



# Higher symmetries of the Schrödinger operator in Newton–Cartan geometry



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## ABSTRACT

We establish several relationships between the non-relativistic conformal symmetries of Newton–Cartan geometry and the Schrödinger equation. In particular we discuss the algebra  $\mathfrak{sch}(d)$  of vector fields conformally-preserving a flat Newton–Cartan spacetime, and we prove that its curved generalisation generates the symmetry group of the covariant Schrödinger equation coupled to a Newtonian potential and generalised Coriolis force. We provide intrinsic Newton–Cartan definitions of Killing tensors and conformal Schrödinger–Killing tensors, and we discuss their respective links to conserved quantities and to the higher symmetries of the Schrödinger equation. Finally we consider the role of conformal symmetries in Newtonian twistor theory, where the infinite-dimensional algebra of holomorphic vector fields on twistor space corresponds to the symmetry algebra  $\mathfrak{nc}(3)$  on the Newton–Cartan spacetime.

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## 1. Introduction and theorems

One of the chief concerns of mathematical physics is to put the ubiquitous symmetries of nature on a geometrical footing. On a Riemannian manifold  $(M, g)$ , the mathematical setting for relativistic physics, the central objects in the study of symmetries are Killing vectors and Killing tensors. The former captures our intuition of what is meant by a continuous symmetry: if some continuous transformation leaves a system unchanged then that transformation is a symmetry of the system. Geometrically this transformation is implemented by pushing-forward the metric  $g$  along the integral curves of a vector field, i.e.  $X$  is a *Killing vector* iff

$$(\mathcal{L}_X g)_{a_1 a_2} = 0, \quad (1)$$

where  $\mathcal{L}_X$  is the Lie derivative along  $X$ . The intuitive concepts of translational and rotational invariance, for instance, find geometrical guises as Killing vectors. In accordance with Noether's theorem, Killing vectors usefully correspond to conserved quantities of the motion.

More exotic are so-called “hidden” symmetries. Unlike with Killing vectors there is no intuitive picture of the metric  $g$  being pushed-forward along integral curves on  $M$ ; instead we must extend our viewpoint to include the cotangent bundle  $T^*M$ . A hidden symmetry is a rank- $n$  symmetric contravariant tensor field  $X$  which preserves the metric in the sense that

$$\{X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n}, g^{b_1 b_2} p_{b_1} p_{b_2}\} = 0, \quad (2)$$

where  $(x^a, p_b) \in T^*M$  and  $\{, \}$  is the canonical Poisson structure on the cotangent bundle. Whilst the *Killing tensor*  $X$  does not generate transformations via integral curves on  $M$ , Eq. (2) exploits the fact that the complete lift

$$\hat{X} = nX^{a_1 \dots a_{n-1} b} p_{a_1} \dots p_{a_{n-1}} \frac{\partial}{\partial x^b} - \frac{\partial X^{a_1 \dots a_n}}{\partial x^b} p_{a_1} \dots p_{a_n} \frac{\partial}{\partial p_b}$$

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of  $X$  does generate transformations on  $T^*M$  [1]. The associated conserved quantity is the Hamiltonian  $X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n}$ , and the existence of such conserved quantities is often essential to the integrability of geodesic motion. For instance, the Kerr metric describing a spinning black hole admits a rank-two Killing tensor, and the associated conserved quantity “Carter’s constant” allows one to determine the orbits (see e.g. [2]). Eq. (2) can be concisely restated in the language of the Schouten bracket as  $\mathcal{L}_X g = 0$ .

Also of interest are the conformal cousins of Killing vectors and Killing tensors, arising when we relax Eqs. (1)–(2) and allow  $X$  to generate conformal transformations of the metric. The defining condition on the vector  $X$  to be a *conformal Killing vector* or the tensor  $X$  to be a *conformal Killing tensor* then becomes

$$\mathcal{L}_X g^{a_1 \dots a_{n+1}} = k^{(a_1 \dots a_{n-1}} g^{a_n a_{n+1})}$$

where respectively  $\mathcal{L}$  refers to the Lie bracket or the Schouten bracket, and  $k$  is a tensor to be determined. The associated “conserved quantities” are conserved only necessarily along null geodesics. Conformal Killing tensors have appeared in many places in mathematical physics; for example, as will be relevant in this paper, conformal Killing tensors generate the higher symmetry operators of the Laplacian [3].

Thus far we have only mentioned Riemannian geometry; the chief aim of this paper is to build upon the work of [4–6] in extending the above ideas to the field of Newton–Cartan geometry. In particular, a contribution of this paper will be to establish some intrinsic definitions of Killing tensors and “conformal” Schrödinger–Killing tensors using a Newton–Cartan Hamiltonian formalism and prove two theorems relating the resulting tensors to the symmetries of the Schrödinger equation. This Hamiltonian formalism will be in agreement with the Eisenhart–Duval lift to a Bargmann structure, and the Killing tensors defined will be in accord with the well-known study of hidden symmetries in non-relativistic physics, but will be written in the Newton–Cartan language.

Newton–Cartan geometry is what results when one takes a non-relativistic limit of general relativity and is the mathematical formalism behind non-relativistic gravitation [7]. This kind of geometry, mathematically remarkable because unlike in Riemannian geometry the connection is non-metric, is of interest to condensed matter physicists whose theories are non-relativistic. Furthermore Newton–Cartan geometry has attracted recent attention in attempts to establish a non-relativistic version of the AdS/CFT correspondence [5,8,9]. A self-contained introduction to Newton–Cartan spacetimes will be provided in section two.

A rank- $n$  symmetry  $\mathcal{D}$  of a linear differential operator  $\Delta$  is a linear differential operator of order  $n$  which obeys

$$\Delta \mathcal{D} = \delta \Delta \tag{3}$$

for some (otherwise irrelevant) linear differential operator  $\delta$ . We will be concerned with relating the geometrical non-relativistic symmetries (such as Schrödinger–Killing tensors) to symmetries in the sense of (3).

In section three we will discuss the well-known Schrödinger algebra spanned by vectors which are a non-relativistic analogue of the conformal Killing vectors of flat spacetime. We will take the definition of these *Schrödinger–Killing* vectors in a general Newton–Cartan spacetime and proceed to prove the following theorem.

**Theorem 1.** *The first-order symmetries of the Schrödinger equation*

$$\hat{\Delta} \psi := i \partial_t \psi - \frac{1}{2m} \delta^{jk} (-i \partial_j + m A_j) (-i \partial_k + m A_k) \psi - m V \psi = 0 \tag{4}$$

have the Schrödinger–Killing vectors of the Newton–Cartan spacetime with Galilean coordinates  $(t, x^i)$  and non-vanishing connection components

$$\Gamma_{tt}^i = \delta^{ij} \partial_j V \quad \text{and} \quad \Gamma_{jt}^i = \Gamma_{tj}^i = \delta_{jl} \epsilon^{ilk} \partial_k \Omega$$

as their principal symbols, where  $\Omega(x^j)$  is a function satisfying  $d\Omega = \star^3 dA$ .

The operator  $\hat{\Delta}$  is the covariant Schrödinger equation exhibited in [10]. One can view **Theorem 1** as the statement of a duality between the geometrical properties of a curved spacetime and the symmetries of an ordinary Schrödinger equation coupled to potentials.

The non-relativistic analogues of (“conformal”) Killing tensors will then be discussed in section four, where we will introduce a Newton–Cartan Hamiltonian formalism in agreement with other approaches involving Bargmann lifts (see e.g. [11,6]). The Hamiltonian formalism will then allow us to give natural intrinsic definitions of Killing tensors and Schrödinger–Killing tensors for Newton–Cartan geometry, where the latter are a non-relativistic analogue of conformal Killing tensors. Much like in the Riemannian setting, the Newton–Cartan Killing tensors correspond to conserved quantities.

