EXPECTATIONS IN FINANCIAL MARKETS

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except as declared in the Preface. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution. It does not exceed the prescribed word limit of 60,000 words.

Harkeerit S. Kalsi, March 2023

Summary

Uncertainty pervades financial markets. How financial market participants form expectations when faced with uncertainty is therefore central to the study of financial markets. This thesis contains three chapters each highlighting the role of expectations formation in determining outcomes in financial markets.

The first chapter, coauthored with Nicholas Vause and Nora Wegner, builds a theoretical model of self-fulfilling fire sales motivated by the dash for cash of March 2020. Investment funds fear being hit by a future liquidity shock and can choose to preemptively liquidate their bond holdings. However, funds face uncertainty about how many other funds will choose to preemptively liquidate. If funds wait to see if the liquidity shock crystalises, they risk selling their bonds into a depressed market if other funds have already chosen to liquidate. This creates the risk of a self-fulfilling fire sale where funds choose to preemptively liquidate because they expect that other funds will liquidate. Following the global games literature (Carlsson & van Damme 1993, Morris & Shin 1998), we derive the probability of a self-fulfilling fire sale and extend the model to include a central bank providing a market backstop. The central bank can choose the quantity of assets it is willing to purchase and the discount (relative to the bond return) that it charges to purchase the bond. When we introduce the central bank, we show that if the central bank can credibly commit to (i) set its discount low enough and (ii) the quantity of asset purchases high enough, then it can eliminate self-fulfilling fire sales. Moreover, it can achieve this without having to purchase any bonds. The aggressive policy works via expectations. In particular, it makes the pessimistic beliefs that drive the fire sale impossible for funds to rationalise because the funds know that the central bank stands ready to provide liquidity via asset purchases if needed.

Whereas in the first chapter I assume agents can costlessly absorb all available information, in the second chapter (solo-authored), I follow the rational inattention literature (Sims 2003) and relax the assumption that information is costlessly obtained. This enables me to examine whether fragilities can build up in the financial system simply because agents pay insufficient attention to each other. I build a model where bank values are interdependent within a financial network. However, banks cannot costlessly observe other banks' values. Instead, banks must choose to pay attention to developments in the value of other banks. Because paying attention incurs a cost, banks may optimally choose not to allocate significant attention to certain banks. In the model, banks that believe they have a higher value choose to supply more credit. Therefore, if inattentiveness causes banks to incorrectly infer their own value, their credit supply decisions will be distorted relative to the optimal. I show that banks that are moderately important for determining values in the network cause the greatest distortion in

credit supply because banks do not deem them important enough to pay high levels of attention to, yet they are still important for determining bank values. This suggests that we should be more cautious about dismissing all but the most interconnected banks as being important for ensuring financial stability.

Thus far, agents have known the objective probability distributions relevant for decision making. In the third chapter, coauthored with Harjoat Bhamra and Raman Uppal, we follow Knight (1921) and distinguish between risk (known probabilities) and Knightian uncertainty (unknown probabilities). We argue that geopolitical uncertainty can often be viewed as Knightian uncertainty rather than risk and our objective is to examine the effects of this uncertainty. We do this by constructing a dynamic stochastic equilibrium production model of a world economy with two countries. Each country is characterised by a traded and a non-traded goods sector and a representative investor with Stochastic Differential Utility who is averse to Knightian uncertainty. We model geopolitical uncertainty as a loss of confidence in the correct model for the shocks to the efficiency units of capital where the investors cannot assign probabilities to the alternative models for the shocks. We solve this model in closed form and show how uncertainty operates by reducing households' perceived expected returns on capital which, in turn, distorts portfolio and consumption choice decisions. We then examine the implications of these distortions for trade flows, exchange rates, growth, and the level of social welfare. We show that our model can match stylised facts of the UK economy following the Brexit referendum.

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Contents

Chapter 1

Self-fulfilling fire sales and market backstops

Abstract Motivated by the March 2020 'dash for cash', we build a model in which a potential liquidity shock to bond funds can prompt a self-fulfilling fire sale in the presence of a dealer with limited trading capacity. Following the global games literature, we derive the probability of a self-fulfilling fire sale. Introducing a central bank market backstop policy, we show that if the central bank can credibly commit to (i) set the size of its asset purchases high enough and (ii) its price discount low enough, then it can eliminate self-fulfilling fire sales without having to purchase any bonds. If the central bank acts less aggressively, it can still reduce the probability of a self-fulfilling fire sale. However, in response to the policy, funds choose to hold more bonds, which increases the probability of a self-fulfilling fire sale and reduces the effectiveness of the market backstop.

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1.1 Introduction

Bond markets came under severe stress at the start of the COVID-19 crisis in March 2020. Investors sought to sell bonds in such volumes that dealers ran into capacity constraints and could not accommodate all desired trades. Concerned about the systemic risk posed by fire sales, central banks stepped in when dealers were overwhelmed and acted as backstop buyers of bonds to stabilise markets. This has led to a debate about whether tools for intervening in capital markets in pursuit of financial stability objectives should be part of central banks' permanent policy toolkits. This debate intensified following the Bank of England's gilt market operations in October 2022, when it purchased long-dated gilts at a 'reserve' spread to mid-prices to restore market functioning, which had become impaired as a result of incipient heavy sales by pension funds (Cunliffe 2022).

One concern with such a policy is that if the central bank is expected to provide an effective market backstop, then problems akin to "moral hazard" can arise. The presence of a central bank backstop ensures that investors do not face the negative consequences of fire sales, so they optimally take more aggressive positions in risky assets. However, more aggressive positions would be more difficult to unwind in a crisis. Central bank intervention could therefore make the financial system more fragile thus risking the incurrence of welfare costs associated with financial instability.

This paper develops a theoretical model to examine these issues motivated by the empirical literature studying the "dash for cash" episode of March 2020. Haddad et al. (2021) argue that a key source of the selling pressure was driven by bond mutual funds facing current liquidity shocks and fearing future shocks. This selling pressure drove down bond prices because dealers struggled to intermediate given the magnitude of the selling pressure. However, Haddad et al. (2021) show how announcements by the Federal Reserve alone stabilised corporate bond markets. Kargar et al. (2021) also argue that corporate bond markets stabilised primarily because the announcements immediately reduced funds' demand for cash.[1]

Our model therefore focuses on the behaviour of bond funds in a liquidity crisis. In the model, there is a continuum of risk-neutral funds initially endowed with a portfolio of cash and bonds. Similar to Bernardo & Welch (2004), funds then learn about the possibility of a future liquidity

[1] Other papers that empirically examine various aspects of the "dash for cash" episode include Boyarchenko et al. (2022), Gilchrist et al. (2021), O'Hara & Zhou (2021), and Vissing-Jorgensen (2021). Interestingly, Vissing-Jorgensen (2021) provide evidence that Federal Reserve intervention in Treasury markets actually operated more through purchases than announcements due to the immediate liquidity needs of Treasury sellers. However, prior literature has found a role for announcement effects of Federal Reserve purchases of Treasuries (e.g., Krishnamurthy & Vissing-Jorgensen (2011)) showing that the need for purchases is not in general a special feature of the US Treasury market.

shock where they would be forced to sell their bond holdings to a dealer at a discount to the bond's expected return. The discount is increasing in the dealer's inventory of bonds, either because of risk-aversion (Bernardo & Welch 2004, Morris & Shin 2004) or institutional constraints.

The discount can create an incentive for funds to preemptively sell their bonds before the outcome of the liquidity shock is realised. If a fund thinks that other funds will sell today, it will worry about being hit by a liquidity shock tomorrow and having to sell to a dealer with a large bond inventory charging a high discount. This makes it more attractive for the fund to sell today. If all funds reason in a similar manner, the optimality of the decision to sell today becomes self-fulfilling. But this is not the only possibility. If a fund thinks that all other funds will hold today, it will think that the dealer will have a low bond inventory tomorrow and therefore it will face the low discount if it is hit by the liquidity shock. This makes it more attractive for the fund to hold today and, if all funds reason similarly, the optimality of the decision to hold today becomes self-fulfilling. Therefore, the model features multiple equilibria: an equilibrium where funds hold and an equilibrium where funds panic and cause a self-fulfilling fire sale.

A problem with a model featuring multiple equilibria is that which equilibrium will be realised is determined by a sunspot rather than economic fundamentals. To solve this problem, we employ a global games approach (Carlsson & van Damme 1993, Morris & Shin 1998) to eliminate the multiplicity of equilibria and link the probability of a self-fulfilling fire sale to economic fundamentals. In particular, we show that there will be a self-fulfilling fire sale if and only if the capacity of the dealer to absorb bonds sales is sufficiently low. We also show that the probability of a self-fulfilling fire sale is increasing in the discount charged by the dealer in stressed market conditions and increasing in the probability of the liquidity shock occurring. The probability of a fire sale is also greater if the funds' initial endowment of bonds is higher because the dealer would have to absorb more bonds if the funds chose to unwind their bond positions.

Our next step is to introduce a portfolio choice decision for each fund. If there is small cost of holding bonds, we can show that funds optimally hold fewer bonds as the probability of a fire sale increases. The result is natural: a fire sale prevents the fund from holding the bond to maturity and instead forces it to sell at a discount to the dealer, so a higher probability of fire sales makes holding bonds less attractive. Since bond holdings of the funds itself influences the probability of a fire sale, these two variables are jointly determined in equilibrium.

Our final step is to extend our model to introduce a central bank acting as a backstop buyer of assets. We allow the central bank to provide additional capacity to absorb bond sales so that the total capacity consists of both central bank and dealer capacity. The central bank can also decide the price at which it is willing to purchase bonds.

We show that by committing to act aggressively, the central bank can completely eliminate the possibility of a self-fulfilling fire sale. Moreover, it does not actually have to purchase any bonds to eliminate the self-fulfilling fire sale. The aggressive policy works via market expectations. In particular, it makes the pessimistic beliefs that drive the self-fulfilling fire sale impossible for funds to rationalise because funds know that the central bank stands ready to act as a market backstop. If the central bank does not act aggressively enough, however, we show that funds will hold more bonds in response to policies designed to reduce the probability of a fire sale. This acts to increase the probability of a fire sale and partially offsets the effect of the policy.

It thus appears clear that the central bank should choose to act aggressively. However, we put forward two arguments as to why a central bank may not want to pursue an aggressive policy. First, whilst a commitment to act aggressively rules out self-fulfilling fire sales without the need for asset purchases, the central bank will be forced to act if there is a fire sale due to the subsequent crystallisation of the liquidity shock rather than self-fulfilling beliefs. The central bank may therefore wish to act less aggressively to trade off the benefit of reducing self-fulfilling fire sales with the cost of intervening in other fire sale events. Second, if market participants expect an aggressive policy, they will choose to hold more bonds. If the central bank suddenly reneges on its policy or market participants lose faith in the ability of the central bank to act aggressively, we show that the probability of self-fulfilling fire sales can increase relative to doing nothing. This highlights the importance of a consistent and credible central bank policy.

Related literature Our paper builds upon the financial market run literature initiated by Bernardo & Welch (2004) and Morris & Shin (2004). We have a similar setup to Bernardo & Welch (2004) in that funds have an incentive to preemptively sell because they are worried about future liquidity shocks and having to sell in depressed market conditions. Bernardo & Welch (2004) generate a downward sloping demand curve for risky assets via a risk-averse market maker. We do not microfound the dealer's demand curve so that we can extend the model more easily to consider policy. The main methodological difference with Bernardo & Welch (2004) is that their equilibrium concept is Nash equilibrium. Their equilibrium features a mixed strategy Nash equilibrium where investors choose just the right probability of selling to equate the payoff from selling today and holding. We view the ability of investors to choose just the right probabilities somewhat at odds with crisis periods which are infrequent and chaotic. Instead, our equilibrium features the possibility that funds panic and all sell when fundamentals are weak but otherwise all hold. We see this as a better fit to a crisis episode.

In Morris & Shin (2004), there is also a risk-neutral trading sector and a risk-averse market making sector implying a downward sloping demand curve for risky assets. Traders may choose to preemptively sell their risky assets because they are worried about prices falling such that they hit a loss limit and they lose their job. Similar to our model, they devise a global game and show that there exists a threshold equilibrium where traders hold if fundamentals are good enough and sell if they are bad enough. The motivation for preemptively selling in our paper is a potential future liquidity shock, which is more tailored to the dash for cash episode. We also extend our model to allow us to investigate the effects of the central bank providing a market backstop.

More generally, this paper is related to the literature on runs in the financial system. Diamond & Dybvig (1983) pioneered the literature by showing how the liquidity mismatch inherent in banking make banks vulnerable a self-fulfilling run. The model in Diamond & Dybvig (1983) features another equilibrium with no bank run and it is unclear which equilibrium will be realised. Therefore, Goldstein & Pauzner (2005) examine a global game variant of a bank run which eliminates the multiplicity and links the probability of bank runs to economic fundamentals. Allen et al. (2018) add a government to the model of Goldstein & Pauzner (2005) to study how government guarantees affect the probability of runs and welfare. The structure of our paper follows the development of the literature for bank runs: we first consider a model with multiple equilibria, then we employ global games techniques to eliminate the multiplicity of equilibria, and finally we extend the model to study the effects of policy.

The structure of open-ended mutual funds can also make them vulnerable to runs. As argued by Chen et al. (2010), open-ended funds allow investors to redeem their investments on a daily basis whilst investing in illiquid assets. Crucially, investors are paid based on the most recent net asset value whilst the trades are conducted in the following days. A large number of withdrawals, which forces the fund to quickly sell some of its assets, can thus depress the net asset value for those remaining in the fund which creates an incentive for them to also withdraw. Goldstein et al. (2017) show that this force is stronger in funds investing in more illiquid corporate bonds. Morris et al. (2017) provide evidence to show that asset managers hoard cash in response to redemptions which can exacerbate fire sales initiated by the redemptions themselves. These papers all examine the interaction between the fund and its investors whereas our paper examines the interaction between the funds.

Our paper is also related to two recent papers studying central bank policy in response to the "dash for cash" episode. Choi & Yorulmazer (2022) examine the role of a central bank as a market maker of last resort. They employ a cash-in-the-market framework (Allen &

Gale 1994, 1998) and model the central bank policy as a promise to inject future cash into the market. They show that this promise encourages private-sector dealers to make markets today. Interestingly, by committing to act more aggressively in the future, the central bank can reduce its asset purchases and eliminate self-fulfilling pessimistic equilibria. The results of Choi & Yorulmazer (2022) therefore mirror our own results regarding the aggressiveness of central bank action within a different framework. Eisenbach & Phelan (2022) also study the "dash for cash" episode as a market run. However, they focus on showing that a flight to safety can trigger a dash for cash in times of stress. Our focus is on how the probability of fire sales, the portfolio decision of funds, and central bank policy all interact.

1.2 Self-fulfilling fire sales

In this section, we fix the initial portfolio of the bond funds. The funds are faced with the risk of a future liquidity shock and are able to preemptively sell their bond holdings today. We first show how multiple equilibria can arise due to self-fulfilling beliefs. If funds believe that future market conditions will be poor, they choose to preemptively sell their bonds which makes market conditions poor. If funds believe market conditions will be good, they choose to hold their bonds which makes the future market conditions good. We then consider a global game variation on the model by introducing some uncertainty in the capacity of dealers to absorb bonds. We show that the global game has a unique equilibrium where self-fulfilling fire sales only occur when dealer capacity is sufficiently low.

1.2.1 Multiple equilibria

There are three dates $t = 0, 1, 2$ and a continuum of risk-neutral bond funds of measure 1. The funds are initially endowed with x units of bonds and $1-x$ units of cash. Cash returns 1 in each period. Bonds have an expected return $R > 1$ at $t = 2$ but cannot be liquidated before $t = 2$. Before $t = 2$, bonds can only be sold to dealers who will purchase the bond at a discounted price $p = R - \delta$ where $\delta > 0$. Dealers face difficulties holding large quantities of bonds so δ increases with the number of bonds held. This can be microfounded by assuming a risk averse dealer sector as in Bernardo & Welch (2004) and Morris & Shin (2004) or by appealing to balance sheet constraints.[2] In particular, assume that $\delta = \delta_L$ when the quantity of bonds held is less

[2]For example, during the dash for cash, the leverage ratio that dealers must satisfy seemingly became a binding constraint (Duffie 2020). Dealers also face a risk-weighted capital constraint that limits their ability to hold bonds on their balance sheet.

than $K \in [0, 1]$ and $\delta = \delta_H$ when the quantity of bonds held is greater than K where $\delta_L < \delta_H$.

At $t = 0$, dealers first set the price they are willing to pay for bonds p_0. Since their bond holdings are initially zero, they set $p_0 = R - \delta_L$. At $t = 0$, funds learn of the possibility of being hit by a liquidity shock at $t = 1$ with probability q independent of the bond return. They can choose to sell their bond holdings at $t = 0$ at price p_0 to prepare for the liquidity shock or hold the asset. If they face the liquidity shock, they are forced to sell their bonds at $t = 1$ at price p_1. If they do not face the liquidity shock, they can hold the bonds through to maturity at $t = 2$ and earn the return R.

A key assumption in our analysis is that prices do not fully adjust within the period due to selling pressure within the same period. In particular, all selling at $t = 0$ takes place at $p_0 = R - \delta_L$, after which the price adjusts for $t = 1$. We motivate this by assuming that the fund may submit their sell order at $p_0 = R - \delta_L$ but not take into account that prices may have adjusted by the time that their sell order is executed.[3] Indeed, Haddad et al. (2021) report that during the dash for cash there was evidence of "liquidity inversion" where assets that are normally more liquid experienced price discounts greater than assets that are typically more illiquid. They argue that funds did not take into account real time price adjustments when making their trading decisions due to the speed at which the crisis took place. This assumption can create an incentive to preemptively sell bonds: funds believe that selling at $t = 0$ guarantees a high price whereas waiting until $t = 1$ carries the risk of being forced to sell at a low price if the liquidity shock hits and other funds have already sold to the dealer.[4]

Given this assumption and since the funds are atomistic, they take p_0 as given and choose to sell at $t = 0$ if

$$p_0 > qp_1(\tilde{\delta}) + (1 - q)R \tag{1.1}$$

where $\tilde{\delta} \in \{\delta_L, \delta_H\}$ is a fund's belief about the discount applied by the dealer for any trades at

[3] Alternatively, the existence of a small friction in dealer price setting so that dealers do not instantaneously adjust the price in response to selling pressure would justify this assumption.

[4] Our assumption is similar to the assumption from the market-run literature (Bernardo & Welch 2004, Morris & Shin 2004) that execution order is not perfectly sequential. In these models, when investors submit a sell order at $t = 0$, they join a queue of investors wishing to sell at $t = 0$. Because investors do not know their exact position in the queue, the models assume that investors believe they will be in the middle of the queue. Waiting to sell until $t = 1$ ensures a position at the rear of the queue. Since investors further down the queue receive a lower price, this gives an incentive to preemptively sell. In our model, our assumption that funds do not take into account within-period price adjustment is equivalent to assuming that all funds believe they will be at the front of the queue rather than the middle. Therefore, the underlying reason for pre-emptive selling in our model is identical to the market-run literature.

$t = 1$. Using $p = R - \delta$, we can write (1.1) as

$$q\tilde{\delta} > \delta_L. \tag{1.2}$$

Thus, the choice of an individual fund to sell depends on whether it expects the dealer to be under strain or not. We look for an equilibrium where $\tilde{\delta} = \delta$, that is, beliefs coincide with outcomes.

Proposition 1.1. *An equilibrium with $\tilde{\delta} = \delta_L$ always exists. If (i) $q\delta_H > \delta_L$ and (ii) $x > K$ both hold, then a second equilibrium exists with $\tilde{\delta} = \delta_H$.*

Proof. If $\tilde{\delta} = \delta_L$, then from (1.2) the condition for selling is $q > 1$. Therefore, an individual fund has no incentive to sell. Since all funds are identical, no fund sells. This means that the bond holdings of the dealer going into $t = 1$ are zero. This implies $\delta = \delta_L$ so we have an equilibrium.

Suppose instead that $\tilde{\delta} = \delta_H$. If $q\delta_H \leq \delta_L$, then from (1.2) we see that no fund chooses to sell. This means that the bond holdings of the dealer remain zero which implies $\delta = \delta_L$. Therefore, this cannot be an equilibrium. If $q\delta_H > \delta_L$, then an individual fund wishes to sell. Therefore, all funds choose to sell and the dealer absorbs x assets. If $x \leq K$, then we still have $\delta = \delta_L$ so this is not an equilibrium. If $x > K$, then the dealer will set $\delta = \delta_H$ for period 1. This gives the other equilibrium. $\square$

From Proposition 1.1, we see that an equilibrium with no fire selling always exists. If funds expect future market conditions to be good, there is no incentive to preemptively sell bonds. Because funds do not preemptively sell their bonds, market conditions indeed are good.

However, we also see that there is potential for an equilibrium with self-fulfilling fire sales to exist. In this equilibrium, funds sell their bonds at $t = 0$ leading to fire sale prices at $t = 1$ simply because funds expect fire sale prices at $t = 1$. The first condition for the fire selling equilibrium to exist is that it must be sufficiently likely that funds are hit by a liquidity shock and then have to sell their bonds at depressed prices. This condition gives the incentive for funds to sell at $t = 0$ and potentially put the dealer under strain. But even if this condition is satisfied, we still require that the total quantity of bonds the dealer absorbs be large enough so that they are indeed put under strain when funds choose to sell.

1.2.2 Unique threshold equilibrium via global games

An undesirable feature of the multiple equilibria result in Proposition 1.1 is that the model is silent about how likely the fire sale equilibrium is to arise. Following the global games literature (Carlsson & van Damme 1993, Morris & Shin 1998), we now introduce a small amount of private noise to the model which eliminates the multiplicity. The result is that we can link the probability of a self-fulfilling fire sale to economic fundamentals.

Now assume that funds face uncertainty about the dealer capacity K. Let the prior distribution of K be uniform on $[0, 1]$ and denote the density function by $f(\cdot)$. K is also independent of the liquidity shock and the bond return. At $t = 0$, each fund i observes a private signal

$$z_i = K + e_i \tag{1.3}$$

where e_i is a noise term which is uniformly distributed on $[-\varepsilon, \varepsilon]$ where $\varepsilon > 0$ is small.[5] This assumption reflects the fact that funds face some uncertainty related to the true capacity of the dealer bank because they do not know the exact details of the dealer's balance sheet position.

After observing their private signal, funds once again have a choice whether to sell their bond holdings at p_0 or hold. Let $a \in \{\text{sell}, \text{hold}\}$ denote the action. Denote the proportion of funds selling by s. Each fund's payoff $u(a, s, K)$ is a function of the action taken, the proportion of funds selling, and the dealer capacity. We can therefore write the payoffs as

$$u(\text{sell}, s, K) = p_0 = R - \delta_L \tag{1.4}$$

$$u(\text{hold}, s, K) = qp_1(\delta) + (1-q)R = \begin{cases} q(R-\delta_L) + (1-q)R & \text{if } sx \leq K \\ q(R-\delta_H) + (1-q)R & \text{if } sx > K. \end{cases} \tag{1.5}$$

Each fund holds x bonds so, if a proportion s sell, the dealer will have to absorb sx bonds. If sx exceeds K, the dealer sets $p_1 = R - \delta_H$. Defining $\pi(s, K) \equiv u(\text{sell}, s, K) - u(\text{hold}, s, K)$ as the payoff gain from selling, we have

$$\pi(s, K) = \begin{cases} -(1-q)\delta_L & \text{if } sx \leq K \\ q\delta_H - \delta_L & \text{if } sx > K. \end{cases} \tag{1.6}$$

Notice that if $q\delta_H \leq \delta_L$ or $x \leq K$, the dominant strategy is to hold. If both of these conditions fail, then the decision whether to sell or hold depends on a fund's belief about s. We therefore

[5] The results go through for a general density function for the noise whenever the density becomes concentrated around zero (see Morris & Shin (2003)).

need to derive this belief.

To make progress, we look for an equilibrium where funds follow a switching strategy of the form:

$$a = \begin{cases} \text{hold} & \text{if } z_i > z^* \\ \text{sell} & \text{if } z_i \leq z^* \end{cases} \tag{1.7}$$

where z^* is a threshold value to be determined. We are then able to derive the distribution of s for a fund whose private signal is exactly the threshold z^*.

Lemma 1.1. *Suppose that funds follow the switching strategy around z^* as in (1.7). Then the density of s conditional on observing a signal equal to z^* is uniform over* $[0, 1]$.

Proof. The proof follows Morris et al. (2017). When the true dealer capacity is K, the signals $\{z_i\}$ are distributed uniformly over $[K - \varepsilon, K + \varepsilon]$. Funds with signals $z_i < z^*$ choose to sell. Hence,

$$s = \frac{z^* - (K - \varepsilon)}{2\varepsilon}.$$

To derive the distribution of s conditional on z^*, we derive the cumulative distribution function. In particular, we compute the probability that $s < b$ conditional on z^*. Define K_0 as

$$b = \frac{z^* - (K_0 - \varepsilon)}{2\varepsilon} \implies K_0 = z^* + \varepsilon - 2\varepsilon b.$$

Thus, $s < b$ if and only if $K > K_0$. We therefore need the probability of $K > K_0$ conditional on z^*. Fund i's posterior density over K conditional on z^* is $[z^* - \varepsilon, z^* + \varepsilon]$. Therefore,

$$\Pr(K > K_0 | z^*) = \frac{z^* + \varepsilon - K_0}{2\varepsilon} = \frac{z^* + \varepsilon - (z^* + \varepsilon - 2b\varepsilon)}{2\varepsilon} = b.$$

This is the cumulative distribution function of the uniform distribution. □

Given Lemma 1.1, we can now determine the equilibrium of the game. We focus on the case when $\varepsilon \to 0$, that is, the private noise vanishes. This implies that the only role of the private noise is to break down the perfect coordination that generates multiple equilibria and allows direct comparison with Proposition 1.1.

Proposition 1.2. *Let $\varepsilon \to 0$. If $q\delta_H \leq \delta_L$ or $x < K$, all funds have a dominant strategy to hold. If $q\delta_H > \delta_L$ and $x > K$, then there is an equilibrium where all funds hold if $K > K^*$ and* *all funds sell if* $K < K^*$ *where*

$$K^* = \left[\frac{q\delta_H - \delta_L}{q(\delta_H - \delta_L)}\right] x \equiv \alpha x < x. \tag{1.8}$$

K^* *is the ex-ante probability of a self-fulfilling fire sale occurring and* α *is the ex-ante probability of a self-fulfilling fire sale occurring conditional on* $x > K$*.*

Proof. Consider a fund observing a signal z_i. Fund i's expectation of K is then z_i. If $z_i \geq x$, from (1.6) we see that it is always optimal for fund i to hold regardless of the value of s. Therefore, since as $\varepsilon \to 0$ we have $z_i \to K$, we have that the funds have a dominant strategy to hold whenever $x \leq K$. It is also evident from (1.6) that funds have a dominant strategy to hold when $q\delta_H \leq \delta_L$.

If $z_i < x$, then fund i's optimal decision to sell or hold depends on the value of s. To make progress, consider a fund observing signal z^*. From Lemma 1.1, we know that the fund believes the density of s is uniform on $[0,1]$ if all funds play the threshold strategy according to (1.7). Therefore, for $z^* < x$ we have

$$\int_0^1 \pi(s, z^*) ds = -(1-q)\delta_L \frac{z^*}{x} + \left(1 - \frac{z^*}{x}\right)(q\delta_H - \delta_L).$$

If $s < z^*/x$, then the fund believes that the dealer faces no stress. Since s is uniform on $[0,1]$ conditional on z^*, the fund believes that the probability of no stress is z^*/x. This gives the first term. The second term follows because the fund believes that the probability of the dealer being under stress is $1 - z^*/x$.

The threshold value z^* must satisfy $\int_0^1 \pi(s, z^*) ds = 0$. This yields

$$z^* = \left[\frac{q\delta_H - \delta_L}{q(\delta_H - \delta_L)}\right] x.$$

To complete the argument, we need to show that if $z_i < z^*$, fund i prefers to sell, and if $z_i > z^*$, fund i prefers to hold. The details are given in Appendix A.

