

Balanced truncation of k -positive systems

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Abstract—This paper considers balanced truncation of discrete-time Hankel k -positive systems, characterized by Hankel matrices whose minors up to order k are nonnegative. Our main result shows that if the truncated system has order k or less, then it is Hankel totally positive (∞ -positive), meaning that it is a sum of first order lags. This result can be understood as a bridge between two known results: the property that the first-order truncation of a positive system is positive ($k = 1$), and the property that balanced truncation preserves state-space symmetry. It provides a broad class of systems where balanced truncation is guaranteed to result in a minimal internally positive system.

Index Terms—total positivity, k -positivity, external positivity, positive systems, model order reduction, internal positivity, non-negative matrix factorization.

I. INTRODUCTION

Model order reduction aims at facilitating analysis, design, and implementation of systems by finding simple lower order approximations. Standard techniques such as balanced truncation provide qualitatively good approximations in reproducing the input-output behaviour. But it is widely unclear in which cases these approximations can be realized through the parallel interconnection of first order lags only. Such approximations, also known as relaxation systems [1], have been of considerable interest as they are passive, externally (input-output) positive and – as recently shown – often admit sparse, scalable optimal controllers [2]. While balanced truncation and optimal Hankel norm approximation are known to preserve this property (in continuous-time) for any order [3], it is an open question in which cases this property can be gained from non-relaxation systems. Here, we provide a first answer by showing that balanced truncation of so-called *Hankel k -positive single-input-single-output (SISO) systems* yields such approximations, if the reduced order is no larger than k .

In discrete-time, Hankel k -positive systems are defined as systems whose Hankel operator has a k -positive matrix representation, i.e., all its minors of order up to k are nonnegative. For example, Hankel 1-positive systems correspond to the well-known (strictly proper) externally (input-output) positive systems [4].

As recently discovered in [5], under a mild multiplicity assumption, Hankel k -positive systems are dominated by relaxation systems of order k , i.e., after a partial fraction

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decomposition, the sum corresponding to the k largest poles in magnitude has a relaxation system structure. This forms a bridge between externally positive (one dominant first order lag) and relaxation systems (sum of first order lags). By our main result, balanced truncation preserves the structure of the dominant parts of the system. Specifically, external positivity is preserved by truncation of SISO systems to order 1.

Many externally positive systems are modelled by internally positive realizations, i.e., system matrices with nonnegative entries. This property is appealing in scalable stability analysis [6–9] and enjoyed by many compartmental models, e.g., within bio-chemistry, economics, or transportation, [6, 10]. Several methods have been suggested to preserve internal positivity in the reduction process, [11–13]. Unfortunately, even for relaxation systems these methods often yield conservative results, which can be outperformed by balanced truncation to much lower orders (see Section VI and [14, 15] for examples). In contrast, for the class of Hankel k -positive SISO systems, we show that balanced truncation preserves internal positivity. In the multi-input-multi-output (MIMO) case, our results remain valid as long as the Hankel operator is symmetric and its representation matrix is k -positive. We believe that the framework of k -positivity also provides a natural extension beyond that. The fact that balanced truncation to order 1 preserves internal positivity also for MIMO systems, [14], is an indication.

The paper is organized as follows. In the preliminaries, we review discrete-time systems and the relationship between Kung’s algorithm and balanced truncation, which is essential in the proof of our main result. Then, we summarize the relevant parts of the k -positivity theory from [5]. Section IV contains our main results on the truncation of Hankel k -positive systems. Finally, we discuss extensions to MIMO systems and conclude with an illustrative example.

II. PRELIMINARIES

A. Notation

1) *Sets*: The sets of nonnegative reals and integers are denoted by $\mathbb{R}_{\geq 0} = [0, \infty)$ and $\mathbb{Z}_{\geq 0} = \mathbb{N}_0$, respectively. For $k, l \in \mathbb{Z}$, we write $(k : l) := \{k, k + 1, \dots, l\}$, $k \leq l$.

2) *Matrices*: For real-valued matrices $X = (x_{ij}) \in \mathbb{R}^{n \times m}$, including vectors $x = (x_i) \in \mathbb{R}^n$, we say that X is *nonnegative*, $X \geq 0$ or $X \in \mathbb{R}_{\geq 0}^{n \times m}$, if all elements $x_{ij} \in \mathbb{R}_{\geq 0}$; we use the corresponding notation for positive matrices. If $X \in \mathbb{R}^{n \times n}$, then $\sigma(X) = \{\lambda_1(X), \dots, \lambda_n(X)\}$ denotes its *spectrum*, where the eigenvalues are ordered by descending absolute value, i.e., $\lambda_1(X)$ is the eigenvalue with the largest magnitude, counting multiplicity. In case that the magnitude of two eigenvalues coincides, we sub-sort them by decreasing real part. A matrix X is called *reducible*, if there exists a