We will then proceed to prove the following theorem, a non-relativistic analogue of Eastwood’s identification of the higher symmetries of the Laplacian as conformal Killing tensors [3].

**Theorem 2.** *The higher symmetries of the free Schrödinger equation*

$$i \partial_t \psi = - \frac{1}{2m} \delta^{ij} \partial_i \partial_j \psi$$

are linear differential operators which have the Schrödinger-Killing tensors of the flat Galilean Newton–Cartan spacetime

$$h = \delta^{ij} \partial_i \partial_j \quad \theta = dt \quad \Gamma_{bc}^a = 0$$

as their principal symbols.

The higher symmetries of the free Schrödinger equation are well known [12]; the novel element here is the correspondence with a special kind of tensor in Newton–Cartan geometry.

In Section 5 we will discuss the links between non-relativistic symmetries and the Newtonian twistor theory introduced in [13], where Newton–Cartan geometry in  $(3 + 1)$  dimensions is constructed on the moduli space of a family of rational curves in a complex manifold  $PT_\infty = \mathcal{O} \oplus \mathcal{O}(2)$ , the total space of a rank-two holomorphic vector bundle over  $\mathbb{C}P^1$ . In particular we will prove the following theorem, giving a twistorial answer to the question of what is the non-relativistic analogue of a conformal Killing vector.

**Theorem 3.** *The global holomorphic sections of  $T(PT_\infty)$  are in one-to-one correspondence with elements of  $\mathfrak{cnc}(3)$ , a Lie algebra of vector fields preserving Newton–Cartan geometry with  $h = \delta^{ij} \partial_i \partial_j$  and  $\theta = dt$  on  $M$ .*

Both  $\mathfrak{cnc}(3)$  and  $H^0(PT_\infty, T(PT_\infty))$  are infinite-dimensional Lie algebras, and the former was introduced in [4]. The significance of this result comes from its relativistic counterpart, where the global sections of the twistor space’s tangent bundle are in one-to-one correspondence with the conformal Killing vectors of the spacetime [14]. We will then proceed to discuss two subalgebras of  $H^0(PT_\infty, T(PT_\infty))$ , the expanded Schrödinger algebra and the CGA [15].

## 2. Newton–Cartan geometry

Newton–Cartan spacetimes are the non-relativistic analogues of Lorentzian manifolds in general relativity: they are the geometrical setting for non-relativistic physics [7]. Just like in general relativity we have a four-dimensional manifold playing the role of the spacetime, and particles travel on geodesics of a torsion-free connection. There is a metric too, though unlike in general relativity the connection and the metric are independent quantities.

**Definition.** A Newton–Cartan spacetime (NC) is a quadruplet  $(M, h, \theta, \nabla)$  where

- $M$  is a  $(d + 1)$ -dimensional manifold;
- $h$  is a symmetric tensor field of valence  $\binom{2}{0}$  with signature  $(0 + \dots +)$  (so rank  $d$ ) called the *metric*;
- $\theta$  is a closed one-form spanning the kernel of  $h$  called the *clock*;
- and  $\nabla$  is a torsion-free connection satisfying  $\nabla h = 0$  and  $\nabla \theta = 0$ .

We emphasise that  $\nabla$  must be specified independently of the metric and clock. Since  $\theta$  is closed we can always locally write  $\theta = dt$  for some function  $t : M \rightarrow \mathbb{R}$ . This function is then taken as a coordinate on the time axis, a one-dimensional submanifold over which the NC is fibred. We call the fibres *spatial slices* and when restricted to such a slice the metric  $h$  is a more familiar signature  $(+\dots+)$   $d$ -metric. Throughout this paper the indices  $a, b, c$  will run from 0 to  $d$ .

The field equations for NC gravity arise as the Newtonian limit of the Einstein equations [16]. They are

$$R_{ab} = 4\pi G \rho \theta_a \theta_b$$

where  $R_{ab}$  is the Ricci tensor associated to  $\nabla$ ;  $G$  is Newton’s constant; and  $\rho : M \rightarrow \mathbb{R}$  is the mass density. Alongside the field equations we have the Trautman condition [4]

$$h^{a[b} R_{(de)a}^{c]} = 0 \tag{5}$$

for  $R_{bcd}^a$  the Riemann tensor of  $\nabla$ . This ensures that there always exist potentials for the connection components, which is needed if we are to make contact with Newtonian physics; accordingly connections which satisfy (5) are referred to as *Newtonian* connections.

The field equations imply that  $h$  is flat on spatial slices, so we can always introduce *Galilean* coordinates  $(t, x^i)$  such that

$$h = \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \quad \text{and} \quad \theta = dt$$

for  $i = 1, 2, \dots, d$ . For notational convenience we can then raise and lower purely spatial indices with  $\delta^{ij}$  and  $\delta_{ij}$ .

Only connections compatible with  $\theta$  and  $h$  are allowed; one can show [17] that the most general such connection has components

$$\Gamma_{bc}^a = \frac{1}{2} h^{ad} (\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}) + \partial_{(b} \theta_{c)} U^a + \theta_{(b} F_{c)d} h^{ad}$$

where

- $U^a$  is any vector field satisfying  $\theta(U) = 1$ ;
- $F_{ab}$  is any two-form;
- and  $h_{ab}$  is uniquely determined by  $h^{ab} h_{bc} = \delta_c^a - \theta_c U^a$  and  $h_{ab} U^b = 0$ .

The possible connections are then parametrised by a choice of  $(U, F)$ . The Trautman condition (5) is equivalent to the statement that  $F$  is closed, and hence for a Newtonian connection we can write  $F = dA$ . Thus we will in future refer to a Newton–Cartan spacetime as a quintuplet  $(M, h, \theta, U, A)$ , implicitly considering a Newtonian connection. Clearly there is a gauge symmetry in  $A$ , as we can always shift

$$A \mapsto A + d\chi$$

for any function  $\chi$  on  $M$ .

There is a further redundancy in this description; there exist *Milne boosts* which can be thought of as gauge transformations of  $(U, F)$  which leave  $\Gamma_{bc}^a$  unchanged [4]. Usually we will gauge-fix to  $U = \partial_t$ , which can be implemented for any initial choice of  $(U, F)$ .

In  $d = 3$  the most general vacuum Newton–Cartan spacetime satisfying (5) then has

$$\Gamma_{tt}^i = \delta^{ij} \partial_j V \quad \text{and} \quad \Gamma_{jt}^i = \Gamma_{ij}^i = \delta_{jl} \epsilon^{ilk} \partial_k \Omega$$

$$\text{where } \delta^{ij} \partial_i \partial_j V + 2\delta^{ij} \partial_i \Omega \partial_j \Omega = 0 \quad \text{and} \quad \delta^{ij} \partial_i \partial_j \Omega = 0,$$

with all other connection components vanishing. For reference, the corresponding two-form  $F$  is given by

$$F = -dV \wedge dt + \epsilon_{ijk} \delta^{kl} \partial_l \Omega \, dx^i \wedge dx^j.$$

The geodesic equations suggest that we should interpret the function  $V$  as the Newtonian (gravitational) potential and the function  $\Omega$  as a potential for generalised (spatially-varying) Coriolis forces.

### 3. First-order symmetries and Killing vectors

#### 3.1. Review: the Schrödinger algebra

A menagerie of symmetry algebras relevant in Newton–Cartan geometry is discussed in [4]; we will here provide a brief review of those relevant to this work, following that paper.

