New Singularities in Unexpected Places

John D. Barrow* and Alexander A. H. Graham†
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Wilberforce Road, Cambridge, CB3 0WA, UK

May 18, 2015

Abstract

Spacetime singularities have been discovered which are physically much weaker than those predicted by the classical singularity theorems. Geodesics evolve through them and they only display infinities in the derivatives of their curvature invariants. So far, these singularities have appeared to require rather exotic and unphysical matter for their occurrence. Here we show that a large class of singularities of this form can be found in a simple Friedmann cosmology containing only a scalar-field with a power-law self-interaction potential. Their existence challenges several preconceived ideas about the nature of spacetime singularities and impacts upon the end of inflation in the early universe.

A striking feature of relativistic cosmology is the prediction that past and future singularities can occur. Originally, singularities were defined by the existence of incomplete geodesics, and a variety of sufficient conditions for geodesic incompleteness were established by a series of important theorems from 1965-1972 [1]. More recently, by using the Einstein equations, new types of physical singularities have been identified which can occur at finite time and are unaccompanied by geodesic incompleteness [2, 3, 4]. Many quantities, such as the density and the expansion rate, which diverge at traditional 'big bang' singularities, remain finite whilst other physical quantities, like the pressure, diverge in finite proper time. The simplest example of what is termed a 'sudden' singularity occurs in the zero-curvature Friedmann universe with scale factor $a(t)$ and Hubble rate $H = \dot{a}/a$, containing matter with density ρ and pressure p. The field equations are ($8\pi G = 1 = c$)

$$\begin{aligned} 3H^2 &= \rho, && (1) \\ \dot{\rho} &= -3H(\rho + p), && (2) \\ \ddot{a} &= -\frac{(\rho + 3p)a}{6}, && (3) \end{aligned}$$

These equations permit there to be a finite time, t_s, at which a, H, and ρ all remain finite, in accord with Eq. (1), but where $p, \dot{\rho}$ and $\ddot{a}$ all become infinite, in accord with Eqs. (2)-(3). The key to their existence is in not assuming any functional link between p and ρ, nor any boundedness condition on p, and this freedom allows an acceleration singularity $\ddot{a} \to \infty$ to arise at finite time as $t \to t_s$ because of a divergence in the matter pressure, $p \to \infty$. Here is an explicit example. On the time interval $0 \leq t \leq t_s$, we can choose a solution for the scale factor $a(t)$ of the form

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^n, \quad (4)$$

where $a_s \equiv a(t_s)$, q and n are positive constants. If $t \to t_s$ from below then $a \to a_s$, $H \to H_s$ and $\rho \to \rho_s > 0$, where a_s, H_s, and ρ_s are all finite, but $p \to \infty$ and $\ddot{a} \to -\infty$ whenever $1 < n < 2$ and $0 < q \leq 1$. As $t \to 0$ we

*Email: J.D.Barrow@damtp.cam.ac.uk
†Email: A.A.H.Graham@damtp.cam.ac.uk

have a big bang singularity with $\rho \to \infty$ and $a(t) \propto t^q$ but, as $t \to t_s$ is approached from below, a sudden singularity occurs with $\ddot{a} \to -\infty$ but a and $\dot{a}$ finite.

Nothing singular happens to geodesics as $t \to t_s$ [5] and we always have $\rho + 3p > 0$ because $\ddot{a}/a < 0$. These singularities are notable because they obey all the classical energy conditions bar the dominant energy condition, in contrast to most other exotic singularities discovered so far [6]. We can also create a divergence in any higher $(N+1)^{st}$ order time-derivative of the scale factor, with all lower-order derivatives staying finite, by choosing $n \in (N, N+1)$ for integer $N > 2$ in Eq. (4). Adding the curvature term to the Friedmann equation makes no difference to these conclusions, and Eq. (4) is actually a leading-order approximation to part of the general solution of the Einstein equations [7].

Notice that no equation of state linking ρ and p has been assumed, and in fact the relation between them in these solutions tends to be pathological, as P diverges at finite ρ. It is natural to ask whether this type of finite-time singularity can occur when there is a physically motivated choice of p and ρ which doesn't allow them to be independent variables? We shall show that the answer to this question is 'yes'.