Letting $\varepsilon \to 0$ implies that the threshold value of z^* corresponds to a threshold value of K^*. Therefore, funds will hold if $K > K^*$ and we will get a self-fulfilling fire sale when $K < K^*$. Since the prior distribution of K is uniform on $[0,1]$, K^* is also the *ex-ante* probability of a self-fulfilling fire sale. Finally, $\Pr(\text{fire sale}|x > K) = \alpha x / x = \alpha$. □

It is instructive to contrast Propositions 1.1 and 1.2. In both models, if $q\delta_H \leq \delta_L$ or $x < K$, then

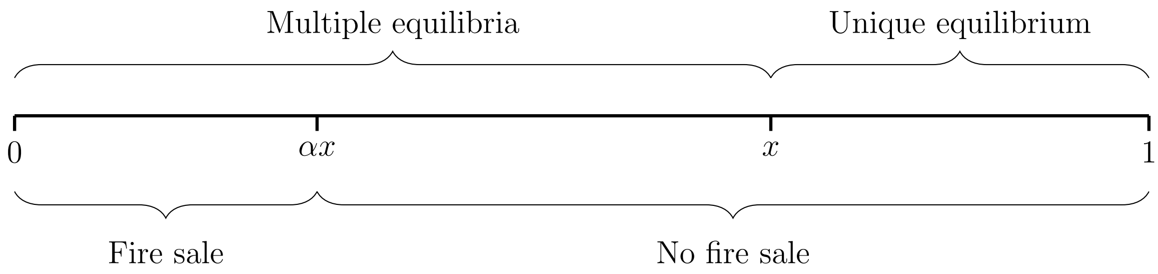


Figure 1.1: Equilibrium outcomes for different values of K

there is only one equilibrium where all funds hold.[6] The value of the global games approach arises when $q\delta_H > \delta_L$ and $x > K$, in which case there are multiple equilibria in the framework without private noise. The global games approach tells us which one of these equilibria will arise as a function of economic fundamentals. Figure 1.1 summarises the results of Propositions 1.1 and 1.2 by depicting the equilibrium outcomes as a function of K, with Proposition 1.1 above the bold line and Proposition 1.2 below it.

We can also see how the *ex-ante* probability of the fire sale equilibrium changes as economic fundamentals change.

Corollary 1.1. *The following are true:*

1. $\partial K^*/\partial x > 0$. *A higher value of x means that funds hold more bonds at the outset. This means that there are more bonds that could potentially be sold to the dealer, which increases the probability that dealer will be under stress. The incentive to sell today increases, so a fire sale is ex-ante more likely.*

2. $\partial K^*/\partial q > 0$. *A higher value of q means it is more likely that a fund will get hit by a liquidity shock. The incentive to sell today increases, so a fire sale is ex-ante more likely.*

3. $\partial K^*/\partial \delta_H > 0$. *As the severity of the stressed state increases, the incentive to sell today increases. The ex-ante probability of a fire sale occurring increases.*

1.3 Optimal portfolio choice

In the analysis thus far, we have taken the choice of the quantity of bonds held as given. In this section, we endogenise this choice and examine the interaction between the probability of fire sales K^* and the optimal portfolio choice.

[6]The ex-ante probability of $x = K$ is zero so we ignore this case.

Suppose that before funds receive their private signals, they make their choice of x. In the current setup, funds are risk-neutral. Therefore, they will simply choose to hold either all cash or all bonds depending on which has a higher expected payoff. To break this result, assume that there is a penalty for variance in the portfolio $c(x) = \gamma x^2$ where $\gamma > 0$.[7] Note that the binary action choice at $t = 0$ combined with no time discounting implies that the introduction of this cost does not affect the optimal choice of a fund to sell or hold in section 1.2. If they sell their entire bond holdings at $t = 0$, they gain some benefit $c > 0$ for no longer holding bonds. If they hold their bonds into $t = 1$, they then either sell their bonds or they mature giving the same benefit c in either case. Therefore, it is optimal to sell if $p_0 x + c > q(p_1 x + c) + (1 - q)(Rx + c)$ which simplifies to equation (1.1).

We focus on an equilibrium where the *ex-ante* probability of a fire sale is strictly positive. Following Proposition 1.2, we therefore must have that (i) $q\delta_H > \delta_L$ and (ii) $x > K$. The former condition we can simply assume. However, since we are now endogenising x, we cannot simply assume that $x > K$. Instead, we guess that $x > K$ and show that we can find an equilibrium where our guess is indeed true.

An individual fund thus takes the aggregate bond holding x (and therefore K^*) as given. Bonds are available in perfectly elastic supply with the price normalised to 1. The fund therefore solves the problem

$$\max_{x \in [0,1]} \quad (1 - x) + x \left[\int_0^{K^*} (R - \delta_L) f(K) dK + \int_{K^*}^1 (q(R - \delta_L) + (1 - q)R)) f(K) dK \right] - \gamma x^2. \tag{1.9}$$

The first term in the square brackets gives the payoff when a self-fulfilling fire sale occurs and funds all sell their entire bond holdings at $p_0 = R - \delta_L$. This outcome occurs for values of K between 0 and K^*. The second term gives the payoff when everyone holds. With probability q, the funds face a liquidity shock at $t = 1$ and are forced to sell their bond holdings. Since the stock of bonds held on the dealer's balance sheet going into $t = 1$ is zero, the price p_1 is $R - \delta_L$. With probability $1 - q$, the funds are able to hold the funds through to maturity and obtain the expected return R. These outcomes occur for values of K between K^* and 1.

The solution to (1.9) gives the optimal x as a function of K^*. Proposition 1.2 derived K^* as a function of x. An equilibrium is thus an (x_e, K_e^*) pair where x_e solves (1.9) conditional on K_e^*

[7] Note that the variance of the portfolio is $\text{Var}[(1 - x) + xR_B]$ where R_B is the overall return on holding bonds. The overall bond return has three sources of stochasticity: the dealer capacity K which determines the occurrence of a fire sale, the liquidity shock, and the bond return itself. Using the rules for the variance operator, the portfolio variance is $x^2 \text{Var}[R_B]$ which motivates the quadratic penalty term. For simplicity, we assume that $\text{Var}[R_B]$ is a constant γ, although strictly speaking γ depends on other model parameters.

and x_e implies $K^* = K_e^*$ according to (1.8).

Proposition 1.3. *Conditional on K^*, aggregate fund bond holdings are*

$$x = \frac{R - 1 - \delta_L(q + K^*(1-q))}{2\gamma}. \tag{1.10}$$

There exists a unique equilibrium pair (x_e, K_e^*) which simultaneously satisfies (1.8) and (1.10):

$$x_e = \frac{R - 1 - \delta_L q}{2\gamma + \delta_L \alpha(1-q)}, \tag{1.11}$$

$$K_e^* = \alpha x_e. \tag{1.12}$$

Proof. First guess that $x > K$ so that it is possible for a self-fulfilling fire sale to occur. We can then solve the fund's problem in (1.9). Using the fact that the prior distribution of K is uniform over $[0,1]$, we can write the term in the square brackets as

$$K^*(R - \delta_L) + (1 - K^*)(q(R - \delta_L) + (1-q)R).$$

This simplifies to

$$R - \delta_L(K^*(1-q) + q).$$

The fund problem can thus be written as

$$\max_{x \in [0,1]} \quad (1 - x) + x(R - \delta_L(q + K^*(1-q))) - \gamma x^2.$$

Assuming the parameters are such that an interior solution exists, there is a unique maximum given by (1.10). Since all funds are *ex-ante* identical, this is the optimal x for all funds and therefore the aggregate fund bond holdings. Substituting $K^* = \alpha x$ from (1.8) into (1.10) and solving for x gives x_e which then immediately implies $K_e^* = \alpha x_e$. Finally, we need to verify our guess that $x > K$. We can always ensure this guess is correct by choosing R to be suitably large. $\square$

The first result in Proposition 1.3 is that aggregate fund bond holdings are decreasing in K^*. This is because a fire sale prevents the fund from holding the bond to maturity and instead forces it to sell at a discount to the dealer. Therefore, a higher probability of fire sales makes holding bonds less attractive. Holding K^* fixed, the effect of K^* on x is attenuated by (i) a reduction in δ_L and (ii) an increase in q. A reduction in δ_L makes fire sales less costly so changes in the probability of fire sales have less of an effect on bond holdings. An increase in q makes

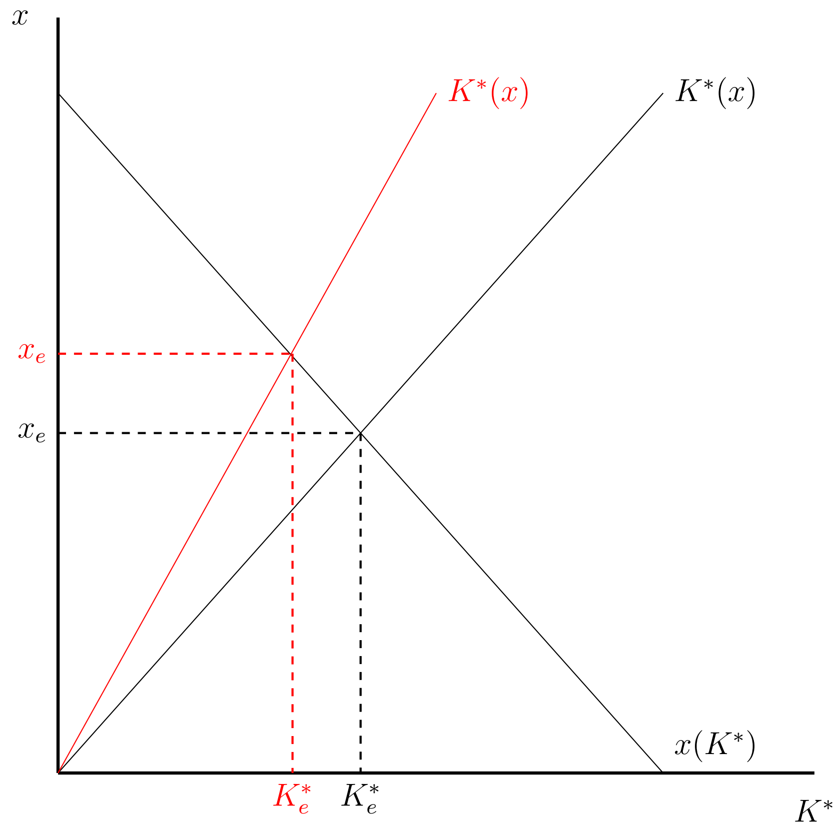


Figure 1.2: Joint determination of x and K^* in equilibrium. A reduction in δ_H reduces the *ex-ante* probability of a fire sale.

it less likely that funds will be able to hold their bonds to maturity due to the $t = 1$ liquidity shock so the fire sale at $t = 0$ becomes less relevant.

We can build graphical intuition for the second result in Proposition 1.3 by drawing equations (1.8) and (1.10), which characterise equilibrium, in (K^*, x) space. In particular, notice that (1.8) is an upward sloping line passing through the origin with slope $1/\alpha$. Equation (1.10), is a downward sloping line with vertical intercept $(R - 1 - \delta_L q)/2\gamma$ and slope $-\delta_L(1 - q)/2\gamma$. The equilibrium is depicted by the black lines in Figure 1.2 with the intersection giving the unique mutually consistent (x, K^*) pair.

We can also perform comparative statics exercises with the aid of our diagram. Suppose that there a reduction in the discount the dealer applies to the bond in stressed market conditions: a reduction in δ_H. From equations (1.8) and (1.10), we see that this increases the slope of the $K^*(x)$ curve and does not affect the $x(K^*)$ curve. The new equilibrium is shown in red in Figure 1.2. The new equilibrium is associated with a lower *ex-ante* probability of fire sales and funds holding a higher quantity of bonds. Notice that the fall in K^* would have been greater if funds did not have the ability to adjust their portfolio in response to the change in δ_H. When

δ_H falls, funds are *ex-ante* less likely to face a fire sale. Bonds are therefore more attractive so funds choose to hold more bonds. But if funds hold more bonds, they increase the probability of facing a fire sale. This partially undoes the effect the fall in δ_H has on reducing the probabilty of fire sales.

1.4 Central bank provision of a market backstop

We now extend our global games framework to study the central bank acting as a backstop buyer of assets. We show that, if the central bank commits to act aggressively enough, then it is able to completely eliminate the possibility of self-fulfilling fire sales. If the central bank does not act with sufficient force, perhaps due to lack of credibility or political constraints, then it can reduce but not entirely eliminate the probability of a self-fulfilling fire sale. The effectiveness of an insufficiently forceful policy is further reduced when funds have time to reoptimise their portfolios in response to central bank policy. This would be the case if the use of market backstops was part of the central bank's permanent policy toolkit.

It thus appears that the central bank should simply commit to act aggressively to completely eliminate the possibility of self-fulfilling fire sales. Using our model, we highlight some reasons why this aggressive policy stance may be undesirable. We complete this section with a discussion of alternative policies that could reduce the probability of self-fulfilling fire sales within the context of our model.

1.4.1 Using market backstops to reduce the probability of fire sales

We introduce a market backstop policy by allowing the central bank to provide additional capacity for absorbing bond sales through an asset purchasing facility. Denote this capacity by K_{CB}. Consistent with the central bank acting as a backstop buyer, the central bank may wish to purchase bonds at a discount $\delta_{CB} \in (\delta_L, \delta_H]$. This mirrors the principle set out by Bagehot (1873) for central banks acting as a lender of last resort. Notice that this specification of policy includes the case of the central bank setting an asset price floor at $R - \delta_{CB}$. This would be achieved by the central bank setting K_{CB} such that $K + K_{CB} = 1$. The market backstop policy is therefore fully described by the pair (K_{CB}, δ_{CB}).

With a central bank, the payoffs become

$$u(\text{sell}, s, K) = R - \delta_L \tag{1.13}$$

$$u(\text{hold}, s, K) = \begin{cases} q(R-\delta_L) + (1-q)R & \text{if } sx \in [0, K] \\ q(R-\delta_{CB}) + (1-q)R & \text{if } sx \in (K, K+K_{CB}] \\ q(R-\delta_H) + (1-q)R & \text{if } sx \in (K+K_{CB}, 1]. \end{cases} \tag{1.14}$$

Under the assumption, discussed in section 1.2.1, that funds do not take into account real time price adjustments during the crisis, funds that choose to preemptively liquidate at $t = 0$ receive the payoff $R - \delta_L$. Funds who choose to hold can sell to the dealer or the central bank at $t = 1$ if hit by the liquidity shock. If $sx \leq K$, then the dealer enters $t = 1$ under no stress and offering to purchase bonds at the discount δ_L. If $K < sx \leq K + K_{CB}$, then the dealer enters $t = 1$ under stress. The backstop policy is thus in operation and the central bank enters period $t = 1$ offering to purchase bonds at the discount δ_{CB}. If $sx > K + K_{CB}$, then the central bank's capacity to absorb bonds has been exceeded in $t = 0$ and the only option for a fund requiring liquidity is to sell to the dealer under stress at the discount δ_H.

The payoff gain from selling is

$$\pi(s, K) = \begin{cases} -(1-q)\delta_L & \text{if } sx \in [0, K] \\ q\delta_{CB} - \delta_L & \text{if } sx \in (K, K+K_{CB}] \\ q\delta_H - \delta_L & \text{if } sx \in (K+K_{CB}, 1]. \end{cases} \tag{1.15}$$

Following similar steps to Proposition 1.2, we can prove the following result.

Proposition 1.4. *Let $\varepsilon \to 0$. Suppose that $q\delta_H > \delta_L$ and $x > K$. If $K + K_{CB} > x$ and $q\delta_{CB} \leq \delta_L$, then funds have a dominant strategy to hold. If $q\delta_{CB} > \delta_L$, then there is an equilibrium where all funds hold if $K > \tilde{K}^*$ and all funds sell if $K < \tilde{K}^*$ where*

$$\tilde{K}^* = \begin{cases} \left[\frac{q\delta_{CB}-\delta_L}{q(\delta_{CB}-\delta_L)}\right] x & \textit{if } K + K_{CB} > x \\ \left[\frac{q\delta_H-\delta_L}{q(\delta_H-\delta_L)}\right] x - \left[\frac{\delta_H-\delta_{CB}}{\delta_H-\delta_L}\right] K_{CB} & \textit{if } K + K_{CB} < x. \end{cases} \tag{1.16}$$

$\tilde{K}^*$ is decreasing in K_{CB} and increasing in δ_{CB}.

Proof. See Appendix A. □

The first point to note about Proposition 1.4 is that by assuming that $q\delta_H > \delta_L$ and $x > K$ we provide the conditions for a possible fire sale in the model without a central bank. If a

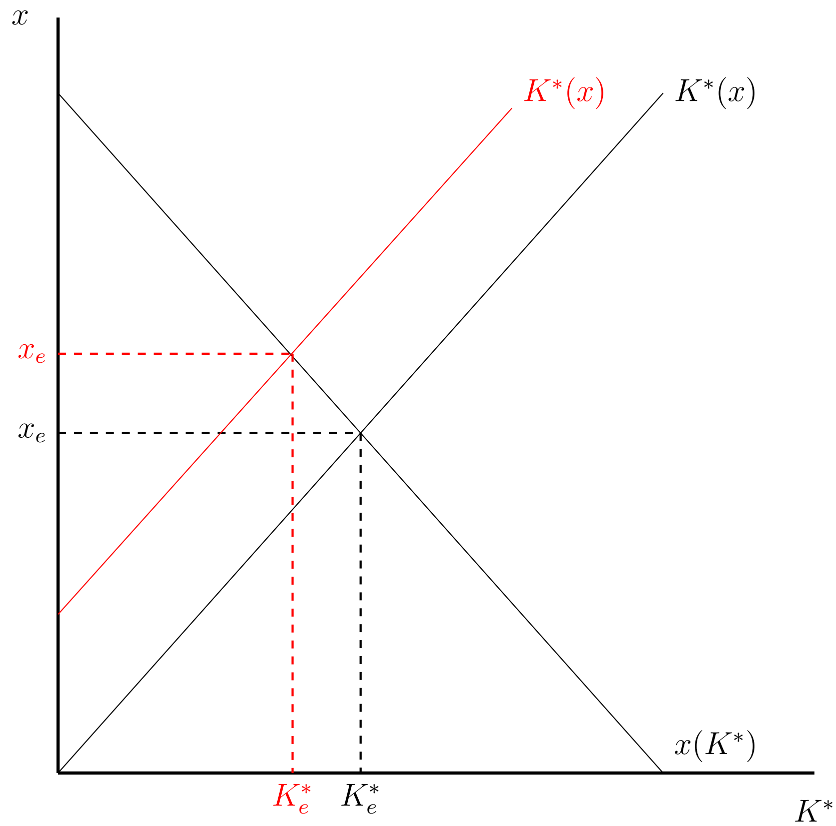


Figure 1.3: An increase in K_{CB} and/or a reduction in δ_{CB} when $K + K_{CB} < x$

central bank dislikes fire sales, we therefore potentially have scope for policy action to reduce the probability of a self-fulfilling fire sale occurring.

The central bank can eliminate the possibility of fire sales entirely by acting aggressively. For example, if it sets $K_{CB} = 1 - K$ and $\delta_{CB} \leq \delta_L/q$, funds have a dominant strategy to hold for all values of x. This result corresponds with the theoretical result in Choi & Yorulmazer (2022) where the central bank may wish to act aggressively to rule out all equilibria except the good one with no fire sales. Notice that the central bank facility needs to be sufficiently large *and* the discount needs to be small enough to eliminate the fire sale outcome entirely: they are not substitutable. The reason is straightforward. If the capacity is large but the discount δ_{CB} is also large, then selling to the central bank is still a bad outcome so the central bank does not remove the incentive to preemptively sell. If the discount δ_{CB} is small but the capacity is small, then the central bank's purchasing facility is close to irrelevant and funds will still put positive weight on the possibility of having to sell to the dealer at discount δ_H. Also notice that the discount δ_{CB} does not need to equal δ_L to eliminate the possibility of fire sales.

Importantly, the central bank only needs to credibly announce a policy for it to be effective. Moreover, the facility is not used at $t = 0$ because all funds choose to hold. This matches the empirical evidence from the dash for cash where the announcement of the Federal Reserve's corporate bond purchase programme alone calmed corporate bond markets (Haddad et al. 2021). It also fits with the experience of Euro area sovereign debt markets after Mario Draghi's "whatever it takes" speech; Draghi's promise calmed markets without the ECB having to buy any bonds. Indeed, this relates to the general idea that asset purchases for financial stability purposes should be "catalytic" in the sense that they restore the functioning of private markets rather than involving considerable direct intervention (Cecchetti & Tucker 2021).

Now consider the case when $q\delta_{CB} > \delta_L$ so the fire sale outcome is not ruled out. If central bank capacity is sufficiently large, $K + K_{CB} > x$, the central bank is essentially setting an asset price floor at $R - \delta_{CB}$. The expression for $\tilde{K}^*$ is identical to K^* in Proposition 1.2 with δ_H replaced by δ_{CB}. A reduction in δ_{CB} therefore reduces $\tilde{K}^*$. Further changes in K_{CB} have no effect.

If the central bank capacity is smaller such that $K + K_{CB} < x$, then the central bank does not rule out the possibility of a fund having to sell to a stressed dealer. In this case, both an increase in K_{CB} and a reduction in δ_{CB} reduce the probability of a self-fulfilling fire sale. Thus, reducing the discount δ_{CB} and increasing the size of the facility K_{CB} can substitute for each other. Moreover, the policies are complementary in the sense that $|\partial K^*/\partial K_{CB}|$ is larger if δ_{CB} is lower and $|\partial K^*/\partial \delta_{CB}|$ is larger if K_{CB} is larger.

Portfolio reoptimisation Now suppose that the central bank announces the policy (K_{CB}, δ_{CB}) ahead of time such that funds are able to optimally choose their value of x in response to the policy.

Note that the optimal portfolio choice problem is not directly affected by the central bank providing a market backstop. The only channel through which central bank policy affects the portfolio choice problem is through influencing the probability of a fire sale $\tilde{K}^*$, which the atomistic funds take as given when choosing x.

Consider first a reduction in δ_{CB} when $K + K_{CB} = 1$, that is, the central bank sets an asset price floor at $R - \delta_{CB}$. The equilibrium outcome is identical to Figure 1.2 showing a reduction in δ_H. The endogenous response of funds to the policy partially offsets the effect of changing δ_{CB} on the probability of a self-fulfilling fire sale.

Next consider the case when $K + K_{CB} < x$. A reduction in δ_{CB} or an increase in K_{CB} will reduce $\tilde{K}^*$ and the effect is independent of the value of x. This is shown graphically in Figure 1.3 as a leftward shift in the $K^*(x)$ curve. We again have a reduction in the equilibrium probability of a fire sale which is partially offset by the endogenous increase in x.

To summarise the discussion, we have seen that if the central bank commits to act with sufficient force, it can completely eliminate self-fulfilling fire sales as a potential equilibrium outcome. Moreover, the central bank will not actually have to purchase any bonds at $t = 0$ for the policy to be effective. If it acts less aggressively, it can still reduce the probability of a self-fulfilling fire sale. However, if the central bank does not take into account the effect of its policy on funds' portfolio choice, it will think its policy is more effective at reducing the probability of fire sales than it actually is. It therefore appears as if central banks have a simple decision: act aggressively to rule out the fire sale outcome. In the next section, however, we discuss some potential pitfalls with such a policy.

1.4.2 Potential problems with providing a market backstop

Costs of central banks holding assets Thus far, we have implicitly assumed that the central bank only cares about reducing the probability of a self-fulfilling fire sale. If enacting a policy (K_{CB}, δ_{CB}) is costless, then the central bank will optimally act aggressively to reduce the probability of a self-fulfilling fire sale to zero. However, in practice, a central bank may not wish to hold certain assets on its balance sheet. Reasons may include political economy considerations, with any losses on purchased assets ultimately falling to the taxpayer. Purchasing bonds may therefore impose a cost on the central bank. We can represent this cost with a central bank cost function $\mathcal{C}(Q_{CB}, \delta_{CB})$ where Q_{CB} denotes the quantity of bonds purchased by the central bank. The cost function is increasing in Q_{CB} and decreasing in δ_{CB}.

This cost, combined with the potential liquidity shock at $t = 1$, implies that acting aggressively to eliminate the self-fulfilling fire sale in $t = 0$ may no longer be optimal. To see this, note that when a central bank is choosing the policy (K_{CB}, δ_{CB}) before the funds make any decisions, its ex-ante expected cost is

$$\mathcal{C}(Q_{CB}^{fs}, \delta_{CB})\tilde{K}^*(K_{CB}, \delta_{CB}) + \mathcal{C}(Q_{CB}^{ls}, \delta_{CB})(1 - \tilde{K}^*(K_{CB}, \delta_{CB}))q \tag{1.17}$$

where the dependence of $\tilde{K}^*$ on policy is made explicit. With probability $\tilde{K}^*$, there is a self-fulfilling fire sale at $t = 0$ and the central bank purchases Q_{CB}^{fs} bonds at discount δ_{CB}. With probability $(1 - \tilde{K}^*)q$, there is no self-fulfilling fire sale at $t = 0$ followed by a liquidity shock at $t = 1$. In this case, a central bank committed to providing a market backstop would purchase Q_{CB}^{ls} bonds at discount δ_{CB}. Evidently, Q_{CB}^{fs} and Q_{CB}^{ls} are weakly increasing in K_{CB} but may be less than K_{CB} if bond sales do not exhaust the central bank's capacity to absorb bonds.

An aggressive backstop policy (high K_{CB} and low δ_{CB}) ensures that $\tilde{K}^* = 0$. With no cost

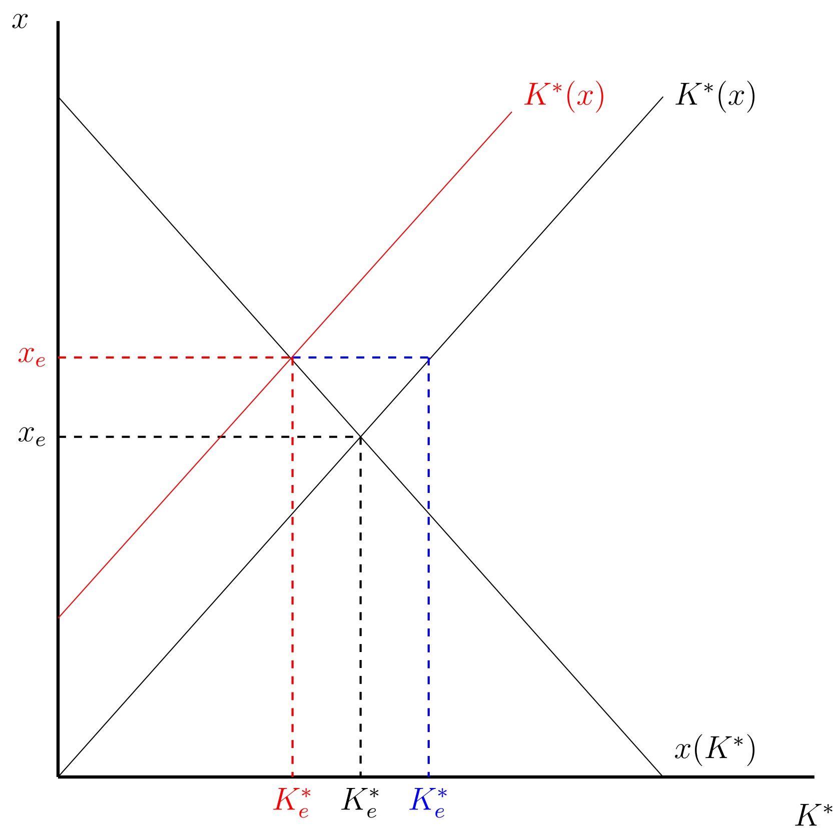


Figure 1.4: Shifts in market expectations can increase the probability of self-fulfilling fire sales

of purchasing bonds and a benefit to reducing the probability of self-fulfilling fire sales, it is clearly optimal to pursue an aggressive policy. However, from equation (1.17), we now see that there is a positive ex-ante expected cost from an aggressive policy, $\mathcal{C}(Q^{ls}_{CB}, \delta_{CB})q$, due to the presence of the liquidity shock. Depending on the cost function $\mathcal{C}(\cdot, \cdot)$ and the benefit of reducing the probability of self-fulfilling fire sales, it may now be optimal for a central bank to pursue a less aggressive policy. Such a policy would trade off an increased probability of self-fulfilling fire sales at $t = 0$ against the reduced cost of intervening aggressively when there are immediate liquidity needs at $t = 1$. Put differently, an aggressive backstop policy is no longer a "free lunch" when there are (i) costs to intervening in markets and (ii) fire sales driven by fundamental liquidity needs rather than self-fulfilling beliefs.

