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Boltzmann, Mach and Wittgenstein's Vienna

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Abstract. Focusing particularly on the work of Ernst Mach, this paper explores some of the epistemological ground shared between Ludwig Boltzmann and Mach, and offers an account of links between Mach's published work in psychophysical investigations of the relations between physical and physiological phenomena and his unpublished lectures in physics, to indicate that Mach's understanding of the scope of the sciences and philosophy – and his intent to move between inner and outer – was as expansive as Wittgenstein's, and that both aimed to develop a thoroughly consistent approach.

1. False positivisms

This paper represents a contribution to the important task of bringing interdisciplinary approaches to the history and philosophy of physics by thinking about Ludwig Wittgenstein. With a starting point in the history of physics rather than the philosophy of science or the history of philosophy, it will develop a somewhat indirect approach to Wittgenstein shaped strongly by Allan Janik and Stephen Toulmin's enormously suggestive and valuable but also sometimes problematic book *Wittgenstein's Vienna* [1]. Their aim was to distill from amongst the many literary, philosophical and scientific figures of the Viennese environment in which Ludwig Wittgenstein grew up, critical elements that helped shape his thinking particularly strongly. Amongst far more colourful figures like Karl Kraus and Otto Weininger, the Viennese physicists Ludwig Boltzmann and Ernst Mach figured importantly in Janik and Toulmin's account, but they played very different roles. Along with Heinrich Hertz's focus on a particular form of conceptual clarity, Boltzmann helped provide a positive stimulus towards Wittgenstein's philosophical development through his focus on models and atomism – clearly important to the "logical atomism" that Wittgenstein referred to in his *Tractatus Logico-Philosophicus*. On the other hand, Ernst Mach stands as a foil, representing an empiricism that Wittgenstein had to leave behind. This paper will not dispute the positive role that Janik and Toulmin give Boltzmann's work, though one should complement their reading with Andrew Wilson's account of the significance of Boltzmann's philosophy of language for Wittgenstein, as well as John Preston's nuanced reading of the distinctions between Hertz's work and Wittgenstein's project [2]. But it will question Janik and Toulmin's account of Mach, arguing that Mach was a much more complex and wide-ranging anti-metaphysicist than he is usually thought to have been, and showing that the breadth and scope of his work is reminiscent of the uncompromising generality of Wittgenstein's philosophical approach.

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There is a second dimension of Janik and Toulmin's account, which has more to do with the reception of Wittgenstein's thought than its generation. They describe Wittgenstein as being widely misread and misinterpreted in the 1920s and 1930s, a process of misreading that helped stimulate Wittgenstein's return to philosophical reflection in the 1940s (see for example [1], p. 145). It is notable that outside Cambridge, Wittgenstein was taken up most enthusiastically by the circle of logical positivists who met first in the name of the Ernst Mach Verein in Vienna. Logical positivists may indeed have misread Wittgenstein in ascribing to him a form of positivism that neglected the critical last sections of the *Tractatus*. However, the material discussed here will suggest that they read Ernst Mach in a similarly selective way, something which can be described as the problem of false positivisms. Understanding fully the subtle relations between physics and philosophy in the first decades of the twentieth century requires getting a better understanding of the complex legacies of Ernst Mach's thought. A good foundation for this can be found in Erik Banks's subtle account of Mach's philosophy as a form of neutral monism, later developed further by Wittgenstein's mentor, Bertrand Russell [3]. Since 2014 a resurgence of interest in Mach has been prompted in part by the centenary of his death and this has offered many valuable new perspectives [4]. Amongst them, John Preston has contributed a careful accounting of similarities and differences in Mach and Wittgenstein's philosophical perspectives, especially in their account of the limits of logical necessity [5]. In contrast to analysts probing for philosophical connections, the approach in this paper focuses rather more on displaying subject matter common to both men and their concern with the analysis of sensation, representation and interpretation.