**Definition.** The *expanded Schrödinger algebra*  $\widetilde{\mathfrak{sch}}(d)$  (for the flat case) is the Lie algebra of vector fields which conformally preserve the metric and clock in the sense that

$$\mathcal{L}_X h^{ab} = fh^{ab} \tag{6}$$

$$\mathcal{L}_X \theta_a = g\theta_a \tag{7}$$

and effect projective transformations of the connection such that

$$\mathcal{L}_X \Gamma_{bc}^a = \delta_{(b}^a \phi_{c)}, \tag{8}$$

with functions  $(f, g)$  and a one-form  $\phi_a$  constrained by

$$\mathcal{L}_X \nabla h = 0 \quad \text{and} \quad \mathcal{L}_X \nabla \theta = 0, \tag{9}$$

and where

$$h = \delta^{ij} \partial_i \partial_j \quad \theta = dt \quad \Gamma_{bc}^a = 0 \tag{10}$$

is the flat Newton–Cartan spacetime.

Condition (8) ensures that the unparametrised geodesics are unaltered by the transformation, whilst (6) and (7) are the non-relativistic analogue of the conformal Killing equations. As an aside we note that the infinite-dimensional algebra of vector fields obeying only (6)–(7) is known as  $\mathfrak{cgal}(d)$ , the *conformal Galilean algebra*.

We can solve the system (6)–(9) for the NC (10) and find that  $X \in \widetilde{\mathfrak{sch}}(d)$  iff

$$X = (\alpha t^2 + \beta t + \gamma) \partial_t + (\omega_j^i x^j + \alpha t x^i + \mu x^i + v^i t + \rho^i) \partial_i \tag{11}$$

for  $(\alpha, \beta, \gamma, \mu, v^i, \rho^i) \in \mathbb{R}^{4+2d}$  and  $\delta_{ik} \omega_j^i = \omega_{kj} \in \mathfrak{so}(d)$ . The dimension of  $\widetilde{\mathfrak{sch}}(d)$  is therefore  $\frac{1}{2}(d^2 + 3d + 8)$ .

**Definition.** The *Schrödinger algebra*  $\mathfrak{sch}(d)$  is the Lie subalgebra of  $\widetilde{\mathfrak{sch}}(d)$  defined by the additional condition  $f + g = 0$ .

This amounts to setting  $\beta = 2\mu$  in (11); we thus have that  $X \in \mathfrak{sch}(d)$  iff

$$X = (\alpha t^2 + 2\mu t + \gamma) \partial_t + (\omega_j^i x^j + \alpha t x^i + \mu x^i + v^i t + \rho^i) \partial_i. \tag{12}$$

Physically, this algebra contains translations  $(\gamma, \rho^i)$ , spatial rotations  $(\omega_j^i)$ , boosts  $(v^i)$ , a special-conformal transformation  $(\alpha)$ , and a dilation  $(\mu)$ . The dimension is now  $\frac{1}{2}(d^2 + 3d + 6)$ .

This algebra is named “Schrödinger” because of its well-known link (see e.g. [5]) to the *free-particle* Schrödinger equation: a first-order linear differential operator  $\mathcal{D} = S^a(x)\partial_a + s(x)$  commutes with  $\Delta = i\partial_t + \frac{1}{2m}\delta^{ij}\partial_i\partial_j$  in the sense that

$$\Delta\mathcal{D} = \delta\Delta$$

for some linear differential operator  $\delta$  iff  $S^a\partial_a \in \mathfrak{sch}(d)$ .

In the remainder of this section we will generalise this statement, proving [Theorem 1](#).

### 3.2. Schrödinger–Killing vectors on curved spacetimes

Eqs. (6)–(9) defining the expanded Schrödinger algebra  $\widetilde{\mathfrak{sch}}(d)$  make sense for a curved Newton–Cartan spacetime as well as a flat one: we simply use

$$h = \delta^{ij}\partial_i\partial_j \quad \theta = dt \quad \Gamma_{tt}^i = \delta^{ij}\partial_j V \quad \Gamma_{jt}^i = \Gamma_{ij}^i = \delta_{ji}\epsilon^{ilk}\partial_k\Omega \tag{13}$$

with all other connection components vanishing instead of (10). In the following definition we will bypass the *expanded* version of these vectors and impose the  $f + g = 0$  constraint from the beginning.

**Definition.** A *Schrödinger–Killing vector* of a curved Newton–Cartan spacetime (13) is a vector field  $X$  obeying (6)–(9) and  $f + g = 0$ .

In order to prove [Theorem 1](#) it will be useful to write out in more detail Eqs. (6)–(9) on (13). Thus we collect for reference

$$\partial_i X^t = 0 \tag{14}$$

$$\partial^i X^j + \partial^j X^i = \partial_t X^t \delta^{ij} \tag{15}$$

$$\partial_t \partial_t X^i + X^j \partial_j \partial^i V + 2\partial^i V \partial_t X^t + 2\epsilon_{jk}^i \partial^k \Omega \partial_t X^j - \partial^j V \partial_j X^i = 0 \tag{16}$$

$$\epsilon_{ijk} \partial^j \partial_t X^i + 2X^j \partial_j \partial_k \Omega + 2\partial_k \Omega \partial_t X^t + \partial_k X^j \partial_j \Omega - \partial^j X_k \partial_j \Omega = 0. \tag{17}$$

(Recall that spatial indices are raised and lowered throughout with Kronecker deltas.)

**Example.** Take the (3 + 1)-dimensional Newton–Cartan spacetime with the linear Newtonian potential  $V = z$ , adopting  $x^i = (x, y, z)$ . The Riemann tensor vanishes, so we expect the symmetry group to be of maximal dimension. Solving (14)–(17) yields

$$X = (\alpha t^2 + 2\mu t + \gamma) \partial_t + (\omega_j^i x^j + \alpha t x^i + \mu x^i + v^i t + \rho^i) \partial_i + \frac{1}{2}\omega^{xz} t^2 \partial_x + \frac{1}{2}\omega^{yz} t^2 \partial_y - \left(\frac{2}{3}\alpha t^3 + 2\mu t^2\right) \partial_z.$$

We thus indeed find a twelve-dimensional algebra, though the vectors come with some additional terms which result from the strange choice of coordinates.

**Example.** The Schrödinger–Killing vectors of the (3 + 1)-dimensional Newton–Cartan spacetime with  $V = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$  and  $\Omega = 0$  are

$$X = \gamma \partial_t + \omega_j^i x^j \partial_i$$

for  $\gamma$  a constant and  $\omega_{jk} \in \mathfrak{so}(3)$ . The presence of a point mass at the origin has reduced the symmetry algebra to just time translations and spatial rotations.

### 3.3. Symmetries of the covariant Schrödinger operator

In this subsection we will consider the first-order symmetries of the operator

$$\hat{\Delta} = i\partial_t - \frac{1}{2m}\delta^{jk}(-i\partial_j + mA_j)(-i\partial_k + mA_k) - mV,$$

where  $V$  and  $A_i$  depend on space only. That is to say, we will seek first-order linear differential operators

$$\mathcal{D} = S^a(x^b)\partial_a + s(x^b)$$

which obey

$$\hat{\Delta}\mathcal{D} = \delta\hat{\Delta} \tag{18}$$

for some (otherwise irrelevant) linear differential operator  $\delta$ .

