A homotopy theorem for incremental stability

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***Abstract*—A theorem is proved to verify incremental stability of a feedback system via a homotopy from a known incrementally stable system, making no assumptions on existence of an extended space, well-posedness or causality. A first corollary of that result is that incremental stability may be verified by separation of Scaled Relative Graphs, correcting two assumptions in [1, Theorem 2]. Necessity of this correction is demonstrated by example. A second corollary provides an incremental version of the classical IQC stability theorem. Finally it is shown how to relax assumptions of incremental boundeness on the operators.**

I. Introduction

There are two standard approaches to verifying absolute stability of a feedback interconnection: the first, introduced by Zames [2], is to work in an extended space, which includes unstable signals, and show that all signals are, in fact, stabilized by the feedback. The second, pioneered by Megretski and Rantzer [3], is based upon a *homotopy* argument: it is shown that a known stable system can be perturbed continuously to produce the desired feedback interconnection, in such a way that stability is never lost. Such an argument has two ingredients: firstly, one shows that a system remains stable under a small perturbation, provided a gain bound is satisfied. Secondly, one must verify such a gain bound along the entire path of perturbations. In the original theorems of Megretski and Rantzer [3], [4], it was shown that perturbations which were small in the gap metric [5] preserved stability, and (soft) *Integral Quadratic Constraints (IQCs)* were used to verify gain bounds.

In this paper, we offer an alternative approach: an *incremental* homotopy argument is developed (Theorem 2), where gain bounds are replaced by incremental gain bounds, and the gap metric is replaced by a minor modification of the incremental small gain theorem (Theorem 1). In a similar vein to the incremental versions of the small gain theorem [6], the stronger assumption of incremental gain allows well-posedness and causality assumptions to be weakened. While in a standard homotopy argument, well-posedness and causality of the feedback interconnection must be assumed along the entire path, requiring reference to an extended space, in the incremental setting, only incremental boundedness of the operators must be assumed, which may be verified without any extended space. The setting of Theorem 2 is more general than the typical IQC setting, as neither operator is assumed to be linear nor time-invariant.

Theorem 2 is a standalone and corrected version of an incremental stability theorem recently proved by the authors in [1], in the context of a graphical stability criterion based on the *Scaled Relative Graph (SRG)*, a graphical representation of an operator introduced by Ryu, Hannah, and Yin [7]. In [1], it was shown that the SRG generalizes the Nyquist diagram of an LTI operator to an arbitrary nonlinear operator, and a generalization of the Nyquist criterion was given in [1, Theorem 2]. As a first corollary of Theorem 2, we reprove [1, Theorem 2], correcting two technical assumptions. This theorem has also been recently generalized by Chen, Khong, and Sepulchre [8], removing a technical assumption, however that generalization relies on an extended space and assumptions of well-posedness and causality, which are not required here. For examples of the application of Theorem 2 and detailed comparisons with other methods of stability verification, we refer the reader to the examples of [1].

As a second corollary to Theorem 2, we obtain an incremental version of the IQC stability theorem of Megretski and Rantzer [3]. This corollary is closely related to [9, Theorem 7.40], but does not make any assumptions of causality, nor rely on an extended space. It has been shown that incremental gain bounds can be verified using closely related *differential* IQCs [10]. Incremental IQCs are used in the study of periodic solutions by Jonsson, Chung-Yao Kao, and Megretski [11], in the study of neural networks by Gronqvist and Rantzer [12] and in system identification by van Waarde and Sepulchre [13]. It has recently been shown that well-posedness and causality assumptions can be relaxed in the non-incremental setting [14], [15].

The remainder of this note is structured as follows. In Section II, we introduce necessary notation and preliminary results. In Section III, we prove our main result (Theorem 2), a general incremental homotopy theorem. In Section IV, it is shown how incremental stability may be verified using separation of SRGs, and in Section V, an incremental IQC stability theorem is given. Finally, in Section VI, it is shown how assumptions of incremental boundedness may be relaxed in exchange for well-posedness and causality, giving a middle ground between the incremental approach of Theorem 2 and standard non-incremental arguments.

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II. Preliminaries

A. Signals and systems

Let $\mathcal{F}$ denote the space of all functions mapping the interval $[0,\infty)$ into $\mathbb{R}^n$. Let L_2^n be the space of equivalence classes of trajectories $u \in \mathcal{F}$ and satisfying

$$\|u\| := \left(\int_0^\infty u(t)^\top u(t)\ \mathrm{d}t\right)^{\frac{1}{2}} < \infty,$$

under the equivalence $u \sim y \iff \|u - y\| = 0$. We will abuse terminology in the usual way and say that a trajectory $u \in \mathcal{F}$ *belongs to* L_2^n when it belongs to an equivalence class in L_2^n. For the remainder of this note, we will drop the dimension n, and simply denote L_2^n by L_2, where n is arbitrary. Given an element $x \in L_2$, we let $\hat{x}$ denote its Fourier transform.

