$\widehat{\mathcal{D}}$-modules on rigid analytic spaces

Andreas Bode

Peterhouse
University of Cambridge
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Abstract

Following the notion of p-adic analytic differential operators introduced by Ardakov–Wadsley, we establish a number of properties for coadmissible $\widehat{\mathcal{D}}$-modules on rigid analytic spaces. Our main result is a $\widehat{\mathcal{D}}$-module analogue of Kiehl's Proper Mapping Theorem, considering the 'naive' pushforward from $\widehat{\mathcal{D}}_X$-modules to $f_*\widehat{\mathcal{D}}_X$-modules for proper morphisms $f : X \to Y$. Under assumptions which can be naturally interpreted as a certain properness condition on the cotangent bundle, we show that any coadmissible $\widehat{\mathcal{D}}_X$-module has coadmissible higher direct images. This implies among other things a purely geometric justification of the fact that the global sections functor in the rigid analytic Beilinson–Bernstein correspondence preserves coadmissibility, and we are able to extend this result to arbitrary twisted $\widehat{\mathcal{D}}$-modules on analytified partial flag varieties.

Our results rely heavily on the study of completed tensor products for p-adic Banach modules, for which we provide several new exactness criteria. We also show that the main results of Ardakov–Wadsley on the algebraic structure of $\widehat{\mathcal{D}}$ still hold without assuming the existence of a smooth Lie lattice. For instance, we prove that the global sections $\widehat{\mathcal{D}}_X(X)$ form a Fréchet–Stein algebra for *any* smooth affinoid X.

To my parents
my sister
my brother

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text.
It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text.

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It is unclear to me whether mathematical research has any potential to make one a better person (rejecting naive sado-masochistic notions that any form of struggle or challenging endeavour ‘builds character’), but it certainly can make one a very grateful person.

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Contents

Chapter 1

Introduction

Let K be a complete non-archimedean field of mixed characteristic with discrete valuation ring R, uniformizer π and residue characteristic p. Schneider and Teitelbaum initiated in [40, 41, 43, 44] the study of locally analytic K-representations of p-adic groups. These can be analysed via the representation theory of the distribution algebra $D(G, K)$ and its subalgebra $\widehat{U(\mathfrak{g})}$ of distributions supported at the identity.
In order to obtain a geometric interpretation of $\widehat{U(\mathfrak{g})}$-modules, Ardakov and Wadsley introduced in [5, 6] the notion of a $\widehat{\mathcal{D}}_X$-module on a rigid analytic space, which allows for a localization result analogous to the classical Beilinson–Bernstein theorem on algebraic flag varieties.

Let us first briefly recall the classical, i.e. algebraic Localization Theorem due to Beilinson–Bernstein and Brylinski–Kashiwara.

Theorem 1.1 (see [9, 17])**.** *Let G be a reductive algebraic group over an algebraically closed field of characteristic 0, with Borel subgroup B and Lie algebra $\mathfrak{g}$. Let $X = G/B$ be the flag variety of G and $\mathcal{D}_X$ the sheaf of differential operators on X. Then the global section functor $\mathcal{M} \mapsto \mathcal{M}(X)$ defines an equivalence of categories*

$$\Gamma : \ \mathcal{D}_X\text{-mod} \to U(\mathfrak{g})_0\text{-mod}$$

between the category of (quasi-coherent) $\mathcal{D}_X$-modules and representations of $\mathfrak{g}$ with trivial central character, such that coherent $\mathcal{D}_X$-modules correspond to finitely generated $U(\mathfrak{g})_0$-modules.

The quasi-inverse is given by **localization**, by associating to a $U(\mathfrak{g})_0$-module M the sheaf on X given by the sheafification of

$$U \mapsto \mathcal{D}_X(U) \otimes_{\mathcal{D}_X(X)} M.$$

For non-trivial central characters, a 'twisted' version of this equivalence holds, provided the character is regular and dominant. For singular characters, a derived equivalence can be established, or an analogue of the above on the partial (parabolic) flag variety G/P (see [7]).

The Beilinson–Bernstein equivalence opened up a fruitful interplay between algebraic geometry and the representation theory of algebraic groups and is often viewed as the starting point of modern geometric representation theory. It found immediate application in the proof of the Kazhdan–Lusztig conjectures, but its benefits are much more obvious than that. In more basic terms, note the clear parallel to the study of quasi-coherent $\mathcal{O}_Y$-modules on an affine scheme Y. For this reason, we sometimes say that by the above theorem, the flag variety is $\mathcal{D}$-affine. Following this line of thought, the Beilinson–Bernstein equivalence allows us to study $U(\mathfrak{g})$-modules locally on X much in the same way that an A-module M can be studied locally by considering the corresponding quasi-coherent sheaf on $\operatorname{Spec} A$.

This thesis can be viewed as a contribution towards the development of a theory of analytic differential operators $\widehat{\mathcal{D}}$ on rigid analytic K-spaces, following the definitions of Ardakov–Wadsley in [5]. We will recall the precise definition later, but only note for now that $\widehat{\mathcal{D}}$ comprises differential operators of a more analytic nature, allowing infinite order operators with a suitable convergence condition.
This notion allows us to view $\widehat{\mathcal{D}}$ as a quantization of analytic functions on the cotangent bundle, analogously to the algebraic situation.

One particular feature of the rigid analytic setting is that there is no reasonable notion of a quasi-coherent module. Neither is the category of coherent $\widehat{\mathcal{D}}$-modules well-behaved, as $\widehat{\mathcal{D}}$ itself is not a coherent sheaf of rings.
Instead, we will work in the following framework: we will show (Theorem 5.5, generalizing [5, Theorem 6.4]) that for every smooth affinoid K-space U, $\widehat{\mathcal{D}}(U)$ is a Fréchet–Stein algebra as defined by Schneider and Teitelbaum [44]. Thus $\widehat{\mathcal{D}}(U) = \varprojlim D_n(U)$ is an inverse limit of Noetherian Banach algebras with flat connecting maps, and we consider the abelian category of *coadmissible* modules (obtained from inverse limits of finitely generated $D_n(U)$-modules with suitable localization properties) as a suitable replacement of the category of coherent modules. Indeed, interpreting $\widehat{\mathcal{D}}_X$ as a quantization of analytic functions on the cotangent bundle T^*X, the notion of coadmissibility matches directly the structure of coherent modules on rigid analytic vector bundles.

Accordingly, Ardakov established a (G-equivariant) rigid analytic Beilinson–Bernstein correspondence between coadmissible $\widehat{U(\mathfrak{g})}$-modules with trivial central character and coadmissible

$\widehat{\mathcal{D}}_X$-modules on X, the analytification of the flag variety (see [1, Theorem 6.4.8]).

In this thesis, we will mainly be concerned with the question of $\widehat{\mathcal{D}}$-module pushforwards (direct images and higher direct image functors) along proper morphisms. Note that the correspondence above implies that the pushforward $f_* = \Gamma(X, -)$ for $f : X \to \operatorname{Sp} K$ preserves coadmissibility. We will vastly generalize this statement by giving conditions when a proper morphism $f : X \to Y$ of smooth rigid analytic K-spaces gives rise to a pushforward preserving coadmissibility. Two versions of our results can be phrased as follows.

Theorem 1.2. *Let $f : X \to Y$ be a proper morphism of smooth rigid analytic K-spaces such that Y admits an admissible covering (Y_i) by affinoid subspaces with the property that there exists a Lie algebroid $\mathscr{L}$ such that $\mathscr{L}|_{X_i}$ is free on each $X_i = f^{-1}Y_i$, together with an epimorphism of Lie algebroids $\mathscr{L} \to \mathcal{T}_X$.*
Then $f_*\widehat{\mathcal{D}}_X$ is a sheaf on Y whose sections over admissible open affinoid subspaces are Fréchet–Stein algebras, and if $\mathcal{M}$ is a coadmissible $\widehat{\mathcal{D}}_X$-module, then $\mathrm{R}^j f_*\mathcal{M}$ is a coadmissible $f_*\widehat{\mathcal{D}}_X$-module for each $j \geq 0$.

Corollary 1.3. *Let X be a proper smooth rigid analytic K-space. If $\mathcal{T}_X$ is generated by global sections, then $\widehat{\mathcal{D}}_X(X)$ is a Fréchet–Stein algebra and $\mathrm{R}^j\Gamma(X, \mathcal{M})$ is a coadmissible $\widehat{\mathcal{D}}_X(X)$-module for each $j \geq 0$ and any coadmissible $\widehat{\mathcal{D}}_X$-module $\mathcal{M}$.*

We can view these statements as $\widehat{\mathcal{D}}$-module analogues of Kiehl's Proper Mapping Theorem for coherent $\mathcal{O}$-modules [32], which indeed provides both intuition and practical tools for the proof of the above and will be discussed in great detail in chapter 4.
The conditions formulated in our results can be interpreted as certain properness conditions on the cotangent bundle in a natural way. Following Ardakov–Wadsley, all results are proved in the more general context of Lie algebroids.

In particular, a more general version of our results enables us to treat the twisted sheaves $\widehat{\mathcal{D}}^\lambda$ and their analogues on partial flag varieties in a unified manner, and realize this part of the Beilinson–Bernstein theorem as a special case of a more general geometric phenomenon.

One may note that it is much more common to describe a $\mathcal{D}$-module pushforward sending $\mathcal{D}_X$-modules to $\mathcal{D}_Y$-modules via transfer bimodules, usually in a derived sense. While this theory is not yet entirely developed for $\widehat{\mathcal{D}}$-modules, we expect that one can use our results above to define such a functor for any proper morphism between smooth rigid analytic K-spaces and establish a version of the Proper Mapping Theorem on the derived category of $\widehat{\mathcal{D}}$-modules with values in complete bornological K-spaces of convex type, a category which already appeared in [2] and in a more general form in [8]. We also mention that this would naturally extend the functor from [6] for the case of closed embeddings.

We identify three main ingredients for our proofs, corresponding to the chapters 3, 4 and 5, as explained below. These can be summarized as the study of exactness properties for completed tensor products of p-adic Banach modules (chapter 3), a generalization of Kiehl's arguments in [32] to a wider class of Noetherian Banach K-algebras which we call strictly NB algebras (chapter 4), and a generalization of the structure theory of $\widehat{\mathcal{D}}_X$ in [5] (chapter 5). Many properties of the sheaf $\widehat{\mathcal{D}}_X$ were previously only known under the additional assumption that X admits a smooth Lie lattice, which can be thought of as a smoothness condition on an R-model for X. We remove this assumption and show the following.

(i) For any smooth affinoid K-space X, $\widehat{\mathcal{D}}_X(X)$ is a Fréchet–Stein algebra.

(ii) For any affinoid subdomain U of X, the restriction map $\widehat{\mathcal{D}}_X(X) \to \widehat{\mathcal{D}}_X(U)$ is c-flat.

While this makes little difference in practice (as subdomains admitting smooth Lie lattices form a basis for the weak G-topology), it is reassuring that these basic properties only rest on features of the geometry over K. It might also help in defining analogues of common $\mathcal{D}$-module theoretic functors intrinsically rather than locally.

Structure

We now describe the content of each chapter in turn.

In chapter 2, we summarize all relevant ideas from non-archimedean functional analysis, G-topologies and rigid analytic geometry, including coherent modules and various results on Čech and sheaf cohomology. We also give a detailed treatment of rigid analytic vector bundles and locally free sheaves. Unlike most sources, we construct rigid analytic vector bundles as rigid analytic K-spaces in order to make our heuristics of quantization more concrete.

In chapter 3, we give a number of technical results concerning completed tensor products and strict morphisms of p-adic Banach modules over a Banach algebra A. By analyzing the π-torsion of the tensor product of unit balls (and the corresponding higher Tor groups), we obtain various new conditions under which the functor $B\widehat{\otimes}_A-$ preserves exactness of a given sequence. This includes necessary and sufficient conditions in the case of strict short exact sequences of Banach A-modules.

In chapter 4, we discuss Kiehl's Proper Mapping Theorem [32] for coherent $\mathcal{O}$-modules in rigid

analytic geometry. Since large parts of the proof as given in [32] can be adapted for our purposes, we will carefully verify that these statements remain true for a wider class of algebras than considered in the original paper – instead of only allowing affinoid algebras, we will talk about a class of (possibly non-commutative) Noetherian Banach algebras which we call strictly NB algebras. The proofs in this chapter are almost identical to those in [32], but more general and sometimes cleaner due to our insistence on a discrete valuation.
We note that a similar approach was taken in [31], where the arguments in [32] are generalized in a slightly different direction, with different applications in mind.

In chapter 5, we introduce Fréchet–Stein algebras and coadmissible modules, and show that $\widehat{\mathcal{D}}(X)$ is a Fréchet–Stein algebra for any smooth affinoid X. In fact, if X is a smooth affinoid K-space, we can construct an inverse system of sheaves $\mathcal{D}_n$ with

$$\widehat{\mathcal{D}}_X(U) = \varprojlim \mathcal{D}_n(U)$$

satisfying the conditions in the Fréchet–Stein definition for *all* admissible open affinoids U simultaneously. We summarize this by saying that $\widehat{\mathcal{D}}_X$ is a global Fréchet–Stein sheaf. A more general result will later yield the same for spaces X_i as in Theorem 1.2.
We similarly generalize statements on c-flatness and vanishing higher cohomology and recall basic properties of coadmissible modules.

In chapter 6, we present the proof of a Proper Mapping Theorem for $\widehat{\mathcal{D}}$-modules. The proof comes naturally in two parts. We first apply the arguments in chapter 4 to show that affinoid sections are coadmissible, using chapter 5 to reduce the problem to finitely generated modules over strictly NB algebras. We then use results from chapter 3 to get the required localization properties. We stress in particular that the localization part here is in some sense more elementary than for $\mathcal{O}$-modules (but it relies on Kiehl's Proper Mapping Theorem).
We conclude the chapter by giving a geometric interpretation of our results, viewing $\widehat{\mathcal{D}}$ as the quantization of analytic functions on the cotangent bundle. From this perspective, all conditions which we impose can be seen as requiring the existence of a morphism between certain vector bundles, and insisting that this map be proper. In this sense the conditions intuitively generalize the situation of the flag variety, where the corresponding moment map is known to be proper (restrict [18, Proposition 3.1.34] to the cotangent bundle). We provide a number of naturally occuring examples where our theorems apply, obviously including twisted sheaves $\widehat{\mathcal{D}}^\lambda$ on analytic (partial) flag varieties.

Notation

Throughout, K will be a complete non-archimedean field of mixed characteristic with discrete valuation ring R and uniformizer π.
Given an R-module M, we will write $\widehat{M}$ for the π-adic completion of M, and abbreviate $\widehat{M} \otimes_R K$ to $\widehat{M}_K$.
For a semi-normed K-module M, $\widehat{M}$ will denote the completion of M with respect to this semi-norm, while M° will denote the elements with semi-norm less than or equal to 1.
If $i = (i_1, \ldots, i_m) \in \mathbb{N}^m$ is a multi-index, we write $|i| = i_1 + i_2 + \cdots + i_m$, and abbreviate the expression

$$X_1^{i_1} X_2^{i_2} \ldots X_m^{i_m}$$

to X^i. Similarly, we will sometimes shorten $K[X_1, \ldots, X_m]$ to $K[X]$ when the number of polynomial variables is understood.

Chapter 2

Background material

We begin by recalling some elementary facts concerning the analysis of Banach algebras and Banach modules, Grothendieck topologies and rigid analytic geometry.

2.1 Basic functional analysis

We first collect some standard results about Banach spaces. Suitable references for a more detailed treatment are e.g. [16, 19, 22]. Most references assume vector spaces to be real or complex, but the arguments remain valid over K. The books [13] and [42] describe explicitly the case of a non-archimedean ground field, though often in much greater generality than we need.

A (non-archimedean) **norm** on a K-vector space V is a function $|-| : V \to \mathbb{R}_{\geq 0}$ satisfying the following:

(i) $|x| = 0$ if and only if $x = 0$.

(ii) $|ax| = |a| \cdot |x|$ for any $a \in K$, $x \in V$.

(iii) $|x + y| \leq \max\{|x|, |y|\}$ for any $x, y \in V$.

If only the second and third condition are satisfied, we call $|-|$ a **semi-norm**.

We will call $|V| \setminus \{0\} = (\mathrm{Im}\,|-|) \setminus \{0\}$ the **value set** of $(V,\ |-|)$. For example, the absolute value turns the field K itself into a normed K-vector space, and its value set is discrete by assumption. In fact, $|K^*| = |\pi|^{\mathbb{Z}}$.

If a normed K-vector space V is complete with respect to the (metric) topology induced by $|-|$, we call V a **Banach space**.

Lemma 2.1 (see [42, Proposition 3.1])**.** *A K-linear map between two semi-normed K-vector spaces $\phi : V \to W$ is continuous if and only if it is bounded in the sense that there exists a real number $c \geq 0$ such that $|\phi(v)| \leq c \cdot |v|$ for all $v \in V$.*

In particular, two norms $|{-}|_1$, $|{-}|_2$ on V will be equivalent if and only if the identity map $(V,\ |{-}|_1) \to (V,\ |{-}|_2)$ and its inverse are both bounded, i.e. there exist integers a and b such that

$$\pi^a V_1^\circ \subseteq V_2^\circ \subseteq \pi^b V_1^\circ.$$

We briefly discuss sums, subspaces and quotients.

Lemma 2.2 (see [13, Definition 2.1.5/1, Proposition 2.1.5/6])**.** *If $(V,\ |{-}|_V)$, $(W,\ |{-}|_W)$ are two normed K-vector spaces, then*

$$|v \oplus w| = \max\{|v|_V, |w|_W\} \quad (v \in V,\ w \in W)$$

defines a norm on the direct sum $V \oplus W$. If V and W are Banach, this turns $V \oplus W$ into a Banach space.

If W is a K-vector subspace of a normed K-vector space V, then the restriction of the norm naturally makes W a normed vector space. We call this the **subspace norm** on W.

Lemma 2.3 (see [22, Fact 1.5])**.** *Let W be a subspace of a Banach K-vector space V. Then W equipped with the subspace norm is Banach if and only if W is a closed subspace of V.*

Similarly, the quotient vector space V/W can be equipped with a **quotient** or **residue semi-norm** by setting

$$|x + W|_{V/W} = \inf_{w \in W} |x + w|_V.$$

The quotient semi-norm is a norm if and only if the subspace W is closed in V (see [13, Proposition 1.1.6/1, Proposition 2.1.2/1]).

Lemma 2.4 (see [22, Proposition 1.35])**.** *If V is Banach and W is a closed subspace, then V/W with the quotient norm is a Banach space.*

Given two normed spaces V and W, the space of continuous K-linear maps $\mathrm{Hom}_K(V, W)$ is equipped with the **sup norm**, given by

$$|\phi| = \begin{cases} 0 & \text{if } V = 0 \\ \sup_{x \neq 0} \frac{|\phi(x)|}{|x|} & \text{otherwise} \end{cases}$$

for any $\phi \in \mathrm{Hom}_K(V, W)$. This turns $\mathrm{Hom}_K(V, W)$ into a normed K-vector space. It is Banach if W is Banach (see [42, Proposition 3.3]).

It is worth pointing out at this point that the category of Banach K-spaces with continuous morphisms is not an abelian category (see [30, Example 8.3.7.(iii)]), but only a quasi-abelian category in the sense of Schneiders (see [45]).

The following result, usually called the Open Mapping Theorem, is well-known and will be referred to quite frequently.

Theorem 2.5 (see [22, Theorem 2.25], [42, Proposition 8.6])**.** *A continuous surjection between two Banach K-spaces is open, i.e. maps open sets to open sets.*

Corollary 2.6 (see [42, Corollary 8.7])**.** *Let V be a K-vector space with two Banach norms $|-|_1$, $|-|_2$ such that the identity map $(V, |-|_1) \to (V, |-|_2)$ is continuous. Then $|-|_1$ and $|-|_2$ are equivalent.*

The strong triangle inequality ensures that for any semi-normed K-vector space V, the unit ball V° is an R-submodule. The notions of lattice and associated (Minkowski) gauge semi-norm formalize the relation between semi-norms and their unit balls.

Definition 2.7. *A **lattice** of a K-vector space V is an R-submodule $L \subseteq V$ such that $L \otimes_R K \cong V$. Given a lattice L, we define its **gauge semi-norm** by*

$$\begin{aligned} |-|_L : \quad V &\to \mathbb{R}_{\geq 0} \\ x &\mapsto |x|_L = \inf_{\substack{a \in K \\ x \in aL}} |a|. \end{aligned}$$

It can be checked easily that this defines a semi-norm on V (see [42, p. 8]) and becomes a norm if and only if L is π-adically separated, since the kernel of $|-|_L$ is $\cap \pi^n L$.

Lemma 2.8. *Let $(V, |-|)$ be a normed K-vector space with unit ball V°. Then V° is a lattice of V, and $|-|$ is equivalent to the gauge norm $|-|_{V^\circ}$.*

Proof. The first statement is obvious from the definition, the second is the content of [42, Lemma 2.2.i].

□

Thus we can define a norm up to equivalence just by specifying the unit ball. Moreover, we can replace any norm on V by an equivalent one whose value set is $|K^*|$ (by discreteness of $|K^*|$, the infimum in Definition 2.7 is attained).

We will make repeated use of the following standard result.

Lemma 2.9. *Let V be a normed K-vector space such that the value set $|V \setminus \{0\}|$ is discrete. If W is a vector subspace of V then the unit ball of the quotient V/W under the quotient semi-norm is given by the image of the unit ball V° of V under the natural projection. Moreover, V/W has the same value set as V.*

Proof. By definition of the quotient semi-norm,

$$|x + W|_q = \inf_{y \in x+W} |y|$$

for any $x \in V$. Suppose $|x+W|_q \leq 1$. If $|x+W|_q < 1$, then some element of $x+W$ is contained in V°. If $|x + W|_q = 1$, discreteness of the value set implies that the infimum in the definition above is attained, i.e. there is again some $y \in V^o \cap (x + W)$. □

Note that even if the value set of V is not discrete, we can consider the lattice $p(V^\circ)$ in V/W, where $p : V \to V/W$ is the natural projection. While the associated gauge semi-norm might not be equal to the quotient semi-norm, it is still equivalent, as the unit ball of the quotient semi-norm $(V/W)^\circ$ satisfies $p(V^\circ) \subseteq (V/W)^\circ \subseteq \pi^{-1}p(V^\circ)$.

Given an arbitrary R-module M, we can obviously interpret the image of M under the natural morphism $M \to M \otimes_R K$ as a lattice, i.e. as the unit ball of some semi-norm.

Lemma 2.10. *The kernel of the morphism $M \to M \otimes K$ is the π-torsion submodule of M,*

$$\pi\text{-tor}(M) = \{x \in M : \ \pi^n x = 0 \text{ for some } n \geq 0\}.$$

Proof. Naturally, every π-torsion element is contained in the kernel, as $x \otimes 1 = \pi^n x \otimes \pi^{-n}$.

Consider the short exact sequence of R-modules

$$0 \to R \to K \to K/R \to 0.$$

Tensoring with M yields an exact sequence

$$0 \to \mathrm{Tor}_1^R(M, K/R) \to M \to M \otimes K \to M \otimes_R K/R \to 0.$$

But every element of $\mathrm{Tor}_1^R(M, K/R)$ is π-torsion, e.g. by considering a free resolution of M and noting that every element of K/R is π-torsion. □

Given a normed K-vector space V, we can form its **completion**. This is a Banach K-space $\widehat{V}$ together with a morphism $\iota : V \to \widehat{V}$, satisfying the following universal property. If $\phi : V \to W$ is any morphism from V to a Banach K-space W, then there exists a unique morphism $\phi' : \widehat{V} \to W$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \widehat{V} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi'} \\ & & W \end{array}$$

commute.

Theorem 2.11. *Let V be a K-vector space and let L be a π-adically separated lattice. Writing V_L for V equipped with the gauge norm $|-|_L$, and $\widehat{L}$ for the π-adic completion of L, then $\widehat{V_L}$ is canonically isomorphic to $\widehat{L} \otimes_R K$.*

Proof. Let W be a Banach K-space and $\phi : V \to W$ a continuous linear map. Since ϕ is bounded, this restricts to a morphism of R-modules

$$\phi^\circ : L = V^\circ \to \pi^n W^\circ$$

for some integer n, satisfying $\phi^\circ \otimes_R K = \phi$.
By completeness of W, $\pi^n W^\circ$ is π-adically complete, so ϕ° factors through the π-adic completion $\widehat{L}$. Tensoring with K shows that $\widehat{L} \otimes K$ satisfies the universal property of $\widehat{V_L}$.

□

2.2 Banach algebras and Banach modules

A K-algebra A is called a **normed K-algebra** if it is a normed K-space satisfying the continuity condition on multiplication

$$|a \cdot b| \leq |a| \cdot |b| \quad \forall a, b \in A.$$

In particular, the unit ball A° is an R-subalgebra.
If A is in fact a Banach space, we call it a **K-Banach algebra**. We will mainly be interested in the case of a Noetherian Banach algebra A with the property that A° is also Noetherian.

Lemma 2.12. *Let A be a normed K-algebra such that its unit ball A° is a (left, right, left and right) Noetherian R-algebra. Then A is (left, right, left and right) Noetherian and $\widehat{A}$ is a Noetherian Banach K-algebra.*

Proof. If I is a (left, right, two-sided) ideal of A, then $I \cap A^\circ$ is an ideal of A° with the property that

$$(I \cap A^\circ) \otimes_R K = I.$$

Thus I is finitely generated by Noetherianity of A°.
For the second statement, combine the above with [11, 3.2.3.(vi)] and Theorem 2.11. □

For the remainder of this section, 'module' will always mean 'left module', 'Noetherian' will mean 'left Noetherian', etc. Corresponding results for right modules hold mutatis mutandis.

Let A be a normed K-algebra.
An A-module M will be called a **semi-normed A-module** if it is a semi-normed K-space such that

$$|a \cdot m| \leq |a| \cdot |m| \quad \forall a \in A,\ m \in M.$$

In particular, the unit ball M° of a semi-normed A-module M is an A°-module.
If M is in fact Banach, we call it a **Banach module**.

Note that if $(A, |-|)$ is a normed K-algebra, then the gauge semi-norm associated to A° turns A into a normed K-algebra with value set $|K^*|$, which is equivalent to $(A, |-|)$ by Lemma 2.8. Given a normed A-module M, we can define an A°-lattice to be an A°-submodule L spanning M as a K-vector space, and get the corresponding notion of a gauge semi-norm

$$|m|_L = \inf_{\substack{a \in A \\ m \in aL}} |a|.$$

All statements concerning lattices and gauge semi-norms in the previous section thus find natural analogues in the category of (semi-)normed A-modules.

Now let A be a Noetherian Banach K-algebra.

Proposition 2.13 (see [13, Proposition 3.7.2/2])**.** *If M is a finitely generated Banach A-module, then every A-submodule of M is closed. In particular, any ideal of A is closed.*

Lemma 2.14. *Any finitely generated A-module M can be equipped with a complete A-module norm, and all such norms are equivalent.*
Moreover, if A° is Noetherian, then the unit ball M° is a finitely generated A°-module for any such norm on M.

Proof. The first part of the statement is a straightforward consequence of the Open Mapping Theorem and can be found e.g. in [13, Proposition 3.7.3/3].

As A° is assumed to be Noetherian, the property of having a finitely generated unit ball is preserved under replacing a norm by an equivalent one. Now a surjection $p : A^r \to M$ for some integer r provides M with a complete quotient norm by Lemma 2.4 and Proposition 2.13. Its unit ball is finitely generated (even if the value set of A is not discrete, the unit ball of the quotient norm is certainly contained in the finitely generated A°-module $\pi^{-1}p((A^\circ)^r)$). By the first part of the lemma, this proves the result. □

2.3 Affinoid algebras and affinoid spaces

For $m \in \mathbb{N}$ we define the mth **Tate algebra** over K by

$$T_m = K\langle X_1, X_2, \ldots, X_m\rangle = \left\{ \sum_{i\in\mathbb{N}^m} a_i X^i : a_i \in K,\ |a_i| \to 0 \text{ as } |i| \to \infty \right\},$$

where $X^i = X_1^{i_1} X_2^{i_2} \ldots X_m^{i_m}$ and $|i| = \sum_j i_j$ for any m-tuple $i = (i_1, \ldots, i_m)$, as mentioned in the introduction.
Defining addition and multiplication as for formal power series turns T_m into a commutative Noetherian Banach algebra with respect to the norm

$$|| \sum a_i X^i || = \sup_i |a_i|.$$

We think of T_m as the algebra of convergent power series on the unit ball $\mathbb{B}^m(\overline{K})$.

Definition 2.15. *A K-algebra A is called an **affinoid K-algebra** if it is isomorphic to a quotient of some Tate algebra T_m.*

To any affinoid algebra A we can associate an **affinoid space** $\mathrm{Sp}\, A$, whose points are the maximal ideals of A. As for affine schemes, morphisms of affinoid K-spaces $f : \mathrm{Sp}\, A \to \mathrm{Sp}\, B$ are those maps which are induced by algebra morphisms $f^\# : B \to A$. We will discuss the topology of $\mathrm{Sp}\, A$ later in this chapter.
Given $x \in \mathrm{Sp}\, A$, we denote the corresponding maximal ideal in A by $\mathfrak{m}_x$.
A surjection $T_m \to A$ equips A with a quotient norm by Lemma 2.4 and Proposition 2.13, called the residue norm. If A is reduced, this is equivalent to the sup-norm

$$|f| = \sup_{x\in \mathrm{Sp}\, A} |f(x)|,$$

where $|f(x)|$ is the absolute value of f in the finite field extension $A/\mathfrak{m}_x$ of K. In this sense, we can think of elements of A as functions on $\mathrm{Sp}\, A$, equipped with the sup norm. Using [12, Theorem 3.1/15], we can replace sup by max in the expression above.

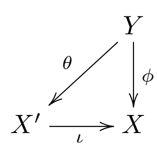
We note that any morphism between affinoid algebras equipped with some residue norm is continuous. In particular, all residue norms on an affinoid algebra A are equivalent (see [12, Proposition 3.1/20]).

We also point out that, since K is assumed to be discretely valued, any residue norm on an affinoid algebra A satisfies the conditions of Lemma 2.9. Thus, the unit ball A° is a quotient of $R\langle X_1, \ldots, X_m\rangle$ for some m, and is therefore a Noetherian R-algebra (note that $R\langle X_1, \ldots, X_m\rangle$ is Noetherian by [12, Remark 7.3/1]).

We call an R-subalgebra $\mathcal{A}$ of an affinoid K-algebra A an **affine formal model** if $\mathcal{A}$ is of topologically finite type over R and $\mathcal{A} \otimes_R K = A$. A surjection $R\langle X_1, \ldots, X_m\rangle \to \mathcal{A}$ then gives rise to a residue norm on A with unit ball $\mathcal{A}$. Therefore an R-subalgebra of A is an affine formal model if and only if it is the unit ball of some residue norm on A.

We now discuss various notions of distinguished subsets of affinoid spaces.

Definition 2.16 (see [12, Definition 3.3/9])**.** *A subset $U \subseteq X$ of an affinoid K-space is called an* ***affinoid subdomain*** *if there exists a morphism of affinoid K-spaces $\iota : X' \to X$ such that $\iota(X') \subseteq U$ and the following universal property holds:*
Any morphism $\phi : Y \to X$ of affinoid K-spaces satisfying $\phi(Y) \subseteq U$ admits a unique factorization through ι, i.e. there exists a unique $\theta : Y \to X'$ such that the diagram



commutes.

One can show that in this situation, X' can in fact be identified with U as a subset of X (see [12, Lemma 3.3/10]).

We mention briefly some special types of affinoid subdomain (see [12, Definition 3.3/7]). Let $X = \operatorname{Sp} A$ be an affinoid K-space, and let $f_1, \ldots, f_r, g_1, \ldots, g_s \in A$. Define

$$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = \\ \{x \in X : |f_i(x)| \leq 1, |g_j(x)| \geq 1,\ i = 1, \ldots, r,\ j = 1, \ldots, s\}.$$

This is a an affinoid subdomain of X, and we have an isomorphism of affinoid K-spaces

$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) \cong \operatorname{Sp} B$ for

$$B = A\langle X_1, \ldots X_r, Y_1, \ldots, Y_s\rangle/(X_i - f_i,\ g_j Y_j - 1,\ i = 1, \ldots, r,\ j = 1, \ldots, s).$$

We call any affinoid subdomain of this form a **Laurent subdomain** (the special case when all exponents are positive is usually called a Weierstrass domain).

If $f_0, f_1, \ldots, f_r \in A$ have no common zeros (i.e. generate the unit ideal), we can also define the **rational subdomain**

$$X\left(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}\right) = \{x \in X : |f_i(x)| \leq |f_0(x)|\ \forall i\}$$
$$\cong \operatorname{Sp}(A\langle X_1, \ldots, X_r\rangle/(f_0 X_i - f_i, i = 1, \ldots, r)).$$

We give without proof two key results, which reduce many questions about affinoid subdomains to ones about rational and Laurent subdomains.

Theorem 2.17 (see [12, Theorem 3.3/20]). *Let X be an affinoid K-space and $U \subseteq X$ an affinoid subdomain. Then U is a finite union of rational subdomains of X.*

Proposition 2.18 (see [12, Proposition 3.3/16]). *Let X be an affinoid K-space and $U \subseteq X$ a rational subdomain. Then there exists a Laurent subdomain $U' \subseteq X$ such that U is contained in U' as a Weierstrass domain.*

2.4 Grothendieck topologies and rigid analytic spaces

As the Zariski topology on affinoid spaces is too coarse for our purposes, we introduce a Grothendieck topology or G-topology, i.e. a categorical generalization of a topology.

Definition 2.19 (see [12, Definition 5.1/1]). *A **Grothendieck topology** $\mathscr{T}$ consists of a category $\operatorname{Cat}\mathscr{T}$ and a set $\operatorname{Cov}\mathscr{T}$ of families $(U_i \to U)_{i \in I}$ of morphisms in $\operatorname{Cat}\mathscr{T}$, called coverings, such that the following hold:*

(i) If $\phi : U \to V$ is an isomorphism in $\operatorname{Cat}\mathscr{T}$, then $(\phi) \in \operatorname{Cov}\mathscr{T}$.

(ii) If $(U_i \to U)_{i \in I}$ and $(V_{ij} \to U_i)_{j \in J}$ for $i \in I$ belong to $\operatorname{Cov}\mathscr{T}$, then the same is true for the composition $(V_{ij} \to U_i \to U)_{i \in I, j \in J}$.

(iii) If $(U_i \to U)_{i \in I}$ is in $\operatorname{Cov}\mathscr{T}$ and if $V \to U$ is a morphism in $\operatorname{Cat}\mathscr{T}$, then the fibre products $U_i \times_U V$ exist in $\operatorname{Cat}\mathscr{T}$, and $(U_i \times_U V \to V)_{i \in I}$ belongs to $\operatorname{Cov}\mathscr{T}$.

A category $\mathcal{X}$ endowed with a Grothendieck topology (i.e. $\mathcal{X} = \mathrm{Cat}\mathscr{T}$) is called a **site**.

Given a set X, a Grothendieck topology on X will be a Grothendieck topology $\mathscr{T}$ such that

(i) $\mathrm{Cat}\mathscr{T}$ has as objects some collection of subsets of X and as morphisms the corresponding inclusion morphisms;

(ii) $\mathrm{Cov}\mathscr{T}$ consists of families $(U_i \to U)_{i\in I}$ where each $(U_i)_{i\in I}$ is a set-theoretic covering of U.

We then call $(X, \mathscr{T})$ a **G-topological space**. We will sometimes equate $(X, \mathscr{T})$ with the site $\mathrm{Cat}\mathscr{T}$ when no confusion is possible.
For a G-topological space $(X, \mathscr{T})$ and $U \in \mathrm{Cat}\mathscr{T}$, we will from now on reserve the terminology 'covering of U' (or '$\mathscr{T}$-covering') for a set $(U_i)_{i\in I}$ such that $(U_i \to U)_{i\in I} \in \mathrm{Cov}\mathscr{T}$, and use 'set-theoretic covering' if we merely want to indicate that $\cup U_i = U$.

As a first example, consider the **weak** G-topology $\mathscr{T}_w$ on an affinoid K-space X, which is defined by taking

(i) $\mathrm{Cat}\mathscr{T}_w$ to consist of the affinoid subdomains of X, and

(ii) $\mathrm{Cov}\mathscr{T}_w$ to consist of all *finite* set-theoretic coverings $(U_i \to U)_{i\in I}$ with $U, U_i \in \mathrm{Cat}\mathscr{T}_w$ for all $i \in I$.

We often write X_w for the site of $(X, \mathscr{T}_w)$.

Given a site $\mathcal{X}$, we can introduce the notion of a sheaf on $\mathcal{X}$.

Definition 2.20 (see [12, Definition 5.1/2])**.** *Let $\mathcal{C}$ be a category admitting fibre products. A **presheaf** with values in $\mathcal{C}$ on a site $\mathcal{X}$ is a contravariant functor $\mathscr{F} : \mathcal{X} \to \mathcal{C}$.*
*We call $\mathscr{F}$ a **sheaf** if the diagram*

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_i \times_U U_j)$$

is exact for any covering $(U_i \to U)_{i\in I}$ of $U \in \mathcal{X}$.

If $\mathcal{C}$ is the category of abelian groups, rings, modules, etc., we also call $\mathscr{F}$ a sheaf of abelian groups, rings, modules, etc.

Given a set X and two set-theoretic coverings $\mathfrak{U} = (U_i)_{i\in I}$, $\mathfrak{V} = (V_j)_{j\in J}$ of X, recall that $\mathfrak{V}$ is called a **(set-theoretic) refinement** of $\mathfrak{U}$ if there exists a function $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for every $j \in J$. In this case we say that $\mathfrak{U}$ admits $\mathfrak{V}$ as a refinement.

We now define the strong Grothendieck topology on an affinoid K-space.

Definition 2.21 (see [12, Definition 5.1/4]). *Let X be an affinoid K-space.*

(i) A subset $U \subseteq X$ is called ***admissible open*** *if it can be written as a set-theoretic union of affinoid subdomains $U = \cup U_i$ such that for all morphisms of affinoid K-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subseteq U$, the set-theoretic covering $(\phi^{-1}(U_i))_i$ of Z admits a set-theoretic refinement that is a covering in Z_w.*
Write $X_{\text{rig}} = \text{Cat}\mathscr{T}_{\text{rig}}$ for the category of admissible open subsets of X with inclusion morphisms.

(ii) Suppose V, V_j are admissible open subsets of X for $j \in J$, and $V = \cup V_j$. The family $(V_j \to V)_{j \in J}$ of inclusions is called an ***admissible covering*** *if for each morphism of affinoid K-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subseteq V$, the set-theoretic covering $(\phi^{-1}(V_j))_j$ of Z admits a set-theoretic refinement which is a covering in Z_w.*
Write $\text{Cov}\mathscr{T}_{\text{rig}}$ for the set of admissible coverings.

Then $\mathscr{T}_{\text{rig}}$ is a G-topology on X, called the ***strong Grothendieck topology****.*

We will use the strong Grothendieck topology for most parts of this thesis, but it will sometimes be convenient to define a sheaf on X_w and then refer to [12, Corollary 5.2/5] to extend it uniquely to a sheaf on X_{rig} – we will do this repeatedly without explicitly saying so.

For instance, we can define a presheaf $\mathcal{O}_X$ on X_w by assigning to each affinoid subdomain $U = \text{Sp}\, B$ the corresponding affinoid algebra B. By Tate's Theorem (see [12, Theorem 4.3/1]), this is a sheaf on X_w and thus extends to a sheaf on X_{rig}, called the **structure sheaf** of X (which we still denote by $\mathcal{O}_X$).

To summarize, we can now associate to an affinoid K-algebra A the affinoid K-space $X = \text{Sp}\, A$, equipped with the strong G-topology $\mathscr{T}_{\text{rig}}$, and the structure sheaf $\mathcal{O}_X$ on $X_{\text{rig}} = (X, \mathscr{T}_{\text{rig}})$.
Then by the above and [12, Proposition 4.1/1], the pair $(X_{\text{rig}}, \mathcal{O}_X)$ is a **locally G-ringed K-space** in the sense of [12, Definition 5.3/1].
We now define general rigid analytic K-spaces to be locally G-ringed K-spaces which are locally of this form.

Definition 2.22 (see [12, Definition 5.3/4]). *A* ***rigid analytic K-space*** *is a locally G-ringed K-space $((X, \mathscr{T}), \mathcal{O}_X)$ such that the following hold:*

(i) $\emptyset$ and X are objects in $\text{Cat}\mathscr{T}$.