permutation matrix $P = (P_1 \ P_2)$ so that $P_2^T X P_1 = 0$; otherwise X is *irreducible*. We call X *Hankel*, if it is constant along its anti-diagonals. Further, X is *positive semidefinite*, $X \succeq 0$, if $X = X^T$ and $\sigma(X) \subset \mathbb{R}_{\geq 0}$. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted by I_n and the *Moore-Penrose pseudo-inverse* of $X \in \mathbb{R}^{n \times m}$ by X^\dagger . A *(consecutive) j -minor* of X in $\mathbb{R}^{n \times m}$ is a minor which is constructed of (consecutive) j columns and j rows of X . The submatrix with rows $I \subset (1 : n)$ and columns $J \subset (1 : m)$ is written as $X_{\{I, J\}}$.

3) *Functions*: We consider functions $g : \mathbb{Z} \rightarrow \mathbb{R} \cup \{\pm\infty\}$. *Nonnegative functions* $g : \mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ are written as $g \geq 0$ and snapshots as $g(i : j) := (g(i) \ \dots \ g(j))^T$.

The *indicator function* of $S \subset \mathbb{Z}$ is $\mathbb{1}_S(t) := \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}$.

It defines the *unit impulse function* $\delta(t) := \mathbb{1}_{\{0\}}(t)$ and the *step function* $s(t) = \mathbb{1}_{\mathbb{Z}_{\geq 0}}(t)$. The set of all *absolutely summable functions* is denoted by ℓ_1 and the set of *bounded functions* by ℓ_∞ .

B. Linear discrete-time systems

We consider *linear discrete-time time-invariant systems*

$$\begin{aligned} x(t+1) &= Ax(t) + bu(t), \\ y(t) &= cx(t), \end{aligned}$$

with $A \in \mathbb{R}^{n \times n}$, $b, c^T \in \mathbb{R}^n$. The output $y(t) = g(t) = cA^{t-1}b$ corresponding to initial state $x(0) = 0$ and input $u = \delta$ is called the *impulse response*. The transfer function is given by $G(z) = c(zI_n - A)^{-1}b$. It can be written as

$$G(z) = \sum_{t=0}^{\infty} g(t)z^{-t} = \frac{r \prod_{i=1}^m (z - z_i)}{\prod_{j=1}^n (z - p_j)}, \quad (1)$$

where $m < n$, $r \in \mathbb{R}$, p_i and z_i are referred to as *poles* and *zeros*, which are both sorted in the same way as the eigenvalues of a matrix. The triple (A, b, c) is also called a *realization* of G . We always assume that $\{z_1, \dots, z_m\} \cap \{p_1, \dots, p_n\} = \emptyset$, in which case the realization is called *minimal*. We also assume asymptotic stability, i.e., $|p_1|, \dots, |p_n| < 1$. Then, for $u \in \ell_\infty$ with $u(\tau) = u(\tau)(1 - s(\tau))$ and $t \geq 0$, the Hankel operator associated to the system is defined by

$$(\mathcal{H}_g u)(t) := \sum_{\tau=-\infty}^{-1} g(t - \tau)u(\tau) = \sum_{\tau=1}^{\infty} g(t + \tau)u(-\tau). \quad (2)$$

If we set $x_0 = \sum_{\tau=-\infty}^{-1} A^{\tau+1}bu(\tau)$, then $(\mathcal{H}_g u)(t)$ equals the impulse response to (A, x_0, c) . The operator is the limit (for $j \rightarrow \infty$) of the finite truncated matrix representations $\mathcal{H}_g^j u = H_g(1, j)u(-1 : -j)$, where

$$H_g(t, j) := \begin{pmatrix} g(t) & g(t+1) & \dots & g(t+j-1) \\ g(t+1) & g(t+2) & \dots & g(t+j) \\ \vdots & \vdots & \ddots & \vdots \\ g(t+j-1) & g(t+j) & \dots & g(t+2j-2) \end{pmatrix}.$$

C. Balanced truncation

Given a minimal system realization (A, b, c) of $G(z)$, let

$$\mathcal{C}^N(A, b) := (b \ Ab \ \dots \ A^{N-1}b) \quad (3a)$$

$$\mathcal{O}^N(A, c) := \begin{pmatrix} c^T & A^T c^T & \dots & A^{T(N-1)} c^T \end{pmatrix}^T \quad (3b)$$

denote the *finite-time controllability* and *observability* operators. Accordingly, we define the *(finite-time) controllability*, *observability* and *cross-Gramian* by

$$P(N) := \mathcal{C}^N(A, b)\mathcal{C}^N(A, b)^T, \quad P = \lim_{N \rightarrow \infty} P(N), \quad (4a)$$

$$Q(N) := \mathcal{O}^N(A, c)^T \mathcal{O}^N(A, c), \quad Q = \lim_{N \rightarrow \infty} Q(N), \quad (4b)$$