There are in fact very simple examples that can arise in the study of inflation in the early stages of the universe, or the universe's more recent phase of dark-energy driven acceleration. They arise for the simple case of a cosmological scalar field, ϕ, with a positive power-law self-interaction potential, $V(\phi)$, contributing a density and pressure

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \text{ and } p = \frac{1}{2}\dot{\phi}^2 - V(\phi),$$

with

$$V(\phi) = A\phi^n, \ A > 0. \tag{5}$$

The field equations (1)-(3) are therefore

$$\begin{aligned} 3H^2 &= \frac{1}{2}\dot{\phi}^2 + V(\phi), && (6) \\ \ddot{\phi} &= -3H\dot{\phi} - An\phi^{n-1}, && (7) \\ 2\dot{H} &= -\dot{\phi}^2. && (8) \end{aligned}$$

When n is a positive *even* integer they describe the classic model of large-field inflation in a potential with a single minimum [8]. When n is a positive *odd* integer the universe appears to recollapse under the influence of the scalar field (for the $n = 1$ case see Ref. [9]). We will be interested in the case where $n > 0$, with n *not* an integer.

We first examine the case where $0 < n < 1$. We choose initial conditions so that the universe is expanding initially and $\phi_0 > 0$, but the value of $\dot{\phi}_0$ is unconstrained. It is not difficult to see how the system evolves in time. Since $\dot{\phi}_0 > 0$ both terms on the right-hand side of Eq. (7) are negative, so in finite time $\dot{\phi}$ becomes negative. Hence, in finite time the scalar field starts to decrease, and since $\dot{\phi}$ continues to decrease, as the second term on the right-hand side of Eq. (7) increases as ϕ decreases, it will reach $\phi = 0$ in finite time. When this happens $\dot{\phi}$ can also be shown to be finite and strictly negative using Eq. (6), but from Eq. (7) we have that $\ddot{\phi} \to -\infty$ as $\phi \to 0$. From Eqs. (6)-(8), we see that H and $\dot{H}$ are both finite at this point but $\ddot{H}$ diverges because

$$\ddot{H} = -\dot{\phi}\ddot{\phi} \to -\infty \text{ as } \phi \to 0. \tag{9}$$

This divergence is not a scalar polynomial curvature singularity [10], as both H and $\dot{H}$ are finite at this point. For our spatially-flat Friedmann universe, the Ricci scalar, R, may be written as

$$R = 6(2H^2 + \dot{H}), \tag{10}$$

which is clearly finite as $\phi \to 0$. However, higher scalar derivatives of the curvature (like $\partial_a R \partial^a R$ or $\Box R$) are not regular since

$$\dot{R} = 6(4H\dot{H} + \ddot{H}) \to -\infty \text{ as } \phi \to 0. \tag{11}$$

It is easy to check also that these singularities satisfy all the classical energy conditions in the vicinity of the singularity. This is the first example of a finite-time singularity for a simple and realistic matter model.

Similar singularities can also be shown to exist when $n > 1$, although in this case only higher-order derivatives of ϕ will diverge at the singularity. This may be seen as differentiating Eq. (7) once gives

$$\dddot{\phi} - 9H^2\dot{\phi} - \frac{3}{2}\dot{\phi}^3 - 3HV'(\phi) + V''(\phi)\dot{\phi} = 0. \tag{12}$$

For $1 < n < 2$, every term except the first and last on the left-hand side is finite as $\phi \to 0$, so $\dddot{\phi} \to \infty$ as $\phi \to 0$. This means that the first divergence in the scale factor occurs at fourth order in its derivatives, since

$$\dddot{H} = -\ddot{\phi}^2 - \dot{\phi}\dddot{\phi} \to \infty \ \text{ as } \ \phi \to 0. \tag{13}$$

Hence, $\Box R$ and higher derivatives of the curvature are divergent on approach to this singularity.

It is not difficult to generalise these conclusions to scalar-field potentials of the form of Eq. (4) with arbitrarily large non-integer values of n. If $k < n < k+1$, where k is a positive integer, then as $\phi \to 0$ we have $\phi^{(k+2)} \to (-1)^{k+1} \times \infty$, with all lower derivatives of ϕ finite. This implies that the first divergence of the Hubble rate occurs for the $(k+2)^{th}$ derivative: $H^{(k+2)} \to (-1)^{k+1} \times \infty$ as $\phi \to 0$. But if n is an integer these singularities *never* occur because $V(\phi)$ is smooth at $\phi = 0$. By similar arguments any potential which is not smooth at $\phi = 0$ should create singularities of a similar type.