(ii) Let $(U_i \to U)_{i \in I}$ be a $\mathscr{T}$-covering, and V a subset of U such that $V \cap U_i$ is an object of $\text{Cat}\mathscr{T}$ for all $i \in I$. Then V is an object of $\text{Cat}\mathscr{T}$.

(iii) Suppose U and U_i for all $i \in I$ are objects in $\mathrm{Cat}\mathscr{T}$ *such that $(U_i)_{i\in I}$ is a set-theoretic covering of U. If $(U_i)_{i\in I}$ admits a set-theoretic refinement which is a $\mathscr{T}$-covering of U, then $(U_i \to U)_{i\in I}$ is a $\mathscr{T}$-covering.*

(iv) X admits a $\mathscr{T}$-covering $(X_i \to X)_{i\in I}$ such that for each $i \in I$, $(X_i, \mathcal{O}_X|_{X_i})$ is isomorphic as a locally G-ringed K-space to $((\mathrm{Sp}\, A_i)_{\mathrm{rig}}, \mathcal{O}_{\mathrm{Sp}\, A_i})$ for some affinoid K-algebra A_i.

A morphism of rigid K-spaces is a morphism in the sense of locally G-ringed K-spaces.

If $((X, \mathscr{T}), \mathcal{O}_X)$ is a rigid analytic K-space, we will call elements of $\mathrm{Cat}\mathscr{T}$ the **admissible open** subsets of X and elements of $\mathrm{Cov}\mathscr{T}$ the **admissible coverings**, just as in the case of affinoid K-spaces equipped with the strong G-topology.

2.5 Coherent modules

Given an affinoid K-space $X = \mathrm{Sp}\, A$ and a finitely generated A-module M, we can construct the associated $\mathcal{O}_X$-module $\tilde{M}$ by setting

$$\tilde{M}(U) = \mathcal{O}_X(U) \otimes_A M$$

for any affinoid subdomain $U \subseteq X$, giving a sheaf on X_w by [12, Corollary 4.3/11], and then extending to the strong G-topology.

For a general rigid analytic K-space X, we say an $\mathcal{O}_X$-module $\mathcal{M}$ is **coherent** if there is some admissible covering by affinoid spaces $X_i = \mathrm{Sp}\, A_i$ such that for any i,

$$\mathcal{M}|_{X_i} \cong \tilde{M}_i$$

for some finitely generated A_i-module M_i (see [12, Remark 6.1/3]).
Kiehl showed that if one admissible affinoid covering realizes $\mathcal{M}$ as a coherent module, then so does any admissible affinoid covering:

Theorem 2.23 (see [12, Theorem 6.1/4])**.** *Let $X = \mathrm{Sp}\, A$ be an affinoid K-space, and $\mathcal{M}$ an $\mathcal{O}_X$-module. Then $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module if and only if it is associated to a finite A-module.*

We denote the category of coherent $\mathcal{O}_X$-modules on a rigid analytic K-space X by $\mathrm{Coh}(X)$.

If $\mathcal{M}$, $\mathcal{M}'$ are two $\mathcal{O}_X$-modules on some rigid analytic K-space X, we can form the sheaf

$\mathcal{H}om(\mathcal{M}, \mathcal{M}')$ by writing

$$\mathcal{H}om(\mathcal{M}, \mathcal{M}')(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}|_U, \mathcal{M}'|_U)$$

for any admissible open affinoid subspace U.
In the case where X is affinoid and $\mathcal{M}$, $\mathcal{M}'$ are coherent, $\mathcal{H}om(\mathcal{M}, \mathcal{M}')$ is the $\mathcal{O}$-module associated to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$. This implies that for arbitrary X, $\mathcal{H}om(\mathcal{M}, \mathcal{M}')$ is coherent whenever $\mathcal{M}$ and $\mathcal{M}'$ are.

2.6 Rigid analytification

We note that there is a functor $(-)^{\mathrm{an}}$ from the category of K-schemes of locally finite type to rigid analytic K-spaces (see [12, 5.4], where the same example as below is discussed). We call $\mathbf{X}^{\mathrm{an}}$ the **rigid analytification** of a K-scheme $\mathbf{X}$. Since similar constructions will be used repeatedly in what is to follow, we describe this functor explicitly in the case $\mathbf{X} = \mathbb{A}_K^m$.
As a set, $(\mathbb{A}_K^m)^{\mathrm{an}}$ consists of the closed points of $\mathbb{A}_K^m$, and as locally G-ringed spaces

$$(\mathbb{A}_K^m)^{\mathrm{an}} = \bigcup_{n=0}^{\infty} \mathrm{Sp}\, K\langle \pi^n X_1, \ldots, \pi^n X_m \rangle.$$

By this we mean the following. The natural inclusions

$$\begin{aligned} K\langle \pi^n X\rangle &\to K\langle \pi^{n-1} X\rangle \\ X_i &\mapsto X_i \end{aligned}$$

give rise to an inverse system of affinoid algebras, and thus dually to an inductive system of affinoid K-spaces $U_n := \mathrm{Sp}\, K\langle \pi^n X\rangle$. Recall that $U_0 = \mathrm{Sp}\, T_m$ can be interpreted as the unit ball $\mathbb{B}^m(\overline{K})$ and hence agrees set-theoretically with the closed points in the unit ball of $\mathbb{A}_K^m$. Thus, by rescaling the variables, U_n admits a natural interpretation as the ball of radius $|\pi|^{-n}$, and the connecting morphisms $U_{n-1} \to U_n$ correspond to the natural embeddings. Taking the direct limit in the category of rigid analytic K-spaces, we thus obtain the rigid analytic K-space $(\mathbb{A}_K^m)^{\mathrm{an}}$ whose underlying set consists of the closed points of $\mathbb{A}_K^m$.

In particular, global sections of the structure sheaf are given by

$$\begin{aligned} \Gamma((\mathbb{A}_K^m)^{\mathrm{an}}, \mathcal{O}) &= \varprojlim K\langle \pi^n X_1, \ldots, \pi^n X_m\rangle \\ &= \left\{ \sum a_i X^i :\ a_i \in K,\ |a_i||\pi|^{-|i|n} \to 0 \text{ as } |i| \to \infty\ \forall n \geq 0 \right\}. \end{aligned}$$

In other words, the (globally) analytic functions defined on the whole of m-space are obtained by considering analytic functions on balls of larger and larger radius.

It is instructive to rewrite the above expression slightly. Write

$$R_n := R[\pi^n X] \subset K[X]$$

for the R-subalgebra generated by $\pi^n X_1, \ldots, \pi^n X_m$.
Forming the π-adic completion of R_n and inverting π, it is straightforward to prove (see Theorem 2.11) that

$$K\langle \pi^n X\rangle = \widehat{R_n} \otimes_R K,$$

so we can write:

$$\Gamma((\mathbb{A}_K^m)^{\text{an}}, \mathcal{O}) = \varprojlim(\widehat{R[\pi^n X]} \otimes_R K).$$

We will return to expressions of this form in chapter 5.

Write $U_n = \operatorname{Sp} K\langle \pi^n X\rangle$ and $V = (\mathbb{A}_K^m)^{\text{an}}$. What do coherent modules on V look like? If $\mathcal{M}$ is a coherent $\mathcal{O}_V$-module, Theorem 2.23 tells us that $\mathcal{M}|_{U_n}$ is associated to some finitely generated $K\langle \pi^n X\rangle$-module M_n. Thus we obtain an inverse system of modules M_n satisfying the following for each $n \geq 0$:

(i) M_n is a finitely generated $K\langle \pi^n X\rangle$-module.

(ii) The natural restriction morphism $M_{n+1} \to M_n$ induces an isomorphism

$$\mathcal{O}_V(U_n) \otimes_{\mathcal{O}_V(U_{n+1})} M_{n+1} \cong M_n.$$

The global sections $\mathcal{M}(V)$ will then be obtained as the inverse limit $\varprojlim M_n$. These structures will provide some intuition for the notions of Fréchet–Stein algebras and coadmissible modules in chapter 5.

Note that $(\mathbb{A}_K^m)^{\text{an}}$ is not an affinoid space (as the global sections of the structure sheaf are not Noetherian), so the analytification of an affine scheme is not necessarily affinoid.

We fix again $V = (\mathbb{A}_K^m)^{\text{an}}$, $U_n = \operatorname{Sp} K\langle \pi^n x\rangle$.

Proposition 2.24. *Let $X = \operatorname{Sp} A$ be an affinoid K-space. Then the set of morphisms $X \to V$ is in natural bijection with the set of K-algebra morphisms $K[x_1, \ldots, x_m] \to A$.*

Proof. This is [12, Lemma 5.4/2] together with [12, Definition 5.4/3]. □

Corollary 2.25. *Let $X = \operatorname{Sp} A$ be an affinoid K-space and write V_A for the direct product space $X \times_{\operatorname{Sp} K} V$. Then the set of morphisms $V_A \to V_A$ is in natural bijection with the set of K-algebra morphisms*

$$A[x_1, \dots, x_m] \to \varprojlim A\langle \pi^n x_1, \dots, \pi^n x_m \rangle.$$

Proof. By definition, a morphism $s : V_A \to V_A$ is given by an inductive system of morphisms $s_n : X \times U_n \to V_A$. Since $X \times U_n = \operatorname{Sp} A\langle \pi^n x\rangle$ is affinoid, this corresponds to a projective system of morphisms $A[x] \to A\langle \pi^n x\rangle$ by the previous proposition, and hence to a morphism $A[x] \to \varprojlim A\langle \pi^n x\rangle$. □

2.7 Cohomology

Let $\mathcal{X}$ be a site and $U \in \mathcal{X}$. For each covering $\mathfrak{U} = (U_i \to U)_{i \in I}$, we define the nth Čech cohomology of a sheaf $\mathcal{F}$ on $\mathcal{X}$ with respect to this covering, denoted $\check{\mathrm{H}}^n(\mathfrak{U}, \mathcal{F})$, as the cohomology of the usual Čech complex. Further we set

$$\check{\mathrm{H}}^n(U, \mathcal{F}) = \varinjlim \check{\mathrm{H}}^n(\mathfrak{U}, \mathcal{F}),$$

where the limit is taken over all coverings of U.
We also define the sheaf cohomology functor $\mathrm{H}^n(U, -)$ as the nth derived functor of the global sections functor $\Gamma(U, -)$.
The following theorems show how to relate these two cohomology theories.

Theorem 2.26 (see [46, Tag 03F9])**.** *Let $\mathcal{X}$ be a site, $\mathcal{F}$ a sheaf on $\mathcal{X}$. Let $\mathcal{S}$ be a collection of elements of $\mathcal{X}$ such that the following is satisfied:*

(i) $\mathcal{S}$ is closed under taking finite intersections: if U_1, U_2 are elements of $\mathcal{S}$, then so is $U_1 \times U_2$.

(ii) Each covering of an element of $\mathcal{X}$ admits a refinement by elements of $\mathcal{S}$.

(iii) $\check{\mathrm{H}}^n(U, \mathcal{F}) = 0$ for any $n > 0$, $U \in \mathcal{S}$.

Then for any $U \in \mathcal{X}$ the canonical homomorphism

$$\check{\mathrm{H}}^n(U, \mathcal{F}) \to \mathrm{H}^n(U, \mathcal{F})$$

is an isomorphism for $n \geq 0$.

Taking X an affinoid K-space, $\mathcal{S}$ the collection of all affinoid subdomains, we can now invoke [12, Theorem 4.3/10, Corollary 4.3/11] to deduce that

$$\check{\mathrm{H}}^n(X, \mathcal{M}) = \mathrm{H}^n(X, \mathcal{M}) = 0 \text{ for } n > 0$$

for any coherent $\mathcal{O}_X$-module $\mathcal{M}$ (see [12, Corollary 6.2/7]).

Theorem 2.27 (see [46, Tag 03F7])**.** *Let $\mathcal{X}$ be a site and $\mathfrak{U} = (U_i \to U)_{i\in I}$ be a covering. Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$ such that $\mathrm{H}^n(V, \mathcal{F}) = 0$ for any $n > 0$ and V any finite intersection of U_is. Then*

$$\check{\mathrm{H}}^n(\mathfrak{U}, \mathcal{F}) \cong \mathrm{H}^n(U, \mathcal{F})$$

for any $n \geq 0$.

Let X be a separated rigid K-space (i.e., the diagonal is a closed embedding, [12, Definition 6.3/2]) and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then [12, Proposition 6.3/4] states that the intersection of two affinoid subspaces in X is again affinoid. Letting $\mathfrak{U}$ be an admissible covering of X by affinoid spaces, we have seen above that $\mathrm{H}^n(V, \mathcal{M}) = 0$ for any $n > 0$ and V a finite intersection of elements of $\mathfrak{U}$. Thus we can apply Theorem 2.27 to get that sheaf cohomology and Čech cohomology (with respect to $\mathfrak{U}$) coincide for coherent $\mathcal{O}_X$-modules.

2.8 Locally free coherent sheaves

Let X be a rigid analytic K-space.

Definition 2.28. *A coherent sheaf $\mathcal{M}$ on X is called **locally free** if there exists an admissible covering $(X_i)_{i\in I}$ of X such that $\mathcal{M}|_{X_i}$ is free for each i, i.e. is isomorphic to a direct sum of finitely many copies of $\mathcal{O}_{X_i}$.*

Most authors will use 'rigid analytic vector bundle' as synonymous terminology (e.g. [23], or [25] for line bundles). We however will introduce a separate notion such that a rigid analytic vector bundle on X is a rigid analytic K-space E in its own right (together with a morphism $E \to X$), rather than a sheaf on X. We will then establish the correspondence between vector bundles in our sense and locally free coherent sheaves analogously to [26, Exercise 5.18].

First, we give an alternative description of locally free coherent modules which will be used repeatedly in chapter 5.

Proposition 2.29. *Let $\mathcal{M}$ be a coherent sheaf on an affinoid space $X = \mathrm{Sp}\, A$. Then $\mathcal{M}$ is locally free if and only if $M = \mathcal{M}(X)$ is a finitely generated projective A-module.*

Proof. By Theorem 2.23, $\mathcal{M}$ is the associated $\mathcal{O}_X$-module to the finitely generated A-module $M = \mathcal{M}(X)$.
Suppose that M is projective. Then by [15, Theorem II.5.2/1], there exist $f_1, \ldots, f_n \in A$ generating the unit ideal such that the localization $M[f_i^{-1}] = A[f_i^{-1}] \otimes_A M$ is free over $A[f_i^{-1}]$ for each $i = 1, \ldots, n$.

Now the affinoid subdomains

$$X_i = X\left(\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}\right),\ i = 1, \ldots, n$$

form an admissible covering of X (called a rational covering in [12, p. 84]), and since $f_1, \ldots, f_n$ generate the unit ideal, we know that the natural restriction map $A \to \mathcal{O}_X(X_i)$ factors through $A[f_i^{-1}]$. Thus

$$\mathcal{M}(X_i) = \mathcal{O}_X(X_i) \otimes_{A[f_i^{-1}]} A[f_i^{-1}] \otimes_A M$$

is free by the above, so $\mathcal{M}|_{X_i}$ is free for each i. Thus $\mathcal{M}$ is locally free, as required.

Conversely, if $\mathcal{M}$ is locally free, consider the stalks $\mathcal{M}_x = \mathcal{O}_{X,x} \otimes_A M$ for $x \in X$ (see [12, 4.1]), which are free $\mathcal{O}_{X,x}$-modules by assumption.
By [12, Proposition 4.1/2], the stalk

$$\mathcal{O}_{X,x} = \varinjlim_{\substack{U \in X_w \\ x \in U}} \mathcal{O}_X(U)$$

is a local ring with maximal ideal $\mathfrak{m}_x\mathcal{O}_{X,x}$, and its completion $\widehat{\mathcal{O}_{X,x}}$ is isomorphic to the $\mathfrak{m}_x$-adic completion of A, $\widehat{A_{\mathfrak{m}_x}}$.
Thus,

$$\widehat{\mathcal{O}_{X,x}} \otimes_A M \cong \widehat{A_{\mathfrak{m}_x}} \otimes_A M$$

is a free $\widehat{A_{\mathfrak{m}_x}}$-module, and since $\widehat{A_{\mathfrak{m}_x}}$ is faithfully flat over the localization $A_{\mathfrak{m}_x}$ (by [15, Proposition III.3.5/9]), it follows that $M_{\mathfrak{m}_x}$ is a free $A_{\mathfrak{m}_x}$-module for each $x \in X$ (see [15, Corollary III.3.5/2]). Thus M is projective by [15, Theorem II.5.2/1]. □

Denote the full subcategory of $\mathrm{Coh}(X)$ consisting of locally free coherent sheaves of rank m by $\mathrm{LF}_m(X)$. Note that the functor

$$\mathrm{Coh}(X) \to \mathrm{Coh}(X)$$
$$\mathcal{E} \mapsto \mathcal{E}^\vee := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$$

preserves $\mathrm{LF}_m(X)$ for every $m \geq 0$, since $\mathcal{H}om(\mathcal{O}_X^m, \mathcal{O}_X) \cong \mathcal{O}_X^m$.

Definition 2.30. *A* ***rigid analytic vector bundle*** *of rank m on a rigid analytic K-space X is a pair (E, ρ) where E is a rigid analytic K-space and $\rho : E \to X$ is a morphism, satisfying the following:*

(i) There exists an admissible covering $(U_i)_{i \in I}$ *of* X *and isomorphisms* $\alpha_i : E \times_X U_i \to (\mathbb{A}_K^m)^{\mathrm{an}} \times U_i$ *for each* $i \in I$ *such that there is a commutative diagram*

$$\begin{array}{ccc} E \times_X U_i & \xrightarrow{\rho \times_X U_i} & U_i \\ \downarrow{\scriptstyle \alpha_i} & \nearrow{\scriptstyle p_2} & \\ (\mathbb{A}_K^m)^{\mathrm{an}} \times U_i & & \end{array}$$

where p_2 *is the natural projection on the second factor.*

(ii) For any admissible open affinoid $V = \operatorname{Sp} A$ *contained in* $U_{ij} = U_i \cap U_j$*, the isomorphism*

$$\alpha_j \circ \alpha_i^{-1} : (\mathbb{A}_K^m)^{\mathrm{an}} \times V \to E \times_X V \to (\mathbb{A}_K^m)^{\mathrm{an}} \times V$$

corresponds under Corollary 2.25 to a linear *morphism of* A*-algebras*

$$\theta : A[x_1, \ldots, x_m] \to \varprojlim A\langle \pi^n x_1, \ldots, \pi^n x_m \rangle$$

in the sense that there exist $a_{uv} \in A$ *for* $u, v \in \{1, \ldots, m\}$ *such that* $\theta(x_v) = \sum_u a_{uv} x_u$ *for each* v*.*

A morphism of vector bundles $\phi : (E_1, \rho_1) \to (E_2, \rho_2)$ *is a morphism of rigid analytic spaces* $\phi : E_1 \to E_2$ *such that*

$$\begin{array}{ccccc} E_1 & & \xrightarrow{\phi} & & E_2 \\ & {\scriptstyle \rho_1}\searrow & & \swarrow{\scriptstyle \rho_2} & \\ & & X & & \end{array}$$

commutes, and ϕ *is locally linear in the sense that there exists an admissible affinoid covering* (U_i)*,* $U_i = \operatorname{Sp} A_i$*, such that both* $\mathcal{E}|_{U_i}$ *and* $\mathcal{E}'|_{U_i}$ *are trivial for each* i *with trivializing morphisms* α_i *and* β_i*, respectively, with the diagram*

$$\begin{array}{ccc} E_1 \times_X U_i & \xrightarrow{\alpha_i} & (\mathbb{A}_K^m)^{\mathrm{an}} \times U_i \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ E_2 \times_X U_i & \xrightarrow{\beta_i} & (\mathbb{A}_K^m)^{\mathrm{an}} \times U_i \end{array}$$

inducing a morphism $\psi = \beta_i \circ \phi \circ \alpha_i^{-1} : (\mathbb{A}_K^m)^{\mathrm{an}} \times U_i \to (\mathbb{A}_K^m)^{\mathrm{an}} \times U_i$ *corresponding to a linear morphism of* A*-algebras* $A[x] \to \varprojlim A\langle \pi^n x \rangle$ *under Corollary 2.25.*

We denote the category of rigid analytic vector bundles of rank m on X by $\mathcal{VB}_m(X)$. We will sometimes refer to a vector bundle (E, ρ) simply by E.

We will now describe an equivalence of categories between $\mathrm{LF}_m(X)$ and $\mathcal{VB}_m(X)$. In order to do this, we will first associate a vector bundle $\mathcal{V}(\mathcal{E})$ to a *free* $\mathcal{O}_U$-module $\mathcal{E} \cong \mathcal{O}_U^m$ on an affinoid U, and then extend this to a functor $\mathrm{LF}_m(X) \to \mathcal{VB}_m(X)$ via standard glueing procedures. We finally verify that this functor is indeed an equivalence.

First, consider the free $\mathcal{O}_U$-module $\mathcal{E} \cong \mathcal{O}_U^m$ on an affinoid K-space $U = \operatorname{Sp} B$. Fix a residue norm on B, whose unit ball we denote by $\mathcal{B}$. Let $\{e_1, \dots, e_m\}$ be a free generating set in $\mathcal{E}(U)$, giving a natural isomorphism of B-algebras $S = \operatorname{Sym}_B \mathcal{E}(U) \to B[x_1, \dots, x_m]$ by sending e_i to x_i. For any integer $n \geq 0$, denote by $\widehat{S}_n(e)$ the completion of S with respect to the norm whose unit ball is identified with $\mathcal{B}[\pi^n x]$. Thus we have a natural isomorphism $\widehat{S}_n(e) \cong B\langle \pi^n x \rangle$, so that $\widehat{S}_n(e)$ is an affinoid K-algebra. This gives an isomorphism $\alpha^n : \operatorname{Sp} \widehat{S}_n(e) \to \operatorname{Sp} B\langle \pi^n x\rangle$, and the natural morphism $B \to S \to \widehat{S}_n(e)$ gives rise to a morphism of affinoid K-spaces $\rho^n : \operatorname{Sp} \widehat{S}_n(e) \to U$, fitting into a commutative diagram

$$\begin{array}{ccc} \operatorname{Sp} \widehat{S}_n(e) & \xrightarrow{\rho^n} & U \\ {\scriptstyle \alpha^n}\downarrow & \nearrow{\scriptstyle p_2} & \\ \operatorname{Sp} K\langle \pi^n x \rangle \times U & & \end{array}$$

where p_2 is again the projection onto the second factor.

The natural map $\widehat{S}_{n+1}(e) \to \widehat{S}_n(e)$ gives rise to an inductive system of affinoid spaces, whose limit we denote by $V(\mathcal{E})$. This is a rigid analytic K-space, and the α^n give rise to an isomorphism $\alpha : V(\mathcal{E}) \to (\mathbb{A}_K^m)^{\mathrm{an}} \times U$, while the ρ_n induce a morphism $V(\mathcal{E}) \to U$, realizing $(V(\mathcal{E}), \rho)$ as a rigid analytic vector bundle of rank m on U.

We quickly verify that another choice of generating set produces the same vector bundle. If $f_1, \dots, f_m$ is another free generating set of $\mathcal{E}(U)$, then there exist integers a and b such that

$$\sum \pi^a \mathcal{B} e_i \subseteq \sum \mathcal{B} f_i \subseteq \sum \pi^b \mathcal{B} e_i,$$

so that the morphism $\operatorname{Sp} \widehat{S}_n(e) \to \operatorname{Sp} \widehat{S}_{n+b-a}(e)$ factors through $\operatorname{Sp} \widehat{S}_n(f)$ for each n. Therefore the limits agree (up to natural isomorphism), similarly for all the associated morphisms.

Likewise, our construction does not depend on the choice of unit ball $\mathcal{B}$, as all residue norms on B are equivalent.

Note that if $U' \subseteq U$ is an affinoid subdomain of U, the above construction naturally induces a commutative diagram

$$\begin{array}{ccccc}
V(\mathcal{E}|_{U'}) & \longrightarrow & (\mathbb{A}^m)^{\mathrm{an}} \times U' & \longrightarrow & U' \\
\downarrow & & \downarrow & & \downarrow \\
V(\mathcal{E}) & \longrightarrow & (\mathbb{A}^m)^{\mathrm{an}} \times U & \longrightarrow & U
\end{array}$$

as any free generating set of $\mathcal{E}(U)$ gives rise to a free generating of $\mathcal{E}(U')$.

Now let $\mathcal{E} \in \mathrm{LF}_m(X)$ be an arbitrary locally free sheaf of rank m on a rigid analytic K-space X, and let $(U_i)_{i\in I}$ be an admissible covering of X by affinoid subspaces $U_i = \operatorname{Sp} B_i$ such that $\mathcal{E}|_{U_i}$ is free. Note that the above construction yields rigid analytic vector bundles $V(\mathcal{E}|_{U_i})$ of rank m on U_i for each $i \in I$, such that the diagram

$$\begin{array}{ccccc}
V(\mathcal{E}|_{U_i}) & \xrightarrow{\alpha_i} & (\mathbb{A}^m)^{\mathrm{an}} \times U_i & \longrightarrow & U_i \\
\uparrow & & \uparrow & & \uparrow \\
 & \overset{\alpha_i}{\nearrow} & (\mathbb{A}^m)^{\mathrm{an}} \times V & \searrow & \\
V(\mathcal{E}|_V) & & & & V \\
 & \underset{\alpha_j}{\searrow} & (\mathbb{A}^m)^{\mathrm{an}} \times V & \nearrow & \\
\downarrow & & \downarrow & & \downarrow \\
V(\mathcal{E}|_{U_j}) & \xrightarrow{\alpha_j} & (\mathbb{A}^m)^{\mathrm{an}} \times U_j & \longrightarrow & U_j
\end{array}$$

commutes for any admissible open affinoid V contained in $U_i \cap U_j$. As the automorphism $\alpha_j \circ \alpha_i^{-1}$ of $(\mathbb{A}^m)^{\mathrm{an}} \times V$ corresponds to a change of free generating set for $\mathcal{E}(V)$, this corresponds to a linear morphism, viewing $\mathcal{E}(V)$ as the degree 1 part of $\operatorname{Sym} \mathcal{E}(V)$.
We will now carefully glue the $V(\mathcal{E}|_{U_i})$ along the fibres of intersections $U_i \cap U_j$ to obtain a vector bundle $V(\mathcal{E})$.

Since $U_{ij} = U_i \cap U_j$ is an admissible open subspace of U_i, the inverse image $\rho_i^{-1} U_{ij}$ is an admissible open subspace of $V(\mathcal{E}|_{U_i})$, and if $(V_r)_r$ is an affinoid covering of U_{ij}, then $(\rho_i^{-1} V_r) = (V(\mathcal{E}|_{V_r}))$ gives an admissible covering of $\rho_i^{-1} U_{ij}$. Applying [12, Proposition 5.3/6] to such a covering, we obtain a commutative diagram

$$\begin{array}{ccccc}
V(\mathcal{E}|_{U_i}) & \xrightarrow{\alpha_i} & (\mathbb{A}^m)^{\mathrm{an}} \times U_i & \longrightarrow & U_i \\
\uparrow & & \uparrow & & \uparrow \\
\rho_i^{-1}U_{ij} & \longrightarrow & (\mathbb{A}^m)^{\mathrm{an}} \times U_{ij} & \longrightarrow & U_{ij} \\
\downarrow{\scriptstyle \psi_{ij}} & & & & \downarrow \\
\rho_j^{-1}U_{ij} & \longrightarrow & (\mathbb{A}^m)^{\mathrm{an}} \times U_{ij} & \longrightarrow & U_{ij} \\
\downarrow & & \downarrow & & \downarrow \\
V(\mathcal{E}|_{U_j}) & \xrightarrow{\alpha_j} & (\mathbb{A}^m)^{\mathrm{an}} \times U_j & \longrightarrow & U_j
\end{array}$$

where ψ_{ij} is an isomorphism for each i and j, satisfying the cocycle condition.

Now we can invoke [12, Proposition 5.3/5] and again [12, Proposition 5.3/6] to obtain a rigid analytic K-space $V(\mathcal{E})$ together with a morphism $\rho : V(\mathcal{E}) \to X$. The data of U_i and α_i for $i \in I$ then make $V(\mathcal{E})$ a rigid analytic vector bundle of rank m on X.
If we had chosen a different covering (V_j), considering admissible open affinoids inside intersections of the form $U_i \cap V_j$ shows that we would obtain the same vector bundle up to canonical isomorphism.

Let $\phi : \mathcal{E} \to \mathcal{E}'$ be a morphism in $\mathrm{LF}_m(X)$, and let (U_i) be an affinoid covering such that both $\mathcal{E}|_{U_i}$ and $\mathcal{E}'|_{U_i}$ are free for each i. Then ϕ gives rise to morphisms $\mathrm{Sym}\,\mathcal{E}'^{\vee}|_{U_i} \to \mathrm{Sym}\,\mathcal{E}^{\vee}|_{U_i}$ and hence naturally to a linear morphism $V(\mathcal{E}^{\vee}|_{U_i}) \to V(\mathcal{E}'^{\vee}|_{U_i})$. Again, [12, Proposition 5.3/6] allows to glue these pieces to obtain a morphism $V(\mathcal{E}^{\vee}) \to V(\mathcal{E}'^{\vee})$.
We thus have constructed a contravariant functor $V : \mathrm{LF}_m(X) \to \mathcal{VB}_m(X)$ and a covariant functor $\mathcal{V} : \mathcal{E} \mapsto V(\mathcal{E}^{\vee})$ from $\mathrm{LF}_m(X)$ to $\mathcal{VB}_m(X)$.

Proposition 2.31. *The functor $\mathcal{V} : \mathrm{LF}_m(X) \to \mathcal{VB}_m(X)$ is an equivalence of categories.*

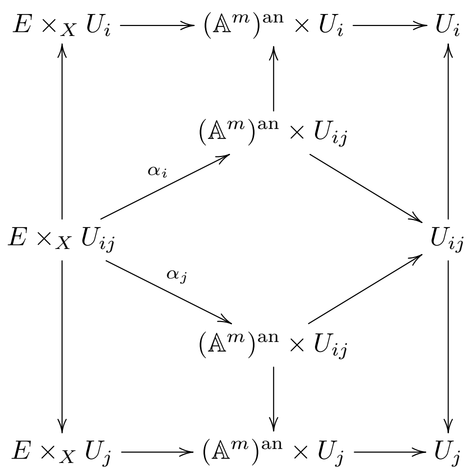
Proof. We show that $\mathcal{V}$ is fully faithful and essentially surjective.
For $\mathcal{E}, \mathcal{E}' \in \mathrm{LF}_m(X)$, $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ is given by the global sections of $\mathcal{H}om(\mathcal{E}, \mathcal{E}')$. Similarly, we can view $\mathrm{Hom}(\mathcal{V}(\mathcal{E}), \mathcal{V}(\mathcal{E}'))$ as the global sections of the sheaf $\mathcal{H}om(\mathcal{V}(\mathcal{E}), \mathcal{V}(\mathcal{E}'))$ given by

$$U \mapsto \mathrm{Hom}_{\mathcal{VB}_n(U)}(\mathcal{V}(\mathcal{E}|_U), \mathcal{V}(\mathcal{E}'|_U)).$$

The functor $\mathcal{V}$ (for varying base spaces U) now induces a morphism of sheaves $\mathcal{H}om(\mathcal{E}, \mathcal{E}') \to \mathcal{H}om(\mathcal{V}(\mathcal{E}), \mathcal{V}(\mathcal{E}'))$, which locally reduces to the identity morphism on $\mathcal{H}om(\mathcal{O}_U^m, \mathcal{O}_U^m)$. Thus we obtain an isomorphism of sheaves, and a fortiori an isomorphism between their global sections.

This proves that $\mathcal{V}$ is a fully faithful functor.

Now let $E \in \mathcal{VB}_m(X)$, and let (U_i) be an admissible covering of X trivializing E. We thus have a commutative diagram



where the $\psi_{ij} = \alpha_j \circ \alpha_i^{-1}$ are isomorphisms satisfying the cocycle condition. By fully faithfulness of $\mathcal{V}$, they correspond to isomorphisms $\phi_{ij} : \mathcal{O}_{U_{ij}}^m \to \mathcal{O}_{U_{ij}}^m$ satisfying the cocycle condition. Glueing the free sheaves $\mathcal{O}_{U_i}$ along the ϕ_{ij} now produces a locally free coherent sheaf $\mathcal{E}$ on X such that $E \cong \mathcal{V}(\mathcal{E})$ by construction.
Hence $\mathcal{V}$ is essentially surjective and thus an equivalence of categories. □

If X is a rigid analytic K-space, we define its **tangent sheaf** $\mathcal{T}_X$ by setting $\mathcal{T}_X(U) = \mathrm{Der}_K(\mathcal{O}_X(U))$ for any admissible open affinoid subspace $U \subseteq X$. If $\mathcal{T}_X$ is locally free, we say that X is **smooth** and call the associated vector bundles $T^*X := V(\mathcal{T})$ and $TX := \mathcal{V}(\mathcal{T})$ the cotangent bundle and the tangent bundle of X.

Chapter 3

Completed tensor products and strict morphisms

We now record some elementary facts on completed tensor products and discuss strict morphisms, a class of morphism which behaves well under completion. Later we investigate how exact sequences behave under the functor $B\widehat{\otimes}_A -$. The standard results in this chapter can mostly be found in [13, 1.1.9, 2.1.7], while most of the results in the last two sections are new.

Throughout this chapter, A will denote a normed (unital, not necessarily commutative) K-algebra. In particular, A will contain a field with non-trivial valuation, i.e. every normed A-module is a normed K-vector space in a natural way, so that we can invoke Lemma 2.1 whenever we talk about continuous morphisms between normed A-modules.
Moreoever, we will assume for simplicity that $|A| \setminus \{0\} = |K^*|$, as mentioned in the previous chapter.

3.1 Definitions and basic properties

Given a normed right A-module M and a normed left A-module N, the tensor product $M \otimes_A N$ is equipped with a semi-norm given by

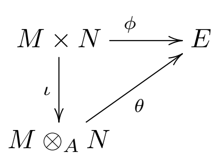
$$||x|| = \inf \{\max_i |m_i| \cdot |n_i|\},$$

where the infimum is taken over all expressions $x = \sum_i m_i \otimes n_i \in M \otimes N$. We call this the tensor product semi-norm on $M \otimes N$.

Definition 3.1 (see [13, 2.1.7])**.** *Given a normed right A-module M and a normed left A-module N, the* ***completed tensor product*** *of M and N, written $M\widehat{\otimes}_A N$, is the completion of the semi-normed space $M \otimes_A N$ with respect to the tensor product semi-norm.*

A priori, $M\widehat{\otimes}_A N$ is just a Banach K-space, but it will naturally inherit the structure of a left Banach B-module if M is a normed (B, A)-bimodule for some normed K-algebra B. As usual, this gives the completed tensor product an A-module structure if A is commutative – as we will deal with non-commutative algebras later, we will be careful to formulate our results in full generality.

We briefly describe the completed tensor product in terms of a universal property (see [12, Appendix B]).
We endow $M \times N$ with the product semi-norm $|(m, n)| = |m| \cdot |n|$ and call a bounded K-linear morphism $\phi : M \times N \to E$ into a semi-normed K-vector space E **A-balanced** if $\phi(ma, n) = \phi(m, an)$ for all $m \in M$, $n \in N$, $a \in A$. Then $M \otimes_A N$ and $M\widehat{\otimes}_A N$ can be characterized by the following universal properties.
If E is a semi-normed K-vector space and $\phi : M \times N \to E$ is an A-balanced morphism, then there exists a unique bounded K-linear morphism $\theta : M \otimes_A N \to E$ such that ϕ factors as



where $\iota : M \times N \to M \otimes_A N$ is the canonical morphism $M \times N \to M \otimes_A N$. By the universal property of completions, the completed tensor product $M\widehat{\otimes}_A N$ satisfies the analogous universal property for A-balanced maps $M \times N \to E$, where E is a *Banach* K-vector space.
This immediately implies the following lemma – note that by an isomorphism of normed K-vector spaces we will always mean a linear homeomorphism, likewise for isomorphisms of normed A-modules.

Lemma 3.2 (see [13, Propositions 2.1.8/5 and 2.1.7/4])**.** *Let M_1, M_2 be normed right A-modules, N_1, N_2 normed left A-modules, and suppose we have isomorphisms of normed A-modules $\theta : M_1 \to M_2$, $\phi : N_1 \to N_2$. Then the induced map*

$$\theta \otimes \phi : M_1 \otimes_A N_1 \to M_2 \otimes_A N_2$$

is a linear homeomorphism. In particular, $M_1\widehat{\otimes}_A N_1$ is linearly homeomorphic to $M_2\widehat{\otimes}_A N_2$.

Moreover, the canonical morphism

$$M_1 \widehat{\otimes}_A N_1 \to \widehat{M_1} \widehat{\otimes}_{\widehat{A}} \widehat{N_1}$$

is an isomorphism.

In studying the tensor product semi-norm on $M \otimes N$, it will be crucial to describe the unit ball in terms of the unit balls M°, N°.
We could not find an explicit reference for the following result, even though it is certainly well-known amongst experts.

Lemma 3.3. *Let M be a normed right A-module, N a normed left A-module such that $|M| \setminus \{0\}$ and $|N| \setminus \{0\}$ are discrete. Suppose that at least one of the two modules has value set equal to $|K^*|$. Then the unit ball of $M \otimes_A N$ under the tensor product semi-norm is the image of $M^\circ \otimes_{A^\circ} N^\circ$ under the canonical map.*

Proof. Suppose without loss of generality that $|M| \setminus \{0\} = |K^*|$, then

$$\{x \in \mathbb{R}_{>0} : \ x = |m| \cdot |n|, \ m \in M, \ n \in N\} = |N| \setminus \{0\}$$

is discrete. Let $x \in M \otimes N$ be in the unit ball. Then there exists an expression $x = \sum m_i \otimes n_i$ such that $|m_i| \cdot |n_i| \leq 1$ for all i – if $|x| < 1$, this follows trivially from the definition of the semi-norm, and if $|x| = 1$, discreteness implies that we can replace the infimum in the definition of $|x|$ by a minimum.
Obviously we can assume $m_i \neq 0$ for each i, and hence $|m_i| > 0$.
We will now show that for each i, there exists some integer $k_i \in \mathbb{Z}$ such that $|\pi^{-k_i} m_i| \leq 1$ and $|\pi^{k_i} n_i| \leq 1$, and thus

$$x = \sum \pi^{-k_i} m_i \otimes \pi^{k_i} n_i$$

is in the image of $M^\circ \otimes N^\circ$.
For each i, $|m_i| > 0$ implies that there exists some integer $k_i \in \mathbb{Z}$ such that $|m_i| = |\pi^{k_i}|$ by assumption on the value set of M. Thus $|\pi^{-k_i} m_i| = 1$, and

$$|\pi^{k_i} n_i| = |m_i| \cdot |n_i| \leq 1.$$

The result follows. □

Using Lemma 2.8 and functoriality of the tensor product (which follows from the universal property), we obtain the following more general result.

Corollary 3.4. *Let M be a normed right A-module and N a normed left A-module. Then the tensor product semi-norm on $M \otimes_A N$ is equivalent to the gauge semi-norm associated to the lattice which is the image of the canonical morphism $M^\circ \otimes_{A^\circ} N^\circ \to M \otimes_A N$.*

We are now ready to introduce the notion of a strict morphism.
Note that the theory of semi-normed A-modules involves some topological subtleties.
Given a continuous A-module morphism $\phi : M \to N$, it is not necessarily true that image and kernel of the induced map $\widehat{\phi} : \widehat{M} \to \widehat{N}$ can be obtained by completing the image and kernel of ϕ.
Moreover, there is no isomorphism theorem in the category of topological A-modules. We naturally have an isomorphism of abstract A-modules

$$M/\ker\phi \to \operatorname{Im}\phi,$$

but we have no guarantee that this is actually an isomorphism of semi-normed A-modues, i.e. a linear homeomorphism.
The notion of a strict morphism addresses this problem. It is possible to view this as a particular instance of Schneiders' theory of quasi-abelian categories [45], but we would gain little from such a generalization at this point.
To ease notation, we will, for any morphism $\phi : M \to N$ of semi-normed A-modules, reserve $\operatorname{Im}\phi$ for the image of ϕ equipped with the subspace semi-norm, and write $\operatorname{Coim}\phi$ for $M/\ker\phi$ with the quotient semi-norm.