Expectations of market participants If the backstop policy is not clear, consistent, and credible, central bank action may increase the probability of self-fulfilling fire sales.

Suppose, for example, that market participants expect the central bank to enact a policy (K_{CB}, δ_{CB}) consistent with the equilibrium given in red in Figure 1.4. Next suppose at the start of the crisis period at $t = 0$, expectations shift so that the relevant $K^*(x)$ curve is the black line (in this case representing no backstop policy). The market run game played between the funds would therefore take place with the higher red x_e as the bond portfolio share. Following the blue line across in the example in Figure 1.4, we see that the probability of a self-fulfilling fire sale actually increases relative to the scenario where expectations remained anchored on the equilibrium in black.

Why might market expectations shift in this manner? One possible reason is if the central bank initially promises the policy (K_{CB}, δ_{CB}) and market participants believe the promise is credible. Expectations could then shift if the central bank explicitly reneges on its policy commitment or market participants suddenly believe that the central bank will no longer follow through on its promise. Thus, any central bank committed to providing a market backstop must ensure that announced policy is consistent and can be credibly implemented in a crisis.

In the absence of an explicit policy commitment, markets may instead infer the policy (K_{CB}, δ_{CB}) from central bank actions or statements. If this inference turns out to be incorrect, market expectations could shift at the onset of the crisis. Our observation thus cautions against the principle of "constructive ambiguity" advocated by some in the context of lender of last resort policies (see, for example, George (1994)). The idea of constructive ambiguity is that the central bank can avoid moral hazard by making access to emergency support facilities uncertain. However, such ambiguity creates the possibility that market participants' expectations will become overly optimistic about the extent of central bank support and we see from Figure 1.4

that this can increase the risk of financial instability. This point is therefore similar to Hanson et al. (2020) who show how misperceptions about central bank interventions in asset markets can lead to sudden price declines if investors overestimate the aggressiveness of central bank intervention.

Incorrect expectations of market participants could even force the central bank to provide a market backstop when it would prefer not to. Suppose that the costs of providing a backstop are deemed too high such that the central bank would not want to provide one: the central bank would prefer the black equilibrium in Figure 1.4. However, if funds expect the policy (K_{CB}, δ_{CB}), they will act in a manner consistent with the red equilibrium in Figure 1.4. At the onset of the crisis, the central bank therefore has a choice. It could not intervene and face an increased probability of self-fulfilling fire sales relative to the black equilibrium. Alternatively, it could choose to intervene consistent with market participants' expectations and lower the probability of a self-fulfilling fire sale at the cost of providing a market backstop. Depending on the objective function of the central bank, it may choose to provide a backstop in line with the expectations of market participants.

1.4.3 Alternative policies

Given the difficulties with providing a market backstop, the central bank may wish to consider using alternative policies to reduce the probability of fire sales. Indeed, Hauser (2021) identifies three steps to strengthen market functioning: (1) reforms to improve resilience of financial institutions to liquidity shocks, (2) strengthened market-wide infrastructure, and (3) central bank backstops. We now provide a brief discussion on steps (1) and (2) within the context of our model.

Improving resilience to liquidity shocks The first fundamental driver of the self-fulfilling fire sale is the fear of being hit by a future liquidity shock. It follows that policies that reduce the severity of the liquidity shock should reduce the probability of a self-fulfilling fire sale.

In our current setup, the liquidity shock requires funds to shift to a 100% cash portfolio which does not allow us to consider a reduction in the severity of a liquidity shock. Therefore, suppose instead that a fund hit by a liquidity shock only has to sell a fraction f of their bond holdings. The liquidity shock requiring 100% cash is the case when $f = 1$. One can easily show that equation (1.1), which gives the condition for funds to optimally sell at $t = 0$, becomes

$$p_0 > qfp_1(\tilde{\delta}) + (1 - qf)R. \tag{1.18}$$

A reduction in f, which we can interpret as a reduction in the severity of the liquidity shock, therefore has the same effect as a reduction in q. We thus see from Corollary 1.1 that a reduction in f will reduce the probability of a self-fulfilling fire sale (other factors equal) because it reduces the incentive to preemptively sell bonds.

Enhancing liquidity management tools could reduce the severity of any given liquidity shock and therefore reduce the probability of a self-fulfilling fire sale. One such tool could be the use of redemption gates that limit redemptions for a short period of time. By design, this reduces the fraction of bonds the fund needs to sell when hit by a liquidity shock. Another possible tool could be swing pricing. This tool reduces run-like incentives on open-ended funds by forcing redeeming investors to internalise the trading costs associated with selling instead of passing it onto those still invested in the fund. By reducing the incentive to run on the fund, swing pricing should reduce the number of bonds the fund needs to sell for any given liquidity shock.

Strengthening market-wide infrastructure The second fundamental driver of the self-fulfilling fire sale is the presence of a constrained dealer which funds have to sell to. Policies to improve the ability of the dealer to intermediate would therefore increase the expected price of bonds at $t = 1$ and reduce the incentive to preemptively sell at $t = 0$. In the aftermath of the dash for cash episode, post-GFC financial regulations have been highlighted as a contributor to dealer capital constraints (Bessembinder et al. 2018, Dick-Nielsen & Rossi 2019). In particular, the leverage ratio, which is typically a non-binding backstop, may have become binding during the dash for cash and prevented dealers from taking more bonds onto their balance sheets (Breckenfelder & Ivashina 2021). Policymakers could therefore consider ways to relax binding capital constraints at times of stress. Alternatively, they could reduce the capital-ratio impact for dealers of new trades by mandating central clearing.[8] This means that dealers could net buy and sell trades regardless of the trading counterparties and only record net buys on the balance sheet. Both of these policies could improve the ability of dealers to intermediate and therefore reduce the fear of market participants that they may have to sell to a dealer under stress.

Alternatively, policymakers could seek to break the fund-dealer relationship by promoting all-to-all trading platforms. This would allow market participants to trade with dealers and each

[8] See, for example, Duffie (2020) who proposes central counterparty clearing for the US Treasury market.

other. If the liquidity shock at $t = 1$ is an idiosyncratic rather than an aggregate shock, there will be q funds that are sellers of bonds and $1 - q$ funds that are natural buyers of bonds. In situations where dealers were constrained and could not intermediate between these buyers and sellers, such a policy would increase the expected price of bonds at $t = 1$ and reduce the incentive to preemptively sell.

1.5 Conclusion

Following the "dash for cash" in March 2020, central banks stepped into secondary markets and purchased bonds to meet their financial stability objectives. However, there are only a few papers theoretically examining the use of central bank asset purchases as a financial stability tool. This paper helps to fill this gap by introducing a central bank which provides a market backstop in a model of self-fulfilling fire sales. We show that by committing to act aggressively, the central bank can eliminate the probability of a self-fulfilling fire sale. However, a central bank may wish to act less aggressively to trade off the benefit of reducing self-fulfilling fire sales with the cost of intervening in other fire sale events. Furthermore, if market participants expect the central bank to act aggressively and the central bank does not act in accordance with expectations, then we show that the probability of self-fulfilling fire sales can increase relative to the baseline scenario of the central bank credibly ruling out the provision of a market backstop.

Our model is highly stylised and thus intended as a conceptual overview of how a central bank may provide a market backstop and to highlight some potential pitfalls with doing so. There remain many open questions on the details of designing a market backstop. Such issues include which assets to buy, how to best unwind any purchases, and how to deal with market dysfunction when financial stability objectives conflict with monetary policy. We leave such questions for future research.

Chapter 2

Inattention in Financial Networks

Abstract I build a model where bank values are interdependent within a financial network. However, banks cannot costlessly observe other banks' values. Instead, banks must choose to pay attention to developments in the value of other banks. Because paying attention incurs a cost, banks may optimally choose not to allocate significant attention to certain banks. In the model, banks that believe they have a higher value choose to supply more credit. Therefore, if inattentiveness causes banks to incorrectly infer their own value, their credit supply decisions will be distorted relative to the optimal. I show that banks that are moderately important for determining values in the network cause the greatest distortion in credit supply because banks do not deem them important enough to pay high levels of attention to, yet they are still important for determining bank values. This suggests that we should be more cautious about dismissing all but the most interconnected banks as being important for ensuring financial stability.

JEL classification: D84, D85, G01, G21, G28.

Keywords: Inattention, Financial networks, Financial crises, Bank lending.

2.1 Introduction

> *"But in today's global lending environment, how does a US bank examiner judge the credit quality of, say, a loan to a Russian bank, and hence of the loan portfolio of that bank? That in turn, would require vetting the Russian bank's counterparties and those counterparties' counterparties, all to judge the soundness of a single loan"*
> *– Alan Greenspan (2014, p.114)*

The banking system is complex and interconnected. The complexity of the banking system suggests that paying attention to all developments in the banking system is highly demanding. Deteriorations in the financial health of a banking institution could therefore go unnoticed. Due to the interconnected nature of the banking system, this deterioration will affect the financial health of other banks. Therefore, a bank failing to pay sufficient attention to developments in the banking system as a whole could believe that its financial position is stronger than it actually is. If this causes the bank to take on more risk than it would have done with full information, inattention could contribute to the build-up of financial fragilities and thus sow the seeds of a financial crisis.

In this paper, I ask whether the network structure of the banking system can affect the attention banks pay to developments in other areas of the system. Is it possible, for example, that some network structures are more "complex" and thus more difficult to pay attention to than others? Secondly, I examine whether these attention choices can lead to a build-up of financial fragility. In particular, how can banks' attention choices distort their credit supply decisions? And even if attention choices matter for individual credit supply decisions, do they matter for aggregate credit supply?

To think about developments in the banking system going unnoticed, we must allow banks to choose how much attention to allocate to different parts of the financial system. I therefore build a theoretical model combining the inattention literature (Sims 2003) with the financial networks literature. In my model, banks are connected within a financial network. However, they are unable to perfectly observe the value of the underlying assets banking system of other banks within the network. Instead, banks only observe a noisy signal on these assets which they can use to form their beliefs about the value of other banks and, since values are interdependent through the financial network, form a belief about their own value. Importantly, banks are able to *choose* the noisiness of the signals they receive, that is, they can pay more attention to certain banks. In the model, a bank's optimal credit supply decision depends on its own value, which creates an incentive for a bank to use resources to pay attention to the value of other banks so it can make a better decision. However, paying attention incurs a cost, so banks must choose how to allocate their attention optimally to different banks in the network.

The first result is that the amount of attention that bank i pays to bank j increases if (i) bank j's underlying assets are more uncertain and (ii) bank j is in a more important position in the network for bank i as measured by the weighted directed walks from j to i. Thus, the importance of bank j to bank i captures not only any direct interdependencies between i and j, but also indirect interdependencies where bank j affects another bank k which then affects bank i. This is a familiar result from the literature on attention choice except that the "importance" of paying more attention is linked to the structure of the network.

I then explore the implications of these attention choices for aggregate credit supply by examining the impact of shocks to the underlying value of banks' assets. Such shocks could be viewed as shocks to the credit quality of the banks' loans. If there were no attention problem, these shocks would be perfectly observed, banks would update their beliefs about their own value correctly, and they would be able to adjust their credit supplies optimally. When there is an attention choice, banks do not necessarily pay full attention to the shock and therefore their credit supply decision will, in general, be distorted from the optimal supply under full attention. The role of the network structure in determining this distortion to credit supply is the central focus of this paper. This is because I view this distortion as a fragile state which is susceptible to a sudden revision of beliefs. An unusual event, such as losses on highly rated mortgage-backed securities, can focus attention and cause a sudden revision in banks' views of their own health. Banks consequently adjust their supply of credit to the economy such that the distortion is eliminated. If this adjustment is downwards, the sudden revision of beliefs causes a credit crunch.

As a first step to understanding the network determinants of the distortion to credit supply, I examine the effect of a shock to the underlying assets of a single bank j. I show that the shock to j's assets causes the largest distortion to aggregate credit supply when j is of moderate importance to several banks in the network. The intuition is simple. If bank j is unimportant, banks choose not to pay much attention to bank j. But shocks to bank j do not distort aggregate credit supply because j is unimportant for determining the values of other banks. If bank j is important, shocks to bank j matter for determining the values of other banks. But bank j does not cause a *distortion* in credit supply because banks pay a high amount of attention to bank j. It follows that bank j causes the largest distortions when it is of moderate importance because it still matters enough for determining the values of other banks but not enough that high levels attention are paid to it.

Next, I examine the effect of idiosyncratic shocks hitting multiple banks. We can then see whether the effect of positive and negative shocks cancels out leaving the distortion to aggregate credit supply unchanged. I show that networks with a large number of moderately important banks tend to produce larger credit supply distortions, in line with the analysis of the single shock case. I also show that having a range of banks in the network of differing importance *independently* increases the likelihood of large distortions. This is because there will be occasions where shocks to the banks' underlying assets of a particular sign all concentrate on the moderately important banks (which do distort credit supply) whereas shocks of the opposite sign concentrate on the low or high important banks (which do not distort credit supply). The moderately important banks will then all distort aggregate credit supply in the same direction and this will not be offset by distortions of the opposite sign. The result, in this case, is a large distortion in aggregate credit supply.

I finally discuss some implications of the model for regulating the financial system, including ring-fencing and central counterparty clearing. The common thread behind such policy implications is that we should be cognizant of the effect of our policies on the network structure of the financial system because certain network structures do not create the incentives for adequate private monitoring. The results also suggest we should be cautious about dismissing all but the most interconnected banks as being important for ensuring financial stability.

Literature Review My paper links the inattention literature with the literature on financial networks.

The literature on financial networks emphasises that banks and other financial institutions are interconnected in various ways that cause their values to become interdependent. Debt contracts between banks can lead to cascades of defaults if one institution is forced to default (Allen & Gale (2000), Eisenberg & Noe (2001)) or can lead to the revaluation of interbank liabilities due to solvency distress (Bardoscia et al. (2019), Abduraimova & Nahai-Williamson (2021)). Equity claims of banks on each other results in bank values becoming interdependent (Elliott et al. (2014), Jackson & Pernoud (2021)). Even in the absence of contracts between banks, common asset exposures can cause bank values to become interdependent (Acharya & Yorulmazer 2007). Much of the literature examines how these interconnections propagate shocks through the network and how this can lead to bank failures. This paper departs from the assumption that the shocks are perfectly observed and instead examines how the network structure shapes the attention paid to shocks within the network.

Many ideas in the inattention literature can be traced back to the seminal work of Sims (2003) who formalised the idea that agents have limited processing capacity and therefore must choose what information to pay attention to. Since this work, models of attention choice been employed to explain a range of phenomena including the real effects of monetary policy shocks

(Maćkowiak & Widerholt 2009), the existence of underdiversified portfolios (Van Nieuwerburgh & Veldkamp 2010), and belief polarisation (Nimark & Sundaresan 2019). The closest paper in this literature to my work is Kohlhas & Walther (2021) who show how asymmetric attention to different components of output can explain the coexistence of extrapolation and underreactions in survey data. My paper builds upon a similar theoretical framework but shows how the network structure can be a reason for asymmetric attention.

More generally, this paper is related to the literature on the build-up of fragility in the financial system. The literature can broadly be divided into two views (Stein 2013). The first view emphasises the role of preferences and beliefs. For example, Gennaioli et al. (2012) build a model where investors neglect risks due to behavioural biases and therefore believe certain securities are safer than they actually are. When investors recognise these risks and revise their beliefs, it causes a flight to safety. The second view emphasises the role of agency frictions and incentives. For example, Rajan (1994) argues that bank managers with short-term objectives to maximise current earnings have an incentive to pursue a liberal credit policy which undermines future credit quality. This paper sits in the former category: fragilities build up because financial institutions do not pay sufficient attention to changes to the health of other institutions within the financial system.

2.2 Model

2.2.1 Financial network

There is a set $N = \{1, \dots, n\}$ of banks and the book value[1] V_i of bank $i \in N$ is

$$V_i = x_i + \sum_j w_{ij} V_j. \tag{2.1}$$

The first term x_i is the value of bank i's assets minus liabilities that are external to the network. In what follows, I call x_j the *primitive (net) assets* of bank j. The second term follows Elliott et al. (2014) by capturing the interdependencies between bank values in a linear model. For any $i, j \in N$, the number $w_{ij} \geq 0$ and $w_{ii} = 0$ for each i. The larger the value of w_{ij}, the more important bank j is for directly determining the value of bank i. I allow $w_{ij} \neq w_{ji}$ so the $\{w_{ij}\}$ define a weighted and directed network. I further restrict $\sum_i w_{ij} < 1$ which will ensure that banks' values are well-defined. We can interpret w_{ij} is interpreted as the equity share bank i

[1] Several equivalent terms are often used in the literature such as equity, net worth, or capital. I use these terms interchangeably in the paper.

has in bank j. The fact that $\sum_i w_{ij} < 1$ rather than $\sum_i w_{ij} \leq 1$ therefore means that there some outside shareholders of bank j.

Note that, in general, the interdependencies between banks could be nonlinear, reflecting the variety of potential contractual agreements between banks. Debt contracts, for example, do not induce interdependencies when banks can meet the face value of their debt. However, when banks cannot meet the face value of their debt, they ration their counterparties in proportion to their claims. As discussed in detail by Elliott et al. (2014), this induces the linear dependencies in (2.1). We can thus think about the model applying to a banking system for which the primary form of interdependence operates through unsecured interbank lending where there is a risk of counterparty default. Alternatively, we could view the linear dependencies as a convenient reduced form approximation for the range of possible nonlinear interdependencies between banks.

Equation (2.1) can be written in matrix form as

$$V = x + WV \tag{2.2}$$

where $V = (V_1, \ldots, V_n)^\top$, $x = (x_1, \ldots, x_n)^\top$, and

$$W = \begin{pmatrix} 0 & w_{12} & \cdots & w_{1n} \\ w_{21} & 0 & \cdots & w_{2n} \\ \vdots & & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & 0 \end{pmatrix}.$$

Under the assumption that $\sum_i w_{ij} < 1$, the spectral radius of W is less than 1 (Berman & Plemmons 1979, p.37). This implies that the matrix $L \equiv (I - W)^{-1}$ is well-defined and

$$L \equiv (I - W)^{-1} = \sum_{i=0}^{\infty} W^i \tag{2.3}$$

where the sum is the Neumann series associated with W (Stachurski 2016, p.69). We can thus write the value of each bank as

$$V = (I - W)^{-1} x = Lx. \tag{2.4}$$

The matrix $(I-W)^{-1}$ is similar to Leontief's (1951) input-output analysis. Equation (2.4) shows that bank i's value does not just depend on its own primitive assets but also the primitive assets of its neighbours, the assets of its neighbours' neighbours, and so on. For example, the value of bank 1 is given by

$$V_1 = l_{11}x_1 + l_{12}x_2 + \cdots + l_{1n}x_n, \tag{2.5}$$

which in general depends on the primitive assets of all other banks in the network. The weights $\{l_{11}, \ldots, l_{1n}\}$ can be interpreted in terms of the underlying network stucture. Recall that $(I - W)^{-1} = I + W + W^2 + W^3 + \cdots$ and note that W^k calculates the number of walks of length k between any two nodes in a network where here the walks are weighted by the $\{w_{ij}\}$ (see Jackson (2008, p.24)). Thus the weight l_{1j} is the number of directed walks one can take from bank j to bank 1 in the network weighted by the importance of those walks. These weights will play an important role in the solution to the optimal attention problem.

2.2.2 Noisy signals

In theory, it is mechanically straightforward to compute the book value of a bank: compute a bank's assets minus its liabilities. In practice, however, whilst some bank assets are easy to value (e.g., US Treasury bonds), banks also hold more illiquid assets such as corporate or emerging market bonds that are more difficult to value. Moreover, banks hold substantial amounts of loans on their balance sheets that are not traded at all and therefore have no market price. Such assets are valued using accounting rules that are subject to discretion. In fact, there is evidence that banks overstate the value of distressed assets by using accounting discretion (Huizinga & Laeven 2012) or continuing to lend to weak firms to avoid realising losses (Peek & Rosengreen 2005). Such factors can make the true value of a bank difficult to determine. Indeed, there is some evidence that banks are more opaque than non-financial firms (Morgan (2002), Flannery et al. (2013)).[2] Despite the opaque nature of banks, empirical evidence does however suggest that bankers, as insiders, have some knowledge of each other's financial position (see, for example, Afonso et al. (2011)).

These observations motivate the information structure in this paper. First, I assume that $x_j = \mu_j + \sigma_j u_j$ where $u_j \sim N(0, 1)$ to capture the prior uncertainty on bank j's balance sheet and μ_j is the prior mean. For simplicity, I take u_j to be independent of u_k for $j \neq k$.[3] Second, I assume that the x_j are not directly observable to bank i for $j \neq i$ to capture the opacity of

[2] See also Rajan (1994) for a theoretical model where the opacity of a bank's loan portfolio allows bank managers to manipulate current earnings and thereby shape outsiders' perceptions of their competence.

[3] In practice, u_j and u_k could be correlated because j and k have common asset exposures. Introducing correlation between bank j and bank k would give bank i an incentive to pay more attention to bank j because that also gives information about bank k. Because the focus of this paper is on how attention choices are shaped by the network structure, it is natural to consider the case of no correlation.

bank balance sheets. Instead, bank i observes a noisy signal on x_j,

$$z_{ij} = x_j + q_{ij}\varepsilon_j, \tag{2.6}$$

where $\varepsilon_j \sim N(0,1)$ with ε_j independent of ε_k for $j \neq k$. The noise of the signal is parameterised by $q_{ij} \in \mathbb{R}_+$ where a higher value of q_{ij} corresponds to a noisier signal. I assume that each bank can perfectly observe its own primitive assets at zero cost so that $q_{ii} = 0$ for all i.[4]

In the next two sections, I explain how bank i optimally conditions its lending decision on its value V_i. This creates a need for bank i to use the noisy signals on the other banks' primitive assets to learn about its own value. Importantly, I follow the attention choice literature and assume that banks are able to *choose* the noisiness of their signals by choosing the value of q_{ij}. As we shall see, this implies that banks optimally choose to receive less noisy signals about banks that are more important for inferring their own value.

2.2.3 Attention choice

If we take q_{ij} to be exogenous, we have a noisy information framework. This paper instead follows the approach of the attention choice literature by allowing banks to choose the information they receive. In the context of this model, banks can choose $\{q_{i1}, \dots, q_{in}\}$ which corresponds to bank i choosing the variance of the signals for each x_j. In what follows, it will be useful to instead think about the bank choosing

$$m_{ij} = \frac{\sigma_j^2}{\sigma_j^2 + q_{ij}^2} \in (0,1] \tag{2.7}$$

so a bank's choice of q_{ij} maps one-to-one into its choice for m_{ij}. For example, choosing a signal with no noise is equivalent to choosing $q_{ij} = 0$ or $m_{ij} = 1$.

By choosing a higher m_{ij}, bank i can obtain better information about bank j's primitive assets. However, obtaining better information comes at a cost. This cost can be thought of as the resources spent learning about the financial situation of bank j. For example, analysts could be employed to track earnings reports, announcements of loan loss provisions, or changes in the exposure of the loan portfolio to different regions in an attempt to have a better understanding of the financial situation of a particular bank. To obtain closed-form solutions, I take a similar approach to Kohlhas & Walther (2021) and assume that there is a cost of paying attention

[4]For the sake of brevity, in the remainder of the paper I implicitly assume that all references to all banks $j = 1, \dots, n$ exclude the case when $j = i$.

given by

$$C(m_i) = \frac{1}{2}\sum_{j\neq i}\kappa_{ij}m_{ij}^2. \tag{2.8}$$

The attention cost function is convex in m_{ij} to capture the idea that it becomes increasingly hard for bank i to learn about bank j: it is easy to look up bank j's financial information on its balance sheet but much more difficult to absorb all available information to arrive at an informed judgement of the financial situation of bank j.[5] The parameters κ_{ij} capture other factors that determine how difficult it is for bank i to learn about bank j.

Once bank i has made its attention choices $\{m_{i1}, \ldots, m_{in}\}$, banks observe the n signals on the x_j and update their beliefs about x_j using Bayes' rule. In particular,

$$E[x_j|z_{ij}] = m_{ij}z_{ij} + (1 - m_{ij})\mu_j \tag{2.9}$$

$$\text{Var}[x_j|z_{ij}] = \sigma_j^2(1 - m_{ij}) \tag{2.10}$$

which is a standard result for Gaussian signals (see Veldkamp (2011)). Since V_i is a function of $\{x_1, \ldots, x_n\}$, bank i can then form a belief about its own value. Notice that as bank i becomes completely inattentive to bank j ($q_{ij} \to \infty$ and $m_{ij} \to 0$), no weight is placed on the signal and all the weight is placed on the prior. As attention increases, more weight is placed on the signal and the variance declines.

Finally, bank i makes its optimal choice. Bank i's realised utility after all uncertainty is resolved is given by

$$U_i = -(a_i^* - a_i)^2 - C(m_i) \tag{2.11}$$

where a_i^* is bank i's ideal action, that is, the action it would take under full information (no inattention problem). Thus, bank i incurs a quadratic loss for choosing an action a_i which deviates from the ideal action a_i^*. I take a_i as bank i's lending and a_i^* as bank i's ideal quantity of lending. I also assume that $a_i^* = \omega V_i$, that is, bank i's ideal quantity of lending scales up linearly with bank value. I justify these assumptions in the next section with a simple bank model.

Bank i also incurs the cost of paying attention $C(m_i)$. If $C(m_i) = 0$, then bank i would set

[5] Damodaran (2012, Chp.21) discusses the difficulties in valuing financial service firms. For an analyst to conduct a complete asset-based valuation of a bank, they would likely require access to a bank's internal data. The impossibility of obtaining this data could be captured in this model by restricting $q_{ij} \in [\bar{q}, \infty]$ where $\bar{q} > 0$ so bank i always faces some uncertainty about the value of x_j. This would change little of substance qualitatively, so I allow $q_{ij} \in \mathbb{R}_+$.

$m_{ij} = 1$ for all $j = 1, \ldots, n$ and therefore would know the x_j with certainty. This is optimal because a_i^* depends on V_i, or equivalently the x_j's, so bank i would be able to set $a_i = a_i^*$. The presence of a positive cost will therefore, in general, make it suboptimal for bank i to pay full attention and i's signals will not perfectly reveal the value of x_j. This will result in the optimal action under full information, a_i^*, differing from the action under inattention, a_i, as bank i trades off the cost of paying attention against the benefit of being able to choose an action close to a_i^*.

2.2.4 Bank model

There are two assumptions that require discussion: (1) the quadratic loss; (2) the form of the optimal action a_i^*.

I motivate allowing the optimal action to depend on bank capital by the empirical evidence in Gambacorta & Shin (2018). Using data on banking institutions that hold over 70% of global banking assets, they show that higher levels of bank capital are associated with higher bank lending.[6] The mechanism they identify is that better capitalised banks have lower funding costs which allows them to lend more.[7]

I provide a simplified bank model to capture this empirical result and deliver the two required assumptions. Banks make profits from the spread of the expected return received on loans compared to the cost of funding (net interest income). Denote the interest rate bank i charges on loans by R_l and assume that bank i takes this as given. The loan is repaid with exogenous probability p so bank i's expected return is pR_l. Let the cost of funding be given by the function $R_d(a_i/V_i)$ with $R_d'(\cdot) > 0$ and $R_d''(\cdot) > 0$ where a_i is the quantity of loans bank i makes. Thus, better capitalised banks have a lower cost of funding as in Gambacorta & Shin (2018).

Bank i's profit is thus

$$\pi_i(a_i) = a_i \left(pR_l - R_d(a_i/V_i)\right). \tag{2.12}$$

I assume that banks adjust a_i independently of V_i, that is, they do not finance new lending with

[6] The authors emphasise that it is accounting measures of equity (i.e., book values) that matter for their results. Their findings are statistically insignificant when they repeat the analysis with market-based measures of equity.

