

Mathematical study of long-range interactions in kinetic equations

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This dissertation is submitted on March, 2025 for the degree of Doctor of Philosophy

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted, or is being concurrently submitted, for any degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

Amélie Justine Loher
March, 2025

Abstract

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On the one hand, we are concerned with the dynamics of gas particles, captured by a partial differential equation named after Ludwig Boltzmann. On the other hand, we study the dynamics of charged particles, described through an equation derived by Lev Landau. A solution of these equations encodes the density of gas or charged particles, respectively, which at a given time are located at a certain point in space and travel at a certain velocity. Both equations are written with a transport operator on the left hand side, which accounts for the trajectories travelled by a particle in absence of any external forces. The right hand side describes the fluctuations in velocity resulting from particle collisions. We investigate when these equations derived from physical principles are rigorously justifiable.

We focus on long-range interactions, where there is a high frequency of interactions at small deviation angle. These particle interactions induce fluctuations in velocity that generate a Lévy process, which explains the regularisation effect that takes place. We capture this regularisation effect on the behaviour of solutions in form of a priori estimates.

Boltzmann In the case of the Boltzmann equation, we assume the underlying potential to be moderately soft, and we work in a conditional regime, where we assume that solutions admit bounded mass, energy and entropy. In this regime, the Boltzmann equation can be written as a transport operator and a non-local diffusion operator with rough coefficients that are elliptic on average.

In the first part, we explain why solutions are Hölder continuous. We derive the Harnack inequalities for solutions to the Boltzmann equation, and we discuss Schauder theory: the gain of regularity in Hölder spaces from the coefficients onto the solution. In particular, a bootstrapping argument permits to deduce that solutions to the Boltzmann equation are smooth, provided that the associated macroscopic quantities such as mass, energy and entropy are bounded. Therefore, if a singularity forms and the equation was to blow up, we would observe this phenomenon on the level of physically relevant quantities. Finally, we discuss how to obtain bounds on the asymptotic behaviour of the fundamental

solution associated to the equation. In short: we evolve around the study of local regularity properties of solutions to non-local hypoelliptic equations.

In a second part, we discuss the behaviour of solutions on the boundary. To this end, we introduce a suitable notion of solutions that is weak enough to be constructed in the space-homogeneous case, and moreover, is able to treat physically reasonable boundary assumptions. We show that these suitable weak solutions to the inhomogeneous non-cutoff Boltzmann equation with moderately soft potentials generate any pointwise polynomial velocity decay.

Landau Finally in the last part, we move away from hypoelliptic structures, but we consider the most singular of all long-range interactions: we study the space-homogeneous Landau equation with Coulomb interactions. We construct smooth solutions with initial data in sub-critical Lebesgue spaces, first global in time but in a perturbative regime, then local in time for large data. Then, building on the work of Guillen-Silvestre, we construct global smooth solutions for rough, slowly decaying initial data in critical Lebesgue spaces, and finally, we discuss how to treat the super-critical regime.

Acknowledgements

First and foremost I want to thank Clément Mouhot. Clément knew exactly how he had to supervise me. His way of supervising matched my way of working. He let me develop independently; he in fact let me do whatever I wanted. In return, he wanted me to be proactive. There were months where we would not see or talk to each other. And there were other periods where we met every other day. Clément always knew when he had to step in. He was extremely supportive, he listens, and he laid the foundation of my mathematical thinking. In difficult periods he knew how to explain to me why I am not wasting my time, he knew how to prevent me from turning in circles, but foremost, he valued my work even when I stopped believing in it. He is able to turn the vaguest thought into an ingenious idea, he knows how to make one's mind speak, he knew how to read me, and he is of an untouchable humanity. *On ne sait jamais ce qu'on fait pour les autres. D'ailleurs, on ne fait pas. On est.*[1]

I am honoured to thank Pierre Raphaël and Luis Silvestre for accepting to examine this overly long thesis.

I also want to thank Annalaura Rebucci. It is hard to describe the role she played in my personal and mathematical life. She has an impressive personality, through whom I have hope, for myself, but for all women in mathematics more generally. I don't know anyone with such an irresistible energy as hers.

Next, I would like to thank William Golding for bringing coffee capsules to my office in exchange of a coffee machine, from which arose not only a fruitful collaboration, but also a friendship. It was through William that I met Maria Pia Gualdani, whom I want to thank for her hospitality at UT Austin, her support, and her endless kindness.

I thank Cyril Imbert for our collaboration at the DMA at ENS, Paris, and I am honoured to have collaborated with Laurent Desvillettes. I am grateful to François Golse, an extraordinary character and thoroughly impressive mathematician and I want to thank László Székheldyhidi, who generously invited me several times to the Max-Planck-Institute in Leipzig.

Let me also mention Lucas Ertzbischoff; Isabelle Gallagher; Alexis Vasseur; Helge Dietert; Amit Einav, who was never short of suggestions; Jo Evans; Christoph Kehle;

[1] Françoise Giroud, Leçons particulières.

Antonin Chodron de Courcel; Michael Struwe; Ben Fehrman, who invited me to my first talk in my life. And I want to thank Scott Armstrong, who opened up my perspectives.

I want to thank all researchers, all secretaries and all staff at DPMMS, in particular I would like to thank the head, Ivan Smith; and Julia Blackwell, the heart of the department. I also thank the other Ph.D. students around me, in particular Samuel, Lennie, and Robert.

I now turn to the cornerstone of my being, who is my family, without which I would never have come that far. I am endlessly grateful to Valérie, for her continuous accompaniment, et pour notre comédie française. I thank Alfons, for his unending encouragement, and for teaching me to not crack under pressure under any circumstance. I thank the angelic Alice, the most talented woman I know, for being wonderfully supportive with her surprising intuition. I thank Frédéric and Arthur, for showing me that no aim is too high. And I thank Jeremia, for his touching thoughtfulness. I thank the diplomatic Alessandro and the authentic Lea. I am grateful to Jocelyne Adrienne, whom I reencountered in Josée's tales; and I also want to thank Josée and Richard, through whom I was introduced to Parisian intellectuals of the '68, for the cafés au Luxembourg, for the dîners au-dessus du Pont Traversé, for explaining to me l'automatisme surréaliste, for Ramuz and Alyn, and for the impressive artwork du "grand" Richard. Finally, I want to thank Michael and Maria, who literally enabled me to fly.

Let me conclude by thanking all my friends; in particular Noëlle, for her constancy, for her profundity and for sharing our lives; and Daria, for her inspiring character, who openly carries her heart in her hands, who makes sure I never stop laughing, and for whom I have an unbounded admiration. And finally R, for his genuineness; *wir existieren ja nur in Beweisen, und verlieren alles wieder an die Finsternis.*

Was wir das Vollendete in der Kunst nennen,
bringt nur von neuem das Unvollendete in Gang.

Ingeborg Bachmann

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Notation

We use the floor function $\lfloor a \rfloor$ for $a \in \mathbb{R}$ to denote the greatest integer $k \in \mathbb{Z}$ such that $k \leq a$. The positive part a_+ of $a \in \mathbb{R}$ is denoted by $a_+ = \max\{a, 0\} \geq 0$. Correspondingly, for the negative part a_- we write $a_- = \min\{a, 0\} \leq 0$. In particular, any a can be written as $a = a_+ + a_-$. We further use the abbreviation $a \lesssim b$ for $a, b \in \mathbb{R}$ if there exists a constant $C > 0$ such that $a \leq Cb$. Similarly, $a \gtrsim b$ denotes $a \geq Cb$ for some $C > 0$. Moreover, we say that $a \lesssim_d b$, if $a \leq Cb$ where $C = C(d)$. Finally, $a \sim b$ if $a \lesssim b$ and $a \gtrsim b$.

We denote by $B_r(v) \subset \mathbb{R}^d$ a ball centred at $v \in \mathbb{R}^d$ with radius $r > 0$. The volume of the unit sphere of $\mathbb{R}^d$ is denoted by ω_{d-1}. We use the Japanese bracket convention to denote $\langle v \rangle := (1 + |v|^2)^{\frac{1}{2}}$. We denote by $\mathbf{1}_d$ the d-vector of 1's, by $\mathbf{0}_d$ the d-vector of 0's, by Id_d the $d \times d$ identity matrix, and by $\mathbb{O}$ a zero block of arbitrary size.

We write $z = (t, x, v)$ for an element of $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. In general, our domains $Q \subset \mathbb{R}^{1+2d}$ are cylinders in time, space and velocity. If we slice the cylinder in time, we denote it by $Q^t \subset \mathbb{R}^{2d}$, that is $Q^t = \big(\{t\} \times \mathbb{R}^d \times \mathbb{R}^d\big) \cap Q$. We let $n = 2s + 2d(s+1)$ with $s \in (0, 1)$ denote the total dimension respecting the scaling of the equation.

For a given domain $\Omega \subset \mathbb{R}^d$ and a given $0 < s < 1$, we denote by $\dot{H}^s(\Omega)$ the space that is equipped with the semi-norm

$$\|f\|^2_{\dot{H}^s(\Omega)} := \int_\Omega \int_\Omega \frac{|f(v) - f(w)|^2}{|v - w|^{d+2s}} \,\mathrm{d}w \,\mathrm{d}v.$$

The space $H^s(\Omega)$ is equipped with the norm $\|f\|^2_{H^s(\Omega)} := \|f\|^2_{\dot{H}^s(\Omega)} + \|f\|^2_{L^2(\Omega)}$. The space $H^s_0(\Omega) \subset H^s(\Omega)$ is defined as the closure of the space of smooth functions in $\mathbb{R}^d$ with compact support contained in Ω, where the closure is taken with respect to the $H^s(\Omega)$ norm. We denote the dual of $H^s_0(\Omega)$ by $H^{-s}(\Omega) \subset H_0^{-s}(\Omega)$.

Finally, the transport operator $\mathcal{T}$ along $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ is denoted by

$$\mathcal{T} := \partial_t + v \cdot \nabla_x.$$

The bilinear form associated to the non-local operator $\mathcal{L}$ with a kernel $K \geq 0$ is given by

$$\mathcal{E}(f, g) = -\int_{\mathbb{R}^d} \mathcal{L}f(v) g(v) \,\mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(f(v) - f(w)\big) g(v) K(v, w) \,\mathrm{d}w \,\mathrm{d}v. \tag{1}$$

CHAPTER 1

INTRODUCTION

1.1 Kinetic theory

1.1.1 Mesoscopic scale

According to Newton's laws, in a system of interacting particles, the motion of each particle is encoded by a Hamiltonian describing its location and velocity. The number of particles in a physical entity is often intractable; and it turns out to be more useful to apply Newton's laws to the infinitesimal volume element, in which case one ends up with hydrodynamic equations, such as Euler's equations for non-viscous fluids or Navier-Stokes equations for viscous fluids.

There is an intermediate regime between the microscopic description of the particle trajectories, and the macroscopic description for the evolution of fluid-dynamical quantities: at the so-called *mesoscopic scale*, the idea is to study the dynamics of the *time-, space-, and velocity-dependent density of all particles*, [28, 156]. More precisely, from the Hamiltonian system encoding the trajectories of all particles we extract the Liouville equation on the joint probability density over the space and velocity domain. In a certain scaling limit, known as the *Boltzmann-Grad limit*, where the number of all particles is sufficiently large, and the volume of each particle sufficiently small, we end up with an equation capturing the evolution of the probability density for a single particle, provided that we assume the joint probability densities to decouple in the limit (*molecular chaos*).

At this intermediate level, we have on the one hand an equation that is more tractable than the microscopic description. On the other hand, we keep more information than at the macroscopic level, and the equations at the intermediate scale turn out to be richer in structure than their fluid-dynamical limit. The dynamics of particles at this mesoscopic scale is the content of *kinetic theory*.

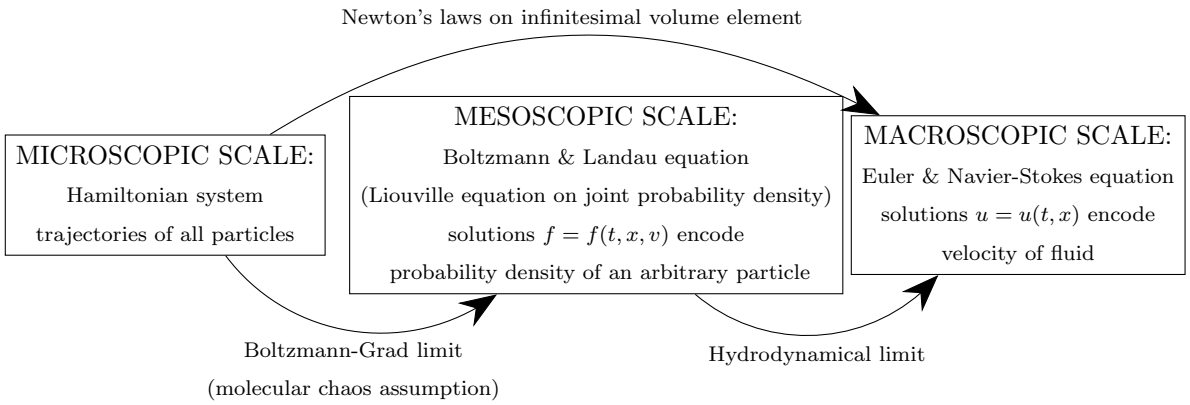


Figure 1.1: Different scales at which we can describe particle dynamics. Further discussions on these scales can be found in [19].

1.1.2 Particle interactions

The probability density of the gas is constant along particle paths that are determined by Newton's laws. In absence of any external forces, particles travel in straight lines. This is described with a transport operator, which accounts for the temporal evolution and the spatial movement, acting on the density f at time $t \geq 0$ and at location $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$:

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = 0.$$

Here ∂_t denotes the partial time derivative and ∇_x the spatial gradient. In order to account for the interaction between particles, we have two options:

Either we model the interaction through a mean-field self-generated force acting on the particles. This is the collision-less regime, introduced by Jeans in an astronomical context [122], where typically the equations are of the form

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) - \nabla_x \Psi(t, x) \cdot \nabla_v f(t, x, v) = 0, \qquad \Psi(t, x) = V(x) *_x \rho(t, x),$$

with interaction potential V and local density ρ of the particles given by

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) \, \mathrm{d}v.$$

The non-linearity enters through the dependence of the force Ψ on the macroscopic quantity ρ associated to the density f. One instance is the gravitational potential, studied by Jeans [122]; another instance is Coulomb interactions, introduced by Vlasov [189]. In both cases the potential is given by

$$V(x) = \frac{C}{|x|},$$

but in the former case, $C \in \mathbb{R} \setminus \{0\}$ depends on the gravitational constant and the point mass, whereas in the latter case, $C \in \mathbb{R} \setminus \{0\}$ depends on the charge of the particles and on the permeability of the medium. These are known as Vlasov-Poisson equations.

Alternatively, a binary interaction between particles is modelled as a *collision operator*, which we denote by $Q(f, f)$, so that the equation takes the form

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f, f)(t, x, v).$$

The collision operator Q, which will be discussed in detail further below (1.3), is derived from the conservation of energy and momentum of the underlying system subject to a suitable interaction potential. Common interaction potentials are *inverse power laws*

$$V(x) = \frac{C}{|x|^{\nu-1}}, \qquad \nu \in [d-1, +\infty]; \tag{1.1}$$

if $\nu = d - 1$, the underlying potential is a *Coulomb interaction*; in case that "$\nu = +\infty$", we formally model the interaction of *hard spheres*.

The choice between these two approaches is driven by the interaction range. Long-range interactions are better modelled by a mean-field; whereas interactions at a sufficiently short range are described by a collisional model. In three dimensions, an interaction based on a collision between particles can be shown to dominate the mean-field only if $\nu > 3$; for $\nu \in [2, 3)$, the collision term can be neglected with regard to the mean-field [27]. The borderline case where $\nu = 3$ is called Manev interaction. On longer time scales, binary collisions still play a non-negligible role even for $\nu \in [2, 3]$; they can be viewed as a correction to the mean-field.

Later, Kac [124] and McKean [157] aimed to derive the spatially inhomogeneous collisional model from a many-particle jump process by introducing a molecular chaos assumption, which they prove to propagate under an ergodicity hypothesis; we refer the reader to [162] and references therein.

In this thesis, we are concerned with the collisional model; even though some of the results below do allow for an additional drift term, which could under suitable assumptions account for the mean-field of collision-less models.

1.2 Boltzmann

The most fundamental collisional model is the Boltzmann equation discovered by Maxwell in 1867 [156], which led Boltzmann to introduce a measure for the likelihood of a macrostate in terms of microstates: the *entropy* [28]. He realised that if we observe a causality in the macrostates then the entropy has to increase in proportion to the volume of the microstates. The resulting Boltzmann equation describes the dynamics of a dilute gas from a statistical

viewpoint, and is given by

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q\big(f,f\big)(t,x,v), \tag{1.2}$$

where $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ encodes the probability density of the gas particles, which at any given time $t \in \mathbb{R}$ have location $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. Here and in the sequel, we abuse notation by abbreviating $f(v) = f(t,x,v)$. The right hand side Q denotes the Boltzmann collision operator, whose explicit form is given by

$$Q(f,f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \big[f(w_*)f(w) - f(v_*)f(v)\big] B(|v-v_*|, \cos\theta)\, \mathrm{d}\sigma\, \mathrm{d}v_*, \tag{1.3}$$

where

$$\cos\theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma,$$

and where v, v_* are the post-, and w, w_* the pre-collisional velocities, specifically,

$$w = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad w_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

These velocities are solutions of the algebraic system corresponding to the conservation of momentum and kinetic energy, parametrised by $\sigma \in \mathbb{S}^{d-1}$. The rate of change in velocities is captured in the cross-section B, which we discuss next.

B counts the frequencies of particle collisions for an underlying interaction potential. Determining B involves solving Newton's equations from the conservation of energy and momentum dynamically. In case of inverse power-law potentials as in (1.1), it turns out that the *kinetic collision kernel*, denoted by Φ, decouples from the *angular collision kernel*, b, so that the cross-section is of the form

$$B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|) b(\cos\theta). \tag{1.4}$$

If "$\nu = +\infty$", which formally corresponds to the case of hard spheres, then $\Phi(r) = r$, and $b(\cos\theta) = const.$ For $\nu \in [d-1, +\infty)$, we find for the kinetic collision kernel

$$\Phi(r) = r^\gamma, \qquad \gamma = \frac{\nu - (2d-1)}{\nu - 1} \in \left[\frac{-d}{d-2}, 1\right),$$

and for the angular collision kernel, as $\theta \sim 0$,

$$b(\cos\theta) \sim \frac{1}{\sin^{d-2}(\theta)\theta^{1+2s}}, \qquad s = \frac{d-1}{2(\nu-1)} \in \left(0, \frac{d-1}{2(d-2)}\right].$$

We will see below that $2s$ determines the order of the equation. Notably, inverse power

laws satisfy in any dimension

$$\gamma + 4s = 1.$$

In the case that $\nu = d - 1$ corresponding to Coulomb interactions the Boltzmann kernel is ill-defined if $d \leq 3$: in two dimensions, we cannot make sense of the Boltzmann kernel; in three dimensions, B is not integrable; but in higher dimensions, $d \geq 4$, B is well-defined. In 1936, Landau derived a collision operator for *charged particles* [144] by refining the computations of Maxwell in [156]; in particular, Landau's collision operator (see (1.40) below) is well-defined even for Coulomb interactions when $\nu = d - 1$. Formally, it corresponds to the so called *grazing collision limit of the Boltzmann operator*, with the highest frequency in collisions occurring at small deviation angle.

1.2.1 Potentials

We then distinguish

i. Hard potentials: $\gamma > 0$,

ii. Maxwell molecules: $\gamma = 0$,

iii. Moderately soft potentials: $\gamma < 0$, and $\gamma + 2s \geq 0$,

iv. Very soft potentials: $\gamma + 2s < 0$.

The potential determines the frequencies of interactions at a specific range.

1.2.2 Grad's cutoff assumption

The angular collision kernel is integrable over the sphere only in case of hard spheres. In all other cases, the collision kernel B in (1.4) is singular, and thus Grad [89] introduced what is now known as *Grad's cutoff assumption*:

$$\int_{\mathbb{S}^{d-1}} b(k \cdot \sigma) \, \mathrm{d}\sigma < \infty. \tag{1.5}$$

This assumption is reasonable for hard spheres. For soft potentials the regularising properties of the collision kernel are related to its singularity and since we aim to exploit its smoothing properties, we *do not cut off the singularity.* The cutoff is not physical; in fact it cuts off all long-range interactions. Past literature mainly assumed a cutoff, until the collision kernel (1.3), tested against a solution and integrated over the velocity domain, was related to the H^s_v-norm in the non-cutoff case in [3]: they discovered the coercivity of the collision operator. This was subsequently used to study the space-homogeneous Boltzmann equation. But it was only with Silvestre's work [177] that the non-cutoff kernel

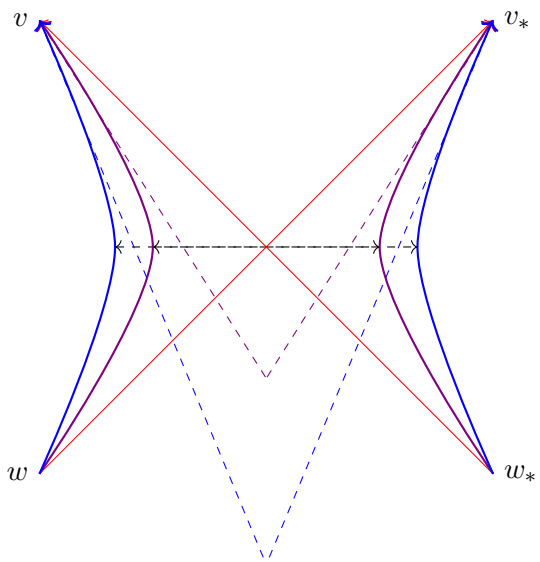


Figure 1.2: A collision of two particles with pre-collisional velocities w, w_* that change their velocities into v, v_*, respectively. The longer the range of interaction, the smaller the deviation angle.

was investigated from a completely different angle, without relying on Fourier analysis, where the regularisation exploits the coercivity properties stemming from the boundedness of certain macroscopic quantities associated to the equation, culminating in the work of Imbert-Silvestre [118].

1.2.3 Long-range interactions

Each interaction is classified by its range: the longer the interaction range, the higher is the proportion of collisions at small deviation angle (named *grazing collisions*). The most extreme case where particles only interact upon contact is the model of hard spheres. This is the shortest range interaction. As the interaction range becomes larger, the deviation angle becomes smaller; which happens more frequently in soft potentials (i.e. $\gamma < 0$). The collision operator B counts the frequencies of each interaction, which explains why the non-cutoff collision operator generates a Lévy process instead of a pure jump process. We focus our attention on *long-range interactions*, and wish to exploit their smoothing properties.

1.2.4 Summary

We summarise: in this thesis we are concerned with the non-cutoff Boltzmann collision kernel for moderately soft potentials, which is given in (1.3), where

$$B(r, \cos\theta) = r^\gamma b(\cos\theta), \quad b(\cos\theta) \sim_{\theta\sim 0} \left|\sin\left(\frac{\theta}{2}\right)\right|^{-(d-1)-2s}, \tag{1.6}$$

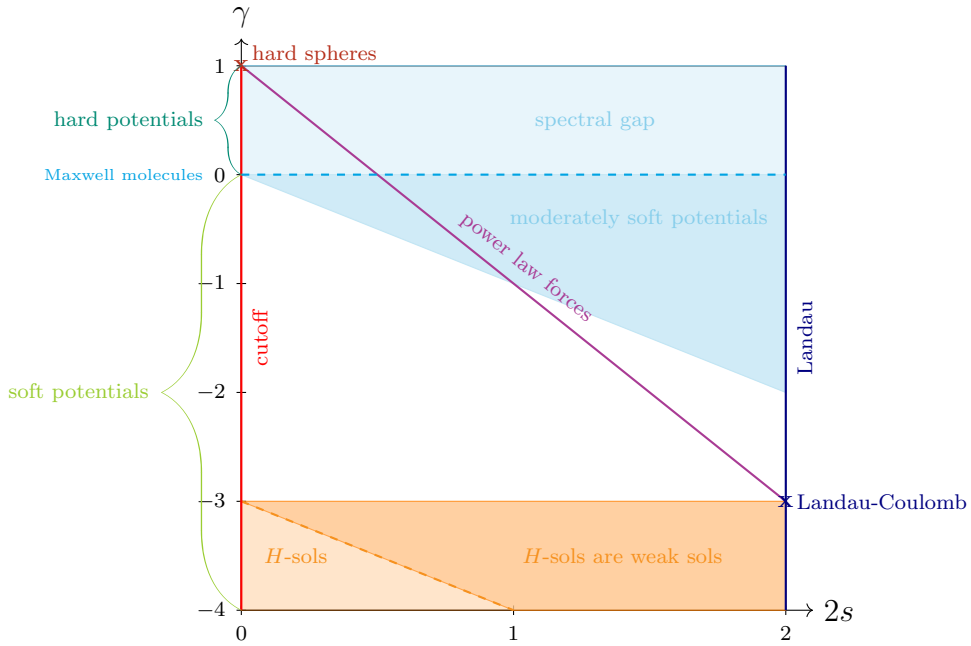


Figure 1.3: A summary of the parameters γ defining the integrability in relative velocities, and $2s \in (0, 2]$ defining the angular integrability in the Boltzmann collision kernel, for inverse power-law potentials (1.1) with $\nu \in [2, \infty]$, in $d = 3$. Power law forces satisfy $\gamma + 4s = 1$; these are the only physical meaningful potentials. The range of parameters away from this line is mathematical playground. (We credit Villani's conference talk at the Festum Pi for a sketch of this figure.)

with parameters $\gamma \in (-3, 1)$ and $s \in (0, 1)$ if $d = 3$, and where

$$\cos\theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad \sin\left(\frac{\theta}{2}\right) := \frac{w - v}{|w - v|} \cdot \sigma.$$

This implies that there exists constants $C_\pm$ such that

$$0 < C_- \, |\theta|^{-d+1-2s} \leq b(\cos\theta) \leq C_+ \, |\theta|^{-d+1-2s} \, .$$

More precisely, with no loss in generality, we can reformulate this as

$$b(\cos\theta) = |v - w|^{-(d-1)-2s} \, |v - v_*|^{d-2-\gamma} \, |v - w_*|^{\gamma+2s+1} \, \tilde{b}(\cos\theta), \tag{1.7}$$

where $\tilde{b}(\cos\theta)$ is such that $0 < \tilde{b}_- < \tilde{b}(\cos\theta) < \tilde{b}_+$ for constants $\tilde{b}_\pm > 0$, see [177]. We restrict our analysis to *moderately soft potentials*, which means we consider $0 \leq \gamma + 2s \leq 2$ throughout this thesis, and moreover, we are interested in *long-range interactions*, which occur when we do *not cut off* the singularity in the kernel at sufficiently small deviation angles θ.

1.3 Bird's-eye view

This thesis continues parts of the recent programme [118] in kinetic theory to study the regularisation effects of long-range interactions stemming from the singularity in the collision kernel. Up to present, most works in the literature on the Boltzmann equation were concerned with the equation under Grad's cutoff assumption (1.5), since it was not clear how to make sense of the singular kernel in (1.3). This assumption is physically reasonable for hard spheres, in which case the interaction range is minimal and the particles interact only upon contact, so that the effect of the singularity is negligible. When one is interested in soft potentials, however, information is lost under Grad's cutoff assumption: all long-range interactions are lost. This is physically not justified. In [3] they discovered the coercivity of the non-cutoff kernel. When the interaction range is longer, the deviation angle is smaller, leading to the effect that the action on the velocity fluctuations in the particles smooths out the density in the velocity variable. This is precisely what we aim to study.

The first work in this direction originates in Luis Silvestre's article [177], who quantified the ellipticity of the kernel (1.3) for soft potentials without cutoff assumption in terms of the boundedness of certain macroscopic quantities associated to the probability density of the particles. To this end, Silvestre introduced the *conditional regime*, where these macroscopic quantities, specifically the mass, energy, and entropy, respectively given for some $(t,x) \in (0,T) \times \Omega$ by

$$\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad \int_{\mathbb{R}^d} f(t,x,v)\,|v|^2\,\mathrm{d}v, \qquad \int_{\mathbb{R}^d} f(t,x,v)\log f(t,x,v)\,\mathrm{d}v, \tag{1.8}$$

are assumed to be bounded (see (1.20)). In this conditional regime, he deduced that the solution itself is a-priori bounded; which in turn determined a notion of ellipticity on the collision kernel (1.3). This was the starting point of his programme with Cyril Imbert, where they deduce that any given solution of the Boltzmann equation submitted to the macroscopic assumption is Hölder continuous, satisfies Schauder estimates, and, through a bootstrapping mechanism, is thus smooth [118]. Their results are crucially based on the hydrodynamical bounds (stated below in (1.20)). The boundedness of the solution to the Boltzmann equation implied by the conditional regime also permits to deal with the non-linearity in the equation: the non-linearity is absorbed as the supremum of the solution convolved with a velocity weight (and thus as the quantities from (1.8)) in the ellipticity constants. The non-linearity plays its role in the first step to quantify the ellipticity, and in the last step to bootstrap the Schauder theory. But the core of the regularity programme established by Imbert-Silvestre on the non-linear Boltzmann equation is ultimately *a linear theory on rough coefficients*.

It seems currently out of reach to relax the conditional regime (1.20) to a physically justifiable assumption. There are settings where mass, energy and entropy defined in (1.8) are naturally bounded, for instance in the space-homogeneous case, when the density $f = f(t, v)$ does not depend on the location x, since in this case, these quantities are conserved by the equation. Yet it is a substantial physical simplification to restrict the study to the space-homogeneous setting. In the space-inhomogeneous case, the macroscopic assumption (see (1.20)) would be reasonable if one weakened it to hold for the L^1 norm in the spatial variable x, since these quantities are conserved by (1.2), in contrast to the quantities in (1.8) defined at any point $x \in \mathbb{R}^d$. We don't think there is any reason to expect the existence of a global classical solution to the space-inhomogeneous Boltzmann equation with moderately soft potentials that admits bounded mass, energy and entropy pointwise in space x; could it be conceivable from a physical standpoint that a gas density admits points in space, where the energy blows up? This is unclear to us; but the reason why the programme established by Imbert-Silvestre [118] is interesting in any case can be explained by its physical implications: if a solution to the space-inhomogeneous Boltzmann equation forms a singularity in finite time, then this would be observable in the macroscopic relevant quantities given in (1.8).

Our thesis connects to this regularity programme by providing constructive proof methods where Imbert-Silvestre use non-quantitative arguments. We establish *quantitative* methods to prove Hölder continuity, Schauder estimates, and Harnack inequalities. We further establish the strong Harnack inequality, not previously known, which states a linear bound on the supremum of the solution in the past by the infimum in the future, provided that the equation is satisfied in the whole velocity space; from which we deduce the asymptotic behaviour of the fundamental solution to the linearised equation. These are all *interior regularity results*: we assume the equation to hold in some domain, within which we demonstrate regularity properties away from the boundary. All our proofs are constructive, and rely on the (average) divergence form of the equation. In this sense, we extend *Hilbert's 19th problem to hypoelliptic equations*, apart, of course, from the fact that we need to assume existence of global solutions to the inhomogeneous model. We refer to Section 1.5 for more details. We mention briefly, however, that *hypoellipticity* - a concept introduced by Hörmander [111] - refers to a property common in kinetic collisional models, where the vector fields of the equation span the whole space: the velocity gradient commuted with the transport operator produces the spatial gradient. Hypoellipticity is responsible for the transfer of regularity gained in the velocity variable onto the spatial variable, and thus the smoothing in kinetic models occurring from the fluctuations in velocity impacts the spatial variable as well. This regularity transfer occurs along trajectories built from the underlying vector fields in the equation: the velocity gradient and the transport operator. The construction of these trajectories is explicit and

it is thus a constructive approach to study the gain of regularity in hypoelliptic equations. The first attempt in this direction was a work of Guerand-Mouhot for a second-order hypoelliptic equation [99], where they follow the vector fields along a piecewise affine path. We developed this approach for the non-local hypoelliptic case in [152], which we then improved collaboratively in [10] by solving the control problem associated to the equation that resulted in a smooth curve along which regularity can be transferred from the velocity variable onto the spatial variable. This line of research, where the *trajectorial structure of hypoelliptic equations* is exploited, is in our opinion the most interesting novelty in this thesis, with potential for further developments.

In a second part (see Part II), we address another aspect of the regularity results in Imbert-Silvestre (and of Part I below). These results hold merely for *classical solutions*. If we are interested in bridging the gap from a-priori estimates to the construction of solutions, then it would be desirable to work with a notion of solution that is sufficiently *weak* to facilitate existence results. In the spirit of the weak solutions to the cutoff Boltzmann equation constructed by DiPerna-Lions [68], we consider a non-linear change of variables based on convex functionals. Together with *Cyril Imbert*, we introduced a suitable notion of weak solutions that uses the integrability from the macroscopic bounds (1.20) combined with an entropy production estimate. In the spatially homogeneous case, these solutions are known to exist since the work of Villani on H-solutions [186]. In the spatially inhomogeneous case, these solutions may be defined on bounded spatial domains $\Omega \subset \mathbb{R}^d$ equipped with boundary conditions introduced by Maxwell [156]. We show that suitable weak solutions of the Boltzmann equation with moderately soft potentials on bounded space domains *generate a decay at any polynomial rate* in velocity at an explicit temporal decay rate. To complete the regularity programme for weak solutions, it remains an open problem to establish Hölder continuity and Schauder estimates for our notion of weak solutions.

Demonstrating estimates on objects that are not known to exist is precisely the content of *a-priori estimates*. It is justified since such estimates are necessary to construct the objects themselves, at least on a finite time horizon, typically through limiting procedures, compactness arguments and fixed point theorems. The first two parts of the thesis are not concerned with existence proofs, but merely focus on the regularity theory.

In Part III, we move to the particle interaction with the longest range: the Landau equation. Collisions at small deviation angle are called *grazing collisions*; these produce a regularisation effect, since the velocity fluctuations are not pure jump processes but the particles' velocities change according to a Lévy process. The Landau collision operator can be obtained as the grazing collision limit of the Boltzmann operator, and it is thus the interaction with the highest frequency in long-range interactions; in Figure 1.3 it is represented by the right edge of the square. We are concerned with the physically most

relevant case, which are Coulomb interactions. For the spatially homogeneous equation, we construct global smooth solutions for general, rough initial data with arbitrarily slow decay. This is a series of works with *William Golding*, *Maria Gualdani*, and *Laurent Desvillettes*.

1.4 Carleman's reformulation of Boltzmann

The Boltzmann equation is a kinetic integro-differential equation with a bounded, non-negative source term. To see this, we split the collision operator Q into two parts

$$Q(f,f) = Q_1(f,f) + Q_2(f,f),$$

where

$$Q_1(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(w_*) \left[f(w) - f(v) \right] B(|v - v_*|, \cos\theta) \, \mathrm{d}\sigma \, \mathrm{d}v_*,$$

and

$$Q_2(f,f) = f(v) \left(\int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left[f(w_*) - f(v_*) \right] B(|v - v_*|, \cos\theta) \, \mathrm{d}\sigma \, \mathrm{d}v_* \right).$$

Then, as shown in [177, Lemma 4.1], Q_1 can be rewritten as

$$Q_1(f,f) = \int_{\mathbb{R}^d} \left[f(t,x,w) - f(t,x,v) \right] K_f(t,x,v,w) \, \mathrm{d}w,$$

where the kernel K_f implicitly depends on the solution f and is given by

$$\begin{aligned} K_f(v,w) &= 2^{d-1} |w - v|^{-1} \int_{w' \perp w - v} f(v + w') B(r, \cos\theta) r^{-d+2} \, \mathrm{d}w' \\ &= 2^{d-1} |w - v|^{-(d+2s)} \int_{w' \perp w - v} f(v + w') \, |w'|^{\gamma + 2s + 1} \, \tilde{b}(\cos\theta) \, \mathrm{d}w', \end{aligned} \tag{1.9}$$

with $r^2 = |w - v|^2 + |w'|^2 = |v - v_*|^2$, $\cos\theta = \frac{w' - (v-w)}{|w' - (v-w)|} \cdot \frac{w' + (w-v)}{|w' + (w-v)|}$, $w_* = v + w'$, and $v_* = w + w'$. The function $\tilde{b}$ is bounded from above and below by positive constants, see (1.7). The proof is based on a change of variables to so-called *Carleman coordinates* [188] – see also [177]. This reformulation parametrises the collision by choosing the pre-collisional velocities as new variables. In this representation, Q_1 is a non-local operator with a non-negative kernel K_f; and the term Q_2 is a non-negative source term due to *cancellation effects*, which imply that Q_2 writes as

$$Q_2(f,f) = f(v) \left(\int_{\mathbb{R}^d} f(v - w') \tilde{B}(|w'|) \, \mathrm{d}w' \right),$$

where

$$\tilde{B}(r) := \int_{\mathbb{S}^{d-1}} \left(\frac{2^{\frac{d}{2}}}{(1-\sigma\cdot e)^{\frac{d}{2}}} B\left(\frac{\sqrt{r}}{\sqrt{1-\sigma\cdot e}}, \cos\theta \right) - B(r,\cos\theta) \right) \mathrm{d}\sigma > 0. \tag{1.10}$$

Here e is any unit vector, and $\cos\theta = \sigma \cdot e$. The cancellation lemma can for instance be found in [177, Lemma 5.1] or [3, Lemma 1]. The idea is that near the singularity $v_* \sim w_*$, so that $f(v_*) - f(w_*)$ cancels the singularity; therefore Q_2 becomes a bounded lower order term (with respect to Q_1).

1.4.1 Integro-differential equation with non-negative source

One arrives at the integro-differential form of the Boltzmann equation with a non-negative source term: if f solves (1.2), then

$$\begin{aligned} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) &= Q_1(f,f) + Q_2(f,f) \\ &= \int_{\mathbb{R}^d} \big(f(t,x,w) - f(t,x,v)\big) K_f(t,x,v,w)\,\mathrm{d}w + \Lambda_f f(t,x,v), \end{aligned} \tag{1.11}$$

where K_f is given by (1.9), and Λ_f is determined by

$$\Lambda_f := \int_{\mathbb{R}^d} f(v-w')\tilde{B}(|w'|)\,\mathrm{d}w' \geq 0, \tag{1.12}$$

with $\tilde{B}$ defined in (1.10).

This shows that the Boltzmann equation can be reformulated as a non-linear kinetic integro-differential equation of the form

$$\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) = \mathcal{L}f(t,x,v) + h(t,x,v), \quad t\in\mathbb{R},\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \tag{1.13}$$

where $h = h(t,x,v)$ is a real measurable scalar field and where $\mathcal{L}$ is defined by

$$\mathcal{L}f(t,x,v) := \mathrm{PV}\int_{\mathbb{R}^d} \big[f(t,x,w) - f(t,x,v)\big] K(t,x,v,w)\,\mathrm{d}w, \tag{1.14}$$

with a non-negative measurable kernel $K : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty)$. Here, PV denotes the Cauchy principal value, that is for any $G : \mathbb{R}^d \to \mathbb{R}$,

$$\mathrm{PV}\int_{\mathbb{R}^d} G(w)\,\mathrm{d}w = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B_\varepsilon(0)} G(w)\,\mathrm{d}w.$$

The coefficient K and the source h both depend on the solution itself, $K = K_f$, $h = h_f$, stressing the non-linear character of the equation. It turns out that in the kinetic theory of gases, equation (1.13) captures the dynamics of uncharged particles. We aim to derive

properties on solutions to the Boltzmann equation for moderately soft potentials without Grad's cutoff assumption by studying the *kinetic integro-differential equation* given in (1.13).

1.4.2 Integro-differential equation in divergence form on average

Another reformulation of the Boltzmann equation, which proves useful if we want to exploit the divergence structure hidden in the operator, is the following. If we reconsider the term Q_2, then a suitable change of variables (see [115, Lemma A.9]) together with (1.9) yields

$$\begin{aligned} Q_2(f,f) &= f(v)\left(\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \big(f(w_*)-f(v_*)\big)B(|v-v_*|,\cos\theta)\,\mathrm{d}\sigma\,\mathrm{d}v_*\right) \\ &= f(v)\Bigg(2^{d-1}\int_{\mathbb{R}^d}\int_{w'\perp w-v} |w-v|^{-1}\,B(r,\cos\theta)r^{-d+2} \\ &\qquad\qquad \times \big(f(v+w')-f(w+w')\big)\,\mathrm{d}w'\,\mathrm{d}w\Bigg) \\ &= f(v)\left(\int_{\mathbb{R}^d}\big(K_f(v,w)-K_f(w,v)\big)\,\mathrm{d}w\right). \end{aligned} \tag{1.15}$$

Thus any solution of (1.2) satisfies

$$\begin{aligned} &\partial_t f(t,x,v)+v\cdot\nabla_x f(t,x,v) \\ &\quad= \int_{\mathbb{R}^d}\big(f(t,x,w)-f(t,x,v)\big)K_f(t,x,v,w)\,\mathrm{d}w \\ &\qquad + f(t,x,v)\left(\int_{\mathbb{R}^d}\big(K_f(t,x,v,w)-K_f(t,x,w,v)\big)\,\mathrm{d}w\right) \\ &\quad= \frac{1}{2}\int_{\mathbb{R}^d}\big(f(t,x,w)-f(t,x,v)\big)\big(K_f(t,x,v,w)+K_f(t,x,w,v)\big)\,\mathrm{d}w \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^d}\big(f(t,x,w)+f(t,x,v)\big)\big(K_f(t,x,v,w)-K_f(t,x,w,v)\big)\,\mathrm{d}w \\ &\quad= \int_{\mathbb{R}^d}\left[f(t,x,w)K_f(t,x,v,w)-f(t,x,v)K_f(t,x,w,v)\right]\,\mathrm{d}w. \end{aligned} \tag{1.16}$$

The strength of this reformulation (1.16) is to emphasise that *on average, the Boltzmann equation is an equation in divergence form*: a non-local operator of the form (1.14) is deemed to be in divergence form, if its kernel K is symmetric in the sense $K(v,w)=K(w,v)$, and it is in non-divergence form, if $K(v,v+w)=K(v,v-w)$. This can be explained by performing for any sufficiently smooth $f,g:\mathbb{R}^d\to\mathbb{R}$ an "integration by parts": if $K^{sym}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}_+$ enjoys the divergence-from symmetry, $K^{sym}(v,w)=K^{sym}(w,v)$,

then

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(v)-f(w))K^{sym}(v,w)g(v)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(w)-f(v))K^{sym}(w,v)g(w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(v)-f(w))K^{sym}(v,w)(g(v)-g(w))\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(v)K^{sym}(v,w)(g(v)-g(w))\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}$$

In particular, if we choose $g = const$, then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(v)-f(w))K^{sym}(v,w)g(v)\,\mathrm{d}w\,\mathrm{d}v = 0.$$

A kernel in non-divergence form cannot be "integrated by parts" in this way. The kernel K_f in (1.16) does not satisfy the pointwise symmetry $K_f(v,w) \neq K_f(w,v)$, yet if we integrate (1.16) over $v \in \mathbb{R}^d$, the right hand side vanishes due to the unevenness of the integrand. In other words: Q_1 is not in divergence form, but Q is. We say that K_f is in *divergence form on average*.

1.5 Hilbert's 19th problem

1.5.1 Divergence form

The reformulation of the Euler-Lagrange equations associated with a convex energy minimising functional into a uniformly elliptic divergence-form differential equation proved to be a key step in De Giorgi's and Nash's solution to Hilbert's nineteenth problem. Hilbert observed the existence of elliptic partial differential equations arising as Euler-Lagrange equations of convex energy functionals that admit only analytic solutions [109]. The content of Hilbert's nineteenth problem was to understand whether the analyticity of solutions to elliptic partial differential equations is a general principle: are minimisers of smooth convex energy functionals always analytic?

1.5.2 Elliptic equations

The study of the behaviour of solutions to partial differential equations is a field known as *regularity theory*. To derive local interior regularity properties of given solutions, the idea is to zoom in onto small scales, so as to capture information on the solutions at small scales. In the context of Euler-Lagrange equations, the ellipticity stems from the convexity

of the functional that gives rise to an energy of the form

$$\mathcal{E}(g) = \int_\Omega F(\nabla_v g)\,\mathrm{d}v,$$

where F is a smooth function $F : \mathbb{R}^d \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$ an open convex bounded subset, and a local minimiser $g \in H^1(\Omega)$ of $\mathcal{E}$ over $H^1(\Omega)$ satisfies

$$\mathcal{E}(g) \leq \mathcal{E}(g + \varphi)$$

for any $\varphi \in C_c^\infty(\Omega)$. In particular, assuming that $D^2F(p) \leq \Lambda I$ for fixed $\Lambda > 0$, and any $p \in \mathbb{R}^d$, one can show that any $g \in H^1(\Omega)$ that is a local minimiser of $\mathcal{E}$ satisfies

$$-\nabla_v \cdot (DF(\nabla_v g)) = 0, \qquad v \in \Omega. \tag{1.17}$$

To see this, let $\varepsilon > 0$ and $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_\Omega F(\nabla_v g + \varepsilon \nabla_v \varphi)\,\mathrm{d}v \geq \int_\Omega F(\nabla_v g)\,\mathrm{d}v.$$

By Taylor-expanding F, we find for any $\varepsilon > 0$

$$\int_\Omega DF(\nabla_v g) \cdot \nabla_v \varphi\,\mathrm{d}v \geq -\varepsilon\Lambda \int_\Omega |\nabla_v \varphi|^2\,\mathrm{d}v,$$

which implies

$$\int_\Omega DF(\nabla_v g) \cdot \nabla_v \varphi\,\mathrm{d}v \geq 0.$$

Upon replacing φ by $-\varphi$, we deduce

$$\int_\Omega DF(\nabla_v g) \cdot \nabla_v \varphi\,\mathrm{d}v \leq 0,$$

so that g is a weak solution to (1.17).

Upon expanding the divergence in (1.17), we see that g satisfies for $i, j = 1, \ldots, d$

$$-(\partial_{ij}^2 F)(\nabla_v g) : \partial_{v_i}\partial_{v_j} g = 0, \tag{1.18}$$

provided that g is sufficiently regular. One instance of such an equation that was of interest to the community at the beginning of the 20th century was the minimal surface equation,

$$-\nabla_v \cdot \left(\frac{\nabla_v g}{\sqrt{1 + |\nabla_v g|^2}} \right) = 0,$$

which was the associated Euler-Lagrange equation of the area functional

$$\mathcal{E}(g) = \int_\Omega \sqrt{1 + |\nabla_v g|^2}\, \mathrm{d}v.$$

It was known that solutions are C^∞. An easier example is the Laplace equation,

$$\nabla_v \cdot (\nabla_v g) = 0,$$

whose associated operator is the Dirichlet functional

$$\mathcal{E}(g) = \int_\Omega |\nabla_v g|^2 \, \mathrm{d}v.$$

Solutions are known to be analytic as a result of the theory of harmonic functions. Hilbert raised the question if it was a general principle that equations arising as Euler-Lagrange formulations of smooth convex energy minimising functionals admit analytic solutions. This became *Hilbert's 19th problem*.

In 1904, Bernstein [25] showed that any smooth solution is analytic. Then, in 1934, Schauder was able to construct a theory which transferred regularity in Hölder spaces from the coefficients onto the solution [176]: if for some $\alpha > 0$ we know $\nabla_v g \in C^\alpha_{loc}$ so that $(\partial^2_{ij} F)(\nabla_v g) \in C^\alpha_{loc}$, then $g \in C^{2+\alpha}_{loc}$. This argument can be *bootstrapped* to yield smooth solutions $g \in C^\infty_{loc}$. The only missing piece to Hilbert's 19th problem was thus to *bridge the gap* between Lebesgue integrability and Hölder continuity. De Giorgi and Nash independently resolved this question in 1956 [56, 168].

The first step was the reformulation of (1.17) into a *divergence form equation with rough coefficients*: to this end we differentiate (1.17) with respect to v_i for all $1 \leq i \leq d$, which yields

$$-\nabla_v \cdot \left(\partial^2_{ij} F(\nabla_v g) \nabla_v f\right) = 0,$$

where $f = \partial_{v_i} g$. Thus we arrive at

$$-\nabla_v \cdot (A(v) \nabla_v f(v)) = 0, \tag{1.19}$$

with $A = \partial^2_{ij} F(\nabla_v g)$. This is now a linear second order partial differential equation in divergence form with rough coefficients for solutions depending only on one variable $v \in \mathbb{R}^d$. The diffusion coefficients A are assumed to be measurable as a means to deal with the non-linearity. We expect Sobolev- and Hölder-regularity of f under the hypothesis that A is bounded and elliptic: we assume there is $0 < \lambda < \Lambda$ such that

$$\lambda I \leq A(v) \leq \Lambda I, \qquad \forall v \in \mathbb{R}^d,$$

which translates into a uniform convexity assumption on F.

De Giorgi's approach further splits into two steps, now known as *first and second De Giorgi Lemma*. We can diagrammatically summarise these ideas:

$$\text{given } f \in L^2 \text{ solving (1.19)} \xrightarrow[\text{1st De Giorgi Lemma}]{} f \in L^\infty \xrightarrow[\text{2nd De Giorgi Lemma}]{} f \in C^\alpha_{loc}$$
$$\xrightarrow[\text{Definition of } A]{} A \in C^\alpha_{loc} \xrightarrow[\text{Schauder}]{} g \text{ solving (1.18) is in } C^{2+\alpha} \xrightarrow[(1.19)]{} \partial_{v_i} g = f \in C^{1+\alpha}_{loc}$$
$$\xrightarrow[\text{Definition of } A]{} A \in C^{1+\alpha}_{loc} \xrightarrow[\text{Schauder}]{} f \in C^{2+\alpha}_{loc} \to \dots \xrightarrow[\text{bootstrap}]{} f \in C^\infty_{loc} \xrightarrow[\text{Bernstein}]{} f \text{ analytic.}$$

1.5.3 Hypoelliptic equations

In this thesis, we discuss an extension of Hilbert's 19th problem to non-linear kinetic integro-differential equations that naturally arise in physics. We aim to adapt theories that lead to the solution of Hilbert's 19th problem to equations of the form (1.13):

- this is an equation of *fractional order* $2s \in (0, 2)$,
- it is *kinetic*, which means that solutions depend on time, space and velocity;
- it is *degenerate*, since the smoothing operator $\mathcal{L}f$ acts merely in velocity (a property that is common in kinetic equations);
- it is *hypoelliptic*, since the vector fields in the equation span the whole space, or in other words, the commutator of the transport operator $\mathcal{T}$ with the velocity gradient ∇_v recovers the lost direction ∇_x.

This last property nourishes our hope to transfer regularity gained in velocity through $\mathcal{L}$ onto the spatial variable. The degeneracy and the non-locality of (1.13) are the reasons why the study of this equation is at the same time interesting and difficult. In general,

- the equation is even *non-linear*, due to the dependence of the coefficients and the source on the solution itself;

however, we treat the non-linearity as a *rough linearity*.

For this class of equations, we establish a quantitative analogue of De Giorgi's theory. We then develop a constructive approach to Schauder estimates. Combined with previous results, we can deduce that a given solution in a physically meaningful regime is smooth. Finally, we demonstrate a Harnack inequality, that is a linear bound on the supremum in the past by the infimum in the future. This proves to be sufficiently strong in order to deduce information on the fundamental solution of (1.13). All these results are proved constructively: the proof methods are quantitative.

1.6 Ellipticity of collision kernels

1.6.1 Conditional regime

In either formulation, (1.11) and (1.16), we see that the coefficients K_f in (1.9) and Λ_f in (1.12) depend on the solution, inducing the non-linear character of the equation. Without any further assumptions on the solution itself, it seems out of reach to treat the non-linearity in the equation. In order to make quantitative statements on the coefficients, we work in a conditional regime that was introduced in [177]. We assume throughout this thesis that the solutions we consider satisfy the following bounds on certain hydrodynamical quantitites: for any $t \in (0, T)$ and any x in the relevant domain (either a bounded subset $x \in \Omega \subset \mathbb{R}^d$, or f is periodic in x), there exists m_0, M_0, E_0, H_0, such that

$$\begin{aligned} 0 < m_0 \leq \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v \leq M_0, \\ \int_{\mathbb{R}^d} f(t,x,v)\,|v|^2\,\mathrm{d}v \leq E_0, \\ \int_{\mathbb{R}^d} f(t,x,v)\log f(t,x,v)\,\mathrm{d}v \leq H_0. \end{aligned} \tag{1.20}$$

In other words, the mass is lower bounded and the mass, the energy, the entropy are upper bounded. Silvestre showed [177, Theorem 1.2], that for solutions in this regime there is a constant $C_0 > 0$ depending only on $\gamma, s, d, m_0, M_0, E_0, H_0$ such that

$$\|f\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d)} \leq C_0.$$

This conditional a-priori bound on the solution enables us to quantify an ellipticity class on the coefficients K_f and Λ_f, where the non-linearity is treated by absorbing it in the macroscopic quantities [177].

The assumption (1.20) is naturally satisfied in the spatially homogeneous case where $f = f(t, v)$ is independent of x. But to assume (1.20) to be satisfied for any point x in space is not guaranteed by natural a-priori bounds from physics: these quantities are not conserved by the space-inhomogeneous equation. It is still interesting to study this conditional regime: using (1.20) to derive smoothness properties of the density f means that a singularity contradicting the conditional regularity of f would be observable in a violation of (1.20).

1.6.2 Ellipticity class of the non-local kernel K

The regularity of integro-differential equations relies on the conditions that are assumed on the kernel of the non-local operator (1.14). A widely studied non-local diffusion operator

is the fractional Laplacian (see (2.2) below, and [64] for a recent review). Based on this, a commonly used assumption is that the kernel is pointwisely bounded from above and below by the kernel of the fractional Laplacian, and satisfies the pointwise symmetry assumptions, $K(v,w) = K(w,v)$ and $K(v, v+w) = K(v, v-w)$, [54, 133, 134]. More general assumptions are discussed in [74, 127, 128].

In the case of kinetic equations, the ellipticity class of the kernel is guided by the kernel featured in the Boltzmann equation. Under the assumption (1.20), we find that the kernel K_f in (1.9) enjoys a certain notion of ellipticity that we outline below in (1.22)-(1.30).

Even though the kernel depends in general on time and space we will often omit to write out this dependency for the sake of brevity. Let $s \in (0,1)$, let $\varphi \in H^s(\mathbb{R}^d)$ and let $\Omega \subseteq \mathbb{R}^d \times \mathbb{R}^d$. For any $(t,x) \in \mathbb{R}_+ \times \Omega$, *under the assumption that* (1.20) *is satisfied*, we can show that there exists $c_0, \lambda_0, \Lambda_0, \mu_0 > 0$ depending on $\gamma, s, d, m_0, M_0, E_0, H_0$ such that K_f given in (1.9) and Λ_f given in (1.12) satisfy the statements described in the following subsections:

1.6.2.1 Cancellation

For $(t,x) \in [0,T] \times \Omega$, we have the *cancellation lemma* (see [3]):

$$\forall v \in \mathbb{R}^d : \int_{\mathbb{R}^d} (K_f(v,v') - K_f(v',v)) \, \mathrm{d}v' = c_b \int_{\mathbb{R}^d} f(v_*) \, |v - v_*|^\gamma \, \mathrm{d}v_*, \tag{1.21}$$

with γ from (1.6) and

$$c_b = \int_{\mathbb{S}^{d-1}} \left\{ \frac{2^{\frac{d+\gamma}{2}}}{(1+\sigma\cdot\mathbf{e})^{\frac{d+\gamma}{2}}} - 1 \right\} b(\sigma\cdot\mathbf{e}) \, \mathrm{d}\sigma > 0,$$

for any $\mathbf{e} \in \mathbb{S}^{d-1}$. This implies that Λ_f given in (1.12) is non-negative, and satisfies

$$0 \leq \Lambda_f \leq \Lambda_0 \langle v \rangle^{\gamma_+}. \tag{1.22}$$

1.6.2.2 Coercivity

Under the condition (1.20), we know that for any fixed $(t,x) \in [0,T] \times \Omega$ there exists a set $\Xi(v) \subset \mathbb{R}^d$ for every $v \in \mathbb{R}^d$ such that $\Xi(v)$ is a symmetric cone, and such that for $v' - v \in \Xi(v)$ we have

$$K_f(v,v') \geq \lambda_0 \langle v \rangle^{\gamma+2s+1} \, |v - v'|^{-d-2s}, \tag{1.23}$$

with $\lambda_0 = \lambda_0(d, m_0, M_0, E_0, H_0) > 0$. Moreover, the Lebesgue measure of the cone in $\mathbb{S}^{d-1}$ is lower bounded:

$$\left| \Xi(v) \cap \mathbb{S}^{d-1} \right| \geq c_0 \langle v \rangle^{-1}, \tag{1.24}$$

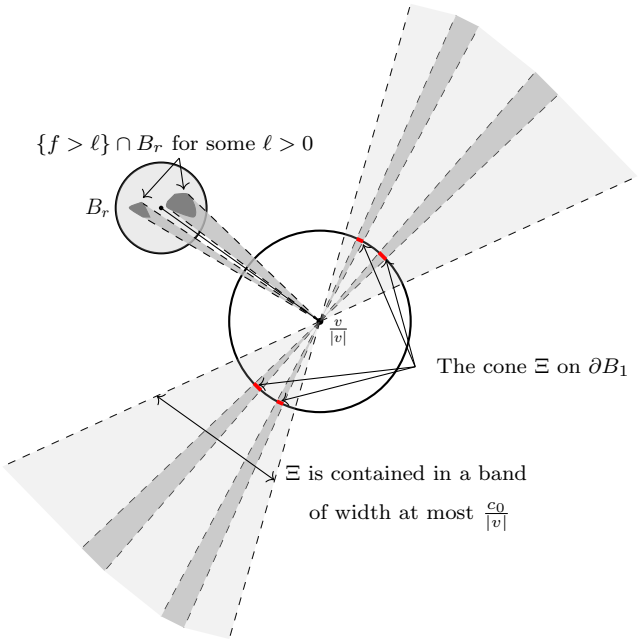


Figure 1.4: An illustration of the coercivity cone exhibited by Silvestre [177]. Due to the macroscopic assumption (1.20), the solution f is strictly positive in a set of strict positive measure, which is used to construct the non-degeneracy cone $\Xi(v)$. The figure is inspired from [177].

for some positive constant $c_0 = c_0(d, m_0, M_0, E_0, H_0)$. The set $\Xi(v)$ is called the *cone of non-degeneracy* of $K_f(v, v')$.

We can ensure that the following estimate holds,

$$|\Xi(v) \cap \{B_3 \setminus B_2\}| \geq c_0 \langle v \rangle^{-1}, \tag{1.25}$$

where B_r is a ball around the origin of radius $r > 0$. Thus, on average, we have a coercivity estimate:

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \big(\varphi(v) - \varphi(v')\big)^2 K_f(v, v') \, \mathrm{d}v' \, \mathrm{d}v \geq \lambda_0 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\varphi(v) - \varphi(v')|^2}{|v - v'|^{d+2s}} \, \mathrm{d}v \, \mathrm{d}v'. \tag{1.26}$$

1.6.2.3 Upper bound

For $(t, x) \in [0, T] \times \Omega$ there exists a postive constant Λ_0 depending on m_0, M_0, E_0, H_0, d such that for any $\varphi \in H^s(\mathbb{R}^d)$

$$\forall v \in \mathbb{R}^d, \ \forall r > 0, \qquad \int_{B_r(v)} K_f(v, v') \, |v' - v|^2 \, \mathrm{d}v' \leq \Lambda_0 \langle v \rangle^{\gamma + 2s} r^{2-2s}, \tag{1.27}$$

where $B_r(v)$ is the ball centred at $v \in \mathbb{R}^d$ with radius r.

1.6.2.4 Symmetry

Third, we discuss the symmetry of the kernel. Note that the kernel K_f given in (1.9) satisfies a pointwise symmetry of the form $K_f(v, v+w) = K_f(v, v-w)$. This symmetry assumption gives rise to a non-local operator in non-divergence form. However, the kernel *does not satisfy a pointwise divergence form symmetry*:

$$K_f(v,w) \neq K_f(w,v), \tag{1.28}$$

for which the local analogue would be an operator in divergence form. This implies that the non-local operator (1.14) with kernel K_f is not necessarily self-adjoint in L^2. As a consequence of the cancellation (1.21), it does, however, satisfy a *weak divergence form symmetry*:

$$\forall v \in \mathbb{R}^d \quad \left| \mathrm{PV} \int_{\mathbb{R}^d} \big(K_f(v,w) - K_f(w,v)\big)\,\mathrm{d}w \right| \leq \Lambda_0 \langle v \rangle^{\gamma_+}, \tag{1.29}$$

and if $s \geq \frac{1}{2}$ we have for all $r > 0$

$$\forall v \in \mathbb{R}^d \quad \left| \mathrm{PV} \int_{B_r(v)} (v-w)\big(K_f(v,w) - K_f(w,v)\big)\,\mathrm{d}w \right| \leq \Lambda_0 r^{1-2s}. \tag{1.30}$$

These conditions mean that the anti-symmetric part is not singular.

1.6.2.5 Remarks

We remark that the coercivity (1.26) together with the weak divergence-form property (1.29) imply for $\varphi \in H^s(\mathbb{R}^d)$

$$\begin{aligned} \lambda_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\varphi(v) - \varphi(w)|^2}{|v-w|^{d+2s}}\,\mathrm{d}v\,\mathrm{d}w \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(\varphi(v) - \varphi(w)\big) K_f(v,w) \varphi(v)\,\mathrm{d}w\,\mathrm{d}v \\ + \Lambda_0 \langle v \rangle^{\gamma_+} \|\varphi\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{1.31}$$

Moreover the upper bound (1.27) is equivalent to

$$\begin{aligned} \forall v \in \mathbb{R}^d, \forall r > 0 \quad & \int_{B_{2r}(v) \setminus B_r(v)} K_f(v,w)\,\mathrm{d}w \leq \Lambda_0 \langle v \rangle^{\gamma+2s} r^{-2s}, \\ \forall w \in \mathbb{R}^d, \forall r > 0 \quad & \int_{B_{2r}(w) \setminus B_r(w)} K_f(v,w)\,\mathrm{d}v \leq \Lambda_0 \langle w \rangle^{\gamma+2s} r^{-2s}. \end{aligned} \tag{1.32}$$

We also note that (1.30) holds for $s \in \big(0, \frac{1}{2}\big)$ as a consequence of (1.29) and (1.27).

The coercivity (1.26) and the upper bound (1.32) are inspired from the work by Imbert-Mouhot-Silvestre on the regularity of solutions of the non-cutoff Boltzmann equation conditional to the macroscopic bounds in (1.20), see [115–118, 120, 177]. The assumptions

(1.21)-(1.30) are satisfied for any solution f of (1.2) in $(0,T)\times\mathbb{R}^d\times\mathbb{R}^d$, such that (1.20) holds. For the justifications, we refer the reader to [177, Lemma 5.1] for (1.21), to [177, Lemma 4.8] for (1.23), to [46, Theorem 1.2] or [115, Lemma A.6] for (1.26), to [115, Lemma 3.5, Lemma 3.6, Lemma 3.7] for (1.27), (1.29) and (1.30), respectively; to [115, Proposition 3.3, Lemma 3.4, Lemma 3.5] for (1.31) and (1.32).

Finally we remark that in Part I, since we are merely concerned with *interior local* regularity results, we will not in general need to take care of the velocity dependence in the assumptions above. We will, however, take these into account in Part II, when we derive decay estimates for weak solutions.

1.7 Weak solutions

1.7.1 Invariances

Here we introduce the kinetic cylinders that provide the domain of solutions to (1.13). Let f solve (1.13). Then for $r\in[0,1]$ the scaled function $f_r(t,x,v)=f(r^{2s}t,r^{1+2s}x,rv)$ can be seen to satisfy

$$\partial_t f_r + v\cdot\nabla_x f_r = \mathcal{L}_r f_r + h_r,$$

where $\mathcal{L}_r$ is the non-local integro-differential operator associated to the scaled kernel

$$K_r(t,x,v,w)=r^{d+2s}K(r^{2s}t,r^{1+2s}x,rv,rw),$$

and the source term is scaled as

$$h_r(t,x,v)=r^{2s}h(r^{2s}t,r^{1+2s}x,rv).$$

For $r\in[0,1]$ the scaled kernel K_r satisfies (1.22)-(1.30). Moreover,

$$\|h_r\|_{L^\infty(Q_1)}\le r^{2s}\|h\|_{L^\infty(Q_1)}\le\|h\|_{L^\infty(Q_1)}.$$

Furthermore, the equation is invariant under Galilean transformations, which, we recall, are given by $z\to z_0\circ z=(t_0+t,x_0+x+tv_0,v_0+v)$ with $z_0=(t_0,x_0,v_0)\in\mathbb{R}^{1+2d}$. If f is a solution of (1.13) then its Galilean transformation $f_{z_0}(z)=f(z_0\circ z)$ solves

$$\partial_t f_{z_0} + v\cdot\nabla_x f_{z_0} = \mathcal{L}_{z_0} f_{z_0} + h_{z_0},$$

where $\mathcal{L}_{z_0}$ is the non-local integro-differential operator associated to the translated kernel

$$K_{z_0}(t,x,v,w)=K(z_0\circ z,v_0+w),$$

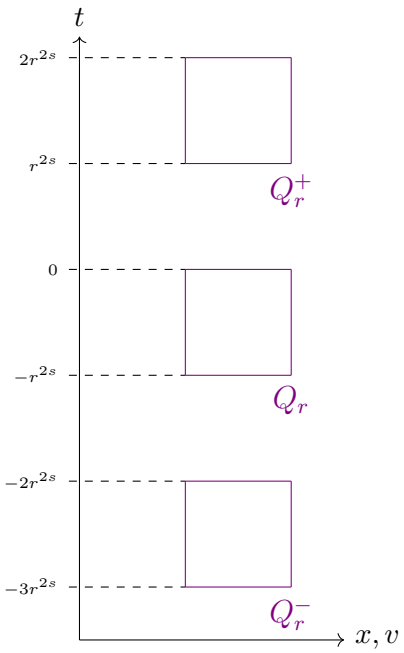


Figure 1.5: An illustration of the kinetic cylinders defined above for $r > 0$.

and with source

$$h_{z_0}(t, x, v) = h(z_0 \circ z).$$

Again the modified kernel K_{z_0} satisfies (1.22)-(1.30) provided that K does, and h_{z_0} is bounded provided that h is.

In view of these invariances we define *kinetic cylinders*

$$Q_r(z_0) := \left\{(t, x, v) : -r^{2s} \le t - t_0 \le 0, |v - v_0| < r, |x - x_0 - (t - t_0)v_0| < r^{1+2s}\right\},$$

for $r > 0$ and $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$, see Figure 1.5. For later reference, we also introduce the cylinder shifted to the past

$$Q_r^-(z_0) := Q_r\big(z_0 - (2r^{2s}, 2r^{2s}v_0, 0)\big),$$

so that in particular for $z_0 = 0$

$$Q_r^- := Q_r(-2r^{2s}, 0, 0) = (-3r^{2s}, -2r^{2s}] \times B_{r^{1+2s}} \times B_r.$$

Similarly the cylinder shifted to the future is denoted as

$$Q_r^+(z_0) := Q_r\big(z_0 + (2r^{2s}, 2r^{2s}v_0, 0)\big).$$

Finally, if we merely use the space and velocity domain of a cylinder, we denote it as $Q_r^t := B_{r^{1+2s}} \times B_r$ where $t \in \mathcal{I}$ for some specified interval $\mathcal{I} \subset \mathbb{R}$.

1.7.2 Notion of solutions

In Part I, we work with a notion of solution that requires sufficient integrability so that the differential operators in (1.2) are weakly defined.

Definition 1.1 (Weak solutions)**.** Let $0 < s < 1$. Assume $0 \leq K$ satisfies (1.26)-(1.30) and for $\bar{R} > 0$, let $\mathcal{I} := (t_0, t_1)$ for $t_0 < t_1$ and $\mathcal{I} \times \Omega_x := \big(-(\frac{\bar{R}}{2})^{2s}, 0\big) \times B_{(\frac{\bar{R}}{2})^{1+2s}}$ and $\Omega_v \subseteq \mathbb{R}^d$. We say that $f : (t_0, t_1) \times \Omega_x \times \mathbb{R}^d \to \mathbb{R}$ is a solution of (1.13)-(1.14) in $(t_0, t_1) \times \Omega_x \times \Omega_v$ for a source $h \in L^2\big((t_0, t_1) \times \Omega_x \times \Omega_v\big)$, if

1. $f \in C^0(\mathcal{I}, L^2(\Omega_x \times \Omega_v)) \cap L^2(\mathcal{I} \times \Omega_x, L^\infty(\mathbb{R}^d \setminus \Omega_v) + H^s\big(\mathbb{R}^d\big))$, where we equip $L^\infty(\mathbb{R}^d \setminus \Omega_v) + H^s\big(\mathbb{R}^d\big)$ with the norm

$$\|f\|_{L^\infty(\mathbb{R}^d \setminus \Omega_v) + H^s(\mathbb{R}^d)} = \inf \Big\{ \|f_1\|_{L^\infty(\mathbb{R}^d \setminus \Omega_v)} + \|f_2\|_{H^s(\mathbb{R}^d)} : f = f_1 + f_2,\ f_1 = 0 \text{ in } \Omega_v \Big\}.$$

2. $\mathcal{T} f \in L^2(\mathcal{I} \times \Omega_x, H^{-s}(\Omega_v))$,

3. for all non-negative $\varphi \in L^2(\mathcal{I} \times \Omega_x, H^s(\mathbb{R}^d))$ such that for every t, x the support of $\varphi(t, x, \cdot)$ is compactly contained in Ω_v if $\Omega_v \subsetneq \mathbb{R}^d$ we have

$$\int_{(t_0,t_1)\times\Omega_x\times\Omega_v} (\mathcal{T} f)\varphi \,\mathrm{d}z + \int_{(t_0,t_1)} \int_{\Omega_x} \mathcal{E}(f, \varphi) \,\mathrm{d}x \,\mathrm{d}t - \int_{(t_0,t_1)\times\Omega_x\times\Omega_v} h\varphi \,\mathrm{d}z \leq 0.$$

A function f is a super-solution of (1.13)-(1.14) in $\mathcal{I} \times \Omega_x \times \Omega_v$ if $-f$ is a sub-solution in $\mathcal{I} \times \Omega_x \times \Omega_v$. It is a solution if it is a sub- and super-solution.

For given solutions in this sense we derive three different types of Harnack inequalities, we derive Hölder continuity, we establish a Schauder theory, and we show bounds on the fundamental solution of the linearised equation. It is currently out of reach to prove existence of such solutions globally in time, since we work in the conditional regime where (1.20) is assumed. But the results of the programme by Imbert-Mouhot-Silvestre [118] imply that singularity formation in finite time is observable via the macroscopic quantities in (1.20).

1.7.3 Results

The following collection of results are all statements about the *interior local regularity* of solutions to (1.13) (and thus to (1.2)). We assume the equation to be satisfied in some domain, within which we deduce properties of the solution away from the boundary. The domains are kinetic time-, space-, and velocity-dependent cylinders. Due to the scaling and translation invariance explained in Subsection 1.7.1, the size or the centring of the cylinder is arbitrary as long as it is contained in the solution domain. We invite the reader to not

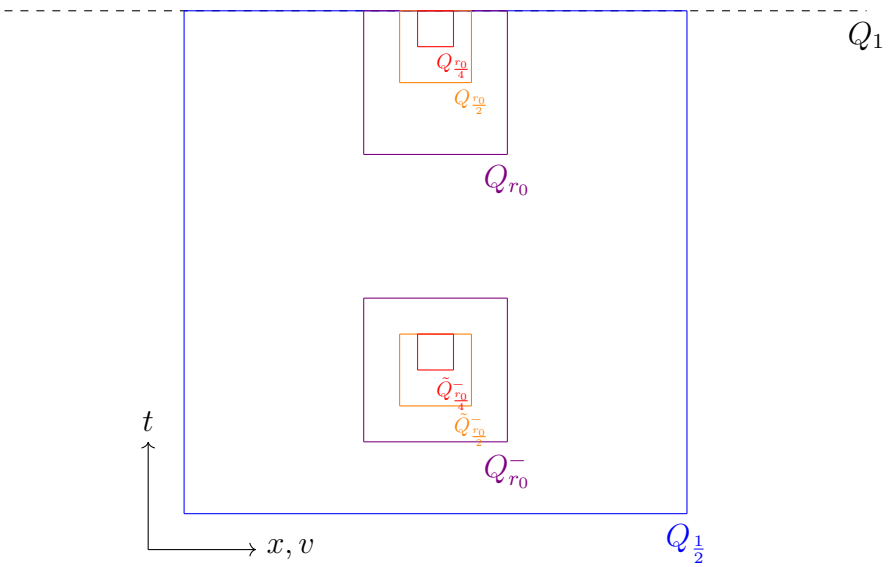


Figure 1.6: For a solution in Q_1, we prove Hölder continuity (see Theorem 3.1) in $Q_{\frac{1}{2}}$. The Intermediate Value Lemma, Theorem 1.2, relates the mass of large values in Q_{r_0} to the mass of small values in $Q^-_{r_0}$ in the past. The weak Harnack inequality, Theorem 1.1, relates the infimum over $Q_{\frac{r_0}{2}}$ with some integrability in $\tilde{Q}^-_{\frac{r_0}{2}}$, and the strong Harnack inequality, Theorem 1.4 (as well as the not-so-strong Harnack inequality, Theorem 3.2) relates the infimum over $Q_{\frac{r_0}{4}}$ with the supremum over $\tilde{Q}^-_{\frac{r_0}{4}}$ in the past. Note that all cylinders depend on $s \in (0,1)$.

fix their attention on the precise size, or the centring of the cylinders in the statements that follow, but instead to consult the illustration of Figure 1.6 to understand the main principle: the cylinders are contained in the solution domain and the separation between the cylinders must be respected. Recall that $2s \in (0,2)$ is the order of the equation (1.13); it appears in the ellipticity of the non-local kernel K, see (1.27).

1.7.3.1 De Giorgi

Our first result recovers the weak Harnack inequality for kinetic integro-differential equations established first by Imbert-Silvestre [115, Theorem 1.6]. This result is in particular applicable to solutions of the Boltzmann equation with moderately soft potentials. Our method, in contrast to the one employed by Imbert-Silvestre, is *quantitative*. For details we refer the reader to Chapter 3.

Theorem 1.1 (Weak Harnack inequality)**.** *Let f be a non-negative super-solution to* (1.13)-(1.14) *in $[-3,0] \times B_1 \times B_1$ with a non-negative kernel K satisfying* (1.26)-(1.30)*. Then there is $C > 0$ and $\zeta > 0$ depending on $s, d, \lambda_0, \Lambda_0$ such that for $r_0 > 0$ sufficiently small, the weak Harnack inequality is satisfied:*

$$\left(\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^\zeta(z)\,\mathrm{d}z \right)^{\frac{1}{\zeta}} \leq C \left(\inf_{Q_{\frac{r_0}{2}}} f + \|h\|_{L^\infty(Q_1)} \right), \tag{1.33}$$

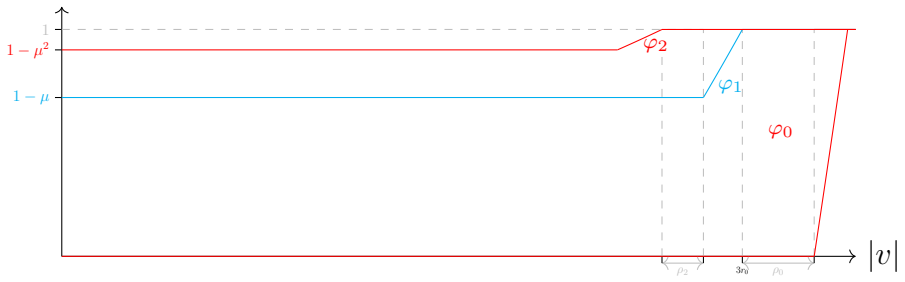


Figure 1.7: The barriers $\varphi_0, \varphi_1, \varphi_2$ in the Intermediate Value Lemma, Theorem 1.2, for $r_0 < \frac{1}{3}$, $\mu < 1$ and fixed $x \in B_{(3r_0)^{1+2s}}$.

where $\tilde{Q}^-_{\frac{r_0}{2}} := Q_{\frac{r_0}{2}}\Big(\big(-\frac{5}{2}r_0^{2s} + \frac{1}{2}(\frac{r_0}{2})^{2s}, 0, 0\big)\Big)$, *and where* $\inf_Q g$ *is the greatest lower bound of* g *over* Q.

By quantifying the proof of the Harnack inequalities, we extend the structure of the De Giorgi argument to the kinetic setting. In particular, we establish the Second De Giorgi Lemma, also known as Intermediate Value Lemma, for kinetic integro-differential equations with a non-locality exponent $s \in (0,1)$. To the best of our knowledge this is the first non-local Intermediate Value Lemma in the kinetic setting for the full range $s \in (0,1)$. The barrier functions φ_0 and φ_2 are defined in detail in Subsection 3.5.4, where we prove Theorem 1.2. For now, we content ourselves with an illustration of the barriers in Figure 1.7, so that we can figuratively make sense of the following statement.

Theorem 1.2 (Intermediate Value Lemma)**.** *Let* f *be a sub-solution to* (1.13)-(1.14) *in* $[-3,0] \times B_1 \times B_1$ *under assumptions* (1.26)-(1.30) *with* $0 \le f \le 1$ *in* $(-3,0] \times B_1 \times \mathbb{R}^d$. *Given* $\delta_1, \delta_2 \in (0,1)$ *there exists* $\mu \lesssim (\delta_1\delta_2)^3$ *and* $\nu \gtrsim (\delta_1\delta_2)^5$ *such that if* $\|h\|_{L^\infty(Q_{3r_0})} \lesssim \mu^2$ *for* $r_0 > 0$ *sufficiently small, and if*

$$\left|\{f \le 0\} \cap Q^-_{r_0}\right| \ge \delta_1 \left|Q^-_{r_0}\right|, \qquad \left|\{f > 1-\mu^2\} \cap Q_{r_0}\right| \ge \delta_2 \left|Q_{r_0}\right|, \tag{1.34}$$

(where $|E|$ *denotes the Lebesgue measure of the set* $E \subset \mathbb{R}^{2d+1}$*), then* f *satisfies*

$$\left|\{\varphi_0 < f < \varphi_2\} \cap (-3,0] \times B_{(\frac{1}{2})^{1+2s}} \times B_\rho\right| \ge \nu \left|Q_{\frac{1}{2}}\right|, \tag{1.35}$$

where $\rho \sim (\delta_1\delta_2)^{-\frac{5}{4s}}$ *and where the barriers* φ_0, φ_2 *are defined in* (3.29) *and illustrated in Figure 1.7. Recall*

$$Q^-_{r_0} = \big(-3r_0^{2s}, -2r_0^{2s}\big] \times B_{r_0^{1+2s}} \times B_{r_0}.$$

Theorem 1.2 is the main result of Chapter 3.

Remark 1.1. In contrast to elliptic equations, there is the additional role of time that has to be accounted for in our setting. The weak Harnack inequality is a statement about super-solutions, and thus it is expected that information in the future controls information

in the past. This is true in both, the parabolic and the hypoelliptic case. However, in contrast to the parabolic case, where the past and future cylinders are allowed to touch each other [97], we recall Figure 1.6 in the hypoelliptic case, where we see that the past cylinder over which we control some integrability is separated by a *gap in time* from the future cylinder over which we take the infimum. This gap in time is characteristic of hypoelliptic equations: the reason is that the trajectories along which we transfer regularity in the velocity variable onto the spatial variable also evolve in time, and therefore there has to be a non-trivial gap in time for the trajectories to be non-degenerate. (The idea is to connect two domains by trajectories; but the mapping from a domain onto the trajectory has a determinant that is inverse proportional to the size of the gap in time and thus blows up as the gap tends to zero.)

There are two ideas in Chapter 3 that we believe are worth mentioning. The first is the construction of trajectories along which we can transfer regularity from the velocity variable onto the spatial variable. To this end, we solve the control problem associated to the underlying vector fields in the equation: through the hypoellipticity we find a smooth curve that connects *any two points* in a given domain. In particular we can connect the two sets of positive measure where $\{f \leq 0\}$ and where $\{f > 1 - \mu^2\}$. This is the first step to prove Theorem 1.2. The second idea forms the second step in the proof of Theorem 1.2: it exploits the *non-local cross terms*. In the spirit of De Giorgi, we test equation (1.13) with $(f-\ell)_+$ for some $\ell > 0$. This creates cross-terms: any f writes as $f = (f-\ell)_+ + (f-\ell)_- + \ell$, and the bilinear form in (1) becomes

$$\mathcal{E}(f, (f-\ell)_+) = \mathcal{E}((f-\ell)_+, (f-\ell)_+) + \mathcal{E}((f-\ell)_-, (f-\ell)_+).$$

The first term will be related through the coercivity (1.26) to the H^s_v norm of $(f-l)_+$. The second is the *cross term*, which is particular, since it is *signed*:

$$\begin{aligned} &\mathcal{E}((f-\ell)_-, (f-\ell)_+) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} ((f-\ell)_-(v) - (f-\ell)_-(w))(f-\ell)_+(v) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f-\ell)_-(w)(f-\ell)_+(v) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \geq 0. \end{aligned} \tag{1.36}$$

Recall that we denote by $(f-l)_- = \min\big(f-l, 0\big) \leq 0$. The appearance of this cross term is a *purely non-local* phenomenon: this term vanishes for a local operator. The boundedness of this term is the reason why a solution to a differential equation of order $2s \in (0,2)$ (which a-priori is just in H^s) satisfies Theorem 1.2 (the conclusion of which clearly fails for a step function, which is in H^s for $s < \frac{1}{2}$).

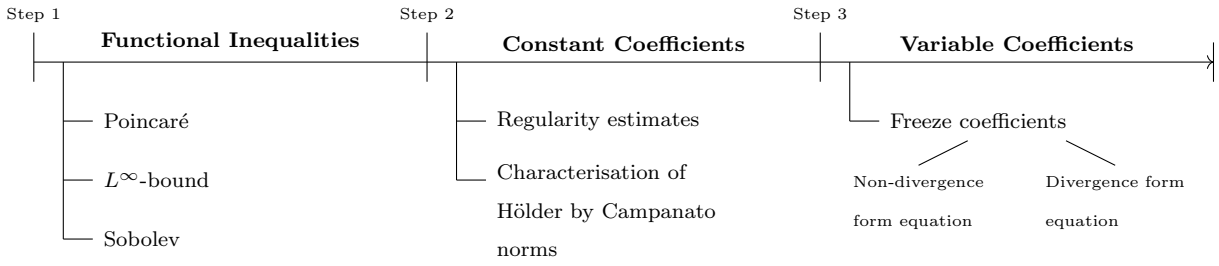


Figure 1.8: Strategy for the Schauder theory. Details are carried out in Chapter 4.

1.7.3.2 Schauder

Chapter 4 is concerned with Schauder theory: the gain of regularity in Hölder spaces from the coefficients onto the solution. The precise statement can be found in **Theorem 4.2** and **Theorem 4.3** below. We paraphrase the main idea here:

Theorem 1.3 (Schauder for kinetic integro-differential equations)**.** *Let* $0 < \gamma < \min(1, 2s)$*. Assume* K *is a non-negative kernel that is elliptic with upper bound* (1.32) *and coercivity* (1.26)*, that is Hölder continuous (in the sense that it satisfies* (4.8) *for some* $A_0 > 0$*) for* $\alpha = \frac{2s}{1+2s}\gamma$ *and each* $z \in Q_1$*. Moreover, suppose that either the kernel is in pointwise non-divergence form* (4.7) *or else in weak divergence form* (1.29)*,* (1.30) *with additionally supposing the higher order Hölder continuity (in the sense of* (4.9)*) for* $\alpha = \frac{2s}{1+2s}\gamma$ *and each* $z \in Q_1$*. Then any solution* $f \in C^\gamma_\ell([-1,0] \times B_1 \times \mathbb{R}^d)$ *of* (1.13) *in* Q_1 *satisfies*

$$\|f\|_{C^{2s+\alpha}_\ell(Q_{1/4})} \leq C\big(\|f\|_{C^\gamma_\ell([-1,0]\times B_1\times\mathbb{R}^d)} + \|h\|_{C^\alpha_\ell(Q_1)}\big),$$

for some constant $C = C(d, s, \lambda_0, \Lambda_0, A_0)$*.*

This result is new for equations in (weak) divergence form. The only result on Schauder estimates for non-local hypoelliptic equations is due to Imbert-Silvestre [117], where they assume a pointwise non-divergence form (4.7). In contrast to their method, which relies on compactness arguments, our proof is *quantitative*. Our approach is sketched in Figure 1.8; it uses the characterisation of Hölder spaces by Campanato spaces [40], and the interested reader is referred to Chapter 4 for details.

1.7.3.3 Harnack

Chapter 5 is devoted to the proof of the a-priori inequality. We stress that the equation has to be satisfied for velocities in the whole Euclidean space, which is not the usual understanding in the theory of local Harnack inequalities. It turns out that the strong Harnack inequality in a classical regularity theoretical sense (even though it is debatable what this classical sense should be, since the Harnack inequalities were established for *local*

equations) fails for kinetic non-local equations [130]. However, from a physical perspective, a solution to the Boltzmann equation only makes sense to be defined in such a way that the equation holds for $v \in \mathbb{R}^d$. For these solutions, we prove:

Theorem 1.4 (Strong Harnack for Boltzmann)**.** *Let $T > 0$. Suppose $0 \leq f$ satisfies (1.20). Let f be a classical solution of (1.2) in $(0,T) \times \mathbb{R}^d \times \mathbb{R}^d$ (in the sense of Definition 5.2) with a cross-section B given by (1.6) for $0 \leq \gamma + 2s \leq 2$, and with initial data $f_0(x,v) = f(0,x,v)$, which in case that $\gamma \leq 0$ is supposed to admit for any $q > 0$ a constant $N_q > 0$ such that $f_0(x,v) \leq N_q(1+|v|)^{-q}$ for all $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. Then for any $0 < r_0 < \frac{1}{6}$, there exists $C > 0$ such that*

$$\sup_{(\tau_0,\tau_1)\times Q^t_{\frac{r_0}{4}}} f \leq C \inf_{(\tau_2,\tau_3)\times Q^t_{\frac{r_0}{4}}} f,$$

where $0 < \tau_0 < \tau_1 < \tau_2 < \tau_3$ such that $\tau_1 - \tau_0 = \tau_2 - \tau_3 = \left(\frac{r_0}{4}\right)^{2s}$ and $\tau_2 - \tau_1 \geq r_0^{2s}$, and where

$$Q^t_{\frac{r_0}{4}} := \left\{(x,v) \in \mathbb{R}^{2d} : |x| < \left(\frac{r_0}{4}\right)^{1+2s}, |v| < \frac{r_0}{4}\right\}$$

is the kinetic cylinder sliced in time. Here $\sup_Q g$ denotes the least upper bound of g over Q. The constant C depends only on $s, d, \gamma, m_0, M_0, E_0, H_0$, and in case that $\gamma \leq 0$, it depends additionally on N_q.

The space domain is only assumed to be $\mathbb{R}^d$ so that the decay estimates of [121] and the regularity properties from [118] are applicable. This is the first strong Harnack inequality for the Boltzmann equation derived in the literature. The idea is to improve the energy estimate for (1.2) by deriving a *local bound* on the non-local tail, by testing against a concave level set function in such a way that the equation is only localised on the upper level sets. On the lower level sets, we exploit the (average) divergence structure. And for the cross terms (1.36) we use the sign given by the concavity of the test function. Combining this improved energy estimate, which relies on the aforementioned local tail bound, with a De Giorgi argument in L^1 results in a *local first De Giorgi lemma* for solutions to the non-local equation (1.13), which implies the strong Harnack inequality. As a consequence, we can deduce upper and lower bounds on the fundamental solution to the linearised equation by following Aronson's method.

1.7.3.4 Aronson

In Chapter 6, as a consequence of Theorem 1.4, we derive polynomial upper and exponential lower bounds on the fundamental solution of (1.11) with coefficients K_h and Λ_h given by (1.9) and (1.12), respectively, for a fixed function $h \geq 0$ that satisfies (1.20). To give sense to the next three theorems, we assume existence of a non-negative measurable function

J, which is the fundamental solution of (1.11) linearised around a fixed function h, by which we mean that J solves (1.11) with coefficients K_h and Λ_h, connecting a given point $(t,x,v) \in \mathbb{R}^{1+2d}$ with $(\tau,y,w) \in \mathbb{R}^{1+2d}$, in the sequel denoted by

$$J(t,x,v;\tau,y,w) = J(t-\tau, x-y-(t-\tau)w, v-w) =: J\big((\tau,y,w)^{-1} \circ (t,x,v)\big),$$

where $\circ$ denotes the Galilean translation, that is $(t_0,x_0,v_0)\circ(t,x,v) = (t_0+t, x_0+x+tv_0, v_0+v)$, which respects the translation invariance of (1.2). Then, we deduce, on the one hand, polynomial upper bounds.

Theorem 1.5 (Polynomial upper bounds on the fundamental solution)**.** *Let $x,v,y_0,w_0 \in \mathbb{R}^d$, and $0 \le \tau_0 < \sigma < T$. Let J be the fundamental solution of* (1.11) *in $[0,T]\times\mathbb{R}^{2d}$ with coefficients K_h and Λ_h given by* (1.9) *and* (1.12)*, respectively, for a fixed function $h \ge 0$ that satisfies* (1.20)*. Assume J satisfies* (6.1)*,* (6.2) *and* (6.3)*. Then there exists $C>0$ depending on $s,\gamma,d,m_0,M_0,E_0,H_0$ (note that these constants refer to the mass, energy and entropy of h), such that*

$$\begin{aligned} &J(\sigma,x,v;\tau_0,y_0,w_0) \\ &\quad \le C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \left[1 + \frac{\max\left\{|v-w_0|^{2s}, |x-y_0-(\sigma-\tau_0)(v-w_0)|^{\frac{2s}{1+2s}}\right\}}{\sigma-\tau_0}\right]^{-\frac{s}{4s}}. \end{aligned} \tag{1.37}$$

On the other hand, we derive an exponential lower bound.

Theorem 1.6 (Exponential lower bounds on the fundamental solution)**.** *Let $x,v,y_0,w_0 \in \mathbb{R}^d$, and $0 \le \tau_0 < \sigma < T$. Let J be the fundamental solution of* (1.11) *in $[0,T]\times\mathbb{R}^{2d}$ with coefficients K_h and Λ_h given by* (1.9) *and* (1.12)*, respectively, for a fixed function $h \ge 0$ that satisfies* (1.20)*. Assume J satisfies* (6.1)*,* (6.2) *and* (6.3)*. Then there exists $C>0$ depending on $s,\gamma,d,m_0,M_0,E_0,H_0$ (note that these constants refer to the mass, energy and entropy of h), such that*

$$J(\sigma,x,v;\tau_0,y_0,w_0) \ge C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \exp\left\{-C\left(\frac{|x-y_0-(\sigma-\tau_0)w_0|^{2s}}{(\sigma-\tau_0)^{1+2s}} + \frac{|v-w|^{2s}}{\sigma-\tau_0}\right)\right\}.$$

Remark 1.2.

1. This is the first time that upper and lower bounds on the fundamental solution of a kinetic non-local equation are derived. In the case of a second-order kinetic equation, we refer the reader to [141, 143].

2. Both, the upper and lower bounds in Theorem 1.5 and Theorems 1.6, respectively, are not necessarily optimal in space and velocity, however the temporal decay rate is sharp.

3. The existence of a fundamental solution operator for hypoelliptic non-local equations has recently been established by Auscher-Imbert-Niebel [16, 17].

4. We refer to the analogue statements in **Theorem 6.1** and **Theorem 6.2** for kinetic integro-differential equations in divergence form.

The proof makes use of the *diagonal bounds* that are directly implied from a fully local first De Giorgi lemma for solutions to (1.11) (see Proposition 5.1). To show the upper bound in Theorem 1.5, we construct a decay function, which solves the equation and which can be compared to the solution itself, reminiscent of Aronson's method [12]. To show the lower bound in Theorem 1.6, we iteratively apply the strong Harnack inequality from Theorem 1.4.

1.8 Suitable weak solutions

1.8.1 Notion of solutions

In the second part of the thesis, Part II, we study the behaviour of solutions to the Boltzmann equation on bounded space domains, which really is what we physically want to consider. To this end, we introduce a significantly weaker notion of solution, which is so weak that it only assumes the integrability of the solution that we obtain from the conditional regime that we work in, that is the assumptions on the boundedness of the mass, energy and entropy of the system. In particular, since in the space-homogeneous case these quantities are conserved by the equation, the conditional regime is actually unconditional, and our notion of solution is suitable in the sense that they match the framework of H-solutions constructed by Villani [186]. But on top of this, we can deal with bounded space domains, in case that the density of the gas particles depends on the spatial dimension x as well. We then equip the equation (1.2) with boundary conditions that were introduced by Maxwell [156], (see (v) below).

The idea is to study the evolution in time of some L^{q_0}-Lebesgue norm of the function $(f - A)_+$, where $A = A(t, v)$ encapsulates the time and velocity decay of the solution f, and where L^{q_0} is the integrability that we get from the conditional regime (1.20) and the entropy production (which we will discuss below). It is convenient to consider a general function $\varphi = \varphi(t, x, v, r)$ that is convex in r instead of r^{q_0}. For a bounded domain $\Omega \subset \mathbb{R}^d$

we then compute formally:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(t,x,v,f)\,\mathrm{d}x\,\mathrm{d}v &= \iint_{\mathbb{R}^d\times\Omega}\partial_r\varphi(t,x,v,f)Q(f,f)(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\\ &\quad+\iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(t,x,v,f)\,\mathrm{d}x\,\mathrm{d}v\\ &\quad-\iint_{\Gamma}\varphi(t,x,v,f)v\cdot n\,\mathrm{d}S(x)\,\mathrm{d}v.\end{aligned}\tag{1.38}$$

We then use the formulation in (1.16) for Q, we add and subtract as an integrand

$$\mathcal{D}_\varphi(f,f') := \varphi(t,x,v',f') - \varphi(t,x,v,f) - \partial_r\varphi(t,x,v,f)\,(f'-f)\,,$$

which due to the convexity of φ is a non-negative quantity; and we use the cancellation property (1.21), so that we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(f)\,\mathrm{d}x\,\mathrm{d}v &= -\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}x\,\mathrm{d}v\\ &\quad+c_b\iint_{\mathbb{R}^d\times\Omega}\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}x\,\mathrm{d}v\\ &\quad+\iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(f)\,\mathrm{d}x\,\mathrm{d}v-\iint_{\Gamma}\varphi(f)(v\cdot n)\,\mathrm{d}S(x)\,\mathrm{d}v,\end{aligned}$$

where c_b is from (1.21) and

$$\Phi(f) := \partial_r\varphi(t,v,f)f - \varphi(t,x,v,f);$$

(details are carried out in Subsection 7.1.2). We supplement the equation with an initial condition $f(0,x,v) = f_0(x,v)$ for $(x,v)\in\Omega\times\mathbb{R}^d$, and with a boundary condition. In order to describe the latter, we let n denote the outward unit normal vector on the boundary and we denote by Γ the domain that considers boundary points in space, that is $\Gamma := \partial\Omega\times\mathbb{R}^d$, by Γ_- the incoming part of the boundary, and by Γ_+ the outgoing part of the boundary, that is

$$\Gamma_- := \{(x,v)\in\Gamma : v\cdot n(x) < 0\},$$

and

$$\Gamma_+ := \{(x,v)\in\Gamma : v\cdot n(x) > 0\}.$$

We discuss the following boundary conditions,

(i). In-flow: $f(t,x,v)|_{\Gamma_-} = f_b(t,x,v)$ for a given function f_b.

(ii). Bounce-back: $f(t,x,v) = f(t,x,-v)$ on Γ.

(iii). Specular reflection: $f(t,x,v) = f(t,x,\mathcal{R}_x v)$ with $\mathcal{R}_x v = v - 2(v\cdot n)n$.

(iv). Diffuse reflection:

$$f(t,x,v)|_{\Gamma_-} = c_\mu \mu(v) \int_{\Gamma_+} f(t,x,v')(v'\cdot n)\,\mathrm{d}v',$$

where $\mu(v) = e^{-|v|^2}$ is the Maxwellian, and c_μ is a normalisation constant, such that $c_\mu \int_{\Gamma_+} \mu(v')v'\cdot n\,\mathrm{d}v' = 1$.

(v). Maxwell reflection:

$$f(t,x,v)|_{\Gamma_-} = (1-\iota) f(t,x,\mathcal{R}_x v) + \iota c_\mu \mu(v) \int_{\Gamma_+} f(t,x,v')(v'\cdot n)\,\mathrm{d}v',$$

where $\iota : \partial\Omega \to [0,1]$ is the accommodation coefficient.

The physically most relevant case are Maxwell reflections studied in [156]: it mimics the ideal behaviour of an elastic collision with a wall, where no absorption or emission of particles occur on the boundary.

1.8.2 Results

We show that with these boundary conditions, suitable weak solutions in the sense described above to the spatially inhomogeneous Boltzmann equation are pointwisely decaying at *any polynomial rate* in velocity. Even for classical solutions, such a decay estimate was not known before for moderately soft potentials. This is based on a collaboration with *C. Imbert*.

Theorem 1.7 (Conditional decay estimates)**.** *Let the parameters $\gamma \in (-d, 1]$ and $s \in (0,1)$ of the non-cutoff collision kernel satisfy $\gamma + 2s \in (0,1)$, Let $f_{\mathrm{in}} \geq 0$ (and $f_b \geq 0$) denote the initial data (and the boundary data in the in-flow case).*

Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ domain, let $T \in (0, +\infty]$ and $f \geq 0$ be a suitable weak sub-solution of the Boltzmann equation (1.2) *in $(0,T)\times\Omega\times\mathbb{R}^d$ (in the sense of Definition 7.1) with either in-flow, bounce-back, specular / diffuse / Maxwell reflection boundary conditions satisfying in $\mathcal{D}'((0,T))$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega\times\mathbb{R}^d} f \ln f \,\mathrm{d}x\,\mathrm{d}v + \int_\Omega D(f)\,\mathrm{d}x \leq 0,$$

where $D(f) \geq 0$ denotes the entropy production (given in equation (7.6) *below). We also assume that f satisfies condition* (1.20) *on mass, energy and entropy density functions.*

Then for all $q \geq 0$, if (only in case of in-flow or diffuse / Maxwell reflection)

$$f_b(t,x,v) \leq C_b \left(1 + t^{-\beta_q}\right) \langle v\rangle^{-q} \quad \text{a.e. in } (0,T)\times\Gamma_-,$$

then there exist $C_q > 0$ *and* $\beta_q > 1$*, depending on* $m_0, M_0, E_0, s, d, \gamma$*,* C_b *and* q *such that the solution* f *satisfies,*

$$f(t,x,v) \leq C_q \left(1 + t^{-\beta_q}\right) \langle v \rangle^{-q} \quad a.e.\ in\ (0,T) \times \Omega \times \mathbb{R}^d.$$

Recall $\langle v \rangle = \left(1 + |v|^2\right)^{\frac{1}{2}}$.

Remark 1.3 (Entropy production). The definition of the entropy production $D(f)$ is recalled in Subsection 7.2 below, see (7.6).

Remark 1.4 (Integrability of solutions). The integrability of these suitable weak subsolutions to (1.2) stems from the entropy production estimate (see Lemma 7.2): it implies that $f \in L^1((0,T) \times \Omega; L^{p_0}_{k_0}(\mathbb{R}^d))$ for $\frac{1}{p_0} = 1 - \frac{2s}{d}$ and $k_0 = \gamma + 2s - \frac{2s}{d}$. Consequently, when combined with the hydrodynamical bounds (1.20), we obtain that the functions f we work with are such that $f \in L^{q_0}_{k_0^*}(0,T) \times \Omega \times \mathbb{R}^d)$ with $q_0 = \frac{d+2s}{d}$ and some $k_0^* \in (0,2)$.

Remark 1.5 (Algebraic time decay in v). For $q \in [0, q_{\mathrm{nsl}}]$ (with q_{nsl} given in (vii) below), the time decay exponent is given by the formula $\beta_q = \frac{d}{2s}$. For large q's, it is proportional to a power of q. Precisely, $\beta_q \propto \left(1 + \frac{1}{2s}\right)^{\frac{q}{2s}}$.

Remark 1.6 (Large times). We emphasise that the estimates hold true uniformly in time, even in the case $T = +\infty$. This is important in the conditional regularity programme. Obtaining bounds that are uniform in time will imply that the large time behaviour can be studied in the class of regular solutions, see in particular the work by L. Desvillettes and C. Villani [61].

Remark 1.7 (Generation of moments). We emphasise that the generation of any pointwise moments was unknown in the case of moderately soft potentials, even for classical solutions. In [121], C. Imbert, C. Mouhot and L. Silvestre established pointwise generation of moments with $q \in [0, q_{\mathrm{nsl}}]$ for classical solutions on the torus with rapid decay. But to the best of our knowledge, Theorem 1.7 is the first result showing *the generation of any pointwise moments.*

Again it is the cross term (1.36) that makes the whole proof work: it is the reason for an extra term with a good sign. It proves to be sufficient to show the generation of some amount of polynomial decay, by which we recover decay estimates of [121] but now for suitable weak solutions. To improve this decay, we exploit the non-linearity in the collision operator, which due to the decay that is already generated can be shown to yield an improved cross term with the good sign. The improvement is sufficient to deduce a better polynomial decay, which can be iterated to yield the generation of *any polynomial decay.*

1.9 Landau

Finally, in the last part of the thesis, Part III, we look at the potential with the highest frequency of long-range interactions: using methods developed for the Boltzmann equation in [67, 68, 149], one can construct renormalised solutions to the Landau equation and show rigorously that the Boltzmann equation converges to the Landau equation in a grazing collision limit [2, 60, 184]; even though Lev Landau derived the equation in 1936 [144] by refining Maxwell's computation [156] to describe the dynamics of *charged particles*. We also refer the reader to [147]. We are concerned with the only physically relevant case for the Landau operator, which is the case of Coulomb potentials.