**Proof of Theorem 1.** If we calculate the left-hand-side of (18) then we get  $\mathcal{D}\hat{\Delta}$ , which is already in the right form, and some additional operator terms. These additional terms arrange themselves into  $\hat{\Delta}$  iff

$$\partial_t S^t = 0 \tag{19}$$

$$\partial^i S^j + \partial^j S^i = \delta^{ij} \partial_t S^t \tag{20}$$

$$-im\partial_t S^i - imA^i \partial_j S^j + imS^j \partial_j A^i + imA^i \partial_t S^t = \partial^i s \tag{21}$$

$$\frac{i}{2m} \partial_i \partial^i s - A^i \partial_i s - (iS^j \partial_j + i\partial_t S^t) \left( \frac{i}{2} \partial_i A^i - \frac{m}{2} A^i A_i - mV \right) = \partial_t s. \tag{22}$$

In order to prove Theorem 1 we must find the conditions on  $S^a$  such that one can always find  $s$  solving these equations. To that end we use (21) to rewrite  $\frac{i}{2m} \partial_i \partial^i s - A^i \partial_i s$  in (22) in terms of  $S^a$  only. Then (21)–(22) have the form

$$\Sigma = ds$$

for  $\Sigma_a = \Sigma_a(S^b, A^i, V)$ . By the Poincaré lemma the conditions we are looking for are

$$d\Sigma = 0. \tag{23}$$

Explicit calculation reveals that (19), (20) and (23) are then exactly Eqs. (14)–(17) defining Schrödinger-Killing vectors with  $d\Omega = \star^3 dA$ , completing the proof of Theorem 1.  $\square$

Note that the gauge symmetry

$$A_i \longmapsto A_i + \partial_i \chi$$

has not here been fixed. The Schrödinger-Killing vectors of the curved NC spacetime are the symmetries of the whole gauge equivalence class of operators  $\hat{\Delta}$ .

#### 4. Higher symmetries and killing tensors

##### 4.1. Non-relativistic killing tensors and conserved quantities

In this subsection we will define the non-relativistic analogues of Killing tensors by exhibiting Newton–Cartan geodesics as the projection of the integral curves of a Hamiltonian vector field on the cotangent bundle. This Hamiltonian formalism is an intrinsic Newton–Cartan analogue of the Eisenhart–Duval lift discussed in, say, [6].

**Lemma 1.** *Geodesics of the Newton–Cartan spacetime  $(M, h, \theta, F)$  with connection components*

$$\Gamma_{bc}^a = \frac{1}{2} h^{ad} (\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}) + \partial_{(b} \theta_{c)} U^a + \theta_{(b} F_{c)d} h^{ad} \tag{24}$$

and with  $F = dA$  are the projection from  $T^*M$  to  $M$  of the integral curves of the geodesic spray

$$\mathfrak{g} = \left( \frac{1}{2} \partial_a h^{cd} \Pi_c \Pi_d + h^{cd} \Pi_c \partial_a A_d - \partial_a U^b \Pi_b - U^b \partial_a A_b \right) \frac{\partial}{\partial p_a} + (U^a - h^{ab} \Pi_b) \frac{\partial}{\partial x^a}$$

(where  $\Pi_a := p_a + A_a$  and  $(x^a, p_b) \in T^*M$ ), which is the Hamiltonian vector field associated to

$$\mathcal{H} = \frac{1}{2} h^{ab} \Pi_a \Pi_b - U^a \Pi_a.$$

The proof of this lemma is straightforward (but tedious); we omit it for brevity.

This Hamiltonian (and therefore also the following definitions) is Milne-boost invariant.

**Definition.** A rank- $n$  Killing tensor of a Newton–Cartan spacetime  $(M, h, \theta, U, F)$  is a symmetric contravariant tensor field  $X^{a_1 \dots a_n}$  such that functions  $\chi_m^{a_1 \dots a_m}$  on  $M$  can be found obeying

$$\left\{ X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} + \sum_{m=0}^{n-1} \chi_m^{a_1 \dots a_m} p_{a_1} \dots p_{a_m}, \mathcal{H} \right\} = 0, \tag{25}$$

where  $\{, \}$  is the canonical Poisson structure on  $T^*M$ . The quantity

$$X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} + \sum_{m=0}^{n-1} \chi_m^{a_1 \dots a_m} p_{a_1} \dots p_{a_m}$$

is constant along geodesics.

Here we have provided an intrinsic Newton–Cartan definition of the usual concept of a hidden symmetry, entirely in line with the familiar concept from classical dynamics.

Taking  $n = 1$  in (25) we arrive at the conditions

$$\mathcal{L}_X h = 0 \tag{26}$$

$$\mathcal{L}_X U - h(\mathcal{L}_X A, \cdot) = -h(d\chi_0, \cdot) \tag{27}$$

$$(\mathcal{L}_X A)(U) = d\chi_0(U). \tag{28}$$

Solving (26)–(28) on a given Newton–Cartan spacetime will give us the Killing vectors of that spacetime.

**Example.**  $X = X^a \partial_a$  solves (26)–(28) with

$$h = \delta^{ij} \partial_i \partial_j \quad \theta = dt \quad U = \partial_t \quad A = 0 \tag{29}$$

and is thus a non-relativistic Killing vector of the flat Newton–Cartan spacetime iff

$$X = \gamma \partial_t + (\omega_j^i x^j + v^i t + \rho^i) \partial_i$$

for any ten constants  $(\gamma, v^i, \rho^i, \omega_{ij} \in \mathfrak{so}(3))$ . Such vectors generate the Galilean group.

**Example.** The  $(3 + 1)$ -dimensional Newton–Cartan spacetime

$$h = \delta^{ij} \partial_i \partial_j \quad \theta = dt \quad U = \partial_t \quad A = -(\delta_{lk} x^l x^k)^{-\frac{1}{2}} dt$$

modelling the Kepler problem (where  $b$  is a constant) admits the following three rank-two non-relativistic Killing tensors

$$X^{ij} = \lambda^l x^k \delta_{lk} \delta^{ij} - \lambda^i x^j \quad X^{it} = X^{tt} = 0$$

(for  $\lambda^i \in \mathbb{R}^3$ ). The lower order terms are

$$\chi_1^a = 0 \quad \text{and} \quad \chi_0 = \frac{\lambda^i \delta_{ij} x^j}{(\delta_{lk} x^l x^k)^{\frac{1}{2}}},$$

and the three associated conserved quantities together form the famous Laplace–Runge–Lenz vector (see e.g. [2]).

#### 4.2. Schrödinger–Killing Tensors

In generalising the Schrödinger algebra  $\mathfrak{sch}(d)$  to the case of Schrödinger–Killing tensors we will again make use of the Hamiltonian formalism introduced above. The following definition is, in the Hamiltonian formalism, a natural way to define a notion of a conformal Killing tensor.

**Definition.** A Schrödinger–Killing tensor of a Newton–Cartan spacetime  $(M, h, \theta, U, F)$  is a symmetric contravariant tensor field  $X^{a_1 \dots a_n}$  for which functions  $\chi_m^{a_1 \dots a_m}$  on  $M$  can be found obeying

$$\left\{ X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} + \sum_{m=0}^{n-1} \chi_m^{a_1 \dots a_m} p_{a_1} \dots p_{a_m}, \mathcal{H} \right\} = \sum_{m=0}^{n-1} (f_m^{a_1 \dots a_m} p_{a_1} \dots p_{a_m}) \mathcal{H}, \tag{30}$$

where  $f_m^{a_1 \dots a_m}$  are symmetric tensor fields determined in terms of  $(X^{a_1 \dots a_n}, \chi_m^{a_1 \dots a_m})$ .

A Killing tensor as defined above is a special case of a Schrödinger–Killing tensor.