$$X(N) := \mathcal{C}^N(A, b)\mathcal{O}^N(A, c), \quad X = \lim_{N \rightarrow \infty} X(N), \quad (4c)$$

respectively. We call (A, b, c) a *finite-time balanced realization* if $P(N) = Q(N)$ is diagonal with decreasing diagonal entries, called the *finite-time Hankel singular values*. Note that with

$$H_g(1, N) := \mathcal{O}^N(A, c)\mathcal{C}^N(A, b), \quad (5)$$

it holds that

$$\begin{aligned} X(N)^2 &= \mathcal{C}^N(A, b)H_g(1, N)\mathcal{O}^N(A, c) \\ &= \mathcal{C}^N(A, b)H_g(1, N)^T \mathcal{O}^N(A, c) = P(N)Q(N). \end{aligned}$$

Therefore,

$$\lambda_i(H_g(1, N)) = \lambda_i(X(N)), \quad 1 \leq i \leq n \quad (6)$$

and if (A, b, c) is finite-time balanced then $X(N)$ is diagonal. An analogous terminology is used in the limit case where we drop the finite-time prefix and replace $H_g(1, N)$ by \mathcal{H}_g .

There always exists a (finite-time) balanced realization (A, b, c) of $G(z)$, and a (finite-time) balanced truncated system approximation $G_r(z)$ of order r is then given by the realization $(A_{(1:r)}, b_{(1:r)}, c_{(1:r)})$.

D. Kung's algorithm

Note that $H_g(2, N) = \mathcal{O}^N(A, c)A\mathcal{C}^N(A, b)$ and for a minimal realization, we have

$$\text{rank } H_g(1, N) = \text{rank } \mathcal{O}^N(A, c) = \text{rank } \mathcal{C}^N(A, b) = \min\{n, N\}.$$

Assume $N \geq n$. If $H_g(1, N) = LR$ is a rank-revealing factorization, then the image of $\mathcal{O}^N(A, c)$ equals the image of L , i.e. $\mathcal{O}^N(A, c)S = L$ for some nonsingular matrix S , and $S^{-1}\mathcal{C}^N(A, b) = R$. We set $\tilde{c} = L_{(:,1)} = cS$ and $\tilde{b} = R_{(:,1)} = S^{-1}b$. The matrices $\mathcal{O}^N(A, c)$ and L are left-invertible, while $\mathcal{C}^N(A, b)$ and R are right-invertible. Therefore,

$$\begin{aligned} \tilde{A} &= L^\dagger H_g(2, N)R^\dagger \\ &= S^{-1}\mathcal{O}^N(A, c)^\dagger \mathcal{O}^N(A, c)A\mathcal{C}^N(A, b)\mathcal{C}^N(A, b)^\dagger S \\ &= S^{-1}AS, \end{aligned}$$

i.e., the triple $(\tilde{A}, \tilde{b}, \tilde{c})$ is similar to (A, b, c) and

$$\tilde{\mathcal{O}}(N) = \mathcal{O}^N(A, c)S = L, \quad \tilde{\mathcal{C}}(N) = S^{-1}\mathcal{C}^N(A, b) = R.$$

If L and R are chosen from a singular value decomposition $H_g(1, N) = U(N)\Sigma(N)V(N)^T$ as $L = U(N)\Sigma(N)^{\frac{1}{2}}$ and $R = \Sigma(N)^{\frac{1}{2}}V(N)^T$, then

$$\tilde{Q} = LL^T = \Sigma(N) = R^T R = \tilde{P},$$

i.e. the realization is finite-time balanced. This approach is known as *Kung's algorithm*, [16], see also [17, p. 74]. Note

that $H_g(1, N)$ is symmetric and therefore $U(N)$ and $V(N)$ are equal up to the column signs.

We denote by $(A_r(N), b_r(N), c_r(N))$ the truncation of $(\tilde{A}, \tilde{b}, \tilde{c})$ to an r -th order approximation. By the convergence of the Gramians (4) it follows that $(A_r(N), b_r(N), c_r(N))$ converges also for $N \rightarrow \infty$.

Proposition 1. For $G(z)$ and $N > n$, $(A_r(N), b_r(N), c_r(N))$ is a finite-time balanced truncated approximation of $G(z)$ and

$$(A_r, b_r, c_r) := \lim_{N \rightarrow \infty} (A_r(N), b_r(N), c_r(N))$$

is a balanced truncated approximation.

III. k -POSITIVITY THEORY

Let us now introduce the framework of k -positivity, which has been studied extensively in the monograph [18]. We begin by a discussion of finite dimensional matrices and continue with recent results on Hankel operators, whose approximation is the main subject of this work.

A. k -positive matrices

A remarkable feature of nonnegative matrices is the Perron-Frobenius theorem [19, 20].

Proposition 2 (Perron-Frobenius). Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$.