The Landau equation with Coulomb potentials is the most singular of all long-range interactions. It is a fundamental model describing the statistical behaviour of charged particles in collisional plasma, written as

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \mathcal{Q}(f,f)(t,x,v) \qquad (t,x,v) \in (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (1.39)$$

where t is the time variable, x the spatial variable and v is the velocity variable; where $f : \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$ is the unknown distribution function that encodes the density of charged particles, whose dynamics are captured by (1.39), and $\mathcal{Q}$ is the collision operator given by

$$\mathcal{Q}(f,f)(v) := \frac{1}{8\pi} \nabla_v \cdot \left(\int_{\mathbb{R}^3} \frac{\Pi(v-v_*)}{|v-v_*|} \left\{ f_* \nabla f - f \nabla f_* \right\} \, \mathrm{d}v_* \right), \quad (1.40)$$

with the standard convention $f = f(v),\ f_* = f(v_*)$, and with the matrix $\Pi(z)$ that projects any vector onto $z^\perp$,

$$\Pi(z) := \mathrm{Id} - \frac{\mathrm{z} \otimes \mathrm{z}}{|\mathrm{z}|^2}.$$

The mathematical investigation of equation (1.41) has witnessed remarkable activity over the past three decades, leading to significant advancements and culminating in the recent breakthrough result by Guillen and Silvestre in [100].

1.9.1 Coulomb

In this thesis we are merely concerned with the spatially homogeneous case, where we assume the density of the particles to be independent of the spatial variable x, that is $f = f(t,v)$. While this is physically less relevant than the inhomogeneous model, the study of this homogeneous equation already provides several mathematical challenges while at the same time being more tractable. Thus, in the sequel, we will focus our attention on $f = f(t,v)$. Then (1.39) may be rewritten as the following quasi-linear, divergence-form

parabolic equation:

$$\partial_t f(t,v) = \nabla \cdot (A[f]\nabla f - \nabla a[f] f)\,, \tag{1.41}$$

where the coefficients A and a are given by

$$A[f] = \frac{\Pi(v)}{8\pi|v|} * f, \qquad a[f] = (-\Delta)^{-1} f = \frac{1}{4\pi|v|} * f. \tag{1.42}$$

It is also possible to rewrite (1.41) in non-divergence form as:

$$\partial_t f(t,v) = A[f] : \nabla^2 f + f^2, \tag{1.43}$$

which highlights one major challenge in studying this equation, namely the *competition between the reaction term and the nonlinear, non-local diffusion term.* Written in non-divergence form, (1.43) suggests a comparison to the semi-linear heat equation, which is known to blow-up in finite time. This comparison turns out to be misleading. Informally, whenever the reaction term is large, the coefficients $A[f]$ also become large. But it is not clear how to express this quantitatively. The known coercivity of the diffusion coefficients $A[f]$ is not sufficient to bound the reaction term. Eventually, it turned out that the reformulation in non-divergence form (1.43) or even in divergence form (1.41) is not the way we should think about the equation, even if from a regularity point of view, it looks more amenable than the non-local collision operator $\mathcal{Q}$ in (1.40). In a ground-breaking work [100], Guillen and Silvestre manage to find a new Lyapunov functional: they show that the Fisher information is monotonically decreasing. It is unusual to find a new Lyapunov functional for an equation that is as fundamental and well-studied as the Landau equation with Coulomb potentials, whose proof is on top of this non-trivial.

The Fisher information has been studied before in connection with kinetic equations; in particular for the Boltzmann and the Landau equation with Maxwell molecules, which corresponds to $\nu = 2d - 1$ in (1.1). It was known that the Fisher information is monotone decreasing for Maxwell molecules since the works of [185, 187]. It was not expected at the time that the same should be true for soft potentials. The ingeniosity of Guillen-Silvestre was to work with the collision operator (1.40), and to lift the variables $v, v_* \in \mathbb{R}^3$ onto a vector in $\mathbb{R}^6$. In these lifted variables, the collision operator boils down to a *linear, elliptic, non-degenerate* operator composed with a projection of a distribution function in $\mathbb{R}^6$ over its marginals. Global well-posedness of such an equation is standard parabolic regularity theory. But this doesn't tell you much about the original solution of (1.41), since the tensor product is not preserved along the flow of the equation. However, the second ingeniosity was to study the Fisher information in the lifted variables, which turns out to control the Fisher information of the projection over the marginals. They thus argue that if the Fisher information in $\mathbb{R}^6$ decreases along the direction of the lifted collision

operator, then the Fisher information of the projection over the marginals decreases along the direction of the original collision operator $\mathcal{Q}$. Interestingly, the proof of the decrease of the lifted Fisher information eventually boiled down to some kind of log-Sobolev inequality on the sphere.

After this exposition, it might seem unnecessary to continue further discussions on the well-posedness of equation (1.41). We argue however that first the articles in this thesis have to be temporally contextualised (the articles based on Chapter 8 appeared before Guillen-Silvestre [100]), and second that the overreaching aim of this thesis are *low regularity regimes in long-range interactions*. What we discuss below, is first, a local in time existence result for rough initial data in sub-critical norms, which was at the time the strongest result in the literature for Coulomb interactions and as such served as a building block to deduce the global well-posedness from the monotonicity of the Fisher information. Second, we push the global existence result of Guillen-Silvestre from smooth, rapidly decaying initial data with bounded Fisher information to rough, slowly decaying initial data in critical, and eventually, super-critical Lebesgue spaces, without any assumption on the Fisher information of the initial datum.

1.9.2 Maxwellians

We recall that (1.41) formally conserves mass, momentum, and energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} f(t,v)\,\mathrm{d}v = 0. \tag{1.44}$$

Moreover, solutions to (1.41) have decreasing Boltzmann entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f\log(f)\,\mathrm{d}v \leq 0. \tag{1.45}$$

Consequently, the steady states of (1.41) may be characterised explicitly: $f \in L^1_2$ is a steady state of (1.41) if and only if f is a Maxwellian distribution, given by

$$\mu_{\rho,u,\theta}(v) = \frac{\rho}{(2\theta\pi)^{\frac{3}{2}}} e^{-\frac{|v-u|^2}{2\theta}},$$

where $\rho \in \mathbb{R}^+$ is the total mass, $u \in \mathbb{R}^3$ is the mean velocity, and $\theta \in \mathbb{R}^+$ is the temperature of the plasma.

Next, because (1.41) is translation invariant and enjoys a two-parameter scaling invariance, we will assume throughout Part III that our initial datum $f_0(v)$ satisfies the

normalisation

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_0(v) \, \mathrm{d}v = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \tag{1.46}$$

which fixes the corresponding Maxwellian with same mass, momentum and energy as

$$\mu(v) := (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Finally, the conserved quantities and decreasing entropy suggest $L^1_2 \cap L\log(L)$ to be a natural functional space for the problem. In this setting, the existence of very weak solutions to (1.39), typically termed H-solutions, was shown in [186]. These H-solutions satisfy the natural estimates (1.44) and (1.45).

1.9.3 Aim

The contributions that are most relevant to our work from an analytical point of view, can be summarised, albeit incompletely, as follows:

i. The existence and uniqueness of smooth solutions for short times, Chapter 8, based on [81],

ii. Global existence and uniqueness of smooth solutions with initial data near Maxwellians. Most relevant here are the ones by Carrapatoso-Mischler [44], Desvillettes-He-Jiang [62], our result with Golding-Gualdani discussed in Chapter 8 based on [83], along with Guo [101] and related references.

iii. Conditional uniqueness and regularity, focusing on conditions ensuring the global well-posedness of solutions for arbitrarily large times. This line of research includes investigations by Silvestre [178], Gualdani-Guillen [93, 94], Alonso-Bagland-Desvillettes-Lods [8], Fournier [75], and Chern-Gualdani [52].

iv. Partial regularity [86, 87].

v. Study of modified models that pertain the same difficulties of the Landau equation but seem analytically more tractable, explored in works by Gressman-Krieger-Strain [90, 136] and Gualdani-Guillen [93].

vi. Efforts directed at ruling out blow-up profiles initiated with the work of Bedrossian-Gualdani-Snelson [23], culminating in the recent work of Guillen-Silvestre [100]. Noteworthy is the recent proof of blow-up for a modified Landau equation by Chen [49].

In [100], they consider C^1 initial data with Maxwellian upper bounds and show the existence and uniqueness of a global-in-time classical solution to (1.41). Their proof relies

on a novel approach to the equation that enables them to prove that the Fisher information, defined via

$$i(f) := \int_{\mathbb{R}^3} \frac{|\nabla f|^2}{f} \, \mathrm{d}v = 4 \left\| \sqrt{f} \right\|_{\dot{H}^1}^2 ,$$

is monotone decreasing. Then, by a standard Sobolev embedding, the Fisher information controls the L^3 norm uniformly in time. Consequently, the coefficients $A[f]$ and $\nabla a[f]$ are bounded[1] for all times $t \geq 0$. Combined with the known lower bound on $A[f]$, (1.41) becomes a uniformly parabolic equation. Global regularity then follows via several approaches. One approach, taken in [100], is to use the maximum principle to propagate bounds on the initial data. Another approach, taken here, is to use parabolic regularisation estimates to show smoothing effects for less regular initial data (see Theorem 9.2). The important point is that the a-priori estimate on the Fisher information rigorously justifies that (1.41) is a parabolic equation with bounded (uniformly in time) measurable coefficients. This satisfactorily addresses the well-posedness theory for (1.41) for smooth and rapidly decaying initial data.

By contrast, the global well-posedness for rough, slowly decaying initial data, meaning initial data with bounded mass, temperature, and Boltzmann entropy, remains unknown. Notably, regularity and uniqueness for H-solutions, first constructed by Villani in [186] and later revisited by Desvillettes in [58], are still unknown.

For (1.41) all $L^p(\mathbb{R}^3)$ norms with p strictly greater than $\frac{3}{2}$ are sub-critical, in the sense that, for $g \geq 0$,

$$\frac{\int_{\mathbb{R}^3} g^{p+1} \, \mathrm{d}v}{\int_{\mathbb{R}^3} \left|\nabla g^{p/2}\right|^2 \, \mathrm{d}v} \lesssim \left(\int_{\mathbb{R}^3} g^{\frac{3}{2}} \, \mathrm{d}v \right)^{\frac{2}{3}} .$$

The case when $p = \frac{3}{2}$ is critical, whereas the super-critical case is for p below $\frac{3}{2}$. The value $p = \frac{3}{2}$ is a threshold for the boundedness of the coefficients ($A[f]$ can be unbounded for $f \in L^{\frac{3}{2}}$) and also a threshold for time integrability of $\|f(t)\|_{L^\infty}$. Consequently, $p = \frac{3}{2}$ is the threshold at which most known analytical techniques break down and no longer apply.

1.9.4 Results

We are concerned with the well-posedness of the *space-homogeneous* Landau equation for *initial data in low regularity regimes*. We study initial data in the sub-critical, the critical, and super-critical regime, see Chapter 8, Chapter 9 and Chapter 10, respectively.

The strongest result with quantitative and optimal smoothing rates is obtained in the critical regime (and also applies in the sub-critical regime), which is the following:

Theorem 1.8. *(Global-in-time existence in the sub- and critical regime) Fix* $\frac{3}{2} \leq p < \infty$,

[1] Technically, $\nabla a[f]$ belongs to $L^\infty(0,T;L^p)$ with $\frac{3}{2} < p < +\infty$ if $f \in L^\infty(0,T;L^3)$.

$m > 6$, and $m_0 \geq 9/2$, and initial data f_{in} such that

$$f_{in} \in L^p_{m_0} \cap L^1_m \quad \text{and} \quad f_{in} \geq \frac{a}{\langle v \rangle^k}, \quad \text{for some } a > 0 \text{ and } k \leq \frac{m_0 - 3}{p-1}. \tag{1.47}$$

Then, (1.41) *admits a global-in-time smooth solution $f : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^+$ with initial data f_{in}. Moreover, f has decreasing in time Fisher information, which satisfies the short-time smoothing estimate:*

$$\int_{\mathbb{R}^3} \frac{|\nabla f|^2}{f} \, \mathrm{d}v \leq \frac{C(a, p, k, m_0, f_{in})}{t} \qquad \text{for} \qquad 0 < t < 1. \tag{1.48}$$

The local well-posedness is demonstrated by propagating a Lebesgue norm for the corresponding regime for some amount of time. Within this time horizon, we use a De Giorgi argument to show that the solution is essentially bounded; which is enough to show the smoothness of the solution. A limiting procedure makes sure that the initial data is attained continuously. To continue the solution globally, we show that our solution satisfies at some positive time t_0 all conditions of Guillen-Silvestre [100, Theorem 1.2], which yields a global smooth solution. We can then patch together our local solution with Guillen-Silvestre's global solution by the uniqueness result of Fournier [75].

In the super-critical regime, we can construct a global weak solution, that is a strong solution for all strictly positive times. We don't show, however, optimal smoothing estimates up to initial time, and thus we cannot show that the initial data is achieved continuously.

Theorem 1.9 (Super-critical regime)**.** *Let f_{in} be a non-negative function such that*

$$\int_{\mathbb{R}^3} f_{in}(v) \left(1, |v|^{18}, \ln f_{in}\right) \, dv < +\infty.$$

The Landau-Coulomb equation with f_{in} as initial datum has a global weak solution such that for any time $t \geq \varepsilon$ (for all $\varepsilon > 0$ as small as one wishes), this weak solution is a strong solution and its Fisher information is monotone decreasing in time. More precisely, for any $t > 0$,

$$\sup_{[t, +\infty)} \int_{\mathbb{R}^3} \left| \nabla \sqrt{f(t, v)} \right|^2 \, dv \leq C' \, t^{-\frac{9}{2}},$$

where $C' > 0$ only depends on the mass, energy, entropy and L^1_{18} norm (that is the L^1 norm weighted with 18 velocity moments) of the initial datum.

Note that the decay rate of the Fisher information in Theorem 1.9 is not optimal.

1.10 Outline

The first Part I is concerned with a-priori interior local regularity estimates for kinetic integro-differential equations. All results derived in this part are sufficiently strong to be applicable to the non-cutoff Boltzmann equation for moderately soft potentials.

- Chapter 2 states preliminaries, mainly focussing on the fractional Kolmogorov equation which plays a crucial role throughout the thesis.
- Chapter 3 discusses Hölder continuity mainly based on [152], using the trajectories constructed in [10],
- Chapter 4 explains Schauder theory based on [150],
- Chapter 5 establishes the strong Harnack inequalities based on [153, 154],
- Chapter 6 deduces bounds on the fundamental solution to the linearised equation based on [153, 154].

The second Part II consists only of

- Chapter 7 which is based on [113],

with *C. Imbert*, in which we deal with a-priori boundary regularity estimates of the Boltzmann equation without cutoff for moderately soft potentials. We show that suitable weak solutions decay at any polynomial rate. The last Part III discusses the construction of solutions to the space-homogeneous Landau equation for initial data in low regularity regimes. We study initial data

- in the sub-critical regime, see Chapter 8, based on [81, 83],
- which we improved to the critical regime as outlined in Chapter 9 based on [84],
- and eventually even to the super-critical regime in Chapter 10 based on [63].

These are collaborations with *W. Golding, M. Gualdani* and *L. Desvillettes.*

Part I

Interior regularity for hypoelliptic non-local equations

CHAPTER 2

KOLMOGOROV

2.1 Fractional Kolmogorov equation

If we want to understand a partial differential equation with rough coefficients, then it proves to be useful to consider the case of constant coefficients first. It turns out that equation (1.13) with constant coefficients admits a fundamental solution, which has explicitly been written down in the limiting case $s = 1$ by Andrey Kolmogorov in [135], after whom the equation was hitherto named. Since we deal with the fractional case where $s \in (0, 1)$, we denote the following

$$\partial_t f + v \cdot \nabla_x f = (-\Delta_v)^s f + h, \tag{2.1}$$

as the *fractional Kolmogorov equation*, with source term $h \in L^2(\mathbb{R}_+ \times \mathbb{R}^{2d})$, and where $(-\Delta_v)^s$ denotes the fractional Laplacian, defined as

$$(-\Delta_v)^s f(v) = C(d, s) \int_{\mathbb{R}^d} \frac{f(v) - f(w)}{|v - w|^{d+2s}} \, \mathrm{d}w. \tag{2.2}$$

2.1.1 Fundamental solution

Equation (2.1) admits a fundamental solution, given through an implicit representation,

$$J_0(t, x, v) = C(d, s) t^{-\frac{2d(1+s)}{2s}} \mathcal{J}_0 \left(\frac{x}{t^{1+\frac{1}{2s}}}, \frac{v}{t^{\frac{1}{2s}}} \right), \tag{2.3}$$

where $\mathcal{J}_0$ reads explicitly in Fourier variables

$$\hat{\mathcal{J}}_0(\varphi, \xi) = \exp\left(- \int_0^1 |\xi - \tau\phi|^{2s} \, \mathrm{d}\tau \right).$$

In the limiting case where $s = 1$, the fundamental solution is given explicitly through

$$J(t,x,v) = \left(\frac{\sqrt{3}}{2\pi t^2}\right)^d \exp\Big(\frac{-3\left|x+\frac{tv}{2}\right|}{t^3} - \frac{|v|^2}{4t}\Big), \quad t>0,$$

and $J = 0$ for $t \leq 0$.

2.1.2 Gain of integrability

We can use the representation of the fundamental solution to derive a gain of integrability in Lebesgue spaces on the solution to the equation with rough coefficients. To match the integrability level that we get from the standard energy estimate, we generally use a gain of integrability in L^2. This proves to be the key preliminary estimate for the First De Giorgi Lemma 3.4. However, for the Strong Harnack inequality, we need local estimates on the tail quantities, which we only control in L^1. Thus, for the proof of Proposition 5.1, we require a gain of integrability in L^1.

For this, we consider the fractional Kolmogorov equation (2.1), which plays the role of the constant coefficient equation for (1.13), with a right hand side $h = h_1 + h_2$ for some $h_2 \in L^1\big([0,\tau]\times\mathbb{R}^{2d}\big)$ and some h_1 such that

$$\left\|(-\Delta_v)^{-\frac{s+\epsilon}{2}} h_1\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} < +\infty$$

for $0 \leq \epsilon < s$. The role of the ϵ is the number of derivatives that we can put on the fundamental solution so that these are absorbed in the time integrability. We prove:

Proposition 2.1 (Gain of integrability)**.** *Let $0 \leq f$ solve (or be a sub-solution of)* (2.1) *in $[0,\tau]\times\mathbb{R}^{2d}$, with $f(0,x,v) = f_0(x,v) \in L^1\cap L^2(\mathbb{R}^{2d})$, and with $h = h_1 + h_2$ where $h_1, h_2 \in L^1\cap L^2([0,\tau]\times\mathbb{R}^{2d})$. Then, for any $2 \leq p_2 < 2 + \frac{2s}{d(1+s)}$ there holds*

$$\begin{aligned}\|f\|_{L^{p_2}([0,\tau]\times\mathbb{R}^{2d})} \lesssim & \tau^{\frac{1}{q}-\alpha_{0,q}} \left\|(-\Delta_v)^{-\frac{s}{2}} h_1\right\|_{L^2([0,\tau]\times\mathbb{R}^{2d})} + \tau^{\frac{1}{q}+\frac{1}{2}-\alpha_{0,q}} \|h_2\|_{L^2([0,\tau]\times\mathbb{R}^{2d})} \\ & + \tau^{\frac{1}{2}-\alpha_{0,q}} \|f_0\|_{L^2(\mathbb{R}^{2d})},\end{aligned} \tag{2.4}$$

where $\frac{1}{q} = \frac{1}{p_2} + \frac{1}{2}$ and

$$\alpha_{0,q} = d\left(1+\frac{1}{s}\right)\left(1-\frac{1}{q}\right) + \frac{1}{2}.$$

Moreover, for any $0 \leq \epsilon < s$ and any $1 \leq p_1 < 1 + \frac{s-\epsilon}{s+2d(s+1)+\epsilon}$, if

$$\left\|(-\Delta_v)^{-\frac{s+\epsilon}{2}} h_1\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} < +\infty,$$

then there holds

$$\|f\|_{L^{p_1}([0,\tau]\times\mathbb{R}^{2d})} \lesssim \tau^{\frac{1}{p_1}-\alpha_{\epsilon,p_1}} \left\|(-\Delta_v)^{-\frac{s+\epsilon}{2}} h_1\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} + \tau^{\frac{1}{p_1}+\frac{1}{2}-\alpha_{0,p_1}} \|h_2\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} + \tau^{\frac{1}{p_1}+\frac{1}{2}-\alpha_{0,p_1}} \|f_0\|_{L^1(\mathbb{R}^{2d})},$$

where

$$\alpha_{\epsilon,p_1} = d\left(1+\frac{1}{s}\right)\left(1-\frac{1}{p_1}\right)+\frac{s+\epsilon}{2s}.$$

Proof. Equation (2.1) admits a fundamental solution, given by

$$\begin{aligned} 0 &\le f(t,x,v) \\ &\le \int_{\mathbb{R}^{2d+1}} \left[h_1(t',x',v')+h_2(t',x',v')\right] J\left(t-t',x-x'-(t-t')v',v-v'\right)\,\mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}v' \\ &\quad + \int_{\mathbb{R}^{2d}} f_0(x',v') J\left(t,x-x'-(t-t')v',v-v'\right)\,\mathrm{d}x'\,\mathrm{d}v', \end{aligned}$$

where $(t,x,v)\in(0,\tau)\times\mathbb{R}^{2d}$ and

$$J(t,x,v) = \frac{C_d}{t^{d+\frac{d}{s}}}\mathcal{J}\left(\frac{x}{t^{1+\frac{1}{2s}}},\frac{v}{t^{\frac{1}{2s}}}\right), \qquad \hat{\mathcal{J}}(\varphi,\xi) = \exp\left(-\int_0^1 |\xi-\tau\varphi|^{2s}\,\mathrm{d}\tau\right). \tag{2.5}$$

We remark that for any $r\ge 1$ there holds for $t>0$ and for any $\epsilon\ge 0$

$$\begin{aligned} \|J(t,\cdot,\cdot)\|_{L^r(\mathbb{R}^{2d})} &= t^{-d(1+\frac{1}{s})(1-\frac{1}{r})}\|\mathcal{J}\|_{L^r(\mathbb{R}^{2d})}, \\ \left\|(-\Delta)_v^{\frac{s+\epsilon}{2}} J(t,\cdot,\cdot)\right\|_{L^r(\mathbb{R}^{2d})} &= t^{-d(1+\frac{1}{s})(1-\frac{1}{r})-\frac{s+\epsilon}{2s}}\left\|(-\Delta)_v^{\frac{s+\epsilon}{2}}\mathcal{J}\right\|_{L^r(\mathbb{R}^{2d})}. \end{aligned}$$

(The factor $t^{-d(1+\frac{1}{s})}$ comes from the prefactor in (2.5). The fractional derivative $(-\Delta)_v^{\frac{s+\epsilon}{2}}$ gives a factor in time of $t^{-\frac{s+\epsilon}{2s}}$. Finally the L^r norm in space and velocity scales as $t^{\frac{d}{r}(\frac{2+2s}{2s})}$.) In particular for $r=p^*:=\frac{2d(1+s)+2s}{2d(1+s)+s}\in(1,2)$ we deduce

$$\begin{aligned} \|J(t,\cdot,\cdot)\|_{L^{p^*}(\mathbb{R}^{2d})} &= t^{\frac{1}{2}-\frac{1}{p^*}}\|\mathcal{J}\|_{L^{p^*}(\mathbb{R}^{2d})}, \\ \left\|(-\Delta)_v^{\frac{s}{2}} J(t,\cdot,\cdot)\right\|_{L^{p^*}(\mathbb{R}^{2d})} &= t^{-\frac{1}{p^*}}\left\|(-\Delta)_v^{\frac{s}{2}}\mathcal{J}\right\|_{L^{p^*}(\mathbb{R}^{2d})}. \end{aligned}$$

We define the modified convolution

$$f *_t g(x,v) := \int_{\mathbb{R}^{2d}} f(x',v')g(x-x'-tv',v-v')\,\mathrm{d}x'\,\mathrm{d}v'.$$

Note that the modified convolution satisfies the usual Young inequality independent of t:

$$\|f *_t g\|_{L^r_{x,v}} \le \|f\|_{L^p_{x,v}}\|g\|_{L^q_{x,v}}$$

for $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. We split $f=\tilde{f}_0+f_1+f_2$ with

$$\begin{aligned}
\tilde{f}_0(t,\cdot,\cdot) &:= f_0 *_t J(t,\cdot,\cdot),\\
f_1^{(\epsilon)}(t,\cdot,\cdot) &:= \int_0^t (-\Delta)_v^{-\frac{s+\epsilon}{2}} h_1 *_{(t-t')} (-\Delta)_v^{\frac{s+\epsilon}{2}} J(t-t',\cdot,\cdot)\,\mathrm{d}t',\\
f_2(t,\cdot,\cdot) &:= \int_0^t h_2 *_{(t-t')} J(t-t',\cdot,\cdot)\,\mathrm{d}t'.
\end{aligned}$$

By Young's inequality we get for

$$\alpha_{0,p_1} = d\left(1+\frac{1}{s}\right)\left(1-\frac{1}{p_1}\right)+\frac{1}{2}$$

and for

$$\alpha_{\epsilon,p_1} = d\left(1+\frac{1}{s}\right)\left(1-\frac{1}{p_1}\right)+\frac{s+\epsilon}{2s}$$

we find

$$\begin{aligned}
\left\|\tilde{f}_0(t,\cdot,\cdot)\right\|_{L^p(\mathbb{R}^{2d})} &\le \|f_0\|_{L^1(\mathbb{R}^{2d})}\,\|J(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}\,\|\mathcal{J}\|_{L^p(\mathbb{R}^{2d})}\,t^{\frac{1}{2}-\alpha_0},\\
\left\|f_1^{(\epsilon)}(t,\cdot,\cdot)\right\|_{L^p(\mathbb{R}^{2d})} &\le \int_0^t \left\|(-\Delta)_v^{-\frac{s+\epsilon}{2}} h_1(t')\right\|_{L^1(\mathbb{R}^{2d})} \left\|(-\Delta)_v^{\frac{s+\epsilon}{2}} J(t-t',\cdot,\cdot)\right\|_{L^p(\mathbb{R}^{2d})}\,\mathrm{d}t'\\
&= \int_0^t \left\|(-\Delta)_v^{-\frac{s+\epsilon}{2}} h_1(t')\right\|_{L^1(\mathbb{R}^{2d})} \left\|(-\Delta)_v^{\frac{s+\epsilon}{2}} \mathcal{J}\right\|_{L^p(\mathbb{R}^{2d})} (t-t')^{-\alpha_\epsilon}\,\mathrm{d}t',\\
\|f_2(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^{2d})} &\le \int_0^t \|h_2(t')\|_{L^1(\mathbb{R}^{2d})}\,\|J(t-t',\cdot,\cdot)\|_{L^p(\mathbb{R}^{2d})}\,\mathrm{d}t'\\
&= \int_0^t \|h_2(t')\|_{L^1(\mathbb{R}^{2d})}\,\|\mathcal{J}\|_{L^p(\mathbb{R}^{2d})}\,(t-t')^{\frac{1}{2}-\alpha_0}\,\mathrm{d}t'.
\end{aligned}$$

Since $p_1\left(\frac{1}{2}-\alpha_{0,p_1}\right)>-1$ we get that $\tilde{f}_0\in L^{p_1}([0,\tau]\times\mathbb{R}^{2d})$ and

$$\left\|\tilde{f}_0\right\|_{L^{p_1}([0,\tau]\times\mathbb{R}^{2d})} \le C\tau^{\frac{1}{p_1}+\frac{1}{2}-\alpha_0}\,\|f_0\|_{L^1(\mathbb{R}^{2d})}.$$

For $f_1^{(\epsilon)}$ and f_2 we apply Young's inequality again and get

$$\begin{aligned}
\left\|f_1^{(\epsilon)}\right\|_{L^{p_1}([0,\tau]\times\mathbb{R}^{2d})} &\le C\left\|(-\Delta)_v^{-\frac{s+\epsilon}{2}} h_1\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})}\,\tau^{\frac{1}{p_1}-\alpha_{\epsilon,p_1}},\\
\|f_2\|_{L^{p_1}([0,\tau]\times\mathbb{R}^{2d})} &\le C\,\|h_2\|_{L^1([0,\tau]\times\mathbb{R}^{2d})}\,\tau^{\frac{1}{p_1}+\frac{1}{2}-\alpha_{0,p_1}},
\end{aligned}$$

provided that $\frac{1}{p_1}-\alpha_{\epsilon,p_1}>0$. Equivalently, we require

$$1\le p_1 < 1+\frac{s-\epsilon}{s+2d(s+1)+\epsilon}.$$

Similarly, we prove (2.4). Let $q \in [1, p^*)$ be such that $\frac{1}{p_2} = \frac{1}{q} - \frac{1}{2}$. By Young's inequality we get for $\alpha_{0,q} = d\left(1 + \frac{1}{s}\right)\left(1 - \frac{1}{q}\right) + \frac{1}{2} < \frac{1}{p^*} < \frac{1}{q}$

$$\begin{aligned}
\left\|\tilde{f}_0(t,\cdot,\cdot)\right\|_{L^{p_2}(\mathbb{R}^{2d})} &\leq \|f_0\|_{L^2(\mathbb{R}^{2d})} \|J(t,\cdot,\cdot)\|_{L^q(\mathbb{R}^{2d})} = \|f_0\|_{L^2(\mathbb{R}^{2d})} \|\mathcal{J}\|_{L^q(\mathbb{R}^{2d})} t^{\frac{1}{2}-\alpha_{0,q}}, \\
\left\|f_1^{(0)}(t,\cdot,\cdot)\right\|_{L^{p_2}(\mathbb{R}^{2d})} &\leq \int_0^t \left\|(-\Delta)_v^{-\frac{s}{2}} h_1(t')\right\|_{L^2(\mathbb{R}^{2d})} \left\|(-\Delta)_v^{\frac{s}{2}} J(t-t',\cdot,\cdot)\right\|_{L^q(\mathbb{R}^{2d})} \mathrm{d}t' \\
&= \int_0^t \left\|(-\Delta)_v^{-\frac{s}{2}} h_1(t')\right\|_{L^2(\mathbb{R}^{2d})} \left\|(-\Delta)_v^{\frac{s}{2}} \mathcal{J}\right\|_{L^q(\mathbb{R}^{2d})} (t-t')^{-\alpha_{0,q}} \mathrm{d}t', \\
\|f_2(t,\cdot,\cdot)\|_{L^{p_2}(\mathbb{R}^{2d})} &\leq \int_0^t \|h_2(t')\|_{L^2(\mathbb{R}^{2d})} \|J(t-t',\cdot,\cdot)\|_{L^q(\mathbb{R}^{2d})} \mathrm{d}t' \\
&= \int_0^t \|h_2(t')\|_{L^2(\mathbb{R}^{2d})} \|\mathcal{J}\|_{L^q(\mathbb{R}^{2d})} (t-t')^{\frac{1}{2}-\alpha_{0,q}} \mathrm{d}t'.
\end{aligned}$$

Since $p_2\left(\frac{1}{2} - \alpha_{0,q}\right) > -1$ we get that $\tilde{f}_0 \in L^{p_2}([0,\tau] \times \mathbb{R}^{2d})$ and

$$\left\|\tilde{f}_0\right\|_{L^{p_2}([0,\tau]\times\mathbb{R}^{2d})} \leq C \|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

For $f_1^{(0)}$ and f_2 we apply Young's inequality again and get

$$\begin{aligned}
\left\|f_1^{(0)}\right\|_{L^{p_2}([0,\tau]\times\mathbb{R}^{2d})} &\leq C \left\|(-\Delta)_v^{-\frac{s}{2}} h_1\right\|_{L^2([0,\tau]\times\mathbb{R}^{2d})} \tau^{\frac{1}{q}-\alpha_{0,q}}, \\
\|f_2\|_{L^{p_2}([0,\tau]\times\mathbb{R}^{2d})} &\leq C \|h_2\|_{L^2([0,\tau]\times\mathbb{R}^{2d})} \tau^{\frac{1}{2}-\alpha_{0,q}+\frac{1}{q}}.
\end{aligned}$$

This implies (2.4).

□

2.1.3 Scaling

Similarly to Proposition 2.1 of [114] we have

Lemma 2.1. *Given $h \in L^\infty(\mathbb{R} \times \mathbb{R}^{2d})$ with compact support in time, the function*

$$f(t,x,v) = \int_{\mathbb{R}\times\mathbb{R}^{2d}} h(\tilde{t},\tilde{x},\tilde{v}) J(t-\tilde{t}, x-\tilde{x}-(t-\tilde{t})v, v-\tilde{v}) \,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v} =: J *_{kin} h(z)$$

solves (2.1) *in $\mathbb{R} \times \mathbb{R}^{2d}$. Moreover, for all $z_0 \in \mathbb{R} \times \mathbb{R}^{2d}$ and $r > 0$ there holds*

$$\left\|J *_{kin} \mathbb{1}_{Q_r(z_0)}\right\|_{L^\infty(Q_r(z_0))} \leq C r^{2s},$$

for some universal constant C depending on d.

Proof. Given $z = (t,x,v) \in Q_r(z_0)$ we compute the scaling of the fundamental solution

stemming from the parabolicity of the equation

$$\begin{aligned} J *_{\text{kin}} \mathbb{1}_{Q_r(z_0)}(t,x,v) &= \int_{Q_r(z_0)} J(t-\tilde{t}, x-\tilde{x}-(t-\tilde{t})v, v-\tilde{v})\,\mathrm{d}\tilde{z} \\ &= r^{2s}\int_{Q_1(z_0)} J\left(\frac{t}{r^{2s}}-\bar{t}, \frac{x}{r^{1+2s}}-\bar{x}-\left(\frac{t}{r^{2s}}-\bar{t}\right)v, \frac{v}{r}-\bar{v}\right)\mathrm{d}\bar{z} \\ &= r^{2s} J *_{\text{kin}} \mathbb{1}_{Q_1(z_0)}\left(\frac{t}{r^{2s}}, \frac{x}{r^{1+2s}}, \frac{v}{r}\right), \end{aligned}$$

and conclude. □

2.2 Preliminaries

Here we collect technical arguments that are used throughout different parts of the thesis further below.

2.2.1 Iteration argument

Finally, we use a standard iteration argument, see for example [79, Lemma 8.18].

Lemma 2.2. *Let $\phi : [0,T] \to \mathbb{R}$ be a non-negative bounded function. Suppose that for $0 \le r < R \le T$ we have*

$$\phi(r) \le A(R-r)^{-\alpha} + \delta\phi(R)$$

for some $A, \alpha > 0$, $0 \le \delta < 1$. Then there exists a constant c depending on α, δ such that for $0 \le r < R \le T$ we have

$$\phi(r) \le cA(R-r)^{-\alpha}.$$

Proof. We consider for some $\tau \in (0,1)$ a sequence given by

$$\begin{cases} r_0 := r, \\ r_{k+1} := r_k + (1-\tau)\tau^k(R-r), \quad k \ge 0. \end{cases}$$

Since

$$\sum_{k=1}^{\infty} \tau^k = \frac{\tau}{1-\tau},$$

we have that $r_k < R$ and proceeding by induction, we deduce

$$\phi(r_0) \le \delta^k \phi(r_k) + A(1-\tau)^{-\alpha}(R-r)^{-\alpha}\sum_{i=0}^{k-1}\delta^i\tau^{-i\alpha}.$$

Choosing τ such that $\delta\tau^{-\alpha} < 1$ and letting $k \to \infty$ implies

$$\phi(r) \leq cA(R-r)^{-\alpha}.$$

□

2.2.2 Non-local product rule

The second observation is a non-local analogue of the product rule.

Lemma 2.3 (Commutator estimates [115, Lemmata 4.10, 4.11])**.** *Let K be a non-negative kernel satisfying (1.32). Let D be a closed set and Ω open such that $D \Subset \Omega \subset B_{\bar{R}/2} \subset \mathbb{R}^d$ for $0 < \bar{R} \leq 2$. Let φ be a smooth function with support in D, let $f \in H^s(\Omega) \cap L^\infty(\mathbb{R}^d)$ and let $\rho = \frac{dist(D, \mathbb{R}^d \setminus \Omega)}{2}$. We write*

$$\mathcal{L}[\varphi f] - \varphi \mathcal{L} f = h_1 + h_2,$$

where h_1, h_2 are given by

$$\begin{aligned} h_1(v) &= \int_{\mathbb{R}^d \setminus B_\rho(v)} f(w)\big(\varphi(w) - \varphi(v)\big) K(v,w)\,\mathrm{d}w, \\ h_2(v) &= \int_{B_\rho(v)} f(w)\big(\varphi(w) - \varphi(v)\big) K(v,w)\,\mathrm{d}w. \end{aligned}$$

Then, by construction $h_2 = 0$ outside Ω, and there holds

$$\|h_1\|_{L^2(\mathbb{R}^d \setminus \Omega)} \leq \Lambda \rho^{-2s} \|\varphi\|_{L^\infty} \|f\|_{L^2(D)}. \tag{2.6}$$

Moreover, if $s \in (0, 1/2)$, there holds

$$\|h_2\|_{L^2(\mathbb{R}^d)} \leq \Lambda \rho^{1-2s} \|\varphi\|_{C^1} \|f\|_{L^2(\Omega)}. \tag{2.7}$$

Else if $s \in [1/2, 1)$ and if additionally K satisfies (1.28), then there exists $h_{22}, h_{23} \in L^2(\mathbb{R}^d)$ such that

$$h_2 = h_{22} + (-\Delta_v)^{\frac{s}{2}} h_{23},$$

with

$$\begin{aligned} \|h_{22}\|_{L^2(\mathbb{R}^d)} &\leq \Lambda \rho^{2-2s} \|\varphi\|_{C^2} \|f\|_{L^2(\Omega)} + \Lambda \rho^{1-s} \|\varphi\|_{C^1} \|f\|_{H^s(\Omega)}, \\ \|h_{23}\|_{L^2(\mathbb{R}^d)} &\leq \Lambda \rho^{1-s} \|\varphi\|_{C^1} \|f\|_{L^2(\Omega)}. \end{aligned} \tag{2.8}$$

We reprove this lemma to make the dependence on ρ in (2.7) and (2.8) precise. We need it for the optimal exponent on the radius in Proposition 5.1.

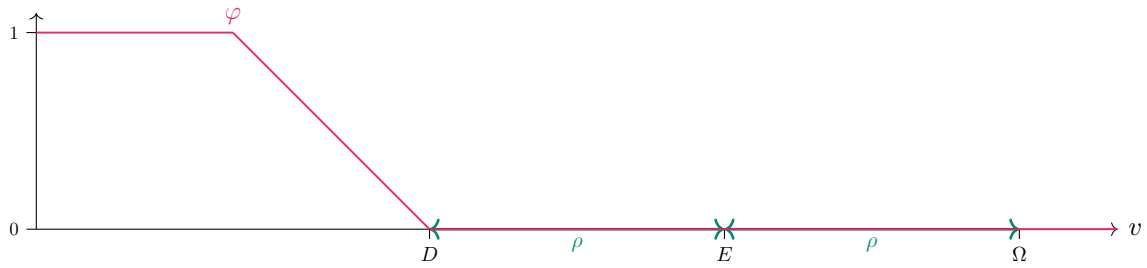


Figure 2.1: Illustration of the smooth cutoff φ from Lemma 2.3 with which we commute f, and the supports of the cutoff and the local contribution of the commutator. Note $D = \operatorname{supp} \varphi$, and $E = \operatorname{supp} h_2$.

Proof. We let $E = D + B_\rho$ so that $D \Subset E \Subset \Omega$ with $\operatorname{dist}(D, \mathbb{R}^d \setminus E) = \rho$ and $\operatorname{dist}(E, \mathbb{R}^d \setminus \Omega) = \rho$.

To bound h_1, we notice that $\varphi(v) = 0$ for $v \notin D$. Thus if $v \notin \Omega \supset D$, then

$$h_1 = \int_{\mathbb{R}^d \setminus B_\rho(v)} f(w)\varphi(w)K(v,w)\,\mathrm{d}w = \int_D f(w)\varphi(w)K(v,w)\,\mathrm{d}w.$$

Therefore, using Cauchy-Schwarz, (1.32) and Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus\Omega} h_1^2\,\mathrm{d}v &= \int_{\mathbb{R}^d\setminus\Omega} \left(\int_D f(w)\varphi(w)K(v,w)\,\mathrm{d}w \right)^2 \mathrm{d}v \\
&\le \|\varphi\|_{L^\infty}^2 \int_{\mathbb{R}^d\setminus\Omega} \left(\int_D f^2(w)K(v,w)\,\mathrm{d}w \right) \left(\int_D K(v,w)\,\mathrm{d}w \right) \mathrm{d}v \\
&\le \Lambda\rho^{-2s} \|\varphi\|_{L^\infty}^2 \int_D f^2(w) \int_{|v-w|\ge 2\rho} K(v,w)\,\mathrm{d}v\,\mathrm{d}w \\
&\le \Lambda^2\rho^{-4s} \|\varphi\|_{L^\infty}^2 \|f\|_{L^2(D)}^2 .
\end{aligned}$$

This yields (2.6).

To bound h_2 we first consider $s \in (0, 1/2)$. We use Cauchy-Schwarz, (1.32) for $s < 1/2$

and Fubini

$$
\begin{aligned}
&\|h_2\|_{L^2(\mathbb{R}^d)}^2 \\
&= \int_E \left(\int_{B_\rho(v)} f(w)\big(\varphi(w)-\varphi(v)\big)K(v,w)\,\mathrm{d}w \right)^2 \mathrm{d}v \\
&\le \int_E \left(\int_{B_\rho(v)} f^2(w)\,|\varphi(w)-\varphi(v)|\,K(v,w)\,\mathrm{d}w \right) \left(\int_{B_\rho(v)} |\varphi(w)-\varphi(v)|\,K(v,w)\,\mathrm{d}w \right) \mathrm{d}v \\
&\le \|\varphi\|_{C^1}^2 \int_E \left(\int_{B_\rho(v)} f^2(w)\,|v-w|\,K(v,w)\,\mathrm{d}w \right) \left(\int_{B_\rho(v)} |v-w|\,K(v,w)\,\mathrm{d}w \right) \mathrm{d}v \\
&\le \Lambda \rho^{1-2s} \|\varphi\|_{C^1}^2 \int_\Omega f^2(w) \int_{E\cap B_\rho(w)} |v-w|\,K(v,w)\,\mathrm{d}v\,\mathrm{d}w \\
&\le \Lambda^2 \rho^{2-4s} \|\varphi\|_{C^1}^2 \|f\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

This yields (2.7).

Second we consider $s \in [1/2, 1)$. Due to (1.28), we find

$$
h_2 = \underbrace{\frac{1}{2} \int_{B_\rho(v)} f(w)\big(\varphi(w)-\varphi(v)\big)\big(K(v,w)+K(w,v)\big)\,\mathrm{d}w}_{=:h_2^{\mathrm{sym}}} .
$$

We estimate h_2^{sym} by duality. Let $g \in H^s(\mathbb{R}^d)$. Then, since $\operatorname{supp} h_2^{\mathrm{sym}} \subseteq E$, we have

$$
\begin{aligned}
&\int_E h_2^{\mathrm{sym}}(v) g(v)\,\mathrm{d}v \\
&\quad = \int_E \int_{B_\rho(v)} g(v) f(w)\big(\varphi(w)-\varphi(v)\big)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad = \frac{1}{2} \int_\Omega \int_{\Omega\cap|v-w|<\rho} f(v)\big(g(v)-g(w)\big)\big(\varphi(w)-\varphi(v)\big)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad + \frac{1}{2} \int_\Omega \int_{\Omega\cap|v-w|<\rho} g(v)\big(f(w)-f(v)\big)\big(\varphi(w)-\varphi(v)\big)K(v,w)\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
$$

Thus by Cauchy-Schwarz, (1.32) and Theorem 3.4

$$
\begin{aligned}
&\int_E h_2^{\mathrm{sym}}(v)g(v)\,\mathrm{d}v \\
&\leq \|f\|_{L^2(\Omega)} \left\{ \int_\Omega \left(\int_{\Omega\cap|v-w|<\rho} \big(g(v)-g(w)\big)\big(\varphi(w)-\varphi(v)\big)K(v,w)\,\mathrm{d}w \right)^2 \mathrm{d}v \right\}^{\frac{1}{2}} \\
&\quad + \|g\|_{L^2(\Omega)} \left\{ \int_\Omega \left(\int_{\Omega\cap|v-w|<\rho} \big(f(w)-f(v)\big)\big(\varphi(w)-\varphi(v)\big)K(v,w)\,\mathrm{d}w \right)^2 \mathrm{d}v \right\}^{\frac{1}{2}} \\
&\leq \|f\|_{L^2(\Omega)} \left\{ \int_\Omega \left(\int_{\Omega\cap|v-w|<\rho} \big(g(v)-g(w)\big)^2 K(v,w)\,\mathrm{d}w \right) \right. \\
&\qquad\qquad \left. \times \left(\int_{\Omega\cap|v-w|<\rho} |\varphi(w)-\varphi(v)|^2 K(v,w)\,\mathrm{d}w \right) \mathrm{d}v \right\}^{\frac{1}{2}} \\
&\quad + \|g\|_{L^2(\Omega)} \left\{ \int_\Omega \left(\int_{\Omega\cap|v-w|<\rho} \big(f(w)-f(v)\big)^2 K(v,w)\,\mathrm{d}w \right) \right. \\
&\qquad\qquad \left. \times \left(\int_{\Omega\cap|v-w|<\rho} |\varphi(w)-\varphi(v)|^2 K(v,w)\,\mathrm{d}w \right) \mathrm{d}v \right\}^{\frac{1}{2}} \\
&\leq \Lambda^{\frac{1}{2}}\rho^{1-s}\,\|f\|_{L^2(\Omega)}\,\|\varphi\|_{C^1} \left\{ \int_\Omega \int_{\Omega\cap|v-w|<\rho} \big(g(v)-g(w)\big)^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v \right\}^{\frac{1}{2}} \\
&\quad + \Lambda^{\frac{1}{2}}\rho^{1-s}\,\|g\|_{L^2(\Omega)}\,\|\varphi\|_{C^1} \left\{ \int_\Omega \int_{\Omega\cap|v-w|<\rho} \big(f(w)-f(v)\big)^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v \right\}^{\frac{1}{2}} \\
&\leq \Lambda\rho^{1-s}\,\|\varphi\|_{C^1} \Big(\|f\|_{L^2(\Omega)}\,\|g\|_{H^s(\Omega)} + \|g\|_{L^2(\Omega)}\,\|f\|_{H^s(\Omega)} \Big).
\end{aligned}
$$

This estimate implies that there exists h_{22}, h_{23} such that

$$
h_2 = h_2^{\mathrm{sym}} = h_{22}^{\mathrm{sym}} + (-\Delta_v)^{\frac{s}{2}} h_{23}^{\mathrm{sym}},
$$

with

$$
\|h_{22}\|_{L^2(\mathbb{R}^d)} \leq \Lambda\rho^{1-s}\,\|\varphi\|_{C^1}\,\|f\|_{H^s(\Omega)}\,, \qquad \|h_{23}\|_{L^2(\mathbb{R}^d)} \leq \Lambda\rho^{1-s}\,\|\varphi\|_{C^1}\,\|f\|_{L^2(\Omega)}\,.
$$

□

CHAPTER 3

De Giorgi

3.1 Introduction

This chapter is based on [152], but we incorporate some ideas from [10].

3.1.1 Problem formulation

The question we raise is whether solutions to (1.13)-(1.14) satisfy Hölder continuity and Harnack inequalities. The weak Harnack inequality and Hölder continuity have been obtained by Cyril Imbert and Luis Silvestre in [115], yet part of their proof relies on an argument by contradiction. Our aim is to develop a constructive method to deduce the results.

3.1.2 Contribution

Our contribution consists in a quantitative proof of regularity for non-local kinetic equations of type (1.13)-(1.14). Our results are applicable to the non-cutoff Boltzmann equation. We prove Harnack inequalities and Hölder continuity. The technical assumptions that are required for these statements were discussed above in (1.26)-(1.30). In short they define a suitable notion of ellipticity on the non-local coefficients of (1.13). The analogue statement for local equations requires the coefficients to be uniformly elliptic, that is the eigenvalues of the diffusion matrix lie in $[\lambda, \Lambda]$ for given $0 < \lambda \leq \Lambda$. The parameter $s \in (0, 1)$ arises from the non-locality of (1.13) and is hidden in the conditions on the kernel. The first main result is the weak Harnack inequality derived in **Theorem 1.1**. In particular, this implies Hölder continuity.

Theorem 3.1 (Hölder continuity)**.** *Let f be a non-negative weak solution of* (1.13)-(1.14) *in $[-3, 0] \times B_1 \times B_1$ with a non-negative kernel K satisfying* (1.26)-(1.30)*. Assume f is*

essentially bounded in $(-3,0]\times B_1\times\mathbb{R}^d$. *Then* f *is Hölder continuous in* $Q_{\frac{1}{2}}$ *with Hölder exponent* $\alpha\in(0,1)$ *depending on* $s,d,\lambda,\Lambda,\|h\|_{L^\infty(Q_1)}$ *with*

$$\begin{aligned}[f]_{C^\alpha(Q_{\frac{1}{2}})} &:= \sup_{z_1,z_2\in Q_{\frac{1}{2}},z_1\neq z_2}\frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha}\\ &\leq C\big(1+\|h\|_{L^\infty(Q_1)}\big)\Big(\|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)}+\|h\|_{L^\infty(Q_1)}\Big),\end{aligned}\tag{3.1}$$

where C *depends on* d,s,λ *and* Λ.

Finally, we are able to show a *non-linear* strong Harnack inequality.

Theorem 3.2 (Not-so-strong Harnack inequality)**.** *Let* f *be a non-negative solution to* (1.13)-(1.14) *in* $[-3,0]\times B_1\times B_1$ *with a non-negative kernel* K *satisfying* (1.26)-(1.30). *Assume* $0\leq f\leq 1$ *in* $(-3,0]\times B_1\times\mathbb{R}^d$. *Then for any* $p\in\left[2,2+\frac{2s}{d(1+s)}\right)$ *there is* $C>0$ *and* $\beta\in(0,1)$ *depending on* s,d,λ,Λ,p *such that for* $r_0<\frac{1}{3}$

$$\sup_{\tilde{Q}^-_{\frac{r_0}{4}}} f\leq C\left(\inf_{Q_{\frac{r_0}{4}}} f+\|h\|_{L^\infty(Q_1)}\right)^\beta,\tag{3.2}$$

where $\tilde{Q}^-_{\frac{r_0}{4}}:=Q_{\frac{r_0}{4}}\big((-\frac{5}{2}r_0^{2s}+\frac{1}{2}(\frac{r_0}{2})^{2s},0,0)\big)$.

The proof of the Weak, not-so-strong Harnack, and the Hölder continuity relies on the **Intermediate Value Lemma, Theorem 1.2**. We view Theorem 1.2 as our main contribution in this work. It is the analogue of the Second De Giorgi Lemma for kinetic integro-differential equations with a non-locality exponent $s\in(0,1)$. This completed the full De Giorgi theory in the kinetic setting. Recall Figure 1.6, where we illustrated the domains of our main results.

Finally, from these results we are able to deduce a statement on the solution of the Boltzmann equation (1.2) in the conditional regime (1.20), where the mass density of the gas is supposed to be uniformly bounded from above and below, and the energy and the entropy are both assumed to be uniformly bounded above. It can be checked that K_f satisfies (1.26)-(1.30) with λ depending on the mass, energy and entropy of f and Λ depending on the mass, energy and $\|f\|_{L^\infty}$. We refer the interested reader to Section 3 and Lemma A.6 of [115]. Therefore Theorem 3.1 implies:

Theorem 3.3 (Imbert-Silvestre)**.** *Let* $s\in(0,1)$ *and* $\gamma\in(-d,1]$ *be such that* $0\geq\gamma+2s\leq 2$. *Let* f *be a non-negative solution to the Boltzmann equation* (1.2) *for* $(t,x,v)\in(-3,0]\times B_1\times B_1$. *Assume that* f *is essentially bounded in* $(-3,0]\times B_1\times\mathbb{R}^d$ *and that there exists* M_0,M_1,E_1 *such that* $M_0\leq M(t,x)\leq M_1$, $E(t,x)\leq E_1$ *for all* $(t,x)\in(-3,0]\times B_1$. *Then* f *is Hölder continuous in* $Q_{1/2}:=\big(-(1/2)^{2s},0\big]\times B_{(1/2)^{1+2s}}\times B_{1/2}$

and for $\alpha \in (0,1)$ *there holds*

$$[f]_{C^\alpha(Q_{1/2})} \leq C,$$

where C *depends on* $s, d, \|f\|_{L^\infty}, M_0, M_1, E_1$.

Remark 3.1. The entropy H satisfies a uniform bound from above in terms of M_1, E_1 and $\|f\|_{L^\infty}$. Thus we do not require the additional assumption $H \leq H_1$ in Theorem 3.3.

3.1.3 Further remarks

Our equation involves a transport term, which transfers some regularity of the velocity variable to the space variable. It also involves a non-local diffusion in the v variable. Our motivation to study the regularity of this type of equation is linked to the question of well-posedness for smooth classical solutions of the inhomogeneous Boltzmann equation without cut-off. There are linear kinetic equations whose solutions in the hydrodynamic limit are described by a fractional diffusion [30, 158, 159]. Indeed diffusion limits of the linear Boltzmann equation with a heavy-tailed distribution of infinite variance as an equilibrium distribution give rise to a fractional diffusion equation [159]. Such heavy-tailed distribution functions arise in astrophysical plasmas [160] or also in granular gases through dissipative collision mechanisms [72]. However, the only source of fractional diffusion at the kinetic level stems from long range interactions of the Boltzmann collision kernel.

In the limit case $s \to 1$, equation (1.13) models the local kinetic Fokker-Planck equation, whose study is motivated by applications to the Landau equation [85]. For the local case, there is a non-constructive method discussed in [85]. A constructive proof first appeared in [190] for ultraparabolic equations that has further been developed in [98] to local kinetic Fokker-Planck type equations. The construction is based on a Poincaré-type inequality and Kruzhkov's method [137]. A novel constructive approach for local kinetic Fokker-Planck type equations has been devised by Guerand and Mouhot in [99]. Their method relies on trajectories. For general $s \in (0,1)$, Cyril Imbert and Luis Silvestre (together with Clément Mouhot in [120, 121]) made important contributions in a series of papers [115, 117, 120, 121, 177] that culminated in the final work [118]. In [177], Silvestre proves for a certain range of s that any solution f is a-priori essentially bounded provided that the mass, energy and entropy defined above (1.20) satisfy uniform bounds from above and that the mass is bounded below. With an additional non-degeneracy assumption in [115], they obtain the a-priori inequality with a quantitative argument when $s \in (0, \frac{1}{2})$. Note that their method, which uses barrier functions, can be extended to $s \in (0,1)$ under the additional symmetry assumption $K(v, v+w) = K(v, v-w)$, cf. section 7 in [115]. When $s \in (\frac{1}{2}, 1)$ they prove an Intermediate Value Lemma using an argument by contradiction. In [117], Imbert and Silvestre derive Schauder estimates for kinetic equations, which can then be bootstrapped for the non-cutoff Boltzmann equation to obtain smooth solutions

[118]. Let us also mention a work by Stokols [181], which combines the method of [85] with fractional estimates from [35] to obtain a non-constructive proof of Hölder continuity in the non-local case. The assumptions on the kernel in [181] are stronger than ours: the kernel is assumed to satisfy pointwise upper and lower bounds in (v, w), as well as the symmetry conditions $K(v, w) = K(w, v)$ and $K(v, v + w) = K(v, v - w)$. We need more general assumptions to be able to apply our results to the Boltzmann equation.

The paper presents a constructive proof of Hölder continuity and Harnack inequalities using a De Giorgi-type argument. De Giorgi methods were originally established for non-linear elliptic equations by De Giorgi [56]. Moser then showed how to deduce a Harnack inequality [165, 166]. The Harnack inequalities are local regularity results. In particular, the weak Harnack inequality implies Hölder continuity [80]. By using an argument based on trajectories, we derive the Intermediate Value Lemma, which in turn implies Hölder continuity for the non-cutoff Boltzmann equation. In order to cover the full range of non-locality parameter $s \in (0, 1)$ we have to carefully exploit the cross-term arising in the energy estimate similar to Section 4 in [35]. This is a purely non-local effect. To the best of our knowledge, this is the first Intermediate Value Lemma for non-local kinetic equation covering the full range $s \in (0, 1)$.

The not-so-strong Harnack inequality is obtained by combining the weak Harnack inequality with De Giorgi's first lemma. The strong Harnack inequality for non-local equations has been discussed in [32, 33, 48] for the elliptic case and in [22, 50, 51, 126] for the parabolic case. Most of these works consider non-local operators with symmetric kernels that verify a pointwise upper and lower bound. In general the strong Harnack inequality fails for non-local equations, if the function is not assumed to be non-negative [125]. Instead one gets an additional term from the tail of the negative part of the function on the right hand side [126]. The parabolic Harnack inequality without tail term has so far only been proved using probabilistic tools such as heat kernel estimates [50, 51]. There are no analogous results in the hypoelliptic case. In [128], they prove the Harnack inequality with a tail term on the right hand side for non-local parabolic equations. Under certain assumptions [128, Conditions (a) and (b) on p. 47] the non-local tail term can be bounded by a local quantity, which could then be absorbed on the left hand side of the inequality via a standard covering argument, yielding the strong Harnack inequality. However, these additional assumptions do not naturally follow. But from the weak Harnack inequality it is a straight-forward consequence to prove a weaker non-linear strong Harnack inequality for essentially bounded solutions in the sense of Theorem 3.2.

3.1.4 Strategy

The core of the proof relies on the following method:

$$\left.\begin{aligned} \text{Energy estimate} &\xrightarrow[0\le f\le 1]{} \text{First De Giorgi Lemma} \\ \text{weak Poincaré inequality} &\xrightarrow[0\le f\le 1]{} \text{Second De Giorgi Lemma} \end{aligned}\right\} \longrightarrow \text{weak Harnack.}$$

Note that in the final statement we require our function to be essentially bounded for v in $\mathbb{R}^d$. The reason behind this is to be able to take care of the tails of the non-local operator. The energy estimate and the weak Poincaré inequality hold for any sub-solution of the equation, whereas for the First and Second De Giorgi Lemma we need to assume that $f \le 1$ almost everywhere.

In Section 3.2 we fix the notation, discuss definitions of weak solutions for (1.13) and state a result of boundedness in $H^s \times H^s$ of the bilinear form (3.3).

For De Giorgi's First Lemma we prove an energy estimate in Section 3.3 and gain integrability in Lebesgue spaces using estimates on the fundamental solution of the fractional Kolmogorov equation. We deduce De Giorgi's First Lemma 3.4 by a classical *De Giorgi iteration* in Section 3.4.

De Giorgi's Second Lemma, Theorem 1.2, is proved in Section 3.5 by first proving a weak Poincaré inequality in L^1 using the trajectorial approach of [10]. The difficulties that arise for small values of s are dealt with by introducing suitable level functions inspired from [35].

We deduce Hölder continuity, Theorem 3.1, and the Harnack inequalities, Theorem 1.1 and Theorem 3.2, in Section 3.6. Hölder continuity follows by standard arguments either from the weak Harnack inequality or by the measure-to-pointwise estimate. To prove the Harnack inequalities, we use a covering argument, which we adapt from [99]. The geometric construction for the covering had to account for the specific scaling of the fractional diffusion. Once we have obtained the weak Harnack inequality, we combine it with the First De Giorgi Lemma to deduce the non-linear strong Harnack inequality.

3.2 The bilinear form

We introduce the bilinear form associated to the operator $\mathcal{L}$ in (1.14) for $(t,x) \in \mathbb{R}^{1+d}$

$$\begin{aligned} \mathcal{E}\big(\varphi, g\big)(t,x) &:= -\int \big(\mathcal{L}\varphi\big)(t,x,v) g(t,x,v)\,\mathrm{d}v \\ &= \lim_{\varepsilon\to 0} \iint_{|v-w|>\varepsilon} K(t,x,v,w)\big[\varphi(t,x,v) - \varphi(t,x,w)\big] g(t,x,v)\,\mathrm{d}w\,\mathrm{d}v. \end{aligned} \tag{3.3}$$

In the sequel, we will abuse notation by ignoring the limit as $\varepsilon \to 0$ and understanding some integrals in the principal value sense. Moreover, even though K can depend on time and space in general, we will subsequently often omit to write out this dependency explicitly. The following theorem has first been proved in [115, Corollary 5.2]. We give a different proof for the anti-symmetric part of the operator.

Theorem 3.4. *Let $\bar{R} > 0$, $\Lambda > 0$ and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be an non-negative kernel satisfying* (1.27), (1.29) *and* (1.30). *Then for $f, g \in H^s(\mathbb{R}^d)$ supported in $B_{\frac{\bar{R}}{2}}$ there holds*

$$\mathcal{E}(f,g) \leq C \left\|f\right\|_{H^s(\mathbb{R}^d)} \left\|g\right\|_{H^s(\mathbb{R}^d)},$$

where C depends on s, d and Λ.

Proof. By density of C^∞ in Sobolev spaces it suffices to consider smooth f, g. We divide the proof into the symmetric and anti-symmetric part of the kernel: we write

$$\begin{aligned}\mathcal{E}(f,g) = \underbrace{\frac{1}{2}\int_{B_{\bar{R}}}\int_{\mathbb{R}^d} \big(f(v)-f(w)\big)\big(g(v)-g(w)\big)K(v,w)\,\mathrm{d}w\,\mathrm{d}v}_{=:I_1(f,g)} \\ + \underbrace{\frac{1}{2}\int_{B_{\bar{R}}}\int_{\mathbb{R}^d} \big(f(v)-f(w)\big)\big(g(v)+g(w)\big)K(v,w)\,\mathrm{d}w\,\mathrm{d}v}_{=:I_2(f,g)},\end{aligned}$$

where the integrals are understood in a principal value sense.

For I_1 we use Young's inequality and we dyadically decompose the domain of integration which gives for $h = f$ or $h = g$

$$I_1(h,h) = \sum_{k=-\infty}^{+\infty} P(2^k), \tag{3.4}$$

where for $r > 0$

$$P(r) := \int_{\Sigma_r} |h(v)-h(w)|^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v,$$

with $\Sigma_r = \big\{(v,w) \in B_{\bar{R}} \times \mathbb{R}^d : r \leq |v-w| < 2r\big\}$. Let $m = \frac{v+w}{2}$ and consider an intermediate point $u \in B_{\frac{r}{4}}(m)$. We use $|h(v)-h(w)|^2 \leq 2\,|h(v)-h(u)|^2 + 2\,|h(u)-h(w)|^2$

to bound

$$\begin{aligned}
P(r) &\lesssim \frac{1}{r^d}\int_{\Sigma_r}\int_{B_{\frac{r}{4}}(m)} K(v,w)\Big(\,|h(v)-h(u)|^2+|h(u)-h(w)|^2\,\Big)\,\mathrm{d}u\,\mathrm{d}w\,\mathrm{d}v\\
&\le \frac{1}{r^d}\int_{\tilde\Sigma_r}|h(v)-h(u)|^2\left(\int_{\Omega_{v,u}}K(v,w)\,\mathrm{d}w\right)\mathrm{d}u\,\mathrm{d}v\\
&\quad+\frac{1}{r^d}\int_{\tilde\Sigma_r}|h(u)-h(w)|^2\left(\int_{\Omega_{w,u}}K(v,w)\,\mathrm{d}v\right)\mathrm{d}u\,\mathrm{d}w,
\end{aligned}$$

where we used Fubini's theorem and write $\tilde\Sigma_r := \big\{(v,w)\in B_{\bar R}\times\mathbb{R}^d : \frac{r}{4}\le|v-w|<\frac{5r}{4}\big\}$ and $\Omega_{v,w}$ for the set containing the u corresponding to any pair (v,w). We note that $\Omega_{v,w}\subset B_{2r}(v)\setminus B_r(v)$. Using (1.32) we get

$$\begin{aligned}
P(r) &\lesssim \frac{\Lambda}{r^{d+2s}}\left(\int_{\tilde\Sigma_r}|h(v)-h(u)|^2\,\mathrm{d}u\,\mathrm{d}v+\int_{\tilde\Sigma_r}|h(u)-h(w)|^2\,\mathrm{d}u\,\mathrm{d}w\right)\\
&\le \Lambda\left(\int_{\tilde\Sigma_r}\frac{|h(v)-h(u)|^2}{|v-u|^{d+2s}}\,\mathrm{d}u\,\mathrm{d}v+\int_{\tilde\Sigma_r}\frac{|h(u)-h(w)|^2}{|u-w|^{d+2s}}\,\mathrm{d}u\,\mathrm{d}w\right)\\
&\lesssim \Lambda\,\|h\|_{H^s}^2\,.
\end{aligned}$$

We can apply this final estimate to each term in the sum (3.4) and use Cauchy-Schwarz inequality to obtain

$$I_1(f,g)\lesssim\Lambda\,\|f\|_{H^s(\mathbb{R}^d)}\,\|g\|_{H^s(\mathbb{R}^d)}\,. \tag{3.5}$$

For I_2 we distinguish the far and near part for $0<R<\frac{\bar R}{2}$ to be determined below

$$I_2(f,g)=\underbrace{\int_{B_{\bar R}}\int_{B_{\bar R}\cap|v-w|<R}\dots}_{=:I_{21}(f,g)}+\underbrace{\int_{B_{\bar R}}\int_{|v-w|>R}\dots}_{=:I_{22}(f,g)}. \tag{3.6}$$

We rewrite I_{21} with Fubini's theorem

$$\begin{aligned}
I_{21}(f,g)=&\underbrace{\int_{B_{\bar R}\times B_{\bar R}}\big(K(v,w)-K(w,v)\big)f(v)g(v)\chi_{|v-w|<R}\,\mathrm{d}w\,\mathrm{d}v}_{=:I_{211}(f,g)}\\
&+\underbrace{\int_{B_{\bar R}\times B_{\bar R}}\big(K(v,w)-K(w,v)\big)f(v)g(w)\chi_{|v-w|<R}\,\mathrm{d}w\,\mathrm{d}v}_{=:I_{212}(f,g)}\,.
\end{aligned}$$

Then for I_{211} we use the cancellation assumption (1.29) and (1.30). We get

$$\begin{aligned} I_{211}(f,g) &\lesssim \int_{B_{\bar{R}}} \left| \mathrm{PV} \int_{B_{\bar{R}}} \big(K(v,w) - K(w,v)\big)\,\mathrm{d}w \right| |f(v)g(v)|\,\mathrm{d}v \\ &\lesssim \Lambda \int_{B_{\bar{R}}} |f(v)g(v)|\,\mathrm{d}v \\ &\leq \Lambda \,\|f\|_{L^2(B_{\bar{R}})}\,\|g\|_{L^2(B_{\bar{R}})}\,. \end{aligned} \tag{3.7}$$

For I_{212} we use Taylor's theorem to write

$$\begin{aligned} I_{212}(f,g) = \int_{B_{\bar{R}}} \int_{B_{\bar{R}}} &\big[K(v,w) - K(w,v)\big]\chi_{|v-w|<R} f(v) \\ &\times \Bigg[g(v) + \nabla g(v)\cdot(w-v) \\ &\quad + \frac{1}{2}(w-v)^T \left(\int_0^1 \mathrm{D}^2 g(\tau w + (1-\tau)v)\,\mathrm{d}\tau \right)(w-v) \Bigg]\,\mathrm{d}w\,\mathrm{d}v. \end{aligned} \tag{3.8}$$

We can bound the first term as above in (3.7). For the second term, in case that $s \in \left(0, \frac{1}{2}\right)$, we use (1.27) so that

$$\begin{aligned} \int_{B_{\bar{R}}} \left| \int_{B_{\bar{R}}} \big[K(v,w) - K(w,v)\big]\chi_{|v-w|<R}(w-v)\,\mathrm{d}w \right| |f(v)\nabla g(v)|\,\mathrm{d}v \\ &\lesssim \Lambda R^{1-2s} \int_{B_R} |f(v)\nabla g(v)|\,\mathrm{d}v \\ &\leq \Lambda R^{1-2s}\,\|f\|_{L^2(B_{\bar{R}})}\,\|\nabla g\|_{L^2(B_{\bar{R}})} \\ &\leq \Lambda R^{1-2s}\,\|f\|_{L^2(B_{\bar{R}})}\,\|g\|^{\frac{1}{2}}_{L^2(B_{\bar{R}})}\,\big\|\mathrm{D}^2 g\big\|^{\frac{1}{2}}_{L^2(B_{\bar{R}})} \\ &\leq \Lambda\,\|f\|_{L^2(B_{\bar{R}})}\,\|g\|^{1-s}_{L^2(B_{\bar{R}})}\,\|g\|^{s}_{H^2(B_{\bar{R}})}\,. \end{aligned}$$

We used the Gagliardo-Nirenberg inequality and chose $R^2 = \frac{\|g\|_{L^2}}{\big\|\mathrm{D}^2 g\big\|_{L^2}}$. Else if $s \in \big[1/2, 1\big)$, we use (1.30), so that with this choice of R we get

$$\begin{aligned} \int_{B_{\bar{R}}} \left| \int_{B_{\bar{R}}} \big[K(v,w) - K(w,v)\big]\chi_{|v-w|<R}(w-v)\,\mathrm{d}w \right| |f(v)\nabla g(v)|\,\mathrm{d}v \\ &\lesssim \Lambda\big(1 + R^{1-2s}\big) \int_{B_{\bar{R}}} |f(v)\nabla g(v)|\,\mathrm{d}v \\ &\leq \Lambda\big(1 + R^{1-2s}\big)\,\|f\|_{L^2(B_{\bar{R}})}\,\|\nabla g\|_{L^2(B_{\bar{R}})} \\ &\leq \Lambda\big(1 + R^{1-2s}\big)\,\|f\|_{L^2(B_{\bar{R}})}\,\|g\|^{\frac{1}{2}}_{L^2(B_{\bar{R}})}\,\big\|\mathrm{D}^2 g\big\|^{\frac{1}{2}}_{L^2(B_{\bar{R}})} \\ &\leq \Lambda\,\|f\|_{L^2(B_{\bar{R}})} \Big(\|g\|_{L^2(B_{\bar{R}})} + \|g\|^{1-s}_{L^2(B_{\bar{R}})}\,\|g\|^{s}_{H^2(B_{\bar{R}})} \Big). \end{aligned}$$

For the last term in (3.8), we have with the Cauchy-Schwarz inequality and (1.27)

$$
\begin{aligned}
&\iint_{B_{\bar R}\times B_{\bar R}} \big[K(v,w)-K(w,v)\big]\chi_{|v-w|<R} f(v)(w-v)^T \\
&\qquad\qquad \times\left(\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\right)(w-v)\,\mathrm{d}w\,\mathrm{d}v \\
&\le \|f\|_{L^2(B_{\bar R})} \\
&\quad\times\left\{\int_{B_{\bar R}}\left|\int_{B_{\bar R}}\big[K(v,w)-K(w,v)\big]\chi_{|v-w|<R}\,|w-v|^2\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\,\mathrm{d}w\right|^2\mathrm{d}v\right\}^{\frac12} \\
&\le \|f\|_{L^2(B_{\bar R})}\left\{\int_{B_{\bar R}}\left|\int_{B_{\bar R}}|K(v,w)-K(w,v)|\,\chi_{|v-w|<R}\,|w-v|^2\,\mathrm{d}w\right|\right. \\
&\quad\left.\times\left|\int_{B_{\bar R}}|K(v,w)-K(w,v)|\,\chi_{|v-w|<R}\,|w-v|^2\left|\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\right|^2\mathrm{d}w\right|\mathrm{d}v\right\}^{\frac12} \\
&\le \Lambda R^{1-s}\,\|f\|_{L^2(B_{\bar R})} \\
&\quad\times\left\{\iint_{B_{\bar R}\times B_{\bar R}}|K(v,w)-K(w,v)|\,\chi_{|v-w|<R}\,|w-v|^2\int_0^1\left|\mathrm{D}^2 g(\tau w+(1-\tau)v)\right|^2\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}v\right\}^{\frac12}.
\end{aligned}
$$

For $\tau\in[0,\frac12)$ we perform the change of variable $v\to\tau w+(1-\tau)v$, whereas for $\tau\in[\frac12,1]$ we perform the change of variable $w\to\tau w+(1-\tau)v$. We get in the former case with (1.27) and Fubini's theorem

$$
\begin{aligned}
&\int_{B_{\bar R}}\int_{B_{\bar R}}|K(v,w)-K(w,v)|\,\chi_{|v-w|<R}\,|w-v|^2\int_0^{\frac12}\left|\mathrm{D}^2 g(\tau w+(1-\tau)v)\right|^2\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\le\int_0^{\frac12}\int_{B_{\bar R}}\int_{B_{\tilde R}}|K(\tilde v,w)-K(w,\tilde v)|\,\chi_{|\tilde v-w|<R}\,|w-\tilde v|^2\left|\mathrm{D}^2 g(\tilde v)\right|^2(1-\tau)^{-3}\,\mathrm{d}\tilde v\,\mathrm{d}w\,\mathrm{d}\tau \\
&\qquad\le C\Lambda R^{2-2s}\left\|\mathrm{D}^2 g\right\|_{L^2(B_{\tilde R})},
\end{aligned}
$$

where $0<\tilde R$ is chosen such that $B_{(1-\tau)\bar R}(\tau w)\subset B_{\tilde R}$ for all $w\in B_{\bar R}$ and $\tau\in[0,\frac12)$. In the latter case we have similarly

$$
\begin{aligned}
&\int_{B_{\bar R}}\int_{B_{\bar R}}|K(v,w)-K(w,v)|\,\chi_{|v-w|<R}\,|w-v|^2\int_{\frac12}^1\left|\mathrm{D}^2 g(\tau w+(1-\tau)v)\right|^2\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\le\int_{\frac12}^1\int_{B_{\bar R}}\int_{B_{\tilde R}}|K(v,\tilde w)-K(\tilde w,v)|\,\chi_{|v-\tilde w|<R}\,|\tilde w-v|^2\left|\mathrm{D}^2 g(\tilde w)\right|^2\tau^{-3}\,\mathrm{d}\tilde w\,\mathrm{d}v\,\mathrm{d}\tau \\
&\qquad\le C\Lambda R^{2-2s}\left\|\mathrm{D}^2 g\right\|_{L^2(B_{\tilde R})},
\end{aligned}
$$

when we choose $0 < \tilde{R}$ such that $B_{\tau\bar{R}}((1-\tau)v) \subset B_{\tilde{R}}$ for all $v \in B_{\bar{R}}$. Thus we get overall

$$\begin{aligned}\int_{B_{\bar{R}}}\int_{B_{\bar{R}}} \big[K(v,w)-K(w,v)\big]\chi_{|v-w|<R} f(v)(w-v)^T \int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau (w-v)\,\mathrm{d}w\,\mathrm{d}v \\ \leq C\Lambda^2 R^{2-2s}\,\|f\|_{L^2(B_{\bar{R}})}\,\big\|\mathrm{D}^2 g\big\|_{L^2(B_{\tilde{R}})} \\ \leq C\Lambda\,\|f\|_{L^2(B_{\bar{R}})}\,\|g\|^{1-s}_{L^2(B_{\bar{R}})}\,\|g\|^{s}_{H^2(B_{\bar{R}})}\,.\end{aligned}$$

Therefore we have obtained

$$I_{21}(f,g) \lesssim_\Lambda \|f\|_{L^2(B_{\bar{R}})}\,\|g\|_{L^2(B_{\bar{R}})} + \|f\|_{L^2(B_{\bar{R}})}\,\|g\|^{1-s}_{L^2(B_{\bar{R}})}\,\|g\|^{s}_{H^2(B_{\bar{R}})}\,. \tag{3.9}$$

For the far part I_{22} in (3.6) we apply Taylor's theorem

$$\begin{aligned}I_{22}(f,g) &= \int_{B_{\bar{R}}}\int_{\mathbb{R}^d\setminus B_R(v)} \big[f(v)-f(w)\big]\big[g(v)+g(w)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &= \int_{B_{\bar{R}}}\int_{\mathbb{R}^d\setminus B_R(v)} K(v,w)\big[f(v)-f(w)\big]\Bigg[2g(v)+\nabla g(v)\cdot(w-v) \\ &\qquad + \frac{1}{2}(w-v)^T\cdot\left(\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\right)\cdot(w-v)\Bigg]\,\mathrm{d}w\,\mathrm{d}v.\end{aligned}$$

The trickiest term to estimate is the last one. We get with Fubini and Cauchy-Schwarz

$$\begin{aligned}&\int_{B_{\bar{R}}}\int_{\mathbb{R}^d\setminus B_R(v)} K(v,w)\big[f(v)-f(w)\big]\,|w-v|^2\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}v \\ &\leq \|f\|_{L^2(B_{\bar{R}})}\left\{\int_{B_{\bar{R}}}\left|\int_{\mathbb{R}^d\setminus B_R(v)} K(v,w)\,|w-v|^2\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\,\mathrm{d}w\right|^2\mathrm{d}v\right\}^{\frac{1}{2}} \\ &\quad + \|f\|_{L^2(\mathbb{R}^d\setminus B_{\bar{R}})}\left\{\int_{\mathbb{R}^d}\left|\int_{B_{\bar{R}}\setminus B_R(w)} K(v,w)\,|w-v|^2\int_0^1 \mathrm{D}^2 g(\tau w+(1-\tau)v)\,\mathrm{d}\tau\,\mathrm{d}v\right|^2\mathrm{d}w\right\}^{\frac{1}{2}} \\ &\leq \Lambda R^{1-s}\,\|f\|_{L^2(B_{\bar{R}})}\left\{\int_{B_{\bar{R}}}\int_{\mathbb{R}^d\setminus B_R(v)} K(v,w)\,|w-v|^2\int_0^1 \big|\mathrm{D}^2 g(\tau w+(1-\tau)v)\big|^2\,\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}v\right\}^{\frac{1}{2}} \\ &\quad + \Lambda R^{1-s}\,\|f\|_{L^2(\mathbb{R}^d\setminus B_{\bar{R}})}\left\{\int_{B_{\bar{R}}}\int_{\mathbb{R}^d\setminus B_R(v)} K(v,w)\,|w-v|^2\right. \\ &\qquad\qquad \left.\times\int_0^1 \big|\mathrm{D}^2 g(\tau w+(1-\tau)v)\big|^2\,\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}v\right\}^{\frac{1}{2}}.\end{aligned}$$

With the same change of variables as above we find for $\tilde{R}$ sufficiently large such that

$B_{(1-\tau)\bar{R}}(\tau w) \subset B_{\tilde{R}}$ for all $w \in \mathbb{R}^d \setminus B_R(v)$ and $\tau \in [0, \frac{1}{2}]$

$$\begin{aligned}\int_{B_{\bar{R}}} \int_{\mathbb{R}^d \setminus B_R(v)} K(v,w) \left|w-v\right|^2 \int_0^{\frac{1}{2}} \left|\mathrm{D}^2 g(\tau w + (1-\tau)v)\right|^2 \mathrm{d}\tau \,\mathrm{d}w \,\mathrm{d}v \\ \leq C \int_{B_{\tilde{R}}} \int_{\mathbb{R}^d \setminus B_R(\tilde{v})} K(\tilde{v},w) \left|w-\tilde{v}\right|^2 \left|\mathrm{D}^2 g(\tilde{v})\right|^2 \mathrm{d}w \,\mathrm{d}\tilde{v} \\ \leq C\Lambda R^{2-2s} \left\|\mathrm{D}^2 g\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Similarly

$$\begin{aligned}\int_{B_{\bar{R}}} \int_{\mathbb{R}^d \setminus B_R(v)} K(v,w) \left|w-v\right|^2 \int_{\frac{1}{2}}^1 \left|\mathrm{D}^2 g(\tau w + (1-\tau)v)\right|^2 \mathrm{d}\tau \,\mathrm{d}w \,\mathrm{d}v \\ \leq C \int_{B_{\tilde{R}}} \int_{\mathbb{R}^d \setminus B_R(v)} K(v,\tilde{w}) \left|\tilde{w}-v\right|^2 \left|\mathrm{D}^2 g(\tilde{w})\right|^2 \mathrm{d}\tilde{w} \,\mathrm{d}v \\ \leq C\Lambda R^{2-2s} \left\|\mathrm{D}^2 g\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus with our choice of R we get

$$\begin{aligned}\int_{B_{\bar{R}}} \int_{\mathbb{R}^d \setminus B_R(v)} K(v,w) \big[f(v) - f(w)\big] \left|w-v\right|^2 \int_0^1 \mathrm{D}^2 g(\tau w + (1-\tau)v) \,\mathrm{d}\tau \,\mathrm{d}w \,\mathrm{d}v \\ \leq C\Lambda R^{2-2s} \left\|f\right\|_{L^2(\mathbb{R}^d)} \left\|\mathrm{D}^2 g\right\|_{L^2(\mathbb{R}^d)} \\ \leq C\Lambda \left\|f\right\|_{L^2(\mathbb{R}^d)} \left\|g\right\|^{1-s}_{L^2(\mathbb{R}^d)} \left\|g\right\|^s_{H^2(\mathbb{R}^d)}.\end{aligned} \tag{3.10}$$

Overall we find

$$\begin{aligned}&I_{22}(f,g) \\ &\lesssim \int_{B_{\bar{R}}} \left(\int_{\mathbb{R}^d \setminus B_R(v)} \big(f(v) - f(w)\big)^2 K(v,w) \,\mathrm{d}w \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d \setminus B_R(v)} K(v,w) \,\mathrm{d}w \right)^{\frac{1}{2}} g(v) \,\mathrm{d}v \\ &\quad + \int_{B_{\bar{R}}} \left(\int_{\mathbb{R}^d \setminus B_R(v)} \big(f(v) - f(w)\big)^2 K(v,w) \,\mathrm{d}w \right)^{\frac{1}{2}} \\ &\qquad\qquad \times \left(\int_{\mathbb{R}^d \setminus B_R(v)} \left|w-v\right|^2 K(v,w) \,\mathrm{d}w \right)^{\frac{1}{2}} \left|\nabla g(v)\right| \,\mathrm{d}v \\ &\quad + C\Lambda R^{2-2s} \left\|f\right\|_{L^2(\mathbb{R}^d)} \left\|\mathrm{D}^2 g\right\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \Lambda R^{-s} \left\|f\right\|_{H^s} \left\|g\right\|_{L^2} + \Lambda R^{1-s} \left\|f\right\|_{H^s} \left\|\nabla g\right\|_{L^2} + \Lambda R^{2-2s} \left\|f\right\|_{L^2(\mathbb{R}^d)} \left\|\mathrm{D}^2 g\right\|_{L^2(\mathbb{R}^d)} \\ &\leq \Lambda R^{-s} \left\|f\right\|_{H^s} \left\|g\right\|_{L^2} + \Lambda R^{1-s} \left\|f\right\|_{H^s} \left\|g\right\|^{\frac{1}{2}}_{L^2} \left\|\mathrm{D}^2 g\right\|^{\frac{1}{2}}_{L^2} + \Lambda R^{2-2s} \left\|f\right\|_{L^2(\mathbb{R}^d)} \left\|\mathrm{D}^2 g\right\|_{L^2(\mathbb{R}^d)} \\ &\leq \Lambda \left\|f\right\|_{H^s} \left\|g\right\|^{1-\frac{s}{2}}_{L^2} \left\|g\right\|^{\frac{s}{2}}_{H^2} + \Lambda \left\|f\right\|_{L^2(\mathbb{R}^d)} \left\|g\right\|^{1-s}_{L^2(\mathbb{R}^d)} \left\|g\right\|^s_{H^2(\mathbb{R}^d)}.\end{aligned}$$

We used the same proof as for the symmetric part to deduce the third inequality, repeatedly

the Cauchy Schwarz inequality, the upper bound (1.27), the Gagliardo-Nirenberg inequality and the choice of R as above.

Now we use Littlewood-Paley theory inspired from the proof of Theorem 4.1 in [115]. We denote with Δ_i the Littlewood-Paley projectors defined by

$$\Delta_i f(z) = \int_{\xi : 2^i < |\xi| < 2^{i+1}} \hat{f}(\xi) e^{2\pi i \xi \cdot z} \, \mathrm{d}\xi,$$

where $\hat{f}$ is the Fourier transform of f. We decompose $f = \sum_{i=0}^{\infty} \Delta_i f$, using the convention that all low modes are contained in Δ_0 so that the index $i \geq 0$. Note that for $s \geq 0$

$$\|\Delta_i f\|_{H^s} \approx 2^{is} \|\Delta_i f\|_{L^2} .$$

Moreover, we bound as in [115] (for a justification see [18])

$$\|\Delta_i g\|_{L^2}^{1-s} \|\Delta_i g\|_{H^2}^{s} \lesssim 2^{si} \|\Delta_i g\|_{H^s} .$$

and

$$\|\Delta_i g\|_{L^2}^{1-\frac{s}{2}} \|\Delta_i g\|_{H^2}^{\frac{s}{2}} \lesssim \|\Delta_i g\|_{H^s} .$$

Then

$$\begin{aligned}
I_2(f,g) &= \sum_{i,j} I_2(\Delta_i f, \Delta_j g) \\
&\lesssim \sum_{j \leq i} \|\Delta_i f\|_{L^2} \|\Delta_j g\|_{L^2}^{1-s} \|\Delta_j g\|_{H^2}^{s} + \sum_{i<j} \|\Delta_j g\|_{L^2} \|\Delta_i f\|_{L^2}^{1-s} \|\Delta_i f\|_{H^2}^{s} \\
&\quad + \sum_{i,j} \Big(\|\Delta_i f\|_{H^s} \|\Delta_j g\|_{L^2}^{1-\frac{s}{2}} \|\Delta_j g\|_{H^2}^{\frac{s}{2}} + \|\Delta_i f\|_{L^2} \|\Delta_j g\|_{L^2} \Big) \\
&\lesssim \sum_{i,j} \Big(2^{-s|i-j|} \|\Delta_i f\|_{H^s} \|\Delta_j g\|_{H^s} + \|\Delta_i f\|_{H^s} \|\Delta_j g\|_{H^s} \Big) + \|f\|_{L^2} \|g\|_{L^2} \\
&\leq \sum_{k=0}^{\infty} 2^{-sk} \left(\sum_{i=0}^{\infty} \|\Delta_i f\|_{H^s} \|\Delta_{i+k} g\|_{H^s} + \|\Delta_{i+k} f\|_{H^s} \|\Delta_i g\|_{H^s} \right) \\
&\quad + \|f\|_{H^s} \|g\|_{H^s} + \|f\|_{L^2} \|g\|_{L^2} \\
&\leq \sum_{k=0}^{\infty} 2^{1-sk} \left(\sum_{i=0}^{\infty} \|\Delta_i f\|_{H^s}^2 \right)^{\frac{1}{2}} \left(\sum_{i=0}^{\infty} \|\Delta_i g\|_{H^s}^2 \right)^{\frac{1}{2}} \\
&\quad + \|f\|_{H^s} \|g\|_{H^s} + \|f\|_{L^2} \|g\|_{L^2} \\
&\lesssim \|f\|_{H^s} \|g\|_{H^s} .
\end{aligned} \tag{3.11}$$

Note that splitting the sum into $j \leq i$ and $i < j$ works since the adjoints of the corresponding integral operators satisfy the same bounds as in (3.9) and (3.10) due to the

cancellation assumptions. Thus we conclude with (3.5) and (3.11). □

To motivate the definition of solutions in Definition 1.1, we recall Lemma 5.6 of [115], which states that the operator $\mathcal{L}$ is bounded from the space $L^\infty\big(\mathbb{R}^d \setminus B_{\bar{R}/2}\big) + H^s(\mathbb{R}^d)$ to $H^{-s}(B_{\bar{R}})$, where the former space is equipped with the norm

$$\|f\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar{R}/2})+H^s(\mathbb{R}^d)} = \inf\Big\{ \|f_1\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar{R}/2})} + \|f_2\|_{H^s(\mathbb{R}^d)} : f = f_1 + f_2,\ f_1 = 0 \text{ in } B_{\bar{R}/2}\Big\}.$$

Lemma 3.1. *Let $\varphi \in H^s(\mathbb{R}^d)$ with supp φ compactly contained in $B_{\bar{R}/2}$. Let K be a kernel with* (1.27), (1.29), (1.30). *Then for $f \in L^\infty(\mathbb{R}^d \setminus B_{\bar{R}/2}) + H^s(\mathbb{R}^d)$ there exists C depending on s, d, Λ and supp φ so that*

$$\mathcal{E}(f,\varphi) \le C\,\|f\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar{R}/2})+H^s(\mathbb{R}^d)}\,\|\varphi\|_{H^s(\mathbb{R}^d)}\,.$$

Proof. The proof comes from Lemma 5.6 in [115]. It suffices to consider smooth f, φ just as above. We write $f = f_1 + f_2$ with f_1 and f_2 as in the definition of the norm for the space $L^\infty(\mathbb{R}^d \setminus B_{\bar{R}/2}) + H^s(\mathbb{R}^d)$. By Theorem 3.4 we have $|\mathcal{E}(f_2,\varphi)| \le C\,\|f_2\|_{H^s(\mathbb{R}^d)}\,\|\varphi\|_{H^s(\mathbb{R}^d)}$. For $\mathcal{E}(f_1,\varphi)$ we get

$$\begin{aligned}
|\mathcal{E}(f_1,\varphi)| &= \left| \lim_{\varepsilon\to 0} \iint_{|w-v|>\varepsilon} \big[f_1(w) - f_1(v)\big]\varphi(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \right| \\
&= \lim_{\varepsilon\to 0} \int_{\text{supp }\varphi} \int_{\mathbb{R}^d\setminus B_\varepsilon(v)} f_1(w)K(v,w)\,\mathrm{d}w\varphi(v)\,\mathrm{d}v \\
&= \int_{\text{supp }\varphi} \int_{\mathbb{R}^d\setminus B_\delta(v)} f_1(w)K(v,w)\,\mathrm{d}w\varphi(v)\,\mathrm{d}v,
\end{aligned}$$

where $\delta = \text{dist}\,\big(\text{supp }\varphi, \mathbb{R}^d \setminus B_{\bar{R}/2}\big)$. Using the upper bound on the kernel (1.27), we find

$$|\mathcal{E}(f_1,\varphi)| \le \Lambda\delta^{-2s}\,\|f_1\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar{R}/2})} \int_{\text{supp }\varphi} \varphi(v)\,\mathrm{d}v \le C\,\|f_1\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar{R}/2})}\,\|\varphi\|_{H^s}\,.$$

□

3.3 Energy and integrability

The following two lemmas are used to extract information from the equation.

Lemma 3.2 (Local energy estimate)**.** *Let f be a non-negative sub-solution to* (1.13)-(1.14) *in $Q_{\frac{\bar{R}}{2}}$ for some $\bar{R} \ge 1$ where $0 \le K$ satisfies* (1.26)-(1.30). *Let $Q_r(z_0) \subset Q_R(z_0) \subset Q_{\frac{\bar{R}}{2}}$ with $0 < r < R < \min\Big\{1, \frac{\bar{R}}{2}\Big\}$ be given. Assume $0 \le f \le 1$ a.e. in $\big(-R^{2s} + t_0, t_0\big] \times$*

$B_{R^{1+2s}}(x_0+t_0v_0)\times\mathbb{R}^d$. *Then there holds*

$$\sup_{\tau\in(-r^{2s}+t_0,t_0)}\int_{Q_r^\tau(z_0)}f^2(\tau,x,v)\,\mathrm{d}x\,\mathrm{d}v+\int_{Q_r(z_0)}\int_{B_r(z_0)}\frac{|f(w)-f(v)|^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}z$$
$$\lesssim_{\lambda,\Lambda,v_0}\int_{Q_R^{-r^{2s}+t_0}(z_0)}f_0^2(x,v)\,\mathrm{d}x\,\mathrm{d}v+(R-r)^{-2}\left|\{f>0\}\cap Q_R(z_0)\right|+\int_{Q_R(z_0)}h^2\,\mathrm{d}z,$$

where $z_0=(t_0,x_0,v_0)$, $f_0(x,v)=f\left(-r^{2s}+t_0,x,v\right)$ *and*

$$Q_r^\tau(z_0)=\left\{(x,v)\in\mathbb{R}^{2d}:(\tau,x,v)\in Q_r(z_0)\right\}.$$

Proof. For $t\in\left(-r^{2s}+t_0,t_0\right]$ we let $\psi\in C^\infty(\mathbb{R}^{2d})$ be a cut-off function with $0\le\psi\le1$, such that $\psi=0$ in $Q_r^t(z_0)$ and $\psi=1$ outside $Q_R^t(z_0)$. We test (1.13) with the truncated function $(f-\psi)_+=\sup\left(0,f-\psi\right)$. We notice that $f=(f-\psi)_++(f-\psi)_-+\psi$ and $\nabla_x(f-\psi)_+^2=2(f-\psi)_+\left(\nabla_xf-\nabla_x\psi\right)$. This yields for a.e. $t\in(-r^{2s}+t_0,t_0]$

$$\begin{aligned}\int_{\mathbb{R}^{2d}}h(f-\psi)_+\,\mathrm{d}x\,\mathrm{d}v&\ge\int_{\mathbb{R}^{2d}}\mathcal{T}f(f-\psi)_+\,\mathrm{d}x\,\mathrm{d}v+\int_{\mathbb{R}^d}\mathcal{E}(f,(f-\psi)_+)\,\mathrm{d}x\\&=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}}(f-\psi)_+^2\,\mathrm{d}x\,\mathrm{d}v+\int_{\mathbb{R}^d}\mathcal{E}((f-\psi)_+,(f-\psi)_+)\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^d}\mathcal{E}((f-\psi)_-,(f-\psi)_+)\,\mathrm{d}x+\int_{\mathbb{R}^d}\mathcal{E}(\psi,(f-\psi)_+)\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^{2d}}(f-\psi)_+\mathcal{T}\psi\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

Due to the coercivity of K (1.31) we know that

$$\mathcal{E}\left((f-\psi)_+,(f-\psi)_+\right)+\Lambda\left\|(f-\psi)_+\right\|_{L^2}^2\ge\lambda\left\|(f-\psi)_+\right\|_{H_v^s}^2.$$

Moreover we notice that

$$\begin{aligned}\mathcal{E}\left((f-\psi)_-,(f-\psi)_+\right)&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[(f-\psi)_-(v)-(f-\psi)_-(w)\right](f-\psi)_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\&=-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f-\psi)_-(w)(f-\psi)_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\ge0,\end{aligned}$$

since $(f-\psi)_+\cdot(f-\psi)_-=0$. This term will be fully exploited for the Second De Giorgi Lemma, Theorem 1.2. Finally to bound $\mathcal{E}(\psi,(f-\psi)_+)$ we distinguish the symmetric and

the skew-symmetric part

$$
\begin{aligned}
&-\mathcal{E}\big(\psi,(f-\psi)_+\big)\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f-\psi)_+(v)\big[\psi(w)-\psi(v)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[(f-\psi)_+(v)-(f-\psi)_+(w)\big]\big[\psi(w)-\psi(v)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[(f-\psi)_+(v)+(f-\psi)_+(w)\big]\big[\psi(w)-\psi(v)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
$$

Realising that $|g_+(v)-g_+(w)|\le\big(\chi_{\{g>0\}}(v)+\chi_{\{g>0\}}(w)\big)\,|g_+(v)-g_+(w)|$ we find

$$
\begin{aligned}
&-\mathcal{E}^{\mathrm{sym}}\big(\psi,(f-\psi)_+\big)\\
&\quad:=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[(f-\psi)_+(v)-(f-\psi)_+(w)\big]\big[\psi(w)-\psi(v)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad\le\frac{1}{8}\mathcal{E}^{\mathrm{sym}}\big((f-\psi)_+,(f-\psi)_+\big)\\
&\qquad+\iint\big[\chi_{\{f-\psi>0\}}(v)+\chi_{\{f-\psi>0\}}(w)\big]\,|\psi(w)-\psi(v)|^2\,K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad\le\frac{1}{8}\,\|(f-\psi)_+\|_{H^s}^2+\|\psi\|_{C^1_v}^2\int\chi_{\{f-\psi>0\}}(v)\,\mathrm{d}v,
\end{aligned}
$$

where we used Theorem 3.4 and the upper bound (1.32). For the skew-symmetric part we have

$$
\begin{aligned}
&-\mathcal{E}^{\mathrm{skew}}(\psi,(f-\psi)_+)\\
&\quad:=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[(f-\psi)_+(v)+(f-\psi)_+(w)\big]\big[\psi(w)-\psi(v)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad=\frac{1}{4}\iint\big[\psi(w)-\psi(v)\big]\big[(f-\psi)_+(v)+(f-\psi)_+(w)\big]\big[K(v,w)-K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v\\
&\quad=\frac{1}{2}\iint\big[\psi(w)-\psi(v)\big](f-\psi)_+(v)\big[K(v,w)-K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v\\
&\quad=\frac{1}{2}\int(f-\psi)_+(v)\left(\int\big[\psi(w)-\psi(v)\big]\big[K(v,w)-K(w,v)\big]\,\mathrm{d}w\right)\mathrm{d}v.
\end{aligned}
$$

We estimate

$$
\begin{aligned}
&\mathrm{PV}\int\big[\psi(w)-\psi(v)\big]\big[K(v,w)-K(w,v)\big]\,\mathrm{d}w=\int_{B_r(v)}\cdots+\int_{\mathbb{R}^d\setminus B_r(v)}\ldots\\
&\quad\le\int_{B_r(v)}(w-v)\cdot\nabla_v\psi(v)\big[K(v,w)-K(w,v)\big]\,\mathrm{d}w\\
&\qquad+\int_{B_r(v)}|w-v|^2\left\|D_v^2\psi\right\|_{L^\infty}|K(v,w)-K(w,v)|\,\mathrm{d}w+\Lambda r^{-2s}\,\|\psi\|_{L^\infty}\\
&\quad\le\Lambda r^{1-2s}\,\|\psi\|_{C^1_v}+\Lambda r^{2-2s}\,\|\psi\|_{C^2_v}+\Lambda r^{-2s}\,\|\psi\|_{L^\infty}.
\end{aligned}
$$

For the tail we used the upper bound (1.32). For the near part we used (1.27) to bound

$$\int_{B_r(v)} |v-w|^2 \, |K(v,w) - K(w,v)| \, \mathrm{d}w \lesssim \Lambda r^{2-2s}.$$

Moreover, in case that $s \in (0, 1/2)$ we again get by (1.27)

$$\int_{B_r(v)} |v-w| \, |K(v,w) - K(w,v)| \, \mathrm{d}w \lesssim \Lambda r^{1-2s},$$

and in case that $s \in [1/2, 1)$ we use the cancellation assumptions (1.30)

$$\left| \mathrm{PV} \int_{B_r(v)} (v-w)\big(K(v,w) - K(w,v)\big) \, \mathrm{d}w \right| \lesssim \Lambda r^{1-2s}.$$

Thus we find

$$\begin{aligned} -\mathcal{E}^{\mathrm{skew}}(\psi, (f-\psi)_+) &\leq C\Big(r^{1-2s} \|\psi\|_{C^1_v} + r^{2-2s} \|\psi\|_{C^2_v} + r^{-2s} \|\psi\|_{L^\infty} \Big) \int (f-\psi)_+(v) \, \mathrm{d}v \\ &\leq C \|\psi\|_{C^2_v} \int (f-\psi)_+(v) \, \mathrm{d}v, \end{aligned}$$

so that

$$\begin{aligned} -\mathcal{E}(\psi, (f-\psi)_+) \leq & \frac{1}{8} \|(f-\psi)_+\|^2_{H^s} + \|\psi\|^2_{C^1_v} \int \chi_{\{f-\psi>0\}}(v) \, \mathrm{d}v \\ & + C \|\psi\|_{C^2_v} \int (f-\psi)_+(v) \, \mathrm{d}v. \end{aligned}$$

Gathering the estimates we obtained so far we get for any $t \in (-r^{2s} + t_0, t_0]$

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} (f-\psi)^2_+(t,x,v) \, \mathrm{d}x \, \mathrm{d}v + \lambda \int_{-r^2 s + t_0}^{t_0} \int_{\mathbb{R}^d} \|(f-\psi)_+\|^2_{H^s_v} \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \int_{\mathbb{R}^{2d}} (f-\psi)^2_+(r^{2s} + t_0, x, v) \, \mathrm{d}x \, \mathrm{d}v + C \|\psi\|_{C^2_v} \int_{-r^2 s + t_0}^{t_0} \int_{\mathbb{R}^{2d}} (f-\psi)_+(v) \, \mathrm{d}z \\ &\quad + \|\psi\|^2_{C^1_v} \left| \{f - \psi > 0\} \cap (-r^{2s} + t_0, t_0] \right| + C \frac{v_0 + R}{(R-r)^{1+2s}} \int_{-r^2 s + t_0}^{t_0} \int_{\mathbb{R}^{2+}} (f-\psi)_+(v) \, \mathrm{d}z \\ &\quad + C\Lambda \int_{-r^2 s + t_0}^{t_0} \int_{\mathbb{R}^{2d}} (f-\psi)^2_+(v) \, \mathrm{d}z + \int_{Q_R(z_0)} h^2 \, \mathrm{d}z. \end{aligned}$$

Since $0 \leq (f-\psi)_+ \leq 1$ the L^1 norm and L^2 norm are bounded by

$$\left| \{f - \psi > 0\} \cap (-r^{2s} + t_0, t_0] \right| \leq \left| \{f > 0\} \cap Q_R(z_0) \right|.$$

Moreover, we chose ψ such that $\|D_v \psi\|_{L^\infty} \lesssim \frac{1}{R-r}$, $\|D^2_v \psi\|_{L^\infty} \lesssim \frac{1}{(R-r)^2}$ and $\|D_x \psi\|_{L^\infty} \lesssim \frac{1}{(R-r)^{1+2s}}$. Finally we note that $(f-\psi)_+ = f$ in $Q_r(z_0)$, $(f-\psi)_+ \leq f$ everywhere and $(f-\psi)_+ = 0$ outside $Q_R(z_0)$ and conclude. □

Lemma 3.3 (Local gain of integrability)**.** *Let f be a non-negative sub-solution of* (1.13)-(1.14) *in $Q_R(z_0)$ with a non-negative kernel satisfying* (1.26)-(1.30) *for some $\bar{R} \geq 1$. Let $Q_r(z_0) \subset Q_R(z_0) \subset Q_{\frac{\bar{R}}{2}}$ for some $0 < r < R \leq 1$. Assume $0 \leq f \leq 1$ a.e. in $\big(-R^{2s} + t_0, t_0\big] \times B_{R^{1+2s}}(x_0 + t_0 v_0) \times \mathbb{R}^d$. Given $p \in \Big[2, 2 + \frac{2s}{d(1+s)}\Big)$, f satisfies*

$$\begin{aligned} \|f\|^2_{L^p(Q_r(z_0))} \lesssim_{\lambda,\Lambda,v_0} (R-r)^{-2} \int_{Q_R^t(z_0)} f_0^2(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ + (R-r)^{-4}\,|\{f>0\} \cap Q_R(z_0)| + (R-r)^{-2} \int_{Q_R(z_0)} h^2\,\mathrm{d}z, \end{aligned} \tag{3.12}$$

where $z_0 = (t_0, x_0, v_0)$, $f_0(x,v) = f\big(-r^{2s} + t_0, x, v\big)$ and

$$Q_r^\tau(z_0) = \big\{(x,v) \in \mathbb{R}^{2d} : (\tau, x, v) \in Q_r(z_0)\big\}.$$

Proof. We first recall a clever observation made by Imbert and Silvestre [115]. If $f(t,x,v) = 0$ then $\mathcal{T} f = 0$ a.e. and $\mathcal{L} f(v) \geq 0$. In particular any sub-solution f of (1.13) satisfies

$$\mathcal{T} f \leq \mathcal{L} f - \left(\int_{\mathbb{R}^d} f(w) K(v,w)\,\mathrm{d}w\right)\chi_{\{f=0\}} + h = \Big(\mathcal{L} f\Big)\chi_{\{f>0\}} + h.$$

Now we write $\mathcal{I} := (-R^{2s} + t_0, t_0]$ and for $t \in \mathcal{I}$ we let $\varphi_1 : \mathbb{R}^{2d} \to [0,1]$ be a smooth function such that $\varphi_1 = 1$ on $Q_r^t(z_0) := B_{r^{1+2s}}(x_0 + (t-t_0)v_0) \times B_r(v_0)$ and $\varphi_1 = 0$ outside $Q^t_{\frac{R+r}{4}}(z_0)$. Consider $g := f\varphi_1$. There holds

$$\begin{aligned} \partial_t g + v \cdot \nabla_x g + (-\Delta)_v^s g &= \big(\mathcal{T} f\big)\varphi_1 + f\big(\mathcal{T}\varphi_1\big) + (-\Delta)_v^s g \\ &\leq \big(\mathcal{L} f\big)\chi_{\{f>0\}}\varphi_1 + h\varphi_1 + f\big(\mathcal{T}\varphi_1\big) + (-\Delta)_v^s g. \end{aligned}$$

Note that we have by the energy estimate of Lemma 3.2

$$\|f\|^2_{L^2_{t,x}\dot{H}^s_v(Q_{\frac{R+r}{4}}(z_0))} \leq CN,$$

where

$$N = \int_{Q_R^t(z_0)} f_0^2(x,v)\,\mathrm{d}x\,\mathrm{d}v + \mathcal{C}\left(\frac{R+r}{4}, R\right)|\{f>0\} \cap Q_R(z_0)| + \int_{Q_R(z_0)} h^2\,\mathrm{d}z$$

with $\mathcal{C}(r,R) = (R-r)^{-2}$. Moreover, since $f \in H^s_v$ with support in $Q_R(z_0)$ we know by

Theorem 3.4 that $\mathcal{L}f \in H_v^{-s}(\mathbb{R}^d)$ and

$$\begin{aligned}\int_{B_{\frac{R+r}{4}}(v_0)} (\mathcal{L}f)\chi_{\{f>0\}}\varphi_1(v)\,\mathrm{d}v &= \int_{B_{\frac{R+r}{4}}(v_0)} \int_{B_{\frac{r}{4}}(v)} \dots \,\mathrm{d}w\,\mathrm{d}v + \int_{B_{\frac{R+r}{4}}(v_0)} \int_{\mathbb{R}^d\setminus B_{\frac{r}{4}}(v)} \dots \,\mathrm{d}w\,\mathrm{d}v \\ &\le C\mathcal{C}^{1/2}\left(r, \frac{R+r}{4}\right) \|f\|_{H_v^s(B_{\frac{R+r}{2}}(v_0))} \\ &\quad + C\Lambda\left(\frac{r}{4}+v_0\right)^{-2s} \int_{B_{\frac{R+r}{4}}(v_0)} \chi_{\{f>0\}}\varphi_1(v)\,\mathrm{d}v.\end{aligned}$$

Further we have

$$\begin{aligned}\|g\|^2_{L^2_{t,x}\dot{H}^s_v(\mathcal{I}\times\mathbb{R}^{2d})} &\le C\,\|f\|^2_{L^2_{t,x}\dot{H}^s_v(Q_{\frac{R+r}{4}}(z_0))} + C\Big(\frac{r}{4}+v_0\Big)^{-2s}\|g\|^2_{L^2} \\ &\le C\,\|f\|^2_{L^2_{t,x}\dot{H}^s_v(Q_{\frac{R+r}{4}}(z_0))} + C\Big(\frac{r}{4}+v_0\Big)^{-2s}\|f\|^2_{L^2(Q_R(z_0)}\,.\end{aligned}$$

Therefore g satisfies

$$\partial_t g + v\cdot\nabla_x g + (-\Delta)^s_v g \le \big(\mathcal{L}f\big)\chi_{\{f>0\}}\varphi_1 + h\varphi_1 + f\big(\mathcal{T}\varphi_1\big) + (-\Delta)^s_v g =: H,$$

with right hand side $H = \big(\mathcal{L}f\big)\chi_{\{f>0\}}\varphi_1 + h\varphi_1 + f\big(\mathcal{T}\varphi_1\big) + (-\Delta)^s_v g \in L^2_{t,x}H^{-s}_v(\mathcal{I}\times\mathbb{R}^{2d})$. The energy estimate 3.2 implies

$$\begin{aligned}\|H\|^2_{L^2_{t,x}H^{-s}_v(\mathcal{I}\times\mathbb{R}^{2d})} &\le C\mathcal{C}\Big(r,\frac{R+r}{4}\Big)\|f\|^2_{L^2_{t,x}(\dot{H}^s_v(Q_{\frac{R+r}{4}}(z_0)))} + C\Big(\frac{r}{4}+v_0\Big)^{-4s}|\{f>0\}\cap Q_R(z_0)| \\ &\quad + \|h\|^2_{L^2(Q_R(z_0))} + \|g\|^2_{L^2_{t,x}(\dot{H}^s_v(\mathcal{I}\times\mathbb{R}^{2d}))} + \frac{(|v_0|+1)^2}{(R-(\frac{R+r}{4}))^{4s}}\|f\|^2_{L^2(Q_R(z_0))} \\ &\lesssim_{\lambda,\Lambda,v_0} \mathcal{C}\Big(r,\frac{R+r}{4}\Big)N + \mathcal{C}^2\Big(r,\frac{R+r}{4}\Big)\|f\|^2_{L^2(Q_R(z_0))}\,.\end{aligned}$$

It remains to quantify the measure of the difference between solution and sub-solution. Note that for (x,v) outside $Q^t_{\frac{R+r}{4}}(z_0)$ there holds $g=0$ and thus $\mathcal{T}g=0$. Moreover, outside $Q^t_{\frac{R+r}{4}}(z_0)$

$$\begin{aligned}|(-\Delta)^s_v g(t,x,v)| &= C\left|\int_{B_{r+\frac{R-r}{2}}(v_0)} g(t,x,w)\,|w-v|^{-(d+2s)}\,\mathrm{d}w\right| \\ &\le C\left(\,|v|-v_0-\frac{R+r}{2}\right)^{-d-2s}\Big|B_{\frac{R+r}{4}}(v_0)\Big|\,\chi_{\left\{|x|\le\left(\frac{R+r}{4}\right)^{1+2s}+x_0+(t-t_0)v_0\right\}}.\end{aligned}$$

Thus $\mathcal{T}g+(-\Delta)^s_v g = (-\Delta)^s_v g \in L^2$ outside $B_{(\frac{R+r}{4})^{1+2s}}(x_0+(t-t_0)v_0)\times B_{\frac{R+r}{2}}(v_0)$. Consider now a second smooth cut-off function $0\le\varphi_2\le 1$ such that $\varphi_2 = 1$ in $B_{(\frac{R+r}{4})^{1+2s}}(x_0+(t-$

$t_0)v_0) \times B_{\frac{R+r}{2}}(v_0)$ and $\varphi_2 = 0$ outside $Q^t_{\frac{3(R+r)}{4}}(z_0)$. We let

$$\tilde{H} = H\varphi_2 + (1-\varphi_2)(-\Delta)^s_v g.$$

We claim $\tilde{H} \in L^2_{t,x}H^{-s}_v(\mathcal{I}\times\mathbb{R}^{2d})$. Indeed $(1-\varphi_2)(-\Delta)^s_v g \in L^2_{t,x}H^{-s}_v$. Moreover, if a function $a \in H^s$ then also $a\varphi_2 \in H^s$. Thus by duality it also follows that $H\varphi_2 \in L^2_{t,x}H^{-s}_v$. With this definition there holds in $\mathcal{I}\times\mathbb{R}^{2d}$

$$\mathcal{T}g + (-\Delta)^s_v g \leq \tilde{H}$$

with equality outside $Q^t_{\frac{3(R+r)}{4}}(z_0)$.

