Strategic Consensus[1]

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Abstract

We study equilibrium retention rules in a dynamic common agency game. The decision to reappoint or not is made by a committee consisting of two principals: the retention decision is uncertain if the two principals disagree. We demonstrate that all equilibrium paths exhibit what we call *strategic consensus*: the agent takes actions that satisfy the performance standards of all principals on the one hand, and all principals lower their standards such that the agent wants to satisfy them on the other. This result applies both to economies with sub- and super-additive costs of providing utilities to the principals.

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I. Introduction

This paper studies a dynamic common agency game. Two principals delegate some task to a common agent who has an incentive to divert resources. The only incentive instrument available to the principals is the decision to reappoint the agent or not. This decision is made by a committee consisting of the principals. Each principal supports reappointment if, and only if, the agent has satisfied a pre-announced performance standard. Importantly, the committee decision is uncertain if the two principals disagree about reappointing the agent: we refer to this as governance uncertainty. Governance uncertainty arises for a number of reasons. First, where decision-making is governed by simple rules such as the majority rule, governance uncertainty is related to random turnout by the principals. This interpretation is natural if we think of the two principals as different groups of voters and the agent as an elected politician. Second, in many cases, committee decisions are made behind closed doors, and outcomes depend on a variety of unpredictable factors. This interpretation is natural in relation to promotion decisions in organizations in which such

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decisions are not governed by rules but by power relationships. Promotion decisions in university departments might be one example of this.

We say that a sequence of performance standards displays *strategic consensus* if the agent prefers to meet both standards at all times, both principals support his reappointment, and he is reappointed with certainty. Thus, an equilibrium with strategic consensus insures the agent against random committee decisions, and similarly insures each principal against "partisan" outcomes whereby the agent seeks to please the other principal only. The question is whether strategic consensus can arise at subgame perfect Nash equilibrium. The main result of the paper is that there are possibly many equilibrium paths but *all* of them display strategic consensus. Importantly, this result is true independently of whether the cost function, which defines the cost to the agent of providing utility levels to the two principals, is sub- or super-additive. While it might be expected that sub-additive costs would lead to strategic consensus simply because it is cheaper for the agent to satisfy the performance standards jointly than separately, it is surprising that the same result obtains with super-additive costs, where it is *more* costly to generate utilities jointly than separately. Uncertainty regarding the reappointment decision is necessary for strategic consensus to arise at equilibrium. If one principal is always decisive, then all equilibrium paths would display "partisan" outcomes and the agent would only provide utility to the decisive principal.

Our model is related to other common agency models in the literature, such as Bernheim and Winston (1986), Dixit et al. (1997), and Bergemann and Välimäki (2003). A key difference is that the principals in our model only have one control instrument at their disposal – the retention rule – and, more importantly, governance uncertainty prevents them from applying this tool with certainty: the consent of both is needed to secure reappointment with certainty. Thus, from the point of view of the agent, the principals are not "perfect substitutes" and that is what prevents Bertrand-like underbidding in our setting. One consequence of this is that the principals, in contrast to, for example, Dixit et al. (1997), are able to retain part of the surplus even if the cost function is additive. Our model is also related to the work by Banks and Sundaram (1993, 1998) and others on retention rules in agency problems with one long-lived principal and a series of short-lived agents. We add to this literature by showing that equilibrium retention rules with two principals display strategic consensus.

The rest of the paper is organized as follows. In section II, we present the model and discuss the main assumptions regarding governance uncertainty and the properties of the cost function. In section III, we state and prove the main result about strategic consensus (Theorem 1). In section IV, we explore the implications of Theorem 1. We show that there exist many possible equilibrium utility allocation with sub-additive costs (Proposition 1), while the utility allocation, under some mild additional assumptions, is unique with super-additive costs (Proposition 2). Proposition 3 demonstrates that governance uncertainty is necessary for strategic consensus. In section V, we briefly discuss a political economy application.

II. The Model

We consider a dynamic common agency. Time is discrete and open ended, $t = 0, 1, \ldots$. Two principals ($i = 1, 2$) delegate a task or a number of tasks to a common agent each period. The agent ($i = 0$) has an incentive to divert resources away from the task, and the two principals need to provide appropriate incentives to avoid this. The only incentive instrument available to the principals is the decision whether to reappoint the agent or to replace him with someone else. We denote per period utilities by u_{it}. All utilities are discounted with the discount rate $\beta \in (0,1)$ and life-time utility is given by

$$V_{i0} = \sum_{t=0}^{\infty} \beta^t u_{it}; \quad i \in \{0,1,2\}.$$

Each period, the agent has access to a pool of resources R. Some of these he may use to provide utility to the two principals; the rest he keeps for himself. Denoting the cost of providing utility to the principals by c_t, we can write the agent's payoff in period t as

$$u_{0t} = R - c_t$$

if appointed and $u_{0t} = 0$ otherwise. It is clear that, absent further incentives, the agent would want to choose $c_t = 0$.

An important feature of the model is the *cost function.* The cost function, $C(x_{1t}, x_{2t})$, shows the minimum cost of providing utilities $u_{1t} \geq x_{1t}$ and $u_{2t} \geq x_{1t}$ to principal 1 and 2, respectively, at time t, where $(x_{1t}, x_{2t}) \in \Re^2_+$. We assume that $C_1(x_{1t}) = C(x_{1t}, 0)$, $C_2(x_{2t}) = C(0, x_{2t})$, and $C(0,0) = C_i(0) = 0$, $i = 1, 2$. In the analysis, we make use of the following additional assumptions.

Assumption 1 *The cost function is monotonically increasing in each argument, i.e.,*

$$\textbf{(M)} \quad x_t > x'_t \Rightarrow C(x_t) > C(x'_t)$$

where $x_t = (x_{1t,} x_{2t})$. *Further,* $\lim_{x_i \to \infty} C(x_1, x_2) = \infty$.

Assumption 2 *The cost function is continuous, i.e.,*

$$\textbf{(K)} \quad C(x_{1t,} x_{2t}) \in \mathcal{C}^1.$$

Assumption 3 *The cost function is sub-additive*

$$[\mathbf{C}^+] \quad C(x_{1t}, x_{2t}) \leq C_1(x_{1t}) + C_2(x_{2t}).$$

Alternatively,

Assumption 4 *The cost function is super-additive*

$$[\mathbf{C}^-] \quad C(x_{1t}, x_{2t}) > C_1(x_{1t}) + C_2(x_{2t}).$$

Assumption 1 says that the agent cannot deliver higher utilities to the principals without incurring additional costs. Assumption 2 says that the cost function cannot have "jumps" and rules out fixed costs. Both of these assumptions can be relaxed. Assumptions 3 and 4 specify whether it is cheaper or more expensive to provide utility to the principals jointly or separately. Both cases are common in practical applications. One illustrative example is promotion decisions within a university department. Often the senior faculty is split between those that prioritize teaching and those that prioritize research. Junior faculty members (the agent) can then devote their time and effort to teaching and/or research. The cost function is subadditive if excellence in teaching makes it easier to produce excellent research and vice versa. The cost function is super-additive if, on the other hand, achieving excellence in teaching makes it harder to also achieve excellence in research and vice versa.