[7] Michelangeli & Sette (2016) also show using a randomised control trial that better capitalised banks increase mortgage supply. These empirical results can be explained theoretically by appealing to bank capital solving a moral hazard problem as in Holmstrom & Tirole (1997) or Allen et al. (2011). Bank capital gives banks "skin in the game" which increases a bank's incentive to monitor its borrowers. This relaxes the constraint on the bank's ability to obtain funding and thus lending increases.

new equity. This matches the evidence in Adrian & Shin (2010) who show that banks engage in more lending by taking on more short-term debt, namely undertaking repo transactions, which leaves V_i unaffected.[8] Therefore, maximising (2.12) with respect to a_i yields the first order condition

$$pR_l = R_d\left(\frac{a_i^*}{V_i}\right) + \frac{a_i^*}{V_i}R_d'\left(\frac{a_i^*}{V_i}\right). \tag{2.13}$$

There is thus an optimal ratio of a_i^*/V_i which is independent of i. That is, we can define $\omega = a_i^*/V_i$ so that a_i^* is a fixed multiple of V_i. Moreover, since R_d is increasing and convex, the right-hand side of the equation is increasing in ω. The left-hand side is constant so there is a unique optimal ratio ω that solves this problem. Finally, note that ω is increasing in the expected return pR_l. Thus, banks that are better capitalised lend more. This is exactly the mechanism highlighted empirically by Gambacorta & Shin (2018): better capitalised banks lend more because they have lower funding costs.

The assumption of a quadratic loss function is a relatively weak assumption because any smooth loss function is locally quadratic. The use of second order approximations to justify quadratic loss is standard in the attention choice literature to obtain tractability (see, for example, Maćkowiak & Widerholt (2009)).

The loan rate R_l is determined in equilibrium. Following Dell'Ariccia et al. (2014), I assume that banks face a negatively sloped demand function for loans, $D(R_l)$.[9] Loan supply is given by $S(R_l) = \sum_{i=1}^n a_i$ and R_l adjusts in equilibrium to ensure that $D(R_l) = S(R_l)$.

2.3 Solving the attention choice problem

The timing in the model is summarised in figure 2.1. Each bank i makes its information choice taking into account the attention cost in (2.8). Each bank then observes the signals and updates their belief about each x_j using Bayes' rule as in (2.9) and (2.10). Bank i then chooses its action a_i to maximise the expected value of (2.11) conditional on its information set. This in turn determines the equilibrium loan interest rate and quantity. All uncertainty in the model is then resolved which determines bank funding costs and therefore profits according to (2.12).

[8] Adrian & Shin (2010) argue that the margin of adjustment in balance sheet size is through short-term debt because banks target a particular leverage ratio in order to obtain a particular credit rating. A key reason for this is presumably because credit ratings affect funding costs. This ties in with the evidence from Gambacorta & Shin (2018) and suggests that banks understand the implications of their level of capitalisation on their funding costs.

[9] See den Haan et al. (2007) for empirical evidence supporting a downward sloping loan demand function.

The idea behind resolving uncertainty is that the difficulty in valuing a bank's assets must eventually subside as the payoffs from these assets are realised.

To summarise, the bank faces uncertainty about its capital position and uses resources to learn about it. Once it has an estimate of its capital, it can decide how much it is able to lend. If the bank's lending decisions result in it being too weakly capitalised, the bank realises that it will face a higher cost of funding which will hurt bank profits. Wishing to avoid a higher cost of funding, the bank will attempt to not lend too much relative to its capital.

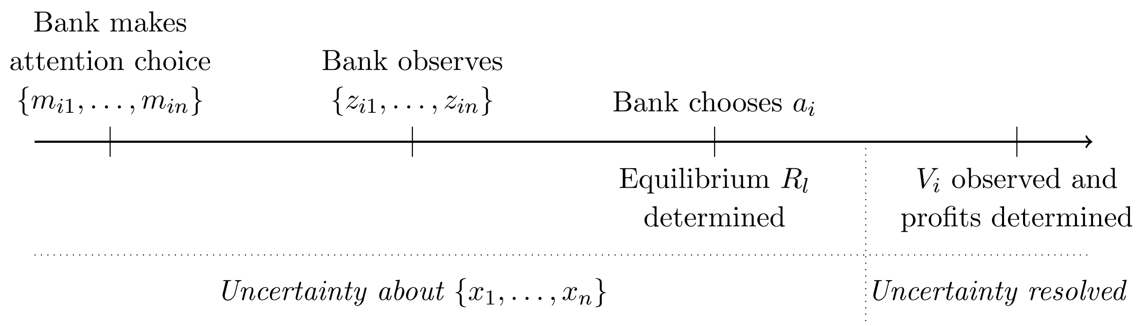


Figure 2.1: Timing of bank actions (above) and outcomes (below)

We solve the attention choice problem by backwards induction. This means that the information choice made by bank i is made anticipating the effect that such a choice will have on bank i's optimal action. The solution to the attention choice problem is given by Proposition 2.1.

Proposition 2.1. *The optimal attention bank i pays to bank j is given by*

$$m_{ij} = \omega^2 \frac{\sigma_j^2 l_{ij}^2}{\kappa_{ij}} \tag{2.14}$$

for $j = 1, \ldots, n$ when there is an interior solution. Otherwise, $m_{ij} = 1$. Thus, bank i pays more attention to bank j if bank j's assets are more volatile (σ_j^2) and if bank j is more important in the network (l_{ij}^2) as measured by the weighted directed walks from j to i.

Proof. In the final stage after bank i has observed its signals, it chooses its action a_i to solve

$$\min_{a_i} E[(a_i^* - a_i)^2 | \Omega_i] \tag{2.15}$$

where $\Omega_i = \{z_{i1}, \ldots, z_{in}\}$ is bank i's information set. The solution is $a_i = E[a_i^* | \Omega_i]$.

Thus at the information choice stage, bank i will want to choose $\{m_{i1}, \ldots, m_{in}\}$ to solve the problem

$$\min_{\{m_{i1},\ldots,m_{in}\}} \underbrace{E[(a_i^* - E(a_i^*|\Omega_i))^2]}_{\text{Var}[a_i^*|\Omega_i]} + \frac{1}{2}\sum_{j\neq i}\kappa_{ij}m_{ij}^2 \tag{2.16}$$

subject to the constraint that $m_{ij} \leq 1$ for all $j \neq i$. Next note that

$$\begin{aligned}
\text{Var}[a_i^*|\Omega_i] &= \text{Var}[\omega V_i|\Omega_i] \\
&= \omega^2 \text{Var}\left[\sum_{j=1}^{n} l_{ij}x_j \middle| \Omega_i\right] \\
&= \omega^2 \sum_{j=1}^{n} l_{ij}^2 \text{Var}[x_j|\Omega_i] \\
&= \omega^2 \sum_{j=1}^{n} l_{ij}^2 \sigma_j^2 (1 - m_{ij}).
\end{aligned}$$

where the final line uses (2.10). The objective function is clearly convex, so there is a unique solution to (2.16). The interior solution is given by the first order conditions

$$-\omega^2 l_{ij}^2 \sigma_j^2 + \kappa_{ij} m_{ij} = 0$$

and (2.14) follows immediately. □

The result in Proposition 2.1 is a familiar result in the attention choice literature. The term $\sigma_j^2 l_{ij}^2$ shows that agents pay more attention to signals that are (i) more volatile and (ii) more important for their decisions. In this model, the importance of a signal is determined by the structure of the network. The term $1/\kappa_{ij}$ shows that bank i will pay less attention to bank j if there are factors that make j more difficult to pay attention to. The term ω^2 shows that bank i will pay more attention if learning about V_i is more important for determining the optimal action. However, it does not affect the attention that bank i pays to bank j relative to another bank k.

2.4 Attention choice and credit supply

I now investigate the implications of the attention choice problem on bank credit supply. The main theme is that inattention combined with certain network structures can generate a distortion in credit supply relative to the full information benchmark. I follow the discussion in

Brunnermeier & Oehmke (2009) and view a situation where there is a distortion in credit supply as a fragile state because it represents a build-up of hidden information caused by inattention. The release of this hidden information can suddenly be triggered by an unusual event, such as losses on highly rated mortgage-backed securities, that focuses agents' attention and causes them to re-evaluate their valuations.[10] As banks revise their beliefs, they will adjust their individual credit supplies accordingly. This revision of beliefs can therefore lead to a credit crunch, which is defined as a leftward shift in the supply curve for bank loans unrelated to credit rationing considerations or changes in the safe real interest rate (Bernanke & Lown 1991, p.207).

2.4.1 Credit supply distortion

I consider shocks to the primitive assets x_j of each bank j. Such shocks can be viewed as shocks to the credit quality of bank j's loans which would imply a change in the fair valuation of bank j's assets. The realisation of these shocks can be written as $x_j = \mu_j + \tau_j$ where τ_j captures the deviation of bank j's primitive assets from the prior mean. When the banking system is hit by the shocks to the primitive assets of each bank, there are two effects on credit supply. The first effect is that a shock to x_j will clearly affect the value of all banks *directly* connected to bank j. But it will also affect the value of other banks that are *indirectly* connected to bank j within the network. Since each bank's optimal credit supply scales linearly with its value, the shock to x_j is able to affect the optimal credit supply a_i^* of each bank $i \in \{1, \ldots, n\}$ and therefore can potentially have a significant effect on aggregate credit supply. The second effect follows from the fact that the network structure determines optimal attention choices. If bank i does not pay sufficient attention to the shock to x_j owing to its position in the network, then this will cause bank i to incorrectly infer its own value V_i. This could potentially create a distortion in actual credit supply a_i relative to the optimal a_i^*.

To enable a clearer discussion of the role of the network structure, I make the following assumption:

Assumption 2.1. *The realisation of the noise draw ε_j in (2.6) is equal to zero for $j = 1, \ldots, n$.*

To understand the role of assumption 2.1, recall from (2.6) that bank i's signal on bank j's primitive assets under assumption 2.1 will be $z_{ij} = x_j$. That is, all banks will receive the same signal on the value of bank j's primitive assets. This implies that differences in posterior beliefs

[10] In terms of the parameters of the model, we could think about the event as a reduction in κ_{ij} for all i, j pairs such that the banks can then pay attention to all signals at low cost.

are driven solely by differing attention choices rather than idiosyncratic noise. This alters nothing of substance and simply makes it easier to observe the impact of optimal attention choices.

I can now state Proposition 2.2 which gives the distortion of credit supply relative to the full information benchmark due to the presence of inattention.

Proposition 2.2. *Define* $\Delta_i = a_i - a_i^*$ *as the distortion of bank* i*'s action from the optimal action under full information. Define* $\Delta = \sum_{i=1}^n \Delta_i$ *as the aggregate distortion to credit supply. Under assumption 2.1, we have*

$$\Delta_i = \omega \left[\sum_{j=1}^n l_{ij} \tau_j (m_{ij} - 1) \right]. \tag{2.17}$$

and

$$\Delta = \omega \sum_{j=1}^n \tau_j \left[\sum_{i=1}^n l_{ij} (m_{ij} - 1) \right]. \tag{2.18}$$

Proof. Bank i's action a_i is given by

$$\begin{aligned} a_i &= E[a_i^* | \Omega_i] \\ &= E[\omega V_i | \Omega_i] \\ &= \omega E \left[\sum_{j=1}^n l_{ij} x_j \middle| \Omega_i \right] \\ &= \omega \sum_{j=1}^n l_{ij} [m_{ij} z_{ij} + (1 - m_{ij}) \mu_j] \end{aligned}$$

where the final line uses (2.9). Using the fact that $z_{ij} = \mu_j + \tau_j$, we can write the optimal action as

$$a_i = \omega \sum_{j=1}^n l_{ij} [\mu_j + m_{ij} \tau_j].$$

The full information outcome is equivalent to bank i setting $m_{ij} = 1$ for all j (at zero cost) and therefore

$$a_i^* = \omega \sum_{j=1}^n l_{ij} [\mu_j + \tau_j]$$

from which equation (2.17) follows. The aggregate distortion to credit supply is given by

$$\begin{aligned}\Delta &= \sum_{i=1}^{n} \Delta_i \\ &= \omega \sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij} \tau_j (m_{ij} - 1) \\ &= \omega \sum_{j=1}^{n} \sum_{i=1}^{n} l_{ij} \tau_j (m_{ij} - 1)\end{aligned}$$

from which equation (2.18) follows. □

2.4.2 Single shock

To understand the implications of Proposition 2.2, I first consider a single shock to the primitive assets of bank j.

Assumption 2.2. *(Single shock assumption)* $\tau_j \neq 0$ *and* $\tau_k = 0$ *for all* $k \neq j$

Under assumption 2.2, we can write the distortion to aggregate credit supply from Proposition 2.2 as

$$\Delta = \omega \times \underbrace{\tau_j}_{\lesseqgtr 0} \times \underbrace{\left(\overbrace{l_{1j}(m_{1j} - 1)}^{\text{Effect on bank 1}} + \cdots + \overbrace{l_{ij}(m_{ij} - 1)}^{\text{Effect on bank } i} + \cdots + \overbrace{l_{nj}(m_{nj} - 1)}^{\text{Effect on bank n}} \right)}_{<0}. \tag{2.19}$$

First note that the sign of the distortion is given by the sign of τ_j: a negative shock causes aggregate credit supply to be higher than the full information benchmark and vice versa. The role of the network in determining the magnitude of this distortion is given by the term in brackets. The network structure influences the distortion to aggregate credit supply *directly* through the l_{ij} terms and *indirectly* because the attention choices m_{ij} are a function of the network.

To understand the role of the network structure, consider the effect of a change in bank j's importance for bank i as measured by l_{ij}. If attention choices were fixed, we can see that an increase in l_{ij} will worsen the distortion of aggregate credit supply operating through the distortion of bank i's credit supply. However, from Proposition 2.1, we see that there will also be an offsetting effect because bank i will pay more attention to bank j. The other parameters in equation (2.19) are unchanged, so the effect of changes in l_{ij} on aggregate credit supply is given by the function

$$f(l_{ij}) = \omega \tau_j l_{ij} \left[\chi_{ij} l_{ij}^2 - 1 \right]. \tag{2.20}$$

where $\chi_{ij} \equiv \omega^2 \sigma_j^2 / \kappa_{ij}$. From the plot of this function in figure 2.2 (drawn for $\tau_j < 0$), we see that the distortionary effect due to the shock to bank j on aggregate credit supply is higher when it is of moderate importance to bank i. The intuition is simple. When shocks to bank

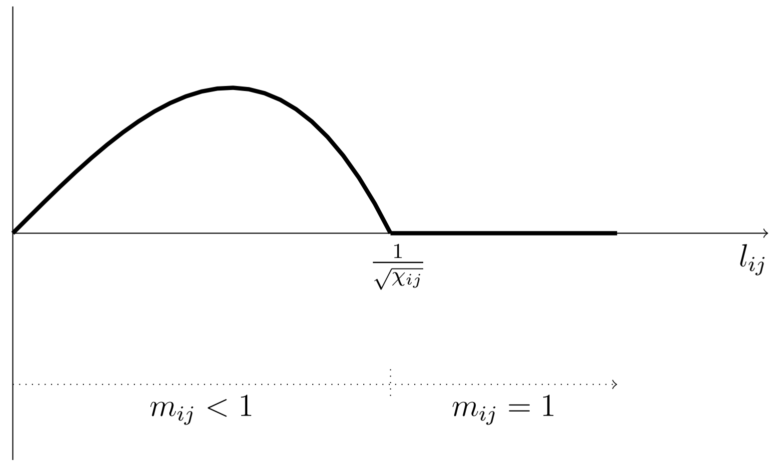


Figure 2.2: Bank j's distortionary effect on aggregate credit supply as bank j's importance to bank i varies (no attention reallocation channel)

j are not important for determining bank i's value, bank i pays little attention to shocks to bank i, but it does not matter that bank i pays little attention. When shocks to bank j are important for determining bank i's value, bank i pays close attention and the adjustments to credit supply are closer to optimal. However, when bank j is of moderate importance to bank i, shocks to bank j still matter for determining bank i's value, but bank i does not pay sufficient attention to the shocks and therefore i's beliefs remain too closely anchored to the prior mean. This then creates a distortion in credit supply from the optimal.

Considering now the effect of a given shock to bank j on aggregate credit supply, we observe that bank j will have the largest effect on distortions to aggregate credit supply if it is of moderate importance to several banks within the network.

It is worth contrasting this result with the networks that would create the largest change in aggregate credit supply under full information. Using the expression for a_i^* in the proof of

Proposition 2.2, we can write

$$\sum_{i=1}^{n} a_i^* = \omega \left[\tau_j (l_{1j} + \cdots + l_{nj}) + \sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij} \mu_j \right]. \tag{2.21}$$

This shows that shocks to bank j will have a large effect on aggregate credit supply under full information when the value of the direct and indirect walks emanating from j, $\sum_{i=1}^{n} l_{ij}$, is large. This is equivalent to saying that j must have a large Bonacich centrality (Bonacich 1987). Comparing (2.19) with (2.21), we see that the term in the brackets in (2.19) is similar to $\sum_{i=1}^{n} l_{ij}$ except that the l_{ij} terms are weighted by $(1 - m_{ij})$ before being summed over j. Thus, banks that cause the largest impacts on aggregate credit supply under full information are different from the banks where distortions to credit supply build-up due to insufficient attention.

The comparison also suggests defining

$$D_j = \sum_{i=1}^{n} l_{ij}(m_{ij} - 1) \tag{2.22}$$

as j's *distortion centrality.* Letting

$$M = \begin{pmatrix} 0 & (m_{12} - 1) & \cdots & (m_{1n} - 1) \\ (m_{21} - 1) & 0 & \cdots & (m_{2n} - 1) \\ \vdots & & \ddots & \vdots \\ (m_{n1} - 1) & (m_{n2} - 1) & \cdots & 0 \end{pmatrix}$$

encode the information about each bank's attention choice, we can write the vector of distortion centralities for a network W as

$$D(W) = [L \circ M]' \mathbf{1}_{n \times 1} \tag{2.23}$$

where $\circ$ is the Hadamard product and $\mathbf{1}_{n \times 1}$ is an $n \times 1$ vector of 1's.

2.4.3 Multiple shocks

We have seen in the previous section how a single shock distorts aggregate credit supply. Now I relax assumption 2.2 and allow for shocks to the primitive assets of multiple banks. Allowing for multiple shocks allows us to examine the extent to which positive and negative shocks cancel out in the aggregate. This question has similarities to the production networks literature that examines whether idiosyncratic productivity shocks "wash out" in the aggregate or instead can cause fluctuations in aggregate economic activity (Acemoglu et al. 2012).

To this end, I replace assumption 2.2 with the assumption that the shocks are independent identically distributed random variables.

Assumption 2.3. *(iid shocks)* $\{\tau_1, \ldots, \tau_n\}$ *are iid random variables with mean* μ_τ *and variance* σ_τ^2

Note that in Proposition 2.2, the distortion to aggregate credit supply was computed given particular realisations of $\{\tau_1, \ldots, \tau_n\}$. Under assumption 2.3, we explicitly consider the shocks as random variables before their value has been realised. The aggregate distortion Δ is now a random variable and we therefore compute the mean and variance of the distortion to aggregate credit supply.

Proposition 2.3. *Under assumption 2.3, the mean and variance of the distortion to aggregate credit supply,* $\Delta = \omega \sum_{j=1}^{n} \tau_j D_j$, *is*

$$E[\Delta] = \omega \mu_\tau \sum_{j=1}^{n} D_j \tag{2.24}$$

$$\text{Var}[\Delta] = \omega^2 \sigma_\tau^2 n [\sigma_D^2 + \bar{D}^2] \tag{2.25}$$

where $\bar{D}$ *is the mean and* σ_D^2 *is the variance of the vector* $D(W)$.

Proof. The derivation of $E[\Delta]$ is immediate. For the variance, we have

$$\text{Var}[\Delta] = \omega^2 \sigma_\tau^2 \sum_{j=1}^{n} D_j^2.$$

Noting that

$$\sigma_D^2 = \left(\frac{1}{n} \sum_{j=1}^{n} D_j^2 \right) - \bar{D}^2$$

where $\bar{D} = (1/n) \sum_{j=1}^{n} D_j$ gives the result. □

The expression for $E[\Delta]$ is a natural analogue of (2.19) in the single shock case. The sign of the mean distortion is determined by the mean of the τ_j shocks and the distortion centralities of each node determine the magnitude of the distortion to aggregate credit supply. Therefore, networks where there are several banks of moderate importance to other banks in the network will have a higher mean distortion of aggregate credit supply.

The expression for Var[Δ] tells us that networks with (i) a large average distortion centrality, $\bar{D}$, and (ii) a large dispersion of distortion centralities, σ_D^2, are more prone to experiencing large distortions in aggregate credit supply. The reason for (i) is evident. The reason for (ii) relates to whether the network is such that the distortions caused by the shocks offset each other. When there is a large dispersion in banks' distortion centralities, positive and negative distortions are less likely to offset each other, which results in larger distortions to aggregate credit supply. Thus, for the same average distortion centrality, networks with a range of nodes of differing importance to other nodes in the network will tend to produce larger distortions in aggregate credit supply.

Examples To further understand this result, it is helpful to consider the example networks in Figure 2.3. To focus on the role of the network structure, suppose that $\sigma_j = \sigma$ for all j, $\kappa_{ij} = \kappa$ for all i, j pairs. Table 2.1 computes σ_D^2, $\bar{D}$, and Var[Δ] for the example networks.[11] The networks are constructed so that the sum of the Bonacich centralities of each node are similar.[12] This goes some way to ensuring that the results are driven by the pattern of the links in the network rather than simply because the links in the network are weak overall.

First, consider the complete network where $l_{ij} = l$ for all i, j pairs. Given the symmetry of the parameters, the attention choices m_{ij} are the same across all i, j pairs and therefore $D_j = D$ for all j. In this case, $\sigma_D^2 = 0$ and all the variation in Δ due to the network is driven by the average distortion centrality, which equals D. With this example, we can see why σ_D^2 has an independent effect on Var[Δ]. Given a realisation of shocks $(\tau_1, \ldots, \tau_n)$, the distortion to aggregate credit supply is $\Delta = \omega D \sum_j \tau_j$. From the summation, we therefore see that the positive and negative distortions caused by the shocks will tend to offset each other. Table 2.1 makes it clear, however, that the variation in Δ can still be large if D is large.

Next, consider the star network with one very important central node and several less important nodes. This star network therefore has a low $\bar{D}$ (because the nodes are either not important or very important to the network) and a low σ_D^2 (because all the distortion centralities are low). Therefore, the star network will tend not to produce large distortions in aggregate credit supply. Intuitively, shocks to the central node do, of course, matter considerably for optimal credit supply decisions. But they do not cause *distortions* from the optimal because nodes pay

[11] For the calculations, I take $\omega = 0.75$, $\kappa = 1$, $\sigma = 1$, $\mu_\tau = 0$, and $\sigma_\tau^2 = 1$.

[12] The sums are 15.22, 15.00, and 14.82 for, respectively, the complete network, line network, and star network.

Network	σ_D^2	$\bar{D}$	Var[Δ]
Complete	0	-1.05	4.34
Line	0.05	-0.85	3.04
Star	0.006	-0.17	0.14

Table 2.1: Computing Var[Δ] for the example networks in Figure 2.3

close to full attention to any shocks hitting the central node.[13] Little attention is paid to the peripheral nodes, but this does not matter for the aggregate credit supply distortion because the peripheral nodes are unimportant for determining bank values in the network.

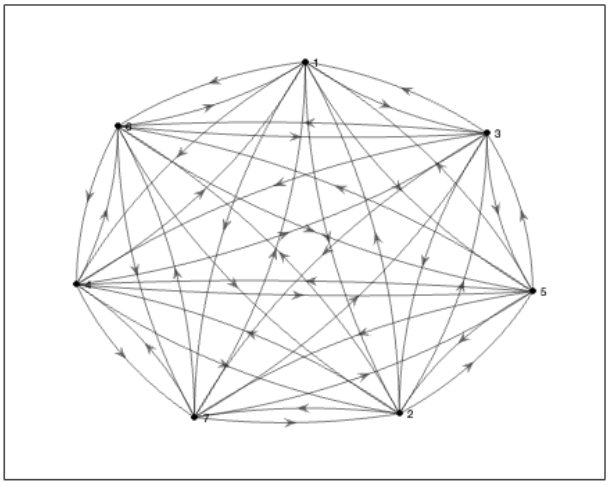
Finally, consider the line network. Notice that the central node is of moderate importance to the end nodes (the distance between the nodes is 3) whereas the two end nodes are unimportant to each other (the distance between the nodes is 6). It is clear, therefore, that the central node is of moderate importance to more nodes in the network compared to nodes towards the end. The distortion centrality is thus highest for the central node and decreases as one moves towards the end nodes. It follows that, compared to the star network, $\bar{D}$ and σ_D^2 will be higher. Therefore, the line network will tend to produce larger distortions in aggregate credit supply. Intuitively, σ_D^2 matters because whenever banks of moderate importance to the network experience negative shocks, this will contribute to a positive distortion to credit supply. This is not offset by the negative distortions caused by positive shocks hitting the other banks in the network because these banks have low distortion centralities. In this example, the result is a large positive distortion to aggregate credit supply.

2.5 Policy implications

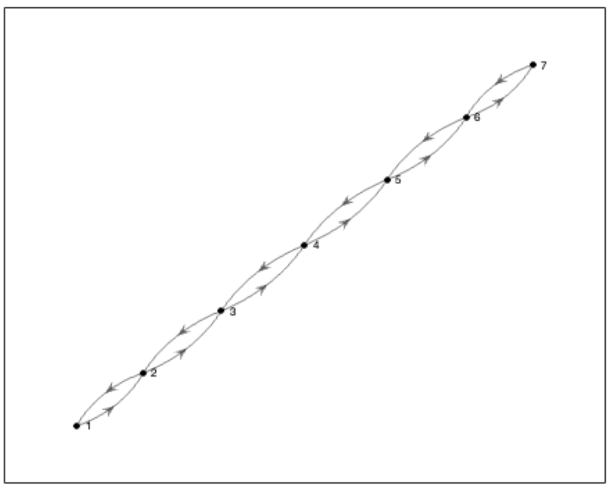
This paper has a number of potential implications for regulating the financial system. The model, however, is highly stylised so the discussion below naturally will not take into account all considerations relevant for formulating real-world policy.

This paper highlights how the network structure of the financial system can increase the likelihood that participants do not pay attention to changes in the health of other participants in the system. In particular, we cannot necessarily rely on markets to pay adequate attention when there are large numbers of participants in the system that are moderately important for

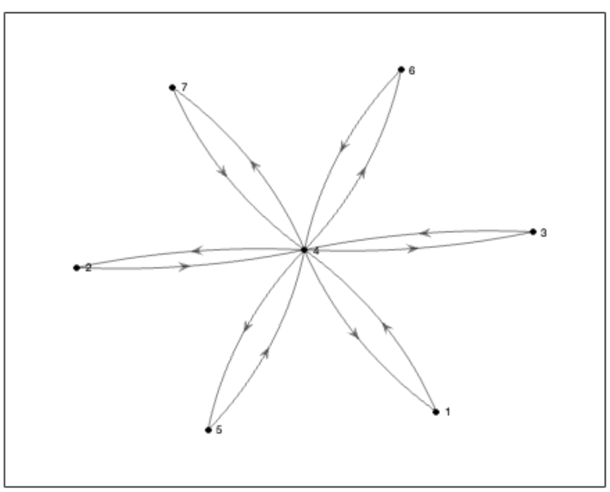
[13]This fits with the empirical evidence in Docking et al. (1997). They examine the effect of a bank announcing an increase in its loan loss provisions. They show that there is an information transfer effect for other nonannouncing banks with the exception of important money-center banks. They conjecture that this is because analysts pay considerable attention to these influential money-center banks and therefore have a good understanding of the true value of money-center banks.



$$W_{\mathrm{comp}} = \begin{pmatrix} 0 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 \\ 0.09 & 0 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 \\ 0.09 & 0.09 & 0 & 0.09 & 0.09 & 0.09 & 0.09 \\ 0.09 & 0.09 & 0.09 & 0 & 0.09 & 0.09 & 0.09 \\ 0.09 & 0.09 & 0.09 & 0.09 & 0 & 0.09 & 0.09 \\ 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0 & 0.09 \\ 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0 \end{pmatrix}$$



$$W_{\mathrm{line}} = \begin{pmatrix} 0 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0 \end{pmatrix}$$



$$W_{\mathrm{star}} = \begin{pmatrix} 0 & 0 & 0 & 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.99 & 0 & 0 & 0 \\ 0.02 & 0.02 & 0.02 & 0 & 0.02 & 0.02 & 0.02 \\ 0 & 0 & 0 & 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.99 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2.3: Network examples

determining values in the network. This implies that policymakers should take into account the impact of financial system reforms on the structure of the system and how this in turn affects the incentives for financial institutions to monitor each other.