Using Proposition 2.1 we deduce

$$\|g\|^2_{L^p(\mathcal{I}\times\mathbb{R}^{2d})} \lesssim \left\|\tilde{H}\right\|^2_{L^2_{t,x}H^{-s}_v(\mathcal{I}\times\mathbb{R}^{2d})} \leq \|H\|^2_{L^2_{t,x}H^{-s}_v(\mathcal{I}\times\mathbb{R}^{2d})}.$$

We conclude by noting that $0 \leq f \leq 1$ a.e. so that the L^1 and the L^2 norm are bounded by $|\{f>0\}\cap Q_R(z_0)|$. □

3.4 First lemma of De Giorgi

Lemma 3.4. *Let $Q_r(z_0) \subset Q_R(z_0)$ with $0<r<R\leq 1$ and $t_0 < r^{2s}$. Let f be a non-negative sub-solution of (1.13)-(1.14) in $Q_R(z_0)$ with a non-negative kernel K satisfying (1.26)-(1.30). Assume $0\leq f\leq 1$ a.e. in $(-R^{2s}+t_0, t_0]\times B_{R^{1+2s}}(x_0+(t-t_0)v_0)\times\mathbb{R}^d$. Then there exists $\varepsilon_0>0$ so that for all $\varepsilon<\varepsilon_0$ there holds: if*

$$\int_{Q_R(z_0)} f^2 \,\mathrm{d}z \leq \varepsilon,$$

then for any $2<p<2+\frac{2s}{d(1+s)}$ there exists a large constant $C>1$ depending only on s,d and p such that

$$f \leq \min\left\{ C\Big[1+(R-r)\left\|h\right\|_{L^\infty(Q_R(z_0))}\Big]^{\frac{\beta_1}{2}}(R-r)^{-\beta_1}\varepsilon^{\beta_2}, \frac{1}{2}\right\} \quad \text{a.e. in } Q_r(z_0),$$

where $\beta_1 = \frac{2p}{2p-2} > 0$ and $\beta_2 = \frac{p-2}{2(2p-2)} > 0$.

Proof of Lemma 3.4. We do a De Giorgi iteration, as was done in [115, Lemma 6.6] and [96, Lemma 3.8]. For an intuitive description of this technique, we refer the reader to [36]. Let $\tilde{\tau} := r^{2s} - t_0$ and $\hat{\tau} := R^{2s} - t_0$. Then $0 < \tilde{\tau} < \hat{\tau}$. Let $L>0$ to be determined and

consider

$$l_k = L(1-2^{-k}),$$
$$r_k = r + 2^{-k}(R-r),$$
$$t_k = -\tilde{\tau} - 2^{-k}(\hat{\tau}-\tilde{\tau}).$$

Define $Q_k(z_0) = (t_k, t_0] \times B_{r_k^{1+2s}}(x_0 + (t_k - t_0)v_0) \times B_{r_k}(v_0)$ and

$$A_k := \int_{Q_k(z_0)} (f-l_k)_+^2 \,\mathrm{d}z.$$

Then $A_0 \le \varepsilon_0$ by assumption. We want to prove that $A_k \to 0$ as $k \to +\infty$. This then proves the lemma.

We observe that $(f-l_{k+1})_+$ is a sub-solution of (1.13)-(1.14) with source term

$$\tilde{h}(t,x,v) = \chi_{\{f(v)>l_{k+1}\}} h(t,x,v),$$

and such that $0 \le (f-l_{k+1})_+ \le 1$ a.e. in $(-R^{2s}+t_0, t_0] \times B_{R^{1+2s}}(x_0 + (t-t_0)v_0) \times \mathbb{R}^d$. With Chebyschev's inequality we can estimate

$$|\{f > l_{k+1}\} \cap Q_k(z_0)| = \left|\left\{(f-l_k)_+ > 2^{-k-2}L\right\} \cap Q_k(z_0)\right| \le 2^{2k+4} L^{-2} A_k. \tag{3.13}$$

Thus

$$\left\|\tilde{h}\right\|^2_{L^2(Q_k(z_0))} \le C \left\|h\right\|^2_{L^\infty(Q_R(z_0))} 2^{2k+4} L^{-2} A_k.$$

We now pick $t_{k+\frac{1}{2}} \in [t_k, t_{k+1}]$ such that

$$\int_{Q_k^{t_{k+\frac{1}{2}}}(z_0)} (f-l_k)_+^2 \,\mathrm{d}x\,\mathrm{d}v \le \frac{1}{t_{k+1}-t_k} \int_{Q_k(z_0)} (f-l_k)_+^2 \,\mathrm{d}z \le C 2^k A_k.$$

Then we apply Lemma 3.3 and (3.13) to find

$$\begin{aligned}
&\left(\int_{t_{k+\frac{1}{2}}+t_0}^{t_0} \int_{Q_{k+1}^t(z_0)} (f-l_{k+1})_+^p \,\mathrm{d}z \right)^{\frac{2}{p}} \\
&\quad \le \frac{2^{4k+4}}{(R-r)^4} \left|\{f > l_{k+1}\} \cap Q_k\right| + C\frac{2^k}{(R-r)^2} A_k + C \left\|h\right\|^2_{L^\infty(Q_R(z_0))} \frac{2^{2k+4}}{(R-r)^2 L^2} A_k \qquad (3.14) \\
&\quad \le \frac{2^{6k+8}}{(R-r)^4 L^2} A_k + C\frac{2^k}{(R-r)^2} A_k + C \left\|h\right\|^2_{L^\infty(Q_R(z_0))} \frac{2^{2k+4}}{(R-r)^2 L^2} A_k \\
&\quad \le C 2^{6k} (R-r)^{-4} L^{-2} \left[1 + (R-r)^2 \left\|h\right\|^2_{L^\infty(Q_R(z_0))}\right] A_k,
\end{aligned}$$

where we used $l_{k+1} \ge l_k$.

Using $Q_{k+1} \subset Q_k$ and Chebyschev's inequality we get

$$\begin{aligned}
A_{k+1} &\leq \left(\int_{Q_{k+1}(z_0)} (f - l_{k+1})_+^p \, \mathrm{d}z \right)^{\frac{2}{p}} |\{f > l_{k+1}\} \cap Q_{k+1}(z_0)|^{1-\frac{2}{p}} \\
&\lesssim_{s,v_0} 2^{6k}(R-r)^{-4}L^{-2} \left[1 + (R-r)^2 \|h\|^2_{L^\infty(Q_R(z_0))}\right] A_k \left|\{f > l_{k+1}\} \cap Q_k(z_0)\right|^{1-\frac{2}{p}} \\
&\lesssim_{s,v_0} 2^{6k}(R-r)^{-4}L^{-2} \left[1 + (R-r)^2 \|h\|^2_{L^\infty(Q_R(z_0))}\right] A_k \left(2^{2k+4}L^{-2}A_k\right)^{1-\frac{2}{p}} \\
&\lesssim_{s,v_0} 2^{8k}(R-r)^{-4} \left[1 + (R-r)^2 \|h\|^2_{L^\infty(Q_R(z_0))}\right] L^{-2(1+\frac{p-2}{p})} A_k^{1+\frac{p-2}{p}}.
\end{aligned}$$

To determine how large we can pick L, we consider the sequence $A_k^* := A_0 Q^{-k}$ for some $Q > 1$ to be determined and for $k \geq 0$. Then we require A_k^* to satisfy the reverse non-linear recurrence, that is

$$A_{k+1}^* \geq 2^{8k}(R-r)^{-4} \left[1 + (R-r)^2 \|h\|^2_{L^\infty(Q_R(z_0))}\right] L^{-2(1+\frac{p-2}{p})} A_k^{1+\frac{p-2}{p}}. \tag{3.15}$$

Equivalently,

$$1 \geq 2^{8k}(R-r)^{-4} \left[1 + (R-r)^2 \|h\|^2_{L^\infty(Q_R(z_0))}\right] L^{-2(1+\frac{p-2}{p})} A_0^{\frac{p-2}{p}} Q^{-k(\frac{p-2}{p})} Q.$$

Choose Q such that $2^8 Q^{-\frac{p-2}{p}} \leq 1$, that is $2^8 \leq Q^{\frac{p-2}{p}}$. Then (3.15) holds if

$$L^{2(1+\frac{p-2}{p})} \geq (R-r)^{-4} \left[1 + (R-r)^2 \|h\|^2_{L^\infty(Q_R(z_0))}\right] A_0^{\frac{p-2}{p}} Q.$$

Thus, if we choose

$$L = A_0^{\frac{p-2}{2(2p-2)}} (R-r)^{-\frac{2p}{2p-2}} 2^{\frac{4p^2}{(p-2)(2p-2)}} \left[1 + (R-r)\|h\|_{L^\infty(Q_R(z_0))}\right]^{\frac{p}{2p-2}},$$

then, since $\frac{p-2}{p} > 0$ and $A_0 \leq \varepsilon$, we deduce $A_k \to 0$ as $k \to +\infty$. In particular, for almost every $z \in Q_r(z_0)$

$$f \leq L.$$

And therefore for almost every $z \in Q_r(z_0)$ there holds

$$\begin{aligned}
f &\leq A_0^{\frac{p-2}{2(2p-2)}} (R-r)^{-\frac{2p}{2p-2}} 2^{\frac{4p^2}{(p-2)(2p-2)}} \left[1 + (R-r)\|h\|_{L^\infty(Q_R(z_0))}\right]^{\frac{p}{2p-2}} \\
&= \|f\|^{\frac{p-2}{2p-2}}_{L^2(Q_R(z_0))} (R-r)^{-\frac{2p}{2p-2}} 2^{\frac{4p^2}{(p-2)(2p-2)}} \left[1 + (R-r)\|h\|_{L^\infty(Q_R(z_0))}\right]^{\frac{p}{2p-2}}.
\end{aligned}$$

□

Remark 3.2. First we remark that if instead of $f \leq 1$ a.e. we just assume f to be essentially

bounded, then the gain of integrability in Lemma 3.3 would yield

$$\begin{aligned}
&\|f\|^2_{L^p(Q_r(z_0))}\\
&\quad\lesssim_{\lambda,\Lambda,v_0} (R-r)^{-2}\int_{Q_R^t(z_0)} f_0^2(x,v)\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad+(R-r)^{-4}\left(\sup_{\Omega_{t,x}(R,z_0)\times\mathbb{R}^d} f\right)^2 |\{f>0\}\cap Q_R(z_0)| + (R-r)^{-2}\int_{Q_R(z_0)} h^2\,\mathrm{d}z,
\end{aligned}$$

where $\Omega_{t,x}(R,z_0) := (-R^{2s}+t_0,t_0]\times B_{R^{1+2s}}(z_0)$. Using this we can further note that the nonlinear relation between the L^∞ and the L^2 norm of the function can be linearised at the price of a non-local tail term. Indeed, going back to (3.14), if for some $\delta\in(0,1]$ we choose L such that

$$L^2 \geq \delta\left(\sup_{z\in\Omega_{t,x}(R,z_0)\times\mathbb{R}^d} f + (R-r)\,\|h\|_{L^\infty(Q_R(z_0))}\right)^2,$$

then

$$\left(\int_{t_{k+\frac12}+t_0}^{t_0}\int_{Q_{k+1}^t(z_0)} (f-l_{k+1})_+^p\,\mathrm{d}z\right)^{\frac{2}{p}} \leq C2^{6k}(R-r)^{-4}\delta^{-1}A_k.$$

Thus we deduce

$$\begin{aligned}
A_{k+1} &\leq \left(\int_{Q_{k+1}(z_0)} (f-l_{k+1})_+^p\,\mathrm{d}z\right)^{\frac{2}{p}} |\{f>l_{k+1}\}\cap Q_{k+1}(z_0)|^{1-\frac{2}{p}}\\
&\lesssim_{s,v_0} 2^{6k}(R-r)^{-4}\delta^{-1}A_k\big(2^{2k+4}L^{-2}A_k\big)^{1-\frac{2}{p}}\\
&\lesssim_{s,v_0} 2^{8k}(R-r)^{-4}\delta^{-1}L^{-\frac{2(p-2)}{p}}A_k^{1+\frac{p-2}{p}}.
\end{aligned}$$

The same barrier argument as before shows that we have to choose L such that

$$L^{2(\frac{p-2}{p})} \geq (R-r)^{-4}\delta^{-1}A_0^{\frac{p-2}{p}}Q.$$

Therefore, we pick

$$L = \delta^{\frac12}\left(\sup_{z\in\Omega_{t,x}(R,z_0)\times\mathbb{R}^d} f + (R-r)\,\|h\|_{L^\infty(Q_R(z_0))}\right) + A_0^{\frac12}(R-r)^{-\frac{2p}{p-2}}2^{\frac{4p^2}{(p-2)^2}}\delta^{-\frac{p}{2(p-2)}},$$

and we deduce that for almost every $z\in Q_r(z_0)$ there holds

$$\begin{aligned}
f \leq& C(R-r)^{-\frac{2p}{p-2}}\delta^{-\frac{p}{2(p-2)}}\,\|f\|_{L^2(Q_R(z_0))} + \delta^{\frac12}\sup_{z\in\Omega_{t,x}(R,z_0)\times\mathbb{R}^d} f\\
&+\delta^{\frac12}(R-r)\,\|h\|_{L^\infty(Q_R(z_0))}.
\end{aligned}\tag{3.16}$$

To derive a linear relation between the L^∞ and the L^2 norm of f, we would need to absorb the non-local tail term on the left hand side. Thus we require an estimate of the tail in terms of local quantities. If we were able to replace the tail by the essential supremum on any finite ball in the velocity domain, then using the standard iteration argument of Lemma 2.2, we would have obtained for almost every $z \in Q_r(z_0)$

$$f \leq C(R-r)^{-\frac{2p}{p-2}} \|f\|_{L^2(Q_R(z_0))} + C \|h\|_{L^\infty(Q_R(z_0))}.$$

Combined with the weak Harnack inequality (1.33), this would allow us to derive the (proper) strong Harnack inequality, in the sense that

$$\sup_{\tilde{Q}^-_{\frac{r_0}{4}}(z_0)} f \leq C \inf_{Q_{\frac{r_0}{4}}(z_0)} f + C \|h\|_{L^\infty(Q_R(z_0))}.$$

We refer to Section 3.6.1 below. Note, however, that it is not clear that the non-local tail in (3.16) can be bounded by local quantities.

3.5 Second lemma of De Giorgi

3.5.1 Hypoelliptic Poincaré

This is where we move along trajectories in order to obtain a hypoelliptic Poincaré-type inequality with an error term. The idea comes from Guerand and Mouhot [99], and was then adapted to the non-local case in [152]. In these works, we were using piecewise affine trajectories that followed the vector fields. Since then, the construction was optimised: instead of following a piecewise affine trajectory, we built a smooth curve constructed from the control problem associated to the underlying hypoelliptic structure of the equation. This was done in the collaborative work [10]. We treated the local and non-local case for any number of Hörmander commutators. Here we prove the hypoelliptic Poincaré inequality for (1.13), by following the construction in [10], in the case of one commutator only. However, we need to account for the non-symmetric kernel of the non-local operator (1.14) that we treat here.

Proposition 3.1. *Let $R \geq 1$ be sufficiently large. Let f be a non-negative sub-solution to (1.13)-(1.14) in $\Omega := (-9, 0] \times Q_R^t$ with a non-negative kernel K. Then, for any non-negative $\varphi \in C_c^\infty(Q_{R/3}^t)$ averaging to one on $Q_{R/3}^t$, for any $\varepsilon > 0$, and any $\varepsilon_0 \in (0, s)$,*

there holds

$$
\begin{aligned}
&\left\|\left(f-\langle f\rangle_{Q_1^-}\right)_+\right\|_{L^1(Q_1)}\\
&\lesssim \int_{B_{\frac{R}{3}}}\int_0^1\int_{\Omega_{t,x}}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[f(y,w'))-f(y,w)\right]\left[K(w,w'))-K(w',w)\right]\right.\\
&\qquad\qquad\left.\times\left[\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w))+\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\right|\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0\\
&+\int_{B_{\frac{R}{3}}}\int_0^1\int_{\Omega_{t,x}}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[f(y,w')-f(y,w)\right]K(w,w')\right.\\
&\qquad\qquad\left.\times\left[\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w))-\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\right|\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0\\
&+\int_{\Omega_{t,x}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(w')-f(w)|}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}w\\
&+\int_0^1\int_{\Omega_{t,x}}\int_{B_{\frac{R}{3}}}\left|\int_{B_R} r^{-\frac{sd}{d+s}-\varepsilon}h\left(\sigma,y,w\right)\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w))\,\mathrm{d}w\right|\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}x_0\,\mathrm{d}r.
\end{aligned}
\tag{3.17}
$$

where $(\Phi^r_\pm)_v^{-1}$ *is an affine smooth function (constructed in* (3.21)*), and where* $Q_1^- := Q_1(-2,0,0) = (-3,-2]\times Q_1^t$, $\langle f\rangle_{Q_1^-} = \fint_{Q_1^-} f = \frac{1}{|Q_1^-|}\int_{Q_1^-} f$, $\Omega_{t,x} := (-9,0]\times B_{R^{1+2s}}$.

This proposition allows us to pick up an intermediate set in De Giorgi's Second Lemma, Theorem 1.2. The proof of the inequality (3.17) does not use De Giorgi's First Lemma.

Remark 3.3. We could simplify the statement of this lemma, but we kept it intentionally as it is written, in view of applying it in the proof of the Intermediate Value Theorem, Theorem 1.2.

3.5.2 Construction of the trajectories

Let $\Omega := (-9,0]\times Q_3^t$. We denote by Q^+, Q^- and Q^0 the cylinders

$$
\begin{aligned}
Q^+ &:= Q_1 = (-1,0]\times B_1\times B_1,\\
Q^- &:= Q_1^- := (-3,-2]\times B_1\times B_1,\\
Q^0 &:= \left(-\frac{7}{4},-\frac{5}{4}\right]\times B_1\times B_1.
\end{aligned}
$$

Given three points $z_+\in Q^+, z_0\in Q^0$ and $z_-\in Q^-$, we want to construct two paths $r\to\Gamma_+(r)$ and $r\to\Gamma_-(r)$ for $r\in[0,1]$ such that (see Figures 3.1 and 3.2)

$$
\begin{aligned}
&\Gamma_+(r) = \left(T_+(r),X_+(r),V_+(r)\right)^T, \qquad \Gamma_+(0)=z_+, \qquad \Gamma_+(1)=z_0,\\
&\Gamma_-(r) = \left(T_-(r),X_-(r),V_-(r)\right)^T, \qquad \Gamma_-(0)=z_-, \qquad \Gamma_-(1)=z_0,
\end{aligned}
$$

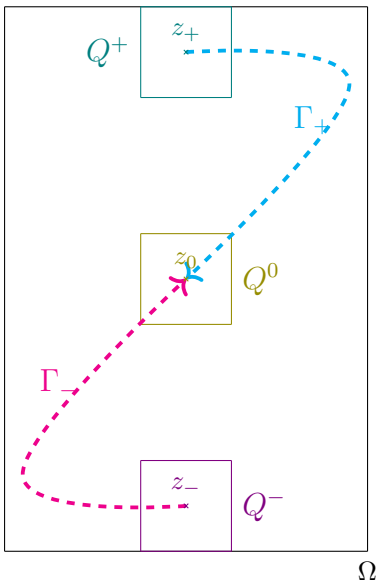


Figure 3.1: Construction of the trajectories. The curve Γ_+ connects any point $z_+ \in Q^+$ to some intermediate point $z_0 \in Q^0$, whereas the curve Γ_- connects any point $z_- \in Q^-$ to some intermediate point $z_0 \in Q^0$.

where

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}r} V_\pm(r) = \mathsf{m}_\pm^{(0)} g_0''(r) + \mathsf{m}_\pm^{(1)} g_1''(r), \\ \frac{\mathrm{d}}{\mathrm{d}r} X_\pm(r) = \delta_\pm V_\pm(r), \\ \frac{\mathrm{d}}{\mathrm{d}r} T_\pm(r) = \delta_\pm, \end{cases} \tag{3.18}$$

where $\delta_\pm := t_0 - t_\pm$ and the control functions $g_i \in C^2((0,1])$ with $g_i(0) = g_i'(0) = 0$ for $i = 0, 1$, and $\mathsf{m}_\pm^{(i)} \in \mathbb{R}^d$ for $i = 0, 1$. Solving the differential equations yields

$$\begin{cases} V_\pm(r) = v_\pm + \mathsf{m}_\pm^{(0)} g_0'(r) + \mathsf{m}_\pm^{(1)} g_1'(r), \\ X_\pm(r) = x_\pm + r\delta_\pm v_\pm + \delta_\pm \left[\mathsf{m}_\pm^{(0)} g_0(r) + \mathsf{m}_\pm^{(1)} g_1(r) \right], \\ T_\pm(r) = r t_0 + (1-r) t_\pm. \end{cases} \tag{3.19}$$

Let us denote by $\mathbf{1}_d$ the d-vector of 1's, $\mathbf{0}_d$ the d-vector of 0's, Id_d the $d \times d$ identity matrix, $\mathbb{O}$ a zero block of arbitrary size, and

$$\mathsf{M}_\pm := \begin{pmatrix} \mathsf{m}_\pm^{(0)} \\ \mathsf{m}_\pm^{(1)} \end{pmatrix}, \quad \mathsf{W}(r) := \begin{pmatrix} g_0(r)\mathrm{Id}_d & g_1(r)\mathrm{Id}_d \\ g_0'(r)\mathrm{Id}_d & g_1'(r)\mathrm{Id}_d \end{pmatrix},$$
$$\mathsf{W}_\pm^\delta(r) := \begin{pmatrix} \delta_\pm \mathrm{Id}_d & \mathbb{O} \\ \mathbb{O} & \mathrm{Id}_d \end{pmatrix} \mathsf{W}(r), \quad \mathsf{Y}_\pm = \begin{pmatrix} \mathsf{y}_\pm^{(0)} \\ \mathsf{y}_\pm^{(1)} \end{pmatrix} := \begin{pmatrix} x_0 - x_\pm - \delta_\pm v_\pm \\ v_0 - v_\pm \end{pmatrix}.$$

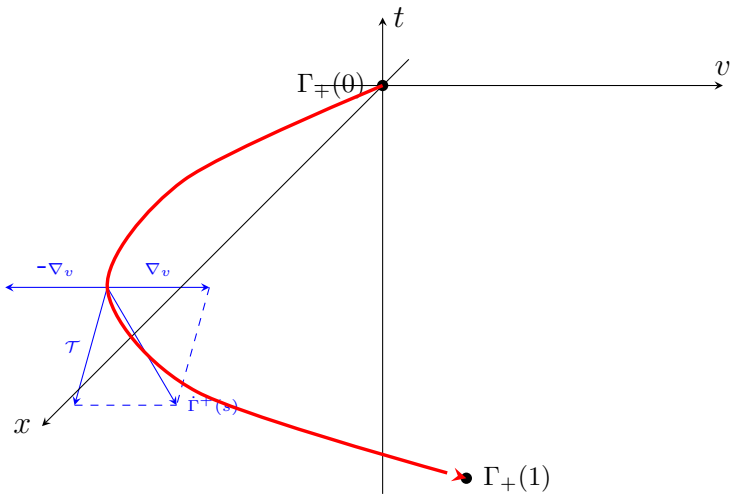


Figure 3.2: Construction of the trajectories. The curve Γ_+ connects any point $\Gamma_+(0) = z_+ \in Q^+$ to some intermediate point $\Gamma_+(1) = z_0 \in Q^0$ along the vector fields $\mathcal{T}$ and ∇_v.

The boundary conditions $\Gamma_\pm(1) = z_0$ impose

$$\mathsf{W}^\delta_\pm(1)\mathsf{M}_\pm = \mathsf{Y}_\pm \quad \Longrightarrow \quad \mathsf{M}_\pm = \mathsf{W}^\delta_\pm(1)^{-1}\mathsf{Y}_\pm, \tag{3.20}$$

provided that the Wronskian matrix $\mathsf{W}(1)$ is invertible at $s = 1$. We deduce

$$\begin{pmatrix} X_\pm(r) \\ V_\pm(r) \end{pmatrix} = \mathsf{W}^\delta_\pm(r)\mathsf{M}_\pm + \begin{pmatrix} \mathrm{Id}_d & \delta_\pm r\mathrm{Id}_d \\ \mathbb{O} & \mathrm{Id}_d \end{pmatrix} \begin{pmatrix} x_\pm \\ v_\pm \end{pmatrix}.$$

Using (3.20) we obtain

$$\begin{aligned} \begin{pmatrix} X_\pm(r) \\ V_\pm(r) \end{pmatrix} &= \mathsf{W}^\delta_\pm(r)\mathsf{W}^\delta_\pm(1)^{-1}\mathsf{Y}_\pm + \begin{pmatrix} \mathrm{Id}_d & \delta_\pm r\mathrm{Id}_d \\ \mathbb{O} & \mathrm{Id}_d \end{pmatrix} \begin{pmatrix} x_\pm \\ v_\pm \end{pmatrix} \\ &= \mathsf{W}^\delta_\pm(r)\big[\mathsf{W}^\delta_\pm(1)\big]^{-1} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \\ &\quad - \mathsf{W}^\delta_\pm(r)\big[W^\delta_\pm(1)\big]^{-1} \begin{pmatrix} x_\pm + \delta_\pm v_\pm \\ v_\pm \end{pmatrix} + \begin{pmatrix} \mathrm{Id}_d & \delta_\pm r\mathrm{Id}_d \\ \mathbb{O} & \mathrm{Id}_d \end{pmatrix} \begin{pmatrix} x_\pm \\ v_\pm \end{pmatrix} \\ &= W^\delta_\pm(r)\big[W^\delta_\pm(1)\big]^{-1} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \\ &\quad + \big(\mathsf{T}^\delta_\pm(r) - \mathsf{W}^\delta_\pm(r)[\mathsf{W}^\delta_\pm(1)]^{-1}\mathsf{T}^\delta_\pm(1)\big) \begin{pmatrix} x_\pm \\ v_\pm \end{pmatrix} \\ &= \mathfrak{A}^r \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + \mathfrak{B}^r =: \Phi^r_\pm\big((x_0, v_0)^T\big), \end{aligned} \tag{3.21}$$

where

$$\mathsf{T}^{\delta}_{\pm}(r) := \begin{pmatrix} \mathrm{Id}_d & \delta_{\pm} r \mathrm{Id}_d \\ \mathbb{O} & \mathrm{Id}_d \end{pmatrix};$$

in particular, $\Phi^r_{\pm}$ is an affine function with matrix $\mathfrak{A}^r := \mathsf{W}^{\delta}_{\pm}(r)[\mathsf{W}^{\delta}_{\pm}(1)]^{-1}$ and a vector $\mathfrak{B}^r$ that depend only on r and $\delta_{\pm}, x_{\pm}, v_{\pm}$. We still have to prove that the matrix $\mathfrak{A}^r$ is invertible for $r \in (0,1]$. If so, given $r \in (0,1]$, the derivative along the velocity variable $V_{\pm}(r)$ of the inverse is

$$\nabla_v \left(\Phi^r_{\pm}\right)^{-1} = \mathsf{W}^{\delta}_{\pm}(1)\left[\mathsf{W}^{\delta}_{\pm}(r)\right]^{-1} \begin{pmatrix} \mathbf{0}_d \\ \mathbf{1}_d \end{pmatrix}.$$

We choose

$$g_i(r) := (1+\alpha_i)^{-1}(2+\alpha_i)^{-1} r^{2+\alpha_i}, \tag{3.22}$$

$i = 0, 1$, with $\alpha_0, \alpha_1 \in (-1, 0)$ and $\alpha_0 \neq \alpha_1$. Then the Wronskian matrix is invertible for all $s \neq 0$:

$$\begin{aligned} \det \mathsf{W}(r) &= \left(g_0(r) g_1'(s) - g_0'(r) g_1(r)\right)^d \\ &= \frac{(\alpha_1 - \alpha_0)^d}{(1+\alpha_0)^d (1+\alpha_1)^d (2+\alpha_0)^d (2+\alpha_1)^d} r^{(3+\alpha_0+\alpha_1)d} \neq 0. \end{aligned} \tag{3.23}$$

Our choice of two linearly independent control functions g_0 and g_1 is guided by ensuring the Wronskian matrix to be invertible. Moreover,

$$\begin{aligned} \mathsf{W}^{\delta}_{\pm}(1) &:= \begin{pmatrix} \frac{\delta_{\pm}\mathrm{Id}_d}{(1+\alpha_0)(2+\alpha_0)} & \frac{\delta_{\pm}\mathrm{Id}_d}{(1+\alpha_1)(2+\alpha_1)} \\ \frac{\mathrm{Id}_d}{(1+\alpha_0)} & \frac{\mathrm{Id}_d}{(1+\alpha_1)} \end{pmatrix}, \\ \mathsf{W}^{\delta}_{\pm}(r)^{-1} &:= \frac{1}{\delta_{\pm} \det \mathsf{W}(s)} \begin{pmatrix} \frac{r^{1+\alpha_1}\mathrm{Id}_d}{1+\alpha_1} & \frac{-\delta_{\pm} r^{2+\alpha_1}\mathrm{Id}_d}{(1+\alpha_1)(2+\alpha_1)} \\ -\frac{r^{1+\alpha_0}\mathrm{Id}_d}{1+\alpha_0} & \frac{\delta_{\pm} r^{2+\alpha_0}\mathrm{Id}_d}{(1+\alpha_0)(2+\alpha_0)} \end{pmatrix}, \end{aligned}$$

which finally yields

$$\begin{aligned} \nabla_v \left(\Phi^r_{\pm}\right)^{-1} &= \mathsf{W}^{\delta}_{\pm}(1)\left[\mathsf{W}^{\delta}_{\pm}(r)\right]^{-1} \begin{pmatrix} \mathbf{0}_d \\ \mathbf{1}_d \end{pmatrix} \\ &= \frac{1}{(\alpha_0 - \alpha_1)} \begin{pmatrix} \delta_{\pm}(r^{-1-\alpha_1} - r^{-1-\alpha_0})\mathbf{1}_d \\ \left((2+\alpha_1) r^{-1-\alpha_1} - (2+\alpha_0) r^{-1-\alpha_0}\right)\mathbf{1}_d \end{pmatrix} \\ &= O(r^{-1-\alpha_0}) + O(r^{-1-\alpha_1}), \end{aligned}$$

which remains integrable for $\alpha_0, \alpha_1 \in (-1, 0)$ with $\alpha_0 \neq \alpha_1$. Our choice of control functions g_0 and g_1 implies that all their derivatives up to order two are integrable on $s \in [0, 1]$, which implies that the trajectories are bounded (with integrable tangent vector field).

3.5.3 Proof of the Poincaré inequality

We consider the three cylinders $Q^+, Q^-, Q^0 \subset \Omega$ as in Figure 3.1, and a sub-solution f of (1.13). Let $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ be a non-negative function in the variables (x, v) with compact support in any time-slice of Q^0 and such that $⨍_{(Q^0)^t} \varphi \,\mathrm{d}x\,\mathrm{d}v = 1$. Then

$$\begin{aligned}
⨍_{Q^+} \Big(f(z_+) - \langle f\rangle_{Q^-}\Big)_+ \mathrm{d}z_+ &\le ⨍_{Q^+}⨍_{Q^-} \Big(f(z_+) - f(z_-)\Big)_+ \mathrm{d}z_-\,\mathrm{d}z_+ \\
&\le ⨍_{Q^+}⨍_{Q^-} \Big(f(z_+) - \langle f\varphi\rangle_{Q^0}\Big)_+ \mathrm{d}z_-\,\mathrm{d}z_+ \\
&\quad + ⨍_{Q^+}⨍_{Q^-} \Big(\langle f\varphi\rangle_{Q^0} - f(z_-)\Big)_+ \mathrm{d}z_-\,\mathrm{d}z_+ \\
&\le ⨍_{Q^+} \Big\{ \underbrace{⨍_{Q^0} \big(f(z_+) - f(z_0)\big)\varphi(x_0,v_0)\,\mathrm{d}z_0}_{=:\mathcal{I}^+} \Big\}_+ \mathrm{d}z_+ \\
&\quad + ⨍_{Q^-} \Big\{ \underbrace{⨍_{Q^0} \big(f(z_0) - f(z_-)\big)\varphi(x_0,v_0)\,\mathrm{d}z_0}_{=:\mathcal{I}^-} \Big\}_+ \mathrm{d}z_-.
\end{aligned} \tag{3.24}$$

where $z_+ = (t_+, x_+, v_+), z_- = (t_-, x_-, v_-)$ and $z_0 = (t_0, x_0, v_0)$. Note that $t_- < t_0 < t_+$. We now use the trajectories constructed in Section 3.5.2 to estimate the right hand side.

Using the chain rule, (1.13), (3.19) with the choice (3.22) we get

$$\begin{aligned}
\mathcal{I}^\pm &= \pm ⨍_{Q^0} \Big(f(z_\pm) - f(z_0)\Big)\,\varphi(x_0,v_0)\,\mathrm{d}z_0 \\
&= \mp ⨍_{Q^0}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f\big(\Gamma_\pm(s)\big)\,\varphi(x_0,v_0)\,\mathrm{d}r\,\mathrm{d}z_0 \\
&= \mp\delta_\pm ⨍_{Q^0}\int_0^1 \delta_\pm\,(\mathcal{T}f)\,\big(\Gamma_\pm(s)\big)\,\varphi(x_0,v_0)\,\mathrm{d}r\,\mathrm{d}z_0 \\
&\qquad \mp ⨍_{Q^0}\int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}r}V(r)\Big)\cdot(\nabla_v f)\,(\Gamma_\pm(r))\,\varphi(x_0,v_0)\,\mathrm{d}r\,\mathrm{d}z_0 \\
&\le \underbrace{\pm ⨍_{Q^0}\int_0^1 \delta_\pm\,[\mathcal{L}f + h]\,\big(\Gamma_\pm(r)\big)\,\varphi(x_0,v_0)\,\mathrm{d}s\,\mathrm{d}z_0}_{=:\mathcal{I}_1^\pm} \\
&\qquad \underbrace{\mp ⨍_{Q^0}\int_0^1 \sum_{i=0}^{1} r^{\alpha_i}\mathsf{m}_\pm^{(i)}\cdot(\nabla_v f)\,(\Gamma_\pm(r))\,\varphi(x_0,v_0)\,\mathrm{d}s\,\mathrm{d}z_0}_{=:\mathcal{I}_2^\pm}.
\end{aligned} \tag{3.25}$$

Note that the only difference between the two terms $\mathcal{I}^+$ and $\mathcal{I}^-$ is the role of z_0: in the former case it is the past variable, in the latter it is the future variable.

We now abuse notation and write $Q^0_{t,x} := \left(-\frac{7}{4}, -\frac{5}{4}\right] \times B_1$. Using that φ vanishes outside B_1, we find

$$\begin{aligned}
\mathcal{I}_1^{\pm} &= \pm \fint_{Q^0} \int_0^1 \delta_{\pm} \left[\mathcal{L}f + h\right] \left(\Gamma_{\pm}(r)\right) \varphi(x_0, v_0) \, \mathrm{d}r \, \mathrm{d}z_0 \\
&= \pm \frac{1}{|Q^0|} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \delta_{\pm} \mathcal{L}f \left(rt_0 + (1-r)t_{\pm}, \Phi^r_{\pm}(x_0, v_0)\right) \varphi(x_0, v_0) \, \mathrm{d}s \, \mathrm{d}z_0 \\
&\quad \pm \fint_{Q^0} \int_0^1 \delta_{\pm} h \left(rt_0 + (1-r)t_{\pm}, \Phi^r_{\pm}(x_0, v_0)\right) \varphi(x_0, v_0) \, \mathrm{d}r \, \mathrm{d}z_0.
\end{aligned} \tag{3.26}$$

Then we abuse notation by writing $\Phi^r_{\pm} := \left((\Phi^r_{\pm})_x, (\Phi^r_{\pm})_v\right)$, we abbreviate $\tau := rt_0 + (1-r)t_{\pm}$, $f((\Phi^r_{\pm})_v(v_0)) = f(\tau, \Phi^r_{\pm}(x_0, v_0))$ and $f(w') = f(\tau, (\Phi^r_{\pm})_x(x_0), w')$, so that we find for the non-local term

$$\begin{aligned}
&\pm \frac{1}{|Q^0|} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \delta_{\pm} \mathcal{L}f \left(\tau, \Phi^r_{\pm}(x_0, v_0)\right) \varphi(x_0, v_0) \, \mathrm{d}s \, \mathrm{d}x_0 \, \mathrm{d}v_0 \, \mathrm{d}t_0 \\
&= \pm \frac{1}{|Q^0|} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \delta_{\pm} \left[f(w') - f\left((\Phi^r_{\pm})_v(v_0)\right)\right] \\
&\qquad \times K(\tau, \Phi^r_{\pm}(x_0, v_0), w') \, \mathrm{d}w' \varphi(x_0, v_0) \, \mathrm{d}r \, \mathrm{d}z_0 \\
&= \pm \frac{1}{2\,|Q^0|} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \delta_{\pm} \left[f(w') - f\left((\Phi^r_{\pm})_v(v_0)\right)\right] K(\tau, \Phi^r_{\pm}(x_0, v_0), w') \\
&\qquad \times \left(\varphi(x_0, v_0) - \varphi(x_0, (\Phi^r_{\pm})_v^{-1}(w'))\right) \, \mathrm{d}w' \, \mathrm{d}r \, \mathrm{d}z_0 \\
&\quad \pm \frac{1}{2\,|Q^0|} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \delta_{\pm} \left[f(w') - f\left((\Phi^r_{\pm})_v(v_0)\right)\right] K(\tau, \Phi^r_{\pm}(x_0, v_0), w') \\
&\qquad \times \left(\varphi(x_0, v_0) + \varphi(x_0, (\Phi^r_{\pm})_v^{-1}(w'))\right) \, \mathrm{d}w' \, \mathrm{d}r \, \mathrm{d}z_0.
\end{aligned} \tag{3.27}$$

We then change variables $(x_0, v_0) \mapsto (y, w) := \Phi^r_{\pm}(x_0, v_0)$ for r, t_0 fixed ($\Phi^r_{\pm}$ is the affine map defined in (3.21), which is invertible for $r \neq 0$). This yields for the first term in (3.27):

$$\begin{aligned}
&\pm \frac{1}{2\,|Q^0|} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \delta_{\pm} \left[f(w') - f\left((\Phi^r_{\pm})_v(v_0)\right)\right] K(\tau, \Phi^r_{\pm}(x_0, v_0), w') \\
&\qquad \times \left(\varphi(x_0, v_0) - \varphi(x_0, (\Phi^r_{\pm})_v^{-1}(w'))\right) \, \mathrm{d}w' \, \mathrm{d}r \, \mathrm{d}x_0 \, \mathrm{d}v_0 \, \mathrm{d}t_0 \\
&= \pm \frac{1}{2\,|Q^0|} \int_{(\mathrm{Id} \otimes \Phi^r_{\pm})(Q^0_{t,x} \times \mathbb{R}^d)} \int_0^1 \int_{\mathbb{R}^d} \delta_{\pm} \left[f(w') - f(w)\right] K(\tau, y, w, w') \\
&\qquad \times \left(\varphi\left(\left(\Phi^r_{\pm}\right)^{-1}(y, w)\right) - \varphi\left(\left(\Phi^r_{\pm}\right)^{-1}(y, w')\right)\right) \, \mathrm{d}w' \frac{\mathrm{d}r \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}t_0}{|\det \mathfrak{A}^r|}
\end{aligned}$$

we change back variables in x and use that $(\Phi^r_\pm)_v(\mathbb{R}^d) = \mathbb{R}^d$

$$= \pm\frac{1}{2\,|Q^0|}\int_{Q^0_{t,x}\times\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\delta_\pm\left[f(w')-f\left(w\right)\right]K(\tau,(\Phi^r_\pm)_x(x_0),w,w')$$
$$\times\left[\varphi\big(x_0,(\Phi^r_\pm)_v^{-1}(w)\big)-\varphi\big(x_0,(\Phi^r_\pm)_v^{-1}(w')\big)\right]\,\mathrm{d}w'\frac{\mathrm{d}r\,\mathrm{d}x_0\,\mathrm{d}w\,\mathrm{d}t_0}{|\det\mathfrak{A}^r_v|}.$$

For the second term in (3.27), we find

$$\pm\frac{1}{2\,|Q^0|}\int_{Q^0_{t,x}\times\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\delta_\pm\left[f(w')-f\left((\Phi^r_\pm)_v(v_0)\right)\right]K(\tau,\Phi^r_\pm(x_0,v_0),w')$$
$$\times\left(\varphi(x_0,v_0)+\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w'))\right)\,\mathrm{d}w'\,\mathrm{d}r\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}t_0$$
$$= \pm\frac{1}{2\,|Q^0|}\int_{(\mathrm{Id}\otimes\Phi^r_\pm)(Q^0_{t,x}\times\mathbb{R}^d)}\int_0^1\int_{\mathbb{R}^d}\delta_\pm\left[f(w')-f\left(w\right)\right]K(\tau,y,w,w')$$
$$\times\left[\varphi((\Phi^r_\pm)^{-1}(y,w)+\varphi((\Phi^r_\pm)^{-1}(y,w'))\right]\frac{\mathrm{d}w'\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}t_0}{|\det\mathfrak{A}^r|},$$

we use $(\Phi^r_\pm)_v(\mathbb{R}^d) = \mathbb{R}^d$ to exploit the symmetry of the integrand

$$= \pm\frac{1}{4\,|Q^0|}\int_{(\mathrm{Id}\otimes(\Phi^r_\pm)_x\otimes\mathrm{Id})(Q^0_{t,x}\times\mathbb{R}^d)}\int_0^1\int_{\mathbb{R}^d}\delta_\pm\left[f(w')-f(w)\right]\left[K(w,w')-K(w',w)\right]$$
$$\times\left[\varphi((\Phi^r_\pm)^{-1}(y,w))+\varphi((\Phi^r_\pm)^{-1}(y,w'))\right]\frac{\mathrm{d}w'\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}t_0}{|\det\mathfrak{A}^r|}$$

we reverse the change in variables in x

$$= \pm\frac{1}{4\,|Q^0|}\int_{Q^0_{t,x}\times\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\delta_\pm\left[f((\Phi^r_\pm)_x(x_0),w'))-f((\Phi^r_\pm)_x(x_0),w')\right]$$
$$\times\left[K(w,w'))-K(w',w)\right]$$
$$\times\left[\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w))+\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w))\right]\frac{\mathrm{d}w'\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}x_0\,\mathrm{d}t_0}{|\det\mathfrak{A}^r_v|}.$$

Finally, for the source term, we use the same change of variables in velocity only, so that abbreviating $\tau = rt_0 + (1-r)t_\pm$

$$\pm\fint_{Q^0}\int_0^1\delta_\pm h\left(\tau,\Phi^r_\pm(x_0,v_0)\right)\,\varphi(x_0,v_0)\,\mathrm{d}r\,\mathrm{d}z_0$$
$$= \pm\fint_{(\mathrm{Id}\otimes\mathrm{Id}\times(\Phi^r_\pm)_v)(Q^0)}\int_0^1\delta_\pm h\left(\tau,(\Phi^r_\pm)_x(x_0),w)\right)\varphi(x_0,(\Phi^r_\pm)_v^{-1}(w))\frac{\mathrm{d}r\,\mathrm{d}t_0\,\mathrm{d}x_0\,\mathrm{d}w}{|\det\mathfrak{A}^r_v|}.$$

We then use that the mapping $\mathfrak{A}$ from (3.21) satisfies

$$\frac{1}{|\det \mathfrak{A}_v^r|} \approx r^{-d(1+\alpha_0)} + r^{-d(1+\alpha_1)},$$

so that for $\mathcal{I}_1^\pm$ from (3.26) we find

$$\begin{aligned}
\mathcal{I}_1^\pm \leq \pm C \int_{Q_{t,x}^0 \times \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \delta_\pm \sum_{i=0}^1 r^{-d(1+\alpha_i)} \left[f((\Phi_\pm^r)_x(x_0), w')) - f((\Phi_\pm^r)_x(x_0), w) \right] \\
\times [K(w, w')) - K(w', w)] \\
\times \left[\varphi(x_0, (\Phi_\pm^r)_v^{-1}(w)) + \varphi(x_0, (\Phi_\pm^r)_v^{-1}(w')) \right] \mathrm{d}w' \, \mathrm{d}w \, \mathrm{d}r \, \mathrm{d}x_0 \, \mathrm{d}t_0 \\
\pm C \int_{Q_{t,x}^0 \times \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \delta_\pm \sum_{i=0}^1 r^{-d(1+\alpha_i)} \left[f(w') - f(w) \right] K(\tau, (\Phi_\pm^r)_x(x_0), w, w') \\
\times \left(\varphi\big(x_0, (\Phi_\pm^r)_v^{-1}(w)\big) - \varphi\big(x_0, (\Phi_\pm^r)_v^{-1}(w')\big) \right) \mathrm{d}w' \, \mathrm{d}r \, \mathrm{d}x_0 \, \mathrm{d}w \, \mathrm{d}t_0 \\
\pm \int_0^1 \fint_{Q_{t,x}^0 \times (\Phi_\pm^r)_v(B_1)} \delta_\pm \sum_{i=0}^1 r^{-d(1+\alpha_i)} h\left(\tau, (\Phi_\pm^r)_x(x_0), w\right) \varphi(x_0, (\Phi_\pm^r)_v^{-1}(w)) \, \mathrm{d}r \, \mathrm{d}t_0 \, \mathrm{d}x_0 \, \mathrm{d}w.
\end{aligned}$$

Now let us turn to $\mathcal{I}_2^\pm$ in (3.25). We find, with the same change of variables, for any $\varepsilon_0 > 0$

$$\begin{aligned}
\mathcal{I}_2^\pm &= \mp \int_0^1 \fint_{Q^0} \sum_{i=0}^1 r^{\alpha_i} \mathsf{m}_\pm^{(i)} \cdot \nabla_v f(\Gamma_\pm(s)) \varphi(x_0, v_0) \, \mathrm{d}r \, \mathrm{d}x_0 \, \mathrm{d}v_0 \, \mathrm{d}t_0 \\
&= \mp \frac{1}{|Q^0|} \sum_{i=0}^1 \int_0^1 \int_{(\Phi_\pm^r \otimes \mathrm{Id})(Q_{t,x}^0 \times \mathbb{R}^d)} r^{\alpha_i} \mathsf{m}_\pm^{(i)} \cdot \nabla_v f\left(st_0 + (1-s)t_\pm, y, w\right) \\
&\qquad \times \varphi\left(\left(\Phi_\pm^r\right)^{-1}(y, w) \right) \frac{\mathrm{d}r \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}t_0}{|\det \mathfrak{A}^r|} \\
&= \mp \frac{1}{|Q^0|} \sum_{i=0}^1 \int_0^1 \int_{(\Phi_\pm^s \otimes \mathrm{Id})(Q_{t,x}^0 \times \mathbb{R}^d)} r^{\alpha_i} \mathsf{m}_\pm^{(i)} \cdot \nabla_v (-\Delta_v)^{-\frac{1-s+\varepsilon_0}{2}} f\left(rt_0 + (1-r)t_\pm, y, w\right) \\
&\qquad \times (-\Delta_v)^{\frac{1-s+\varepsilon_0}{2}} \left\{ \varphi\left[\left(\Phi_\pm^r\right)^{-1}(y, w) \right] \right\} \frac{\mathrm{d}r \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}t_0}{|\det \mathfrak{A}^r|} \\
&= \mp \frac{1}{|Q^0|} \sum_{i=0}^1 \int_0^1 \int_{Q_{t,x}^0 \times \mathbb{R}^d} r^{\alpha_i} \mathsf{m}_\pm^{(i)} \cdot \left(\nabla_v (-\Delta_v)^{-\frac{1-s+\varepsilon_0}{2}} f \right)(\Gamma_\pm(r)) \\
&\qquad \times \left\{ (-\Delta_v)^{\frac{1-s+\varepsilon_0}{2}} \left\{ \varphi\left[\left(\Phi_\pm^r\right)^{-1}(y, w) \right] \right\} \right\}_{|(y,w) = \Phi_\pm^r(x_0, v_0)} \mathrm{d}r \, \mathrm{d}z_0.
\end{aligned}$$

We then observe that, first,

$$\left\{(-\Delta_v)^{\frac{1-s+\varepsilon_0}{2}}\left\{\varphi\left[\left(\Phi_\pm^r\right)^{-1}(y,w)\right]\right\}\right\}_{|(y,w)=\Phi_\pm^r(x_0,v_0)} \lesssim \left\|\nabla_v\left(\Phi_\pm^r\right)^{-1}\right\|_\infty^{1-s+\varepsilon_0} \lesssim \left(\sum_{j=0}^1 r^{-1-\alpha_j}\right)^{1-s+\varepsilon_0},$$

and second,

$$\int_{\mathbb{R}^d}\left|\left(\nabla_v(-\Delta_v)^{-\frac{1-s+\varepsilon_0}{2}}f\right)(\Gamma_\pm(r))\right|\,\mathrm{d}z_0 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left|f(w')-f((\Phi_\pm^r)_v(v_0))\right|}{|w'-(\Phi_\pm^r)_v(v_0)|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}z_0.$$

We deduce

$$\mathcal{I}_2^\pm \le \frac{C}{|Q^0|}\int_0^1\int_{Q^0_{t,x}\times\mathbb{R}^d}\sum_{i,j=0}^1 r^{\alpha_i+s(1+\alpha_j)-(1+\varepsilon_0)(\alpha_j+1)}\int_{\mathbb{R}^d}\frac{\left|f(w')-f((\Phi_\pm^r)_v(v_0))\right|}{|w'-(\Phi_\pm^r)_v(v_0)|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}r\,\mathrm{d}z_0.$$

It follows from (3.24) and (3.25) that we are left with estimating

$$\mathcal{J} := \fint_{Q^\pm}\left\{\mathcal{I}_1^\pm\right\}_+\mathrm{d}z_\pm + \fint_{Q^\pm}\left\{\mathcal{I}_2^\pm\right\}_+\mathrm{d}z_\pm.$$

The previous estimates imply

$$\begin{aligned}
\mathcal{J} \lesssim{}& \int_{Q^\pm}\int_{Q^0_{t,x}}\Bigg|\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\sum_{i=0}^1 r^{-d(1+\alpha_i)}\left[f((\Phi_\pm^r)_x(x_0),w'))-f((\Phi_\pm^r)_x(x_0),w)\right]\\
&\times\left[K(w,w'))-K(w',w)\right]\\
&\times\left[\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))+\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\,\mathrm{d}r\Bigg|\,\mathrm{d}x_0\,\mathrm{d}t_0\,\mathrm{d}z_\pm\\
&+\int_{Q^\pm}\Bigg|\int_{Q^0_{t,x}\times\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\sum_{i=0}^1 r^{-d(1+\alpha_i)}\left[f(w')-f(w)\right]K(w,w')\\
&\times\left[\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))-\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}x_0\,\mathrm{d}t_0\Bigg|\,\mathrm{d}z_\pm\\
&+\int_{Q^\pm}\int_{Q^0_{t,x}\times\mathbb{R}^d}\int_0^1\sum_{i,j=0}^1 r^{\alpha_i+(s-1-\varepsilon_0)(1+\alpha_j)}\int_{\mathbb{R}^d}\frac{\left|f(w')-f((\Phi_\pm^r)_v(v_0))\right|}{|w'-(\Phi_\pm^r)_v(v_0)|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}z_0\,\mathrm{d}z_\pm\\
&+\int_{Q^\pm}\fint_{Q^0_{t,x}}\Bigg|\int_0^1\fint_{(\Phi_\pm^r)_v(B_1)}\sum_{i=0}^1 r^{-d(1+\alpha_i)}\\
&\times h\left(\tau,(\Phi_\pm^r)_x(x_0),w\right)\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))\,\mathrm{d}w\,\mathrm{d}r\Bigg|\,\mathrm{d}t_0\,\mathrm{d}x_0\,\mathrm{d}z_\pm.
\end{aligned}\tag{3.28}$$

We now choose all α_i's close to $-\frac{d}{d+s} \in (-1,0)$, pairwise distinct.

Then (3.28) simplifies into, for some ε as small as wanted, and any $\varepsilon_0 \in (0,s)$

$$\begin{aligned}
\mathcal{J} \lesssim & \int_{Q^\pm} \int_{Q^0_{t,x}} \Bigg| \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon} \left[f((\Phi^r_\pm)_x(x_0), w')) - f((\Phi^r_\pm)_x(x_0), w) \right] \\
& \qquad \times \left[K(w,w')) - K(w',w) \right] \\
& \qquad \times \left[\varphi(x_0, (\Phi^r_\pm)_v^{-1}(w)) + \varphi(x_0, (\Phi^r_\pm)_v^{-1}(w')) \right] \mathrm{d}w' \, \mathrm{d}w \, \mathrm{d}r \Bigg| \mathrm{d}x_0 \, \mathrm{d}t_0 \, \mathrm{d}z_\pm \\
& + \int_{Q^\pm} \int_{Q^0_{t,x}} \Bigg| \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon} \left[f(w') - f(w) \right] K(w,w') \\
& \qquad \times \left[\varphi(x_0, (\Phi^r_\pm)_v^{-1}(w)) - \varphi(x_0, (\Phi^r_\pm)_v^{-1}(w')) \right] \mathrm{d}w' \, \mathrm{d}w \, \mathrm{d}r \Bigg| \mathrm{d}x_0 \, \mathrm{d}t_0 \, \mathrm{d}z_\pm \\
& + \int_{Q^\pm} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 r^{-1+\frac{s(s-\varepsilon_0)}{d+s}} \int_{\mathbb{R}^d} \frac{\left| f(w') - f((\Phi^r_\pm)_v(v_0)) \right|}{|w' - (\Phi^r_\pm)_v(v_0)|^{d+s-\varepsilon_0}} \mathrm{d}w' \, \mathrm{d}z_0 \, \mathrm{d}z_\pm \\
& + \int_{Q^\pm} \fint_{Q^0_{t,x}} \Bigg| \int_0^1 \fint_{(\Phi^r_\pm)_v(B_1)} r^{-\frac{sd}{d+s}-\varepsilon} \\
& \qquad \times h\left(\tau, (\Phi^r_\pm)_x(x_0), w\right) \varphi(x_0, (\Phi^r_\pm)_v^{-1}(w)) \, \mathrm{d}w \, \mathrm{d}r \Bigg| \mathrm{d}t_0 \, \mathrm{d}x_0 \, \mathrm{d}z_\pm .
\end{aligned}$$

We are now in a position to use not only the intermediate variable z_0, but also the future/past variables $z_\pm$ for a change of variables, since we are now integrating in both Q^0 and $Q_\pm$. Note that it was not possible to use the integration in $Q_\pm$ before because of the positive value around the Q^0 integral.

For given $r_0 \in (0,1)$, we split $\mathcal{J}$ as follows,

$$\begin{aligned}
& \int_{Q^\pm} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^1 \Big(\cdots \Big) \, \mathrm{d}r \, \mathrm{d}z_0 \, \mathrm{d}z_\pm \\
& \quad = \underbrace{\int_{Q^\pm} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_0^{r_0} \Big(\cdots \Big) \, \mathrm{d}r \, \mathrm{d}z_0 \, \mathrm{d}z_\pm}_{=: \mathcal{J}_1^\pm} + \underbrace{\int_{Q^\pm} \int_{Q^0_{t,x} \times \mathbb{R}^d} \int_{r_0}^1 \Big(\cdots \Big) \, \mathrm{d}r \, \mathrm{d}z_0 \, \mathrm{d}z_\pm}_{=: \mathcal{J}_2^\pm} .
\end{aligned}$$

The two changes of variables on each part are represented in Figure 3.3.

The part $\mathcal{J}_2^\pm$ is controlled by using the same change of variables $\Phi^s_\pm$ as before, and we additionally map t_0 onto $\tau = rt_0 + (1-r)t_\pm$ with the mapping $\phi^r_\pm(t_0) := rt_0 + (1-r)t_\pm$:

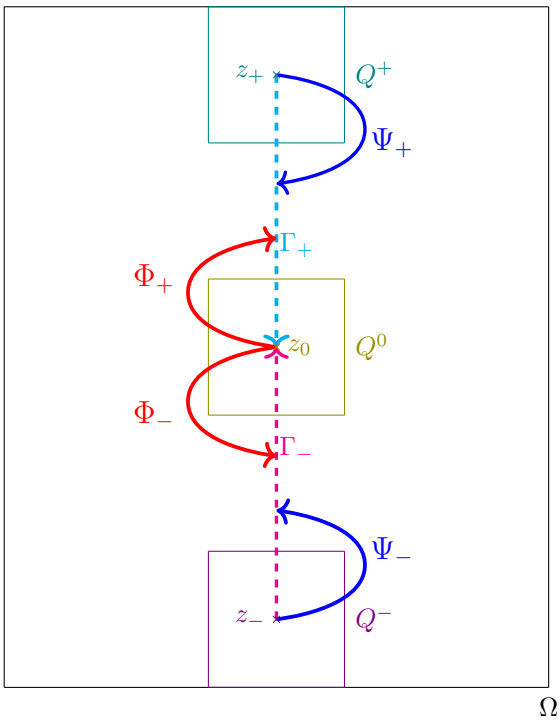


Figure 3.3: The change of variables that we use in the proof for some fixed $r \in (0,1)$. For $r \in (0, r_0)$ we use $\Psi_\pm$ that map $z_\pm$ onto $\Gamma_\pm$, whereas for $r \in (r_0, 1)$ we use $\Phi_\pm$ that map z_0 onto $\Gamma_\pm$.

using that r avoids the singularity at zero, we find

$$
\begin{aligned}
\mathcal{J}_2^\pm \lesssim & \int_{Q^\pm} \int_{r_0}^1 \int_{(\phi_\pm^r \otimes (\Phi_\pm^r)_x)(Q^0_{t,x})} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon} \left[f(y,w')) - f(y,w) \right] \right. \\
& \qquad \times \left[K(w,w')) - K(w',w) \right] \\
& \qquad \left. \times \left[\varphi((\Phi_\pm^r)^{-1}(y,w)) + \varphi((\Phi_\pm^r)^{-1}(y,w')) \right] \, \mathrm{d}w' \, \mathrm{d}w \right| \frac{\mathrm{d}y \, \mathrm{d}\sigma \, \mathrm{d}r \, \mathrm{d}z_\pm}{r \left| \det \mathfrak{A}_x^r \right|} \\
& + \int_{Q^\pm} \int_{r_0}^1 \int_{(\phi_\pm^r \otimes (\Phi_\pm^r)_x)(Q^0_{t,x})} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon} \left[f(w') - f(w) \right] K(w,w') \right. \\
& \qquad \left. \times \left[\varphi((\Phi_\pm^r)^{-1}(y,w)) - \varphi((\Phi_\pm^r)^{-1}(y,w')) \right] \, \mathrm{d}w' \, \mathrm{d}w \right| \frac{\mathrm{d}y \, \mathrm{d}\sigma \, \mathrm{d}r \, \mathrm{d}z_\pm}{r \left| \det \mathfrak{A}_x^r \right|} \\
& + \int_{Q^\pm} \int_{(\phi_\pm^r \otimes \Phi_\pm^r)(Q^0_{t,x} \times \mathbb{R}^d)} \int_{r_0}^1 r^{-1+\frac{s(s-\varepsilon_0)}{d+s}-\varepsilon} \int_{\mathbb{R}^d} \frac{|f(w') - f(w)|}{|w' - w|^{d+s-\varepsilon_0}} \frac{\mathrm{d}w' \, \mathrm{d}r \, \mathrm{d}\sigma \, \mathrm{d}y \, \mathrm{d}w \, \mathrm{d}z_\pm}{r \left| \det \mathfrak{A}^r \right|} \\
& + \int_{Q^\pm} \int_{r_0}^1 \fint_{(\phi_\pm^r \otimes (\Phi_\pm^r)_x)(Q^0_{t,x})} \left| \fint_{(\Phi_\pm^r)_v(B_1)} r^{-\frac{sd}{d+s}-\varepsilon} \right. \\
& \qquad \left. \times h\left(\sigma, y, w\right) \varphi((\Phi_\pm^r)^{-1}(y,w)) \, \mathrm{d}w \right| \frac{\mathrm{d}\sigma \, \mathrm{d}y \, \mathrm{d}r \, \mathrm{d}z_\pm}{r \left| \det \mathfrak{A}_x^r \right|}
\end{aligned}
$$

using that $|\det \mathfrak{A}^r|$ is uniformly bounded below

$$
\begin{aligned}
\lesssim_{r_0} \int_{r_0}^1 \int_{\Omega_{t,x}} \Bigg| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & [f(y,w')) - f(y,w)]\,[K(w,w')) - K(w',w)] \\
& \times \left[\varphi((\Phi_\pm^r)^{-1}(y,w)) + \varphi((\Phi_\pm^r)^{-1}(y,w'))\right]\, \mathrm{d}w'\,\mathrm{d}w \Bigg| \mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r \\
+ \int_{r_0}^1 \int_{\Omega_{t,x}} \Bigg| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & [f(w') - f(w)]\,K(w,w') \\
& \times \left[\varphi((\Phi_\pm^r)^{-1}(y,w)) - \varphi((\Phi_\pm^r)^{-1}(y,w'))\right]\, \mathrm{d}w'\,\mathrm{d}w \Bigg| \mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r \\
+ \int_{r_0}^1 \int_{\Omega_{t,x}} \Bigg| \int_{(\Phi_\pm^r)_v(B_1)} & h\,(\sigma,y,w)\,\varphi((\Phi_\pm^r)^{-1}(y,w))\,\mathrm{d}w \Bigg| \mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}r.
\end{aligned}
$$

To control the part $\mathcal{J}_1^\pm$ we parametrise $\Gamma_\pm(r)$ by the $z_\pm$ coordinates, for $r \in [0, r_0]$ and $t_0, t_\pm$ all fixed:

$$
(t_\pm, x_\pm, v_\pm) \to \Gamma_\pm(r) =: (t_0 r + (1-r)t_\pm, \Psi_\pm^r(x_\pm, v_\pm),)
$$

Since r does not approach 1, we can prove that this change of variables in not singular. The mappings $\Psi_\pm^r$ are determined by solving (3.21) for $x_\pm, v_\pm$ instead of x_0, v_0. It yields $\Psi_\pm^r\,(x_\pm, v_\pm)^T = \mathfrak{a}_\pm^r\,(x_\pm, v_\pm)^T + \mathfrak{b}_\pm^r$ with the matrix $\mathfrak{a}_\pm^r$ and vector $\mathfrak{b}_\pm^r$ given by

$$
\mathfrak{a}_\pm^r := \left(\mathsf{T}_\pm(r) - \mathsf{W}_\pm^\delta(r)\left[\mathsf{W}_\pm^\delta(1)\right]^{-1}\mathsf{T}_\pm(1)\right) \qquad \mathfrak{b}_\pm^r := \mathsf{W}_\pm^\delta(r)\left[\mathsf{W}_\pm^\delta(1)\right]^{-1}(x_0, v_0)^T,
$$

which depend only on r and x_0, v_0. Since we have computed above that $\mathsf{W}(r) = \mathcal{O}\left(r^{3d+d\sum_{i=0}^1 \alpha_i}\right)$ it follows that $\mathsf{W}_\pm^\delta(r) = \mathcal{O}\left(r^{3d+d\sum_{i=0}^1 \alpha_i}\right)$ and thus $\mathsf{W}_\pm^\delta(r)$ goes to zero as $r \to 0$. Since $\mathsf{T}_\pm^\delta$ is a lower triangular matrix with unitary diagonal it is invertible for all r, and finally $\mathfrak{a}_\pm^r$ is invertible with uniformly bounded inverse on $r \in [0, r_0]$ for $r_0 > 0$ small enough. We apply this change of variables to estimate $\mathcal{J}_1^\pm$, after exchanging the

integration domains:

$$\begin{aligned}
\mathcal{J}_1^{\pm} \lesssim & \int_{B_1}\int_0^{r_0}\int_{(\psi_\pm^r\otimes(\Psi_\pm^r)_x)(Q_{t,x}^0)}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[f(y,w'))-f(y,w)\right]\right. \\
&\qquad\times\left[K(w,w'))-K(w',w)\right] \\
&\qquad\left.\times\left[\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))+\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\right|\frac{\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0}{(1-r)\left|\det(\mathfrak{a}_\pm^r)_x\right|} \\
&+\int_{B_1}\int_0^{r_0}\int_{(\psi_\pm^r\otimes(\Psi_\pm^r)_x)(Q_{t,x}^\pm)}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[f(w')-f(w)\right]K(w,w')\right. \\
&\qquad\left.\times\left[\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))-\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\right|\frac{\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0}{(1-r)\left|\det(\mathfrak{a}_\pm^r)_x\right|} \\
&+\int_0^{r_0}\int_{\mathbb{R}^d}\int_{(\psi_\pm^r\otimes\Psi_\pm^r)(Q^\pm)} r^{-1+\frac{s(s-\varepsilon_0)}{d+s}-\varepsilon}\int_{\mathbb{R}^d}\frac{|f(w')-f(w)|}{|w'-w|^{d+s-\varepsilon_0}}\frac{\mathrm{d}w'\,\mathrm{d}r\,\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}v_0}{(1-r)\left|\det\mathfrak{a}_\pm^r\right|} \\
&+\int_0^{r_0}\int_{(\psi_\pm^r\otimes(\Psi_\pm^r)_x\otimes\mathrm{Id})(Q^\pm)}\fint_{Q_{t,x}^0}\left|\fint_{(\Phi_\pm^r)_v(B_1)} r^{-\frac{sd}{d+s}-\varepsilon}\right. \\
&\qquad\left.\times h\left(\sigma,y,w\right)\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))\,\mathrm{d}w\right|\frac{\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}t_0\,\mathrm{d}x_0\,\mathrm{d}v_\pm\,\mathrm{d}r}{(1-r)\left|\det(\mathfrak{a}_\pm^r)_x\right|}
\end{aligned}$$

using that $\left|\det\mathfrak{a}_\pm^r\right|$ is uniformly bounded below, $r^{-1+\frac{s(s-\varepsilon_0)}{d+s}-\varepsilon}\in L^1(0,r_0)$ for $\varepsilon_0<s$ and ε sufficiently small,

$$\begin{aligned}
\lesssim_{r_0} & \int_{B_1}\int_0^{r_0}\int_{\Omega_{t,x}}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[f(y,w'))-f(y,w)\right]\left[K(w,w'))-K(w',w)\right]\right. \\
&\qquad\left.\times\left[\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))+\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\right|\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0 \\
&+\int_{B_1}\int_0^{r_0}\int_{\Omega_{t,x}}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[f(y,w')-f(y,w)\right]K(w,w')\right. \\
&\qquad\left.\times\left[\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))-\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w'))\right]\,\mathrm{d}w'\,\mathrm{d}w\right|\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0 \\
&+\int_{\Omega_{t,x}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(w')-f(w)|}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}w \\
&+\int_0^{r_0}\int_{\Omega_{t,x}}\int_{B_1}\left|\fint_{(\Phi_\pm^r)_v(B_1)} r^{-\frac{sd}{d+s}-\varepsilon}h\left(\sigma,y,w\right)\varphi(x_0,(\Phi_\pm^r)_v^{-1}(w))\,\mathrm{d}w\right|\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}x_0\,\mathrm{d}r.
\end{aligned}$$

This concludes the proof of Proposition 3.1.

3.5.4 Intermediate Value Lemma

In this section we prove the Second De Giorgi Lemma, also known as the Intermediate Value Lemma, stated in Theorem 1.2. The proof exploits the cross-term arising in the energy estimate. The idea originates from Bass and Kassmann [21] and was further employed by Caffarelli, Chan and Vasseur [35]. However, this is the first time it is used in the context of kinetic equations, and also the first time it appears for kernels as general as the non-cutoff Boltzmann kernel that in particular are not necessarily symmetric.

Let $0 < \mu < 1$. We take a cut-off $\varphi \in C_c^\infty(\mathbb{R}^d)$ in x such that $0 \le \varphi \le 1$ and where $\varphi = 1$ in $B_{(3r_0)^{1+2s}}$ and $\varphi = 0$ outside $B_{(9r_0)^{1+2s}}$. Denote $\phi = 1 - \varphi$. We further define for $\rho_0 \ge 1$ and $0 < \rho_2 \le \sqrt{8} - \sqrt{2}$ to be determined

$$
\begin{aligned}
F_0(v) &= \frac{1}{r_0} \sup\left(- r_0, \frac{\inf(0, |v|^2 - (1 + (3+\rho_0)^2) r_0^2)}{r_0} \right), \\
F_1(v) &= \frac{1}{r_0} \sup\left(- r_0, \frac{\inf(0, |v|^2 - 9 r_0^2)}{r_0} \right), \\
F_2(v) &= \frac{1}{r_0} \sup\left(- r_0, \frac{\inf(0, |v|^2 - (\sqrt{8} - \rho_2)^2 r_0^2)}{r_0} \right).
\end{aligned}
$$

These functions are Lipschitz, equal to 0 outside $B_{\sqrt{1+(3+\rho_0)^2} r_0}$, B_{3r_0} and $B_{(\sqrt{8}-\rho_2) r_0}$ respectively and equal to -1 in $B_{(3+\rho_0) r_0}$, $B_{2\sqrt{2} r_0}$ and $B_{\sqrt{((\sqrt{8}-\rho_2)^2 - 1) r_0}}$. Then we use the following three cut-offs:

$$
\begin{aligned}
\varphi_0(x,v) &= \phi(x) + 1 + F_0(v), \\
\varphi_1(x,v) &= \phi(x) + 1 + \mu F_1(v), \\
\varphi_2(x,v) &= \phi(x) + 1 + \mu^2 F_2(v).
\end{aligned}
\tag{3.29}
$$

We refer the reader to Figure 1.7 to visualise (3.29).

Proof of Theorem 1.2. Let f be a sub-solution to (1.13)-(1.14) in $[-3, 0] \times Q_1^t$ satisfying (1.34).

Step 1: Improved energy estimate. We test (1.13) with $(f - \varphi_1)_+$ similar to the proof of the energy estimate that we had before. We get for $-3 < T_1 < T_2 < 0$

$$
\begin{aligned}
&\iint (f - \varphi_1)_+^2 \, \mathrm{d}x \, \mathrm{d}v \Big|_{T_1}^{T_2} + \int_{T_1}^{T_2} \int \mathcal{E}\big((f - \varphi_1)_+, (f - \varphi_1)_+\big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \le - \int_{T_1}^{T_2} \int \mathcal{E}\big(\varphi_1, (f - \varphi_1)_+\big) \, \mathrm{d}x \, \mathrm{d}t - \int_{T_1}^{T_2} \int \mathcal{E}\big((f - \varphi_1)_-, (f - \varphi_1)_+\big) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_{T_1}^{T_2} \iint (f - \varphi_1)_+ h \, \mathrm{d}z.
\end{aligned}
\tag{3.30}
$$

Note that here holds

$$(f-\varphi_1)_+(v) \le f - 1 + \mu \le \mu.$$

Therefore by our assumption on the source term we have

$$\int_{T_1}^{T_2} \iint (f-\varphi_1)_+ h \,\mathrm{d}z \le C\mu^3.$$

Furthermore we estimate

$$\begin{aligned}-\mathcal{E}^{\mathrm{sym}}\big(\varphi_1, (f-\varphi_1)_+\big) \le& \frac{1}{8}\mathcal{E}^{\mathrm{sym}}\big((f-\varphi_1)_+, (f-\varphi_1)_+\big) \\ &+ \iint |\varphi_1(v)-\varphi_1(w)|^2 K(v,w)\big(\chi_{B_{3r_0}}(v)+\chi_{B_{3r_0}}(w)\big)\,\mathrm{d}w\,\mathrm{d}v.\end{aligned}$$

The first term is absorbed on the left in (3.30). For the second one there holds

$$\begin{aligned}&2\iint |\varphi_1(v)-\varphi_1(w)|^2 K(v,w)\chi_{B_{3r_0}}(v)\,\mathrm{d}w\,\mathrm{d}v \\ &\le 4\mu^2 \iint |F_1(v)-F_1(w)|^2 K(v,w)\chi_{B_{3r_0}}(v)\,\mathrm{d}w\,\mathrm{d}v \\ &\le C\mu^2 \iint |v-w|^2 K(v,w)\chi_{B_{3r_0}}(v)\,\mathrm{d}w\,\mathrm{d}v \\ &\le C\Lambda\mu^2,\end{aligned}$$

using that F_1 is Lipschitz and the upper bound (1.27) on K. For the skew-symmetric part we note that $\mathcal{E}^{\mathrm{skew}} = 0$ for $f \le \varphi_1$. Thus

$$\begin{aligned}&-\mathcal{E}^{\mathrm{skew}}\big(\varphi_1, (f-\varphi_1)_+\big) \\ &= \frac{1}{2}\iint \big(\varphi_1(w)-\varphi_1(v)\big)(f-\varphi_1)_+(v)\big(K(v,w)-K(w,v)\big)\,\mathrm{d}w\,\mathrm{d}v \\ &= \frac{1}{2}\iint \big(\varphi_1(w)-\varphi_1(v)\big)(f-\varphi_1)_+(v)\big(K(v,w)-K(w,v)\big)\chi_{B_{3r_0}}(v)\,\mathrm{d}w\,\mathrm{d}v \\ &= \iint \mu\big(F_1(w)-F_1(v)\big)(f-\varphi_1)_+(v)\big(K(v,w)-K(w,v)\big)\chi_{B_{3r_0}}(v)\,\mathrm{d}w\,\mathrm{d}v \\ &\le \mu\int \left|\mathrm{PV}\int (v-w)\big(K(v,w)-K(w,v)\big)\,\mathrm{d}w\right|(f-\varphi_1)_+(v)\chi_{B_{3r_0}}(v)\,\mathrm{d}v \\ &\le C\mu\Lambda\int (f-\varphi_1)_+(v)\chi_{B_{3r_0}}(v)\,\mathrm{d}v.\end{aligned}$$

We used the cancellation assumptions (1.30) and the definition of F_1. Thus

$$-\mathcal{E}^{\mathrm{skew}}\big(\varphi_1, (f-\varphi_1)_+\big) \le C\mu^2.$$

We deduce

$$-\mathcal{E}\big(\varphi_1,(f-\varphi_1)_+\big)\le C\mu^2.$$

Finally we write due to the cancellation assumptions (1.29)

$$\begin{aligned}
&\mathcal{E}\big((f-\varphi_1)_+,(f-\varphi_1)_+\big)\\
&\quad\ge \iint \big((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)\big)^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\qquad-\iint (f-\varphi_1)_+^2(v)\left|\int \big(K(v,w)-K(w,v)\big)\,\mathrm{d}w\right|\mathrm{d}v\\
&\quad\ge \iint \big((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)\big)^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v-\Lambda\mu^2.
\end{aligned}$$

Thus we obtain from (3.30) for $t\in(T_1,T_2)$

$$\begin{aligned}
&\iint_{Q^t_{3r_0}}(f-\varphi_1)_+^2\,\mathrm{d}x\,\mathrm{d}v\Big|_{T_1}^{T_2}\\
&\qquad+\int_{T_1}^{T_2}\int_{Q^t_{3r_0}}\int_{\mathbb{R}^d}\big((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)\big)^2K(v,w)\,\mathrm{d}w\,\mathrm{d}z\\
&\qquad+\int_{T_1}^{T_2}\int_{B_{(3r_0)^{1+2s}}}\mathcal{E}\big((f-\varphi_1)_-,(f-\varphi_1)_+\big)\,\mathrm{d}x\,\mathrm{d}t\le C\mu^2(T_2-T_1).
\end{aligned}\tag{3.31}$$

In particular, we deduce for $-3<T_1<T_2<0$:

$$-\int_{T_1}^{T_2}\iiint (f-\varphi_1)_+(v)(f-\varphi_1)_-(w)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\le C\mu^2\big(T_2-T_1\big).$$

Step 2: Weak Poincaré for the intermediate cut-off. We want to use the hypoelliptic Poincaré inequality of Proposition 3.1 for the intermediate cut-off $(f-\varphi_1)_+$. We note that in $(-3,0]\times Q^t_{3r_0}$ there holds $\nabla_x\varphi_1=0$, thus

$$\mathcal{T}(f-\varphi_1)_+=\mathcal{T}f\chi_{f>\varphi_1}\le\big[\mathcal{L}f+h\big]\chi_{f>\varphi_1}\le\mathcal{L}(f-\varphi_1)_++\underbrace{\big(\mu\mathcal{L}F+h\big)\chi_{f>\varphi_1}}_{=:\tilde h}.$$

Therefore $(f-\varphi_1)_+$ is a sub-solution of (1.13) in $(-3,0]\times Q^t_{3r_0}$ with source term $\tilde h$. Then, if we denote by $\Omega_{t,x}(3r_0)$ the domain $\Omega_{t,x}(3r_0):=(-(3r_0)^{2s},0]\times B_{(3r_0)^{1+2s}}$, the hypoelliptic Poincaré inequality (3.17) gives

$$
\begin{aligned}
&\big\|\big[(f-\varphi_1)_+ - \langle (f-\varphi_1)_+\rangle_{Q^-_{r_0}}\big]_+\big\|_{L^1(Q_{r_0})}\\
&\lesssim \int_{B_{r_0}}\int_0^1\int_{\Omega_{t,x}(3r_0)}\bigg|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[(f-\varphi_1)_+(y,w')) - (f-\varphi_1)_+(y,w)\right]\\
&\qquad\times\left[K(w,w')) - K(w',w)\right]\left[\psi(x_0,w)+\psi(x_0,w')\right]\,\mathrm{d}w'\,\mathrm{d}w\bigg|\,\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0\\
&+\int_{B_{r_0}}\int_0^1\int_{\Omega_{t,x}(3r_0)}\bigg|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r^{-\frac{sd}{d+s}-\varepsilon}\left[(f-\varphi_1)_+(y,w') - (f-\varphi_1)_+(y,w)\right]\\
&\qquad\times K(w,w')\left[\psi(x_0,w)-\psi(x_0,w')\right]\,\mathrm{d}w'\,\mathrm{d}w\bigg|\,\mathrm{d}y\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}x_0\\
&+\int_{\Omega_{t,x}(3r_0)\times\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}w\\
&+\int_0^1\int_{\Omega_{t,x}(3r_0)}\int_{B_{r_0}}\bigg|\int_{B_{3r_0}} r^{-\frac{sd}{d+s}-\varepsilon}\tilde{h}\,(\sigma,y,w)\,\psi(x_0,w)\,\mathrm{d}w\bigg|\,\mathrm{d}\sigma\,\mathrm{d}y\,\mathrm{d}x_0\,\mathrm{d}r.
\end{aligned}
\tag{3.32}
$$

where $\psi=\varphi\circ(\mathrm{Id}\otimes(\Phi^r_\pm)^{-1}_v)$ with $\Phi^r_\pm$ given in (3.21). In particular, $|D_v\psi|\,,|D^2_v\psi|\approx r^{-\frac{s}{d+s}}$. In the sequel we write $n=2s+d(2+2s)$ for the total dimension of the kinetic cylinder such that $|Q_{r_0}|\sim r_0^n$. On the one hand, we have by assumption (1.34)

$$
\begin{aligned}
\fint_{Q^-_{r_0}}(f-\varphi_1)_+\,\mathrm{d}z &\le \mu\frac{\big|\{f>\varphi_1\}\cap Q^-_{r_0}\big|}{\big|Q^-_{r_0}\big|}\le \mu\frac{\big|\{f>\varphi_0\}\cap Q^-_{r_0}\big|}{\big|Q^-_{r_0}\big|}\\
&\le \mu\left(1-\frac{\big|\{f\le\varphi_0\}\cap Q^-_{r_0}\big|}{\big|Q^-_{r_0}\big|}\right)\le \mu-\mu\delta_1,
\end{aligned}
$$

so that with (1.34)

$$
\begin{aligned}
&\big\|\big[(f-\varphi_1)_+ - \langle (f-\varphi_1)_+\rangle_{Q^-_{r_0}}\big]_+\big\|_{L^1(Q_{r_0})}\\
&\qquad\ge\int_{Q_{r_0}\cap\{f>\varphi_2\}}\big[\big(f-\varphi_1\big)_+-\mu(1-\delta_1)\big]_+\,\mathrm{d}z\\
&\qquad\ge\int_{\{f>\varphi_2\}\cap Q_{r_0}}\big[\big(\varphi_2-\varphi_1\big)_+-\mu(1-\delta_1)\big]_+\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad\ge\big(\mu\delta_1-\mu^2\big)\delta_2 r_0^n.
\end{aligned}
\tag{3.33}
$$

Note that $\varphi_2-\varphi_1=\mu-\mu^2$ in Q_{r_0}.

Step 3-(i): Upper bounds on the symmetric fractional gradient.

We introduce the following indicator functions

$$\begin{aligned}
\chi_{ii}(v,w) &:= \chi_{\varphi_0(v)<f(v)<\varphi_2(v)}\chi_{\varphi_0(w)<f(w)<\varphi_2(w)},\\
\chi_i(v,w) &:= (\chi_{\varphi_0(v)<f(v)<\varphi_2(v)} + \chi_{\varphi_0(w)<f(w)<\varphi_2(w)})(1-\chi_{ii}(v,w)),\\
\chi_a(v,w) &:= \chi_{f(v)>\varphi_2(v)}\chi_{f(w)>\varphi_2(w)},\\
\chi_b(v,w) &:= (\chi_{f(w)<\varphi_0(w)} + \chi_{f(v)<\varphi_0(v)})(1-\chi_i(v,w)),
\end{aligned} \tag{3.34}$$

to distinguish when at least one of v, w are such that f is in the intermediate set, when both are such that f is above the intermediate set, and when one of them is such that f is below the intermediate set. Our goal is to bound for fixed x_0, y, r, σ,

$$\begin{aligned}
\mathcal{I}^{sym} &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right] K(w,w') \left[\psi(x_0,w) - \psi(x_0,w')\right] \mathrm{d}w'\,\mathrm{d}w\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\dots)(\chi_{ii}+\chi_i)(w,w')\,\mathrm{d}w'\,\mathrm{d}w + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\dots)\chi_b(w,w')\,\mathrm{d}w'\,\mathrm{d}w\\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\dots)\chi_a(w,w')\,\mathrm{d}w'\,\mathrm{d}w\\
&=: \mathcal{I}_i^{sym} + \mathcal{I}_b^{sym} + \mathcal{I}_a^{sym}.
\end{aligned}$$

If at least one of the functions is in the intermediate set, then we bound using if necessary Fubini, and in any case Cauchy-Schwarz,

$$\begin{aligned}
\mathcal{I}_i^{sym} &= C\int_{\varphi_0<f<\varphi_2}\int_{\mathbb{R}^d} \left[(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right] K(w,w') \left[\psi(w) - \psi(w')\right] \mathrm{d}w'\,\mathrm{d}w\\
&\le C\int_{\varphi_0<f<\varphi_2}\left(\int_{\mathbb{R}^d} \left|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right|^2 K(w,w')\,\mathrm{d}w'\right)^{\frac12}\\
&\qquad \times\left(\int_{\mathbb{R}^d} |\psi(w)-\psi(w')|^2 K(w,w')\,\mathrm{d}w'\right)^{\frac12}\mathrm{d}w
\end{aligned}$$

using that $|D_v^s\psi| \approx r^{-\frac{s^2}{d+s}}$ and Cauchy-Schwarz again

$$\begin{aligned}
&\le Cr^{-\frac{s^2}{2(d+s)}}\left|\varphi_0<f<\varphi_2\right|^{\frac12}\\
&\qquad\times\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right|^2 K(w,w')\,\mathrm{d}w'\,\mathrm{d}w\right)^{\frac12}
\end{aligned}$$

and finally by (3.31)

$$\le C\mu r^{-\frac{s^2}{2(d+s)}}\left|\varphi_0<f<\varphi_2\right|^{\frac12}.$$

If one of them is below the intermediate set, we distinguish the singular from the

non-singular set: for any $\rho > 0$

$$\begin{aligned}\mathcal{I}_b^{sym} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)]\, K(w,w') \\ &\qquad \times [\psi(x_0,w) - \psi(x_0,w')]\, \chi_b(w,w')\chi_{|w-w'|<\rho r_0}\, \mathrm{d}w'\, \mathrm{d}w \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)]\, K(w,w') \\ &\qquad \times [\psi(x_0,w) - \psi(x_0,w')]\, \chi_b(w,w')\chi_{|w-w'|>\rho r_0}\, \mathrm{d}w'\, \mathrm{d}w \\ &=: \mathcal{I}_{b,sin}^{sym} + \mathcal{I}_{b,nsin}^{sym}.\end{aligned}$$

Note that on $\chi_b(w,w')$ at least one f is such that $f > \varphi_2$, and thus by definition of φ_2, at least one of w, w' is in $B_{2\sqrt{2}r_0}$. In particular, if $|w-w'| < \rho r_0$ then $w, w' \in B_{(2\sqrt{2}+\rho)r_0}$. By definition of φ_0, there is thus a non-trivial gap between φ_0 and φ_1, and thus assuming with no loss in generality $f(w') < \varphi_0(w')$ and $f(w) > \varphi_2(w)$

$$\begin{aligned}\mathcal{I}_{b,sin}^{sym} &= C\int_{B_{2\sqrt{2}r_0}}\int_{B_{(2\sqrt{2}+\rho)r_0}} (f-\varphi_1)_+(w)\chi_{f>\varphi_2}(w)\chi_{f\le\varphi_0}(w')K(w,w') \\ &\qquad \times [\psi(x_0,w') - \psi(x_0,w)]\, \chi_{|w-w'|<\rho r_0}\, \mathrm{d}w'\, \mathrm{d}w, \\ &\le Cr^{-\frac{s^2}{2(d+s)}} \int_{B_{2\sqrt{2}r_0}} (f-\varphi_1)_+(w)\chi_{f>\varphi_2}(w) \\ &\qquad \times \left(\int_{B_{(2\sqrt{2}+\rho)r_0}} \chi_{f\le\varphi_0}(w')K(w,w')\chi_{|w-w'|<\rho r_0}\, \mathrm{d}w'\right)^{\frac{1}{2}} \mathrm{d}w,\end{aligned}$$

using that $(f-\varphi_1)_+ \le \mu$ and Cauchy-Schwarz again

$$\begin{aligned}&\le C\mu^{\frac{1}{2}} r_0^{\frac{d}{2}} r^{-\frac{s^2}{2(d+s)}} \\ &\qquad \times \left(\int_{B_{2\sqrt{2}r_0}\cap\{f>\varphi_2\}} (f-\varphi_1)_+(w) \int_{B_{(2\sqrt{2}+\rho)r_0}} \chi_{f\le\varphi_0}(w')K(w,w')\, \mathrm{d}w'\, \mathrm{d}w\right)^{\frac{1}{2}},\end{aligned}$$

using the gap between φ_0 and φ_1

$$\begin{aligned}&\le C\mu^{\frac{1}{2}} r_0^{\frac{d}{2}} r^{-\frac{s^2}{2(d+s)}} \\ &\qquad \times \left(\int_{B_{2\sqrt{2}r_0}\cap\{f>\varphi_2\}} (f-\varphi_1)_+(w) \int_{B_{(2\sqrt{2}+\rho)r_0}} (-(f-\varphi_1)_-(w'))K(w,w')\, \mathrm{d}w'\, \mathrm{d}w\right)^{\frac{1}{2}},\end{aligned}$$

using the improved energy estimate (3.31)

$$\le C\mu^{\frac{3}{2}} r_0^d r^{-\frac{s^2}{2(d+s)}}.$$

The non-singular set we bound initially as above

$$\mathcal{I}^{sym}_{b,nsin} \leq Cr^{-\frac{s^2}{2(d+s)}} \int_{B_{2\sqrt{2}r_0}} (f-\varphi_1)_+(w)\chi_{f>\varphi_2}(w) \\ \times \left(\int_{\mathbb{R}^d} \chi_{f\leq\varphi_0}(w')K(w,w')\chi_{|w-w'|>\rho r_0}\,\mathrm{d}w' \right)^{\frac{1}{2}} \,\mathrm{d}w,$$

but then integrating out the kernel, and using $(f-\varphi_1)_+ \leq \mu$

$$\leq C\mu r^{-\frac{s^2}{2(d+s)}}\rho^{-s}r_0^{d-s}.$$

We then pick ρ in such a way that $\frac{1}{\rho^s} \sim (\delta_1\delta_2)^{\frac{5}{4}}$. Thus

$$\mathcal{I}^{sym}_{b,sin} \leq Cr_0^{d-s}(\delta_1\delta_2)^{\frac{5}{4}}\mu r^{-\frac{s^2}{2(d+s)}}.$$

It remains to bound the case when both are above the level set. In particular $w, w' \in B_{2\sqrt{2}r_0}$ and $\varphi_1(w) = \varphi_1(w') = 1-\mu$,

$$\begin{aligned}
\mathcal{I}^{sym}_a &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right] K(w,w') \\
&\quad \times \left[\psi(x_0,w) - \psi(x_0,w')\right] \chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_2)_+(w') - (f-\varphi_2)_+(w)\right] K(w,w') \\
&\quad \times \left[\psi(x_0,w) - \psi(x_0,w')\right] \chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\varphi_2(w') - \varphi_2(w)\right] K(w,w') \\
&\quad \times \left[\psi(x_0,w) - \psi(x_0,w')\right] \chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&\leq Cr_0^{\frac{d}{2}}r^{-\frac{s^2}{2(d+s)}} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|(f-\varphi_2)_+(w') - (f-\varphi_2)_+(w)\right|^2 K(w,w')\,\mathrm{d}w'\,\mathrm{d}w \right)^{\frac{1}{2}} \\
&\quad + C\mu^2 r_0^{\frac{d}{2}} r^{-\frac{s^2}{2(d+s)}}
\end{aligned}$$

using the energy estimate for $(f-\varphi_2)_+ \leq \mu^2$

$$\leq C\mu^2 r^{-\frac{s^2}{2(d+s)}}.$$

We conclude

$$\begin{aligned}
\mathcal{I}_{sym} \leq & C\mu r_0^{\frac{d}{2}} r^{-\frac{s^2}{2(d+s)}} \left|\varphi_0 < f < \varphi_2\right|^{\frac{1}{2}} + C\mu^{\frac{3}{2}} r_0^d r^{-\frac{s^2}{2(d+s)}} \\
& + Cr_0^{d-s}(\delta_1\delta_2)^{\frac{5}{4}}\mu r^{-\frac{s^2}{2(d+s)}} + Cr_0^d\mu^2 r^{-\frac{s^2}{2(d+s)}}.
\end{aligned} \tag{3.35}$$

Step 3-(ii): Upper bounds on the anti-symmetric fractional gradient.