These singularities have a number of remarkable properties. They are remarkably weak in that they exhibit no divergence of the curvature on approach to the singularity and all polynomial curvature invariants are finite: the only divergence occurs in derivatives of the curvature. Due to the weakness of these singularities the spacetime remains geodesically complete. To our knowledge they are the first examples of such weak singularities in a Friedmann spacetime with a realistic matter model.

Evolving these spacetimes beyond $\phi = 0$ is not always simple, as in some cases (for instance $n = \frac{1}{2}$) the matter model breaks down beyond the singular point $\phi = 0$, since the naive evolution would push ϕ to become strictly negative, which makes the expansion rate become complex. However, this is only the case for some choices of n, and there are many choices (e.g. $n = \frac{1}{3}$) for which $V(\phi)$ is always real-valued. Numerical evidence suggests that if $V(\phi)$ is negative-definite for $\phi < 0$ the spacetime collapses to a 'big crunch' singularity, while if $V(\phi)$ is positive-definite for $\phi < 0$ no such collapse occurs. Instead, the universe passes through $\phi = 0$ an infinite number of times. These spacetimes illustrate that the distinction between singular and non-singular spacetimes is by no means clear-cut, and there are many spacetimes, sourced by realistic matter, which are geodesically complete yet possess observables which can evolve to finite-time singularities.

The formation of these singularities is completely generic. Indeed, once can rigorously show that they form from any homogeneous and isotropic initial data when $0 < n < 1$. They can also be shown to be stable to small perturbations using standard perturbation theory [11], and one can adapt the arguments above to show they form in expanding, homogeneous universes with large anisotropies. They are even stable to quantum corrections since one can construct examples where the divergence only occurs at an arbitrarily high order of the scale factor derivative [12, 13]. This is in contrast to most other examples of weak singularities discovered so far.

Finally, the model studied in this essay offers an intriguing alternative to conventional models of inflation. If we choose initial conditions so that the system starts high enough up the potential, and its early evolution is potential-dominated, then inflation occurs as usual for as long as the strong-energy condition is violated. Inflation ends as $\phi \to 0$, whereupon the universe enters the reheating phase [14]. However, the evolution will differs from standard models of inflation when the system reaches $\phi = 0$ deep in the reheating phase. When that happens, the spacetime develops the weak singularity described in this essay. Since predictions for the power spectrum of the cosmic microwave background (CMB) are insensitive to the behaviour at reheating these models will give the same predictions for CMB observables as conventional large-field inflation models. Indeed, our monomial potentials with $n < 2$ give a better fit to current CMB data than those with large integer values of n [15]. However, these models ultimately have very different dynamics from conventional reheating models. They bring inflation to a singular but timely end.

References

[1] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, (CUP, Cambridge, 1973)

[2] J. D. Barrow, G. J. Galloway and F. J. Tipler, *Mon. Not. Roy. Astron. Soc.* **223**, 835-844 (1986)

[3] J. D. Barrow, *Class. Quantum Grav.* **21**, L79-L82 (2004)

[4] J. D. Barrow, *Class. Quantum Grav.* **21**, 5619-5622 (2004)

[5] L. Fernández-Jambrina and R. Lazkoz, *Phys. Rev. D* **70**, 121503(R) (2004)

[6] R. R. Caldwell, *Phys. Lett. B* **545**, 23 (2002)

[7] J. D. Barrow, S. Cotsakis and A. Tsokaros, *Class. Quantum Grav.* **27**, 165017 (2010)

[8] A. D. Linde, *Phys. Lett. B* **129**, 177-181 (1983)

[9] N. Kaloper and A. Padilla, *Phys. Rev. Lett.* **114**, 101302 (2015)

[10] G. F. R. Ellis and B. Schmidt, *Gen. Rel. Grav.* **10**, 989-997 (1979)

[11] J. D. Barrow and S. Z. W. Lip, *Phys. Rev. D* **80**, 043518 (2009)

[12] J. D. Barrow, A. B. Batista, J. C. Fabris and S. Houndjo, *Phys. Rev. D* **78,** 123508 (2008)

[13] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **595**, 1-8 (2004) and *Phys. Rev. D* **70**, 103522 (2004)

[14] L. Kofman, A. Linde and A. A. Starobinsky, *Phys. Rev. Lett.* **73**, 3195-3198 (1994)

[15] P. A. R. Ade et al, *Astron. Astrophys.* **571**, A22 (2014)