2. Boltzmann and Mach in Vienna

As a teenager in Vienna, Wittgenstein read avidly and was notably stirred enough by Ludwig Boltzmann's collection of popular lectures to consider studying with Boltzmann before eventually deciding to travel to Berlin instead to study at the Technische Hochschule. Boltzmann had been pursuing an on-again, off-again relationship with Vienna, spending time in both Munich and Leipzig but in 1902 he returned to the Austrian capital. His inaugural lecture on mechanics is famous for treating himself as his own predecessor and thereby neglecting the customary practice of honoring the immediately previous Chair holder, who might have been regarded as Ernst Mach [6]. Yet the character of this snub has been overplayed, since in the inaugural lecture he gave for his other course on natural philosophy, Boltzmann praised Mach by saying he would do Mach the honour of taking seriously the way that Mach paired strictures against taking any theory as absolutely true with the recognition that few were absolutely false either, and his view that theories were commonly developed in an evolutionary process that winnowed the useful from the inessential. In this manner Boltzmann addressed himself to the further development of Mach's own ideas by noting how much Mach's laconic statement that he did not believe in atoms had stimulated Boltzmann's thought [7]. This is characteristic of the relationship between these two men, one of stimulating, profitable opposition. They differed remarkably in their stances towards mechanics with Boltzmann propagating its further development in atomistic terms through statistical mechanics, while Mach often warned against taking mechanics as a fundamental science and argued that atomism was fruitful only if the hypothetical status of the models developed was remembered. Nevertheless, they shared a great deal in their epistemological approach.

In the same year that Boltzmann published his popular writings, including his inaugural lectures alongside an essay on the kind of heroic genius that would be required to develop aeronautics and thereby helping the young Wittgenstein decide the course of his studies [8], Mach published a book that was based on the lectures he had begun giving in Vienna when he first took up his Chair in the History and Theory of the Inductive Sciences in 1895. In *Knowledge and Error*, Mach drew together the impulses that had previously been represented in the partial perspectives of his books on mechanics, sensation and heat, in an account subtitled *Sketches on the Psychology of Enquiry* [9]. Where his earlier works had largely been framed as critical contributions to their concrete subject matter, this book had a more general purview and more distinctly philosophical cast. Its chapters

encompassed subjects like reflex, will and the ego, but also offered the first systematic study of the nature of thought experiments, as well as reflections on the relations between physiological and metrical space, and psychology and geometry.

If one asks how Boltzmann and Mach understood themselves and were perceived by their contemporaries in this period, one can note that Boltzmann liked to describe himself as classical for his allegiance to mechanics, but unlike Karl Przibram, who sketched him in a toga as 'the natural philosopher', most of his contemporaries were reluctant to accept that description of someone whose advocacy of atomism was so partisan and contested [10]. For those who studied physics around the turn of the century in Vienna, Boltzmann's work was clearly more important technically than Mach's, but Mach was understood to have offered extraordinarily significant and also culturally resonant criticisms, especially of absolutes in the sciences. Students like Paul Ehrenfest or Erwin Schrödinger surely read Mach's well-known books on mechanics and heat but they trained themselves in the statistical techniques Boltzmann had offered with the result that Boltzmann was often interpreted through a conceptual framework modulated by Mach's concerns; a revealing example of how difficult it might be to trace this kind of relationship concerns an article in which Christian Joas and Shaul Katzir emphasise Schrödinger's debt to Boltzmann without noting that his private notes described Boltzmann's work in the language of relative and absolute that Mach had provided [11]. Mach was far more significant outside the physics discipline, having attracted the attention of literary and artistic figures as well as intellectuals both within Vienna and Europe more generally. It is remarkable, however, that Mach proved intensely controversial, attracting stringent critics even as he won enthusiasts.