To estimate the anti-symmetric gradient, we use the same indicator functions in (3.34) with the goal to bound

$$\begin{aligned}\mathcal{I}^{skew} &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)]\,[K(w,w') - K(w',w)] \\ &\qquad \times [\psi(x_0,w) + \psi(x_0,w')]\,\mathrm{d}w'\,\mathrm{d}w \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\dots)(\chi_{ii}+\chi_i)(w,w')\,\mathrm{d}w'\,\mathrm{d}w + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\dots)\chi_b(w,w')\,\mathrm{d}w'\,\mathrm{d}w \\ &\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\dots)\chi_a(w,w')\,\mathrm{d}w'\,\mathrm{d}w \\ &=: \mathcal{I}_i^{skew} + \mathcal{I}_b^{skew} + \mathcal{I}_a^{skew}.\end{aligned}$$

For the intermediate set, we use the symmetry,

$$\begin{aligned}\mathcal{I}_i^{skew} &= 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f-\varphi_1)_+(w)\big(K(w',w) - K(w,w')\big) \\ &\qquad \times \big(\psi(w') + \psi(w)\big)(\chi_i + \chi_{ii})(w,w')\,\mathrm{d}w'\,\mathrm{d}w \\ &= 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f-\varphi_1)_+(w)\chi_{\varphi_0<f<\varphi_2}(w) \\ &\qquad \times \big(K(w',w) - K(w,w')\big)\big(\psi(w') + \psi(w)\big)\,\mathrm{d}w'\,\mathrm{d}w \\ &\quad + 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f-\varphi_1)_+(w)\chi_{f>\varphi_2}(w)\big(K(w',w) - K(w,w')\big) \\ &\qquad \times \big(\psi(w') + \psi(w)\big)\chi_{\varphi_0<f<\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w\end{aligned}$$

we Taylor-expand ψ,

$$
\begin{aligned}
&\leq 4 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f-\varphi_1)_+(w) \chi_{\varphi_0<f<\varphi_2}(w) \psi(w) \big(K(w',w) - K(w,w')\big) \,\mathrm{d}w \,\mathrm{d}w \\
&\quad + 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f-\varphi_1)_+(w) \chi_{\varphi_0<f<\varphi_2}(w) D_v\psi(w) \\
&\qquad \times (w-w') \big(K(w',w) - K(w,w')\big) \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + C \left\| D_v^2 \psi \right\|_{L^\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f-\varphi_1)_+(w) \chi_{\varphi_0<f<\varphi_2}(w) \left|w-w'\right|^2 \left|K(w',w)\right| \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + 4 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f-\varphi_2)_+(w) \psi(w) \big(K(w',w) - K(w,w')\big) \chi_{\varphi_0<f<\varphi_2}(w') \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f-\varphi_2)_+(w) \chi_{f>\varphi_2}(w) D_v\psi(w) \cdot (w-w') \\
&\qquad \times \big(K(w',w) - K(w,w')\big) \chi_{\varphi_0<f<\varphi_2}(w') \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + C \left\| D_v^2 \psi \right\|_{L^\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f-\varphi_2)_+(w) \chi_{\varphi_0<f<\varphi_2}(w) \left|w-w'\right|^2 \left|K(w',w)\right| \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi_2-\varphi_1)_+(w) \chi_{f>\varphi_2}(w) \big(K(w',w) - K(w,w')\big) \\
&\qquad \times \big(\psi(w') + \psi(w)\big) \chi_{\varphi_0<f<\varphi_2}(w') \,\mathrm{d}w' \,\mathrm{d}w
\end{aligned}
$$

we use the cancellations (1.29), (1.30), and the upper bound (1.32), as well as the fact that the domains are bounded, and $f - \varphi_2 \leq \mu^2$; for the second term, we Taylor expand ψ in w'

$$
\begin{aligned}
&\leq C(\Lambda_0, r_0) \mu \left|\{\varphi_0(v) < f(v) < \varphi_2(v)\}\right| + C(\Lambda_0) \mu^2 \\
&\quad + 4 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi_2(w) - \varphi_1(w)) \chi_{f>\varphi_2}(w) \\
&\qquad \times \big(K(w',w) - K(w,w')\big) \psi(w') \chi_{\varphi_0<f<\varphi_2}(w') \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + 4 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi_2(w) - \varphi_1(w)) \chi_{f(w)>\varphi_2(w)} \\
&\qquad \times (w-w') \big(K(w',w) - K(w,w')\big) \cdot D_v\psi(w') \chi_{\varphi_0<f<\varphi_2}(w') \,\mathrm{d}w' \,\mathrm{d}w \\
&\quad + C \left\| D_v^2 \psi \right\|_{L^\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi_2(w) - \varphi_1(w)) \chi_{f>\varphi_2}(w) \\
&\qquad \times \left|w-w'\right|^2 \left|K(w',w) - K(w,w')\right| \chi_{\varphi_0<f<\varphi_2}(w') \,\mathrm{d}w' \,\mathrm{d}w
\end{aligned}
$$

we note $\varphi_2 - \varphi_1 = \mu - \mu^2$ on $B_{\sqrt{((\sqrt{8}-\rho_2)^2-1)}r_0}$ and $(\varphi_2 - \varphi_1)(w) = \mu - \mu^2 \left(\frac{|w|^2 - (\sqrt{8}-\rho)^2 r_0^2}{r_0^2}\right)$ in $B_{(\sqrt{8}-\rho_2)r_0} \setminus B_{\sqrt{((\sqrt{8}-\rho_2)^2-1)}r_0}$, and we use $|w|^2 = |w-w'|^2 + |w'|^2 + 2(w-w')\cdot w'$

$$
\begin{aligned}
&\leq C(r_0)\mu(1 + \|D_v\psi\|_{L^\infty} + \left\|D_v^2\psi\right\|_{L^\infty}) \left|\{\varphi_0(v) < f(v) < \varphi_2(v)\}\right| + C\mu^2 \\
&\quad + 4\mu^2 \int_{\mathbb{R}^d} |w'|^2 \psi(w') \chi_{\varphi_0 < f < \varphi_2}(w') \int_{\mathbb{R}^d} \big(K(w',w) - K(w,w')\big)\,\mathrm{d}w\,\mathrm{d}w' \\
&\quad + 4\mu^2 \int_{\mathbb{R}^d} \left(\psi(w')w' + D_v\psi(w')\,|w'|^2\right) \chi_{\varphi_0 < f < \varphi_2)}(w') \\
&\qquad \times \int_{\mathbb{R}^d} (w-w')\big(K(w',w) - K(w,w')\big)\,\mathrm{d}w\,\mathrm{d}w' \\
&\leq C(\Lambda_0, r_0) r^{-\frac{s}{d+s}} \mu \left|\{\varphi_0 < f < \varphi_2\}\right| + C(\Lambda_0)\mu^2.
\end{aligned}
$$

The other parts are simpler. They are essentially bounded like $\mathcal{I}_b^{sym}$ and $\mathcal{I}_a^{sym}$, respectively. If one of them is below the intermediate set, we distinguish the singular from the non-singular set: for any $\rho > 0$

$$
\begin{aligned}
\mathcal{I}_b^{skew} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right] \left[K(w,w') - K(w',w)\right] \\
&\qquad \times \left[\psi(w) + \psi(w')\right] \chi_b(w,w') \chi_{|w-w'|<\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right] \left[K(w,w') - K(w',w)\right] \\
&\qquad \times \left[\psi(w) + \psi(w')\right] \chi_b(w,w') \chi_{|w-w'|>\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w \\
&=: \mathcal{I}_{b,sin}^{skew} + \mathcal{I}_{b,nsin}^{skew}.
\end{aligned}
$$

For the singular part, we use the same arguments as in the symmetric case, and additionally the fact that the integrand is signed, so that

$$
\begin{aligned}
\mathcal{I}_{b,sin}^{skew} &\leq C \int_{B_{2\sqrt{2}r_0}} \int_{B_{(2\sqrt{2}+\rho)r_0}} (f-\varphi_1)_+(w) \chi_{f>\varphi_2}(w) \chi_{f\leq\varphi_0}(w') K(w',w) \\
&\qquad \times \left[\psi(w') + \psi(w)\right] \chi_{|w-w'|<\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w,
\end{aligned}
$$

using that ψ is uniformly bounded, picking $\rho = \rho_0$, the gap between φ_0 and φ_1 and (3.31)

$$
\begin{aligned}
&\leq -C \int_{B_{2\sqrt{2}r_0}} \int_{B_{(2\sqrt{2}+\rho_0)r_0}} (f-\varphi_1)_-(w')(f-\varphi_1)_+(w) K(w',w) \chi_{|w-w'|<\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w, \\
&\leq -C\mu^2.
\end{aligned}
$$

The non-singular set we bound initially as above but then integrating out the kernel, and

using $(f-\varphi_1)_+ \le \mu$

$$\mathcal{I}^{skew}_{b,nsin} \le C\mu\rho^{-s}r_0^{d-s}.$$

We then recall that $\rho = \rho_0$ is such that $\frac{1}{\rho^s} \sim (\delta_1\delta_2)^{\frac{5}{4}}$, so that

$$\mathcal{I}^{skew}_{b} \le Cr_0^{d-s}(\delta_1\delta_2)^{\frac{5}{4}}\mu + C\mu^2.$$

Finally, it remains to bound the case when both are above the level set. In particular $w, w' \in B_{2\sqrt{2}r_0}$ and $\varphi_1(w) = \varphi_1(w') = 1-\mu$,

$$\begin{aligned}
\mathcal{I}^{skew}_{a} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)\right]\left[K(w,w') - K(w',w)\right] \\
&\qquad \times \left[\psi(w)+\psi(w')\right]\chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-\varphi_2)_+(w') - (f-\varphi_2)_+(w)\right]\left[K(w,w') - K(w',w)\right] \\
&\qquad \times \left[\psi(w)+\psi(w')\right]\chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\varphi_2(w') - \varphi_2(w)\right]\left[K(w,w') - K(w',w)\right] \\
&\qquad \times \left[\psi(w)+\psi(w')\right]\chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&= 2\int_{\mathbb{R}^d}(f-\varphi_2)_+(w)\int_{\mathbb{R}^d}\left[K(w',w) - K(w,w')\right] \\
&\qquad \times \left[\psi(w)+\psi(w')\right]\chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&\quad + C\mu^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(w'-w)\left[K(w,w') - K(w',w)\right] \\
&\qquad \times \left[\psi(w)+\psi(w')\right]\chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w
\end{aligned}$$

we Taylor-expand ψ, use $(f-\varphi_2)_+ \le \mu^2$ and the cancellations (1.29), (1.30), and the upper bound (1.32)

$$\begin{aligned}
&\le C\mu^2 r_0^d + 2\int_{\mathbb{R}^d}(f-\varphi_2)_+(w)D_v\psi(w) \\
&\qquad \times \int_{\mathbb{R}^d}(w-w')\left[K(w',w) - K(w,w')\right]\chi_{f>\varphi_2}(w)\chi_{f>\varphi_2}(w')\,\mathrm{d}w'\,\mathrm{d}w \\
&\quad + C\left\|D_v^2\psi\right\|_{L^2}\mu^2 r_0^{2-2s} + C\mu^2\left\|D_v\psi\right\|_{L^\infty} r_0^{2-2s} \\
&\le C(r_0)r^{-\frac{s}{d+s}}\mu^2.
\end{aligned}$$

We conclude

$$\mathcal{I}^{skew} \le C(\Lambda_0, r_0)r^{-\frac{s}{d+s}}\mu\left|\{\varphi_0 < f < \varphi_2\}\right| + Cr_0^{d-s}(\delta_1\delta_2)^{\frac{5}{4}}\mu + C(r_0)r^{-\frac{s}{d+s}}\mu^2. \tag{3.36}$$

Step 3-(iii): Upper bounds on the lower order fractional gradient. We now use the same methods to bound

$$\begin{aligned}\mathcal{I}^{\varepsilon_0} &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}w \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\dots)(\chi_{ii}+\chi_i)(w,w')\,\mathrm{d}w'\,\mathrm{d}w + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\dots)\chi_b(w,w')\,\mathrm{d}w'\,\mathrm{d}w \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\dots)\chi_a(w,w')\,\mathrm{d}w'\,\mathrm{d}w \\ &=: \mathcal{I}_i^{\varepsilon_0} + \mathcal{I}_b^{\varepsilon_0} + \mathcal{I}_a^{\varepsilon_0}.\end{aligned}$$

For the intermediate set we just use Cauchy-Schwarz

$$\begin{aligned}\mathcal{I}_i^{\varepsilon_0} &= \int_{\{\varphi_0<f<\varphi_2\}}\int_{\mathbb{R}^d} \frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}w \\ &= \int_{\{\varphi_0<f<\varphi_2\}} (f-\varphi_1)_+(w)\int_{\{f<\varphi_0\}} \frac{1}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}w \\ &\quad + \int_{\{\varphi_0<f<\varphi_2\}}\int_{\{f>\varphi_0\}\cap B_{(3+\rho)r_0}} \frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|}{|w'-w|^{d+s-\varepsilon_0}}\,\mathrm{d}w'\,\mathrm{d}w \\ &\le C\mu\,|\{\varphi_0<f<\varphi_2\}|\,r_0^{-s+\varepsilon_0} \\ &\quad + \int_{\{\varphi_0<f<\varphi_2\}}\left(\int_{\{f>\varphi_0\}\cap B_{(3+\rho)r_0}} \frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|^2}{|w'-w|^{d+2s}}\,\mathrm{d}w'\right)^{\frac12} \\ &\qquad \times\left(\int_{\{f>\varphi_0\}\cap B_{(3+\rho)r_0}} \frac{1}{|w'-w|^{d-2\varepsilon_0}}\,\mathrm{d}w'\right)^{\frac12}\mathrm{d}w \\ &\le C\mu\,|\{\varphi_0<f<\varphi_2\}|\,r_0^{-s+\varepsilon_0} + Cr_0^{\varepsilon_0}\,|\{\varphi_0<f<\varphi_2\}|^{\frac12}\,\mu,\end{aligned}$$

where the last inequality uses the coercivity (1.26) and the energy estimate (3.31).

Below the level set, we distinguish again the non-singular from the singular part: for any $\rho>0$

$$\begin{aligned}\mathcal{I}_{b,sin}^{\varepsilon_0} &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|}{|w'-w|^{d+s-\varepsilon_0}}\chi_b(w,w')\chi_{|w-w'|<\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w \\ &\le C\rho^{\varepsilon_0}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|(f-\varphi_1)_+(w') - (f-\varphi_1)_+(w)|^2}{|w'-w|^{d+2s}}\chi_b(w,w')\chi_{|w-w'|<\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w\right)^{\frac12}\end{aligned}$$

using $f-\varphi_1\le\mu$, picking $\rho=\rho_0$ and using the gap between φ_0 and φ_1

$$\le C\rho^{\varepsilon_0}\mu^{\frac12}\left(\int_{\{f>\varphi_1\}}\int_{\{f<\varphi_0\}} \frac{-(f-\varphi_1)_-(w')(f-\varphi_1)_+(w)}{|w'-w|^{d+2s}}\chi_{|w-w'|<\rho r_0}\,\mathrm{d}w'\,\mathrm{d}w\right)^{\frac12}$$

using the coercivity and the improved energy estimate (3.31)

$$\leq C\rho_0^{\varepsilon_0}\mu^{\frac{3}{2}}.$$

On the other hand

$$\mathcal{I}_{b,nsin}^{\varepsilon_0} \leq C\rho^{-s+\varepsilon_0}\int_{\mathbb{R}^d}(f-\varphi_1)_+\,\mathrm{d}w \leq C\rho^{-s+\varepsilon_0}\mu r_0^d.$$

We recall that $\rho = \rho_0$ and thus $\frac{1}{\rho^s} \sim (\delta_1\delta_2)^{\frac{5}{4}}$. If we pick $\varepsilon_0 = \frac{s}{6}$ then we conclude

$$\mathcal{I}_b^{\varepsilon_0} \leq C(\delta_1\delta_2)^{-\frac{5}{24}}\mu^{\frac{3}{2}} + C(\delta_1\delta_2)^{\frac{25}{24}}\mu r_0^d.$$

Finally if both are above the level set, we use that $w, w' \in B_{(\sqrt{8}-\rho_2)r_0}$, so that by Cauchy-Schwarz

$$\begin{aligned}
\mathcal{I}_a^{\varepsilon_0} &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(f-\varphi_2)_+(w') - (f-\varphi_2)_+(w)|}{|w'-w|^{d+s-\varepsilon_0}}\chi_a(w,w')\,\mathrm{d}w\,\mathrm{d}w' \\
&\leq Cr_0^{\varepsilon_0+\frac{d}{2}}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(f-\varphi_2)_+(w') - (f-\varphi_2)_+(w)|^2}{|w'-w|^{d+2s}}\chi_a(w,w')\,\mathrm{d}w\,\mathrm{d}w'\right)^{\frac{1}{2}} \\
&\leq Cr_0^{\varepsilon_0+\frac{d}{2}}\mu^2.
\end{aligned}$$

We conclude

$$\mathcal{I}^{\varepsilon_0} \leq C\mu\,|\{\varphi_0 < f < \varphi_2\}|^{\frac{1}{2}} + C(\delta_1\delta_2)^{-\frac{5}{24}}\mu^{\frac{3}{2}} + C(\delta_1\delta_2)^{\frac{25}{24}}\mu r_0^d + Cr_0^{\frac{s}{6}+\frac{d}{2}}\mu^2. \tag{3.37}$$

Step 3-(iv): Upper bounds on the source $\tilde{h}$. Finally, we estimate the source term $\tilde{h}$ in the Poincaré-inequality (3.32). We fix r, x_0, y, σ and bound

$$\left|\int_{B_{3r_0}}\tilde{h}(\sigma,y,w)\psi(x_0,w)\,\mathrm{d}w\right|.$$

The first part consists of

$$\begin{aligned}
&\int_{B_{3r_0}}\mathcal{L}\varphi_1(v)\chi_{f>\varphi_1}(v)\psi(v)\,\mathrm{d}v \\
&\quad= \int_{\{\varphi_1<f<\varphi_2\}}\int\big(\varphi_1(w)-\varphi_1(v)\big)K(v,w)\psi(v)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad+\int_{\{f>\varphi_2\}}\int_{\{\varphi_0<f<\varphi_2\}}\big(\varphi_1(w)-\varphi_1(v)\big)K(v,w)\psi(v)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad+\int_{\{f>\varphi_2\}}\int_{\{f<\varphi_0\}}\big(\varphi_1(w)-\varphi_1(v)\big)K(v,w)\psi(v)\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned} \tag{3.38}$$

The last term uses the gap between φ_2 and φ_1, and the gap between φ_1 and φ_0, Cauchy-Schwarz, the Lipschitz continuity of φ_1 and remembering that $\varphi_1 - \varphi_1' \sim \mu$, $f - \varphi_1 \leq \mu$ and the energy estimate (3.31)

$$\begin{aligned}
&\int_{\{f>\varphi_2\}} \int_{\{f<\varphi_0\}} \big(\varphi_1(w) - \varphi_1(v)\big) K(v,w)\psi(v)\,\mathrm{d}w\,\mathrm{d}v \\
&\leq C\mu^{-1} \int_{\{f>\varphi_2\}} \int_{\{f<\varphi_0\}} (f-\varphi_1)_+(v)\,|\varphi_1(w) - \varphi_1(v)|\,K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\leq -\frac{C}{\mu\delta_2^2} \int_{\{f>\varphi_2\}} \int_{\{f<\varphi_0\}} (f-\varphi_1)_+(v)(f-\varphi_1)_-(w)\,|\varphi_1(w) - \varphi_1(v)|\,K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\leq \frac{C}{\mu\delta_2^2} \int_{\{f>\varphi_2\}} \left(- \int_{\{f<\varphi_0\}} (f-\varphi_1)_+^2(v)(f-\varphi_1)_-(w) K(v,w)\,\mathrm{d}v \right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\int |\varphi_1(w) - \varphi_1(v)|^2\, K(v,w)\,\mathrm{d}v \right)^{\frac{1}{2}} \mathrm{d}w \qquad (3.39) \\
&\leq \frac{C}{\delta_2^2} \int_{\{f>\varphi_2\}} \left(- \int (f-\varphi_1)_+^2(v)(f-\varphi_1)_-(w) K(v,w)\,\mathrm{d}v \right)^{\frac{1}{2}} \mathrm{d}w \\
&\leq C r_0^d \frac{\mu^{\frac{1}{2}}}{\delta_2^2} \left(- \iint (f-\varphi_1)_+(v)(f-\varphi_1)_-(w) K(v,w)\,\mathrm{d}v\,\mathrm{d}w \right)^{\frac{1}{2}} \\
&\leq C \frac{\mu^{\frac{3}{2}}}{\delta_2^2} r_0^d.
\end{aligned}$$

Moreover, if in either variable we integrate over the set $\{\varphi_0 < f < \varphi_2\}$ we split into the singular and the non-singular part. For the singular part, we Taylor-expand φ_1, to find by (1.30) and the upper bound (1.32)

$$\begin{aligned}
&\int_{\{\varphi_0<f<\varphi_2\}} \int_{|v-w|<\rho_2 r_0} \psi(v)\big(\varphi_1(w) - \varphi_1(v)\big) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad = \mu \int_{\{\varphi_0<f<\varphi_2\}} \psi(v) D_v F(v) \int_{|v-w|<\rho_2 r_0} (w-v) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\quad + C\mu\rho_2^{2-2s} \left\| D_v^2 F \right\| |\{\varphi_0 < f < \varphi_2\}| \\
&\qquad \leq C\mu\rho_2^{1-2s} |\{\varphi_0 < f < \varphi_2\}| + C\mu\rho_2^{2-2s} |\{\varphi_0 < f < \varphi_2\}|.
\end{aligned}$$

Moreover, we notice that φ_2 is defined in such a way that if $f(v) > \varphi_2(v)$ and $|v-w| < \rho_2 r_0$ then $\varphi_1(w) = \varphi_1(v) = 1 - \mu$.

For the non-singular part, using that ψ is uniformly bounded, and the upper bound (1.32)

$$\begin{aligned}
&\int_{\{\varphi_0<f<\varphi_2\}} \int_{|v-w|>\rho_2 r_0} \psi(v)\big(\varphi_1(w) - \varphi_1(v)\big) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\qquad \leq C\mu\rho_2^{1-2s} |\{\varphi_0 < f < \varphi_2\} \cap B_{3r_0}|.
\end{aligned} \qquad (3.40)$$

Similarly, we get

$$\begin{aligned}\int_{\{\varphi_2<f\}}\int_{\{\varphi_0<f<\varphi_2\}\cap|v-w|>\rho_2 r_0}\psi(v)\big(\varphi_1(w)-\varphi_1(v)\big)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ \leq C\mu\rho_2^{1-2s}\left|\{\varphi_0<f<\varphi_2\}\cap B_{5r_0}\right|.\end{aligned}\tag{3.41}$$

The second part of the source term $\tilde{h}$ in (3.32) satisfies by assumption

$$\int_{B_{3r_0}} h\chi_{f>\varphi_1}(v)\psi(v)\,\mathrm{d}v\leq C\mu^2 r_0^n.\tag{3.42}$$

Gathering these observations in (3.39) and (3.42) we find

$$\begin{aligned}\left|\int_{B_{3r_0}}\tilde{h}(\sigma,y,w)\psi(x_0,w)\,\mathrm{d}w\right|&\leq Cr_0^d\frac{\mu^{\frac{3}{2}}}{\delta_2^2}+Cr_0^d\mu^2+C\left|\{\varphi_0<f<\varphi_2\}\cap Q_{3r_0}\right|\\ &\leq Cr_0^d\frac{\mu^{\frac{3}{2}}}{\delta_2^2}+C\left|\{\varphi_0<f<\varphi_2\}\cap Q_{3r_0}\right|^{\frac{1}{2}}.\end{aligned}\tag{3.43}$$

Step 4: Conclusion. Now we assemble all the pieces. Recall that $n=2s+d(2+2s)$. Then (3.32) gives with (3.33), (3.35), (3.36), (3.37), (3.43)

$$\begin{aligned}\big(\mu\delta_1-\mu^2\big)\delta_2 r_0^n\leq&C\int_0^1 r^{-\frac{s(2d+s)}{2(d+s)}}\left(\mu r_0^{\frac{n}{2}}\left|\{\varphi_0<f<\varphi_2\}\right|^{\frac{1}{2}}+\mu^{\frac{3}{2}}r_0^n+r_0^n(\delta_1\delta_2)^{\frac{5}{4}}\mu+r_0^n\mu^2\right)\mathrm{d}r\\ &+C\int_0^1 r^{-\frac{sd}{d+s}}\left(r^{-\frac{s}{d+s}}\mu\left|\{\varphi_0<f<\varphi_2\}\right|+r_0^{n-s}(\delta_1\delta_2)^{\frac{5}{4}}\mu+r_0^n r^{-\frac{s}{d+s}}\mu^2\right)\mathrm{d}r\\ &+C\mu r_0^{\frac{n}{2}}\left|\{\varphi_0<f<\varphi_2\}\right|^{\frac{1}{2}}+C(\delta_1\delta_2)^{-\frac{5}{24}}r_0^n\mu^{\frac{3}{2}}+C(\delta_1\delta_2)^{\frac{25}{24}}\mu r_0^n+Cr_0^n\mu^2\\ &+C\int_0^1 r^{-\frac{sd}{d+s}}\left(r_0^n\frac{\mu^{\frac{3}{2}}}{\delta_2^2}+r_0^{\frac{n}{2}}\left|\{\varphi_0<f<\varphi_2\}\cap Q_{3r_0}\right|^{\frac{1}{2}}\right)\mathrm{d}r\\ \leq&Cr_0^{\frac{n}{2}}\left|\{\varphi_0<f<\varphi_2\}\right|^{\frac{1}{2}}+C\mu^{\frac{3}{2}}r_0^n+r_0^n(\delta_1\delta_2)^{\frac{5}{4}}\mu\\ &+C(\delta_1\delta_2)^{-\frac{5}{24}}\mu^{\frac{3}{2}}r_0^n+C(\delta_1\delta_2)^{\frac{25}{24}}\mu r_0^n+r_0^n\frac{\mu^{\frac{3}{2}}}{\delta_2^2}\\ \leq&Cr_0^{\frac{n}{2}}\left|\{\varphi_0<f<\varphi_2\}\right|^{\frac{1}{2}}+C_0(\delta_1\delta_2)^{-\frac{5}{24}}\mu^{\frac{3}{2}}r_0^n+C(\delta_1\delta_2)^{\frac{25}{24}}\mu r_0^n+C_1r_0^n\frac{\mu^{\frac{3}{2}}}{\delta_2^2}\end{aligned}$$

for some universal constants $C_0, C_1\geq 0$. Since $\delta_1,\delta_2<1$, we can make sure with no loss in generality that $C(\delta_1\delta_2)^{\frac{1}{24}}\leq\frac{1}{4}$, by noticing that the power can be adjusted by choosing a different ε_0 in (3.37). Moreover, if we choose μ such that

$$\mu^{\frac{1}{2}}\leq\frac{1}{16}\min\left\{C_0(\delta_1\delta_2)^{\frac{29}{24}},C_1\delta_1\delta_2^3\right\}\lesssim(\delta_1\delta_2)^3,$$

then we conclude

$$\mu\delta_1\delta_2 r_0^n \le C\mu r_0^n \left|\{\varphi_0 < f < \varphi_2\}\right|^{\frac{1}{2}} + \frac{1}{8}\mu\delta_1\delta_2 r_0^n + \frac{1}{4}\delta_1\delta_2\mu r_0^n.$$

so that

$$\left|\{\varphi_0 < f < \varphi_2\}\right| \ge \frac{25}{64}\mu^2\delta_1^2\delta_2^2 r_0^n \sim (\delta_1\delta_2)^5 r_0^n.$$

□

3.5.5 Measure-to-Pointwise estimate

Lemma 3.5 (Measure-to-pointwise Lemma)**.** *Let $\delta \in (0,1)$ and $r_0 < \frac{1}{3}$. Then there is $\theta = \theta(\delta) \sim \delta^{6(1+\delta^{-5})} > 0$ such that any sub-solution f of (1.13)-(1.14) in $[-3,0] \times Q_1^t$ with (1.26)-(1.30), with a source term h such that $\|h\|_{L^\infty(Q_1)} \le \theta$ and so that $f \le 1$ in $(-3,0] \times B_{2^{-(1+2s)}} \times \mathbb{R}^d$ and*

$$\left|\{f \le 0\} \cap Q_{r_0}^-\right| \ge \delta \left|Q_{r_0}^-\right| \tag{3.44}$$

satisfies

$$f \le 1 - \theta \quad \text{in } Q_{\frac{r_0}{2}},$$

where $Q_{r_0}^- := Q_{r_0}(-2r_0^{2s}, 0, 0)$ and $[-3,0] \times Q_1^t = [-3,0] \times B_1 \times B_1$ as before.

Proof. By Lemma 3.4 we know that there is a $\delta_0 > 0$ such that for any $r > 0$ any sub-solution f on Q_{2r} satisfying $\int_{Q_r} f_+^2 \le \delta_0 |Q_r|$ there holds $f \le \frac{1}{2}$ in $Q_{\frac{r}{2}}$. Therefore, we define $\nu, \mu > 0$ as in the proof of Theorem 1.2 with $\delta_1 = \delta$ and $\delta_2 = \delta_0$.

We consider $f_k := \mu^{-2k}\left[f - (1-\mu^{2k})\right]$ for $k \ge 0$. Note that f_k is a sub-solution to (1.13)-(1.14) for each $k \ge 0$ with (1.27)-(1.30) and with a source term of L^∞ norm less than θ as long as $k \le 1 + \frac{1}{\nu}$ since by assumption $\|h\|_{L^\infty} \le \theta$ so that $\|h\|_{L^\infty} \le \mu^{2(1+\frac{1}{\nu})}$. Moreover, $\{\varphi_0 < f_i < \varphi_2\} \cap \{\varphi_0 < f_j < \varphi_2\} = \emptyset$ for all $i \ne j$ and each f_k satisfies (3.44).

In case that $\int_{Q_{r_0}} (f_k)_+^2 \le \delta_0 |Q_{r_0}|$ there holds $f_k \le \frac{1}{2}$ in $Q_{\frac{r_0}{2}}$, and we conclude $f \le 1 - \theta$ with $\theta = \frac{\mu^{2k}}{2}$. Thus it suffices to consider $1 \le k_0 \le 1 + \frac{1}{\nu}$ such that $\int_{Q_{r_0}} (f_k)_+^2 > \delta_0 |Q_{r_0}|$ for any $0 \le k \le k_0$. Then for $0 \le k \le k_0 - 1$ there holds

$$\begin{aligned}
&\left|\{f_k \ge 1 - \mu^2\} \cap Q_{r_0}\right| = \left|\{f_{k+1} \ge 0\} \cap Q_{r_0}\right| \ge \int_{Q_{r_0}} (f_{k+1})_+^2 > \delta_0 \left|Q_{r_0}\right|, \\
&\left|\{f_k \le 0\} \cap Q_{r_0}^-\right| \ge \left|\{f \le 0\} \cap Q_{r_0}^-\right| \ge \delta \left|Q_{r_0}^-\right|,
\end{aligned}$$

where we used the fact that $f \le 0$ implies $f_k \le 0$ for all $k \ge 0$. Thus we can apply Theorem 1.2 for our choice of δ_1, δ_2 so that we get for some $\rho > 0$

$$\left|\{\varphi_0 < f_k < \varphi_2\} \cap (-3,0] \times Q_\rho^t\right| \ge \nu \left|(-3,0] \times Q_\rho^t\right|.$$

Since these sets are all disjoint, we get by summing these estimates

$$\left|(-3,0]\times Q_\rho^t\right| \geq \sum_{k=0}^{k_0-1} \left|\{\varphi_0 < f_k < \varphi_2\}\cap (-3,0]\times Q_\rho^t\right| \geq k_0\nu \left|(-3,0]\times Q_\rho^t\right|.$$

Thus $k_0 \leq \frac{1}{\nu}$ and we deduce with $f_{k_0+1} \leq \frac{1}{2}$ in $Q_{\frac{r_0}{2}}$

$$f \leq 1 - \frac{\mu^{2k_0+2}}{2} \leq 1 - \frac{\mu^{\frac{2+2\nu}{\nu}}}{2} \quad \text{in } Q_{\frac{r_0}{2}}.$$

This yields the claim for $\theta(\delta) := \frac{\mu^{\frac{2+2\nu}{\nu}}}{2} \sim \delta^{6(1+\delta^{-5})}$. □

3.6 Weak and not-so-strong Harnack inequality and Hölder continuity

3.6.1 Harnack Inequalities

We follow Section 4.1 in [99]. We consider a non-negative super-solution f to (1.13)-(1.14) for $h = 0$ on $(-3,0]\times Q_1^t$ so that (1.26)-(1.30) holds for $\bar{R} = 2$. Then Lemma 3.5 applied to the sub-solution $g := 1 - \frac{f}{M}$ implies for any $\delta \in (0,1)$ and $M \sim \delta^{-6(1+\delta^{-5})}$ that $\forall Q_r(z) \subset (-3,0]\times Q_1^t$ so that $Q^+_{\frac{r}{2}}(z) \subset (-3,0]\times Q_1^t$

$$\frac{|\{f > M\}\cap Q_r(z)|}{|Q_r(z)|} > \delta \implies \inf_{Q^+_{\frac{r}{2}}(z)} f \geq 1,$$

where we recall $Q_r^+(z) = Q_r(z + (2r^{2s}, 2r^{2s}v, 0))$ for $z = (t,x,v)$. This implies using the layer-cake representation that if $\inf_{Q_{\frac{r}{2}}} f < 1$, then

$$\begin{aligned} &\frac{|\{f > M\}\cap Q_r^-|}{|Q_r^-|} \lesssim \delta(M) = \left(\frac{1}{\ln(1+M)}\right)^{\frac{1}{6}} \\ &\implies \int_{Q_r^-} \big(\ln(1+f)\big)^{\frac{1}{7}} \lesssim 1. \end{aligned} \tag{3.45}$$

We can improve this logarithmic integrability as follows. We pick $r_0 < \frac{1}{3}$ and consider the sequence of cylinders

$$\mathcal{Q}^k := Q_{\frac{r_0}{2}+\alpha_k}\left(-\frac{5}{2}r_0^{2s} + \frac{1}{2}\big(\frac{r_0}{2}+\alpha_k\big)^{2s}, 0, 0\right) \quad \text{where} \quad \alpha_k := \frac{r_0}{2}7^{1-k}.$$

Then $\mathcal{Q}^1 = Q^-_{r_0}$ and $\mathcal{Q}^k \to \tilde{Q}^-_{\frac{r_0}{2}} := Q_{\frac{r_0}{2}}\big((-\frac{5}{2} + \frac{1}{2\cdot 2^{2s}})r_0^{2s}, 0, 0)\big)$ as $k \to \infty$. Note that these cylinders satisfy $\tilde{Q}^-_{\frac{r_0}{2}} \subset \mathcal{Q}^k \subset \bar{\mathcal{Q}}^k \subset \mathring{\mathcal{Q}}^{k-1} \subset Q^-_{r_0}$ for all $k \geq 1$. We claim as in [99] that for $\delta_0 > 0$ to be determined, for any non-negative super-solution f with $\inf_{Q_{\frac{r}{2}}} f < 1$, there holds

$$\forall k \geq 1, \quad \frac{\big|\{f \geq M^k\} \cap \mathcal{Q}^k\big|}{|\mathcal{Q}^k|} \leq \delta_0 \Big(\frac{1}{35m}\Big)^{(2d+2s(d+1))k}, \tag{3.46}$$

where $M \sim \delta^{-6(1+\delta^{-5})}$ with $\delta := \delta_0 \Big(\frac{1}{35m}\Big)^{2d+2s(d+1)}$ and $m \geq 3$. If we can show that (3.46) holds, then we deduce with the layer-cake representation that there exists $\zeta \gtrsim \delta_0^6 > 0$ such that $\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^\zeta \,\mathrm{d}z \lesssim 1$, which in turn yields by linearity

$$\left(\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^\zeta(z) \,\mathrm{d}z \right)^{\frac{1}{\zeta}} \lesssim \inf_{Q_{\frac{r_0}{2}}} f. \tag{3.47}$$

We can now derive a non-linear Harnack inequality using De Giorgi's first Lemma 3.4. Indeed, assume f is a solution of (1.13) such that $0 \leq f \leq 1$ a.e. and write

$$\inf_{Q_{\frac{r_0}{2}}} f =: \varepsilon_1 \leq 1.$$

Then due to (3.47) we have

$$\left(\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^2(z) \,\mathrm{d}z \right)^{\frac{1}{\zeta}} = \left(\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^\zeta(z) f^{2-\zeta}(z) \,\mathrm{d}z \right)^{\frac{1}{\zeta}} \leq \left(\int_{\tilde{Q}^-_{\frac{r_0}{2}}} f^\zeta(z) \,\mathrm{d}z \right)^{\frac{1}{\zeta}} \lesssim \inf_{Q_{\frac{r_0}{2}}} f = \varepsilon_1.$$

Thus by Lemma 3.4 with $\varepsilon_0 = \varepsilon_1^\zeta$

$$\sup_{\tilde{Q}^-_{\frac{r_0}{4}}} f \leq C\varepsilon_1^{\zeta\beta_2} = C\Big(\inf_{Q_{\frac{r_0}{2}}} f \Big)^{\zeta\beta_2}, \tag{3.48}$$

where $\beta_2 = \frac{p-2}{2(2p-2)} \in (0,1)$ and $\tilde{Q}^-_{\frac{r_0}{4}} = Q_{\frac{r_0}{4}}\big((-\frac{5}{2} + \frac{1}{2\cdot 2^{2s}})r_0^{2s}, 0, 0\big)$.

This implies the Weak (1.33) and Not-so-Strong (3.2) Harnack inequality (with $\beta = \zeta\beta_2$) for any non-negative $\tilde{f}$ by applying the previous estimate to $f = \tilde{f} + (1+t)\,\|h\|_{L^\infty}$.

We now prove (3.46) inductively. The case $k = 1$ holds by (3.45). We define

$$A_{k+1} := \{f > M^{k+1}\} \cap \mathcal{Q}^{k+1}$$

and denote

$$\mathfrak{C}_r[z] := z \circ Q_{2r}\left(\Big(\frac{1}{2}(2r)^{2s}, 0, 0\Big) \right) = z \circ \left(-\frac{1}{2}(2r)^{2s}, \frac{1}{2}(2r)^{2s} \right] \times B_{(2r)^{1+2s}} \times B_{2r}.$$

Just as in [99] we construct $z_l = (t_l, x_l, v_l) \in \mathcal{Q}^{k+1}$ and $r_l > 0$, $l \geq 1$, $m \geq 3$, $n \geq 1$ so that

$$\begin{aligned}
&\forall l \geq 1, r_l \in \Big(0, \frac{\alpha_{k+1}}{5m \cdot n}\Big),\\
&\forall l \geq 1, |A_{k+1} \cap \mathfrak{C}_{5m\cdot r_l}[z_l]| \leq \delta_0 \, |\mathfrak{C}_{5m\cdot r_l}[z_l]|\,,\\
&\forall l \geq 1, |A_{k+1} \cap \mathfrak{C}_{r_l}[z_l]| > \delta_0 \, |\mathfrak{C}_{r_l}[z_l]|\,,\\
&\mathfrak{C}_{m\cdot r_l}, l \geq 1, \text{are disjoint cylinders},\\
&A_{k+1} \text{ is covered by the family } \big(\mathfrak{C}_{5m\cdot r_l}[z_l]\big)_{l\geq 1}.
\end{aligned}$$

For these cylinders, we have

$$\mathfrak{C}^+_{r_l}[z_l] := z_l \circ Q^+_{2r_l}\Big((\frac{1}{2}(2r_l)^{2s}, 0, 0)\Big) = z_l \circ \Big(\frac{3}{2}(2r_l)^{2s}, \frac{5}{2}(2r_l)^{2s}\Big] \times B_{(2r_l)^{1+2s}} \times B_{2r_l}.$$

We remark that for $m \geq 5^{\frac{1}{2s}}$, there holds $\mathfrak{C}_{r_l}[z_l]^+ \subset \mathfrak{C}_{m\cdot r_l}[z_l]$. Moreover, we pick $n \geq 1$ so that $\forall k \geq 1$

$$\big(7^k + 7\big)^{2s} - \big(7^k + 1\big)^{2s} \geq \Big(\frac{2}{n}\Big)^{2s}.$$

Then by choice of z_l and r_l we have that $\mathfrak{C}_{5m\cdot r_l}[z_l] \subset \mathcal{Q}^k$. Note that for example, if $s \geq \frac{1}{2}$ we can choose $n = 1$.

As in [99], we prove that the family of cylinders $\mathfrak{C}_r[z]$ with $z \in \mathcal{Q}^{k+1}$, $r \in \big(0, \frac{\alpha_{k+1}}{5m\cdot n}\big)$, so that $|A_{k+1} \cap \mathfrak{C}_{5m\cdot r}[z]| \leq \delta_0 \, |\mathfrak{C}_{5m\cdot r}[z]|$ and $|A_{k+1} \cap \mathfrak{C}_r[z]| > \delta_0 \, |\mathfrak{C}_r[z]|$ cover A_{k+1}. By induction hypothesis there holds for all $r \in \big(\frac{\alpha_{k+1}}{5m\cdot n}, \alpha_{k+1}\big)$

$$\begin{aligned}
|A_{k+1} \cap \mathfrak{C}_r[z]| \leq |A_k \cap \mathfrak{C}_r[z]| &\leq \big|A_k \cap \mathcal{Q}^k\big|\\
&\leq \delta_0 \Big(\frac{1}{35m}\Big)^{(2d+2s(d+1))k} \big|\mathcal{Q}^k\big| \leq \delta_0 \, |\mathcal{C}_r[z]|\,.
\end{aligned} \tag{3.49}$$

If $z \in A_{k+1}$ is not covered by $\mathcal{F}$ then the continuous positive function $\varphi(r) = \frac{|A_{k+1} \cap \mathfrak{C}_r[z]|}{|\mathfrak{C}_r[z]|}$ on $(0, +\infty)$ satisfies $\varphi(r) \leq \delta_0$ or $\varphi(5m \cdot n \cdot r) > \delta_0$ for all $r \in \big(0, \frac{\alpha_{k+1}}{5m\cdot n}\big)$. By (3.49) and by continuity, we must have $\varphi(r) \leq \delta_0$ for all $r \in \big(0, \frac{\alpha_{k+1}}{5m\cdot n}\big)$. Lebesgue's differentiation theorem implies in the limit $r \to 0$ there holds $z \notin A_{k+1}$. Thus $\mathcal{F}$ covers A_{k+1}.

In particular, A_{k+1} is covered by the family $\mathcal{F}'$ of cylinders $\mathfrak{C}_{m\cdot r}[z]$ with $z \in \mathcal{Q}^{k+1}$, $r \in \big(0, \frac{\alpha_{k+1}}{5m\cdot n}\big)$, so that $|A_{k+1} \cap \mathfrak{C}_{5m\cdot r}[z]| \leq \delta_0 \, |\mathfrak{C}_{5m\cdot r}[z]|$ and $|A_{k+1} \cap \mathfrak{C}_r[z]| > \delta_0 \, |\mathfrak{C}_r[z]|$. Vitali's covering lemma gives us the existence of a countable sub-family, denoted $\big(\mathfrak{C}_{r_l}[z_l]\big)_{l\geq 1}$, such that $\big(\mathfrak{C}_{5m\cdot r_l}[z_l]\big)_{l\geq 1}$ covers A_{k+1} and the $(\mathfrak{C}_{m\cdot r_l}[z_l])_{l\geq 1}$ are disjoint.

This finishes the construction of the covering with the desired properties. We can then apply Lemma 3.5 to each $\mathfrak{C}_{r_l}[z_l]$ to obtain $\mathfrak{C}_{r_l}[z_l]^+ \subset A_k$ and the $(\mathfrak{C}_{r_l}[z_l]^+)_{l\geq 1}$ are disjoint

since $\mathfrak{C}_{r_l}[z_l]^+ \subset \mathfrak{C}_{m\cdot r_l}[z_l]$. This yields for the right choice of δ_0, i.e. $\delta_0 \leq \left(\frac{4}{1225m^2}\right)^{2d+2s(d+1)}$,

$$\begin{aligned}
|A_{k+1}| &\leq \sum_{l\geq 1} |A_{k+1} \cap \mathfrak{C}_{5m\cdot r_l}[z_l]| \leq \delta_0 \sum_{l\geq 1} |C_{5m\cdot r_l}[z_l]| \\
&\leq (5m)^{2d(s+1)+2s}\delta_0 \sum_{l\geq 1} |\mathfrak{C}_{r_l}[z_l]| \leq (5m)^{2d(s+1)+2s}\delta_0 \sum_{l\geq 1} \left|\mathfrak{C}_{r_l}[z_l]^+\right| \\
&\leq (5m)^{2d(s+1)+2s}\delta_0 \left|A_k\right| \leq (5m)^{2d(s+1)+2s}\delta_0^2 \Big(\frac{1}{35m}\Big)^{(2d+2s(d+1))k} \left|\mathcal{Q}^k\right| \\
&\leq \delta_0 \Big(\frac{1}{35m}\Big)^{(2d+2s(d+1))(k+1)} \left|\mathcal{Q}^{k+1}\right|.
\end{aligned}$$

This proves (3.46) and thus concludes the proof.

3.6.2 Hölder continuity

Obtaining (3.1) is a standard argument, see [183] and [99]. Let f be a weak solution to (1.13)-(1.14). For given $r_0 < \frac{1}{3}$ small enough, Lemma 3.5 implies

$$\operatorname*{osc}_{Q_{r_0}} f \leq \left(1 - \frac{\theta}{2}\right) \max\left\{ \operatorname*{osc}_{Q_1} f, e^{2(3+2^5)} \left\|h\right\|_{L^\infty(Q_2)} \right\}. \tag{3.50}$$

To see this we apply Lemma 3.5 rescaled to Q_2 to either F or $-F$ depending on which of these satisfy (3.44), where F is defined by

$$F := \frac{2f - (\sup_{Q_1} f + \inf_{Q_1} f)}{\max\left\{ \operatorname*{osc}_{Q_1} f, e^{2(3+2^5)} \left\|h\right\|_{L^\infty(Q_2)} \right\}}.$$

We want to prove $\forall z_0 \in Q_1, \forall r \in (0, r_0)$,

$$\operatorname*{osc}_{Q_r(z_0)} f \leq r^\alpha e^{2(3+2^5)} \big(1 + \left\|h\right\|_{L^\infty(Q_2)}\big) \max\left\{ \operatorname*{osc}_{Q_1} f, e^{2(3+2^5)} \left\|h\right\|_{L^\infty(Q_2)} \right\}, \tag{3.51}$$

where we chose $\alpha \in (0,1)$ so that $1 - \frac{\theta}{2} = r_0^\alpha$. From there we can deduce Hölder regularity by choosing $z, z' \in Q_1$ so that $|z - z'| \leq r_0$.

In order to prove (3.51) we proceed iteratively. We define for $n \in \mathbb{N} \setminus \{0\}$ a sequence of solutions to (1.13)-(1.14) in Q_1 by

$$f_n(s,y,w) := \frac{2\big(1 - \frac{\theta}{2}\big)^{1-n} f\big(t_0 + r_0^{2sn}s, x_0 - r_0^{2sn}sv_0 + r_0^{(1+2s)n}y, v_0 + r_0^n w\big)}{\max\left\{ \operatorname*{osc}_{Q_1} f, e^{2(3+2^5)} \left\|h\right\|_{L^\infty(Q_2)} \right\}}.$$

We prove by induction on $n \geq 1$ that $\operatorname{osc}_{Q_1} f_n \leq 2e^{2(3+2^5)}\big(1 + \left\|h\right\|_{L^\infty(Q_2)}\big)$. The case $n = 1$

is clear by definition of f_1. Using (3.50) and the induction hypothesis, we get

$$\begin{aligned}\operatorname*{osc}_{Q_1} f_n &= \left(1-\frac{\theta}{2}\right)^{-1} \operatorname*{osc}_{Q_{r_0}} f_{n-1} \le \max\left\{\operatorname*{osc}_{Q_1} f_{n-1}, e^{2(3+2^5)}\|h\|_{L^\infty(Q_2)}\right\} \\ &\le 2e^{2(3+2^5)}\big(1+\|h\|_{L^\infty(Q_2)}\big).\end{aligned}$$

This implies (3.51) if we choose $\alpha \in (0,1)$ so that $1-\frac{\theta}{2} = r_0^\alpha$. Indeed for $z_0 \in Q_1$ and $r \in (0, r_0)$ we write $r = r_0^n$ for some $n \in \mathbb{N}\setminus\{0\}$. Then

$$\begin{aligned}\operatorname*{osc}_{Q_r(z_0)} f &= \frac{1}{2} r_0^{\alpha(n-1)} \max\left\{\operatorname*{osc}_{Q_1} f, e^{2(3+2^5)}\|h\|_{L^\infty(Q_2)}\right\} \operatorname*{osc}_{Q_r(z_0)} f_n \\ &= \frac{1}{2} r_0^{\alpha(n-1)} r^\alpha \max\left\{\operatorname*{osc}_{Q_1} f, e^{2(3+2^5)}\|h\|_{L^\infty(Q_2)}\right\} \operatorname*{osc}_{Q_1} f_{n+1} \\ &\le r_0^{\alpha n} e^{2(3+2^5)}\left(1+\|h\|_{L^\infty(Q_2)}\right) \max\left\{\operatorname*{osc}_{Q_1} f, e^{2(3+2^5)}\|h\|_{L^\infty(Q_2)}\right\} \\ &= r^\alpha e^{2(3+2^5)}\left(1+\|h\|_{L^\infty(Q_2)}\right) \max\left\{\operatorname*{osc}_{Q_1} f, e^{2(3+2^5)}\|h\|_{L^\infty(Q_2)}\right\}.\end{aligned}$$

CHAPTER 4

SCHAUDER

4.1 Introduction

This chapter is based on [150].

4.1.1 Problem formulation

We consider functions $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ solving either a kinetic Fokker-Planck-type equation in divergence form

$$\partial_t f + v \cdot \nabla_x f = \sum_{1 \le i,j \le d} \partial_{v_i}\big(a^{ij} \partial_{v_j} f\big) + \sum_{1 \le i \le d} b^i \partial_{v_i} f + cf + h, \tag{4.1}$$

or in non-divergence form

$$\partial_t f + v \cdot \nabla_x f = \sum_{1 \le i,j \le d} a^{ij} \partial^2_{v_i v_j} f + \sum_{1 \le i \le d} b^i \partial_{v_i} f + cf + h, \tag{4.2}$$

with diffusion coefficients $A = \big(a^{ij}(t,x,v)\big)_{i,j=1,\dots,d}$, lower order terms $B = (b^i)_{i=1,\dots,d}$, c, and source term h, or a fractional analogue (1.13),

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f + h,$$

where

$$\mathcal{L} f(t,x,v) = \int_{\mathbb{R}^d} \big[f(t,x,v') - f(t,x,v)\big] K(t,x,v,v') \, \mathrm{d}v',$$

for some non-negative kernel $K = K(t,x,v,v')$. The integral is to be understood in a principal value sense. The solutions are functions of time, space and velocity $f = f(t,x,v)$. In the local case (4.2), we assume A to be uniformly elliptic and Hölder continuous. Similarly, in the non-local case (1.13) we require a suitably defined ellipticity condition

on K as well as Hölder continuity. In both cases, we also assume our source term h and the lower order terms B, c to be Hölder continuous. Our goal is to establish Schauder estimates for solutions of (4.1), (4.2) and (1.13), which means that we want to quantify the gain of regularity stemming from the equation.

The equation is set, as above, in the *kinetic cylinder*

$$\begin{aligned} Q_R(z_0) := \{z = (t, x, v) : -R^{2s} \leq t - t_0 \leq 0, |v - v_0| < R, \\ |x - x_0 - (t - t_0)v_0| < R^{1+2s}\} \end{aligned} \tag{4.3}$$

for some $R > 0$ and $z_0 = (t_0, x_0, v_0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. The parameter $s \in (0, 1)$ determines the non-locality of the operator $\mathcal{L}$ in (1.14). For the local case set $s = 1$. This choice of domain is motivated by the underlying Lie group structure of (4.1), (4.2), (1.13). We recall, in fact, equation (1.13) scales as

$$(t, x, v) \to (r^{2s}t, r^{1+2s}x, rv) =: (t, x, v)_r = z_r. \tag{4.4}$$

This coincides with the scaling of the local analogues (4.1) and (4.2) for $s = 1$. Furthermore, these equations (4.1), (4.2) and (1.13) verify a Galilean invariance:

$$(t_1, x_1, v_1) \circ (t_2, x_2, v_2) \to (t_1 + t_2, x_1 + x_2 + t_2v_1, v_1 + v_2), \tag{4.5}$$

for any two points $z_1, z_2 \in \mathbb{R}^{1+2d}$.

The notion of Hölder continuity that we work with takes these invariances into account. On the one hand, the Hölder norm in the velocity variable coincides with the usual notion of Hölder regularity, whereas the regularity in time and space directions is adjusted according to the scaling (4.4). On the other hand, we choose a Hölder norm with respect to a distance that is left-invariant by the underlying Lie group structure (4.5). We introduce the kinetic Hölder spaces in Definition 4.2 below.

Before we state our main results, it is important to discuss the assumptions that define the ellipticity class and the Hölder continuity of the coefficients. We want our results in the local case to be applicable to the inhomogeneous Landau equation. In the fractional case, we work with a kernel general enough so that $\mathcal{L}f$ includes the non-cutoff Boltzmann collision operator.

4.1.2 Assumptions and result: the local (non-fractional) case

We consider (4.1) and (4.2), and we assume uniform ellipticity on the divergence coefficients, that is for some $\lambda_0 > 0$ there holds

$$\forall (t,x,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d, \quad \sum_{1 \le i,j \le d} a^{i,j}(t,x,v)\xi_i \xi_j \ge \lambda_0 |\xi|^2 . \tag{4.6}$$

Moreover, we work with coefficients A, B, c and source term h that are Hölder continuous in the sense of the kinetic Hölder regularity defined below in Definition 4.2.

Theorem 4.1 (Schauder estimate for kinetic Fokker-Planck equations)**.** *Let $\alpha \in (0,1)$ be given. Let $m \ge 3$ be some integer. Suppose $A \in C_\ell^{m-3+\alpha}(Q_1)$ satisfies (4.6) for some $\lambda_0 > 0$ and assume $B, c, h \in C_\ell^{m-3+\alpha}(Q_1)$. Let f solve (4.1) or (4.2) in Q_1. In the former case, we further assume $\nabla_v A \in C_\ell^{m-3+\alpha}(Q_1)$. Then there holds*

$$\|f\|_{C_\ell^{m-1+\alpha}(Q_{1/4})} \le C\Big(\|f\|_{L^\infty(Q_1)} + \|h\|_{C_\ell^{m-3+\alpha}(Q_1)} \Big),$$

for some C depending on $d, \lambda_0, \alpha, \|A\|_{C_\ell^{m-3+\alpha}}, \|B\|_{C_\ell^{m-3+\alpha}}, \|c\|_{C_\ell^{m-3+\alpha}}$, and for the divergence form case also on $\|\nabla_v A\|_{C_\ell^{m-3+\alpha}}$.

In particular, we recover Theorem 3.9 of Imbert and Mouhot [114] when $m = 3$ and Theorem 2.12 of Henderson-Snelson [103] when $m \in \{3, 4\}$. More generally, our approach is robust enough to cover higher order hypoelliptic equations, or also Dini-regular coefficients; we refer to Theorem 4.5 and Theorem 4.6 in Appendix 4.7.

4.1.3 Assumptions and results: the non-local (fractional) case

For the non-local equation (1.13), we specify the following notion of ellipticity and Hölder continuity. Let $s \in (0,1)$ and $0 < \lambda_0 < \Lambda_0$ be given. To be consistent with the previous work of Imbert-Silvestre [117], we consider a non-negative kernel $K = K(t,x,v,v')$ that maps (t,x,v) into a non-negative Radon measure $K_{(t,x,v)}$ in $\mathbb{R}^d \setminus \{0\}$ with

$$K_{(t,x,v)}(w) := K(t,x,v,v+w).$$

Then, for all $r > 0$, we assume the upper bound (1.32) and for any $\varphi \in C^2(B_{2r})$ the coercivity condition (1.31). Moreover, we either work with the following symmetry condition, which is the non-local analogue of *non-divergence form* equations

$$K(w) = K(-w). \tag{4.7}$$

Or else, if we consider the *divergence form* analogue instead, we impose a weak form of symmetry in the sense of the cancellation conditions (1.29), (1.30).

Finally we want K to be Hölder continuous with exponent $\alpha \in (0, +\infty)$: given $z_1 = (t_1, x_1, v_1)$ and $z_2 = (t_2, x_2, v_2)$ we have for any $r > 0$

$$\int_{B_r} |K_{z_1}(w) - K_{z_2}(w)| \, |w|^2 \, \mathrm{d}w \leq A_0 r^{2-2s} d_\ell(z_1, z_2)^\alpha, \tag{4.8}$$

where d_ℓ denotes the kinetic distance defined below in Definition 4.1. In the divergence form case, we require in addition to (4.8) for any $r > 0$

$$\left| \mathrm{PV} \int_{B_r} w\big(K_{z_1}(w) - K_{z_2}(w)\big) \, \mathrm{d}w \right| \leq A_0 r^{1-2s} d_\ell(z_1, z_2)^\alpha. \tag{4.9}$$

For integro-differential equations in non-divergence form we recover Theorem 1.6 of [117].

Theorem 4.2 (Imbert-Silvestre [117, Theorem 1.6])**.** *Let $0 < \gamma < \min(1, 2s)$. Assume K is a non-negative kernel that is elliptic* (1.32), (1.26)*, in non-divergence form* (4.7)*, and Hölder continuous in the sense that it satisfies* (4.8) *for $\alpha = \frac{2s}{1+2s}\gamma$ and each $z \in Q_1$. Then any solution $f \in C_l^\gamma([-1, 0] \times B_1 \times \mathbb{R}^d)$ of* (1.13) *in Q_1 satisfies*

$$\|f\|_{C_\ell^{2s+\alpha}(Q_{1/4})} \leq C\big(\|f\|_{C_\ell^\gamma([-1,0]\times B_1 \times \mathbb{R}^d)} + \|h\|_{C_\ell^\alpha(Q_1)} \big),$$

for some constant $C = C(d, s, \lambda_0, \Lambda_0, A_0)$.

For divergence-form kinetic integro-differential equations we establish the following result.

Theorem 4.3 (Schauder estimates for kinetic integro-differential equations in divergence form)**.** *Let $0 < \gamma < \min(1, 2s)$. Assume K is a non-negative kernel that is elliptic and Hölder continuous in the sense that it satisfies* (1.32), (1.26)*, the (weak) divergence form symmetry* (1.29) *and* (1.30)*, and is Hölder continuous* (4.8) *and* (4.9) *for $\alpha = \frac{2s}{1+2s}\gamma$ and each $z \in Q_1$. Then any solution $f \in C_l^\gamma([-1, 0] \times B_1 \times \mathbb{R}^d)$ of* (1.13) *in Q_1 satisfies*

$$\|f\|_{C_\ell^{2s+\alpha}(Q_{1/4})} \leq C\big(\|f\|_{C_\ell^\gamma([-1,0]\times B_1 \times \mathbb{R}^d)} + \|h\|_{C_\ell^\alpha(Q_1)} \big),$$

for some constant $C = C(d, s, \lambda_0, \Lambda_0, A_0)$.

Remark 4.1. We want to emphasise that Theorems 4.1 and 4.2 are applicable to the inhomogeneous Landau and the Boltzmann equation without cut-off, respectively. On the one hand, we recall the Landau equation given by (1.39) for general potentials reads

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left(\int_{\mathbb{R}^d} a(v - w)\big[f(w)\nabla f(v) - f(v)\nabla f(w)\big] \, \mathrm{d}w \right), \tag{4.10}$$

where

$$a(z) = a_{d,\gamma} |z|^{\gamma+2} \Big(I - \frac{z \otimes z}{|z|^2} \Big),$$

for $\gamma \geq -d$, $a_{d,\gamma} > 0$. It can be rewritten in divergence (4.1) or non-divergence form (4.2) for suitable coefficients A, B, c, as stated, for example, on page one in [103]. The Boltzmann equation, on the other hand, is given by (1.2). Using Carleman coordinates and the cancellation lemma, we can rewrite this as (1.13), for some specific kernel K.

Whenever the macroscopic quantities in (1.20) are satisfied, we can check that the coefficients for Landau and the kernel for Boltzmann satisfy the ellipticity assumptions made in Subsection 4.1.2 and Subsection 4.1.3, respectively. In particular, any Hölder continuous solution f of (4.10) or (1.2) with mass, energy and entropy bounded above, and mass bounded below, satisfies the Schauder estimate in Theorem 4.1 or 4.2, respectively. We refer the reader to [103, Theorem 1.2] for the Landau equation, and [118, Section 4] for the Boltzmann equation.

4.1.4 Contribution

Our contribution consists of a quantified and unified approach to Schauder estimates for kinetic equations with either non-fractional or fractional coefficients, in either non-divergence or divergence form. In this respect it improves upon the previous results on kinetic Schauder estimates in the local case by Imbert-Mouhot [114] and Henderson-Snelson [103], and in the non-local case by Imbert-Silvestre [117]. Indeed, on the one hand, in the non-fractional case we manage to gain two orders of Hölder regularity at any stage $m \geq 3$. On the other hand, we establish Schauder estimates for *divergence form equations* in Theorem 4.1 and 4.3, which, to the best of our knowledge, is a novelty in the fractional case. Moreover, our approach is fully quantitative, which, in the fractional case, is in contrast to the blow-up argument used in [117]. Finally, in the non-fractional case, the method is robust enough to deal with any hypoelliptic operator, and it works even more generally for Dini-regular coefficients, see Theorem 4.5 and Theorem 4.6, respectively.

4.1.5 Literature

All the works on Schauder estimates have to be classified according to the notion of Hölder continuity that is used and the assumptions on the coefficients that are made.

In the local case, there is the work by Imbert and Mouhot [114], which adapts Krylov's approach [138] to the kinetic setting. Furthermore, in [103], Henderson-Snelson discuss how to derive a C^∞-smoothing estimate for the Landau equation by iteratively applying their Schauder estimates. There are also two articles [69, 105] for kinetic Fokker-Planck equations, which assume less regularity in time, and deduce partial Schauder estimates

for space and velocity only. Their goal is to reduce the regularity assumptions needed on time. Note, however, that the Hölder norms defined in [69, 105] differ from our notion of Hölder continuity.

In the non-local case, the work that inspired us most is from Imbert and Silvestre [117]. In particular, the definition of kinetic Hölder spaces, the notion of distance and degree of a kinetic polynomial all stem from their seminal contribution on regularity for the non-cutoff Boltzmann equation [115–118]. Their approach to Schauder estimates consists of first proving a Liouville-type theorem, then using a blow-up argument. Their work is inspired from Ros-Oton-Serra [174], who have used these techniques for non-local operators that are generators of any stable and symmetric Lévy process. Note, however, that this method is non-constructive, as it relies on a compactness argument. The structure of this argument comes from Simon [180], who used a scaling argument to derive a Liouville theorem for general hypo-elliptic operators, from which he deduces the Schauder estimate by a compactness argument.

We follow Campanato's approach. This method was first established for elliptic equations. A nice reference is the book by Giaquinta and Martinazzi [79, Chapter 5]. The idea is to use the scaling stemming from a combination of a Poincaré inequality, Sobolev and regularity estimates on the constant coefficient equation. In contrast, Simon's scaling argument [180, Lemma 1] replaces the Sobolev inequality and regularity estimates by a reasoning of Hörmander [111, Theorem 3.7] based on the closed graph theorem and the homogeneity of the operator. Through the characterisation of Hölder norms by Campanato norms, we replace the blow-up argument of Simon by a constructive method.

4.1.6 Strategy

We consider a solution of either the local or non-local equation, and freeze coefficients: the part which solves a constant coefficient equation with zero source term is considered separately from the rest. The latter can be viewed as a *lower order source term* with the expected bounds due to the Hölder continuity of the coefficients. For the constant coefficient solution, we subtract a certain polynomial constructed from the vector fields of the equation of degree up to the order of our equation, such that we have a zero-averaged function. We then apply Poincaré's inequality repeatedly as long as the zero-average condition is satisfied and the integrand is orthogonal to the kernel of the Poincaré inequality, that is one order higher than the equation itself. We then use an L^∞-bound and Sobolev's embedding. But then, since we consider a solution to a constant coefficient equation, regularity estimates yield a bound uniform in the Hölder norm of the coefficients. The left hand side will be a higher order Campanato norm, which characterises Hölder norms. The gain of regularity arises from the scaling of the equation. We recall the illustration of these steps in Figure 1.8.

Section 4.2 introduces the notion of Hölder spaces that we work with. We state the equivalence of Hölder and Campanato norms in Theorem 4.4, whose proof is postponed to the Appendix 4.8. In Section 4.3 we assemble tools that are setting the framework for Campanato's approach. In particular, we derive regularity estimates 4.3 for the constant coefficient equation. Section 4.4 is devoted to the proof of Campanato's inequality. Section 4.5 proves the Schauder estimates in the non-fractional case, whereas Section 4.6 treats the fractional case. We want to emphasise, whenever a statement holds both in the local and the non-local case, we will state the non-local result and we ask the reader to set $s = 1$ to obtain the local analogue.

4.2 Preliminaries

4.2.1 Definition of kinetic Hölder spaces

To define the Hölder spaces that we are working with, we first need to understand the underlying Lie group structure of (4.1), (4.2) and (1.13). These equations are invariant under Galilean transformations (4.5), in the sense that if f solves (4.1), (4.2) or (1.13) then $f(z_1 \circ z_2)$ is also a solution of the respective equation with a translated right hand side and a translated kernel. The translated kernel will still be elliptic. Furthermore, both equations are invariant under scaling (4.4) for a rescaled right hand side. The rescaled kernel will again be elliptic. The notion of distance that we introduce respects these invariances. It has been used by Imbert-Silvestre [117, Def. 2.1] before.

Definition 4.1 (Kinetic distance)**.** For $z_1 = (t_1, x_1, v_1), z_2 = (t_2, x_2, v_2) \in \mathbb{R}^{1+2d}$ we define

$$d_l(z_1, z_2) := \min_{w \in \mathbb{R}^d} \Big\{ \max \Big[|t_1 - t_2|^{\frac{1}{2s}}, |x_1 - x_2 - (t_1 - t_2)w|^{\frac{1}{2s}}, |v_1 - w|, |v_2 - w| \Big] \Big\}.$$

Moreover we define

$$\|z\| = \max \Big\{ |t|^{\frac{1}{2s}}, |x|^{\frac{1}{1+2s}}, |v| \Big\}.$$

This is not a norm in the mathematical sense.

Remark 4.2. This notion of distance should not be confused with the distance function towards the grazing set as introduced in [102, Def. 1], which apart from the name does not have any connection to this distance here.

Let us observe that this distance is left invariant in the sense that $d_\ell(z \circ z_1, z \circ z_2) = d_\ell(z_1, z_2)$ for any $z, z_1, z_2 \in \mathbb{R}^{1+2d}$. We can also reformulate it as d_ℓ being the infimum value of $r > 0$ such that both z_1, z_2 belong to $Q_r(z_0)$ for some $z_0 \in \mathbb{R}^{1+2d}$. Other equivalent

formulations are

$$d_\ell(z_1,z_2) \sim \left\|z_2^{-1}\circ z_1\right\| \sim \left\|z_1^{-1}\circ z_2\right\| \\ \sim \inf_{w\in\mathbb{R}^d} |t_2-t_1|^{\frac{1}{2s}} + |x_2-x_1-(t_2-t_1)w|^{\frac{1}{1+2s}} + |v_1-w| + |v_2-w|\,.$$

For more remarks on this distance we refer the reader to [117, Section 2].

In addition to the kinetic distance, we use the notion of kinetic degree of a monomial $m_j \in \mathbb{R}[t,x,v]$ introduced in [117, Subsection 2.2] as

$$\deg_{\text{kin}} m_j = 2s\cdot j_0 + (1+2s)\left(\sum_{i=1}^{d} j_i\right) + \sum_{i=d+1}^{2d} j_i = 2s\cdot j_0 + (1+2s)\cdot|J_1| + |J_2| =: |J|\,,$$

where we denote a multi-index $j\in\mathbb{N}^{1+2d}$ with $j=(j_0,J_1,J_2)$ where $J_1=(j_1,\dots,j_d)$ and $J_2=(j_{d+1},\dots,j_{2d})$. Under scaling a monomial m_j behaves as

$$m_j(z_R) = R^{2sj_0}t^{j_0}R^{(1+2s)|J_1|}x^{J_1}R^{|J_2|}v^{J_2} = R^{|J|}z^j, \quad R>0,$$

and its degree is precisely $|J| = 2sj_0 + (1+2s)\,|J_1| + |J_2|$. We denote with $\mathcal{P}_k$ the space of k degree polynomials. Note that in the non-local case k is in the discrete set $k\in\mathbb{N}+2s\mathbb{N}$, and we will write $k = 2s\cdot k_0 + (1+2s)\cdot k_1 + k_2$ for $k_0,k_1,k_2\in\mathbb{N}$. An element $p\in\mathcal{P}_k$ is written as

$$p(t,x,v) = \sum_{\substack{j\in\mathbb{N}^{1+2d},\\ |J|\le k}} a_j m_j(z). \tag{4.11}$$

The sum is taken over $j_0\in[0,k_0], |J_1|\in[0,k_1], |J_2|\in[0,k_2]$. We will abbreviate this and write $|J|\le k$. In the local case there is no ambiguity.

Our notion of Hölder continuity leans on [114, Def. 2.2] and [117, Def. 2.3].

Definition 4.2 (Hölder spaces)**.** Given an open set $\Omega\subset\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ and $\beta\in(0,\infty)$ we say that $f:\Omega\to\mathbb{R}$ is $C^\beta_\ell(\Omega)$ at a point $z_0\in\mathbb{R}^{1+2d}$ if there is a polynomial $p\in\mathbb{R}[t,x,v]$ with kinetic degree $\deg_{\text{kin}}\, p<\beta$ and a constant $C>0$ such that

$$\forall r>0 \quad \|f-p\|_{L^\infty(Q_r(z_0)\cap\Omega)} \le Cr^\beta. \tag{4.12}$$

When this property holds at every point $z_0\in\Omega$ we say that $f\in C^\beta_\ell(\Omega)$. The semi-norm $[f]_{C^\beta_\ell(\Omega)}$ is the smallest C such that (4.12) holds for all $z_0\in\Omega$. We equip $C^\beta_\ell(\Omega)$ with the norm

$$\|f\|_{C^\beta_\ell(\Omega)} = \|f\|_{L^\infty(\Omega)} + [f]_{C^\beta_\ell(\Omega)}.$$

Remark 4.3. This definition coincides with the definition of [114, Def. 2.2]. As the authors

point out, it is equivalent to ask that for any $z \in \Omega$

$$|f(z) - p(z)| \leq C d_\ell(z, z_0)^\beta.$$

We can further rephrase Hölder regularity of f at z_0 due to the left-invariance as follows [117]. For any $z \in \mathbb{R}^{1+2d}$ such that $z_0 \circ z \in \Omega$ we have

$$|f(z_0 \circ z) - p_{z_0}(z)| \leq C \|z\|^\beta,$$

where $p_{z_0}(z) = p(z_0 \circ z)$. The polynomial p_{z_0} will be the expansion of f at z_0.

Hölder spaces can also be characterised in terms of Campanato spaces. These have been introduced by Campanato himself [38–40] in the elliptic context. We adapt his notion to the kinetic setting.

Definition 4.3 (Higher order Campanato spaces)**.** Let $\Omega \subset \mathbb{R}^{1+2d}$ be an open subset. For $1 \leq p \leq \infty,\ \lambda \geq 0,\ k \geq 0$ we define the Campanato space $\mathcal{L}_k^{p,\lambda}(\Omega)$ as

$$\mathcal{L}_k^{p,\lambda}(\Omega) := \left\{ f \in L^p(\Omega) : \sup_{z \in \Omega, r>0} r^{-\lambda} \inf_{P \in \mathcal{P}_k} \int_{Q_r(z) \cap \Omega} |f - P|^p \, \mathrm{d}z < +\infty \right\} \tag{4.13}$$

where $\mathcal{P}_k$ is the space of polynomials of kinetic degree less or equal k. We endow it with the seminorm

$$[f]^p_{\mathcal{L}_k^{p,\lambda}} := \sup_{z \in \Omega, r>0} r^{-\lambda} \inf_{P \in \mathcal{P}_k} \int_{Q_r(z) \cap \Omega} |f - P|^p \, \mathrm{d}z \tag{4.14}$$

and the norm

$$\|f\|_{\mathcal{L}_k^{p,\lambda}} = [f]_{\mathcal{L}_k^{p,\lambda}} + \|f\|_{L^p}. \tag{4.15}$$

Remark 4.4.

i. We observe that for the local case $k \in \mathbb{N}$, whereas in the non-local case $k \in \mathbb{N} + 2s\mathbb{N}$.

ii. Campanato's spaces are most commonly known for $k = 0$. Such spaces have been used for Schauder estimates in the elliptic context [79]. To gain higher Hölder continuity ($k \geq 1$) the equation was just differentiated. This would not work as easily for our equations. Note that a method inspired from Campanato's approach with $k = 0$ has been developed for partial Schauder estimates in the kinetic setting in [69]. However, we have not seen any Schauder estimates that use the higher order Campanato spaces directly, neither in the elliptic nor the parabolic case, and certainly not in the hypoelliptic case.

The next subsection states a characterisation of Hölder continuity in terms of Campanato's norms.

4.2.2 Relation between Hölder and Campanato spaces

Hölder spaces can be characterised through Campanato spaces, and vice versa. This equivalence has been established by Campanato himself in [38] for the lowest order Campanato space, and in [40] for higher order Campanato spaces. Following Campanato's arguments, we can show the following relation between Campanato and Hölder spaces defined in 4.2 and 4.3. We refer the reader to the proof in Appendix 4.8.

Theorem 4.4 (Campanato)**.** *Let $\tilde{z}_0 \in \mathbb{R}^{1+2d}$ and $R > 0$, and write $\Omega = Q_R(\tilde{z}_0)$. Then, for $n+kp < \lambda \leq n+(k+1)p$ and $\beta = \frac{\lambda-n}{p}$ we have $\mathcal{L}^{p,\lambda}_k(\Omega) \cong C^\beta_\ell(\bar{\Omega})$, where $n = 2s+2d(s+1)$.*

Remark 4.5. For the local case, setting $s = 1$ yields the same result.

4.2.3 Differential operators

In this section, we show how to relate Hölder norms to kinetic differential operators. We reprove Lemma 2.7 of [117] to make our paper self-contained.

Lemma 4.1 (Imbert-Silvestre [117, Lemma 2.7])**.** *Let $D = \mathcal{T}, D = \nabla_x$ or $D = \nabla_v$. Let $f \in C^\beta_\ell(Q)$ for $\beta \in (0,\infty)$ and Q some kinetic cylinder. Then $D^l f \in C^{\beta-k}_\ell(Q)$ where k is the kinetic degree of D^l, $l \in \mathbb{N}$, and*

$$[D^l f]_{C^{\beta-k}_\ell(Q)} \leq C[f]_{C^\beta_\ell(Q)}.$$

Proof. Let $z_1, z_2 \in Q$. Since $f \in C^\beta_\ell(Q)$ there exists a polynomial p with degree $k = \deg_{\text{kin}} \mathrm{p} < \beta$ so that for $z \in Q$ with $\|z\| \leq d_l(z_1, z_2) = r$

$$\begin{aligned} |f(z_1 \circ z) - p(z_1 \circ z)| &\leq Cr^\beta, \\ |f(z_2 \circ z) - p(z_2 \circ z)| &\leq Cr^\beta. \end{aligned} \tag{4.16}$$

We can compute that

$$p(z_1 \circ z) = f(z_1) + \mathcal{T} f(z_1)t + \nabla_x f(z_1) \cdot x + \nabla_v f(z_1) \cdot v + \dots$$

By equivalence of norms in finite dimensional spaces, we know that if $\sup_{|z|\leq 1} |p(z)| \leq C_0$ then the coefficients of p denoted by a_j will satisfy $\sup_j |a_j| \leq CC_0$ for some constant C depending on k and n. Scaling this argument yields together with (4.16)

$$\left|D^l f(z_1) - D^l f(z_2)\right| r^k \leq Cr^\beta,$$

where D^l is the differential operator of degree k. □

We will need a similar estimate for the fractional operator (1.14). We start with a global bound, see [117, Lemma 3.6] for kernels in non-divergence form (4.7).

Lemma 4.2. *Assume $0 < \alpha < \min(1, 2s)$. For any non-negative kernel K satisfying (1.32), and either satisfy (4.7) or (1.29), (1.30). Then for $f \in C_\ell^{2s+\alpha}(\mathbb{R}^{1+2d})$ there holds*

$$[\mathcal{L}f]_{C_\ell^\alpha(\mathbb{R}^{2d+1})} \leq C[f]_{C_\ell^{2s+\alpha}(\mathbb{R}^{2d+1})}.$$

Proof. Let $z, \xi \in \mathbb{R}^{1+2d}$. We know that

$$|f(z \circ \xi) - p_z(\xi)| \leq [f]_{C_\ell^{2s+\alpha}} \|\xi\|^{2s+\alpha}. \tag{4.17}$$

We need to estimate

$$\begin{aligned}\mathcal{L}f(z \circ \xi) - \mathcal{L}f(z) = &\int_{\mathbb{R}^d} \big[f(z \circ \xi \circ (0,0,v'-v-\xi_v)) - f(z \circ \xi)\big] K(z \circ \xi, v')\,\mathrm{d}v' \\ &- \int_{\mathbb{R}^d} \big[f\big(z \circ (0,0,v'-v)\big) - f(z)\big] K(z, v')\,\mathrm{d}v'.\end{aligned}$$

We distinguish the close and the far part. Let $R > 0$ and write for ease of notation $\phi = (0,0,v'-v-\xi_v)$ and $\psi = (0,0,v'-v)$ for $\xi = (\xi_t, \xi_x, \xi_v)$.

If we assume symmetry in the non-divergence form (4.7), then we can symmetrise the integral and remove the principal value. We find

$$\begin{aligned}&\mathrm{PV} \int_{B_R(v)} \big[f(z \circ \psi) - f(z)\big] K(z, v')\,\mathrm{d}v' \\ &= \frac{1}{2} \int_{B_R(v)} \big[f(z \circ \psi) + f(z \circ -\psi) - 2f(z)\big] K(z, v')\,\mathrm{d}v' \\ &\leq \frac{1}{2} \int_{B_R(v)} |f(z \circ \psi)) - p_z(\psi)|\, K(z, v')\,\mathrm{d}v' + \frac{1}{2} \int_{B_R(v)} |p_z(\psi) - f(z)|\, K(z, v')\,\mathrm{d}v' \\ &\quad + \frac{1}{2} \int_{B_R(v)} |f(z \circ -\psi) - p_z(-\psi)|\, K(z, v')\,\mathrm{d}v' + \frac{1}{2} \int_{B_R(v)} |p_z(-\psi) - f(z)|\, K(z, v')\,\mathrm{d}v' \\ &\leq [f]_{C_\ell^{2s+\alpha}} \int_{B_R(v)} |v'-v|^{2s+\alpha} K(z, v')\,\mathrm{d}v' + \big|\nabla_v^2 f(z)\big| \int_{B_R(v)} |v'-v|^2 K(z, v')\,\mathrm{d}v' \\ &\lesssim_\Lambda [f]_{C_\ell^{2s+\alpha}} R^\alpha + \big|\nabla_v^2 f(z)\big|\, R^{2-2s}.\end{aligned}$$

We used that $p_z(\psi) = f(z) + \nabla_v f(z) \cdot (v'-v) + (v'-v)^T \cdot \nabla_v^2 f(z) \cdot (v'-v)$. Any higher order terms vanish since $\deg p < 2s + \alpha$. The first order term in v vanishes due to (4.7). For the second order term, we use (1.32). All estimates are independent of $z \in \mathbb{R}^{1+2d}$ so

that we similarly obtain

$$\mathrm{PV}\int_{B_R(v+\xi_v)}\big[f(z\circ\xi\circ\phi)-f(z\circ\xi)\big]K(z\circ\xi,v')\,\mathrm{d}v'\lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell}R^\alpha+\big|\nabla_v^2 f(z\circ\xi)\big|\,R^{2-2s}.$$

Therefore

$$\begin{aligned}\int_{B_R(v+\xi_v)}\big[f(z\circ\xi\circ\phi)-f(z\circ\xi)\big]K(z\circ\xi,v')\,\mathrm{d}v'-\int_{B_R(v)}\big[f(z\circ\psi)-f(z)\big]K(z,v')\,\mathrm{d}v'\\ \lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell}R^\alpha+\big|\nabla_v^2 f(z\circ\xi)-\nabla_v^2 f(z)\big|\,R^{2-2s}\\ \lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell}R^\alpha+\|\xi\|^{2s+\alpha-2}R^{2-2s}[f]_{C^{2s+\alpha}_\ell}.\end{aligned}$$

We used Lemma 4.1 for the last inequality. Choosing $R=\|\xi\|$ therefore yields

$$\begin{aligned}\int_{B_R(v+\xi_v)}\big[f(z\circ\xi\circ\phi)-f(z\circ\xi)\big]K(z\circ\xi,v')\,\mathrm{d}v'-\int_{B_R(v)}\big[f(z\circ\psi)-f(z)\big]K(z,v')\,\mathrm{d}v'\\ \lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell}R^\alpha.\end{aligned}$$

If, instead of (4.7), we assume (1.29) and (1.30), then we bound

$$\begin{aligned}&\left|\mathrm{PV}\int_{B_R(v)}\big[f(z\circ\psi)-f(z)\big]K(z,v')\,\mathrm{d}v'\right|\\ &=\left|\mathrm{PV}\int_{B_R(v)}\big[f(z\circ\psi)-p_z(\psi)-\big(f(z)-p_z(\psi)\big)\big]K(z,v')\,\mathrm{d}v'\right|\\ &\le\left|\mathrm{PV}\int_{B_R(v)}|f(z\circ\psi)-f(z)|\,K(z,v')\,\mathrm{d}v'\right|\\ &\quad+\left|\mathrm{PV}\int_{B_R(v)}D_vf(z)\cdot\big(v-v'\big)K(z,v')\,\mathrm{d}v'\right|\\ &\quad+\left|\mathrm{PV}\int_{B_R(v)}\big|D_v^2f(z)\big|\,|v-v'|^2\,K(z,v')\,\mathrm{d}v'\right|\\ &\le[f]_{C^{2s+\alpha}_\ell}\int_{B_R(v)}|v'-v|^{2s+\alpha}\,K(z,v')\,\mathrm{d}v'+C\Lambda\,|D_vf(z)|\,R^{1-2s}+C\Lambda\,\big|D_v^2f(z)\big|\,R^{2-2s}\\ &\lesssim_\Lambda[f]_{C^{2s+\alpha}_\ell}R^\alpha+|D_vf(z)|\,R^{1-2s}+\big|D_v^2f(z)\big|\,R^{2-2s}.\end{aligned}$$

We again used (1.32) and (4.17). The same computations yield

$$\begin{aligned}\left|\mathrm{PV}\int_{B_R(v+\xi_v)}\big[f(z\circ\xi\circ\phi)-f(z\circ\xi)\big]K(z\circ\xi,v')\,\mathrm{d}v'\right|\\ \lesssim_\Lambda[f]_{C^{2s+\alpha}_\ell}R^\alpha+|D_vf(z\circ\xi)|\,R^{1-2s}+\big|D_v^2f(z\circ\xi)\big|\,R^{2-2s},\end{aligned}$$

so that as before, we obtain with Lemma 4.1

$$\begin{aligned}\int_{B_R(v+\xi_v)} \big[f(z\circ\xi\circ\phi)-f(z\circ\xi)\big]K(z\circ\xi,v')\,\mathrm{d}v' &- \int_{B_R(v)} \big[f(z\circ\psi)-f(z)\big]K(z,v')\,\mathrm{d}v'\\ &\lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell} R^\alpha + |\nabla_v f(z\circ\xi)-\nabla_v f(z)|\,R^{1-2s} + \big|\nabla_v^2 f(z\circ\xi)-\nabla_v^2 f(z)\big|\,R^{2-2s}\\ &\lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell} R^\alpha + \|\xi\|^{2s+\alpha-1} R^{1-2s}[f]_{C^{2s+\alpha}_\ell} + \|\xi\|^{2s+\alpha-2} R^{2-2s}[f]_{C^{2s+\alpha}_\ell}\\ &\lesssim_\Lambda [f]_{C^{2s+\alpha}_\ell} R^\alpha,\end{aligned}$$

since $\|\xi\| = R$.

For the far part we do not need to distinguish non-divergence form from divergence form. In both cases we separate the integral into different terms

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_R(v+\xi_v)} \big[f(z\circ\xi\circ\phi)-f(z\circ\xi)\big]K(z\circ\xi,v')\,\mathrm{d}v' &\\ - \int_{\mathbb{R}^d\setminus B_R(v)} \big[f(z\circ\psi)-f(z)\big]K(z,v')\,\mathrm{d}v' &\le \sum_{i=1}^5 I_i,\end{aligned}$$

with

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^d\setminus B_R(v+\xi_v)} \big|f(z\circ\xi\circ\phi)-p_{z\circ\phi}(\phi^{-1}\circ\xi\circ\phi)\big|\,K(z\circ\xi,v')\,\mathrm{d}v',\\ I_2 &= \int_{\mathbb{R}^d\setminus B_R(v+\xi_v)} |f(z\circ\xi)-p_z(\xi)|\,K(z\circ\xi,v')\,\mathrm{d}v',\\ I_3 &= \int_{\mathbb{R}^d\setminus B_R(v+\xi_v)} \big|p_{z\circ\phi}(\phi^{-1}\circ\xi\circ\phi)-p_z(\xi)\big|\,K(z\circ\xi,v')\,\mathrm{d}v',\\ I_4 &= \int_{\mathbb{R}^d\setminus B_R(v)} |p_{z\circ\psi}(\xi)-p_z(\xi)-f(z\circ\psi)+f(z)|\,K(z,v')\,\mathrm{d}v',\\ I_5 &= \int_{\mathbb{R}^d\setminus B_R(v)} |p_{z\circ\psi}(\xi)-p_z(\xi)|\,K(z,v')\,\mathrm{d}v'.\end{aligned}$$

Using that

$$\begin{aligned}&\big|f(z\circ\xi\circ\phi)-p_{z\circ\phi}(\phi^{-1}\circ\xi\circ\phi)\big|\\ &\quad\le [f]_{C^{2s+\alpha}_\ell}\big\|\phi^{-1}\circ\xi\circ\phi\big\|^{2s+\alpha}\\ &\quad\le [f]_{C^{2s+\alpha}_\ell}\Big(\|\xi\| + |v'-v-\xi_v|^{1/(1+2s)}\,\|\xi_t\|^{2s/(1+2s)}\Big)^{2s+\alpha},\end{aligned}$$

we bound the first term with (1.32) by

$$\begin{aligned}I_1 &\le C[f]_{C^{2s+\alpha}_\ell}\left(\|\xi\|^{2s+\alpha}R^{-2s} + \|\xi\|^{\frac{2s(2s+\alpha)}{1+2s}}\int_{\mathbb{R}^d\setminus B_R(v+\xi_v)} |v'-v-\xi_v|^{\frac{2s+\alpha}{1+2s}}\,K(z\circ\xi,v')\,\mathrm{d}v'\right)\\ &\le C[f]_{C^{2s+\alpha}_\ell}R^{-2s}\Big(\|\xi\|^{2s+\alpha} + \|\xi\|^{\frac{2s(2s+\alpha)}{1+2s}}\,R^{\frac{2s+\alpha}{1+2s}}\Big).\end{aligned}$$

For I_2 we get

$$I_2 \leq C[f]_{C_\ell^{2s+\alpha}} \|\xi\|^{2s+\alpha} R^{-2s}.$$

We further notice that I_4 is the same as I_5 without the lowest order term of $p_{z\circ\psi} - p_z$. To estimate I_5 we write $p_z(\xi) = \sum a_j(z) m_j(\xi)$. Note that by Lemma 4.1 the coefficients a_j satisfy

$$[a_j]_{C_\ell^{2s-j+\alpha}} \leq C[f]_{C_\ell^{2s+\alpha}},$$

where j is the degree of the corresponding monomial. Thus

$$I_5 \leq C[f]_{C_\ell^{2s+\alpha}} \Big(R^\alpha + R^{\alpha-1} \|\xi\| + R^{\alpha-2s} \|\xi\|^{2s} + R^{\alpha-2} \|\xi\|^2 \Big),$$

and

$$I_4 \leq C[f]_{C_\ell^{2s+\alpha}} \Big(R^{\alpha-1} \|\xi\| + R^{\alpha-2s} \|\xi\|^{2s} + R^{\alpha-2} \|\xi\|^2 \Big).$$

For I_3 we notice that $\phi^{-1} \circ \xi \circ \phi^{-1} = \big(\xi_t, \xi_x + \xi_t(v' - v - \xi_v), \xi_v\big)$. Apart from the space variable this coincides with ξ. But since we only conisder polynomial expansion up to order $2s + \alpha < 2s + 1$ the space variable won't appear, so that in fact $|I_3| = |I_5|$. We now choose $R = \|\xi\|$ so that all terms are bounded by $I_i \leq C[f]_{C_\ell^{2s+\alpha}} \|\xi\|^\alpha$ for all $i = 1, \dots, 5$. □

To localise Lemma 4.2 we follow the proof of Imbert and Silvestre in [117, Lemma 3.7]. Note, however, that we also cover the non-divergence form symmetry (1.29)-(1.30).

Lemma 4.3 (Imbert-Silvestre [117, Lemma 3.7])**.** *Let $0 < \alpha \leq \gamma < \min(1, 2s)$ and let K satisfy* (1.32) *and either* (4.7) *or* (1.29),(1.30)*. Then*

$$[\mathcal{L}f]_{C_\ell^\alpha(Q_{\frac{1}{2}})} \leq C\Big([f]_{C_\ell^{2s+\alpha}(Q_{\frac{1}{2}})} + [f]_{C_\ell^\gamma((-1,0]\times B_1\times\mathbb{R}^d)} \Big),$$

for some C depending on n, s, Λ_0 and A_0.

Proof. We write $\mathcal{L}f(z) = \tilde{\mathcal{L}}f(z) + C(z)$ where $\tilde{\mathcal{L}}f(z)$ corresponds to the non-local operator as in (1.14) with kernel $\tilde{K}(v, v') = \mathbb{1}_{B_\rho(v)}(v') K(v, v')$ and $C(z)$ corresponds to $\mathcal{L}f$ with kernel $\big[1 - \mathbb{1}_{B_\rho(v)}(v')\big] K(v, v')$ for some small $\rho > 0$. Then by Lemma 4.2 we have

$$[\tilde{\mathcal{L}}f]_{C_\ell^\alpha(Q_{\frac{1}{2}})} \leq C[f]_{C_\ell^{2s+\alpha}(Q_1)}.$$

Now we consider $z_0, z \in Q_{\frac{1}{2}}$ such that $z_0 \circ z \in Q_{\frac{1}{2}}$. If we write $\phi = (0, 0, v' - v - v_0)$ and

$\psi = (0,0,v'-v)$ we have for $K(w) = K(v, v+w)$

$$\begin{aligned}
&C(z_0 \circ z) - C(z) \\
&\qquad = \int_{\mathbb{R}^d \setminus B_\rho(v+v_0)} \big[f(z_0 \circ z \circ \phi) - f(z_0 \circ z)\big] K(z_0 \circ z, v')\,\mathrm{d}v' \\
&\qquad\quad - \int_{\mathbb{R}^d \setminus B_\rho(v)} \big[f(z \circ \psi) - f(z)\big] K(z, v')\,\mathrm{d}v' \\
&\qquad = \int_{\mathbb{R}^d \setminus B_\rho} \big[f(z) - f(z_0 \circ z)\big] K(w)\,\mathrm{d}w \\
&\qquad\quad - \int_{\mathbb{R}^d \setminus B_\rho} \big[f\big(z \circ (0,0,w)\big) - f\big(z_0 \circ z \circ (0,0,w)\big)\big] K(w)\,\mathrm{d}w \\
&\qquad \le C\Lambda_0 \rho^{-2s} [f]_{C^\gamma_\ell} d_\ell(z, z_0 \circ z)^\alpha \\
&\qquad\quad + C[f]_{C^\gamma_\ell} \int_{\mathbb{R}^d \setminus B_\rho} d_\ell\big(z \circ (0,0,w), z_0 \circ z \circ (0,0,w)\big)^\gamma K(w)\,\mathrm{d}w,
\end{aligned}$$

since $\alpha \le \gamma$. But now we compute

$$\begin{aligned}
d_\ell(z \circ (0,0,w), z_0 \circ z \circ (0,0,w)) &= \big\|(0,0,w)^{-1} \circ z^{-1} \circ z_0^{-1} \circ z \circ (0,0,w)\big\| \\
&= d_\ell((z_0 \circ z)^{-1}, z) - (0, t_0 w, 0) \\
&\lesssim d_\ell(z, z_0 \circ z) + |t - t_0|^{\frac{1}{1+2s}} |w|^{\frac{1}{1+2s}} \\
&\lesssim d_\ell(z, z_0 \circ z)^{\frac{2s}{1+2s}} \big(1 + |w|^{\frac{1}{1+2s}}\big).
\end{aligned}$$

Therefore, since $\alpha \le \frac{2s}{1+2s}$ and since K satisfies (1.32) we find

$$C(z_0 \circ z) - C(z) \le C\Lambda_0 [f]_{C^\gamma_\ell} \rho^{-2s} d_\ell(z, z_0 \circ z)^\alpha.$$

This concludes the proof. □

4.2.4 Interpolation

We also have an interpolation inequality, see [117, Prop. 2.10]. Unlike the other preliminary results that we have stated in Subsection 4.2.3, the proof of the following proposition is verbatim the same as in [117, Prop. 2.10]. For the sake of self-containment we recall it in Appendix 4.9.

Proposition 4.1 (Imbert-Silvestre [117, Prop. 2.10])**.** *Given $\beta_1 < \beta_2 < \beta_3$ so that $\beta_2 = \theta\beta_1 + (1-\theta)\beta_3$, then for any $f \in C^{\beta_3}_\ell(Q_1)$ there holds*

$$[f]_{C^{\beta_2}_\ell(Q_1)} \le [f]^\theta_{C^{\beta_1}_\ell(Q_1)} [f]^{1-\theta}_{C^{\beta_3}_\ell(Q_1)} + [f]_{C^{\beta_1}_\ell(Q_1)}.$$

In particular for all $\varepsilon > 0$

$$[f]_{C^{\beta_2}_{\ell}(Q_1)} \leq C(\varepsilon)[f]_{C^{\beta_1}_{\ell}(Q_1)} + \varepsilon[f]_{C^{\beta_3}_{\ell}(Q_1)}.$$

4.3 Toolbox

Campanato's approach is a scaling argument, consisting of a clever combination of several tools that permit to gain as much regularity as can be gained from the equation. In short, we combine Poincaré's inequality with Sobolev embedding, and close the argument with regularity estimates. In this section we assemble the tools that are used in both the non-fractional and the fractional case.

4.3.1 Functional inequalities

We observe, similar to the elliptic case in [78], that for $f \in W^{m,p}(Q_R(z_0))$ there exists a unique polynomial $p_{m-1} = p_{m-1}(z_0, R, f, z)$ of degree less or equal to $m-1$ so that

$$⨍_{Q_R(z_0)} D^{\phi}(f - p_{m-1})\,\mathrm{d}z = 0 \qquad \forall \alpha \text{ with } |\Phi| \leq m-1. \tag{4.18}$$

Note that $m \in \mathbb{N} + 2s\mathbb{N}$ and D^{ϕ} is a kinetic differential whose order is in the discrete set $\mathbb{N} + 2s\mathbb{N}$ as well. Indeed, the polynomial is given by

$$p_{m-1}(z) = \sum_{\psi \in \mathbb{N}^{1+2d}, |\Psi| \leq m-1} \frac{c_{\psi}}{\psi!}(z - z_0)^{\psi}$$

with

$$c_{\psi} = \sum_{\substack{\phi \in \mathbb{N}^{1+2d}, \\ 2|\Phi| \leq m-1-|\Psi|}} c_{\psi,\phi} R^{-n+2|\phi|} \int_{Q_R(z_0)} D^{\psi+2\phi} f\,\mathrm{d}z,$$

where $n = 2s + 2d(s+1)$. Recall that for $\psi = (\psi_0, \Psi_1, \Psi_2) \in \mathbb{N}^{1+2d}$ we denote by $|\Psi|$ the size of ψ respecting the scaling, i.e. $|\Psi| = 2s \cdot \psi_0 + (1+2s)\,|\Psi_1| + |\Psi_2|$. Here $\psi!$ denotes the element-wise operation $\psi! = \psi_0!\psi_1! \cdots \psi_{2d}!$.

The idea is to use (4.18) in order to apply the standard Poincaré-inequality [79, Prop 3.12] to $D^{\phi}(f - p_{m-1})$ for $|\Phi| = 0, \ldots, m-1$. Moreover, we have for any non-negative function $f \in L^2(Q_r(z_0))$

$$\int_{Q_r(z_0)} f^2\,\mathrm{d}z \leq C r^n \left\|f\right\|^2_{L^{\infty}(Q_r(z_0))}, \tag{4.19}$$

where $r > 0$ and $n = 2s + 2d(s+1)$. Combined with Sobolev's embedding and regularity estimates, we obtain an estimate commonly referred to as Campanato's (first) inequality, which will be the first tool to tackle the Schauder estimates. For reference, in the elliptic case, Campanato's first inequality reads

$$\int_{B_r} |u|^2 \, \mathrm{d}x \leq C\Big(\frac{r}{R}\Big)^d \int_{B_R} |u|^2 \, \mathrm{d}x,$$

for a solution $u : \mathbb{R}^d \to \mathbb{R}$ of a second order elliptic equation, see [79, Section 5].

4.3.2 Regularity estimates

The second key step are regularity estimates for the constant coefficient equation. We consider solutions f of the constant coefficient Kolmogorov equation

$$\partial_t f + v \cdot \nabla_x f - A^0 \Delta_v f = h \tag{4.20}$$

in $Q_R(z_0)$ for some $z_0 \in \mathbb{R}^{1+2d}$ and $R > 0$. Here A^0 is some constant such that $A^0 \geq \lambda_0$ with λ_0 from (4.6). The fractional analogue reads

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L}_0 f = h, \tag{4.21}$$

where $\mathcal{L}_0$ is the non-local operator (1.14) associated to a non-negative, translation-invariant kernel K_0 such that

$$\frac{\lambda_0}{|w|^{d+2s}} \leq K_0(w) \leq \frac{\Lambda_0}{|w|^{d+2s}}, \tag{4.22}$$

and $K_0(w) = K_0(-w)$ is independent of z. We derive inductive regularity estimates relying on Bouchut's Proposition 4.4, which captures the regularising effect of the transport operator in the space variable. For the sake of brevity we will introduce the notation $|D|^\gamma := (-\Delta)^{\frac{\gamma}{2}}$ for any $\gamma \geq 0$.

Proposition 4.2 (Local (non-fractional) regularity estimates)**.** *Let f be a non-negative solution in $Q_R(z_0)$ of (4.20) with $s = 1$. Let $l \in \mathbb{N}_0$, $0 < r < R \leq 1$ and write $\delta := R - r > 0$. Then there holds*

$$\big\|D^{l+2} f\big\|_{L^2(Q_r(z_0))} \leq C\delta^{-2(l+2)}\Big(\|f\|_{L^2(Q_R(z_0))} + \big\|D^l h\big\|_{L^2(Q_R(z_0))} \Big),$$

where D^l is a pseudo-differential of order $l \geq 0$, and $C = C(n, \lambda_0)$. In particular, if $h = 0$, then

$$\Big\| |D_v|^{l+2} f \Big\|_{L^2(Q_r(z_0))} + \Big\| |D_t|^{\frac{l+2}{2}} f \Big\|_{L^2(Q_r(z_0))} + \Big\| |D_x|^{\frac{l+2}{3}} f \Big\|_{L^2(Q_r(z_0))} \lesssim \delta^{-2(l+2)} \|f\|_{L^2(Q_R(z_0))}.$$

For the fractional case, the right hand side involves a norm on the whole velocity space.

Proposition 4.3 (Non-local (fractional) regularity estimates)**.** *Let $\gamma > 0$ be such that $\gamma < \min(1, 2s)$. Suppose $f \in C^{\gamma}_{\ell}(Q^v_R(z_0) \times \mathbb{R}^d)$ is a non-negative solution in $Q_R(z_0)$ of (4.21) with $s \in (0,1)$. Let $l \in \mathbb{N}_0$, $0 < r < R \leq 1$ and write $\delta = R - r > 0$. Then there holds*

$$\left\| D^{(l+2)s} f \right\|_{L^2(Q_r(z_0))} \leq C\delta^{-2(l+2)} \Big(\|f\|_{C^{\gamma}_{\ell}(Q^v_R(z_0)\times\mathbb{R}^d)} + \left\| D^{ls} h \right\|_{L^2(Q_R(z_0))} \Big), \tag{4.23}$$

where D^{ls} is a pseudo-differential of order $ls \geq 0$ and $C = C(n, s, \Lambda_0, \lambda_0)$.

Remark 4.6. i. The proof of Proposition 4.2 is similar to the proof of Proposition 4.3. In fact, for Step 1 in the proof Proposition 4.3, we can just set $s = 1$ and obtain the global version of the energy estimate for the non-fractional case. Steps 2, 3 and 4 are much simpler for the non-fractional case: it suffices to localise with some smooth cut-off $\theta \in C^{\infty}_c(Q_R(z_0))$, and then consider the equation solved by $g := f\theta$. Since the equation solved by f is non-fractional, g solves an equation with a right hand side that can be bounded by $\|f\|_{L^2(Q_R(z_0))}$ using the induction hypothesis. Since this case is comparatively simpler, we will focus on the proof of Proposition 4.3.

ii. With slightly more work, we would possibly also be able to deduce a similar result for a general diffusion coefficient that is uniformly elliptic and satisfies $D^l A \in L^2(Q_R(z_0))$ with $l \in \mathbb{N}_0$ as in the statement. For our purposes, the constant coefficient case suffices.

The proof builds upon the work of Alexandre and Bouchut [1, 29]. In particular, we will make use of the following proposition[29, Proposition 1.1].

Proposition 4.4 (Bouchut)**.** *Assume that $f, S \in L^2(\mathbb{R}^{1+2d})$ satisfy*

$$\partial_t f + v \cdot \nabla_x f = S,$$

and $|D_v|^{\beta} f \in L^2(\mathbb{R}^{1+2d})$ for some $\beta \geq 0$. Then $|D_x|^{\frac{\beta}{1+\beta}} f \in L^2(\mathbb{R}^{1+2d})$, and

$$\left\| |D_x|^{\frac{\beta}{1+\beta}} f \right\|_{L^2(\mathbb{R}^{1+2d})} \leq C \left\| |D_v|^{\beta} f \right\|^{\frac{1}{1+\beta}}_{L^2(\mathbb{R}^{1+2d})} \|S\|^{\frac{\beta}{1+\beta}}_{L^2(\mathbb{R}^{1+2d})}, \tag{4.24}$$

for some universal constant $C > 0$.

We recall the proof of Proposition 4.4 in Appendix 4.10.

Proof of Proposition 4.3. With no loss of generality, we assume $A^0 = 1$ and $K_0(w) = \frac{1}{|w|^{d+2s}}$ (otherwise we can either perform a constant change of variable or just use the pointwise bounds on the kernel). We start with global estimates, and then we localise the result.

Step 1: Global estimate. Assume for now that f solves (4.21) on $\mathbb{R}^{1+2d}$ with a source term $h \in L^2(\mathbb{R}^{1+2d})$, that is

$$\mathcal{T} f + |D_v|^{2s} f = h. \tag{4.25}$$

To prove the global statement (4.23) in its full generality, we will need to assume that $|D_v|^{ls} h, |D_x|^{\frac{ls}{1+2s}} h, |D_t|^{\frac{ls}{2s}} h \in L^2(\mathbb{R}^{1+2d})$.