Definition 3.5 (see [13, Definition 1.1.9/1])**.** *A continuous linear map ϕ between two semi-normed K-vector spaces $G \to H$ is called **strict** if the natural morphism $\operatorname{Coim}\phi \to \operatorname{Im}\phi$ is a homeomorphism.*

In practice, we often use the following equivalent property as a definition.

Lemma 3.6 (see [13, Lemma 1.1.9/2])**.** *Let $\phi : M \to N$ be a continuous morphism between two semi-normed (left) A-modules. Then ϕ is strict if and only if there exists some integer a satisfying the following:*
For any $x \in M$ with $|\phi(x)| \leq 1$, there exists $y \in M$ such that $\phi(x) = \phi(y)$ and $|y| \leq |\pi|^a$, i.e. $N^\circ \cap \operatorname{Im}\phi \subseteq \phi(\pi^a M^\circ)$.

Proof. ϕ is strict if and only if the natural bijection $\alpha : \operatorname{Coim}\phi \to \operatorname{Im}\phi$ and its inverse are both continuous, i.e. bounded (Lemma 2.1). Now α inherits continuity from ϕ, and α^{-1} is bounded if and only if the above property holds. □

This implies for instance that strictness is preserved under taking finite direct sums.

In the setting of Banach spaces, the Open Mapping Theorem allows for the following criterion.

Lemma 3.7. *Let $f : M \to N$ be a continuous morphism of Banach spaces. Then f is strict if and only if the image of f is closed in N.*

Proof. If f is strict, then $\operatorname{Im} f \cong M/\ker f$ as normed spaces. Since the kernel is closed in M by continuity and M is Banach, this turns $\operatorname{Im} f$ (with the subspace norm) into a complete subspace of N (by Lemma 2.4). Thus $\operatorname{Im} f$ is closed in N by Lemma 2.3.
Conversely, if $\operatorname{Im} f$ is closed in N, it is itself Banach by Lemma 2.3, and the surjection $M \to \operatorname{Im} f$ is strict by the Open Mapping Theorem (Theorem 2.5). □

This means in particular that any exact sequence of Banach spaces with continuous differentials consists of strict morphisms, as each image is closed (being equal to the kernel of the next map).

The key property of strict morphisms is the following.

Proposition 3.8 (see [13, 1.1.9/4, 5])**.** *If M, N are semi-normed (left) A-modules and $\phi : M \to N$ is strict, then the completion*

$$\widehat{\phi} : \widehat{M} \to \widehat{N}$$

is also strict and has kernel $\widehat{\ker\ \phi}$ and image $\widehat{\operatorname{Im}\ \phi}$.
In particular, an exact sequence consisting of strict morphisms of semi-normed A-modules remains exact after completion.

We also note that strict surjections behave well under tensor products.

Theorem 3.9 (see [13, 2.1.8/6])**.** *If $\phi_1 : M_1 \to N_1$, resp. $\phi_2 : M_2 \to N_2$ are strict surjective morphisms of normed right, resp. left A-modules (and A contains a field with a non-trivial valuation), then the morphism*

$$\phi_1 \otimes \phi_2 : M_1 \otimes_A M_2 \to N_1 \otimes_A N_2$$

is surjective and strict with respect to the corresponding tensor product semi-norms.
Thus $\phi_1 \widehat{\otimes} \phi_2$ is still surjective.

Since the codomain of a strict surjection is (up to equivalence) equipped with the quotient semi-norm, we can summarize the above by saying that the tensor semi-norm of two quotients is equivalent to the quotient semi-norm from the corresponding tensor product.

As noted earlier, the condition that A contains a field with non-trivial valuation is necessary,

but will automatically be satisfied in all cases we are considering, as we will only take tensor products over unital K-algebras.
We also note that the reference states this theorem with ‘epimorphism’ instead of ‘surjective morphism’, but the proof makes it clear that this is understood to be an epimorphism in some category in which the two notions coincide (presumedly the category of *semi*-normed A-modules – see chapter 2 for more details on the quotient semi-norm).

Lemma 3.10. *Let $\phi : A \to B$ be a morphism of affinoid K-algebras, and let $\mathcal{A} \subset A$, $\mathcal{B} \subset B$ be affine formal models. Then $\mathcal{B}' := \phi(\mathcal{A}) \cdot \mathcal{B}$ is an affine formal model of B.*

Proof. Let $\alpha : T_m \to A$, $\beta : T_n \to B$ be surjections of K-algebras such that $\alpha(R\langle X_1, \dots, X_m\rangle) = \mathcal{A}$ and $\beta(R\langle Y_1, \dots Y_n\rangle) = \mathcal{B}$. Note that

$$T_m \widehat{\otimes}_K T_n \cong K[X_1, \dots, X_m] \widehat{\otimes}_K K[Y_1, \dots Y_n] \cong T_{m+n},$$

and the morphism $\theta : T_{m+n} \to B$ given by

$$T_{m+n} \cong T_m \widehat{\otimes}_K T_n \to A \widehat{\otimes}_K B \to B$$

gives rise to a residue norm with unit ball $\phi(\mathcal{A}) \cdot \mathcal{B}$. □

In particular, there exists a residue norm which makes B a Banach A-module, and it thus follows from the definition that we can view any Banach B-module as a Banach A-module.

The composition of strict morphisms is not necessarily strict, but at least we have the following result.

Lemma 3.11. *Let*

$$L \xrightarrow{f} M \xrightarrow{g} N$$

be a composition of continuous morphisms between semi-normed A-modules. Suppose that f and g are strict and that at least one of the following is satisfied:

(i) f is surjective.

(ii) g is injective.

Then the composition gf is also strict.

Proof. Let a, b be integers such that $M^\circ \cap \operatorname{Im} f \subseteq f(\pi^a L^\circ)$ and $N^\circ \cap \operatorname{Im} g \subseteq g(\pi^b M^\circ)$. We will show that in both cases, $N^\circ \cap \operatorname{Im} gf \subseteq gf(\pi^{a+b} L^\circ)$.
Let $x \in L$ satisfy $|gf(x)| \leq 1$. By definition, there exists some $y \in M$ such that $g(y) = gf(x)$

and $|y| \leq |\pi|^b$.
If f is surjective, then $y \in \operatorname{Im} f$, so that $\pi^{-b} y \in M^\circ \cap \operatorname{Im} f$. Thus there exists $z \in L$ such that $f(z) = y$ and $|z| \leq |\pi|^{a+b}$. But then $gf(z) = g(y) = gf(x)$, proving strictness of gf.
Similarly, if g is injective, we know that $y = f(x)$, so again $\pi^{-b} y \in M^\circ \cap \operatorname{Im} f$. So by strictness of f, there exists some $z \in L$ such that $|z| \leq |\pi|^{a+b}$ and $f(z) = y$. Then $gf(z) = g(y) = gf(x)$ yields the result. □

Lemma 3.12. *Let A and B be Noetherian Banach K-algebras. Let N be a finitely generated left Banach A-module, and let M be a Banach (B, A)-bimodule such that it is finitely generated as a left B-module. Then the natural morphism*

$$M \otimes_A N \to M \widehat{\otimes}_A N,$$

is an isomorphism, i.e. the tensor semi-norm is a norm with respect to which $M \otimes N$ is already complete.

Proof. This is a straightforward generalization of [13, Proposition 3.7.3/6].
By Noetherianity of A and Lemma 2.14, the norm on N is equivalent to one induced by a surjection $p : A^r \to N$ for some integer r. The map p is then strict by definition.
Theorem 3.9 then implies that the map $M \otimes p : M \otimes_A A^r \to M \otimes_A N$ is a strict surjection of semi-normed left B-modules, i.e. the tensor semi-norm on $M \otimes N$ is equivalent to the quotient semi-norm induced by $M \otimes p$.
Now $M \otimes A^r \cong M^r$ is a finitely generated left Banach B-module. Therefore the kernel of $M \otimes p$ is closed by Proposition 2.13, making $M \otimes N$ a Banach B-module by Lemma 2.4. □

3.2 Completed tensor products and short exact sequences

For the remainder of this chapter, we let A and B be normed unital K-algebras such that B is also a normed right A-module via a contractive morphism $\phi : A \to B$. In particular, the unit ball B° is also a right A°-module. We also assume that both A and B have discrete value sets equal to $|K^*|$.
Let $\mathcal{B}$ be a ring such that the map $\phi^\circ : A^\circ \to B^\circ$ factors as $A^\circ \to \mathcal{B} \to B^\circ$. We assume that the morphism $\mathcal{B} \to B^\circ$ is surjective and induces an isomorphism $\mathcal{B} \otimes_R K \cong B$. There is no harm in taking $\mathcal{B} = B^\circ$ in this section, but we will require our results in the more general setting later.

We will now be concerned with the question under which conditions the functor $B \widehat{\otimes}_A -$ preserves the exactness of a given sequence.
We will first restrict ourselves to short exact sequences, before generalizing our results to other

cochain complexes in the next section.

As might be expected, we obtain some reasonable results in the case when the given sequence consists of strict morphisms. We briefly describe the general strategy of our arguments. Given a normed left A-module M, we analyse the R-module $\mathcal{B} \otimes_{A^\circ} M^\circ$, which determines the tensor product semi-norm on $B \otimes M$ by Lemma 3.3. To pass from $\mathcal{B} \otimes M^\circ$ to the actual unit ball of $B \otimes M$, recall Lemma 2.10.
Thus we will rephrase questions about strictness in terms of the corresponding 'R-model tensors' $\mathcal{B} \otimes M^\circ$ and their π-torsion.

As many of the arguments will involve properties of R-modules 'up to bounded π-torsion', we introduce the following language (see [5, 3.4]).
Write $\mathcal{BT}$ for the category of R-modules which are bounded π-torsion, i.e. are killed by some power of π. Consider the quotient abelian category

$$\mathcal{Q} = R\text{-mod}/\mathcal{BT},$$

with the natural quotient functor $q : R\text{-mod} \to \mathcal{Q}$.
For example, if $f : M \to N$ is a morphism of R-modules such that both kernel and cokernel of f are objects in $\mathcal{BT}$, then $q(f)$ is an isomorphism between $q(M)$ and $q(N)$ (we will sometimes suppress the functor q and write instead 'f induces an isomorphism between M and N in $\mathcal{Q}$').
In particular, if $a \in \mathbb{N}$, then the morphism $\pi^a : M \to M$ given by multiplication by π^a induces an isomorphism in $\mathcal{Q}$ for any R-module M, as both kernel and cokernel of the map (in R-mod) are annihilated by π^a.

Lemma 3.13. *Let $f : M \to N$ be a morphism of R-modules inducing an isomorphism in $\mathcal{Q}$. Then f induces an isomorphism in $\mathcal{Q}$ between $\pi-\mathrm{tor}(M)$ and $\pi-\mathrm{tor}(N)$.*

Proof. By definition, there exists some positive integers a and b such that π^a annihilates the kernel of f and π^b annihilates the cokernel of f.
Restricting to $\pi-\mathrm{tor}(M)$, f thus induces an isomorphism between $\pi-\mathrm{tor}(M)$ and $f(\pi-\mathrm{tor}(M))$ in $\mathcal{Q}$. Now let $x \in N$ be π-torsion, i.e. there exists n such that $\pi^n x = 0$. By assumption, there exists some $y \in M$ with $f(y) = \pi^b x$, and hence $\pi^{n-b} y \in \ker f$. Thus $\pi^{a+n-b} y = 0$, so $y \in \pi-\mathrm{tor}(M)$. Therefore $\pi^b \cdot \pi-\mathrm{tor}(N) \subseteq f(\pi-\mathrm{tor}(M)) \subseteq \pi-\mathrm{tor}(N)$, finishing the proof. $\square$

Lemma 3.14. *Let M be a normed left A-module with two equivalent norms $|-|_1$, $|-|_2$, with respective unit balls M_1°, M_2°. Then $\mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_1^\circ)$ is isomorphic to $\mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_2^\circ)$ in $\mathcal{Q}$ for each $s \geq 0$.*

Proof. By equivalence of norms, there exist positive integers a and b such that

$$\pi^a M_1^\circ \subseteq M_2^\circ \subseteq \pi^{-b} M_1^\circ,$$

inducing $\mathcal{B}$-module morphisms

$$f_1 : \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_1^\circ) \xrightarrow{\pi^a} \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_2^\circ) \xrightarrow{\pi^b} \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_1^\circ)$$

and

$$f_2 : \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_2^\circ) \xrightarrow{\pi^b} \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_1^\circ) \xrightarrow{\pi^a} \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_2^\circ)$$

both of which are simply multiplication by π^{a+b} by functoriality. Thus the kernel of

$$\pi^b : \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_2^\circ) \to \mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M_1^\circ)$$

is annihilated by some positive power of π by looking at f_2, likewise for the cokernel by looking at f_1. □

Lemma 3.15. *Suppose*

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is an exact sequence of R-modules, N has bounded π-torsion, and $L \in \mathcal{BT}$, i.e. there exists a positive integer a such that $\pi^a L = 0$. Then M has bounded π-torsion.

Proof. Let b be an integer such that π^b annihilates every π-torsion element of N, and let $x \in M$ be a π-torsion element. Then its image $g(x)$ is π-torsion in N, so $\pi^b g(x) = 0$, and $\pi^b x$ has some preimage y in L. By assumption $\pi^a y = 0$, so $\pi^{a+b} x = 0$. □

Lemma 3.16. *Let*

$$0 \to L \to M \to N \to 0$$

be a strict short exact sequence of normed left A-modules. Assume that tensoring with B yields a short exact sequence

$$0 \longrightarrow B \otimes_A L \xrightarrow{f} B \otimes_A M \xrightarrow{g} B \otimes_A N \longrightarrow 0.$$

Then this sequence is strict with respect to the tensor semi-norms if and only if the following condition is satisfied:
The induced map

$$\pi\text{−tor}(\mathcal{B} \otimes M^\circ) \to \pi\text{−tor}(\mathcal{B} \otimes N^\circ)$$

is an epimorphism in $\mathcal{Q}$, i.e. there exists a non-negative integer r such that for any π-torsion element $x \in \mathcal{B} \otimes N^\circ$, $\pi^r x$ is the image of some π-torsion element of $\mathcal{B} \otimes M^\circ$.

Proof. Without loss of generality (using Lemmas 3.13 and 3.14), we can assume that L and N are equipped with the subspace and quotient norm, respectively, and that M (and hence L and N) has discrete value set equal to $|K^*|$.

Thus we have a short exact sequence

$$0 \to L^\circ \to M^\circ \to N^\circ \to 0$$

Note that the map $M^\circ \to N^\circ$ is indeed surjective since M has discrete value set (Lemma 2.9). Thus we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & \mathcal{B} \otimes_{A^\circ} L^\circ & \xrightarrow{f^\circ} & \mathcal{B} \otimes_{A^\circ} M^\circ & \xrightarrow{g^\circ} & \mathcal{B} \otimes_{A^\circ} N^\circ & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B \otimes_A L & \xrightarrow{f} & B \otimes_A M & \xrightarrow{g} & B \otimes_A N & \longrightarrow & 0 \end{array}$$

The kernel of each vertical arrow consists of the π-torsion submodule by Lemma 2.10, and by Lemma 3.3, the images of the vertical maps are the unit balls of the terms in the second row with respect to the tensor semi-norms. Given an element x in some term in the first row, we will denote its image under the vertical map by $\overline{x}$.
Note that by Theorem 3.9, g is always strict.

Suppose now that the condition stated in the Lemma is satisfied, and let $x \in B \otimes L$ such that $|f(x)| \leq 1$, i.e. there exists some $y \in \mathcal{B} \otimes M^\circ$ such that $\overline{y} = f(x)$. Note that $g(\overline{y}) = 0$ implies that $g^\circ(y)$ is π-torsion in $\mathcal{B} \otimes N^\circ$. Then $\pi^r g^\circ(y)$ is the image of some π-torsion element z of $\mathcal{B} \otimes M^\circ$ by assumption. Thus $\pi^r y - z$ is in the kernel of g° and hence has a preimage u in $\mathcal{B} \otimes L^\circ$.

Since z is π-torsion, $\overline{\pi^r y - z} = \pi^r f(x)$, and by commutativity of the diagram and injectivity of f, we have $\overline{u} = \pi^r x$, so $|\pi^r x| \leq 1$ in $B \otimes L$. Thus $\operatorname{Im} f \cap (B \otimes M)^\circ \subseteq f(\pi^{-r}(B \otimes L)^\circ)$, proving that f is strict.

Conversely, suppose the map f is strict, i.e. there exists an integer r such that $\operatorname{Im} f \cap (B \otimes M)^\circ \subseteq f(\pi^{-r}(B \otimes L)^\circ)$. Without loss of generality, we can choose r to be non-negative. Now let $x \in \mathcal{B} \otimes N^\circ$ be a π-torsion element. By surjectivity, there exists some $y \in \mathcal{B} \otimes M^\circ$ such that $x = g^\circ(y)$. Then $\overline{y} \in \ker g$ has norm ≤ 1 in $B \otimes M$, and hence $\overline{y} \in \operatorname{Im} f \cap (B \otimes M)^\circ$. By definition of r, $\pi^r \overline{y}$ has a preimage in $(B \otimes L)^\circ$, and thus a preimage z in $\mathcal{B} \otimes L^\circ$.

Thus $\pi^r y - f^\circ(z) \in \mathcal{B} \otimes M^\circ$ is π-torsion, and $\pi^r x = g^\circ(\pi^r y - f^\circ(z))$ is the image of a π-torsion element. □

We obtain as an immediate consequence two corollaries.

Corollary 3.17. *Let*

$$0 \to L \to M \to N \to 0$$

be as in Lemma 3.16. Assume that the $\mathcal{B}$*-module* $\mathcal{B} \otimes_{A^\circ} N^\circ$ *has bounded* π*-torsion, i.e. there exists an integer* a *such that if* $x \in \mathcal{B} \otimes N^\circ$*,* $\pi^n x = 0$ *for some* n*, then* $\pi^a x = 0$*.*
Then the sequence

$$0 \longrightarrow B \otimes_A L \xrightarrow{f} B \otimes_A M \xrightarrow{g} B \otimes_A N \longrightarrow 0$$

is strict with respect to the tensor semi-norms.

Proof. If $x \in \pi\text{-tor}(\mathcal{B} \otimes N^\circ)$, then $\pi^a x = 0$ is the image of some π-torsion element of $\mathcal{B} \otimes M^\circ$. □

Corollary 3.18. *Let*

$$0 \to L \to M \to N \to 0$$

be as in Lemma 3.16, and assume moreover that $\mathcal{B}$ *is a flat right* A°*-module.*
Then the sequence

$$0 \longrightarrow B \otimes L \xrightarrow{f} B \otimes M \xrightarrow{g} B \otimes N \longrightarrow 0$$

is strict with respect to the tensor semi-norms.

Proof. By flatness, the morphism

$$\mathcal{B} \otimes_{A^\circ} N^\circ \to \mathcal{B} \otimes_{A^\circ} (N^\circ \otimes_R K) = B \otimes_A N$$

is an injection, so that $\mathcal{B} \otimes N^\circ$ is π-torsionfree. Hence the claim follows from Corollary 3.17. □

3.3 Completed tensor products and cochain complexes

Finally, we need a variant of the results above to deal with more general cochain complexes. For this, we keep the set-up of the previous section, assuming additionally that A° and $\mathcal{B}$ are left Noetherian rings and that B is flat as a right A-module.
A standard example would be the case of A and B being affinoid algebras equipped with some residue norms, $\mathcal{B} = B^\circ$, and $A \to B$ the restriction morphism realizing $\operatorname{Sp} B$ as an affinoid

subdomain of $\operatorname{Sp} A$ (see [12, Corollary 4.1/5]).
Consider a cochain complex $(C^\bullet, \partial)$ of left Banach A-modules. We assume without loss of generality that $|C^j| = |K^*|$, and we suppose that for each j, the differential ∂^j is strict.

It is worth pointing out that we do not assume A and B to be themselves complete at this point.
We will state all our results for the case of left A-modules – the corresponding statements for right A-modules can be proved mutatis mutandis.
Note that it follows from Lemma 3.7 that the images $\operatorname{Im} \partial^j$ are closed in C^{j+1}.

Unless explicitly stated otherwise, the modules $\operatorname{Im} \partial^{j-1}$ and $\ker \partial^j$ are equipped with the subspace norms induced from the normed A-module C^j. All tensor products will be equipped with the corresponding tensor semi-norms, and we will write e.g. $\operatorname{Coim}(B \otimes \partial^j)$ when we equip the tensor product $B \otimes \operatorname{Coim} \partial^j$ with the quotient semi-norm induced from $B \otimes C^j \to B \otimes \operatorname{Coim} \partial^j$, or $\operatorname{Im}(B \otimes \partial^{j-1})$ when we equip the tensor product $B \otimes \operatorname{Im} \partial^{j-1}$ with the subspace semi-norm inherited from $B \otimes_A C^j$. We will sometimes abbreviate $\operatorname{Im} \partial^j$ to Im when it is obvious which term in the complex we are considering.
When we say that two semi-normed B-modules are isomorphic, it will be as topological modules, except when we say explicitly that we consider them as abstract B-modules.

Since $\ker \partial^j$ is closed in C^j, it is Banach by Lemma 2.3, and $\operatorname{Im} \partial^{j-1}$ is assumed to be closed. Hence Lemma 2.4 implies that the quotient norm induced from the short exact sequence

$$0 \to \operatorname{Im} \partial^{j-1} \to \ker \partial^j \to \mathrm{H}^j(C^\bullet) \to 0$$

turns $\mathrm{H}^j(C^\bullet)$ into a Banach A-module.
With this choice of norm, the short exact sequence above consists of strict morphisms by definition.

Proposition 3.19. *Suppose that for each j, the following is satisfied:*

(i) The module $\mathcal{B} \otimes_{A^\circ} \mathrm{H}^j(C^\bullet)^\circ$ has bounded π-torsion.

(ii) The morphism $B \otimes_A \ker \partial^j \to B \otimes_A C^j$ is strict.

Then the complex $B \otimes_A C^\bullet$ consists of strict morphisms, and the canonical morphism

$$B \widehat{\otimes}_A \mathrm{H}^j(C^\bullet) \to \mathrm{H}^j(B \widehat{\otimes}_A C^\bullet)$$

is an isomorphism for each j.

Proof. By assumption, $\mathcal{B} \otimes \mathrm{H}^j(C^\bullet)^\circ$ has bounded π-torsion, so we can apply Corollary 3.17 to see that
$$0 \to B \otimes_A \operatorname{Im} \partial^{j-1} \to B \otimes_A \ker \partial^j \to B \otimes_A \mathrm{H}^j(C^\bullet) \to 0$$
is strict exact.

Now the map $B \otimes_A C^{j-1} \to B \otimes_A C^j$ factors as
$$B \otimes_A C^{j-1} \to B \otimes \operatorname{Coim} \partial^{j-1} \cong B \otimes \operatorname{Im} \partial^{j-1} \to B \otimes \ker \partial^j \to B \otimes C^j,$$
where the first map is a strict surjection by Theorem 3.9, the second map is a homeomorphism by strictness of ∂^{j-1} and Lemma 3.2, and the third map is a strict injection by the above. Since we also assume that the fourth map is a strict injection, Lemma 3.11 now implies that $B \otimes C^\bullet$ consists of strict morphisms.

We have also seen above that the sequence
$$0 \to B \otimes \operatorname{Im} \partial^{j-1} \to B \otimes \ker \partial^j \to B \otimes \mathrm{H}^j(C^\bullet) \to 0$$
is strict exact, so that its completion
$$0 \to B \widehat{\otimes} \operatorname{Im} \partial^{j-1} \to B \widehat{\otimes} \ker \partial^j \to B \widehat{\otimes} \mathrm{H}^j(C^\bullet) \to 0$$
is also exact by Proposition 3.8. We will now identify the first two terms with the corresponding images and kernels in the complex $B \widehat{\otimes} C^\bullet$, i.e. we show that the vertical arrows in the commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B \widehat{\otimes} \operatorname{Im} & \longrightarrow & B \widehat{\otimes} \ker & \longrightarrow & B \widehat{\otimes} \mathrm{H}^j(C^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Im}(B \widehat{\otimes} \partial^{j-1}) & \longrightarrow & \ker(B \widehat{\otimes} \partial^j) & \longrightarrow & \mathrm{H}^j(B \widehat{\otimes} C^\bullet) & \longrightarrow & 0 \end{array}$$
are isomorphisms, completing the proof.

Note that by strictness of $B \otimes \partial$, we can invoke Proposition 3.8 to identify $\operatorname{Im}(B \widehat{\otimes} \partial^{j-1})$ and $\ker(B \widehat{\otimes} \partial^j)$ with the completions of the image and kernel of $B \otimes \partial$, respectively.
But now we have natural isomorphisms of normed B-modules
$$B \widehat{\otimes} \operatorname{Im} \cong B \widehat{\otimes} \operatorname{Coim} \cong \operatorname{Coim}(\widehat{B \otimes \partial^{j-1}}) \cong \operatorname{Im}(\widehat{B \otimes \partial^{j-1}}).$$

We will explain each of these isomorphisms in turn. The first isomorphism is due to the strictness of ∂^{j-1} (and applying Lemma 3.2). The second isomorphism follows from Theorem 3.9 applied to the strict surjection $C^{j-1} \to \operatorname{Coim} \partial^{j-1}$. For the third isomorphism, note that $B \otimes \partial^{j-1}$ is a strict morphism by the above, so

$$\operatorname{Coim}(B \otimes \partial^{j-1}) \cong \operatorname{Im}(B \otimes \partial^{j-1})$$

by definition, and completion gives the desired isomorphism.
This proves that the first vertical arrow is an isomorphism.

Similarly, since we assume that $B \otimes \ker \partial^j \to B \otimes C^j$ is strict, we have

$$B \otimes \ker \partial^j \cong \ker(B \otimes \partial^j),$$

and hence

$$B \widehat{\otimes} \ker \partial^j \cong \ker \widehat{(B \otimes \partial^j)}.$$

Now Proposition 3.8 implies again

$$B \widehat{\otimes} \ker \partial^j \cong \ker(B \widehat{\otimes} \partial^j),$$

proving that the second vertical arrow is an isomorphism.

By exactness of the rows of the diagram, it therefore follows that the third arrow is also an isomorphism, and

$$B \widehat{\otimes} \mathrm{H}^j(C^\bullet) \cong \mathrm{H}^j(B \widehat{\otimes} C^\bullet)$$

as required. □

Corollary 3.20. *Suppose that $\mathcal{B}$ is a flat right A°-module. Then $B \otimes_A C^\bullet$ consists of strict morphisms, and*

$$B \widehat{\otimes}_A \mathrm{H}^j(C^\bullet) \to \mathrm{H}^j(B \widehat{\otimes} C^\bullet)$$

is an isomoprhism for each j.

Proof. As in the proof of Corollary 3.18, $\mathcal{B} \otimes \mathrm{H}^j(C^\bullet)^\circ$ is π-torsionfree.
The sequence

$$0 \to \ker \partial^j \to C^j \to \operatorname{Coim} \partial^j \to 0$$

is trivially strict, so applying Corollary 3.18 shows that the conditions of Proposition 3.19 are satisfied. □

Our next results can be viewed as an extension of this result to the case when $\mathcal{B}$ is not flat. It turns out that we can establish analogous statements as long as $\mathcal{B}$ is sufficiently close to being flat in the sense that all corresponding Tor groups should have bounded π-torsion.

To simplify notation, we will from now on abbreviate $\mathrm{Tor}_s^{A^\circ}(\mathcal{B}, M)$ to $T_s(M)$ for any left A°-module M.
Note that for $s \geq 1$, this is always a π-torsion module, as

$$T_s(M) \otimes_R K = \mathrm{Tor}_s^A(B, M \otimes_R K) = 0,$$

since K is flat over R, and B is assumed to be flat over A (see [47, Corollary 3.2.10] and Lemma 2.10).

Theorem 3.21. *Suppose that for large enough j, $T_s((\mathrm{Coim}\,\partial^j)^\circ)$ and $T_s((\ker \partial^j)^\circ)$ have bounded π-torsion for all $s \geq 0$. Suppose further that for all j, the following is satisfied:*

(i) $T_s\left(\mathrm{H}^j(C^\bullet)^\circ\right)$ has bounded π-torsion for all $s \geq 0$.

(ii) $T_s\left((C^j)^\circ\right)$ has bounded π-torsion for all $s \geq 0$.

Then the complex $B \otimes C^\bullet$ consists of strict morphisms, and the canonical morphism

$$B \widehat{\otimes}_A \mathrm{H}^j(C^\bullet) \to \mathrm{H}^j(B \widehat{\otimes} C^\bullet)$$

is an isomorphism for each j.

Proof. Consider again the strict short exact sequence

$$0 \to \ker \partial^j \to C^j \to \mathrm{Coim}\,\partial^j \to 0.$$

In the light of Proposition 3.19 and Corollary 3.17, it is now enough to show that $\mathcal{B} \otimes (\mathrm{Coim}\,\partial^j)^\circ$ has bounded π-torsion for each j.
We will in fact show the following stronger statement: for each j and each $s \geq 0$, $T_s((\mathrm{Coim}\,\partial^j)^o)$ and $T_s((\ker \partial^j)^\circ)$ have bounded π-torsion.
We will argue inductively on j. The statement is true for sufficiently large j (and arbitrary s) by assumption. Let us now assume we have proved the statement for $T_s((\mathrm{Coim}\,\partial^{j+1})^\circ)$ and $T_s((\ker \partial^{j+1})^\circ)$ for each s.

Consider the following long exact sequence

$$\cdots \to T_{s+1}((\mathrm{H}^{j+1}(C^\bullet))^\circ) \to T_s((\mathrm{Im}\,\partial^j)^\circ) \to T_s((\ker \partial^{j+1})^\circ) \to \ldots,$$

obtained from the natural short exact sequence.
As noted earlier, $T_{s+1}((\mathrm{H}^{j+1}(C^\bullet))^\circ)$ is a π-torsion module, since B is flat over A, and it has bounded π-torsion by assumption.
Thus, $T_{s+1}((\mathrm{H}^{j+1}(C^\bullet))^\circ) \in \mathcal{BT}$ is annihilated by some positive power of π.
By inductive hypothesis, $T_s((\ker \partial^{j+1})^\circ)$ has bounded π-torsion, so Lemma 3.15 now implies that $T_s((\operatorname{Im} \partial^j)^\circ)$ has bounded π-torsion for each $s \geq 0$.

But now $(\operatorname{Im} \partial^j)^\circ$ and $(\operatorname{Coim} \partial^j)^\circ$ are unit balls of equivalent norms on $\operatorname{Im} \partial^j$, so by Lemma 3.14, $T_s((\operatorname{Coim} \partial^j)^\circ)$ is isomorphic to $T_s((\operatorname{Im} \partial^j)^\circ)$ in $\mathcal{Q}$, and has therefore bounded π-torsion for each $s \geq 0$ by Lemma 3.13.
In particular, $T_s((\operatorname{Coim} \partial^j)^\circ) \in \mathcal{BT}$ for $s \geq 1$ by flatness of B.
Now applying Lemma 3.15 to the part of the long exact sequence

$$\cdots \to T_{s+1}((\operatorname{Coim} \partial^j)^\circ) \to T_s((\ker \partial^j)^\circ) \to T_s((C^j)^\circ) \to \dots$$

shows that $T_s((\ker \partial^j)^\circ)$ has bounded π-torsion for all $s \geq 0$. □

We would like to highlight two particular instances of the above theorem.

Corollary 3.22. *Suppose that $\mathcal{B} \otimes_{A^\circ} \widehat{A^\circ}$ carries the structure of a (left) Noetherian ring, and assume that for each j, the following is satisfied:*

(i) $\mathrm{H}^j(C^\bullet)$ is a finitely generated $\widehat{A}$-module.

(ii) $T_s((C^j)^\circ)$ has bounded π-torsion for each $s \geq 0$.

Then $B \otimes_A C^\bullet$ consists of strict morphisms and the natural morphism

$$\widehat{B} \otimes_{\widehat{A}} \mathrm{H}^j(C^\bullet) \to \mathrm{H}^j(B \widehat{\otimes}_A C^\bullet)$$

is an isomorphism of $\widehat{B}$-modules.

Proof. Note that by Theorem 2.11, we can identify the unit ball $\widehat{A}^\circ$ of $\widehat{A}$ with the π-adic completion of A°, which is Noetherian by [11, 3.2.3.(vi)].
By Lemma 2.12, $\widehat{A}$ is a Noetherian Banach algebra such that $\widehat{A}^\circ$ is Noetherian, so by Lemma 2.14, $\mathrm{H}^j(C^\bullet)^\circ$ is a finitely generated $\widehat{A^\circ}$-module. Now

$$T_s(\mathrm{H}^j(C^\bullet)^\circ) = \operatorname{Tor}_s^{A^\circ}(\mathcal{B}, \mathrm{H}^j(C^\bullet)^\circ) \cong \operatorname{Tor}_s^{\widehat{A^\circ}}(\mathcal{B} \otimes_{A^\circ} \widehat{A}^\circ, \mathrm{H}^j(C^\bullet)^\circ)$$

by [47, Proposition 3.2.9], as $\widehat{A^\circ}$ is flat over A° by [11, 3.2.3.(iv)].
By Noetherianity of $\widehat{A}^\circ$, $\mathrm{H}^j(C^\bullet)^\circ$ now admits a free resolution of finitely generated $\widehat{A}^\circ$-modules,

so that each

$$\mathrm{Tor}_s^{\widehat{A}^\circ}(\mathcal{B} \otimes \widehat{A}^\circ, \mathrm{H}^j(C^\bullet)^\circ)$$

is a finitely generated left $\mathcal{B} \otimes \widehat{A^\circ}$-module, as we assume this ring to be Noetherian.
So by Noetherianity, the π-torsion submodule is also finitely generated, and thus $T_s(\mathrm{H}^j(C^\bullet)^\circ)$ has in fact bounded π-torsion for each $s \geq 0$. Now apply the theorem above.

For the last isomorphism, note that we have

$$\widehat{B}\widehat{\otimes}_{\widehat{A}}\mathrm{H}^j(C^\bullet) \cong B\widehat{\otimes}_A\mathrm{H}^j(C^\bullet) \cong \mathrm{H}^j(B\widehat{\otimes}_A C^\bullet)$$

by Lemma 3.2 and the above, and we can remove the completion symbol over the first tensor product by Lemma 3.12. □

Corollary 3.23. *Suppose that both A and B are Banach algebras. Assume that for each j, the following is satisfied:*

(i) $\mathrm{H}^j(C^\bullet)$ *is a finitely generated A-module.*

(ii) $T_s((C^j)^\circ)$ *has bounded π-torsion for each $s \geq 0$.*

Then $B \otimes_A C^\bullet$ consists of strict morphisms and $B \otimes_A \mathrm{H}^j(C^\bullet) \cong \mathrm{H}^j(B\widehat{\otimes}C^\bullet)$.

Proof. This is just a special case of the above: $\widehat{A^\circ} = A^\circ$, as we assume A to be complete, so $\mathcal{B} \otimes_{A^\circ} \widehat{A^\circ} = \mathcal{B}$, which is Noetherian by assumption. □

Chapter 4

Kiehl's Proper Mapping Theorem

Kiehl's Proper Mapping Theorem, as given in [32] (see [12, sections 6.3, 6.4] for an account in English), establishes the coherence of higher direct images of coherent $\mathcal{O}_X$-modules along a proper morphism $f : X \to Y$. As we will employ the same techniques in the setting of coadmissible $\widehat{\mathcal{D}}_X$-modules, we give a detailed account of the key elements of the proof in [32], all suitably generalized for our purposes. We will not be too concerned with those parts of the proof which do not lend themselves to such a generalization.
First, we will recall the notion of a proper morphism in order to state the theorem. The proof then relies on a number of general results concerning strictly completely continuous morphisms, and an analysis of the Čech complex.

We begin with the notion of a proper morphism of rigid analytic spaces.

Definition 4.1 (see [12, Definition 6.3/6]). *Let $f : X \to Y$ be a morphism of rigid analytic spaces with Y being affinoid, and let $U \subseteq U' \subseteq X$ be open affinoid subspaces. We say U is **relatively compact** in U' (with respect to Y), or U lies in the interior of U' with respect to Y, if the map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(U')$ gives rise to a surjection*

$$\theta : \mathcal{O}_Y(Y)\langle x_1, \ldots, x_l\rangle \to \mathcal{O}_X(U')$$

for some integer l, such that

$$U \subset \{x \in U' : |f_i(x)| < 1\},$$

where f_i is the image of x_i under θ.

Definition 4.2 (see [12, Definition 6.3/8]). *A morphism $f : X \to Y$ between rigid analytic varieties is **proper** if it is separated and there exists an admissible affinoid covering* $(\operatorname{Sp} A_i)_{i\in I}$

of Y such that for all $i \in I$, $X_i = f^{-1}(\mathrm{Sp}\, A_i)$ has two finite admissible affinoid coverings (U_{ij}), (V_{ij}) with V_{ij} being relatively compact in U_{ij} with respect to $\mathrm{Sp}\, A_i$ for each j.

We quickly summarize basic examples and properties.

(i) Properness is stable under base change. If $f : X \to Y$ is a proper morphism and $g : Z \to Y$ is another morphism of rigid analytic K-spaces, then the corresponding map $X \times_Y Z \to Y \times_Y Z = Z$ is also proper (see [13, Lemma 9.6.2/1]). In particular, if $U \subseteq Y$ is an admissible open subspace, then the restriction $f|_{f^{-1}U} : f^{-1}U \to U$ is proper.

(ii) Properness is local on the base. A morphism $f : X \to Y$ is proper if and only there exists an admissible open covering (Y_i) of Y such that the restrictions $X \times_Y Y_i \to Y_i$ are all proper. (see [13, Proposition 9.6.2/3]).

(iii) A morphism of affinoid K-spaces $f : \mathrm{Sp}\, B \to \mathrm{Sp}\, A$ is **finite** if the corresponding algebra morphism $A \to B$ turns B into a finitely generated A-module. More generally, a morphism of rigid analytic K-spaces is finite if it is locally of this form. By [13, Proposition 9.6.2/5], any finite morphism is proper.
In particular, closed immersions are proper.

(iv) As in the work of Raynaud [36], we can view any (quasi-separated, quasi-paracompact) rigid analytic K-space as the generic fibre of some admissible **formal R-scheme** (see [12, Theorem 8.4/3]). If $f : X \to Y$ is a morphism of admissible formal R-schemes, then $f^{\mathrm{rig}} : X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ is a proper morphism between their generic fibres if and only if f is proper (see [34, Theorem 3.1]). We will not use the theory of formal schemes in this thesis, but note only as a consequence that the composition of proper morphisms is again proper (see [34, Corollary 3.2]).

(v) If $f : X \to Y$ is a morphism of K-schemes of locally finite type, then f^{an} is a proper morphism of rigid analytic K-spaces if and only if f is proper in the sense of algebraic geometry (see [33, Satz 2.16]).

(vi) In particular, if X is a K-scheme of finite type which is projective, e.g. $\mathbb{P}^n_K$ or a partial flag variety of a reductive algebraic group over K, then X^{an} is proper over the point $\mathrm{Sp}\, K$, and hence the projection $X^{\mathrm{an}} \times Y \to Y$ is a proper morphism for any rigid analytic K-space Y by the above.

A standard example of a proper morphism is thus what might be called a **projective** morphism, by which we mean a morphism of rigid analytic K-spaces $f : X \to Y$ which factors as a closed immersion $X \to (\mathbb{P}^n_K)^{\mathrm{an}} \times Y$ followed by the canonical projection. Since f is the composition of two proper morphisms, it is proper. As in algebraic geometry, most naturally occuring proper morphisms will be projective.

As properness is local on the base, we will often restrict our attention to the case when $Y = \operatorname{Sp} A$ is itself affinoid and satisfies the condition in Definition 4.2, i.e. we have two finite admissible affinoid coverings $\mathfrak{U} = (U_i)$, $\mathfrak{V} = (V_i)$ of X such that V_i is relatively compact in U_i for each i. Thus there exists a commutative diagram

$$\begin{array}{ccc} A\langle x_1, \ldots, x_l \rangle & & \\ \downarrow \theta_i & \searrow^{h_i} & \\ \mathcal{O}_X(U_i) & \xrightarrow{\text{res}} & \mathcal{O}_X(V_i) \end{array}$$

such that the map θ_i is surjective and

$$|h_i(x_j)|_{\sup} < 1$$

for any $j = 1, \ldots, l$.
In particular, $h_i(x_j)$ is topologically nilpotent in $\mathcal{O}_X(V_i)$ for each j (it follows from the comment after Definition 2.15 that this notion is independent of the choice of norm on $\mathcal{O}_X(V_i)$ - see [12, Corollary 3.1/18]).
Moreover, writing $U_{i_1 \ldots i_j}$ for the finite intersection $U_{i_1} \cap \cdots \cap U_{i_j}$, it follows from separatedness that all $U_{i_1 \ldots i_j}$ and $V_{i_1 \ldots i_j}$ are admissible open affinoid subspaces of X, and that $V_{i_1 \ldots i_j}$ is relatively compact in $U_{i_1 \ldots i_j}$ with respect to Y.