- 1) $\lambda_1(A) \geq 0$.
- 2) If $\lambda_1(A)$ has algebraic multiplicity m_0 , then A has m_0 linearly independent nonnegative eigenvectors related to $\lambda_1(A)$.
- 3) If A is irreducible, then $m_0 = 1$, $\lambda_1(A) > 0$ and A has a strictly positive eigenvector related to $\lambda_1(A)$.

Obviously, all 1-minors of a nonnegative matrix A are nonnegative. A generalization of this property is provided through the concept of *total positivity* [18], which is central in our further approach. To introduce it, we need the notion of an r -th compound matrix. Consider the set of sorted r -tuples of $\{1, \dots, n\}$ given by

$$\mathcal{I}_{n,r} := \{v = \{v_1, \dots, v_r\} : 1 \leq v_1 < v_2 < \dots < v_r \leq n\},$$

where $\mathcal{I}_{n,r}$ is ordered lexicographically. The (i, j) -th entry of the r -th multiplicative compound matrix $X_{[r]} \in \mathbb{R}^{\binom{n}{r} \times \binom{m}{r}}$ to $X \in \mathbb{R}^{n \times m}$ is then defined by $\det(X_{(I,J)})$, where I is the i -th and J is the j -th element in $\mathcal{I}_{n,r}$ and $\mathcal{I}_{m,r}$, respectively. For example, if $X \in \mathbb{R}^{3 \times 3}$, then $X_{[r]}$ reads

$$\begin{pmatrix} \det(X_{\{1,2\},\{1,2\}}) & \det(X_{\{1,2\},\{1,3\}}) & \det(X_{\{1,2\},\{2,3\}}) \\ \det(X_{\{1,3\},\{1,2\}}) & \det(X_{\{1,3\},\{1,3\}}) & \det(X_{\{1,3\},\{2,3\}}) \\ \det(X_{\{2,3\},\{1,2\}}) & \det(X_{\{2,3\},\{1,3\}}) & \det(X_{\{2,3\},\{2,3\}}) \end{pmatrix}.$$

By the Cauchy-Binet formula [21], one can show the following properties (see e.g. [22, Chapter 6]).

Lemma 1. Let $X \in \mathbb{R}^{n \times p}$, $Y \in \mathbb{R}^{p \times m}$ and $r \in \mathbb{Z}_{\geq 1}$.

- i) $(XY)_{[r]} = X_{[r]}Y_{[r]}$.
- ii) If $p = n$, then $\sigma(X_{[r]}) = \{\prod_{i \in I} \lambda_i(X) : I \in \mathcal{I}_{n,r}\}$. Moreover, if for $i \in I$ the columns v_i of $V_I \in \mathbb{C}^{n \times r}$ are eigenvectors of X corresponding to λ_i , then $C_r(V_I)$ is an eigenvector of $X_{[r]}$ corresponding to $\prod_{i \in I} \lambda_i(X)$.

$$\text{iii) } (X^T)_{[r]} = X_{[r]}^T.$$

iv) If $X \succeq 0$, then $X_{[r]} \succeq 0$.

Definition 1. Let $X \in \mathbb{R}^{n \times m}$ and $k \leq \min\{m, n\}$. Then, X is called (strictly) k -positive if all j -minors of X are (positive) nonnegative for $1 \leq j \leq k$. If $k = \min\{m, n\}$, we call X (strictly) totally positive.

By Lemma 1 and Proposition 2, it holds therefore for strictly k -positive $X \in \mathbb{R}^{n \times n}$ that X is a nonnegative matrix with $\lambda_1(X) > \dots > \lambda_k(X) > 0$. This extends the result on the Perron-Frobenius eigenvalue $\lambda_1(X)$. In particular, we have the following important properties [23].

Lemma 2. Let $(S)HP_k \subset \mathbb{R}^{n \times n}$ denote the set of all (strictly) k -positive Hankel matrices. Then,

- i. HP_k is a proper convex cone.
- ii. SHP_k lies densely in HP_k .
- iii. If $X_1 \in HP_{k_1}$ and $X_2 \in HP_{k_2}$, then
 - a) $\lambda_1(X_1) \geq \dots \geq \lambda_{k_1}(X_1) \geq 0$.
 - b) $X_1 + X_2 \in HP_{\min\{k_1, k_2\}}$

B. Hankel k -positive systems

Next, we review LTI systems with $G(z)$ given by (1), whose Hankel operator representation matrix (2) is k -positive. These systems are the main interest of this paper. The results stated here can be found in [5].

Definition 2 (Hankel k -positivity). $G(z)$ is called *Hankel (strictly) k -positive* if $H_g(1, N)$ is (strictly) k -positive for all $N \geq k$. We say that $G(z)$ is *Hankel (strictly) totally positive* if $k = \infty$.

In case of $k = 1$, this means that $g \geq 0$. As such system map nonnegative inputs to nonnegative outputs, they are also called *externally positive*. An important sub-class of externally positive systems is formed through so-called internal positivity.