First note that testing (4.25) with $\bar{f}$ yields

$$\| |D_v|^s f \|^2_{L^2(\mathbb{R}^{1+2d})} \leq \|h\|_{L^2(\mathbb{R}^{1+2d})} \|f\|_{L^2(\mathbb{R}^{1+2d})}.$$

Second, we note that any solution f of (4.25) satisfies

$$\mathcal{T}\big(|D_x|^{\frac{ls}{1+2s}} f\big) = - |D_v|^{2s} |D_x|^{\frac{ls}{1+2s}} f + |D_x|^{\frac{ls}{1+2s}} h.$$

Then Bouchut's Proposition 4.4 applied to $|D_x|^{\frac{ls}{1+2s}} f$ yields for $\beta = 2s \geq 0$

$$\begin{aligned}\left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2(\mathbb{R}^{1+2d})} \lesssim{}& \left\| |D_v|^{2s} |D_x|^{\frac{ls}{1+2s}} f \right\|_{L^2(\mathbb{R}^{1+2d})} \\ &+ \left\| |D_v|^{2s} |D_x|^{\frac{ls}{1+2s}} f \right\|^{\frac{1}{1+2s}}_{L^2(\mathbb{R}^{1+2d})} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|^{\frac{2s}{1+2s}}_{L^2(\mathbb{R}^{1+2d})}.\end{aligned}$$

Now we use Hölder's inequality in Fourier variables (k, ξ) of (x, v) to bound

$$\begin{aligned}\left\| |D_v|^{2s} |D_x|^{\frac{ls}{1+2s}} f \right\|_{L^2(\mathbb{R}^{1+2d})} &= \left\| |D_x|^{\frac{\theta\cdot(l+2)s}{1+2s}} |D_v|^{(1-\theta)\cdot(l+2)s} f \right\|_{L^2(\mathbb{R}^{1+2d})} \\ &\leq \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|^{\theta}_{L^2(\mathbb{R}^{1+2d})} \left\| |D_v|^{(l+2)s} f \right\|^{1-\theta}_{L^2(\mathbb{R}^{1+2d})},\end{aligned}$$

where $\theta = \frac{l}{l+2}$. Thus

$$\begin{aligned}\left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2} \lesssim{}& \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|^{\theta}_{L^2} \left\| |D_v|^{(l+2)s} f \right\|^{1-\theta}_{L^2} \\ &+ \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|^{\frac{\theta}{1+2s}}_{L^2} \left\| |D_v|^{(l+2)s} f \right\|^{\frac{1-\theta}{1+2s}}_{L^2} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|^{\frac{2s}{1+2s}}_{L^2},\end{aligned}$$

from which we deduce by dividing by $\left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|^{\frac{\theta}{1+2s}}$ and using Young's inequality for some $\varepsilon \in (0,1)$

$$\begin{aligned}&\left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2} \\ &\lesssim \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|^{\frac{2s\theta}{1+2s-\theta}}_{L^2} \left\| |D_v|^{(l+2)s} f \right\|^{\frac{(1-\theta)(1+2s)}{1+2s-\theta}}_{L^2} + \left\| |D_v|^{(l+2)s} f \right\|^{\frac{1-\theta}{1+2s-\theta}}_{L^2} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|^{\frac{2s}{1+2s-\theta}}_{L^2} \\ &\leq \varepsilon^{\frac{1+2s-\theta}{2s\theta}} \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2} + C_\varepsilon \left\| |D_v|^{(l+2)s} f \right\|_{L^2} + \left\| |D_v|^{(l+2)s} f \right\|^{\frac{1-\theta}{1+2s-\theta}}_{L^2} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|^{\frac{2s}{1+2s-\theta}}_{L^2}.\end{aligned}$$

Finally, absorbing the first term on the right hand side to the left hand side and using $\theta = \frac{l}{l+2}$ we have

$$\left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2} \lesssim \left\| |D_v|^{(l+2)s} f \right\|_{L^2} + \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^{\frac{1}{1+sl+2s}} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{s(l+2)}{1+2s+sl}}. \tag{4.26}$$

Third, we test (4.25) with $|D_x|^{\frac{(l+1)2s}{1+2s}} f$. Then

$$\left\| |D_v|^s D_x^{\frac{(l+1)s}{1+2s}} f \right\|_{L^2} \leq \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{1}{2}} \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2}^{\frac{1}{2}}. \tag{4.27}$$

Finally, we test (4.25) with

$$\left(\delta + |D_v|^{2(l+1)} + |D_x|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |D_v|^{2j} |D_x|^{\frac{2l+2-2j}{1+2s}} \right)^s \bar{f} + |D_t|^l \partial_t \bar{f}$$

for some small $\delta \in (0,1)$. We get

$$\begin{aligned}
&\int \left\{ \left(\delta + |D_v|^{2(l+1)} + |D_x|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |D_v|^{2j} |D_x|^{\frac{2l+2-2j}{1+2s}} \right)^s + |D_t|^l \partial_t \right\} \bar{f} \\
&\qquad \times \left(|D_v|^{2s} f + \partial_t f \right) \mathrm{d}z \\
&= -\int \left\{ \left(\delta + |D_v|^{2(l+1)} + |D_x|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |D_v|^{2j} |D_x|^{\frac{2l+2-2j}{1+2s}} \right)^s + |D_t|^l \partial_t \right\} \bar{f} v \cdot \nabla_x f \, \mathrm{d}z \\
&\quad + \int \left\{ \left(\delta + |D_v|^{2(l+1)} + |D_x|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |D_v|^{2j} |D_x|^{\frac{2l+2-2j}{1+2s}} \right)^s + |D_t|^l \partial_t \right\} \bar{f} h \, \mathrm{d}z \\
&=: I_1 + I_2.
\end{aligned} \tag{4.28}$$

For the left hand side we find

$$\begin{aligned}
&\int \left\{ \left(\delta + |D_v|^{2(l+1)} + |D_x|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |D_v|^{2j} |D_x|^{\frac{2l+2-2j}{1+2s}} \right)^s + |D_t|^l \partial_t \right\} \bar{f} \cdot \left(|D_v|^{2s} f + \partial_t f \right) \mathrm{d}z \\
&\gtrsim \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^2 + \left\| |D_t|^{\frac{l}{2}} \partial_t f \right\|_{L^2}^2 \\
&\qquad + \sum_{j=1}^{l} \left\| |D_v|^{(j+1)s} |D_x|^{\frac{(l+1-j)s}{1+2s}} f \right\|_{L^2}^2 + \left\| |D_v|^s |D_x|^{\frac{(l+1)s}{1+2s}} f \right\|_{L^2}^2.
\end{aligned}$$

On the other hand we get with (4.26)

$$
\begin{aligned}
I_2 &\lesssim \|f\|_{L^2}\,\|h\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_v|^{ls} h\right\|_{L^2} + \left\||D_x|^{\frac{(l+2)s}{1+2s}} f\right\|_{L^2}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2} \\
&\quad + \left\||D_t|^{\frac{ls}{2s}}\,\partial_t f\right\|_{L^2}\left\||D_t|^{\frac{ls}{2s}} h\right\|_{L^2} \\
&\quad + \sum_{j=1}^{l}\left\||D_v|^{(j+1)s}\,|D_x|^{\frac{(l+1-j)s}{1+2s}} f\right\|_{L^2}\left\||D_v|^{(j-1)s}\,|D_x|^{\frac{(l+1-j)s}{1+2s}} h\right\|_{L^2} \\
&\lesssim \|f\|_{L^2}\,\|h\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_v|^{ls} h\right\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2} \\
&\quad + \left\||D_v|^{(l+2)s} f\right\|_{L^2}^{\frac{1}{1+2s+sl}}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2}^{1+\frac{(l+2)s}{1+2s+sl}} + \left\||D_t|^{\frac{ls}{2s}}\,\partial_t f\right\|_{L^2}\left\||D_t|^{\frac{ls}{2s}} h\right\|_{L^2} \\
&\quad + \sum_{j=1}^{l}\left\||D_v|^{(j+1)s}\,|D_x|^{\frac{(l+1-j)s}{1+2s}} f\right\|_{L^2}\left\||D_v|^{(j-1)s}\,|D_x|^{\frac{(l+1-j)s}{1+2s}} h\right\|_{L^2} \\
&\lesssim \|f\|_{L^2}\,\|h\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_v|^{ls} h\right\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2} \\
&\quad + \left\||D_v|^{(l+2)s} f\right\|_{L^2}^{\frac{1}{1+2s+ls}}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2}^{1+\frac{(l+2)s}{1+2s+sl}} + \left\||D_t|^{\frac{ls}{2s}}\,\partial_t f\right\|_{L^2}\left\||D_t|^{\frac{ls}{2s}} h\right\|_{L^2} \\
&\quad + \sum_{j=1}^{l}\left\||D_v|^{(j+1)s}\,|D_x|^{\frac{(l+1-j)s}{1+2s}} f\right\|_{L^2}\left\||D_v|^{ls} h\right\|_{L^2}^{\frac{j-1}{l}}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2}^{\frac{l+1-j}{l}} \\
&\lesssim \|f\|_{L^2}\,\|h\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_v|^{ls} h\right\|_{L^2} + \left\||D_v|^{(l+2)s} f\right\|_{L^2}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2} \\
&\quad + \left\||D_v|^{(l+2)s} f\right\|_{L^2}^{\frac{1}{1+2s+ls}}\left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2}^{1+\frac{(l+2)s}{1+2s+ls}} + \left\||D_t|^{\frac{ls}{2s}}\,\partial_t f\right\|_{L^2}\left\||D_t|^{\frac{ls}{2s}} h\right\|_{L^2} \\
&\quad + \left(\left\||D_v|^{ls} h\right\|_{L^2} + \left\||D_x|^{\frac{ls}{1+2s}} h\right\|_{L^2}\right)\sum_{j=1}^{l}\left\||D_v|^{(j+1)s}\,|D_x|^{\frac{(l+1-j)s}{1+2s}} f\right\|_{L^2},
\end{aligned}
\tag{4.29}
$$

where in the second last inequality we again used Hölder in Fourier and Young's inequality. Note that the last sum can be absorbed on the left hand side of (4.28) eventually.

For I_1 we Fourier-transform $(t,x,v)\to(\eta,k,\xi)$ so that we get

$$\begin{aligned}
I_1 &= -\mathrm{Re}\Big\langle \Big\{ \Big(\delta + |D_v|^{2(l+1)} + |D_x|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |D_v|^{2j}\, |D_x|^{\frac{2l+2-2j}{1+2s}} \Big)^s + |D_t|^l\, \partial_t \Big\} \bar{f}, v\cdot\nabla_x f \Big\rangle \\
&= -\mathrm{Re}\Big\langle \Big\{ \Big(\hat{\delta} + |\xi|^{2(l+1)} + |k|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |\xi|^{2j}\, |k|^{\frac{2l+2-2j}{1+2s}} \Big)^s + |\eta|^{l+1} \Big\} \bar{\hat{f}}, k_i \partial_{\xi_i} \hat{f} \Big\rangle \\
&= 2s\ \mathrm{Re}\Bigg\langle \Big(\hat{\delta} + |\xi|^{2(l+1)} + |k|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |\xi|^{2j}\, |k|^{\frac{2l+2-2j}{1+2s}} \Big)^{s-1} \\
&\qquad \times \Big((l+1)\,|\xi|^{2l} + \sum_{j=1}^{l} j\,|k|^{\frac{2l+2-2j}{1+2s}}\,|\xi|^{2j-2}\Big)\xi_i \bar{\hat{f}}, k_i \hat{f} \Bigg\rangle \\
&\quad + \mathrm{Re}\Big\langle \Big\{ \Big(\hat{\delta} + |\xi|^{2(l+1)} + |k|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |\xi|^{2j}\, |k|^{\frac{2l+2-2j}{1+2s}} \Big)^s + |\eta|^{l+1} \Big\} \partial_{\xi_i} \bar{\hat{f}}, k_i \hat{f} \Big\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
I_1 &= s\ \mathrm{Re}\Bigg\langle \Big(\hat{\delta} + |\xi|^{2(l+1)} + |k|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |\xi|^{2j}\, |k|^{\frac{2l+2-2j}{1+2s}} \Big)^{s-1} \\
&\qquad \times \Big((l+1)\,|\xi|^{2l} + \sum_{j=1}^{l} j\,|k|^{\frac{2l+2-2j}{1+2s}}\,|\xi|^{2j-2}\Big)\xi_i \bar{\hat{f}}, k_i \hat{f} \Bigg\rangle \\
&\lesssim \int \bar{\hat{f}}\hat{f}\cdot \Big(\,|k|^{\frac{2(l+1)}{1+2s}} + \sum_{j=1}^{l} |\xi|^{2j}\,|k|^{\frac{2l+2-2j}{1+2s}}\Big)^{s-1} \Big(\,|\xi|^{2l} + \sum_{j=1}^{l} |k|^{\frac{2l+2-2j}{1+2s}}\,|\xi|^{2j-2}\Big)\,|\xi|\,|k|\ \mathrm{d}z.
\end{aligned}$$

We claim that we can bound

$$I_1 \lesssim \int \bar{\hat{f}}\hat{f}\cdot \sum_{j=1}^{l} |\xi|^{2(j-1)s+s}\,|k|^{\frac{2ls+3s-2(j-1)s}{1+2s}}\ \mathrm{d}z + \int \bar{\hat{f}}\hat{f}\cdot |\xi|^{2ls+s}\,|k|^{\frac{3s}{1+2s}}\ \mathrm{d}z. \tag{4.30}$$

Indeed, if $|\xi|\sim|k|$ then one can check that the homogeneity is kept. Else assume first that $|\xi|\ll|k|$. Then we have

$$I_1 \lesssim \int \bar{\hat{f}}\hat{f}\cdot|\xi| \sum_{j=1}^{l} |k|^{\frac{2l+2-2j}{1+2s}}\,|\xi|^{2j-2}\,|k|^{\frac{2(l+1)(s-1)}{1+2s}+1}\ \mathrm{d}z = \int \bar{\hat{f}}\hat{f}\cdot \sum_{j=1}^{l} |k|^{\frac{2ls-2j+4s+1}{1+2s}}\,|\xi|^{2j-1}\ \mathrm{d}z.$$

Comparing the exponents of $|\xi|$ and $|k|$ gives $2l$ conditions that need to be satisfied,

$$2j-1 \geq 2(j-1)s+s, \qquad 2ls+4s-2j+1 \leq 2ls-2(j-1)s+3s, \quad \forall j\in\{1,\dots,l\},$$

which holds since $s \leq 1$. Now assume on the other hand that $|k| \ll |\xi|$. Then we have

$$I_1 \lesssim \int \bar{\hat{f}} \hat{f} \cdot |\xi|^{2l+1+2(l+1)(s-1)} |k| \, \mathrm{d}z = \int \bar{\hat{f}} \hat{f} \cdot |\xi|^{2ls-1+2s} |k| \, \mathrm{d}z.$$

Thus we need

$$2ls + 2s - 1 \leq 2ls + s, \qquad 1 \geq \frac{3s}{1+2s},$$

which both clearly holds for $s \leq 1$.

Thus we estimate

$$\begin{aligned} I_1 &\lesssim \int \bar{\hat{f}} \hat{f} \cdot \sum_{j=1}^{l} |\xi|^{2(j-1)s+s} |k|^{\frac{2ls+3s-2(j-1)s}{1+2s}} \, \mathrm{d}z + \int \bar{\hat{f}} \hat{f} \cdot |\xi|^{2ls+s} |k|^{\frac{3s}{1+2s}} \, \mathrm{d}z \\ &\lesssim \left\| |D_v|^{ls} |D_x|^{\frac{2s}{1+2s}} f \right\|_{L^2} \left\| |D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} f \right\|_{L^2} \\ &\quad + \sum_{j=1}^{l} \left\| |D_v|^{js} |D_x|^{\frac{ls+2s-js}{1+2s}} f \right\|_{L^2} \left\| |D_v|^{(j-1)s} |D_x|^{\frac{ls+3s-js}{1+2s}} f \right\|_{L^2}. \end{aligned} \tag{4.31}$$

Now we use Hölder in Fourier variables again. Eventually we want to use (4.27) and (4.26) in order to get a right hand side in terms of our source term. Thus, for each $j \in \{1, \dots, l\}$ we will look for parameters $\theta_j \in (0,1)$ such that we can express the right hand side of (4.31) in terms of

$$\begin{aligned} &\left\| \left(|D_v|^{s} |D_x|^{\frac{(l+1)s}{1+2s}} \right)^{1-\theta_j} \left(|D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} \right)^{\theta_j} f \right\|_{L^2} \\ &\qquad \leq \left\| |D_v|^{s} |D_x|^{\frac{(l+1)s}{1+2s}} f \right\|_{L^2}^{1-\theta_j} \left\| |D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} f \right\|_{L^2}^{\theta_j}. \end{aligned}$$

Indeed, then we get with (4.27) and (4.26)

$$\begin{aligned} &\left\| \left(|D_v|^{s} |D_x|^{\frac{(l+1)s}{1+2s}} \right)^{1-\theta_j} \left(|D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} \right)^{\theta_j} f \right\|_{L^2} \\ &\leq \left\| |D_v|^{s} |D_x|^{\frac{(l+1)s}{1+2s}} f \right\|_{L^2}^{1-\theta_j} \left\| |D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} f \right\|_{L^2}^{\theta_j} \\ &\lesssim \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2}^{\frac{1-\theta_j}{2}} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{1-\theta_j}{2}} \left\| |D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} f \right\|_{L^2}^{\theta_j} \\ &\lesssim \left(\left\| |D_v|^{(l+2)s} f \right\|_{L^2} + \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^{\frac{1}{1+(l+2)s}} \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{(l+2)s}{1+(l+2)s}} \right)^{\frac{1-\theta_j}{2}} \\ &\qquad \times \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{1-\theta_j}{2}} \left\| |D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}} f \right\|_{L^2}^{\theta_j}. \end{aligned} \tag{4.32}$$

We start by using (4.32) for the first term on the right hand side of (4.31). We can

write

$$\Big\| |D_v|^{ls} \, |D_x|^{\frac{2s}{1+2s}} \, f \Big\|_{L^2} = \Big\| \big(\, |D_v|^{s} \, |D_x|^{\frac{(l+1)s}{1+2s}} \, \big)^{1-\theta_{l+1}} \big(\, |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, \big)^{\theta_{l+1}} f \Big\|_{L^2}$$

for $\theta_{l+1} = \frac{l-1}{l}$. Thus with Young's inequality

$$\begin{aligned}
&\Big\| |D_v|^{ls} \, |D_x|^{\frac{2s}{1+2s}} \, f \Big\|_{L^2} \Big\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \Big\|_{L^2} \\
&\lesssim \Big(\Big\| |D_v|^{(l+2)s} \, f \Big\|_{L^2} + \Big\| |D_v|^{(l+2)s} \, f \Big\|_{L^2}^{\frac{1}{1+(l+2)s}} \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{\frac{(l+2)s}{1+(l+2)s}} \Big)^{\frac{1-\theta_{l+1}}{2}} \\
&\quad \times \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{\frac{1-\theta_{l+1}}{2}} \Big\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \Big\|_{L^2}^{\theta_{l+1}+1} \\
&\lesssim \Big\| |D_v|^{(l+2)s} \, f \Big\|_{L^2}^{\frac{4}{3}} \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{\frac{2}{3}} + \Big\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \Big\|_{L^2}^{\frac{8(\theta_{l+1}+1)}{5+3\theta_l}} \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{\frac{2(1-\theta_{l+1})}{5+3\theta_{l+1}}} \\
&\quad + \Big\| |D_v|^{(l+2)s} \, f \Big\|_{L^2}^{\frac{2}{1+(l+2)s}} \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{\frac{2(l+2)s}{1+(l+2)s}} \\
&\quad + \Big\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \Big\|_{L^2}^{\frac{4(\theta_{l+1}+1)}{3+\theta_{l+1}}} \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{\frac{2(1-\theta_{l+1})}{3+\theta_{l+1}}} \\
&\lesssim \varepsilon \Big\| |D_v|^{(l+2)s} \, f \Big\|_{L^2}^{2} + \varepsilon \Big\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \Big\|_{L^2}^{2} + C_\varepsilon \Big\| |D_x|^{\frac{ls}{1+2s}} \, h \Big\|_{L^2}^{2},
\end{aligned}$$

for $\varepsilon \in (0,1)$.

Similarly we can proceed with the remaining terms on the right hand side of (4.31). Note that for each $j \in \{1, \dots, l\}$ we can absorb the term $\Big\| |D_v|^{(j+1)s} \, |D_x|^{\frac{ls+s-js}{1+2s}} \, f \Big\|_{L^2}$ on the left hand side. Thus we write for each $j \in \{2, \dots, l\}$

$$\Big\| |D_v|^{(j-1)s} \, |D_x|^{\frac{ls+3s-js}{1+2s}} \, f \Big\|_{L^2} = \Big\| \big(\, |D_v|^{s} \, |D_x|^{\frac{(l+1)s}{1+2s}} \, \big)^{1-\theta_j} \big(\, |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, \big)^{\theta_j} f \Big\|_{L^2},$$

where $\theta_j = \frac{j-2}{l}$. Then by (4.32) and Young's inequality

$$
\begin{aligned}
&\sum_{j=2}^{l} \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2} \left\| |D_v|^{(j-1)s} \, |D_x|^{\frac{ls+3s-js}{1+2s}} \, f \right\|_{L^2} \\
&\quad \lesssim \sum_{j=2}^{l} \Big(\left\| |D_v|^{(l+2)s} \, f \right\|_{L^2} + \left\| |D_v|^{(l+2)s} \, f \right\|_{L^2}^{\frac{1}{1+(l+2)s}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{(l+2)s}{1+(l+2)s}} \Big)^{\frac{1-\theta_j}{2}} \\
&\qquad \times \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{1-\theta_j}{2}} \left\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \right\|_{L^2}^{\theta_j} \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2} \\
&\quad \lesssim \left\| |D_v|^{(l+2)s} \, f \right\|_{L^2}^{\frac{4}{3}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2}{3}} + \left\| |D_v|^{(l+2)s} \, f \right\|_{L^2}^{\frac{2}{1+(l+2)s}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(l+2)s}{1+(l+2)s}} \\
&\qquad + \sum_{j=2}^{l} \Bigg[\left\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \right\|_{L^2}^{\frac{8\theta_j}{5+3\theta_j}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(1-\theta_j)}{5+3\theta_j}} \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2}^{\frac{8}{5+3\theta_j}} \\
&\qquad + \left\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \right\|_{L^2}^{\frac{4\theta_j}{3+\theta_j}} \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2}^{\frac{4}{3+\theta_j}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(1-\theta_j)}{3+\theta_j}} \Bigg] \\
&\quad \lesssim \left\| |D_v|^{(l+2)s} \, f \right\|_{L^2}^{\frac{4}{3}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2}{3}} + \left\| |D_v|^{(l+2)s} \, f \right\|_{L^2}^{\frac{2}{1+(l+2)s}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(l+2)s}{1+(l+2)s}} \\
&\qquad + \sum_{j=2}^{l} \Bigg[\left\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \right\|_{L^2}^{\frac{16\theta_j}{1+7\theta_j}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(1-\theta_j)}{1+7\theta_j}} \\
&\qquad + \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(1-\theta_j)}{9-\theta_j}} \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2}^{\frac{16}{9-\theta_j}} \\
&\qquad + \left\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \right\|_{L^2}^{\frac{8\theta_j}{1+3\theta_j}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(1-\theta_j)}{1+3\theta_j}} \\
&\qquad + \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2}^{\frac{8}{5-\theta_j}} \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{\frac{2(1-\theta_j)}{5-\theta_j}} \Bigg] \\
&\quad \lesssim \varepsilon \left\| |D_v|^{(l+2)s} \, f \right\|_{L^2}^{2} + \varepsilon \left\| |D_v|^{(l+1)s} \, |D_x|^{\frac{s}{1+2s}} \, f \right\|_{L^2}^{2} + \varepsilon \sum_{j=2}^{l} \left\| |D_v|^{js} \, |D_x|^{\frac{ls+2s-js}{1+2s}} \, f \right\|_{L^2}^{2} \\
&\qquad + C_\varepsilon \left\| |D_x|^{\frac{ls}{1+2s}} \, h \right\|_{L^2}^{2},
\end{aligned}
$$

for some $\varepsilon \in (0,1)$.

Finally, the only remaining term is when $j = 1$ in (4.31), which we estimate using

(4.27) and (4.26)

$$\begin{aligned}\left\| |D_v|^s |D_x|^{\frac{(l+1)s}{1+2s}} f \right\|_{L^2} \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2} &\lesssim \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{1}{2}} \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2}^{\frac{3}{2}} \\ &\lesssim \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{1}{2}} \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^{\frac{3}{2}} \\ &\quad + \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^{\frac{1}{2}+\frac{3(l+2)s}{2(1+(l+2)s)}} \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^{\frac{3}{2(1+(l+2)s)}} \\ &\lesssim \varepsilon \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^2 + C_\varepsilon \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^2 .\end{aligned}$$

Therefore, we have shown

$$\begin{aligned}I_1 \lesssim & \varepsilon \left\| |D_v|^{(l+2)s} f \right\|_{L^2}^2 + \varepsilon \left\| |D_v|^{(l+1)s} |D_x|^{\frac{s}{1+2s}}_{L^2} f \right\|^2 \\ & + \varepsilon \sum_{j=2}^{l} \left\| |D_v|^{js} |D_x|^{\frac{ls+2s-js}{1+2s}} f \right\|_{L^2}^2 + C_\varepsilon \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^2 .\end{aligned} \tag{4.33}$$

We combine (4.28), (4.29) and (4.33) to get

$$\begin{aligned}&\left\| |D_v|^{(l+2)s} f \right\|_{L^2}^2 + \left\| |D_t|^{\frac{ls}{2s}} \partial_t f \right\|_{L^2}^2 + \sum_{j=1}^{l} \left\| |D_v|^{(j+1)s} |D_x|^{\frac{ls+s-js}{1+2s}} f \right\|_{L^2}^2 + \left\| |D_v|^s |D_x|^{\frac{ls+s}{1+2s}} f \right\|_{L^2}^2 \\ &\qquad \lesssim \|f\|_{L^2}^2 + \|h\|_{L^2}^2 + \left\| |D_v|^{ls} h \right\|_{L^2}^2 + \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^2 + \left\| |D_t|^{\frac{ls}{2s}} h \right\|_{L^2}^2 .\end{aligned}$$

Thus, by (4.26) we have

$$\left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 + \|h\|_{L^2}^2 + \left\| |D_v|^{ls} h \right\|_{L^2}^2 + \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^2 + \left\| |D_t|^{\frac{ls}{2s}} h \right\|_{L^2}^2 .$$

We conclude

$$\begin{aligned}&\left\| |D_v|^{(l+2)s} f \right\|_{L^2}^2 + \left\| |D_t|^{\frac{ls}{2s}} \partial_t f \right\|_{L^2}^2 + \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2}^2 \\ &\qquad \lesssim \|f\|_{L^2}^2 + \|h\|_{L^2}^2 + \left\| |D_v|^{ls} h \right\|_{L^2}^2 + \left\| |D_x|^{\frac{ls}{1+2s}} h \right\|_{L^2}^2 + \left\| |D_t|^{\frac{ls}{2s}} h \right\|_{L^2}^2 .\end{aligned}$$

Step 2: Local estimates. Let $0 < r < R$ and let $\delta = R - r > 0$ be from the statement of the theorem. With no loss in generality set $z_0 = (0,0,0)$ and assume f solves (4.21) in $Q_R(z_0)$. We introduce a smooth function $\theta = \theta(t,x,v) \in C_c^\infty(\mathbb{R}^{1+2d})$ such that $\theta = 1$ in Q_r and $\theta = 0$ outside Q_R, and so that $|D_v|\,\theta \lesssim \delta^{-1}, |D_x|^{\frac{1}{1+2s}}\,\theta \lesssim \delta^{-1}, |D_t|^{\frac{1}{2s}}\,\theta \lesssim \delta^{-1}$. Then we let

$$g = f\theta,$$

so that g satisfies

$$\begin{aligned}\mathcal{T}g + |D_v|^{2s} g &= f\big(\mathcal{T}\theta\big) + |D_v|^{2s} g - \big(\,|D_v|^{2s} f\big)\theta \\ &= f\big(\mathcal{T}\theta\big) + h\theta + h_1 + h_{22} + |D_v|^{s} h_{23} =: H\end{aligned} \tag{4.34}$$

in $\mathbb{R}^{1+2d}$, where h_1, h_{22}, h_{23} are as in the commutator estimates of Lemma 2.3.

Step 3-(i): The base case. We start with $l = 0$. The global case for $l = 0$ gives

$$\big\||D_v|^{2s} g\big\|_{L^2} + \|\partial_t g\|_{L^2} + \Big\||D_x|^{\frac{2s}{1+2s}} g\Big\|_{L^2} + \Big\||D_v|^{s} |D_x|^{\frac{ls+s}{1+2s}} g\Big\|_{L^2}^2 \lesssim \|H\|_{L^2}\,.$$

Therefore it remains to bound the right hand side. We have by the standard energy estimate (see [99, Proposition 9] for $s = 1$ and Proposition 3.2 for the fractional case $s \in (0,1)$)

$$\||D_v|^{s} f\|_{L^2(Q_r)} \lesssim \|h\|_{L^2(Q_R)} + \delta^{-1} \|f\|_{L^2_{t,x} L^\infty_v (Q^v_R \times \mathbb{R}^d)}\,.$$

Moreover, we see

$$\|f\mathcal{T}\theta\|_{L^2} \lesssim \delta^{-2} \|f\|_{L^2(Q_R)}\,.$$

The remaining part is a commutator of the form

$$|D_v|^{2s} g - \big(\,|D_v|^{2s} f\big)\theta = \int_{\mathbb{R}^d} f(w) \frac{\big(\theta(v) - \theta(w)\big)}{|v-w|^{d+2s}}\,\mathrm{d}w.$$

Just as in Lemma 2.3 we write

$$h_1 = \int_{\mathbb{R}^d \setminus B_r(v)} f(w) \frac{\big(\theta(v) - \theta(w)\big)}{|v-w|^{d+2s}}\,\mathrm{d}w,$$

and

$$h_2 = \int_{B_r(v)} f(w) \frac{\big(\theta(v) - \theta(w)\big)}{|v-w|^{d+2s}}\,\mathrm{d}w.$$

Then we get for any $v \in \mathbb{R}^d$

$$\|h_1\|_{L^2} \lesssim \delta^{-2} \|f\|_{L^2_{t,x} L^\infty_v (Q^v_R \times \mathbb{R}^d)}\,.$$

Moreover, as the proof of Lemma 2.3 shows, we can write $h_2 = h_{22} + |D_v|^{s} h_{23}$ for some h_{22}, h_{23} that satisfy

$$\|h_{22}\|_{L^2} \lesssim \delta^{-2} \|f\|_{L^2_{t,x} H^s_v (Q_R)}\,,$$

and

$$\|h_{23}\|_{L^2} \lesssim \delta^{-2} \|f\|_{L^2_{t,x,v} (Q_R)}\,.$$

Thus

$$\begin{aligned}\|H\|_{L^2} &\lesssim \delta^{-2}\Big(\|f\|_{L^2_{t,x}H^s_v(Q_R)} + \|f\|_{L^2_{t,x}L^\infty_v(Q^v_R\times\mathbb{R}^d)} \Big) + \|h\|_{L^2(Q_R)} \\ &\lesssim \delta^{-3}\,\|f\|_{L^2_{t,x}L^\infty_v(Q^v_R\times\mathbb{R}^d)} + \|h\|_{L^2(Q_R)}\,.\end{aligned}$$

Finally, since $g = f$ in Q_r we conclude

$$\begin{aligned}\|\partial_t f\|_{L^2(Q_r)} + \left\||D_v|^{2s} f\right\|_{L^2(Q_r)} + \left\||D_x|^{\frac{2s}{1+2s}} f\right\|_{L^2(Q_r)} + \left\||D_v|^s\,|D_x|^{\frac{ls+s}{1+2s}} f\right\| \\ \lesssim \delta^{-3}\,\|f\|_{L^2_{t,x}L^\infty_v(Q^v_R\times\mathbb{R}^d)} + \|h\|_{L^2(Q_R)}\,.\end{aligned}$$

Step 3-(ii): The general case. Now let $l \in \mathbb{N}_0$. We proceed by induction. Let $l \geq 1$ and assume the conclusion holds for $l-1$, that is we have

$$\begin{aligned}&\left\||D_t|^{\frac{(l+1)s}{2s}} f\right\|_{L^2(Q_r)} + \left\||D_v|^{(l+1)s} f\right\|_{L^2(Q_r)} \\ &\quad + \left\||D_x|^{\frac{(l+1)s}{1+2s}} f\right\|_{L^2(Q_r)} + \sum_{j=0}^{l-1} \left\||D_v|^{(j+1)s}\,|D_x|^{\frac{ls+s-js}{1+2s}} f\right\|^2_{L^2(Q_r)} \\ &\lesssim \delta^{-2(l+1)}\,\|f\|_{C^\gamma_\ell(Q^v_R\times\mathbb{R}^d)} \\ &\qquad + \delta^{-(l+1)s}\Big(\left\||D_v|^{(l-1)s} h\right\|_{L^2(Q_R)} + \left\||D_x|^{\frac{(l-1)s}{1+2s}} h\right\|_{L^2(Q_R)} + \left\||D_t|^{\frac{l-1}{2}} h\right\|_{L^2(Q_R)} \Big).\end{aligned}$$

where $\delta = R - r$.

The same procedure as for f on $\mathbb{R}^{1+2d}$ in Step 1 will yield

$$\begin{aligned}&\left\||D_v|^{(l+2)s} g\right\|_{L^2} + \left\||D_t|^{\frac{l}{2}}\,\partial_t g\right\|_{L^2} + \left\||D_x|^{\frac{(l+2)s}{1+2s}} g\right\|_{L^2} + \sum_{j=0}^{l} \left\||D_v|^{(j+1)s}\,|D_x|^{\frac{ls+s-js}{1+2s}} g\right\|^2_{L^2} \\ &\qquad \lesssim \left\||D_v|^{ls} H\right\|_{L^2} + \left\||D_x|^{\frac{ls}{1+2s}} H\right\|_{L^2} + \left\||D_t|^{\frac{l}{2}} H\right\|_{L^2}\,.\end{aligned}$$

To estimate the right hand side of (4.34), we first notice by induction hypothesis

$$\begin{aligned}&\left\||D_v|^{ls}\,(f\mathcal{T}\theta)\right\|_{L^2} + \left\||D_x|^{\frac{ls}{1+2s}}\,(f\mathcal{T}\theta)\right\|_{L^2} + \left\||D_t|^{\frac{l}{2}}\,(f\mathcal{T}\theta)\right\|_{L^2} \\ &\lesssim \delta^{-2s}\Big(\left\||D_v|^{ls} f\right\|_{L^2(Q_{2r})} + \left\||D_x|^{\frac{ls}{1+2s}} f\right\|_{L^2(Q_{2r})} + \left\||D_t|^{\frac{l}{2}} f\right\|_{L^2(Q_{2r})} \Big) \\ &\lesssim \delta^{-(2l+2)}\,\|f\|_{C^\gamma_\ell(Q_R\times\mathbb{R}^d)} \\ &\qquad + \delta^{-(l+2)s}\Big(\left\||D_v|^{(l-2)s} h\right\|_{L^2(Q_R)} + \left\||D_x|^{\frac{(l-2)s}{1+2s}} h\right\|_{L^2(Q_R)} + \left\||D_t|^{\frac{l-2}{2}} h\right\|_{L^2(Q_R)} \Big).\end{aligned}$$

Second, we have

$$\left\||D_v|^{ls} H\right\|_{L^2} \leq C\,\|f\|_{L^\infty(\mathbb{R}^d)} \left\||D_v|^{(l+2)s}\,\theta\right\|_{L^2} \lesssim \delta^{-(l+2)s}\,\|f\|_{L^\infty(Q^v_R\times\mathbb{R}^d)}\,.$$

Moreover, we get

$$\begin{aligned}
&\left\| |D_x|^{\frac{ls}{1+2s}} h_1 \right\|_{L^2} \\
&= \left\| |D_x|^{\frac{ls}{1+2s}} \int_{\mathbb{R}^d \setminus B_r(v)} f(w) \frac{\theta(t,x,v) - \theta(t,x,w)}{|v-w|^{d+2s}} \,\mathrm{d}w \right\|_{L^2} \\
&\lesssim [f]_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} \left\| \int_{\mathbb{R}^d \setminus B_r(v)} \int_{\mathbb{R}^d} \frac{\theta(t,x,v) - \theta(t,x,w) - \theta(t,y,v) + \theta(t,y,w)}{|v-w|^{d+2s} \, |x-y|^{d+\frac{ls-\gamma}{1+2s}}} \,\mathrm{d}y \,\mathrm{d}w \right\|_{L^2} \\
&\lesssim \delta^{-(l+2)s} \, \|f\|_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} \,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\| |D_t|^{\frac{ls}{2s}} h_1 \right\|_{L^2} \\
&= \left\| |D_t|^{\frac{ls}{2s}} \int_{\mathbb{R}^d \setminus B_r(v)} f(w) \frac{\theta(t,x,v) - \theta(t,x,w)}{|v-w|^{d+2s}} \,\mathrm{d}w \right\|_{L^2} \\
&\lesssim [f]_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} \left\| \int_{\mathbb{R}^d \setminus B_r(v)} \int_{\mathbb{R}^d} \frac{\theta(t,x,v) - \theta(t,x,w) - \theta(s,x,v) + \theta(s,x,w)}{|v-w|^{d+2s} \, |t-s|^{d+\frac{ls-\gamma}{2s}}} \,\mathrm{d}s \,\mathrm{d}w \right\|_{L^2} \\
&\lesssim \delta^{-(l+2)s} \, \|f\|_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} \,.
\end{aligned}$$

Finally using the induction hypothesis

$$\begin{aligned}
\left\| |D_x|^{\frac{ls}{1+2s}} h_{22} \right\|_{L^2} + \left\| |D_t|^{\frac{ls}{2s}} h_{22} \right\|_{L^2} &\lesssim \delta^{-2s} \Big(\left\| |D_x|^{\frac{ls}{1+2s}} f \right\|_{L^2_{t,x} H^s_v(Q_R)} + \left\| |D_t|^{\frac{ls}{2s}} f \right\|_{L^2_{t,x} H^s_v(Q_R)} \Big) \\
&\lesssim \delta^{-2(l+2)} \, \|f\|_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} + \delta^{-(l+2)s} \left\| |D|^{ls} \, h \right\|_{L^2(Q_R)} ,
\end{aligned}$$

as well as

$$\begin{aligned}
\left\| |D_x|^{\frac{ls}{1+2s}} h_{23} \right\|_{L^2} + \left\| |D_t|^{\frac{ls}{2s}} h_{23} \right\|_{L^2} &\lesssim \delta^{-2s} \Big(\left\| |D_x|^{\frac{ls}{1+2s}} f \right\|_{L^2(Q_R)} + \left\| |D_t|^{\frac{ls}{2s}} f \right\|_{L^2(Q_R)} \Big) \\
&\lesssim \delta^{-2(l+2)} \, \|f\|_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} + \delta^{-(l+2)s} \left\| |D|^{ls} \, h \right\|_{L^2(Q_R)} .
\end{aligned}$$

We combine all these estimates for the right hand side and use that $f = g$ in B_r so that

$$\begin{aligned}
&\left\| |D_v|^{(l+2)s} f \right\|_{L^2(Q_r)} + \left\| |D_t|^{\frac{l}{2}} \, \partial_t f \right\|_{L^2(Q_r)} + \left\| |D_x|^{\frac{(l+2)s}{1+2s}} f \right\|_{L^2(Q_r)} \\
&\qquad \lesssim \delta^{-2(l+2)} \, \|f\|_{C^\gamma_\ell(Q^v_R \times \mathbb{R}^d)} + \delta^{-(l+2)s} \left\| |D|^{ls} \, h \right\|_{L^2(Q_R)} .
\end{aligned}$$

This concludes the proof. □

4.4 Campanato

4.4.1 Local (non-fractional) Campanato's inequality

Let $0 < r < R \leq 1$ and $z_0 \in \mathbb{R}^{1+2d}$ such that $Q_R(z_0) \subset Q_1$. Assume f solves (4.20) for some constant diffusion coefficient A satisfying (4.6) and zero source term $h = 0$. As the coefficients A are constant, there is no distinction between the non-divergence and divergence form. Moreover, note that in this case we can assume $f \in C^\infty$ as we can approximate f with a smooth solution by mollification respecting the Lie group structure. We want to combine (4.19) with the regularity estimates in Proposition 4.2 to infer Campanto's inequality. Together with Poincaré's inequality this constitutes Campanato's approach to Schauder estimates.

We have by combining the estimate (4.18) and the Poincaré inequality [79, Proposition 3.12] applied m-times, and then a fractional Poincaré inequality in the final step, see for example [110, Equation 1.2] or [172, page 241]

$$\begin{aligned}
&\int_{Q_r(z_0)} |f - p_{m-1}|^2 \,\mathrm{d}z \\
&\quad \leq Cr^{2m} \int_{Q_r(z_0)} |D_v^m f|^2 \,\mathrm{d}z + Cr^{6\lfloor \frac{m}{3} \rfloor} \int_{Q_r(z_0)} \left| D_x^{\lfloor \frac{m}{3} \rfloor} (f - p_{m-1}) \right|^2 \mathrm{d}z \\
&\qquad + Cr^{4\lfloor \frac{m}{2} \rfloor} \int_{Q_r(z_0)} \left| D_t^{\lfloor \frac{m}{2} \rfloor} (f - p_{m-1}) \right|^2 \mathrm{d}z \\
&\qquad + Cr^{2\left(2\lfloor \frac{i}{2} \rfloor + 3\lfloor \frac{j}{3} \rfloor + k\right)} \int_{Q_r(z_0)} \sum_{\substack{i,j,k \geq 0 \\ i+j+k=m}} \left| D_t^{\lfloor \frac{i}{2} \rfloor} D_x^{\lfloor \frac{j}{3} \rfloor} D_v^k (f - p_{m-1}) \right|^2 \mathrm{d}z \\
&\quad \leq Cr^{2m} \int_{Q_{2r}(z_0)} |D^m f|^2 \,\mathrm{d}z,
\end{aligned} \tag{4.35}$$

where D^m is a derivative in time, space or velocity of order m. To control the right hand side, we use (4.19), Sobolev's embedding for some k sufficiently large depending on n, and the regularity estimates of Proposition 4.2 to get

$$\begin{aligned}
\int_{Q_{2r}(z_0)} |D^m f|^2 \,\mathrm{d}z &\leq Cr^n \left\| D^m f \right\|^2_{L^\infty(Q_{2r}(z_0))} \\
&\leq Cr^n \left\| f \right\|^2_{H^k(Q_{R/2}(z_0))} \leq C(n, R, k) r^n \left\| f \right\|^2_{L^2(Q_R(z_0))}.
\end{aligned} \tag{4.36}$$

Thus we deduce

$$\left\| f - p_{m-1} \right\|^2_{L^2(Q_r(z_0))} \leq Cr^{n+2m} \left\| f \right\|^2_{L^2(Q_R(z_0))},$$

where $C = C(n, s, R)$. This inequality is Campanato's (second) inequality. Now, dividing

by r^{n+2m} yields the Campanato norm on the left hand side:

$$[f]^2_{\mathcal{L}^{2,\lambda}_{m-1}(Q_r(z_0))} = r^{-\lambda} \|f - p_{m-1}\|^2_{L^2(Q_r(z_0))} \leq C \|f\|^2_{L^2(Q_R(z_0))},$$

where

$$\lambda = n + 2m.$$

Remark 4.7. As a consequence of (4.36), we deduce that the only smooth solutions of (4.20) with constant coefficients that grow at most polynomially at infinity are kinetic polynomials. Indeed, if we assume that a solution f of (4.20) in $\mathbb{R}^{1+2d}$ satisfies

$$\sup_{Q_R} f(z) \leq MR^{m-1}, \qquad \forall R \geq 1,$$

for some constant $M > 0$ and $m \geq 1$, then as before we get with Poincaré's inequality, Sobolev embedding, and the regularity estimates for any $r > 0$

$$\begin{aligned}\int_{Q_r} |f - p_{m-1}|^2 \, \mathrm{d}z &\leq Cr^{2m} \int_{Q_{2r}} |D^m f|^2 \, \mathrm{d}z \\ &\leq Cr^{n+2m} \|D^m f\|^2_{H^k(Q_{2r})} \leq C\Big(\frac{r}{R}\Big)^{n+2m} \|f\|^2_{L^2(Q_R)},\end{aligned}$$

where p_{m-1} is some kinetic polynomial of degree $m-1$. Due to the growth assumption on f, we thus find

$$\int_{Q_r} |f - p_{m-1}|^2 \, \mathrm{d}z \leq C(r,n) R^{-n-2m} R^{2m-2+n},$$

which tends to 0 as $R \to \infty$. Thus $f = p_{m-1}$ in Q_r. Since $r > 0$ was arbitrary, we deduce f is a polynomial of degree at most $m-1$ in $\mathbb{R}^{1+2d}$. In other words, a generalisation of Liouville's theorem holds. Note that a Liouville-type theorem has been used to derive Schauder estimates in the elliptic case by [180, Lemma 1] and in the hypoelliptic case by [117, Theorem 4.1].

4.4.2 Non-local (fractional) Campanato's inequality

As before, we want to combine the observation in (4.19) with the energy estimates derived in the last subsection to infer Campanto's inequality. Let $0 < r < R \leq 1$ and $z_0 \in \mathbb{R}^{1+2d}$ such that $Q_R(z_0) \subset Q_1$. We consider the constant coefficient equation (4.21) with zero source term in $Q_R(z_0)$. We have by combining (4.18) and the fractional Poincaré inequality,

see [172, page 241], [110, equation (1.2)], or [167, Section 1],

$$
\begin{aligned}
&\int_{Q_r(z_0)} |f - p_{2s}|^2 \, \mathrm{d}z \\
&\leq Cr^{2s} \Bigg(\int_{Q_r(z_0)} \left| D_t^{\frac{s}{2s}} (f - p_{2s}) \right|^2 \, \mathrm{d}z + \int_{Q_r(z_0)} \left| D_x^{\frac{s}{1+2s}} (f - p_{2s}) \right|^2 \, \mathrm{d}z \\
&\qquad + \int_{Q_r(z_0)} |D_v^s (f - p_{2s})|^2 \, \mathrm{d}z \Bigg) \\
&\leq Cr^{4s} \Bigg(\int_{Q_r(z_0)} \left| D_t^{\frac{2s}{2s}} (f - p_{2s}) \right|^2 \, \mathrm{d}z + \int_{Q_r(z_0)} \left| D_x^{\frac{2s}{1+2s}} (f - p_{2s}) \right|^2 \, \mathrm{d}z \\
&\qquad + \int_{Q_r(z_0)} \left| D_v^{2s} (f - p_{2s}) \right|^2 \, \mathrm{d}z + \int_{Q_r(z_0)} \left| D_t^{\frac{s}{2s}} D_x^{\frac{s}{1+2s}} (f - p_{2s}) \right|^2 \, \mathrm{d}z \\
&\qquad + \int_{Q_r(z_0)} \left| D_t^{\frac{s}{2s}} D_v^s (f - p_{2s}) \right|^2 \, \mathrm{d}z + \int_{Q_r(z_0)} \left| D_v^s D_x^{\frac{s}{1+2s}} (f - p_{2s}) \right|^2 \, \mathrm{d}z \Bigg) \\
&\leq Cr^{6s} \int_{Q_r(z_0)} \left| D^{3s} f \right|^2 \, \mathrm{d}z,
\end{aligned}
\tag{4.37}
$$

where D^{3s} is a differential of order $3s$ in time, space, or velocity. We use (4.19), Sobolev's embedding for some k sufficiently large depending on s and n, and the energy estimates of Proposition 4.3 to get

$$
\begin{aligned}
\int_{Q_r(z_0)} \left| D^{3s} f \right|^2 \, \mathrm{d}z &\leq r^n \left\| D^{3s} f \right\|^2_{L^\infty(Q_r(z_0))} \\
&\leq Cr^n \left\| f \right\|^2_{H^k(Q_{R/2}(z_0))} \leq C(n, s, R, k) r^n \left\| f \right\|^2_{C_\ell^\gamma(Q_R^v(z_0) \times \mathbb{R}^d)}.
\end{aligned}
$$

This can be seen as a non-local analogue of Campanto's inequality. Thus we deduce

$$
\left\| f - p_{2s} \right\|^2_{L^2(Q_r(z_0))} \leq Cr^{n+6s} \left\| f \right\|^2_{C_\ell^\gamma(Q_R^v(z_0) \times \mathbb{R}^d)},
$$

with $C = C(n, s, R)$. Therefore, dividing by r^{n+6s} yields the Campanato norm on the left hand side:

$$
[f]^2_{\mathcal{L}^{2,\lambda}_{2s}(Q_r(z_0))} = r^{-\lambda} \left\| f - p_{2s} \right\|^2_{L^2(Q_r(z_0))} \leq C \left\| f \right\|^2_{C_\ell^\gamma(Q_R^v(z_0) \times \mathbb{R}^d)},
$$

where

$$
\lambda = n + 6s.
$$

4.5 Campanato's approach: the local (non-fractional) case

We freeze coefficients (also known as Korn's trick) to derive Schauder estimates for the general case. Let f classically solve (4.1) or (4.2). Suppose $A = A(t,x,v)$ satisfies (4.6) and $h \in C_\ell^{m-3+\alpha}(Q_1)$. Assume that the diffusion coefficient satisfies $A \in C_\ell^{m-3+\alpha}(Q_1)$. For the divergence form equation (4.1) we additionally require $\nabla_v A \in C_\ell^{m-3+\alpha}(Q_1)$.

Similarly to [114] we consider $0 < \rho \leq \frac{1}{2}$ to be determined and we pick $z_0, z_1 \in Q_1$ and $0 < r \leq 1$ such that $z_1 \in Q_r(z_0)$ and

$$[f]_{C_\ell^{m-1+\alpha}(Q_{\frac{1}{4}})} \leq 2\frac{\left|f(z_1) - p_{m-1}^{z_0}[f](z_1)\right|}{r^{m-1+\alpha}}.$$

We recall that the Taylor expansion of f at z_0 of kinetic degree $m-1$ is given by

$$p_{m-1}^{z_0}[f](z) = \sum_j \frac{a_j(z_0)}{j!}\big(t-t_0\big)^{j_0}\big(x_1-(x_0)_1\big)^{j_1}\cdots\big(x_d-(x_0)_d\big)^{j_d}\big(v_1-(v_0)_1\big)^{j_{d+1}}\cdots\big(v_d-(v_0)_d\big)^{j_{2d}}$$

where we require $j_0 \leq \lfloor \frac{m-1}{2} \rfloor$, $j_1 + \cdots + j_d \leq \lfloor \frac{m-1}{3} \rfloor$ and $j_{d+1} + \cdots + j_{2d} \leq m-1$. The coefficients can be computed and are given by

$$a_j(z_0) = (\partial_t + v\cdot\nabla_x)^{j_0}\partial_{x_1}^{j_1}\cdots\partial_{x_d}^{j_d}\partial_{v_1}^{j_{d+1}}\cdots\partial_{v_d}^{j_{2d}} f(z_0).$$

If $r \geq \rho$, we have, using Lemma 4.1,

$$\begin{aligned}
[f]_{C_\ell^{m-1+\alpha}(Q_{1/4})} &\leq 2\rho^{-(m-1+\alpha)}\Bigg\{2\left\|f\right\|_{L^\infty(Q_r(z_0))} + \sum_j \Big[\rho^{2j_0}\left\|(\partial_t + v\cdot\nabla_x)^{j_0} f\right\|_{L^\infty} \\
&\qquad + \rho^{3(j_1+\cdots+j_d)}\left\|\partial_{x_1}^{j_1}\cdots\partial_{x_d}^{j_d} f\right\|_{L^\infty} \\
&\qquad + \rho^{(j_{d+1}+\cdots+j_{2d})}\left\|\partial_{v_1}^{j_{d+1}}\cdots\partial_{v_d}^{j_{2d}} f\right\|_{L^\infty}\Big]\Bigg\} \\
&\leq \frac{1}{4}[f]_{C_\ell^{m-1+\alpha}(Q_1)} + C(\rho)\left\|f\right\|_{L^\infty(Q_1)}.
\end{aligned}$$

4.5.1 Non-divergence form

Now we consider $r \leq \rho$ and a solution f of (4.2). Let $\eta \in C_c^\infty(\mathbb{R}^{1+2d})$ be a cut-off with $0 \leq \eta \leq 1$, such that $\eta = 1$ in $Q_\rho(z_0)$ and $\eta = 0$ outside $Q_{2\rho}(z_0)$. Let $\tilde{f} = f\eta$. With no loss of generality we set $z_0 = (0,0,0)$. We denote with $p_2^{(z_0)}[f]$ the Taylor polynomial of f at z_0 with kinetic degree less or equal to 2. To approximate the general case by the constant

coefficient case we split

$$\tilde{f} - p_2^{(0)}[\tilde{f}] = g_1 + g_2,$$

where g_1 solves

$$\partial_t g_1 + v \cdot \nabla_x g_1 - \sum_{i,j} a_{(0)}^{i,j} \partial_{v_i v_j}^2 g_1 = 0,$$

for $a_{(0)}^{i,j} = a^{i,j}(z_1)$. Then g_2 solves

$$\partial_t g_2 + v \cdot \nabla_x g_2 - \sum_{i,j} a_{(0)}^{i,j} \partial_{v_i v_j}^2 g_2 = \tilde{h} - p_{m-3}^{(0)}\big[\tilde{h}\big],$$

where

$$\begin{aligned} \tilde{h} :=& \left[\sum_{i,j} \Big(a^{i,j} - a_{(0)}^{i,j} \Big) \partial_{v_i v_j}^2 f \right] \eta + \sum_i \big(b^i \eta - 2 a_{(0)}^{i,j} \partial_{v_i} \eta \big) \partial_{v_i} f \\ &+ \sum_{i,j} \big(c\eta + \mathcal{T}\eta - a_{(0)}^{i,j} \partial_{v_i v_j}^2 \eta \big) f + h\eta. \end{aligned} \tag{4.38}$$

For g_1 we have by Subsection 4.4.1

$$\int_{Q_r} \Big| g_1 - p_{m-1}^{(0)}[g_1] \Big|^2 \, \mathrm{d}z \le C \Big(\frac{r}{R} \Big)^{n+2m} \int_{Q_R} |g_1|^2 \, \mathrm{d}z \le C \Big(\frac{r}{R} \Big)^{n+2m} \int_{Q_R} |f|^2 \, \mathrm{d}z.$$

For g_2 we first perform a change of variables $g_2^{(0)}(t,x,v) := g_2\big(t, (A^{(0)})^{-\frac{1}{2}} x, (A^{(0)})^{-\frac{1}{2}} v\big)$ where $A^{(0)}$ is the constant diffusion coefficient $A^{(0)} = \big(a_{(0)}^{i,j}\big)_{i,j}$. Then $g_2^{(0)}$ solves

$$\begin{aligned} &\left(\partial_t + v \cdot \nabla_x - \sum_{i,j} \partial_{v_i v_j}^2 \right) g_2^{(0)}(t,x,v) \\ &\quad = \left(\partial_t + v \cdot \nabla_x - \sum_{i,j} a_{(0)}^{i,j} \partial_{v_i v_j}^2 \right) g_2\Big(t, (A^{(0)})^{-\frac{1}{2}} x, (A^{(0)})^{-\frac{1}{2}} v \Big) \\ &\quad = \Big(\tilde{h} - \tilde{h}(0,0,0) \Big) \Big(t, (A^{(0)})^{-\frac{1}{2}} x, (A^{(0)})^{-\frac{1}{2}} v \Big) \\ &\quad =: \Big(\tilde{h}^{(0)} - \big(p_{m-3}^{(0)}[\tilde{h}] \big)^{(0)}(0,0,0) \Big)(t,x,v). \end{aligned} \tag{4.39}$$

Thus, using the scaling of the fundamental solution as stated in Lemma 2.1,

$$\int_{Q_r} \Big| g_2^{(0)} \Big|^2 \, \mathrm{d}z \le C r^n \Big\| g_2^{(0)} \Big\|_{L^\infty(Q_r)}^2 \le C r^{n+2m+2\alpha-2} [\tilde{h}^{(0)}]_{C_\ell^{m-3+\alpha}(Q_r)}^2.$$

Since $\Big\| g_2^{(0)} \Big\|_{L^2} \sim \|g_2\|_{L^2}$ and $[\tilde{h}^{(0)}]_{C_\ell^{m-3+\alpha}}^2 \sim [\tilde{h}]_{C_\ell^{m-3+\alpha}}^2$ up to a constant depending on $A^{(0)}$,

we thus find

$$\begin{aligned}\inf_{p\in\mathcal{P}_{m-1}}\int_{Q_r}|f-p|^2\,\mathrm{d}z &\le \int_{Q_r}\Big|f-p_2^{(0)}[f]-p_{m-1}^{(0)}[g_1]\Big|^2\,\mathrm{d}z\\ &\le C\Big(\frac{r}{R}\Big)^{n+2m}\int_{Q_R}|f|^2\,\mathrm{d}z + Cr^{n+2m+2\alpha-2}[\tilde{h}]^2_{C_\ell^{m-3+\alpha}(Q_r)}\\ &\le C\Big(\frac{r}{R}\Big)^{n+2m+2\alpha-2}\bigg(\int_{Q_R}|f|^2\,\mathrm{d}z + [\tilde{h}]^2_{C_\ell^{m-3+\alpha}(Q_r)}\bigg).\end{aligned}$$

Equivalently,

$$[f]_{\mathcal{L}^{2,n+2m+2\alpha-2}_{m-1}(Q_r)} \le C\,\|f\|_{L^2(Q_R)} + C[\tilde{h}]_{C_\ell^{m-3+\alpha}(Q_r)}.$$

Thus by the characterisation of Campanato-norms with Hölder-norms in Theorem 4.4 we have

$$[f]_{C_\ell^{m-1+\alpha}(Q_r)} \le C\,\|f\|_{L^2(Q_R)} + C[\tilde{h}]_{C_\ell^{m-3+\alpha}(Q_r)}.$$

Since $A, B, c, h \in C_\ell^{m-3+\alpha}(Q_1)$ we therefore obtain

$$\begin{aligned}[f]_{C_\ell^{m-1+\alpha}(Q_{1/4})} &\le [f]_{C_\ell^{m-1+\alpha}(Q_r)} \le C\,\|f\|_{L^2(Q_R)} + C[\tilde{h}]_{C_\ell^{m-3+\alpha}(Q_r)}\\ &\le C\,\|f\|_{L^2(Q_\rho(z_0))} + \Big[\sum_{i,j}\big(a^{i,j}_{(0)}-a^{i,j}\big)\partial^2_{v_iv_j}\tilde{f}\Big]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\quad + C[b^i\partial_{v_i}f]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))} + C(\rho)[\partial_{v_i}f]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\quad + C[cf]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))} + C(\rho)[f]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\quad + [h]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\le C\,\|f\|_{L^2(Q_\rho(z_0))} + C\rho^{m-3+\alpha}[D_v^2f]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\quad + C\rho^{m-3+\alpha}[D_vf]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))} + C\rho^{m-3+\alpha}[f]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\quad + C(\rho)[f]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))} + C(\rho)[D_vf]_{C_\ell^{m-3+\alpha}(Q_\rho(z_0))}\\ &\quad + [h]_{C_\ell^{m-3+\alpha}(Q_\rho)}\\ &\le C\,\|f\|_{L^2(Q_1)} + \frac{1}{4}[f]_{C_\ell^{m-1+\alpha}(Q_{2\rho}(z_0))} + C(\rho)\,\|f\|_{L^\infty(Q_{2\rho}(z_0))}\\ &\quad + C_1\rho^{m-1+\alpha}[f]_{C_\ell^{m-1+\alpha}(Q_{2\rho}(z_0))} + [h]_{C_\ell^{m-3+\alpha}(Q_\rho)},\end{aligned} \tag{4.40}$$

where we have used Lemma 4.1 and Proposition 4.1. Choosing ρ such that

$$C_1\rho^{m-1+\alpha} + \frac{1}{4} \le \frac{1}{2},$$

we find for some $\beta > 0$

$$[f]_{C_\ell^{m-1+\alpha}(Q_{\rho/4})} \le C\rho^{-\beta}\,\|f\|_{L^\infty(Q_\rho)} + C[h]_{C_\ell^{m-3+\alpha}(Q_\rho)} + \frac{1}{2}[f]_{C_\ell^{m-1+\alpha}(Q_{2\rho}(z_0))}. \tag{4.41}$$

A standard covering argument (see Lemma 2.2) implies

$$[f]_{C_\ell^{m-1+\alpha}(Q_{\rho/4})} \leq C \|f\|_{L^\infty(Q_\rho)} + C[h]_{C_\ell^{m-3+\alpha}(Q_\rho)}, \tag{4.42}$$

where C depends on n, m, α, λ_0, and the Hölder norms of all coefficients.

4.5.2 Divergence form

The case of divergence form equations follows is similar by modifying the $\tilde{h}$ in (4.38) as follows

$$\begin{aligned}\tilde{h} :=& \left[\sum_{i,j} \partial_{v_i}\Big(a^{i,j} - a^{i,j}_{(0)}\Big)\partial_{v_j} f\right]\eta + \sum_i \big(b^i\eta - 2a^{i,j}_{(0)}\partial_i\eta\big)\partial_{v_i} f \\ &+ \sum_{i,j}\big(c\eta + \mathcal{T}\eta - a^{i,j}_{(0)}\partial^2_{v_iv_j}\eta\big) f + h\eta.\end{aligned}$$

Note that for (4.40) we will require $\nabla_v A \in C^{m-3+\alpha}(Q_1)$.

4.6 Campanato's approach: the non-local (fractional) case

We consider a solution f of (1.13) in $C^\gamma_\ell([-1,0]\times B_1\times\mathbb{R}^d)$ and assume that the non-negative kernel satisfies the ellipticity assumptions (1.32), (1.26) and the Hölder condition (4.8). Moreover, we further assume that it either satisfies the non-divergence form symmetry (4.7), or that it verifies the divergence form symmetry (1.29), (1.30), and the additional Hölder condition (4.9).

Let $\eta \in C^\infty_c((-1,0]\times B_1\times\mathbb{R}^d)$ so that $\eta = 1$ in $Q_{\frac{3}{4}}$ and $\eta = 0$ outside Q_1. Let $\tilde{f} = f\eta$. We freeze coefficients and write $K_0(w) = K(0,0,0,w)$ for the constant coefficient kernel; its corresponding operator $\mathcal{L}_0$ satisfies (4.22). We compute for any $z \in Q_{\frac{1}{2}}$

$$\mathcal{T}\tilde{f} - \mathcal{L}_0\tilde{f} = h\eta + A\cdot\eta + B$$

with

$$A(z) := \int_{\mathbb{R}^d}\big(f(w) - f(v)\big)\big[K(t,x,v,w) - K_0(w)\big]\,\mathrm{d}w$$

and

$$B(z) := \int_{\mathbb{R}^d}\big(\eta(v) - \eta(w)\big)f(w)K_0(w)\,\mathrm{d}w.$$

We write

$$\tilde{f} - p[\tilde{f}] = g_1 + g_2,$$

where g_1 solves

$$\mathcal{T} g_1 - \mathcal{L}_0 g_1 = 0,$$

and with

$$p[f] := f(z_0) + (t - t_0)\big(\mathcal{T} f(z_0) - \mathcal{L}_0 f(z_0)\big)$$

for some $z_0 \in \mathbb{R}^{1+2d}$. With no loss of generality set $z_0 = (0,0,0)$. In particular, g_2 solves

$$\begin{aligned}
\mathcal{T} g_2 - \mathcal{L}_0 g_2 &= \mathcal{T}(\tilde{f} - p[\tilde{f}] - g_1) - \mathcal{L}_0(\tilde{f} - p[\tilde{f}] - g_1) \\
&= h \cdot \eta + A \cdot \eta + B - \mathcal{T}\tilde{f}(z_0) + \mathcal{L}_0 \tilde{f}(z_0) \\
&= h \cdot \eta + A \cdot \eta + B - \big(h \cdot \eta + A \cdot \eta + B\big)(z_0) \\
&= \tilde{h} - \tilde{h}(z_0),
\end{aligned}$$

where $\tilde{h} := h\eta + A\eta + B$. For g_1 we find with Subsection 4.4.2

$$\int_{Q_r} \left| g_1 - p_{2s}^{(0)}[g_1] \right|^2 \mathrm{d}z \leq C r^{n+6s} \left\| g_1 \right\|^2_{C_\ell^\gamma(Q_1^v \times \mathbb{R}^d)} \leq C r^{n+6s} \left\| f \right\|^2_{C_\ell^\gamma(Q_1^v \times \mathbb{R}^d)},$$

where $r > 0$ is such that $Q_r \subset Q_{1/2}$. For g_2 we first perform a change of variables $g_2^{(0)}(t,x,v) := g_2\big(t, \kappa_0^{-\frac{1}{2s}} x, \kappa_0^{-\frac{1}{2s}} v\big)$ where κ_0 is such that $K_0(w) = \frac{\kappa_0}{|w|^{d+2s}}$. Then $g_2^{(0)}$ solves

$$\begin{aligned}
\Big(\partial_t + v \cdot \nabla_x + (-\Delta_v)^s\Big) g_2^{(0)}(t,x,v) &= \Big(\partial_t + v \cdot \nabla_x + \mathcal{L}_0\Big) g_2\Big(t, \kappa_0^{-\frac{1}{2s}} x, \kappa_0^{-\frac{1}{2s}} v\Big) \\
&= \Big(\tilde{h} - \tilde{h}(0,0,0)\Big)\Big(t, \kappa_0^{-\frac{1}{2}} x, \kappa_0^{-\frac{1}{2}} v\Big) \\
&=: \Big(\tilde{h}^{(0)} - \tilde{h}^{(0)}(0,0,0)\Big)(t,x,v).
\end{aligned}$$

Thus by Lemma 2.1

$$\int_{Q_r} \left| g_2^{(0)} \right|^2 \mathrm{d}z \leq C r^n \left\| g_2^{(0)} \right\|^2_{L^\infty(Q_r)} \leq C r^{n+4s+2\alpha} \big[\tilde{h}^{(0)}\big]^2_{C_\ell^\alpha(Q_r)}.$$

Since $\left\| g_2^{(0)} \right\|_{L^2} \sim \left\| g_2 \right\|_{L^2}$ and $\big[\tilde{h}^{(0)}\big]^2_{C_\ell^\alpha} \sim [\tilde{h}]^2_{C_\ell^\alpha}$ up to a constant depending on κ_0, we thus find

$$\begin{aligned}
\inf_{p \in \mathcal{P}_{m-1}} \int_{Q_r} \left| \tilde{f} - p \right|^2 \mathrm{d}z &\leq \int_{Q_r} \left| \tilde{f} - p[\tilde{f}] - p_{2s}^{(0)}[g_1] \right|^2 \mathrm{d}z \\
&\leq C r^{n+6s} \left\| f \right\|^2_{C_\ell^\gamma(Q_1^v \times \mathbb{R}^d)} + C r^{n+4s+2\alpha} [\tilde{h}]^2_{C_\ell^\alpha(Q_r)} \\
&\leq C r^{n+4s+2\alpha} \left(\left\| f \right\|^2_{C_\ell^\gamma(Q_1^v \times \mathbb{R}^d)} + [\tilde{h}]^2_{C_\ell^\alpha(Q_r)} \right).
\end{aligned}$$

In the last inequality we used $\alpha < s$ since $\alpha = \frac{2s}{1+2s}\gamma < \frac{2s}{1+2s} \min(1, 2s)$. Equivalently,

$$[f]_{\mathcal{L}_{2s}^{2,n+4s+2\alpha}(Q_r)} \leq C \left\| f \right\|^2_{C_\ell^\gamma(Q_1^v \times \mathbb{R}^d)} + C [\tilde{h}]_{C_\ell^\alpha(Q_r)}.$$

Thus by the characterisation of Campanato norms with Hölder norms in Theorem 4.4 we have

$$[f]_{C_\ell^{2s+\alpha}(Q_r)} \leq C \|f\|^2_{C_\ell^\gamma(Q_1^v\times\mathbb{R}^d)} + C[\tilde{h}]_{C_\ell^\alpha(Q_r)}.$$

It remains to bound the C_ℓ^α-norm of $\tilde{h} = h\eta + A\eta + B$. We claim

$$[A]_{C_\ell^\alpha(Q_{\frac{1}{2}})} \lesssim A_0\big(\|f\|_{C_\ell^{2s+\alpha}(Q_1)} + \|f\|_{C_\ell^\gamma((-1,0]\times B_1\times\mathbb{R}^d)} \big). \tag{4.43}$$

Indeed we write $A(z_1) - A(z_2) = I_1 + I_2$ with

$$\begin{aligned} I_1 &= \int \big(f(z_2\circ(0,0,w)) - f(z_2)\big)\big[K_{z_1}(w) - K_{z_2}(w)\big]\,\mathrm{d}w, \\ I_2 &= \int \big(f(z_1\circ(0,0,w)) - f(z_1) - f(z_2\circ(0,0,w)) + f(z_2)\big)\big[K_{z_1}(w) - K_0(w)\big]\,\mathrm{d}w. \end{aligned}$$

For I_1 we distinguish the far and the close part and write I_{11} and I_{12} respectively. Then for the far part there holds with (4.8)

$$\begin{aligned} |I_{11}| &\leq \|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)} \int_{|w|\geq 1} |K_{z_1}(w) - K_{z_2}(w)|\,\mathrm{d}w \\ &\lesssim A_0 \|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)}\, d_l(z_1,z_2)^\alpha. \end{aligned}$$

For the close part we have in case of the *non-divergence form symmetry* (4.7) and Lemma 4.1

$$\begin{aligned} |I_{12}| &\leq \int_{|w|\leq 1} \Big| f(z_2\circ(0,0,w)) - p_{2s}^{z_2\circ(0,0,w)}[f] \Big| \,|K_{z_1}(w) - K_{z_2}(w)|\,\mathrm{d}w \\ &\quad + \frac{1}{2}\int_{|w|\leq 1} \Big| p_{2s}^{z_2\circ(0,0,w)}[f] + p_{2s}^{z_2\circ(0,0,-w)}[f] - f(z_2) \Big| \,|K_{z_1}(w) - K_{z_2}(w)|\,\mathrm{d}w \\ &\lesssim [f]_{C_\ell^{2s+\alpha}} \int_{|w|\leq 1} |w|^{2s+\alpha}\,|K_{z_1}(w) - K_{z_2}(w)|\,\mathrm{d}w \\ &\quad + \big|D_v^2 f\big| \int_{|w|\leq 1} |w|^2\,|K_{z_1}(w) - K_{z_2}(w)|\,\mathrm{d}w \\ &\lesssim A_0 \|f\|_{C_\ell^{2s+\alpha}}\, d_\ell(z_1,z_2)^\alpha. \end{aligned}$$

If instead we assume the *divergence form symmetry* (1.29) and (1.30) we get

$$\begin{aligned}
|I_{12}| &\le \operatorname{PV}\left|\int_{|w|\le 1}\Big[f(z_2\circ(0,0,w)) - p_{2s}^{z_2\circ(0,0,w)}[f]\Big]\,[K_{z_1}(w)-K_{z_2}(w)]\,\mathrm{d}w\right| \\
&\quad + \operatorname{PV}\left|\int_{|w|\le 1}\Big[p_{2s}^{z_2\circ(0,0,w)}[f] - f(z_2)\Big]\,[K_{z_1}(w)-K_{z_2}(w)]\,\mathrm{d}w\right| \\
&\lesssim [f]_{C_\ell^{2s+\alpha}}\int_{|w|\le 1}|w|^{2s+\alpha}\,|K_{z_1}(w)-K_{z_2}(w)|\,\mathrm{d}w \\
&\quad + |D_v f|\,\Big|\operatorname{PV}\int_{|w|\le 1} w\big(K_{z_1}(w)-K_{z_2}(w)\big)\,\mathrm{d}w\Big| \\
&\quad + \big|D_v^2 f\big|\int_{|w|\le 1}|w|^2\,|K_{z_1}(w)-K_{z_2}(w)|\,\mathrm{d}w \\
&\lesssim A_0\,\|f\|_{C_\ell^{2s+\alpha}}\,d_\ell(z_1,z_2)^\alpha,
\end{aligned}$$

by assumption (4.8) and (4.9).

To estimate I_2 we can use Lemma 4.3. This proves the claim.

We further claim

$$[B]_{C_\ell^\alpha(Q_{\frac12})} \lesssim \|f\|_{C_\ell^\gamma((-1,0]\times B_1\times\mathbb{R}^d)}\,. \tag{4.44}$$

For $z_1, z_2 \in Q_r$ we compute $B(z_2) - B(z_1) = J_1 + J_2$ with

$$\begin{aligned}
J_1 &= \int_{|w|>r/4}\big[\eta(z_1\circ(0,0,w)) - \eta(z_1) - \eta(z_2\circ(0,0,w)) + \eta(z_2)\big] \\
&\qquad \times f\big(z_1\circ(0,0,w)\big)K_0(w)\,\mathrm{d}w, \\
J_2 &= \int_{|w|>r/4}\big[\eta(z_2\circ(0,0,w)) - \eta(z_2)\big]\big[f(z_1\circ(0,0,w)) - f(z_2\circ(0,0,w))\big]K_0(w)\,\mathrm{d}w.
\end{aligned}$$

Since η is smooth we can apply Lemma 4.3 and get

$$|J_1| \le \|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)}\,d_l(z_1,z_2)^\alpha.$$

For J_2 we have

$$|J_2| \le 2\,\|\eta\|_{L^\infty}\,[f]_{C_\ell^\gamma}\int_{|w|>r/4} d_l(z_1\circ(0,0,w), z_2\circ(0,0,w))^\gamma K_0(w)\,\mathrm{d}w.$$

Since $\alpha = \frac{2s\gamma}{1+2s}$ we have

$$\begin{aligned}
|J_2| &\lesssim [f]_{C^\gamma_\ell} \int_{|w|>r/4} d_l(z_1 \circ (0,0,w), z_2 \circ (0,0,w))^\gamma K_0(w)\,\mathrm{d}w \\
&\lesssim [f]_{C^\gamma_\ell} \int_{|w|>r/4} \big(d_l(z_1,z_2) + |t_1 - t_2|^{\frac{1}{1+2s}}\, |w|^{\frac{1}{1+2s}}\big)^\gamma K_0(w)\,\mathrm{d}w \\
&\lesssim [f]_{C^\gamma_\ell} \int_{|w|>r/4} \big(1 + |w|^{\frac{\gamma}{1+2s}}\big) d_l(z_1,z_2)^{\frac{2s\gamma}{1+2s}} K_0(w)\,\mathrm{d}w \\
&\lesssim_\Lambda [f]_{C^\gamma_\ell} d_l(z_1,z_2)^{\frac{2s\gamma}{1+2s}} \le [f]_{C^\gamma_\ell} d_l(z_1,z_2)^\alpha,
\end{aligned}$$

where we used the upper bound on K_0 (1.32). This proves the second claim (4.44).

Thus we deduce by combining (4.43) and (4.44) that

$$[f]_{C^{2s+\alpha}_\ell(Q_{\frac14})} \le (1+A_0)\,\|f\|_{C^\gamma_\ell((-1,0]\times B_1\times\mathbb{R}^d)} + A_0\,\|f\|_{C^{2s+\alpha}_\ell(Q_1)} + \|h\|_{C^\alpha_\ell(Q_1)}\,. \tag{4.45}$$

Without loss in generality we can assume that $A_0 < 1$, otherwise we scale the equation initially. We then translate and scale the estimate (4.45) and we get for any $r > 0$ and $z_0 \in Q_1$ such that $Q_r(z_0) \subset Q_1$

$$\begin{aligned}
r^{2s+\alpha}[f]_{C^{2s+\alpha}_\ell(Q_{\frac r4}(z_0))} &\lesssim r^\alpha A_0 r^{2s+\alpha}[f]_{C^{2s+\alpha}_\ell(Q_r(z_0))} + r^\gamma [f]_{C^\gamma_\ell((-1,0]\times B_1\times\mathbb{R}^d)} \\
&\quad + \|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)} + r^{2s+\alpha}[h]_{C^\alpha_\ell(Q_r(z_0))} + r^{2s}\,\|h\|_{L^\infty(Q_r(z_0))} \\
&\le A_0 r^{2s+\alpha}[f]_{C^{2s+\alpha}_\ell(Q_r(z_0))} + \|f\|_{C^\gamma_\ell((-1,0]\times B_1\times\mathbb{R}^d)} \\
&\quad + r^{2s+\alpha}[h]_{C^\alpha_\ell(Q_r(z_0))} + r^{2s}\,\|h\|_{L^\infty(Q_r(z_0))}\,.
\end{aligned}$$

The same covering argument as outlined for (4.42) in Subsection 4.5.1 implies then

$$\|f\|_{C^{2s+\alpha}_\ell(Q_{\frac14})} \le C\Big(\,\|h\|_{C^\alpha_\ell(Q_1)} + \|f\|_{C^\gamma_\ell((-1,0]\times B_1\times\mathbb{R}^d)}\Big),$$

where C depends on $s, d, \lambda_0, \Lambda_0, A_0$.

4.7 Appendix: Hypoelliptic operators

4.7.1 Toolbox

In this section, we briefly outline that our approach is robust enough to deal with general second order Kolmogorov equations of the form

$$\begin{aligned}\mathscr{L}f(t,x) := &\sum_{N-d\le i,j\le N} a_{i,j}(t,x)\partial_{x_ix_j}f(t,x)\\ &+\sum_{1\le i,j\le N}\tilde{b}_{i,j}x_j\partial_{x_i}f(t,x)-\partial_t f(t,x)\\ &+\sum_{N-d\le i\le N} b_i(t,x)\partial_i f(t,x)+c(t,x)f(t,x)=h,\end{aligned} \tag{4.46}$$

where $z=(t,x)=(t,x_0,x_1,\dots,x_\kappa)\in\mathbb{R}^{1+N}$, $\kappa\ge 1$ is the number of commutators, and $1\le d\le N$. The velocity variable corresponds to the last entry $x_\kappa\in\mathbb{R}^d$. The diffusion matrix $A(z)=\big(a_{i,j}(z)\big)_{N-d\le i,j\le N}$ is symmetric with real measurable entries, and uniformly elliptic (4.6). The matrix $\tilde{B}=\big(\tilde{b}_{i,j}\big)_{1\le i,j\le N}$ has constant entries and satisfies suitable assumptions such that the *principal part operator* $\mathscr{K}$ of $\mathscr{L}$ with respect to the kinetic degree, given by

$$\mathscr{K}f(t,x)=\sum_{N-d\le i,j\le N}\partial_{x_ix_j}f(t,x)+\sum_{1\le i,j\le N}\tilde{b}_{i,j}x_j\partial_{x_i}f(t,x)-\partial_t f(t,x), \tag{4.47}$$

is hypoelliptic, i.e. any distributional solution of $\mathscr{K}f=h$ is smooth whenever $h\in C^\infty$. In particular, this assumption coincides with $\tilde{B}$ having constant real entries and taking the form

$$\tilde{B}=\begin{pmatrix} * & \tilde{B}_1 & 0 & \dots & 0\\ * & * & \tilde{B}_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \dots & \tilde{B}_\kappa\\ * & * & * & \dots & * \end{pmatrix}, \tag{4.48}$$

where each $\tilde{B}_i$ is a $d_{i-1}\times d_i$ matrix of rank d_i with $d:=d_\kappa\ge d_{\kappa-1}\ge\cdots\ge d_0\ge 1$ and $\sum_{i=0}^{\kappa}d_i=N$. For further discussion on this operator, we refer the reader to [142, Section 1 and 2]. We remark that the principal part operator $\mathscr{K}$ is still invariant under Galilean transformation (4.5). Moreover, $\mathscr{K}$ is invariant under the scaling given by

$$(t,x_0,\dots,x_\kappa)\to(r^2t,r^3x_0,\dots,r^{2\kappa+1}x_{\kappa-1},rx_\kappa)=:z_r, \tag{4.49}$$

for $r>0$, where $\kappa\ge 1$ is the number of commutators, *if and only if* all the $*$-blocks in $\tilde{B}$ are zero [142, Proposition 2.2]. We denote the scaling invariant principal part by $\mathscr{K}_0$, and emphasise that it is of the form (4.47) with the matrix $\tilde{B}$ as in (4.48) where all the $*$-entries are zero. The cylinders will be defined respecting the scaling invariance, similar

as above (4.3).

We briefly sketch how to obtain Schauder estimates for a solution f of (4.46) in Q_1. Note that the kinetic distance and the corresponding Hölder norms have to be defined more generally taking into account the scaling (4.49).

First, the regularity estimates will be replaced by an argument of Hörmander [111, Theorem 3.7] as follows. Any solution f of $\mathscr{K} f = 0$ satisfies for $l \geq 1$

$$\left\| D^l f \right\|_{L^\infty(Q_r(z_0))} \leq C(l, N) \left\| f \right\|_{L^2(Q_R(z_0))}, \tag{4.50}$$

where D^l is a differential of order l. Indeed, let δ be a mutli-index such that $|\delta| = l \geq 1$. Let $G \subset L^2(Q_R(z_0))$ be defined as

$$G := \left\{ g \in L^2(Q_R(z_0)) \cap C^\infty(Q_R(z_0)) : \mathscr{K} g = 0 \text{ in } Q_R(z_0) \right\}.$$

Due to the hypoellipticity of $\mathscr{K}$ the subspace G is closed in $L^2(Q_R(z_0))$. Define $\mathcal{B} : G \to C^0(Q_r(z_0))$ by $\mathcal{B} g = D^\delta g|_{Q_r(z_0)}$ for δ such that $|\delta| = l \geq 0$. Then $\mathcal{B}$ has closed graph in $G \times C^0(Q_r(z_0))$, and thus, by virtue of the closed graph theorem we conclude (4.50). Then we derive Campanato's inequality (4.36) just as above in Subsection 4.4.1.

Second, the principal part operator $\mathscr{K}_0$ admits an explicit fundamental solution given in [9, Equation (2.7)]. In particular, it satisfies for $r > 0$

$$\mathscr{K}_0 f_r = r^2 \big(\mathscr{K}_0 f \big)_r, \tag{4.51}$$

where f_r denotes the rescaled function $f_r(z) := f(z_r)$. Note that we do not require the scaling invariance to deduce the Schauder estimates. To see this, we denote the fundamental solution of $\mathscr{K}$ by Γ and the fundamental solution of $\mathscr{K}_0$ by Γ_0, respectively. Then we can use the upper bound on Γ by Γ_0, stated in [142, Theorem 3.1],

$$\Gamma(z) \leq a \Gamma_0(z), \tag{4.52}$$

for some $a > 0$. Due to (4.51) we then have the good scaling for g_2, where g_2 comes from the splitting of our solution $f - p_2^{(0)}[f] = g_1 + g_2$ as done in Section 4.5 above, with the polynomial $p_2^{(0)}[f]$ given in (4.54) below. Alternatively, we can directly consider the scaling of the full matrix $\tilde{B}$ in (4.48). According to [142, Remark 3.2] and [143, Remark 2.4], the $*$-blocks in (4.48) scale to some higher power of r than the superdiagonal blocks. Thus, using $\tilde{B} = \tilde{B}_0 + \tilde{B} - \tilde{B}_0$, where $\tilde{B}_0$ corresponds to $\tilde{B}$ with all $*$-blocks equal to zero, we rewrite

$$\mathscr{K} = \mathscr{K}_0 + \sum_{1 \leq i,j \leq N} \big(\tilde{b}_{i,j} - \tilde{b}^0_{i,j} \big) x_j \partial_{x_i} f,$$

so that

$$\mathscr{K}_0^a g_2 = \tilde{h} + \sum_{1\leq i,j\leq N} \big(\tilde{b}^0_{i,j} - \tilde{b}_{i,j}\big) x_j \partial_{x_i} f$$

with $\mathscr{K}_0^a$ as in (4.57) but where $\tilde{B}_0$ replaces $\tilde{B}$, and with $\tilde{h}$ given in (4.58). The right hand side can be bounded as in Section 4.5 above, since the term $\sum_{1\leq i,j\leq N} \big(\tilde{b}_{i,j} - \tilde{b}^0_{i,j}\big) x_j \partial_{x_i} f$ scales like a lower order term due to [142, Remark 3.2] and [143, Remark 2.4]. The details of the splitting are done for Dini-continuous coefficients in Subsection 4.7.3 below.

4.7.2 Hölder coefficients

We have assembled the toolbox required for the Schauder estimates, and the argument of Section 4.5 goes through (with suitable modifications as outlined above in Subsection 4.7.1), so that we derive

Theorem 4.5 (Schauder estimate for Kolmogorov operators)**.** *Let $\alpha \in (0,1)$ be given. Let $m \geq 3$ be some integer. Let f solve (4.46) in Q_1. Suppose $A \in C_\ell^{m-3+\alpha}(Q_1)$ satisfies (4.6) for some $\lambda_0 > 0$, where $d = d_0$, and assume $B, c, h \in C_\ell^{m-3+\alpha}(Q_1)$. We further assume that the principal part operator $\mathscr{K}$ defined in (4.47) is hypoelliptic, i.e. $\tilde{B}$ is of the form (4.48). Then there holds*

$$\|f\|_{C_\ell^{m-1+\alpha}(Q_{1/4})} \leq C\Big(\|f\|_{L^\infty(Q_1)} + \|h\|_{C_\ell^{m-3+\alpha}(Q_1)} \Big),$$

for some C depending on $N, \lambda_0, \alpha, \|A\|_{C_\ell^{m-3+\alpha}}, \|B\|_{C_\ell^{m-3+\alpha}}, \|c\|_{C_\ell^{m-3+\alpha}}$.

Similar to Subsection 4.5.2 the divergence form case just follows by realising that any divergence form equation can be written in non-divergence form plus an additional lower order term, provided that $\nabla_{x_\kappa} A \in C_\ell^{m-3+\alpha}(Q_1)$. Finally, we can derive a Schauder-type estimate under less stringent assumptions assuming Dini-regularity instead of Hölder regularity, inspired from [171].

4.7.3 Dini coefficients

We point out a structural peculiarity when we consider more generally Dini-regular coefficients A, B, c and source h. We denote by ω_g the modulus of continuity of a function g on a subset $Q \subset \mathbb{R}^{1+N}$, given by

$$\omega_g(\ln r) := \sup_{\substack{z_1,z_2\in Q \\ d_\ell(z_1,z_2)<r}} \big|g(z_1) - g(z_2)\big|.$$

A function g is said to be Dini-continuous in Q if

$$\int_0^1 \frac{\omega_g(\ln r)}{r}\,\mathrm{d}r = \int_{-\infty}^0 \omega_g(\rho)\,\mathrm{d}\rho < +\infty.$$

We aim to show:

Theorem 4.6. *Let f solve (4.46) in Q_1 such that A is a symmetric, uniformly elliptic matrix with real measurable entries, and suppose $\tilde{B}$ has constant entries. Assume that the principal part operator $\mathscr{K}$ (4.47) is hypoelliptic, i.e. $\tilde{B}$ is of the form (4.48). Suppose that the coefficients A, B, c and the source h are Dini-regular. Then, for any $z, z_0 \in \mathbb{R}^{1+N}$ such that $d_\ell(z, z_0) < 1/2$, f satisfies*

$$\begin{aligned}
&\left|D^2 f(z) - D^2 f(z_0)\right| \\
&\leq C\left(\int_{-\infty}^{\ln d_\ell(z,z_0)} \omega_A(\xi)\,\mathrm{d}\xi + d_\ell(z,z_0)\int_{\ln d_\ell(z,z_0)}^0 \omega_A(\xi)e^{-\xi}\,\mathrm{d}\xi\right)\sum_{i,j}\sup_{Q_1}\left|\partial^2_{v_i v_j} f\right| \\
&\quad + C\left(d_\ell(z,z_0) + \int_{-\infty}^{\ln d_\ell(z,z_0)} \omega_c(\xi)\,\mathrm{d}\xi + d_\ell(z,z_0)\int_{\ln d_\ell(z,z_0)}^0 \omega_c(\xi)e^{-\xi}\,\mathrm{d}\xi\right)\sup_{Q_1}\left|f\right| \\
&\quad + C\left(\int_{-\infty}^{\ln d_\ell(z,z_0)} \omega_B(\xi)\,\mathrm{d}\xi + d_\ell(z,z_0)\int_{\ln d_\ell(z,z_0)}^0 \omega_B(\xi)e^{-\xi}\,\mathrm{d}\xi\right)\sum_{i}\sup_{Q_1}\left|\partial_{v_i} f\right| \\
&\quad + C\int_{-\infty}^{\ln d_\ell(z,z_0)} \omega_h(\xi)\,\mathrm{d}\xi + Cd_\ell(z,z_0)\int_{\ln d_\ell(z,z_0)}^0 \omega_h(\xi)e^{-\xi}\,\mathrm{d}\xi + Cd_\ell(z,z_0)\sup_{Q_1}|h|\,.
\end{aligned} \tag{4.53}$$

Here D^2 is a differential of order 2, and $C = C(N, \lambda_0)$.

In particular we recover Theorem 1.6 of [171].

Remark 4.8. Theorem 4.6 suggests that Dini continuity is the suitable notion of regularity for Schauder estimates. In particular, in Theorem 4.5, we see that Hölder regular solutions f are *fixed points* of the Schauder estimates.

For this purpose, we consider $0 < \rho \leq 1$ to be determined and a solution f of (4.2) in Q_1. Let $\eta \in C_c^\infty(\mathbb{R}^{1+2d})$ be a cut-off with $0 \leq \eta \leq 1$, such that $\eta = 1$ in Q_ρ and $\eta = 0$ outside $Q_{2\rho}$. Let $\tilde{f} = f \cdot \eta$. With no loss in generality we set $z_0 = (0,0,0)$. We denote with $p_2^{(z_0)}[f]$ the Taylor polynomial of f at z_0 with kinetic degree less or equal to 2, given by

$$\begin{aligned}
p_2^{z_0}[f](z) = f(z_0) &+ \sum_{N-d\leq i\leq N} \partial_{x_i} f(z_0)\big(z^{(i)} - z_0^{(i)}\big) \\
&+ \frac{1}{2}\sum_{N-d\leq i,j\leq N} \partial^2_{x_i x_j} f(z_0)\big(z^{(i)} - z_0^{(i)}\big)\big(z^{(j)} - z_0^{(j)}\big) \\
&+ \left[\sum_{1\leq i,j\leq N} \tilde{b}_{i,j} x_j \partial_{x_i} f(z_0) - \partial_t f(z_0)\right](t - t_0),
\end{aligned} \tag{4.54}$$

where $z^{(i)}$ denotes the element at index i. We then write

$$\tilde{f} - p_2^{(0)}[\tilde{f}] = \tilde{f} - \tilde{f}_k + \tilde{f}_k - p_2^{(0)}[\tilde{f}], \tag{4.55}$$

where each f_k solves

$$\mathscr{K}^a f_k = \tilde{h}(0,0,0), \tag{4.56}$$

in $\mathcal{Q}_k := Q_{\rho^k}$, with the constant coefficient operator $\mathscr{K}^a$ given by

$$\mathscr{K}^a := \sum_{N-d\le i,j\le N} a_{i,j}^{(0)} \partial^2_{x_i x_j} + \sum_{1\le i,j\le N} \tilde{b}_{i,j} x_j \partial_{x_i} - \partial_t \tag{4.57}$$

for $a_{(0)}^{i,j} = a^{i,j}(z_0)$, and the right hand side $\tilde{h}$ given by

$$\begin{aligned} \tilde{h} := & \sum_{N-d\le i,j\le N} \Big(a_{i,j}^{(0)} - a_{i,j}\Big)\partial^2_{x_i x_j} f \cdot \eta \\ & + \sum_{N-d\le i\le N} \big(2a_{i,j}^{(0)} \partial_{x_i}\eta - b_i \eta\big)\partial_{x_i} f + (-c\eta + \mathscr{K}_a \eta) f + h \cdot \eta. \end{aligned} \tag{4.58}$$

In particular, there holds

$$\mathscr{K}^a\big(\tilde{f}_k - \tilde{f}_{k+1}\big) = 0, \qquad \text{in } \mathcal{Q}_{k+1}, \tag{4.59}$$

and

$$\mathscr{K}^a\big(\tilde{f} - \tilde{f}_k\big) = \tilde{h} - \tilde{h}(0,0,0), \qquad \text{in } \mathcal{Q}_k. \tag{4.60}$$

On the one hand, we first perform a constant change of variables to rewrite $\mathscr{K}^a$ in terms of $\mathscr{K}$, as was done in (4.39). Then, due to (4.60), the upper bound of the fundamental solution (4.52) and the scaling (4.51), which extends Lemma 2.1, we bound for any $k \ge 1$

$$\int_{\mathcal{Q}_{k+1}} \Big|\tilde{f} - \tilde{f}_k\Big|^2 \,\mathrm{d}z \le C\rho^{(n+4)(k+1)} \sup_{\mathcal{Q}_{k+1}} \big|\tilde{h} - \tilde{h}(0,0,0)\big|^2 \le C\rho^{(n+4)(k+1)} \omega^2_{\tilde{h}}(\rho^{k+1}).$$

Since $\tilde{f}_k = \tilde{f}_0 + \sum_{l=0}^{k-1} \tilde{f}_{l+1} - \tilde{f}_l$ we thus find

$$\begin{aligned}
&\left(\rho^{-(n+6)(k+1)} \int_{\mathcal{Q}_{k+1}} \left|\tilde{f} - \tilde{f}_k\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{l=0}^{k-1} \rho^{-(n+6)(l+1)} \int_{\mathcal{Q}_{l+1}} \left|\tilde{f}_{l+1} - \tilde{f}_l\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} \\
&\leq \left\{\sum_{l=0}^{k-1} \rho^{-(n+6)(l+1)} \left(\int_{\mathcal{Q}_{l+1}} \left|\tilde{f}_{l+1} - \tilde{f}\right|^2 + \left|\tilde{f} - \tilde{f}_l\right|^2 \mathrm{d}z\right)\right\}^{\frac{1}{2}} \\
&\leq C \sum_{l=0}^{k-1} \frac{\omega_{\tilde{h}}(\ln \rho^{l+1})}{\rho^{l+1}} \\
&\leq C \int_{\ln \rho}^{0} \omega_{\tilde{h}}(\xi) e^{-\xi} \,\mathrm{d}\xi.
\end{aligned} \tag{4.61}$$

On the other hand, we note that $p_2^{(0)}[\tilde{f}] = \lim_{k\to\infty} \tilde{f}_k$. Indeed, $p_2^{(0)}[\tilde{f}]$ is the Taylor polynomial of $\tilde{f}$, so that

$$\sup_{\mathcal{Q}_k} \left(\tilde{f} - p_2^{(0)}[\tilde{f}]\right) = o(\rho^{2k}),$$

and we also refer to [171, Equation (5.16)]. Moreover, due to (4.60) we can use (4.51) so that overall we find

$$\begin{aligned}
\left|\tilde{f}_k(z) - p_2^{(0)}[\tilde{f}](z)\right| &\leq \sup_{\mathcal{Q}_k} \left|\tilde{f}_k - \tilde{f}\right| + \sup_{\mathcal{Q}_k} \left|\tilde{f} - p_2^{(0)}[\tilde{f}]\right| \\
&\leq C\rho^{2k} \sup_{\mathcal{Q}_k} \left|\tilde{h} - \tilde{h}(0,0,0)\right| + o(\rho^{2k}) \\
&\leq C\rho^{2k} \omega_{\tilde{h}}(\ln \rho^k) + o(\rho^{2k}) \\
&\leq o(\rho^{2k}).
\end{aligned}$$

Therefore, we may write

$$\tilde{f}_k - p_2^{(0)}[\tilde{f}] = \sum_{l=k}^{\infty} \tilde{f}_l - \tilde{f}_{l+1}. \tag{4.62}$$

Due to (4.56), Subsection 4.4.1 (suitably making the replacements for the more general equation as outlined in Subsection 4.7.1), (4.62), (4.60) and (4.51), we then find for

$\tilde{f}_k - p_2^{(0)}[\tilde{f}]$,

$$\begin{aligned}
&\int_{\mathcal{Q}_{k+1}} \left|\tilde{f}_k - p_2^{(0)}[\tilde{f}] - p_2^{(0)}\big[\tilde{f}_k - p_2^{(0)}[\tilde{f}]\big]\right|^2 \mathrm{d}z \\
&\quad\le C\Big(\frac{\rho^{k+1}}{\rho^k}\Big)^{n+6} \int_{\mathcal{Q}_k} \left|\tilde{f}_k - p_2^{(0)}[\tilde{f}]\right|^2 \mathrm{d}z \\
&\quad= C\Big(\frac{\rho^{k+1}}{\rho^k}\Big)^{n+6} \sum_{l=k}^{\infty} \int_{\mathcal{Q}_k} \left|\tilde{f}_l - \tilde{f}_{l+1}\right|^2 \mathrm{d}z \\
&\quad\le C\Big(\frac{\rho^{k+1}}{\rho^k}\Big)^{n+6} \sum_{l=k}^{\infty} \left(\int_{\mathcal{Q}_l} \left|\tilde{f}_l - \tilde{f}\right|^2 \mathrm{d}z + \int_{\mathcal{Q}_l} \left|\tilde{f} - \tilde{f}_{l+1}\right|^2 \mathrm{d}z \right) \\
&\quad\le C\Big(\frac{\rho^{k+1}}{\rho^k}\Big)^{n+6} \sum_{l=k}^{\infty} \rho^{l(n+4)} \omega_{\tilde{h}}^2(\ln \rho^l) \\
&\quad\le C\rho^{(k+1)(n+6)} \rho^{-2k} \sum_{l=k}^{\infty} \omega_{\tilde{h}}^2(\ln \rho^l),
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\left(\rho^{-(n+6)(k+1)} \int_{\mathcal{Q}_{k+1}} \left|\tilde{f}_k - p_2^{(0)}[\tilde{f}] - p_2^{(0)}\big[\tilde{f}_k - p_2^{(0)}[\tilde{f}]\big]\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} \\
&\qquad\qquad \le C\rho^{-k} \sum_{l=k}^{\infty} \omega_{\tilde{h}}(\ln \rho^l) \le C\rho^{-k} \int_{-\infty}^{\ln \rho} \omega_{\tilde{h}}(\xi)\, \mathrm{d}\xi.
\end{aligned} \tag{4.63}$$

Thus due to (4.55), (4.61) and (4.63) we conclude

$$\left(\rho^{-(n+6)(k+1)} \int_{\mathcal{Q}_{k+1}} \left|\tilde{f} - p_2^{(0)}[\tilde{f}]\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} \le C\rho^{-(k+1)} \int_{-\infty}^{\ln \rho} \omega_{\tilde{h}}(\xi)\, \mathrm{d}\xi + C \int_{\ln \rho}^{0} \omega_{\tilde{h}}(\xi) e^{-\xi}\, \mathrm{d}\xi.$$

The right hand side will further be bounded using the explicit form of $\tilde{h}$ in (4.58):

$$\begin{aligned}
&\rho^{-(k+1)} \int_{-\infty}^{\ln \rho} \omega_{\tilde{h}}(\xi)\, \mathrm{d}\xi + \int_{\ln \rho}^{0} \omega_{\tilde{h}}(\xi) e^{-\xi}\, \mathrm{d}\xi \\
&\qquad\lesssim \left(\rho^{-(k+1)} \int_{-\infty}^{\ln \rho} \omega_A(\xi)\, \mathrm{d}\xi + \int_{\ln \rho}^{0} \omega_A(\xi) e^{\xi}\, \mathrm{d}\xi\right) \sum_{1\le i,j\le d_0} \sup_{Q_1} \left|\partial_{x_i x_j}^2 f\right| \\
&\qquad\quad + \left(1 + \rho^{-(k+1)} \int_{-\infty}^{\ln \rho} \omega_c(\xi)\, \mathrm{d}\xi + \int_{\ln \rho}^{0} \omega_c(\xi) e^{-\xi}\, \mathrm{d}\xi\right) \sup_{Q_1} \left|f\right| \\
&\qquad\quad + \left(\rho^{-(k+1)} \int_{-\infty}^{\ln \rho} \omega_B(\xi)\, \mathrm{d}\xi + \int_{\ln \rho}^{0} \omega_B(\xi) e^{-\xi}\, \mathrm{d}\xi\right) \sum_{1\le i\le d_0} \sup_{Q_1} \left|\partial_{x_i} f\right| \\
&\qquad\quad + \rho^{-(k+1)} \int_{-\infty}^{\ln \rho} \omega_h(\xi)\, \mathrm{d}\xi + \int_{\ln \rho}^{0} \omega_h(\xi) e^{-\xi}\, \mathrm{d}\xi + \sup_{Q_1} |h|\,.
\end{aligned}$$

For the left hand side we find for z, z_0 such that $d_\ell(z, z_0) \leq 1/2$ upon choosing $\rho = d_\ell(z, z_0)$

$$\begin{aligned}\frac{\left|D^2 f(z) - D^2 f(z_0)\right|^2}{d_\ell(z, z_0)^2} &\leq C[f]^2_{C_\ell^{2+1-}(Q_\rho)} \leq C \inf_{p \in \mathcal{P}_2} \rho^{-(n+6)} \int_{Q_\rho} \left|\tilde{f} - p\right|^2 \mathrm{d}z \\ &\leq C\rho^{-(n+6)} \int_{Q_\rho} \left|\tilde{f} - p_2^{(0)}[\tilde{f}]\right|^2 \mathrm{d}z,\end{aligned}$$

where we used Lemma 4.1 and the characterisation of Campanato norms in Theorem 4.4. This concludes the proof of (4.53).

4.8 Appendix: Relation between Hölder and Campanato spaces

This section is devoted to the proof of the equivalence between kinetic Campanato and Hölder spaces, as stated in Theorem 4.4. We follow Campanato's arguments from [40]. We recall the notation $\Omega(z_0, r) := \Omega \cap Q_r(z_0)$ for any subset $\Omega \subset \mathbb{R}^n$. Throughout this section we will denote $\Omega = Q_R(\tilde{z}_0)$ as in the statement of Theorem 4.4.

4.8.1 Auxiliary result

We start with a preliminary lemma, which in the elliptic case has first been derived by De Giorgi [40, Lemma 2.1].

Lemma 4.4. *For a polynomial $P \in \mathcal{P}_k$, a real number $q \geq 1$, $z_0 \in \mathbb{R}^{1+2d}$, and $\rho > 0$ there exists a constant c such that*

$$\left|(\partial_t + v \cdot \nabla_x)^{j_0} \partial_{x_1}^{j_1} \cdots \partial_{x_d}^{j_d} \partial_{v_1}^{j_{d+1}} \cdots \partial_{v_d}^{j_{2d}} P(z)\big|_{z=z_0}\right|^q \leq \frac{c}{\rho^{n+|J|q}} \int_{Q_\rho(z_0)} |P(z)|^q \, \mathrm{d}z,$$

where $|J| = 2s \cdot j_0 + (1+2s)\,|(j_1, \ldots, j_d)| + |(j_{d+1}, \ldots, j_{2d})|$.