We assume that the reappointment decision is made by a committee consisting of the two principals. The committee meets at the end of every period to decide if the agent is reappointed or not. Before each meeting, the two principals independently announce *performance standards* specifying how much utility the agent needs to generate for each of them to gain their support at the meeting. We do not need to specify the details of how the committee reaches decisions. The following assumptions are sufficient for our purposes. If both principals agree that the agent should either be retained or dismissed, the outcome of the meeting is clear: the agent is either retained or dismissed. In contrast, if the two principals disagree, the outcome is not clear and depends on many factors, most of which are hard for the agent to predict ex ante. We capture this by introducing the following assumption.

Assumption 5 (Governance Uncertainty) *Suppose the two principals disagree about whether or not the agent should be reappointed. The ex-ante probability that principal* 1 ((2)) *is decisive in the sense that his preference becomes that of the committee is equal to* p $((1-p))$. *We assume that* $p \in (0,1)$ *is constant over time.*

The assumption implies that the agent can secure reappointment with certainty only by satisfying the performance standards of both principals; by satisfying one standard alone, the agent runs the risk of not being reappointed. In a sense, uncertainty in rewards arises from uncertainty about which of the two principals will have the "casting vote", or final say, in the only reward available: reappointment. There is no aggregate uncertainty, as one of the principals will have the casting vote for sure. For this reason, it is sufficient to deal with p rather than specifying p_1 and p_2, where p_i is the probability that principal i is decisive.

The game between the agent and the two principals unfold over time as follows. At the beginning of each period, the two principals announce the performance standards that the agent need to satisfy to get their support at the next meeting. The standards are chosen by the two principals non-cooperatively and simultaneously and are denoted $x_t = (x_{1t}, x_{2t})$. The agent observes the standards and determines whether to comply, and if so, how many standards to meet. We denote the set of actions available to the agent by $A = \{(00), (10), (01), (11)\}$ with elements $a_t = (00)$ (meet neither standard); $a_t = (10)$ (meet principal 1's standard only); $a_t = (01)$ (meet principal 2's standard only); and $a_t = (11)$ (meet both standards). At the end of the period, a new meeting is held and a decision regarding reappointment is made. The decision is based on the observed utility generated by the action taken by the agent relative to the performance standards announced at the beginning of the period. If there is agreement, the outcome of the meeting follows from this; in case of disagreement, the outcome is determined by p. If the agent is not reappointed, he is replaced by another similar agent. After the meeting, the game continues to the next period where a similar sequence of events takes place.

Formally, the game described above is a dynamic game with absorbing states and perfect information. The natural solution concept is subgame perfect Nash equilibrium. Strategies in a subgame perfect Nash equilibrium can depend in complex ways on the history of the whole game. We restrict attention to stationary equilibria: subgame perfect Nash equilibria where strategies depend only on within the period histories. Where we refer to "stationary equilibrium" in what follows this is what we mean. In addition to this, we need to specify how ties are broken. We assume that the agent if indifferent between two or more actions (which are then preferred to the remaining ones) chooses the action that maximizes reappointment chances.[4]

We can motivate this tie-breaking rule by saying that the agent has lexicographic preferences. He cares about the expected rent generated by keeping his appointment and chooses actions accordingly, but when two actions yield the same utility, he prefers to take the action that maximizes reappointment chances. Alternatively, we could assume that the agent simply picks action $a_t = (11)$ if indifference between this and any of the other three actions and that he always goes with, say, principal 1, if indifference between $a_t = (10)$ and $a_t = (01)$. This and other similar rules would yield the same outcomes. Thus, there is nothing special about the "maximize reelection chances rule:" it simply determines how the tie is broken in each case.

III. Strategic Consensus

We can now state the main Theorem. It demonstrates that all stationary equilibrium paths displays *strategic consensus:* the agent prefers to meet both standards at all times, both principals support his reappointment, and he is reappointed with certainty.

Theorem 1 (Strategic Consensus) *Assume that* $\beta \in (0,1)$. *Let* $x_t = (x_{1t}, x_{2t})$ *be a pair of performance standards set by the two principals for period* t *and define* $X = \{x_t\}_{t=0}^{\infty}$ *as a sequence of such standards. Let* a_t^* *be the action implemented by the agent in period* t*; define* $V_{0t}(a_t)$ *as the agent's payoff.*

1. *A stationary subgame perfect Nash equilibrium exists.*

2. *Suppose* (**M**), (**K**) *and* ($\mathbf{C}^+$) *hold. Along any stationary equilibrium path,* X *satisfies*

$$(\mathbf{SC}^+) \quad V_{0t}(11) = V_{0t}(00) \geq \max\{V_{0t}(10), V_{0t}(01)\}.$$

   Any sequence X *satisfying* ($\mathbf{SC}^+$) *is a stationary subgame perfect Nash equilibrium in performance standards. Along any stationary equilibrium path, the agent chooses* $a_t^* = (11)$ *at every* t *and he is reappointed for sure.*

3. *Suppose* (**M**), (**K**) *and* ($\mathbf{C}^-$) *hold. Along any stationary equilibrium path,* X *satisfies*

$$(\mathbf{SC}^-) \quad V_{0t}(11) = V_{0t}(10) = V_{0t}(01) > V_{0t}(00).$$

   Any sequence X *satisfying* ($\mathbf{SC}^-$) *is a stationary subgame perfect Nash equilibrium in performance standards. Along any stationary equilibrium path, the agent chooses* $a_t^* = (11)$ *at every* t *and he is reappointed for sure.*

Corollary 1 *Every stationary subgame perfect Nash equilibrium path displays strategic consensus at each* t.

[4] For $p = \frac{1}{2}$, we assume that he chooses $a_t^* = (10)$ if $V_{0t}(10) = V_{0t}(01) > \max[V_{0t}(11), V_{0t}(00)]$.