By an *operator* on a domain $D \subseteq L_2$ we will mean a single-valued map $H : D \to L_2$. The domain D will also be denoted $\mathrm{dom}\,(H)$. We will associate an operator H with its *relation* or *graph*, defined as $\{(u, y) \mid y = H(u)\} \subseteq L_2 \times L_2$, and denote the two in the same way. Scalar multiplication, summation, and inversion of relations are defined as follows:

$$\alpha H := \{(u, \alpha y) \mid y = H(u)\} \tag{1}$$

$$H_1 + H_2 := \{(u, y + z) \mid y = H_1(u), z = H_2(u)\} \tag{2}$$

$$H^{-1} := \{(y, u) \mid y = H(u)\}. \tag{3}$$

We note that the inverse of an operator may be multivalued, and therefore not necessarily an operator. However, these relational operations are always well defined.

Given an operator $H : L_2 \to L_2$, we define the *gain* of H on L_2, denoted $\|H\|$, to be the smallest $\gamma > 0$ such that there exists $\beta \in \mathbb{R}$ such that, for all inputs $u \in L_2$,

$$\|H(u)\| \leq \gamma \|u\| + \beta. \tag{4}$$

If the gain of an operator is finite, the operator is said to be *bounded*. If $\beta = 0$, the operator is said to have *finite gain with zero offset*.

The *incremental gain* of H on L_2 is defined to be

$$\sup_{u_1, u_2 \in L_2, u_1 \neq u_2} \frac{\|H(u_1) - H(u_2)\|}{\|u_1 - u_2\|}. \tag{5}$$

If the incremental gain of an operator is finite, the operator is said to be *incrementally bounded* or *incrementally stable*. If an operator derives from a dynamical system, incremental boundedness is equivalent to asymptotic stability of any input/output trajectory, under reachability and observability assumptions [16].

Consider the negative feedback interconnection of two operators, H_1 and H_2, defined by the equations

$$e = u - H_2(y) \tag{6}$$

$$y = H_1(e) \tag{7}$$

and illustrated in Figure 1. We make the standing assumption that this negative feedback interconnection defines a (single-valued) operator from some (possibly empty) domain $D \subseteq L_2$ to L_2, mapping u to y. We denote the relation of this operator by $[H_1, H_2] := \{(u, y) \mid$ there exits a unique e s.t. (6) – (7) are satisfied$\}$.

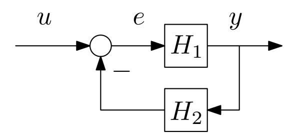


Fig. 1. Negative feedback interconnection of $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$.

We will make use of the following two technical lemmas.

Lemma 1. *Given operators $H_1, H_2 : L_2 \to L_2$,*

$$[H_1, H_2] = (H_1^{-1} + H_2)^{-1}.$$

Proof. Adopting the notation of Figure 1 and applying the definitions of relational inverse and sum gives

$$H_1^{-1} = \{(y, e) \mid y = H_1(e)\}$$

$$H_1^{-1} + H_2 = \{(y, e + z) \mid y = H_1(e), z = H_2(y)\}$$

$$(H_1^{-1} + H_2)^{-1} = \{(e + z, y) \mid y = H_1(e), z = H_2(y)\}.$$

Setting $u = e + z$, we arrive at the definition of $[H_1, H_2]$. ☐

Lemma 2. *Given $\tau, \nu \geq 0$ and operators $H_1, H_2 : L_2 \to L_2$,*

$$[H_1, (\tau + \nu)H_2] = [[H_1, \tau H_2], \nu H_2]. \tag{8}$$

Proof. Note that $(H^{-1})^{-1} = H$, and, given a single-valued operator, $(\tau + \nu)H = \tau H + \nu H$. We then have:

$$\begin{aligned}[H_1, (\tau + \nu)H_2] &= (H_1^{-1} + (\tau + \nu)H_2)^{-1} \\ &= (H_1^{-1} + \tau H_2 + \nu H_2)^{-1} \\ &= ((H_1^{-1} + \tau H_2)^{-1})^{-1} + \nu H_2)^{-1} \\ &= [[H_1, \tau H_2], \nu H_2].\end{aligned}$$

☐

The following theorem is a modified version of the classical incremental small gain theorem [6, Theorem 30, p. 184]. It differs from the classical statement of the theorem in that the operators are defined only on L_2, rather than an extended space, and are not required to map 0 to 0. It is closely related to the incremental gap robustness result of Georgiou and Smith [5, Theorem 1].

Theorem 1 (Incremental small gain theorem)**.** *Let $H_1, H_2 : L_2 \to L_2$ be operators with incremental gain bounds of γ_1, γ_2 respectively. If $\gamma_1\gamma_2 < 1$, then for any $u \in L_2$, there exist unique $e, y \in L_2$ satisfying the feedback interconnection* (6)–(7).