Definition 3 (Internal positivity). $G(z)$ has an internally positive realization (A, b, c) if A, b and c are nonnegative.

There exists several sufficient certificates for external positivity [24, 25]. Fortunately, also in case of $k > 1$, we do not need to check all minors of $H_g(1, N)$, but it suffices to verify external positivity of the so-called j -th compound system $G_{[j]}(z)$ with $g_{[j]}(t) := \det(H_g(t, j))$, $1 \leq j \leq k$.

Proposition 3. For $G(z)$ and $k \leq n$, the following are equivalent:

- 1) $G(z)$ is Hankel k -positive.
- 2) $G_{[j]} \geq 0$ is externally positive, $1 \leq j \leq k$.
- 3) $H_g(1, k-1) \succ 0$, $H_g(2, k-1) \succeq 0$ and $G_{[k]}$ is externally positive.

Note that $G_{[j]}$ are of finite order as $G_{[j]}$ has the realization $(A_{[j]}, \mathcal{C}^j(A, b)_{[j]}, \mathcal{O}^j(A, c)_{[j]})$.

Example 1. The simplest example of a Hankel totally positive system is $G(z) = \sum_{i=1}^n \frac{r_i}{z-p_i}$ with $r_i, p_i \geq 0$. Indeed, for each system $(p_i, r_i, 1)$, it holds for $j \geq 2$ that $\text{rank}(\mathcal{O}(j)) = 1$, which is why $C_j(\mathcal{O}(j)) = 0$ and thus $g_{[j]} = 0$. First

order externally positive systems are therefore Hankel totally positive and by Lemma 2 also their sums.

First order systems are indeed the prototypes of k -positivity.

Proposition 4. *Let $G(z) = \sum_{i=1}^n \frac{r_i}{z-p_i}$ have distinct poles and be Hankel k -positive with $n \geq k \geq 2$. Then,*

$$G(z) = \frac{r_1}{z-p_1} + G_r(z), \quad (7)$$

where $G_r(z)$ is Hankel $k-1$ -positive, $r_1 > 0$ and $p_1 \geq 0$.

In particular, a repeated application of Proposition 4 implies that $G(z) = \sum_{i=1}^k \frac{r_i}{z-p_i} + G_r(z)$ with $r_i > 0$, $p_i \geq 0$, with $G_r(z)$ only containing poles of smaller magnitude. The dominant dynamics of $G(z)$ are, therefore, Hankel totally positive. For $k = n$, we have the following necessary and sufficient decomposition known from relaxation systems.

Corollary 1. *$G(z)$ is Hankel totally positive if and only if $G(z) = \sum_{i=1}^n \frac{r_i}{z-p_i}$, where $r_i > 0$ and $p_i \geq 0$.*

In other words, Hankel k -positivity is a framework that quantifies the transit from external positivity – one dominant first order lag – to relaxation systems – sums of first order lags.

IV. REDUCTION OF k -POSITIVE HANKEL OPERATORS

Next, we look into balanced truncation of k -positive Hankel operators. We start with state-space symmetric systems as an intermediate step. Then, we treat the totally positive case, before we finally prove Theorem 1 as our general main result.

A. Balanced truncation of state-space symmetric systems

Definition 4. A realization (A, b, c) is called *state-space symmetric* if $A = A^T$ and $b^T = c$.

The following characterizations of state-space symmetric systems holds.

Proposition 5. *Let $G(z)$ be of order n . Then the following are equivalent:*

- 1) $G(z)$ has a state-space symmetric minimal realization.
- 2) $H_g(1, n) \succ 0$.
- 3) $G(z) = \sum_{i=1}^n \frac{r_i}{z-p_i}$ with $r_i > 0$ and $p_i \in \mathbb{R}$ for all i .
- 4) If (A, b, c) is a minimal realization of $G(z)$ with cross-Gramian X , then $\sigma(X) \subset \mathbb{R}_{>0}$.
- 5) $G(z)$ has a balanced state-space symmetric minimal realization.

Proof. 4) \Rightarrow 2) Since $X = \lim_{N \rightarrow \infty} \mathcal{C}(N)\mathcal{O}(N)$ and $\sigma(X) \subset \mathbb{R}_{>0}$, there is an $N > n$, such that

$$\mathbb{R}_{>0} \supset \sigma(\mathcal{C}(N)\mathcal{O}(N)) = \sigma(\mathcal{O}(N)\mathcal{C}(N)) \setminus \{0\}.$$

Hence, $\mathcal{O}(N)\mathcal{C}(N) = H_g(1, N) \succeq 0$ and as such its principle sub-matrix $H_g(1, n) \succeq 0$. Since $H_g(1, n)$ is non-singular, $H_g(1, n) \succ 0$.

2) \Rightarrow 5) If $H_g(1, n) \succ 0$ then it has a symmetric SVD $H_g(1, n) = U\Sigma U^T$ and the balanced realization obtained by Kung's algorithm is symmetric.