One example of such a reform would be the 'ring-fencing' of retail banking activities for large UK banks. This policy was introduced to protect retail banking activity from shocks in the other areas of the banking system. This paper highlights an additional implication of this policy because the act of splitting the network into separate components also changes the strength of the direct and indirect interdependencies between banks. If the reform is enacted such that the number of banks with high distortion centralities falls, then it also reduces the risk that distortions in credit supply can build up due to inattention. Another example of a reform is central counterparty clearing (CCP). One argument for CCP is greater transparency of risks (see, for example, Haldane (2009)). The model formalises this idea because a CCP system creates a star network in a particular market with the CCP acting as the important central node.[14] In the model, this network structure is less prone to distorting credit supply decisions because participants can focus their attention on the important central participant. The model also supports the claim that the lengthening of financial intermediation chains in the lead up to the 2008 financial crisis were harmful to financial stability (Shin 2010). Such intermediation chains can be viewed as line networks. This is a structure that can generate a large value for σ_D^2 which independently contributes to large distortions in aggregate credit supply.

It is worth contrasting these points with the existing networks literature on systemic risk. Much focus is (rightly) placed on large financial institutions that are too big or interconnected to fail. This paper does not contradict this point; after all, there is no default or bank failure in this model so we cannot even discuss issues of too interconnected to fail. Rather, the focus in this paper is on the role the network plays in the build up of financial fragilities, where the fragility in this model is the distortion to credit supply. In other words, this paper focuses on the potential *sources* of a shock rather than the *propagation* of a given shock. In this sense, my paper complements the existing financial networks literature on system risk rather than overturning the policy implications stemming from it. The additional financial stability implication of the paper is that one should be wary of ignoring all but the most interconnected banks because fragilities arising from inattention are most likely to build up in moderately important banks.

[14] Strictly speaking, the financial institutions represented in the network are banks which have a different business model to CCPs. It should, however, be clear that the intuition extends to this case.

2.6 Conclusion

This paper combines the attention choice and financial networks literature to investigate whether the structure of the financial system can play a role in the build-up of financial fragility due to inattention. I show that banks that are of moderate importance to other banks in the network are most prone to causing distortions in credit supply from the optimal level. The reason is because banks do not deem such banks to be important enough to pay high levels of attention to, but they are still important for determining bank values. This insight has implications for reforms of the financial system that change the network structure of the financial system, such as ring-fencing or central counterparty clearing. It also suggests that we should be more cautious about dismissing all but the most interconnected banks as being important for ensuring financial stability.

A limitation of this paper is that the opacity only refers to the balance sheets of the banks in the network and not the links in the networks themselves. For banks with publicly traded equity, this may not be hugely limiting because there are methods for estimating the position of banks in the network (see Diebold & Yilmaz (2014)). However, generalising the model to non-bank financial institutions, for example, where the opacity of the links is of first-order importance would be problematic. Such an extension would be challenging to incorporate whilst allowing for an attention choice. This is because network effects are nonlinear and tractability in the attention choice literature requires a linear-quadratic setup. Investigating the implications of the opacity of the network linkages themselves could provide an interesting avenue for future research.

Chapter 3

The Economic Consequences of Geopolitical Uncertainty

Abstract We argue that geopolitical uncertainty can often be viewed as Knightian uncertainty rather than risk and our objective is to examine the effects of this uncertainty. We do this by constructing a dynamic stochastic equilibrium production model of a world economy with two countries. Each country is characterised by a traded and a nontraded goods sector and a representative investor with Stochastic Differential Utility who is averse to Knightian uncertainty. We model geopolitical uncertainty as a loss of confidence in the correct model for the shocks to the efficiency units of capital where the investors cannot assign probabilities to the alternative models for the shocks. We solve this model in closed form and show how uncertainty operates by reducing households' perceived expected returns on capital which, in turn, distorts portfolio and consumption choice decisions. We then examine the implications of these distortions for trade flows, exchange rates, growth, and the level of social welfare. We show that our model can match stylised facts of the UK economy following the Brexit referendum.

JEL classification: D51, D81, E44, E71, F41, G11

Keywords: Uncertainty, investment, financial markets, portfolio choice, trade, real exchange rate.

3.1 Introduction

In recent years, the economic environment has been characterized by a substantial increase in geopolitical uncertainty. In the U.S., Trump's trade war with China, his withdrawal from the nuclear treaty with Iran, and his on-and-off engagement with North Korea are only some of the erratic policy swings that have increased uncertainty. In China, Xi Jinping is fighting back with retaliatory tariffs against U.S. goods. In the U.K., the vote in 2016 to leave the European Union has led to heightened uncertainty about the terms under which Brexit will ultimately be negotiated. In other European countries, traditional political parties have lost power, while populist and nationalist political parties on the extreme right and left have increased in prominence leading to higher uncertainty about future government policy. The Covid-19 pandemic has exacerbated this uncertainty.

Our objective in this paper is to determine the economic consequences of geopolitical uncertainty. In particular, what are the consequences for the consumption and portfolio decisions of households? And what do these decisions imply for trade flows, exchange rates, the growth rate of the economy, and the level of social welfare?

To answer these questions, we construct a dynamic stochastic equilibrium model of a world economy with two countries and solve the model in closed form. Each country is characterized by two production sectors: one sector produces a tradable good, as in Cox et al. (1985), where physical capital is subject to exogenous shocks. The other sector produces a good that is not tradable. Each country has a representative investor with Epstein-Zin-Weil preferences modelled in continuous time, as in Duffie & Epstein (1992).

We model geopolitical uncertainty as a loss in confidence in the probability model for the exogenous shocks to physical capital. Due to this loss in confidence, investors consider alternative probability models for the shocks (Hansen & Sargent 2008). We assume that the investors do not know the probability distribution over the alternative probability models. In other words, investors face Knightian uncertainty (Knight 1921) with respect to the alternative probability models. This modelling choice is motivated by the observation that it is difficult to put probabilities on the state of the business environment during times of geopolitical uncertainty. Indeed, Bloom et al. (2019) point out that a significant portion of uncertainty due to Brexit can be viewed as Knightian uncertainty. In a Bank of England speech, Haskel (2019) argues that the uncertainty due to Brexit is perhaps not amenable to applying probabilities.

We show that an increase in geopolitical uncertainty has the effect of reducing investors' subjective beliefs of expected returns downwards and this channel is distinct from an increase in the volatility of the shocks to physical capital. Due to lower perceptions of expected returns, investors tilt their portfolio away from regions where there is high geopolitical uncertainty and this distorts the consumption-wealth ratio relative to the case of no geopolitical uncertainty. The distortions to the portfolio and consumption choices of investors have implications for trade flows, exchange rates, the growth rate of the economy, and the level of social welfare. We show that our model can match stylised facts of the UK economy following the Brexit referendum.

We now provide a review of the literature including an explanation of the difference between risk and uncertainty.

Risk and uncertainty This paper is related to the extensive literature on decision making when outcomes are no longer certain. Often we simply refer to such decisions as decisions under uncertainty. However, this terminology is too imprecise for our purposes. Instead, we follow Knight (1921) and distinguish between risk and uncertainty.

Let $X = \{x_1, x_2, \dots, x_n\}$ be a fixed, finite set of possible outcomes. We say that the decision maker faces a choice under *risk* if she knows the objective probability distribution $p = (p_1, p_2, \dots, p_n)$ corresponding to the outcomes with $\sum_{i=1}^{n} p_i = 1$. A set of possible outcomes combined with an objective probability distribution gives us a *lottery*, which we can write as $L = (x_1, p_1; x_2, p_2, \dots; x_n, p_n)$. The most well-known theory of choice under risk is expected utility theory due to von Neumann & Morgenstern (1947). They showed that, under certain axioms, the utility from any lottery can be written in expected utility form

$$U(L) = \sum_{i=1}^{n} p_i u(x_i).$$

We say a decision maker faces choice under *(Knightian) uncertainty* if she does not know the objective probability distribution p. Savage (1954) showed that we can model decision makers as if they were expected utility maximisers but where they replace the objective probability distribution with a subjective probability distribution. Put differently, decision makers reduce uncertainty to subjective risk. Savage's theory is called subjective expected utility theory.

In a famous experiment, Ellsberg (1961) provided a critique to subjective expected utility theory known as the Ellsberg paradox. In its simplest form, the decision maker faces a choice between two urns. Urn A contains 50 red balls and 50 black balls whilst urn B contains an unknown fraction of red and black balls. The decision-maker is asked to select an urn to draw a ball from and she wins a prize if the ball is red. Most decision-makers choose to draw a ball from urn A. According to subjective utility theory, this implies that the decision-maker must think there are less than 50 red balls in urn B. The rules are then changed so that the winning ball is now black. Most decision-makers still choose to draw a ball from urn A. If subjective utility theory were correct, then this implies that the decision-maker must think there are more than 50 red balls in urn B. Clearly, a valid subjective probability distribution does not exist. We refer to the situation when no subjective probability distribution over the outcomes exists as choice under *ambiguity*. Choice under ambiguity is thus a special case of choice under Knightian uncertainty.[1]

A leading explanation for the Ellsberg paradox is the maximin expected utility model of Gilboa & Schmeidler (1989).[2] To see how this model can explain the Ellsberg paradox, consider the urn example from above and suppose that decision-maker knows the set of possible subjective probabilities for the number of red balls in urn B is given by the interval $\Delta = [0.4, 0.6]$. That is, we assume that the decision-maker has a set of different prior models for the number of red balls in urn B. Decision-makers are assumed to be completely agnostic about the correct model in Δ. Instead of forming some subjective probability distribution over the models in Δ, they choose the worst possible model in Δ to base their decisions on. Therefore, when the winning ball is red, they base their decisions on the belief that there are 40 red balls in urn B; when the winning ball is black, they base their decisions on the belief that there are 60 red balls in urn B. Clearly, this can explain why decision-makers usually choose urn A both times.

The maximin expected utility model has been applied to macroeconomics by Hansen & Sargent (2001, 2008) with the development of robust control theory. Decision-makers are assumed to have a reference model for the economy, but they are unsure whether it is the correct model. They therefore consider alternative models for the economy but are penalised for considering models that are far from the reference model, as measured by relative entropy (or the Kullback-Leibler divergence). They choose the worst possible model for the economy but are prevented from being overly pessimistic by the penalty term. The preferences used in robust control theory are not the same as those in Gilboa & Schmeidler (1989). However, Maccheroni et al. (2006) show that the preferences used in robust control theory and formulation by Gilboa & Schmeidler (1989) are a special case of a more general class of variational preferences.[3] In this paper, we draw upon the robust control theory approach to modelling choice under Knightian

[1]In what follows, our terminology (somewhat loosely) equates choice under ambiguity to choice under Knightian uncertainty. This follows the approach in much of the literature (see Nishimura & Ozaki (2017)) and is justified by the fact that the interesting case of choice under Knightian uncertainty is when uncertainty is not reducible to (subjective) risk.

[2]Other explanations of Ellsberg paradox include Choquet expected utility Schmeidler (1986, 1989), failure of the reduction of compound lotteries (Segal 1987, 1990), and the smooth ambiguity aversion model (Klibanoff et al. 2005). See Dhami (2016) for a survey.

[3]See Strzalecki (2011) and Hansen et al. (2006) for more on the links between different types of preference specifications.

uncertainty.

Applications of Knightian uncertainty in macroeconomics and finance Robust control theory has been applied to asset pricing. Hansen et al. (1999) introduce a concern for model misspecification into a robust permanent income model. They show that a preference for robustness can offer an explanation for the equity premium puzzle (Mehra & Prescott 1985) by substituting for risk aversion. Maenhout (2004) solves the portfolio choice problem explicitly for the case of a riskless and single risky asset and shows that one can maintain a realistic value for risk aversion whilst generating a 4-6% equity premium. Uppal & Wang (2003) allow ambiguity to differ across multiple risky assets which allows them to study the implications of ambiguity for portfolio diversification. Bhamra & Uppal (2019) build on this framework to argue that the costs of household familiarity biases can be large when one accounts for the general equilibrium effects of portfolio underdiversification. This paper recasts this framework in the context of geopolitical uncertainty and extends it to allow us to study the implications of geopolitical uncertainty for trade flows, exchange rates, growth rates, and the level of social welfare.

There is also a literature that remains closer to the Gilboa & Schmeidler (1989) formulation. Epstein & Wang (1994) introduce uncertainty aversion into the Lucas (1978) asset pricing model and show that this can lead to equilibria indeterminancy. Chen & Epstein (2002) extend the framework to the continuous time. Nishimura & Ozaki (2004) and Nishimura & Ozaki (2007) analyse the effects of introducing Knightian uncertainty into (respectively) search models and a model of irreversible investment. Ilut & Schneider (2014) show that Knightian uncertainty can help to explain business cycle movements in macroeconomic aggregates. Caballero & Krishnamurthy (2008) and Caballero & Simsek (2013) explain how Knightian uncertainty can be important for understanding financial crises.

Political uncertainty This paper is related to the literature examining the effect of political uncertainty on financial markets. Pástor & Veronesi (2012, 2013) develop a general equilibrium models to study the effect of political uncertainty on stock prices. Kelly et al. (2016) measure how uncertainty around major political events is priced in markets for financial options. Dai & Zhang (2019) provide a survey. Our paper differs in that it relates political uncertainty to Knightian uncertainty and analyses the effect of this uncertainty on financial markets in an equilibrium model.

3.2 A Model of Geopolitical Uncertainty and Growth

3.2.1 Production

We model an economy with 2 countries or regions. Each country produces two goods: a nontraded good that is country-specific and a traded good that is produced in both countries. We take the traded good to be the numeraire. We first discuss the production of the common traded good.

The date-t value of the capital stock located within country $n \in \{1, 2\}$ is given by $K_{n,t}$ and the output flow by

$$Y_{n,t} = \alpha K_{n,t}, \tag{3.1}$$

for some constant technology level α. The level of the capital stock located in country n can be increased by investing at the rate $I_{n,t}$. Thus, the capital accumulation in country n's capital stock is given by:

$$dK_{n,t} = I_{n,t}\, dt + \sigma\, K_{n,t}\, dZ_{n,t}, \tag{3.2}$$

where σ, the volatility of the exogenous shock to country n's capital stock, is constant over time and across countries. The term $dZ_{n,t}$ is the increment in a standard Brownian motion under the physical probability measure $\mathbb{P}$. The term $dZ_{n,t}$ can be interpreted as a stand-in for an exogenous shock to the efficiency units of capital, for example, due to technological innovation or changes in regulations.[4]

Furthermore, $dZ_{n,t}$ is country-specific, which creates ex-post heterogeneity across countries. To ease the exposition, we assume the correlation between shocks to the capital stocks for countries 1 and 2 is zero i.e.,

$$E_t[dZ_{1,t} dZ_{2,t}] = 0, \tag{3.3}$$

where $E_t[\cdot]$ denotes the date-t expectation under the physical probability measure $\mathbb{P}$.[5]

The flow of output of the traded good produced in country n is divided between its investment

[4] Introducing the source of stochasticity in the capital accumulation equation is standard in the macrofinance literature (see, for example, Brunnermeier & Sannikov (2014) and Fernández-Villaverde et al. (2019)).

[5] The assumptions (i) $\alpha_n = \alpha$ for all n, (ii) $\sigma_n = \sigma$ for all n, and (iii) $E_t[dZ_{1,t} dZ_{2,t}] = 0$ can be relaxed relatively trivially. The assumptions are made to ease the exposition.

flow and dividend flow:

$$Y_{n,t} = I_{n,t} + D_{n,t}. \tag{3.4}$$

We can therefore rewrite the capital accumulation equation for country n's capital stock as

$$dK_{n,t} = \big(\alpha K_{n,t} - D_{n,t}\big)dt + \sigma\, K_{n,t}\, dZ_{n,t}. \tag{3.5}$$

There are 2 non-traded goods, one for each country. The date-t output flow of the nontraded good, which is specific to country n, is denoted by the constant Y_n^{non}. The date-t price of the nontraded good produced by country n (relative to the price of the numeraire) is denoted by $P_{n,t}$.

3.2.2 Investment Opportunities

There are 2 representative investors, one for each country, indexed by $h \in \{1, 2\}$. The representative investors of each country can invest their wealth in two classes of assets. The first is a risk-free bond, which has an interest rate r that we assume is exogenous and constant over time. This assumption is necessary to obtain closed-form solutions and we discuss this assumption in more detail in section 3.3.4.

Let $B_{h,t}$ denote the stock of wealth invested by household h in the risk-free asset at date t. The bond is continuously rolled over and so

$$B_{h,t+dt} = e^{r\,dt} B_{h,t}. \tag{3.6}$$

Hence, we obtain

$$\frac{dB_{h,t}}{B_{h,t}} = r\,dt. \tag{3.7}$$

Additionally, investors can invest in the 2 risky firms, where firm n is located in country n. Firm n produces output flow according to (3.1). We use $K_{hn,t}$ to denote the stock of wealth owned by the representative investor of country h which is invested in the capital stock of country n. Given that the financial wealth owned by the representative investor of country h, $W_{h,t}$, is held in the risk-free asset and in each of the firms $n \in \{1, 2\}$, we have that the total financial wealth of representative investor h is given by

$$W_{h,t} = B_{h,t} + \sum_{n=1}^{2} K_{hn,t}. \tag{3.8}$$

Let $\omega_{hn,t}$ denote the proportion of a household's wealth invested in risky firm n at time t and $\pi_{h,t} = \sum_{n=1}^{2} \omega_{hn,t}$ denote the proportion of household h's wealth allocated to all risky assets at date t. Moreover, let $x_{hn,t} = \omega_{hn,t}/\pi_{h,t}$ denote the weight of risky asset n in the portfolio consisting of only risky assets. The representative investor for country h consumes the traded good at the rate $C_{h,t}$ and the nontraded good produced in country h at the rate $C_{h,t}^{\text{non}}$. As shown in the appendix,[6] the dynamic intertemporal budget constraint for $h \in \{1, 2\}$ can be written as

$$\frac{dW_{h,t}}{W_{h,t}} = (1 - \pi_{h,t})r dt + \pi_{h,t} \sum_{n=1}^{2} x_{hn,t}(\alpha dt + \sigma dZ_{n,t}) + \left(\frac{P_{h,t} Y_h^{\text{non}}}{W_{h,t}} - \frac{P_{h,t} C_{h,t}^{\text{non}}}{W_{h,t}} \right) dt - \frac{C_{h,t}}{W_{h,t}} dt. \tag{3.9}$$

The first term is the change in wealth due to bond holdings. The second term is the change in wealth due to holdings of risky assets. The third term is zero in equilibrium because $P_{h,t}$ adjusts to ensure the market for the nontraded good clears ($Y_h^{\text{non}} = C_{h,t}^{\text{non}}$). We introduce the nontraded good into the budget constraint so we can derive the investor h's optimal choice of $C_{h,t}^{\text{non}}$. The final term is the reduction in wealth due to consumption.

3.2.3 Representative investors

We first explain the optimisation problem of the representative investor in each country in the absence of the concern that the probability model $\mathbb{P}$ for the shocks to the capital stock of the 2 countries might be misspecified. We then explain how we model investors' concern for model misspecification when they face Knightian uncertainty with respect to alternative models.

Optimisation problem in the absence of Knightian uncertainty The utility of each representative investors is modelled by standard Epstein & Zin (1989) preferences, but set in continuous time as in Duffie & Epstein (1992). In the absence of Knightian uncertainty, an investor's date-t utility level, $U_{h,t}$, is defined by an intertemporal aggregation of date t consumption flow, $C_{h,t}$, and the date-t certainty-equivalent of date $t + dt$ utility

$$U_{h,t} = \mathcal{A}(C_{h,t}^{\text{ind}}, \mu_{h,t}[U_{h,t+dt}]), \tag{3.10}$$

[6] All proofs can be found in Appendix B.

where $C^{\text{ind}}_{h,t}$ is the Cobb-Douglas consumption index, defined over the consumption of the traded and nontraded goods by

$$C^{\text{ind}}_{h,t} = C^{\theta}_{h,t}(C^{\text{non}}_{h,t})^{1-\theta}, \tag{3.11}$$

and $\mathcal{A}(\cdot,\cdot)$ is the standard time aggregator,

$$\mathcal{A}(a,b) = \left[(1-e^{-\delta dt})a^{1-\frac{1}{\psi}} + e^{-\delta dt}b^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{3.12}$$

in which $\delta > 0$ is the rate of time preference, $\psi > 0$ is its elasticity of intertemporal substitution, and $\mu_{h,t}[U_{h,t+dt}]$ is the date-t certainty equivalent of $U_{h,t+dt}$. Note that, in order to focus on the effect of Knightian uncertainty, we have assumed that representative investors for all countries have the same preference parameters.

In the absence of model misspecification, a household would choose its consumption rate of the traded good $C_{h,t}$, its consumption rate of the nontraded good $C^{\text{non}}_{h,t}$, its asset-allocation policy $\pi_{h,t}$, and its portfolio-diversification policy $\boldsymbol{x}_{h,t} = (x_{h1,t},\ldots,x_{hN,t})^\top$ to maximize its utility

$$\sup_{C_{h,t},C^{\text{non}}_{h,t}} \mathcal{A}\left(C^{\text{ind}}_{h,t}, \sup_{\pi_{h,t},\,\boldsymbol{x}_{h,t}} \mu_{h,t}[U_{h,t+dt}]\right), \tag{3.13}$$

where expectations are computed using the physical probability model $\mathbb{P}$.

Modelling Knightian uncertainty In our model, the representative investors are not fully confident in the physical probability model $\mathbb{P}$ for the shocks to the capital stock of the 2 countries, $d\boldsymbol{Z}_t = (dZ_{1,t}, dZ_{2,t})^\top$. Instead, they treat $\mathbb{P}$ as a "reference model" obtained from the result of some imperfect estimation process. They acknowledge their uncertainty by recognising that a number of alternative probability models could in fact be the true probability model for $d\boldsymbol{Z}_t$. We assume that the investors cannot specify a probability distribution over the alternative models for $d\boldsymbol{Z}_t$. That is, the investors face Knightian uncertainty with respect to the *alternative probability models* for $d\boldsymbol{Z}_t$.

We show in the appendix that the alternative probability models are fully characterised by the vector $\boldsymbol{\nu}_{h,t} = (\nu_{h1,t}, \nu_{h2,t})^\top \in \mathbb{R}^2$. The vector $\boldsymbol{\nu}_{h,t} = \mathbf{0}$ corresponds to the reference model $\mathbb{P}$. Values of $\nu_{hn,t}$ further from zero correspond to models for $dZ_{n,t}$ that are further from the reference model. By Girsanov's theorem,[7] we show that different probability models for $dZ_{n,t}$

[7] See Hansen & Sargent (2001) for a discussion of Girsanov's theorem in the context of the robust control literature. Girsanov's theorem is also widely used in finance in the context of risk-neutral pricing where the expected value of future cashflows is calculated using probabilities adjusted for risk (risk-neutral probabilities)

correspond to different adjustments to each investor's perceptions of expected return on capital for the traded-good sector for country n. In particular, an investor using the model $\nu_{hn,t}$ perceives an expected return of $\alpha + \nu_{hn,t}$ compared to the objective expected return α.

Each representative investor selects one alternative probability model to use as the basis for her decisions. We denote this alternative model by $\mathbb{Q}^{\boldsymbol{\nu}_h}$. In the next subsection, we explain how the investor chooses $\mathbb{Q}^{\boldsymbol{\nu}_h}$.

Optimisation problem with Knightian uncertainty We capture the investors' response to Knightian uncertainty inspired by the approach taken in the robust control theory literature Hansen & Sargent (2001, 2008). We replace the maximization of the standard certainty-equivalent with the combined maximization and minimization of the *modified certainty-equivalent* $\mu^{\boldsymbol{\nu}}_{h,t}$:

$$\sup_{C_{h,t}, C^{\text{non}}_{h,t}} \mathcal{A}\left(C^{\text{ind}}_{h,t}, \sup_{\pi_{h,t}, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} \mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]\right), \tag{3.14}$$

where we *define* the modified certainty-equivalent by

$$\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = \widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] + U_{h,t} L_{h,t} dt, \tag{3.15}$$

where $\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]$ is

$$u_\gamma\left(\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]\right) = E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t[u_\gamma\left(U_{h,t+dt}\right)]. \tag{3.16}$$

and where $u_\gamma(\cdot)$ is the static utility index defined by the power utility function

$$u_\gamma(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0,\, \gamma \neq 1 \\ \ln x, & \gamma = 1. \end{cases} \tag{3.17}$$

The first term in the definition of the modified certainty equivalent is simply the standard certainty equivalent of date $t + dt$ utility computed using the probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$. The second term is a penalty term $L_{h,t}$ that penalises the investor for choosing a probability model further from the reference model. This captures the idea that the investor will typically not want to throw away all the useful information contained in her estimate for $\mathbb{P}$.

as opposed to objective probabilities. Such probabilities adjust for risk by putting more weight on bad outcomes relative to the objective probabilities. In our model, we will see in section 3.3 that investors will evaluate the future using a more pessimistic probability measure due to concern about model misspecification rather than as an adjustment for risk.

We define the loss function as[8]

$$L_{h,t} = \frac{1}{2}\frac{1}{\sigma^2}\frac{\theta}{[1-\theta+\gamma\theta]}\sum_{n=1}^{2}\phi_{hn}\nu_{hn,t}^2. \tag{3.18}$$

The $\nu_{hn,t}^2$ terms penalize the investor for choosing a model that is far from the reference model. These penalties are weighted by confidence parameters $\phi_{hn} \in [0,\infty)$. When $\phi_{hn} \to \infty$, investor h is fully confident in the reference model for country n and the penalty incurred by any deviation from the reference model is infinitely large. When $\phi_{hn} = 0$, investor h has no confidence in the reference model for country n and any deviation from the reference model incurs zero penalty. Finally, we scale the loss function by the constant $\theta/(2\sigma^2[1-\theta+\gamma\theta])$.[9]

For intuition, we can visualize the set of alternative models corresponding to a confidence-weighted penalty less than or equal to R^2 via an ellipse centred on the origin with semi-axis lengths $R\sigma/\sqrt{\phi_{h1}}$ and $R\sigma/\sqrt{\phi_{h2}}$. We think of times of higher geopolitical uncertainty in country 1 as times when it is more reasonable for an investor to contemplate the idea that the reference model $\mathbb{P}$ as applied to country 1 may not be the true model for $dZ_{1,t}$. The increase in geopolitical uncertainty is captured by a fall in ϕ_{h1}. This widens the ellipse and thus increases the set of alternative models for representative investor of country h with penalty less than or equal to R^2 (see figure 3.1).[10] Note that ϕ_{21} and ϕ_{11} do not have to be identical, so the representative investors of countries 1 and 2 can consider differing sets of models for a given confidence-weighted penalty.

In what follows, it will be useful to use

$$\kappa_{hn} = \frac{\phi_{hn}}{1+\phi_{hn}}$$

as our measure of geopolitical uncertainty. Larger values of $\kappa_{hn} \in [0,1]$ correspond to higher confidence in the reference measure; that is, lower amounts of geopolitical uncertainty.

[8] In the appendix, we motivate the definition of the loss function via the Kullback-Leibler divergence.

[9] The scaling does not affect the analytical results in the paper. Indeed, we could redefine the uncertainty parameters as $\tilde{\phi}_{hn} = \frac{\theta}{\sigma^2[1-\theta+\gamma\theta]}\phi_{hn}$ and proceed with no changes to the analytical results. The scaling only matters if we were to do comparative statics exercises on θ, σ, and γ, or if we wished to undertake a rigorous quantitative analysis.

[10] This intuition highlights the similarity with the specification in Drechsler (2013) which limits deviations from the reference model via an entropy constraint rather than an entropy penalty. Time-variation in uncertainty is incorporated through time-variation of the bound on the constraint. We have a similar setup via the entropy penalty approach except that for our purposes it is not necessary to explicitly model time variation in ϕ_{hn}.