We prove the Theorem with a series of Lemmas. We begin by introducing some notation. Denote for each action $a_t \in A$, the agent's payoff by $V_{0t}(a_t)$ and write

$$V_{0t}(00) = R; \tag{1}$$

$$V_{0t}(10) = R - C_1(x_{1t}) + p\beta V_{t+1}; \tag{2}$$

$$V_{0t}(01) = R - C_2(x_{2t}) + (1-p)\beta V_{t+1}; \tag{3}$$

$$V_{0t}(11) = R - C(x_{1t}, x_{2t}) + \beta V_{t+1}. \tag{4}$$

where $V_{t+1} > 0$ is the value of being reappointed at time $t+1$. Note that the agent is only reappointed with some probability (p or $1-p$) if he chooses to be "partisan" and satisfy one of the standards only.

Now, suppose, in some period t, that the two principals announce the standards $x_t = \{x_{1t}, x_{2t}\}$. Given these standards, the agent chooses an action from the set $\{a_t \in A : \arg\max_{a_t \in A} V_{0t}(a_t)\}$. If the agent is indifferent between two or more actions in this set, he chooses the action that maximizes reappointment chances as discussed above. This is anticipated by the two principals when they, simultaneously, set their standards at the beginning of the period. With these preliminary remarks we can state the first Lemma.

Lemma 2 *Suppose that* [**M**] *and* [**K**] *hold. If the performance standards* $x_t = \{x_{1t}, x_{2t}\}$ *constitute a subgame perfect Nash equilibrium at time t, then* x_t *must satisfy*

$$(E0) \quad V_{0t}(11) \geq \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\}.$$

Proof: We argue by contradiction. Suppose that $\widetilde{x}_t = \{\widetilde{x}_{1t}, \widetilde{x}_{2t}\}$ constitutes a stationary subgame perfect Nash equilibrium in performance standards *and* that

$$V_{0t}(11) < \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\}$$

at time t. There are four separate cases to consider. We show in each case that at least one of the two principals has an incentive to deviate from $\widetilde{x}_t$, leading to the required contradiction.

1. Suppose that

   $$V_{0t}(10) = \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\} > V_{0t}(11)$$

   or that

   $$V_{0t}(10) = V_{0t}(00) = \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\} > V_{0t}(11).$$

   Rewrite (2) and (4) to get

   $$V_{0t}(10) - V_{0t}(11) = C(x_{1t}, x_{2t}) - C(x_{1t}, 0) - (1-p)\beta V_{t+1}.$$

   By [**M**] and [**K**], there must exist a $x'_{2t} > 0$ such that

   $$C(\widetilde{x}_{1t}, x'_{2t}) - C(\widetilde{x}_{1t}, 0) - (1-p)\beta V_{t+1} < 0.$$

   This implies that principal 2 can gain at least $x'_{2t} > 0$ by announcing the standard x'_{2t} instead of $\widetilde{x}_{2t}$.

2. Suppose instead that

$$V_{0t}(01) = \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\} > V_{0t}(11)$$

or that

$$V_{0t}(01) = V_{0t}(00) = \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\} > V_{0t}(11).$$

By an argument similar to the previous case, there must exist a $x'_{1t} > 0$ such that

$$C(x'_{1t}, \widetilde{x}_{2t}) - C(0, \widetilde{x}_{2t}) - p\beta V_{t+1} < 0.$$

This implies that principal 1 can gain at least $x'_{1t} > 0$ by announcing the standard x'_{1t} instead of $\widetilde{x}_{1t}$.

3. Suppose that

$$V_{0t}(00) = \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\} > V_{0t}(11).$$

Rewrite equations (1) and (2) to get

$$V_{0t}(00) - V_{0t}(10) = C(x_{1t}, 0) - p\beta V_{t+1}.$$

By [**M**] and [**K**], there must exist a $x''_{1t} > 0$ such that

$$C(x''_{1t}, 0) - p\beta V_{t+1} < 0.$$

This implies that principal 1 can at least gain $x''_{1t} > 0$ by announcing the standard x''_{1t} instead of $\widetilde{x}_{1t}$. A similar argument can be made for principal 2, who can gain at least $x''_{2t} > 0$.

4. Suppose that

$$V_{0t}(10) = V_{0t}(01) = \max\{V_{0t}(10), V_{0t}(01), V_{0t}(00)\} > V_{0t}(11)$$

or

$$V_{0t}(10) = V_{0t}(01) = V_{0t}(00) > V_{0t}(11).$$

We need to consider two sub-cases. First, suppose the agent chooses $a_t = (10)$. We can then repeat the argument from case 1 to show that there exists a deviation for principal 2. Second, suppose the agent chooses $a_t = (01)$ and we can repeat the argument from case 2 to show that there exists a deviation for principal 1

Lemma 3 *A pair of performance standards $x_t = (x_{1t}, x_{2t})$ is a stationary subgame perfect Nash equilibrium at time t if, and only if*

$$(\mathbf{E1}) \quad V_{0t}(11) = \max\{V_{0t}(01), V_{0t}(00)\};$$

$$(\mathbf{E2}) \quad V_{0t}(11) = \max\{V_{0t}(10), V_{0t}(00)\}.$$

Proof: Suppose that $p \geq \frac{1}{2}$. The (per-period) payoff of principal 1 is

$$\begin{aligned} u_{1t} &= x_{1t} \quad \text{if } \max\{V_{0t}(11), V_{0t}(10)\} \geq \max\{V_{0t}(01), V_{0t}(00)\}; \\ u_{1t} &= 0 \quad \text{otherwise.} \end{aligned}$$

The period payoff of principal 2 is

$$\begin{aligned} u_{2t} &= x_{2t} \quad \text{if} \begin{cases} V_{0t}(11) \geq \max\{V_{0t}(10), V_{0t}(00), V_{0t}(01)\} \\ V_{0t}(01) > \max\{V_{0t}(10), V_{0t}(00), V_{0t}(11)\} \\ V_{0t}(01) = V_{0t}(00) > \max\{V_{0t}(10), V_{0t}(11)\} \end{cases} ; \\ u_{2t} &= 0 \quad \text{otherwise.} \end{aligned}$$

Recall that $C(x_{1t}, x_{2t})$ and $C_i(x_{it})$ are monotonically increasing in their arguments by [**M**]. Suppose that $\widetilde{x}_t$ is a (stationary subgame perfect Nash) equilibrium. Then, by Lemma 2, (**E0**) is satisfied by $\widetilde{x}_t$. It follows that the payoff of principal 1 is maximized by the standard, x_{1t}, that satisfies **(E1)**, and that the payoff of principal 2 is maximized by the standard, x_{2t}, that satisfies **(E2).** Finally, notice that if **(E1)** and **(E2)** are satisfied by a set of performance standards at time t, then these standards constitute a stationary subgame perfect Nash Equilibrium. This completes the proof for the case with $p \geq \frac{1}{2}$. The proof for the case where $p < \frac{1}{2}$ is similar and is omitted.$\diamondsuit$

The following two Lemmas explore the implications of assumptions ($\mathbf{C}^+$) and ($\mathbf{C}^-$), respectively.