5) \Rightarrow 3) By symmetry of the realization we have $G(z) =$

$b^T(zI - A)^{-1}b$. If $S^TAS = \text{diag}(p_1, \dots, p_n) \subset \mathbb{R}^{n \times n}$ is the spectral decomposition of A and $S^Tb = \tilde{b}$, then $G(z) = b^T S(zI - S^TAS)^{-1} S^T b = \sum_{i=1}^n \frac{\tilde{b}_i^2}{z-p_i}$.

3) \Rightarrow 1) If $c = [\sqrt{r_1}, \dots, \sqrt{r_n}]$, $b = c^T$, and $A = \text{diag}(p_1, \dots, p_n)$, then $G(z) = c(zI - A)^{-1}b$. Hence we have a symmetric minimal realization.

1) \Rightarrow 4) If $A = A^T$ and $b = c^T$, then all Gramians are equal, $P = Q = X$. In particular $X \succ 0$, if the realization is minimal. \square

The last item in Proposition 5 yields the following property of balanced truncation.

Corollary 2. *Balanced truncation preserves state-space symmetry, i.e., all truncated models are state-space symmetric.*

In fact, this property is also shared by optimal Hankel-norm approximation [3].

B. Balanced truncation of totally positive Hankel operator

A comparison with Corollary 1 reveals that state-space symmetric systems fulfil many of the requirements necessary for Hankel total positivity. However, there is an important difference, which manifests itself as follows.

Corollary 3. *For $G(z)$, the following are equivalent:*

- 1) $G(z)$ is Hankel totally positive.
- 2) $H_g(1, n) \succ 0$ and $H_g(2, n) \succeq 0$.
- 3) $G(z) = \sum_{i=1}^n \frac{r_i}{z-p_i}$ with $r_i > 0$ and $p_i \geq 0$ for all i .
- 4) $G(z)$ has an internally positive state-space symmetric realization.
- 5) $G(z)$ has a balanced minimal state-space symmetric realization (A, b, c) with $A \succeq 0$.

Proof. 1) \Rightarrow 2) By definition, total positivity implies $H_g(1, n) \succeq 0$ and $H_g(2, n) \succeq 0$. Since G has order n , it follows that $H_g(1, n)$ is nonsingular.

2) \Rightarrow 5) Since $H_g(1, n) \succ 0$, we can factorize $H_g(1, n) = LL^T$ to obtain a balanced symmetric minimal realization, where $b = c^T$ is the first column of L and $A = L^\dagger H_g(2, n)(L^\dagger)^T \succeq 0$.

5) \Rightarrow 4) This is obvious.

4) \Rightarrow 3) As in item 3 of Proposition 5, we obtain the partial fraction expansion of G where now additionally $p_i \geq 0$, since $A \succeq 0$.

5) \Rightarrow 1) This has been discussed in Example 1. \square

The equivalence of the first two items has already been noted in [26, Theorem 4.4], but since we use realization theory, its proof is greatly simplified and also provides an alternative proof of Corollary 1. The last item in Corollary 3 implies the following property of balanced truncation [3].

Proposition 6. *Let $G(z)$ be Hankel totally positive. Then, balanced truncation yields Hankel totally positive approximations.*

C. Balanced truncation of Hankel k -positive systems

While the previous results have well-known analogues for continuous-time systems [1, 3, 14], the general case is our main result. It exploits the next lemma.

Lemma 3. Let $G(z)$ be Hankel k -positive with $k \leq n$. If (A, b, c) is a minimal realization of $G(z)$ with cross-Gramian X , then $\lambda_1(X), \dots, \lambda_k(X) > 0$.

Proof. Using Lemma 2, it follows for $N \geq k$ that $\lambda_1(H_g(1, N)), \dots, \lambda_k(H_g(1, N)) > 0$. Since $\lambda_i(\mathcal{H}_g) = \lim_{N \rightarrow \infty} \lambda_i(H_g(1, N))$, by the continuity of the eigenvalues (see e.g. [21]), the result follows because $\text{rank}(\mathcal{H}_g) = n$ and $\lambda_i(X) = \lambda_i(\mathcal{H}_g)$. \square

Theorem 1. Let $G(z)$ be Hankel k -positive and $r \leq k$. Then, if $\sigma_r(\mathcal{H}_g) \neq \sigma_{r+1}(\mathcal{H}_g)$, balanced truncation to order r yields an asymptotically stable Hankel totally positive approximation.

Proof. Since Hankel k -positivity implies Hankel r -positivity, $r \leq k$, it suffices to consider the case $k = r$. It is known that balanced truncation to order k preserves asymptotic stability, if $\sigma_k(\mathcal{H}_g) > \sigma_{k+1}(\mathcal{H}_g)$ (e.g. [27]).