In this situation (i.e. when the covering $(\operatorname{Sp} A_i)$ in Definition 4.2 consists of a single affinoid) we say that $f : X \to Y$ is **elementary proper**.
We can now state Kiehl's Proper Mapping Theorem.

Theorem 4.3 ([32, Satz 2.6, Satz 3.5])**.** *Let $f : X \to Y$ be a proper morphism of rigid analytic K-spaces, and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then $\mathrm{R}^j f_* \mathcal{M}$ is a coherent $\mathcal{O}_Y$-module for all $j \geq 0$.*
Equivalently, if f is elementary proper, with $Y = \operatorname{Sp} A$, the following is satisfied for all $j \geq 0$:

(i) $\mathrm{H}^j(X, \mathcal{M})$ is a finitely generated A-module.

(ii) For any affinoid subdomain $\operatorname{Sp} B = U \subseteq Y$, the natural morphism

$$B \otimes_A \mathrm{H}^j(X, \mathcal{M}) \to \mathrm{H}^j(f^{-1}U, \mathcal{M})$$

is an isomorphism.

4.1 The category Ban_A and strictly completely continuous morphisms

Before turning to the proof of Theorem 4.3, we need to establish some terminology. The proof we are going to present in the next two sections follows that of [32], but we will verify all statements for the larger class of algebras we are considering.
Throughout, A will be a (not necessarily commutative) unital Noetherian Banach K-algebra, whose norm is determined by an R-algebra A° as its unit ball, which we assume to be also Noetherian. We summarize this by saying that A is a **strictly Noetherian Banach (NB) K-algebra**.
As before, we assume for simplicity that $|A| \setminus \{0\} = |K^*|$.
Note affinoid K-algebras are obvious examples of strictly NB algebras.

The module category we will be working with consists of all (left) Banach A-modules, together with continuous A-module morphisms. We call this category Ban_A.
We recall the following facts from chapter 2.

(i) An A-module morphism between normed spaces is continuous if and only if it is bounded (Lemma 2.1).

(ii) Any surjection in Ban_A is open (Open Mapping Theorem, Theorem 2.5).

(iii) Any finitely generated A-module is in Ban_A, equipped with a canonical topology (Lemma 2.14).

(iv) Ban_A is additive. Given two objects M, N, their direct sum $M \oplus N$ carries the structure of a Banach A-module with respect to the max norm (Lemma 2.2).

Note that for any $M, N \in \mathrm{Ban}_A$, the space of morphisms

$$\mathrm{Ban}_A(M,N) = \mathrm{Hom}_A^{\mathrm{cts}}(M,N)$$

is itself a left A-module, and may be equipped with the supremum norm

$$|f|_{\sup} := \sup_{x \neq 0} \frac{|f(x)|}{|x|}.$$

When we speak of a sequence of morphisms f_i converging to some $f \in \mathrm{Ban}_A(M,N)$, we mean uniform convergence, i.e. convergence with respect to the supremum norm.

We also need to define topologically free modules. Given an indexing set S, consider the A-module

$$\oplus_{s\in S} Ae_s,$$

equipped with the direct sum (maximum) norm, where $|ae_s| = |a|$ for each $s \in S$, $a \in A$. Its completion

$$F_S := \widehat{\oplus}_{s\in S} Ae_s$$

lies in Ban_A and satisfies the following universal property.

Proposition 4.4. *For M in Ban_A and any map $f : S \to M$ such that the set $\{|f(s)| : s \in S\}$ is bounded in $\mathbb{R}$, there exists a unique morphism $\phi : F_S \to M$ in Ban_A satisfying $\phi(e_s) = f(s)$ for each $s \in S$.*
Moreover, the operator norm of ϕ is $|\phi| = \sup_{s\in S} |f(s)|$.

Proof. By the universal property of (abstract) free modules, there exists a unique A-module morphism extending f, given by

$$\begin{aligned} \theta : \oplus_{s\in S} Ae_i &\to M \\ \sum a_s e_s &\mapsto \sum a_s f(s). \end{aligned}$$

Moreover, $|\theta(\sum a_s e_s)| = |\sum a_s f(s)| \leq \max |a_s||f(s)|$ for any finite sum $\sum a_s e_s$, so that θ is continuous by the boundedness assumption, with operator norm $|\theta| = \sup |f(s)|$. By continuity, θ extends uniquely to a continuous map ϕ between the completions $F_S \to M$, and $|\phi| = |\theta|$. □

We call F_S the **topologically free module** over S or the topologically free module (topologically) generated by S.
The following corollary follows immediately from the proposition above.

Corollary 4.5. *For any $M \in \mathrm{Ban}_A$, there exists a topologically free module $F \in \mathrm{Ban}_A$ and a surjection*

$$p : F \to M.$$

Lastly, we need to introduce a special kind of morphism in the category Ban_A.

Definition 4.6. *A morphism $f : M \to N$ in Ban_A is called **strictly completely continuous** if f is the limit of morphisms $f_i : M \to N$ in Ban_A such that $f_i(M^\circ)$ is a finitely generated A°-module for each i.*

It follows from Noetherianity of A° that this notion does not depend on a particular choice of norm on M, but only on its equivalence class.
We mention here that Kiehl phrases this definition slightly differently in [32, Definition 1.1],

since he does not assume K to be discretely valued (in particular, affinoid K-algebras might have a unit ball which is not Noetherian). It is easy to check that the two definitions are equivalent in Ban_A, where A is some strictly NB algebra.

We discuss one example which will feature in our proof of Theorem 4.3.

Lemma 4.7. *Let $F = \widehat{\bigoplus}_S Ae_s$ be a topologically free A-module over S. If $f : F \to M$ is a morphism in Ban_A such that for any $\epsilon > 0$ there are only finitely many $s \in S$ such that $|f(e_s)| \geq \epsilon$, then f is strictly completely continous.*

Proof. For any $\epsilon > 0$, denote by S_ϵ the finite set of $s \in S$ such that $|f(e_s)| \geq \epsilon$.
Given $s \in S$, consider the continuous A-module morphism

$$\begin{aligned} g_s : \quad F &\longrightarrow \quad M \\ \sum a_j e_j &\mapsto a_s f(e_s), \end{aligned}$$

i.e. we only consider the e_s part of f. Now we set for any $n \in \mathbb{N}$

$$f_n = \sum_{s \in S_{1/n}} g_s,$$

a continuous A-module morphism such that

$$f_n(F^\circ) \subseteq \sum_{s \in S_{1/n}} A^\circ f(e_s)$$

is a finitely generated A°-module.
It thus remains to show that the f_n tend to f. By Proposition 4.4, $|f - f_n| = \sup_{s \in S} |f(e_s) - f_n(e_s)|$. Now if $s \in S_{1/n}$, then $f(e_s) = f_n(e_s)$, and if s is not in $S_{1/n}$, then $f_n(e_s) = 0$ and $|f(e_s) - f_n(e_s)| < 1/n$ by construction. Thus $|f - f_n| < 1/n$, proving the result. □

Corollary 4.8. *Let $f : A\langle x_1, \ldots, x_n \rangle \to M$ be a morphism in Ban_A such that $f(x^i)$ tends to zero as $|i| \to \infty$. Then f is strictly completely continuous.*

We briefly record the following properties.

Lemma 4.9. *Let $f : M \to N$ be a strictly completely continuous morphism in Ban_A, and let L, G be in Ban_A. Then the following holds:*

(i) For any morphism $g : N \to G$ in Ban_A, the composition gf is strictly completely continuous.

(ii) For any morphism $h : L \to M$ in Ban_A*, the composition fh is strictly completely continuous.*

Proof. Let $(f_i : M \to N)$ be a sequence of morphisms in Ban_A as in Definition 4.6.

(i) For any continuous morphism g, the compositions gf_i converge to gf, and since $f_i(M^\circ)$ is finitely generated, so is $gf_i(M^\circ)$: if $f_i(M^\circ)$ is generated by $n_1, \dots, n_r$, then $gf_i(M^\circ)$ is generated by $g(n_1), \dots, g(n_r)$.

(ii) Since $|(f - f_i)h| \leq |f - f_i| \cdot |h|$, we know that $f_i h$ converges to fh. Since h is continuous, boundedness implies that there exists some integer a such that

$$h(L^\circ) \subseteq \pi^a M^\circ,$$

and thus $f_i h(L^\circ) \subseteq \pi^a f_i(M^\circ)$, a finitely generated A°-module by definition of the f_i (multiplication by π^a establishes an isomorphism $f_i(M^\circ) \cong \pi^a f_i(M^\circ)$). By Noetherianity of A°, $f_i h(L^\circ)$ is thus a finitely generated A°-module.

□

Lemma 4.10. *Let $f_1 : M_1 \to N_1, \dots, f_r : M_r \to N_r$ be a finite set of strictly completely continuous morphisms in* Ban_A*. Then the finite direct sum*

$$\oplus_{i=1}^r f_i : \oplus M_i \to \oplus N_i$$

is also a strictly completely continuous morphism in Ban_A*.*

Proof. The modules $\oplus M_i$ and $\oplus N_i$ are in Ban_A and $\oplus f_i$ is a morphism in Ban_A, as Ban_A is an additive category.
For each i, let f_i be the limit of A-module morphisms g_{ij} such that $g_{ij}(M_i^\circ)$ is finitely generated for each $j \in \mathbb{N}$. Then clearly $\oplus f_i$ is the uniform limit of $(\oplus_i g_{ij})_j$, and moreover

$$(\oplus_i g_{ij})(\oplus M_i^\circ) = \oplus_i g_{ij}(M_i^\circ)$$

is a finitely generated A°-module for any j, as required. □

4.2 Schwartz' Theorem and consequences

We will use the class of strictly completely continuous morphisms in the proof of Kiehl's Proper Mapping Theorem by applying Theorem 4.12, which is known as Schwartz' Theorem. First, we need a definition.

Definition 4.11. *Let N be an object of* Ban_A*, and let M be a submodule of N. We say M is* ***closed and of finite index*** *in N if M is a closed submodule such that the quotient module N/M is a finitely generated A-module.*

Theorem 4.12 ([32, Satz 1.2])**.** *Let $f : M \to N$ be a surjection in* Ban_A*, and let $g : M \to N$ be a strictly completely continuous homomorphism of A-modules. Then* $\mathrm{Im}(f+g)$ *is closed and of finite index in N.*

Before turning to the proof of Theorem 4.12, note that we have the following easy properties concerning submodules which are closed and of finite index.

Lemma 4.13. *Let N be in* Ban_A *and let M be some A-submodule of N. Suppose there exists some morphism*

$$f : N \to G$$

in Ban_A *such that $f(M)$ is closed and of finite index in G, and M contains the kernel of f. Then M is closed and of finite index in N.*

Proof. By continuity of f, we know that $f^{-1}(f(M))$ is closed in N. But $f^{-1}(f(M)) = M$, because M contains the kernel of f. Moreover, as abstract A-modules we have isomorphisms

$$N/M \cong (N/\ker f)/(M/\ker f) \cong f(N)/f(M) \leq G/f(M),$$

which is finitely generated by Noetherianity of A. □

Lemma 4.14. *Let N be in* Ban_A *and let M be some A-submodule of N. Suppose M contains some A-module M' which is closed and of finite index in N. Then M is itself closed and of finite index in N.*

Proof. Since M' is closed in N, the quotient semi-norm on N/M' is actually a complete norm (Lemma 2.4), and it follows from Lemma 2.14 that this gives rise to the canonical topology on the finitely generated A-module N/M', i.e. N/M' equipped with the quotient norm is in Ban_A. Now apply the above lemma to the natural projection $p : N \to N/M'$, noting that $p(M)$ is closed in N/M', as every A-submodule of a finitely generated A-module (with the canonical topology) is closed by Proposition 2.13, while finite generation of the quotient $(N/M')/p(M)$ follows directly from finite generation of N/M'. □

Lemma 4.15. *Let M and N be modules in* Ban_A *such that M is closed and of finite index in N. Let $f : N \to G$ be a surjection in* Ban_A*. Then $f(M)$ is closed and of finite index in G.*

Proof. By Lemma 4.14, the submodule $M + \ker f$ is closed and of finite index in N. Since $f(M) = f(M + \ker f)$, we can assume without loss of generality that $\ker f \subseteq M$.

By the Open Mapping Theorem (Theorem 2.5), f is open. By assumption, the set complement $N \setminus M$ is open in N, so $f(N \setminus M)$ is open in G. But since f is surjective and $\ker f \subseteq M$, we have

$$f(N \setminus M) = G \setminus f(M),$$

so that $f(M)$ is a closed submodule of G.
Moreover, we have the following isomorphisms as abstract A-modules

$$G/f(M) \cong (N/\ker f)/(M/\ker f) \cong (N/M),$$

which is finitely generated, since N/M is a finitely generated A-module by assumption. □

The content of the following lemma can be summarized as: small continuous displacements of surjections are still surjective.

Lemma 4.16 ([12, Lemma 1.3])**.** *Let $f : M \to N$ be a surjection in* Ban_A*. Then there exists a real number $c > 0$ such that for any $\epsilon \in \mathrm{Ban}_A(M, N)$ with $|\epsilon| < c$ (again with respect to the supremum norm), the map $f - \epsilon$ is still surjective.*

Proof. By the Open Mapping Theorem, f is open. Hence there exists a bounded K-linear map (not necessarily A-linear) $b : N \to M$ such that $fb = \mathrm{id}_N$.
Set $c = 1/|b|$. If $|\epsilon| < c$, define

$$g_n = \sum_{i=0}^{n} (b\epsilon)^i b \in \mathrm{Hom}_K(N, M).$$

Since $\mathrm{Hom}_K(N, M)$ is Banach (see [42, Proposition 3.3]), the g_n converge to the element

$$g = \sum_{i=0}^{\infty} (b\epsilon)^i b \in \mathrm{Hom}_K(N, M),$$

a continuous K-linear map from N to M, satisfying

$$(f - \epsilon)g = \mathrm{id}_N.$$

In particular, $f - \epsilon$ has to be surjective. □

Proof of Theorem 4.12. Since g is strictly completely continuous, we have a sequence of homomorphisms $g_i : M \to N$ converging to g such that each $g_i(M^\circ)$ is a finitely generated A°-module. Note in particular that for each i, the image $g_i(M)$ is a finitely generated A-module.
By Lemma 4.16, we can choose i large enough such that $f - (g_i - g)$ is surjective. We set

$h = f - (g_i - g)$, and note that $f + g = h + g_i$.
Let $K = \ker g_i$, which is closed by continuity of g_i and of finite index in M, since $M/K \cong g_i(M)$ as abstract A-modules. Thus by surjectivity of h, Lemma 4.15 implies that $h(K)$ is closed and of finite index in N. But now $h(K) = (h + g_i)(K)$ by definition of K, and $(h + g_i)(K)$ is contained in $(h + g_i)(M)$. Thus by Lemma 4.14, $(h + g_i)(M)$ is closed and of finite index in N, as required. □

We finally need an analogue of Satz 1.4 and Korollar 1.5 in [32].

Theorem 4.17 ([32, Satz 1.4])**.** *Let $f : M \to N$ be a morphism in Ban_A. Suppose that N is a closed submodule of some $G \in \mathrm{Ban}_A$ via the injection $j : N \to G$ such that the composition jf is strictly completely continuous. Then there exists a topologically free A-module F and a surjection $p : F \to M$ in Ban_A such that fp is strictly completely continuous.*

Proof. Note that by Corollary 4.5, there exists some topologically free A-module F and a surjection $p : F \to M$ in Ban_A, and by Lemma 4.9, the composition jfp is strictly completely continuous. Thus, replacing M by F, we can from now on assume that M is topologically free, and we only need to show that f is strictly completely continuous.

Since jf is strictly completely continuous, it is the limit of morphisms $h_i : M \to G$ such that each $h_i(M^\circ)$ is a finitely generated A°-module. The general idea of the proof is now as follows: we would like to replace the h_i by morphisms which factor through N, exhibiting f as a strictly completely continuous morphism. Using the freeness of M, it will be enough to specify suitable images of topological generators of M.

Write $M = \widehat{\bigoplus}_{s\in S} A e_s$ and $M^\circ = \widehat{\bigoplus}_{s\in S} A^\circ e_s$. Given $0 < \epsilon < 1$, choose i such that $|jf - h_i| \le \epsilon$. Let $y_1, \dots, y_r$ be generators of $h_i(M^\circ)$ over A°, and write

$$h_i(e_s) = \sum_{t=1}^{r} a_t^s y_t$$

for $a_t^s \in A^\circ$, $s \in S$, $t = 1, \dots, r$.
Since $y_t \in h_i(M^\circ)$, we can choose $x_t \in M^\circ$ such that $h_i(x_t) = y_t$, and set $z_t = f(x_t) \in N$.

Define

$$f_s = \sum_{t=1}^{r} a_t^s z_t \in N$$

and consider the map

$$\begin{aligned} S &\to N \\ s &\mapsto f_s. \end{aligned}$$

Since $a_t^s \in A^\circ$ for each $s \in S$, $t = 1, \dots r$, we have $|f_s| \leq \max_t |z_t|$ for each $s \in S$, so that this map has bounded image. Hence we can invoke Proposition 4.4 to obtain a morphism $\phi : M \to N$ in Ban_A with the property that $\phi(e_s) = f_s$ for each $s \in S$. We write ϕ_ϵ when we want to stress that the morphism depends on a value ϵ.

Note that for every $0 < \epsilon < 1$, $\phi_\epsilon(M^\circ) \subseteq \sum_{t=1}^r A^\circ z_t$ is finitely generated by Noetherianity of A°, and we claim that $|f - \phi_\epsilon| \leq \epsilon$. For each $s \in S$, we have

$$\begin{aligned} |f(e_s) - \phi_\epsilon(e_s)|_N &= |jf(e_s) - j\phi_\epsilon(e_s)|_G \\ &\leq \max\{|jf(e_s) - h_i(e_s)|, |h_i(e_s) - j\phi_\epsilon(e_s)|\} \\ &= \max\left\{|jf(e_s) - h_i(e_s)|,\ |\sum a_t^s(y_t - j(z_t))|\right\} \\ &\leq \max\{\epsilon, |y_t - j(z_t)| :\ t = 1, \dots, r\} \\ &= \max\{\epsilon, |h_i(x_t) - jf(x_t)| :\ t = 1, \dots, r\}, \end{aligned}$$

and hence $|f(e_s) - \phi_\epsilon(e_s)| \leq \epsilon$.
Therefore Proposition 4.4 implies that $|f - \phi_\epsilon| \leq \epsilon$.
Thus we have shown that f is the limit of morphisms $\phi_\epsilon : M \to N$ with the property that $\phi_\epsilon(M^\circ)$ is a finitely generated A°-module, proving that f is strictly completely continuous. □

Theorem 4.18 ([32, Korollar 1.5]). *Let $f : M \to N$ be a surjection in Ban_A, and let $g : M \to N$ be another morphism in Ban_A. Suppose N is a closed submodule of some $G \in \mathrm{Ban}_A$ via the injection $j : N \to G$, and suppose that the composition jg is strictly completely continuous. Then $\mathrm{Im}(f + g)$ is closed and of finite index in N.*

Proof. By Theorem 4.17, there exists a topologically free module F and a surjection $p : F \to M$ in Ban_A such that gp is strictly completely continuous. By surjectivity of p, we have that fp is still surjective, so Theorem 4.12 implies that $\mathrm{Im}(fp + gp) = \mathrm{Im}((f + g) \circ p)$ is closed and of finite index in N. But since p is surjective, this is the same as $\mathrm{Im}(f + g)$, and the result follows. □

Note that the same result holds for $\mathrm{Im}(f - g)$. If jg is strictly completely continuous, written as the limit of some $(h_i)_i$, then $j \circ (-g)$ is strictly completely continuous, as it is the limit of $(-h_i)_i$.

These results will be applied in the proof of Theorem 4.3 using the following observation, which in [31] is attributed to Cartan–Serre.

Proposition 4.19. *Let $C^\bullet$, $D^\bullet$ be two cochain complexes in Ban_A, and let $\alpha = (\alpha_i \in \mathrm{Ban}_A(C^i, D^i))$ be a quasi-isomorphism. Assume further that for each i there exists $F^i \in \mathrm{Ban}_A$ together with a surjection $\beta_i : F^i \to C^i$ such that $\alpha_i\beta_i$ is a stricly completely continuous morphism of A-modules. Then $\mathrm{H}^i(D^\bullet)$ is a finitely generated A-module.*

Proof. This proof can be found in [32] as part of the proof of Satz 2.5 and Satz 2.6. In a slight abuse of notation, all differentials will be denoted by the same letter d.
Let G^i be the inverse image of $Z^i(C^\bullet) = \ker d \subseteq C^i$ in F^i. Note that $Z^i(C^\bullet)$ is closed in C^i, so it is complete when equipped with the subspace norm. Similarly it follows from continuity that G^i is closed in F^i, and hence an object in Ban_A by Lemma 2.3.
We wish to apply Theorem 4.18 to

$$\begin{aligned} G^i \oplus D^{i-1} &\to Z^i(D^\bullet) \\ (a, b) &\mapsto d(b) = (\alpha_i\beta_i(a) + d(b)) - \alpha_i\beta_i(a). \end{aligned}$$

We will verify the conditions of Theorem 4.18.

Firstly, we claim that the map

$$\begin{aligned} f : G^i \oplus D^{i-1} &\to Z^i(D^\bullet) \\ (a, b) &\to \alpha_i\beta_i(a) + d(b) \end{aligned}$$

is a surjection in Ban_A.
We have already shown that each of the modules appearing is an object in Ban_A (recall that Ban_A is closed under taking finite direct sums with the corresponding max norm, i.e. is additive), and since α_i, β_i and d are all bounded, f is clearly also bounded. For surjectivity, note that we assume that α_i induces an isomorphism of cohomology groups, and hence the map

$$\begin{aligned} Z^i(C^\bullet) \oplus D^{i-1} &\to Z^i(D^\bullet) \\ (a, b) &\mapsto \alpha_i(a) + d(b) \end{aligned}$$

is surjective. Since β_i is surjective, it follows that the restriction $\beta_i|_{G^i} : G^i \to Z^i(C^\bullet)$ is surjective by definition of G^i, proving that the composition $f : G^i \oplus D^{i-1} \to Z^i(D^\bullet)$ is also surjective, as required.

Secondly, we need to show that the map

$$\begin{aligned} g : G^i \oplus D^{i-1} &\to Z^i(D^\bullet) \\ (a,b) &\mapsto \alpha_i\beta_i(a) \end{aligned}$$

is strictly completely continuous after composition with the injection $j : Z^i(D^\bullet) \to D^i$.
Again, it is straightforward to see that g is a morphism in Ban_A. Note that it fits into the commutative diagram

$$\begin{array}{ccccc} G^i \oplus D^{i-1} & & \xrightarrow{\quad g \quad} & & Z^i(D^\bullet) \\ \downarrow{\scriptstyle p} & & & & \\ G^i & & & & \downarrow{\scriptstyle j} \\ \downarrow{\scriptstyle \iota} & & & & \\ F^i & \xrightarrow{\beta_i} & C^i & \xrightarrow{\alpha_i} & D^i \end{array}$$

where the bottom row is strictly completely continuous by assumption, and the map p is the projection onto the first factor.
By Lemma 4.9, the composition $\alpha_i\beta_i\iota p$ is strictly completely continuous.
It follows by commutativity of the diagram that the composition jg is a strictly completely comtinuous morphism of A-modules, as required.

We can therefore apply Theorem 4.18 (and the remark after its proof) to conclude that $\operatorname{Im} d = \operatorname{Im}(f-g)$ is closed and of finite index in $Z^i(D^\bullet)$, i.e.

$$\mathrm{H}^i(D^\bullet) = Z^i(D^\bullet)/d(D^{i-1})$$

is a finitely generated A-module. □

Corollary 4.20. *In the situation of Proposition 4.19, $D^\bullet$ is a cochain complex with strict morphisms.*

Proof. By the above, $\operatorname{Im} d^{j-1}$ is a closed subspace of $Z^j(D^\bullet)$, which is in turn a closed subspace of D^j by continuity. Thus we can apply Lemma 3.7 to show that d^{j-1} is strict for each j. □

We briefly sketch how the above results are used in the proof of Theorem 4.3.
Without loss of generality, we can consider the case of an elementary proper morphism $f : X \to Y = \operatorname{Sp} A$, so that X is covered by two finite admissible open coverings $\mathfrak{U} = (U_i)$ and $\mathfrak{V} = (V_i)$ with V_i relatively compact in U_i with respect to Y for each i.
Applying Proposition 4.19 to the natural restriction map $\check{C}^\bullet(\mathfrak{U}, \mathcal{M}) \to \check{C}^\bullet(\mathfrak{V}, \mathcal{M})$ shows that $\check{\mathrm{H}}^j(\mathfrak{V}, \mathcal{M})$ is a finitely generated A-module for each $j \geq 0$, and this is isomorphic to $\mathrm{H}^j(X, \mathcal{M})$

by Theorem 2.27. We will apply the same argument for $\widehat{\mathcal{D}}$-modules later.
To prove the localizaton property, Kiehl argues by induction on the Krull dimension of A, using a variant of the Formal Function Theorem (see [32, Theorem 3.4]) for the inductive step. This strategy will find no analogue in our $\widehat{\mathcal{D}}$-module version of the result, which is why we omit further details.

4.3 An immediate consequence: Stein factorization

We remain in the setting where $f : X \to Y = \mathrm{Sp}A$ is an elementary proper morphism of rigid analytic K-spaces. By Kiehl's Proper Mapping Theorem, $B = \mathcal{O}_X(X)$ is finitely generated as an A-module, so a fortiori is of topologically finite type as an A-algebra. Hence B is an affinoid algebra and the corresponding map $A \to B$ induces a factorization

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathrm{Sp}B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} \\ & & \mathrm{Sp}A \end{array}$$

Using the same coverings (U_i) and (V_i), we see that g is a proper morphism (separatedness follows from [13, Proposition 9.6.2/4]), and h is a morphism of finite type by definition. This is the elementary version of **Stein factorization**, a more general version of which can be obtained by glueing.

Proposition 4.21 (see [13, Proposition 9.6.3/5])**.** *Let $f : X \to Y$ be a proper morphism of rigid analytic K-spaces. Then there exists a rigid analytic K-space Z and a factorization*

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} \\ & & Y \end{array}$$

where g is a surjective proper morphism with connected fibres and $g_*\mathcal{O}_X \cong \mathcal{O}_Z$, and where h is finite.

Chapter 5

Fréchet–Stein algebras and coadmissible modules

In this chapter we will introduce the sheaf of differential operators $\widehat{\mathcal{D}}_X$ on a rigid analytic space as the Fréchet completion of the enveloping algebra of the tangent sheaf. As in [5], we will treat this as one particular manifestation of a more general contruction by considering Fréchet completions of enveloping algebras of **Lie algebroids**.

We will show that these are well-behaved in the sense that we obtain Fréchet–Stein algebras over every affinoid subspace, and we define the notion of coadmissible modules as the analogue of coherent modules in this setting. We will also construct sheaves $\mathcal{D}_n = \mathscr{U}_n(\mathcal{T})$ and M_n exhibiting the structures of $\widehat{\mathcal{D}}_X$ and a coadmissible module $\mathcal{M}$ not just section by section, but in the category of sheaves.

Most of these results can already be found in [5], under additional smoothness assumptions for a lattice of the Lie algebroid. The main part of this chapter will be concerned with removing these conditions from our results.

5.1 The enveloping algebra $U(L)$ and its Fréchet completion $\widehat{U(L)}$

For the entirety of this chapter, A will denote an affinoid K-algebra with affine formal model $\mathcal{A}$.

Our first fundamental object of study will be the enveloping algebra $U(L)$ of an $(R, \mathcal{A})$-Lie algebra L, as described in [37]. It can be thought of as a natural generalization of the universal

enveloping algebra of a Lie algebra.
Let S be a commutative base ring and B a commutative S-algebra. Then a B-module L is a (S,B)**-Lie algebra** if L is also an S-Lie algebra equipped with a B-linear Lie algebra homomorphism

$$\rho : L \to \mathrm{Der}_S(B),$$

called the anchor map, satisfying

$$[x, by] = b[x, y] + \rho(x)(b)y$$

for $x, y \in L$, $b \in B$, i.e. the Lie bracket respects the B-action via a Leibniz rule.
A standard example will be the (K, A)-Lie algebra $L = \mathrm{Der}_K(A)$, which is isomorphic to the global sections of the tangent sheaf $\mathcal{T}_X(X)$ for $X = \mathrm{Sp}\, A$.

Given an (S, B)-Lie algebra L, Rinehart defined in [37] the enveloping algebra $U_B(L)$, which comes equipped with two canonical injections

$$i_B : B \to U_B(L),\ \ i_L : L \to U_B(L)$$

and satisfies the following universal property.

Proposition 5.1 (see [5, 2.1])**.** *Let T be an associative S-algebra together with an S-algebra morphism $j_B : B \to T$ and an S-Lie algebra morphism $j_L : L \to T$, satisfying*

$$j_L(bx) = j_B(b)j_L(x)\ \forall b \in B,\ \ x \in L$$

and

$$[j_L(x), j_B(b)] = j_B(\rho(x)(b))\ \forall b \in B,\ \ x \in L.$$

Then there exists a unique S-algebra morphism $\phi : U_B(L) \to T$ such that $j_B = \phi \circ i_B$ and $j_L = \phi \circ i_L$.

Note that $U_B(L)$ comes equipped with a natural degree filtration, setting $F_0 = B$, $F_1 = B+L$, $F_i = F_1 \cdot F_{i-1}$ for $i \geq 2$.
The following analogue of the Poincaré–Birkhoff–Witt Theorem is also due to Rinehart.

Theorem 5.2 ([37, Theorem 3.1])**.** *Let L be an (S, B)-Lie algebra which is finitely generated as a B-module. Then the morphism*

$$\mathrm{Sym}_B\, L \to \mathrm{gr}\, U_B(L)$$

is surjective, and is an isomorphism if L is projective.

In particular, if B is Noetherian and L is finitely generated, then $U_B(L)$ is a Noetherian K-algebra.

Lemma 5.3. *Let L be an (S, B)-Lie algebra which is a finitely generated projective B-module. Then $U_B(L)$ is a flat left B-module.*

Proof. By [14, III. 6.6, Corollary to Theorem 1], $\operatorname{Sym}_B L$ is a projective B-module, so each graded piece $(\operatorname{Sym}_B L)_n$ is projective and hence flat. The short exact sequence

$$0 \to F_{n-1}U_B(L) \to F_nU_B(L) \to (\operatorname{Sym}_B L)_n \to 0$$

then ensures inductively that $F_nU_B(L)$ is a flat B-module for each n, and since tensor products commute with direct limits, $U_B(L)$ is also flat. □

In analogy to the procedure of analytification (see section 2.5), we will be studying the following structure.

Given a (K, A)-Lie algebra L which is finitely generated as an A-module, a **lattice** $\mathcal{L}$ is defined to be a finitely generated $\mathcal{A}$-submodule of L such that $\mathcal{L} \otimes_R K = L$. We call $\mathcal{L}$ an **$(R, \mathcal{A})$-Lie lattice** if moreover $\mathcal{L}$ is closed under the Lie bracket and the $\mathcal{L}$-action on A induced by ρ preserves $\mathcal{A}$ (in particular, $\mathcal{L}$ is an $(R, \mathcal{A})$-Lie algebra). In this case $\pi^n\mathcal{L}$ is an $(R, \mathcal{A})$-Lie lattice for any non-negative integer n, and we can form the **Fréchet completion**

$$\widehat{U(L)} = \varprojlim \widehat{U_{\mathcal{A}}(\pi^n\mathcal{L})}_K.$$

It turns out that the Fréchet completion is independent of the choice of formal model $\mathcal{A}$ and Lie lattice $\mathcal{L}$, as shown in [5, section 6.2].

We look at an easy example.

Any finite-dimensional K-Lie algebra $\mathfrak{g}$ is a (K, K)-Lie algebra with ρ being the zero map. Choosing an ordered K-basis $x_1, \ldots, x_m$ such that the R-span of the x_i is closed under the Lie bracket, we get

$$\widehat{U(\mathfrak{g})} = \{\sum_{i \in \mathbb{N}^m} a_i x^i : \ a_i \in K, \ |a_i||\pi|^{-|i|n} \to 0 \text{ as } |i| \to \infty \ \forall n \geq 0\},$$

the Arens–Michael envelope of $\mathfrak{g}$. This algebra, which is closely related to the representation theory of the associated p-adic Lie group, was already studied in [38], [39].

We can think of this as a non-commutative version of analytic functions on $\mathfrak{g}^*$.

Under suitable smoothness conditions, the algebras $\widehat{U(L)}$ behave indeed quite similarly to the algebra of analytic functions on $(\mathbb{A}^m)^{\mathrm{an}}$, say.

Definition 5.4. *A K-algebra U is called a (left, two-sided)* ***Fréchet–Stein algebra*** *if $U = \varprojlim U_n$ is an inverse limit of countably many (left, two-sided) Noetherian Banach K-algebras U_n, such that for every n the following is satisfied:*

(i) The morphism $U_{n+1} \to U_n$ makes U_n a flat U_{n+1}-module (on the right, on both sides).

(ii) The morphism $U_{n+1} \to U_n$ has dense image.

From now on, we will understand 'Fréchet–Stein' to mean 'two-sided Fréchet–Stein' throughout.
It is not difficult to see that $\widehat{U(\mathfrak{g})}$ is in fact a Fréchet–Stein algebra, and that the same holds in general with $\mathfrak{g}$ being replaced by any (K, A)-Lie algebra which is a free finitely generated A-module. The only non-trivial ingredient for this is Rinehart's theorem as given in Theorem 5.2, which allows for explicit calculations.

Later, we want to construct a sheaf $\widehat{\mathcal{D}}_X$ on a smooth space X, whose sections over affinoid subspaces $V = \operatorname{Sp} B$ are precisely $\widehat{U_B(\mathcal{T}(V))}$, where $\mathcal{T}$ is the tangent sheaf. Again, Rinehart's theorem and the results above look promising, since projectiveness of sections of the tangent sheaf corresponds to smoothness of X, so we should expect to get Fréchet–Stein algebras as sections over every affinoid.
Note however that this result is not immediate. Roughly, we want to apply Rinehart's theorem not to the projective module $\mathcal{T}(V)$ itself, but rather to a lattice inside it – which might not be a priori a projective $\mathcal{A}$-module. In the free module case, this was not a difficulty, as we could simply choose the $\mathcal{A}$-module generated by the free generators (scaled down suitably to make it a Lie lattice). This argument has been generalized in the obvious way to the case when $\mathcal{L}$ is a projective $\mathcal{A}$-module, see [5, chapter 6].
We now present a proof which avoids this subtlety.

Theorem 5.5. *Let L be a (K, A)-Lie algebra which is a finitely generated projective A-module, and let $\mathcal{L}$ be any $(R, \mathcal{A})$-Lie lattice in L. Then*

$$\widehat{U(L)} = \varprojlim \widehat{U_{\mathcal{A}}(\pi^n \mathcal{L})}_K$$

is a Fréchet–Stein algebra.

Before we turn to the proof, we need to establish some lemmas. Throughout, U_n will denote the image of $U(\pi^n \mathcal{L})$ in $U_A(L)$, i.e. the $\mathcal{A}$-subalgebra generated by $\mathcal{A}$ and $\pi^n \mathcal{L}$.
Note that by [5, Lemma 2.5], we have $\widehat{U(\pi^n \mathcal{L})}_K \cong \widehat{U_n}_K$, and we can therefore think of $\widehat{U(\pi^n \mathcal{L})}_K$

as the completion of $U_A(L)$ with respect to the semi-norm with unit ball U_n (see Theorem 2.11). Once U_n is π-adically separated, this semi-norm is in fact a norm.

Our proof relies mainly on the following result.

Proposition 5.6 ([21, Lemma 5.3.9, Proposition 5.3.10]). *Let V_1 be a (not necessarily commutative) π-torsionfree, π-adically separated left (resp. right) Noetherian R-algebra, and let V_0 be an R-subalgebra of $V_1 \otimes_R K$ containing V_1, equipped with an exhaustive increasing filtration $F_0 \subseteq F_1 \subseteq \dots$ by R-submodules such that the following is satisfied:*

(i) For each $i, j \geq 0$, $F_i F_j \subseteq F_{i+j}$.

(ii) $F_0 = V_1$.

(iii) The associated graded algebra $\operatorname{gr} V_0$ is finitely generated over its zeroth graded piece V_1 by central elements.

Then $\widehat{V_1}_K$ and $\widehat{V_0}_K$ are left (resp. right) Noetherian K-algebras, and the natural map $\widehat{V_1}_K \to \widehat{V_0}_K$ makes $\widehat{V_0}_K$ a flat right (resp. left) $\widehat{V_1}_K$-module.

In the reference, this result is only given for the case $R = \mathbb{Z}_p$ and only for left Noetherian algebras, but the proof naturally generalizes. We will verify that all conditions of Proposition 5.6 are satisfied for $V_0 = U_n$, $V_1 = U_{n+1}$ for sufficiently large n.

Lemma 5.7. *If L is a free A-module, then U_n is π-adically separated for sufficiently large n.*

Proof. Let $\partial_1, \partial_2, \dots, \partial_m$ be an ordered A-basis of L, suitably rescaled such that $\oplus \mathcal{A}\partial_i$ is an $(R, \mathcal{A})$-Lie lattice. We are first going to assume that $\mathcal{L} = \oplus \mathcal{A}\partial_i$.
In this case it follows immediately that U_0 (and hence any U_n for $n \geq 0$) is π-adically separated. Identifying $U_A(L)$ as a K-vector space with the space of ordered polynomial expressions in the ∂_i with coefficients in A by Rinehart's theorem, U_0 corresponds to the subset consisting of polynomials with coefficients in $\mathcal{A}$, which is π-adically separated since $\mathcal{A}$ is.
Now let $\mathcal{L}$ be an arbitrary $(R, \mathcal{A})$-Lie lattice. Since $\mathcal{L}$ is finitely generated, there exists some integer n such that

$$\pi^n \mathcal{L} \subseteq \oplus \mathcal{A}\partial_i,$$

and thus U_n is contained in the $\mathcal{A}$-subalgebra of $U_A(L)$ generated by $\mathcal{A}$ and $\oplus \mathcal{A}\partial_i$. Therefore U_n is π-adically separated by the first part of the proof. □

In fact, we can go further and drop the freeness condition.