Lemma 4 *Conditions* (**E1**), (**E2**), *and* ($\mathbf{C}^+$) *hold at t if, and only if*

$$V_{0t}(11) = V_{0t}(00) \geq \max\{V_{0t}(10), V_{0t}(01)\}.$$

Proof: Note that **($\mathbf{C}^+$)** implies that

$$(\mathbf{C0}^+) \quad V_{0t}(11) + V_{0t}(00) \geq V_{0t}(10) + V_{0t}(01)$$

at any t. We prove the Lemma by contradiction. Suppose $V_{0t}(11) > V_{0t}(00)$. Condition **(E2)** implies that

$$V_{0t}(11) = V_{0t}(10).$$

Substitute into **($\mathbf{C0}^+$)** to get that

$$V_{0t}(00) \geq V_{0t}(01).$$

Combing this with **(E1)** yields

$$V_{0t}(11) \leq V_{0t}(00).$$

This is a contradiction, so $V_{0t}(11)$ cannot be greater than $V_{0t}(00)$. It follows directly from (**E1**) that $V_{0t}(11)$ cannot be smaller than $V_{0t}(00)$. Finally, $V_{0t}(11) = V_{0t}(00)$ is compatible with **($\mathbf{C0}^+$)**, **(E1),** and **(E2)** only if $V_{0t}(10) \leq V_{0t}(00)$ and $V_{0t}(01) \leq V_{0t}(00)$. $\diamondsuit$

The next Lemma considers the case of super-additive costs

Lemma 5 *Conditions* (**E1**), (**E2**), *and* ($\mathbf{C}^-$) *hold at t if, and only if*

$$V_{0t}(11) = V_{0t}(10) = V_{0t}(01) > V_{0t}(00).$$

Proof: Note that ($\mathbf{C}^-$) implies that

$$(\mathbf{C0}^-) \quad V_{0t}(11) + V_{0t}(00) < V_{0t}(10) + V_{0t}(01)$$

at any t. We begin by proving that $V_{0t}(11) = V_{0t}(10)$. This is done by contradiction. First, suppose that $V_{0t}(11) > V_{0t}(10)$. **(E2)** implies that

$$V_{0t}(00) > V_{0t}(10).$$

Combining this with ($\mathbf{C0}^-$) implies that

$$V_{0t}(11) < V_{0t}(01).$$

However, **(E1)** implies that $V_{0t}(11) \geq V_{0t}(01)$. This is a contradiction, so $V_{0t}(11)$ cannot be greater than $V_{0t}(10)$. Second, suppose that $V_{0t}(10) > V_{0t}(11)$. **(E2)** implies that

$$V_{0t}(11) \geq V_{0t}(10);$$

This is a contradiction, and so $V_{0t}(10)$ cannot be greater than $V_{0t}(11)$. We conclude that $V_{0t}(10) = V_{0t}(11)$. The proof that $V_{0t}(01) = V_{0t}(11)$ is similar and omitted. Finally, $V_{0t}(11) = V_{0t}(10) = V_{0t}(01)$ is compatible with ($\mathbf{C0}^-$) only if $V_{0t}(11) = V_{0t}(10) = V_{0t}(01) > V_{0t}(00)\diamondsuit$

The last Lemma establishes that a stationary subgame perfect equilibrium exists.

Lemma 6 *A stationary subgame perfect equilibrium exists for* $\beta \in (0, 1]$.

Proof: Suppose first that ($\mathbf{C}^+$) holds. In this case, a stationary equilibrium $\hat{x} = \{\hat{x}_1, \hat{x}_2\}$ satisfies ($\mathbf{SC}^+$) at every t. This implies

$$\frac{R - C(\hat{x}_1, \hat{x}_2)}{1 - \beta} = R; \tag{5}$$

and that

$$R \geq \max[\frac{R - C(\hat{x}_1, 0)}{(1 - p\beta)}, \frac{R - C(0, \hat{x}_2)}{(1 - (1 - p)\beta)}].$$

Equation (5) rewrites as

$$C(\hat{x}_1, \hat{x}_2) = \beta R.$$

Equilibrium levels of $\hat{x}$ satisfy

$$C(\hat{x}_1, \hat{x}_2) = \beta R \tag{6}$$

and

$$R \leq \min[\frac{C(\hat{x}_1, 0)}{p\beta}, \frac{C(0, \hat{x}_2)}{(1 - p)\beta}]. \tag{7}$$

It follows from conditions **(C^+),** (M) and (K) that there exists a solution to equations (6) and (7).

Suppose instead that **(C^-)** holds. In this case, a stationary equilibrium $\bar{x} = \{\bar{x}_1, \bar{x}_2\}$ must satisfy

$$\frac{R - C(\bar{x}_1, \bar{x}_2)}{1-\beta} = \frac{R - C(\bar{x}_1, 0)}{1 - p\beta} \tag{8}$$

and

$$\frac{R - C(\bar{x}_1, \bar{x}_2)}{1-\beta} = \frac{R - C(0, \bar{x}_2)}{1 - (1-p)\beta} \tag{9}$$

along with

$$\frac{R - C(\bar{x}_1, \bar{x}_2)}{1-\beta} > R. \tag{10}$$

Define the quantities $x_{11}, x_{12}, x_{21}, x_{22}$ as solutions to equations (8) and (9) when $x_1 = 0$ and $x_2 = 0$ respectively. Then,

$$\frac{R - C(x_{11}, 0)}{1-\beta} = \frac{R - C(x_{11}, 0)}{1 - p\beta};$$

$$\frac{R - C(0, x_{21})}{1-\beta} = \frac{R}{1 - p\beta}$$

$$\frac{R - C(x_{12}, 0)}{1-\beta} = \frac{R}{1 - (1-p)\beta}$$

$$\frac{R - C(0, x_{22})}{1-\beta} = \frac{R - C(0, x_{22})}{1 - (1-p)\beta}.$$

Solving these equations yields

$$R = C(x_{11}, 0) = C(0, x_{22});$$

in addition,

$$x_{12} \leq x_{11};$$

and

$$x_{21} \leq x_{22}$$

whenever $\beta \in (0,1)$. It follows that a solution to equations (8) and (9) exists.