To prove total positivity assume first that G is strictly Hankel k -positive. As before let $\sigma_i(H_g(1, N))$ denote the i -th singular value of $H_g(1, N)$. Then $\sigma_i(H_g(1, N))$ converges to $\sigma_i(\mathcal{H}_g)$ for $N \rightarrow \infty$. Hence, for sufficiently large N , we have $\sigma_k(H_g(1, N)) > \sigma_{k+1}(H_g(1, N))$. Since $G(z)$ is Hankel k -positive, all $H_g(1, N)$ are k -positive and thus $\sigma_i(H_g(1, N)) = \lambda_i(H_g(1, N))$ for $i = 1, \dots, k$. Let $u_1(N), \dots, u_k(N)$ be a corresponding set of orthonormal eigenvectors and define $U_j(N) = [u_1(N), \dots, u_j(N)] \in \mathbb{R}^{N \times j}$ for $1 \leq j \leq k$. Then, following Kung's algorithm described in subsection II-D, a balanced truncated approximation is given by

$$A_k(N) = \Sigma_k(N)^{-\frac{1}{2}} U_k(N)^T H_g(2, N) U_k(N) \Sigma_k(N)^{-\frac{1}{2}},$$

$c_k(N) = b_k(N)^T$ equal to the 1st row of $\Sigma_k(N)^{-\frac{1}{2}} U_k(N)$. It is evident, that $A_k(N)$ is symmetric. In view of Corollary 3, we need to show that $A_k(N) \succeq 0$. This follows from Sylvester's criterion, if

$$\det(U_j(N)^T H_g(2, N) U_j(N)) > 0 \quad (8)$$

for all $j = 1, \dots, k$. By Lemma 1 the compound matrix $C_j(U_j(N))$ is an eigenvector of the positive matrix $C_j(H_g(1, N))$ corresponding to the eigenvalue

$$\lambda_1(C_j(H_g(1, N))) = \prod_{i=1}^j \lambda_i(H_g(1, N)) > 0.$$

Hence we can assume that $C_j(U_j(N))$ is positive (see also Remark 1 below). Together with the positivity of $H_g(2, N)$ we have

$$\begin{aligned} 0 &< C_j(U_j(N))^T C_j(H_g(2, N)) C_j(U_j(N)) \\ &= C_j(U_j(N))^T H_g(2, N) U_j(N) \\ &= \det(U_j(N)^T H_g(2, N) U_j(N)), \end{aligned}$$

which is (8).

We conclude that the N -balanced reduced system is strictly totally positive. By Lemma 2, the result follows also for the non-strict case. Finally, letting $N \rightarrow \infty$ yields the corresponding statements for \mathcal{H}_g . \square

Thus, systems with k -positive Hankel operators have approximations that naturally correspond to their characteristic

dominant dynamics. In particular, we want to single out the following important case for $k = 1$.

Corollary 4. Let $G(z)$ be externally positive. Then, its first order balanced truncated approximation is externally positive.

Remark 1. 1) A word on the assumption $C_j(U_j(N)) > 0$ in the previous proof might be helpful. By Lemma 1 there exist eigenvectors $\tilde{u}_1(N), \dots, \tilde{u}_k(N)$, forming a matrix $\tilde{U}_j(N)$, such that $C_j(\tilde{U}_j(N)) > 0$ for all $j \leq k$. These eigenvectors may differ from $u_1(N), \dots, u_k(N)$, but span the same space. Therefore, $U_k(N) = \tilde{U}_k(N)S$ where S is an orthogonal matrix. This transformation amounts to a similarity transformation of the reduced system.

2) If we drop the assumption that $\sigma_r(\mathcal{H}_g) > \sigma_{r+1}(\mathcal{H}_g)$ then the reduced system might not be asymptotically stable. Moreover, our proof does not guarantee total positivity of every balanced truncated approximation to order k , although it still holds true that there exists such a truncation.

V. MULTI-INPUT-MULTI-OUTPUT SYSTEMS

It is easy to see that our results extend to MIMO systems with symmetric Hankel operators, i.e., $\mathcal{H}_g = \mathcal{H}_g^T$. However, the following result for internally positive systems suggests that we can even leap beyond that.

Theorem 2. Let (A, B, C) be an internally positive MIMO system. Then, there exists an asymptotically stable, internally positive, balanced truncated first order approximation.

Proof. Let P and Q be the controllability and observability Gramians of (A, B, C) . Obviously, $P, Q \in \mathbb{R}_{>0}^{n \times n}$ and thus $PQ \in \mathbb{R}_{>0}^{n \times n}$, too. Balancing the system via a state-space transformation $x = T\xi$ yields $T^{-1}PQT = \text{diag}(\Sigma^2, 0)$, where $\Sigma = \text{diag}(\sigma_1 I_{k_1}, \dots, \sigma_N I_{k_N})$, containing the Hankel singular values $\sigma_1 > \dots > \sigma_N$, with corresponding multiplicities k_1, \dots, k_N . Hence, the columns of T are eigenvectors of PQ and by Proposition 2 there exists a nonnegative right-eigenvector v_1 to the largest eigenvalue σ_1 , i.e. $PQv_1 = \sigma_1 v_1$ with $T = (v_1, \dots, v_n)$. Analogously, there is a nonnegative left-eigenvector w_1 with $T^{-1} = (w_1, \dots, w_n)^T$. If $k_1 = 1$, the asymptotic stability of the reduced system of order 1 is given by nonnegative $B_1 = w_1^T B$ and $C_1 = C v_1 \geq 0$ as well as $A_1 = w_1^T A v_1$, where A_1 is positive in discrete-time and negative in continuous-time.