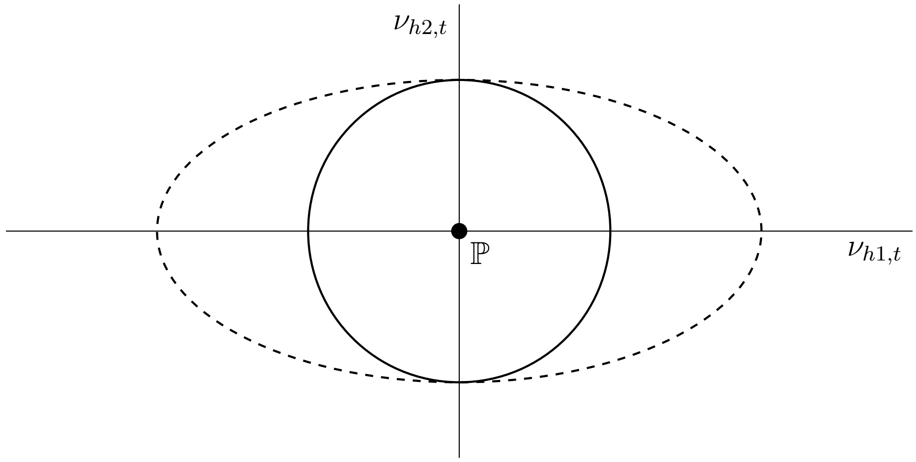


Figure 3.1: Increase in geopolitical uncertainty in region 1 increases the set of models with confidence-weighted penalty less than or equal to R^2

3.3 Individual Portfolio and Consumption Choices

In this section we solve the household's stochastic control problem and identify the effect of geopolitical uncertainty on the household's portfolio and consumption choices. We also provide a discussion and justification of the assumption of a constant exogenous interest rate.

3.3.1 The household decision problem

The optimal consumption and portfolio policies can be identified by solving the Hamilton-Jacobi-Bellman equation,

$$\begin{aligned} 0 = \sup_{C_{h,t}, C_{h,t}^{\text{non}}} \Bigg(\delta\, u_\psi \left(\frac{C_{h,t}^{\theta} (C_{h,t}^{\text{non}})^{1-\theta}}{J_{h,t}} \right) \\ + \sup_{\pi_{h,t}, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} \frac{1}{dt} \left(E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] + L_{h,t} dt \right) \Bigg), \end{aligned} \tag{3.19}$$

where $J_{h,t}$ denotes investor h's time-t value function. Under the Ansatz $J_{h,t} = \xi_h W_{h,t}^{\theta}$, where ξ_h is a constant, (3.19) can be written as

$$0 = \sup_{C_{h,t}, C_{h,t}^{\text{non}}} \left(\frac{\delta}{\theta}\, u_\psi \Big(\frac{C_{h,t}^{\theta} (C_{h,t}^{\text{non}})^{1-\theta}}{\xi_h W_{h,t}^{\theta}} \Big) + \Big(\frac{P_{h,t} Y_h^{\text{non}}}{W_{h,t}} \Big) - \Big(\frac{C_{h,t} + P_{h,t} C_{h,t}^{\text{non}}}{W_{h,t}} \Big) + MV_h \right), \tag{3.20}$$

where

$$MV_h = \sup_{\pi_{h,t},\, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dR^p_{h,t}}{dt} \right] - \frac{1}{2} \left[1 - \theta + \gamma\theta\right] \frac{1}{dt} E_t[(dR^p_{h,t})^2] + \frac{L_{h,t}}{\theta} \tag{3.21}$$

and $dR^p_{h,t}$ is the stochastic process describing the portfolio return of investor h.

The expression in (3.21) is investor h's mean-variance utility from the portfolio choice problem. The first term says that utility is increasing in the expected change in the portfolio's return, $dR^p_{h,t}$, because higher returns on the portfolio generate higher levels of wealth. Notice that the expectation is computed using the investor's alternative probability model $\mathbb{Q}^{\boldsymbol{\nu}_h}$. The second term penalises the investor for variance in the portfolio's return with the penalty for variance equal to $\frac{1}{2}[1 - \theta + \gamma\theta]$ (we explain this below). Notice that the expectation can be computed under the objective measure $\mathbb{P}$ because the second order terms are deterministic given that our underlying source of randomness is a Brownian motion. The final term is the penalty for choosing an alternative model $\mathbb{Q}^{\boldsymbol{\nu}_h}$ that is far from the reference model $\mathbb{P}$, as explained above.

The penalty for variance can be best understood by first considering the case when $\theta = 1$ for which the value function is linear in wealth. If $\theta = 1$, then the penalty for variance is the familiar $\frac{\gamma}{2}$ which is increasing in the coefficient of relative risk aversion γ. When $\theta < 1$, the value function is becomes strictly concave in wealth. The key change is the additional penalty term for variance $1 - \theta$ which does not depend on relative risk aversion. To obtain some loose intuition for this term, recall that the source of randomness in wealth is the Brownian motion whose fluctuations are so "large" that these fluctuations actually make a deterministic contribution to the value function $J_{h,t}$. Since $J_{h,t}$ is a concave function in wealth, this deterministic contribution arising due to fluctuations in the Brownian motion will cause $J_{h,t}$ to go down regardless of whether the fluctuations are positive or negative.[11]

3.3.2 Optimal portfolio choice

The optimal choices for beliefs, the share of the risky portfolio allocated to each risky asset, and the total share of wealth allocated to all risky assets are obtained by solving the first-order conditions from (3.20) for $\boldsymbol{\nu}_{h,t}$, $\boldsymbol{x}_{h,t}$, and $\pi_{h,t}$, respectively.

[11] This is essentially an intuitive (and necessarily imprecise) explanation for the presence of a second order term in Ito's lemma.

The optimal adjustment to expected returns due to geopolitical uncertainty is

$$\nu_{hn,t} = -\frac{(\alpha - r)}{1 + \phi_{hn}}. \tag{3.22}$$

To understand this expression, first notice that $\nu_{hn,t} < 0$. That is, household h chooses to make its portfolio decisions according to a model with lower expected returns compared to the reference model for country n. This pessimistic behaviour is a consequence of household h's aversion to the Knightian uncertainty surrounding the alternative models for $d\boldsymbol{Z}_t$. Household h's pessimism is counterbalanced by the confidence it has in its reference model for country n where we link confidence to the amount of geopolitical uncertainty. We see that the less confidence household h has in the reference model for country n (i.e., smaller ϕ_{hn}/higher geopolitical uncertainty), the more they adjust expected returns downwards. On the other hand, if $\phi_{hn} \to \infty$ implying that there is full confidence in the reference model due to zero geopolitical uncertainty, then the adjustment vanishes. In our model, geopolitical uncertainty thus acts in a similar way to risk in that it makes holding shares of the capital stock less attractive. But the channel is distinct to risk because uncertainty acts through the first moment and does not rely on curvature of the utility function to operate.

The optimal weights in the portfolio consisting of only risky assets are given by

$$x_{h1} = \frac{\kappa_{h1}}{\kappa_{h1} + \kappa_{h2}} \tag{3.23}$$

$$x_{h2} = \frac{\kappa_{h2}}{\kappa_{h1} + \kappa_{h2}} \tag{3.24}$$

If an investor is equally confident in the reference model applied to each country, then because the assets have identical first and second moments, the investor would optimally place equal weight on both risky assets in the portfolio consisting of only risky assets. However, if the investor becomes less confident in the reference model applied to country 1 relative to country 2, then the investor tilts her portfolio away from country 1 because she dislikes the geopolitical uncertainty associated with country 1. In the extreme case, if the investor has no confidence in the reference model for country 1, $\kappa_{h1} = 0$, she invests in only the risky asset for country 2. Note that, if we make the reasonable assumption that investors are more confident in the model for their own country, then our model can generate home bias (French & Poterba 1991).

The proportion of household h's assets held in risky assets is given by

$$\pi_h = \frac{1}{1 - \theta + \gamma\theta} \left(\frac{\alpha - r}{\sigma^2/2} \right) \left[\frac{1}{2}(\kappa_{h1} + \kappa_{h2}) \right] \tag{3.25}$$

The first two terms give the standard optimal Merton weight (Merton 1969, 1971) where the penalty for variance is $1-\theta+\gamma\theta$. The second term shows how geopolitical uncertainty causes the investor to reduce the share of wealth allocated to risky assets. Observe that κ_{h1} and κ_{h2} affect the optimal π_h symmetrically.

3.3.3 Optimal consumption

Define household h's total expenditure on the traded and nontraded good as $C_{h,t}^{\text{tot}} = C_{h,t} + P_{h,t}C_{h,t}^{\text{non}}$. The first-order conditions from (3.20) yield

$$\frac{C_{h,t}^{\text{tot}}}{W_{h,t}} = \frac{\psi\delta + (1-\psi)\theta MV_h}{\theta[\psi + (1-\psi)\theta]}, \tag{3.26}$$

where MV_h evaluated at the optimal portfolio weights is given by

$$MV_h = r + \frac{1}{(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[\frac{1}{2}(\kappa_{h1}+\kappa_{h2})\right]. \tag{3.27}$$

MV_h is household h's mean-variance utility from the portfolio choice problem evaluated using the probability model $\mathbb{Q}^{\boldsymbol{\nu}_h}$. We observe that the presence of geopolitical uncertainty reduces MV_h. The effect on the consumption-wealth ratio depends on the magnitude of the elasticity of intertemporal substitution, ψ.

To understand the role of ψ, note that we can interpret MV_h as the quality of the investment opportunities available to the investor. When the quality of investment opportunities declines, there is a negative income effect that acts to reduce consumption. But there is also an offsetting substitution effect: investors will choose to increase consumption because investment opportunities are less attractive. When $\psi > 1$, the investor is less averse to temporal variation in consumption and the substitution effect dominates. An increase in geopolitical uncertainty, which reduces the quality of investment opportunities, therefore increases the consumption-wealth ratio. When $\psi < 1$, the income effect dominates so an increase in geopolitical uncertainty reduces the consumption-wealth ratio. When $\psi = 1$, the income and substitution effects cancel out and the consumption-wealth ratio does not depend on geopolitical uncertainty.

In the discussion that follows, we consider the case $\psi > 1$ so that an increase in geopolitical uncertainty increases the consumption-wealth ratio. Bansal & Yaron (2004) show that an elasticity of intertemporal substitution greater than 1 is required for matching the negative correlation between consumption volatility and price-dividend ratios seen in the data. Note, however, that there is considerable debate in the literature about the correct value of ψ. For example, Vissing-Jørgensen (2002), Vissing-Jørgensen & Attanasio (2003), and Gruber (2013) all estimate ψ to be considerably greater than 1.5. Hall (1988) estimates $\psi < 1$, although Bansal & Yaron (2004) show that ignoring fluctuating economic uncertainty biases Hall's estimator for ψ downwards.

3.3.4 Constant exogenous interest rate assumption

We now provide further discussion and justification for our constant exogenous interest rate assumption.

In a full general equilibrium analysis, we would endogenise the interest rate by imposing that the risk-free bond must be in zero net-supply: $\sum_{h=1}^{2} B_{h,t} = 0$. Using the fact that $B_{h,t} = (1 - \pi_{h,t})W_{h,t}$, we can use equation (3.25) to write the market clearing condition as

$$0 = \left[1 - \frac{1}{1-\theta+\gamma\theta}\left(\frac{\alpha - r_t}{\sigma^2/2}\right)\bar{\kappa}_1\right] W_{1,t} + \left[1 - \frac{1}{1-\theta+\gamma\theta}\left(\frac{\alpha - r_t}{\sigma^2/2}\right)\bar{\kappa}_2\right] W_{2,t} \tag{3.28}$$

where $\bar{\kappa}_h \equiv \frac{1}{2}(\kappa_{h1} + \kappa_{h2})$ and the interest rate is potentially time-varying. Defining investor 1's share of total wealth as

$$\eta_t = \frac{W_{1,t}}{W_{1,t} + W_{2,t}}, \tag{3.29}$$

we can solve for the equilibrium interest rate

$$r_t = \alpha - \frac{(1-\theta+\gamma\theta)(\sigma^2/2)}{\bar{\kappa}_1\eta_t + \bar{\kappa}_2(1-\eta_t)}. \tag{3.30}$$

The term in the numerator of the fraction gives the standard result that the risk premium, $\alpha - r_t$, increases with the variance of the full-diversified portfolio, $\sigma^2/2$, and the penalty for risk, $1-\theta+\gamma\theta$. An increase in geopolitical uncertainty perceived by either investor reduces the attractiveness of investing in risky assets and therefore increases the risk premium. The greater the share of wealth of a particular investor, the greater influence its perceptions of geopolitical uncertainty have on the risk-free rate.

Note that if $\bar{\kappa}_1 = \bar{\kappa}_2 \equiv \bar{\kappa}$, then the interest rate will indeed be constant (and endogenously determined). However, there is no reason to expect this to be the case. In general, the interest rate will be time-varying because it depends on the wealth share η_t which will vary stochastically over time. Allowing a stochastic interest rate would mean that we would need to introduce r_t (or, equivalently, η_t) as an additional state variable. As we now explain, this would add unnecessary complexity to the analysis.

First, the inclusion of an additional state variable would imply that we can no longer obtain closed-form solutions. The optimal consumption and portfolio policies can be derived in terms of the value function, which now also depends on r_t. After substituting the optimal solutions back into the Hamilton-Jacobi-Bellman equation, the problem of finding the value function would reduce to solving a nonlinear partial differential equation. Solving this equation would require numerical methods and therefore the economic mechanisms present in the model would be less transparent.

Moreover, allowing a stochastic interest rate would have little quantitative implication for the portfolio choice problem. The optimal portfolio weight would now also include an intertemporal hedging component to capture the fact that investors would want to hold more of a risky asset that hedges against unfavourable movements in the risk-free rate. It is thus evident that the quantitative importance of this term depends on the correlation between risky assets and the risk-free rate. However, as noted by Bhamra & Uppal (2019), this correlation is small over long-horizons and thus we would expect the effect of the intertemporal hedging component to be small. Therefore, we make the assumption that the interest rate is constant and exogenously given to allow us to obtain closed-form solutions.

3.4 Capital flows, real exchange rates, and growth

3.4.1 Return on wealth

The return on wealth is given by

$$\frac{dW_{h,t}}{W_{h,t}} = \left[r + \frac{1}{1-\theta+\gamma\theta} \left(\frac{\alpha - r}{\sigma} \right)^2 (\kappa_{h1} + \kappa_{h2}) - \frac{C_{h,t}}{W_{h,t}} \right] dt + \frac{(\alpha - r)}{\sigma(1-\theta+\gamma\theta)} \sum_{n=1}^{2} \kappa_{hn} dZ_{n,t}. \tag{3.31}$$

An increase in geopolitical uncertainty (a reduction in κ) affects the return on wealth through three channels. Firstly, it reduces the growth rate of wealth because investors reduce the share of their portfolio allocated to risky assets. Secondly, an increase in uncertainty increases the consumption-wealth ratio and therefore reduces the growth rate of wealth. Thirdly, an increase in uncertainty reduces the volatility of wealth because the investor holds more of their wealth in the risk-free bond.

3.4.2 Net asset position and current account

The net foreign asset position for region h is given by the value of assets owned abroad by the representative investor for region h less the assets of region h held by foreigners. Hence, the net foreign asset position for region 1 is

$$N_{1,t} = \omega_{12} W_{1,t} - \omega_{21} W_{2,t}. \tag{3.32}$$

Substituting in the solutions for the optimal portfolio weights yields

$$N_{1,t} = \frac{(\alpha - r)}{\sigma^2(1 - \theta + \gamma\theta)} \left(\kappa_{12} W_{1,t} - \kappa_{21} W_{2,t}\right). \tag{3.33}$$

For intuition, consider the case when $W_{1,t} = W_{2,t}$. Then $N_{1,t} > 0$ if and only if $\kappa_{12} > \kappa_{21}$. That is, the confidence that the representative household for region 1 has in the model for region 2 must exceed the confidence the representative household for region 2 has in the model for region 1.

The current account for region h is defined as the change in the net asset position of the economy. Hence, the current account for region 1 is

$$dN_{1,t} = \frac{(\alpha - r)}{\sigma^2(1 - \theta + \gamma\theta)} \left[\kappa_{12} W_{1,t} \frac{dW_{1,t}}{W_{1,t}} - \kappa_{21} W_{2,t} \frac{dW_{2,t}}{W_{2,t}}\right]. \tag{3.34}$$

To understand the effect of uncertainty on the current account, suppose that $\kappa_{hn} = \kappa_{jn} = \kappa$ for $n \in \{1, 2\}$ and $W_{1,t} = W_{2,t}$. Since the two regions are initially symmetric, the current account is zero. Now consider an increase in uncertainty in region 1, which in our model corresponds to $\kappa_{11} < \kappa$ and $\kappa_{21} < \kappa$.

If $\kappa_{11} = \kappa_{21} < \kappa$, then $dN_{1,t} > 0$ on impact because investors in region 2 invest less in region 1's capital stock while investors in region 1 do not change their investment in region 2's capital stock. Both countries also experience the same reduction in the growth rate of wealth because they reduce their reduction in region 1's capital stock by the same amount. Therefore, in expectation, there are no further changes in the current account.

Suppose instead that investors in region 2 become more uncertain in region 1's economy *relative* to investors in region 1. That is, suppose that $\kappa_{21} < \kappa_{11} < \kappa$. The effect on the current account on impact is the same as the case of the symmetric increase in uncertainty in region 1. But now there is an additional effect on the current account over time because investors in region 2 will invest less in risky assets than investors in region 1. Therefore, region 2's wealth will fall relative to region 1 resulting in an even larger current account surplus for region 1.

3.4.3 Growth rate

The growth rate is a weighted average of the rates of wealth accumulation. The weights are the proportion of domestic and foreign wealth held in the capital of each region, respectively.

$$\frac{dK_{1,t}}{K_{1,t}} = \frac{\omega_{11}W_{1,t}}{\omega_{11}W_{1,t} + \omega_{21}W_{2,t}}\frac{dW_{1,t}}{W_{1,t}} + \frac{\omega_{21}W_{2,t}}{\omega_{11}W_{1,t} + \omega_{21}W_{2,t}}\frac{dW_{2,t}}{W_{2,t}} \tag{3.35}$$

$$\frac{dK_{2,t}}{K_{2,t}} = \frac{\omega_{12}W_{1,t}}{\omega_{12}W_{1,t} + \omega_{22}W_{2,t}}\frac{dW_{1,t}}{W_{1,t}} + \frac{\omega_{22}W_{2,t}}{\omega_{12}W_{1,t} + \omega_{22}W_{2,t}}\frac{dW_{2,t}}{W_{2,t}}. \tag{3.36}$$

Once again, consider the effect of an increase in uncertainty in region 1. The first observation is that this will reduce the growth rate in region 1 *and* region 2 because $dW_{h,t}/W_{h,t}$ falls for $h \in \{1, 2\}$. There is a further effect on the growth rate of region 1 because investors shift a proportion of their wealth held in the capital stock of region 1 to risk-free bonds.

3.4.4 Real exchange rate

The real exchange rate, giving the price of the composite good consumer by investor j in terms of the composite good consumed by investor h, is given by

$$S_{h,j,t} = \frac{P^{\text{com}}_{h,t}}{P^{\text{com}}_{j,t}} \tag{3.37}$$

$$= \left[\frac{Y^{\text{non}}_j}{Y^{\text{non}}_h}\frac{W_{h,t}}{W_{j,t}}\frac{\psi\delta + (1-\psi)\theta MV_h}{\psi\delta + (1-\psi)\theta MV_j}\right]^{1-\theta}. \tag{3.38}$$

The first term $Y^{\text{non}}_j/Y^{\text{non}}_h$ appears because an increase in the quantity of the domestic good produced must reduce the domestic price level to achieve market clearing. The second and third terms are equal to $C^{\text{tot}}_{h,t}/C^{\text{tot}}_{j,t}$ with the optimal consumption-wealth ratio. These terms appear because, if we were to hold prices fixed and increase total consumption by Δ, a household will want to increase consumption of the domestic good by $(1-\theta)\Delta$. But the quantity of the domestic good is fixed, so the domestic price level must rise to clear the market.

To understand the effect that a change in uncertainty has on the exchange rate on impact, first set $\kappa_{hn} = \kappa_{jn}$ for $n \in \{1, 2\}$. Because the uncertainty parameters are the only source of heterogeneity in the third term of (3.38), this term drops out. Now consider a decrease in the confidence applied to the reference model of country 1. If $\kappa_{h1} = \kappa_{j1}$ still holds, then there will be no effect on the real exchange rate. However, if $\kappa_{h1} \neq \kappa_{j1}$, then the consumption choices of household h and j will be distorted by differing amounts and uncertainty will have an impact on the exchange rate. For example, suppose that $\kappa_{h1} > \kappa_{j1}$. Then $MV_h > MV_j$

and $C^{\text{tot}}_{h,t}/W_{h,t} < C^{\text{tot}}_{j,t}/W_{j,t}$. On impact, wealth is unchanged so $C^{\text{tot}}_{j,t}$ must increase by a greater proportion that $C^{\text{tot}}_{h,t}$. To clear the market for the nontraded good, $P^{\text{tot}}_{j,t}$ must therefore rise by a greater proportion than $P^{\text{com}}_{h,t}$. Thus, $S_{h,j,t}$ falls: a real depreciation for region h and a real appreciation for region j.

We can also consider the impact on the evolution of the real exchange rate over time. Taking logs of $S_{h,j,t}$, taking the differential, and using the expression for $dW_{h,t}/W_{h,t}$ yields

$$\begin{aligned}\frac{E_t[d\log S_{h,j,t}]}{dt} =&(1-\theta)\left[\frac{(\alpha-r)^2}{\sigma^2(1-\theta+\gamma\theta)}\sum_{n=1}^{2}(\kappa_{hn}-\kappa_{jn})+\left(\frac{C^{\text{tot}}_{j,t}}{W_{j,t}}-\frac{C^{\text{tot}}_{h,t}}{W_{h,t}}\right)\right.\\ &\left.-\frac{(\alpha-r)^2}{2\sigma^2(1-\theta+\gamma\theta)^2}\sum_{n=1}^{2}\left(\kappa^2_{hn}-\kappa^2_{jn}\right)\right]\end{aligned}$$

Consider the same example of an increase in uncertainty for region 1 such that $\kappa_{h1} > \kappa_{j1}$. The first two terms show that the increase in uncertainty increases the growth of the exchange rate, that is, region h experiences a real appreciation over time after the initial real depreciation. The reason for this is that the increase in uncertainty increases $dW_{1,t}/W_{h,t}$ relative to $dW_{2,t}/W_{2,t}$ because (i) region 1 investors hold a higher share of their portfolio in risky assets; (ii) region 2's consumption-wealth ratio increases relative to region 1's consumption-wealth ratio. The third term is a downward adjustment to the growth of the exchange rate due to the second-order term from Ito's lemma. We can show that the third term is always dominated by the first term when $\gamma > 1$.

The role of the real exchange rate in our model is therefore to induce "expenditure switching" to ensure goods market equilibrium (see Itskhoki (2021)). The contribution of our model is to show how geopolitical uncertainty, by distorting consumption demand, can initiate this standard expenditure switching mechanism.

3.5 Welfare

We now derive expressions to calculate the welfare effect of an increase in geopolitical uncertainty. In doing so, we follow Kahneman et al. (1997) in distinguishing between an investor's decision utility and experienced utility. An investor's decision utility is the objective the household seeks to maximise when making its portfolio and consumption decisions, that is,

$$U_{h,t} = \mathcal{A}(C^{\text{ind}}_{h,t}, \mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]). \tag{3.39}$$

An investor's experienced utility is the investor's actual well-being as a function of her choices, that is,

$$U_{h,t} = \mathcal{A}(C^{\text{ind}}_{h,t}, \mu_{h,t}[U_{h,t+dt}]) \tag{3.40}$$

evaluated at her optimal choices.

3.5.1 Mean-variance welfare

The mean-variance utility achieved by a household is given by evaluating household utility under objective beliefs at the portfolio choices derived using the household's decision utility:

$$U^h_{MV} = r + \frac{1}{1-\theta+\gamma\theta}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[1 - (1-\mu_{h,\kappa})^2 - \sigma^2_{h,\kappa}\right] \tag{3.41}$$

where

$$\mu_{h,\kappa} = \frac{1}{2}\sum_{n=1}^{2} \kappa_{hn}, \tag{3.42}$$

$$\sigma_{h,\kappa} = \sqrt{\frac{1}{2}\sum_{n=1}^{2} \kappa^2_{hn} - \mu^2_{h,\kappa}}. \tag{3.43}$$

When there is no uncertainty, $\mu_{h,\kappa} = 1$ and $\sigma_{h,\kappa} = 0$. We therefore see that geopolitical uncertainty reduces mean-variance welfare.

The overall effect of geopolitical uncertainty on welfare has two components. The first is the portfolio-tilting effect in which differences in uncertainty between the countries distort $x_{hn,t}$ from the $1/N$ portfolio weight, as seen in (3.23) and (3.24), implying that $\sigma_{h,\kappa} > 0$. The second is that the presence of uncertainty (irrespective of country) distorts the asset-allocation decision of households, as seen in (3.25) and captured in (3.41) by the deviation of $\mu_{h,k}$ from 1.

When there is an increase in uncertainty in country 1 such that $|\kappa_{h1} - \kappa_{h2}|$ increases, the portfolio tilting and the distorted asset-allocation effects act in the same direction to reduce welfare. When uncertainty in country 1 increases such that $|\kappa_{h1} - \kappa_{h2}|$ decreases, the portfolio tilting effect mitigates the negative welfare effect of the distortion in the asset-allocation decision. However, as shown in the Appendix, the negative asset-allocation effect outweighs the positive portfolio tilting effect. Therefore, the overall effect of an increase in geopolitical uncertainty on mean-variance welfare is always negative.

3.5.2 Lifetime welfare

We simplified the Hamilton-Jacobi-Bellman equation (3.19) under the Ansatz $U_{h,t} = \xi_h W_{h,t}^{\theta}$. We now verify our guess by solving for ξ_h. In the Appendix, we show that

$$\xi_h = \left[\frac{\psi\delta}{\psi\delta + (1-\psi)\left(U_h^{MV} - c_h\right)} \right]^{\frac{1}{1-\frac{1}{\psi}}} \left(c_h^{\theta} (Y_h^{\text{non}})^{1-\theta} \right), \tag{3.44}$$

where $c_h = C_{h,t}/W_{h,t}$. Mean-variance welfare depended on only the portfolio-diversification and asset-allocation decisions of a household. In contrast, lifetime welfare depends also on the household's consumption choice. To understand the expression for ξ_h, observe that increasing current consumption on the traded good directly increases utility through the term c_h^{θ} but comes at the expense of reduced savings. The negative impact on savings is captured by the term $U_h^{MV} - c_h$ in the denominator of the first component. The term $U_h^{MV} - c_h$ is the risk-adjusted expected return on a household's wealth, net of consumption. To obtain its impact on lifetime welfare, this expected return needs to be capitalized, as shown in the first term of (3.44). The capitalized value depends on the intertemporal aspects of the household's preferences, that is, the rate of time preference, δ, and the elasticity of intertemporal substitution, ψ.

3.6 Conclusion

In this paper, we have constructed a dynamic stochastic equilibrium model with traded and nontraded production sectors. We solve the model in closed form and find that increased geopolitical uncertainty reduces household investment in risky assets and distorts the consumption-wealth ratio. We have shown that for an increase in geopolitical uncertainty in country A to have an effect on trade flows and the exchange rate, we need investors in one country to reduce their confidence in the reference model for country A by more than investors in the other country. We have also demonstrated, using the example of Brexit uncertainty, that our model can match qualitative features of the UK economy following the Brexit referendum.

This paper emphasises the importance of distinguishing between the effect of risk and Knightian uncertainty. Although an increase in risk and Knightian uncertainty generate the same effects in the model, the channels through which they operate are distinct. In particular, Knightian uncertainty operates through reducing perceived expected returns and therefore does not require risk aversion in order to operate. This is important to remember when thinking about the effects of geopolitical uncertainty where the "uncertainty" is often better viewed as Knightian uncertainty rather than risk.

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Appendix A

Appendix to Chapter 1

A.1 Proof of Proposition 1.2 continued

In this appendix, we complete the proof of Proposition 1.2 by showing that if $z_i < z^*$, fund i chooses to sell, and if $z_i > z^*$, fund i chooses to hold. The proof is similar to that presented in Morris & Shin (2004).