In the period in which Wittgenstein finished his engineering studies in Berlin, travelled to Manchester and began flying kites, Mach was attacked first by the socialist Vladimir Lenin and then by the physicist Max Planck. These attacks came from radically different perspectives and they indicate the challenge of understanding the protean ways in which Mach's work was understood. Lenin criticised Mach for idealism and solipsism, but only took up this battle because of the importance Mach's work was beginning to assume for socialists in Russia and notably for Alexander Bogdanov. This was a battle for the heart of socialism fought on the grounds of science and philosophy that led Lenin to devote sections to the discussion of space and time in his 1908 book *Materialism and Empirio-criticism* [12]. The second attack on Mach came from articles the physicist Max Planck published in 1909 and 1910 [13]. In sharp contrast to Lenin, Planck regarded Mach as being too narrowly empirical, too closely focused on the grounds for knowledge in sensual perception. Planck's explicit targets were Mach's anti-atomism and his view that the aim of all science was merely the economical description of phenomena. Planck argued that this had never provided a reliable guide to the formation of new knowledge and branded Mach a false prophet. Picking up the religious language that infused what seemed to many a surprisingly vehement attack, Mach replied that if belief in the reality of atoms was so important, he would decline the communion of the faithful in preference for freedom of thought [14]. Although Planck did not mention relativity, and commentators have usually focused only on his explicit targets, one of his unstated aims was probably to separate the success of relativity from the Machian epistemology that might otherwise be regarded as having provided its foundation; Planck celebrated the extent to which relativity offered a new understanding of invariants that were unchanged in coordinate transformations. In other words, contra Mach, Planck was intent on showing that relative space and time could lead to new absolutes.

How could such diverse views emerge around the work of one man? To understand this the breadth of Mach's research and the variety of perspectives he encompassed within his emphasis on sensation and economy need to be examined. After completing his doctorate on electrical discharge and induction in Vienna in 1860, Mach began his career with lectures on physics for medical students. His research traced physical phenomena between motion and the senses, and the mind and the senses, absorbing but also critiquing what Gustav Fechner called psychophysics [15]. Like Hermann von Helmholtz, Mach moved between physiology, physics and psychology, offering scientific perspectives on aesthetics and perception. Several elements of Mach's research in this period informed both his

rather complex philosophical stance towards sensations as the basis of knowledge, and his later critiques of Newton. As will be shown later, some of them were also subsequently picked up by Wittgenstein, sometimes in second or third hand treatments.

Firstly, Mach was intimately concerned with how the relative motion of a source and the observer affected perception of sound or light. It is notable that in these cases observed differences in perception could be resolved by a better understanding of the physical phenomena, as long as the critical role of relative motion was recognised. One does not need to know much about the ear or eye to understand the shift in tone or wavelength that results from the Doppler effect, for example. At this time Mach also pursued a psycho-physical parallelism and so expected to find that some effects that seemed largely psychological would actually be found to have a physical correlate. One is the very different perception one has of particular chord changes if one concentrates on the high or low note when the first chord is sounded, a phenomena called accommodation. As Alex Hui has shown, Mach thought that shifting attention in this way would lead to changes in the ear, making the physiology of the ear critical to understanding what is heard in the different instances [16]. Yet nearly a decade's research failed to disclose such a physiological basis and Hui argues this led Mach to focus on the importance of distinctively historical explanations, coming to believe that significant features of our hearing had to be learned; they depended upon cultural development.

Other phenomena taught something very different. In 1865 Mach discovered what is known as Mach bands [17]. The set-up is illustrated in figure 1. If either disks 1a or 1b are spun rapidly, the physical phenomena might lead one to expect that a varying field of white and greys with two sharp changes would be seen. In addition, however, one sees two bands, bright lines at beta and at gamma that Mach depicted in 1c (the image is designed to mimic for his readers the physiological effect experienced by those observing the spinning disk), or when the gradient between dark and light was reversed in 2a and 2b, the dark lines depicted in 2c. In this case Mach had disclosed an effect that was physiological rather than being either simply physical or dependent on psychological attention or cultural learning, and his explanation was both relational and evolutionary.