If $k_1 > 1$, it could happen that A_1 is only marginally stable. But since the reduced system of order k_1 (belonging to all σ_1) is asymptotically stable, there must exist at least one asymptotically stable first order approximation. Further, by Proposition 2 we conclude the reducibility of PQ and thus the internal positivity of each first order approximation. \square

VI. EXAMPLE

We consider an illustrative example to demonstrate how Hankel k -positivity emerges from relaxation systems as well as to show that Hankel k -positive systems do not allow for

much larger Hankel totally positive approximations than up to order k . To this end, let

$$G_k(z) = \sum_{j=1}^6 \frac{1}{z - \frac{1}{10^{-j}}} - \frac{r_k}{z - 0.3}, r_k \geq 0$$

where the parameter vector $r = (r_1, r_2, r_3, r_4, r_5, r_6, r_7)$,

$$r = (6 \quad 1.1538 \quad 0.3125 \quad 0.0769 \quad 0.0132 \quad 0.0011 \quad 0)$$

contains the threshold values up to which $G_k(z)$ is Hankel k -positive. Note, e.g., that by Corollary 1, $G_k(z)$ cannot be Hankel totally positive for $r_k > 0$. For each r_k , the largest orders o_k for which balanced truncation of $G_k(z)$ yields a relaxation system are then contained in the vector $o = (1 \ 2 \ 4 \ 5 \ 6 \ 6 \ 6)$. This demonstrates that the positivity degree may be quite sharp for determining a priori the largest truncation order for which Hankel totally positive approximations can be expected.

It follows from Corollary 1 that o_k also determines the order up to which balanced truncation returns an internally positive realizable approximation, which is independent of a particular system realization. In contrast, [11–13] require internally positive realizations to begin with, which leads to internally positive approximations with conservative errors, even after the reduction of only a few states [14, 15]. For example, applying [11] for obtaining a fifth order approximation of $G_7(z)$ with realization $A = \text{diag}(0.9, \dots, 0.4)$, $b^T = c = (1 \ \dots \ 1)$, simply removes the dynamics of the fastest pole, resulting in a relative H_∞ -error of $6.8 \cdot 10^{-2}$. Balanced truncation to order 2, however, has only an error of $8.8 \cdot 10^{-3}$.

Our example, further, suggests that small imperfections, e.g., in the measurement of the impulse response may make it impossible to identify a truly underlying Hankel totally positive system. Then, our results indicate that balanced truncation may be used to damp the contribution of these imperfections by finding a nearby Hankel totally positive approximation.

Finally, note that systems such as $G_k(z)$ can be found as the linear part of a perceptron within neural networks [5].

VII. CONCLUSION

In this work, we have addressed the problem of finding reduced order models that consist of a parallel interconnection of first order lags. While approximating a system with a relaxation or an internally positive system generally requires new algorithms, our results show that for the class of Hankel k -positive systems it suffices to use balanced truncation. Interestingly, this proves that balanced truncation yields approximations, which are of the same form as the dominant system dynamics. So far, this has only been observed for the reduction of relaxation systems [3]. In particular, reduction of an externally positive system to order 1 will always provide an internally positive approximation, which often outperforms specialized internally positivity preserving reduction methods. Our example, further, indicates that the Hankel positivity degree is often close to the largest possible order for which balanced truncation yields a relaxation system.

Nonetheless, our results also face limitations: (i) for large system, it may be computationally difficult to verify Hankel k -positivity, (ii) our results mainly apply to systems with symmetric Hankel operators. In the future, we hope to overcome the first limitation through extensions to the class of Hankel internally k -positive systems, i.e., systems with internally positive compound systems. In particular, as this class requires A to be k -positive, it will connect to recent investigations of autonomous internally k -positive systems in [28–31]. Concerning the second limitation, our result on the reduction of internally positive systems to order 1 indicates that extensions to systems with non-symmetric Hankel operator are plausible.

In the future, it would be interesting to extend these results to the Toeplitz operator. Another important question is whether our results also extend to optimal low-rank Hankel approximations. The example in [32] suggests an affirmative answer. In particular, this would result in so-called *completely positive* approximations [33] with the attractive feature of having a rank-revealing nonnegative matrix factorization [34].

Finally note that our results can also be readily extended to continuous-time systems.

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