We need to compute the probability density function of s conditional on fund i observing private signal z_i and all funds using the threshold strategy z^*. When the true market maker capacity is K, the signals $\{z_i\}$ are distributed uniformly over $[K - \varepsilon, K + \varepsilon]$. Funds with signals $z_i < z^*$ choose to sell. Hence,

$$s = \frac{z^* - (K - \varepsilon)}{2\varepsilon}.$$

To derive the conditional distribution of s, we derive the cumulative distribution function. In particular, we compute the conditional probability that $s < b$: $G(\cdot|z_i, z^*)$. Define K_0 as

$$b = \frac{z_i - (K_0 - \varepsilon)}{2\varepsilon} \implies K_0 = z_i + \varepsilon - 2\varepsilon b.$$

Thus, $s < b$ if and only if $K > K_0$. We therefore need the probability of $K > K_0$ conditional on z_i. Fund i's posterior density over K conditional on z_i is $[z_i - \varepsilon, z_i + \varepsilon]$. Therefore,

$$\Pr(K > K_0|z_i) = \frac{z_i + \varepsilon - K_0}{2\varepsilon} = \frac{z_i + \varepsilon - (z^* + \varepsilon - 2b\varepsilon)}{2\varepsilon} = \frac{z_i - z^*}{2\varepsilon} + b.$$

Thus,

$$G(b|z_i, z^*) = \begin{cases} 0 & \text{if } \frac{z_i - z^*}{2\varepsilon} + b < 0 \\ 1 & \text{if } \frac{z_i - z^*}{2\varepsilon} + b > 1 \\ \frac{z_i - z^*}{2\varepsilon} + b & \text{otherwise.} \end{cases}$$

Consider the case where $z_i < z^*$ (the case where $z_i > z^*$ follows an analagous argument). We need to show that fund i prefers to sell. The conditional density over the half-open interval $s \in [0, 1)$ is given by

$$g(s|z_i, z^*) = \begin{cases} 0 & \text{if } s < \frac{z^* - z_i}{2\varepsilon} \\ 1 & \text{if } s \geq \frac{z^* - z_i}{2\varepsilon} \end{cases}$$

with an atom at $s = 1$ with mass $\frac{z^* - z_i}{2\varepsilon}$.

Now recall that the payoff gain from selling compared to holding for fund i after observing z_i is

$$\pi(s, z_i) = \begin{cases} -(1-q)\delta_L & \text{if } sx \leq z_i \\ q\delta_H - \delta_L & \text{if } sx > z_i. \end{cases}$$

The function $\pi(s, z_i)$ is increasing in s and decreasing in z_i. Moreover, if we impose the parameter restriction $q\delta_H > \delta_L$, the payoff gain is negative for $sx \leq z_i$ and positive for $sx > z_i$. Note that the density $g(s|z_i, z^*)$ can be obtained from the uniform density by transferring weight from the interval $\left[0, \frac{z^* - z_i}{2\varepsilon}\right]$ to $s = 1$. Therefore,

$$\begin{aligned} 0 &= \int_0^1 \pi(s, z^*) ds \\ &< \int_0^1 \pi(s, z^*) g(l|z_i, z^*) ds \\ &< \int_0^1 \pi(s, z_i) g(l|z_i, z^*) ds. \end{aligned}$$

Thus, fund i strictly prefers to sell when $z_i < z^*$.

A.2 Proof of Proposition 1.4

The proof is similar to Proposition 1.2 so we only provide a sketch of the argument.

Consider a fund observing a signal z_i and thus its expectation of K is z_i. We cannot have $z_i > x$ because as $\varepsilon \to 0$ this would imply $K > x$. Therefore, $z_i < x$.

The first case to consider is when $z_i + K_{CB} \geq x$ and $q\delta_{CB} \leq \delta_L$. Then $\pi(s, z_i) \leq 0$ for all s: funds have a dominant straetgy to hold. As $\varepsilon \to 0$, $z_i \to K$ so that funds have a dominant strategy to hold whenever $K + K_{CB} > x$.

The second case is $z_i + K_{CB} \geq x$ and $q\delta_{CB} > \delta_L$. Now the sign of $\pi(s, z_i)$ depends on the value of s. To proceed, consider the fund observing the threshold signal z^*. From Lemma 1.1, we know that $s|z^* \sim U[0,1]$. If $s < z^*/x$, the fund believes the dealer will be under no stress. Otherwise the fund believes it will sell to the central bank at discount δ_{CB}. Since $z^* + K_{CB} \geq x$, the fund does not believe it will ever sell to the dealer at discount δ_H even if $s = 1$. Therefore, the analysis is identical to Proposition 1.2 with $\delta_H = \delta_{CB}$.

The final case is $z_i + K_{CB} < x$. The sign of $\pi(s, z_i)$ depends on the value of s so we consider the indifference condition of a fund observing signal z^*. There are three possibilities: (i) sell to the dealer under no stress with probability z^*/x; (ii) sell to the central bank with probability K_{CB}/x; (iii) sell to the dealer under stress with probability $1 - (z^* + K_{CB})/x$. Therefore,

$$\int_0^1 \pi(s, z^*)ds = -(1-q)\delta_L \frac{z^*}{x} + (q\delta_{CB} - \delta_L)\frac{K_{CB}}{x} + (q\delta_H - \delta_L)\left(1 - \frac{z^* + K_{CB}}{x}\right).$$

The threshold value z^* (or equivalently K^* as $\varepsilon \to 0$) sets this expression to zero. This gives the result in Proposition 1.4. A similar argument to Proposition 1.2 establishes the optimality of selling (holding) if $z_i < z^*$ ($z_i \geq z^*$).

By inspection and Corollary 1.1, we can see that $\tilde{K}^*$ is increasing in δ_{CB}. For $K + K_{CB} < x$, it is evident that $\tilde{K}^*$ is decreasing in K_{CB}. Letting $K_{CB} \to x - K$ and performing some simple algebra shows that the expression for $\tilde{K}^*$ when $K + K_{CB}$ converges to the expression for $\tilde{K}^*$ when $K + K_{CB} > x$.

Appendix B

Appendix to Chapter 3

B.1 The intertemporal budget constraint

Proposition B.1. *The intertemporal budget constraint can be written as*

$$\frac{dW_{h,t}}{W_{h,t}} = (1-\pi_{h,t})rdt + \pi_{h,t}\sum_{n=1}^{N} x_{hn,t}(\alpha dt + \sigma dZ_{n,t}) + \frac{P_{h,t}Y_h^{\text{non}}}{W_{h,t}}dt - \frac{C_{h,t}+P_{h,t}C_{h,t}^{\text{non}}}{W_{h,t}}dt, \quad \text{(B.1)}$$

or using vector notation

$$\frac{dW_{h,t}}{W_{h,t}} = \left[r + \pi_{h,t}(\boldsymbol{x}_{h,t}^{\top}\boldsymbol{\alpha} - r) + \frac{P_{h,t}Y_h^{\text{non}}}{W_{h,t}} - \frac{C_{h,t}+P_{h,t}C_{h,t}^{\text{non}}}{W_{h,t}}\right]dt + \sigma\pi_{h,t}\boldsymbol{x}_{h,t}^{\top}d\boldsymbol{Z}_t \quad \text{(B.2)}$$

where $\boldsymbol{x}_{h,t} = (x_{h1,t},\ldots,x_{hn,t})^{\top}$, $\boldsymbol{\alpha} = \mathbf{1}\alpha$, *and* $d\boldsymbol{Z}_t = (dZ_{1,t},\ldots,dZ_{n,t})^{\top}$.

Proof. Start from the definition of wealth

$$W_{h,t} = B_{h,t} + \sum_{n=1}^{N} K_{hn,t}$$

Taking the differential yields

$$
\begin{aligned}
dW_{h,t} &= dB_{h,t} + \sum_{n=1}^{N} dK_{hn,t} \\
&= rdtB_{h,t} + \sum_{n=1}^{N} [(\alpha K_{hn,t} - D_{hn,t})dt + \sigma K_{hn,t} dZ_{n,t}] \\
&= rdtB_{h,t} + \sum_{n=1}^{N} [(\alpha dt + \sigma dZ_{n,t}) K_{hn,t} - D_{hn,t} dt] \\
&= rdt(1 - \pi_{h,t}) W_{h,t} + \pi_{h,t} \sum_{n=1}^{N} x_{hn,t} (\alpha dt + \sigma dZ_{n,t}) W_{h,t} - \sum_{n=1}^{N} D_{hn,t}
\end{aligned}
$$

Now add and subtract $P_{h,t} Y_h^{\text{non}}$ and use the fact that $P_{h,t}$ must always adjust to ensure $Y_h^{\text{non}} = C_{h,t}^{\text{non}}$.[1]

$$
dW_{h,t} = rdt(1 - \pi_{h,t}) W_{h,t} + \pi_{h,t} \sum_{n=1}^{N} x_{hn,t} (\alpha dt + \sigma dZ_{n,t}) W_{h,t} + P_{h,t} Y_h^{\text{non}} - \sum_{n=1}^{N} D_{hn,t} - P_{h,t} C_{h,t}^{\text{non}}
$$

Dividing through by $W_{h,t}$ and noting that dividends finance the consumption of the traded good gives the result. □

B.2 The certainty equivalent

Definition B.1. *A certainty equivalent amount of a risky quantity is the equivalent risk-free amount in static utility terms, i.e.*

$$
u_\gamma \left(\mu_{h,t} \left[U_{h,t+dt} \right] \right) = E_t [u_\gamma (U_{h,t+dt})], \tag{B.3}
$$

where $u_\gamma(\cdot)$ is the static utility index defined by the power utility function

$$
u_\gamma(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1 \\ \ln x, & \gamma = 1, \end{cases} \tag{B.4}
$$

and the conditional expectation $E_t[\cdot]$ is defined relative to a reference probability measure $\mathbb{P}$.

Proposition B.2. *The date-t certainty equivalent of household h's date-$t + dt$ utility is given*

[1] We assume that the investor does not internalise the fact her decisions can influence $P_{h,t}$ and instead takes the equilibrium price as given when solving her optimisation problem.

by

$$\mu_{h,t}[U_{h,t+dt}] = E_t[U_{h,t+dt}] - \frac{1}{2}\,\gamma\,U_{h,t}\,E_t\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]. \tag{B.5}$$

Proof. The definition of the certainty equivalent in (B.3) implies that

$$\mu_{h,t}[U_{h,t+dt}] = E_t\left[U_{h,t+dt}^{1-\gamma}\right]^{\frac{1}{1-\gamma}} = E_t\left[U_{h,t}^{1-\gamma} + d(U_{h,t}^{1-\gamma})\right]^{\frac{1}{1-\gamma}}. \tag{B.6}$$

Applying Ito's Lemma, we obtain

$$d(U_{h,t}^{1-\gamma}) = (1-\gamma)U_{h,t}^{-\gamma}dU_{h,t} - \frac{1}{2}(1-\gamma)\gamma U_{h,t}^{-\gamma-1}(dU_{h,t})^2 \tag{B.7}$$

$$= (1-\gamma)U_{h,t}^{1-\gamma}\left[\frac{dU_{h,t}}{U_{h,t}} - \frac{1}{2}\gamma\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]. \tag{B.8}$$

Therefore,

$$\mu_{h,t}[U_{h,t+dt}] = E_t\left[U_{h,t+dt}^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \tag{B.9}$$

$$= U_{h,t}\left(E_t\left[1 + (1-\gamma)\left[\frac{dU_{h,t}}{U_{h,t}} - \frac{1}{2}\gamma\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right]\right)^{\frac{1}{1-\gamma}} \tag{B.10}$$

$$= U_{h,t}\left(1 + (1-\gamma)\left[E_t\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right]\right)^{\frac{1}{1-\gamma}}. \tag{B.11}$$

Hence, expanding the above expression, and using the notation $g = o(dt)$ to denote that $g/dt \to 0$ as $dt \to 0$, one obtains

$$\mu_{h,t}[U_{h,t+dt}] = U_{h,t}\left(1 + E_t\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right) + o(dt), \tag{B.12}$$

which, in the continuous-time limit, leads to the result. □

B.3 Motivating the loss function

Each alternative probability measure considered by an investor is connected to an alternative model of the economy via Girsanov's Theorem. Define the alternative probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ as follows

Definition B.2. *The alternative probability measure* $\mathbb{Q}^{\boldsymbol{\nu}_h}$ *is defined by*

$$\mathbb{Q}^{\boldsymbol{\nu}_h}(A) = E[1_A \xi_{h,T}], \tag{B.13}$$

where E *is the expectation under* $\mathbb{P}$, A *is an event realized at date* T, *and* $\xi_{h,t}$ *is the exponential martingale (under the reference probability measure* $\mathbb{P}$*) given by*

$$\frac{d\xi_{hn,t}}{\xi_{hn,t}} = \frac{1}{\sigma}\nu_{hn,t} dZ_{n,t} \tag{B.14}$$

Applying Girsanov's Theorem, we see that under the alternative probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$, the evolution of Region n's capital stock is given by the following alternative model:

$$dK_{n,t} = [(\alpha_n + \nu_{hn,t})K_{n,t} - D_{n,t}]dt + \sigma K_{n,t} dZ^{\boldsymbol{\nu}_h}_{n,t}, \tag{B.15}$$

where $Z^{\boldsymbol{\nu}_h}_{n,t}$ is a standard Brownian motion under $\mathbb{Q}^{\boldsymbol{\nu}_h}$, such that

$$E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t [dZ^{\boldsymbol{\nu}_h}_{n,t} dZ^{\boldsymbol{\nu}_h}_{m,t}] = 0 \qquad n \neq m. \tag{B.16}$$

The rate of information loss stemming from using $\mathbb{Q}^{\boldsymbol{\nu}_h}$ instead of $\mathbb{P}$ is given by relative entropy per unit time of the reference probability measure $\mathbb{P}$ with respect to the alternative probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ (also known as the Kullback-Leibler divergence from $\mathbb{Q}^{\boldsymbol{\nu}_h}$ to $\mathbb{P}$), i.e.,

$$D^{KL}[\mathbb{P}|\mathbb{Q}^{\boldsymbol{\nu}_h}] = \frac{1}{dt} E_t \left[\left(\frac{d\xi_{h,t}}{\xi_{h,t}} \right)^2 \right]. \tag{B.17}$$

The above expression reduces to $D^{KL}[\mathbb{P}|\mathbb{Q}^{\boldsymbol{\nu}_h}] = \frac{1}{\sigma^2}\sum_{n=1}^{N} \nu^2_{hn,t}$. Thus, for $N = 2$, the set of alternative models corresponding to information losses less than or equal to R^2 can be visualized in (ν_{h1}, ν_{h2})-space as a circle of radius $R\sigma$ centered on the origin.

We want to weight information losses less severely when the economic environment becomes more uncertain. Motivated by the Kullback-Leibler divergence, we therefore *define* the confidence-weighted information loss rate as follows

$$L_{h,t} = \frac{\theta}{2\sigma^2[1 - \theta + \gamma\theta]} \sum_{n=1}^{N} \phi_{hn} \nu^2_{hn,t} \tag{B.18}$$

where $\phi_{hn} = [0, \infty)$ measures how confident the representative investor for Region h is in the reference model as applied to region n and we scale by $\theta/2[1 - \theta + \gamma\theta]$.

B.4 The Bellman Equation

We state the Hamilton-Jacobi-Bellman equation as the following proposition.

Proposition B.3. *The optimal consumption, portfolio, and alternative probability model choices of the representative household for region h are given by the following Hamilton-Jacobi-Bellman equation*

$$\begin{aligned}0 = &\sup_{C_{h,t}, C^{\text{non}}_{h,t}} \left(\delta\, u_\psi \left(\frac{C^{\theta}_{h,t} (C^{\text{non}}_{h,t})^{1-\theta}}{J_{h,t}} \right) \right. \\ &\left. + \sup_{\pi_{h,t}, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} \frac{1}{dt} \left(E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] + L_{h,t} dt \right) \right),\end{aligned} \tag{B.19}$$

where $J_{h,t}$ denotes investor h's time-t value function.[2]

Proof. The value function $J_{h,t}$ satisfies

$$J^{1-\frac{1}{\psi}}_{h,t} = (1 - e^{-\delta dt})(C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} + e^{-\delta dt} \left(\mu^{\boldsymbol{\nu}}_{h,t}[J_{h,t+dt}] \right)^{1-\frac{1}{\psi}}, \tag{B.20}$$

where for ease of notation sup and inf have been suppressed. Now,

$$\begin{aligned} \left(\mu^{\boldsymbol{\nu}}_{h,t}[J_{h,t+dt}] \right)^{1-\frac{1}{\psi}} &= \left(J_{h,t} \left(1 + E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] + L_{h,t} dt \right) \right)^{1-\frac{1}{\psi}} && \text{(B.21)} \\ &= J^{1-\frac{1}{\psi}}_{h,t} \left(1 + E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] + L_{h,t} dt \right)^{1-\frac{1}{\psi}} && \text{(B.22)} \\ &= J^{1-\frac{1}{\psi}}_{h,t} \left(1 + \left(1 - \frac{1}{\psi} \right) \left(E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] + L_{h,t} dt \right) \right) && \text{(B.23)} \end{aligned}$$

Substituting into (B.20) and using the series definition of the exponential function yields

$$\begin{aligned} J^{1-\frac{1}{\psi}}_{h,t} &= \delta (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} dt \\ &+ J^{1-\frac{1}{\psi}}_{h,t} \left(1 + \left(1 - \frac{1}{\psi} \right) \left(E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] + L_{h,t} dt \right) \right) - \delta J^{1-\frac{1}{\psi}}_{h,t} dt + o(dt), \end{aligned} \tag{B.24}$$

from which we obtain (B.19). □

[2]The optimal choices must also satisfy a transversality condition which we verify in section B.8.

Proposition B.4. *Under the Ansatz $J_{h,t} = \xi_h W_{h,t}^{\theta}$, where ξ_h is a constant, the Hamilton-Jacobi-Bellman equation reduces to*

$$0 = \sup_{C_{h,t}, C_{h,t}^{\text{non}}} \left(\frac{\delta}{\theta} u_{\psi} \left(\frac{C_{h,t}^{\theta} (C_{h,t}^{\text{non}})^{1-\theta}}{\xi_h W_{h,t}^{\theta}} \right) + \left(\frac{P_{h,t} Y_h^{\text{non}}}{W_{h,t}} \right) - \left(\frac{C_{h,t} + P_{h,t} C_{h,t}^{\text{non}}}{W_{h,t}} \right) + MV_h \right), \quad \text{(B.25)}$$

where

$$MV_h = \sup_{\pi_{h,t}, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dR_{h,t}^p}{dt} \right] - \frac{1}{2} [1 - \theta + \gamma\theta] \frac{1}{dt} E_t \left[(dR_{h,t}^p)^2 \right] + \frac{L_{h,t}}{\theta}, \quad \text{(B.26)}$$

and, under the probability model $\mathbb{Q}^{\boldsymbol{\nu}_h}$,

$$dR_{h,t}^p = \left[r + \pi_{h,t} (\boldsymbol{x}_h^{\top} \boldsymbol{\alpha} - r) + \pi_{h,t} \boldsymbol{x}_{h,t}^{\top} \boldsymbol{\nu}_{h,t} \right] dt + \sigma \pi_{h,t} \sum_{n=1}^{2} x_{hn,t} dZ_{n,t}^{\mathbb{Q}^{\boldsymbol{\nu}_h}}. \quad \text{(B.27)}$$

Proof. Under the probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$, the process for wealth is (using Girsanov's theorem):

$$\frac{dW_{h,t}}{W_{h,t}} = dR_{h,t}^p + \frac{P_{h,t} Y_h^{\text{non}}}{W_{h,t}} dt - \frac{C_{h,t} + P_{h,t} C_{h,t}^{\text{non}}}{W_{h,t}} dt, \quad \text{(B.28)}$$

where

$$dR_{h,t}^p = [r + \pi_{h,t} (\boldsymbol{x}_h^{\top} \boldsymbol{\alpha} - r) + \pi_{h,t} \boldsymbol{x}_{h,t}^{\top} \boldsymbol{\nu}_{h,t}] dt + \sigma \pi_{h,t} \sum_{n=1}^{2} x_{hn,t} dZ_{n,t}^{\mathbb{Q}^{\boldsymbol{\nu}_h}}. \quad \text{(B.29)}$$

By Ito's lemma

$$\frac{dJ_{h,t}}{J_{h,t}} = \theta \left(\frac{dW_{h,t}}{W_{h,t}} - \frac{1}{2}(1-\theta) \left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right). \quad \text{(B.30)}$$

Thus,

$$\begin{aligned} &\frac{1}{dt} \left(E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dJ_{h,t}}{J_{h,t}} \right] - \frac{1}{2} \gamma E_t \left[\left(\frac{dJ_{h,t}}{J_{h,t}} \right)^2 \right] \right) = \\ &\theta \left(E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dR_{h,t}^p}{dt} \right] - \frac{1}{2} (1 - \theta + \gamma\theta) \frac{1}{dt} E_t [(dR_{h,t}^p)^2] + \left(\frac{P_{h,t} Y_h^{\text{non}}}{W_{h,t}} \right) - \left(\frac{C_{h,t} + P_{h,t} C_{h,t}^{\text{non}}}{W_{h,t}} \right) \right), \end{aligned} \quad \text{(B.31)}$$

which implies the HJB equation can be written as

$$
\begin{aligned}
0 = \sup_{C_{h,t}, C^{\text{non}}_{h,t}} & \left(\frac{\delta}{\theta} u_\psi \left(\frac{C^\theta_{h,t} (C^{\text{non}}_{h,t})^{1-\theta}}{\xi_h W^\theta_{h,t}} \right) + \left(\frac{P_{h,t} Y^{\text{non}}_h}{W_{h,t}} \right) - \left(\frac{C_{h,t} + P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}} \right) \right. \\
& \left. + \sup_{\pi_{h,t}, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t \left[\frac{dR^p_{h,t}}{dt} \right] - \frac{1}{2} (1 - \theta + \gamma\theta) \frac{1}{dt} E_t[(dR^p_{h,t})^2] + \frac{L_{h,t}}{\theta} \right).
\end{aligned}
\tag{B.32}
$$

□

B.5 Optimal portfolio choice

The FOC's for the traded and nontraded good are given by

$$
\begin{aligned}
\delta \left(\frac{C^\theta_{h,t} (C^{\text{non}}_{h,t})^{1-\theta}}{\xi_h W^\theta_{h,t}} \right)^{1-\frac{1}{\psi}} &= \frac{C_{h,t}}{W_{h,t}} \\
\frac{(1-\theta)}{\theta} \delta \left(\frac{C^\theta_{h,t} (C^{\text{non}}_{h,t})^{1-\theta}}{\xi_h W^\theta_{h,t}} \right)^{1-\frac{1}{\psi}} &= \frac{P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}},
\end{aligned}
\tag{B.33}
$$

respectively. Therefore,

$$
\frac{C_{h,t} + P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}} = \frac{\delta}{\theta} \left(\frac{C^\theta_{h,t} (C^{\text{non}}_{h,t})^{1-\theta}}{\xi_h W^\theta_{h,t}} \right)^{1-\frac{1}{\psi}}
\tag{B.34}
$$

and

$$
\begin{aligned}
\frac{C_{h,t}}{W_{h,t}} &= \theta \frac{C_{h,t} + P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}} \\
\frac{P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}} &= (1-\theta) \frac{C_{h,t} + P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}}.
\end{aligned}
\tag{B.35}
$$

Market clearing for the nontraded good in Region h implies

$$
C^{\text{non}}_{h,t} = Y^{\text{non}}_h
\tag{B.36}
$$

Hence, the HJB reduces to

$$
\frac{C_{h,t} + P_{h,t} C^{\text{non}}_{h,t}}{W_{h,t}} = \frac{\psi\delta + (1-\psi) M V_h}{\theta[\psi + (1-\psi)\theta]},
\tag{B.37}
$$

where

$$MV_h = \sup_{\pi_{h,t},\, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dR^p_{h,t}}{dt} \right] - \frac{1}{2} \left[1 - \theta + \gamma\theta \right] \frac{1}{dt} E_t[(dR^p_{h,t})^2] + \frac{L_{h,t}}{\theta} \tag{B.38}$$

From (B.38), we can see that the optimal choice of personal beliefs is given by the following optimization problem

$$\inf_{\boldsymbol{\nu}_{h,t}} E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dR_{p,h,t}}{dt} \right] + \frac{1}{\theta} L_{h,t}, \tag{B.39}$$

which reduces to the linear-quadratic optimisation problem

$$\inf_{\boldsymbol{\nu}_{h,t}} \pi_{h,t} \boldsymbol{x}_{h,t}^\top \boldsymbol{\nu}_{h,t} + \frac{1}{2\sigma^2(1-\theta+\gamma\theta)} \sum_{n=1}^{2} \phi_{hn} \nu_{hn,t}^2 \tag{B.40}$$

The FOC for $\nu_{hn,t}$ is

$$\nu_{hn,t} = -\sigma^2(1-\theta+\gamma\theta) \frac{1}{\phi_{hn}} \pi_{h,t} x_{hn,t} \tag{B.41}$$

In vector-matrix form

$$\boldsymbol{\nu}_{h,t} = -\sigma^2(1-\theta+\gamma\theta) \mathbf{1}^\top D_{\phi,h} \boldsymbol{\omega}_{h,t} \tag{B.42}$$

where

$$D_{\phi,h} = \mathrm{diag}\left(\frac{1}{\phi_{h1}}, \frac{1}{\phi_{h2}} \right) \tag{B.43}$$

Therefore,

$$\inf_{\boldsymbol{\nu}_{h,t}} \pi_{h,t} \boldsymbol{x}_{h,t}^\top \boldsymbol{\nu}_{h,t} + \frac{1}{2\sigma^2(1-\theta+\gamma\theta)} \sum_{n=1}^{2} \phi_{hn} \nu_{hn,t}^2 = -\frac{1}{2} \sigma^2 (1-\theta+\gamma\theta) \pi_{h,t}^2 \sum_{n=1}^{2} \frac{x_{hn,t}^2}{\phi_{hn}} \tag{B.44}$$

Hence,

$$\begin{aligned} &\inf_{\boldsymbol{\nu}_{h,t}} E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}} \left[\frac{dR_{p,h,t}}{dt} \right] - \frac{1}{2} \frac{1}{dt} (1-\theta+\gamma\theta) E_t[(dR_{p,h,t})^2] + L_{h,t} \\ &= r + \pi_{h,t}(\boldsymbol{x}_h^\top \boldsymbol{\alpha} - r) - \frac{1}{2} \sigma^2 (1-\theta+\gamma\theta) \pi_{h,t}^2 \sum_{n=1}^{2} \frac{x_{hn,t}^2}{\phi_{hn}} - \frac{1}{2} \sigma^2 (1-\theta+\gamma\theta) \pi_{h,t}^2 \boldsymbol{x}_{h,t}^\top \boldsymbol{x}_{h,t}, \end{aligned} \tag{B.45}$$

We now define

$$V_{\phi,h} = D_{\phi,h} + I \tag{B.46}$$

Therefore,

$$\begin{aligned}
&\inf_{\nu_{h,t}} E_t^{\mathbb{Q}^{\nu_h}}\left[\frac{dR_{p,h,t}}{dt}\right] - \frac{1}{2}\frac{1}{dt}(1-\theta+\gamma\theta)E_t[(dR_{p,h,t})^2] + L_{h,t} \\
&= r + \pi_{h,t}(\boldsymbol{x}_h^\top \boldsymbol{\alpha} - r) - \frac{1}{2}\sigma^2(1-\theta+\gamma\theta)\pi_{h,t}^2 \boldsymbol{x}_{h,t}^\top V_{\phi,h}\boldsymbol{x}_{h,t}
\end{aligned} \tag{B.47}$$

Using $\boldsymbol{\omega}_{h,t} = \pi_{h,t}\boldsymbol{x}_{h,t}$, the optimal portfolio choice problem can be written as

$$\sup_{\boldsymbol{\omega}_{h,t}} \boldsymbol{\omega}_{h,t}^\top(\boldsymbol{\alpha} - r\mathbf{1}) - \frac{1}{2}\sigma^2(1-\theta+\gamma\theta)\boldsymbol{\omega}_{h,t}^\top V_{\phi,h}\boldsymbol{\omega}_{h,t} \tag{B.48}$$

The solution is

$$\boldsymbol{\omega}_{h,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} V_{\phi,h}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) \tag{B.49}$$

Since $\mathbf{1}^\top \boldsymbol{x}_{h,t} = 1$, we have

$$\pi_{h,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} \mathbf{1}^\top V_{\phi,h}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) \tag{B.50}$$

Therefore,

$$\boldsymbol{x}_{h,t} = \frac{V_{\phi,h}^{-1}(\boldsymbol{\alpha} - r\mathbf{1})}{\mathbf{1}^\top V_{\phi,h}^{-1}(\boldsymbol{\alpha} - r\mathbf{1})} \tag{B.51}$$

Note that

$$V_{\phi,h}^{-1} = \operatorname{diag}\left(\frac{\phi_{h1}}{1+\phi_{h1}}, \frac{\phi_{h2}}{1+\phi_{h2}}\right) \tag{B.52}$$