Additionally, if $\bar{x}$ satisfies equations (8) and (9) then restriction (10) holds for all $\beta \in (0,1)$. To show that an equilibrium exists for all $\beta \in (0,1)$, rewrite (8) and (9) as

$$R\theta = (1+\theta)C(\bar{x}_1, \bar{x}_2) - C(\bar{x}_1, 0);$$

$$R\eta = (1+\eta)C(\bar{x}_1, \bar{x}_2) - C(0, \bar{x}_2);$$

where $\theta = \frac{(1-p)\beta}{1-\beta}$ and $\eta = \frac{p\beta}{1-\beta}$. Adding the two equations, we obtain

$$(\theta + \eta)(R - C(\bar{x}_1, \bar{x}_2)) - C(\bar{x}_1, \bar{x}_2) = C(\bar{x}_1, \bar{x}_2) - C(\bar{x}_1, 0) - C(0, \bar{x}_2) > 0 \tag{11}$$

by **[C-]**. Note also that $\theta + \eta = \frac{\beta}{1-\beta}$ and that (11) implies

$$C(\bar{x}_1, \bar{x}_2) < \beta R$$

as assumed$\diamondsuit$

IV. Interpretation and Implications

Theorem 1 shows that all stationary equilibrium paths displays *strategic consensus.* Competition between the two principals implies that the performance standards are sufficiently low that the agent chooses to meet both to get the reappointment reward with certainty. While this outcome, perhaps, is to be expected when the cost function is sub-additive and it is cheaper for the agent to satisfy the standards jointly than separately, it is surprising that the same result obtains with super-additive costs. In this case, the fact that it is *more* expensive to satisfy the standards jointly than separately suggests that "partisan" outcomes would be more likely. This intuition is wrong for the following reason. Whenever the agent is willing to implement a "partisan" outcome, one of the principals would receive zero payoff and have an incentive to prevent this from happening. By reducing his standard, he can induce the agent to implement a "partisan" outcome in his favor. This logic continues until the standards are such that the agent is just willing to implement $a_t = (11)$ and the result is strategic consensus.

Although all stationary equilibrium paths displays strategic consensus, the distribution of payoffs depends critically on the properties of the cost function. We explore this in a series of propositions and corollaries, most of which follow directly from Theorem 1. The first proposition unpacks condition ($\mathbf{SC}^{+}$) and shows how the surplus of the agency is divided between the two principals and the agent in an economy with sub-additive costs.

Proposition 1 (Sub-additive Costs) *Condition* ($\mathbf{SC}^{+}$) *is equivalent to*

$$(\mathbf{SC}_{\mathbf{1}}^{+})\quad C(x_{1t}, x_{2t}) = \beta R;$$

$$(\mathbf{SC}_{\mathbf{2}}^{+})\quad C_1(x_{1t}) \geq \beta p R;$$

$$(\mathbf{SC}_{\mathbf{3}}^{+})\quad C_2(x_{2t}) \geq \beta(1-p)R.$$

Along all stationary equilibrium paths, the agent receives payoffs $(1-\beta)R$ *per period.*

Proof: The value of appointment starting from any period t is $V_{0t} = \max[V_{0t}(01), V_{0t}(10), V_{0t}(11$ We obtain from Lemma 4 and equation (1) that $V_{0t} = V_{0t}(00) = R$. This implies that

$$V_{0t+1} = R.$$

We obtain, from Theorem 1 and equations (1) to (4), that

$$V_{0t}(11) = V_{0t}(00) \Rightarrow C(x_{1t}, x_{2t}) = \beta R;$$

and that

$$V_{0t}(00) \geq V_{0t}(10) \Rightarrow C_1(x_{1t}) \geq \beta p R;$$

$$V_{0t}(00) \geq V_{0t}(01) \Rightarrow C_2(x_{2t}) \geq \beta(1-p)R.$$

The agent's per period payoff is $R - C(x_{1t}, x_{2t}) = (1-\beta)R$

Corollary 2 (Additive Costs) *Suppose* $C(x_{1t}, x_{2t}) = C_1(x_{1t}) + C_2(x_{2t})$*. Then, the agency has a unique stationary equilibrium,* $x_{1t} = x_1^*$ *and* $x_{2t} = x_2^*$*, with*

$$C_1(x_1^*) = \beta p R;$$

$$C_2(x_2^*) = \beta(1-p)R.$$

In economies with sub-additive costs, the indifference condition $V_{0t}(11) = V_{0t}(00)$ determines how the surplus of the agency is split between the agent and the two principals combined. The agent *always* gets $(1-\beta)R$ per period, while the remaining share of resources, βR, is devoted to the task of generating utilities to the two principals, who, in the aggregate, receive utils corresponding to at least βR. Importantly, this distribution of resources is unaffected by uncertainty. Thus, strategic consensus provides the agent with "full insurance" against random appointment decisions.

Strategic consensus also provides each principal with insurance against "partisan" choices. Corollary 2 demonstrates that this insurance is actuarially fair when the cost function is additive. Principal 1 would get x_1^{**} satisfying $C_1(x_1^{**}) = \beta R$ if $p = 1$; similarly, principal 2 would get x_2^{**} satisfying $C_2(x_2^{**}) = \beta R$ if $p = 0$. Their equilibrium payoffs achieve the expected cost of best and worst (getting 0) outcomes. When the cost function is sub-additive, Proposition 1 shows that the actuarially fair payoffs define a lower bound on the payoffs received, in equilibrium, by each principal. Furthermore, economies with *strictly* sub-additive costs exhibits multiple equilibria in performance standards at each t, and any sequence of selections from these generates stationary paths of equilibria. The locus of stage game equilibria defines a (stationary) utility-possibility frontier subject to agency constraints as follows:

$$\mathcal{U}_c = \{(u_{1t}, u_{2t}) : C(u_{1t}, u_{2t}) = \beta R, C_1(u_{1t}) \geq p\beta R, C_2(u_{2t}) \geq (1-p)\beta R\}.$$

An increase in resource availability shifts the utility frontier out. This is because R raises the value of being appointed at each t, and the principals can for this reason demand more of their agent. A similar argument holds for the discount factor β: an increase in β shifts the frontier out because the principals can demand more of a more farsighted agent. The equilibrium allocations on the utility frontier cannot be Pareto ranked. The cost function increases in both arguments, so that a move from one equilibrium to another must make one principal worse off. The frontier $\mathcal{U}_c$ thus describes the trade-off between principals. It is possible to select equilibria, of course, by choosing one that maximizes a weighted sum of utilities, or any other social welfare function. Corollary 2 shows that equilibrium utility allocation is unique and equal to $\{u_{1t}, u_{2t}\}$=$\{C_1^{-1}[p\beta R]$,$C_2^{-1}[(1-p)\beta R]\}$ when the cost function is additive, and any equilibrium allocation what arises with sub-additive costs (weakly) Pareto dominates this allocation.