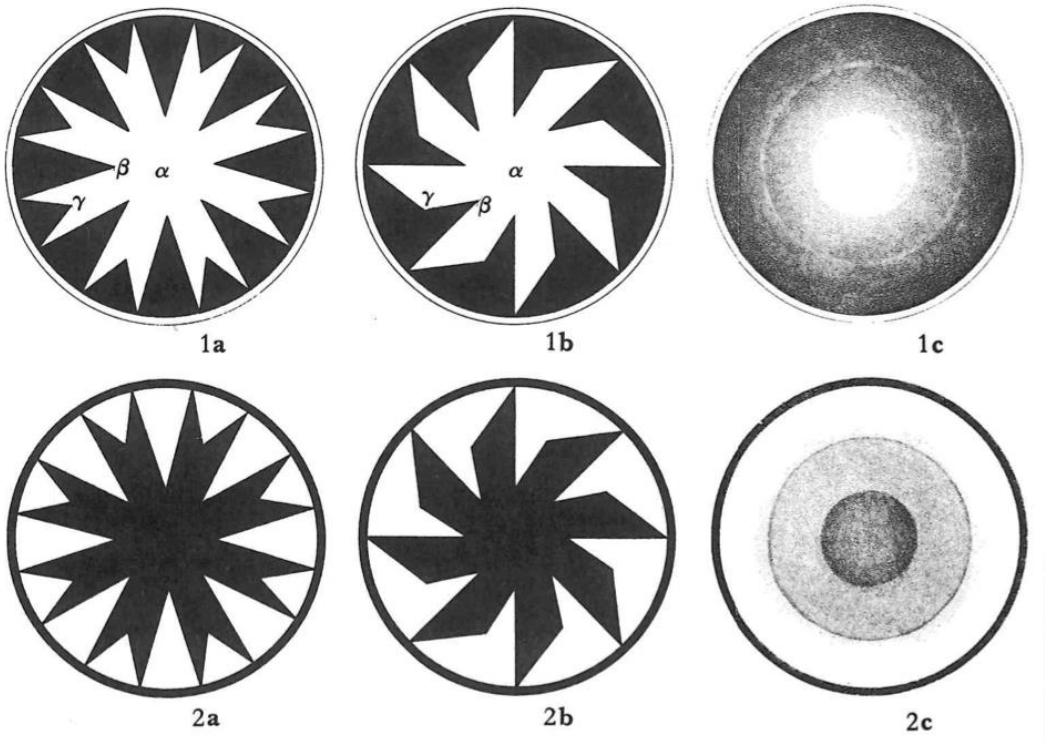

Figure 1. The disks (Mach's figures 1a and b, 2a and b) that in motion produce the physiological effect experienced as Mach bands, as illustrated in his figures 1c and 2c [13]

As Paul Pojman has shown, Mach thought perception resulted from the comparison of stimuli. Those near the mean of the surroundings become effaced, while those above or below are disproportionately brought into prominence as a result of the evolutionary significance of the ability to detect variation and change. If one did not perceive relations but differences in illumination, one and the same object perceived in the same surroundings under diverse light intensities would become unrecognisable [18]. Thus Mach argued that the world is not perceived directly, for that would amount to chaos. *Sensations by themselves can have no organic meaning,* instead we have evolved senses that perceive contrasts, relations of perception. There was a similar point in Mach's exploration of the relations between what he described as physiological and geometrical space, using several rather simple diagrams to examine the variety of ways in which physical phenomena could be perceived in a thorough analysis of the relations between sensation and representation – work he also regarded as preliminary to a full explanation when he published it in 1886 (see figure 2). One example concerns the play of light and shadow on bent a visiting card, which in monocular vision could be inverted to see the bent edge as depressed instead of raised, in Mach's fig. 24; another is the plane quadrilateral or two optically possible tetrahedrons of his fig. 29, while several pages later he repeated the visiting card image in his fig. 31 to illustrate how such changes in perception led to an apparent change in position as well as form [19].