Proof. Let $\mathcal{T}_k \subset \mathcal{P}_k$ be the subset of k-degree polynomials such that

$$\sum_{|J| \leq k} |a_j|^2 = 1, \tag{4.64}$$

where we recall that a_j are the coefficients of an element $p \in \mathcal{P}_k$, which can be written as in (4.11). Let $\mathcal{F}$ denote the class of measurable functions $f : \mathbb{R}^n \to [0, 1]$ compactly supported on Q_1 such that $\int_{\mathbb{R}^n} f(z)\,\mathrm{d}z \geq A$, where $A = |Q_\rho(z_0)|\,\rho^{-n}$. Let $\gamma(A) = \inf_{P \in \mathcal{T}_k, f \in \mathcal{F}} \int_{Q_1} |P(z)|^q f(z)\,\mathrm{d}z$. We want to show that

$$\gamma(A) = \min_{P \in \mathcal{T}_k, f \in \mathcal{F}} \int_{Q_1} |P(z)|^q f(z)\,\mathrm{d}z. \tag{4.65}$$

For any integer m there exists $P_m \in \mathcal{T}_k$ and $f_m \in \mathcal{F}$ such that

$$\gamma(A) \leq \int_{Q_1} |P_m(z)|^q \, f_m(z) \, \mathrm{d}z < \gamma(A) + \frac{1}{m}. \tag{4.66}$$

Due to the normalisation (4.64) we can extract a subsequence $\{P_\nu\}$ of $\{P_m\}$ converging uniformly on compact subsets of $\mathbb{R}^n$ to $P^* \in \mathcal{T}_k$. Similarly, since $0 \leq f \leq 1$ we can extract another subsequence $\{f_\mu\}$ of $\{f_\nu\}$ converging weakly in $L^2(Q_1)$ to some $f^* \in \mathcal{F}$. The subsequence will still satisfy (4.66), so that taking the limit yields

$$\gamma(A) = \int_{Q_1} |P^*(z)|^q \, f^*(z) \, \mathrm{d}z.$$

This proves the claim (4.65). It follows that $\gamma(A) > 0$. Moreover, for z_0 and ρ such that $Q_\rho(z_0) \subset Q_1$, and for $P \in \mathcal{T}_k$ there holds

$$\gamma(A) \leq \int_{Q_\rho(z_0)} P(z) \, \mathrm{d}z.$$

since $|Q_\rho(z_0)| \geq A\rho^n$. If $P \in \mathcal{P}_k$ then $P(z) \cdot \left\{ \sum_{|J| \leq k} |a_j|^2 \right\}^{-\frac{1}{2}} \in \mathcal{T}_k$ and thus

$$\left\{ \sum_{|J| \leq k} |a_j|^2 \right\}^{\frac{q}{2}} \leq \frac{1}{\gamma(A)} \int_{Q_\rho(z_0)} |P(z)|^q \, \mathrm{d}z,$$

or also

$$|a_j|^q \leq \frac{1}{\gamma(A)} \int_{Q_\rho(z_0)} |P(z)|^q \, \mathrm{d}z, \quad \forall \, |J| \leq k. \tag{4.67}$$

Now let $P \in \mathcal{P}_k$. Denote with $(s, y, w) = T(t, x, v)$ the transformation respecting the Lie group structure

$$\tilde{z} := (s, y, w) = \left(\frac{t - t_0}{\rho^{2s}}, \frac{x - x_0 - (t - t_0)v_0}{\rho^{1+2s}}, \frac{v - v_0}{\rho} \right) = \left(z_0^{-1} \circ z \right)_{\frac{1}{\rho}}.$$

Then

$$\begin{aligned} \int_{Q_\rho(z_0)} |P(z)|^q \, \mathrm{d}z &= \rho^n \int_{T(Q_\rho(z_0))} \left| P(\rho^{2s} s + t_0, \rho^{1+2s} y + x_0 + (t - t_0)v_0, \rho w + v_0) \right|^q \, \mathrm{d}\tilde{z} \\ &= \rho^n \int_{T(Q_\rho(z_0))} \left| P\big(z_0 \circ \tilde{z}_\rho\big) \right|^q \, \mathrm{d}\tilde{z}. \end{aligned} \tag{4.68}$$

We note that $T(Q_\rho(z_0)) \subset Q_1$, $|T(Q_\rho(z_0))| = \rho^{-n} \int_{Q_\rho(z_0)} \mathrm{d}z \geq A$ and for $J_1 := (j_1, \ldots, j_d)$,

$J_2 := (j_{d+1}, \dots, j_{2d})$

$$P\big(z_0 \circ \tilde{z}_\rho\big) = \sum_{|J|\le k} \frac{(\partial_t + v\cdot\nabla_x)^{j_0}\partial_{x_1}^{j_1}\cdots\partial_{x_d}^{j_d}\partial_{v_1}^{j_{d+1}}\cdots\partial_{v_d}^{j_{2d}}P(z)|_{z=z_0}}{j!}\rho^{2s\cdot j_0}\rho^{(1+2s)\cdot|J_1|}\rho^{|J_2|}\tilde{z}^j.$$

Equations (4.67) and (4.68) then imply

$$\begin{aligned}&\Big|(\partial_t + v\cdot\nabla_x)^{j_0}\partial_{x_1}^{j_1}\cdots\partial_{x_d}^{j_d}\partial_{v_1}^{j_{d+1}}\cdots\partial_{v_d}^{j_{2d}}P(z)|_{z=z_0}\Big|^q \\ &\quad\le \frac{(j!)^q}{\rho^{n+q[2sj_0+(1+2s)|J_1|+|J_2|]}\gamma(A)}\int_{Q_\rho(z_0)}|P(z)|^q\,\mathrm{d}z \quad \forall j.\end{aligned}$$

□

4.8.2 Expansion of f

We let $f \in \mathcal{L}_k^{q,\lambda}(\Omega)$. For all $z_0 \in \bar{\Omega}$ and for all $\rho \in [0, \operatorname{diam}\Omega]$ we show the existence of a unique polynomial $P_k(z, z_0, \rho, f)$ such that

$$\inf_{p\in\mathcal{P}_k}\int_{\Omega(x_0,\rho)}|f(z)-p(z)|^q\,\mathrm{d}z = \int_{\Omega(x_0,\rho)}|f(z)-P_k(z,z_0,\rho,f)|^q\,\mathrm{d}z. \tag{4.69}$$

Indeed, $P_k(z, z_0, \rho, f)$ is the kinetic Taylor expansion of f at z_0. Let $P \in \mathcal{P}_k$ and write

$$P(z) = \sum_{j\in\mathbb{N}^{1+2d},|J|\le k}\frac{a_j(z_0)}{j!}(z-z_0)^j.$$

We denote

$$h(\{a_j\}) = \|f - P\|_{L^q(\Omega(z_0,\rho))},$$

where $\Omega(z_0, \rho) = Q_R(\tilde{z}_0) \cap Q_\rho(z_0)$ with $Q_R(\tilde{z}_0)$ as in the statement of Theorem 4.4. Note that h is a non-negative continuous real function of the coefficients of P. The infimum of h will be attained in a compact set containing the origin, so that the existence of P_k follows standardly. The uniqueness of P_k follows by uniform convexity of the Lebesgue spaces L^q. We will denote the coefficients of $P_k(z, z_0, \rho, f)$ with $a_j(z_0, \rho)$. Note that they are given by

$$a_j(z_0,\rho,f) = (\partial_t + v\cdot\nabla_x)^{j_0}\partial_{x_1}^{j_1}\cdots\partial_{x_d}^{j_d}\partial_{v_1}^{j_{d+1}}\cdots\partial_{v_d}^{j_{2d}}P_k(z,z_0,\rho,f)\big|_{z=z_0}. \tag{4.70}$$

Lemma 4.5. *For $f \in \mathcal{L}_k^{q,\lambda}(\Omega)$ there exists a constant $c(q, \lambda) > 0$ such that for any $z_0 \in \Omega$ and $0 < \rho \le$ diam Ω and $l \in \mathbb{N}_0$ there holds*

$$\int_{\Omega(z_0,\rho 2^{-(l-1)})}\big|P_k(z,z_0,\rho 2^{-l},f) - P_k(z,z_0,\rho 2^{-l-1},f)\big|^q\,\mathrm{d}z \le c2^{-l\lambda}\rho^\lambda[f]^q_{\mathcal{L}_k^{q,\lambda}}.$$

Proof. For all $z \in \Omega\big(z_0, \rho 2^{-(l-1)}\big)$ there holds

$$\begin{aligned}&\big|P_k(z, z_0, \rho 2^{-l}, f) - P_k(z, z_0, \rho 2^{-l-1}, f)\big|^q \\ &\leq 2^q \left|P_k(z, z_0, \rho 2^{-l}, f) - f(z)\right|^q + 2^q \left|P_k(z, z_0, \rho 2^{-(l-1)}, f) - f(z)\right|^q\end{aligned}$$

Thus

$$\begin{aligned}\int_{\Omega(z_0, \rho 2^{-(l-1)})} \big|P_k(z, z_0, \rho 2^{-l}, f) - P_k(z, z_0, \rho 2^{-l-1}, f)\big|^q \, \mathrm{d}z &\leq 2^q [f]^q_{\mathcal{L}^{q,\lambda}_k} \big(2^{-l\lambda}\rho^\lambda + 2^{(-l-1)\lambda}\rho^\lambda\big) \\ &= 2^q(1 + 2^{-\lambda})2^{-l\lambda}\rho^\lambda [f]^q_{\mathcal{L}^{q,\lambda}_k}.\end{aligned}$$

□

Lemma 4.6. *Suppose $f \in \mathcal{L}^{q,\lambda}_k(\Omega)$. Then for any $z_0, z_1 \in \bar{\Omega}$ and for any multi-index l such that $|L| = k$ with $|L| = 2s \cdot l_0 + (1 + 2s)\,|L_1| + |L_2|$ there holds*

$$|a_l(z_0, 2d_\ell(z_0, z_1), f) - a_l(z_1, 2d_\ell(z_0, z_1), f)|^q \leq c 2^{q+1+\lambda} [f]^q_{\mathcal{L}^{q,\lambda}_k} d_\ell(z_0, z_1)^{\lambda - n - kq}, \tag{4.71}$$

where d_ℓ is the kinetic distance defined in 4.1.

Proof. Let $z_0, z_1 \in \bar{\Omega}$. We write $\rho = d_\ell(z_0, z_1)$ and $I_\rho = \Omega(z_0, 2\rho) \cap \Omega(z_1, 2\rho)$. Then we have

$$\begin{aligned}&|P_k(z, z_0, 2\rho, f) - P_k(z, z_1, 2\rho, f)|^q \\ &\leq 2^q \,|P_k(z, z_0, 2\rho, f) - f(z)|^q + 2^q \,|P_k(z, z_1, 2\rho, f) - f(z)|^q \,.\end{aligned}$$

Integrating over $\Omega(z_0, \rho) \subset I_\rho$ we obtain

$$\begin{aligned}&\int_{\Omega(z_0,\rho)} |P_k(z, z_0, 2\rho, f) - P_k(z, z_1, 2\rho, f)|^q \, \mathrm{d}z \\ &\leq 2^q \int_{\Omega(z_0,\rho)} |P_k(z, z_0, 2\rho, f) - f(z)|^q \, \mathrm{d}z + 2^q \int_{\Omega(z_0,\rho)} |P_k(z, z_1, 2\rho, f) - f(z)|^q \, \mathrm{d}z \\ &\leq 2^{q+\lambda+1} \rho^\lambda [f]^q_{\mathcal{L}^{q,\lambda}_k}.\end{aligned} \tag{4.72}$$

On the other hand, by (4.70), and Lemma 4.4 applied to $P(z) = P_k(z, z_0, 2\rho, f) - P_k(z, z_1, 2\rho, f)$ and since the k-th derivative of a polynomial of degree k is constant, we have

$$\begin{aligned}&\big|a_l\big(z_0, 2d_\ell(z_0, z_1), f\big) - a_l\big(z_1, 2d_\ell(z_0, z_1), f\big)\big|^q \\ &\leq c\rho^{-(n+kq)} \int_{\Omega(z_0,\rho)} |P_k(z, z_0, 2\rho, f) - P_k(z, z_1, 2\rho, f)|^q \, \mathrm{d}z.\end{aligned} \tag{4.73}$$

Finally, the combination of (4.72) and (4.73) implies (4.71) and concludes the proof. □

Lemma 4.7. *Let $f \in \mathcal{L}_k^{q,\lambda}(\Omega)$. Then there exists a constant c such that for all $z_0 \in \bar{\Omega}, 0 < \rho \leq$ diam Ω, $i \in \mathbb{N}$ and multi-index $l \in \mathbb{N}^{1+2d}$ with $|L| \leq k$ there holds*

$$\left|a_l(z_0,\rho,f) - a_l(z_0,\rho 2^{-i},f)\right| \leq c[f]_{\mathcal{L}_k^{q,\lambda}} \sum_{m=0}^{i-1} 2^{m\left(\frac{n+|L|q-\lambda}{q}\right)} \rho^{\frac{\lambda-n-|L|q}{q}}.$$

Proof. We have

$$\left|a_l(z_0,\rho,f) - a_l(z_0,\rho 2^{-i},f)\right| \leq \sum_{m=0}^{i-1} \left|a_l(z_0,\rho 2^{-m},f) - a_l(z_0,\rho 2^{-m-1},f)\right|.$$

Using the relation (4.70) and applying Lemma 4.4 to $P_k(z,z_0,\rho 2^{-m},f) - P_k(z,z_0,\rho 2^{-m-1},f)$ we get

$$\begin{aligned}
&\left|a_l(z_0,\rho,f) - a_l(z_0,\rho 2^{-i},f)\right| \\
&\quad \leq c\rho^{-\frac{n}{q}-|L|} \sum_{m=0}^{i-1} 2^{(m+1)\left(\frac{n}{q}+|L|\right)} \\
&\qquad \times \left[\int_{\Omega(z_0,\rho 2^{-m-1})} \left|P_k(z,z_0,\rho 2^{-m},f) - P_k(z,z_0,\rho 2^{-m-1},f)\right|^q \,\mathrm{d}z\right]^{\frac{1}{q}}.
\end{aligned}$$

We conclude using Lemma 4.5. □

Now we can prove the following useful lemma.

Lemma 4.8. *Let $f \in \mathcal{L}_k^{q,\lambda}(\Omega)$ such that $n + \tilde{k}q < \lambda \leq n + (\tilde{k}+1)q$ where $0 \leq \tilde{k} \leq k$. Then there exists functions $\{g_j(z_0)\}$ for $j \in \mathbb{N}^{1+2d}$ with $|J| \leq \tilde{k}$ such that for all $0 < \rho \leq$ diam $\bar{\Omega}, z_0 \in \bar{\Omega}$ there holds*

$$|a_j(z_0,\rho,f) - g_j(z_0)| \leq c(\lambda,q,k,n,B)\rho^{\frac{\lambda-n-|J|q}{q}}[f]_{\mathcal{L}_k^{q,\lambda}}. \tag{4.74}$$

As a consequence, there holds

$$\lim_{\rho\to 0} a_j(z_0,\rho,f) = g_j(z_0), \tag{4.75}$$

uniformly with respect to z_0.

Proof. We show that the sequence $\{a_j(z_0,\rho 2^{-i},f)\}$ converges in the limit $i \to \infty$. Let i_1, i_2 be two non-negative integers and assume without loss in generality that $i_2 > i_1$. With Lemma 4.7 we obtain

$$\left|a_j(z_0,\rho 2^{-i_2},f) - a_j(z_0,\rho 2^{-i_1},f)\right| \leq c[f]_{\mathcal{L}_k^{q,\lambda}} \sum_{m=i_1}^{i_2-1} 2^{m\left(\frac{n+|J|q-\lambda}{q}\right)} \rho^{\frac{\lambda-n-|J|q}{q}}.$$

Since $|J| \leq p = \tilde{k}$ and $\lambda > n + \tilde{k}q$ the series $\sum_{m=0}^{\infty} 2^{m\left(\frac{n+|J|q-\lambda}{q}\right)}$ converges. Thus $\{a_j(z_0, \rho 2^{-i}, f)\}$ is a Cauchy sequence and hence converges as $i \to \infty$.

We now show that the limit is uniform in ρ. Let ρ_1 and ρ_2 be such that $0 < \rho_1 \leq \rho_2 \leq \operatorname{diam} \Omega$. With Lemma 4.4 we get

$$\begin{aligned}
&\left|a_j(z_0, \rho_1 2^{-i}, f) - a_j(z_0, \rho_2 2^{-i}, f)\right|^q \\
&\leq c \frac{2^{i(n+|J|q)}}{\rho_1^{n+|J|q}} \int_{\Omega(z_0, \rho_1 2^{-i})} \left|P_k(z, z_0, \rho_1 2^{-i}, f) - P_k(z, z_0, \rho_2 2^{-i}, f)\right|^q \, \mathrm{d}z \\
&\leq c \frac{2^{i(n+|J|q)}}{\rho_1^{n+|J|q}} \Bigg[\int_{\Omega(z_0, \rho_1 2^{-i})} \left|P_k(z, z_0, \rho_1 2^{-i}, f) - f(z)\right|^q \, \mathrm{d}z \\
&\qquad\qquad + \int_{\Omega(z_0, \rho_2 2^{-i})} \left|P_k(z, z_0, \rho_2 2^{-i}, f) - f(z)\right|^q \, \mathrm{d}z \Bigg] \\
&\leq c 2^q \frac{\rho_1^\lambda + \rho_2^\lambda}{\rho_1^{n+|J|q}} 2^{-i(\lambda - n - |J|q)} [f]_{\mathcal{L}_k^{q,\lambda}} \to 0,
\end{aligned}$$

as $i \to \infty$ since $\lambda - n - |J|\, q > 0$.

Thus for $z_0 \in \bar{\Omega}, 0 < \rho \leq \operatorname{diam}(\Omega)$ and $|J| \leq \tilde{k}$ we can take

$$g_j(z_0) = \lim_{i \to \infty} a_j(z_0, \rho 2^{-i}, f). \tag{4.76}$$

The sequence $g_j(z_0)$ is well-defined in $\bar{\Omega}$. Since the series $\sum_{m=0}^{\infty} 2^{m\left(\frac{n+|J|q-\lambda}{q}\right)}$ converges, we deduce from Lemma 4.7

$$\left|a_j(z_0, \rho, f) - a_j(z_0, \rho 2^{-i}, f)\right| \leq c [f]_{\mathcal{L}_k^{q,\lambda}} \rho^{\frac{\lambda - n - |J|q}{q}}. \tag{4.77}$$

Combining (4.76) and (4.77) yields the result. □

4.8.3 The function $g_j(z_0)$

We have the following theorem.

Theorem 4.7. *Let $f \in \mathcal{L}_k^{q,\lambda}(\Omega)$ with $n + kq < \lambda$. Then the functions $g_j(z_0)$ with $|J| = k$ are Hölder continuous in $\bar{\Omega}$ and for any $z_1, z_2 \in \bar{\Omega}$ there holds*

$$|g_j(z_1) - g_j(z_2)| \leq c [f]_{\mathcal{L}_k^{q,\lambda}} d_\ell(z_1, z_2)^{\frac{\lambda - n - kq}{q}}. \tag{4.78}$$

Proof. Take $z_1, z_2 \in \bar{\Omega}$ such that $\rho = d_\ell(z_1, z_2) \leq \frac{\operatorname{diam} \Omega}{2}$. Then

$$|g_j(z_1) - g_j(z_2)| \leq |g_j(z_1) - a_j(z_1, 2\rho)| + |g_j(z_2) - a_j(z_2, 2\rho)| + |a_j(z_1, 2\rho) - a_j(z_2, 2\rho)|\,.$$

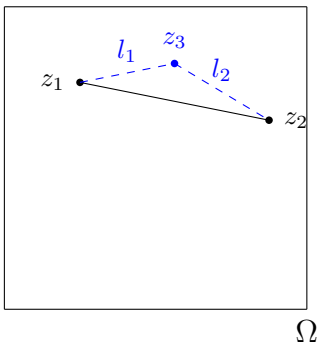


Figure 4.1: In case that $d_l(z_1, z_2) > \frac{\operatorname{diam}\Omega}{2}$ we construct a polygon with side lengths l_1, l_2 such that $l_1, l_2 \leq \frac{\operatorname{diam}\Omega}{2}$.

On the one hand, by (4.74) we have

$$|g_j(z_1) - a_j(z_1, 2\rho)| \leq c2^{\frac{\lambda-n-kq}{q}} \rho^{\frac{\lambda-n-kq}{q}} [f]_{\mathcal{L}_k^{q,\lambda}},$$

and

$$|g_j(z_2) - a_j(z_2, 2\rho)| \leq c2^{\frac{\lambda-n-kq}{q}} \rho^{\frac{\lambda-n-kq}{q}} [f]_{\mathcal{L}_k^{q,\lambda}}.$$

On the other hand (4.71) implies

$$|a_j(z_1, 2\rho) - a_j(z_2, 2\rho)| \leq c2^{\frac{q+1+\lambda}{q}} \rho^{\frac{\lambda-n-kq}{q}} [f]_{\mathcal{L}_k^{q,\lambda}}.$$

This yields the result in case that $d_\ell(z_1, z_2) \leq \frac{\operatorname{diam}\Omega}{2}$.

In case that $d_l(z_1, z_2) > \frac{\operatorname{diam}\Omega}{2}$ we can construct a polygon contained in $\bar{\Omega}$ with extremal points z_1 and z_2 and with sides of length smaller or equal to $\frac{\operatorname{diam}\Omega}{2}$, see Figure 4.1. The length of the sides can be bounded by diam Ω uniformly with respect to z_1 and z_2. Thus to conclude it suffices to apply (4.78) to all points at the end of the sides of such a polygonal. □

For the sequel, we denote by (0) the d-tuple $(0, \ldots, 0)$ and by e_i the vector in $\mathbb{R}^d$ with the i-th coordinate equal to 1 and else 0. We also note that any polynomial degree $k \in \mathbb{N} + 2s\mathbb{N}$ can be written as $k = 2s \cdot k_0 + (1 + 2s) \cdot k_1 + k_2$ with $k_0, k_1, k_2 \in \mathbb{N}$.

Theorem 4.8. *Let $f \in \mathcal{L}_k^{q,\lambda}(\Omega)$ with $k_0, k_1, k_2 \geq 1$ and $n + kq < \lambda$. Then for any multi-index $j \in \mathbb{N}^{1+2d}$ such that $|J| \leq k$ the function g_j has a first partial derivative in Ω, and for any $z \in \Omega$ and $i = 1, \ldots, d$ there holds*

$$\begin{aligned}
\mathcal{T} g_j(z) &= g_{(j_0+1, J_1, J_2)}(z), && j_0 \leq k_0 - 1, |J_1| \leq k_1, |J_2| \leq k_2 \\
\frac{\partial g_j(z)}{\partial x_i} &= g_{(j_0, J_1+e_i, J_2)}(z), && j_0 \leq k_0, |J_1| \leq k_1 - 1, |J_2| \leq k_2 \\
\frac{\partial g_j(z)}{\partial v_i} &= g_{(0, J_1, J_2+e_i)}(z), && j_0 = 0, |J_1| \leq k_1, |J_2| \leq k_2 - 1.
\end{aligned} \tag{4.79}$$

Proof. For this proof we omit the dependency on f in the coefficients $a_j(z_0,\rho,f)$ and $P_k(z,z_0,\rho,f)$ and simply write $a_j(z_0,\rho)$ and $P_k(z,z_0,\rho)$, respectively.

Step 1. We will start proving the first line. We consider $j=(j_0,J_1,J_2)$ for $j_0\le k_0-1, |J_1|=k_1, |J_2|=k_2$. Theorem 4.7 proves that $g_{(k_0,J_1,J_2)}$ is Hölder continuous in a classical sense for $|J_1|=k_1, |J_2|=k_2$ and in particular continuous. Thus we may assume that $g_{(j_0+\delta,J_1,J_2)}$ is continuous in $\bar\Omega$ for $\delta=1,\dots,k_0-j_0$. Let $z_0\in\Omega$ and ρ be such that $B_{|\rho|}(z_0)\subset\Omega$. By (4.70) we have

$$\begin{aligned}&\frac{a_j\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-a_j(z_0,2\,|\rho|)}{\rho}\\&=\frac{D^j\big[P_k\big(z,z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-P_k(z,z_0,2\,|\rho|)\big]}{\rho}\\&\qquad-\sum_{\delta=1}^{k_0-j_0}\frac{(-1)^\delta}{\delta!}\rho^{\delta-1}a_j\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big).\end{aligned}\tag{4.80}$$

With Lemma 4.4 and (4.72) we obtain

$$\begin{aligned}&\left|\frac{D^j\big[P_k\big(z,z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-P_k(z,z_0,2\,|\rho|)\big]}{\rho}\right|^q\\&\le c|\rho|^{-n-|J|q}\int_{\Omega(z_0,|\rho|)}\left|P_k\big(z,z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-P_k(z,z_0,2\,|\rho|)\right|^q\mathrm{d}z\\&\le c2^{q+\lambda+1}\,|\rho|^{\lambda-n-|J|q}\,[f]_{\mathcal{L}^{q,\lambda}_k}.\end{aligned}\tag{4.81}$$

Moreover, for $1\le\delta\le k_0-j_0$ there holds

$$\begin{aligned}&\left|a_{(j_0+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-g_{(j_0+\delta,J_1,J_2)}(z_0)\right|\\&\qquad\le\left|a_{(j_0+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-g_{(j_0+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0))\big)\right|\\&\qquad\quad+\left|g_{(j_0+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0))\big)-g_{(j_0+\delta,J_1,J_2)}(z_0)\right|.\end{aligned}\tag{4.82}$$

Using (4.74) we can estimate the first term on the right hand side of (4.82) by

$$\begin{aligned}&\left|a_{(j_0+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-g_{(j_0+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0))\big)\right|\\&\qquad\le c2^{\frac{\lambda-n-(|J|+2s\delta)}{q}}\,|\rho|^{\frac{\lambda-n-(|J|+2s\delta)}{q}}\,[f]_{\mathcal{L}^{q,\lambda}_k}.\end{aligned}\tag{4.83}$$

From (4.82) and (4.83) and since by induction hypothesis $g_{(j_0+\delta,J_1,J_2)}$ are continuous for $\delta=1,\dots,k_0-j_0$ we have

$$\lim_{\rho\to0}a_{(j+\delta,J_1,J_2)}\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)=g_{(j_0+\delta,J_1,J_2)}(z_0)\qquad\delta=1,\dots,k_0-j_0.\tag{4.84}$$

Thus from (4.80), (4.81) and (4.84) we deduce that

$$\lim_{\rho\to 0}\frac{a_j\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-a_j(z_0,2\,|\rho|)}{\rho}=g_{(j_0+1,J_1,J_2)}(z_0),$$

uniformly in z_0. Thus if we can show that

$$\lim_{\rho\to 0}\frac{g_j\big(z_0+(\rho,(0),(0))\big)-g_j(z_0)}{\rho}=\lim_{\rho\to 0}\frac{a_j\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)-a_j(z_0,2\rho)}{\rho},\quad(4.85)$$

then we can conclude the proof of the first line of (4.79). We first notice that by (4.74)

$$\begin{aligned}&\left|\frac{g_j\big(z_0+(\rho,(0),(0))\big)-a_j\big(z_0+(\rho,(0),(0)),2\,|\rho|\,\big)}{\rho}\right|\\&\qquad\le c2^{\frac{\lambda-n-|J|q}{q}}\,|\rho|^{\frac{\lambda-n}{q}-|J|-1}\,[f]_{\mathcal{L}^{q,\lambda}_k},\end{aligned}\quad(4.86)$$

and

$$\left|\frac{g_j(z_0)-a_j(z_0,2\,|\rho|)}{\rho}\right|\le c2^{\frac{\lambda-n-|J|q}{q}}\,|\rho|^{\frac{\lambda-n}{q}-|J|-1}\,[f]_{\mathcal{L}^{q,\lambda}_k}.\quad(4.87)$$

Thus with the triangle inequality (4.86) and (4.87) imply (4.85), which in turn implies the first line of(4.79).

Step 2. To prove the second statement in (4.79) we proceed as in Step 1. Now we consider $j_0=1,\dots,k_0,|J_1|\le k_1-1$ and $|J_2|=k_2$. We have shown that g_j is continuous for $j_0=1,\dots,k_0,|J_1|=k_1,|J_2|=k_2$. Assume then that $g_{(j_0,J_1+\delta e_i,J_2)}$ is continuous in $\bar\Omega$ for $\delta=1,\dots,k_1-|J_1|$. We again have by (4.70)

$$\begin{aligned}&\frac{a_j\big(z_0+\rho(0,e_i,(0)),2\,|\rho|\,\big)-a_j(z_0,2\,|\rho|)}{\rho}\\&=\frac{D^j\big[P_k\big(z,z_0+\rho(0,e_i,(0)),2\,|\rho|\,\big)-P_k(z,z_0,2\,|\rho|)\big]}{\rho}\\&\quad-\sum_{\delta=1}^{k_1-|J_1|}\frac{(-1)^\delta}{\delta!}\rho^{\delta-1}a_j\big(z_0+\rho(0,e_i,(0)),2\,|\rho|\,\big).\end{aligned}\quad(4.88)$$

The proof is exactly the same if we replace $(\rho,(0),(0))$ with $\rho(0,e_i,(0))$, k_0-j_0 with $k_1-|J_1|$ and instead of $2s\delta$ in the exponent of (4.83) we get $(1+2s)\delta$.

Step 3. To deduce the final statement in (4.79) the ideas are the same but the statement only holds for $j_0=0$ since $\mathcal{T}$ and D_v do not commute. Therefore it was important to prove the first statement first, since now we know that g_j is continuous for $j_0=0,|J_1|\le k_1$ and $|J_2|=k_2$. We now assume that $g_{(j_0,J_1,J_2+\delta e_i)}$ is continuous in $\bar\Omega$ for $\delta=1,\dots,k_2-|J_2|$. Replacing $(\rho,(0),(0))$ with $\rho(0,(0),e_i)$, k_0-j_0 with $k_2-|J_2|$ and $2s\delta$ in the exponent of (4.83) with δ, but otherwise proceeding as above, we conclude.

Finally, combining the argument for the continuity of g_j in all three steps yields the improvement in ranges of $|J_1|$ and $|J_2|$ as stated in the theorem. □

As a corollary of Theorem 4.7 and 4.8 we get

Theorem 4.9. *Let $f \in \mathcal{L}^{q,\lambda}_k(\Omega)$ with $n + kq < \lambda$. Then the function $g_{(0)} \in C^\beta_\ell(\bar{\Omega})$ where $\beta = \frac{\lambda - n}{q}$ and there holds*

$$\mathcal{T}^{j_0} D_x^{J_1} D_v^{J_2} g_{(0)}(z) = g_j(z) \qquad \forall z \in \Omega,\ \forall\, |J| \le k.$$

Recall $j = (j_0, J_1, J_2) \in \mathbb{N}^{1+2d}$ and $|J| = 2s \cdot j_0 + (1+2s) \cdot |J_1| + |J_2|$.

Remark 4.9. For $f \in \mathcal{L}^{q,\lambda}_k(\Omega)$ with $n + (k+1)q < \lambda$ we deduce from (4.78) that g_j with $|J| = k$ are constant and thus by Theorem 4.9, $g_{(0)}$ is a polynomial of kinetic degree at most k.

4.8.4 Comparing the Hölder norm and the Campanato norm

Theorem 4.10. *Let $f \in \mathcal{L}^{q,\lambda}_k(\Omega)$ with $n + kq < \lambda \le n + (k+1)q$. Then $f \in C^\beta_\ell(\Omega)$ where $\beta = \frac{\lambda - n}{q}$ and there holds*

$$[f]_{C^\beta_\ell} \le c [f]_{\mathcal{L}^{q,\lambda}_k}. \tag{4.89}$$

If $\lambda > n + (k+1)q$ then f is a polynomial of kinetic degree at most k.

Proof. Due to Theorem 4.9 and Remark 4.9 it suffices to show that $f(z) = g_{(0)}(z) = \lim_{\rho \to 0} a_{(0)}(z, \rho)$ for almost every $z \in \Omega$. Then (4.89) follows from (4.78) in Theorem 4.7 and Taylor's formula.

Since $f \in L^q(\Omega)$ there holds for almost every $z_0 \in \Omega$

$$\lim_{\rho \to 0} \frac{1}{|\Omega(z_0, \rho)|} \int_{\Omega(z_0,\rho)} |f(z) - f(z_0)|^q \, \mathrm{d}z = 0. \tag{4.90}$$

Now let $z_0 \in \Omega$ be such that (4.90) holds. Then for almost every $z \in \Omega$ we have

$$\begin{aligned} \big|a_{(0)}(z_0, \rho) - f(z_0)\big|^q \le c \Big(& \big|P_k(z, z_0, \rho) - a_{(0)}(z_0, \rho)\big|^q \\ & + |P_k(z, z_0, \rho) - f(z)|^q + |f(z) - f(z_0)|^q \Big). \end{aligned}$$

Integrating this inequality over $\Omega(z_0,\rho)$ yields

$$\begin{aligned}\left|a_{(0)}(z_0,\rho)-f(z_0)\right|^q \le{}& \frac{c}{A_1\rho^n}\int_{\Omega(z_0,\rho)}\left|P_k(z,z_0,\rho)-a_{(0)}(z_0,\rho)\right|^q\,\mathrm{d}z\\ &+\frac{c}{A_1\rho^n}\int_{\Omega(z_0,\rho)}|P_k(z,z_0,\rho)-f(z)|^q\,\mathrm{d}z\\ &+\frac{c}{A_1\rho^n}\int_{\Omega(z_0,\rho)}|f(z)-f(z_0)|^q\,\mathrm{d}z.\end{aligned}\tag{4.91}$$

By definition of $\mathcal{L}_k^{q,\lambda}$ we have

$$\frac{c}{R^{-n}\left|Q_R(\tilde{z}_0)\right|\rho^n}\int_{\Omega(z_0,\rho)}|P_k(z,z_0,\rho)-f(z)|^q\,\mathrm{d}z \le c\frac{\rho^{\lambda-n}}{R^{-n}\left|Q_R(\tilde{z}_0)\right|}[f]_{\mathcal{L}_k^{q,\lambda}} \xrightarrow[\rho\to 0]{} 0.$$

Due to (4.90) the last integral in (4.91) vanishes as well in the limit $\rho\to 0$. Finally there holds

$$\frac{c}{R^{-n}\left|Q_R(\tilde{z}_0)\right|\rho^n}\int_{\Omega(z_0,\rho)}\left|P_k(z,z_0,\rho)-a_{(0)}(z_0,\rho)\right|^q\,\mathrm{d}z \le c(n,q,k)\sum_{\substack{j\in\mathbb{N}^{1+2d},\\ |J|\le k}}|a_j(z_0,\rho)|^q\,\rho^{|J|q}.$$

Due to (4.75) this integral vanishes in the limit $\rho\to 0$, so that (4.91) gives for almost every $z_0\in\Omega$

$$\lim_{\rho\to 0} a_{(0)}(z_0,\rho) = f(z_0).$$

Equivalently, there holds $f(z)=g_{(0)}(z)$ almost everywhere in Ω. □

Proof of Theorem 4.4. If $f\in\mathcal{L}_k^{p,\lambda}(\Omega)$, then Theorem 4.10 yields $f\in C_\ell^\beta(\Omega)$ and the Hölder semi-norm is bounded above by the Campanato semi-norm (4.89).

Conversely, let $f\in C_\ell^\beta(\bar{\Omega})$ and $P\in\mathcal{P}_k$ where $k=\deg_{\mathrm{kin}}\mathrm{P}<\beta$. For $z\in Q_r(z_0)\cap\Omega$ we have

$$|f(z)-P(z)| \le [f]_{C_\ell^\beta} r^\beta.$$

Thus for $\beta=\frac{\lambda-n}{p}$ there holds

$$\frac{1}{r^\lambda}\int_{Q_r(z_0)\cap\Omega}|f(z)-P(z)|^p\,\mathrm{d}z \le C[f]^p_{C_\ell^\beta} r^{p\beta-\lambda+n} = C[f]^p_{C_\ell^\beta}.$$

□

4.9 Appendix: Interpolation inequality for Hölder spaces

For the sake of completeness, we prove Lemma 4.1 following the arguments of Imbert-Silvestre [117, Proposition 2.10].

Proof of Lemma 4.1. It suffices to prove the statement for β_3 sufficiently close to β_1. Thus we assume that there exists only one element $\bar{\beta} \in \mathbb{N} + 2s\mathbb{N}$ such that $\bar{\beta} \in [\beta_1, \beta_3)$. We know that if $p_z^i \in \mathcal{P}_{\beta_i}$ is the polynomial expansion of f at z of order less than β_i for all $i \in \{1, \dots, 3\}$, then for all $z \circ \xi \in Q_1$

$$\left|f(z \circ \xi) - p_z^i(\xi)\right| \leq [f]_{C_\ell^{\beta_i}} \|\xi\|^{\beta_i}, \quad i = 1, 2, 3. \tag{4.92}$$

The polynomials p_z^i are of increasingly higher order. We assume that the difference of degree of homogeneity of p_z^1 and p_z^3 is at most one, so that p_z^2 coincides with either p_z^1 or p_z^3, depending on whether $\bar{\beta} \geq \beta_2$ or $\bar{\beta} < \beta_2$. If there is no $\bar{\beta}$ then all three polynomials coincide. Let us first assume therefore that there is exactly one $\bar{\beta}$. We have by subtracting (4.92) for $i = 1, 3$ from each other

$$\left|p_z^3(\xi) - p_z^1(\xi)\right| \leq [f]_{C_\ell^{\beta_1}} \|\xi\|^{\beta_1} + [f]_{C_\ell^{\beta_3}} \|\xi\|^{\beta_3}. \tag{4.93}$$

For any $R \in (0, 1]$ and $z \in Q_1$ we pick $\xi_1 \in Q_1$ such that $\|\xi_1\| \leq R$ and whenever $d_\ell(\xi_1, \xi) < cR$, then $\|\xi\| \leq R$ and $z \circ \xi \in Q_1$ with some universal constant c. From (4.93) we then have

$$\sup_{\xi : d_\ell(\xi_1, \xi) \leq cR} \left|p_z^3(\xi) - p_z^1(\xi)\right| \leq [f]_{C_\ell^{\beta_1}} R^{\beta_1} + [f]_{C_\ell^{\beta_3}} R^{\beta_3}.$$

Since $p_z^3 - p_z^1$ is homogeneous of degree $\bar{\beta}$ we get by scaling

$$\sup_{\xi : d_l((\xi_1)_{R^{-1}}, \xi) \leq c} \left|p_z^3(\xi) - p_z^1(\xi)\right| \leq [f]_{C_\ell^{\beta_1}} R^{\beta_1 - \bar{\beta}} + [f]_{C_\ell^{\beta_3}} R^{\beta_3 - \bar{\beta}}.$$

Using the triangle inequality from [117, Prop. 2.2] we can assure that whenever $|\xi| \leq 1$ then $d_\ell\big((\xi_1)_{R^{-1}}, \xi\big) \leq C$ for some universal constant C. Since all norms on the space of polynomials are equivalent, we have

$$\begin{aligned} \left\|p_z^3 - p_z^1\right\| &= \sup_{\xi : \|\xi\| \leq 1} \left|p_z^3(\xi) - p_z^1(\xi)\right| \leq C \sup_{\xi : d_\ell\left((\xi_1)_{R^{-1}}, \xi\right) \leq c} \left|p_z^3(\xi) - p_z^1(\xi)\right| \\ &\leq C[f]_{C_\ell^{\beta_1}} R^{\beta_1 - \bar{\beta}} + C[f]_{C_\ell^{\beta_3}} R^{\beta_3 - \bar{\beta}}. \end{aligned}$$

For

$$R = \left(\frac{[f]_{C_\ell^{\beta_1}}}{[f]_{C_\ell^{\beta_3}}} \right)^{\frac{1}{\beta_3 - \beta_1}}$$

we obtain

$$\left\| p_z^3 - p_z^1 \right\| \leq C [f]_{C_\ell^{\beta_1}}^{\bar{\theta}} [f]_{C_\ell^{\beta_3}}^{1-\bar{\theta}} + [f]_{C_\ell^{\beta_1}},$$

where $\bar{\beta} = \bar{\theta}\beta_1 + (1-\bar{\theta})\beta_3$.

Therefore we can estimate $f - p_z^2$. Assume first $\beta_2 \leq \bar{\beta}$. Then $p_z^2 = p_z^1$ and

$$\left| f(z \circ \xi) - p_z^2(\xi) \right| \leq \begin{cases} [f]_{C_\ell^{\beta_1}} \|\xi\|^{\beta_1}, \\ [f]_{C_\ell^{\beta_3}} \|\xi\|^{\beta_3} + \left([f]_{C_\ell^{\beta_1}}^{\bar{\theta}} [f]_{C_\ell^{\beta_3}}^{1-\bar{\theta}} + [f]_{C_\ell^{\beta_1}} \right) \|\xi\|^{\bar{\beta}}. \end{cases}$$

Now if $\|\xi\| \geq R$ then

$$[f]_{C_\ell^{\beta_1}} \|\xi\|^{\beta_1} \leq [f]_{C_\ell^{\beta_1}}^{\theta} [f]_{C_\ell^{\beta_3}}^{1-\theta} \|\xi\|^{\beta_2}.$$

Else if $\|\xi\| < R$

$$[f]_{C_\ell^{\beta_3}} \|\xi\|^{\beta_3} + \left([f]_{C_\ell^{\beta_1}}^{\bar{\theta}} [f]_{C_\ell^{\beta_3}}^{1-\bar{\theta}} + [f]_{C_\ell^{\beta_1}} \right) \|\xi\|^{\bar{\beta}} \leq [f]_{C_\ell^{\beta_1}}^{\theta} [f]_{C_\ell^{\beta_3}}^{1-\theta} \|\xi\|^{\beta_2} + [f]_{C_\ell^{\beta_1}} \|\xi\|^{\bar{\beta}}.$$

Thus we conclude $|f(z \circ \xi) - p_z^2(\xi)| \leq [f]_{C_\ell^{\beta_1}}^{\theta} [f]_{C_\ell^{\beta_3}}^{1-\theta} \|\xi\|^{\beta_2} + [f]_{C_\ell^{\beta_1}} \|\xi\|^{\beta_2}$.

In case that $\bar{\beta} < \beta_2$

$$\left| f(z \circ \xi) - p_z^2(\xi) \right| \leq \begin{cases} [f]_{C_\ell^{\beta_1}} \|\xi\|^{\beta_1} + \left([f]_{C_\ell^{\beta_1}}^{\bar{\theta}} [f]_{C_\ell^{\beta_3}}^{1-\bar{\theta}} + [f]_{C_\ell^{\beta_1}} \right) \|\xi\|^{\bar{\beta}}, \\ [f]_{C_\ell^{\beta_3}} \|\xi\|^{\beta_3}. \end{cases}$$

and we conclude as above.

In case that no $\bar{\beta}$ exists, then all polynomials coincide and we get

$$\left| f(z \circ \xi) - p_z^2(\xi) \right| \leq [f]_{C_\ell^{\beta_1}}^{\theta} [f]_{C_\ell^{\beta_3}}^{1-\theta} \|\xi\|^{\beta_2}.$$

□

4.10 Appendix: Proof of Bouchut's proposition

For the sake of self-containment, we recall the proof of Proposition 4.4 from [29, Proposition 1.1].

Proof of Proposition 4.4. We denote by $\hat{f}(\eta, k, v)$ the Fourier-transform of a solution f of

(5.19) in time t and space x. Then $\hat{f}$ solves

$$i(\eta + v \cdot k)\hat{f} = \hat{S}.$$

We introduce a smoothing sequence $\rho_1 \in C_c^\infty(\mathbb{R}^d)$ in velocity such that

$$\rho_\varepsilon(v) = \frac{1}{\varepsilon^d}\rho_1\Big(\frac{v}{\varepsilon}\Big), \qquad \int \rho_1 \,\mathrm{d}v = 1, \qquad \int v^\alpha \rho_1 = 0 \text{ for } 1 \le |\alpha| < |\beta|\,. \tag{4.94}$$

For fixed (η, k) we decompose

$$\hat{f}(\eta, k, v) = \Big(\rho_\varepsilon *_v \hat{f}\Big)(\eta, k, v) + \Big(\hat{f} - \Big(\rho_\varepsilon *_v \hat{f}\Big)\Big)(\eta, k, v), \tag{4.95}$$

where $*_v$ denotes the convolution in velocity v. Then by the properties of ρ (4.94) we can bound $|1 - \hat{\rho}_\varepsilon| \le C_{d,\beta}\,|\varepsilon v|^\beta$ so that

$$\left\| \Big(\hat{f} - \Big(\rho_\varepsilon *_v \hat{f}\Big)\Big)(\eta, k, \cdot)\right\|_{L^2(\mathbb{R}^d)} \le C_{d,\beta}\varepsilon^\beta \left\||D_v|^\beta\, \hat{f}(\eta, k, \cdot)\right\|_{L^2(\mathbb{R}^d)}. \tag{4.96}$$

For the first term in (4.95) we introduce $\lambda > 0$ such that

$$\big(\lambda + i(\eta + v \cdot k)\big)\hat{f}(\eta, k, v) = \lambda \hat{f}(\eta, k, v) + \hat{S}(\eta, kv).$$

Equivalently,

$$\hat{f}(\eta, k, v) = \frac{\lambda \hat{f}(\eta, k, v) + \hat{S}(\eta, k, v)}{\lambda + i(\eta + v \cdot k)},$$

which yields

$$\Big(\rho_\varepsilon *_v \hat{f}\Big)(\eta, k, v) = \int \frac{\lambda \hat{f}(\eta, k, \xi) + \hat{S}(\eta, k, \xi)}{\lambda + i(\eta + \xi \cdot k)} \rho_\varepsilon(v - \xi)\,\mathrm{d}\xi.$$

Then we bound

$$\begin{aligned}
&\left|\Big(\rho_\varepsilon *_v \hat{f}\Big)(\eta, k, v)\right| \\
&\quad \le \Big(\left\|\hat{f}(\eta, k, \cdot)\,|\rho_\varepsilon(v - \cdot)|^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^d)} + \lambda^{-1}\left\|\hat{S}(\eta, k, \cdot)\,|\rho_\varepsilon(v - \cdot)|^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^d)}\Big) \\
&\qquad \times \left(\int \frac{|\rho_\varepsilon(v - \xi)|}{|1 + i(\eta + \xi \cdot k)\lambda^{-1}|^2}\,\mathrm{d}\xi\right)^{\frac{1}{2}}.
\end{aligned}$$

The last integral is estimated using $|\rho_\varepsilon(v)| \le C_{d,\beta}\varepsilon^{-d}\chi_{|v|\le\varepsilon}$, and decomposing $\xi = \tilde{\xi}\frac{k}{|k|} + \xi^\perp$

with $\xi^\perp \cdot k = 0$, so that

$$\int \frac{|\rho_\varepsilon(v-\xi)|}{|1+i(\eta+\xi\cdot k)\lambda^{-1}|^2}\,\mathrm{d}\xi \le C_{d,\beta}\frac{1}{\varepsilon}\int \frac{\chi_{|\frac{v\cdot k}{|k|}-\tilde\xi|<\varepsilon}}{\left|1+i(\eta+\tilde\xi\cdot k)\lambda^{-1}\right|^2}\,\mathrm{d}\tilde\xi \le C_{d,\beta}\frac{\lambda}{\varepsilon\,|k|}.$$

Thus

$$\left\| \left(\rho_\varepsilon *_v \hat f\right)(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)} \le C_{d,\beta}\Big(\frac{\lambda}{\varepsilon\,|k|}\Big)^{\frac12}\Big(\left\|\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)} + \lambda^{-1}\left\|\hat S(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}\Big).$$

Choosing

$$\lambda = \frac{\left\|\hat S(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}}{\left\|\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}}$$

yields

$$\left\| \left(\rho_\varepsilon *_v \hat f\right)(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)} \le \frac{C_{d,\beta}}{\sqrt{\varepsilon\,|k|}}\left\|\hat f(\eta,k,\cdot)\right\|^{\frac12}_{L^2(\mathbb{R}^d)}\left\|\hat S(\eta,k,\cdot)\right\|^{\frac12}_{L^2(\mathbb{R}^d)}. \tag{4.97}$$

Combining (4.95) with (4.96) and (4.97) yields

$$\left\|\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)} \le \frac{C_{d,\beta}}{\sqrt{\varepsilon\,|k|}}\left\|\hat f(\eta,k,\cdot)\right\|^{\frac12}_{L^2(\mathbb{R}^d)}\left\|\hat S(\eta,k,\cdot)\right\|^{\frac12}_{L^2(\mathbb{R}^d)} + C_{d,\beta}\varepsilon^\beta\left\||D_v|^\beta\,\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}.$$

We finally optimise ε so that

$$\left\|\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)} \le \left(\frac{1}{|k|}\left\|\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}\left\|\hat S(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}\right)^{\frac{\beta}{1+2\beta}}\left\||D_v|^\beta\,\hat f(\eta,k,\cdot)\right\|^{\frac{1}{1+2\beta}}_{L^2(\mathbb{R}^d)}.$$

Dividing by $\left\|\hat f(\eta,k,\cdot)\right\|^{\frac{\beta}{1+2\beta}}_{L^2(\mathbb{R}^d)}$ yields

$$\left\|\hat f(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)} \le \left(\frac{1}{|k|}\left\|\hat S(\eta,k,\cdot)\right\|_{L^2(\mathbb{R}^d)}\right)^{\frac{\beta}{1+\beta}}\left\||D_v|^\beta\,\hat f(\eta,k,\cdot)\right\|^{\frac{1}{1+\beta}}_{L^2(\mathbb{R}^d)},$$

which concludes the proof of (4.24) after integrating over (η,k). □

CHAPTER 5

HARNACK

5.1 Introduction

This chapter is based on [153] and [154].

5.1.1 Problem formulation

Given a solution to (1.13) in some bounded subset of $\mathbb{R}^{1+d}$ in time and space, and on $\mathbb{R}^d$ in velocity, we derive a linear bound on the local supremum of this solution by its local infimum in the future. Such a relation is known as the strong Harnack inequality for solutions of (1.13). As a consequence, we deduce polynomial upper and exponential lower bounds on the fundamental solution of (1.13) away from the initial time.

Equation (1.13) is an integro-differential equation. The non-locality measures fluctuations in velocity for a local time and space. In particular, the non-local operator takes into account values of the function outside any local velocity domain. The regularity of solutions to (1.13) has first been investigated by Imbert-Silvestre [115, 118]. In particular, we have seen in Chapter 3 that super-solutions of (1.13) satisfy the weak Harnack inequality, Theorem 1.1, that is some local average of a power of the solution in a past cylinder is bounded by the local infimum in a future cylinder. This is even true for solutions in bounded velocity domains. The proof of the weak Harnack inequality uses a gain of integrability in form of a bound on the supremum by the L^2 norm of the solution (an L^2-L^∞ bound), see Lemma 3.4. Combining the weak Harnack with the L^2-L^∞ bound suggests that solutions of (1.13) satisfy the strong Harnack inequality. Intuitively, however, it is not clear that we expect the supremum of a solution to (1.13) to be bounded by a local quantity, since the values in the whole velocity domain affect the local supremum. In fact, the strong Harnack inequality fails for kinetic equations satisfied in bounded velocity domains [130]. So far only a *nonlinear* relation between the supremum and the L^2 norm

has been established for (1.13) in Lemma 3.4, which can be made linear at the cost of *non-local tail terms* that precisely take into account function values of the solution outside the domain in which (1.13) holds. We refer the reader to Remark 3.2. As a consequence of this nonlinear L^2-L^∞ bound in Lemma 3.4, and the weak Harnack inequality, Theorem 1.1, we deduced a nonlinear upper bound on the local supremum, which we called the "not-so-strong Harnack inequality", see Theorem 3.2. Here, however, we aim to establish a *linear* L^2-L^∞ bound *without tail terms*, so that we improve the "not-so-strong Harnack inequality" of Theorem 3.2 to a strong Harnack inequality. We show that if (1.13) is satisfied for any $v \in \mathbb{R}^d$, then the strong Harnack inequality holds. Such a statement is of independent interest since it captures the local regularity of solutions to a class of hypoelliptic integro-differential equations, and moreover, it bears interesting consequences, such as quantitative bounds on the fundamental solution.

5.1.2 Main results

In this chapter, we capture the local behaviour of solutions to non-local hypoelliptic equations. We derive the strong Harnack inequality for solutions of (1.13), and consequentially, we deduce polynomial upper bounds and exponential lower bounds on the fundamental solution of (1.13). The kernels of the non-local operator (1.14) that we are interested in would ideally correspond to the setting that was introduced in Subsection 1.6.2: there the ellipticity class of the kernel was prescribed through physical considerations of gas dynamics modelled by the Boltzmann equation without cutoff assumptions. In particular, the notion of ellipticity, originally introduced in [177], generalises the commonly used pointwise bounds on the kernel of the non-local operator. The idea is to relate the non-local operator (1.14) to the fractional Laplacian in an average sense. We will reuse this notion of ellipticity here: we assume a coercivity condition in integral form (1.26) and an integral upper bound (1.32). However, in contrast to the setting considered Subsection 1.6.2, we will first derive the results by making a pointwise symmetry assumption stated in (1.28), which is the non-local analogue of local operators in divergence form. We will then see further below, in Subsection 5.6, how to adapt the ideas in this first section to the Boltzmann equation. We show that solutions to (1.13) of this class satisfy the strong Harnack inequality, *provided that the equation is satisfied for any* $v \in \mathbb{R}^d$.

Theorem 5.1 (Strong Harnack inequality)**.** *Let* $f \in L^2([-3,0] \times B_1; H^s(\mathbb{R}^d))$ *be a non-negative solution of* (1.13)-(1.14) *in* $[-3,0] \times B_1 \times \mathbb{R}^d \subset \mathbb{R}^{1+2d}$ *in the sense of Definition 5.1, with a non-negative kernel* K *of order* $2s \in (0,2)$ *that satisfies for given* $\lambda_0, \Lambda_0 > 0$, *a coercivity condition on average* (1.26)*, an upper bound on average* (1.32)*, and a pointwise divergence form symmetry* (1.28)*, and with a non-negative source term* $0 \leq h \in L^\infty([-3,0] \times B_1 \times \mathbb{R}^d)$. *Then there is* $C > 0$ *depending on* $s, d, \lambda_0, \Lambda_0$ *such that for any*

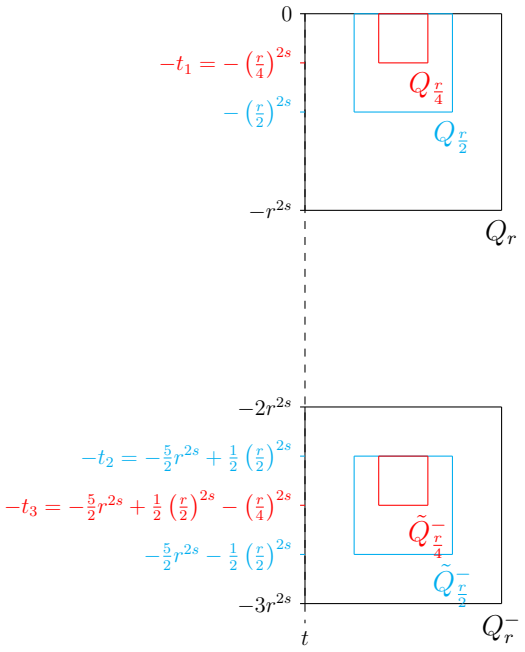


Figure 5.1: The kinetic cylinder Q_r and its past cylinder Q_r^- for some $r > 0$. The dashed vertical line represents the timeline. The blue cylinder $Q_{\frac{r}{2}}$ and its corresponding past $\tilde{Q}^-_{\frac{r}{2}}$ represent the domains appearing in the weak Harnack inequality, Theorem 1.1. The red cylinder $Q_{\frac{r}{4}}$ and its corresponding past $\tilde{Q}^-_{\frac{r}{4}}$ are the domains appearing in the strong Harnack inequality, Theorem 5.1.

$0 < r_0 < \frac{1}{6}$ *the strong Harnack inequality is satisfied:*

$$\sup_{\tilde{Q}^-_{\frac{r_0}{4}}} f \leq C \inf_{Q_{\frac{r_0}{4}}} f + C \left\| h \right\|_{L^\infty([-3,0]\times Q_1^t)}, \tag{5.1}$$

where $\tilde{Q}^-_{\frac{r_0}{4}} := [-t_3, -t_2] \times B_{\left(\frac{r_0}{4}\right)^{1+2s}} \times B_{\frac{r_0}{4}}$ *with* $t_2 = \frac{5}{2}r_0^{2s} - \frac{1}{2}\left(\frac{r_0}{2}\right)^{2s} \leq t_3 = t_2 + \left(\frac{r_0}{4}\right)^{2s}$, *and* $Q_{\frac{r_0}{4}} := [-t_1, 0] \times B_{\left(\frac{r_0}{4}\right)^{1+2s}} \times B_{\frac{r_0}{4}}$ *with* $t_1 = \left(\frac{r_0}{4}\right)^{2s}$, *see Figure 5.1. In particular, any solution* f *of* (1.13) *with zero source term* $h = 0$ *satisfies*

$$\sup_{\tilde{Q}^-_{\frac{r_0}{4}}} f \leq C \inf_{Q_{\frac{r_0}{4}}} f.$$

Figure 5.1 illustrates a non-local to local bridge: we recover the same domains for the Harnack inequalities as in the local case [99, Theorem 5].

5.1.3 Contribution

On the one hand, Theorem 5.1 is the first strong Harnack inequality for *hypoelliptic* equations without additional tail error term. We have announced the result in [151], yet the derivation of the local tail bound [151, Proposition 3.2] that is sketched there

contains an issue in Step 4, as it would require the supremum to be linear. The method proposed in this article circumvents this issue, by deriving a local tail bound on upper level sets, exploiting the sign of the cross terms, so that it can be directly incorporated into the local boundedness result, if we perform a De Giorgi iteration based on a gain of integrability in L^1. Provided that the equation is satisfied for all $v \in \mathbb{R}^d$, the strong Harnack inequality improves the "not-so-strong Harnack inequality" previously obtained in [152, Theorem 1.3], which stated a non-linear relation between the supremum in the past and the infimum in the present. This non-linear relation could be made linear at the prize of a non-local tail term, as was outlined in [152, Remark 4.2], and thus the key to the strong Harnack inequality (5.1) lies in capturing the behaviour of the non-local tail term, which we introduce for any $0 < r < R$ and any $v_0 \in \mathbb{R}^d$ as

$$\forall v \in B_r(v_0): \qquad \mathbf{T}(f; r, R, v_0) := \int_{\mathbb{R}^d \setminus B_R(v_0)} f(w) K(v, w) \, \mathrm{d}w. \tag{5.2}$$

The tail measures the non-singular part of the non-local diffusion operator $\mathcal{L}$ defined in (1.14), see also [65, 66]. In particular, we want to derive a bound on the tail in terms of local quantities only. This was first achieved for parabolic equations in [131, Theorem 1.9]. In contrast to the parabolic case, we prove a local tail bound *on upper level sets*, so that when we derive the energy estimate and the gain of integrability on level set functions, we can directly incorporate this local tail bound to obtain purely local estimates, which can then be iterated with a De Giorgi argument. This yields a *linear* relation between the L^∞ and the L^1 norm of a *solution* (rather than merely a sub-solution) to (1.13) as stated in the following proposition, which in turn implies the strong Harnack inequality, but which is of independent interest. We emphasise that for the tail bound to hold on level set functions, we require (1.13) to hold for all $v \in \mathbb{R}^d$.

Proposition 5.1 (L^1-L^∞ bound)**.** *Let $R > 0$ and $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$. Let $f \in L^2([-3, 0] \times B_1; H^s(\mathbb{R}^d))$ be a non-negative solution of* (1.13)-(1.14) *in $[-R^{2s} + t_0, t_0] \times B_{R^{1+2s}}(x_0 + (t - t_0)v_0) \times \mathbb{R}^d$ in the sense of Definition 5.1, with a kernel satisfying the coercivity condition* (1.26)*, the upper bound* (1.32) *and the symmetry* (1.28) *with an essentially bounded source term $h \geq 0$. Then for any $\delta \in (0, 1)$ there exists a large constant $C > 1$ depending on $\delta, s, d, \lambda_0, \Lambda_0$ such that*

$$\text{for a.e. } z_1 \in Q_{\frac{R}{8}}(z_0): \qquad f(z_1) \leq C R^{-(2d(1+s)+2s)} \int_{Q_R(z_0)} f(z) \, \mathrm{d}z + \delta R^{2s} \left\| h \right\|_{L^\infty(Q_R(z_0))}.$$

To the best of our knowledge, Proposition 5.1 is the first occurrence of the First De Giorgi Lemma for hypoelliptic integro-differential equations, where the supremum is bounded linearly by the L^1 norm without any tail terms. Even in the non-local parabolic case, where the strong Harnack inequality in [131, Theorem 1.1] naturally implies this

linear L^1-L^∞ bound, such a statement has not been derived directly. In particular, a tail bound on upper level sets has not been deduced before, not even for parabolic equations. The strength of this statement lies in the fact that it describes the local behaviour of a solution to a non-local equation in terms of local quantities only. The weakness, however, is that this result applies only to functions solving (1.13) for any $v \in \mathbb{R}^d$. Even though this limits the interest of the statement in itself, it does not affect the consequences on the fundamental solution derived in Theorem 6.1 and Theorem 6.2. Moreover, if we consider global solutions of the non-cutoff Boltzmann equation, then these satisfy pointwise decay estimates [121], which prove to be sufficient to deduce a boundedness condition on the antisymmetric part of the kernel, which in turn is sufficient to derive the results of this first section for functions solving the Boltzmann equation for any $v \in \mathbb{R}^d$, see Subsection 5.6 below. In short, concerning applications to a physically relevant kinetic model, the global in v assumption seems justifiable.

5.1.4 Literature

For *local* equations, the strong Harnack inequality simply follows from the weak Harnack inequality combined with the L^2-L^∞ bound in the first De Giorgi Lemma [56, 164]. We refer the reader to [85, Theorem 4] and [99, Theorem 5] for the proof of the strong Harnack inequality for local kinetic equations.

For *non-local elliptic* equations, a strong Harnack inequality was derived in [54]: first they show a local L^2-L^∞ bound on the solution with a non-local tail term, then they use the weak Harnack inequality for elliptic non-local equations and a bound on the non-local tail term in terms of local quantities only, to obtain the full Harnack inequality.

For *non-local parabolic* equations, on the other hand, until recently there had only been probabilistic methods to prove a strong Harnack inequality without any tail term [22, 192–194]. In [22, 194], they consider pointwisely bounded kernels, and establish the strong Harnack inequality using the corresponding stochastic jump process and heat kernel estimates. In [192, 193] they consider nonlocal Dirichlet forms on doubling metric measure spaces, and characterise Harnack Inequalities in terms of conditions on the underlying geometry and assumptions on the kernel. The first *non-stochastic* approach for parabolic equations was derived in Kassmann-Weidner [131]. Their idea is to first improve the local boundedness result from [127] by bounding the local L^∞ norm of a solution by the sum of its local L^2 norm and a non-local tail term in L^1. To this end, they introduce two strategies, both relying on a "tail corrector term", essentially the L^1 norm in time of the tail (5.2) with a negative sign, which has the property that it can be added to a solution to the equation such that the sum of these two terms remains a solution to the equation, and thus the tail corrector term has the effect of cancelling the tail terms arising through the equation. As a second step, they improve the weak Harnack inequality from [128] by

bounding the L^1 tail by the local infimum of the solution. Such a tail bound is contained in Proposition 5.2 below, if we consider a solution that is constant in x, so that we reduce the hypoelliptic case to a parabolic setting, and if we set $l = 0$. Combining both steps yields the strong Harnack inequality without any tail term.

Finally we remark that the strong Harnack inequality fails in the *hypoelliptic* case if the equation is satisfied merely in a bounded velocity domain, [130].

5.1.5 Strategy

The first step is to derive a tail bound in terms of local quantities: we show that the L^1 norm in time and space of the non-local tail (5.2) is bounded by a local Lebesgue norm. We work on level set functions, so that this step is compatible with the local boundedness result that we aim for in a second step. This introduces two main complications; firstly, the fact that a level set function is not a super-solution of the equation anymore, and secondly, the appearance of cross terms, that is when the test function and the non-locality are on different sides of the level l:

$$\chi_{f>l}(v)\chi_{f<l}(w).$$

Note that the appearance of these cross terms is a purely non-local effect; in the local case, these terms naturally vanish. We test the equation with a root of the level set function. Due to the concavity of the test function, this gives rise to a signed term if both the local and the non-local function are above the level. Moreover, on the upper level set, the tail has the good sign as well, since we work with super-solutions. The error terms occur on the lower level set and through the cross terms. If we localise only on the upper level set, however, then, the term that arises when the solution and the test function are both in the lower level set, vanishes due to the divergence form symmetry. Moreover, the cross terms have the good sign by virtue of localising only on the upper level set.

The goal in a second step is to derive a linear L^1-L^∞ bound by incorporating the tail bound into the energy estimate and the gain of integrability. This yields local estimates that we can iterate à la De Giorgi. To this end, we work with solutions, which is in contrast to the local case, where a linear L^1-L^∞ bound holds for mere sub-solutions. It might be interesting to investigate whether it is sufficient to work with *sub-solutions* in the non-local case to deduce a linear local boundedness result without tail terms.

5.1.6 Outline

This chapter continues by stating some preliminaries in Section 5.2. We then bound the non-local tail on the upper level set in terms of a local Lebesgue norm in Section 5.3.

Section 5.4 demonstrates Proposition 5.1. We conclude the proof of Theorem 5.1 in Section 5.5. Finally, we show in Section 5.6 how to relax the pointwise divergence-form symmetry that is assumed in Theorem 5.1 in order to prove the strong Harnack inequality for the non-cutoff Boltzmann equation with moderately soft potentials.

5.2 Preliminaries

5.2.1 Solutions

The results in this subsection are stated in a simplified manner since we are only concerned with symmetric kernels in the sense of (1.28). Here we summarise two observations on the non-local kernel K of (1.14) which are also used later in this paper. First we recall the boundedness in H^2 in velocity of the bilinear functional associated to the non-local operator $\mathcal{L}$ from (1.14), stated in Theorem 3.4, which in particular also motivates our notion of solutions to (1.13).

Definition 5.1 (Notion of solutions)**.** Assume $0 \leq K$ satisfies (1.26)-(1.28) and for $\bar{R} > 0$, let $(t_0, t_1) \times \Omega_x := \big(-(\frac{\bar{R}}{2})^{2s}, 0\big) \times B_{(\frac{\bar{R}}{2})^{1+2s}}$ and $\Omega_v \subseteq \mathbb{R}^d$. We say that $f : (t_0, t_1) \times \Omega_x \times \mathbb{R}^d \to \mathbb{R}$ is a solution of (1.13)-(1.14) in $(t_0, t_1) \times \Omega_x \times \Omega_v$ for a source $h \in L^2\big((t_0, t_1) \times \Omega_x \times \Omega_v\big)$, if

1. $f \in C^0((t_0, t_1), L^2(\Omega_x \times \Omega_v)) \cap L^2((t_0, t_1) \times \Omega_x, L^\infty(\mathbb{R}^d) \cap H^s\big(\mathbb{R}^d\big))$,
2. $\mathcal{T} f \in L^2((t_0, t_1) \times \Omega_x, \dot{H}^{-s}(\Omega_v))$,
3. f is periodic in x,
4. $\forall t_0 \leq \tau_0 \leq \tau_1 \leq t_1$ there holds $\int_{\Omega_v} \int_{\Omega_x} f(\tau_0, x, v) \,\mathrm{d}x \,\mathrm{d}v = \int_{\Omega_v} \int_{\Omega_x} f(\tau_1, x, v) \,\mathrm{d}x \,\mathrm{d}v < +\infty$,
5. for all non-negative $\varphi \in L^2((t_0, t_1) \times \Omega_x, \dot{H}^s(\Omega_v))$ such that $\forall (t, x) \in (t_0, t_1) \times \Omega_x$, $\varphi(t, x, \cdot)$ is supported in Ω_v if $\Omega_v \subsetneq \mathbb{R}^d$ we have

$$\int_{(t_0,t_1)\times\Omega_x\times\Omega_v} (\mathcal{T} f)\varphi \,\mathrm{d}z + \int_{(t_0,t_1)} \int_{\Omega_x} \mathcal{E}(f, \varphi) \,\mathrm{d}x \,\mathrm{d}t - \int_{(t_0,t_1)\times\Omega_x\times\Omega_v} h\varphi \,\mathrm{d}z = 0.$$

5.2.2 Weak Harnack

We make use of the weak Harnack inequality, Theorem 1.1 derived in Chapter 3. The proof of the weak Harnack inequality follows the De Giorgi method: in a first step one shows the local gain of regularity from L^2 to L^∞, then in a second step from L^∞ to C^α. The weak Harnack inequality follows in the end from the C^α regularity by using a covering argument, see Chapter 3.

5.3 Tail bound on upper level sets

In this section we derive the first step to the strong Harnack inequality: we show a bound on the non-local tail term in L^1 by a local quantity. The idea, inspired from [131], is to test the equation against $f^{-(1-\zeta)}$ for $\zeta \in (0,1)$ and where $f > 0$ solves (1.13), with the aim of obtaining a local bound on the L^1 norm of the tail (5.2). If we work with super-solutions, then the tail has the good sign just by virtue of the definition of $\mathcal{L}$ (1.14). Moreover, for $\zeta < 1$, the concavity of the test function gives a signed term. Finally, if we work with non-negative super-solutions, then we end up with the local tail bound in L^1. We then want to incorporate this L^1 tail bound in the next step into the derivation of a local boundedness result. However, this boundedness result is established on level set functions in the spirit of De Giorgi. Note that even though $f - l$ is a solution to (1.13) for any $l \in \mathbb{R}_+$, the positive part $(f-l)_+$ is merely a sub-solution. The tail bound in this section, however, works for super-solutions only. Thus, in order to exploit the L^1 tail bound directly in the energy estimate (5.20), we test with the level set function to some inverse power, that is $(f-l)_+^{-(1-\zeta)}$ for any $l \in \mathbb{R}_+$, and for the sake of obtaining a local bound, we localise with some cutoff. The non-locality in velocity introduces cross terms, that is terms when $f(v)$ and $f(w)$ are on different sides of the level. Even though these terms have a sign, it is not clear a-priori that they have the good sign, since we work with super-solutions. However, *if we only localise above the level set*, the cross terms end up with the good sign. Moreover, the term that arises when both, the function and the test function, are below level l, vanishes, due to the divergence form symmetry (1.28). This is why, in fact, the only contribution from the non-local operator stems from the term above level l. Note that the appearance of cross terms is a purely non-local effect. In order to relate the non-local operator to the equation, we need to assume (1.13) to be satisfied for all $v \in \mathbb{R}^d$.[1]

Proposition 5.2 (Non-local to local bound)**.** *For given $R > 0$, $z_0 \in \mathbb{R}^{1+2d}$, $l \in \mathbb{R}_+$, and any non-negative super-solution f of* (1.13) *in $[-R^{2s}+t_0, t_0] \times B_{R^{1+2s}}(x_0 + (t-t_0)v_0) \times \mathbb{R}^d$ with a non-negative source term h, such that f is periodic in space, and either $f(-R^{2s}+t_0) \geq l$ or*

$$\int_{\mathbb{R}^d} \int_{B_{R^{1+2s}}(x_0+(t-t_0)v_0)} f(-R^{2s}+t_0)\,\mathrm{d}x\,\mathrm{d}v \geq \int_{\mathbb{R}^d} \int_{B_{R^{1+2s}}(x_0+(t-t_0)v_0)} f(t_0)\,\mathrm{d}x\,\mathrm{d}v,$$

and such that the non-negative kernel satisfies (1.32)-(1.28)*, we can find for any $\zeta \in (0,1)$,*

[1] We could instead of this global assumption also assume the following: if (1.13) is satisfied for $v \in B_{\tilde{R}}$, then $\tilde{R}$ has to be sufficiently large such that for any $w \in \{f > l\}$ we can find a $\rho \in (0, \tilde{R}]$ such that $B_\rho(w) \cap \{f(v) < l\} \subset B_{\tilde{R}}$. In that case, we also need to assume a condition on the oscillation of the kernel. However, it is not clear that $\tilde{R} < +\infty$.

a constant $C > 0$ depending on ζ, s, d, Λ_0 such that

$$\begin{aligned}\int_{Q_{\frac{R}{2}}(z_0)} \int_{\mathbb{R}^d \setminus B_R(v_0)} (f-l)_+(w)\chi_{f>l}(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ \leq CR^{-2s} \sup_{Q_R(z_0)} (f-l)_+^{1-\zeta} \int_{Q_R(z_0)} (f-l)_+^{\zeta}\,\mathrm{d}z.\end{aligned} \tag{5.3}$$

Remark 5.1. For $l = 0$ we could relate the local L^ζ norm on the right hand side to the local infimum using the weak Harnack inequality of Theorem 1.1, if we additionally assume (1.26). Moreover, for $l = 0$, it would suffice to suppose (1.13) to hold on a bounded subset of $\mathbb{R}^d$.

For the proof, we collect preliminary estimates in the following lemma.

Lemma 5.1. *Let $R > 0$ and $v_0 \in \mathbb{R}^d$. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be some smooth function with support in $B_R(v_0)$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be non-negative, $l \geq 0$, $f_{l,\varepsilon} = (f-l)_+ + \varepsilon$ for some $\varepsilon > 0$ and $\zeta \in (0,1)$. Then there holds:*

i. First, we have for any v, w outside $f(v) \leq l \cap f(w) \leq l$

$$\begin{aligned}\big[f(v)-f(w)\big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big] \\ \leq -\frac{4(1-\zeta)}{\zeta^2}\frac{\big[f(v)-f(w)\big]}{\big[f_{l,\varepsilon}(v)-f_{l,\varepsilon}(w)\big]}\Big(f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big)^2.\end{aligned} \tag{5.4}$$

ii. Second, for any v, w outside $f(v) \leq l \cap f(w) \leq l$

$$\begin{aligned}\min\Big\{f_{l,\varepsilon}^{-(1-\zeta)}(v), f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big\}\,|f(v)-f(w)| \\ \leq \frac{2}{\zeta}\frac{\big[f(v)-f(w)\big]}{\big[f_{l,\varepsilon}(v)-f_{l,\varepsilon}(w)\big]} \max\Big\{f_{l,\varepsilon}^{\frac{\zeta}{2}}(v), f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big\}\,\Big|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|.\end{aligned} \tag{5.5}$$

iii. Third,

$$\begin{aligned}\frac{1}{2}\left|\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v) - \left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right|^2 - \big(\eta(v)-\eta(w)\big)^2 f_{l,\varepsilon}^{\zeta}(w) \\ \leq \min\big\{\eta^2(v), \eta^2(w)\big\}\left|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right|^2.\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\max\big\{\eta^2(v), \eta^2(w)\big\}\left|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right|^2 \\ \leq 2\left|\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v) - \left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right|^2 + 2\big(\eta(v)-\eta(w)\big)^2 f_{l,\varepsilon}^{\zeta}(v).\end{aligned} \tag{5.7}$$

iv. Fourth, for any v, w in $f(v) > l \cap f(w) > l$

$$\begin{aligned} &\min\{(f-l)_+(v), (f-l)_+(w)\} \left| f_{l,\varepsilon}^{-(1-\zeta)}(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w) \right| \\ &\quad \leq \frac{\zeta}{2(1-\zeta)} \left| f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w) \right| \min\left\{ f_{l,\varepsilon}^{\frac{\zeta}{2}}(v), f_{l,\varepsilon}^{\frac{\zeta}{2}}(w) \right\}. \end{aligned} \tag{5.8}$$

Proof of Lemma 5.1. i. We define for $\xi > 0$

$$F(\xi) := \xi^{-(1-\zeta)},$$

so that

$$F'(\xi) = -(1-\zeta)\xi^{-(2-\zeta)} \leq 0.$$

Then we compute using Jensen's inequality for all v, w outside $f(v) \leq l \cap f(w) \leq l$

$$\begin{aligned} &\big[f(v) - f(w)\big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big] \\ &\qquad = \big[f(v) - f(w)\big]\Big[F\big(f_{l,\varepsilon}(v)\big) - F\big(f_{l,\varepsilon}(w)\big)\Big] \\ &\qquad = \big[f(v) - f(w)\big] \int_{f_{l,\varepsilon}(w)}^{f_{l,\varepsilon}(v)} F'(\xi)\,\mathrm{d}\xi \\ &\qquad = -(1-\zeta)\big[f(v) - f(w)\big] \int_{f_{l,\varepsilon}(w)}^{f_{l,\varepsilon}(v)} \xi^{-(2-\zeta)}\,\mathrm{d}\xi \\ &\qquad \leq -(1-\zeta)\frac{\big[f(v) - f(w)\big]}{\big[f_{l,\varepsilon}(v) - f_{l,\varepsilon}(w)\big]} \left(\int_{f_{l,\varepsilon}(w)}^{f_{l,\varepsilon}(v)} \xi^{-\frac{(2-\zeta)}{2}}\,\mathrm{d}\xi \right)^2 \\ &\qquad = -\frac{4(1-\zeta)}{\zeta^2} \frac{\big[f(v) - f(w)\big]}{\big[f_{l,\varepsilon}(v) - f_{l,\varepsilon}(w)\big]} \Big(f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w) \Big)^2. \end{aligned}$$

This proves (5.4).

ii. To prove (5.5) we denote by

$$\mathcal{F}'(\xi) := \sqrt{-F'(\xi)} = (1-\zeta)^{\frac{1}{2}} \xi^{-\frac{(2-\zeta)}{2}},$$

so that

$$\mathcal{F}(\xi) = \frac{2(1-\zeta)^{\frac{1}{2}}}{\zeta} \xi^{\frac{\zeta}{2}}.$$

Then, for all v, w outside $f(v) \leq l \cap f(w) \leq l$, we note by Cauchy's mean value

theorem

$$\begin{aligned}\frac{\zeta}{2(1-\zeta)^{\frac{1}{2}}}&\frac{|f(v)-f(w)|}{\Big|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|}\\ &=\frac{|f(v)-f(w)|}{\big|\mathcal{F}\big(f_{l,\varepsilon}(v)\big)-\mathcal{F}\big(f_{l,\varepsilon}(w)\big)\big|}\\ &\le\frac{|f(v)-f(w)|}{|f_{l,\varepsilon}(v)-f_{l,\varepsilon}(w)|}\max\left\{\frac{1}{\mathcal{F}'(f_{l,\varepsilon}(v))},\frac{1}{\mathcal{F}'(f_{l,\varepsilon}(w))}\right\}.\end{aligned}\tag{5.9}$$

Upon multiplying (5.9) by $\min\big\{f_{l,\varepsilon}^{-(1-\zeta)}(v),f_{l,\varepsilon}^{-(1-\zeta)}(w)\big\}$ we deduce

$$\begin{aligned}&\frac{\zeta}{2}\min\Big\{f_{l,\varepsilon}^{-(1-\zeta)}(v),f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big\}\frac{|f(v)-f(w)|}{\Big|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|}\\ &\le\frac{|f(v)-f(w)|}{|f_{l,\varepsilon}(v)-f_{l,\varepsilon}(w)|}\max\left\{f_{l,\varepsilon}^{\frac{(2-\zeta)}{2}}(v),f_{l,\varepsilon}^{\frac{(2-\zeta)}{2}}(w)\right\}\min\Big\{f_{l,\varepsilon}^{-(1-\zeta)}(v),f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big\}\\ &\le\frac{|f(v)-f(w)|}{|f_{l,\varepsilon}(v)-f_{l,\varepsilon}(w)|}\max\left\{f_{l,\varepsilon}^{\frac{\zeta}{2}}(v),f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right\},\end{aligned}$$

or equivalently (5.5).

iii. To prove (5.6) we assume with no loss in generality that $\eta(v)\le\eta(w)$. Then we simply write

$$\eta(v)f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-\eta(w)f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)=\eta(v)\Big(f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big)+f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big(\eta(v)-\eta(w)\Big).$$

This implies

$$\Big|\eta(v)f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-\eta(w)f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|^2\le 2\eta^2(v)\Big|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|^2+2f_{l,\varepsilon}^{\zeta}(w)\Big(\eta(v)-\eta(w)\Big)^2.$$

Similarly, to prove (5.7) we note

$$\eta(w)\Big(f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big)=\eta(v)f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-\eta(w)f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)+\Big(\eta(w)-\eta(v)\Big)f_{l,\varepsilon}^{\frac{\zeta}{2}}(v),$$

so that

$$\eta^2(w)\Big|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|^2\le 2\Big|\eta(v)f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-\eta(w)f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big|^2+2\Big(\eta(w)-\eta(v)\Big)^2f_{l,\varepsilon}^{\zeta}(v).$$

iv. To prove (5.8), if we use

$$\tilde{\mathcal{F}}(\xi):=-\xi^{-\frac{\zeta}{2}\frac{1}{(1-\zeta)}},\qquad\text{with}\qquad\tilde{\mathcal{F}}'(\xi):=\frac{\zeta}{2}\frac{1}{1-\zeta}\xi^{\frac{-(2-\zeta)}{2(1-\zeta)}},$$

so that for all v, w in $f(v) > l \cap f(w) > l$, we note by Cauchy's mean value theorem

$$\begin{aligned}\frac{\left|f_{l,\varepsilon}^{-(1-\zeta)}(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\right|}{\left|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v) - f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right|} &= \frac{\left|f_{l,\varepsilon}^{-(1-\zeta)}(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\right|}{\left|\tilde{\mathcal{F}}\big(f_{l,\varepsilon}^{-(1-\zeta)}(w)\big) - \tilde{\mathcal{F}}\big(f_{l,\varepsilon}^{-(1-\zeta)}(v)\big)\right|} \\ &\leq \max\left\{\frac{1}{\tilde{\mathcal{F}}'\big(f_{l,\varepsilon}^{-(1-\zeta)}(v)\big)}, \frac{1}{\tilde{\mathcal{F}}'\big(f_{l,\varepsilon}^{-(1-\zeta)}(w)\big)}\right\} \\ &\leq \frac{2(1-\zeta)}{\zeta}\max\left\{f_{l,\varepsilon}^{-\frac{(2-\zeta)}{2}}(v), f_{l,\varepsilon}^{-\frac{(2-\zeta)}{2}}(w)\big)\right\}.\end{aligned}$$

Multiplied by $\min\{(f-l)_+(v), (f-l)_+(w)\}$ yields (5.8).

□

Proof of Proposition 5.2. Our aim is to get a local bound on the tail on the upper level set. To this end we do a concavity estimate on the upper level sets by testing the equation with the level set function of the solution to some inverse power. This creates *cross* terms, which have to be dealt with. We localise only on the upper level set, which implies that these cross terms have the good sign, and which makes the terms on the lower level set vanish due to the divergence structure.

With no loss in generality, we may assume $h = 0$. Indeed, if $h \geq 0$, then f is a super-solution to (1.13) with $h = 0$ as well. Moreover, due to the translation invariance, we may, with no loss in generality, consider $z_0 = (0, 0, 0)$.

To justify the computations below with our notion of solutions, we replace the variable kernel by the kernel of the fractional Laplacian in a neighbourhood of the origin with a smooth cutoff, so that we end up with the fractional Kolmogorov equation with a bounded source term. This equation admits a solution, say f_δ, represented by a convolution with the fundamental solution. In particular, $f_\delta \in \dot{H}_v^{\frac{3s}{2}}$ and we may test the equation with a function in $\dot{H}_v^{\frac{s}{2}}$. We prove the a-priori estimate on f_δ, all integrals are uniformly convergent, and all estimates are uniform in δ, so that we can pass to the limit in the end. The equation is linear, so that the solution is unique, and the limit is uniquely determined through the limit equation, which corresponds to the equation with rough coefficients. Details are carried out in the Appendix 5.7.

For $l \in \mathbb{R}_+$ we consider $\psi_l \geq 0$ given by

$$\psi_l(t, x, v) := \eta^2(t, x, v)\chi_{f>l}(t, x, v) + \chi_{f<l}(t, x, v),$$

where $\eta \in C_c^\infty(\mathbb{R}^{2d+1})$ with $0 \leq \eta \leq 1$ such that $\eta = 1$ in $Q_{\frac{3R}{4}}$ and $\eta = 0$ outside $Q_{\frac{7R}{8}}$. In particular, for any $v, w \in \mathbb{R}^d$ there holds:

i. If $f(v), f(w) > l$ then $\psi_l(v) = \eta^2(v)$ and $\psi_l(w) = \eta^2(w)$.

ii. If $f(v), f(w) < l$ then $\psi_l(v) = \psi_l(w) = 1$.

iii. If $f(v) > l > f(w)$ then $\psi_l(v) = \eta^2(v) \leq 1 = \psi_l(w)$.

iv. If $f(w) > l > f(v)$ then $\psi_l(w) = \eta^2(w) \leq 1 = \psi_l(v)$.

Then we test (1.13) with $f_{l,\varepsilon}^{-(1-\zeta)}\psi_l$ for any $l \in \mathbb{R}_+$ and $\varepsilon > 0$. Note that the concavity of the test function is the reason for the sign on the cross term, and on the squared concave term in the upper level set, since it obeys the structure of the super-solution. If we introduce

$$\Phi'\left(f_{l,\varepsilon}^{\zeta}\right)(z) := \begin{cases} \eta^2(v), & \text{if } f(z) > l, \\ 1, & \text{if } f(z) < l, \end{cases}$$

for some linear η with compact support, then Φ' is concave, and moreover

$$\frac{1}{\zeta}\mathcal{T}\Phi\left(f_{l,\varepsilon}^{\zeta}\right) = f_{l,\varepsilon}^{-(1-\zeta)}\Phi'\left(f_{l,\varepsilon}^{\zeta}\right)\mathcal{T}f.$$

Step 1: Non-local operator.

We first consider the non-local operator $\mathcal{E}$ for fixed (t, x). Due to the divergence structure of K (1.28), we find

$$\begin{aligned}
&\mathcal{E}\left(f, \psi f_{l,\varepsilon}^{-(1-\zeta)}\right) \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v) - f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\psi_l(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad = \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v) - f(w)\big]\Big[\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(v) - \big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(w)\Big]\big[K(v,w) + K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}$$

We then split $\mathcal{E}$ into three parts: we distinguish when $f(v), f(w) > l$, when $f(v), f(w) < l$ and when either $f(v) > l > f(w)$ or $f(w) > l > f(v)$. We write

$$\begin{aligned}
&\mathcal{E}\left(f, \psi f_{l,\varepsilon}^{-(1-\zeta)}\right) \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v) - f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\psi_l(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v) - f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\psi_l(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v) - f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\psi_l(v)K(v,w)\chi_{f(v)<l}\chi_{f(w)<l}\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v) - f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\psi_l(v)K(v,w)\big[\chi_{f(v)>l>f(w)} + \chi_{f(w)>l>f(v)}\big]\,\mathrm{d}w\,\mathrm{d}v \\
&\quad =: \mathcal{E}_{up} + \mathcal{E}_{low} + \mathcal{E}_{cross}.
\end{aligned}$$

Note that the indicator functions that we added do not change the symmetry of each integrand.

Step 1-i: Non-locality on the lower level set.

The term vanishes due to the divergence structure of (1.13) assumed through (1.28). Concretely, we see by choice of ψ_l, in particular through observation ii, that

$$\begin{aligned}\mathcal{E}_{low} &= \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v)-f(w)\big]\Big[\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\phi\big)(v)-\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(w)\Big]\\ &\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(v)<l}\chi_{f(w)<l}\,\mathrm{d}w\,\mathrm{d}v\\ &= 0.\end{aligned}\tag{5.10}$$

Step 1-ii: Cross non-locality.

Due to the concavity of the test function, the cross term has a sign: by iii

$$\begin{aligned}\mathcal{E}_{cross} &= \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v)-f(w)\big]\Big[\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(v)-\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(w)\Big]\\ &\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\big[\chi_{f(v)>l>f(w)}+\chi_{f(w)>l>f(v)}\big]\,\mathrm{d}w\,\mathrm{d}v\\ &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v)-f(w)\big]\Big[\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(v)-\big(f_{l,\varepsilon}^{-(1-\zeta)}\psi_l\big)(w)\Big]\\ &\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(v)>l>f(w)}\,\mathrm{d}w\,\mathrm{d}v\\ &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v)-f(w)\big]\Big[\eta^2(v)(f-l+\varepsilon)^{-(1-\zeta)}(v)-\varepsilon^{-(1-\zeta)}\Big]\\ &\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(v)>l>f(w)}\,\mathrm{d}w\,\mathrm{d}v\\ &\leq 0.\end{aligned}\tag{5.11}$$

Step 1-iii: Non-locality on the upper level set.

It remains to bound $\mathcal{E}_{up}$. For fixed time and space, we find due to i

$$\begin{aligned}\mathcal{E}_{up} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big[f(v)-f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v\\ &= \int_{B_R}\int_{B_R}\big[f(v)-f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\int_{B_R}\int_{\mathbb{R}^d\setminus B_R}\big[f(v)-f(w)\big]f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v\\ &=: \mathcal{E}_{up}^{loc}+\mathcal{E}_{up}^{tail}.\end{aligned}\tag{5.12}$$

For $\mathcal{E}_{up}^{tail}$, we note that there is no singularity, due to the choice of η, which vanishes outside

$B_{\frac{7R}{8}}$. Therefore, by (1.32)

$$\begin{aligned}
\mathcal{E}_{up}^{tail} &= \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} \big[f(v)-f(w)\big] f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad - \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w) f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \qquad (5.13)\\
&\le \Lambda_0\left(\frac{R}{8}\right)^{-2s}\int_{B_R}((f-l)_+ +\varepsilon)^{\zeta}(v)\chi_{f>l}(v)\,\mathrm{d}v \\
&\quad - \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w) f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}$$

In particular, the tail of $(f-l)_+$ has a negative sign.

Moreover, we find for the local contribution of $\mathcal{E}_{up}$

$$\begin{aligned}
\mathcal{E}_{up}^{loc} &= \int_{B_R}\int_{B_R} \big[f(v)-f(w)\big] f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v)K(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{4}\int_{B_R}\int_{B_R} \Big[f(v)-f(w)\Big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\eta^2(w)\Big] \\
&\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{4}\int_{B_R}\int_{B_R} \Big[f(v)-f(w)\Big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\eta^2(w)\Big] \\
&\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(v)>l}\chi_{f(w)>l}\big[\chi_{f(v)>f(w)}+\chi_{f(w)>f(v)}\big]\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{2}\int_{B_R}\int_{B_R} \Big[f(v)-f(w)\Big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v)\eta^2(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\eta^2(w)\Big] \qquad (5.14)\\
&\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(v)>l}\chi_{f(w)>l}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{2}\int_{B_R}\int_{B_R} \Big[f(v)-f(w)\Big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v) - f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big]\eta^2(v) \\
&\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(v)>l}\chi_{f(w)>l}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \frac{1}{2}\int_{B_R}\int_{B_R} \big[f(v)-f(w)\big] f_{l,\varepsilon}^{-(1-\zeta)}(w)\big[\eta^2(v)-\eta^2(w)\big] \\
&\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(v)>l}\chi_{f(w)>l}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&=: I_{sign} + I_{sym}.
\end{aligned}$$

The term I_{sign} has a sign due to the concavity of the test function, whereas in I_{sym} the cutoff η absorbs some of the singularity of the kernel K. To quantitatively bound I_{sign}

and I_{sym} we use the estimates of Lemma 5.1. For I_{sign} we use (5.4), (5.6), and (1.32)

$$
\begin{aligned}
I_{sign} &= \frac{1}{2}\int_{B_R}\int_{B_R}\big[f(v)-f(w)\big]\Big[f_{l,\varepsilon}^{-(1-\zeta)}(v)-f_{l,\varepsilon}^{-(1-\zeta)}(w)\Big]\eta^2(v)\\
&\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\le -\frac{2(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right)^2\eta^2(v)\\
&\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\le \frac{2(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}f_{l,\varepsilon}^{\zeta}(w)\big(\eta(v)-\eta(w)\big)^2\\
&\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \qquad (5.15)\\
&\quad-\frac{1-\zeta}{\zeta^2}\int_{B_R}\int_{B_R}\left[\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\eta\right)(v)-\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\eta\right)(w)\right]^2\\
&\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\le C(\zeta,\Lambda_0)R^{-2s}\int_{B_R}(f-l+\varepsilon)^{\zeta}(w)\chi_{f(w)>l}\,\mathrm{d}w\\
&\quad-\frac{1-\zeta}{\zeta^2}\int_{B_R}\int_{B_R}\left[\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\eta\right)(v)-\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\eta\right)(w)\right]^2\\
&\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
$$

This was the good term. Concerning the unsigned term I_{sym}, we use (5.5), Young's

inequality, (5.7) and (1.32) so that for any $\delta_0 \in (0,1)$

$$
\begin{aligned}
I_{sym} &= \frac{1}{2}\int_{B_R}\int_{B_R} \big[f(v)-f(w)\big] f_{l,\varepsilon}^{-(1-\zeta)}(w)\big[\eta^2(v)-\eta^2(w)\big] \\
&\qquad\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\le \frac{1}{\zeta}\int_{B_R}\int_{B_R} f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\left[f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)\right]\big|\eta^2(w)-\eta^2(v)\big| \\
&\qquad\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\le C(\delta_0,\zeta)\int_{B_R}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\,|\eta(w)-\eta(v)|^2\left[K(v,w)+K(w,v)\right]\mathrm{d}w\,\mathrm{d}v \\
&\quad + \frac{\delta_0}{2}\int_{B_R}\int_{B_R}\max\{\eta^2(v),\eta^2(w)\}\left[f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)\right]^2 \\
&\qquad\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\le C(\delta_0,\zeta)\int_{B_R}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\,|\eta(w)-\eta(v)|^2 K(v,w)\chi_{f>l}(w)\;\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \delta_0\int_{B_R}\int_{B_R} f_{l,\varepsilon}^{\zeta}(v)\,|\eta(w)-\eta(v)|^2 K(v,w)\chi_{f>l}(v)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \delta_0\int_{B_R}\int_{B_R}\left[\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)-\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)\right]^2 \\
&\qquad\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\le C(\delta_0,\zeta,\Lambda_0)R^{-2s}\int_{B_R}(f-l+\varepsilon)^{\zeta}(v)\chi_{f>l}(v)\,\mathrm{d}v \\
&\quad + \delta_0\int_{B_R}\int_{B_R}\left[\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\eta\right)(w)-\left(f_{l,\varepsilon}^{\frac{\zeta}{2}}\eta\right)(v)\right]^2 \\
&\qquad\qquad \times \big[K(v,w)+K(w,v)\big]\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
\tag{5.16}
$$

In particular, the last term can be absorbed with the signed term of I_{sign} in (5.15) for δ_0

sufficiently small and for any $\zeta < 1$, so that we conclude from (5.14), (5.15), and (5.16)

$$\begin{aligned}
\mathcal{E}_{up}^{loc} &= I_{sign} + I_{sym} \\
&\leq C(\delta_0, \zeta, \Lambda_0) R^{-2s} \int_{B_R} (f - l + \varepsilon)^{\zeta}(v) \chi_{f>l}(v) \, \mathrm{d}v \\
&\quad - \frac{1-\zeta}{\zeta^2} \int_{B_R} \int_{B_R} \left[\left(f_{l,\varepsilon}^{\frac{\zeta}{2}} \eta \right)(v) - \left(f_{l,\varepsilon}^{\frac{\zeta}{2}} \eta \right)(w) \right]^2 \\
&\qquad\qquad \times \big[K(v,w) + K(w,v) \big] \chi_{f(w)>f(v)>l} \, \mathrm{d}w \, \mathrm{d}v \\
&\quad + \delta_0 \int_{B_R} \int_{B_R} \left[\left(f_{l,\varepsilon}^{\frac{\zeta}{2}} \eta \right)(w) - \left(f_{l,\varepsilon}^{\frac{\zeta}{2}} \eta \right)(v) \right]^2 \\
&\qquad\qquad \times \big[K(v,w) + K(w,v) \big] \chi_{f(w)>f(v)>l} \, \mathrm{d}w \, \mathrm{d}v \\
&\leq C(\delta_0, \zeta, \Lambda_0) R^{-2s} \int_{B_R} (f - l + \varepsilon)^{\zeta}(v) \chi_{f>l}(v) \, \mathrm{d}v.
\end{aligned} \tag{5.17}$$

Step 1-iv: Bound on the non-locality.

We conclude from (5.10), (5.11), (5.12), (5.13), and (5.17)

$$\begin{aligned}
\mathcal{E}\left(f, \psi f_{l,\varepsilon}^{-(1-\zeta)} \right) &= \mathcal{E}_{up} + \mathcal{E}_{low} + \mathcal{E}_{cross} \\
&\leq C(\zeta, \Lambda_0) R^{-2s} \int_{B_R} (f - l + \varepsilon)^{\zeta}(v) \chi_{f>l} \, \mathrm{d}v \\
&\quad - \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} (f-l)_+(w) \eta^2(v) f_{l,\varepsilon}^{-(1-\zeta)}(v) K(v,w) \chi_{f(v)>l} \, \mathrm{d}w \, \mathrm{d}v.
\end{aligned} \tag{5.18}$$

Note that this estimate holds true for any (t, x).

Step 2: Transport.

We start with the time derivative. We denote by $t_1 := -R^{2s}$ and by $t_2 = 0$, so that

$$\begin{aligned}
&\int_{-R^{2s}}^{0} \int_{B_{R^{1+2s}}} \int_{\mathbb{R}^d} \partial_t f(z) \psi_l(z) f_{l,\varepsilon}^{-(1-\zeta)}(z) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_{\mathbb{R}} \int_{B_{R^{1+2s}}} \int_{\mathbb{R}^d} \partial_t (f-l)_+(z) \eta^2(z) f_{l,\varepsilon}^{-(1-\zeta)}(z) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \varepsilon^{-(1-\zeta)} \int_{t_1}^{t_2} \int_{B_{R^{1+2s}}} \int_{\mathbb{R}^d} \partial_t (f-l)_-(z) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

The first term we integrate by parts. For the second term, we use that since $f - l =$

$(f-l)_+ + (f-l)_-$ and since f conserves mass, there holds

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)_-(t_2) - (f-l)_-(t_1))\,\mathrm{d}x\,\mathrm{d}v \\
&= \iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)(t_2) - (f-l)(t_1))\,\mathrm{d}x\,\mathrm{d}v - \iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)_+(t_2) - (f-l)_+(t_1))\,\mathrm{d}x\,\mathrm{d}v \\
&= -\iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)_+(t_2) - (f-l)_+(t_1))\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

Now, we distinguish for any x, v the case when $f(t_1) \geq \min(l, f(t_2))$, or $f(t_1) < \min(l, f(t_2)) \leq \max(l, f(t_2))$. In the former case, if $f(t_2) > l$, then $f(t_1) \geq l$, and the integral vanishes. Else, $f(t_2) \leq l$, so that $f(t_1) \geq f(t_2)$ and $(f-l)_-(t_2) - (f-l)_-(t_1) \leq f(t_2) - l + l - f(t_1) \leq 0$. Thus,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)_-(t_2) - (f-l)_-(t_1))\,\mathrm{d}x\,\mathrm{d}v \leq 0.$$

In the latter case, when $f(t_1) \leq \max(l, f(t_2))$, if $f(t_2) \geq l$ then $f(t_1) \leq f(t_2)$ and $(f-l)_+(t_1) - f(t_2) + l \leq f(t_1) - f(t_2) \leq 0$ so that

$$\begin{aligned}
\iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)_-(t_2) - (f-l)_-(t_1))\,\mathrm{d}x\,\mathrm{d}v &= -\iint_{\mathbb{R}^d\times\mathbb{R}^d} ((f-l)_+(t_2) - (f-l)_+(t_1))\,\mathrm{d}x\,\mathrm{d}v \\
&\leq 0.
\end{aligned}$$

Else if $f(t_2) < l$, then $f(t_1) \leq l$ and the integral vanishes. Combining this with the divergence theorem, we get

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{B_{R^{1+2s}}} \int_{\mathbb{R}^d} \partial_t (f-l)_-(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_{B_{R^{1+2s}}} \int_{\mathbb{R}^d} \Big[(f-l)_-(t_2, x, v) - (f-l)_-(t_1, x, v)\Big]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leq 0.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\partial_t f(z)\psi_l(z)f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\frac{1}{\zeta}\int_{\mathbb{R}}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\partial_t\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)\eta^2(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\varepsilon^{-(1-\zeta)}\int_{t_1}^{t_2}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\partial_t(f-l)_-(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\frac{1}{\zeta}\int_{\mathbb{R}}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)\partial_t\eta^2(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\varepsilon^{-(1-\zeta)}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\Big[(f-l)_-(t_2,x,v)-(f-l)_-(t_1,x,v)\Big]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le CR^{-2s}\int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\int_{B_R}\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Moreover, we use that f is periodic in space, so that denoting by n the normal vector on $\partial B_{R^{1+2s}}$

$$\begin{aligned}
&\int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}v\cdot\nabla_x f(z)\psi_l(z)f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}v\cdot\nabla_x(f-l)_+(z)\eta^2(z)f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\quad+\varepsilon^{-(1-\zeta)}\int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}v\cdot\nabla_x(f-l)_-(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\frac{1}{\zeta}\int_{-R^{2s}}^{0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}v\cdot\nabla_x\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)\eta^2(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\quad+\varepsilon^{-(1-\zeta)}\int_{-R^{2s}}^{0}\int_{\mathbb{R}^d}\int_{\partial B_{R^{1+2s}}}(f-l)_-(z)v\cdot n\,\mathrm{d}S(x)\,\mathrm{d}v\,\mathrm{d}t.
\end{aligned}$$

The last term vanishes due to the periodicity in x. The second last term we integrate by parts, and use $|v\cdot\nabla_x\eta|\sim R^{-2s}$

$$\begin{aligned}
&\frac{1}{\zeta}\int_{-R^{2s}}^{0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}v\cdot\nabla_x\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)\eta^2(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\frac{1}{\zeta}\int_{-R^{2s}}^{0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)v\cdot\nabla_x\eta^2(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le CR^{-2s}\int_{Q_R}\big((f-l)_+ +\varepsilon\big)^{\zeta}(z)\,\mathrm{d}z.
\end{aligned}$$

Overall, we find

$$\int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\mathcal{T}f(z)\psi_l(z)f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ \leq CR^{-2s}\int_{Q_R}\left((f-l)_+ + \varepsilon\right)^{\zeta}(z)\,\mathrm{d}z. \tag{5.19}$$

Step 3: Relation of transport and non-locality.

Finally we use the equation. By (1.13), (5.19), and (5.18) we see

$$\begin{aligned}0 &\leq \int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\mathcal{T}f(z)\psi_l(z)f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}v\,\mathrm{d}x + \int_{-R^{2s}}^{0}\int_{B_{R^{1+2s}}}\mathcal{E}\left(f,\psi_l f_{l,\varepsilon}^{-(1-\zeta)}\right)\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C(\zeta)R^{-2s}\int_{Q_R}((f-l)_+ + \varepsilon)^{\zeta}(z)\,\mathrm{d}z\\ &\quad - \int_{Q_R}\int_{\mathbb{R}^d\setminus B_R}(f-l)_+(w)\eta^2(z)\phi(t)f_{l,\varepsilon}^{-(1-\zeta)}(v)K(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}z.\end{aligned}$$

or rearranged,

$$\int_{Q_{\frac{3R}{4}}}\int_{\mathbb{R}^d\setminus B_R}(f-l)_+(w)f_{l,\varepsilon}^{-(1-\zeta)}(v)K(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}z \leq CR^{-2s}\int_{Q_R}((f-l)_+ + \varepsilon)^{\zeta}(z)\,\mathrm{d}z.$$

This implies (5.3), if we take out the infimum of $(f-l+\varepsilon)^{-(1-\zeta)}$ over $z\in Q_{\frac{3R}{4}}\cap\chi_{f>l}$ on the left hand side, divide by the infimum, use $(\inf g)^{-1}=\sup g^{-1}$, and let $\varepsilon\to 0$, so that

$$\int_{Q_{\frac{3R}{4}}}\int_{\mathbb{R}^d\setminus B_R}(f-l)_+(w)\chi_{f>l}(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \leq CR^{-2s}\sup_{Q_{\frac{3R}{4}}\cap f>l}(f-l)^{1-\zeta}\int_{Q_R}(f-l)_+^{\zeta}\,\mathrm{d}z.$$

□

5.4 De Giorgi in L^1

We aim to derive De Giorgi's first lemma, which is typically a bound from L^2 to L^∞. Such a boundedness result has been derived in [152, Lemma 4.1] before, however, with a tail quantity on the right hand side. We aim to derive a fully local boundedness result. Since we saw in the previous section how to bound the tail in L^1, we perform the De Giorgi iteration in L^1. The result is Proposition 5.1.