If  $n = 1$  we have

$$\mathcal{L}_X h = f_0 h \tag{31}$$

$$\mathcal{L}_X U - h(\mathcal{L}_X A, \cdot) = f_0 U - h(d\chi_0, \cdot) \tag{32}$$

$$(\mathcal{L}_X A)(U) = d\chi_0(U). \tag{33}$$

Using the flat Newton–Cartan spacetime (29) reduces this definition to that of  $\mathfrak{sch}(d)$  above.

In order to prove Theorem 2 we will display in more detail the conditions describing the Schrödinger–Killing tensors of the flat NC. The defining condition (30) becomes the coupled family of equations

$$-\partial^i X^{a_1 \dots a_n} p_i p_{a_1} \dots p_{a_n} = \frac{1}{2} \delta^{ij} f_{n-1}^{a_1 \dots a_{n-1}} p_i p_j p_{a_1} \dots p_{a_{n-2}} \tag{34}$$

$$\partial_t X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} - \partial^i \chi_{n-1}^{a_1 \dots a_{n-1}} p_i p_{a_1} \dots p_{a_{n-1}} = -f_{n-1}^{a_1 \dots a_{n-1}} p_t p_{a_1} \dots p_{a_{n-1}} + \frac{1}{2} \delta^{ij} f_{n-2}^{a_1 \dots a_{n-2}} p_i p_j p_{a_1} \dots p_{a_{n-2}} \tag{35}$$



$$\partial_t \chi_{n-1}^{a_1 \dots a_{n-1}} p_{a_1} \dots p_{a_{n-1}} - \partial^i \chi_{n-2}^{a_1 \dots a_{n-2}} p_i p_{a_1} \dots p_{a_{n-2}} = -f_{n-2}^{a_1 \dots a_{n-2}} p_t p_{a_1} \dots p_{a_{n-2}} + \frac{1}{2} \delta^{ij} f_{n-3}^{a_1 \dots a_{n-3}} p_i p_j p_{a_1} \dots p_{a_{n-3}} \tag{36}$$

⋮

$$\partial_t \chi_2^{a_1 a_2} p_{a_1} p_{a_2} - \partial^i \chi_1^{a_1} p_i p_{a_1} = -f_1^{a_1} p_t p_{a_1} + \frac{1}{2} \delta^{ij} f_0 p_i p_j \tag{37}$$

$$\partial_t \chi_1^{a_1} p_{a_1} - \partial^i \chi_0 p_i = -f_0 p_t \tag{38}$$

$$\partial_t \chi_0 = 0. \tag{39}$$

We can rewrite these concisely using the Schouten brackets of  $X$  with  $h$  and  $U$ , denoted  $\mathcal{L}_X h$  and  $\mathcal{L}_X U$ . They become

$$\mathcal{L}_X h = f_{n-1} h \tag{40}$$

$$\mathcal{L}_{\chi_{n-1}} h - 2\mathcal{L}_X U = f_{n-2} h - 2f_{n-1} U \tag{41}$$

$$\mathcal{L}_{\chi_{n-2}} h - 2\mathcal{L}_{\chi_{n-1}} U = f_{n-3} h - 2f_{n-2} U \tag{42}$$

⋮ etc.

with this pattern continuing on the understanding that for negative  $m$  we have  $f_m = \chi_m = 0$ , and where all indices on the right-hand-side products are symmetrised.

### 4.3. Higher symmetry operators

The higher symmetries of the Laplacian and of various Schrödinger operators have been calculated and are to be found in the literature [18,3,12]. In this subsection we will define such symmetries, following those papers, and then proceed to prove Theorem 2, identifying the higher symmetries of the free Schrödinger operator with the Schrödinger–Killing tensors of the flat Newton–Cartan spacetime.

#### The Laplacian

In [3] Eastwood finds the *higher symmetries* of the Laplacian. These are linear differential operators

$$\mathcal{D} = V_n^{\mu_1 \dots \mu_n} \frac{\partial^n}{\partial \chi^{\mu_1} \partial \chi^{\mu_2} \dots \partial \chi^{\mu_n}} + V_{n-1}^{\mu_1 \dots \mu_{n-1}} \frac{\partial^{n-1}}{\partial \chi^{\mu_1} \partial \chi^{\mu_2} \dots \partial \chi^{\mu_{n-1}}} + \dots + V_1^{\mu_1} \frac{\partial}{\partial \chi^{\mu_1}} + V_0$$

which commute with the Laplacian  $\Delta_L$  in the sense that

$$\Delta_L \mathcal{D} = \delta \Delta_L$$

for some linear differential operator  $\delta$  (determined by  $\mathcal{D}$ ). The functions  $V_p^{\mu_1 \dots \mu_p}$  (for  $0 \leq p \leq n$ ) are the components of totally symmetric rank- $p$  tensor fields on flat spacetime, and the tensor of highest rank is called the *symbol* of the symmetry operator. Eastwood finds that if  $\mathcal{D}$  is a symmetry of the Laplacian then its symbol is a conformal Killing tensor on flat spacetime, i.e.

$$\partial^{(\mu_0} V_n^{\mu_1 \dots \mu_n)} = g^{(\mu_0 \mu_1} k^{\mu_2 \dots \mu_n)} \tag{43}$$

for some rank- $(n - 1)$  tensor field  $k$  (which itself is determined from (43)) and inverse (flat) metric  $g^{\mu\nu}$ . Furthermore, when (43) is satisfied for some symbol  $V_n^{\mu_1 \dots \mu_n}$  one can uniquely solve for lower order operators ( $V_{n-1}^{\mu_1 \dots \mu_{n-1}}, V_{n-2}^{\mu_1 \dots \mu_{n-2}}, \dots, V_0$ ) determined in terms of the symbol such that  $\mathcal{D}$  is a symmetry of the Laplacian.

#### The free Schrödinger operator

The analogous higher symmetries of the free-particle Schrödinger operator

$$\Delta = i\partial_t + \frac{1}{2m} \delta^{ij} \partial_i \partial_j$$

can be found in the literature [12]. Here we will summarise and make use of the approach of [18], where the symmetries of  $\Delta$  in  $d + 1$  dimensions arise as the light-cone reduction of conformal Killing tensors in  $d + 2$  dimensions.

Consider the wave equation in  $d + 2$  dimensions, written in light-cone coordinates  $(x^i, x^+, x^-)$ :

$$\Delta_L \phi = (\delta^{ij} \partial_i \partial_j - 2\partial_+ \partial_-) \phi = 0.$$

Restricting to fields of the form

$$\phi(x^i, x^+, x^-) = \psi(x^+, x^i) \exp \{-imx^-\} \tag{44}$$



reduces the wave equation to

$$\left( i\partial_+ + \frac{1}{2m} \delta^{ij} \partial_i \partial_j \right) \psi(x^+, x') = 0,$$

which is just  $\Delta\psi = 0$  if we identify  $x^+$  with time.

Let  $\mathcal{D}$  be a symmetry of the Laplacian, allowing us to write

$$\Delta_L \mathcal{D}\phi = \delta \Delta_L \phi.$$

Restricting to the ansatz (44) reduces this to

$$\Delta_L \mathcal{D} \left( e^{-imx^-} \psi \right) = \delta e^{-imx^-} \Delta \phi. \tag{45}$$

Applying  $\mathcal{D}$  to  $e^{-imx^-} \psi$  results in a new symmetry operator  $\tilde{\mathcal{D}}$ :

$$\mathcal{D} \left( e^{-imx^-} \psi \right) = e^{-imx^-} \tilde{\mathcal{D}}\psi.$$

The left-hand-side of (45) rearranges into  $\Delta$  iff  $\partial_- \tilde{\mathcal{D}} = 0$ , giving us

$$\Delta \tilde{\mathcal{D}}\psi = \tilde{\delta} \Delta \psi \quad \text{for } \partial_- \tilde{\mathcal{D}} = 0.$$

We can reverse these steps, giving us the statement that the higher symmetries of  $\Delta$  are the operators  $\tilde{\mathcal{D}}$ , arising from conformal Killing tensors.