Proof. Fix $u \in L_2$. Substituting (7) in (6) gives $e = u - H_2(H_1(e))$. Define $K_u(x) := u - H_2(H_1(x))$. Since $H_1, H_2 : L_2 \to L_2$, $K_u : L_2 \to L_2$. We claim that K_u is a contraction on L_2. Indeed, letting $x, \bar{x} \in L_2$, we have

$$\begin{aligned}\|K_u(x) - K_u(\bar{x})\| &= \|u - H_2(H_1(x)) - u + H_2(H_1(\bar{x}))\| \\ &= \|H_2(H_1(\bar{x}) - H_2(H_1(x))\| \\ &\leq \gamma_1 \|H_2(\bar{x}) - H_2(x)\| \\ &\leq \gamma_1\gamma_2 \|x - \bar{x}\|.\end{aligned}$$

Therefore, by the Banach fixed point theorem, there exists a unique solution to $e = K_u(e)$ for each $u \in L_2$. Furthermore, we have $y = H_1(e)$, so existence and uniqueness of y is guaranteed. ☐

B. Scaled Relative Graphs

The Scaled Relative Graph (SRG) is a graphical representation of the gain and phase of an operator. Phase is given by the angle between two signals treated as vectors. For $u, y \in L_2$, this is defined as

$$\angle(u, y) := \arccos \frac{\operatorname{Re}\langle u, y\rangle}{\|u\|\,\|y\|} \in [0, \pi]. \tag{9}$$

We define the SRG for an arbitrary relation, allowing us to talk about the SRG of an operator and its relational inverse on an equal footing. Let $R \subseteq L_2 \times L_2$. We write $u \in R(y)$ if $(u, y) \in R$. Given $u_1, u_2 \in L_2$, $u_1 \neq u_2$, define the set of complex numbers $z_R(u_1, u_2)$ by

$$z_R(u_1, u_2) := \left\{ \frac{\|y_1 - y_2\|}{\|u_1 - u_2\|} e^{\pm j\angle(u_1 - u_2, y_1 - y_2)} \tag{10}\right.$$

$$\left. \middle|\; y_1 \in R(u_1), y_2 \in R(u_2) \right\}. \tag{11}$$

If $u_1 = u_2$ and there are corresponding outputs $y_1 \neq y_2$, then $z_R(u_1, u_2)$ is defined to be $\{\infty\}$. If R is single valued at u_1, $z_R(u_1, u_1)$ is the empty set. The *Scaled Relative Graph* (SRG) of R is then given by

$$\operatorname{SRG}(R) := \bigcup_{u_1, u_2 \in L_2} z_R(u_1, u_2). \tag{12}$$

The SRG of an operator $H : L_2 \to L_2$ is defined to be the SRG of its relation. We refer the reader to [1] for the relationship between the SRG, the Nyquist diagram of a transfer function and the incremental disc of a static nonlinearity, and to [17] for the relationship to the numerical range of a linear operator.

The incremental gain of an operator (on its domain) is the maximum radius of its SRG. Letting $|z|$ denote the magnitude of $z \in \mathbb{C}$, we have the following lemma, the proof of which is immediate from the definition of the SRG.

Lemma 3. *Given an operator $H : D \to L_2$,*

$$\sup_{u_1, u_2 \in D} \frac{\|H(u_1) - H(u_2)\|}{\|u_1 - u_2\|} = \sup_{z \in \operatorname{SRG}(H)} |z|.$$

Given an interconnection of operators, the SRG of the interconnection can be bounded by applying graphical operations to the SRGs of the individual operators. We recall a couple of necessary results in the following lemma, and refer the reader to [7] for the full theory. These graphical operations will be used in the proof of Lemma 5 in Section IV.

An SRG $\mathcal{G}$ is said to satisfy the *chord property* if, for each $z \in \mathcal{G}$, $\lambda z + (1 - \lambda)\bar{z} \in \mathcal{G}$ for all $\lambda \in [0, 1]$.

Lemma 4. *Given $A, B \subseteq L_2 \times L_2$, we have:*

1) $\operatorname{SRG}(A^{-1}) = \{z^{-1} \mid z \in \operatorname{SRG}(A)\}$;.
2) if $\bar{A} \supseteq \operatorname{SRG}(A)$ is any set satisfying the chord property, then $\operatorname{SRG}(A + B) \subseteq \bar{A} + \operatorname{SRG}(B)$.