The De Giorgi argument is based on the realisation that for any sub-solution f of (1.13), the level set function $(f-l)_+$ for any $l\in\mathbb{R}_+$ is still a sub-solution to (1.13). Then the argument relies on three steps. The first is the energy estimate, which is typically derived by testing the equation with the solution itself, and thus yields a bound in L^2. This is required to estimate the coercive term, which for our equation is the $L^2_{t,x}H^s_v$ norm

of the solution. But even then, by Hölder's inequality, the tail term naturally appears in L^1, if we pay the supremum on the test function. Thus, we can use the non-local to local bound of Proposition 5.2, to obtain a fully local energy estimate. The second step is the gain of integrability. This is usually done on the same Lebesgue level as the energy estimate, that is L^2. Since this only works for solutions on the whole space, we need to localise with a cutoff. This, however, naturally makes appear another tail term, which then, would need to be bounded in L^2, which we don't know how to bound. Therefore, we need a gain of integrability in L^1. Then, on the one hand, the tail terms are again bounded by Proposition 5.2. On the other hand, we then also have to bound the $L^1_{t,x}W^{s,1}_v$ norm of the localised level set function, which can be done using Cauchy-Schwarz and the energy estimate. The last step is the proper choice of control quantity, which for us is the L^1 norm of the level set functions, which by Chebyshev's inequality, can be related to the measure of the upper level set.

We dedicate the remainder of this section to the proof of Proposition 5.1. We use the tail bound from Proposition 5.2 to get a local energy estimate. Then we use the gain of integrability in L^1 from Proposition 2.1, by comparing the solution of (1.13) to the fractional Kolmogorov equation (2.1). We bound the right hand side by the local energy estimate, and the tail term again with Proposition 5.2. Finally, we iterate the so gained local a-priori estimate on level set functions.

In this section, we let f solve (1.13) in $[-R^{2s}+t_0, t_0] \times B_{R^{1+2s}}(x_0 + (t-t_0)v_0) \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \int_{B_{R^{1+2s}}(x_0+(t-t_0)v_0)} f(-R^{2s}+t_0)\,\mathrm{d}x\,\mathrm{d}v \geq \int_{\mathbb{R}^d} \int_{B_{R^{1+2s}}(x_0+(t-t_0)v_0)} f(t_0)\,\mathrm{d}x\,\mathrm{d}v.$$

With no loss in generality, we set $z_0 = (0,0,0)$. We, moreover, work with zero source term. Note that the energy estimate, the gain of integrability, and correspondingly the De Giorgi iteration, have to be modified in a suitable way to account for non-zero source.

Step 1: Local energy estimate.

Let $\phi \in C_c^\infty(\mathbb{R}^{1+2d})$ be such that $\phi(t,x,v) = 0$ for (t,x,v) outside $Q_{\frac{R}{4}}$ and $\phi(t,x,v) = 1$ for $(t,x,v) \in Q_{\frac{R}{8}}$. We then test (1.13) with $(f-l)_+\phi^2$, so that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{1+2d}} \mathcal{T} f(z)\phi^2(z)(f-l)_+(z)\,\mathrm{d}z - \int_{\mathbb{R}^{1+2d}} \mathcal{L}f(z)(f-l)_+(z)\phi^2(z)\,\mathrm{d}z \\ &= \frac{1}{2}\int_{\mathbb{R}^{1+2d}} \mathcal{T}(f-l)_+^2(z)\phi^2(z)\,\mathrm{d}z - \int_{\mathbb{R}^{1+2d}} \mathcal{L}f(z)(f-l)_+(z)\phi^2(z)\,\mathrm{d}z. \end{aligned} \tag{5.20}$$

We observe that for any $\frac{R}{4} < r$, if we abbreviate $(f-l)_+ =: f_{l_+}$ and if we denote by

$\Omega_\rho := [-\rho^{2s}, 0] \times B_{\rho^{1+2s}}$, then due to (1.28)

$$
\begin{aligned}
&-\int_{\mathbb{R}^{1+2d}} \mathcal{L}f(z)(f-l)_+(z)\phi^2(z)\,\mathrm{d}z \\
&\quad= \int_{\mathbb{R}^{1+2d}}\int_{\mathbb{R}^d} \big[f(t,x,v)-f(t,x,w)\big]\big((f-l)_+\phi^2\big)(t,x,v)K(t,x,v,w)\,\mathrm{d}w\,\mathrm{d}z \\
&\quad= \frac{1}{2}\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r} \big[f(v)-f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v)-\big(f_{l_+}\phi^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}z \\
&\qquad+ \frac{1}{2}\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r} \big[f(v)-f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v)+\big(f_{l_+}\phi^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+ \int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{\mathbb{R}^d\setminus B_r} \big[f(v)-f(w)\big]\big(f_{l_+}\phi^2\big)(z)K(v,w)\,\mathrm{d}w\,\mathrm{d}z \qquad (5.21)\\
&\quad= \frac{1}{4}\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r} \big[f(v)-f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v)-\big(f_{l_+}\phi^2\big)(w)\Big] \\
&\qquad\qquad\qquad\qquad\times\big[K(v,w)+K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}z \\
&\qquad+ \int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{\mathbb{R}^d\setminus B_r} \big[f(v)-f(w)\big]\big(f_{l_+}\phi^2\big)(z)K(v,w)\,\mathrm{d}w\,\mathrm{d}z \\
&\quad=: \mathcal{E}_{loc}+\mathcal{E}_{tail}.
\end{aligned}
$$

Step 1-i: Transport. We integrate by parts the transport term, and use $|\mathcal{T}\phi| \sim R^{-2s}$.

$$
\begin{aligned}
-\frac{1}{2}\int_{\mathbb{R}^{1+2d}} \mathcal{T}(f-l)_+^2(z)\phi^2(z)\,\mathrm{d}z &= \frac{1}{2}\int_{\mathbb{R}^{1+2d}}(f-l)_+^2(z)\mathcal{T}\phi^2(z)\,\mathrm{d}z \\
&\le CR^{-2s}\int_{Q_{\frac{R}{4}}}(f-l)_+^2(z)\,\mathrm{d}z.
\end{aligned} \qquad (5.22)
$$

Step 1-ii: Tail bound. We use Proposition 5.2 for f, which we assume to be a solution,

in particular a super-solution, so that for $r = \frac{R}{2}$

$$
\begin{aligned}
-\mathcal{E}_{tail} &= \int_{\Omega_{\frac{R}{4}}} \int_{B_r} \int_{\mathbb{R}^d \setminus B_r} \big[f(w) - f(v)\big]\big(f_{l+}\phi^2\big)(z) K(v,w)\,\mathrm{d}w\,\mathrm{d}z \\
&\le \int_{Q_{\frac{R}{4}}} \int_{\mathbb{R}^d \setminus B_{\frac{R}{2}}} \big[f(t,x,w) - l\big]\big(f_{l+}\phi^2\big)(z) K(v,w)\,\mathrm{d}w\,\mathrm{d}z \\
&\le \sup_{Q_{\frac{R}{4}}}(f-l)_+ \int_{Q_{\frac{R}{4}}} \int_{\mathbb{R}^d \setminus B_{\frac{R}{2}}} (f-l)_+(t,x,w) K(v,w) \chi_{f>l}(v)\,\mathrm{d}w\,\mathrm{d}z \\
&\le C R^{-2s} \sup_{Q_{\frac{R}{4}}}(f-l)_+ \sup_{Q_{\frac{R}{2}}}(f-l)_+^{1-\zeta} \int_{Q_{\frac{R}{2}}} (f-l)_+^{\zeta}(z)\,\mathrm{d}z \qquad (5.23) \\
&\le C R^{-2s} \left[\left(\sup_{Q_{\frac{R}{4}}}(f-l)_+ \right)^{2-\zeta} + \left(\sup_{Q_{\frac{R}{2}}}(f-l)_+^{1-\zeta} \right)^{\frac{2-\zeta}{1-\zeta}} \right] \int_{Q_{\frac{R}{2}}} (f-l)_+^{\zeta}(z)\,\mathrm{d}z \\
&\le C R^{-2s} \left(\sup_{Q_{\frac{R}{2}}}(f-l)_+ \right)^{2-\zeta} \int_{Q_{\frac{R}{2}}} (f-l)_+^{\zeta}(z)\,\mathrm{d}z.
\end{aligned}
$$

Step 1-iii: Not-too-non-local operator. What remains to be estimated is the local contribution of the non-local operator,

$$
\mathcal{E}_{loc} = \frac{1}{2} \int_{B_r} \int_{B_r} \big[f(v) - f(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) - \big(f_{l+}\phi^2\big)(w)\Big] K(v,w)\,\mathrm{d}w\,\mathrm{d}z,
$$

where we recall $f_{l+} = (f-l)_+$.

Claim. For $z = (t,x,v)$ and $r = \frac{R}{2}$

$$
\begin{aligned}
\mathcal{E}_{loc} \ge{}& \frac{1}{2} \int_{B_r} \int_{B_r} \Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K(v,w)\,\mathrm{d}v - C r^{-2s} \int_{B_r} f_{l+}^2(v)\,\mathrm{d}v \\
&+ \frac{1}{2} \int_{B_r} \int_{B_r} \big[f_{l-}(v) - f_{l-}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) - \big(f_{l+}\phi^2\big)(w)\Big] K(v,w)\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned} \qquad (5.24)
$$

Indeed, since

$$
f_{l-}(v) - f_{l-}(w) = f(v) - f(w) - \big(f_{l+}(v) - f_{l+}(w)\big), \qquad (5.25)
$$

and

$$
\begin{aligned}
\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 - \big[f_{l+}(v) - f_{l+}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) - \big(f_{l+}\phi^2\big)(w)\Big] \\
= f_{l+}(v) f_{l+}(w) \big[\phi(v) - \phi(w)\big]^2,
\end{aligned}
$$

there holds

$$\begin{aligned}
&\int_{B_r}\int_{B_r}\Big\{\Big[\big(f_{l+}\phi\big)(v)-\big(f_{l+}\phi\big)(w)\Big]^2\\
&\qquad\qquad+\big[f_{l-}(v)-f_{l-}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v)-\big(f_{l+}\phi^2\big)(w)\Big]\Big\}K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&=\int_{B_r}\int_{B_r}\Big\{\Big[\big(f_{l+}\phi\big)(v)-\big(f_{l+}\phi\big)(w)\Big]^2\\
&\qquad\qquad-\big[f_{l+}(v)-f_{l+}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v)-\big(f_{l+}\phi^2\big)(w)\Big]\Big\}K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+\int_{B_r}\int_{B_r}\big[f(v)-f(w)\big]\Big[\big(f_{l+}\phi^2\big)(v)-\big(f_{l+}\phi^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&=\int_{B_r}\int_{B_r}f_{l+}(v)f_{l+}(w)\big[\phi(v)-\phi(w)\big]^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v+2\mathcal{E}_{loc}.
\end{aligned}$$

Thus, using Young's inequality and (1.32), we further bound

$$\begin{aligned}
&\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r}f_{l+}(v)f_{l+}(w)\big[\phi(v)-\phi(w)\big]^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le C\,\|\phi\|_{C_v^1}^2\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r}\Big(f_{l+}^2(v)+f_{l+}^2(w)\Big)\,|v-w|^2\,K(v,w)\,\mathrm{d}w\,\mathrm{d}z\\
&\qquad\le Cr^{-2}r^{2-2s}\int_{Q_{\frac{R}{2}}}f_{l+}^2(z)\,\mathrm{d}z.
\end{aligned}$$

This proves (5.24).

In particular, (5.24) implies for $r=\frac{R}{2}$

$$\begin{aligned}
\mathcal{E}_{loc}&=\iint_{B_r\times B_r}\Big[f(v)-f(w)\Big]\big(f_{l+}\phi^2\big)(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\ge\frac{1}{2}\iint_{B_r\times B_r}\Big[\big(f_{l+}\phi\big)(v)-\big(f_{l+}\phi\big)(w)\Big]^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v-Cr^{-2s}\int_{B_r}f_{l+}^2(v)\,\mathrm{d}v\\
&\quad+\frac{1}{2}\iint_{B_r\times B_r}\big[f_{l-}(v)-f_{l-}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v)-\big(f_{l+}\phi^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\ge\frac{1}{2}\iint_{B_r\times B_r}\Big[\big(f_{l+}\phi\big)(v)-\big(f_{l+}\phi\big)(w)\Big]^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v-Cr^{-2s}\int_{B_r}f_{l+}^2(v)\,\mathrm{d}v\\
&\quad-\iint_{B_r\times B_r}\Big[f_{l-}(v)\big(f_{l+}\phi^2\big)(w)+f_{l-}(w)\big(f_{l+}\phi^2\big)(v)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\ge\frac{1}{2}\iint_{B_r\times B_r}\Big[\big(f_{l+}\phi\big)(v)-\big(f_{l+}\phi\big)(w)\Big]^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v-Cr^{-2s}\int_{B_r}f_{l+}^2(v)\,\mathrm{d}v.
\end{aligned}\tag{5.26}$$

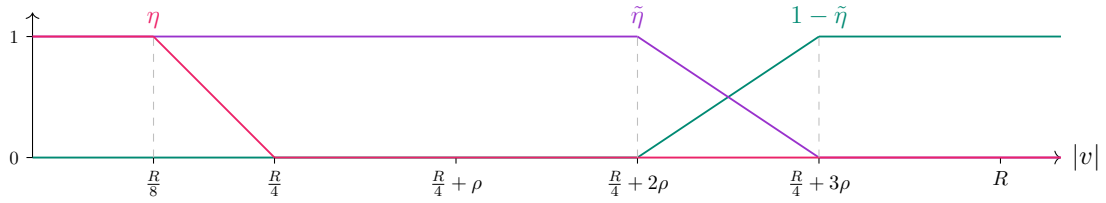


Figure 5.2: The cutoffs $\eta, \tilde{\eta} \in C_c^\infty(\mathbb{R}^d)$ in velocity that are used in the proof of Proposition 5.1.

In the last inequality we used for each $v, w \in \mathbb{R}^d$

$$-f_{l-}(w)\big(f_{l+}\phi^2\big)(v) - f_{l-}(v)\big(f_{l+}\phi^2\big)(w) \geq 0.$$

Step 1-iv: Conclusion. Combining (5.20) and (5.21) with (5.22) of *Step 1-i*, (5.23) of *Step 1-ii*, and (5.26) of *Step 1-iii*, we conclude

$$\begin{aligned}
&\frac{1}{8}\int_{\Omega_{\frac{R}{4}}} \iint_{B_{\frac{R}{2}}\times B_{\frac{R}{2}}} \Big[\big((f-l)_+\phi\big)(v) - \big((f-l)_+\phi\big)(w)\Big]^2 K(t,x,v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq CR^{-2s}\int_{Q_{\frac{R}{4}}} (f-l)_+^2(z)\,\mathrm{d}z + CR^{-2s}\left(\sup_{Q_{\frac{R}{2}}}(f-l)_+\right)^{2-\zeta}\int_{Q_{\frac{R}{2}}}(f-l)_+^{\zeta}(z)\,\mathrm{d}z.
\end{aligned} \tag{5.27}$$

Step 2: Gain of Integrability.

For the gain of integrability, we use Proposition 2.1. We check that $(f-l)_+\phi$ satisfies in $\mathbb{R}^{1+2d}$

$$\begin{aligned}
&\mathcal{T}\big[(f-l)_+\phi\big] + (-\Delta_v)^s\big[(f-l)_+\phi\big] \\
&\qquad \leq \mathcal{L}\big[(f-l)_+\phi\big] + (-\Delta_v)^s\big[(f-l)_+\phi\big] + (f-l)_+\mathcal{T}\phi \\
&\qquad\quad + \left(\int_{\mathbb{R}^d\setminus B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\tilde{\eta}(v) \\
&\qquad\quad + \left(\int_{\mathbb{R}^d} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\big(1-\tilde{\eta}(v)\big) \\
&\qquad\quad + \left(\int_{B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\tilde{\eta}(v),
\end{aligned}$$

where $0 < \rho \leq R/8$, $\tilde{\eta} = 1$ in $B_{\frac{R}{4}+2\rho}$ and $\tilde{\eta} = 0$ outside $B_{\frac{R}{4}+3\rho}$. The only non-local term in the inequality above is on the third line. Note that $\phi(v) - \phi(w) \leq 0$ for v outside $B_{\frac{R}{4}}$ and w outside $B_\rho(v)$, so that the tail term arising from the equation (on the third line) is non-negative only for $v \in B_{\frac{R}{4}}$, where $\tilde{\eta}(v) = 1$. Furthermore, we note since ϕ and $1-\tilde{\eta}$ have disjoint support, there holds $\phi(v)(1-\tilde{\eta}(v)) = 0$ for all $v \in \mathbb{R}^d$. Thus, since

$0 \le 1-\tilde\eta \le 1$, we see that

$$\left(\int_{\mathbb{R}^d} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\big(1-\tilde\eta(v)\big) \le 0.$$

Therefore, there holds for $(t,x,v)\in\mathbb{R}^{1+2d}$

$$\begin{aligned}
&\mathcal{T}\big[(f-l)_+\phi\big] + (-\Delta_v)^s\big[(f-l)_+\phi\big] \\
&\quad\le \mathcal{L}\big[(f-l)_+\phi\big] + (-\Delta_v)^s\big[(f-l)_+\phi\big] + (f-l)_+\mathcal{T}\phi \\
&\qquad + \left(\int_{\mathbb{R}^d\setminus B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\,\mathrm{d}w\right)\chi_{f>l}\phi \\
&\qquad + \left(\int_{B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\tilde\eta(v) \\
&\quad=: h_1 + h_2,
\end{aligned}$$

with

$$\begin{aligned}
h_1 &:= \mathcal{L}\big[(f-l)_+\phi\big] + (-\Delta_v)^s\big[(f-l)_+\phi\big], \\
h_2 &:= (f-l)_+\mathcal{T}\phi + \left(\int_{\mathbb{R}^d\setminus B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\,\mathrm{d}w\right)\chi_{f>l}\phi \\
&\qquad + \left(\int_{B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\tilde\eta(v).
\end{aligned}$$

Proposition 2.1 then implies for any $0<\epsilon<s$ and any $1<p<1+\frac{s-\epsilon}{2d(1+s)+s+\epsilon}$

$$\begin{aligned}
&\big\|(f-l)_+\phi\big\|_{L^p([0,\tau]\times\mathbb{R}^{2d})} \\
&\quad\le \tau^{\frac1p-\alpha_\epsilon}\left\|(-\Delta_v)^{-\frac{s+\epsilon}{2}}h_1\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} + \tau^{\frac1p+\frac12-\alpha_0}\left\|h_2\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} \\
&\qquad + \tau^{\frac1p+\frac12-\alpha_0}\left\|\big[(f-l)_+\phi\big](0)\right\|_{L^1(\mathbb{R}^{2d})},
\end{aligned} \tag{5.28}$$

where $\alpha_\epsilon = d\left(1+\frac1s\right)\left(1-\frac1p\right)+\frac{s+\epsilon}{2s}$ and, in our case, $\tau\sim R^{2s}$.

Step 2-i: Contribution of h_1. To bound h_1, we have to estimate, on the one hand,

$$\begin{aligned}
&\left\|(-\Delta_v)^{-\frac{s+\epsilon}{2}}(-\Delta_v)^s\big[(f-l)_+\phi\big]\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} \\
&\quad= \left\|(-\Delta_v)^{\frac{s-\epsilon}{2}}\big[(f-l)_+\phi\big]\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})} \\
&\quad= \int_{\Omega_{\frac{R}{4}}}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big|(f-l)_+\phi(v)-(f-l)_+\phi(w)\big|}{|v-w|^{d+s-\epsilon}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

On the other hand, we note that $(-\Delta_v)^{-\frac{s+\epsilon}{2}}\mathcal{L}$ is a non-local operator with a kernel $K(v,w)\,|v-w|^{s+\epsilon}$, which satisfies the same ellipticity assumptions as K with an improved

order $s-\epsilon$ (instead of $2s$). In particular, to bound h_1 it suffices to bound for $r=\frac{R}{2}$

$$\begin{aligned}
&\left\|(-\Delta_v)^{-\frac{s+\epsilon}{2}}\mathcal{L}\big[(f-l)_+\phi\big]\right\|_{L^1([0,\tau]\times\mathbb{R}^{2d})}\\
&\quad=\int_{\mathbb{R}^{1+2d}}(-\Delta_v)^{-\frac{s+\epsilon}{2}}\mathcal{L}\big[(f-l)_+\phi\big](z)\,\mathrm{d}z\\
&\quad=\int_{\Omega_{\frac{R}{4}}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\quad=\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{B_r}\cdots+\int_{\Omega_{\frac{R}{4}}\times\mathbb{R}^d\setminus B_r}\int_{B_r}\cdots+\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{\mathbb{R}^d\setminus B_r}\cdots.
\end{aligned}\tag{5.29}$$

Then, we further note, since $\operatorname{supp}\phi\subseteq Q_{\frac{R}{4}}$ and $r=\frac{R}{2}$, by (1.32)

$$\begin{aligned}
&\int_{\Omega_{\frac{R}{4}}\times\mathbb{R}^d\setminus B_r}\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\quad+\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{\mathbb{R}^d\setminus B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\quad=\int_{\Omega_{\frac{R}{4}}\times\mathbb{R}^d\setminus B_{\frac{R}{2}}}\int_{B_{\frac{R}{4}}}\big((f-l)_+\phi\big)(w)K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\qquad+\int_{Q_{\frac{R}{4}}}\int_{\mathbb{R}^d\setminus B_{\frac{R}{2}}}\big((f-l)_+\phi\big)(v)K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\quad\le CR^{-s+\epsilon}\int_{Q_{\frac{R}{4}}}(f-l)_+\,\mathrm{d}z.
\end{aligned}\tag{5.30}$$

Moreover, we introduce $\mathcal{C}_K$ to be such that

$$K^2(v,w)\mathcal{C}_K^{-1}(v,w)\,|v-w|^{2s+d}\sim K(v,w).\tag{5.31}$$

For the Boltzmann equation for instance (see [118]), we would have

$$\mathcal{C}_K(v,w)=\int_{w'\perp w-v}f(v+w')\,|w'|^{\gamma+2s+1}\,\tilde{b}(\cos\theta)\,\mathrm{d}w'.$$

In particular, due to the ellipticity of K in (1.32), we know that

$$\int_{B_r(v)}\mathcal{C}_K(v,w)\,|v-w|^{-d+2\epsilon}\,\mathrm{d}w\le C(\Lambda_0,\epsilon)r^{2\epsilon}.$$

Thus

$$
\begin{aligned}
&\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\le\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{B_r}(\chi_{f>l}(t,x,v)+\chi_{f>l}(t,x,w))\\
&\qquad\qquad\times\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|K(v,w)\,|v-w|^{s+\epsilon}\,\mathrm{d}w\,\mathrm{d}z\\
&\le\int_{\Omega_{\frac{R}{4}}\times B_r}\chi_{f>l}(v)\left(\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|^2K^2(v,w)\mathcal{C}_K^{-1}\,|v-w|^{2s+d}\,\mathrm{d}w\right)^{\frac12}\\
&\qquad\qquad\times\left(\int_{B_r}|v-w|^{-d+2\epsilon}\,\mathcal{C}_K\,\mathrm{d}w\right)^{\frac12}\mathrm{d}z\\
&\quad+\int_{\Omega_{\frac{R}{4}}\times B_r}\chi_{f>l}(w)\left(\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|^2K^2(v,w)\mathcal{C}_K^{-1}\,|v-w|^{2s+d}\,\mathrm{d}v\right)^{\frac12}\\
&\qquad\qquad\times\left(\int_{B_r}|v-w|^{-d+2\epsilon}\,\mathcal{C}_K\,\mathrm{d}v\right)^{\frac12}\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\le 2C(\Lambda_0,\epsilon)r^\epsilon\left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac12}\\
&\qquad\times\left(\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|^2K^2(v,w)C_K^{-1}\,|v-w|^{2s+d}\,\mathrm{d}w\,\mathrm{d}z\right)^{\frac12},
\end{aligned}
$$

By choice of $\mathcal{C}_K$ in (5.31), we then relate the last integral to the H^s_v norm of $(f-l)_+\phi$, which, due to the energy estimate (5.27), is bounded as

$$
\begin{aligned}
&\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|K(v,w)\,|v-w|^{s}\,\mathrm{d}w\,\mathrm{d}z\\
&\quad\le CR^\epsilon\left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac12}\\
&\qquad\times\left(\int_{\Omega_{\frac{R}{4}}\times B_r}\int_{B_r}\left|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\right|^2K(v,w)\,\mathrm{d}w\,\mathrm{d}z\right)^{\frac12}\qquad(5.32)\\
&\quad\le CR^{-s+\epsilon}\left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac12}\|(f-l)_+\phi\|_{L^2(Q_{\frac{3R}{4}})}\\
&\qquad+CR^{-s+\epsilon}\left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac12}\left(\sup_{Q_{\frac{3R}{4}}}(f-l)_+\right)^{\frac{2-\zeta}{2}}\left(\int_{Q_{\frac{3R}{4}}}(f-l)_+^\zeta(z)\,\mathrm{d}z\right)^{\frac12}.
\end{aligned}
$$

Thus we combine (5.29), (5.30) and (5.32) to get

$$\begin{aligned}
&\Big\|(-\Delta_v)^{-\frac{s+\epsilon}{2}}\mathcal{L}\big[(f-l)_+\phi\big]\Big\|_{L^1([0,\tau]\times\mathbb{R}^{2d})}\\
&\quad=\int_{\Omega_{\frac{R}{4}}\times\mathbb{R}^d}\int_{\mathbb{R}^d}\Big|\big((f-l)_+\phi\big)(w)-\big((f-l)_+\phi\big)(v)\Big|\,K(v,w)\,|v-w|^s\,\mathrm{d}w\,\mathrm{d}z\\
&\quad\le CR^{-s+\epsilon}\int_{Q_{\frac{R}{4}}}(f-l)_+\,\mathrm{d}z+CR^{-s+\epsilon}\Big|\{f>l\}\cap Q_{\frac{R}{2}}\Big|^{\frac12}\,\|(f-l)_+\phi\|_{L^2(Q_{\frac{3R}{4}})}\\
&\qquad+CR^{-s+\epsilon}\Big|\{f>l\}\cap Q_{\frac{R}{2}}\Big|^{\frac12}\left(\sup_{Q_{\frac{3R}{4}}}(f-l)_+\right)^{\frac{2-\zeta}{2}}\left(\int_{Q_{\frac{3R}{4}}}(f-l)_+^{\zeta}(z)\,\mathrm{d}z\right)^{\frac12}.
\end{aligned}\tag{5.33}$$

We conclude thus the bound for h_1 using (5.33):

$$\begin{aligned}
&\Big\|(-\Delta_v)^{-\frac{s+\epsilon}{2}}h_1\Big\|_{L^1([0,\tau]\times\mathbb{R}^{2d})}\\
&\quad\le CR^{-s+\epsilon}\,\|(f-l)_+\|_{L^1(Q_{\frac{R}{4}})}+CR^{-s+\epsilon}\Big|\{f>l\}\cap Q_{\frac{R}{2}}\Big|^{\frac12}\,\|(f-l)_+\phi\|_{L^2(Q_{\frac{3R}{4}})}\\
&\qquad+CR^{-s+\epsilon}\Big|\{f>l\}\cap Q_{\frac{R}{2}}\Big|^{\frac12}\left(\sup_{Q_{\frac{3R}{4}}}(f-l)_+\right)^{\frac{2-\zeta}{2}}\left(\int_{Q_{\frac{3R}{4}}}(f-l)_+^{\zeta}(z)\,\mathrm{d}z\right)^{\frac12}.
\end{aligned}\tag{5.34}$$

Step 2-ii: Contribution of h_2. To bound h_2, we distinguish the transport term, and the tail, which is non-singular, from the local contribution, which is singular,

$$\begin{aligned}
h_2:=\underbrace{(f-l)_+\mathcal{T}\phi}_{=:h_2^{\mathrm{transport}}}&+\underbrace{\left(\int_{\mathbb{R}^d\setminus B_\rho(v)}\big(f(w)-l\big)_+K(v,w)\,\mathrm{d}w\right)\chi_{f>l}\phi}_{=:h_2^{\mathrm{tail}}}\\
&+\underbrace{\left(\int_{B_\rho(v)}\big(f(w)-l\big)_+K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w\right)\chi_{f>l}\tilde\eta(v)}_{=:h_2^{\mathrm{loc}}},
\end{aligned}$$

The transport term gives

$$\big\|h_2^{\mathrm{transport}}\big\|_{L^1(\mathbb{R}^{1+2d})}\le CR^{-2s}\,\|(f-l)_+\|_{L^1(Q_{\frac{R}{2}})}.\tag{5.35}$$

Then we bound h_2^{loc} using the commutator estimates of Lemma 2.3: we find for $\rho = \frac{R}{16}$

$$
\begin{aligned}
&\left\|h_2^{\mathrm{loc}}\right\|_{L^1(\mathbb{R}^{1+2d})} \\
&= \int_{Q_{\frac{R}{2}}} \left(\int_{B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w \right) \chi_{f>l}\,\mathrm{d}z \\
&\le \left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac{1}{2}} \left(\int_{Q_{\frac{R}{2}}} \left(\int_{B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\big(\phi(v)-\phi(w)\big)\,\mathrm{d}w \right)^2 \mathrm{d}z \right)^{\frac{1}{2}} \\
&\le CR^{-2s} \left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac{1}{2}} \|(f-l)_+\|_{L^2(Q_{\frac{R}{2}})} \\
&\quad + CR^{-s} \left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac{1}{2}} \|(f-l)_+\|_{L^2_{t,x}H^s_v(Q_{\frac{R}{2}})}\,,
\end{aligned}
$$

which again by the coercivity (1.26) and the energy estimate (5.27) is bounded as

$$
\begin{aligned}
&\left\|h_2^{\mathrm{loc}}\right\|_{L^1(\mathbb{R}^{1+2d})} \\
&\le CR^{-2s} \left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac{1}{2}} \|(f-l)_+\|_{L^2(Q_{\frac{R}{2}})} \\
&\quad + CR^{-2s} \left(\sup_{Q_R}(f-l)_+\right)^{\frac{2-\zeta}{2}} \left(\int_{Q_R}(f-l)_+^\zeta(z)\,\mathrm{d}z\right)^{\frac{1}{2}} |\{f>l\}\cap Q_R|^{\frac{1}{2}}\,.
\end{aligned}
\tag{5.36}
$$

Finally, to bound the contribution of the tail term, we use the bound of Proposition 5.2 on f:

$$
\begin{aligned}
\left\|h_2^{\mathrm{tail}}\right\|_{L^1(\mathbb{R}^{1+2d})} &= \int_{Q_{\frac{R}{4}}} \int_{\mathbb{R}^d\setminus B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\,\mathrm{d}w\chi_{f>l}\phi\,\mathrm{d}z \\
&\le CR^{-2s} \sup_{Q_R}(f-l)_+^{1-\zeta} \int_{Q_R}(f-l)_+^\zeta\,\mathrm{d}z.
\end{aligned}
\tag{5.37}
$$

Step 2-iii: Conclusion. Since $\phi = 1$ in $Q_{\frac{R}{8}}$ and $\phi = 0$ outside $Q_{\frac{R}{4}}$, we conclude from (5.28), (5.34), (5.35), (5.36) and (5.37)

$$
\begin{aligned}
&\|(f-l)_+\phi\|_{L^p(Q_{\frac{R}{8}})} \\
&\le CR^{s+\epsilon}\,\|(f-l)_+(t_0)\|_{L^1(Q^{t_0}_{\frac{R}{4}})} + CR^{-s+\epsilon}\,\|(f-l)_+\|_{L^1(Q_{\frac{R}{2}})} \\
&\quad + CR^{-s+\epsilon} \left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac{1}{2}} \|(f-l)_+\|_{L^2(Q_{\frac{R}{2}})} \\
&\quad + CR^{-s+\epsilon} \left(\sup_{Q_R}(f-l)_+\right)^{\frac{2-\zeta}{2}} \left(\int_{Q_R}(f-l)_+^\zeta(z)\,\mathrm{d}z\right)^{\frac{1}{2}} |\{f>l\}\cap Q_R|^{\frac{1}{2}} \\
&\quad + CR^{-s+\epsilon} \sup_{Q_R}(f-l)_+^{1-\zeta} \int_{Q_R}(f-l)_+^\zeta\,\mathrm{d}z.
\end{aligned}
\tag{5.38}
$$

Let us remark that, in case of non-zero source term, we would have the additional term

$$CR^{s+\epsilon}\|h\|_{L^\infty(Q_R)}|\{f>l\}\cap Q_R| + CR^{\epsilon}\|h\|^{\frac{1}{2}}_{L^\infty(Q_R)}\left(\sup_{Q_R}(f-l)_+\right)^{\frac{1}{2}}|\{f>l\}\cap Q_R|$$

on the right hand side of (5.38).

Step 3: De Giorgi iteration.

Let $L>0$ to be determined and consider

$$l_k = L(1-2^{-k}), \quad r_k = \frac{R}{8} + 2^{-k}\left(R - \frac{R}{8}\right).$$

Define $Q_k = (-r_k^{2s}, 0]\times B_{r_k^{1+2s}}\times B_{r_k}$ and

$$A_k := \int_{Q_k}(f-l_k)_+(z)\,\mathrm{d}z.$$

We want to prove that $A_k\to 0$ as $k\to+\infty$. This then proves

$$f(t,x,v)\le L$$

for $(t,x,v)\in Q_{\frac{R}{8}}$.

Chebyschev's inequality gives

$$\begin{aligned}|\{f>l_{k+1}\}\cap Q_k| &= \left|\left\{(f-l_k)_+ > 2^{-k-2}L\right\}\cap Q_k\right| \\ &\le 2^{k+2}L^{-1}\int_{Q_k}(f-l_k)_+(z)\,\mathrm{d}z = 2^{k+2}L^{-1}A_k.\end{aligned} \tag{5.39}$$

To bound the right hand side of (5.38) we pick $t_{k+\frac{1}{2}}\in\left[-r_k^{2s}, -r_{k+1}^{2s}\right]$ such that

$$\begin{aligned}\int_{Q_k^{t_{k+\frac{1}{2}}}}(f-l_k)_+(t_{k+\frac{1}{2}},x,v)\,\mathrm{d}v\,\mathrm{d}x &\le \frac{1}{t_{k+1}-t_k}\int_{Q_k}(f-l_k)_+(t,x,v)\,\mathrm{d}z \\ &\le C2^kR^{-2s}A_k.\end{aligned}$$

Moreover, we interpolate the L^2 norm between L^1 and L^∞,

$$\|(f-l_{k+1})_+\|_{L^2(Q_{k+1})}\le\|(f-l_{k+1})_+\|^{\frac{1}{2}}_{L^1(Q_{k+1})}\left(\sup_{Q_{k+1}}(f-l_{k+1})_+\right)^{\frac{1}{2}},$$

and we use Hölder's inequality (note $\zeta<1$)

$$\left(\int_{Q_{k+1}}(f-l_{k+1})_+^\zeta\,\mathrm{d}z\right)^{\frac{1}{2}}\le\left(\int_{Q_{k+1}}(f-l_{k+1})_+\,\mathrm{d}z\right)^{\frac{\zeta}{2}}|\{f>l_{k+1}\}\cap Q_{k+1}|^{\frac{1-\zeta}{2}},$$

and finally, we bound

$$\begin{aligned}
&\sup_{Q_{k+1}}(f-l_{k+1})_+^{1-\zeta}\int_{Q_{k+1}}(f-l_{k+1})_+^{\zeta}\,\mathrm{d}z\\
&\leq\left(\sup_{Q_{k+1}}(f-l_{k+1})_+^{1-\zeta}\right)\left(\sup_{Q_{k+1}}(f-l_{k+1})_+^{\zeta}\right)|\{f>l_{k+1}\}\cap Q_{k+1}|\\
&\leq\left[\left(\sup_{Q_{k+1}}(f-l_{k+1})_+^{1-\zeta}\right)^{\frac{1}{1-\zeta}}+\left(\sup_{Q_{k+1}}(f-l_{k+1})_+^{\zeta}\right)^{\frac{1}{\zeta}}\right]|\{f>l_{k+1}\}\cap Q_{k+1}|\\
&\leq\sup_{Q_{k+1}}(f-l_{k+1})_+\,|\{f>l_{k+1}\}\cap Q_{k+1}|\,.
\end{aligned}$$

Then we use the gain of integrability (5.38), Chebyshev (5.39) and $l_{k+1}\geq l_k$ to find

$$\begin{aligned}
\left(\int_{t_{k+\frac{1}{2}}}^{0}\int_{Q_{k+1}^t}(f-l_{k+1})_+^p(z)\,\mathrm{d}z\right)^{\frac{1}{p}}\leq\;&C2^{4k}R^{-s+\epsilon}A_k+C2^{3k}R^{-s+\epsilon}A_k\\
&+C2^{4k}R^{-s+\epsilon}L^{-\frac{1}{2}}\left(\sup_{Q_k}(f-l_k)_+\right)^{\frac{1}{2}}A_k\\
&+C2^{4k}R^{-s+\epsilon}\left(\sup_{Q_k}(f-l_k)_+\right)^{\frac{2-\zeta}{2}}L^{-(1-\frac{\zeta}{2})}A_k\\
&+C2^{4k}R^{-s+\epsilon}\sup_{Q_k}(f-l_k)_+L^{-1}A_k.
\end{aligned}\tag{5.40}$$

Again, for non-zero source term, we have the additional terms

$$\begin{aligned}
&C2^{k+2}R^{-s+\epsilon}\left(R^{2s}\left\|h\right\|_{L^\infty(Q_k)}+R^s\left\|h\right\|_{L^\infty(Q_k)}^{\frac{1}{2}}\left(\sup_{Q_k}(f-l_k)_+\right)^{\frac{1}{2}}\right)L^{-1}A_k\\
&\leq C2^{k+2}R^{-s+\epsilon}\left(2R^{2s}\left\|h\right\|_{L^\infty(Q_k)}+\sup_{Q_k}(f-l_k)_+\right)L^{-1}A_k
\end{aligned}$$

on the right hand side of (5.40). Using $Q_{k+1}\subset Q_k$, Hölder, (5.40) and Chebyschev's

inequality (5.39) we get

$$\begin{aligned}
A_{k+1} &= \int_{Q_{k+1}} (f - l_{k+1})_+(z)\,\mathrm{d}z \\
&\le \left(\int_{Q_{k+1}} (f - l_{k+1})_+^p(z)\,\mathrm{d}z \right)^{\frac{1}{p}} |\{f > l_{k+1}\} \cap Q_{k+1}|^{1-\frac{1}{p}} \\
&\le \left(\int_{t_{k+\frac{1}{2}}}^0 \int_{Q_{k+1}^t} (f - l_{k+1})_+^p(z)\,\mathrm{d}z \right)^{\frac{1}{p}} |\{f > l_{k+1}\} \cap Q_k|^{1-\frac{1}{p}} \\
&\le C 2^{4k} R^{-s+\epsilon} \left(1 + \left(\sup_{Q_R} f \right)^{\frac{1}{2}} L^{-\frac{1}{2}} + \left(\sup_{Q_R} f \right) L^{-1} + \left(\sup_{Q_R} f \right)^{\frac{2-\zeta}{\zeta}} L^{-\frac{2-\zeta}{\zeta}} \right) \\
&\qquad \times L^{-\frac{p-1}{p}} A_k^{1+\frac{p-1}{p}}.
\end{aligned}$$

For any $\delta > 0$, pick

$$L \ge \delta \sup_{Q_R} f,$$

or, in case of non-zero source term,

$$L \ge \delta \left(\sup_{Q_R} f + R^{2s} \sup_{Q_R} h \right).$$

Then for $0 < \delta \le 1$

$$\begin{aligned}
A_{k+1} &\le C 2^{4k} R^{-s+\epsilon} \left(1 + \delta^{-\frac{1}{2}} + \delta^{-1} + \delta^{-\frac{2-\zeta}{\zeta}} \right) L^{-\frac{p-1}{p}} A_k^{1+\frac{p-1}{p}} \\
&\le C 2^{4k} R^{-s+\epsilon} \delta^{-\frac{2-\zeta}{\zeta}} L^{-\frac{p-1}{p}} A_k^{1+\frac{p-1}{p}}.
\end{aligned}$$

To determine how large we can pick L, we consider the sequence $A_k^* := A_0 Q^{-k}$ for some $Q > 1$ to be determined and for $k \ge 0$. Then we require A_k^* to satisfy the reverse non-linear recurrence, that is

$$A_{k+1}^* \ge 2^{4k} R^{-s+\epsilon} \delta^{-\frac{2-\zeta}{\zeta}} L^{-\frac{p-1}{p}} A_k^{1+\frac{p-1}{p}}. \tag{5.41}$$

Equivalently,

$$1 \ge 2^{4k} R^{-s+\epsilon} \delta^{-\frac{2-\zeta}{\zeta}} L^{-\frac{p-1}{p}} A_0^{\frac{p-1}{p}} Q^{-k(\frac{p-1}{p})} Q.$$

Choose Q such that $2^4 Q^{-\frac{p-1}{p}} \le 1$ or $2^4 \le Q^{\frac{p-1}{p}}$. Then (5.41) holds if

$$L^{\frac{p-1}{p}} \ge R^{-s+\epsilon} A_0^{\frac{p-1}{p}} Q \delta^{-\frac{2-\zeta}{\zeta}}.$$

Therefore, we pick

$$L = A_0 R^{-\frac{(s-\epsilon)p}{p-1}} 2^{\frac{4p^2}{(p-1)^2}} \delta^{-\frac{2-\zeta}{\zeta}\frac{p}{p-1}} + \delta \left(\sup_{Q_R} f + R^{2s} \sup_{Q_R} h \right). \tag{5.42}$$

Then since $\frac{p-1}{p} > 0$ and since $A_0 \leq R^{\frac{(s-\epsilon)p}{p-1}} 2^{-\frac{4p^2}{(p-1)^2}} L \delta^{\frac{2-\zeta}{\zeta}\frac{p}{p-1}}$, we deduce inductively

$$A_k \leq 2^{-\frac{4kp}{p-1}} A_0,$$

so that $A_k \to 0$ as $k \to +\infty$. In particular, for almost every $(t,x,v) \in Q_{\frac{R}{8}}$

$$f(z) \leq L = \delta \left(\sup_{Q_R} f + R^{2s} \sup_{Q_R} h \right) + R^{-\frac{(s-\epsilon)p}{p-1}} 2^{\frac{4p^2}{(p-1)^2}} \delta^{-\frac{2-\zeta}{\zeta}\frac{p}{p-1}} \int_{Q_R} f(z) \, \mathrm{d}z.$$

Finally, note that $\sup_{Q_R} f < +\infty$ (which holds due to the non-local L^2-L^∞ estimate, see Lemma 3.4, together with Definition 5.1), so that we can absorb the first term on the left hand side with Lemma 2.2. This concludes the proof of Proposition 5.1. Remarkably, the proof only works for solutions, and not for sub-solutions, which is in contrast to the local case.

5.5 Strong Harnack for equations in divergence form

Proof of the strong Harnack inequality in Theorem 5.1. As a consequence of Proposition 5.1, Young's inequality, the non-negativity of f and the weak Harnack (1.33), we obtain for any $\delta_1 \in (0,1)$

$$\begin{aligned}
\sup_{Q_{\frac{R}{8}}(z_0)} f &\leq C R^{-(2d(1+s)+2s)} \left\| f \right\|_{L^1(Q_R(z_0))} + C R^{2s} \left\| h \right\|_{L^\infty(Q_R(z_0))} \\
&\leq \delta_1 \sup_{Q_R(z_0)} f + C(\delta_1) R^{-\frac{2d(1+s)+2s}{\zeta}} \left(\int_{Q_R(z_0)} f^\zeta \, \mathrm{d}z \right)^{\frac{1}{\zeta}} + C R^{2s} \left\| h \right\|_{L^\infty(Q_R(z_0))} \\
&\leq \delta_1 \sup_{Q_R(z_0)} f + C(\delta_1) R^{-\frac{2d(1+s)+2s}{\zeta}} \inf_{\tilde{Q}_R^+(z_0)} f + C R^{-\frac{2d(1+s)+2s}{\zeta}} \left\| h \right\|_{L^\infty(Q_R(z_0))},
\end{aligned}$$

where we recall $\tilde{Q}_R^+ := Q_R \left(\left(\frac{3}{2} R^{2s} + \frac{1}{2} \left(\frac{R}{2} \right)^{2s}, 0, 0 \right) \right)$. Lemma 2.2 permits to absorb the first term on the right hand side, and we conclude

$$\sup_{Q_{\frac{R}{8}}(z_0)} f \leq C R^{-\frac{2d(1+s)+2s}{\zeta}} \left(\inf_{\tilde{Q}_R^+(z_0)} f + \left\| h \right\|_{L^\infty(Q_R(z_0))} \right).$$

□

5.6 Boltzmann

The aim of this section is to establish the strong Harnack inequality for the Boltzmann equation for moderately soft potentials without cutoff assumption. This section is based on [154]. We prove **Theorem 1.4**.

5.6.1 Notion of solutions

Since we require further regularity properties of the solution to deduce that the skew-symmetric part of K_f is of lower order in the sense of (5.43), we need to work with the following notion of solutions.

Definition 5.2 (Solutions)**.** We say that $f : \big(0, T\big) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a solution of (1.2), if

1. f is non-negative everywhere,
2. f solves (1.2) classically for every (t, x, v),
3. $f \in C^\infty$ in time, space, and velocity,
4. For each value of (t, x) the function f decays rapidly as $|v| \to \infty$, that is for any $q > 0$
$$\lim_{|v|\to\infty} \frac{f(t, x, v)}{(1 + |v|)^q} = 0$$
locally uniformly in (t, x).
5. f is periodic in x,
6. f conserves mass:
$$\forall 0 \le t_1 < t_2 \le T : \iint_{\mathbb{R}^{2d}} f(t_2)\,\mathrm{d}x\,\mathrm{d}v \le \iint_{\mathbb{R}^{2d}} f(t_1)\,\mathrm{d}x\,\mathrm{d}v.$$

This notion of solutions was also used by Imbert-Silvestre in [118].

5.6.2 On the non-divergent part of the kernel

It seems that conditions (1.29) and (1.30) are not sufficient to deduce our main theorem. We also need to assume that the anti-symmetric part is of lower order with respect to the symmetric part: for any $0 < r \le 1$ there is $\alpha > s$ such that

$$\sup_{v\in B_r} \int_{B_r} \frac{|K_f(v, w) - K_f(w, v)|^2}{(K_f(v, w) + K_f(w, v))}\,\mathrm{d}w \le \mu_0 r^{2(\alpha-s)}. \tag{5.43}$$

This condition has been used to derive Hölder continuity of solutions to parabolic nonlocal equations by Kassmann-Weidner [127]. We show that any solution f of (1.2) in $(0,T)\times\mathbb{R}^d\times\mathbb{R}^d$, such that (1.20) holds, satisfies (5.43), provided that, for the case that $\gamma\le 0$, for any $q>0$ there is $N_q>0$ such that $f_0(x,v)=f(0,x,v)\le N_q(1+|v|)^{-q}$ for $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$. The justification uses further regularity properties of solutions to (1.2), derived by Imbert-Mouhot-Silvestre in [115, 118, 121].

We consider a solution f of (1.2) in $(0,T)\times\mathbb{R}^d\times\mathbb{R}^d$, periodic in x, such that (1.20) holds. If $\gamma\le 0$, then we also assume that for any $q>0$ there is $N_q>0$ such that $f_0(x,v)=f(0,x,v)\le N_q(1+|v|)^{-q}$ for $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$. Then for every $q>0$ there is a constant $C_{s,q}>0$ and $\alpha>s$ depending only on $s,d,\gamma,m_0,M_0,E_0,H_0$ and N_q in case that $\gamma\le 0$, such that for any $R>0$, any $\tau_0>0$, any $(t,x)\in(\tau_0,T)\times B_{R^{1+2s}}$ and any $v,w\in\mathbb{R}^d$ with $|v-w|<\min\left\{\frac{R}{2},1\right\}$ there holds

$$|f(v)-f(w)|\le C_{s,q}\,|v-w|^{\alpha}\left(1+|v|\,\right)^{-q}\approx C_{s,q}\,|v-w|^{\alpha}\left(1+|w|\,\right)^{-q}. \tag{5.44}$$

An even stronger statement is derived in [118, Corollary 7.8]. In fact, the constants $C_{s,q}$ and α may depend on N_p for some fixed $p>q$.

We now consider $(t,x,v)\in Q_R$ for $0<R\le 1$. We recall [177, Corollary 4.2]: for any $v,w\in\mathbb{R}^d$ there is a universal constant $C_K>0$ such that

$$K_f(v,w)=C_K\,|v-w|^{-(d+2s)}\int_{w'\perp v-w} f(v+w')\,|w'|^{\gamma+2s+1}\,\mathrm{d}w'.$$

Thus for $v\in B_R$ we take $q>d+2s+\gamma$ and use (5.44), so that

$$\begin{aligned}
&|K_f(v,w)-K_f(w,v)|\\
&\quad\le C_K\,|v-w|^{-(d+2s)}\left|\int_{w'\perp v-w} f(v+w')\,|w'|^{\gamma+2s+1}\,\mathrm{d}w'-\int_{w'\perp w-v} f(w+w')\,|w'|^{\gamma+2s+1}\,\mathrm{d}w'\right|\\
&\quad\le C_K\,|v-w|^{-(d+2s)}\int_{w'\perp v-w}|f(v+w')-f(w+w')|\,|w'|^{\gamma+2s+1}\,\mathrm{d}w'\\
&\quad\le C_K C_{s,q}\,|v-w|^{\alpha-(d+2s)}\int_{w'\perp v-w}\left(1+|w'|\,\right)^{-q}|w'|^{\gamma+2s+1}\,\mathrm{d}w'.
\end{aligned}$$

We then combine this with [177, Lemma 4.8], which shows that for each $v\in B_R$ there is a symmetric subset of the unit sphere $A=A(v)\subset\partial B_1$, with strictly positive measure, and such that for each $\frac{v-w}{|v-w|}\in A$, the kernel satisfies a pointwise lower bound $K_f(v,w)\ge\lambda\,|v-w|^{-(d+2s)}$ for some λ depending only on the hydrodynamical quantities in (1.20) and R. Indeed, Lemma 4.6 in [177] implies that there is $\ell>0$, $m>0$ and $r>0$ such that

$$|\{v:f(v)>\ell\}\cap B_r|\ge m.$$

Thus if we introduce $S := \{v : f(v) > \ell\} \cap B_r$, then $f(v) \geq \ell \chi_S(v)$ and thus $K_f(v,w) \geq \ell K_{\chi_S}(v,w)$. Since $S \subset B_r$, we find on the one hand,

$$\int_{w' \perp v-w} (\chi_S(v+w') + \chi_S(w+w')) \, |w'|^{\gamma+2s+1} \, \mathrm{d}w' \leq 2(R+r)^{\gamma+2s+1} \, |B_r| \, .$$

On the other hand, $|S| \geq m$. In particular, this implies for any $v \in B_R$ that $K_f(v,w) \geq \lambda \, |v-w|^{-(d+2s)}$ whenever $\frac{v-w}{|v-w|} \in A$. We find

$$\begin{aligned}
&\int_{B_R} \frac{|K_f(v,w) - K_f(w,v)|^2}{(K_f(v,w) + K_f(w,v))} \, \mathrm{d}w = \int_{B_R} \frac{|K_f(v,w) - K_f(w,v)|^2}{(K_f(v,w) + K_f(w,v))} (1 - \chi_{K_f(v,w)=0} \chi_{K_f(w,v)=0}) \, \mathrm{d}w \\
&\leq \frac{1}{\ell} C_K^2 C_{s,q}^2 \int_{B_R} \frac{(1 - \chi_{K_f(v,w)=0} \chi_{K_f(w,v)=0})}{|v-w|^{d+2(s-\alpha)}} \\
&\qquad \times \frac{\int_{w' \perp v-w} \left(1 + |w'|\right)^{-q} |w'|^{\gamma+2s+1} \, \mathrm{d}w'}{\int_{w' \perp v-w} (\chi_S(v+w') + \chi_S(w+w')) \, |w'|^{\gamma+2s+1} \, \mathrm{d}w'} \, \mathrm{d}w \\
&\leq \frac{C}{\ell} C_K^2 C_{s,q}^2 \int_0^R \rho^{2(\alpha-s)-1} \int_{\partial B_1} \left(\frac{(1 - \chi_{K_f(v,w)=0} \chi_{K_f(w,v)=0})}{\int_{w' \perp \sigma} (\chi_S(v+w') + \chi_S(w+w')) \, |w'|^{\gamma+2s+1} \, \mathrm{d}w'} \right) \mathrm{d}\sigma \, \mathrm{d}\rho \\
&\leq C(\ell, \lambda, r, m) C_K^2 C_{s,q}^2 R^{2(\alpha-s)}.
\end{aligned}$$

The last inequality follows from the fact that as σ swipes ∂B_1, the hyperplane perpendicular to σ covers the whole space $\mathbb{R}^d$, and using that $|S| \geq m$, we see that at least one of the indicator functions in the denominator is positive; combined with the boundedness of S we get a lower bound on the denominator.

5.6.3 Tail bound on upper level sets

In this section we establish the non-local-to-local bound, similar to Proposition 5.2 above, for the Boltzmann equation. It is in this section that the weak divergence form structure of (1.16) proves useful. Just as in the divergence form case, we derive this non-local-to-local bound on upper level sets. To this end, we consider $l \in \mathbb{R}_+$. We check using (1.16) that if

f solves (1.2), then $f - l$ satisfies

$$\begin{aligned}\partial_t f + v \cdot \nabla_x f &= \frac{1}{2}\int_{\mathbb{R}^d} \big((f-l)(w) - (f-l)(v)\big)\big(K_f(v,w) + K_f(w,v)\big)\,\mathrm{d}w \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d} \big((f-l)(v) + (f-l)(w)\big)\big(K_f(v,w) - K_f(w,v)\big)\,\mathrm{d}w \\ &\quad + l\int_{\mathbb{R}^d} \big(K_f(v,w) - K_f(w,v)\big)\,\mathrm{d}w \\ &\geq \frac{1}{2}\int_{\mathbb{R}^d} \big((f-l)(w) - (f-l)(v)\big)\big(K_f(v,w) + K_f(w,v)\big)\,\mathrm{d}w \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d} \big((f-l)(v) + (f-l)(w)\big)\big(K_f(v,w) - K_f(w,v)\big)\,\mathrm{d}w.\end{aligned} \tag{5.45}$$

The last inequality follows due to the classical cancellation from (1.15) and (1.10) provided that $l \geq 0$.

To derive a tail bound on upper level sets, we use the formulation in (5.45). The idea is to construct a concave test function that localises only on the upper level sets. Then, the weak divergence form structure proves to be sufficient to make the terms on the lower level sets vanish. Moreover, the cross terms end up with a sign.

Proposition 5.3 (Non-local-to-local bound)**.** *For given $R > 0$, $z_0 \in \mathbb{R}^{1+2d}$, $l \in \mathbb{R}_+$, and any non-negative super-solution f of (1.16) in $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ with a kernel K_f that satisfies (1.32), (1.29), (1.29), (5.43), and such that f conserves mass, we can find for any $\zeta \in (0,1)$, a constant $C > 0$ depending on $\zeta, s, d, \Lambda_0, \mu_0$ such that*

$$\begin{aligned}\int_{Q_{\frac{3R}{4}}(z_0)} \int_{\mathbb{R}^d \setminus B_R(v_0)} &(f-l)_+(w)\chi_{f>l}(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\leq CR^{-2s} \sup_{Q_R(z_0)} (f-l)_+^{1-\zeta} \int_{Q_R(z_0)} (f-l)_+^{\zeta}\,\mathrm{d}z.\end{aligned} \tag{5.46}$$

Proof of Proposition 5.4. The proof idea is as above for Propsition 5.2. Our aim is to get a local bound on the tail of a level set function. To this end we do a concavity estimate on level sets by testing the equation with the level set of the solution to some inverse power. This creates *cross* terms, which have the good sign due to the concavity of the test function. Moreover, we only localise on upper level sets, which makes the terms on lower level sets vanish due to the weak divergence structure.

Again we may, with no loss in generality, consider $z_0 = (0,0,0)$; and we again consider $0 \leq \psi$ given by

$$\psi_l(t,x,v) := \eta^2(t,x,v)\chi_{f>l}(t,x,v) + \chi_{f<l}(t,x,v),$$

for $l \in \mathbb{R}_+$, and where $\eta \in C_c^\infty(\mathbb{R}^{2d+1})$ is such that $\eta = 1$ in $Q_{\frac{3R}{4}}$ and $\eta = 0$ outside $Q_{\frac{7R}{8}}$. In particular, for any $v, w \in \mathbb{R}^d$ there holds:

i. If $f(v), f(w) > l$ then $\psi_l(v) = \eta^2(v)$ and $\psi_l(w) = \eta^2(w)$.

ii. If $f(v), f(w) < l$ then $\psi_l(v) = \psi_l(w) = 1$.

iii. If $f(v) > l > f(w)$ then $\psi_l(v) = \eta^2(v) \leq 1 = \psi_l(w)$.

iv. If $f(w) > l > f(v)$ then $\psi_l(w) = \eta^2(w) \leq 1 = \psi_l(v)$.

We test (5.45) with $f_{l,\varepsilon}^{-(1-\zeta)}\psi_l$ for any $l \in \mathbb{R}_+$ and $\varepsilon > 0$.

Step 1: Non-local operator.

We first consider for fixed (t, x) the right hand side of (5.45) after testing with $f_{l,\varepsilon}^{-(1-\zeta)}\psi_l$:

$$\begin{aligned}
\mathcal{E} &:= \mathcal{E}\left(f - l, \psi f_{l,\varepsilon}^{-(1-\zeta)}\right) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)\right] \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)\right] \\
&\qquad \times \left[\psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v) - \psi_l(w) f_{l,\varepsilon}^{-(1-\zeta)}(w)\right] \mathrm{d}w\,\mathrm{d}v.
\end{aligned}$$

We then split $\mathcal{E}$ into three parts: we introduce

$$\begin{aligned}
\chi_{up}(v,w) &:= \chi_{f(v)>l}\chi_{f(w)>l}, \\
\chi_{low}(v,w) &:= \chi_{f(v)<l}\chi_{f(w)<l}, \\
\chi_{cross}(v,w) &:= \chi_{f(v)>l}\chi_{f(w)<l} + \chi_{f(v)<l}\chi_{f(w)>l},
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{E} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)\right] \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{up}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)\right] \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{low}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)\right] \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{cross}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&=: \mathcal{E}_{up} + \mathcal{E}_{low} + \mathcal{E}_{cross}.
\end{aligned} \tag{5.47}$$

Step 1-i.: Non-locality on lower level sets.

As a consequence of the weak divergence form structure of (1.16), we see that the choice of ψ_l implies through observation ii, that

$$\begin{aligned}
\mathcal{E}_{low} &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)\right] \\
&\qquad \times \left[\psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v) - \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(w)\right] \chi_{low}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&= 0.
\end{aligned} \tag{5.48}$$

Step 1-ii.: Cross non-locality.

Moreover, we claim that the cross term has a sign:

$$\begin{aligned}
\mathcal{E}_{cross} &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad\qquad \times \left[\psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v) - \psi_l(w) f_{l,\varepsilon}^{-(1-\zeta)}(w)\right] \chi_{cross}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad\qquad \times \left[\eta^2(v)(f-l+\varepsilon)^{-(1-\zeta)}(v) - \varepsilon^{-(1-\zeta)}\right] \chi_{f(v)>l>f(w)}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad\qquad \times \left[\varepsilon^{-(1-\zeta)} - \eta^2(w)(f-l+\varepsilon)^{-(1-\zeta)}(w)\right] \chi_{f(w)>l>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\leq 0.
\end{aligned} \tag{5.49}$$

The last inequality uses, for the first integrand, that we are restricted to the case that $f(v)-l>0$ and $-(f(w)-l)>0$, combined with the fact that $\eta^2(v)(f-l+\varepsilon)^{-(1-\zeta)}(v) \leq \varepsilon^{-(1-\zeta)}$, since $f(v)>l$ and $\eta^2(v)\leq 1$. Similarly, we argue for the second integrand, after noticing that $(f-l)(v)<0$ and $-(f-l)(w)<0$.

Step 1-iii.: Non-locality on upper level sets.

To bound $\mathcal{E}_{up}$, we first note that $\mathcal{E}_{up}$ is localised in time and space due to the test function. In fact, if we integrate over time and space on the whole space, the support of η restricts it to a local domain. Thus, we find for any $(t,x)\in[-R^{2s},0]\times B_{R^{1+2s}}$ due to i

$$\begin{aligned}
\mathcal{E}_{up} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad\qquad \times \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{up}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_{B_R}\int_{B_R} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad\qquad \times \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{up}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad\qquad \times \psi_l(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{up}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&=: \mathcal{E}_{up}^{loc} + \mathcal{E}_{up}^{tail}.
\end{aligned} \tag{5.50}$$

For $\mathcal{E}_{up}^{tail}$, we note that there is no singularity, due to the choice of η, which vanishes outside

$B_{\frac{7R}{8}}$. Thus we find due to (1.27)

$$\begin{aligned}
\mathcal{E}_{up}^{tail} &= \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} [(f-l)(v)K_f(w,v) - (f-l)(w)K_f(v,w)] \\
&\qquad \times \eta^2(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(v)\eta^2(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)K_f(w,v)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad - \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w)\eta^2(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)K_f(v,w)\chi_{f(v)>l}\chi_{f(w)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\le \Lambda_0\left(\frac{R}{8}\right)^{-2s}\int_{B_R}((f-l)_+ + \varepsilon)^{\zeta}(v)\chi_{f>l}(v)\,\mathrm{d}v \\
&\quad - \int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w)\eta^2(v) f_{l,\varepsilon}^{-(1-\zeta)}(v)K_f(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned} \tag{5.51}$$

In particular, the tail of $(f-l)_+$ has a good sign.

To bound the not-too-non-local non-locality of $\mathcal{E}_{up}$, we exploit the concavity of the test function again, due to which we get a signed coercive term, and we show that the terms that have no sign have a singularity of lower order, and thus can be bounded. Concretely, using that $1 = \chi_{f(w)>f(v)} + \chi_{f(v)>f(w)}$, the symmetry of the integrand, adding and subtracting $f_{l,\varepsilon}^{-(1-\zeta)}(w)\eta^2(v)$, and finally adding and subtracting $f_l(v)K_f(v,w)$ and

$f_l(w)K_f(w,v)$, we find

$$
\begin{aligned}
\mathcal{E}^{loc}_{up} &= \frac{1}{2}\int_{B_R}\int_{B_R}[(f-l)(v)K_f(w,v)-(f-l)(w)K_f(v,w)] \\
&\qquad\times\left[\eta^2(v)f_{l,\varepsilon}^{-(1-\zeta)}(v)-\eta^2(w)f_{l,\varepsilon}^{-(1-\zeta)}(w)\right]\chi_{up}(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_{B_R}\int_{B_R}[(f-l)(v)K_f(w,v)-(f-l)(w)K_f(v,w)] \\
&\qquad\times\left[\eta^2(v)f_{l,\varepsilon}^{-(1-\zeta)}(v)-\eta^2(w)f_{l,\varepsilon}^{-(1-\zeta)}(w)\right]\chi_{up}(v,w)\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_{B_R}\int_{B_R}[(f-l)(v)K_f(w,v)-(f-l)(w)K_f(v,w)] \\
&\qquad\times\left[f_{l,\varepsilon}^{-(1-\zeta)}(v)-f_{l,\varepsilon}^{-(1-\zeta)}(w)\right]\eta^2(v)\chi_{up}(v,w)\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\int_{B_R}\int_{B_R}[(f-l)(v)K_f(w,v)-(f-l)(w)K_f(v,w)] \\
&\qquad\times\left[\eta^2(v)-\eta^2(w)\right]f_{l,\varepsilon}^{-(1-\zeta)}(w)\chi_{up}(v,w)\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_{B_R}\int_{B_R}[f_l(v)-f_l(w)]\left[f_{l,\varepsilon}^{-(1-\zeta)}(v)-f_{l,\varepsilon}^{-(1-\zeta)}(w)\right] \\
&\qquad\times\eta^2(v)K_f(v,w)\chi_{up}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\int_{B_R}\int_{B_R}[K_f(w,v)-K_f(v,w)]\,f_l(v)\left[f_{l,\varepsilon}^{-(1-\zeta)}(v)-f_{l,\varepsilon}^{-(1-\zeta)}(w)\right] \\
&\qquad\times\eta^2(v)\chi_{up}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\int_{B_R}\int_{B_R}[f_l(v)-f_l(w)]\left[\eta^2(v)-\eta^2(w)\right] \\
&\qquad\times f_{l,\varepsilon}^{-(1-\zeta)}(w)K_f(w,v)\chi_{up}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\int_{B_R}\int_{B_R}[K_f(w,v)-K_f(v,w)]\left[\eta^2(v)-\eta^2(w)\right] \\
&\qquad\times f_l(w)f_{l,\varepsilon}^{-(1-\zeta)}(w)\chi_{up}\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&=: I^f_{concave}+I^K_{concave}+I^f_{cutoff}+I^K_{cutoff},
\end{aligned}
\tag{5.52}
$$

where we denoted $f_l = f - l$.

For $I^f_{concave}$ we use (5.4), (5.6), and (1.32)

$$
\begin{aligned}
I^f_{concave} &\le -\frac{4(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}\Big(f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\Big)^2\eta^2(v)K_f(v,w)\chi_{up}(v,w)\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v \\
&\le \frac{4(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}f_{l,\varepsilon}^{\zeta}(w)\big(\eta(v)-\eta(w)\big)^2K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad-\frac{2(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\big)(v)-\big(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\big)(w)\Big)^2K_f(v,w)\chi_{up}(v,w)\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
$$

Then we use the upper bound (1.32). Moreover, in order to absorb the remaining terms by

the signed term, we distinguish the symmetric and the anti-symmetric part by adding and subtracting $K_f(w,v)$, then using Young's inequality and (5.43), so that for δ sufficiently small

$$
\begin{aligned}
&I^f_{concave}\\
&\le CR^{2-2s}\left\|D_v\eta\right\|^2_{L^\infty}\int_{B_R} f^\zeta_{l,\varepsilon}(w)\,\mathrm{d}w\\
&\quad-\frac{(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+\frac{(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(w,v)-K_f(v,w))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\le CR^{-2s}\int_{B_R} f^\zeta_{l,\varepsilon}(w)\,\mathrm{d}w\\
&\quad-\frac{(1-\zeta)}{\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+\delta\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(w,v)+K_f(v,w))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+C(\delta,\zeta)\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2\frac{|K_f(w,v)-K_f(v,w)|^2}{(K_f(v,w)+K_f(w,v))}\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\le CR^{-2s}\int_{B_R} f^\zeta_{l,\varepsilon}(w)\,\mathrm{d}w\\
&\quad-\frac{(1-\zeta)}{2\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+C(\delta,\zeta)\int_{B_R}\eta^2(v)f^\zeta_{l,\varepsilon}(v)\sup_{v\in B_R}\int_{B_R}\frac{|K_f(w,v)-K_f(v,w)|^2}{(K_f(v,w)+K_f(w,v))}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+C(\delta,\zeta)\int_{B_R}\eta^2(w)f^\zeta_{l,\varepsilon}(w)\sup_{w\in B_R}\int_{B_R}\frac{|K_f(w,v)-K_f(v,w)|^2}{(K_f(v,w)+K_f(w,v))}\,\mathrm{d}v\,\mathrm{d}w\\
&\le CR^{-2s}\int_{B_R} f^\zeta_{l,\varepsilon}(w)\,\mathrm{d}w\\
&\quad-\frac{(1-\zeta)}{2\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
\tag{5.53}
$$

To bound I^K_{cutoff} we use $|\eta^2(v)-\eta^2(w)|\le\max\{\eta(v),\eta(w)\}\,|\eta(v)-\eta(w)|$, $(f-l)_+\le$

$f_{l,\varepsilon}$, Young's inequality, (5.43), and (1.32)

$$
\begin{aligned}
I^K_{cutoff} &= \int_{B_R}\int_{B_R} \left[K_f(w,v) - K_f(v,w)\right]\left[\eta^2(v) - \eta^2(w)\right] \\
&\qquad \times (f-l)(w) f_{l,\varepsilon}^{-(1-\zeta)}(w)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\leq \int_{B_R}\int_{B_R} \frac{|K_f(w,v) - K_f(v,w)|^2}{(K_f(v,w) + K_f(w,v))} \\
&\qquad \times \max\{\eta^2(v),\eta^2(w)\} f^{\zeta}_{l,\varepsilon}(w)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\quad + \int_{B_R}\int_{B_R} (K_f(w,v) + K_f(v,w))\,|\eta(v) - \eta(w)|^2\, f^{\zeta}_{l,\varepsilon}(w)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v \\
&\leq C(\mu_0)\int_{B_R} f^{\zeta}_{l,\varepsilon}(w)\,\mathrm{d}w + CR^{2-2s}\,\|D\eta\|^2_{L^\infty}\int_{B_R} f^{\zeta}_{l,\varepsilon}(w)\,\mathrm{d}w.
\end{aligned}
\tag{5.54}
$$

For I^f_{cutoff}, we recall

$$
\begin{aligned}
I^f_{cutoff} &= \int_{B_R}\int_{B_R} [(f-l)(v) - (f-l)(w)] \\
&\qquad \times \left[\eta^2(v) - \eta^2(w)\right] f_{l,\varepsilon}^{-(1-\zeta)}(w) K_f(w,v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v,
\end{aligned}
$$

and we use (5.5), Young's inequality, (5.7), the upper bound (1.32), we add and subtract

$K_f(w,v)$, we use Young's inequality again, and (5.43), so that

$$
\begin{aligned}
I^f_{cutoff} &\le \frac{2}{\zeta}\int_{B_R}\int_{B_R}\left|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right| f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\left|\eta^2(v)-\eta^2(w)\right| K_f(w,v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\le \frac{\delta}{2}\int_{B_R}\int_{B_R}\left|f_{l,\varepsilon}^{\frac{\zeta}{2}}(v)-f_{l,\varepsilon}^{\frac{\zeta}{2}}(w)\right|^2 \max\{\eta^2(v),\eta^2(w)\}K_f(w,v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + C(\delta)\int_{B_R}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\left|\eta(v)-\eta(w)\right|^2 K_f(w,v)\chi_{up}(v,w)\chi_{f(w)>f(v)}\,\mathrm{d}w\,\mathrm{d}v\\
&\le \delta\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 K_f(w,v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + \delta\int_{B_R}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)(\eta(v)-\eta(w))^2 K_f(w,v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + C(\delta)\left\|D_v\eta\right\|_{L^\infty}^2 R^{2-2s}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\,\mathrm{d}w\\
&\le \frac{\delta}{2}\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 (K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + \frac{\delta}{2}\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 (K_f(w,v)-K_f(v,w))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + C(\delta)R^{-2s}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\,\mathrm{d}w\\
&\le \frac{\delta}{2}\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 (K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + C(\delta)\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 \frac{|K_f(v,w)-K_f(w,v)|^2}{(K_f(v,w)+K_f(w,v))}\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + \frac{\delta}{2}\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 (K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + C(\delta)R^{-2s}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\,\mathrm{d}w\\
&\le \delta\int_{B_R}\int_{B_R}\left(\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(v)-\left(\eta f_{l,\varepsilon}^{\frac{\zeta}{2}}\right)(w)\right)^2 (K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + C(\delta)R^{-2s}\int_{B_R} f_{l,\varepsilon}^{\zeta}(w)\,\mathrm{d}w.
\end{aligned}
\tag{5.55}
$$

Finally, to bound $I^K_{concave}$, we recall

$$
\begin{aligned}
I^K_{concave} = \int_{B_R}\int_{B_R} &[K_f(w,v)-K_f(v,w)]\\
&\times\left[f_{l,\varepsilon}^{-(1-\zeta)}(v)-f_{l,\varepsilon}^{-(1-\zeta)}(w)\right] f_l(v)\eta^2(v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v,
\end{aligned}
$$

and we use (5.8), Young's inequality, (5.43), (5.7), (1.32)

$$
\begin{aligned}
&I^K_{concave}\\
&\leq \frac{\zeta}{(1-\zeta)}\int_{B_R}\int_{B_R}|K_f(w,v)-K_f(v,w)|\\
&\qquad\times\left|f^{\frac{\zeta}{2}}_{l,\varepsilon}(v)-f^{\frac{\zeta}{2}}_{l,\varepsilon}(w)\right|f^{\frac{\zeta}{2}}_{l,\varepsilon}(v)\eta^2(v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\leq C(\delta)\int_{B_R}\int_{B_R}\frac{|K_f(w,v)-K_f(v,w)|^2}{(K_f(v,w)+K_f(w,v))}f^{\zeta}_{l,\varepsilon}(v)\eta^2(v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+\frac{\delta}{2}\int_{B_R}\int_{B_R}(K_f(v,w)+K_f(w,v))\left|f^{\frac{\zeta}{2}}_{l,\varepsilon}(v)-f^{\frac{\zeta}{2}}_{l,\varepsilon}(w)\right|\eta^2(v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\leq C\int_{B_R}f^{\zeta}_{l,\varepsilon}(v)\,\mathrm{d}v\\
&\quad+\delta\int_{B_R}\int_{B_R}(K_f(v,w)+K_f(w,v))\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+\delta\int_{B_R}\int_{B_R}(K_f(v,w)+K_f(w,v))\big(\eta(v)-\eta(w)\big)^2f^{\zeta}_{l,\varepsilon}(v)\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\leq C\int_{B_R}f^{\zeta}_{l,\varepsilon}(v)\,\mathrm{d}v+CR^{2-2s}\left\|D_v\eta\right\|^2_{L^\infty}\int_{B_R}f^{\zeta}_{l,\varepsilon}(w)\,\mathrm{d}w\\
&\quad+\delta\int_{B_R}\int_{B_R}(K_f(v,w)+K_f(w,v))\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
\tag{5.56}
$$

If we combine (5.52) with (5.53), (5.54), (5.55), (5.56), we find for δ sufficiently small

$$
\begin{aligned}
\mathcal{E}^{loc}_{up}&\leq CR^{-2s}\int_{B_R}f^{\zeta}_{l,\varepsilon}(w)\,\mathrm{d}w\\
&\quad-\frac{(1-\zeta)}{2\zeta^2}\int_{B_R}\int_{B_R}\Big(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\Big)^2(K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+2\delta\int_{B_R}\int_{B_R}\left(\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(v)-\big(\eta f^{\frac{\zeta}{2}}_{l,\varepsilon}\big)(w)\right)^2(K_f(v,w)+K_f(w,v))\chi_{f(w)>f(v)>l}\,\mathrm{d}w\,\mathrm{d}v\\
&\leq CR^{-2s}\int_{B_R}f^{\zeta}_{l,\varepsilon}(w)\,\mathrm{d}w.
\end{aligned}
\tag{5.57}
$$

In particular, this implies together with (5.50) and (5.51)

$$
\begin{aligned}
\mathcal{E}_{up}\leq&CR^{-2s}\int_{B_R}f^{\zeta}_{l,\varepsilon}(w)\,\mathrm{d}w\\
&-\int_{B_R}\int_{\mathbb{R}^d\setminus B_R}(f-l)_+(w)\eta^2(v)f^{-(1-\zeta)}_{l,\varepsilon}(v)K_f(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}
\tag{5.58}
$$

Step 1-iv.: Non-locality.

We conclude from (5.47), (5.48), (5.49), (5.58),

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{up} + \mathcal{E}_{low} + \mathcal{E}_{cross} \\ &\leq C(\zeta, \Lambda_0, \mu_0) R^{-2s} \int_{B_R} (f - l + \varepsilon)^{\zeta}(v) \chi_{f>l} \, \mathrm{d}v \\ &\quad - \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} (f-l)_+(w) \eta^2(v) f_{l,\varepsilon}^{-(1-\zeta)}(v) K_f(v,w) \chi_{f(v)>l} \, \mathrm{d}w \, \mathrm{d}v.\end{aligned} \tag{5.59}$$

Note that this estimate holds true for any (t, x). Moreover, as remarked above (5.50), since the only positive contribution stems from $\mathcal{E}_{up}$, the estimate in (5.59) is localised in time and space.

Step 2.: Transport.

Finally we use the equation. For the transport operator, we first note that since $f - l = (f-l)_+ + (f-l)_-$ and since f conserves mass, there holds

$$\begin{aligned}&\iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)_-(T) - (f-l)_-(0)) \, \mathrm{d}x \, \mathrm{d}v \\ &\quad = \iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)(T) - (f-l)(0)) \, \mathrm{d}x \, \mathrm{d}v - \iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)_+(T) - (f-l)_+(0)) \, \mathrm{d}x \, \mathrm{d}v \\ &\quad = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)_+(T) - (f-l)_+(0)) \, \mathrm{d}x \, \mathrm{d}v.\end{aligned}$$

Now, we distinguish for any x, v the case when $f(0) \geq \min(l, f(T))$, or $f(0) < \min(l, f(T)) \leq \max(l, f(T))$. In the former case, if $f(T) > l$, then $f(0) \geq l$, and the integral vanishes. Else, $f(T) \leq l$, so that $f(0) \geq f(T)$ and $(f-l)_-(T) - (f-l)_-(0) \leq f(T) - l + l - f(0) \leq 0$. Thus,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)_-(T) - (f-l)_-(0)) \, \mathrm{d}x \, \mathrm{d}v \leq 0.$$

In the latter case, when $f(0) \leq \max(l, f(T))$, if $f(T) \geq l$ then $f(0) \leq f(T)$ and $(f - l)_+(0) - f(T) + l \leq f(0) - f(T) \leq 0$ so that

$$\begin{aligned}\iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)_-(T) - (f-l)_-(0)) \, \mathrm{d}x \, \mathrm{d}v &= - \iint_{\mathbb{R}^d \times \mathbb{R}^d} ((f-l)_+(T) - (f-l)_+(0)) \, \mathrm{d}x \, \mathrm{d}v \\ &\leq 0.\end{aligned}$$

Else if $f(T) < l$, then $f(0) \leq l$ and the integral vanishes. Combining this with the

divergence theorem, integration by parts and using that $|\mathcal{T}\eta| \sim R^{-2s}$, we get

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}^{2d}} \mathcal{T}f(z)\psi_l(z) f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}z \\
&\qquad = \int_{[0,T]\times\mathbb{R}^{2d}} \mathcal{T}f(z)\eta^2(z)\chi_{f>l}(v) f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}z \\
&\qquad\quad + \int_{[0,T]\times\mathbb{R}^{2d}} \mathcal{T}f(z)\chi_{f<l}(z)\varepsilon^{-(1-\zeta)}\,\mathrm{d}z \\
&\qquad = \frac{1}{\zeta}\int_{\mathbb{R}^{1+2d}} \mathcal{T}((f-l)_+ + \varepsilon)^{\zeta}(z)\eta^2(z)\,\mathrm{d}z \\
&\qquad\quad + \int_{[0,T]\times\mathbb{R}^{2d}} \mathcal{T}(f-l)_-(z)\varepsilon^{-(1-\zeta)}\,\mathrm{d}z \\
&\qquad = -\frac{1}{\zeta}\int_{\mathbb{R}^{1+2d}} ((f-l)_+ + \varepsilon)^{\zeta}(z)\mathcal{T}\eta^2(z)\,\mathrm{d}z \\
&\qquad\quad + \int_{\mathbb{R}^{2d}} \Big((f-l)_-(T,x,v) - (f-l)_-(0,x,v)\Big)\varepsilon^{-(1-\zeta)}\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad \le C(\zeta)R^{-2s}\int_{Q_R} ((f-l)_+ + \varepsilon)^{\zeta}(z)\,\mathrm{d}z.
\end{aligned} \tag{5.60}$$

Thus by (5.45), (5.60), and (5.59)

$$\begin{aligned}
0 &\le \int_{[0,T]\times\mathbb{R}^{2d}} \mathcal{T}f(z)\psi_l(z) f_{l,\varepsilon}^{-(1-\zeta)}(z)\,\mathrm{d}z + \int_{[0,T]\times\mathbb{R}^{2d}} \mathcal{E}\left(f-l, \psi_l f_{l,\varepsilon}^{-(1-\zeta)}\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\le C(\zeta)R^{-2s}\int_{Q_R} (f-l+\varepsilon)^{\zeta}(z)\eta^2(z)\chi_{f>l}(z)\,\mathrm{d}z \\
&\quad - \int_{\mathbb{R}^{1+d}}\int_{B_R}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w)\eta^2(z) f_{l,\varepsilon}^{-(1-\zeta)}(v)K_f(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}z,
\end{aligned}$$

or rearranged

$$\begin{aligned}
&\int_{Q_{\frac{3R}{4}}}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w) f_{l,\varepsilon}^{-(1-\zeta)}(v)K_f(v,w)\chi_{f(v)>l}\,\mathrm{d}w\,\mathrm{d}z \\
&\qquad\qquad \le CR^{-2s}\int_{Q_R} (f-l+\varepsilon)^{\zeta}(z)\eta^2(v)\chi_{f>l}(v)\,\mathrm{d}z.
\end{aligned}$$

Step 3.: Conclusion.

This implies (5.46), if we let $\varepsilon \to 0$, take out the infimum of $(f-l)^{-(1-\zeta)}$ over $z \in Q_{\frac{3R}{4}} \cap \chi_{f>l}$ on the left hand side, divide the estimate by the infimum (note $f-l \in L^\infty(Q_R)$), and use $(\inf g)^{-1} = \sup g^{-1}$, so that

$$\int_{Q_{\frac{3R}{4}}}\int_{\mathbb{R}^d\setminus B_R} (f-l)_+(w)\chi_{f>l}(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \le CR^{-2s}\sup_{Q_{\frac{3R}{4}}\cap f>l}(f-l)^{1-\zeta}\int_{Q_R}(f-l)_+^{\zeta}\,\mathrm{d}z.$$

□

5.6.4 Strong Harnack for Boltzmann

We use a De Giorgi argument in L^1 to derive a linear L^1-L^∞ bound for solutions to (1.2).

Proposition 5.4 (L^1-L^∞ bound for Boltzmann)**.** *Let $0 < R \le 1$, $z_0 \in \mathbb{R}^{1+2d}$, and let $f \in L^2([-3,0]\times B_1; L^\infty(\mathbb{R}^d) + H^s(B_1))$ be a non-negative solution to (1.11) in $Q_R(z_0)$ with a kernel K_f and a coefficient Λ_f that satisfy (1.22), (1.26), (1.32), (1.29), (1.30), (5.43). Then there exists a large constant $C > 1$ depending on $s, d, \lambda_0, \Lambda_0, \mu_0$ such that*

$$\text{for a.e. } z_1 \in Q_{\frac{R}{8}}(z_0): \qquad f(z_1) \le CR^{-(2d(1+s)+2s)} \int_{Q_R(z_0)} f(z)\,\mathrm{d}z.$$

Proof. Step 1: Energy estimate. Let $\phi \in C_c^\infty(\mathbb{R}^{1+2d})$ be such that $\phi(t,x,v) = 0$ for (t,x,v) outside $Q_{\frac{R}{4}}$ and $\phi(t,x,v) = 1$ for $(t,x,v) \in Q_{\frac{R}{8}}$. We then test (1.11) with $(f-l)_+\phi^2$, so that if we denote by

$$\mathcal{L}f(v) := \int_{\mathbb{R}^d} \big(f(w) - f(v)\big) K_f(v,w)\,\mathrm{d}w,$$

we get

$$\begin{aligned} &\Lambda_f \int_{\mathbb{R}^{1+2d}} f(z)\varphi^2(z)(f-l)_+(z)\,\mathrm{d}z \\ &\quad \ge \frac{1}{2}\int_{\mathbb{R}^{1+2d}} \mathcal{T}(f-l)_+^2(z)\phi^2(z)\,\mathrm{d}z - \int_{\mathbb{R}^{1+2d}} \mathcal{L}f(z)(f-l)_+(z)\phi^2(z)\,\mathrm{d}z. \end{aligned} \tag{5.61}$$

We observe that for any $\frac{R}{4} < r$, if we abbreviate $(f-l)_+ =: f_{l_+}$ and if we denote by $\Omega_\rho := [-\rho^{2s}, 0] \times B_{\rho^{1+2s}}$, then

$$\begin{aligned} &-\int_{\mathbb{R}^{1+2d}} \mathcal{L}f(z)(f-l)_+(z)\phi^2(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}^{1+2d}}\int_{\mathbb{R}^d} \big[f(t,x,v) - f(t,x,w)\big]\big((f-l)_+\phi^2\big)(t,x,v) K_f(t,x,v,w)\,\mathrm{d}w\,\mathrm{d}z \\ &= \frac{1}{2}\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r} \big[f(v) - f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v) - \big(f_{l_+}\phi^2\big)(w)\Big] K_f(v,w)\,\mathrm{d}w\,\mathrm{d}z \\ &\quad + \frac{1}{2}\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{B_r} \big[f(v) - f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v) + \big(f_{l_+}\phi^2\big)(w)\Big] K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{\mathbb{R}^d\setminus B_r} \big[f(v) - f(w)\big]\big(f_{l_+}\phi^2\big)(z) K_f(v,w)\,\mathrm{d}w\,\mathrm{d}z. \end{aligned} \tag{5.62}$$

Step 1-i.: Transport. We integrate by parts the transport term, and use $\mathcal{T}\phi \sim -R^{-2s}$.

$$\begin{aligned} -\frac{1}{2}\int_{\mathbb{R}^{1+2d}} \mathcal{T}(f-l)_+^2(z)\phi^2(z)\,\mathrm{d}z &= \frac{1}{2}\int_{\mathbb{R}^{1+2d}} (f-l)_+^2(t,x,v)\mathcal{T}\phi^2(t,x,v)\,\mathrm{d}z \\ &\leq CR^{-2s}\int_{Q_{\frac{R}{4}}} (f-l)_+^2(z)\,\mathrm{d}z. \end{aligned} \tag{5.63}$$

Step 1-ii.: Tail bound. We use Proposition 5.3 for f, which we assume to be a solution, in particular a super-solution, so that for $r = \frac{R}{2}$

$$\begin{aligned} &\int_{\Omega_{\frac{R}{4}}}\int_{B_r}\int_{\mathbb{R}^d\setminus B_r} \big[f(w)-f(v)\big]\big(f_{l_+}\phi^2\big)(z)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}z \\ &\quad\leq \sup_{Q_{\frac{R}{4}}}(f-l)_+ \int_{Q_{\frac{R}{4}}}\int_{\mathbb{R}^d\setminus B_{\frac{R}{2}}} (f-l)_+(w)K_f(v,w)\chi_{f>l}(v)\,\mathrm{d}w\,\mathrm{d}z \\ &\quad\leq CR^{-2s}\sup_{Q_{\frac{R}{4}}}(f-l)_+ \sup_{Q_{\frac{R}{2}}}(f-l)_+^{1-\zeta}\int_{Q_{\frac{R}{2}}}(f-l)_+^{\zeta}(z)\,\mathrm{d}z \\ &\quad\leq CR^{-2s}\left(\sup_{Q_{\frac{R}{2}}}(f-l)_+\right)^{2-\zeta}\int_{Q_{\frac{R}{2}}}(f-l)_+^{\zeta}(z)\,\mathrm{d}z. \end{aligned} \tag{5.64}$$

We used $-f(v) < -l$ in the first inequality, and Young's inequality in the last inequality.

Step 1-iii: Not-too-non-local operator. What remains to be estimated is the local contribution of the non-local operator,

$$\begin{aligned} \mathcal{E}_{loc} &:= \frac{1}{2}\int_{B_r}\int_{B_r}\big[f(v)-f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v)-\big(f_{l_+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad+\frac{1}{2}\int_{B_r}\int_{B_r}\big[f(v)-f(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v)+\big(f_{l_+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v =: \mathcal{E}_{loc}^{sym}+\mathcal{E}_{loc}^{skew}, \end{aligned}$$

where we recall $f_{l_+} = (f-l)_+$.

Claim 1. For $z = (t,x,v)$ and $r = \frac{R}{2}$

$$\begin{aligned} \mathcal{E}_{loc}^{sym} &\geq \frac{1}{2}\int_{B_r}\int_{B_r}\Big[\big(f_{l_+}\phi\big)(v)-\big(f_{l_+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}v - Cr^{-2s}\int_{B_r} f_{l_+}^2(v)\,\mathrm{d}v \\ &\quad+\frac{1}{2}\int_{B_r}\int_{B_r}\big[f_{l_-}(v)-f_{l_-}(w)\big]\Big[\big(f_{l_+}\phi^2\big)(v)-\big(f_{l_+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v. \end{aligned} \tag{5.65}$$

See the claim in (5.24) above.

Claim 2. For any $\delta_0 \in (0,1)$, there exists a constant $C > 0$ depending on δ_0 such that

$$\begin{aligned}\mathcal{E}_{loc}^{skew} \geq \frac{1}{2}\int_{B_r}\int_{B_r}\big[f_{l-}(v) - f_{l-}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) + \big(f_{l+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ - CR^{-2s}\int_{B_{\frac{R}{4}}} f_{l+}^2(v)\,\mathrm{d}v \\ - \left(\frac{1}{4}+\delta_0\right)\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v.\end{aligned} \tag{5.66}$$

To prove (5.66), we note

$$\begin{aligned}\big[f_{l+}(v) - f_{l+}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) + \big(f_{l+}\phi^2\big)(w)\Big] \\ = \Big[\big(f_{l+}^2\phi^2\big)(v) - \big(f_{l+}^2\phi^2\big)(w)\Big] + f_{l+}(v)f_{l+}(w)\big[\phi^2(w) - \phi^2(v)\big].\end{aligned}$$

Together with $f_{l-}(v) - f_{l-}(w) = f(v) - f(w) - \big(f_{l+}(v) - f_{l+}(w)\big)$, this implies

$$\begin{aligned}&\int_{B_r}\int_{B_r}\big[f(v) - f(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) + \big(f_{l+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &= \int_{B_r}\int_{B_r}\big[f_{l-}(v) - f_{l-}(w)\big]\Big[\big(g_{l+}\phi^2\big)(v) + \big(f_{l+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + \int_{B_r}\int_{B_r}\left\{\Big[\big(f_{l+}^2\phi^2\big)(v) - \big(f_{l+}^2\phi^2\big)(w)\Big] + f_{l+}(v)f_{l+}(w)\big[\phi^2(w) - \phi^2(v)\big]\right\} \\ &\qquad\qquad \times K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v.\end{aligned} \tag{5.67}$$

Then we use the cancellation (1.29)

$$\begin{aligned}&\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)^2(w) - \big(f_{l+}\phi\big)^2(v)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &= \int_{B_r}\int_{B_r}\big(f_{l+}\phi\big)^2(w)\big[K_f(v,w) - K_f(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v \leq \Lambda_0\int_{B_r}\big(f_{l+}^2\phi^2\big)(w)\,\mathrm{d}w.\end{aligned} \tag{5.68}$$

Moreover, we bound with Young's inequality, on the one hand, and using $-2ab = (a-b)^2 - a^2 - b^2$, on the other hand,

$$\begin{aligned}&f_{l+}(v)f_{l+}(w)\big[\phi^2(v) - \phi^2(w)\big] \\ &\leq \frac{1}{2}\big[f_{l+}^2(v) + f_{l+}^2(w)\big]\phi^2(v) + \frac{1}{2}\big(f_{l+}(v) - f_{l+}(w)\big)^2\phi^2(w) - \frac{1}{2}\big[f_{l+}^2(v) + f_{l+}^2(w)\big]\phi^2(w) \\ &= \frac{1}{2}\big[f_{l+}^2(v) + f_{l+}^2(w)\big]\big[\phi^2(v) - \phi^2(w)\big] + \frac{1}{2}\big(f_{l+}(v) - f_{l+}(w)\big)^2\phi^2(w).\end{aligned}$$

Then, using the proof of (5.7) and Young's inequality, we see for any $\delta_0 \in (0,1)$

$$\begin{aligned}
&\frac{1}{2}\big(f_{l+}(v) - f_{l+}(w)\big)^2\phi^2(w) \\
&\qquad \le \left(\frac{1}{2} + \delta_0\right)\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 + C(\delta_0)\big(\phi(v) - \phi(w)\big)^2 f_{l+}^2(v).
\end{aligned}$$

Thus, by doing a Taylor expansion of ϕ, using the upper bound (1.32) for $s \in (0,1/2)$, or the cancellation (1.30) and the upper bound (1.32) for $s \in [1/2,1)$, and finally using the definition of η, we find

$$\begin{aligned}
&\int_{B_r}\int_{B_r} f_{l+}(v)f_{l+}(w)\big[\phi^2(v) - \phi^2(w)\big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad \le \frac{1}{2}\int_{B_r}\int_{B_r}\big[f_{l+}^2(v) + f_{l+}^2(w)\big]\big[\phi^2(v) - \phi^2(w)\big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\quad + \frac{1}{2}\int_{B_r}\int_{B_r}\big(f_{l+}(v) - f_{l+}(w)\big)^2\phi^2(w)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad \le \frac{1}{2}\int_{B_r}\int_{B_r}\big[f_{l+}^2(v)D_v\phi^2(v) + f_{l+}^2(w)D_v\phi^2(w)\big](v-w)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\quad + \big\|D_v^2\phi^2\big\|_{L^\infty}\int_{B_r}\int_{B_r}\big[f_{l+}^2(v) + f_{l+}^2(w)\big]\,|v-w|^2\,K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\quad + \left(\frac{1}{2} + \delta_0\right)\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\quad + C(\delta_0)\int_{B_r}\int_{B_r}\big(\phi(v) - \phi(w)\big)^2 f_{l+}^2(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad \le \frac{1}{2}\int_{B_r}\int_{B_r} f_{l+}^2(v)D_v\phi^2(v)(v-w)\big[K_f(v,w) - K_f(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad\quad + CR^{2-2s}\big\|D_v^2\phi^2\big\|_{L^\infty}\int_{B_r} f_{l+}^2(v)\,\mathrm{d}v \\
&\qquad\quad + \left(\frac{1}{2} + \delta_0\right)\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad \le CR^{1-2s}\big\|D_v\phi^2\big\|_{L^\infty}\int_{B_r} f_{l+}^2(v)\,\mathrm{d}v + CR^{-2s}\int_{B_r} f_{l+}^2(v)\,\mathrm{d}v \\
&\qquad\quad + \left(\frac{1}{2} + \delta_0\right)\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v
\end{aligned} \tag{5.69}$$

Combining (5.67) with (5.68) and (5.69) yields (5.66).

Thus, from the claims in (5.65) and (5.66), we infer for δ_0 sufficiently small ($\delta_0 < \frac{1}{8}$)

and for $r = \frac{R}{2}$

$$
\begin{aligned}
&\mathcal{E}^{sym}_{loc} + \mathcal{E}^{skew}_{loc} \\
&\quad \geq \frac{1}{2}\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}v - Cr^{-2s}\int_{B_r} f_{l+}^2(v)\,\mathrm{d}v \\
&\qquad + \frac{1}{2}\int_{B_r}\int_{B_r}\big[f_{l-}(v) - f_{l-}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) - \big(f_{l+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad + \frac{1}{2}\int_{B_r}\int_{B_r}\big[f_{l-}(v) - f_{l-}(w)\big]\Big[\big(f_{l+}\phi^2\big)(v) + \big(f_{l+}\phi^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad - CR^{-2s}\int_{B_{\frac{R}{4}}} f_{l+}^2(v)\,\mathrm{d}v \\
&\qquad - \left(\frac{1}{4} + \delta_0\right)\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad \geq \frac{1}{8}\int_{B_r}\int_{B_r}\Big[\big(f_{l+}\phi\big)(v) - \big(f_{l+}\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}v - Cr^{-2s}\int_{B_r} f_{l+}^2(v)\,\mathrm{d}v.
\end{aligned}
\tag{5.70}
$$

Step 1-iv.: Conclusion. Combining (5.20) with (5.63), (5.64), and (5.70), we conclude

$$
\begin{aligned}
&\frac{1}{8}\int_{\Omega_{\frac{R}{4}}}\iint_{B_{\frac{R}{2}}\times B_{\frac{R}{2}}}\Big[\big((f-l)_+\phi\big)(v) - \big((f-l)_+\phi\big)(w)\Big]^2 K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq CR^{-2s}\int_{Q_{\frac{R}{4}}}(f-l)_+^2(z)\,\mathrm{d}z + CR^{-2s}\big(\sup_{Q_{\frac{R}{2}}} f\big)^{2-\zeta}\int_{Q_{\frac{R}{2}}}(f-l)_+^{\zeta}(z)\,\mathrm{d}z \\
&\qquad + \mu_0 \sup_{Q_{\frac{R}{4}}} f \int_{Q_{\frac{R}{4}}}(f-l)_+\,\mathrm{d}z.
\end{aligned}
$$

Note that we used (1.22) to bound Λ_f.

Step 2 & 3: Gain of Integrability & De Giorgi iteration. These steps carry over verbally from Proposition 5.1 above, if we work with the Boltzmann equation in the formulation (1.11). Then we have the following gain of integrability

$$
\begin{aligned}
&\|(f-l)_+\phi\|_{L^p(Q_{\frac{R}{8}})} \\
&\leq CR^{s+\epsilon}\,\|(f-l)_+(t_0)\|_{L^1(Q^{t_0}_{\frac{R}{4}})} + CR^{-s+\epsilon}\,\|(f-l)_+\|_{L^1(Q_{\frac{R}{2}})} \\
&\quad + CR^{-s+\epsilon}\left|\{f>l\}\cap Q_{\frac{R}{2}}\right|^{\frac{1}{2}}\|(f-l)_+\|_{L^2(Q_{\frac{R}{2}})} \\
&\quad + CR^{-s+\epsilon}\left(\sup_{Q_R}(f-l)_+\right)^{\frac{2-\zeta}{2}}\left(\int_{Q_R}(f-l)_+^{\zeta}(z)\,\mathrm{d}z\right)^{\frac{1}{2}}|\{f>l\}\cap Q_R|^{\frac{1}{2}} \\
&\quad + CR^{-s+\epsilon}\sup_{Q_R}(f-l)_+^{1-\zeta}\int_{Q_R}(f-l)_+^{\zeta}\,\mathrm{d}z \\
&\quad + CR^{s+\epsilon}\sup_{Q_R} f\,|\{f>l\}\cap Q_R| + CR^{\epsilon}(\sup_{Q_R} f)^{\frac{1}{2}}\,|\{f>l\}\cap Q_R|^{\frac{1}{2}}\,\|(f-l)_+\|_{L^1}^{\frac{1}{2}},
\end{aligned}
$$

with which we conclude for almost every $(t,x,v) \in Q_{\frac{R}{8}}$

$$f(z) \leq L = \delta(1+R^{2s}) \sup_{Q_R} f + R^{-\frac{(s-\epsilon)p}{p-1}} 2^{\frac{4p^2}{(p-1)^2}} \delta^{-\frac{2-\zeta}{\zeta}\frac{p}{p-1}} \int_{Q_R} f(z)\,\mathrm{d}z.$$

For $R \leq 1$, we absorb the first term on the right hand side with a standard iteration argument, concluding the proof of Proposition 5.4. □

Proof of Theorem 1.4. Let f solve (1.2) in $(0,T) \times \mathbb{R}^d \times \mathbb{R}^d$, and assume (1.20) holds. Then, in particular, K_f and Λ_f given in (1.9) and (1.12) satisfy (1.22)-(1.30). Moreover, due to our notion of solutions in Definition 5.2, also (5.43) is satisfied.

Let $\mathcal{I}^- := (\tau_0, \tau_1)$ and $\mathcal{I}^+ := (\tau_2, \tau_3)$ be two disjoint compactly contained subsets of $(0,T)$ such that $\tau_2 - \tau_1 \geq r_0^{2s}$ for sufficiently small $r_0 < \frac{1}{6}$. As a consequence of Proposition 5.4, Young's inequality and the weak Harnack inequality, Theorem 1.1, we obtain for any $\delta \in (0,1)$ and for $\zeta \in (0,1)$ from the weak Harnack inequality,

$$\begin{aligned}\sup_{\mathcal{I}^- \times Q^t_{\frac{r_0}{4}}} f &\leq C r_0^{-(2d(1+s)+2s)} \|f\|_{L^1(\mathcal{I}^- \times Q^t_{2r_0})} \leq \delta \sup_{\mathcal{I}^- \times Q^t_{\frac{r_0}{4}}} f + C(\delta) r_0^{-\frac{(2d(1+s)+2s)}{\zeta}} \|f\|_{L^\zeta(\mathcal{I}^- \times Q^t_{2r_0})} \\ &\leq \delta \sup_{\mathcal{I}^- \times Q^t_{\frac{r_0}{4}}} f + C(\delta) r_0^{-\frac{(2d(1+s)+2s)}{\zeta}} \inf_{\mathcal{I}^+ \times Q^t_{2r_0}} f.\end{aligned}$$

We recall that there is no source term appearing after applying the weak Harnack inequality, due to the positivity of Q_2 (1.15), which implies that f is a non-negative super-solution to (1.11) with zero source term. Absorbing the first term on the left hand side with a standard iteration argument concludes the proof of Theorem 1.4. □

5.7 Appendix: Approximation argument

Here we justify the approximation argument for the use of the test function constructed in Proposition 5.2.