**Proof of Theorem 2.** To prove Theorem 2 we will consider the conformal Killing equation in  $d + 2$  dimensions with coordinates  $x^\mu = (x^i, x^+, x^-)$ . We will calculate the resulting conditions on  $\mathcal{D}$  and compare them to Eqs. (34)–(39) characterising Schrödinger-Killing tensors.

For a tensor of rank  $n$  the conformal Killing equation is

$$\partial^{(\mu_0} S^{\mu_1 \dots \mu_n)} = g^{(\mu_0 \mu_1} k^{\mu_2 \dots \mu_n)}$$

where the only non-vanishing components of the metric are

$$g_{ij} = \delta_{ij} \quad g_{+-} = g_{-+} = -1.$$

Recall that we are only interested in solutions not-depending on  $x^-$ .

Consider first the case  $(\mu_0 \dots \mu_n) = (a_0 \dots a_n)$ , i.e. no  $(-)$  indices are included. We can then identify

$$g^{ab} = h^{ab} \quad \partial^a = h^{ab} \partial_b,$$

giving us

$$h^{b(a_0} \partial_b S^{a_1 \dots a_n)} = h^{(a_0 a_1} k^{a_2 \dots a_n)}.$$

Writing  $X^{a_1 \dots a_n} = S^{a_1 \dots a_n}$  and  $k^{a_1 \dots a_{n-1}} = -\frac{1}{2} f_{n-1}^{a_1 \dots a_{n-1}}$ , we then have

$$\mathcal{L}_X h = f_{n-1} h,$$

the first of Eqs. (34)–(39) characterising the Schrödinger-Killing tensor  $X^{a_1 \dots a_n}$ . Similarly we can start to set one index to  $(-)$  and the rest to  $(a_1 \dots a_n)$ , giving

$$\begin{aligned} \partial^- S^{a_1 \dots a_n} + n \partial^{(a_1} S^{a_2 \dots a_n)-} &= 2g^{-(a_1} k^{a_2 \dots a_n)} + (n-1)g^{(a_1 a_2} k^{a_3 \dots a_n)-} \\ \Rightarrow \partial_+ S^{a_1 \dots a_n} - n h^{b(a_1} \partial_b S^{a_2 \dots a_n)-} &= 2\delta_+^{(a_1} k^{a_2 \dots a_n)} - (n-1)h^{(a_1 a_2} k^{a_3 \dots a_n)-}. \end{aligned}$$

Again, this equation is the same as (41) with the identifications

$$\chi_{n-1}^{a_1 \dots a_{n-1}} = n S^{-a_1 \dots a_{n-1}} \quad f_{n-2}^{a_1 \dots a_{n-2}} = -2(n-1)k^{-a_1 \dots a_{n-2}}. \tag{46}$$

In fact, all of Eqs. (34)–(39) can be reproduced in this manner from the conformal Killing equation, one for each number of indices set to  $(-)$ , and with similar identifications to (46). The equation with  $q$  indices set to  $(-)$  and the rest set to  $(a_1 \dots a_{n+1-q})$  is

$$\begin{aligned} q \partial^- S^{\dots - a_1 \dots a_{n+1-q}} + (n+1-q) \partial^{(a_1} S^{a_2 \dots a_{n+1-q}) \dots -} \\ = 2 \frac{q}{n} (n+1-q) g^{-(a_1} k^{a_2 \dots a_{n+1-q}) \dots -} + \frac{1}{n} (n+1-q)(n-q) g^{(a_1 a_2} k^{a_3 \dots a_{n+1-q}) \dots -} \\ \Rightarrow q \partial_+ S^{\dots - a_1 \dots a_{n+1-q}} - (n+1-q) h^{b(a_1} \partial_b S^{a_2 \dots a_{n+1-q}) \dots -} \\ = 2 \frac{q}{n} (n+1-q) \delta_+^{(a_1} k^{a_2 \dots a_{n+1-q}) \dots -} - \frac{1}{n} (n+1-q)(n-q) h^{(a_1 a_2} k^{a_3 \dots a_{n+1-q}) \dots -}. \end{aligned}$$

We can then identify

$$\chi_{n-q}^{a_1 \dots a_{n-q}} = \binom{n}{q} S^{a_1 \dots a_{n-q} \dots} \quad \text{for } q \geq 1$$

$$\text{and } f_{n-q}^{a_1 \dots a_{n-q}} = -2 \binom{n-1}{q-1} k^{a_1 \dots a_{n-q} \dots} \quad \text{for } q \geq 1.$$

These equations are now exactly those above characterising Schrödinger–Killing tensors, completing the proof of [Theorem 2](#).  $\square$

With this achieved, a natural question to ask is whether this result extends to curved Newton–Cartan spacetimes and the covariant Schrödinger equation. The situation here remains unclear, just as it does in the case of the curved Riemannian manifold and the Laplacian, and we defer this question to future investigations.

### 5. Conformal symmetries on Newtonian twistor space

#### 5.1. Review: Newtonian twistor theory

Newton–Cartan spacetimes admit a twistorial construction as the moduli space of rational curves in the twistor space. The details of this construction are to be found in [\[13\]](#); here we will provide a short review in order to discuss the role of conformal Newton–Cartan symmetries in the associated twistor theory.

Let  $(\lambda, \hat{\lambda})$  be inhomogeneous coordinates on two patches defined by stereographic projection from the north and south poles of the Riemann sphere  $\mathbb{C}\mathbb{P}^1$ , and let

$$\begin{array}{c} \mathcal{O}(n) \\ \downarrow \\ \mathbb{C}\mathbb{P}^1 \end{array}$$

be a rank-one holomorphic vector bundle over  $\mathbb{C}\mathbb{P}^1$  with patching

$$\hat{s} = \lambda^{-n} s,$$

where  $(s, \hat{s})$  are coordinates on the fibres over the two patches.

The twistor space is the total space of the bundle  $PT_\infty = \mathcal{O} \oplus \mathcal{O}(2)$ , fibred over  $\mathbb{C}\mathbb{P}^1$ . Let  $PT_\infty$  be covered by two patches  $U$  and  $\hat{U}$  with coordinates  $Z^\alpha = (T, Q, \lambda)$  and  $\hat{Z}^\alpha = (\hat{T}, \hat{Q}, \hat{\lambda})$ , where  $(T, Q)$  are coordinates on the fibres and  $\lambda$  is a coordinate on the base  $\mathbb{C}\mathbb{P}^1$ . The patching is then

$$\hat{T} = T \quad \hat{Q} = \lambda^{-2} Q \quad \hat{\lambda} = \lambda^{-1}.$$

We identify the  $(3 + 1)$ -dimensional Newton–Cartan spacetime  $M$  as the moduli space of global sections of  $PT_\infty \rightarrow \mathbb{C}\mathbb{P}^1$ , by means of the double-fibration of the projective spin bundle  $PS'$ .

$$\begin{array}{ccc} & PS' & \\ \nu \swarrow & & \searrow \mu \\ M & & PT_\infty. \end{array}$$

The maps realising the fibrations are

$$\nu(x^a, \lambda) = x^a$$

and

$$\mu : (x^a, \lambda) \mapsto \begin{pmatrix} T \\ Q \\ \lambda \end{pmatrix} = \begin{pmatrix} t \\ \lambda^2(x - iy) - 2\lambda z - (x + iy) \\ \lambda \end{pmatrix}.$$

Global holomorphic data on  $PT_\infty$  can be mapped to  $M$  giving rise to the Galilean structure. The mapping procedure (for the case of vector fields) is discussed concretely in [Section 5.2](#).