The next proposition shows how the surplus is distributed when the cost function is super-additive.

Proposition 2 (Super-additive Costs) *Suppose that* X *is a sequence of stationary performance standards that satisfies condition* ($\mathbf{SC}^-$)*. Then* X *must satisfy*

$$(\mathbf{SC}_1^-) \quad C(x_1, x_2)(1+\theta) - C_1(x_1) = \theta R$$

$$(\mathbf{SC}_2^-) \quad C(x_1, x_2)(1+\eta) - C_2(x_2) = \eta R$$

where $\theta = \frac{(1-p)\beta}{1-\beta}$ *and* $\eta = \frac{p\beta}{1-\beta}$*. The agent receives payoffs* $R - C(x_1, x_2) > (1-\beta)R$ *every period.*

Proof: The value of appointment starting from any period t is

$$V_{0t} = \max[V_{0t}(01), V_{0t}(10), V_{0t}(11), V_{0t}(00)].$$

We obtain from Lemma 5 that $V_{0t} = V_{0t}(11)$ for all t. Iterative, forward substitution, using equation (4), yields

$$V_{0t} = \sum_{k=0}^{\infty} \beta^k (R - C(x_{1t+k}, x_{2t+k})).$$

For sequences of stationary standards, we get

$$V_{0t} = V_{0t+1} = \frac{R - C(x_1, x_2)}{1-\beta}.$$

Substituting for $V_{0t+1} = \frac{R-C(x_1,x_2)}{1-\beta}$, we get that

$$V_{0t}(11) = V_{0t}(10) \Rightarrow (\mathbf{SC}_1^-)$$

and

$$V_{0t}(11) = V_{0t}(01) \Rightarrow (\mathbf{SC}_2^-).$$

Finally, $V_{0t} = \frac{R-C(x_1,x_2)}{1-\beta}$ for all t implies that the agent gets $R - C(x_1, x_2)$ per period. This is strictly greater than $(1-\beta)R$ because $V_{0t}(11) > V_{0t}(00)$ by Lemma 5.

In economies with super-additive costs, the indifference condition $V_{0t}(11) = V_{0t}(10) = V_{0t}(01)$ determines how the surplus of the agency is divided between the principals and the agent. Along stationary equilibrium paths, the agent receives $R - C(x_1, x_2)$, and this is more that he receives along *any* equilibrium path with sub-additive costs. Intuitively, super-additive costs make it costly for the agent to implement consensus outcomes. This enables him to extract more surplus: the principals have, *ceteris paribus*, to lower their standards more to insure that the agent is willing to implement the consensus outcome ($a_t = (11)$) rather than one of the two "partisan" outcomes ($a_t = (10)$ or $a_t = (01)$).

The stage-game utilities of the principals (and the cost to the agent of providing these utilities) are determined by conditions $(\mathbf{SC}_1^-)$ and $(\mathbf{SC}_2^-)$. The next proposition shows that the utility allocation is unique under some, fairly mild, additional assumptions about the properties of the cost function.

Proposition 3 (Uniqueness) *Assume that $C(x_{1t}, x_{2t})$ is differentiable and has continuous partial derivatives. If*

$$(CD) \quad \frac{\partial C}{\partial x_{1t} \partial x_{2t}} > 0$$

then there exists a unique utility allocation $\widehat{x} = (\widehat{x}_1, \widehat{x}_2)$ that satisfies conditions $(\mathbf{SC}_1^-)$ and $(\mathbf{SC}_2^-)$ and the agent receives $R - C(\widehat{x}_1, \widehat{x}_2)$ every period.

Proof: Conditions $(\mathbf{M})$ and $(\mathbf{K})$ imply that there exists a unique $\overline{x}_1 = C_1^{-1}(R)$ that solves $(\mathbf{SC}_1^-)$ for $x_2 = 0$. Likewise, there exists a unique $\overline{x}_2 = C_2^{-1}(R)$ that solves $(\mathbf{SC}_2^-)$ for $x_1 = 0$. Evaluate $(\mathbf{SC}_1^-)$ at $x_1 = 0$ to get that

$$C_2(x_2)(1+\theta) = \theta R.$$

Since $\theta > 0$, the solution to this equation $\widetilde{x}_2 = C_2^{-1}(\frac{\theta}{1+\theta}R)$ satisfies

$$(I2) \quad \widetilde{x}_2 < \overline{x}_2.$$

Evaluate $(\mathbf{SC}_2^-)$ at $x_2 = 0$ to get that

$$C_1(x_1)(1+\eta) = \eta R.$$

Since $\eta > 0$, the solution to this equation $\widetilde{x}_1 = C_1^{-1}(\frac{\eta}{1+\eta}R)$ satisfies

$$(I1) \quad \widetilde{x}_1 < \overline{x}_1$$

The conditions of the Implicit Function Theorem are satisfied, so $(\mathbf{SC}_1^-)$ and $(\mathbf{SC}_2^-)$ define implicit functions $x_2 = \phi_1(x_1)$ and $x_2 = \phi_2(x_1)$, respectively, with partial derivatives given by

$$\frac{\partial \phi_1}{\partial x_1} = \frac{\frac{\partial C}{\partial x_1}(x_1, 0) - (1+\theta)\frac{\partial C}{\partial x_1}(x_1, x_2)}{(1+\theta)\frac{\partial C}{\partial x_2}(x_1, x_2)}$$

$$\frac{\partial \phi_2}{\partial x_1} = \frac{(1+\eta)\frac{\partial C}{\partial x_1}(x_1, x_2)}{\frac{\partial C}{\partial x_2}(0, x_2) - (1+\eta)\frac{\partial C}{\partial x_2}(x_1, x_2)}.$$

The assumption that $\frac{\partial C}{\partial x_{1t} \partial x_{2t}} > 0$ implies that $\frac{\partial \phi_1}{\partial x_1} < 0$ and $\frac{\partial \phi_2}{\partial x_1} < 0$. This and $(I1)$ and $(I2)$ imply that there exist at least one $x = (x_1, x_2)$ that satisfies conditions $(\mathbf{SC}_1^-)$ and $(\mathbf{SC}_2^-)$. A sufficient condition for uniqueness is that $\frac{\partial \phi_2}{\partial x_1} < \frac{\partial \phi_1}{\partial x_1}$ for all $(x_1, x_2) \in \Re_+^2$. A simple manipulation yields that $\frac{\partial \phi_2}{\partial x_1} < \frac{\partial \phi_1}{\partial x_1}$ if, and only if

$$\frac{\partial C}{\partial x_1}(x_1, 0)\left[\frac{\partial C}{\partial x_2}(0, x_2) - \frac{\partial C}{\partial x_2}(x_1, x_2)\right]$$
$$<$$
$$(1+\theta)\frac{\partial C}{\partial x_1}(x_1, x_2)\frac{\partial C}{\partial x_2}(0, x_2) + \eta\frac{\partial C}{\partial x_1}(x_1, 0)\frac{\partial C}{\partial x_2}(x_1, x_2).$$

The left-hand side is negative because $\frac{\partial C}{\partial x_{1t} \partial x_{2t}} > 0$, while condition $(\mathbf{M})$ implies that the right-hand side is positive.