Proof. Property 1 is proved in [7, Theorem 5]. The proof of Property 2 proceeds as follows. Let $(u_1, y_A), (u_2, z_A) \in A, (u_1, y_B), (u_2, z_B) \in B$. Then $(u_1, y_A + y_B), (u_2, z_A + z_B) \in A + B$, and

$$\begin{aligned} w &= \frac{\|y_A + y_B - z_A - z_B\|}{\|u_1 - u_2\|} \\ &\quad \exp(j\angle(y_A + y_B - z_A - z_B, u_1 - u_2)) \in \operatorname{SRG}(A + B), \\ w_A &= \frac{\|y_A - z_A\|}{\|u_1 - u_2\|} \exp(j\angle y_A - z_A, u_1 - u_2) \in \operatorname{SRG}(A), \\ w_B &= \frac{\|y_B - z_B\|}{\|u_1 - u_2\|} \exp(j\angle y_B - z_B, u_1 - u_2) \in \operatorname{SRG}(B). \end{aligned}$$

Then, by direct calculation, $\operatorname{Re}(w) = \operatorname{Re}(w_A) + \operatorname{Re}(w_B)$ and $\operatorname{Im}(w_B) - \operatorname{Im}(w_A) \leq \operatorname{Im}(w) \leq \operatorname{Im}(w_B) + \operatorname{Im}(w_A)$, that is, $w \in w_B + [w_A, \overline{w_A}]$. Since $[w_A, \overline{w_A}] \subseteq \bar{A}$, the claim follows. □

III. Incremental homotopy

The following theorem gives a method for verifying finite incremental gain of a feedback interconnection using a homotopy from a known incrementally bounded operator. The proof proceeds by using the incremental small gain theorem to show small perturbations in the feedback preserve stability, and applies this idea inductively to scale the feedback from 0 to 1.

Theorem 2 (Incremental homotopy)**.** *Let $H_1, H_2 : L_2 \to L_2$ be operators such that*

(i) H_1, H_2 have finite incremental gain;
(ii) there exists $\gamma > 0$ such that, for all $\tau \in [0, 1]$ and all $u_1, u_2 \in \operatorname{dom}([H_1, \tau H_2])$, we have

$$\|y_1 - y_2\| \leq \gamma \|u_1 - u_2\|,$$

where $y_i = [H_1, \tau H_2](u_i), i = 1, 2$.

Then $\operatorname{dom}([H_1, H_2]) = L_2$ and $[H_1, H_2]$ has an incremental gain bound of γ.

Proof. Let γ_1, γ_2 be incremental gain bounds for H_1 and H_2 respectively. We begin by showing that there exists $\nu > 0$ such that $\operatorname{dom}([H_1, \nu H_2]) = L_2$. Indeed, setting $\nu < 1/(\gamma_1\gamma_2)$, we have that $\nu\gamma_1\gamma_2 < 1$, so it follows from Theorem 1 that $\operatorname{dom}([H_1, \nu H_2]) = L_2$.

By assumption, γ is an incremental gain bound for $[H_1, \nu H_2]$. It then follows from Theorem 1 that, for all $\tau \in [0, 1/\gamma\gamma_2)$, $\operatorname{dom}([[H_1, \nu H_2], \tau H_2]) = L_2$. From Lemma 2, $[[H_1, \nu H_2], \tau H_2] = [H_1, (\nu + \tau)H_2]$. Again, by assumption, this operator has an incremental gain bound of γ.

Proceeding inductively, we have that $\operatorname{dom}([H_1, (\nu + k\tau)H_2]) = L_2$ for all $\tau \in [0, 1/\gamma\gamma_2)$ and positive integers k such that $\nu + k\tau \leq 1$, so, in particular, $\operatorname{dom}([H_1, H_2]) = L_2$. The incremental gain bound of γ then follows from Condition (ii) in the theorem statement. □

We note that (ii) implies single-valuedness of $[H_1, \tau H_2]$ for all $\tau \in [0, 1]$. Furthermore, we note that τ must be independent of u_1 and u_2. Necessity of this condition is demonstrated in Example 1. Condition (ii) may be relaxed to allow unstable systems in the loop, for example by encapsulating the unstable operator in a stabilizing loop transform as in classical IQC analysis [4]. In the non-incremental setting, such homotopy arguments are known to guarantee a strong "quadratic" form of topological graph separation. Topological graph separation is a necessary and sufficient condition for feedback stability [18]; enforcing the stronger quadratic separation introduces some conservatism but allows computational methods such as IQC analysis to be applied [4]. We expect such relationships to be true in the non-incremental setting also, but the details remain a question for future research. Closely related results in the differential setting are presented in [19].

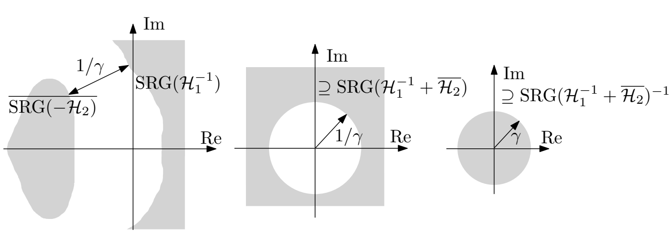
Theorem 2 allows incremental stability of a feedback interconnection to be verified using only information about L_2 signals, without any reference to an extended space containing unbounded signals. This is useful as it allows the use of operator theoretic tools to verify the incremental gain bound of condition (ii). In the following section, we show this may be done graphically using the SRG, and in Section V we verify the incremental gain bound using IQCs.