Lemma 5.8. *Let L be a (K, A)-Lie algebra which is a finitely generated projective A-module, and let $\mathcal{L}$ be any $(R, \mathcal{A})$-Lie lattice in L. Then $U_n \subseteq U_A(L)$ is π-adically separated for sufficiently large n.*

Proof. Since L is a finitely generated A-module, we obtain an associated coherent $\mathcal{O}_X$-module $\mathscr{L}$ on $X = \operatorname{Sp} A$. As L is projective, $\mathscr{L}$ is locally free, so there exists a finite admissible covering (X_i) of X by affinoid subdomains such that $\mathscr{L}|_{X_i}$ is a free $\mathcal{O}_{X_i}$-module (see Proposition 2.29). Write $X_i = \operatorname{Sp} A_i$, and let $\mathcal{A}_i$ be an affine formal model of A_i containing $\mathcal{A}$, which exists by Lemma 3.10. Now $\mathscr{L}(X_i) = A_i \otimes_A L$ is a (K, A_i)-Lie algebra for any i (see [5, Corollary 2.4]). We write $\mathcal{L}_i$ for the image of $\mathcal{A}_i \otimes_{\mathcal{A}} \mathcal{L}$ in $A_i \otimes L$. Replacing $\mathcal{L}$ by $\pi^m \mathcal{L}$ for sufficiently large m such that $\mathcal{L}(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for all i (which can be done as each formal model is topologically of finite type), we can assume that $\mathcal{L}_i$ is a $(R, \mathcal{A}_i)$-Lie lattice inside the free A_i-module $A_i \otimes_A L$.
In particular, if we denote the image of $U_{\mathcal{A}_i}(\pi^n \mathcal{L}_i)$ inside $U_{A_i}(A_i \otimes L)$ by V_i^n, then Lemma 5.7 implies that for each i, V_i^n is π-adically separated for sufficiently large n.
Restriction to each element of the covering yields the following commutative diagram.

$$\begin{array}{ccc} U_{\mathcal{A}}(\pi^n \mathcal{L}) & \xrightarrow{\quad g \quad} & U_A(L) \\ \downarrow & & \downarrow f \\ \oplus_i U_{\mathcal{A}_i}(\pi^n \mathcal{L}_i) & \xrightarrow{\oplus g_i} & \oplus U_{A_i}(A_i \otimes_A L) \end{array}$$

Note that $U_{A_i}(A_i \otimes_A L) \cong A_i \otimes_A U_A(L)$ by [5, Proposition 2.3], and that the morphism f is thus injective by [13, Corollary 8.2.1/5]. Therefore we can identify $U_n = \operatorname{Im} g$ with the image of fg. By commutativity of the diagram, this is contained in $\oplus_i \operatorname{Im}(g_i) = \oplus_i V_i^n$ and hence is π-adically separated for sufficiently large n by Lemma 5.7. □

Proof of Theorem 5.5. By Rinehart, $\operatorname{Sym}(\pi^n \mathcal{L}) \to \operatorname{gr} U(\pi^n \mathcal{L})$ is a surjection, so $\operatorname{gr} U(\pi^n \mathcal{L})$ is Noetherian. Hence $U(\pi^n \mathcal{L})$ is Noetherian by [35, Corollary D.IV.5], making $\widehat{U(\pi^n \mathcal{L})}_K$ a Noetherian Banach algebra by Lemma 2.12. The denseness condition is straightforward, as every term $\widehat{U(\pi^n \mathcal{L})}_K$ contains $U_A(L)$ as a dense subspace. It remains to show flatness of the connecting maps.

As before, let U_n denote the image of $U(\pi^n \mathcal{L})$ in $U_A(L)$, and write $U_0 = U$. Replacing $\mathcal{L}$ by $\pi^m \mathcal{L}$ for sufficiently large m, we can assume that U_n is π-adically separated for all $n \geq 0$. By [5, Lemma 2.5], it is sufficient to prove that $\widehat{U_1}_K \to \widehat{U}_K$ is flat on both sides.
We are going to apply Proposition 5.6, using the same kind of filtration as done in [3, Lemma 3].
Give U the quotient filtration $F_i U$ induced from the surjection $U_{\mathcal{A}}(\mathcal{L}) \to U$, i.e. $F_0 U = \mathcal{A}$, $F_1 U = \mathcal{L} + \mathcal{A}$, $F_i U = F_1 U \cdot F_{i-1} U$ for $i \geq 2$. Now define a new filtration by

$$F_i' U = U_1 \cdot F_i U.$$

Note that U is π-torsionfree (as it is a subring of the K-algebra $U(L)$), π-adically separated by assumption, left and right Noetherian (image of the Noetherian ring $U(\mathcal{L})$), and $U_1 \subseteq U \subseteq U_1 \otimes_R K = U_A(L)$.

We now check the conditions specified in Proposition 5.6, in analogy to the proof of [3, Lemma 3].
Condition (ii) is clear: $F'_0 U = U_1$.
For condition (i) we need to show that $F'_i U \cdot F'_j U \subseteq F'_{i+j} U$. For this it clearly suffices to show that $F_i U \cdot U_1 \subseteq F'_i U$. Since $[\mathcal{L}, \pi\mathcal{L}] \subseteq \pi\mathcal{L}$, we have $[\mathcal{L}, U_1] \subseteq U_1$, i.e.

$$\mathcal{L} \cdot U_1 \subseteq U_1 \cdot \mathcal{L} + U_1.$$

Then inductively $F_i U \cdot U_1 \subseteq F'_i U$ as required.
Condition (iii) requires that $\operatorname{gr}' U$ is finitely generated as an algebra over U_1 by central elements. Since $F_i U \subseteq F'_i U$ for each i, we have algebra morphisms

$$\operatorname{Sym} \mathcal{L} \longrightarrow \operatorname{gr} U_{\mathcal{A}}(\mathcal{L}) \longrightarrow \operatorname{gr} U \xrightarrow{\sigma} \operatorname{gr}' U,$$

with the first two arrows being surjections. If $\mathcal{L}$ is generated by $\partial_1, \ldots, \partial_m$ as an $\mathcal{A}$-module, write $\overline{\partial}_j$ for the symbol of ∂_j in $\operatorname{gr} U$. We claim that $\operatorname{gr}' U$ is generated by the images of the $\overline{\partial}_j$'s over U_1, i.e. it is generated by U_1 and the image of σ.
First, let us establish the following notation: for $u \in U$, write $\overline{u}$ for its symbol in $\operatorname{gr} U$ and $\overline{u}'$ for its symbol in $\operatorname{gr}' U$.
Let

$$x \in \frac{F'_i U}{F'_{i-1} U},$$

and assume without loss of generality that there exist $u \in U_1$, $y \in F_i U$ such that

$$\overline{u \cdot y}' = x.$$

If $y \in U_1 F_{i-1} U = F'_{i-1} U$, we have $x = 0$ and x is obviously in the subalgebra generated by $\operatorname{Im} \sigma$ over U_1. If $y \notin U_1 F_{i-1} U$, both $\overline{y}$ and $\overline{y}'$ live in degree i, and we therefore have $\overline{y}' = \sigma(\overline{y})$. Since $F'_0 U = U_1$ (and $\overline{u \cdot y}'$ also lives in degree i), it follows that

$$x = \overline{u \cdot y}' = \overline{u}' \cdot \overline{y}' = \overline{u}' \cdot \sigma(\overline{y}),$$

proving the claim.
It remains to show that the $\sigma(\overline{\partial_j})$ are central. By commutativity of $\operatorname{Sym} \mathcal{L}$, we are done if we can show that these generators commute with everything in $F'_0 U = U_1$.
But since $[\mathcal{L}, U_1] \subseteq U_1$ in U, we see that for any index j and any $x \in U_1$, $\partial_j x - x\partial_j \in U_1 = F'_0 U$,

so the $\sigma(\overline{\partial_j})$'s commute with U_1 in $\operatorname{gr}' U$.
Hence we can apply Proposition 5.6 to get that the morphism

$$\widehat{U_1}_K \to \widehat{U}_K$$

is flat on the right and on the left.

Thus $\widehat{U(L)}$ is an inverse limit of Noetherian Banach algebras $\widehat{U(\pi^n \mathcal{L})}_K$ with flat connecting morphisms and dense images, as required. □

Note that we have shown that the Fréchet-Stein structure of $\widehat{U(L)}$ is exhibited by *any* $(R, \mathcal{A})$-Lie lattice $\mathcal{L}$, in the sense that for sufficiently large n, $\widehat{U(\pi^n \mathcal{L})}_K$ plays the role of the U_n in Definition 5.4.
Furthermore, each of the terms $\widehat{U(\pi^n \mathcal{L})}_K$ is a strictly NB algebra, as defined in chapter 4.

5.2 The sheaf $\widehat{\mathscr{U}(\mathscr{L})}$

We will now introduce the sheaf analogue of the theory in the previous section. The notion of a (K, A)-Lie algebra gets replaced by that of a Lie algebroid $\mathscr{L}$ on a rigid analytic K-space X, and we will construct the sheaf $\widehat{\mathscr{U}(\mathscr{L})}$ of Fréchet completed enveloping algebras. Our main interest lies in the case where $\mathscr{L} = \mathcal{T}_X$.

Definition 5.9 (see [5, Definition 9.1])**.** *A **Lie algebroid** on X is a pair $(\rho, \mathscr{L})$ such that $\mathscr{L}$ is a locally free $\mathcal{O}_X$-module of finite rank on X_{rig} which is also a sheaf of K-Lie algebras, and $\rho : \mathscr{L} \to \mathcal{T}_X$ is an $\mathcal{O}$-linear map of sheaves of Lie algebras, satisfying*

$$[x, ay] = a[x, y] + \rho(x)(a)y$$

for any $x, y \in \mathscr{L}(U)$, $a \in \mathcal{O}_X(U)$, U an admissible open subset of X.

Note that $\mathcal{T}_X$ is naturally a Lie algebroid on X whenever X is smooth.

Let $X = \operatorname{Sp} A$ be an affinoid K-space and assume that $\mathscr{L}$ is a Lie algebroid on X (the general case will then follow from glueing). We write $L = \mathscr{L}(X)$.
Fix an affine formal model $\mathcal{A} \subset A$ and an $(R, \mathcal{A})$-Lie lattice $\mathcal{L} \subset L$. We will first need to introduce some notions of affinoid subdomains Y of X behaving well with respect to $\mathcal{L}$.

Definition 5.10 (see [5, Definitions 3.1, 3.2])**.** *Let $Y = \operatorname{Sp} B$ be an affinoid subdomain of X, and let $\sigma : A \to B$ be the restriction map. We say U is $\mathcal{L}$-**admissible** if B has an affine formal* *model $\mathcal{B}$ such that $\sigma(\mathcal{A}) \subseteq \mathcal{B}$ and the induced action of $\mathcal{L}$ on B preserves $\mathcal{B}$. We call such a $\mathcal{B}$ an $\mathcal{L}$-**stable** affine formal model.*

We remark that any Y is $\pi^n\mathcal{L}$-admissible for sufficiently large n. Picking any affine formal model $\mathcal{B}'$ for B, $\mathcal{B} := \sigma(\mathcal{A})\mathcal{B}'$ is also an affine formal model by Lemma 3.10, and it is preserved by $\pi^n\mathcal{L}$ for sufficiently large n, as $\mathcal{B}$ is topologically of finite type (see [5, Lemma 3.1]).

The $\mathcal{L}$-admissible affinoid subdomains give rise to the G-topology $X_w(\mathcal{L})$ (see [5, Lemma 3.2]). For most of our purposes, $\mathcal{L}$-admissibility will not be a sufficiently strong property. Recall from Theorem 2.17 and Proposition 2.18 that any affinoid subdomain of X is the finite union of some rational domains, and every rational domain can be obtained by iterating the construction of Laurent domains. This gives rise to the stronger notion of an **$\mathcal{L}$-accessible** subdomain, as introduced in [5]. We briefly recall the definitions, and refer to [5] for the proofs of the essential properties.

Definition 5.11 (see [5, Definition 4.6])**.** *Let Y be a rational subdomain of X. If $Y = X$, we say that it is $\mathcal{L}$-**accessible** in 0 steps. Inductively, if $n \geq 1$ then we say that it is $\mathcal{L}$-accessible in n steps if there exists a chain $Y \subseteq Z \subseteq X$ such that the following is satisfied:*

(i) $Z \subseteq X$ is $\mathcal{L}$-accessible in $(n-1)$ steps,

(ii) $Y = Z(f)$ or $Z(f^{-1})$ for some non-zero $f \in \mathcal{O}(Z)$,

(iii) there is an $\mathcal{L}$-stable affine formal model $\mathcal{C} \subset \mathcal{O}(Z)$ such that $\mathcal{L} \cdot f \subseteq \mathcal{C}$.

A rational subdomain $Y \subseteq X$ is said to be $\mathcal{L}$-accessible if it is $\mathcal{L}$-accessible in n steps for some $n \in \mathbb{N}$.

We will see below (see also [5, Lemma 4.3]) that every $\mathcal{L}$-accessible rational subdomain is $\mathcal{L}$-admissible.

Definition 5.12 (see [5, Definition 4.8])**.** *An affinoid subdomain Y of X is said to be $\mathcal{L}$-**accessible** if it is $\mathcal{L}$-admissible and there exists a finite covering $Y = \cup_{j=1}^r X_j$ where each X_j is an $\mathcal{L}$-accessible rational subdomain of X.*
A finite covering $\{X_j\}$ of X by affinoid subdomains is said to be $\mathcal{L}$-accessible if each X_j is an $\mathcal{L}$-accessible affinoid subdomain of X.

It is shown in [5, Lemma 4.8] that the $\mathcal{L}$-accessible affinoid subdomains of X together with the $\mathcal{L}$-accessible coverings form a Grothendieck topology $X_{\mathrm{ac}}(\mathcal{L})$ on X (in [5], it is assumed that $\mathcal{L}$ is a projective $\mathcal{A}$-module, but this assumption is not used in the proof).

Note that if $Y = \operatorname{Sp} B$ is $\mathcal{L}$-accessible with $\mathcal{L}$-stable affine formal model $\mathcal{B}$, then $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ is a $(R, \mathcal{B})$-Lie algebra, and so its image in $B \otimes_A L = \mathscr{L}(Y)$ (which is known to be obtained by quotienting out by the π-torsion, see Lemma 2.10) is an $(R, \mathcal{B})$-Lie lattice.

We briefly recall some results and notation from [5] relating to $\mathcal{L}$-accessible affinoid subdomains.

The inductive nature of the definition of an accessible rational subdomain suggests that we should consider the basic cases of rational subdomains which are $\mathcal{L}$-accessible in one step. For this purpose, let $f \in A$ be a non-zero element, and choose $a \in \mathbb{N}$ such that $\pi^a f \in \mathcal{A}$. We write

$$u_1 = \pi^a t - \pi^a f, \quad u_2 = \pi^a f t - \pi^a$$

as elements of $A\langle t \rangle$.

We will consider the subdomains

$$X_1 = X(f), \quad X_2 = X(f^{-1}).$$

Note that $X_i = \operatorname{Sp} C_i$, where

$$C_i = A\langle t \rangle / u_i A\langle t \rangle$$

for $i = 1, 2$.

Write $x \cdot f = \rho(x)(f)$ for $x \in L$, and assume that $\mathcal{L} \cdot f \subset \mathcal{A}$.

Proposition 5.13 ([5, Proposition 4.2])**.** *There exist two lifts $\sigma_1, \sigma_2 : \mathcal{L} \to \operatorname{Der}_R(\mathcal{A}\langle t \rangle)$ of the action of $\mathcal{L}$ on $\mathcal{A}$ to $\mathcal{A}\langle t \rangle$, given by*

$$\sigma_1(x)(t) = x \cdot f, \quad \sigma_2(x)(t) = -t^2(x \cdot f)$$

It can be shown (see [5, Lemma 4.3]) that σ_i induces an $\mathcal{L}$-action on C_i which agrees with the action defined via $\mathscr{L}(X) \to \mathscr{L}(X_i)$. Thus X_i is an $\mathcal{L}$-admissible affinoid subdomain, with $\mathcal{L}$-stable affine formal model $\overline{\mathcal{C}_i}$, where

$$\mathcal{C}_i = \mathcal{A}\langle t \rangle / u_i \mathcal{A}\langle t \rangle$$

and

$$\overline{\mathcal{C}_i} = \mathcal{C}_i / \pi\text{−tor}(\mathcal{C}_i).$$

Note this also verifies by an easy inductive argument that every $\mathcal{L}$-accessible rational subdomain is $\mathcal{L}$-admissible.

Lemma 5.14. *Let Y be an affinoid subdomain of X. Then Y is $\pi^n\mathcal{L}$-accessible for sufficiently large n. Any finite affinoid covering of X is $\pi^n\mathcal{L}$-accessible for sufficiently large n.*

Proof. It follows from the definition that it is sufficient to consider the case of a rational subdomain Y of X. By Proposition 2.18, there exists a Laurent subdomain $Z = \operatorname{Sp} C \subseteq X$ such that $Y = \operatorname{Sp} B$ is a Weierstrass subdomain of Z. Thus there exist $f_1, \ldots, f_r$, $g_1, \ldots g_s \in A$ such that

$$Z = X\left(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}\right)$$

and some $h_1, \ldots, h_t \in C$ such that $Y = Z(h_1, \ldots, h_t)$.
We now have a chain of rational subdomains of X of the form

$$X \supseteq X(f_1) \supseteq X(f_1, f_2) \supseteq \ldots X(f_1, \ldots, f_r, g_1^{-1}) \supseteq \cdots \supseteq Z \supseteq Z(h_1) \supseteq \cdots \supseteq Y.$$

For sufficiently large n, $\pi^n \mathcal{L} \cdot f_1 \subseteq \mathcal{A}$, so that $X(f_1)$ is $\pi^n\mathcal{L}$-accessible in one step. In particular, it is $\pi^n\mathcal{L}$-admissible with an $\pi^n\mathcal{L}$-stable affine formal model $\overline{\mathcal{C}_1}$ as described above.
Choosing n such that additionally $\pi^n\mathcal{L} \cdot f_2 \subseteq \overline{\mathcal{C}_1}$ shows that $X(f_1, f_2)$ is also $\pi^n\mathcal{L}$-accessible for large enough n, and working inductively through the finite chain proves that Y is $\pi^n\mathcal{L}$-accessible (in $r + s + t$ steps) for sufficiently large n. □

Many of our arguments will now establish properties first for $\mathcal{L}$-accessible rational subdomains in one step by analyzing the structures above and then argue by induction. The next lemma is the first instance of this strategy.

Lemma 5.15. *Let $Y = \operatorname{Sp} B$ be an $\mathcal{L}$-accessible rational subdomain of X with $\mathcal{L}$-stable affine formal model $\mathcal{B}$. Then*

$$\operatorname{Tor}_s^{\mathcal{A}}\left(\mathcal{B}, U_{\mathcal{A}}(\mathcal{L})\right)$$

has bounded π-torsion for each $s \geq 0$.

Proof. We will abbreviate $\operatorname{Tor}_s^{\mathcal{A}}(\mathcal{B}, U_{\mathcal{A}}(\mathcal{L}))$ to $T_s(\mathcal{B})$.

Suppose Y is $\mathcal{L}$-accessible in n steps. We will argue by induction on n. The case $n = 0$ is straightforward: the statement is trivial for $\mathcal{B} = \mathcal{A}$, but any other $\mathcal{L}$-stable affine formal model (which must contain $\mathcal{A}$ by definition) is the unit ball of some equivalent residue norm, so we are done by Lemma 3.14 and Lemma 3.13.

Now suppose the result holds for $\mathcal{L}$-accessible rational subdomains in $n-1$ steps, and let Y be $\mathcal{L}$-accessible in n steps.
For $s = 0$, note that

$$\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{L}) \cong U_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L})$$

is a Noetherian ring by Theorem 5.2, so has bounded π-torsion.

By definition, there exists a chain $Y \subseteq Z \subseteq X$ such that $Z = \operatorname{Sp} C$ is $\mathcal{L}$-accessible in $n-1$ steps, with $\mathcal{L}$-stable affine formal model $\mathcal{C}$, and there is some non-zero $f \in C$ such that $\mathcal{L} \cdot f \subseteq \mathcal{C}$, and $Y = Z(f) = Z_1$ or $Y = Z(f^{-1}) = Z_2$. The argument now proceeds in the same way for $i = 1$ and $i = 2$.
Recall from the above that we have short exact sequences

$$0 \longrightarrow C\langle t\rangle \xrightarrow{u_i \cdot} C\langle t\rangle \longrightarrow C_i \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{C}\langle t\rangle \longrightarrow \mathcal{C}\langle t\rangle \longrightarrow \mathcal{C}_i \longrightarrow 0,$$

such that $\overline{\mathcal{C}_i} = \mathcal{C}_i/\pi\text{-tor}(\mathcal{C}_i) \subset C_i$ is an $\mathcal{L}$-stable affine formal model.

We will prove the lemma in three steps. First, we will show that $T_s(\mathcal{C}_i)$ has bounded π-torsion for $s \geq 1$, then prove the same for $T_s(\overline{\mathcal{C}_i})$, and finally make the step from this particular affine formal model to an arbitrary $\mathcal{L}$-stable affine formal model.

For the first step, consider the short exact sequence

$$0 \longrightarrow \mathcal{C}\langle t\rangle \xrightarrow{u_i \cdot} \mathcal{C}\langle t\rangle \longrightarrow \mathcal{C}_i \longrightarrow 0,$$

giving rise to the long exact sequence

$$\cdots \to T_s(\mathcal{C}\langle t\rangle) \to T_s(\mathcal{C}\langle t\rangle) \to T_s(\mathcal{C}_i) \to T_{s-1}(\mathcal{C}\langle t\rangle) \to \ldots$$

By [12, Remark 7.3/2], $\mathcal{C}\langle t\rangle$ is flat over $\mathcal{C}$, and hence

$$T_s(\mathcal{C}\langle t\rangle) \cong \mathcal{C}\langle t\rangle \otimes_{\mathcal{C}} T_s(\mathcal{C})$$

by [47, Proposition 3.2.9, Corollary 3.2.10].
Note that for $s \geq 1$, $T_s(\mathcal{C})$ is π-torsion, since C is flat over A (by [12, Corollary 4.1/5]), and by inductive hypothesis, it is thus annihilated by some power π^{n_s}, for some $n_s \in \mathbb{N}$. But then π^{n_s} annihilates $T_s(\mathcal{C}\langle t\rangle)$, and we can apply Lemma 3.15 to deduce that $T_s(\mathcal{C}_i)$ has bounded π-torsion for $s \geq 1$ (for the case $s = 1$, we need to know that $\mathcal{C}\langle t\rangle \otimes_{\mathcal{A}} U_{\mathcal{A}}(\mathcal{L})$ has bounded π-torsion, but this is precisely the Noetherianity argument above).

We now pass from $\mathcal{C}_i$ to $\overline{\mathcal{C}_i}$. Note that $\mathcal{C}_i$ is a Noetherian ring, so its π-torsion S is annihilated by some power π^m. We have the short exact sequence

$$0 \to S \to \mathcal{C}_i \to \overline{\mathcal{C}_i} \to 0,$$

and tensoring with $U(\mathcal{L})$ yields the long exact sequence

$$\dots \to T_s(S) \to T_s(\mathcal{C}_i) \to T_s(\overline{\mathcal{C}_i}) \to T_{s-1}(S) \to \dots$$

Now $T_s(S)$ is annihilated by π^m for every s, and $T_s(\mathcal{C}_i)$ is annihilated by some power of π whenever $s \geq 1$ (it has bounded π-torsion by the above, and $\mathcal{C}_i \otimes_R K = B$ is flat over A). Thus we can again apply Lemma 3.15 to see that $T_s(\overline{\mathcal{C}_i})$ has bounded π-torsion for $j \geq 1$.

Lastly, consider an arbitrary $\mathcal{L}$-stable affine formal model $\mathcal{B}$ of B. This is the unit ball of some residue norm on B, and since $\mathcal{A} \subseteq \mathcal{B}$, this turns B into a Banach A-module. Since all residue norms are equivalent, the above in conjunction with Lemma 3.14 and Lemma 3.13 tells us that $T_s(\mathcal{B})$ has bounded π-torsion for any $s \geq 1$. □

We now fix an $(R, \mathcal{A})$-Lie lattice $\mathcal{L}$ inside L such that the subalgebra U_0 of $U_A(L)$ generated by $\mathcal{A}$ and $\mathcal{L}$ is π-adically separated. Thus U_0 is the unit ball of some norm on $U_A(L)$.

We will now define sheaves of algebras $\mathscr{U}_n(\mathscr{L})$ on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$, and set

$$\widehat{\mathscr{U}(\mathscr{L})}(U) = \varprojlim \mathscr{U}_n(\mathscr{L})(U)$$

for any affinoid subdomain $U \subseteq X$.
Since any affinoid subdomain (and any finite affinoid covering) is in $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ for sufficiently large n by Lemma 5.14, this defines a sheaf on X_w. Hence we can extend $\widehat{\mathscr{U}(\mathscr{L})}$ uniquely to a sheaf on X with respect to the strong Grothendieck topology by [12, Corollary 5.2/5], which we will also denote by $\widehat{\mathscr{U}(\mathscr{L})}$. We will show later that this agrees with the construction of $\widehat{\mathscr{U}(\mathscr{L})}$ in [5].

First, let us define $\mathscr{U}_n(\mathscr{L})$.
Given $U = \operatorname{Sp} B \subseteq X$ a $\pi^n\mathcal{L}$-accessible subdomain, $\mathcal{B}$ a $\pi^n\mathcal{L}$-stable formal model in B, we set

$$\mathscr{U}_n(\mathscr{L})(U) = U_B\widehat{(\mathscr{L}(U))},$$

where the completion is with respect to the semi-norm whose unit ball is the image of $U_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \pi^n\mathcal{L})$ inside $U_B(\mathscr{L}(U)) = U_B(B \otimes_A L)$.

Note that by [5, Proposition 2.3], $\mathscr{U}_n(\mathscr{L})(U)$ is isomorphic to

$$B\widehat{\otimes}_A U_A(L),$$

where $U_A(L)$ is equipped with the norm with unit ball U_n and B is equipped with the residue norm with unit ball $\mathcal{B}$. Since all residue norms on B are equivalent, this also shows that our definiton of $\mathscr{U}_n(\mathscr{L})$ is independent of choices of $\pi^n\mathcal{L}$-stable affine formal models.
Clearly $\mathscr{U}_n(\mathscr{L})$ is a presheaf on $X_w(\pi^n\mathcal{L})$. We write $\mathscr{U}_{X,\pi^n\mathcal{L}}$ when we need to stress the dependency on the ambient space X and the choice of Lie lattice.

Lemma 5.16. *Let $X = \operatorname{Sp} A$ be an affinoid K-space, $\mathcal{L}$ an $(R,\mathcal{A})$-Lie lattice in $L = \mathscr{L}(X)$, and let $Y = \operatorname{Sp} B$ be an $\mathcal{L}$-accessible affinoid subdomain with $\mathcal{L}$-stable affine formal model $\mathcal{B}$. Write $\mathcal{L}'$ for the image of $\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}$ inside $B\otimes_A L = L' = \mathscr{L}(Y)$. Then $\mathcal{L}'$ is an $(R,\mathcal{B})$-Lie lattice and the restriction $\mathscr{U}_{X,\mathcal{L}}|_Y$ is canonically isomorphic to $\mathscr{U}_{Y,\mathcal{L}'}$ on $Y_{\mathrm{ac}}(\mathcal{L}')$.*

Proof. We have already seen above that $\mathcal{L}'$ is a Lie lattice in L'.
If $Z = \operatorname{Sp} C$ is any affinoid subdomain of Y with $Z \in X_{\mathrm{ac}}(\mathcal{L})$, then Z is also $\mathcal{L}'$-accessible by [5, Lemma 4.8.b)]. Since any $\mathcal{L}'$-stable affine formal model is automatically $\mathcal{L}$-stable, we can consider an affine formal model $\mathcal{C}$ of C which is both $\mathcal{L}'$- and $\mathcal{L}$-stable.

Then both $\mathscr{U}_{X,\mathcal{L}}(Z)$ and $\mathscr{U}_{Y,\mathcal{L}'}(Z)$ are defined to be the completion of

$$U(\mathscr{L}(Z)) \cong C\otimes_B U(\mathscr{L}(Y)) \cong C\otimes_A U(\mathscr{L}(X))$$

with respect to the semi-norm with unit ball given by the image of

$$\mathcal{C}\otimes_{\mathcal{B}} U_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{L}) \cong \mathcal{C}\otimes_{\mathcal{A}} U_{\mathcal{A}}(\mathcal{L}),$$

so the two presheaves are canonically isomorphic. □

Theorem 5.17. *$\mathscr{U}_n(\mathscr{L})$ is a sheaf on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ and has vanishing higher Čech cohomology with respect to any covering in $X_{\mathrm{ac}}(\pi^n\mathcal{L})$.*

Proof. Let $U = \operatorname{Sp} B \in X_{\mathrm{ac}}(\pi^n\mathcal{L})$, and let $\mathfrak{V} = (V_i)_i$ be a (finite) $\pi^n\mathcal{L}$-accessible covering of U. By definition, each V_i is the finite union of $\pi^n\mathcal{L}$-accessible rational subdomains, so using [12, Lemma 4.3/2], we can assume without loss of generality that each V_i is rational and $\pi^n\mathcal{L}$-accessible.
By [13, Corollary 8.2.1/5], the Čech complex

$$0 \to B \to \oplus_i \mathcal{O}_X(V_i) \to \oplus_{i,j}\mathcal{O}_X(V_{ij}) \to \dots$$

is exact. Equipping each term with a residue norm induced by a $\pi^n\mathcal{L}$-stable affine formal model (and equipping A with the residue norm with unit ball $\mathcal{A}$) turns this into a complex of Banach A-modules with continuous boundary morphisms. The boundary morphisms are in fact strict by Lemma 3.7.

We now wish to apply Theorem 3.21 (tensoring on the right instead of the left). The Čech complex above is a strict complex of Banach modules over the Banach algebra A, with discrete value sets, and $U_A(L)$ is flat over A, since L is projective (see Lemma 5.3). Since the covering is finite, we have $\check{C}^j(\mathfrak{V}, \mathcal{O}) = 0$ for sufficiently large j. Both A and $U_A(L)$ are Noetherian normed algebras with Noetherian unit balls by Theorem 5.2, and the morphism $A \to U_A(L)$ is contractive by definition of the norm on $U_A(L)$.
Lemma 5.15 now implies that

$$\operatorname{Tor}_s^{\mathcal{A}}\left(\check{C}^j(\mathfrak{V}, \mathcal{O})^\circ, U_{\mathcal{A}}(\pi^n\mathcal{L})\right)$$

has bounded π-torsion for each j and each $s \geq 0$.

Thus all conditions of Theorem 3.21 are satisfied, and the complex

$$0 \to B \otimes_A U_A(L) \to \oplus \mathcal{O}_X(V_i) \otimes_A U_A(L) \to \dots$$

is strict exact and remains exact after completion with respect to the corresponding tensor semi-norms. □

Thus we have constructed a sheaf $\widehat{\mathscr{U}(\mathscr{L})}$ on X_{rig} such that for any affinoid subdomain $V = \operatorname{Sp} B \subseteq X$, we have

$$\widehat{\mathscr{U}(\mathscr{L})}(V) = \varprojlim U(\widehat{\mathscr{L}(V)})$$

where the limit is taken over n, varying the norm on $U(\mathscr{L}(V))$ determined by the unit ball $U(\mathcal{B} \otimes \pi^n\mathcal{L})$ for some affine formal model $\mathcal{B}$.
Thus

$$\widehat{\mathscr{U}(\mathscr{L})}(V) = \widehat{U(\mathscr{L}(V))}$$

for any affinoid subdomain V.
Ardakov and Wadsley have constructed a sheaf satisfying this property for any affinoid subdomain V which admits a *smooth* Lie lattice, and since such subdomains form a base of the topology, it follows by uniqueness of extension (see [5, 9.1]) that our construction agrees with theirs.

We can also relate the cohomology of $\widehat{\mathscr{U}(\mathscr{L})}$ to the cohomologies of $\mathscr{U}_n(\mathscr{L})$. Fixing a finite affinoid covering $\mathfrak{V}$ of X, the terms $\check{C}^j(\mathfrak{V}, \mathscr{U}_n(\mathscr{L}))$ satisfy the Mittag-Leffler property as described in [24, 0.13.2.4] for each j, so [24, 0.13.2.3] implies that $\widehat{\mathscr{U}(\mathscr{L})}$ also has vanishing higher Čech cohomology groups, as we obtain

$$\check{\mathrm{H}}^j(\mathfrak{V}, \varprojlim \mathscr{U}_n(\mathscr{L})) \cong \varprojlim \check{\mathrm{H}}^j(\mathfrak{V}, \mathscr{U}_n(\mathscr{L})) = 0$$

for any $j \geq 1$.

Applying Theorem 2.26, we thus have

$$\mathrm{H}^j(X_{\mathrm{ac}}(\pi^n \mathcal{L}), \mathscr{U}_n(\mathscr{L})) \cong \check{\mathrm{H}}^j(\mathfrak{V}, \mathscr{U}_n(\mathscr{L}))$$

and

$$\mathrm{H}^j(X, \widehat{\mathscr{U}(\mathscr{L})}) \cong \check{\mathrm{H}}^j(\mathfrak{V}, \widehat{\mathscr{U}(\mathscr{L})})$$

for any $j \geq 0$, and both terms are zero for $j \geq 1$.

Suppose X is some rigid analytic K-space (not necessarily affinoid) and $\mathscr{L}$ a Lie algebroid on X. Then we can glue our construction above to obtain a sheaf $\widehat{\mathscr{U}(\mathscr{L})}$. If X is moreover separated, Theorem 2.27 implies that

$$\mathrm{H}^j(X, \widehat{\mathscr{U}(\mathscr{L})}) \cong \check{\mathrm{H}}^j(\mathfrak{V}, \widehat{\mathscr{U}(\mathscr{L})})$$

for any finite affinoid covering $\mathfrak{V}$ and any $j \geq 0$.

Our main example for a Lie algebroid is the tangent sheaf $\mathcal{T}_X$ of X in the case when X is a smooth rigid analytic K-variety. In this case, we write $\widehat{\mathcal{D}}_X = \widehat{\mathscr{U}(\mathcal{T}_X)}$.

In the next section, we briefly discuss another example of a Lie algebroid which will become relevant later.

5.3 Example: Elementary proper pushforwards and pullbacks of a Lie algebroid

In this section, let $f : X \to Y$ be an elementary proper morphism of rigid analytic K-spaces. Replacing f by the first map in its Stein factorization, we will assume that $f_* \mathcal{O}_X = \mathcal{O}_Y$, so that $Y = \operatorname{Sp} A$, where $A = \mathcal{O}_X(X)$.

Suppose throughout that $(\rho, \mathscr{L})$ is a Lie algebroid on X with the additional property that $f_* \mathscr{L}$ is a free $\mathcal{O}_Y$-module, i.e. $L = \mathscr{L}(X)$ is a free A-module, finitely generated by Kiehl's Proper Mapping Theorem.

It follows from Kiehl's Proper Mapping Theorem that these assumptions are all preserved under restriction to $f|_{f^{-1}U} : f^{-1}U \to U$ for any affinoid subdomain $U \subseteq Y$.

Lemma 5.18. *The sheaf $f_*\mathscr{L}$ is a Lie algebroid on Y.*

Proof. By Kiehl's Proper Mapping Theorem, $f_*\mathscr{L}$ is a coherent $\mathcal{O}_Y$-module, and it is free by assumption.
The anchor map $\rho : \mathscr{L} \to \mathcal{T}_X$ gives rise to a Lie algebra action of L on $\mathcal{O}_X(X) = A$. Restricting to an admissible affinoid covering of X, it follows from the definition of a Lie algebroid that L acts via derivations, and that the corresponding map $\rho'(Y) : L \to \mathrm{Der}_K(A)$ satisfies the Leibniz property of an anchor map.
By the remark above, we thus obtain morphisms $\rho'(U) : f_*\mathscr{L}(U) \to \mathcal{T}_Y(U)$ for any affinoid subdomain $U \subseteq Y$, which naturally give rise to an anchor map $\rho' : f_*\mathscr{L} \to \mathcal{T}_Y$, finishing the proof. $\square$

Moreover, we can consider the free coherent $\mathcal{O}_X$-module

$$f^*f_*\mathscr{L} := \mathcal{O}_X \otimes_A L,$$

given by $f^*f_*\mathscr{L}(U) = \mathcal{O}_X(U) \otimes_A L$ for any admissible open subspace U of X.
Note that we do not define the inverse image functor f^* in a general context, as it would rarely preserve coherence. The expression above should rather be considered a notational convenience.

Lemma 5.19. *The sheaf $f^*f_*\mathscr{L}$ is a Lie algebroid on X, and the natural morphism $\iota : f^*f_*\mathscr{L} \to \mathscr{L}$ is a morphism of Lie algebroids.*

Proof. By assumption, $f^*f_*\mathscr{L}$ is a free $\mathcal{O}_X$-module of finite rank.
Now $\rho\iota$ is an $\mathcal{O}_X$-linear morphism $f^*f_*\mathscr{L} \to \mathcal{T}_X$, and it is sufficient to endow $f^*f_*\mathscr{L}$ with a Lie bracket such that $\rho\iota$ is a morphism of sheaves of Lie algebras, satisfying the usual Leibniz property for anchor maps.

We now define a Lie bracket on $f^*f_*\mathscr{L}(U) = \mathcal{O}(U) \otimes_A L$ by

$$[a \otimes x, b \otimes y] := ab \otimes [x, y]_L + \rho\iota(a \otimes x)(b) \otimes y - \rho\iota(b \otimes y)(a) \otimes x$$

for all $a, b \in \mathcal{O}_X(U)$, $x, y \in L$, extending bilinearly.
Note that this is well-defined, as ρ satisfies the Leibniz axiom, i.e. if $b' \in A$, we have

$$\begin{aligned}[a \otimes x, b \otimes b'y] &= ab \otimes [x, b'y]_L + a\rho(x)(b) \otimes b'y - bb'\rho(y)(a) \otimes x \\ &= ab \otimes b'[x, y]_L + ab \otimes \rho(x)(b')y + a\rho(x)(b) \otimes b'y - bb'\rho(y)(a) \otimes x \\ &= [a \otimes x, bb' \otimes y].\end{aligned}$$

The bracket is K-bilinear and anti-symmetric. The Jacobi identity also follows from the Jacobi identity for $[\ ,\ ]_L$. This turns $f^*f_*\mathscr{L}$ into a sheaf of Lie algebras.

Applying the Leibniz axiom for ρ to the expression $[ax, by]$ in the Lie algebra $\mathscr{L}(U)$, it follows that ι, and hence $\rho\iota$, is a morphism of sheaves of Lie algebras.
Lastly, we verify the Leibniz axiom, which again follows quite easily from the definition. Since

$$b'[a \otimes x, b \otimes y] = abb' \otimes [x, y]_L + ab'\rho(x)(b) \otimes y - bb'\rho(y)(a) \otimes x,$$

we have

$$\begin{aligned}[a \otimes x, bb' \otimes y] &= abb' \otimes [x, y]_L + a\rho(x)(bb') \otimes y - bb'\rho(y)(a) \otimes x \\ &= b'[a \otimes x, b \otimes y] + a\rho(x)(b')b \otimes y,\end{aligned}$$

finishing the proof. □

Let us note two things. Firstly, as we will explain in more detail in the next chapter, the global sections $\widehat{\mathscr{U}(f^*f_*\mathscr{L})}(X) = \widehat{U(\mathscr{L}(X))}$ form a Fréchet–Stein algebra by Theorem 5.5 even though X is not assumed to be affinoid.
Secondly, we would like to spell out the above in the particular case of the analytic flag variety. Let $\mathbf{G}$ be a split reductive affine algebraic group scheme over K, $\mathbf{B}$ a Borel subgroup, and $X = (\mathbf{G}/\mathbf{B})^{\mathrm{an}}$ the analytification of the flag variety. X is the analytification of a projective scheme over K, so it satisfies the following:

(i) X is proper over the point $\operatorname{Sp} K$. Since the codomain is a point, the projection map $f : X \to \operatorname{Sp} K$ is then clearly elementary proper.

(ii) $\mathcal{O}_X(X) = K$ and $\mathcal{T}_X(X) = \mathfrak{g}$ by [29, II.1.8] and by GAGA (see [12, Theorem 6.3/11]).

Thus the Lie algebroid $\mathcal{T}_X$ satisfies the conditions above, and $f^*f_*\mathcal{T}_X$ is given by

$$f^*f_*\mathcal{T}_X = \mathcal{O}_X \otimes_K \mathfrak{g},$$

which is a Lie algebroid by the above. The natural morphism $\mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X$ induces a morphism of sheaves of algebras

$$\widehat{\mathscr{U}(\mathcal{O}_X \otimes \mathfrak{g})} \to \widehat{\mathcal{D}}_X.$$

Considering the global sections of this morphism, we recover the usual map

$$\widehat{U(\mathfrak{g})} \to \widehat{\mathcal{D}}_X(X),$$

the Fréchet completed analogue of the classical morphism in the algebraic setting which is at the heart of the Beilinson–Bernstein correspondence.

Note that the morphism $\widehat{U(\mathfrak{g})} \to \widehat{\mathscr{U}(\mathcal{O}_X \otimes \mathfrak{g})} \to \widehat{\mathcal{D}}_X$ can also be interpreted as the quantization of the moment map $T^*X \to X \times \mathfrak{g}^* \to \mathfrak{g}^*$, which is proper since $T^*X \to X \times \mathfrak{g}^*$ is a closed immersion (it is the analytification of the closed immersion $T^*(G/B) = G \times_B \mathfrak{b}^\perp \to G/B \times \mathfrak{g}^*$, see [18, Proposition 1.4.11]). So in this case, the proper morphism $f : X \to \operatorname{Sp} K$ lifts to a proper morphism between rigid analytic vector bundles $T^*X \to V(f_*\mathcal{T}_X)$.
This example will motivate most of the arguments in chapter 6.