The proposition demonstrates that there exists a unique (stationary) utility allocation if the (super-additive) cost function is differentiable and its cross partial derivative is positive. Importantly, for $p \in (0, 1)$ both principals receives positive utility along the stationary equilibrium path. The allocation of payoffs between the two principals, and between the two principals combined and the agent depends on the three parameters of the model: p, R and β.[5] The parameter p captures how likely it is that principal 1 is decisive if the two principals disagree about reappointing the agent. A high value of p allows principal 1 to extract utility from the agent at the expense of principal 2. The more likely it is that principal 1 is decisive, the more attractive it is for the agent to join up with him alone. As a consequence, principal 1 can extract more surplus while the principal 2 must lower his standard to avoid the "partisan" outcome. Although the agent *is* reappointed for sure every period, it is interesting to note that his payoff is not, as in the case of sub-additive

[5] The comparative statics are formally derived in the appendix.

costs, independent of governance uncertainty. This is because changes in p affect the allocation of utilities between the two principals and, consequently, the cost of providing these utilities. Whether the agent would like p to be high or low depends on how fast the marginal cost of providing utility to principal 1 increases relative to that of principal 2.

At least one of the principals is better off if the agent becomes more patient (high β) or if more resources (high R) are made available to generate utility. However, it is not inevitable that both gain despite the fact that the agent devotes more resources in total to the task of generating utilities. For example, if p is close to one, only principal 1 would gain from an increase in either β or R; principal 2 would be worse off.

A final point to stress is that governance uncertainty (as defined by assumption 5) is *necessary* for consensus outcomes to arise in equilibrium. This is demonstrated by the next proposition for the extreme case with $p = 1$.[6]

Proposition 4 (No Uncertainty) *Suppose that* $p = 1$. *The agent satisfies every period the standard* $x_t = x_1^{**}$ *set by principal* 1 *where*

$$C_1(x_1^{**}) = \beta R.$$

Principal 2 *receives no utility, and the agent receives* $(1 - \beta)R$ *per period.*

Proof: Assume that $p = 1$. Suppose that $\widehat{x} = \{\widehat{x}_1, \widehat{x}_2\}$ with $\widehat{x}_2 > 0$ is a stationary equilibrium. We show that this leads to a contradiction. Notice that

$$C(\widehat{x}_1, \widehat{x}_2) > C(\widehat{x}_1, 0)$$

by condition (**M**). Since $p = 1$, the agent can reduce the cost of getting reappointed without scarifying his chance of reappointment by implementing $\{\widehat{x}_1,\ 0\}$ instead of $\{\widehat{x}_1, \widehat{x}_2\}$. Thus, along any stationary equilibrium path $x_2 = 0$. This implies that

$$V_{0t}(10) = V_{0t}(11)$$

and that

$$V_{0t}(01) = V_{0t}(00).$$

Principal 1 sets x_1 such that $V_{0t}(11) = V_{0t}(00)$ which implies

$$C(\widehat{x}_1, 0) = C_1(\widehat{x}_1) = \beta R$$

and the agent gets $(1 - \beta)\, R$ each period.

Intuitively, governance uncertainty ($p \in (0, 1)$) limits how much the agent is willing to do to please one principal alone simply because neither of them can on their own guarantee reappointment. If there is no governance uncertainty, proposition 4 shows that all equilibrium paths exhibits "partisan" choices and that only the decisive principal (principal 1 if $p = 1$) receives any utility; the other principal gets nothing. This is true even with sub-additive costs as long as the cost function is increasing in its arguments.[7]

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[6] The case with $p = 0$ is symmetric.

[7] If the cost function is non-decreasing in its arguments, it is possible that the agent will provide some utility to principal 2 even if $p = 1$, but only because doing so enables him to provide utilities to principal 1 at a lower cost.

V. Discussion and Applications

Theorem 1 has many applications, including some interesting political economy ones. Suppose the principals are two groups of voters and the agent is a politician that is appointed to implement policies on behalf of the electorate.[8] Governance uncertainty arises in the context because voters cannot promise to turn out in full force in elections. This makes the re-election reward uncertain unless the politician meets the demands of all groups of voters by his policy implementation. Outcomes depend on the properties of the political cost function. Sub-additive political costs arise from the fundamental role of public goods. Imagine that the politician wants to provide utility to one group of voters only. He can do this by making transfers (or equivalently, giving tax-cuts) to this group. If he wants to provide utility to both groups, it may be cheaper to provide universal public goods. The fact that public goods can be used to provide utility to everybody at the same time allows the cost function to be sub-additive. A politician with access to (lump sum) taxes and subsidies only would, in contrast, always face an additive cost function. Super-additive political costs arise because of externalities. Suppose, for example, that the politician can provide public services to specific groups of voters but that the production of these services generates pollution that does harm to everybody. This would make the political cost function super-additive. The same is true if local amenities provided to one group of voters create envy among other groups of voters.

Theorem 1 characterizes sequences of stationary subgame perfect Nash equilibria. We conjecture that there exist equilibria supported by non-stationary and history-dependent strategies but leave this matter to further research. In our model, the principals are infinitely lived. This, however, is not important for our results. A sequence of short-lived principals can implement the same utility allocations. What is important is that the agent cares about the future.

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[8] Ferejohn (1986) studies optimal reelection rules in a dynamic political agency with one voter and a series of politicians. In extensions with more than one group of voters, the incumbent politician only needs a minimum winning coalition to get reelected, and he can play the different groups off against each other (Persson and Tabellini, 2000, chapter 9; Aidt and Magris, 2003). With additive costs voters are unable to extract any surplus from their politicians. Turnout uncertainty changes that by making it impossible for politicians to play off groups of voters in this way without making their reelection prospect uncertain.