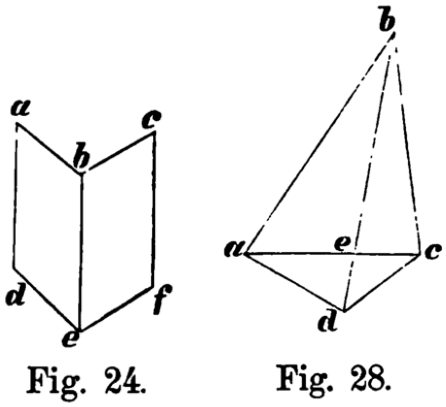

Figure 2. Two of the many illustrations that Mach used to analyse diverse physical, physiological and psychological dimensions of the relations between sensation and representation [15, figs. 24, 28].

Many of these illustrations were subsequently taken up in the 1899 article in which Joseph Jastrow famously concluded his discussion of the mind's eye with an analysis of the image of the rabbit/duck (figure 3), arguing that seeing is very significantly a subjective matter [20]. Wittgenstein later demonstrated similar points in extended passages in *Philosophical Investigations*, writing that his readers could imagine an illustration repeated in a book several times: "In the relevant text something different is in question every time: here a glass cube, there an inverted open box, there a wire frame of that shape, there three boards forming a solid angle" [21]. Mach's work could have provided an example and this helps underline the extent to which, in contrast to Boltzmann and Hertz, Mach and Wittgenstein pursued common subject matter. While Mach's analysis often began with the physiology of perception and then reached to differences in perception, Wittgenstein focused on analysing conceptually the character of diverse interpretations and used his own simple line-drawn version of Jastrow's figure to articulate the distinction between seeing and seeing as.

Summarising this early research it can be seen that Mach encountered such diverse relationships between physiological, psychological and physical phenomena that he hesitated to explain one in terms of the other, while also demonstrating both how contingent these relationships could be and the possibility of varying perception through changes in conditions and attention. Mach thus analysed visual (but also aural and other sensory) phenomena alert to the diverse ways physical, physiological and psychological dimensions conditioned representation; Wittgenstein would later parse very similar phenomena and illustrations philosophically, in order to investigate the extent to which interpretation conditions vision.

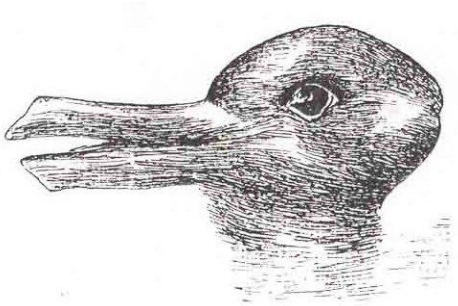

Figure 3. Robert Jastrow's version of the duck-rabbit asked 'Do you see a duck, or a rabbit, or either' [16, fig. 20].

3. Mach's anti-metaphysics

Even when Mach began developing a distinctly anti-metaphysical approach to physics in 1867, a revealing trace of the broader framework to his thought remains, although this has usually escaped the attention of historians of physics and of philosophy. Focusing on the definition of mass, Mach aimed to clarify the distinctions between a priori, empirical and hypothetical elements [22]. He developed a highly abstract philosophical study that is now seen as foundational for the operationalism that Percy Bridgeman helped make famous much later by focusing on length. But it should be noted that Mach argued first that the only feature that can be taken to be a-priori is the law of cause and effect, and specifically commented that whether that depends on a powerful induction or has its ground in our psychic organisation can be left undecided, because in the psychic life also, similar effects follow similar causes. He then offered a ground-breaking definition of mass relating that concept to the procedures by which the mutual acceleration of different bodies could be measured and describing this as an attempt to improve on Newton's formulation by developing a completely scientific treatment of the fundamental laws of mechanics. The rejection of his short paper by the prestigious *Annalen der Physik* reinforced Mach's caution about discussing such ideas with physicist colleagues (it eventually appeared the following year in *Carl's Reportorium*).

But four years later Mach developed this kind of conceptual work further in the notes to a lecture to the Royal Bohemian Society of the Sciences on the root and history of the conservation of work, which was delivered in 1871 and published in 1872 [23]. Manuscripts of his lectures at the University of Prague in the same period offer a still more pointed indication of the intellectual breadth of Mach's approach, first the published notes, then the soul notes.