- $H^0(PT_\infty, T^*(PT_\infty))$  contains one-forms  $k(T)dT$  for holomorphic  $k(T)$ , which correspond to one-forms  $k(t)dt$  on  $M$ . This gives us the conformal clock  $[\theta]$ .
- The zero of  $H^0(PT_\infty, T(PT_\infty)^{\otimes 2})$  corresponds to the conformal structure  $[h]$  containing  $\delta^{ij}\partial_i\partial_j$  on  $M$ .

Both of these conformal factors can be fixed using data on the non-projective twistor space to give  $h = \delta^{ij}\partial_i\partial_j$  and  $\theta = dt$ : see [\[13\]](#) for details. The construction of the affine connection on  $M$  need not concern us here, and we again refer the reader to [\[13\]](#).

5.2. Holomorphic vector fields on  $PT_\infty$  and  $\text{cnc}(3)$

In the nonlinear graviton construction the conformal symmetries of the spacetime are in one-to-one correspondence with holomorphic vector fields on twistor space, that is to say the conformal symmetries on  $M$  arise as global sections of  $T(PT)$  [14].

It is thus natural to ask what the global sections of  $T(PT_\infty)$  correspond to on the Newton–Cartan spacetime. The answer is the following theorem.

**Theorem 3.** *The global sections of  $T(PT_\infty)$  are in one-to-one correspondence with conformal Newton–Cartan vectors  $\text{cnc}(3)$  of the Galilean structure  $h = \delta^{ij} \partial_i \partial_j$  and  $\theta = dt$  on  $M$ .*

The algebra  $\text{cnc}(3)$  is discussed in [4], where it is defined to be the algebra of vector fields on a Newton–Cartan spacetime which generate conformal transformations of  $(h, \theta)$  and null-projective transformations of  $\nabla$ . Both  $\text{cnc}(3)$  and  $H^0(PT_\infty, T(PT_\infty))$  are infinite-dimensional Lie algebras.

We will prove Theorem 3 by directly calculating the holomorphic vector fields  $\beta^\alpha$  on  $PT_\infty$ . The patching

$$\hat{\beta}^\rho = \frac{\partial \hat{Z}^\rho}{\partial Z^\sigma} \beta^\sigma$$

can be expanded to give

$$\begin{aligned} \hat{\beta}^T &= \beta^T \\ \hat{\beta}^Q &= \lambda^{-2} \beta^Q - 2\lambda^{-3} Q \beta^\lambda \\ \hat{\beta}^\lambda &= -\lambda^{-2} \beta^\lambda. \end{aligned}$$

By considering an ansatz in which  $\beta^\alpha$  are arbitrary polynomials in  $(Q, \lambda)$  whose coefficients are arbitrary holomorphic functions of the trivial coordinate  $T$  we find that  $\beta \in H^0(PT_\infty, T(PT_\infty))$  iff

$$\begin{aligned} \beta &= h(T) \frac{\partial}{\partial T} + (a(T) + b(T)Q + c(T)\lambda + d(T)\lambda Q + e(T)\lambda^2) \frac{\partial}{\partial Q} \\ &\quad + \left( f(T) + g(T)\lambda + \frac{1}{2} \lambda^2 d(T) \right) \frac{\partial}{\partial \lambda} \end{aligned} \tag{47}$$

for  $(a, b, c, \dots, h)$  any eight holomorphic functions of  $T$ . These sections form an infinite-dimensional Lie algebra (under the usual commutator).

Pushing this algebra to  $M$  is a two-stage procedure. First we consider an arbitrary vector  $\Lambda \in T(PS')$  and its push-forward to  $PT_\infty$ :

$$(\mu_* \Lambda)^\alpha = \frac{\partial (Z^\alpha)}{\partial x^\Sigma} \Lambda^\Sigma,$$

where  $x^\Sigma = (x^a, \lambda)$  are coordinates on  $\mu^{-1}(U) \subset PS'$ . Thus setting  $\beta^\alpha = (\mu_* \Lambda)^\alpha$  we have

$$\beta^T = \Lambda^t \quad \beta^Q - \frac{\partial(Q)}{\partial \lambda} \beta^\lambda = \Lambda^i \frac{\partial(Q)}{\partial x^i} \quad \beta^\lambda = \Lambda^\lambda,$$

and we can uniquely determine a vector  $\Lambda$  such that  $\Lambda^a$  does not depend on  $\lambda$  (necessary for the next step). The second half of the procedure is to simply push-down  $\Lambda$  to  $X = \nu_* \Lambda$  on  $M$ , giving

$$X = \Lambda^a(x^i, t) \frac{\partial}{\partial x^a}.$$

Doing this for the general global vector (47) yields

$$X = h(t) \frac{\partial}{\partial t} + (\omega_j^i(t) x^j + \chi(t) x^i + \eta^i(t)) \frac{\partial}{\partial x^i}$$

where

$$\begin{aligned} \chi(t) &= b(t) - g(t) \\ \omega_y^x(t) &= ig(t) \\ \omega_x^z(t) &= f(t) + \frac{1}{2} d(t) \\ \omega_y^z(t) &= i \left( \frac{1}{2} d(t) - f(t) \right) \\ \eta^i(t) \frac{\partial(Q)}{\partial x^i} &= a(t) + c(t)\lambda + e(t)\lambda^2, \end{aligned}$$

revealing  $X$  to be an arbitrary element of  $\mathfrak{cnc}(3)$ . The procedure is reversible with no trouble: the  $\partial_\lambda$  component of the lift of  $X$  to  $PS'$  is calculated by requiring that the resulting vector field should descend to a vector field on  $PT_\infty$ . This completes the proof of [Theorem 3](#).  $\square$

Note that the factors of  $i$  above do not prevent  $X$  from being real; it is possible to choose the real and imaginary parts of  $(a, b, c, \dots, h)$  such that  $X$  is any element of the real  $\mathfrak{cnc}(3)$ .

### 5.3. The expanded Schrödinger algebra on $PT_\infty$

The expanded Schrödinger algebra  $\widetilde{\mathfrak{sch}}(3)$  in a finite-dimensional subalgebra of  $\mathfrak{cnc}(3)$ , and so it is natural to ask what characterises the corresponding holomorphic vector fields on  $PT_\infty$ . On  $M$  we pick out the subalgebra (as described in section three) by a requirement that the vector must generate projective transformations of a connection. In this section we will describe an analogous procedure which takes place in twistor space.

In order to have a notion of a projective holomorphic vector field on  $PT_\infty$  we must first establish some kind of affine connection on the twistor space. Using the standard law for transforming connection components we can write down the patching for a new bundle  $\mathfrak{g} \rightarrow PT_\infty$ , whose sections are (the components of) torsion-free affine connections on twistor space. The patching is

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial \hat{Z}^\alpha}{\partial Z^\mu} \frac{\partial Z^\nu}{\partial \hat{Z}^\beta} \frac{\partial Z^\rho}{\partial \hat{Z}^\gamma} \Gamma^{\mu\nu\rho} - \frac{\partial Z^\nu}{\partial \hat{Z}^\beta} \frac{\partial Z^\rho}{\partial \hat{Z}^\gamma} \frac{\partial^2 \hat{Z}^\alpha}{\partial Z^\nu \partial Z^\rho}. \tag{48}$$

One might expect that we could now identify  $H^0(PT_\infty, \mathfrak{g})$  and then proceed to consider the projective vectors of global connections. Unfortunately there is an issue: the bundle  $\mathfrak{g}$  admits no global sections, as can be seen from, for example, the  ${}^\alpha_{\beta\gamma} = \lambda_{\lambda\lambda}$  component of the patching,

$$\hat{\Gamma}^\lambda_{\lambda\lambda} = -\lambda^2 \Gamma^\lambda_{\lambda\lambda} - 4\lambda Q \Gamma^\lambda_{Q\lambda} - 4Q^2 \Gamma^\lambda_{QQ} - 2\lambda. \tag{49}$$

The final term here cannot be (holomorphically) included in any of the other terms, so there can be no global solutions to [\(49\)](#);  $H^0(PT_\infty, \mathfrak{g}) = 0$ .