IV. SRG SEPARATION

Given an SRG $\mathcal{G}$, let $\overline{\mathcal{G}}$ denote the smallest SRG containing $\mathcal{G}$ and satisfying the chord property. We begin with a graphical condition which guarantees finite incremental gain of a feedback interconnection, on its domain (which may not, in general, be all of L_2). Given two regions X_1, X_2 in the extended complex plane, we let $\operatorname{dist}(X_1, X_2)$ denote $\inf_{x_1 \in X_1, x_2 \in X_2} |x_1 - x_2|$.

Lemma 5. *Let H_1, H_2 be operators, and suppose there exists $\gamma > 0$ such that $\operatorname{dist}(\operatorname{SRG}(H_1)^{-1}, -\overline{\operatorname{SRG}(H_2)}) \geq 1/\gamma$. Then, for any $u_i \in \operatorname{dom}([H_1, H_2])$ and $y_i = (H_1^{-1} + H_2)^{-1}(u_i)$, we have $\|y_1 - y_2\| \leq \gamma \|u_1 - u_2\|$.*

Proof. Since $\overline{\operatorname{SRG}(H_2)^{-1}}$ satisfies the chord property by construction, we can apply the SRG sum rule (Lemma 4, Property 2) to obtain a bounding region for $\operatorname{SRG}(H_1^{-1} + H_2)$, according to the geometry in the left and middle plots below. The result then follows by applying the inversion rule (Lemma 4, Property 1) to obtain an SRG with bounded magnitude, illustrated in the right plot below.



□

The separation condition of Lemma 5 guarantees that the feedback interconnection of two operators has an incremental gain bound on its domain. In other words, Condition (ii) of Theorem 2 is satisfied for $\tau = 1$. However, this alone does not guarantee stability, as we also must verify that the domain of the feedback interconnection is L_2. Guaranteeing that the domain is L_2 requires a stronger separation property: the SRGs must remain separated as the feedback is gradually increased from zero. We formalize this separation property as follows.

Definition 1. Two operators H_1 and H_2 are said to have *strictly separated SRGs with margin* $r_{\min} > 0$ if $\operatorname{dist}(\operatorname{SRG}(H_1)^{-1}, -\tau\overline{\operatorname{SRG}(H_2)}) \geq r_{\min}$ for all $\tau \in (0, 1]$. ⌟

The following corollary is proved by showing that strict separation of the SRGs implies the existence of an incremental gain bound on the feedback interconnection which is uniform in the gain τ.

Corollary 1. *Suppose $H_1, H_2 : L_2 \to L_2$ have bounded incremental gain and strictly separated SRGs. Then the feedback interconnection $[H_1, H_2]$ maps L_2 to L_2 and has bounded incremental gain.*

Proof. It follows from strict separation of the SRGs of H_1 and H_2, and Lemma 5, that $1/r_{\min}$ is an incremental gain bound for $[H_1, \nu H_2]$. The result then follows from Theorem 2. □

Corollary 1 corrects the assumptions of [1, Theorem 2, Corollary 1, Corollary 2] in two ways: in [1], only H_1 was was assumed to be incrementally bounded, and the separation of the SRGs was not required to be strict. Strict separation must also be assumed in [1, Theorem 1]. The additional assumptions are satisfied in all the examples of [1], and we refer the reader to those examples for illustrations of the application of Theorem 2, and for comparisons with other stability verification methods. The following example illustrates why strict separation of the SRGs is required to guarantee incremental boundedness.

Example 1. Consider the operator $N : L_2 \to L_2$ given by

$$(Nu)(t) = \phi(u(t))$$

where $\phi(x) = -\arctan(x)$. ϕ is 1-Lipschitz continuous, so N has an incremental gain bound of 1. It follows that its SRG is contained in the closed unit disc. We now observe the following: the SRG of N is, in fact, contained in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, and therefore does not contain the point -1. However, the operator formed by putting N in unity gain negative feedback has infinite incremental gain.