4. Mass and mechanics

Mach took the opportunity of the notes to his public lecture to reprint his discussion of the definition of mass, but also to set out an argument that Einstein later christened Mach's Principle. The central point Mach made was that Newton's first law of inertia was undefined without specifying the actual bodies in relation to which a given body remained at rest or in uniform motion. This usually meant the laboratory or the fixed stars, and Mach insisted that just because usually another room or reference star can be substituted didn't mean we could abstract to an idea of motion in absolute space, independent of the particular material bodies through which we always actually form our understanding of coordinate systems. He used several examples to make this point concrete and described its significance in a way that shows the interesting creative move he was making between description and causation. Mach wrote: "*Do we think that these bodies, without which one cannot describe the motion imagined, are without influence on this motion? Does not that to which we must appeal explicitly or implicitly when one describes a phenomenon belong to the most essential conditions, to the causal nexus of the phenomenon?* [23 p 49]"

When Kirchhoff's 1876 textbook on mechanics described the task of physics as descriptive [24], Mach was to welcome this wholeheartedly, but it can be seen that this was not an argument against the recognition of functional relations. Both Mach and Boltzmann were to join in this advocacy of what John Heilbron has described as descriptionism, though they did so with somewhat different emphases [25].

Already in the manuscript for the lectures he gave *On the Principal Questions of Physics* in the summer semester of 1872, it became obvious that Mach had not stopped at Newton's First Law, but had begun developing more extensive approaches to the fundamental laws of mechanics [26], something he was only to publish in his textbook on mechanics in 1883. There Mach noted that Newton's First and Second Laws were contained in the definitions of force that preceded them, since without force there was no acceleration and consequently only rest or uniform motion in a straight line. Furthermore, he wrote, it was an unnecessary tautology, after establishing acceleration as the measure of force, to say again that change of motion is proportional to the force: "*It would have been enough to say that the definitions premised were not arbitrary mathematical ones, but correspond to properties of bodies experimentally given.*" [27]. Mach's own reorganisation of Newtonian mechanics would begin with the empirical proposition that bodies set opposite one another induce under conditions specified by experimental physics contrary accelerations in the direction of their line of junction, and Mach noted that the law of inertia was contained within this statement. Mach's work to clarify Newton is surely one of the most direct inspirations for Hertz's posthumously published endeavours along related lines [28]. Hertz took precisely the other side of the same relationship and like Mach he avoided force when he took as his fundamental law the empirical finding that every free system persists in its state of rest or of uniform motion in a straightest path. It is no wonder that Mach noted how close their approach was, despite the fact that Hertz began from Kantian presuppositions that he could not share [29], and indeed Janik (returning to the arguments he had first developed with Toulmin) notes the similarity between many of Hertz's and Mach's positions [30], while Preston highlights the distinctions between Hertz's stated aims, the approach pursued in the body of his text, and Wittgenstein's later work, and makes a similar point about the different senses in which Mach and Wittgenstein approached logical and empirical relations [2, 5].

So the first major point is that in the careful parsing of the empirical and logical relations between the fundamental definitions and laws of the physical sciences one can see grounds for thinking there were common pursuits that might have been reflected first in the relations between Mach, Hertz and Wittgenstein, and then between Wittgenstein and the logical positivists who read the *Tractatus* so enthusiastically when it was published in 1922.

But this paper will now go further to suggest that there may also be some relations in uncommon pursuits, and that a similar concern with the self and the inner world in Ernst Mach and Ludwig Wittgenstein might be recognised. While they may have pursued this in different ways, these dimensions of their thought are likely to have been equally troubling for many of their philosophically inclined readers. This paper cannot be very sure about this, but it does seem important to recognise that each aimed at thoroughly general insights, and to suggest that Mach might not have been as worried about the unsayable as good empiricists are taken to be.