Inspection of [\(48\)](#) reveals that if one considers only vectors in the  $Z^A = (T, Q)$  directions then the patching for the relevant connection components *does* admit global sections. Thus it is sensible to decompose the tangent bundle as

$$T(PT_\infty) = \mathfrak{h} \oplus \mathfrak{v}$$

with respect to the fibration  $PT_\infty \rightarrow \mathbb{C}P^1$ , i.e. such that

$$\beta \in \mathfrak{v} \quad \text{iff} \quad d\lambda(\beta) = 0.$$

The general global section of the reduced bundle  $\mathfrak{g}_\mathfrak{v}$  is then

$$\begin{aligned} \Gamma^T_{TT} &= \Sigma(T) & \Gamma^Q_{TQ} &= \Gamma^Q_{QT} = \mathcal{E}(T) \\ \Gamma^Q_{TT} &= \Phi_0(T) + \Phi_1(T)\lambda + \Phi_2(T)\lambda^2 + \Psi(T)Q, \end{aligned} \tag{50}$$

with all other components  $\Gamma^A_{BC}$  set to zero, and where  $(\Sigma, \mathcal{E}, \Phi_0, \Phi_1, \Phi_2, \Psi)$  are six arbitrary holomorphic functions of  $T$ .

In the spirit of the spacetime characterisation of the expanded Schrödinger algebra we will consider, out of all the possibilities in [\(50\)](#), the case in which *all*  $\Gamma^A_{BC} = 0$ . We then define the subalgebra  $S_\mathfrak{v} \subset \mathfrak{v}$  to be the algebra of vertical holomorphic vector fields  $\beta$  obeying

$$\mathcal{L}_\beta \Gamma^A_{BC} = \delta^A_{(B} \kappa_{C)}, \tag{51}$$

for  $\kappa$  a one-form on  $PT_\infty$  to be determined in solving [\(51\)](#). Not unexpectedly,  $\kappa$  is always an element of  $H^0(PT_\infty, T^*(PT_\infty))$ , which is populated only by one-forms  $k(T)dT$  for holomorphic  $k$  corresponding to the conformal clock on  $M$ . The condition [\(51\)](#) sets

$$\begin{aligned} a(T) &= a_0 + a_1 T & b(T) &= b_0 + h_2 T & c(T) &= c_0 + c_1 T \\ d(T) &= d_0 & e(T) &= e_0 + e_1 T & h(T) &= h_0 + h_1 T + h_2 T^2 \end{aligned}$$

for eleven constants  $(a_0, a_1, b_0, c_0, c_1, d_0, e_0, e_1, h_0, h_1, h_2)$ .

To get the twistorial analogue  $\tilde{S}$  of the expanded Schrödinger algebra we must then reintroduce the  $\partial_\lambda$  parts of the vectors. This is done by taking the closure under Lie bracket of

$$S_\mathfrak{v} \oplus \left\{ \left( f(T) + g(T)\lambda + \frac{1}{2}\lambda^2 d_0 \right) \frac{\partial}{\partial \lambda} \right\},$$

which fixes

$$f(T) = f_0 \quad g(T) = g_0$$

for two further constants  $(f_0, g_0)$ . The thirteen-dimensional Lie algebras  $\tilde{S}$  on  $PT_\infty$  and  $\widetilde{\mathfrak{sch}}(3)$  on  $M$  are then in one-to-one correspondence, a subcorrespondence of that in [Theorem 3](#).

5.4. The CGA on  $PT_\infty$

In [15] the authors discuss a particular non-relativistic limit of the conformal algebra, in which one sends  $c \rightarrow \infty$  but scales each generator by an appropriate fact of  $c$  such that the leading term survives. The number of generators is therefore unchanged, and the resulting fifteen-dimensional algebra is known as the CGA (conformal Galilean algebra).

We can realise  $PT_\infty$  as the  $c \rightarrow \infty$  limit of the twistor space  $PT_c$  associated to Minkowski space [13], and so we can take a limit of the (fifteen) holomorphic vector fields on  $PT_c$  in the CGA style to give a representation of the CGA on  $PT_\infty$ . The conformal Killing vectors on Minkowski space  $M_c$  and their resulting limits on  $PT_\infty$  are shown in the following table.

	Vector on $M_c$	Limit on $PT_\infty$
Translations	$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial T}$
	$\frac{\partial}{\partial x}$	$(\lambda^2 - 1) \frac{\partial}{\partial Q}$
	$\frac{\partial}{\partial y}$	$-i(\lambda^2 + 1) \frac{\partial}{\partial Q}$
	$\frac{\partial}{\partial z}$	$-2\lambda \frac{\partial}{\partial Q}$
Dilation	$t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$	$T \frac{\partial}{\partial T} + Q \frac{\partial}{\partial Q}$
Rotations	$x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$	$iQ \frac{\partial}{\partial Q} + i\lambda \frac{\partial}{\partial \lambda}$
	$y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}$	$-i\lambda Q \frac{\partial}{\partial Q} - \frac{i}{2}(\lambda^2 - 1) \frac{\partial}{\partial \lambda}$
	$z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}$	$-\lambda Q \frac{\partial}{\partial Q} - \frac{1}{2}(1 + \lambda^2) \frac{\partial}{\partial \lambda}$
Boosts	$t \frac{\partial}{\partial x} + \frac{x}{c^2} \frac{\partial}{\partial t}$	$(\lambda^2 - 1)T \frac{\partial}{\partial Q}$
	$t \frac{\partial}{\partial y} + \frac{y}{c^2} \frac{\partial}{\partial t}$	$-i(1 + \lambda^2)T \frac{\partial}{\partial Q}$
	$t \frac{\partial}{\partial z} + \frac{z}{c^2} \frac{\partial}{\partial t}$	$-2\lambda T \frac{\partial}{\partial Q}$
Special	$-2t(x \cdot \partial) - (x \cdot x) \frac{1}{c^2} \frac{\partial}{\partial t}$	$-T^2 \frac{\partial}{\partial T} - 2TQ \frac{\partial}{\partial Q}$
	$\frac{2}{c^2} x(x \cdot \partial) - \frac{1}{c^2}(x \cdot x) \frac{\partial}{\partial x}$	$(\lambda^2 - 1)T^2 \frac{\partial}{\partial Q}$
	$\frac{2}{c^2} y(x \cdot \partial) - \frac{1}{c^2}(x \cdot x) \frac{\partial}{\partial y}$	$-i(\lambda^2 + 1)T^2 \frac{\partial}{\partial Q}$
	$\frac{2}{c^2} z(x \cdot \partial) - \frac{1}{c^2}(x \cdot x) \frac{\partial}{\partial z}$	$-2\lambda T^2 \frac{\partial}{\partial Q}$

The CGA on  $PT_\infty$  is a finite-dimensional subalgebra of  $H^0(PT_\infty, T(PT_\infty))$ , giving us another subcorrespondence of that in Theorem 3.

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