To see that $\operatorname{SRG}(N) \subseteq \mathbb{D}$, suppose that there exist $u_1, u_2 \in L_2$, $u_1 \neq u_2$, such that

$$\frac{\|N(u_1) - N(u_2)\|}{\|u_1 - u_2\|} = 1. \tag{13}$$

Since $u_1 \neq u_2$, the set $E = \{t \in \mathbb{R} \mid u_1(t) \neq u_2(t)\}$ has positive measure. Equation (13) implies

$$\begin{aligned}\int_E |u_1(t) - u_2(t)|^2 \, \mathrm{d}t &= \int_E |\phi(u_1(t)) - \phi(u_2(t))|^2 \, \mathrm{d}t \\ &= \int_E \left| \frac{\phi(u_1(t)) - \phi(u_2(t))}{u_1(t) - u_2(t)} \right|^2 |u_1(t) - u_2(t)|^2 \, \mathrm{d}t,\end{aligned}$$

or equivalently

$$\int_E \left(1 - \left| \frac{\phi(u_1(t)) - \phi(u_2(t))}{u_1(t) - u_2(t)} \right|^2 \right) |u_1(t) - u_2(t)|^2 \, \mathrm{d}t = 0.$$

Hence we must have

$$\left| \frac{\phi(u_1(t)) - \phi(u_2(t))}{u_1(t) - u_2(t)} \right| = 1$$

for almost all $t \in E$. But this is impossible, since

$$\left|\frac{\phi(x)-\phi(y)}{x-y}\right| < 1$$

for any $x, y \in \mathbb{R}$ with $x \neq y$. Thus indeed $\mathrm{SRG}(N) \subseteq \mathbb{D}$.

We now show the $(N^{-1}+I)^{-1}$ does not have a finite incremental gain. Indeed, on its domain, we have

$$(N^{-1}+I)^{-1}(u)(t) = \psi^{-1}(u(t)),$$

where $\psi(x) = x - \tan(x)$. However, the function ψ^{-1} is not Lipschitz continuous at $x = 0$, since $\psi'(0) = 0$ so $(\psi^{-1})'(0)$ does not exist. The ratio

$$\frac{\left\|(N^{-1}+I)^{-1}(u_1) - (N^{-1}+I)^{-1}(u_2)\right\|}{\|u_1 - u_2\|}$$

can be made arbitrarily large, for example by taking $u_1(t) = au(t)$, $u_2(t) = u(t)$ for $a, b \neq 0$ small enough, where

$$u(t) := \begin{cases} 1 & t \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

⌟

V. Incremental IQCs

In this section, we give a second method of verifying condition (ii) of Theorem 2: satisfaction of an incremental IQC. This gives a incremental version of the classical IQC stability theorem [3, Theorem 1]. Our theorem is closely related to [9, Theorem 7.40], but does not rely on an extended space or assumptions of causality. The developments closely follow the non-incremental theory of [4], [14].

Corollary 2. *Let $H_1 : L_2 \to L_2$ be a bounded LTI operator, and $H_2 : L_2 \to L_2$ be incrementally bounded. Let $\Pi : j\mathbb{R} \to \mathbb{C}^{n\times n}$ be a Hermitian-valued function with L_∞ entries. Given $y_1, y_2 \in L_2$, define $\Delta\hat{y}(j\omega) := \hat{y}_1(j\omega) - \hat{y}_2(j\omega)$ and $\Delta\hat{H}_2(y)(j\omega) := \hat{H}_2(y_1)(j\omega) - \tau\hat{H}_2(y_2)(j\omega)$. Suppose that, for every $\tau \in [0,1]$ and every $y_1, y_2 \in L_2$, we have*

$$\int_{-\infty}^{\infty} \begin{pmatrix} \Delta\hat{y}(j\omega) \\ \tau\Delta\hat{H}_2(y)(j\omega) \end{pmatrix}^* \Pi(j\omega) \begin{pmatrix} \Delta\hat{y}(j\omega) \\ \tau\Delta\hat{H}_2(y)(j\omega) \end{pmatrix} \mathrm{d}\omega \geq 0, \quad (14)$$

and that there exists $\varepsilon > 0$ such that

$$\begin{pmatrix} \hat{H}_1(j\omega) \\ I \end{pmatrix}^* \Pi(j\omega) \begin{pmatrix} \hat{H}_1(j\omega) \\ I \end{pmatrix} \leq -\varepsilon I \quad (15)$$

for all $\omega \in \mathbb{R}$. Then $[H_1, H_2]$ maps L_2 to L_2 and has bounded incremental gain.

Equations (14) and (15) state that the graph of H_2 and the inverse graph of $-H_1$, respectively, satisfy the incremental IQC described by Π. In contrast to [3, Theorem 1], Corollary 2 does not require any well-posedness assumptions along the homotopy path. Before giving the proof of Corollary 2, we make a technical definition and prove two lemmas, on which the proof relies.

Definition 2. A functional $\sigma : L_2 \to \mathbb{R}$ is said to be *quadratically continuous* if, for every $\varepsilon > 0$, there exists $C > 0$ such that

$$\sigma(y) \leq \sigma(x) + \varepsilon\|x\|^2 + C\|x-y\|^2$$

for all $x, y \in L_2$. ⌟

The class of quadratically continuous functionals that we will use are characterized in the following lemma, which is a special case of [14, Lem. 3.18].