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Appendix

Comparative statics Use condition (SC_1^-) and (SC_2^-) to derive

$$\begin{aligned} &\left[(1+\theta)\frac{\partial C}{\partial x_1}(x_1,x_2)-\frac{\partial C}{\partial x_1}(x_1,0)\right]dx_1+(1+\theta)\frac{\partial C}{\partial x_2}(x_1,x_2)dx_2 \\ = \\ &[R-C(x_1,x_2)]\left[\frac{(1-p)}{(1-\beta)^2}d\beta-\frac{\beta}{1-\beta}dp\right]+\theta dR \end{aligned}$$

and

$$\begin{aligned} &(1+\eta)\frac{\partial C}{\partial x_1}(x_1,x_2)dx_1+\left[(1+\eta)\frac{\partial C}{\partial x_2}(x_1,x_2)-\frac{\partial C}{\partial x_2}(0,x_2)dx_1\right]dx_2 \\ = \\ &[R-C(x_1,x_2)]\left[\frac{p}{(1-\beta)^2}d\beta+\frac{\beta}{1-\beta}dp\right]+\eta dR \end{aligned}$$

Define

$$\begin{aligned} D=\frac{\partial C}{\partial x_1}(x_1,0)&\left[\frac{\partial C}{\partial x_2}(0,x_2)-\frac{\partial C}{\partial x_2}(x_1,x_2)\right] \\ &-(1+\theta)\frac{\partial C}{\partial x_1}(x_1,x_2)\frac{\partial C}{\partial x_2}(0,x_2) \\ &-\eta\frac{\partial C}{\partial x_1}(x_1,0)\frac{\partial C}{\partial x_2}(x_1,x_2)<0. \end{aligned}$$

Use Cramer's rule to calculate:

$$\frac{\partial x_1}{\partial \beta}=\frac{[R-C(.)]}{(1-\beta)^2}\frac{\left[\upsilon\frac{\partial C}{\partial x_2}(x_1,x_2)-(1-p)\frac{\partial C}{\partial x_2}(0,x_2)\right]}{D}$$

$$\frac{\partial x_2}{\partial \beta}=\frac{[R-C(.)]}{(1-\beta)^2}\frac{\left[-\upsilon\frac{\partial C}{\partial x_1}(x_1,x_2)-p\frac{\partial C}{\partial x_1}(x_1,0)\right]}{D}$$

where $\upsilon=\frac{(1-2p)-\beta(1-p)}{1-\beta}$. Clearly, if $\frac{\partial x_1}{\partial \beta}<0$, then $\frac{\partial x_2}{\partial \beta}>0$ and vice versa. For $p=\frac{1-\beta}{2-\beta}$, both derivatives are positive. For $p\to 1$, $\frac{\partial x_1}{\partial \beta}>0$ and $\frac{\partial x_2}{\partial \beta}<0$ because $p(1-\beta)<1$ and assumption (**CD**) implies that $\frac{\partial C}{\partial x_1}(x_1,x_2)>\frac{\partial C}{\partial x_1}(x_1,0)$. Finally,

$$\frac{\partial C}{\partial \beta}=-\frac{[R-C(.)]}{(1-\beta)^2 D}\left[\begin{array}{c} p\frac{\partial C}{\partial x_1}(x_1,0)\frac{\partial C}{\partial x_2}(x_1,x_2) \\ +(1-p)\frac{\partial C}{\partial x_2}(0,x_2)\frac{\partial C}{\partial x_1}(x_1,x_2) \end{array}\right]>0.$$

Using Cramer's rule again, we get

$$\frac{\partial x_1}{\partial R}=\frac{\left[\frac{(1-2p)\beta}{1-\beta}\frac{\partial C}{\partial x_2}(x_1,x_2)-\theta(1+\eta)\frac{\partial C}{\partial x_2}(0,x_2)\right]}{D}$$

$$\frac{\partial x_2}{\partial R}=\frac{\left[-\frac{(1-2p)\beta}{1-\beta}\frac{\partial C}{\partial x_1}(x_1,x_2)-\eta(1+\theta)\frac{\partial C}{\partial x_1}(x_1,0)\right]}{D}.$$

Clearly, if $\frac{\partial x_1}{\partial R} < 0$, then $\frac{\partial x_2}{\partial R} > 0$ and vice versa. For $p = \frac{1}{2}$, both derivatives are positive. For $p \to 1$, $\frac{\partial x_1}{\partial R} > 0$ and $\frac{\partial x_2}{\partial R} < 0$ because $\eta(1+\theta) = -\frac{(1-2p)\beta}{1-\beta} = \frac{\beta}{1-\beta}$ and assumption (**CD**) implies that $\frac{\partial C}{\partial x_1}(x_1, x_2) > \frac{\partial C}{\partial x_1}(x_1, 0)$. Finally,

$$\frac{\partial C}{\partial R} = -\frac{1}{D}\left[\begin{array}{c} \theta(1+\eta)\frac{\partial C}{\partial x_1}(x_1,0)\frac{\partial C}{\partial x_2}(x_1,x_2) \\ +\eta(1+\theta)\frac{\partial C}{\partial x_2}(0,x_2)\frac{\partial C}{\partial x_1}(x_1,x_2) \end{array}\right] > 0.$$

A final application of Cramer's rule yields

$$\frac{\partial x_1}{\partial p} = -\frac{\beta\,[R - C(.)]}{1-\beta}\frac{\left[\frac{2-\beta}{1-\beta}\frac{\partial C}{\partial x_2}(x_1,x_2) - \frac{\partial C}{\partial x_2}(0,x_2)\right]}{D} > 0$$

$$\frac{\partial x_2}{\partial p} = \frac{\beta\,[R - C(.)]}{1-\beta}\frac{\left[\frac{2-\beta}{1-\beta}\frac{\partial C}{\partial x_1}(x_1,x_2) - \frac{\partial C}{\partial x_1}(x_1,0)\right]}{D} < 0.$$

Moreover,

$$\frac{\partial C}{\partial p} = -\frac{\beta\,[R - C(.)]}{1-\beta}\left[\begin{array}{c} \frac{\partial C}{\partial x_1}(x_1,0)\frac{\partial C}{\partial x_2}(x_1,x_2) \\ -\frac{\partial C}{\partial x_2}(0,x_2)\frac{\partial C}{\partial x_1}(x_1,x_2) \end{array}\right].$$

The sign of $\frac{\partial C}{\partial p}$ depends on the curvature of the marginal cost functions.