5.4 Flat localization for $\mathscr{U}_n$

Let $X = \operatorname{Sp} A$ be an affinoid K-space, $\mathcal{A}$ an affine formal model in A and $\mathcal{L}$ a $(R, \mathcal{A})$-Lie lattice in $L = \mathscr{L}(X)$ for a Lie algebroid $\mathscr{L}$.
Let Y be a $\pi^n\mathcal{L}$-accessible affinoid subdomain, and let $\mathscr{U}_n = \mathscr{U}_n(\mathscr{L})$ be the sheaf on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ as constructed earlier. We wish to prove the following result.

Theorem 5.20. *$\mathscr{U}_n(Y)$ is a flat $\mathscr{U}_n(X)$-module on both sides.*

As usual, replacing $\mathcal{L}$ by the Lie lattice $\pi^n\mathcal{L}$, it is sufficient to prove the statement in the case $n = 0$.

This theorem was proved by Ardakov and Wadsley (see [5, Theorem 4.9]) under the additional assumption that $\mathcal{L}$ is smooth (i.e. projective over $\mathcal{A}$). Our proof will be almost entirely as in [5], but we will need to generalize the proof of [5, Proposition 4.3.c)]. The original argument can be interpreted as an instance of Corollary 3.18, which we are going to replace by the more general Corollary 3.17 in order to remove the smoothness assumption.
For large parts, however, we will be able to refer to [5].

Again, we will first prove the result for $\mathcal{L}$-accessible rational subdomains in one step, then inductively for general $\mathcal{L}$-accessible rational subdomains, and finally for arbitrary $Y \in X_{\mathrm{ac}}(\mathcal{L})$.

Let $f \in A$ be non-zero, and write again

$$X_1 = X(f) = \operatorname{Sp} C_1,\ X_2 = X(f^{-1}) = \operatorname{Sp} C_2.$$

Assume $\mathcal{L} \cdot f \subseteq \mathcal{A}$, so that X_i is $\mathcal{L}$-accessible for $i = 1, 2$.

As before, choose $a \in \mathbb{N}$ such that $\pi^a f \in \mathcal{A}$, and set

$$u_1 = \pi^a f - \pi^a t \in \mathcal{A}\langle t\rangle,\ u_2 = \pi^a f t - \pi^a \in \mathcal{A}\langle t\rangle.$$

Recall the notation from before, with $\mathcal{C}_i = \mathcal{A}\langle t\rangle / u_i \mathcal{A}\langle t\rangle$, and $\overline{\mathcal{C}_i} = \mathcal{C}_i/\pi{-}\mathrm{tor}(\mathcal{C}_i)$ an $\mathcal{L}$-stable affine formal model of C_i.
We write $\mathcal{L}_i = \mathcal{A}\langle t\rangle \otimes_{\mathcal{A}} \mathcal{L}$ for the $(R, \mathcal{A}\langle t\rangle)$-Lie algebra with anchor map $1 \otimes \sigma_i$ as defined in Proposition 5.13, and $L_i = \mathcal{L}_i \otimes_R K$.

Lemma 5.21 (see [5, Proposition 4.3.c)])**.** *There exists a short exact sequence*

$$0 \longrightarrow \widehat{U(\mathcal{L}_i)}_K \xrightarrow{u_i\cdot} \widehat{U(\mathcal{L}_i)}_K \longrightarrow \mathscr{U}_0(X_i) \longrightarrow 0$$

of right $\widehat{U(\mathcal{L}_i)}_K$-modules, analogously for the left module structure.

Proof. By definition, the sequence

$$0 \longrightarrow A\langle t\rangle \xrightarrow{u_i\cdot} A\langle t\rangle \xrightarrow{p} C_i \longrightarrow 0$$

is exact.
Equip $A\langle t\rangle$ with a residue norm with unit ball $\mathcal{A}\langle t\rangle$, and C_i with a residue norm with unit ball $\overline{\mathcal{C}_i}$. Since the maps are continuous, Lemma 3.7 implies that this short exact sequence consists of strict morphisms.

Since L is a projective A-module by Proposition 2.29, we know that $U_A(L)$ is a flat A-module by Lemma 5.3. Thus

$$0 \to U(L_i) \to U(L_i) \to U_{C_i}(C_i \otimes L) \to 0$$

is a short exact sequence of right $U(L_i)$-modules, where we have used the isomorphism (from [5, Proposition 2.3])

$$C_i \otimes_A U_A(L) \cong U_{C_i}(C_i \otimes_A L),$$

likewise for the other terms.
The corresponding tensor semi-norms on the first two terms have as unit balls the images of

$$\mathcal{A}\langle t\rangle \otimes_{\mathcal{A}} U_{\mathcal{A}}(\mathcal{L}) \cong U_{\mathcal{A}\langle t\rangle}(\mathcal{L}_i),$$

inside $U(L_i)$. Similarly the unit ball of the tensor semi-norm on $U_{C_i}(C_i \otimes L) \cong C_i \otimes U(L)$ is the image of $\mathcal{C}_i \otimes_{\mathcal{A}} U_{\mathcal{A}}(\mathcal{L}) \cong U_{\mathcal{C}_i}(\mathcal{C}_i \otimes \mathcal{L})$. In particular, its completion is $\mathscr{U}_0(X_i)$.

But now $U_{\mathcal{C}_i}(\mathcal{C}_i \otimes \mathcal{L})$ is Noetherian, so has bounded π-torsion, and we can invoke Corollary 3.17 to see that the completion

$$0 \to \widehat{U(L_i)} \to \widehat{U(L_i)} \to C_i \widehat{\otimes}_A U_A(L) \to 0$$

is exact. By Theorem 2.11, $\widehat{U(L_i)} \cong \widehat{U(\mathcal{L}_i)}_K$, and by the above,

$$C_i \widehat{\otimes}_A U_A(L) \cong \mathscr{U}_0(X_i),$$

as required. □

Proof of Theorem 5.20. First assume that $Y = X_i$, $i = 1, 2$, as in the above discussion.
By [5, Lemma 4.3.b)], $T_i = \widehat{U(\mathcal{L}_i)}_K$ is a flat right $\mathscr{U}_0(X)$-module, and by Lemma 5.21, we have $\mathscr{U}_0(X_i) \cong \widehat{U(\mathcal{L}_i)}_K / u_i \widehat{U(\mathcal{L}_i)}_K$ as a right $\widehat{U(\mathcal{L}_i)}_K$-module. Now the proof of [5, Theorem 4.5] goes through unchanged. Thus we have proved the theorem for the case when Y is a rational subdomain which is $\mathcal{L}$-accessible in one step.

Now suppose Y is a rational subdomain which is $\mathcal{L}$-accessible in n steps, and let $Y \subseteq Z \subseteq X$ be as in Definition 5.11, i.e. $Y = Z(f)$ or $Z(f^{-1})$ for some non-zero $f \in \mathcal{O}(Z)$. We can assume inductively that $\mathscr{U}_0(Z)$ is flat over $\mathscr{U}_0(X)$.
Write $Z = \operatorname{Sp} B$, and let $\mathcal{B}$ be an $\mathcal{L}$-stable affine formal model of B such that $\mathcal{L} \cdot f \subseteq \mathcal{B}$. Then $L' = \mathcal{T}_X(Z) = B \otimes_A L$, and the image of $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ in L', which we will denote by $\mathcal{L}'$, is an $(R, \mathcal{B})$-Lie lattice in L'.

By the argument above, $\mathscr{U}_{Z,\mathcal{L}'}(Y)$ is flat over $\mathscr{U}_{Z,\mathcal{L}'}(Z)$, and by Lemma 5.16, this says that $\mathscr{U}_0(Y)$ is flat over $\mathscr{U}_0(Z)$. Since we assumed that $\mathscr{U}_0(Z)$ is flat over $\mathscr{U}_0(X)$, this shows $\mathscr{U}_0(Y)$ is flat over $\mathscr{U}_0(X)$.

For the case of a general $\mathcal{L}$-accessible affinoid subdomain Y, the argument of [5, Theorem 4.9.a)] now goes through unchanged. □

5.5 Coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-modules

Finally, we define coadmissible modules, which are the analogues of coherent modules for Fréchet–Stein algebras.

Definition 5.22 (see [44, section 3])**.** *A (left) module M of a (left) Fréchet–Stein algebra $U = \varprojlim U_n$ is called (left)* ***coadmissible*** *if $M = \varprojlim M_n$, such that the following is satisfied for every n:*

(i) M_n *is a finitely generated (left)* U_n*-module.*

(ii) *The natural morphism* $U_n \otimes_{U_{n+1}} M_{n+1} \to M_n$ *is an isomorphism.*

We record the following basic results from [44, section 3].

Proposition 5.23. *Let* $M = \varprojlim M_n$ *be a coadmissible* $U = \varprojlim U_n$*-module. Then the following hold:*

(i) *The natural morphism* $U_n \otimes_U M \to M_n$ *is an isomorphism for each* n*.*

(ii) *The system* $(M_n)_n$ *has the Mittag-Leffler property as described in [24, 0.13.2.4].*

(iii) *The category of coadmissible* U*-modules is an abelian category, containing all finitely presented* U*-modules.*

Just as a Fréchet–Stein algebra $B = \varprojlim B_n$ carries a natural Fréchet topology as the inverse limit topology of the Banach norms on B_n, so any coadmissible B-module $M = \varprojlim M_n$ carries a canonical Fréchet topology induced by the canonical Banach module structures on each M_n. An algebra structure on M is said to have **continuous multiplication** if we can choose for each n a Banach B_n-module norm $| - |$ on M_n such that the natural morphism $\iota_n : M \to M_n$ endows M with a semi-norm $| - |_n$ which is submultiplicative, i.e.

$$|xy|_n := |\iota_n(xy)| \leq |\iota_n(x)| \cdot |\iota_n(y)|$$

for any $x, y \in M$.

Lemma 5.24. *Let* $B = \varprojlim B_n$ *be a left Fréchet–Stein* K*-algebra, and let* C *be a* K*-algebra which is also a left coadmissible* B*-module via an algebra morphism* $B \to C$*, with continuous multiplication. Then* C *is a left Fréchet–Stein algebra.*
If M *is a left* C*-module, then it is coadmissible if and only if it is coadmissible as a* B*-module.*

Proof. By assumption, $C_n := B_n \otimes_B C$ is a finitely generated left B_n-module and the canonical Banach norm gives rise to a submultiplicative semi-norm $| - |_n$ on C. Note that the image of $\iota_n : C \to C_n$ is dense by [44, Theorem A]. But then the completion of C with respect to $| - |_n$ is a Banach K-algebra, which as a Banach space is naturally isomorphic to C_n by construction. Thus $C = \varprojlim C_n$ is the limit of Banach K-algebras, and each C_n is left Noetherian, as it is finitely generated over B_n.

Since the functor $C_n \otimes_{C_{n+1}} -$ can be written as

$$(B_n \otimes_{B_{n+1}} C_{n+1}) \otimes_{C_{n+1}} -,$$

flatness follows from flatness of the maps $B_{n+1} \to B_n$.
Since the map $B_{n+1} \to B_n$ has dense image, it follows that $C_{n+1} \to C_n = B_n \otimes_{B_{n+1}} C_{n+1}$ also has dense image for each n. Thus C is a left Fréchet–Stein algebra.

The second part of the statement is now a simplified version of [44, Lemma 3.8].
If M is a left C-module, then

$$M_n := C_n \otimes_C M \cong (B_n \otimes_B C) \otimes_C M = B_n \otimes_B M$$

as a B_n-module, and M_n is finitely generated as a C_n-module if and only if it is finitely generated as a B_n-module, because C_n is finitely generated over B_n. Moreover,

$$\begin{aligned} B_n \otimes_{B_{n+1}} M_{n+1} &\cong (B_n \otimes_{B_{n+1}} C_{n+1}) \otimes_{C_{n+1}} M_{n+1} \\ &\cong C_n \otimes_{C_{n+1}} M_{n+1}, \end{aligned}$$

finishing the proof. □

Given a (K, A)-Lie algebra L which is finitely generated projective over A, finitely generated $U_A(L)$-modules give rise to coadmissible $\widehat{U_A(L)}$-modules in a natural way as follows.
Choose an $(R, \mathcal{A})$-Lie lattice $\mathcal{L}$ in L and write $U_n = U_{\mathcal{A}}(\pi^n \mathcal{L})$. Given a finitely generated $U_A(L)$-module M, we obtain the coadmissible module $\widehat{M} := \varprojlim \left(\widehat{U_n}_K \otimes_{U(L)} M\right)$, which we might call the coadmissible completion of M, similarly to [5, 7.1].

Lemma 5.25. *The functor $M \mapsto \widehat{M}$ is an exact functor from finitely generated $U_A(L)$-modules to coadmissible $\widehat{U_A(L)}$-modules.*

Proof. First note that $\widehat{U_n}_K$ is flat over $U_A(L)$ for each n, since $\widehat{U_n}$ is flat over U_n by [11, 3.2.3.(iv)] and $\widehat{U_n}_K \otimes_{U(L)} M \cong \widehat{U_n} \otimes_{U_n} M$ for each $U(L)$-module M.
If $0 \to M \to M' \to M'' \to 0$ is a short exact sequence of finitely generated $U_A(L)$-modules, flatness of $\widehat{U_n}_K$ over $U_A(L)$ ensures the exactness of

$$0 \to \widehat{U_n}_K \otimes M \to \widehat{U_n}_K \otimes M' \to \widehat{U_n}_K \otimes M'' \to 0$$

for each n, and the result follows from [24, 0.13.2.4]. □

Let $X = \operatorname{Sp} A$ be an affinoid K-space, and $\mathscr{L}$ a Lie algebroid on X. To a given coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(X)$-module M we will now associate a $\widehat{\mathscr{U}(\mathscr{L})}$-module by a form of **localization**. To achieve this, we first need the correct form of tensor product, as discussed in [5, section 7].

Given Fréchet–Stein algebras $U = \varprojlim U_n$ and $V = \varprojlim V_n$ with compatible maps $U_n \to V_n$ and a coadmissible U-module M, we can form a coadmissible V-module

$$V \widehat{\otimes}_U M := \varprojlim (V_n \otimes_{U_n} M_n) \cong \varprojlim (V_n \otimes_U M).$$

In particular, given an affinoid K-space X and a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(X)$-module M, we can form a presheaf $\operatorname{Loc} M$ defined by

$$\operatorname{Loc} M(U) = \widehat{\mathscr{U}(\mathscr{L})}(U) \widehat{\otimes}_{\widehat{\mathscr{U}(\mathscr{L})}(X)} M$$

for each affinoid subdomain $U \subseteq X$.

In this way we obtain a functor from the category of coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(X)$-modules to presheaves of $\widehat{\mathscr{U}(\mathscr{L})}$-modules (we will justify below that $\operatorname{Loc} M$ is indeed a sheaf). Note that for any affinoid subdomain $U \subseteq X$, $\operatorname{Loc} M(U)$ will be a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(U)$-module.

Proposition 5.26. *If $Y \subseteq X$ is an affinoid subdomain of X, then $\widehat{\mathscr{U}(\mathscr{L})}(Y)$ is c-flat over $\widehat{\mathscr{U}(\mathscr{L})}(X)$, i.e. the functor $\widehat{\mathscr{U}(\mathscr{L})}(Y) \widehat{\otimes}_{\widehat{\mathscr{U}(\mathscr{L})}(X)} -$ is an exact functor from coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(X)$-modules to coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(Y)$-modules.*

Proof. Choosing an affine formal model $\mathcal{A}$ and an $(R, \mathcal{A})$-Lie lattice $\mathcal{L}$, we know that Y is $\pi^n \mathcal{L}$-accessible for sufficiently large n. We adopt the notation from previous sections and abbreviate $\mathscr{U}_n(\mathscr{L})$ to $\mathscr{U}_n$ and $\widehat{\mathscr{U}(\mathscr{L})}$ to $\widehat{\mathscr{U}}$.

By Theorem 5.20, $\mathscr{U}_n(Y)$ is then flat over $\mathscr{U}_n(X)$ for all sufficiently large n, and thus flat over $\widehat{\mathscr{U}}(X)$ by [44, Remark 3.2].

Given a short exact sequence $0 \to M \to M' \to M'' \to 0$ of coadmissible $\widehat{\mathscr{U}}(X)$-modules, we have by definition

$$\widehat{\mathscr{U}}(Y) \widehat{\otimes}_{\widehat{\mathscr{U}}(X)} N \cong \varprojlim_n \Big(\mathscr{U}_n(Y) \otimes_{\mathscr{U}_n(X)} (\mathscr{U}_n(X) \otimes_{\widehat{\mathscr{U}}(X)} N) \Big)$$

for N any of the three modules M, M', M''.

Each of the terms in the projective system $(\mathscr{U}_n(Y) \otimes_{\widehat{\mathscr{U}}(X)} N)_n$ carries a natural Banach space structure as a finitely generated $\mathscr{U}_n(Y)$-module such that the images under the connecting morphisms are dense, and so [24, 0.13.2.4] ensures that taking the limit of the exact sequences

$$0 \to \mathscr{U}_n(Y) \otimes_{\widehat{\mathscr{U}}(X)} M \to \mathscr{U}_n(Y) \otimes_{\widehat{\mathscr{U}}(X)} M' \to \mathscr{U}_n(Y) \otimes_{\widehat{\mathscr{U}}(X)} M'' \to 0$$

preserves exactness. □

Our next aim is to establish a module analogue of Theorem 5.17, i.e. $\operatorname{Loc} M$ should be the inverse limit of sheaves on well chosen sites, with vanishing higher cohomology.

We define the analogues of the sheaves $\mathscr{U}_n$ in the module case as follows.
If M is a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(X)$-module and $\mathcal{M} = \operatorname{Loc} M$, we define the presheaf M_n on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ by

$$\begin{aligned} M_n(V) &= \mathscr{U}_n(\mathscr{L})(V) \otimes_{\mathscr{U}_n(\mathscr{L})(X)} (\mathscr{U}_n(\mathscr{L})(X) \otimes_{\widehat{\mathscr{U}(\mathscr{L})}(X)} M) \\ &= \mathscr{U}_n(\mathscr{L})(V) \otimes_{\widehat{\mathscr{U}(\mathscr{L})}(X)} M. \end{aligned}$$

Note that by [5, 7.3],

$$M_n(V) = \mathscr{U}_n(\mathscr{L})(V) \otimes_{\widehat{\mathscr{U}(\mathscr{L})}(V)} \mathcal{M}(V),$$

so $\mathcal{M}(V) = \varprojlim M_n(V)$ by definition of $\widehat{\otimes}$, and $M_n(V)$ is a finitely generated $\mathscr{U}_n(\mathscr{L})(V)$-module for every V in $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ by Proposition 5.23.

Theorem 5.27. *Let $X = \operatorname{Sp} A$ be an affinoid K-space, $\mathscr{L}$ a Lie algebroid on X, and let M be a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(X)$-module. Fix an affine formal model $\mathcal{A}$ of A and an $(R, \mathcal{A})$-Lie lattice $\mathcal{L}$ in $\mathscr{L}(X)$, and let $\mathscr{U}_n(\mathscr{L})$ be as in Theorem 5.17.*
Then the presheaf M_n is a sheaf on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ and has vanishing higher Čech cohomology with respect to any $\pi^n\mathcal{L}$-accessible covering.

Proof. This is a straightforward variant of [13, Corollary 8.2.1/5]. We prove the slightly more general statement that for any finitely generated $\mathscr{U}_n(\mathscr{L})(X)$-module N, the presheaf

$$V \mapsto \mathscr{U}_n(\mathscr{L})(V) \otimes_{\mathscr{U}_n(\mathscr{L})(X)} N$$

is a sheaf on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$ with vanishing higher cohomology groups.
Since N is finitely generated, we have a short exact sequence

$$0 \to N' \to \mathscr{U}_n(\mathscr{L})(X)^{\oplus k} \to N \to 0,$$

where N' is also a finitely generated $\mathscr{U}_n(\mathscr{L})(X)$-module by Noetherianity. Now given a finite $\pi^n\mathcal{L}$-accessible covering $\mathfrak{V}$, Theorem 5.20 implies that we have short exact sequences

$$0 \to \check{C}^\bullet(\mathscr{U}_n \otimes N') \to \check{C}^\bullet(\mathscr{U}_n)^{\oplus k} \to \check{C}^\bullet(\mathscr{U}_n \otimes N) \to 0,$$

where we abbreviate $\check{C}^i_{\mathrm{aug}}(\mathfrak{V}, M)$ to $\check{C}^i(M)$. Taking the corresponding long exact sequence of cohomology, it follows from the vanishing of the cohomology in the middle term (Theorem 5.17) that we obtain isomorphisms

$$\check{\mathrm{H}}^i(\mathscr{U}_n \otimes N) \cong \check{\mathrm{H}}^{i+1}(\mathscr{U}_n \otimes N').$$

Since N' was also finitely generated, it follows by an inductive argument that the augmented Čech complex is exact, as required. □

In particular, we see as before that $\operatorname{Loc} M$ is a sheaf on X_w, extending uniquely to a sheaf on X_{rig}, and has vanishing higher Čech cohomology with respect to any (finite) affinoid covering.

Now let X be a rigid analytic K-space, $\mathscr{L}$ a Lie algebroid on X. In analogy with coherent $\mathcal{O}_X$-modules, we call a $\widehat{\mathscr{U}(\mathscr{L})}$-module $\mathcal{M}$ **coadmissible** if there exists an admissible covering $\mathfrak{U} = (U_i)_i$ of X_w by affinoid spaces such that for each i, the following is satisfied:

(i) $\mathcal{M}(U_i)$ is a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}(U_i)$-module.

(ii) The natural morphism $\operatorname{Loc}(\mathcal{M}(U_i)) \to \mathcal{M}|_{U_i}$ is an isomorphism.

If $\mathcal{M}$ satisfies the above for a certain admissible covering $\mathfrak{U}$, then we say $\mathcal{M}$ is $\mathfrak{U}$-coadmissible.

We also mention a theorem mirroring the classical results for coherent $\mathcal{O}_X$-modules (compare to Theorem 2.23).

Theorem 5.28 (see [5, Theorem 8.4])**.** *Let X be an affinoid K-space, $\mathfrak{U}$ an admissible covering in X_w, and let $\mathcal{M}$ be a $\mathfrak{U}$-coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-module. Then*

$$\mathcal{M} \cong \operatorname{Loc}(\mathcal{M}(X)).$$

The proof is as in [5, Theorem 8.4], where the result is given under the assumption that X admits a smooth Lie lattice.
Note that the theorem implies that if $\mathcal{M}$ is a $\mathfrak{U}$-coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-module on some smooth rigid analytic K-space X, then $\mathcal{M}$ is coadmissible with respect to any affinoid covering.

We slightly generalize these notions in order to extend Lemma 5.24 to coadmissible sheaves.

Definition 5.29. *A sheaf of K-algebras $\mathcal{F}$ on a rigid analytic K-space X is called a* ***global (left) Fréchet–Stein sheaf*** *if there exist*

(i) a collection of sites $(X_n)_{n\in\mathbb{N}}$ on X such that X_n is contained in X_{n+1} for each n, and any $U \in X_w$ is in X_n for sufficiently large n, likewise for X_w-coverings; and

(ii) for each n a sheaf of K-algebras $\mathcal{F}_n$ on X_n together with morphisms $\mathcal{F}_{n+1}|_{X_n} \to \mathcal{F}_n$

such that the sheaf given by $U \mapsto \varprojlim(\mathcal{F}_n(U))$ is isomorphic to $\mathcal{F}$, and the isomorphism $\mathcal{F}(U) \cong \varprojlim(\mathcal{F}_n(U))$ exhibits $\mathcal{F}(U)$ as a (left) Fréchet–Stein algebra for every admissible open affinoid subspace $U \subseteq X$.

An $\mathcal{F}$-module $\mathcal{M}$ is then called coadmissible if for every admissible open affinoid subspace $U \subseteq X$, the following is satisfied:

(i) $\mathcal{M}(U)$ is a coadmissible $\mathcal{F}(U)$-module.

(ii) If V is an affinoid subdomain of U, then $\mathcal{F}(V)\widehat{\otimes}_{\mathcal{F}(U)}\mathcal{M}(U) \cong \mathcal{M}(V)$ via the natural morphism.

The previous discussion then implies that for any Lie algebroid $\mathscr{L}$ on an affinoid K-space X, the sheaf $\widehat{\mathscr{U}(\mathscr{L})}$ is a global Fréchet–Stein sheaf, and Theorem 5.28 shows that the definition of coadmissibility in Definition 5.29 agrees in this case with the earlier one.

Proposition 5.30. *Let X be a rigid analytic K-space and let $\mathcal{F}' = \varprojlim \mathcal{F}'_n$ be a global left Fréchet–Stein sheaf on X.*
Let $\mathcal{F}$ be a sheaf of K-algebras on X which is also a left coadmissible $\mathcal{F}'$-module via a morphism $\theta : \mathcal{F}' \to \mathcal{F}$, with continuous multiplication. Then $\mathcal{F}$ is itself a global left Fréchet–Stein sheaf on X, and an $\mathcal{F}$-module $\mathcal{M}$ is coadmissible if and only if it is coadmissible as an $\mathcal{F}'$-module.

Proof. This is just Lemma 5.24 combined with the natural isomorphism

$$\mathcal{F}(V)\widehat{\otimes}_{\mathcal{F}(U)}M \cong \left(\mathcal{F}'(V)\widehat{\otimes}_{\mathcal{F}'(U)}\mathcal{F}(U)\right)\widehat{\otimes}_{\mathcal{F}(U)}M \cong \mathcal{F}'(V)\widehat{\otimes}_{\mathcal{F}'(U)}M$$

for any coadmissible $\mathcal{F}(U)$-module M, U an admissible open affinoid subspace with affinoid subdomain V (see [5, Proposition 7.4]). □

In this case, we call $\mathcal{F}$ a **coadmissible enlargement** of $\mathcal{F}'$.
A standard example of coadmissible enlargement is given by the following. If $\mathscr{L}' \to \mathscr{L}$ is an epimorphism of Lie algebroids on an affinoid K-space X, then the natural epimorphism $\widehat{\mathscr{U}(\mathscr{L}')} \to \widehat{\mathscr{U}(\mathscr{L})}$ turns $\widehat{\mathscr{U}(\mathscr{L})}$ into a coadmissible enlargement of $\widehat{\mathscr{U}(\mathscr{L}')}$. The proposition above can then be viewed as the sheaf analogue of the remark after Lemma 3.8 in [44].

Chapter 6

A Proper Mapping Theorem for coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-modules

The parallels between coherent $\mathcal{O}$-modules and coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-modules suggest a Proper Mapping Theorem for $\widehat{\mathscr{U}(\mathscr{L})}$-modules similar to Theorem 4.3.
Let $f : X \to Y$ be a proper morphism of rigid analytic K-spaces, and let $\mathscr{L}$ be a Lie algebroid on X. Let $\mathcal{M}$ be a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-module.
We will consider the pushforward $\mathrm{R}^j f_*\mathcal{M}$ as a module over $f_*\widehat{\mathscr{U}(\mathscr{L})}$.
Note that this, rather than a $\mathcal{D}$-module pushforward, is the right approach for a Beilinson–Bernstein style theory on the analytic flag variety (with f being the projection to the point $\operatorname{Sp} K$), as in this case $f_* = \Gamma(X, -)$.

Most of this chapter will be devoted to the special case when $\mathscr{L}$ is a *free* $\mathcal{O}_X$-module. The more general result is then a relatively straightforward corollary.

Theorem 6.1. *Let $f : X \to Y$ be an elementary proper morphism of rigid analytic K-spaces, and let $\mathscr{L}$ be a Lie algebroid on X which is free as an $\mathcal{O}_X$-module. Then $f_*\widehat{\mathscr{U}(\mathscr{L})}$ is a global Fréchet–Stein sheaf on Y, and if $\mathcal{M}$ is a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-module, then $\mathrm{R}^j f_*\mathcal{M}$ is a coadmissible $f_*\widehat{\mathscr{U}(\mathscr{L})}$-module for each $j \geq 0$, i.e. the following is satisfied:*

(i) $\mathrm{R}^j f_*\mathcal{M}(Y)$ is a coadmissible module over $f_*\widehat{\mathscr{U}(\mathscr{L})}(Y)$.

(ii) For any affinoid subdomain U of Y, the natural morphism

$$f_*\widehat{\mathscr{U}(\mathscr{L})}(U)\widehat{\otimes}_{f_*\widehat{\mathscr{U}(\mathscr{L})}(Y)}\mathrm{R}^j f_*\mathcal{M}(Y) \to \mathrm{R}^j f_*\mathcal{M}(U)$$

is an isomorphism.

Recall the Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ & \overset{f}{\searrow} & \downarrow h \\ & & Y = \operatorname{Sp} A \end{array}$$

where g is also elementary proper and h is a finite morphism.

If $U \subseteq Y$ is an affinoid subdomain and $\mathscr{F}$ is any sheaf on X, then $h^{-1}U$ is an affinoid subdomain of Z by [12, Proposition 3.3/13], and

$$\mathrm{R}^j f_* \mathscr{F}(U) = \mathrm{H}^j(f^{-1}U, \mathscr{F}) = \mathrm{H}^j(g^{-1}h^{-1}U, \mathscr{F}) = \mathrm{R}^j g_* \mathscr{F}(h^{-1}U)$$

for each $j \geq 0$.
Therefore, we will assume from now on that f is equal to the first map in its Stein factorization, i.e. $Y = \operatorname{Sp} A$, where $A = \mathcal{O}_X(X)$.
Note that if $U \subseteq Y$ is an affinoid subdomain of Y, then all our assumptions are still satisfied after restricting to $f|_{f^{-1}U} : f^{-1}U \to U$. If $U = \operatorname{Sp} B$, then $\mathcal{O}_X(f^{-1}U) = B$ by Kiehl's Proper Mapping Theorem, $\mathscr{L}|_{f^{-1}U}$ is a free Lie algebroid and $f|_{f^{-1}U}$ is an elementary proper morphism $f^{-1}U \to U$ by the behaviour of relative compactness under direct products (see [12, Lemma 6.3/7.(i)]).

Under our assumptions, the first part of our proof was already hinted at in section 5.3.

- Fréchet–Stein algebras: For any affinoid subdomain $U \subseteq Y$, the algebra

  $$f_* \widehat{\mathscr{U}(\mathscr{L})}(U) = \widehat{\mathscr{U}(\mathscr{L})}(f^{-1}U)$$

  is isomorphic to $\widehat{U(\mathscr{L}(f^{-1}U))}$, and hence is a Fréchet–Stein algebra. The proof of this statement in the next section will in fact exhibit $f_* \widehat{\mathscr{U}(\mathscr{L})}$ as a global Fréchet–Stein sheaf on Y.

Identifying $\mathrm{R}^j f_* \mathcal{M}(U) = \mathrm{H}^j(f^{-1}U, \mathcal{M})$ and writing $L = \mathscr{L}(X)$, we will show the following.

- Global sections: $\mathrm{H}^j(X, \mathcal{M})$ is a coadmissible $\widehat{U(L)}$-module for each $j \geq 0$.

- Localization: For any affinoid subdomain $\operatorname{Sp} B = U \subseteq Y$, the canonical morphism

  $$\widehat{U(B \otimes_A L)} \widehat{\otimes}_{\widehat{U(L)}} \mathrm{H}^j(X, \mathcal{M}) \to \mathrm{H}^j(f^{-1}U, \mathcal{M})$$

  is an isomorphism for each $j \geq 0$.

6.1 Fréchet–Stein algebras: The sheaves $\mathscr{U}_n$ and M_n

Let us abbreviate $\widehat{\mathscr{U}(\mathscr{L})}$ to $\widehat{\mathscr{U}}$, and let $\mathcal{M}$ be a left coadmissible $\widehat{\mathscr{U}}$-module. Analogous arguments will work in the case of right modules.
We begin by constructing sheaves $\mathscr{U}_n$ and M_n such that $\varprojlim \mathscr{U}_n = \widehat{\mathscr{U}}$ and $\varprojlim M_n = \mathcal{M}$, similarly to the previous chapter (even though X is not assumed to be affinoid here).

First, we need to introduce the right Grothendieck topology on X, corresponding to $X_{\mathrm{ac}}(\mathcal{L})$ in the affinoid case.

Let (U_i), (V_i) be affinoid coverings of X as described in Definition 4.2, i.e. for each i, V_i is relatively compact in U_i with respect to Y.
Choose an affine formal model $\mathcal{A}$ inside $A = \mathcal{O}_X(X)$, and let $\mathcal{L}$ be an $(R, \mathcal{A})$-Lie lattice inside $L = \mathscr{L}(X)$. Since $\mathscr{L}$ is assumed to be free, L is a free A-module, so that we can (and will) take $\mathcal{L}$ to be a free $\mathcal{A}$-module.

Now let U be any finite intersection of U_is or V_js. Recall that by Lemma 3.10, each $\mathcal{O}_X(U)$ admits an affine formal model containing the image of $\mathcal{A}$ under the restriction map

$$\mathcal{O}_X(X) \to \mathcal{O}_X(U).$$

Replacing $\mathcal{L}$ by $\pi^n\mathcal{L}$ for suitable n, we can assume that each $\mathcal{O}_X(U)$ admits an affine formal model $\mathcal{B}_U$ that

(i) contains the image of $\mathcal{A}$, and

(ii) is preserved under the action of $\mathcal{L}$ induced via the map $\mathscr{L}(X) \to \mathscr{L}(U)$ (as any affine formal model is topologically of finite type),

where U is any finite intersection of U_is or V_js (here we are using that both coverings are finite, so we are only considering a finite collection of affinoid subspaces U). We adapt the same terminology as in the case of affinoid subdomains and call $\mathcal{B}_U$ an $\mathcal{L}$-stable affine formal model.

Thus

$$\mathcal{L}_U := \mathcal{B}_U \otimes_{\mathcal{A}} \mathcal{L} \subseteq \mathcal{O}_X(U) \otimes_A L = \mathscr{L}(U)$$

is an $(R, \mathcal{B}_U)$-Lie lattice inside $\mathscr{L}(U)$ for any such U.

For each i, we abbreviate $B_i = \mathcal{O}_X(U_i)$, $\mathcal{B}_i = \mathcal{B}_{U_i}$, $L_i = \mathscr{L}(U_i) = B_i \otimes_A L$, $\mathcal{L}_i = \mathcal{L}_{U_i}$, similarly for intersections $U_{i_1 \dots i_j}$.

Recall that for each i, we have defined the G-topology $U_{i,\mathrm{ac}}(\mathcal{L}_i)$ of $\mathcal{L}_i$-accessible subdomains of U_i. Again, replacing $\mathcal{L}$ by $\pi^n\mathcal{L}$ for suitable n and invoking Lemma 3.10, we can assume that each $U_{i_1\dots i_j}$ and each $V_{i_1\dots i_j}$ is $\mathcal{L}_i$-accessible whenever it is a subdomain of U_i, and that $\mathcal{B}_{ij}$ is an $\mathcal{L}_i$-stable affine formal model.

We now define the site $X_{\mathrm{ac}}(\mathcal{L})$ in a slight abuse of notation as the G-topology on X generated by the $U_{i,\mathrm{ac}}(\mathcal{L}_i)$, i.e. the finest G-topology on X inducing on U_i the topology $U_{i,\mathrm{ac}}(\mathcal{L}_i)$ – see [13, 9.1.3]. We won't really use any properties of this topology, it simply provides the right language in order to glue sheaves defined on each $U_{i,\mathrm{ac}}(\mathcal{L}_i)$.

Recall from the previous chapter that we have for each non-negative integer n a sheaf $\mathscr{U}_{n,i}$ on $U_{i,\mathrm{ac}}(\pi^n\mathcal{L}_i)$ given by

$$U \mapsto \mathcal{O}_X(U)\widehat{\otimes}_{B_i}\widehat{U(\pi^n\mathcal{L}_i)}_K,$$

satisfying $\varprojlim \mathscr{U}_{n,i}(U) = \widehat{\mathscr{U}(\mathscr{L})}(U)$ for each affinoid subdomain $U \subseteq U_i$.
On each overlap $U_{ij} = U_i \cap U_j$, we have

$$\begin{aligned}\mathscr{U}_{n,i}|_{U_{ij}} &= (\mathcal{O}_X(U_{ij})\widehat{\otimes}_{B_i}U(L_i))^{\sim}\\ &= \widehat{U(L_{ij})}^{\sim}\\ &= (\mathcal{O}_X(U_{ij})\widehat{\otimes}_{B_j}U(L_j))^{\sim}\\ &= \mathscr{U}_{n,j}|_{U_{ij}},\end{aligned}$$

where we write $M^{\sim}$for the presheaf $V \mapsto \mathcal{O}_X(V)\widehat{\otimes}M$ (in all cases the completion is taken with respect to the semi-norm with unit ball $\mathcal{B}_{ij} \otimes_{\mathcal{B}_i} (\mathcal{B}_i \otimes_{\mathcal{A}} \pi^n\mathcal{L})$, so the completions are indeed equal). Thus the sheaves $\mathscr{U}_{n,i}$ agree on all overlaps and glue to give a sheaf $\mathscr{U}_n$ on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$. Since $\varprojlim \mathscr{U}_{n,i} = \widehat{\mathscr{U}}|_{U_i}$ on each U_i, this implies the equality $\varprojlim \mathscr{U}_n(U) = \widehat{\mathscr{U}}(U)$ for any admissible open subspace U of X.

It follows that $\widehat{\mathscr{U}} \cong \varprojlim \mathscr{U}_n$ is a global Fréchet–Stein sheaf.

There is another way to describe the $\mathscr{U}_n$ which is slightly more constructive, at least for those sections of $\mathscr{U}_n$ we will be concerned with.
Consider the Čech complex $\check{C}^\bullet(\mathfrak{V}, \mathcal{O}_X)$, where $\mathfrak{V} = (V_i)$. As we have seen in chapter 4, this is a finite cochain complex of Banach A-modules with strict morphisms, where each cohomology group is a finitely generated A-module by Kiehl's Proper Mapping Theorem.
Since $\pi^n\mathcal{L}$ is a free $\mathcal{A}$-module, so is $U_{\mathcal{A}}(\pi^n\mathcal{L})$ by Rinehart's Theorem. In particular, applying

Corollary 3.20, the complex

$$\widehat{U(\pi^n\mathcal{L})}_K \widehat{\otimes}_A \check{C}^\bullet(\mathfrak{V}, \mathcal{O}_X) = \check{C}^\bullet(\mathfrak{V}, \mathscr{U}_n)$$

has cohomology

$$\check{\mathrm{H}}^j(\mathfrak{V}, \mathscr{U}_n) = \widehat{U(\pi^n\mathcal{L})}_K \otimes_A \check{\mathrm{H}}^j(\mathfrak{V}, \mathcal{O}_X).$$

This naturally identifies $\mathscr{U}_n(X)$ with $\widehat{U(\pi^n\mathcal{L})}_K$, and $\widehat{\mathscr{U}}(X) = \widehat{U(L)}$ as promised earlier.

Corollary 6.2. *For any affinoid subalgebra $U \subseteq Y$, we have*

$$f_*\widehat{\mathscr{U}}(U) = \widehat{U(\mathscr{L}(f^{-1}U))}.$$

In particular, $f_*\widehat{\mathscr{U}}(U)$ is a Fréchet–Stein algebra, and $f_*\widehat{\mathscr{U}} = \varprojlim f_*\mathscr{U}_n$ is a global Fréchet–Stein sheaf on Y.

Proof. We have seen the proof for $U = Y$ above. Restricting f to $f|_{f^{-1}U} : f^{-1}U \to U$ preserves all assumed properties of the morphism, so that the same argument applies to arbitrary affinoid subdomains U. The last statement follows from Theorem 5.5 and the definition of global Fréchet–Stein sheaf. □

We can also read off from the above discussion that $\check{\mathrm{H}}^j(\mathfrak{V}, \mathscr{U}_n)$ is a finitely generated $\mathscr{U}_n(X)$-module for any $j \geq 0$, which can be seen as a first partial result in the direction of Theorem 6.1.