Lemma 6. *Let $\Pi : j\mathbb{R} \to \mathbb{C}^{n\times n}$ be a Hermitian-valued function with L_∞ entries. Then the functional $\sigma : L_2 \to \mathbb{R}$ defined by*

$$\sigma(x) := \langle x, x\rangle_\Pi$$
$$\langle x, y\rangle_\Pi := \mathrm{Re}\int_{-\infty}^{\infty} \hat{x}^*(j\omega)\Pi(j\omega)\hat{y}(j\omega)\,\mathrm{d}\omega$$

is quadratically continuous.

Proof. Note that $\langle\cdot,\cdot\rangle_\Pi$ defines a bounded Hermitian form.

$$\begin{aligned}
\sigma(x) - \sigma(y) &= \langle y, y\rangle_\Pi - \langle x, x\rangle_\Pi \\
&= \langle y, y\rangle_\Pi - \langle x, y\rangle_\Pi + \langle x, y\rangle_\Pi - \langle x, x\rangle_\Pi \\
&= \langle y-x, y\rangle_\Pi - \langle x, y-x\rangle_\Pi \\
&= \langle y-x, y\rangle_\Pi - \langle y-x, x\rangle_\Pi \\
&\quad + \langle y-x, y\rangle_\Pi + \langle x, y-x\rangle_\Pi \\
&= \langle y-x, y-x\rangle_\Pi + \langle y-x, x\rangle_\Pi + \langle x, y-x\rangle_\Pi.
\end{aligned}$$

Since $\langle\cdot,\cdot\rangle_\Pi$ is bounded, there exists $M \geq 0$ such that $\langle x, y\rangle \leq M\|x\|\|y\|$ for all $x, y \in L_2$. We therefore have

$$\sigma(x) - \sigma(y) \leq M\|x-y\|^2 + 2M\|x\|\|x-y\|. \quad (16)$$

Furthermore, for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(M\|x-y\| - \varepsilon\|x\|\right)^2 = \\
&\frac{1}{\varepsilon}M^2\|x-y\|^2 - 2M\|x\|\|x-y\| + \varepsilon\|x\|^2 \geq 0 \\
&\text{so} \quad 2M\|x\|\|x-y\| \leq \frac{1}{\varepsilon}M^2\|x-y\|^2 + \varepsilon\|x\|^2.
\end{aligned}$$

Combining with (16), we have

$$\sigma(x) - \sigma(y) \leq \left(M + \frac{1}{\varepsilon}M^2\right)\|x-y\|^2 + \varepsilon\|x\|^2. \qquad \square$$

We now show that a quadratically continuous functional can be used to verify incremental boundedness.

Lemma 7. *Consider $H_1, H_2 : L_2 \to L_2$ and assume H_1 has an incremental gain bound of λ. Let $\sigma : L_2 \to \mathbb{R}$ be quadratically continuous with constant C. For $u_i \in L_2$, let $y_i \in [H_1, H_2](u_i)$ and e_i denote $u_i - H_2(y_i)$. Suppose that, for all $u_1, u_2 \in \mathrm{dom}([H_1, H_2])$, we have*

$$\sigma(h_1) \leq -2\varepsilon\|h_1\|^2 \quad (17)$$
$$\sigma(h_2) \geq 0, \quad (18)$$

where $h_1 := (y_1 - y_2, e_1 - e_2)$, $h_2 := (y_1 - y_2, H_2(y_1) - H_2(y_2))$. Then there exists $\lambda > 0$ such that, for all u_1, u_2, we have

$$\|y_1 - y_2\| \leq \sqrt{\frac{C}{\varepsilon}\left(\frac{\lambda^2}{1+\lambda^2}\right)}\|u_1 - u_2\|.$$

Proof. Given two signals x_1, x_2, let $\Delta x := x_1 - x_2$. We have

$$\begin{aligned}0 &\leq \sigma(h_2)\\ &\leq \sigma(h_1) + \varepsilon \|h_1\|^2 + C\|h_2 - h_1\|^2\\ &\leq -\varepsilon\|h_1\|^2 + C\|h_2 - h_1\|^2\\ &= -\varepsilon(\|\Delta y\|^2 + \|\Delta e\|^2) + C(\|\Delta H_2(y) - \Delta e\|^2)\\ &= -\varepsilon(\|\Delta y\|^2 + \|\Delta e\|^2) + C\|\Delta u\|^2\\ &\leq -\varepsilon\left(1 + \frac{1}{\lambda^2}\right)\|\Delta y\| + C\|\Delta u\|^2,\end{aligned}$$