To appreciate this more about his attitude towards the psyche and towards politics needs to be learnt. The 1872 lectures described were part of a course Mach gave each year on the principal questions of physics [26]. They finished with the fundamental laws of mechanics, but began with psychophysics and an argument that sensation was a general property of matter. If it was not regarded as an emergent property of groups of molecules, then Mach noted sensation had to be considered part of the elements themselves. He then asked where the soul was located. Experiments with animals showed it could not be located in the brain: the soul is not so simple, he said, exploring an analogy with the state, which looks like a person. Mach asked his students to consider Bismarck's soul before moving between the soul of the professions, individuals and back to the state. Five years after the Prussian defeat of Austria and shortly after Bismarck had been made Imperial Chancellor of a newly unified Germany, Mach's questions made an abstract point about parts and wholes with direct political meaning without at any point stating his own political stance. Yet while Mach was careful with his students, he had been thoroughly plain in the public lecture mentioned previously [23]. Remember that he had talked on the conservation of work before the Royal Bohemian Society of Science. Mach began by telling his audience that as a child there were two things he had pestered his mother about: why the rulership of a king should be suffered for even a minute and why there should be such inequality

between rich and poor. He had learned that one could either become so accustomed to these circumstances that one forgot to think about them or one could learn to understand them at the hands of history and thereby be able to approach them without hatred. He went on to note something that applied equally to his stance towards physics and politics: "*We should not let go of the guiding hand of history. History has made everything, history can change everything*" [23 pp 1-4 on 3-4; pp 15-18 on 18]. Mach was extremely thorough in his insistence on the nature of abstraction and the centrality of relations across psyche and matter, but also in politics.

In this particular instance Mach's underlying philosophical point was that one attributes souls to others by adding them in thought, by analogy with our direct experience of our own outer and inner sides. Scientific concepts were no different. When he asked his assistant Hajek to bring a prism into the room, one could understand Hajek as an automaton, but that was much more difficult than ascribing him a soul; just as potential theory or Ampere's rules had a practical economical value. Mach's course was dedicated to overcoming the cleft between the physical and the psychic. He wrapped up his introductory lecture with striking methodological advice, which was bold in its epistemological and ontological parity and deliberately philosophical, humdrum and poetic at once. He told his students: "*So in fact we can hope to come to a better understanding of the world if we measure ourselves with the standards of the outer world and the outer world with our own standards, considering it as physical process, but attributing sensation to matter*." He finished the lecture by quoting Friedrich Schiller's lines, "*Do you want to know yourself? Observe how the others act. Do you want to understand others? Look into your own heart*" [26].

One has not typically seen all these different aspects of Ernst Mach's thought, firstly because we didn't get the chance to sit in his classes, and secondly because he addressed physiologists and physicists and those interested in sensation in a series of separate books, in particular following his 1883 *Science of Mechanics* with *Contributions to the Analysis of Sensations* in 1886 [27, 19], but also because many of his readers could not stomach the span he insisted upon, moving ontologically from soul to state to matter, taking the methodological measure of both inner and outer worlds, and exposing current physics to the challenges of historical and philosophical examination. It appears that Mach began to note some of the losses that his separate publications had occasioned when he met Planck's challenge and it is interesting to note that he immediately republished his 1871 public lecture, thereby insisting just how longstanding his basic epistemology was and perhaps reminding his audience that his relational physics was in fact closely linked to an anti-absolutist politics [31].

And Wittgenstein? When Wittgenstein left Vienna and began his philosophical journey, as Boltzmann had suggested he flew heroic kites, and as Mach might have counselled, he undertook experiments in psychology [1]. From 1913 his interlocutor between Gottlob Frege and Bertrand Russell was Philip Jourdain, the London editor of *The Monist*, and the person who translated Mach's public lecture into English in 1911. It is known that when Russell told Wittgenstein he was reading *Contributions to the Analysis of the Sensations* in 1913, Wittgenstein described Mach's manner of expression as horrible [5 p 68], but it may also be fair to say that Mach and Wittgenstein shared three significant features. Each pursued an analysis of sensations, representation and interpretation to extraordinary effect; and both were unusual in taking the scope of the sciences and of philosophy to be equally wide and in trying to be thoroughly consistent.

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