Similarly, consider the sheaf $M_{n,i}$ on $U_{i,\mathrm{ac}}(\pi^n\mathcal{L}_i)$ from Theorem 5.27 given by

$$U \mapsto \mathscr{U}_n(U) \otimes_{\widehat{\mathscr{U}}(U_i)} \mathcal{M}(U_i).$$

Note that then by definition of $\widehat{\otimes}$,

$$\begin{aligned} M_{n,i}(U) &= \mathscr{U}_n(U) \otimes_{\widehat{\mathscr{U}}(U)} (\widehat{\mathscr{U}}(U) \widehat{\otimes}_{\widehat{\mathscr{U}}(U_i)} \mathcal{M}(U_i)) \\ &= \mathscr{U}_n(U) \otimes_{\widehat{\mathscr{U}}(U)} \mathcal{M}(U) \end{aligned}$$

for any $U \in U_{i,\mathrm{ac}}(\pi^n\mathcal{L})$.
Thus $M_{n,i}$ agrees with $M_{n,j}$ on U_{ij}, giving a sheaf M_n on $X_{\mathrm{ac}}(\pi^n\mathcal{L})$. Since $\varprojlim M_{n,i} \cong \mathcal{M}|_{U_i}$, we see that $\varprojlim M_n(U) \cong \mathcal{M}(U)$ for any admissible open subspace U of X.

By Theorem 5.27, $\mathfrak{U}$ resp. $\mathfrak{V}$ are admissible coverings such that if $U \in X_{\mathrm{ac}}(\pi^n\mathcal{L})$ is a finite intersection of sets in $\mathfrak{U}$ resp. $\mathfrak{V}$, then $\mathrm{H}^j(U, M_n) = 0$ for any $j > 0$.

Thus applying Theorem 2.27 gives

$$\check{\mathrm{H}}^j(\mathfrak{U}, M_n) \cong \mathrm{H}^j(X_{\mathrm{ac}}(\pi^n\mathcal{L}), M_n) \cong \check{\mathrm{H}}^j(\mathfrak{V}, M_n)$$

for any $j \geq 0$.

Finally, we will see later that

$$\mathrm{R}^j f_*\mathcal{M}(Y) = \mathrm{H}^j(X, \mathcal{M}) \cong \varprojlim \check{\mathrm{H}}^j(\mathfrak{V}, M_n),$$

so we have found natural candidates exhibiting the coadmissibility of $\mathrm{H}^j(X, \mathcal{M})$.

6.2 Global sections

In particular, we can reduce our problem to a 'Noetherian' setup. For the global sections, we wish to show the following.

(i) For each $j \geq 0$ and each n, $\check{\mathrm{H}}^j(\mathfrak{V}, M_n)$ is a finitely generated $\mathscr{U}_n(X)$-module.

(ii) The natural morphism

$$\mathscr{U}_n(X) \otimes_{\mathscr{U}_{n+1}(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_{n+1}) \to \check{\mathrm{H}}^j(\mathfrak{V}, M_n)$$

is an isomorphism of $\mathscr{U}_n(X)$-modules.

(iii) The natural morphism

$$\check{\mathrm{H}}^j(\mathfrak{V}, \mathcal{M}) \to \varprojlim \check{\mathrm{H}}^j(\mathfrak{V}, M_n)$$

is an isomorphism of $\widehat{\mathscr{U}}(X)$-modules.

The argument for (i) will rest on the discussion in chapter 4 and be analogous to the argument in [32], while (ii) will be established through an application of Theorem 3.21. The last statement (iii) will then follow easily from property (ii) in Proposition 5.23.

Recall the commutative diagram of A-modules

$$\begin{array}{ccc} A\langle x_1, \ldots, x_l\rangle & & \\ \downarrow & \overset{h_{i_1\ldots i_j}}{\searrow} & \\ \mathcal{O}_X(U_{i_1\ldots i_j}) & \longrightarrow & \mathcal{O}_X(V_{i_1\ldots i_j}) \end{array}$$

induced from the definition of properness, where $h_{i_1\ldots i_j}(x_m)$ is topologically nilpotent in $\mathcal{O}_X(V_{i_1\ldots i_j})$ for each $m = 1, \ldots, l$.

Equip $A\langle x_1, \ldots x_l\rangle$ with the natural residue norm (i.e. with unit ball $\mathcal{A}\langle x\rangle$), and recall that we have already chosen residue norms for the other terms given by the unit balls $\mathcal{B}_{U_{i_1\ldots i_j}}$, $B_{V_{i_1\ldots i_j}}$ respectively, which turn the above into a diagram in Ban_A.
Now apply the functor $\mathscr{U}_n(X)\widehat{\otimes}_A-$ to the diagram to obtain

$$\begin{array}{ccc} \mathscr{U}_n(X)\widehat{\otimes}_A A\langle x_1, \ldots, x_l\rangle & & \\ \downarrow \theta' & \searrow^{h'} & \\ \mathscr{U}_n(X)\widehat{\otimes}_A \mathcal{O}(U_{i_1\ldots i_j}) & \longrightarrow & \mathscr{U}_n(X)\widehat{\otimes}_A \mathcal{O}_X(V_{i_1\ldots i_j}) \end{array}$$

which is a commutative diagram in $\mathrm{Ban}_{\mathscr{U}_n(X)}$.
Note that h' is no longer a homomorphism of algebras, but only of left Banach $\mathscr{U}_n(X)$-modules. It inherits from $h_{i_1\ldots i_j}$ the property that

$$h'(x^r) = (\mathscr{U}_n(X)\widehat{\otimes}_A h_{i_1\ldots i_j})(x^r)$$

tends to zero as $|r| \to \infty$, so Corollary 4.8 implies that h' is strictly completely continuous in $\mathrm{Ban}_{\mathscr{U}_n(X)}$.

Now note that

$$\mathscr{U}_n(X)\widehat{\otimes}_A \mathcal{O}_X(U_{i_1\ldots i_j}) = U_A(L)\widehat{\otimes}_A \mathcal{O}_X(U_{i_1\ldots i_j}),$$

where $U_A(L)$ is equipped with the norm with unit ball $U(\pi^n\mathcal{L})$.
Thus $\mathscr{U}_n(X)\widehat{\otimes}_A \mathcal{O}_X(U_{i_1\ldots i_j}) = \mathscr{U}_n(U_{i_1\ldots i_j})$.

The corresponding statement holds for $V_{i_1\ldots i_j}$, and the horizontal map between the two terms is simply the restriction map.

Thus we can read the above diagram as

$$\begin{array}{ccc} \mathscr{U}_n(X)\widehat{\otimes}_A A\langle x_1, \ldots, x_l\rangle & & \\ \downarrow \theta' & \searrow^{h'} & \\ \mathscr{U}_n(U_{i_1\ldots i_j}) & \xrightarrow{\mathrm{res}} & \mathscr{U}_n(V_{i_1\ldots i_j}) \end{array}$$

where h' is strictly completely continuous.

Lemma 6.3. *If $\mathcal{M}$ is a left coadmissible $\widehat{\mathscr{U}}$-module, then $\check{\mathrm{H}}^j(\mathfrak{V}, M_n)$ is a finitely generated $\mathscr{U}_n(X)$-module for all $j \geq 0$.*
The corresponding statement holds for right modules.

Proof. By functoriality, both θ' and h' are maps in $\mathrm{Ban}_{\mathscr{U}_n(X)}$. Likewise, the restriction maps are naturally morphisms in $\mathrm{Ban}_{\mathscr{U}_n(X)}$.

By Theorem 3.9, the map θ' is a strict surjection in $\mathrm{Ban}_{\mathscr{U}_n(X)}$.
We have thus shown that all the maps in the diagram are in $\mathrm{Ban}_{\mathcal{U}_n(X)}$, the arrow on the left is surjective, and h' is strictly completely continuous.

We now verify the conditions of Proposition 4.19 by following the corresponding steps from the proof of Theorem 4.3 as in [32].
Since $M_n(U_{i_1 \dots i_j})$ is finitely generated over $\mathscr{U}_n(U_{i_1 \dots i_j})$, it is equipped with a canonical topology, making it an object in $\mathrm{Ban}_{\mathscr{U}_n(U_{i_1 \dots i_j})}$ and hence a fortiori in $\mathrm{Ban}_{\mathscr{U}_n(X)}$. All the restriction maps are naturally continuous, so the Čech complexes $\check{C}^\bullet(\mathfrak{U}, M_n)$ and $\check{C}^\bullet(\mathfrak{V}, M_n)$ are cochain complexes in $\mathrm{Ban}_{\mathscr{U}_n(X)}$.
By construction, we have

$$M_n(V_{i_1 \dots i_j}) \cong \mathscr{U}_n(V_{i_1 \dots i_j}) \otimes_{\mathscr{U}_n(U_{i_1 \dots i_j})} M_n(U_{i_1 \dots i_j}),$$

so that finite generation induces a commutative diagram in $\mathscr{U}_n(X)$

$$\begin{array}{ccc}
\mathscr{U}_n(U_{i_1 \dots i_j})^{\oplus r} & \longrightarrow & \mathscr{U}_n(V_{i_1 \dots i_j})^{\oplus r} \\
\downarrow & & \downarrow \\
M_n(U_{i_1 \dots i_j}) & \xrightarrow{\text{res}} & M_n(V_{i_1 \dots i_j})
\end{array}$$

where both vertical maps are surjections and r is the size of some finite generating set.

Attaching this to r copies of the previous diagram, we obtain

$$\begin{array}{ccc}
\mathscr{U}_n(X) \widehat{\otimes}_A A\langle x_1, \dots, x_l \rangle^{\oplus r} & & \\
\downarrow & \searrow & \\
\mathscr{U}_n(U_{i_1 \dots i_j})^{\oplus r} & \longrightarrow & \mathscr{U}_n(V_{i_1 \dots i_j})^{\oplus r} \\
\downarrow & & \downarrow \\
M_n(U_{i_1 \dots i_j}) & \longrightarrow & M_n(V_{i_1 \dots i_j})
\end{array}$$

Writing $G_{i_1\dots i_j} := (\mathscr{U}_n(X)\widehat{\otimes}_A A\langle x_1,\dots x_l\rangle)^{\oplus r}$ and $\beta'_{i_1\dots i_j} : G_{i_1\dots i_j} \to M_n(U_{i_1\dots i_j})$ for the surjective morphism on the left hand side of the diagram, we can invoke Lemma 4.10 and Lemma 4.9 to see that

$$\mathrm{res}\circ\beta'_{i_1\dots i_j} : G_{i_1\dots i_j} \to M_n(V_{i_1\dots i_j})$$

is strictly completely continuous in $\mathrm{Ban}_{\mathscr{U}_n(X)}$, by commutativity of the diagram.

Summing over all different $U_{i_1\dots i_j}$, Lemma 4.10 thus implies that

$$\beta_j : F^j := \oplus G_{i_1\dots i_j} \to \oplus M_n(U_{i_1\dots i_j}) = \check{C}^j(\mathfrak{U}, M_n)$$

is a surjection in $\mathrm{Ban}_{\mathscr{U}_n(X)}$ with the property that $\mathrm{res}\circ\beta_j$ is strictly completely continuous. But $\mathrm{res} : \check{C}^j(\mathfrak{U}, M_n) \to \check{C}^j(\mathfrak{V}, M_n)$ induces an isomorphism on the level of cohomology groups, as seen in the previous chapter. Thus we have verified that Proposition 4.19 applies, proving the result.

Tensoring instead on the right, we can repeat the same argument for right $\mathscr{U}_n(X)$-modules. □

It now follows from Corollary 4.20 that $\check{C}^\bullet(\mathfrak{V}, M_n)$ consists of strict morphisms.

In general, we see that the part of Theorem 6.1 which is concerned with certain finiteness properties is still very close to the proof of Theorem 4.3. The only additional difficulties here lie in passing to sheaves $\mathscr{U}_n$ and M_n whose structure is more 'finite' than that of the original sheaves, and analyzing some easy completed tensor products.

Note however that there remains an additional property to be checked which has no counterpart in Theorem 4.3. We need to show that the finite components which we have exhibited match up in the right way, that is to say

$$\mathscr{U}_n(X) \otimes_{\mathscr{U}_{n+1}(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_{n+1}) \cong \check{\mathrm{H}}^j(\mathfrak{V}, M_n).$$

Recall that $\mathscr{U}_n(X) = \widehat{U(\pi^n\mathcal{L})}_K$ is flat over $\mathscr{U}_{n+1}(X)$, so we know that

$$\mathscr{U}_n(X) \otimes_{\mathscr{U}_{n+1}(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_{n+1}) \cong \mathrm{H}^j(\mathscr{U}_n(X) \otimes \check{C}^\bullet(\mathfrak{V}, M_{n+1})).$$

Our first goal will be to show that the isomorphism claimed above can be viewed as a $\widehat{\otimes}$-version of this statement.

Lemma 6.4. *If V is an admissible open affinoid subspace of X, then the natural map*

$$\mathscr{U}_n(X)\widehat{\otimes}_{\mathscr{U}_{n+1}(X)}M_{n+1}(V) \to M_n(V)$$

is an isomorphism of $\mathscr{U}_n(X)$-modules.

Proof. Since $\widehat{\otimes}$ agrees with $\otimes$ for finitely generated modules (see Lemma 3.12), we have the following chain of isomorphisms

$$\begin{aligned}
M_n(V) &= \mathscr{U}_n(V) \otimes_{\mathscr{U}_{n+1}(V)} M_{n+1}(V) \\
&\cong \mathscr{U}_n(V)\widehat{\otimes}_{\mathscr{U}_{n+1}(V)}M_{n+1}(V) \\
&\cong \left(\widehat{U(\pi^n\mathcal{L})}_K\widehat{\otimes}_A\mathcal{O}_X(V)\right)\widehat{\otimes}_{\mathscr{U}_{n+1}(V)}M_{n+1}(V) \\
&\cong \left(\widehat{U(\pi^n\mathcal{L})}_K\widehat{\otimes}_{\widehat{U(\pi^{n+1}\mathcal{L})}_K}\widehat{U(\pi^{n+1}\mathcal{L})}_K\widehat{\otimes}_A\mathcal{O}_X(V)\right)\widehat{\otimes}_{\mathscr{U}_{n+1}(V)}M_{n+1}(V) \\
&\cong \widehat{U(\pi^n\mathcal{L})}_K\widehat{\otimes}_{\widehat{U(\pi^{n+1}\mathcal{L})}_K}\left(\left(\widehat{U(\pi^{n+1}\mathcal{L})}_K\widehat{\otimes}_A\mathcal{O}_X(V)\right)\widehat{\otimes}_{\mathscr{U}_{n+1}(V)}M_{n+1}(V)\right) \\
&= \mathscr{U}_n(X)\widehat{\otimes}_{\mathscr{U}_{n+1}(X)}\left(\mathscr{U}_{n+1}(V)\widehat{\otimes}_{\mathscr{U}_{n+1}(V)}M_{n+1}(V)\right) \\
&\cong \mathscr{U}_n(X)\widehat{\otimes}_{\mathscr{U}_{n+1}(X)}M_{n+1}(V),
\end{aligned}$$

using associativity of the completed tensor product (see [13, Proposition 2.1.7/6]). □

To continue in our proof of Theorem 6.1, we therefore wish to show that

$$\mathscr{U}_n(X)\widehat{\otimes}_{\mathscr{U}_{n+1}(X)}\check{\mathrm{H}}^j(\mathfrak{V}, M_{n+1}) \cong \mathrm{H}^j(\mathscr{U}_n(X)\widehat{\otimes}_{\mathscr{U}_{n+1}(X)}\check{C}^\bullet(\mathfrak{V}, M_{n+1})).$$

This will be achieved by checking all the conditions in Corollary 3.22, where the role of A° is played by $U(\pi\mathcal{L})$, and that of $\mathcal{B}$ by $U(\mathcal{L})$ (by replacing $\mathcal{L}$ by $\pi^n\mathcal{L}$, it is enough to consider throughout the case $n = 0$).

Lemma 6.5. *$U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} \widehat{U(\pi\mathcal{L})}$ carries a natural ring structure, making it a left and right Noetherian ring.*

Proof. By freeness of $\mathcal{L}$, we have a natural injection $U_{\mathcal{A}}(\mathcal{L}) \to U_A(L)$ (by Rinehart's Theorem). Since $\widehat{U(\pi\mathcal{L})}$ is flat over $U(\pi\mathcal{L})$ by [11, 3.2.3.(iv)], we thus can view $U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} \widehat{U(\pi\mathcal{L})}$ as a subset of

$$U_A(L) \otimes_{U(\pi\mathcal{L})} \widehat{U(\pi\mathcal{L})} = U_A(L) \otimes_{U_A(L)} \widehat{U(\pi\mathcal{L})}_K = \widehat{U(\pi\mathcal{L})}_K,$$

identifying it with $U(\mathcal{L}) \cdot \widehat{U(\pi\mathcal{L})}$. We will show that this is a Noetherian subring of $\widehat{U(\pi\mathcal{L})}_K$.

Since $[\mathcal{L}, \pi\mathcal{L}] \subseteq \pi\mathcal{L}$, an easy inductive argument shows that $[\mathcal{L}, U(\pi\mathcal{L})] \subseteq U(\pi\mathcal{L})$, where the

commutator is understood in $U_A(L)$.
Hence we have that for each $\partial \in \mathcal{L}$, the commutator map

$$[\partial, -] : U_A(L) \to U_A(L)$$

preserves $U(\pi\mathcal{L})$, i.e. is a bounded linear map on $U_A(L)$ with unit ball $U(\pi\mathcal{L})$, where we can take 1 as a bound.
Thus passing to the completion, $[\mathcal{L}, \widehat{U(\pi\mathcal{L})}] \subseteq \widehat{U(\pi\mathcal{L})}$, where the commutator is understood in $\widehat{U(\pi\mathcal{L})}_K$. Therefore

$$U(\mathcal{L}) \cdot \widehat{U(\pi\mathcal{L})}$$

is a subring of $\widehat{U(\pi\mathcal{L})}_K$ by another easy induction argument, as required.
Denote this ring by $\mathcal{E}$.

Let $F_\bullet U(\mathcal{L})$ be the usual degree filtration. Then $\mathcal{E}$ is filtered by

$$F'_i\mathcal{E} = F_i U(\mathcal{L}) \cdot \widehat{U(\pi\mathcal{L})},$$

such that the following is satisfied:

(i) $F'_0\mathcal{E} = \widehat{U(\pi\mathcal{L})}$.

(ii) $F'_i\mathcal{E} \cdot F'_j\mathcal{E} \subseteq F'_{i+j}\mathcal{E}$, by reiterating the above commutator expression.

Just as in the proof of Theorem 5.5, we see that the associated graded ring $\mathrm{gr}'\mathcal{E}$ is generated by finitely many central elements over the zeroth piece $\widehat{U(\pi\mathcal{L})}$, which is Noetherian by Rinehart's Theorem and [11, 3.2.3.(vi)].
Thus $\mathcal{E}$ is a Noetherian ring by [35, Corollary D.IV.5]. □

We thus have confirmed that the first condition in Corollary 3.22 is satisfied. It remains to show that the relevant Tor groups have bounded π-torsion.

Write $\mathcal{O}_X(V_{i_1\dots i_j}) = B$, and let $\mathcal{B} = \mathcal{B}_{V_{i_1\dots i_j}}$ be an $\mathcal{L}$-stable affine formal model, as discussed in the previous section.
Denote by $U_\mathcal{B}$ the Noetherian ring $U_\mathcal{B}(\mathcal{B} \otimes_\mathcal{A} \pi\mathcal{L})$ and by $\widehat{U_\mathcal{B}}$ its π-adic completion. Note that this is the unit ball of $\mathscr{U}_1(V_{i_1\dots i_j})$.
Similarly to the above, we have the following Lemma.

Lemma 6.6. $U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} \widehat{U_\mathcal{B}}$ *carries a natural ring structure, making it a left and right Noetherian ring.*

Proof. As before, we identify the tensor product with a certain subset of a K-algebra. Since $U(\mathcal{L})$ is flat over $\mathcal{A}$, we have an injection

$$U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} U(\pi\mathcal{L}) \otimes_{\mathcal{A}} \mathcal{B} = U(\mathcal{L}) \otimes_{\mathcal{A}} \mathcal{B} \to U(\mathcal{L}) \otimes_{\mathcal{A}} B = U_A(L) \otimes_A B.$$

As $U(\pi\mathcal{L}) \otimes \mathcal{B} \cong U(\mathcal{B} \otimes \pi\mathcal{L})$, and $U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})$ is flat over $U(\mathcal{B} \otimes \pi\mathcal{L})$ by [11, 3.2.3.(iv)], this induces an injective map

$$U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L}) \to U(B \otimes L) \otimes_{U(B\otimes L)} U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})_K,$$

and the right hand side is clearly just $U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})_K = \widehat{U_{\mathcal{B}}} \otimes K$. The map above identifies $U(\mathcal{L}) \otimes \widehat{U_{\mathcal{B}}}$ with $U(\mathcal{L}) \cdot \widehat{U_{\mathcal{B}}}$ in this algebra.
Since $\mathcal{B}$ is $\mathcal{L}$-stable, we can repeat the argument in Lemma 6.5 to show this is a Noetherian subring of $U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})_K$. □

Lemma 6.7. *Let N be a finitely generated $\widehat{U_{\mathcal{B}}}$-module.*
Then the module $\mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), N)$ has bounded π-torsion for each $s \geq 0$.
Thus $\mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), \check{C}^j(\mathfrak{V}, M_1)^\circ)$ has bounded π-torsion for each $s \geq 0$ and each j.

Proof. We abbreviate the functor $\mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), -)$ to $T_s(-)$.
By Noetherianity, we have a short exact sequence

$$0 \to N' \to \widehat{U_{\mathcal{B}}}^{\oplus r} \to N \to 0$$

for some integer r and some finitely generated $\widehat{U_{\mathcal{B}}}$-module N'.
We will use this to prove the lemma inductively via the corresponding long exact sequence.

For $s = 0$, we have that

$$U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} N = U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L}) \otimes_{U(\widehat{\mathcal{B}\otimes\pi}\mathcal{L})} N$$

is a finitely generated $U(\mathcal{L}) \otimes U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})$-module and hence has bounded π-torsion by Noetherianity (see Lemma 6.6).

Next, we show that

$$\mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), \widehat{U_{\mathcal{B}}}) = 0$$

for $s \geq 1$.

For this note that by flatness of $U(\mathcal{L})$ and $U(\pi\mathcal{L})$ over $\mathcal{A}$, we have

$$0 = \mathrm{Tor}_s^{\mathcal{A}}(U(\mathcal{L}), \mathcal{B}) = \mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), U(\pi\mathcal{L}) \otimes_{\mathcal{A}} \mathcal{B}),$$

using [47, Proposition 3.2.9]. Therefore, $\mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), U_{\mathcal{B}}) = 0$.
As moreover $\widehat{U_{\mathcal{B}}} = U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})$ is flat over $U(\mathcal{B} \otimes \pi\mathcal{L})$ by [11, 3.2.3.(iv)], we obtain

$$\mathrm{Tor}_s^{U(\pi\mathcal{L})}(U(\mathcal{L}), \widehat{U_{\mathcal{B}}}) = 0$$

for $s \geq 1$.

Thus, the long exact sequence

$$\cdots \to T_s(N') \to T_s\left(\widehat{U_{\mathcal{B}}}\right)^{\oplus r} \to T_s(N) \to T_{s-1}(N') \to \ldots$$

shows that $T_s(N) \to T_{s-1}(N')$ is an injection for all s, and an isomorphism for $s \geq 2$.
So if we suppose that $T_{s-1}(N)$ has bounded π-torsion for any finitely generated $\widehat{U_{\mathcal{B}}}$-module N, this holds in particular for N', proving that $T_s(N)$ has bounded π-torsion as well.
By induction, this finishes the proof of the first statement.

Now $M_1(V_{i_1\ldots i_j})^\circ$ is a finitely generated $\widehat{U_{\mathcal{B}}} = U(\widehat{\mathcal{B} \otimes \pi}\mathcal{L})$-module by Lemma 2.14, so taking the corresponding finite direct sum to form the jth term of the Čech complex proves the result. □

Theorem 6.8. *The natural morphism*

$$\mathscr{U}_n(X) \otimes_{\mathscr{U}_{n+1}(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_{n+1}) \to \mathrm{H}^j(\mathscr{U}_n(X) \widehat{\otimes}_{\mathscr{U}_{n+1}(X)} \check{C}^\bullet(\mathfrak{V}, M_{n+1}))$$

is an isomorphism of $\mathscr{U}_n(X)$-modules for each $n \geq 0$, $j \geq 0$.

Proof. Without loss of generality, we can assume $n = 0$.
Then the theorem is precisely Corollary 3.22 applied to the Čech complex $\check{C}^\bullet(\mathfrak{V}, M_1)$. This is a finite chain complex in $\mathrm{Ban}_{\mathscr{U}_1(X)}$ with strict morphisms, as we have seen above.
By Lemma 6.5, $U(\mathcal{L}) \otimes_{U(\pi\mathcal{L})} \widehat{U(\pi\mathcal{L})}$ is a Noetherian ring, $\check{\mathrm{H}}^j(\mathfrak{V}, M_1)$ is a finitely generated $\mathscr{U}_1(X) = \widehat{U(\pi\mathcal{L})}_K$-module by Theorem 6.3, and Lemma 6.7 ensures bounded π-torsion for each of the Tor groups. Thus Corollary 3.22 states that

$$U_A(L)' \otimes_{U_A(L)} \check{C}^\bullet(\mathfrak{V}, M_1)$$

is a strict complex, where $U(L)$ is equipped with the norm with unit ball $U(\pi\mathcal{L})$ and we write

$U(L)'$ for $U(L)$ with unit ball $U(\mathcal{L})$.
Moreover, Corollary 3.22 also implies that

$$\begin{aligned}\mathscr{U}_0(X)\otimes_{\mathscr{U}_1(X)}\check{\mathrm{H}}^j(\mathfrak{V},M_1) &\cong \widehat{U(L)'}\otimes_{\widehat{U(L)}}\check{\mathrm{H}}^j(\mathfrak{V},M_1)\\ &\cong \mathrm{H}^j(U(L)'\widehat{\otimes}_{U(L)}\check{C}^\bullet(\mathfrak{V},M_1))\\ &\cong \mathrm{H}^j(\mathscr{U}_0(X)\widehat{\otimes}_{\mathscr{U}_1(X)}\check{C}^\bullet(\mathfrak{V},M_1)),\end{aligned}$$

proving the result. □

Corollary 6.9. *The $\widehat{\mathscr{U}}(X)$-module $\varprojlim_n \check{\mathrm{H}}^j(\mathfrak{V},M_n)$ is coadmissible for each $j\geq 0$.*

Proof. Each module $\check{\mathrm{H}}^j(\mathfrak{V},M_n)$ is a finitely generated $\mathscr{U}_n(X)$-module by Theorem 6.3, and

$$\mathscr{U}_n(X)\otimes_{\mathscr{U}_{n+1}(X)}\check{\mathrm{H}}^j(\mathfrak{V},M_{n+1})\cong\check{\mathrm{H}}^j(\mathfrak{V},M_n)$$

by the theorem above combined with the observation that

$$\mathscr{U}_n(X)\widehat{\otimes}_{\mathscr{U}_{n+1}(X)}\check{C}^\bullet(\mathfrak{V},M_{n+1})=\check{C}^\bullet(\mathfrak{V},M_n)$$

by Lemma 6.4. □

Finally, fixing an integer j, we show that $\varprojlim \check{\mathrm{H}}^j(\mathfrak{V},M_n)$ gives indeed the corresponding higher direct image of $\mathcal{M}$.

Proposition 6.10. *For each $j\geq 0$, the canonical morphism of $\widehat{\mathscr{U}}(X)$-modules*

$$\mathrm{H}^j(X,\mathcal{M})\cong\varprojlim\check{\mathrm{H}}^j(\mathfrak{V},M_n)$$

is an isomorphism.

Proof. By Proposition 5.23, each system of terms $(\check{C}^j(\mathfrak{V},M_n))_n$ satisfies the Mittag-Leffler property as described in [24, 0.13.2.4], and by Corollary 6.9, so does the inverse system $(\check{\mathrm{H}}^j(\mathfrak{V},M_n))_n$. Hence we can apply [24, Proposition 0.13.2.3] to deduce that

$$\check{\mathrm{H}}^j(\mathfrak{V},\mathcal{M})=\mathrm{H}^j(\varprojlim\check{C}^\bullet(\mathfrak{V},M_n))\cong\varprojlim\check{\mathrm{H}}^j(\mathfrak{V},M_n)$$

as required. Since $\mathcal{M}$ also has vanishing higher Čech cohomology on affinoids, sheaf cohomology and Čech cohomology agree by Theorem 2.27, as already noted in chapter 5.
Thus we have

$$\mathrm{H}^j(X,\mathcal{M})\cong\check{\mathrm{H}}^j(\mathfrak{V},\mathcal{M})\cong\varprojlim\check{\mathrm{H}}^j(\mathfrak{V},M_n)$$

as required. □

This concludes the proof that $\mathrm{R}^j f_*\mathcal{M}(Y) = \mathrm{H}^j(X, \mathcal{M})$ is a coadmissible $\widehat{\mathscr{U}}(X)$-module.

6.3 Localization

It remains to show that other sections of the sheaf are obtained by localization, that is if $U = \operatorname{Sp} B \subseteq Y$ is an affinoid subdomain, we want to show that

$$\widehat{\mathscr{U}}(f^{-1}U)\widehat{\otimes}_{\widehat{\mathscr{U}}(X)}\mathrm{H}^j(X, \mathcal{M}) \cong \mathrm{H}^j(f^{-1}U, \mathcal{M})$$

via the natural morphism.
Similarly to the above, our strategy will consist in a reduction to the Noetherian components of the coadmissible module and a result about completed tensor products similar to Theorem 6.8. We here show the argument for left coadmissible modules, but all statements also hold for right coadmissible modules mutatis mutandis.

Recall from Lemma 5.18 that $f_*\mathscr{L}$ is a Lie algebroid on Y with $f_*\mathscr{L}(Y) = L$.
Similarly to previous arguments, we will show the desired isomorphism first in the case when U is a $\pi^n\mathcal{L}$-accessible rational subdomain of Y in one step, then for arbitrary $\pi^n\mathcal{L}$-accessible rational subdomains and finally for arbitrary $\pi^n\mathcal{L}$-accessible affinoid subdomains of Y.

Our plan looks as follows.

- Step A: Consider the case of U a rational subdomain which is $\pi^n\mathcal{L}$-accessible in one step. As $B = \mathcal{O}_Y(U)$ can be described as a quotient of $A\langle t\rangle$, we will establish some properties relating to $A\langle t\rangle$ before passing to the quotient.

- Step B: An easy inductive argument extends the result to any $\pi^n\mathcal{L}$-accessible rational subdomain.

- Step C: Passing to suitable coverings and arguing locally, we can generalize to arbitrary $\pi^n\mathcal{L}$-accessible affinoid subdomains. Since any affinoid subdomain is $\pi^n\mathcal{L}$-accessible for sufficiently large n, this finishes the proof.

Step A

Let $x \in A$ be non-zero such that $\pi^n\mathcal{L} \cdot x \subseteq \mathcal{A}$, and consider

$$Y_1 = Y(x) = \operatorname{Sp} B_1,\ Y_2 = Y(x^{-1}) = \operatorname{Sp} B_2$$

with $\pi^n\mathcal{L}$-stable affine formal models $\mathcal{B}_1$ and $\mathcal{B}_2$ constructed as quotients of $\mathcal{A}\langle t\rangle$ as described in chapter 5.

By the definition of $\widehat{\otimes}$ and Proposition 6.10, it will be enough to show that the natural morphism

$$\mathscr{U}_n(f^{-1}Y_i)\otimes_{\mathscr{U}_n(X)}\check{\mathrm{H}}^j(\mathfrak{V},M_n)\to\check{\mathrm{H}}^j(\mathfrak{V}\cap f^{-1}Y_i,M_n)$$

is an isomorphism for $i=1,2$.

First recall that by Kiehl's Proper Mapping Theorem, $\mathcal{O}_X(f^{-1}Y_i)=B_i$, and

$$\mathscr{U}_n(f^{-1}Y_i)=U(\widehat{\mathcal{B}_i\otimes\pi^n}\mathcal{L})_K\cong B_i\widehat{\otimes}_A\mathscr{U}_n(X)$$

by Corollary 6.2.

Considering the right hand side, note that for any admissible open affinoid subspace $V\subseteq X$, we have

$$f^{-1}Y_i\cap V=\operatorname{Sp}B_i\times_{\operatorname{Sp}A}\operatorname{Sp}\mathcal{O}_X(V)=\operatorname{Sp}\left(B_i\widehat{\otimes}_A\mathcal{O}_X(V)\right)$$

by [13, Proposition 7.1.4/4], and hence

$$\begin{aligned}M_n(f^{-1}Y_i\cap V)&=\mathscr{U}_n(f^{-1}Y_i\cap V)\otimes_{\mathscr{U}_n(V)}M_n(V)\\&=\mathscr{U}_n(f^{-1}Y_i\cap V)\widehat{\otimes}_{\mathscr{U}_n(V)}M_n(V)\\&=\left((B_i\widehat{\otimes}_A\mathcal{O}_X(V))\widehat{\otimes}_{\mathcal{O}_X(V)}\mathscr{U}_n(V)\right)\widehat{\otimes}_{\mathscr{U}_n(V)}M_n(V)\\&=(B_i\widehat{\otimes}_A\mathcal{O}_X(V))\widehat{\otimes}_{\mathcal{O}_X(V)}M_n(V)\\&=B_i\widehat{\otimes}_AM_n(V).\end{aligned}$$

Thus

$$\check{C}^\bullet(\mathfrak{V}\cap f^{-1}Y_i,M_n)\cong B_i\widehat{\otimes}_A\check{C}^\bullet(\mathfrak{V},M_n),$$

which in turn can be written as $\mathscr{U}_n(f^{-1}Y_i)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V},M_n)$ by the above.

We thus wish to show that

$$\mathscr{U}_n(f^{-1}Y_i)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{\mathrm{H}}^j(\mathfrak{V},M_n)\cong\mathrm{H}^j(\mathscr{U}_n(f^{-1}Y_i)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V},M_n)).$$

We will prove this isomorphism by a number of lemmas, mainly exploiting the short exact sequence

$$0 \to A\langle t\rangle \to A\langle t\rangle \to B_i \to 0$$

and our study of completed tensor products.

Recall the two $\mathcal{L}$-actions on $\mathcal{A}\langle t\rangle$ defined in chapter 5, which we denoted by σ_i, $i = 1, 2$. We will write

$$\mathscr{U}_n(X)\langle t\rangle_i = U(\widehat{\mathcal{A}\langle t\rangle \otimes_{\mathcal{A}}} \pi^n \mathcal{L})_K$$

for the completed enveloping algebra of the $(R, \mathcal{A}\langle t\rangle)$-Lie algebra arising from the corresponding σ_i.

Lemma 6.11. *The natural map*

$$\mathscr{U}_n(X)\langle t\rangle_i \otimes_{\mathscr{U}_n(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_n) \to \mathrm{H}^j(\mathscr{U}_n(X)\langle t\rangle_i \widehat{\otimes}_{\mathscr{U}_n(X)} \check{C}^\bullet(\mathfrak{V}, M_n))$$

is an isomorphism of left $\mathscr{U}_n(X)\langle t\rangle_i$-modules for each $j \geq 0$.

Proof. We wish to apply Corollary 3.20.

We know that $\check{C}^\bullet(\mathfrak{V}, M_n)$ is a cochain complex of Banach $\mathscr{U}_n(X)$-modules, a fortiori of Banach A-modules, with strict morphisms.

As as right $\mathscr{U}_n(X)$-module, $\mathscr{U}_n(X)\langle t\rangle_i$ is isomorphic to

$$A\langle t\rangle \widehat{\otimes}_A \mathscr{U}_n(X)$$

by [5, Proposition 2.3], which is the completion of $A\langle t\rangle \otimes_A \mathscr{U}_n(X)$ with respect to the tensor product semi-norm with unit ball given by

$$\mathcal{A}\langle t\rangle \otimes_{\mathcal{A}} \widehat{U(\pi\mathcal{L})}.$$

In particular, viewing the natural morphism as a morphism of $A\langle t\rangle$-modules, it can be written as

$$A\langle t\rangle \widehat{\otimes}_A \check{\mathrm{H}}^j(\mathfrak{V}, M_n) \to \mathrm{H}^j(A\langle t\rangle \widehat{\otimes}_A \check{C}^\bullet(\mathfrak{V}, M_n)).$$

Since $\mathcal{A}\langle t\rangle$ is flat over $\mathcal{A}$ by [12, Remark 7.3/2], this is an isomorphism of $A\langle t\rangle$-modules by Corollary 3.20 and hence a bijection. Thus the natural morphism

$$\mathscr{U}_n(X)\langle t\rangle_i \widehat{\otimes}_{\mathscr{U}_n(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_n) \to \mathrm{H}^j(\mathscr{U}_n(X)\langle t\rangle_i \widehat{\otimes} \check{C}^\bullet(\mathfrak{V}, M_n))$$

is an isomorphism of $\mathscr{U}_n(X)\langle t\rangle_i$-modules. ☐

We now fix a finite set of indices $i_1, \ldots, i_j$ and write $V = V_{i_1\ldots i_j}$, $C = \mathcal{O}_X(V)$.

Lemma 6.12. *Let $\mathcal{C}$ be a $\pi^n\mathcal{L}$-stable affine formal model in $C = \mathcal{O}_X(V)$. Then*

$$\mathcal{B}_i \otimes_{\mathcal{A}} U(\widehat{\mathcal{C} \otimes_{\mathcal{A}} \pi^n\mathcal{L}})$$

has bounded π-torsion.

Proof. Define $\mathcal{B}'_i = \mathcal{A}[t]/u_i\mathcal{A}[t]$. Note that the π-adic completion of the short exact sequence

$$0 \longrightarrow \mathcal{A}[t] \xrightarrow{u_i\cdot} \mathcal{A}[t] \longrightarrow \mathcal{B}'_i \longrightarrow 0$$

is obtained by applying the exact functor $\mathcal{A}\langle t\rangle \otimes_{\mathcal{A}[t]} -$ by [20, Theorem 7.2]. In particular, $\widehat{\mathcal{B}'_i} = \mathcal{B}_i$.

Since $\mathcal{B}'_i$ is of finite type over $\mathcal{A}$, the ring $\mathcal{B}'_i \otimes_{\mathcal{A}} \mathcal{C}$ is of finite type over $\mathcal{C}$ and is hence Noetherian. In particular, it has bounded π-torsion.

Tensoring with a flat module preserves the property of bounded π-torsion by Lemma 2.10. Since $\mathcal{B}_i = \widehat{\mathcal{B}'_i}$, it is flat over $\mathcal{B}'_i$ by [20, Theorem 7.2]. Moreover,

$$U(\mathcal{C} \otimes_{\mathcal{A}} \pi\mathcal{L}) \cong \mathcal{C} \otimes_{\mathcal{A}} U(\pi^n\mathcal{L})$$

is flat over $\mathcal{C}$, and $U(\widehat{\mathcal{C} \otimes_{\mathcal{A}} \pi^n\mathcal{L}})$ is flat over $U(\mathcal{C} \otimes_{\mathcal{A}} \pi^n\mathcal{L})$ again by [11, 3.2.3.(iv)].
Thus

$$\mathcal{B}_i \otimes_{\mathcal{A}} U(\widehat{\mathcal{C} \otimes_{\mathcal{A}} \pi^n\mathcal{L}}) \cong \mathcal{B}_i \otimes_{\mathcal{B}'_i} (\mathcal{B}'_i \otimes_{\mathcal{A}} \mathcal{C}) \otimes_{\mathcal{C}} U(\widehat{\mathcal{C} \otimes \pi^n\mathcal{L}})$$

has bounded π-torsion, as required. ☐

Lemma 6.13. *There is a short exact sequence*

$$0 \longrightarrow A\langle t\rangle \widehat{\otimes}_A \mathscr{U}_n(V) \xrightarrow{u_i\cdot} A\langle t\rangle \widehat{\otimes}_A \mathscr{U}_n(V) \longrightarrow B_i \widehat{\otimes}_A \mathscr{U}_n(V) \longrightarrow 0$$

of left $A\langle t\rangle$-modules, analogously for the right module structure.

Proof. This is an easy variant of Lemma 5.21 and [5, Proposition 4.3.c)].
First note that the short exact sequence

$$0 \to A\langle t\rangle \to A\langle t\rangle \to B_i \to 0$$

consists of strict morphisms by Lemma 3.7.