from which the result follows. □

Proof of Corollary 2. Define $\langle\cdot,\cdot\rangle_\Pi$ as in Lemma 6. It follows from that lemma that $\sigma(x) := \langle x, x\rangle_\Pi$ is quadratically continuous. Equation (14) gives condition (18) of Lemma 7. We now show that Equation (15) gives condition (17). Indeed, pre- and post-multiplying with $\hat{u}^*(j\omega)$ and $\hat{u}(j\omega)$, respectively, and integrating over ω, gives

$$\int_{-\infty}^{\infty}\begin{pmatrix}\hat{H}_1(j\omega)\hat{u}(j\omega)\\ \hat{u}(j\omega)\end{pmatrix}^*\Pi(\omega)\begin{pmatrix}\hat{H}_1(j\omega)\hat{u}(j\omega)\\ \hat{u}(j\omega)\end{pmatrix}\mathrm{d}\omega \leq -\varepsilon\|\hat{u}\|^2.$$

Now let $\alpha > 0$ be a gain bound for H_1. Then we can write $-\varepsilon\|u\|^2 \leq -2\bar{\varepsilon}(\|u\|^2 + \|H_1(u)\|^2)$ for some $\bar{\varepsilon} \leq \varepsilon/(2(1+\alpha))$. Since H_1 is LTI, this non-incremental condition is then equivalent to the incremental condition (17).

We finally note that the gain bound given by Lemma 7 depends only on σ and the incremental gain of H_1, and not on τ. The conclusions of the corollary then follow from Theorem 2. □

VI. Relaxing the assumption of incremental boundedness

Much of the existing stability literature focuses on the verification of non-incremental stability, and assumes weaker non-incremental boundedness properties for the components. In exchange, systems must be assumed to be well-posed, in the sense of causality and existence of solutions in an appropriate ambient space, containing L_2 but allowing unbounded signals. In this section, we show that incremental stability can also be verified under these weaker assumptions, subject to the same well-posedness assumptions as in a typical non-incremental analysis. We begin by introducing the extended L_2 space.

Given $T > 0$, denote by $P_T : \mathcal{F} \to \mathcal{F}$ the truncation operator

$$P_T(u)(t) := \begin{cases} u(t) & t < T\\ 0 & t \geq T.\end{cases}$$

The *extended L_2 space*, L_{2e}, is defined as the subset of $\mathcal{F}$ such that $P_T u \in L_2$ for all T. An operator $H : L_2 \to L_2$, or $H_e : L_{2e} \to L_{2e}$, is said to be *causal* if $P_T H P_T = P_T H$ for all $T > 0$.

A negative feedback interconnection is said to be *well-posed* if, for any $u \in L_{2e}$, there exist unique $e, y \in L_{2e}$ satisfying (6)–(7).

Theorem 3. *Suppose*

(i) $H_1, H_2 : L_2 \to L_2$ *have finite gain with zero offset and are causal;*

(ii) $[H_1, \tau H_2]$ *is well-posed and causal for all* $\tau \in (0, 1]$*;*

(iii) there exists $\gamma > 0$ *such that, for all* $\tau \in [0, 1]$ *and all* $u_1, u_2 \in \operatorname{dom}([H_1, \tau H_2])$*, we have*

$$\|y_1 - y_2\| \leq \gamma\|u_1 - u_2\|,$$

where $y_i = [H_1, \tau H_2](u_i), i = 1, 2$.

Then $[H_1, H_2]$ *maps* L_2 *to* L_2 *and has finite incremental gain.*

This theorem provides a middle ground between the incremental Theorem 2 and classical homotopy results such as [3, Theorem 1].

Proof of Theorem 3. The proof mirrors that of Theorem 2, but replacing the incremental small gain theorem with its non-incremental version – see, for example, [6, Theorem 1, p. 41], with the modified condition suggested in Equation (8c) on the same page. □

As in the case of Theorem 2, condition (iii) of Theorem 3 can be verified using SRG separation or incremental IQCs. In the case of SRG separation, we have the following result, the proof of which is similar to Corollary 1. The corresponding result for IQCs is similary to Corollary 2 but incorporates the assumptions of Theorem 3.

Corollary 3. *Suppose* $H_1, H_2 : L_2 \to L_2$ *have finite gain with zero offset. Suppose that* H_1 *and* H_2 *have strictly separated SRGs. Then the feedback interconnection* $[H_1, H_2]$ *maps* L_2 *to* L_2 *and has bounded incremental gain. Suppose*

(i) $H_1, H_2 : L_2 \to L_2$ *have finite gain with zero offset and are causal;*

(ii) $[H_1, \tau H_2]$ *is well-posed and causal for all* $\tau \in (0, 1]$*;*

(iii) H_1 *and* H_2 *have strictly separated SRGs.*

Then $[H_1, H_2]$ *maps* L_2 *to* L_2 *and has finite incremental gain.*

Non-incremental finite gain can be verified using separation of (non-incremental) Scaled Graphs, as shown in [20, Thm. 4].

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