Letting $\boldsymbol{\alpha} = \alpha\mathbf{1}$ as in the text, we have

$$\pi_{h,t} = \frac{(\alpha - r)}{\sigma^2(1-\theta+\gamma\theta)}\left(\frac{\phi_{h1}}{1+\phi_{h1}} + \frac{\phi_{h2}}{1+\phi_{h2}}\right) \tag{B.53}$$

This can be rewritten as

$$\pi_{h,t} = \frac{2(\alpha - r)}{\sigma^2(1-\theta+\gamma\theta)} - \frac{(\alpha - r)}{\sigma^2(1-\theta+\gamma\theta)}\left[\frac{2+\phi_{h1}+\phi_{h2}}{(1+\phi_{h1})(1+\phi_{h2})}\right] \tag{B.54}$$

We also have

$$
\begin{aligned}
x_{h1,t} &= \frac{\frac{\phi_{h1}}{1+\phi_{h1}}}{\frac{\phi_{h1}}{1+\phi_{h1}} + \frac{\phi_{h2}}{1+\phi_{h2}}} \\
x_{h2,t} &= \frac{\frac{\phi_{h2}}{1+\phi_{h2}}}{\frac{\phi_{h1}}{1+\phi_{h1}} + \frac{\phi_{h2}}{1+\phi_{h2}}}
\end{aligned} \tag{B.55}
$$

Substituting the optimal portfolio weights into (B.41) yields the optimal belief

$$
\nu_{hn,t} = -\frac{\alpha - r}{1 + \phi_{hn}}. \tag{B.56}
$$

B.6 Optimal consumption

Substituting the optimal portfolio weights into (B.48) yields

$$
\begin{aligned}
&\sup_{\boldsymbol{\omega}_{h,t}} \boldsymbol{\omega}_{h,t}^{\top}(\boldsymbol{\alpha} - r\mathbf{1}) - \frac{1}{2}\sigma^2(1-\theta+\gamma\theta)\boldsymbol{\omega}_{h,t}^{\top} V_{\phi,h} \boldsymbol{\omega}_{h,t} \\
&\quad = \frac{1}{2\sigma^2(1-\theta+\gamma\theta)}(\boldsymbol{\alpha} - r\mathbf{1})^{\top} V_{\phi,h}^{-1}(\boldsymbol{\alpha} - r\mathbf{1})
\end{aligned} \tag{B.57}
$$

Hence,

$$
MV_h = r + \frac{1}{2\sigma^2(1-\theta+\gamma\theta)}(\boldsymbol{\alpha} - r\mathbf{1})^{\top} V_{\phi,h}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) \tag{B.58}
$$

Letting $\boldsymbol{\alpha} = \alpha\mathbf{1}$, we have

$$
MV_h = r + \frac{1}{2(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \sum_{n=1}^{2} \frac{\phi_{hn}}{1+\phi_{hn}} \tag{B.59}
$$

This can be rewritten as

$$
MV_h = r + \frac{1}{(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 - \frac{1}{2(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[\frac{2+\phi_{h1}+\phi_{h2}}{(1+\phi_{h1})(1+\phi_{h2})}\right] \tag{B.60}
$$

Define household h's total consumption of the traded and nontraded good (with the traded good as the numeraire) as $C_{h,t}^{\text{tot}} = C_{h,t} + P_{h,t}C_{h,t}^{\text{non}}$. Substituting the expression for mean-variance welfare in (B.60) into (B.37) yields the consumption-wealth ratio chosen by household h

$$\frac{C^{\text{tot}}_{h,t}}{W_{h,t}} = \frac{1}{\psi + (1-\psi)\theta}\left(\psi\delta + (1-\psi)\left(r + \frac{1}{(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2\right)\right) \\ - (1-\psi)\frac{1}{2(1-\theta+\gamma\theta)(\psi+(1-\psi)\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2\left[\frac{2+\phi_{h1}+\phi_{h2}}{(1+\phi_{h1})(1+\phi_{h2})}\right] \tag{B.61}$$

B.7 Return on wealth

Under the objective probability model $\mathbb{P}$, we have

$$\begin{aligned}\frac{dW_{h,t}}{W_{h,t}} &= dR^p_{h,t} + \frac{P_{h,t}Y^{\text{non}}_h}{W_{h,t}}dt - \frac{C_{h,t} + P_{h,t}C^{\text{non}}_{h,t}}{W_{h,t}}dt \\ &= [r + \boldsymbol{\omega}^\top_{h,t}(\boldsymbol{\alpha} - r\mathbf{1})]dt + \sigma\boldsymbol{\omega}^\top_{h,t}d\boldsymbol{Z}_t - \frac{C_{h,t}}{W_{h,t}}dt,\end{aligned} \tag{B.62}$$

where the second line imposes equilibrium in the market for the nontraded good and $d\boldsymbol{Z}_t = [dZ_{1,t}, dZ_{2,t}]^\top$. Substituting in the optimal portfolio and consumption choices yields

$$\frac{dW_{h,t}}{W_{h,t}} = \left[r + \frac{1}{\sigma^2(1-\theta+\gamma\theta)}(\boldsymbol{\alpha} - r\mathbf{1})^\top V^{-1}_{\phi,h}(\boldsymbol{\alpha} - r\mathbf{1})\right]dt \\ + \frac{1}{\sigma(1-\theta+\gamma\theta)}(\boldsymbol{\alpha} - r\mathbf{1})^\top V^{-1}_{\phi,h}d\boldsymbol{Z}_t - \theta\left[\frac{\psi\delta + (1-\psi)\theta MV_h}{\theta[\psi + (1-\psi)\theta]}\right]dt. \tag{B.63}$$

For the special case $\boldsymbol{\alpha} = \alpha\mathbf{1}$,

$$\frac{dW_{h,t}}{W_{h,t}} = \left[r + \frac{1}{\sigma^2(1-\theta+\gamma\theta)}(\alpha - r)^2\sum_{n=1}^{2}\frac{\phi_{hn}}{1+\phi_{hn}} - \frac{\psi\delta + (1-\psi)\theta MV_h}{\psi + (1-\psi)\theta}\right]dt \\ + \frac{1}{\sigma(1-\theta+\gamma\theta)}(\alpha - r)\sum_{n=1}^{2}\frac{\phi_{hn}}{1+\phi_{hn}}dZ_{n,t}. \tag{B.64}$$

B.8 Transversality condition

Optimality also requires that the solution satisfies the transversality condition

$$\lim_{T\to\infty} E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t[e^{-\delta T}W^\theta_{h,T}]. \tag{B.65}$$

Following a variation on a standard argument (see Merton (1969)), we show that this condition is satisfied for a sufficiently high δ. In other words, wealth should not grow too fast relative to the rate of time preference.

From Girsanov's theorem, we know that $dZ_{n,t} = dZ_{n,t}^{\boldsymbol{\nu}_h} + \frac{1}{\sigma}\nu_{hn,t}dt$. From equation (B.64), we thus see that $W_{h,t}$ follows

$$\frac{dW_{h,t}}{W_{h,t}} = \mu_h^w dt + \sum_{n=1}^{2} \sigma_{hn}^w dZ_{n,t}^{\boldsymbol{\nu}_h} \tag{B.66}$$

under the alternative probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ where the drift of the process is μ_h^w and the loadings on the Brownian motions are σ_{hn}^w. Notice that μ_h^w is decreasing in δ and σ_{hn}^w for $n \in \{1, 2\}$ do not depend on δ.

Next define a new stochastic process $\hat{Z}_t^{\boldsymbol{\nu}_h}$ by $\hat{Z}_0^{\boldsymbol{\nu}_h} = 0$, and

$$d\hat{Z}_t^{\boldsymbol{\nu}_h} = \frac{1}{\sqrt{\sum_{n=1}^{2}(\sigma_{hn}^w)^2}} \left(\sum_{n=1}^{2} \sigma_{hn}^w dZ_{n,t}^{\boldsymbol{\nu}_h} \right) \tag{B.67}$$

By Levy's theorem (see Back (2017, p.310)), we know that $\hat{Z}_t^{\boldsymbol{\nu}_h}$ is a Brownian motion and that

$$\frac{dW_{h,t}}{W_{h,t}} = \mu_h^w dt + \hat{\sigma}_h^w d\hat{Z}_t^{\boldsymbol{\nu}_h} \tag{B.68}$$

where $\hat{\sigma}_h^w = \sqrt{\sum_{n=1}^{2}(\sigma_{hn}^w)^2}$. We therefore see that $W_{h,t}$ follows a geometric Brownian motion and the solution of this stochastic differential equation is

$$W_{h,t} = W_{h,0} \exp\left(\left(\mu_h^w - \frac{1}{2}(\hat{\sigma}_h^w)^2 \right) t + \hat{\sigma}_h^w \hat{Z}_t^{\boldsymbol{\nu}_h} \right). \tag{B.69}$$

Therefore,

$$E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}[W_{h,T}^\theta] = W_{h,0}^\theta \exp\left(\theta \left(\mu_h^w - \frac{1}{2}(\hat{\sigma}_h^w)^2 \right) T \right) E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}[\exp(\theta \hat{\sigma}_h^w \hat{Z}_T^{\boldsymbol{\nu}_h})]. \tag{B.70}$$

We know that $\hat{Z}_T^{\boldsymbol{\nu}_h} \sim N(0, T)$. Therefore,[3]

$$E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}[\exp(\theta \hat{\sigma}_h^w \hat{Z}_T^{\boldsymbol{\nu}_h})] = \exp\left(\frac{(\theta \hat{\sigma}_h^w)^2}{2} T \right). \tag{B.71}$$

[3] If $X \sim N(\mu, \sigma^2)$, then $E[\exp(X)] = E[\exp(\mu + 0.5\sigma^2)]$.

Combining terms yields

$$E_t^{\mathbb{Q}^{\nu_h}}[W_{h,T}^{\theta}] = W_{h,0}^{\theta} \exp\left(\theta\left(\mu_h^w - (1-\theta)\frac{(\hat{\sigma}_h^w)^2}{2}\right)T\right). \tag{B.72}$$

The transversality condition can now be written as

$$\lim_{T\to\infty} \exp\left(\left[\theta\left(\mu_h^w - (1-\theta)\frac{(\hat{\sigma}_h^w)^2}{2}\right) - \delta\right]T\right) = 0. \tag{B.73}$$

This requires

$$\delta > \theta\left(\mu_h^w - (1-\theta)\frac{(\hat{\sigma}_h^w)^2}{2}\right). \tag{B.74}$$

The left-hand side is increasing in δ whilst the right-hand side is decreasing in δ. This implies there exists a $\bar{\delta}$ such that the transversality condition holds for all $\delta > \bar{\delta}$. We assume that the parameters are such that this condition is always met.

B.9 Growth rate

From the production function

$$\frac{dY_{n,t}}{Y_{n,t}} = \frac{dK_{n,t}}{K_{n,t}}. \tag{B.75}$$

Equilibrium requires the following relationships hold

$$\begin{aligned} \omega_{11}W_{1,t} + \omega_{21}W_{2,t} &= K_{1,t} \\ \omega_{12}W_{1,t} + \omega_{22}W_{2,t} &= K_{2,t} \end{aligned} \tag{B.76}$$

where we drop the time subscripts in the portfolio weights because they are constant over time. Taking differentials yields

$$\begin{aligned} \frac{dK_{1,t}}{K_{1,t}} &= \frac{\omega_{11}W_{1,t}}{\omega_{11}W_{1,t} + \omega_{21}W_{2,t}}\frac{dW_{1,t}}{W_{1,t}} + \frac{\omega_{21}W_{2,t}}{\omega_{11}W_{1,t} + \omega_{21}W_{2,t}}\frac{dW_{2,t}}{W_{2,t}} \\ \frac{dK_{2,t}}{K_{2,t}} &= \frac{\omega_{12}W_{1,t}}{\omega_{12}W_{1,t} + \omega_{22}W_{2,t}}\frac{dW_{1,t}}{W_{1,t}} + \frac{\omega_{22}W_{2,t}}{\omega_{12}W_{1,t} + \omega_{22}W_{2,t}}\frac{dW_{2,t}}{W_{2,t}}. \end{aligned} \tag{B.77}$$

The growth rate is a weighted average of the rates of wealth accumulation. The weights are the proportion of domestic and foreign wealth held in the capital of each region, respectively.

B.10 Net asset position and current account

The net foreign asset position for region h is given by the value of assets owned abroad by the representative investor for region h less the assets of region h held by foreigners. Hence, the net foreign asset position for region 1 is

$$N_{1,t} = \omega_{12} W_{1,t} - \omega_{21} W_{2,t}. \tag{B.78}$$

Substituting in the solutions for the optimal portfolio weights yields

$$N_{1,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} \boldsymbol{e}_2^\top V_{\phi,1}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) W_{1,t} - \frac{1}{\sigma^2(1-\theta+\gamma\theta)} \boldsymbol{e}_1^\top V_{\phi,2}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) W_{2,t}, \tag{B.79}$$

where $\boldsymbol{e}_1 = (1,0)^\top$ and $\boldsymbol{e}_2 = (0,1)^\top$. For the special case $\boldsymbol{\alpha} = \alpha\mathbf{1}$

$$N_{1,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} (\alpha - r) \left[\frac{\phi_{12}}{1+\phi_{12}} W_{1,t} - \frac{\phi_{21}}{1+\phi_{21}} W_{2,t} \right]. \tag{B.80}$$

The analogous expression for region 2 is

$$N_{2,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} (\alpha - r) \left[\frac{\phi_{21}}{1+\phi_{21}} W_{2,t} - \frac{\phi_{12}}{1+\phi_{12}} W_{1,t} \right]. \tag{B.81}$$

The current account is defined as the change in the net asset position of the economy. Taking the differential of (B.79) yields

$$dN_{1,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} \boldsymbol{e}_2^\top V_{\phi,1}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) dW_{1,t} - \frac{1}{\sigma^2(1-\theta+\gamma\theta)} \boldsymbol{e}_1^\top V_{\phi,2}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) dW_{2,t}. \tag{B.82}$$

For the special case $\boldsymbol{\alpha} = \alpha\mathbf{1}$

$$dN_{1,t} = \frac{1}{\sigma^2(1-\theta+\gamma\theta)} (\alpha - r) \left[\frac{\phi_{12}}{1+\phi_{12}} W_{1,t} \frac{dW_{1,t}}{W_{1,t}} - \frac{\phi_{21}}{1+\phi_{21}} W_{2,t} \frac{dW_{2,t}}{W_{2,t}} \right]. \tag{B.83}$$

Foreign direct investment is the purchase of foreign capital. Region 1's holding of capital in region 2 is $\omega_{12} W_{1,t}$. Therefore, region 1's FDI is $\omega_{12} dW_{1,t}$. Similarly, region 2's FDI is $\omega_{21} dW_{2,t}$.

B.11 Real exchange rate

Equilibrium in the nontraded consumption good sector is given by

$$P_{h,t}Y_h^{\text{non}} = (1-\theta)C_{h,t}^{\text{tot}} \tag{B.84}$$

Therefore,

$$\begin{aligned} P_{h,t} &= \left(\frac{1-\theta}{Y_h^{\text{non}}}\right)\frac{C_{h,t}^{\text{tot}}}{W_{h,t}}W_{h,t} \\ &= \left(\frac{1-\theta}{Y_h^{\text{non}}}\right)\left(\frac{\psi\delta+(1-\psi)MV_h}{\theta[\psi+(1-\psi)\theta]}\right)W_{h,t} \end{aligned} \tag{B.85}$$

and so

$$\frac{P_{h,t}}{P_{h',t}} = \frac{Y_{h'}^{\text{non}}}{Y_h^{\text{non}}}\frac{W_{h,t}}{W_{h',t}}\frac{\psi\delta+(1-\psi)MV_h}{\psi\delta+(1-\psi)MV_{h'}} \tag{B.86}$$

The price of the composite good consumed by investor is

$$P_{h,t}^{\text{com}} = \frac{1}{\theta^\theta(1-\theta)^{1-\theta}}P_{h,t}^{1-\theta} \tag{B.87}$$

which is a well-known consequence of the Cobb-Douglas aggregator. Therefore, the real exchange rate, giving the price of the composite good consumer by investor j in terms of the composite good consumed by investor h, is given by

$$\begin{aligned} S_{h,j,t} &= \frac{P_{h,t}^{\text{com}}}{P_{j,t}^{\text{com}}} \\ &= \left[\frac{Y_j^{\text{non}}}{Y_h^{\text{non}}}\frac{W_{h,t}}{W_{j,t}}\frac{\psi\delta+(1-\psi)MV_h}{\psi\delta+(1-\psi)MV_j}\right]^{1-\theta} \end{aligned} \tag{B.88}$$

Now take the logarithm of $S_{h,j,t}$

$$\log S_{h,j,t} = \text{Constant} + (1-\theta)(\log W_{h,t} - \log W_{j,t}) \tag{B.89}$$

Taking differentials and applying Ito's lemma yields

$$d\log S_{h,j,t} = (1-\theta)\left(\frac{dW_{h,t}}{W_{h,t}} - \frac{dW_{j,t}}{W_{j,t}} - \frac{1}{2}\left[\left(\frac{dW_{h,t}}{W_{h,t}}\right)^2 - \left(\frac{dW_{j,t}}{W_{j,t}}\right)^2\right]\right) \tag{B.90}$$

Now note that

$$\frac{1}{dt}E_t\left[\frac{dW_{h,t}}{W_{h,t}} - \frac{dW_{j,t}}{W_{j,t}}\right] = \frac{(\alpha-r)^2}{\sigma^2(1-\theta+\gamma\theta)}\sum_{n=1}^{2}\left(\frac{\phi_{hn}}{1+\phi_{hn}} - \frac{\phi_{jn}}{1+\phi_{jn}}\right) + \frac{\psi\delta+(1-\psi)\theta MV_j}{\psi+(1-\psi)\theta} - \frac{\psi\delta+(1-\psi)\theta MV_h}{\psi+(1-\psi)\theta} \tag{B.91}$$

and

$$\frac{1}{dt}E_t\left[\left(\frac{dW_{h,t}}{W_{h,t}}\right)^2 - \left(\frac{dW_{j,t}}{W_{j,t}}\right)^2\right] = \frac{(\alpha-r)^2}{\sigma^2(1-\theta+\gamma\theta)^2}\sum_{n=1}^{2}\left(\left(\frac{\phi_{hn}}{1+\phi_{hn}}\right)^2 - \left(\frac{\phi_{jn}}{1+\phi_{jn}}\right)^2\right) \tag{B.92}$$

Thus,

$$\begin{aligned} &\frac{1}{dt}E_t[d\log S_{h,j,t}] = \\ &(1-\theta)\Bigg[\frac{(\alpha-r)^2}{\sigma^2(1-\theta+\gamma\theta)}\sum_{n=1}^{2}\left(\frac{\phi_{hn}}{1+\phi_{hn}} - \frac{\phi_{jn}}{1+\phi_{jn}}\right) \\ &+\left(\frac{\psi\delta+(1-\psi)\theta MV_j}{\psi+(1-\psi)\theta} - \frac{\psi\delta+(1-\psi)\theta MV_h}{\psi+(1-\psi)\theta}\right) \\ &-\frac{(\alpha-r)^2}{2\sigma^2(1-\theta+\gamma\theta)^2}\sum_{n=1}^{2}\left(\left(\frac{\phi_{hn}}{1+\phi_{hn}}\right)^2 - \left(\frac{\phi_{jn}}{1+\phi_{jn}}\right)^2\right)\Bigg] \end{aligned} \tag{B.93}$$

We can check that the conditions under which the sign of the first and third terms is always given by the sign of the first term for any change in the uncertainty parameters. Write the first and third terms as

$$\left(\frac{\alpha-r}{\sigma}\right)^2\left[\sum_{n=1}^{2}\frac{\kappa_{hn}}{1-\theta+\gamma\theta} - \frac{\kappa_{jn}}{1-\theta+\gamma\theta} - \frac{1}{2}\left(\left(\frac{\kappa_{hn}}{1-\theta+\gamma\theta}\right)^2 - \left(\frac{\kappa_{jn}}{1-\theta+\gamma\theta}\right)^2\right)\right]. \tag{B.94}$$

Differentiate with respect to $\kappa_{h1}/(1-\theta+\gamma\theta)$ (similar arguments apply for the other uncertainty parameters):

$$\left(\frac{\alpha-r}{\sigma}\right)^2\left[1-\frac{\kappa_{h1}}{1-\theta+\gamma\theta}\right], \tag{B.95}$$

which we require to be positive. This implies $1-\theta+\gamma\theta > \kappa_{h1}$ which will always hold if $1-\theta+\gamma\theta > 1$. This requires that $\gamma > 1$.

B.12 Mean-variance welfare

We calculate welfare under objective beliefs. Mean-variance utility absent of familiarity biases is given by taking $\phi_{hn} \to \infty$ in (B.38) keeping optimal choices constant

$$U_h^{MV} - r = \boldsymbol{\omega}_{h,t}^\top(\boldsymbol{\alpha} - r\mathbf{1}) - \frac{1}{2}\sigma^2(1-\theta+\gamma\theta)\boldsymbol{\omega}_{h,t}^\top\boldsymbol{\omega}_{h,t} \tag{B.96}$$

Substituting in the optimal $\boldsymbol{\omega}_{h,t}$ in (B.49) and setting $\boldsymbol{\alpha} = \alpha\mathbf{1}$ yields

$$\begin{aligned} U_h^{MV} &= r + \frac{1}{(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \mathbf{1}^\top V_{\phi,h}^{-1}\mathbf{1} - \frac{1}{2(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \mathbf{1}^\top V_{\phi,h}^{-1}V_{\phi,h}^{-1}\mathbf{1} \\ &= \frac{1}{2(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \\ &\quad \left[2\left(\frac{\phi_{h1}}{1+\phi_{h1}} + \frac{\phi_{h2}}{1+\phi_{h2}}\right) - \left(\left(\frac{\phi_{h1}}{1+\phi_{h1}}\right)^2 + \left(\frac{\phi_{h2}}{1+\phi_{h2}}\right)^2\right)\right] \end{aligned} \tag{B.97}$$

We can write this in the form

$$U_h^{MV} = r + \frac{1}{(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 + A \tag{B.98}$$

where A is to be determined. We have

$$A = -\frac{1}{2(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[\left(\frac{\phi_{h1}}{1+\phi_{h1}} - 1\right)^2 + \left(\frac{\phi_{h2}}{1+\phi_{h2}} - 1\right)^2\right]. \tag{B.99}$$

Therefore,

$$\begin{aligned} U_h^{MV} &= r + \frac{1}{(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 - \frac{1}{2(1-\theta+\gamma\theta)}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[(\kappa_{h1}-1)^2 + (\kappa_{h2}-1)^2\right] \\ &= r + \frac{1}{1-\theta+\gamma\theta}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[1 - \frac{1}{2}\left((\kappa_{h1}-1)^2 + (\kappa_{h2}-1)^2\right)\right] \\ &= r + \frac{1}{1-\theta+\gamma\theta}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[1 - \left(1 - (\kappa_{h1}+\kappa_{h2}) + \frac{1}{2}(\kappa_{h1}^2 + \kappa_{h2}^2)\right)\right] \\ &= r + \frac{1}{1-\theta+\gamma\theta}\left(\frac{\alpha - r}{\sigma}\right)^2 \left[1 - (1-\mu_{h,\kappa})^2 - \sigma_{\mu,\kappa}^2\right] \end{aligned}$$

where

$$\mu_{h,\kappa} = \frac{1}{2}(\kappa_{h1} + \kappa_{h2}),$$
$$\sigma^2_{h,\kappa} = \frac{1}{2}(\kappa^2_{h1} + \kappa^2_{h2}) - \mu^2_{h,\kappa}.$$

It is also evident from the first line that a reduction in κ_{hn} always reduces mean-variance welfare.

B.13 Lifetime welfare

We start from the recursive equation for welfare

$$J_{h,t} = \mathcal{A}(C^{\text{ind}}_{h,t}, \mu_{h,t}[J_{h,t+dt}]). \tag{B.100}$$

Hence,

$$\begin{aligned}
(J_{h,t})^{1-\frac{1}{\psi}} &= (1-e^{-\delta dt})(C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} + e^{-\delta dt}\left(\mu_{h,t}\left[J_{h,t+dt}\right]\right)^{1-\frac{1}{\psi}} \\
&= \delta dt (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} + (1-\delta dt)\left(J_{h,t}\left(1 + E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right)\right)^{1-\frac{1}{\psi}} \\
&= \delta dt (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} + (1-\delta dt)(J_{h,t})^{1-\frac{1}{\psi}}\left(1 + E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right)^{1-\frac{1}{\psi}} \\
&= \delta dt (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} \\
&\quad + (1-\delta dt)(J_{h,t})^{1-\frac{1}{\psi}}\left(1 + \left(1-\frac{1}{\psi}\right)\left(E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right) + o(dt)\right) \\
&= \delta (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} dt \\
&\quad + (J_{h,t})^{1-\frac{1}{\psi}}\left(1 + \left(1-\frac{1}{\psi}\right)\left(E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right) - \delta dt\right) + o(dt)
\end{aligned}$$

from which it follows that

$$0 = \delta (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} dt + (J_{h,t})^{1-\frac{1}{\psi}}\left(\left(1-\frac{1}{\psi}\right)\left(E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right) - \delta dt\right) + o(dt).$$

Hence, in the continuous time limit, we obtain

$$0 = \delta (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} + (J_{h,t})^{1-\frac{1}{\psi}} \left[\left(1-\frac{1}{\psi}\right) \frac{1}{dt} \left(E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right) - \delta \right]. \tag{B.101}$$

Under the assumption that $J_{h,t} = \xi_h W^{\theta}_{h,t}$, we obtain using Ito's lemma

$$\frac{dJ_{h,t}}{J_{h,t}} = \theta \left(\frac{dW_{h,t}}{W_{h,t}}\right) + \frac{1}{2}\theta(\theta-1)\left(\frac{dW_{h,t}}{W_{h,t}}\right)^2 \tag{B.102}$$

Hence,

$$\begin{aligned}
\left(E_t\left[\frac{dJ_{h,t}}{J_{h,t}}\right] - \frac{1}{2}\gamma E_t\left[\left(\frac{dJ_{h,t}}{J_{h,t}}\right)^2\right]\right) &= E_t\left[\theta\left(\frac{dW_{h,t}}{W_{h,t}}\right) + \frac{1}{2}\theta(\theta-1)\left(\frac{dW_{h,t}}{W_{h,t}}\right)^2\right] - \frac{1}{2}\gamma E_t\left[\theta^2\left(\frac{dW_{h,t}}{W_{h,t}}\right)^2\right] \\
&= \Bigg[\theta[r + (\alpha - r)\pi_{h,t}] + \frac{1}{2}\left[\theta(\theta-1) - \gamma\theta^2\right]\sigma^2 \pi^2_{h,t} \sum_{n=1}^{2} x^2_{hn,t} \\
&\quad + \frac{P_{h,t} Y^{\text{non}}_h}{W_{h,t}} - \frac{C^{\text{tot}}_{h,t}}{W_{h,t}}\Bigg] dt \\
&= \left[U^{MV}_h - c_h\right] dt
\end{aligned}$$

where the final line imposes equilibrium in the nontraded goods sector and uses the definition $c_h = C_{h,t}/W_{h,t}$ which we know to be constant over time.

Therefore,

$$\begin{aligned}
0 &= \delta (C^{\text{ind}}_{h,t})^{1-\frac{1}{\psi}} + (J_{h,t})^{1-\frac{1}{\psi}} \left[\left(1-\frac{1}{\psi}\right)\left(U^{MV}_h - c_h\right) - \delta\right] \\
0 &= \delta \left(\frac{C^{\theta}_{h,t}(C^{\text{non}}_{h,t})^{1-\theta}}{W^{\theta}_{h,t}}\right)^{1-\frac{1}{\psi}} + (\xi_h)^{1-\frac{1}{\psi}} \left[\left(1-\frac{1}{\psi}\right)\left(U^{MV}_h - c_h\right) - \delta\right] \\
0 &= \psi\delta \left(c^{\theta}_h (Y^{\text{non}}_h)^{1-\theta}\right)^{1-\frac{1}{\psi}} - (\xi_h)^{1-\frac{1}{\psi}} \left[\psi\delta + (1-\psi)\left(U^{MV}_h - c_h\right)\right]
\end{aligned}$$

Solving for ξ_h yields

$$\xi_h = \left[\frac{\psi\delta}{\psi\delta + (1-\psi)\left(U^{MV}_h - c_h\right)}\right]^{\frac{1}{1-\frac{1}{\psi}}} \left(c^{\theta}_h (Y^{\text{non}}_h)^{1-\theta}\right). \tag{B.103}$$