Since B_i is flat over A by [12, Corollary 4.1/5], we have

$$\mathrm{Tor}_1^A(B_i, \mathscr{U}_n(V)) = 0,$$

so tensoring with $\mathscr{U}_n(V)$ yields a short exact sequence

$$0 \to A\langle t\rangle \otimes_A \mathscr{U}_n(V) \to A\langle t\rangle \otimes_A \mathscr{U}_n(V) \to B_i \otimes_A \mathscr{U}_n(V) \to 0.$$

Finally, $\mathcal{B}_i \otimes_{\mathcal{A}} U(\widehat{\mathcal{C} \otimes_{\mathcal{A}} \pi^n}\mathcal{L})$ has bounded π-torsion by Lemma 6.12, so that the short exact sequence above consists of strict morphisms and stays exact after completion by Corollary 3.17. □

Lemma 6.14. *Let N be a finitely generated left $\mathscr{U}_n(V)$-module, equipped with a complete norm inducing the canonical topology. Then we have a short exact sequence*

$$0 \longrightarrow A\langle t\rangle \widehat{\otimes}_A N \xrightarrow{u_i \cdot} A\langle t\rangle \widehat{\otimes}_A N \longrightarrow B_i \widehat{\otimes}_A N \longrightarrow 0$$

of left $A\langle t\rangle$-modules, analogously for right modules.

Proof. Since N is finitely generated over the Noetherian algebra $\mathscr{U}_n(V)$, we have a short exact sequence

$$0 \to N' \to \mathscr{U}_n(V)^{\oplus r} \to N \to 0,$$

where N' is another finitely generated Banach module over $\mathscr{U}_n(V)$. By Lemma 3.7, this consists of strict morphisms.

Since $\mathcal{A}\langle t\rangle$ is flat over $\mathcal{A}$ by [12, Remark 7.3/2], we know by Lemma 3.18 that

$$0 \to A\langle t\rangle \widehat{\otimes}_A N' \to A\langle t\rangle \widehat{\otimes}_A \mathscr{U}_n(V)^{\oplus r} \to A\langle t\rangle \widehat{\otimes}_A N \to 0$$

is exact.

Moreover, $B_i \widehat{\otimes}_A N' \cong \mathscr{U}_n(f^{-1}Y_i \cap V) \otimes_{\mathscr{U}_n(V)} N'$ as left B_i-modules, where we could omit the completion symbol on the right hand side by Lemma 3.12. Likewise for the other terms.
Now $f^{-1}Y_i \cap V$ is a rational subdomain of V by [12, Proposition 3.3/13], and is actually $\mathcal{C} \otimes \pi^n \mathcal{L}$-accessible - it is $V(x)$ if $i = 1$, $V(x^{-1})$ if $i = 2$, again by [12, Proposition 3.3/13]. So by Theorem 5.20, we know that

$$0 \to B_i \widehat{\otimes}_A N' \to B_i \widehat{\otimes}_A \mathscr{U}_n(V)^{\oplus r} \to B_i \widehat{\otimes}_A N \to 0$$

is exact.

We thus obtain the following commutative diagram of left $A\langle t\rangle$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\langle t\rangle\widehat{\otimes}_A N' & \longrightarrow & A\langle t\rangle\widehat{\otimes}_A \mathscr{U}_n(V)^{\oplus r} & \longrightarrow & A\langle t\rangle\widehat{\otimes}_A N & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 & & \\ 0 & \longrightarrow & A\langle t\rangle\widehat{\otimes}_A N' & \longrightarrow & A\langle t\rangle\widehat{\otimes}_A \mathscr{U}_n(V)^{\oplus r} & \longrightarrow & A\langle t\rangle\widehat{\otimes}_A N & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow g_2 & & \downarrow h_2 & & \\ 0 & \longrightarrow & B_i\widehat{\otimes}_A N' & \longrightarrow & B_i\widehat{\otimes}_A \mathscr{U}_n(V)^{\oplus r} & \longrightarrow & B_i\widehat{\otimes}_A N & \longrightarrow & 0 \end{array}$$

where each row is exact.
We know from Theorem 3.9 that f_2, g_2 and h_2 are surjections, so we have a long exact sequence

$$0 \to \ker f_1 \to \ker g_1 \to \ker h_1 \to \ker f_2/\operatorname{Im} f_1 \to \ker g_2/\operatorname{Im} g_1 \to \ker h_2/\operatorname{Im} h_1 \to 0.$$

By Lemma 6.13, this becomes

$$0 \to \ker f_1 \to 0 \to \ker h_1 \to \ker f_2/\operatorname{Im} f_1 \to 0 \to \ker h_2/\operatorname{Im} h_1 \to 0,$$

so we immediately get that $\ker h_2 = \operatorname{Im} h_1$. But this argument holds for any finitely generated $\mathscr{U}_n(V)$-module, so in particular for N'. Thus $\ker f_2 = \operatorname{Im} f_1$, and by exactness $\ker h_1 = 0$.

Thus
$$0 \to A\langle t\rangle\widehat{\otimes}_A N \to A\langle t\rangle\widehat{\otimes}_A N \to B_i\widehat{\otimes}_A N \to 0$$
is a short exact sequence. □

Theorem 6.15. *The natural morphism*

$$\mathscr{U}_n(f^{-1}Y_i) \otimes_{\mathscr{U}_n(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_n) \to \mathrm{H}^j(\mathscr{U}_n(f^{-1}Y_i)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V}, M_n))$$

is an isomorphism of $\mathscr{U}_n(f^{-1}Y_i)$-modules for each $j \geq 0$.

Proof. We abbreviate $\check{\mathrm{H}}^j(\mathfrak{V}, M_n)$ to H^j and $\check{C}^\bullet(\mathfrak{V}, M_n)$ to $C^\bullet$.

Since
$$\mathscr{U}_n(f^{-1}Y_i) \cong B_i\widehat{\otimes}_A \mathscr{U}_n(X)$$

as B_i-modules, it is enough to show that the natural morphism

$$B_i\widehat{\otimes}_A H^j \to \mathrm{H}^j(B_i\widehat{\otimes}_A C^\bullet)$$

is an isomorphism of left B_i-modules, or equivalently as $A\langle t\rangle$-modules.

Since $\mathscr{U}_n(f^{-1}Y_i) = \mathscr{U}_n(f_*\mathscr{L})(Y_i)$ is flat over $\mathscr{U}_n(X) = \mathscr{U}_n(f_*\mathscr{L})(Y)$ on the right by Theorem 5.20, we know that

$$\mathrm{Tor}_1^{\mathscr{U}_n(X)}\left(\mathscr{U}_n(f^{-1}Y_i), H^j\right) = 0,$$

so that the short exact sequence from Lemma 5.21,

$$0 \to \mathscr{U}_n(X)\langle t\rangle_i \to \mathscr{U}_n(X)\langle t\rangle_i \to \mathscr{U}_n(f^{-1}Y_i) \to 0$$

remains exact after applying the functor $-\otimes_{\mathscr{U}_n(X)} H^j$, producing a short exact sequence of left $A\langle t\rangle$-modules, which can be written as

$$0 \to A\langle t\rangle\widehat{\otimes}_A H^j \to A\langle t\rangle\widehat{\otimes}_A H^j \to B_i\widehat{\otimes}_A H^j \to 0.$$

By Lemma 6.14, we also have a short exact sequence

$$0 \to A\langle t\rangle\widehat{\otimes}_A C^\bullet \to A\langle t\rangle\widehat{\otimes}_A C^\bullet \to B_i\widehat{\otimes}_A C^\bullet \to 0,$$

of left $A\langle t\rangle$-modules.

This now induces a long exact sequence

$$\dots \to \mathrm{H}^j(A\langle t\rangle\widehat{\otimes}C^\bullet) \to \mathrm{H}^j(A\langle t\rangle\widehat{\otimes}C^\bullet) \to \mathrm{H}^j(B_i\widehat{\otimes}C^\bullet) \to \dots$$

fitting into a commutative diagram

$$\begin{array}{ccccccccc}
A\langle t\rangle\widehat{\otimes}H^j & \xrightarrow{u_i\cdot} & A\langle t\rangle\widehat{\otimes}H^j & \longrightarrow & B_i\widehat{\otimes}H^j & \xrightarrow{\xi} & A\langle t\rangle\widehat{\otimes}H^{j+1} & \xrightarrow{u_i\cdot} & A\langle t\rangle\widehat{\otimes}H^{j+1} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\theta & & \downarrow\cong & & \downarrow\cong \\
\mathrm{H}^j(A\langle t\rangle\widehat{\otimes}C^\bullet) & \longrightarrow & \mathrm{H}^j(A\langle t\rangle\widehat{\otimes}C^\bullet) & \longrightarrow & \mathrm{H}^j(B_i\widehat{\otimes}C^\bullet) & \longrightarrow & \mathrm{H}^{j+1}(A\langle t\rangle\widehat{\otimes}C^\bullet) & \longrightarrow & \mathrm{H}^{j+1}(A\langle t\rangle\widehat{\otimes}C^\bullet)
\end{array}$$

where the vertical maps are isomorphisms as indicated by Lemma 6.11.

The top row is exact (with ξ being the zero map) by the exactness of the short exact sequences above, and the bottom row is exact by construction, so θ is an isomorphism by the

5-lemma (see [47, Exercise 1.3.3]), as required. □

Step B

We now generalize the argument to arbitrary $\pi^n\mathcal{L}$-accessible rational subdomains.

Proposition 6.16. *Let $U \subseteq Y$ be a $\pi^n\mathcal{L}$-accessible rational subdomain of Y. Then the natural morphism*

$$\mathscr{U}_n(f^{-1}U) \otimes_{\mathscr{U}_n(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_n) \to \mathrm{H}^j(\mathscr{U}_n(f^{-1}U)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V}, M_n))$$

is an isomorphism for each $j \geq 0$, and thus

$$\widehat{\mathscr{U}}(f^{-1}U)\widehat{\otimes}_{\widehat{\mathscr{U}}(X)}\mathrm{H}^j(X, \mathcal{M}) \cong \mathrm{H}^j(f^{-1}U, \mathcal{M}).$$

Proof. Let U be $\pi^n\mathcal{L}$-accessible in r steps. Theorem 6.15 proves the case of $r = 1$. We proceed inductively. Let $V \subseteq Y$ be a $\pi^n\mathcal{L}$-accessible rational subdomain in $r-1$ steps, containing U and satisfying the properties in Definition 5.11, so that $U = V(x)$ or $V(x^{-1})$ for a suitable $x \in \mathcal{O}_Y(V)$.
By induction hypothesis, we have

$$\begin{aligned}\mathscr{U}_n(f^{-1}V) \otimes_{\mathscr{U}_n(X)} \check{\mathrm{H}}^j(\mathfrak{V}, M_n) &\cong \mathrm{H}^j(\mathscr{U}_n(f^{-1}V)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V}, M_n))\\ &\cong \mathrm{H}^j(f^{-1}V \cap \mathfrak{V}, M_n).\end{aligned}$$

Moreover, the restriction

$$f|_{f^{-1}V} : f^{-1}V \to V$$

is an elementary proper morphism with trivial Stein factorization, and $\mathscr{L}|_{f^{-1}V}$ is a free Lie algebroid on $f^{-1}V$, i.e. all our assumption remain valid under restriction. But now U is a rational subdomain of V which is $\mathcal{C} \otimes_{\mathcal{A}} \pi^n\mathcal{L}$-accessible in one step, where $\mathcal{C}$ is a suitable affine formal model in $\mathcal{O}_Y(V)$. Thus Theorem 6.15 implies that

$$\mathscr{U}_n(f^{-1}U) \otimes_{\mathscr{U}_n(f^{-1}V)} \check{\mathrm{H}}^j(f^{-1}V \cap \mathfrak{V}, M_n) \cong \mathrm{H}^j(\mathscr{U}_n(f^{-1}U)\widehat{\otimes}_{\mathscr{U}_n(f^{-1}V)}\check{C}^\bullet(f^{-1}V \cap \mathfrak{V}, M_n)).$$

Writing H^j for $\check{\mathrm{H}}^j(\mathfrak{V}, M_n)$, we therefore obtain

$$\begin{aligned}\mathscr{U}_n(f^{-1}U) \otimes_{\mathscr{U}_n(X)} H^j &\cong \mathscr{U}_n(f^{-1}U) \otimes_{\mathscr{U}_n(f^{-1}V)} \mathscr{U}_n(f^{-1}V) \otimes_{\mathscr{U}_n(X)} H^j\\ &\cong \mathscr{U}_n(f^{-1}U) \otimes_{\mathscr{U}_n(f^{-1}V)} \check{\mathrm{H}}^j(f^{-1} \cap \mathfrak{V}, M_n),\end{aligned}$$

and thus

$$\begin{aligned}\mathscr{U}_n(f^{-1}U)\otimes_{\mathscr{U}_n(X)} H^j &\cong \mathrm{H}^j(\mathscr{U}_n(f^{-1}U)\widehat{\otimes}_{\mathscr{U}_n(f^{-1}V)}\check{C}^\bullet(f^{-1}V\cap\mathfrak{V},M_n))\\ &\cong \mathrm{H}^j(\mathscr{U}_n(f^{-1}U)\widehat{\otimes}_{\mathscr{U}_n(f^{-1}V)}\mathscr{U}_n(f^{-1}V)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V},M_n))\\ &\cong \mathrm{H}^j(\mathscr{U}_n(f^{-1}U)\widehat{\otimes}_{\mathscr{U}_n(X)}\check{C}^\bullet(\mathfrak{V},M_n)),\end{aligned}$$

as required. □

Step C

Theorem 6.17. *Let $U\subseteq Y$ be an affinoid subdomain. Then the natural morphism*

$$\widehat{\mathscr{U}}(f^{-1}U)\widehat{\otimes}_{\widehat{\mathscr{U}}(X)}\mathrm{H}^j(X,\mathcal{M})\to \mathrm{H}^j(f^{-1}U,\mathcal{M})$$

is an isomorphism for each $j\geq 0$.

Proof. We know that U is $\pi^n\mathcal{L}$-accessible for sufficiently large n, so there exists a finite covering of U by $\pi^n\mathcal{L}$-accessible rational subdomains (W_i) of Y.
By Corollary 6.9 and Proposition 6.10, $\mathrm{H}^j(X,\mathcal{M})$ is a coadmissible $\widehat{\mathscr{U}}(X)$-module, so that

$$\widehat{\mathscr{U}}(f^{-1}U)\widehat{\otimes}_{\widehat{\mathscr{U}}(X)}\mathrm{H}^j(X,\mathcal{M})$$

is a coadmissible $\widehat{\mathscr{U}}(f^{-1}U)=\widehat{\mathscr{U}(f_*\mathscr{L})}(U)$-module.

We have a natural morphism

$$\mathrm{Loc}\left(\widehat{\mathscr{U}}(f^{-1}U)\widehat{\otimes}_{\widehat{\mathscr{U}}(X)}\mathrm{H}^j(X,\mathcal{M})\right)\to\left(\mathrm{R}^jf_*\mathcal{M}\right)|_U$$

of sheaves of $\widehat{\mathscr{U}(f_*\mathscr{L})}$-modules, and by Proposition 6.16, this becomes an isomorphism after taking sections over any W_i or any finite intersection of W_is.
Considering the corresponding Čech complex therefore forces the map between the global sections also to be an isomorphism, i.e.

$$\widehat{\mathscr{U}}(f^{-1}U)\widehat{\otimes}_{\widehat{\mathscr{U}}(X)}\mathrm{H}^j(X,\mathcal{M})\cong \mathrm{H}^j(f^{-1}U,\mathcal{M}).$$

□

This concludes the proof of Theorem 6.1.

6.4 The Proper Mapping Theorem: General case

We can now state a more general Proper Mapping Theorem.

Proposition 6.18. *Let $f : X \to Y$ be a proper morphism of rigid analytic K-spaces, and let $\mathscr{U}$ be a sheaf of K-algebras on X satisfying the following property:*

There is an admissible affinoid covering (Y_i) of Y, such that there exists a Lie algebroid $\mathscr{L}$ on X whose restriction to $X_i = f^{-1}Y_i$ is free for each i, together with a morphism $\widehat{\mathscr{U}(\mathscr{L})} \to \mathscr{U}$ such that $\mathscr{U}|_{X_i}$ is a left coadmissible enlargement of the global Fréchet–Stein sheaf $\widehat{\mathscr{U}(\mathscr{L})}|_{X_i}$.

Then for any admissible open affinoid subspace U of Y, $f_*\mathscr{U}(U)$ is a left Fréchet–Stein algebra, and if $\mathcal{M}$ is a left $\mathscr{U}$-module that is coadmissible on each X_i (this makes sense by Proposition 5.30), then $\mathrm{R}^j f_*\mathcal{M}$ is coadmissible over $f_*\mathscr{U}$ for each $j \geq 0$, in the sense that for any admissible open affinoid subspace U of Y, the following holds:

(i) $\mathrm{R}^j f_*\mathcal{M}(U)$ *is a coadmissible $f_*\mathscr{U}(U)$-module.*

(ii) If V is an affinoid subdomain of U, then the natural morphism

$$f_*\mathscr{U}(V)\widehat{\otimes}_{f_*\mathscr{U}(U)}\mathrm{R}^j f_*\mathcal{M}(U) \to \mathrm{R}^j f_*\mathcal{M}(V)$$

is an isomorphism.

The analogous statement also holds for right modules provided $\mathscr{U}_{|}X_i$ is a right coadmissible enlargement.

Proof. Let $f = hg$ be the Stein factorization of f, where $g : X \to Z$ satisfies $g_*\mathcal{O}_X \cong \mathcal{O}_Z$. As we have seen earlier, $g_*\mathscr{L}$ is a Lie algebroid on Z and $g_*\widehat{\mathscr{U}(\mathscr{L})} \cong \widehat{\mathscr{U}(g_*\mathscr{L})}$ is a Fréchet completed enveloping algebra on Z. Since h is affinoid, we can again assume that $f = g$, and Theorem 5.28 implies (in conjunction with Lemma 5.24 and Proposition 5.30) that we can assume without loss of generality that Y itself is affinoid, f is elementary proper and $\mathscr{U}$ is a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-module for some free Lie algebroid $\mathscr{L}$ with continuous multiplication.

By Theorem 6.1, we know that $f_*\mathscr{U}$ is then a coadmissible $f_*\widehat{\mathscr{U}(\mathscr{L})}$-module with continuous multiplication, so by Proposition 5.30, $f_*\mathscr{U}(U)$ is a Fréchet–Stein algebra for any affinoid subdomain U of Y.

Similarly, if $\mathcal{M}$ is a coadmissible $\mathscr{U}$-module, Propsoition 5.30 implies that it is also a coadmissible $\widehat{\mathscr{U}(\mathscr{L})}$-module, so again Theorem 6.1 states that $\mathrm{R}^j f_*\mathcal{M}$ is a coadmissible $f_*\widehat{\mathscr{U}(\mathscr{L})}$-module, and thus a fortiori a coadmissible $f_*\mathscr{U}$-module by Proposition 5.30. ☐

As a corollary, we can consider Lie algebroids $\mathscr{L}$ which are not themselves free, but admit an epimorphism $\mathscr{L}' \to \mathscr{L}$ for some free Lie algebroid $\mathscr{L}'$. The reason why we spell this out explicitly is given by the geometric interpretation later.

Corollary 6.19. *Let $f : X \to Y$ be a proper morphism of rigid analytic K-spaces, and let $\mathscr{L}$ be a Lie algebroid on X such that there is an epimorphism $\mathscr{L}' \to \mathscr{L}$ for some free Lie algebroid $\mathscr{L}'$.*
Then the conclusions of Proposition 6.18 hold for $\mathscr{U} = \widehat{\mathscr{U}(\mathscr{L})}$, since the epimorphism $\widehat{\mathscr{U}(\mathscr{L}')} \to \widehat{\mathscr{U}(\mathscr{L})}$ turns the latter into a coadmissible $\widehat{\mathscr{U}(\mathscr{L}')}$-module.

We now revisit our example from chapter 5.
Suppose that $f : X \to Y$ is elementary proper, writing $f = hg$ for the Stein factorization as usual. Let $\mathscr{L}$ be a Lie algebroid on X with the property that $g_*\mathscr{L}$ is free, i.e. $\mathscr{L}(X)$ is a free $\mathcal{O}_X(X)$-module.
In this case $g^*g_*\mathscr{L}$ is a Lie algebroid on X which is free as an $\mathcal{O}_X$-module and comes equipped with a natural morphism

$$g^*g_*\mathscr{L} \to \mathscr{L},$$

as seen in Lemma 5.19.
Thus we can apply Corollary 6.19 as soon as this morphism is an epimorphism. By definition of $g^*g_*\mathscr{L}$, this is equivalent to requiring $\mathscr{L}$ to be generated by global sections.

Corollary 6.20. *Let $f : X \to Y$ be an elementary proper morphism of rigid analytic K-spaces, and let $\mathscr{L}$ be a Lie algebroid on X such that $\mathscr{L}(X)$ is a free $\mathcal{O}_X(X)$-module and $\mathscr{L}$ is generated by global sections.*
Then the conclusions of Proposition 6.18 hold for $\mathscr{U} = \widehat{\mathscr{U}(\mathscr{L})}$.

By glueing we obtain the following more general version.

Corollary 6.21. *Let $f : X \to Y$ be a proper morphism with Stein factorization $f = hg$, and let $\mathscr{L}$ be a Lie algebroid on X such that the following holds:*

(i) $g_*\mathscr{L}$ is locally free.

(ii) The natural morphism $g^*g_*\mathscr{L} \to \mathscr{L}$ is an epimorphism of sheaves on X_{rig}.

Then the conclusions of Proposition 6.18 hold for $\mathscr{U} = \widehat{\mathscr{U}(\mathscr{L})}$.

These formulations will turn out to be the most useful when we are considering sheaves which are given as Fréchet completions of enveloping algebras. For more general cases, e.g. concerning twisted versions of $\widehat{\mathcal{D}}$, we can still refer back to Proposition 6.18. We will discuss a number of examples after giving a more geometric motivation for the conditions we have imposed in our results.

Let $f : X \to Y = \operatorname{Sp} A$ be an elementary proper morphism such that $A = \mathcal{O}_X(X)$, and let $\mathscr{L}$ be a Lie algebroid on X which is a free $\mathcal{O}_X$-module of rank m. In this case the rigid analytic vector bundles

$$V(\mathscr{L}) \cong X \times (\mathbb{A}^m)^{\mathrm{an}},\ V(f_*\mathscr{L}) \cong Y \times (\mathbb{A}^m)^{\mathrm{an}}$$

are trivial, and there is a natural morphism $V(\mathscr{L}) \to V(f_*\mathscr{L})$, which is proper by [13, Lemma 9.6.2/1].
Our Theorem 6.1 can thus be viewed as a quantised version of Kiehl's Theorem 4.3 on trivial vector bundles.

The next result extends this interpretation to the case of Corollary 6.21.

Proposition 6.22. *Let $f : X \to Y$ be a proper morphism of rigid analytic K-spaces with Stein factorization*

$$X \xrightarrow{g} Z \xrightarrow{h} Y,$$

and let $\mathscr{L}$ be a Lie algebroid on X such that the following holds:

(i) $g_*\mathscr{L}$ is locally free.

(ii) The natural morphism $g^*g_*\mathscr{L} \to \mathscr{L}$ is an epimorphism of sheaves on X_{rig}.

Then there is a natural morphism

$$V(\mathscr{L}) \to V(g^*g_*\mathscr{L}),$$

which is a closed immersion, and and a proper morphism

$$V(g^*g_*\mathscr{L}) \to V(g_*\mathscr{L})$$

of rigid analytic K-spaces.
In particular, their composition $V(\mathscr{L}) \to V(g_*\mathscr{L})$ is proper.

Proof. The natural map $\mu : \mathscr{L}' := g^*g_*\mathscr{L} \to \mathscr{L}$ induces a morphism of rigid analytic K-spaces $V(\mu) : V(\mathscr{L}) \to V(\mathscr{L}')$ by functoriality.

We show that this is a closed immersion. Restricting to an admissible affinoid covering (U_i) of X on which both $\mathscr{L}'$ and $\mathscr{L}$ are free, the morphism $\theta_i : \operatorname{Sym} \mathscr{L}'(U_i) \to \operatorname{Sym} \mathscr{L}(U_i)$ is a surjection for each i by assumption.

Choosing a residue norm on $\mathcal{O}_X(U_i)$ with unit ball $\mathcal{B}_i$ and a free generating set $e_1, \ldots, e_m$ of $\mathscr{L}'(U_i)$, endow $\operatorname{Sym} \mathscr{L}'(U_i)$ with the norm with unit ball the $\mathcal{B}_i$-subalgebra generated by the e_j, and endow $\operatorname{Sym} \mathscr{L}(U_i)$ with the corresponding quotient norm via θ_i. In particular, θ_i is strict with respect to these choices of norm by construction.
The completion of $\operatorname{Sym} \mathscr{L}'(U_i)$ is the affinoid algebra $\widehat{S}_0(e)$ constructed in chapter 2, and by strictness this surjects onto the completion of $\operatorname{Sym} \mathscr{L}(U_i)$, which is again affinoid, as it is topologically of finite type over K.
Replacing e_j by $\pi^n e_j$ for varying n, the affinoid spaces $\operatorname{Sp} \widehat{S}_n(e)$ form an admissible covering of $V(\mathscr{L}'|_{U_i})$ by affinoid subspaces, and the surjections between affinoid algebras exhibit $V(\mu)$ as a closed immersion.

Choosing an admissible covering (Z_i) of Z such that $g_*\mathscr{L}|_{Z_i}$ is free of rank m on each i, $g^*g_*\mathscr{L}|_{g^{-1}Z_i}$ is also free of rank m, again inducing a proper morphism

$$\begin{array}{ccc} g^{-1}Z_i \times (\mathbb{A}^m)^{\mathrm{an}} & \longrightarrow & Z_i \times (\mathbb{A}^m)^{\mathrm{an}} \\ \downarrow \cong & & \downarrow \cong \\ V(\mathscr{L}'|_{g^{-1}Z_i}) & \longrightarrow & V(g_*\mathscr{L}|_{Z_i}) \end{array}$$

These glue to give a proper morphism $V(\mathscr{L}') \to V(g_*\mathscr{L})$, and the result follows from the fact that the composition of proper morphisms is proper. $\square$

Thus our assumptions can be interpreted as requiring a vector bundle $V(g_*\mathscr{L})$ on Z together with a proper morphism $V(\mathscr{L}) \to V(g_*\mathscr{L})$.

Our next goal will be to present a number of naturally occuring cases in which our assumptions are satisfied and therefore Proposition 6.18 applies. We reserve the main application, our discussion of analytic partial flag varieties, for the next section.

Example 1: Closed immersions

Let $Y = \operatorname{Sp} A$ be an affinoid K-space and let $\iota : X \to Y$ be a closed immersion of affinoid spaces, i.e. if $X = \operatorname{Sp} B$, then the corresponding morphism of affinoid algebras $A \to B$ is a surjection. This map is proper by [13, Proposition 9.6.2/5] with trivial Stein factorization in the sense that $g = \mathrm{id}_X$ is the identity on X and $h = \iota$ in our usual notation.
In particular, if $\mathscr{L}$ is a Lie algebroid on X, then $g_*\mathscr{L} = \mathscr{L}$, $g^*g_*\mathscr{L} = \mathscr{L}$, and all conditions in Corollary 6.21 are trivially satisfied.
Since all conditions in Corollary 6.21 are local, it follows that the same holds true for arbitrary closed immersions ι.

This is of course not really surprising, as ι is an affinoid morphism, so we could deduce everything in this case simply from Theorem 5.28 (as is tacitly done in [6]).
The same argument works more generally in the case of a finite morphism.

Example 2: Projections

Let $X = \mathbb{P}^{n,\mathrm{an}}$ be the analytification of projective n-space over K, and consider the projection to a point

$$f : \mathbb{P}^{n,\mathrm{an}} \to \operatorname{Sp} K.$$

This is trivially a projective morphism and hence proper, and the Stein factorization in this case is $g = f$, $h = \mathrm{id}_{\operatorname{Sp} K}$.
If $\mathscr{L}$ is a Lie algebroid on X, then $g_*\mathscr{L} = \mathscr{L}(X)$ is a (finite-dimensional) K-vector space, and

$$g^*g_*\mathscr{L} = \mathcal{O}_X \otimes_K \mathscr{L}(X).$$

Thus our assumptions are satisfied if and only if $\mathcal{O}_X \otimes \mathscr{L}(X) \to \mathscr{L}$ is an epimorphism, i.e. if and only if $\mathscr{L}$ is generated by global sections. This is for example the case when $\mathscr{L} = \mathcal{T}_X$ is the tangent sheaf of X, as the tangent sheaf of the K-scheme $\mathbb{P}^n$ is generated by global sections (by considering the Euler short exact sequence), and the GAGA principle [12, Theorems 6.3.12, 6.3.13] ensures that global generation is preserved under analytification.

Example 3: Direct products

Let Y be a smooth rigid analytic K-space and consider the projection $f : \mathbb{P}^{n,\mathrm{an}} \times Y \to Y$, which is again proper. Now $X = \mathbb{P}^{n,\mathrm{an}} \times Y$ is smooth, so the tangent sheaf $\mathcal{T}_X$ is a Lie algebroid on X. By definition of smoothness, Y admits an admissible covering by affinoid subspaces (Y_i) such that $\mathcal{T}_{Y_i}$ is free, and we write $X_i = \mathbb{P}^{n,\mathrm{an}} \times Y_i$. Write $p_1 : X \to \mathbb{P}^{n,\mathrm{an}}$ for the projection onto the first factor. Since

$$\mathcal{T}_X = \mathcal{O}_X \otimes_{p_1^{-1}\mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}} p_1^{-1}\mathcal{T}_{\mathbb{P}^{n,\mathrm{an}}} \oplus \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{T}_Y$$

as in the algebraic case, $\mathcal{T}_X(X_i)$ is a free module over $\mathcal{O}_Y(Y_i) = \mathcal{O}_X(X_i)$, and $\mathcal{T}_{X_i}$ is again generated by global sections.
Thus $f_*\widehat{\mathcal{D}}_X(U)$ is a Fréchet–Stein algebra for any admissible open affinoid subspace $U \subseteq Y$, and $\mathrm{R}^j f_*\mathcal{M}$ is a coadmissible $f_*\widehat{\mathcal{D}}_X$-module for each $j \geq 0$, where $\mathcal{M}$ is any coadmissible $\widehat{\mathcal{D}}_X$-module.

It now seems natural to rewrite an arbitrary projective morphism $f : X \to Y$ as a composition

$$X \to \mathbb{P}^{n,\mathrm{an}} \times Y \to Y,$$

and apply the examples above in turn. Note however that our results behave rather poorly with respect to the composition of proper morphisms. The main reason for this is that if $f : X \to Y$ is proper, then $f_*\mathscr{L}$ is hardly ever a Lie algebroid on Y (even if we assume locally freeness), which is precisely why we had to make the detour via the Stein factorization in our arguments. Still, this strategy should be useful when considering a $\widehat{\mathcal{D}}$-module pushforward via transfer bimodules, analogously to [27, Proposition 1.5.21, Theorem 2.5.1].

6.5 Application: Analytic partial flag varieties

Let $\mathbf{G}$ be a split reductive affine algebraic group scheme over K, and let $G = \mathbf{G}(K)$ with Lie algebra $\mathfrak{g}$. Let $\mathbf{B} \leq \mathbf{G}$ be a Borel subgroup scheme, $\mathbf{P} \leq \mathbf{G}$ a parabolic subgroup scheme and let $\mathbf{X} = \mathbf{G}/\mathbf{P}$ be the partial flag variety. In this section, we will be concerned with coadmissible $\widehat{\mathcal{D}}$-modules on the analytification $X = \mathbf{X}^{\mathrm{an}}$.
By [29, II.1.8], $\mathbf{G}/\mathbf{P}$ is projective, and thus X is proper over $\operatorname{Sp} K$.
More generally, if $\mathbf{P}_1 \leq \mathbf{P}_2$ are two parabolics, $X_i = (\mathbf{G}/\mathbf{P}_i)^{\mathrm{an}}$, then the natural projection morphism $X_1 \to X_2$ is proper by [13, Proposition 9.6.2/4].

Let $\mathbf{R} \leq \mathbf{G}$ be the unipotent radical of $\mathbf{P}$ and $\mathbf{L}$ its Levi factor. Write $\mathfrak{l}$ for the Lie algebra of $L = \mathbf{L}(K)$. Following [7], the natural morphism $\xi : \mathbf{G}/\mathbf{R} \to \mathbf{G}/\mathbf{P}$ turns $\mathbf{G}/\mathbf{R}$ into an $\mathbf{L}$-torsor in the sense of [7, 4.1], where $\mathbf{L}$ acts on $\mathbf{G}/\mathbf{R}$ by right translations.

Define the enhanced tangent sheaf $\widetilde{\mathcal{T}}_{\mathbf{G}/\mathbf{P}} := (\xi_* \mathcal{T}_{\mathbf{G}/\mathbf{R}})^{\mathbf{L}}$, a Lie algebroid on $\mathbf{X}$ (see [7, Definition 4.2], [4, 4.4]).
Applying the analytification functor, we obtain the Lie algebroid $\widetilde{\mathcal{T}}_X$. Since the natural morphism $\mathcal{O}_{\mathbf{X}} \otimes_K \mathfrak{g} \to \widetilde{\mathcal{T}}_{\mathbf{G}/\mathbf{P}}$ is an epimorphism by the same argument as in [4, Proposition 4.8.(a)], it follows from [12, Theorems 6.3/12 and 13] that $\widetilde{\mathcal{T}}_X$ is generated by global sections.

We now set $\widehat{\widetilde{\mathcal{D}}}_X := \widehat{\mathscr{U}(\widetilde{\mathcal{T}}_X)}$, a coadmissible enlargement of $\widehat{\mathscr{U}(\mathcal{O}_X \otimes \mathfrak{g})}$. Applying Corollary 6.20, we obtain the following.

Corollary 6.23. *The global sections $\widehat{\widetilde{\mathcal{D}}}_X(X)$ form a Fréchet–Stein algebra, and if $\mathcal{M}$ is a coadmissible $\widehat{\widetilde{\mathcal{D}}}_X$-module on X, then $\mathrm{R}^j\Gamma(X, \mathcal{M})$ is coadmissible over both $\widehat{\widetilde{\mathcal{D}}}_X(X)$ and $\widehat{U(\mathfrak{g})}$ for each $j \geq 0$.*

Write $\mathfrak{h}$ for a Cartan subalgebra of $\mathfrak{g}$, and let $\lambda \in \mathfrak{h}^*$. The centre of the enveloping algebra $U(\mathfrak{g})$ will be denoted by $Z(\mathfrak{g})$.

The triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ induces the Harish-Chandra morphism $\theta : Z(\mathfrak{g}) \to U(\mathfrak{g}) \to U(\mathfrak{h}) = \operatorname{Sym} \mathfrak{h}$, which allows us to view λ as a character for $Z(\mathfrak{g})$. We let K_λ denote the corresponding one-dimensional $Z(\mathfrak{g})$-representation, and set

$$U^\lambda := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} K_\lambda.$$

We denote the kernel of the surjection $U(\mathfrak{g}) \to U^\lambda$ by $\mathfrak{m}_\lambda$.
Now choose an (R, R)-Lie lattice $\mathfrak{g}_R$ inside $\mathfrak{g}$, and write $U_n = U(\pi^n \mathfrak{g}_R)$. This induces a norm on $U(\mathfrak{g})$, and we let $Z(\mathfrak{g})$ be equipped with the corresponding subspace norm. We define

$$\widehat{U}^\lambda := \varprojlim \left(\widehat{U_n}_K \widehat{\otimes}_{Z(\mathfrak{g})} K_\lambda \right),$$

where the norm on K_λ is given by identification with K.

Lemma 6.24. *The K-algebra $\widehat{U}^\lambda$ is naturally isomorphic to the quotient of $\widehat{U(\mathfrak{g})}$ by the closure of $\mathfrak{m}_\lambda$.*
In particular, $\widehat{U}^\lambda$ is a Fréchet–Stein algebra.

Proof. This is Lemma 5.25 applied to the short exact sequence

$$0 \to \mathfrak{m}_\lambda \to U(\mathfrak{g}) \to U^\lambda \to 0,$$

together with [44, Proposition 3.7]. □

As in section 6.1, note that $\pi^n \mathfrak{g}_R$ determines compatible norms on $U(\widetilde{\mathcal{T}}_X)(U)$ for any admissible open affinoid subspace $U \subset X$, inducing completed sheaves $\widetilde{\mathcal{D}}_n := \mathscr{U}_n(\widetilde{\mathcal{T}}_X)$ such that $\widehat{\widetilde{\mathcal{D}}}_X = \varprojlim \widetilde{\mathcal{D}}_n$ is a global Fréchet–Stein sheaf on X.

Identifying $Z(\mathfrak{l})$ with $\mathbf{L}$-invariant differential operators on $\mathbf{G}/\mathbf{R}$ (see [7, 4.1]), we obtain a natural morphism $Z(\mathfrak{l}) \to \widetilde{\mathcal{D}}_n$ for each n with central image, and we define the sheaf of twisted differential operators on X by

$$\widehat{\mathcal{D}}_X^\lambda = \varprojlim (\widetilde{\mathcal{D}}_n \widehat{\otimes}_{Z(\mathfrak{l})} K_\lambda).$$

Again, Lemma 5.25 shows that the natural morphism $\widehat{\widetilde{\mathcal{D}}}_X \to \widehat{\mathcal{D}}_X^\lambda$ is an epimorphism which turns $\widehat{\mathcal{D}}_X^\lambda$ into a coadmissible enlargement of $\widehat{\widetilde{\mathcal{D}}}_X$, and hence a coadmissible enlargement of $\widehat{\mathscr{U}(\mathcal{O}_X \otimes \mathfrak{g})}$.

Now let $\mathbf{P}_1 \leq \mathbf{P}_2$ be two parabolic subgroups, and consider the proper morphism $f : X_1 \to X_2$, where $X_i = (\mathbf{G}/\mathbf{P}_i)^{\mathrm{an}}$.

Corollary 6.25. *The pushforward* $f_*\widehat{\mathcal{D}}^\lambda_{X_1}$ *is a global Fréchet–Stein sheaf on the partial flag variety* X_2*.*
If $\mathcal{M}$ *is a coadmissible* $\widehat{\mathcal{D}}^\lambda_{X_1}$*-module, then* $\mathrm{R}^j f_*\mathcal{M}$ *is coadmissible over* $f_*\widehat{\mathcal{D}}^\lambda_{X_1}$*, and a fortiori coadmissible over* $f_*\widehat{\mathscr{U}(\mathcal{O}_{X_1}\otimes\mathfrak{g})}$ *for each* $j \geq 0$*.*

Proof. This is the content of Theorem 6.18. □

Note that in the extreme case $\mathbf{P}_2 = \mathbf{G}$, we obtain the following generalization of the first statement in [1, Theorem 6.4.7].

Corollary 6.26. *Let* $\mathbf{P} \leq \mathbf{G}$ *be a parabolic subgroup, and let* $X = (\mathbf{G}/\mathbf{P})^{\mathrm{an}}$*. Then the global sections* $\widehat{\mathcal{D}}^\lambda_X(X)$ *form a Fréchet–Stein algebra, and if* $\mathcal{M}$ *is a coadmissible* $\widehat{\mathcal{D}}^\lambda_X$*-module, then* $\mathrm{R}^j\Gamma(X,\mathcal{M})$ *is coadmissible over both* $\widehat{\mathcal{D}}^\lambda_X(X)$ *and over* $\widehat{U(\mathfrak{g})}$ *for each* $j \geq 0$*.*
As the $\widehat{U(\mathfrak{g})}$*-action factors through* $\widehat{U}^\lambda$*, this makes* $\mathrm{R}^j\Gamma(X,\mathcal{M})$ *a coadmissible* $\widehat{U}^\lambda$*-module for each* $j \geq 0$*.*

As in the algebraic case (see [7, Theorem 4.10]), one expects an isomorphism $f_*\widehat{\mathcal{D}}^\lambda_{X_1} \cong \widehat{\mathcal{D}}^\lambda_{X_2}$. So far, this has only been established in the case of the full flag variety $\mathbf{P}_1 = \mathbf{B}$, $\mathbf{P}_2 = \mathbf{G}$ by Ardakov [1, Theorem 5.3.5.(a)], but we are confident that his argument can be generalized.

In this way our Proper Mapping Theorem recovers and generalizes many finiteness results associated with the Beilinson–Bernstein theorem in [1], and we expect several further applications. Apart from the general theory of $\widehat{\mathcal{D}}$-module pushforward already mentioned, the above strongly suggests that one can study intertwining functors for coadmissible modules analogous to those in [10], in order to obtain further global information on the representation theory of $\widehat{U(\mathfrak{g})}$ from the geometric picture.

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