Let $\eta : \mathbb{R}^d \to \mathbb{R}_+$ be a cutoff, such that $0 \leq \eta \leq 1$ with $\eta = 1$ in B_1, $\eta = 0$ outside B_2 and $\eta(v) = -|v| + 2$ for $v \in B_2 \setminus B_1$. Denote by η_δ the cutoff $\eta_\delta(v) = \eta\big(\frac{v}{\delta}\big) \in C_c^\infty(B_{2\delta})$ satisfying $0 \leq \eta_\delta \leq 1$ with $\eta_\delta = 1$ in B_δ and $\eta_\delta = 0$ outside $B_{2\delta}$. Then we consider

$$K_\delta(v,w) = \eta_\delta(v-w) c_{d,s} |v-w|^{-(d+2s)} + \big(1 - \eta_\delta(v-w)\big) K(v,w),$$

where $c_d > 0$ is the constant such that

$$(-\Delta_v)^s f(v) = c_{d,s} \int_{\mathbb{R}^d} \big(f(w) - f(v)\big) |v-w|^{-(d+2s)}\,\mathrm{d}w.$$

Note that $\eta_\delta \xrightarrow[\delta\to 0]{} 0$, pointwise almost everywhere and in L^p for any $1 \le p < \infty$. Now let f_δ solve

$$\mathcal{T} f_\delta = \mathcal{L}_\delta f_\delta := \int_{\mathbb{R}^d} \big(f_\delta(w) - f_\delta(v)\big) K_\delta(v,w)\,\mathrm{d}w, \tag{5.71}$$

in $[t_1,t_2] \times B_{R^{1+2s}} \times \mathbb{R}^d$, such that f_δ is periodic in x and conserves mass. Note

$$f_\delta \in L^2((t_1,t_2) \times B_{R^{1+2s}}; \dot{H}^s(\mathbb{R}^d))$$

uniformly in δ. To see this, we test (5.71) with $f_\delta \phi^2$ where $\phi \in C_c^\infty([t_1,t_2] \times B_{R^{1+2s}})$ is a cutoff in time and space such that $\phi = 1$ in $\left[\frac{t_1}{2}, \frac{t_2}{2}\right] \times B_{\left(\frac{R}{2}\right)^{1+2s}}$ and $\phi = 0$ outside $[t_1,t_2] \times B_{R^{1+2s}}$. We get

$$\begin{aligned}
&\int_{[t_1,t_2]\times B_{R^{1+2s}} \times \mathbb{R}^d} \mathcal{T} f_\delta f_\delta \phi^2 \,\mathrm{d}z \\
&= \int_{[t_1,t_2]\times B_{R^{1+2s}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(f_\delta(w) - f_\delta(v)\big) f_\delta(v) \phi^2(t,x) K_\delta(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{1}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(f_\delta(w) - f_\delta(v)\big)^2 K_\delta(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \\
&= -\frac{c_d}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta_\delta(v-w) \frac{\big(f_\delta(w) - f_\delta(v)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}z \\
&\quad - \frac{1}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 - \eta_\delta(v-w)) \big(f_\delta(w) - f_\delta(v)\big)^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}z.
\end{aligned}$$

Using that $(1-\eta_\delta) \ge \frac{1}{4}$ outside $B_{\frac{5\delta}{4}}$ and $\eta_\delta \ge \frac{1}{4}$ inside $B_{\frac{7\delta}{4}}$, we find with the coercivity of K (1.26)

$$\begin{aligned}
&\int_{[t_1,t_2]\times B_{R^{1+2s}} \times \mathbb{R}^d} \mathcal{T} f_\delta f_\delta \phi^2 \,\mathrm{d}z \\
&\le -\frac{1}{4}\frac{c_d}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{|v-w|<\frac{7\delta}{4}} \frac{\big(f_\delta(w) - f_\delta(v)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}z \\
&\quad - \frac{1}{4} \cdot \frac{1}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{|v-w|>\frac{5\delta}{4}} \big(f_\delta(w) - f_\delta(v)\big)^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}z \\
&\le -\frac{1}{4}\frac{c_d}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{|v-w|<\frac{7\delta}{4}} \frac{\big(f_\delta(w) - f_\delta(v)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}z \\
&\quad - \frac{1}{4}\frac{\lambda}{2} \int_{[t_1,t_2]\times B_{R^{1+2s}}} \phi^2(t,x) \int_{\mathbb{R}^d} \int_{|v-w|>\frac{5\delta}{4}} \frac{\big(f_\delta(w) - f_\delta(v)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}z.
\end{aligned}$$

Thus

$$\begin{aligned}\sup_{t\in[t_1,t_2]}\int_{B_{(\frac{R}{2})^{1+2s}}\times\mathbb{R}^d} f_\delta^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v + C_{d,s,\lambda}\int_{\left[\frac{t_1}{2},\frac{t_2}{2}\right]\times B_{(\frac{R}{2})^{1+2s}}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\big(f_\delta(w)-f_\delta(v)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}z\\ \leq -\frac{1}{2}\int_{[t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d} v\cdot\nabla_x\big(f_\delta^2\big)\phi^2\,\mathrm{d}z\\ \leq CR^{-2s}\int_{[t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d} f_\delta^2\,\mathrm{d}z.\end{aligned}$$

In particular, $C_{d,s,\lambda}$ and C are uniform constants in δ, thus $f_\delta \in L^2((t_1,t_2)\times B_{R^{1+2s}};\dot{H}^s(\mathbb{R}^d))$ uniformly in δ.

Moreover, we observe that f_δ satisfies

$$\mathcal{T}f_\delta = (-\Delta_v)^s f_\delta + \int_{|v-w|>\delta}\big(f_\delta(w)-f_\delta(v)\big)\big(1-\eta_\delta(v-w)\big)\big(K(v,w)-c_{d,s}\,|v-w|^{-(d+2s)}\big)\,\mathrm{d}w$$

in $[t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d$. This is the Kolmogorov equation with bounded right hand side

$$h_\delta(t,x,v) := \int_{|v-w|>\delta}\big(f_\delta(w)-f_\delta(v)\big)\big(1-\eta_\delta(v-w)\big)\big(K(v,w)-c_{d,s}\,|v-w|^{-(d+2s)}\big)\,\mathrm{d}w.$$

Thus f_δ is given by

$$f_\delta(t,x,v) = h_\delta * J(t,x,v) + f_\delta(t_1,x,v) * J(t_1,x,v),$$

where J is the fundamental solution of the Kolmogorov equation in $[t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d$. Then, we find on the one hand, by Young's inequality and Theorem 3.4

$$\begin{aligned}\|f_\delta\|_{L^2([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d)} &\leq \big\|(-\Delta_v)^{-\frac{s}{2}}h_\delta\big\|_{L^2([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d)}\,\big\|(-\Delta_v)^{\frac{s}{2}}J\big\|_{L^1([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d)}\\ &\leq CR^{-s}\,\|f_\delta\|_{L^2([t_1,t_2]\times B_{R^{1+2s}};\dot{H}^s_v(\mathbb{R}^d))}\\ &\leq CR^{-2s}\,\|f_\delta\|_{L^2([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d))},\end{aligned}$$

uniformly in δ.

On the other hand, since $\|h_\delta\|_{L^2}\leq C\delta^{-2s}\,\|f_\delta\|_{L^2}$, we find

$$f_\delta \in L^2([t_1,t_2]\times B_{R^{1+2s}};\dot{H}^{\frac{3s}{2}}(\mathbb{R}^d))$$

with a constant that degenerates as $\delta\to 0$, and we can use any test function in $L^2_{t,x}\dot{H}^{\frac{s}{2}}_v$ to

give sense to the equation, and thus we can deduce the a-priori estimate on f_δ. We find:

$$\begin{aligned}\int_{Q_{\frac{3R}{4}}}\int_{\mathbb{R}^d\setminus B_R}(f_\delta-l)_+(w)\chi_{f>l}(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ \leq CR^{-2s}\sup_{Q_{\frac{3R}{4}}\cap f_\delta>l}(f_\delta-l)^{1-\zeta}\int_{Q_R}(f_\delta-l)_+^\zeta\,\mathrm{d}z,\end{aligned}\tag{5.72}$$

where C does not depend on δ. Now we want to pass to the limit. Note that as $\delta\to 0$, the coefficients K_δ converge to K pointwise a.e. Prior regularity results imply $f_\delta\in L^\infty_{loc}([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d)$ uniformly in δ, and moreover $f_\delta\in C^\alpha_{loc}([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d)$ uniformly in δ for some $\alpha\in(0,1)$. As a consequence of the uniform estimates, f_δ converges to some $f:[t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d\to\mathbb{R}_+$ pointwise almost everywhere in $[t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d$, weakly in $L^2([t_1,t_2]\times B_{R^{1+2s}};\dot H^s(\mathbb{R}^d))$ by weak compactness, and strongly in $L^1([t_1,t_2]\times B_{R^{1+2s}}\times\mathbb{R}^d)$ by uniform integrability, pointwise convergence and tightness. In particular, $f\geq 0$, f is periodic in space, it conserves mass and it satisfies

$$\int_{Q_{\frac{3R}{4}}}\int_{\mathbb{R}^d\setminus B_R}(f-l)_+(w)\chi_{f>l}(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\leq CR^{-2s}\sup_{Q_{\frac{3R}{4}}\cap f_{\delta,k}>l}(f-l)^{1-\zeta}\int_{Q_R}(f-l)_+^\zeta\,\mathrm{d}z.$$

It remains to check that f satisfies the limit equation

$$\mathcal{T}f=\int_{\mathbb{R}^d}\big(f(w)-f(v)\big)K(v,w)\,\mathrm{d}w.$$

Let $\varphi\in L^2((t_1,t_2)\times B_{R^{1+2s}},\dot H^s(\mathbb{R}^d))$ such that $f_\delta\varphi$ either conserves mass and is periodic in x, or $\varphi(\cdot,\cdot,v)$ has compact support in $(t_1,t_2)\times B_{R^{1+2s}}$. Then f_δ satisfies

$$\int_{(t_1,t_2)\times B_{R^{1+2s}}\times\mathbb{R}^d}(\mathcal{T}f_\delta)\varphi\,\mathrm{d}z+\int_{(t_1,t_2)}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(f_\delta(v)-f_\delta(w)\big)K_\delta(v,w)\varphi(v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t=0.$$

Integrating by parts and using the global convergence of the coefficients pointwise and strongly in L^1, and by weak compactness in $L^2_{t,x}\dot H^s_v$, we see that f satisfies

$$-\int_{(t_1,t_2)\times B_{R^{1+2s}}\times\mathbb{R}^d}f\mathcal{T}\varphi\,\mathrm{d}z+\int_{(t_1,t_2)}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(v)K(v,w)(\varphi(v)-\varphi(w))\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t=0,$$

that is

$$\int_{(t_1,t_2)\times B_{R^{1+2s}}\times\mathbb{R}^d}(\mathcal{T}f)\varphi\,\mathrm{d}z+\int_{(t_1,t_2)}\int_{B_{R^{1+2s}}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(f(v)-f(w)\big)K(v,w)\varphi(v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t=0,$$

where we used that either φ has compact support in $(t_1,t_2)\times B_{R^{1+2s}}$ or else $f\varphi$ conserves mass and is periodic in space.

CHAPTER 6

ARONSON

This chapter is based on [153] and [154].

6.1 Introduction

As a consequence of Theorem 5.1, we derive polynomial upper and exponential lower bounds on the fundamental solution of (1.13) with a non-negative kernel K satisfying (1.26)-(1.28), *provided that the fundamental solution exists*. To give sense to the next three theorems, we thus assume existence of a non-negative measurable function J, which is the fundamental solution of (1.13) connecting a given point $(t, x, v) \in \mathbb{R}^{1+2d}$ with $(\tau, y, w) \in \mathbb{R}^{1+2d}$, in the sequel denoted by

$$J(t, x, v; \tau, y, w) = J(t - \tau, x - y - (t - \tau)w, v - w) =: J\big((\tau, y, w)^{-1} \circ (t, x, v)\big),$$

where $\circ$ denotes the Galilean translation, that is $(t_0, x_0, v_0) \circ (t, x, v) = (t_0 + t, x_0 + x + tv_0, v_0 + v)$, which respects the translation invariance of (1.13). Moreover, we assume that the fundamental solution J has the following properties:

i. For every $t \in \mathbb{R}_+$ there holds the normalisation

$$\int_{\mathbb{R}^{2d}} J(t, x, v) \, \mathrm{d}x \, \mathrm{d}v = 1. \tag{6.1}$$

ii. There holds $J \geq 0$, and for all $(t, x, v), (\tau, y, w) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$ a form of symmetry

$$J\left((\tau, y, w)^{-1} \circ (t, x, v)\right) = J\left((\tau, x, v)^{-1} \circ (t, y, w)\right). \tag{6.2}$$

iii. For any $0 \leq \tau < \sigma < t < T$ and any $(x, v), (y, w) \in \mathbb{R}^{2d}$ the Chapman-Kolmogorov

identity holds

$$\begin{aligned} J(t,x,v;\tau,y,w) &= J\left((\tau,y,w)^{-1}\circ(t,x,v)\right) \\ &= \int_{\mathbb{R}^{2d}} J\left((\sigma,\varphi,\xi)^{-1}\circ(t,x,v)\right) J\left((\tau,y,w)^{-1}\circ(\sigma,\varphi,\xi)\right)\,\mathrm{d}\varphi\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^{2d}} J(t,x,v;\sigma,\varphi,\xi)J(\sigma,\varphi,\xi;\tau,y,w)\,\mathrm{d}\varphi\,\mathrm{d}\xi. \end{aligned} \tag{6.3}$$

Then, we deduce, on the one hand, polynomial upper bounds.

Theorem 6.1 (Polynomial upper bounds on the fundamental solution)**.** *Let $x,v,y_0,w_0\in\mathbb{R}^d$, and $0\le\tau_0<\sigma<T$. Let J be the fundamental solution of* (1.13) *in $[0,T]\times\mathbb{R}^{2d}$ with a vanishing source term $h=0$ and with a non-negative kernel K satisfying the coercivity assumption* (1.26)*, the integral upper bound* (1.32)*, and the divergence form symmetry* (1.28)*. Assume J satisfies* (6.1)*,* (6.2) *and* (6.3)*. Then there exists $C>0$ depending on s,d,λ_0,Λ_0 such that*

$$\begin{aligned} &J(\sigma,x,v;\tau_0,y_0,w_0) \\ &\le C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}}\left[1+\frac{\max\left\{|v-w_0|^{2s},|x-y_0-(\sigma-\tau_0)(v-w_0)|^{\frac{2s}{1+2s}}\right\}}{\sigma-\tau_0}\right]^{-\frac{s}{4s}}. \end{aligned} \tag{6.4}$$

On the other hand, we derive an exponential lower bound.

Theorem 6.2 (Exponential lower bounds on the fundamental solution)**.** *Let $x,v,y_0,w_0\in\mathbb{R}^d$ and $0<\tau_0<\sigma<T$. Let J be the fundamental solution of* (1.13) *in $[0,T]\times\mathbb{R}^{2d}$ with a vanishing source term $h=0$ and with a kernel satisfying the coercivity assumption* (1.26)*, the integral upper bound* (1.32)*, and the divergence form symmetry* (1.28)*. Then there exists a constant C depending on s,d,λ_0,Λ_0 such that*

$$J(\sigma,x,v;\tau_0,y_0,w_0)\ge C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}}\exp\left\{-C\left(\frac{|x-y_0-(\sigma-\tau_0)w_0|^{2s}}{(\sigma-\tau_0)^{1+2s}}+\frac{|v-w|^{2s}}{\sigma-\tau_0}\right)\right\}.$$

Remark 6.1. i. Note that in general, the fundamental solution has no explicit expression, not even in case of constant coefficients. We already discussed, however, the representation of the fundamental solution (2.3) of the fractional Kolmogorov equation (2.1). In particular, the fundamental solution J_0 satisfies (6.1), (6.2), (6.3), and moreover, $0\le\mathcal{J}_0\in C^\infty$ decays polynomially at infinity.

ii. In case of rough coefficients, the existence of a fundamental solution operator has only recently been established by Auscher-Imbert-Niebel [16] for local and non-local

hypoelliptic equations. In particular, their results are applicable to (1.13) with kernels satisfying (1.26)-(1.28).

iii. In Theorem 6.1 and Theorem 6.2, the time decay is optimal, as can be seen by comparing the exponents on time to (2.3). This is a consequence of the optimality of the linear L^1-L^∞ bound of Proposition 5.1 below. However, neither the decay in space nor in velocity is optimal.

Concerning the exponential lower bound of Theorem 6.2, it is not clear to us if we should expect polynomial lower bounds to hold without assuming a pointwise lower bound on the kernel of the non-local operator (1.14).

The polynomial upper bounds in Theorem 6.1 are sub-optimal in two ways. First, a minor improvement could possibly be achieved, if, in Aronson's method (see Section 6.2.3 below), when we define the decay function H in (6.26), we replace $\max\{|v-w_0|, |x-y_0-tw_0|^{\frac{1}{1+2s}}\}$ by the sum $|v-w_0| + \frac{|x-y_0-tw_0|}{t}$. It is conceivable that this leads to an improved decay in space to $\frac{s}{2}$ instead of $\frac{s}{2(1+2s)}$. Second, the most significant sub-optimality arises due to the way we treat the non-singular contribution of the fundamental solution in Aronson's method (see Proposition 6.1). To determine the decay of the non-singular part, we only use the integral upper bounds on the kernel (1.32). In the parabolic literature, the non-singular part is usually treated separately from the singular terms, using a technique referred to as Meyer's decomposition [161]. Conditional to such a decomposition, see (6.5), we obtain the optimal decay at least in velocity, see Theorem 6.3 below. Finally note that in the parabolic setting with kernels that are pointwisely bounded from above, the proof of Theorem 6.1 - which does not rely on Meyer's decomposition - yields a decay that is not so far from optimal, see Appendix 6.5.

For the statement of the conditional upper bounds, we denote by J_ρ the fundamental solution to the cutoff problem

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}_\rho f,$$

where $\mathcal{L}_\rho f$ is defined for any $\rho > 0$ as

$$\mathcal{L}_\rho f(t,x,v) := PV \int_{B_\rho(v)} (f(t,x,w) - f(t,x,v)) K(t,x,v,w)\, \mathrm{d}w.$$

Then we show:

Theorem 6.3 (Conditional upper bounds on the fundamental solution)**.** *Let $x, v, y_0, w_0 \in \mathbb{R}^d$ and $0 < \tau_0 < \sigma < T$. Let J be the fundamental solution of* (1.13) *in $[0,T] \times \mathbb{R}^{2d}$ with a vanishing source term $h = 0$. We assume the kernel is coercive* (1.26)*, satisfies integral*

upper bounds (1.32)*, and is symmetric in divergence form* (1.28)*. If, moreover, there exists a constant* $c > 0$*, such that for every* $\rho > 0$*, there holds*

$$J(\sigma, x, v; \tau_0, y_0, w_0) \leq J_\rho(\sigma, x, v; \tau_0, y_0, w_0) + c(\sigma - \tau_0)\rho^{-(2d(1+s)+2s)}, \tag{6.5}$$

then there exists $C > 0$ *depending on* $c, s, d, \Lambda_0, \lambda_0$ *such that*

$$\begin{aligned} &J(\sigma, x, v; \tau_0, y_0, w_0) \\ &\leq C(\sigma - \tau_0)^{-\frac{2d(1+s)}{2s}} \left[1 + \frac{\max\left\{|v - w_0|^{2s}, |x - y_0 - (\sigma - \tau_0)(v - w_0)|^{\frac{2s}{1+2s}}\right\}}{\sigma - \tau_0}\right]^{-\frac{2d(1+s)+2s}{2s}}. \end{aligned} \tag{6.6}$$

Remark 6.2. As outlined in Remark 6.1, the major sub-optimality in Theorem 6.1 came from the non-singular contribution of the fundamental solution. The condition in (6.5) precisely says that we can consider the non-singular part separately, so that we only need to use Aronson's method on the singular term, for which we get the optimal decay at least in velocity.

In the parabolic literature, (6.5) is established (with a corresponding decay in ρ of order $d + 2s$ instead of $2d(1 + s) + 2s$) using parabolic maximum principles, see [91, 92], *provided that the kernels are pointwisely bounded from above*: there exists $\Lambda > 0$ such that

$$\forall v, w \in \mathbb{R}^d \qquad K(v, w) \leq \Lambda |v - w|^{-(d+2s)}. \tag{6.7}$$

This assumption does not seem to be easily relaxable to integral upper bounds (1.32). Moreover, even if it is conceivable that the methods from the parabolic setting could be carried over, it is not clear that one would get the expected decay of $2d(1 + s) + 2s$ on ρ.

6.1.1 Contribution

As a consequence of the local nature of Proposition 5.1, Theorem 6.1 and Theorem 6.2 are the first occurrence of bounds on the fundamental solution of a kinetic integro-differential equation. Previous results have discussed two-sided estimates either for kinetic equations with a local diffusion operator with rough, but uniformly elliptic coefficients [17, 141, 143], or for non-local parabolic equations, with kernels that satisfy pointwise upper bounds, divergence form symmetry (1.28) and coercivity as in (1.26), see [129]. In this paper, we use Aronson's method, originally conceived for local parabolic equations [12, 13], and establish a strategy based thereon for kinetic integro-differential equations. This yields both the exponential lower bounds from Theorem 6.2, and the polynomial upper bounds from Theorem 6.1.

6.1.2 Literature

Regarding the case of *local parabolic* equations, the study of two-sided estimates on the fundamental solution originated with the papers of Nash in 1958 [168], Aronson in 1967-68 [12, 13], and Fabes and Stroock in 1986[73]. More precisely, on the one hand, the first proof of Gaussian bounds for linear parabolic second-order equations has been derived by Aronson for uniformly elliptic, but rough, diffusion coefficients in divergence form. Aronson's derivation is based on energy estimates and Harnack inequalities, and thus exploits the regularity properties of the solution to the underlying equation. On the other hand, the argument of Nash, Fabes and Stroock proves the Gaussian bounds without referring to the regularity properties of the solution, and instead implies estimates on the local behaviour of the solution as a consequence of the Gaussian bounds. When Nash solved Hilbert's 19th problem on the continuity of solutions to equations with uniformly elliptic rough diffusion coefficients [168], he stated without proof estimates on the fundamental solution. These have been revisited by Fabes and Stroock [73], who combine Nash's sketch of the argument together with Davies' idea of replacing the Euclidean distance by the Riemannian distance associated to the coefficients of the equation at hand [55]. With this, Fabes-Stroock obtain Gaussian upper and lower bounds, from which they further derive the Harnack inequalities.

Variants of these ideas have been applied to *local kinetic* equations, most notably in [141, 143], or also in the recent preprint [17]. For *non-local parabolic* equation, the most recent work that we are aware of is by Kassmann-Weidner [129], who employ ideas from Aronson's method to derive a polynomial upper bound on the fundamental solution.

We end this discussion with a historical remark: in 1958 Nash [168] derived Hölder continuity of parabolic equations in divergence form with rough coefficients, and stated two-sided non-Gaussian estimates for the fundamental solution as a consequence. In 1986, Fabes and Stroock [73] derived two-sided Gaussian estimates directly from Nash's ideas, and derived the Harnack inequalities as a consequence. Since we know how to derive the Harnack inequalities for (1.13), one might investigate the derivation of the Harnack inequalities for (1.13) as a consequence of the bounds on the fundamental solution. We assemble this remark into a collection of open problems:

Open Problems 6.1.

1. Derive the Harnack inequalities of Theorem 1.1 and Theorem 5.1 for solutions to (1.13) as a consequence of the polynomial upper and the exponential lower bounds of Theorem 6.1 and Theorem 6.2. Instead of an exponential lower bound, one might need to assume a polynomial lower bound on the fundamental solution. It is also conceivable that it requires the optimal upper bounds.

2. Is the exponential lower bound on the fundamental solution of (1.13) in Theorem 6.2 optimal for kernels with an average coercivity condition (1.26), an average upper

bound (1.32) and with the divergence form symmetry (1.28)? A polynomial lower bound might require a pointwise lower bound on the kernel.

3. Improve the polynomial upper bounds of Theorem 6.1, or in other words, make Theorem 6.3 unconditional. Again, we think this requires a pointwise upper bound on the kernel (6.7) instead of the average upper bound assumed in (1.32).

4. Does the linear L^1 to L^∞ bound hold for mere *sub-solutions* to (1.13)? The proof method in this article heavily relies on the fact that we work with solutions. However, in the local case, a linear L^1 to L^∞ bound can be derived even for sub-solutions. It seems to us that the sub-solution property is not sufficient in the non-local case to yield a linear L^1 to L^∞ bound.

6.1.3 Strategy

As a consequence of the strong Harnack inequality, Theorem 5.1, we adapt Aronson's method to the non-local kinetic case. In fact, Proposition 5.1 implies directly an on-diagonal upper bound for the fundamental solution to (1.13). For the off-diagonal bounds, the idea is to construct a function H that satisfies

$$\partial_t H + v \cdot \nabla_x H + \int_{\mathbb{R}^d} \left[H^{\frac{1}{2}}(w) - H^{\frac{1}{2}}(v) \right]^2 K(v,w) \, \mathrm{d}w \leq 0.$$

By testing (1.13) with the product of the solution and this function H, we derive an energy estimate on the L^2 norm of $fH^{\frac{1}{2}}$, which in turn permits to determine the decay of the solution from the decay of the function H. Together with the on-diagonal bounds that follow as a consequence of the linear L^1-L^∞ bound of Proposition 5.1, this implies the unconditional and conditional upper bounds on the fundamental solution of Theorem 6.1 and Theorem 6.3.

The exponential lower bound of Theorem 6.2 follows by applying the strong Harnack inequality of Theorem 5.1 repeatedly along a path that connects two given points. This yields a pointwise Harnack inequality, which in turn implies an exponential lower bound on any solution that has a non-vanishing average over some local ball. In particular, it implies the exponential bound on the fundamental solution that connects two given points.

6.2 Upper bound on the fundamental solution

For the sequel we assume that (1.13) under assumptions (1.26)-(1.28) admits a fundamental solution with the properties (6.1), (6.2) and (6.3). We denote it by $J(t,x,v;\tau,\zeta,\xi)$ for $(t,x,v),(\tau,\zeta,\xi) \in \mathbb{R}^{1+2d}$.

6.2.1 On-diagonal bound

The on-diagonal upper bound follows from Proposition 5.1.

Theorem 6.4 (Nash Upper Bound)**.** *Let $x, v, \zeta, \xi \in \mathbb{R}^d$ and $0 \le \tau_0 < t < T$. Let $J(t, x, v; \tau_0, \zeta, \xi)$ be the fundamental solution of* (1.13) *with a non-negative kernel K satisfying* (1.26)-(1.28)*. Then there exists $C > 0$ depending on $s, d, \lambda_0, \Lambda_0$ such that*

$$J(t, x, v; \tau_0, \zeta, \xi) \le C(t - \tau_0)^{-\frac{2d(1+s)}{2s}}. \tag{6.8}$$

Proof. Since $(t, x, v) \to J(t, x, v; \tau_0, \zeta, \xi)$ solves (1.13), we find by Proposition 5.1 with $z = (t, x, v)$, $r^{2s} = \frac{1}{4}(t - \tau_0)$ and $R = 2r$

$$\begin{aligned} J(t, x, v; \tau_0, \zeta, \xi) &\le \sup_{Q_r(z)} J(\cdot, \cdot, \cdot; \tau_0, \zeta, \xi) \\ &\le C(t - \tau_0)^{-\frac{2d(1+s)+2s}{2s}} \int_{Q_{2r}(z)} J(s, y, w; \tau_0, \zeta, \xi) \,\mathrm{d}s \,\mathrm{d}y \,\mathrm{d}w \\ &\le C(t - \tau_0)^{-\frac{2d(1+s)+2s}{2s}} \int_{t-r^{2s}}^{t} \int_{\mathbb{R}^{2d}} J(s, y, w; \tau_0, \zeta, \xi) \,\mathrm{d}s \,\mathrm{d}y \,\mathrm{d}w \\ &\le C(t - \tau_0)^{-\frac{2d(1+s)}{2s}}. \end{aligned}$$

The last inequality follows by using (6.1). □

As a consequence of (6.8) and (6.1) we get for every $0 \le \tau_0 < \sigma < T$ and $\zeta, \xi \in \mathbb{R}^d$

$$\left(\int_{\mathbb{R}^{2d}} J^2(\sigma, x, v; \tau_0, \zeta, \xi) \,\mathrm{d}x \,\mathrm{d}v \right)^{\frac{1}{2}} \le C(\sigma - \tau_0)^{-\frac{d(1+s)}{2s}}, \tag{6.9}$$

and due to (6.2) also for every $x, v \in \mathbb{R}^d$

$$\left(\int_{\mathbb{R}^{2d}} J^2(\sigma, x, v; \tau_0, \zeta, \xi) \,\mathrm{d}\zeta \,\mathrm{d}\xi \right)^{\frac{1}{2}} \le C(\sigma - \tau_0)^{-\frac{d(1+s)}{2s}}. \tag{6.10}$$

6.2.2 Off-diagonal bound

In this section, we derive an off-diagonal bound on the fundamental solution.

Theorem 6.5 (Off-diagonal bound)**.** *Let J be the fundamental solution of* (1.13) *with a non-negative kernel K satisfying* (1.26)-(1.28)*. Then there exists $C > 0$ depending on $d, \lambda_0, \Lambda_0, s$ such that for every $0 < \rho$, $0 \le \tau_0 < \sigma < T$, $x, v, y_0, w_0 \in \mathbb{R}^d$ with $\sigma - \tau_0 \le \frac{\rho^{2s}}{4k}$*

there holds

$$J(\sigma,x,v;\tau_0,y_0,w_0)\le C\left((\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}}\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac14-\frac{\gamma}{12\rho}}+\rho^{-\frac{s}{2}}(\sigma-\tau_0)^{\frac14-\frac{2d(1+s)}{2s}}\right),$$

for

$$\gamma=\frac14\max\left\{|v-w_0|\,,|x-y_0-(\sigma-\tau_0)w_0|^{\frac{1}{1+2s}}\right\},$$

with $|\sigma-\tau_0|^{\frac{1}{2s}}\le 4\gamma$.

To prove Theorem 6.5, we use Aronson's method: we construct a function H that satisfies

$$\mathcal{T}H+\int_{\mathbb{R}^d}\left[H^{\frac12}(v)-H^{\frac12}(w)\right]^2K(v,w)\,\mathrm{d}w\le 0.$$

Testing (1.13) with the product of the solution and H then yields

$$\sup_{t\in(t_0,\sigma)}\int_{\mathbb{R}^{2d}}f^2(t,x,v)H(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\lesssim\int_{\mathbb{R}^{2d}}f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v.\tag{6.11}$$

The decay from the function H determines through this inequality (6.11) how quick solutions increase inside a ball that lies outside the support of the initial data:

Theorem 6.6 (Aronson's bound)**.** *Let* $0<\tau_0<\sigma<T$, $0<\rho$, $\gamma>0$ *such that* $|\sigma-\tau_0|^{\frac{1}{2s}}\le\gamma$, *and* $(y_0,w_0)\in\mathbb{R}^{2d}$. *Let* $f_0=f(\tau_0)\in L^2(\mathbb{R}^{2d})$ *be such that*

$$f(t_0,x,v)=0,\qquad \textit{for }\max\left\{|x-y_0-(\sigma-\tau_0)w_0|^{\frac{1}{1+2s}}\,,|v-w_0|\right\}\le\gamma.$$

Let $f\in L^\infty\big((\tau_0,T)\times\mathbb{R}^{2d}\big)$ *solve* (1.13) *in* $(\tau_0,T)\times\mathbb{R}^{2d}$ *with a non-local operator whose kernel is non-negative and satisfies* (1.26), (1.32), (1.28). *Then there exists* $k\ge1$, *and* $C>0$ *depending on* s,d,λ_0,Λ_0 *such that if* $\sigma-\tau_0<\frac{\rho^{2s}}{4k}$ *there holds*

$$|f(\sigma,y_0,w_0)|\le C\left[(\sigma-\tau_0)^{-\frac{d(1+s)}{2s}}\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac12-\frac{\gamma}{6\rho}}+\rho^{-s}(\sigma-\tau_0)^{\frac12-\frac{d(1+s)}{2s}}\right]\|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

Before proving Theorem 6.6, we complete the proof of the off-diagonal bound in Theorem 6.5 using Theorem 6.6 and the on-diagonal bounds in Theorem 6.4.

Proof of Theorem 6.5. Step 1: First consequence of Theorem 6.6.

Let $0<\rho$, and $0\le\tau_0<\sigma$ be such that $\sigma-\tau_0\le\frac{\rho^{2s}}{4k}$. Let $v,\xi,x,\zeta\in\mathbb{R}^d$. Then we

claim that for $0 \le \tau_0 < t < \sigma$ there holds

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\setminus Q_\gamma^{\tau_0}(t,x,v)} J^2(t,x,v;\tau_0,\zeta,\xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi \\
&\qquad \le C(t-\tau_0)^{-\frac{2d(1+s)}{2s}}\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{2}-\frac{\gamma}{6\rho}} + \rho^{-s}(t-\tau_0)^{\frac{1}{2}-\frac{2d(1+s)}{2s}}, \qquad x,v\in\mathbb{R}^d, \\
&\int_{\mathbb{R}^{2d}\setminus Q_\gamma^{t}(\tau_0,\zeta,\xi)} J^2(t,x,v;\tau_0,\zeta,\xi)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad \le C(t-\tau_0)^{-\frac{2d(1+s)}{2s}}\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{2}-\frac{\gamma}{6\rho}} + \rho^{-s}(t-\tau_0)^{\frac{1}{2}-\frac{2d(1+s)}{2s}}, \qquad \zeta,\xi\in\mathbb{R}^d,
\end{aligned} \tag{6.12}$$

where

$$Q_\gamma^{\tau_0}(t,x,v) := \left\{(\zeta,\xi)\in\mathbb{R}^{2d} : \max\left\{|\zeta - x - (t-\tau_0)v|^{\frac{1}{1+2s}}, |\xi - v|\right\} < \gamma\right\},$$

and similarly,

$$Q_\gamma^{t}(\tau_0,\zeta,\xi) := \left\{(x,v)\in\mathbb{R}^{2d} : \max\left\{|x - \zeta - (\tau_0-t)\xi|^{\frac{1}{1+2s}}, |v - \xi|\right\} < \gamma\right\}.$$

To prove this, we first note

$$\int_{Q_\gamma^{\tau_0}(t,x,v)} J^2(t,x,v;\tau_0,\zeta,\xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi = \int_{Q_\gamma^{0}(0,y,w)} J^2(t,y-(t-\tau_0)w,w;\tau_0,\zeta,\xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi.$$

Then we define

$$f(\sigma,y_0,w_0) := \int_{\mathbb{R}^{2d}\setminus Q_\gamma^{0}(0,y,w)} J(\sigma,y_0,w_0;\tau_0,\zeta,\xi)J(t,y-(t-\tau_0)w,w;\tau_0,\zeta,\xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi.$$

This function f is a non-negative solution of (1.13) for $0 \le \tau_0 < \sigma < T$ with initial condition

$$f(\tau_0,y_0,w_0) = \begin{cases} 0, & \text{if } (y_0,w_0)\in Q_\gamma^0(0,y,w), \\ J(t,y-(t-\tau_0)w,w;\tau_0,y_0,w_0), & \text{if } (y_0,w_0)\in\mathbb{R}^{2d}\setminus Q_\gamma^0(0,y,w), \end{cases}$$

due to (6.3) and (6.2). Setting $(\sigma, y_0, w_0) = (t, y-(t-\tau_0)w, w)$ yields by Theorem 6.6

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\setminus Q_\gamma^0(0,y,w)} J^2(t, y-(t-\tau_0)w, w; \tau_0, \zeta, \xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi \\
&\qquad = f(\sigma, y-(t-\tau_0)w, w) \\
&\qquad \le C(\sigma-\tau_0)^{-\frac{d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{2}-\frac{\gamma}{6\rho}} \|f_0\|_{L^2(\mathbb{R}^{2d})} \\
&\qquad\quad + \rho^{-s}(\sigma-\tau_0)^{\frac{1}{2}-\frac{d(1+s)}{2s}} \|f_0\|_{L^2(\mathbb{R}^{2d})} \\
&\qquad \le C(\sigma-\tau_0)^{-\frac{d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{2}-\frac{\gamma}{6\rho}} \|J(t, y-(t-\tau_0)w, w; \tau_0, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})} \\
&\qquad\quad + \rho^{-s}(\sigma-\tau_0)^{\frac{1}{2}-\frac{d(1+s)}{2s}} \|J(t, y-(t-\tau_0)w, w; \tau_0, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})} \\
&\qquad \le C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{2}-\frac{\gamma}{6\rho}} + \rho^{-s}(\sigma-\tau_0)^{\frac{1}{2}-\frac{2d(1+s)}{2s}}.
\end{aligned}$$

The last inequality follows from the on-diagonal bound (6.10). This yields the first inequality in claim (6.12). Together with (6.2) this also yields the second inequality in (6.12).

Step 2: Second consequence of Theorem 6.6.

We fix $0 \le \tau_0 < \sigma < T$ such that $\sigma - \tau_0 \le \frac{\rho^{2s}}{4k}$ and $x, v, y_0, w_0 \in \mathbb{R}^d$. We consider

$$\gamma = \frac{1}{4}\max\left\{|v-w_0|, |x-y_0-(\sigma-\tau_0)w_0|^{\frac{1}{1+2s}}\right\}, \tag{6.13}$$

and assume $|\sigma-\tau_0|^{\frac{1}{2s}} \le 4\gamma$. Then by (6.3)

$$\begin{aligned}
J(\sigma, x, v; \tau_0, y_0, w_0) &= \int_{\mathbb{R}^{2d}} J\Big(\sigma, x, v; \sigma - \frac{\sigma-\tau_0}{2}, x', v'\Big) J\Big(\sigma - \frac{\sigma-\tau_0}{2}, x', v'; \tau_0, y_0, w_0\Big)\,\mathrm{d}x'\,\mathrm{d}v' \\
&= \underbrace{\int_{\mathbb{R}^{2d}\setminus Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v)} \dots\,\mathrm{d}x'\,\mathrm{d}v'}_{=:J_1} + \underbrace{\int_{Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v)} \dots\,\mathrm{d}x'\,\mathrm{d}v'}_{=:J_2}.
\end{aligned}$$

For J_1 we use (6.9) and (6.12) to get

$$\begin{aligned}
J_1 &\le \left(\int_{\mathbb{R}^{2d}\setminus Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v)} J^2\Big(\sigma, x, v; \sigma - \frac{\sigma-\tau_0}{2}, x', v'\Big)\,\mathrm{d}x'\,\mathrm{d}v' \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^{2d}\setminus Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v)} J^2\Big(\sigma - \frac{\sigma-\tau_0}{2}, x', v'; \tau_0, y_0, w_0\Big)\,\mathrm{d}x'\,\mathrm{d}v' \right)^{\frac{1}{2}} \\
&\le C\left((\sigma-\tau_0)^{-\frac{2d(1+s)}{4s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)} \right)^{\frac{1}{4}-\frac{\gamma}{12\rho}} + \rho^{-\frac{s}{2}} (\sigma-\tau_0)^{\frac{1}{4}-\frac{2d(1+s)}{4s}} \right) (\sigma-\tau_0)^{-\frac{2d(1+s)}{4s}} \\
&\le C\left((\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)} \right)^{\frac{1}{4}-\frac{\gamma}{12\rho}} + \rho^{-\frac{s}{2}} (\sigma-\tau_0)^{\frac{1}{4}-\frac{2d(1+s)}{2s}} \right).
\end{aligned}$$

For J_2 we first note that for $x', v' \in Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v)$ by choice of γ in (6.13) either

$$|v' - w_0| \ge |v - w_0| - |v' - v| \ge 4\gamma - \gamma \ge \gamma,$$

or

$$\begin{aligned}
(4\gamma)^{1+2s} &= |x - y_0 - (\sigma-\tau_0)w_0| \\
&= \left| \left(x - x' - \frac{(\sigma-\tau_0)}{2} v \right) + \left(x' - y_0 - \frac{(\sigma-\tau_0)}{2} w_0 \right) + \frac{(\sigma-\tau_0)}{2}(v - w_0) \right| \\
&\le \left| \gamma^{1+2s} + \left| x' - y_0 - \frac{(\sigma-\tau_0)}{2} w_0 \right| + \frac{1}{2} 4^{2s} \gamma^{2s} 4\gamma \right| \\
&\le (1 + 2^{1+4s})\gamma^{1+2s} + \left| x' - y_0 - \frac{(\sigma-\tau_0)}{2} w_0 \right|,
\end{aligned}$$

since $x' \in Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v)$ and by choice of γ in (6.13), so that

$$\left| x' - y_0 - \frac{(\sigma-\tau)}{2} w_0 \right|^{\frac{1}{1+2s}} \ge (4^{1+2s} - (1 + 2^{1+4s}))^{\frac{1}{1+2s}} \gamma \ge \gamma.$$

In other words, $Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,x,v) \subset \mathbb{R}^{2d} \setminus Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,y_0,w_0)$ so that again by (6.9) and (6.12)

we get

$$
\begin{aligned}
J_2 &\leq \left(\int_{\mathbb{R}^{2d} \setminus Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,y_0,w_0)} J^2 \left(\sigma, x, v; \sigma - \frac{\sigma-\tau_0}{2}, x', v' \right) \mathrm{d}x' \, \mathrm{d}v' \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^{2d} \setminus Q_\gamma^{\frac{\sigma-\tau_0}{2}}(0,y_0,w_0)} J^2 \left(\sigma - \frac{\sigma-\tau_0}{2}, x', v'; \tau_0, y_0, w_0 \right) \mathrm{d}x' \, \mathrm{d}v' \right)^{\frac{1}{2}} \\
&\leq C \left((\sigma-\tau_0)^{-\frac{2d(1+s)}{4s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)} \right)^{\frac{1}{4}-\frac{\gamma}{12\rho}} + \rho^{-\frac{s}{2}} (\sigma-\tau_0)^{\frac{1}{4}-\frac{2d(1+s)}{4s}} \right) (\sigma-\tau_0)^{-\frac{2d(1+s)}{4s}} \\
&\leq C \left((\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)} \right)^{\frac{1}{4}-\frac{\gamma}{12\rho}} + \rho^{-\frac{s}{2}} (\sigma-\tau_0)^{\frac{1}{4}-\frac{2d(1+s)}{2s}} \right),
\end{aligned}
$$

which proves Theorem 6.5. □

It remains to establish Aronson's method.

6.2.3 Aronson's method

The proof of Theorem 6.6 relies on Aronson's method: we aim to construct a suitable function H to derive (6.11).

6.2.3.1 Decay relation

We aim to define a function that satisfies for some $\rho > 0$ and for some constant $c > 0$

$$
\mathcal{T} H(v) + c \underbrace{\int_{B_\rho(v)} \left[H^{\frac{1}{2}}(v) - H^{\frac{1}{2}}(w) \right]^2 \left[K(v,w) + K(w,v) \right] \mathrm{d}w}_{=: \mathcal{L}_\rho H^{\frac{1}{2}}} \leq 0. \tag{6.14}
$$

Then we can derive the following statement.

Proposition 6.1. *Let $0 < \tau_0 < \sigma < T$ and $0 < \rho$. Let $f \in L^\infty\big((\tau_0, T) \times \mathbb{R}^{2d}\big)$ solve (1.13) in $(\tau_0, T) \times \mathbb{R}^{2d}$ with a non-local operator whose kernel is non-negative and satisfies (1.32), (1.28). Then for every bounded function $H : [\tau_0, \sigma] \times \mathbb{R}^{2d} \to [0, \infty)$ such that $H^{\frac{1}{2}} \in L^2\big((\tau_0, \sigma) \times \mathbb{R}^d; H_v^s(\mathbb{R}^d)\big)$ satisfies (6.14) in $(\tau_0, \sigma) \times \mathbb{R}^{2d}$, there exists a constant $C = C(\Lambda_0, s, d)$ such that*

$$
\begin{aligned}
\sup_{t \in (\tau_0, \sigma)} \int_{\mathbb{R}^{2d}} f^2(t,x,v) H(t,x,v) \, \mathrm{d}x \, \mathrm{d}v \leq & \int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v) H(\tau_0,x,v) \, \mathrm{d}x \, \mathrm{d}v \\
& + C \rho^{-2s} \left\| H \right\|_{L^\infty([\tau_0,\sigma] \times \mathbb{R}^{2d})} \left\| f \right\|^2_{L^2([\tau_0,\sigma] \times \mathbb{R}^{2d})}.
\end{aligned} \tag{6.15}
$$

Proof of Proposition 6.1. For $R \geq \max\{2, 2\rho+1\}$ we consider $\varphi_R \in C_c^\infty(\mathbb{R}^{2d})$ with $0 \leq \varphi_R \leq 1$ such that $\varphi_R \equiv 1$ for $(x,v) \in B_{(R-1)^{1+2s}} \times B_{R-1}$ and $\varphi_R \equiv 0$ for (x,v) outside $B_{R^{1+2s}} \times B_R$, with bounded derivatives and such that $|v \cdot \varphi_R| \sim R^{-2s}$. We test (1.13) with $fH\varphi_R^2$ where H satisfies (6.14) over $[\tau_0, \tau_1] \times \mathbb{R}^{2d}$ for $0 \leq \tau_0 \leq \tau_1 \leq \sigma$ and get

$$0 = \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \mathcal{T} f\Big(fH\varphi_R^2\Big)\,\mathrm{d}z - \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \mathcal{L} f\Big(fH\varphi_R^2\Big)\,\mathrm{d}z.$$

Step 1: Transport operator.

First we integrate by parts the transport operator

$$\begin{aligned}
&\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \mathcal{T} f\Big(fH\varphi_R^2\Big)\,\mathrm{d}z \\
&\quad= \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2 H\varphi_R^2\Big|_{t=\tau_0}^{\tau_1}\,\mathrm{d}x\,\mathrm{d}v - \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2 H\varphi_R v\cdot\nabla_x\varphi_R\,\mathrm{d}z \\
&\qquad- \frac{1}{2}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2\varphi_R^2 \mathcal{T} H\,\mathrm{d}z \\
&\quad\geq \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2 H\varphi_R^2\Big|_{t=\tau_0}^{\tau_1}\,\mathrm{d}x\,\mathrm{d}v - CR^{-2s}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2 H\varphi_R\,\mathrm{d}z \\
&\qquad- \frac{1}{2}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2\varphi_R^2 \mathcal{T} H\,\mathrm{d}z.
\end{aligned} \tag{6.16}$$

Step 2: Non-local operator.

Now we deal with the non-local term. We write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big(\mathcal{L} f\big) fH\varphi_R^2\,\mathrm{d}v \\
&= \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}} \big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+ \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}} \big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) + \big(fH\varphi_R^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+ \int_{B_{2R}}\int_{\mathbb{R}^d\setminus B_{2R}} \big[f(w)-f(v)\big]\big(fH\varphi_R^2\big)(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&= \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}} \big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+ \int_{B_{2R}}\int_{\mathbb{R}^d\setminus B_{2R}} \big[f(w)-f(v)\big]\big(fH\varphi_R^2\big)(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&=: \mathcal{I}_{loc} + \mathcal{I}_{tail},
\end{aligned} \tag{6.17}$$

where we used the divergence structure (1.28).

Step 2-i: Tail.

First we note that the tail tends to zero as $R \to \infty$:

$$\begin{aligned}\mathcal{I}_{tail} &= \int_{B_{2R}} \int_{\mathbb{R}^d \setminus B_{2R}} \big[f(w) - f(v)\big]\big(fH\varphi_R^2\big)(v) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\le \int_{B_R} \int_{\mathbb{R}^d \setminus B_{2R}} f(w)\big(fH\varphi_R^2\big)(v) K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\le C\Lambda R^{-2s} \big\|H\varphi_R^2\big\|_{L^\infty} \|f\|_{L^1(B_R)} \|f\|_{L^\infty(\mathbb{R}^d \setminus B_{2R})} \xrightarrow[R\to\infty]{} 0.\end{aligned} \tag{6.18}$$

Step 2-ii: Symmetric not-too-non-local non-locality.

Second, to bound $\mathcal{I}_{loc}$ from (6.17), we first distinguish the non-singular from the singular part. We write

$$\begin{aligned}&\mathcal{I}_{loc} \\ &= \int_{B_{2R}} \int_{B_{2R}} \big[f(w) - f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big] K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &= \int_{B_{2R}} \int_{B_{2R}} \big[f(w) - f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big] K(v,w)\chi_{|v-w|>\rho}\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + \int_{B_{2R}} \int_{B_{2R}} \big[f(w) - f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big] K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}w\,\mathrm{d}v \\ &=: \mathcal{I}_{loc}^{ns} + \mathcal{I}_{loc}^{s}.\end{aligned} \tag{6.19}$$

Then, on the one hand, we crudely bound with (1.32)

$$\begin{aligned}\mathcal{I}_{loc}^{ns} &= \int_{B_{2R}} \int_{B_{2R}} \big[f(w) - f(v)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big] K(v,w)\chi_{|v-w|>\rho}\,\mathrm{d}w\,\mathrm{d}v \\ &\le \int_{B_{2R}} \int_{B_{2R}} \Big[f(w)\big(fH\varphi_R^2\big)(v) + f(v)\big(fH\varphi_R^2\big)(w)\Big] K(v,w)\chi_{|v-w|>\rho}\,\mathrm{d}w\,\mathrm{d}v \\ &\le \int_{B_{2R}} \int_{B_{2R}} \big[f^2(w) + f^2(v)\big]\big(H\varphi_R^2\big)(v) + \big(H\varphi_R^2\big)(w)\big] K(v,w)\chi_{|v-w|>\rho}\,\mathrm{d}w\,\mathrm{d}v \\ &\le C\rho^{-2s} \|H\|_{L^\infty} \int_{B_{2R}} f^2(v)\,\mathrm{d}v.\end{aligned} \tag{6.20}$$

On the other hand, to bound the singular part of $\mathcal{I}_{loc}$, we realise

$$\begin{aligned}&\big[f(v) - f(w)\big]\Big[\big(fH\varphi_R^2\big)(v) - \big(fH\varphi_R^2\big)(w)\Big] \\ &\qquad = \Big[\big(fH^{\frac12}\varphi_R\big)(v) - \big(fH^{\frac12}\varphi_R\big)(w)\Big]^2 - f(v)f(w)\Big[\big(\varphi_R H^{\frac12}\big)(v) - \big(\varphi_R H^{\frac12}\big)(w)\Big]^2,\end{aligned}$$

so that

$$\mathcal{I}^s_{loc} = \int_{B_{2R}}\int_{B_{2R}} f(v)f(w)\Big[\big(\varphi_R H^{\frac12}\big)(v) - \big(\varphi_R H^{\frac12}\big)(w)\Big]^2 K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}v\,\mathrm{d}w$$
$$- \int_{B_{2R}}\int_{B_{2R}} \Big[\big(fH^{\frac12}\varphi_R\big)(v) - \big(fH^{\frac12}\varphi_R\big)(w)\Big]^2 K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}w\,\mathrm{d}v.$$

Now notice that for some $c_0 \geq 1$ there holds

$$f(v)f(w)\Big[\big(\varphi_R H^{\frac12}\big)(v) - \big(\varphi_R H^{\frac12}\big)(w)\Big]^2$$
$$\leq c_0\Big[f^2(v)+f^2(w)\Big]\Big[\big(\varphi_R(v)-\varphi_R(w)\big)^2\big(H(v)+H(w)\big)$$
$$+\Big(H^{\frac12}(v)-H^{\frac12}(w)\Big)^2\Big(\varphi_R^2(v)+\varphi_R^2(w)\Big)\Big].$$

Thus

$$\begin{aligned}
\mathcal{I}^s_{loc} &\leq \int_{B_{2R}}\int_{B_{2R}} f(v)f(w)\Big[\big(\varphi_R H^{\frac12}\big)(v) - \big(\varphi_R H^{\frac12}\big)(w)\Big]^2 K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}v\,\mathrm{d}w\\
&\lesssim_{c_0} \int_{B_{2R}}\int_{B_{2R}} \big[f^2(v)+f^2(w)\big]\big[\varphi_R(v)-\varphi_R(w)\big]^2\\
&\qquad\qquad \times \big[H(v)+H(w)\big]K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + \int_{B_{2R}}\int_{B_{2R}} \big[f^2(v)+f^2(w)\big]\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2\\
&\qquad\qquad \times \Big[\varphi_R^2(v)+\varphi_R^2(w)\Big]K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}w\,\mathrm{d}v\\
&=: \mathcal{I}^{s,\varphi}_{loc} + \mathcal{I}^{s,H}_{loc}.
\end{aligned} \tag{6.21}$$

We start with $\mathcal{I}^{s,\varphi}_{loc}$: we observe that $\varphi_R(v) = \varphi_R(w)$ for $w \in B_\rho(v)$ with $v \in B_{\frac{R-1}{2}}$. Thus

$$\begin{aligned}
\mathcal{I}^{s,\varphi}_{loc} &= \int_{B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} \big[f^2(v)+f^2(w)\big]\big[\varphi_R(v)-\varphi_R(w)\big]^2\big[H(v)+H(w)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad + \int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} \big[f^2(v)+f^2(w)\big]\big[\varphi_R(v)-\varphi_R(w)\big]^2\\
&\qquad\qquad \times \big[H(v)+H(w)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\
&= \int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} \big[f^2(v)+f^2(w)\big]\big[\varphi_R(v)-\varphi_R(w)\big]^2\\
&\qquad\qquad \times \big[H(v)+H(w)\big]K(v,w)\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}$$

Since the integrand is bounded, we obtain as $R \to \infty$

$$\mathcal{I}^{s,\varphi}_{loc} \to 0. \tag{6.22}$$

We continue with $\mathcal{I}_{loc}^{s,H}$: due to (1.28), the integrand is symmetric, so that

$$\begin{aligned}\frac{1}{2}\mathcal{I}_{loc}^{s,H} &= \int_{B_{2R}}\int_{B_{2R}} f^2(v)\big[\varphi_R^2(v)+\varphi_R^2(w)\big]\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2 K(v,w)\chi_{|v-w|<\rho}\,\mathrm{d}w\,\mathrm{d}v\\ &= \int_{B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\big[\varphi_R^2(v)+\varphi_R^2(w)\big]\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\big[\varphi_R^2(v)+\varphi_R^2(w)\big]\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &=: \mathcal{I}_{loc}^{H-decay}+\mathcal{I}_{loc}^{H-error}.\end{aligned}$$

The first term encodes the localised decay of H. The second term tends to zero as $R\to\infty$, as the integrand is bounded. Concretely, since $\varphi_R(v)=\varphi_R(w)$ for any $w\in B_\rho(v)$ and $v\in B_{\frac{R-1}{2}}$ if $R\geq 1+2\rho$, we find

$$\mathcal{I}_{loc}^{H-decay}=2\int_{B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\varphi_R^2(v)\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v.$$

For the second term, we see that the integrand is bounded so that

$$\mathcal{I}_{loc}^{H-error}\leq 2\int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v \xrightarrow[R\to\infty]{} 0.$$

Thus, as $R\to\infty$, we get

$$\mathcal{I}_{loc}^{s,H}\leq 4\int_{\mathbb{R}^d}\int_{B_\rho(v)} f^2(v)\varphi_R^2(v)\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v. \tag{6.23}$$

The combination of (6.19) with (6.20), (6.21), (6.22) and (6.23) yields as $R\to\infty$

$$\begin{aligned}\mathcal{I}_{loc}\leq& 2\int_{\mathbb{R}^d}\int_{B_\rho(v)} f^2(v)\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2\big[K(v,w)+K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v\\ &+C\rho^{-2s}\left\|H\right\|_{L^\infty}\left\|f\right\|_{L^2(\mathbb{R}^d)}^2.\end{aligned} \tag{6.24}$$

Therefore, by letting $R\to\infty$, we conclude from (6.17), (6.18), (6.24)

$$\begin{aligned}&\int_{\mathbb{R}^d}(\mathcal{L}f)fH\,\mathrm{d}v\\ &\quad\leq 2c_0\int_{\mathbb{R}^d}\int_{B_\rho(v)} f^2(v)\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2\big[K(v,w)+K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v\\ &\qquad+C\rho^{-2s}\left\|H\right\|_{L^\infty(\mathbb{R}^d)}\left\|f\right\|_{L^2(\mathbb{R}^d)}^2.\end{aligned} \tag{6.25}$$

Step 3: Conclusion.

We assemble the pieces. Equation (6.16) implies

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_1,x,v)H(\tau_1,x,v)\varphi_R^2\,\mathrm{d}x\,\mathrm{d}v \le \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\varphi_R^2\,\mathrm{d}x\,\mathrm{d}v \\ + CR^{-2s}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)H(z)\varphi_R(x,v)\,\mathrm{d}z \\ + \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)\varphi_R^2(x,v)\mathcal{T}H(z)\,\mathrm{d}z \\ + \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \big(\mathcal{L}f\big)(z)f(z)H(z)\varphi_R^2(x,v)\,\mathrm{d}z,\end{aligned}$$

which as $R\to\infty$ yields together with (6.25)

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_1,x,v)H(\tau_1,x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\quad\le \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v + \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)\mathcal{T}H(z)\,\mathrm{d}z \\ &\qquad + 2c_0\int_{\mathbb{R}^d}\int_{B_\rho(v)} f^2(v)\Big[H^{\frac12}(v)-H^{\frac12}(w)\Big]^2\big[K(v,w)+K(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v \\ &\qquad + C\rho^{-2s}\left\|H\right\|_{L^\infty}\left\|f\right\|^2_{L^2([\tau_0,\tau_1]\times\mathbb{R}^{2d})} \\ &\quad\le \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v + C\rho^{-2s}\left\|H\right\|_{L^\infty}\left\|f\right\|^2_{L^2([\tau_0,\tau_1]\times\mathbb{R}^{2d})},\end{aligned}$$

since by construction H satisfies (6.14) with $c = 2c_0$. This concludes the proof of Proposition 6.1. □

6.2.3.2 Construction of H.

In this section, we construct a function H that satisfies (6.14).

Lemma 6.1. *Let $y_0, w_0 \in \mathbb{R}^d$. Let $0<\rho$ and $0\le\tau_0<\sigma$. Let $k\ge 1$ be such that $\sigma-\tau_0\le\frac{\rho^{2s}}{4k}$. For $(t,x,v)\in[\tau_0,\sigma]\times\mathbb{R}^{2d}$, we define $\delta(t):=2(\sigma-\tau_0)-(t-\tau_0)$ and*

$$H(t,x,v) := e^{-\max\left\{1,\frac{1}{3\rho}\max\left(|v-w_0|,|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}\right)\right\}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}. \tag{6.26}$$

Then there exist a constant $C_1>0$ depending only on s,d,λ_0,Λ_0 such that, if $k>C_1$, then H satisfies (6.14)*, where K is a non-negative kernel satisfying* (1.32)*,* (1.28)*.*

Proof. We verify that H satisfies (6.14) by a case distinction.

i. For (t,x,v) such that $|v-w_0|<2\rho$ and $|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}<3\rho$: then

$$H(t,x,v)=\frac{k\delta(t)}{\rho^{2s}},$$

so that

$$\partial_t H = -\frac{k}{\rho^{2s}}, \qquad v \cdot \nabla_x H = 0, \qquad \mathcal{L}_\rho H^{\frac{1}{2}} = 0.$$

Thus for any $k > 0$ (6.14) is satisfied.

ii. For (t, x, v) such that $|v - w_0| < 2\rho$ and $|x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} > 3\rho$. Then

$$H(t, x, v) = e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)},$$

so that

$$\mathcal{L}_\rho H^{\frac{1}{2}} = 0,$$

and

$$\begin{aligned}
&\mathcal{T} H(t, x, v) \\
&= -\Bigg[\frac{1}{3\rho(1+2s)} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right) |x - y_0 - (\sigma + t - 2\tau_0)w_0|^{-\frac{1+4s}{1+2s}} \\
&\qquad \times (x - y_0 - (\sigma + t - 2\tau_0)w_0) \cdot (v - w_0) \\
&\qquad + \frac{|x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho\delta(t)}\Bigg] H(t, x, v) \\
&\leq C\left[\frac{\rho^{2s}}{3\rho k\delta(t)}(3\rho)^{-2s}(2\rho) - \frac{1}{\delta(t)}\right] e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \\
&\leq C\left[\frac{1}{k\delta(t)} - \frac{1}{\delta(t)}\right] e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \leq 0,
\end{aligned}$$

for any $k \geq 1$. We used the bounds $|v - w_0| < 2\rho, |x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} > 3\rho$, and the fact that $\log a < a$ for $a > 0$.

iii. For (t, x, v) such that $2\rho < |v - w_0| < 3\rho$ and $|x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} < 3\rho$. Then

$$H(t, x, v) = \frac{k\delta(t)}{\rho^{2s}},$$

but for $w \in B_\rho(v)$ it might happen that $|w - w_0| > 3\rho$, in which case

$$H(t, x, w) = e^{-\frac{1}{3\rho}|w-w_0| \log\left(\frac{\rho^{2s}}{k\delta}\right)} \leq \frac{k\delta(t)}{\rho^{2s}} = H(t, x, v),$$

with

$$\nabla_v H^{\frac{1}{2}}(t, x, w) = -\frac{1}{6\rho} \log\left(\frac{\rho^{2s}}{k\delta}\right) \frac{(w - w_0)}{|w - w_0|} e^{-\frac{1}{6\rho}|w-w_0| \log\left(\frac{\rho^{2s}}{k\delta}\right)}.$$

We fix $v_0 \in B_\rho(v)$ with $|v_0 - w_0| = 3\rho$ such that for every $w \in \mathbb{R}^d \setminus B_{3\rho}(w_0)$ there holds $|v_0 - w| \leq 2\,|v - w|$. As Kassmann-Weidner [129] point out, this works for example if $v_0 \in \partial B_{3\rho}(w_0)$ so that v_0 minimises $\operatorname{dist}(v, \mathbb{R}^d \setminus B_{3\rho}(w_0))$. Then $B_\rho(v) \subset B_{2\rho}(v_0)$ and

$H(t,x,v_0) = H(t,x,v)$. Note that

$$\left|H^{\frac{1}{2}}(t,x,v_0) - H^{\frac{1}{2}}(t,x,w)\right|^2 \leq \left|\nabla_v H^{\frac{1}{2}}(t,x,v_0)\right|^2 |v_0 - w|^2$$

since

$$\sup_{w\in\mathbb{R}^d\setminus B_{3\rho}(w_0)} \left|\nabla_v H^{\frac{1}{2}}(t,x,w)\right| = \left|\nabla_v H^{\frac{1}{2}}(t,x,v_0)\right|, \quad \text{for } v_0 \in \partial B_{3\rho}(w_0).$$

Therefore, using the upper bound (1.32) and $|v_0 - w_0| = 3\rho$, as well as $\log a \leq a^{1/2}$, we get

$$\begin{aligned}
\mathcal{L}_\rho H^{\frac{1}{2}}(v) &= \int_{B_\rho(v)} \left(H^{\frac{1}{2}}(v) - H^{\frac{1}{2}}(w)\right)^2 \big(K(v,w) + K(w,v)\big)\,\mathrm{d}w \\
&= \int_{B_\rho(v)\setminus B_{3\rho}(w_0)} \left(H^{\frac{1}{2}}(v) - H^{\frac{1}{2}}(w)\right)^2 \big(K(v,w) + K(w,v)\big)\,\mathrm{d}w \\
&\leq 2\int_{B_\rho(v)\setminus B_{3\rho}(w_0)} \left(H^{\frac{1}{2}}(v_0) - H^{\frac{1}{2}}(w)\right)^2 K(v_0,w)\,\mathrm{d}w \\
&\leq C\int_{B_\rho(v)\setminus B_{3\rho}(w_0)} \left|\nabla_v H^{\frac{1}{2}}(v_0)\right|^2 |v_0 - w|^2 K(v_0,w)\,\mathrm{d}w \\
&\leq C\rho^{2-2s}\frac{1}{36\rho^2}\left(\log\left(\frac{\rho^{2s}}{k\delta}\right)\right)^2 e^{-\frac{1}{3\rho}|v_0-w_0|\log\left(\frac{\rho^{2s}}{k\delta}\right)} \\
&= C\rho^{-2s}\left(\frac{\rho^{2s}}{k\delta}\right)\frac{k\delta}{\rho^{2s}} =: c_1\rho^{-2s},
\end{aligned} \tag{6.27}$$

for some constant $c_1 > 0$ independent of $\rho, \delta(t), k$. Together with

$$\mathcal{T}H(t,x,v) = -\frac{k}{\rho^{2s}},$$

we thus find

$$\mathcal{T}H + \mathcal{L}_\rho H^{\frac{1}{2}} \leq 0$$

for $k \geq c_1$.

iv. For (t,x,v) such that $2\rho < |v - w_0| < 3\rho$ and $|x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} > 3\rho$. Then

$$H(t,x,v) = e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)},$$

but for $w \in B_\rho(v)$ it might happen that $|w - w_0| > |x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} > 3\rho$, in which case

$$H(t,x,w) = e^{-\frac{1}{3\rho}|w-w_0|\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \leq e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} = H(t,x,v).$$

Similar to before, we fix $v_0 \in B_\rho(v)$ with $|v_0 - w_0| = |x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}}$ such that for every $w \in \mathbb{R}^d \setminus B_{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{(1+2s)}}}(w_0)$ it holds $|v_0 - w| \leq 2|v - w|$. For instance, we choose $v_0 \in \partial B_{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{(1+2s)}}}(w_0)$ so that v_0 minimises the distance between v and
$\mathbb{R}^d \setminus B_{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{(1+2s)}}}(w_0)$. Then $B_\rho(v) \subset B_{2\rho}(v_0)$ and $H(t,x,v_0) = H(t,x,v)$. If we denote by $\Omega := B_\rho(v) \setminus B_{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{(1+2s)}}}(w_0)$, then we find similar to before

$$\begin{aligned}
\mathcal{L}_\rho H^{\frac{1}{2}}(v) &= \int_{B_\rho(v)} \left(H^{\frac{1}{2}}(v) - H^{\frac{1}{2}}(w)\right)^2 \left(K(v,w) + K(w,v)\right) \mathrm{d}w \\
&= \int_{\Omega} \left(H^{\frac{1}{2}}(v) - H^{\frac{1}{2}}(w)\right)^2 \left(K(v,w) + K(w,v)\right) \mathrm{d}w \\
&\leq \left|\nabla_v H^{\frac{1}{2}}(v_0)\right|^2 \int_\Omega |v_0 - w|^2 K(v_0, w)\, \mathrm{d}w \\
&\leq C\rho^{2-2s} \frac{1}{36\rho^2} \left(\log\left(\frac{\rho^{2s}}{k\delta}\right)\right)^2 e^{-\frac{1}{3\rho}|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}} \log\left(\frac{\rho^{2s}}{k\delta}\right)} \\
&=: c_2 \frac{1}{k\delta(t)} e^{-\frac{1}{3\rho}|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}} \log\left(\frac{\rho^{2s}}{k\delta}\right)},
\end{aligned}$$

for c_2 independent of $k, \rho, \delta(t)$. Finally, using $\log a < a$ for $a > 0$

$$\begin{aligned}
&\mathcal{T}H(t,x,v) \\
&= -\left[\frac{1}{3\rho(1+2s)} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right) |x - y_0 - (\sigma + t - 2\tau_0)w_0|^{-\frac{1+4s}{1+2s}} \right. \\
&\qquad \times (x - y_0 - (\sigma + t - 2\tau_0)w_0) \cdot (v - w_0) \\
&\quad \left. + \frac{|x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho\delta(t)}\right] H(t,x,v) \\
&\leq C\left[\frac{1}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)\frac{\rho}{\rho^{2s}} - \frac{1}{\delta(t)}\right] e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \\
&=: c_3 \left[\frac{1}{\delta(t)k} - \frac{1}{\delta(t)}\right] e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}.
\end{aligned}$$

Thus $\mathcal{T}H + \mathcal{L}_\rho H^{\frac{1}{2}} \leq 0$, for $k \geq c_2 + c_3$.

v. For (t,x,v) such that $3\rho < |v - w_0|$ and either $|x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} < 3\rho$ or $3\rho < |x - y_0 - (\sigma + t - 2\tau_0)w_0|^{\frac{1}{1+2s}} < |v - w_0|$: Then

$$H(t,x,v) := e^{-\frac{|v-w_0|}{3\rho} \log\left(\frac{\rho^{2s}}{k\delta(t)}\right)},$$

and

$$\mathcal{T}H(t,x,v) = -\frac{|v-w_0|}{3\rho\delta(t)}e^{-\frac{|v-w_0|}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \leq -\frac{c_4}{\delta(t)}e^{-\frac{|v-w_0|}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)},$$

for some $c_4 > 0$ independent of $k, \delta(t), \rho$. Moreover, again as before, using the fact that $|w-w_0| > |v-w_0| - \rho$ we get

$$\begin{aligned}
\mathcal{L}_\rho H^{\frac{1}{2}}(v) &= \int_{B_\rho(v)} \left(H^{\frac{1}{2}}(v) - H^{\frac{1}{2}}(w)\right)^2 \left(K(v,w) + K(w,v)\right) \mathrm{d}w \\
&\leq \sup_{v' \in B_\rho(v)} \left|\nabla_v H^{\frac{1}{2}}(v')\right|^2 \int_{B_\rho(v)} |v-w|^2 K(v,w)\,\mathrm{d}w \\
&\leq C\rho^{2-2s} e^{-\frac{|v-w_0|-\rho}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \left[\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)\right]^2 \frac{1}{(6\rho)^2} \\
&\leq C\rho^{2-2s} e^{-\frac{|v-w_0|}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \left[\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)\right]^2 \left(\frac{\rho^{2s}}{k\delta(t)}\right)^{\frac{1}{3}} \frac{1}{(6\rho)^2} \\
&\leq \frac{c_5}{k\delta(t)} e^{-\frac{|v-w_0|}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)},
\end{aligned}$$

for some $c_5 > 0$ independent of $k, \rho, \delta(t)$. Thus

$$\mathcal{T}H + \mathcal{L}_\rho H^{\frac{1}{2}} \leq \left[-\frac{c_4}{\delta(t)} + \frac{c_5}{k\delta(t)}\right] e^{-\frac{|v-w_0|}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} \leq 0,$$

for $k > \frac{c_5}{c_4}$.

vi. For (t,x,v) such that $3\rho < |v-w_0| < |x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}$. Then

$$H(t,x,v) := e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)},$$

and

$$
\begin{aligned}
&\mathcal{T}H(t,x,v)\\
&= -\Bigg[\frac{1}{3\rho(1+2s)}\log\Big(\frac{\rho^{2s}}{k\delta(t)}\Big)\,|x-y_0-(\sigma+t-2\tau_0)w_0|^{-\frac{1+4s}{1+2s}}\\
&\qquad\qquad\times(x-y_0-(\sigma+t-2\tau_0)w_0)\cdot(v-w_0)\\
&\qquad+\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho\delta(t)}\Bigg]H(t,x,v)\\
&\le C\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{\rho\delta(t)}\left[\frac{\rho^{2s}\,|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{-2s}{1+2s}}}{k}-1\right]H(t,x,v)\\
&\le C\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{\rho\delta(t)}\left[\frac{1}{k}-1\right]e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}\\
&\le C\left[\frac{1}{\delta(t)k}-\frac{1}{\delta(t)}\right]e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}\\
&=: c_6\left[\frac{1}{\delta(t)k}-\frac{1}{\delta(t)}\right]e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}.
\end{aligned}
$$

The last inequality uses $k>1$. Moreover, using $\log a < a^{1/2}$ we denote by

$$\Omega := B_\rho(v)\setminus B_{|x-y_0-(\sigma+t-2\tau_0)w_0|^{1/(1+2s)}}(w_0),$$

and we again take $v_0\in\partial B_{|x-y_0-(\sigma+t-2\tau_0)w_0|^{1/(1+2s)}}(w_0)$, so that

$$
\begin{aligned}
\mathcal{L}_\rho H^{\frac12}(v) &= \int_\Omega\Big(H^{\frac12}(v)-H^{\frac12}(w)\Big)^2\Big(K(v,w)+K(w,v)\Big)\,\mathrm{d}w\\
&\le C\rho^{2-2s}\left|\nabla_v H^{\frac12}(v_0)\right|^2\\
&\le C\rho^{2-2s}\left[\log\Big(\frac{\rho^{2s}}{k\delta(t)}\Big)\right]^2\Big(\frac{1}{6\rho}\Big)^2 e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}\\
&\le C\frac{1}{k\delta(t)}e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}\\
&=: c_7\frac{1}{k\delta(t)}e^{-\frac{|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}}{3\rho}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}.
\end{aligned}
$$

Thus, for (6.14) to hold in this case, we need $k>\frac{c_7}{c_6}$.

Overall, we conclude the proof upon choosing

$$k>\max\left\{1,c_1,c_2+c_3,\frac{c_5}{c_4},\frac{c_7}{c_6}\right\},$$

where $c_i > 0$ for $i = 1, \dots, 7$ are constants independent of ρ and $\delta(t)$. □

6.2.3.3 Proof of Aronson's bound

The existence of a function satisfying (6.14) given by Lemma 6.1 combined with Proposition 6.1 implies Theorem 6.6.

Proof of Theorem 6.6. Step 1: Bounds on H.

We recall the definition of H from (6.26): for $\delta(t) = 2(\sigma - \tau_0) - (t - \tau_0)$ we have defined

$$H(t,x,v) = e^{-\max\left\{1, \frac{1}{3\rho}\max\left(|v-w_0|, |x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}\right)\right\}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)}.$$

Step 1-(i): The lower bound. For

$$(t,x,v) \in [\tau_0, \sigma] \times B_{(2\rho)^{1+2s}}(y_0 + (\sigma + t - 2\tau_0)w_0) \times B_{2\rho}(w_0)$$

we find

$$H(t,x,v) \geq \left(\frac{\rho^{2s}}{k\delta(t)}\right)^{-1} \geq \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{-1}. \tag{6.28}$$

We used $\delta(t) \geq \sigma - \tau_0$ and $\sigma - \tau_0 \leq \frac{\rho^{2s}}{4k}$.

Step 1-(ii): The upper bound. For $t = \tau_0$ and (x, v) such that

$$\max\left\{ |x - y_0 - (\sigma - \tau_0)w_0|^{\frac{1}{1+2s}}, |v - w_0| \right\} \geq \gamma$$

we find

$$H(\tau_0,x,v) \leq \left(\frac{\rho^{2s}}{k\delta(\tau_0)}\right)^{-\frac{\gamma}{3\rho}} \leq \left(\frac{\rho^{2s}}{2k(\sigma-\tau_0)}\right)^{-\frac{\gamma}{3\rho}}. \tag{6.29}$$

We used $\sigma - \tau_0 \leq \frac{\rho^{2s}}{4k}$.

Step 1-(iii): The general upper bound. Note that for $(t, x, v) \in [\tau_0, \sigma] \times \mathbb{R}^{2d}$ we have

$$H(t,x,v) \leq \left(\frac{\rho^{2s}}{k\delta(t)}\right)^{-1} \leq \left(\frac{\rho^{2s}}{2k(\sigma-\tau_0)}\right)^{-1}. \tag{6.30}$$

Step 2: Assembly.

For fixed $(y_0, w_0) \in \mathbb{R}^{2d}$ we denote

$$Q^0_{2\rho}(t, y_0, w_0) := B_{(2\rho)^{1+2s}}(y_0 + (\sigma + t - 2\tau_0)w_0) \times B_{2\rho}(w_0),$$

to obtain with Proposition 6.1, (6.29) and (6.30)

$$\begin{aligned}
&\sup_{t\in(\tau_0,\sigma)} \int_{Q^0_{2\rho}(t,y_0,w_0)} f^2(t,x,v)H(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad\le \sup_{t\in(\tau_0,\sigma)} \int_{\mathbb{R}^{2d}} f^2(t,x,v)H(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad\le \int_{\mathbb{R}^{2d}\setminus Q^0_\gamma(\tau_0,y_0,w_0)} f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad + C\rho^{-2s}\left\|H\right\|_{L^\infty([\tau_0,\sigma]\times\mathbb{R}^{2d})}\left\|f\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})} \\
&\quad\le \left(\frac{\rho^{2s}}{2k(\sigma-\tau_0)}\right)^{-\frac{\gamma}{3\rho}}\left\|f_0\right\|^2_{L^2(\mathbb{R}^{2d})} + C\rho^{-2s}\left(\frac{\rho^{2s}}{2k(\sigma-\tau_0)}\right)^{-1}\left\|f\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})}.
\end{aligned}$$

This implies due to (6.28)

$$\begin{aligned}
&\sup_{t\in(\tau_0,\sigma)} \int_{Q^0_{2\rho}(t,y_0,w_0)} f^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad\le \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)\left(\frac{\rho^{2s}}{2k(\sigma-\tau_0)}\right)^{-\frac{\gamma}{3\rho}}\left\|f_0\right\|^2_{L^2(\mathbb{R}^{2d})} \\
&\qquad\quad + C\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)\left(\frac{\rho^{2s}}{2k(\sigma-\tau_0)}\right)^{-1}\rho^{-2s}\left\|f\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})} \\
&\qquad\le \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{1-\frac{\gamma}{3\rho}}\left\|f_0\right\|^2_{L^2(\mathbb{R}^{2d})} + C\rho^{-2s}\left\|f\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})}.
\end{aligned}$$

Since we assume that (1.13) admits a fundamental solution, we estimate with Young's convolution inequality and with (6.1)

$$\begin{aligned}
\left\|f\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})} \le \left\|f_0 *_v J\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})} &\le (\sigma-\tau_0)\left\|f_0\right\|^2_{L^2(\mathbb{R}^{2d})}\left\|J\right\|^2_{L^\infty([\tau_0,\sigma];L^1(\mathbb{R}^{2d}))} \\
&= (\sigma-\tau_0)\left\|f_0\right\|^2_{L^2(\mathbb{R}^{2d})},
\end{aligned}$$

so that

$$\begin{aligned}
&\sup_{t\in(\tau_0,\sigma)} \int_{Q^0_{2\rho}(t,y_0,w_0)} f^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad\le \left[\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{1-\frac{\gamma}{3\rho}} + \rho^{-2s}(\sigma-\tau_0)\right]\left\|f_0\right\|^2_{L^2(\mathbb{R}^{2d})}.
\end{aligned} \tag{6.31}$$

Then using Proposition 5.1 with $r^{2s} = \frac{\sigma-\tau_0}{2^{2s}}$, $R = 2r$ for $\rho \geq r$ and (6.31)

$$
\begin{aligned}
&|f(\sigma, y_0, w_0)|^2 \\
&\leq \sup_{Q_r(\sigma,y_0,w_0)} f^2 \\
&\leq Cr^{-(2d(1+s)+2s)} \int_{Q_{2r}(\sigma,y_0,w_0)} f^2 \,\mathrm{d}z \\
&\leq Cr^{-(2d(1+s)+2s)+2s} \sup_{t\in(\tau_0,\sigma)} \int_{Q^0_{2\rho}(t,y_0,w_0)} f^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\leq C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{1-\frac{\gamma}{3\rho}} \|f_0\|^2_{L^2(\mathbb{R}^{2d})} + \rho^{-2s}(\sigma-\tau_0)^{1-\frac{2d(1+s)}{2s}} \|f_0\|^2_{L^2(\mathbb{R}^{2d})}\,.
\end{aligned}
$$

This concludes the proof of Theorem 6.6. □

6.2.4 Proof of the polynomial decay

With Theorem 6.4 and Theorem 6.5 at hand, we use standard methods to derive the decay on the fundamental solution.

Proof of Theorem 6.1. Due to the on-diagonal bounds (6.8), it suffices to consider $(\sigma - \tau_0) \leq \frac{\rho^{2s}}{4k}$. We choose

$$
\rho = \frac{\gamma}{12}\left(\frac{1}{4} + \frac{2d(1+s)+2s}{2s}\right)^{-1}. \tag{6.32}
$$

Then the off-diagonal bounds from Theorem 6.5 yield

$$
\begin{aligned}
&J(\sigma, x, v; \tau_0, y_0, w_0) \\
&\lesssim (\sigma-\tau_0)\Big(\max\left\{|v-w_0|\,, |x-y_0-(\sigma-\tau_0)(v-w_0)|^{\frac{1}{1+2s}}\right\}\Big)^{-(2d(1+s)+2s)} \\
&+ (\sigma-\tau_0)^{\frac{1}{4}-\frac{2d(1+s)}{2s}} \left(\max\left\{|v-w_0|\,, |x-y_0-(\sigma-\tau_0)(v-w_0)|^{\frac{1}{1+2s}}\right\}\right)^{-\frac{s}{2}},
\end{aligned}
$$

which concludes the proof of Theorem 6.1. □

Proof of Theorem 6.3. Again, due to the on-diagonal bounds (6.8), it suffices to consider $(\sigma - \tau_0) \leq \frac{\rho^{2s}}{4k}$. Due to (6.5), it suffices to use Aronson's method on J_ρ. In particular, the term arising from the non-singular part in Proposition 6.1 vanishes, that is $\mathcal{I}^{ns}_{loc} = 0$ in (6.19). This implies that in Theorem 6.6 only the first term on the right hand side remains, from which we deduce Theorem 6.5 on J_ρ with an improved right hand side:

$$
J_\rho(\sigma, x, v; \tau_0, y_0, w_0) \leq C(\sigma-\tau_0)^{-\frac{2d(1+s)}{2s}} \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{4}-\frac{\gamma}{12\rho}}.
$$

If we choose ρ as above in (6.32) then together with (6.5) this yields for $(\sigma - \tau_0) \leq \frac{\rho^{2s}}{4k}$

$$J(\sigma, x, v; \tau_0, y_0, w_0)$$
$$\lesssim (\sigma - \tau_0) \max\left\{ |v - w_0|, |x - y_0 - (\sigma - \tau_0)(v - w_0)|^{\frac{1}{1+2s}} \right\}\Big)^{-(2d(1+s)+2s)},$$

which concludes the proof of Theorem 6.3. □

6.3 Lower bound on the fundamental solution

6.3.1 Pointwise Harnack inequality

Theorem 6.7. *Let f solve (1.13) in $[0,T] \times \mathbb{R}^{2d}$ for a non-negative kernel satisfying (1.26)-(1.28) and for zero source $h = 0$. Consider $(\tau_0, y_0, w_0) \in [0,T] \times \mathbb{R}^{2d}$ and $(t, x, v) \in [\tau_0, T] \times \mathbb{R}^{2d}$. Then there exists a constant C depending on $s, d, \lambda_0, \Lambda_0$ and T such that*

$$f(\tau_0, y_0, w_0) \leq f(t, x, v) \exp C \left\{ \frac{|x - y_0 - (t - \tau_0)w_0|^{2s}}{(t - \tau_0)^{1+2s}} + \frac{|v - w_0|^{2s}}{t - \tau_0} + \frac{t}{\tau_0} \right\}.$$

Proof. We consider the set of all cylinders $Q_\sigma = [0, \sigma^{2s}] \times B_{\sigma^{1+2s}} \times B_\sigma$ for $\sigma := \min\left\{1, T^{\frac{1}{2s}}\right\}$. We fix $(\tau_1, y_1, w_1) \in [0,T] \times \mathbb{R}^{2d}$. We consider some sufficiently small $r \leq \sigma$, so that

$$-r^{2s} < t - \tau_1 < 3r^{2s}, \qquad |x - y_1 - (t - \tau_1)w_1| < (12r)^{1+2s}, \qquad |v - w_1| < 12r,$$

is contained in Q_σ. Then by the strong Harnack inequality, Theorem 5.1, there exists a constant $C > 0$ depending on $s, d, \lambda_0, \Lambda_0$ such that

$$f(\tau_1, y_1, w_1) \leq C f(\tau_1 + 3r^{2s}, x, v),$$

for

$$|x - y_1 - (t - \tau_1)w_1| < \left(\frac{r}{2}\right)^{1+2s}, \qquad |v - w_1| < \frac{r}{2}.$$

Equivalently, we can rewrite

$$f(\tau_1, y_1, w_1) \leq C f(t, x, v), \tag{6.33}$$

for

$$0 < t - \tau_1 < 3r^{2s}, \qquad |x - y_1 - (t - \tau_1)w_1|^{\frac{2s}{1+2s}} < \frac{t - \tau_1}{3 \cdot 2^{2s}}, \qquad |v - w_1|^{2s} < \frac{t - \tau_1}{3 \cdot 2^{2s}}. \tag{6.34}$$

We connect (t,x,v) and (τ_0,y_0,w_0) by a path. Let $N\geq 1$ and consider (t^ν,x^ν,v^ν) given by

$$\begin{aligned}
&t^\nu=\tau_0+\frac{\nu}{N}(t-\tau_0),\quad x^\nu=y_0+(t-\tau_0)w_0+\Big(\frac{\nu}{N}\Big)^{\frac{1+s}{s}}\big(x-y_0-(t-\tau_0)w_0\big),\\
&v^\nu=w_0+\Big(\frac{\nu}{N}\Big)^{\frac{1}{s}}(v-w_0).
\end{aligned}$$

Moreover, for each $\nu=0,\dots,N$, the cylinder

$$-r^{2s}<t-t^\nu\leq 3r^{2s},\qquad |x-x^\nu-(t-t^\nu)v^\nu|<(12r)^{1+2s},\qquad |v-v^\nu|<12r,$$

is then contained in Q_σ if

$$r^{2s}\leq\frac{t-\tau_0}{3N}\leq\tau_0. \tag{6.35}$$

Thus by (6.33) there holds

$$f(t^\nu,x^\nu,v^\nu)\leq Cf(t^{\nu+1},x^{\nu+1},v^{\nu+1}),\qquad \nu=0,\dots,N-1, \tag{6.36}$$

if $(t^{\nu+1},x^{\nu+1},v^{\nu+1})$ satisfies (6.34) for each $\nu=0,\dots,N-1$, that is

$$\begin{aligned}
&0<t^{\nu+1}-t^\nu=\frac{t-\tau_0}{N}\leq 3r^{2s},\\
&\big|x^{\nu+1}-x^\nu-(t^{\nu+1}-t^\nu)v^\nu\big|^{\frac{2s}{1+2s}}\leq\frac{1}{N^{\frac{2+2s}{1+2s}}}\,|x-y-(t-\tau_0)w_0|^{\frac{2s}{1+2s}}<\frac{t-\tau_0}{3\cdot 2^{2s}N^{\frac{1}{1+2s}}},\\
&\big|v^{\nu+1}-v^\nu\big|^{2s}\leq\frac{1}{N^2}\,|v-w_0|^{2s}<\frac{t-\tau_0}{3\cdot 2^{2s}N}.
\end{aligned} \tag{6.37}$$

We choose N as the smallest integer satisfying

$$\begin{aligned}
&N\geq\frac{t-\tau_0}{3\sigma^{2s}},\qquad N\geq\frac{t-\tau_0}{3\tau_0},\\
&N\geq\big(3\cdot 2^{2s}\big)^{1+2s}\,\frac{|x-y_0-(t-\tau_0)w_0|^{2s}}{(t-\tau_0)^{1+2s}},\qquad N\geq 3\cdot 2^{2s}\frac{|v-w_0|^{2s}}{t-\tau_0}.
\end{aligned}$$

Then $r^{2s}\leq\sigma^{2s}$, (6.35) and (6.37) are satisfied. Iterating (6.36) yields

$$\begin{aligned}
&f(\tau_0,y_0,w_0)\\
&\leq C^N f(t,x,v)=e^{\log CN}f(t,x,v)\\
&\leq f(t,x,v)\exp C\left\{\frac{|x-y_0-(t-\tau_0)w_0|^{2s}}{(t-\tau_0)^{1+2s}}+\frac{|v-w_0|^{2s}}{t-\tau_0}+\frac{t-\tau_0}{\min\{\sigma^{2s},\tau_0\}}\right\}
\end{aligned} \tag{6.38}$$

for a constant depending on s,d,λ_0,Λ_0. Using that $\frac{(t-\tau_0)}{\sigma^{2s}}+\frac{t-\tau_0}{\tau_0}\leq\frac{t}{\tau_0}+\frac{T}{\tau_0}$ we can also

rephrase this estimate as

$$f(\tau_0, y_0, w_0) \leq f(t, x, v) \exp C' \left\{ \frac{|x - y_0 - (t-\tau_0)w_0|^{2s}}{(t-\tau_0)^{1+2s}} + \frac{|v - w_0|^{2s}}{t - \tau_0} + \frac{t}{\tau_0} \right\},$$

for a constant C' depending on T. □

6.3.2 Sufficient condition for the exponential lower bound

Theorem 6.8. *Let f be a non-negative weak solution of (1.13) in $[0,T] \times \mathbb{R}^{2d}$ with (1.26)-(1.28) that is essentially bounded almost everywhere. Suppose that for some $\alpha > 0$ there holds*

$$\mathcal{M} := \inf_{0<t<T} \int_{\max\left\{|x|^{\frac{2s}{1+2s}}, |v|^{2s}\right\} < \alpha t} f(t, x, v) \, \mathrm{d}x \, \mathrm{d}v > 0. \tag{6.39}$$

Then there exists constants $C_1, C_2 > 0$ such that

$$f(t, x, v) \geq C_1 t^{-\frac{2d(1+s)}{2s}} \exp\left(-C_2 \left(\frac{|x|^{2s}}{t^{1+2s}} + \frac{|v|^{2s}}{t} \right) \right).$$

The constant C_1 depends on $s, d, \alpha, T, \lambda_0, \Lambda_0$ and $\mathcal{M}$, whereas C_2 depends only on $s, d, T, \lambda_0, \Lambda_0$.

Remark 6.3. To draw a connection to the literature on hypoelliptic lower bounds, there has been a result discussing Gaussian lower bounds for solutions to the Boltzmann equation by Imbert-Mouhot-Silvestre in [120]. The authors show that for any $t \in [0,T]$ there exists $a(t), b(t) > 0$ such that any non-negative solution of the Boltzmann equation satisfies

$$f(t, x, v) \geq a(t) e^{-b(t)|v|^2}.$$

We explicitly give expressions for a and b, however, we are only able to treat symmetric kernels due to the tail bound of Section 5.3.

Proof. Consider $y, w \in \mathbb{R}^d$ such that $\max\left\{|y|^{\frac{2s}{1+2s}}, |w|^{2s}\right\} < \alpha t$. Then by (6.38) from Theorem 6.7 there holds

$$f(t/4, y, w) \leq f(t/2, 0, 0) e^{C(\alpha^{1+2s} + \alpha + 1)}. \tag{6.40}$$

Now notice that the measure

$$\left| \left\{ (y, w) \in \mathbb{R}^{2d} : \max\left\{|y|^{\frac{2s}{1+2s}}, |w|^{2s}\right\} < \alpha t \right\} \right| \approx t^{\frac{2d(1+s)}{2s}}.$$

Thus, on the one hand, integrating (6.40) over $\left\{ (y, w) \in \mathbb{R}^{2d} : \max\left\{|y|^{\frac{2s}{1+2s}}, |w|^{2s}\right\} < \alpha t \right\}$

yields due to (6.39)

$$f(t/2,0,0) \geq Ct^{-\frac{2d(1+s)}{2s}} e^{-C(\alpha^{1+2s}+\alpha+1)} \int_{\left\{(y,w)\in\mathbb{R}^{2d}:\max\left\{|y|^{\frac{2s}{1+2s}},|w|^{2s}\right\}<\alpha t\right\}} f(t/4,y,w)\,\mathrm{d}y\,\mathrm{d}w$$
$$\geq C\mathcal{M}t^{-\frac{2d(1+s)}{2s}} e^{-C(\alpha^{1+2s}+\alpha+1)}.$$

On the other hand, again applying Theorem 6.7, we find

$$f(t/2,0,0) \leq f(t,x,v)\exp\left\{C\left(\frac{|x|^{2s}}{t^{1+2s}}+\frac{|v|^{2s}}{t}+1\right)\right\},$$

so that overall we obtained

$$f(t,x,v) \geq Ct^{-\frac{2d(1+s)}{2s}}\exp\left\{-C\left(\frac{|x|^{2s}}{t^{1+2s}}+\frac{|v|^{2s}}{t}\right)\right\}.$$

□

6.3.3 Proof of Theorem 6.2

Proof of Theorem 6.2. Fix $(\tau_0,y_0,w_0)\in[0,T]\times\mathbb{R}^{2d}$ and let $0<\tau_0<\sigma<T$. Consider for $\tau_1<\sigma$ and for some α large enough the function

$$f(\tau_1,y_1,w_1) = M\int_{\max\left\{|x-y_0-(\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}},|v-w_0|^{2s}\right\}<\alpha(\sigma-\tau_0)} J(\sigma,x,v;\tau_1,y_1,w_1)\,\mathrm{d}x\,\mathrm{d}v,$$

where $M\geq 1$ is some constant such that $f(\tau_1,y_1,w_1)\leq M$ for $(\tau_1,y_1,w_1)\in[0,\sigma]\times\mathbb{R}^{2d}$.

Due to the divergence form symmetry (1.28), the non-local operator (1.14) is self-adjoint, so that f is a solution to the adjoint of (1.13) for $(\tau_1,y_1,w_1)\in[0,\sigma)\times\mathbb{R}^{2d}$ with initial values

$$\begin{cases} f(\sigma,y_1,w_1)=M, & \text{if } \max\left\{|y_1-y_0-(\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}},|w_1-w_0|^{2s}\right\}<\alpha(\sigma-\tau_0),\\ f(\sigma,y_1,w_1)=0, & \text{if } \max\left\{|y_1-y_0-(\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}},|w_1-w_0|^{2s}\right\}>\alpha(\sigma-\tau_0).\end{cases}$$

If we set

$$\tilde{f}(\tau_1,y_1,w_1) = \begin{cases} f(\tau_1,y_1,w_1), & \text{if } \tau_1<\sigma,\\ M, & \text{if } \tau_1>\sigma,\end{cases}$$

then $\tilde{f}$ is a non-negative super-solution of the adjoint of (1.13) in $[0,\infty)\times\mathbb{R}^{2d}$ since $\tilde{f}\leq M$ for $(\tau_1,y_1,w_1)\in[0,\infty)\times\mathbb{R}^{2d}$, so that for $\tau_1>\sigma$

$$\mathcal{T}^*\tilde{f}-\mathcal{L}^*\tilde{f} = -\mathcal{L}\tilde{f} = \int[M-\tilde{f}(w)]K(v,w)\,\mathrm{d}w \geq 0.$$

By the weak Harnack inequality, Theorem 1.1, we find for any $\delta > 0$ sufficiently small (such that the gap between τ_0 and $\sigma - \delta$ is sufficiently large)

$$\begin{aligned}
&\int_{\max\left\{|x-y_0-(\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}}, |v-w_0|^{2s}\right\}<\alpha(\sigma-\tau_0)} J(\sigma, x, v; \tau_0, y_0, w_0)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad = f(\tau_0, y_0, w_0) = \tilde{f}(\tau_0, y_0, w_0) \\
&\quad \geq C\left(\int_{\sigma-\delta}^{\sigma+\delta}\int_{\max\left\{|y'-y_0-(\tau'-\tau_0)w_0|^{\frac{2s}{1+2s}}, |w'-w_0|^{2s}\right\}<\alpha(\sigma-\tau_0)} \tilde{f}^{\zeta}(\tau', y', w')\,\mathrm{d}y'\,\mathrm{d}w'\,\mathrm{d}\tau'\right)^{\frac{1}{\zeta}} \\
&\quad > 0,
\end{aligned}$$

where C depends only on $d, s, \lambda_0, \Lambda_0$. The last inequality follows due to the fact that $f(\sigma, y_0, w_0) = M$. Then by Theorem 6.8 we get since $M \geq 1$

$$J(\sigma, x, v; \tau_0, y_0, w_0) \geq C(\sigma-\tau_0)^{-\frac{d(1+s)}{s}} \exp\left\{-C\left(\frac{|x-y_0-(\sigma-\tau_0)w_0|^{2s}}{(\sigma-\tau)^{1+2s}} + \frac{|v-w_0|^{2s}}{\sigma-\tau_0}\right)\right\}.$$

□

6.4 Linearised Boltzmann

We end this article with a short remark on how to adapt Aronson's method for non-local hypoelliptic equations in divergence form to the case of more general kernels that satisfy the ellipticity assumptions (1.26)-(5.43) inspired from the Boltzmann collision kernel.

6.4.1 Results

As a consequence of **Theorem 1.4**, we derive polynomial upper and exponential lower bounds on the fundamental solution of (1.11) with coefficients K_h and Λ_h given by (1.9) and (1.12), respectively, for a fixed function $h \geq 0$ that satisfies (1.20). Recall that we denote by J the fundamental solution of (1.11) linearised around a fixed function h, connecting a given point $(t, x, v) \in \mathbb{R}^{1+2d}$ with $(\tau, y, w) \in \mathbb{R}^{1+2d}$, as

$$J(t, x, v; \tau, y, w) = J(t-\tau, x-y-(t-\tau)w, v-w) =: J\big((\tau, y, w)^{-1} \circ (t, x, v)\big),$$

where $\circ$ denotes the Galilean translation, that is $(t_0, x_0, v_0) \circ (t, x, v) = (t_0 + t, x_0 + x + tv_0, v_0 + v)$. We derive polynomial upper bounds in **Theorem 1.5** and exponential lower bounds in **Theorem 1.6**. These are sub-optimal in space and velocity, however the temporal decay is optimal.

6.4.2 On the polynomial upper bounds

The proof method remains the same as Section 6.2. The on-diagonal bound, Theorem 6.4, follows by Proposition 5.4. The off-diagonal bound, Theorem 6.5 follows, if we are able to derive Aronson's bound in Proposition 6.1 without using the divergence form symmetry of the kernel of the non-local operator. This is contained in Proposition 6.2 below, which is the weak divergence-form analogue of Proposition 6.1. Once we have established Proposition 6.2 and constructed a decay function H satisfying Aronson's condition (6.41), the proof of Theorem 1.5 carries over almost verbally from the proof of Theorem 6.1.

We aim to define a function that satisfies for some $\rho > 0$

$$\begin{aligned} \mathcal{T}H(v) + \frac{1}{2}\int_{B_\rho(v)} \big[H(w) - H(v)\big] K_f(w,v)\,\mathrm{d}w \\ + \frac{H(v)}{2}\int_{\mathbb{R}^d} \left[K_f(v,w) - K_f(w,v)\right]\,\mathrm{d}w \le 0. \end{aligned} \tag{6.41}$$

Then we can derive the following statement.

Proposition 6.2 (Aronson's auxiliary proposition for Boltzmann)**.** *Let $0 < \tau_0 < \sigma < T$ and $0 < \rho$. Let $f \in L^\infty\big((\tau_0, T) \times \mathbb{R}^{2d}\big)$ solve (1.16) in $(\tau_0, T) \times \mathbb{R}^{2d}$ with a non-local operator whose kernel is non-negative and satisfies (1.32), (1.29), (1.30). Then for every bounded function $H : [\tau_0, \sigma] \times \mathbb{R}^{2d} \to [0, \infty)$ such that $H^{\frac{1}{2}} \in L^2\big((\tau_0, \sigma) \times \mathbb{R}^d; H^s_v(\mathbb{R}^d)\big)$ and $D_v H, D_v^2 H \in L^2((\tau_0, \sigma) \times \mathbb{R}^d; L^\infty(\mathbb{R}^d))$, and, moreover, satisfying (6.14) in $(\tau_0, \sigma) \times \mathbb{R}^{2d}$, there exists a constant $C = C(\lambda, \Lambda, s, d)$ such that*

$$\begin{aligned} \sup_{t\in(\tau_0,\sigma)} \int_{\mathbb{R}^{2d}} f^2(t,x,v)H(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \le \int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v \\ + C\rho^{-2s}\left\|H\right\|_{L^\infty([\tau_0,\sigma]\times\mathbb{R}^{2d})}\left\|f\right\|^2_{L^2([\tau_0,\sigma]\times\mathbb{R}^{2d})}. \end{aligned}$$

Proof of Proposition 6.2. For $R \ge \max\{2, 2\rho + 1\}$ we consider $\varphi_R \in C_c^\infty(\mathbb{R}^{2d})$ with $0 \le \varphi_R \le 1$ such that $\varphi_R \equiv 1$ for $(x, v) \in B_{(R-1)^{1+2s}} \times B_{R-1}$ and $\varphi_R \equiv 0$ for (x, v) outside $B_{R^{1+2s}} \times B_R$, with bounded derivatives and such that $|v \cdot \varphi_R| \sim R^{-2s}$. We test (1.16) with $fH\varphi_R^2$ where H satisfies (6.41) over $[\tau_0, \tau_1] \times \mathbb{R}^{2d}$ for $0 \le \tau_0 \le \tau_1 \le \sigma$ and get

$$\begin{aligned} &\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \mathcal{T}f\Big(fH\varphi_R^2\Big)\,\mathrm{d}z \\ &\quad= \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}}\int_{\mathbb{R}^d} \big(f(w) - f(v)\big)K_f(v,w)\Big(fH\varphi_R^2\Big)(v)\,\mathrm{d}w\,\mathrm{d}z \\ &\qquad+ \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f(v)\Big(fH\varphi_R^2\Big)(v)\left(\int_{\mathbb{R}^d}\big(K_f(v,w) - K_f(w,v)\big)\,\mathrm{d}w\right)\mathrm{d}z. \end{aligned}$$

Step 1: Transport operator.

First we integrate by parts the transport operator

$$\begin{aligned}
&\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \mathcal{T}f\Big(fH\varphi_R^2\Big)\,\mathrm{d}z \\
&\quad= \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2H\varphi_R^2\Big|_{t=\tau_0}^{\tau_1}\,\mathrm{d}x\,\mathrm{d}v - \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2H\varphi_R v\cdot\nabla_x\varphi_R\,\mathrm{d}z \\
&\qquad- \frac{1}{2}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2\varphi_R^2\mathcal{T}H\,\mathrm{d}z \qquad (6.42)\\
&\quad\geq \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2H\varphi_R^2\Big|_{t=\tau_0}^{\tau_1}\,\mathrm{d}x\,\mathrm{d}v - CR^{-2s}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2H\varphi_R\,\mathrm{d}z \\
&\qquad- \frac{1}{2}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2\varphi_R^2\mathcal{T}H\,\mathrm{d}z.
\end{aligned}$$

Step 2: Non-local operator.

Now we deal with the non-local term. We write

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big(\tilde{\mathcal{L}}f\big)fH\varphi_R^2\,\mathrm{d}v \\
&:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(f(w)-f(v)\big)K_f(v,w)\Big(fH\varphi_R^2\Big)(v)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\int_{\mathbb{R}^d}\big(f^2H\varphi_R^2\big)(v)\left(\int_{\mathbb{R}^d}\big(K_f(v,w)-K_f(w,v)\big)\,\mathrm{d}w\right)\mathrm{d}v \\
&= \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}}\big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v)-\big(fH\varphi_R^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}}\big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v)+\big(fH\varphi_R^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v \\
&\quad+\int_{\mathbb{R}^d}\big(f^2H\varphi_R^2\big)(v)\left(\int_{\mathbb{R}^d}\big(K_f(v,w)-K_f(w,v)\big)\,\mathrm{d}w\right)\mathrm{d}v \\
&\quad+\int_{B_{2R}}\int_{\mathbb{R}^d\setminus B_{2R}}\big[f(w)-f(v)\big]\big(fH\varphi_R^2\big)(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v.
\end{aligned}$$

Then we note that

$$\begin{aligned}
&\big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v)+\big(fH\varphi_R^2\big)(w)\Big] \\
&\quad=\big[f(w)+f(v)\big]\Big[\big(fH\varphi_R^2\big)(v)-\big(fH\varphi_R^2\big)(w)\Big] \\
&\qquad-2f(v)\big(fH\varphi_R^2\big)(v)+2f(w)\big(fH\varphi_R^2\big)(w),
\end{aligned}$$

and moreover, by Young's inequality

$$\begin{aligned}
&f(w)\Big[\big(fH\varphi_R^2\big)(v)-\big(fH\varphi_R^2\big)(w)\Big] \\
&\quad\leq\frac{1}{2}f^2(w)\Big[\big(H\varphi_R^2\big)(v)-\big(H\varphi_R^2\big)(w)\Big]+\frac{1}{2}f^2(v)\big(H\varphi_R^2\big)(v)-\frac{1}{2}f^2(w)\big(H\varphi_R^2\big)(w).
\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^d}\big(\tilde{\mathcal{L}}f\big)fH\varphi_R^2\,\mathrm{d}v &\le \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}} f^2(w)\Big[\big(H\varphi_R^2\big)(v)-\big(H\varphi_R^2\big)(w)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}}\Big[\big(f^2H\varphi_R^2\big)(w)-\big(f^2H\varphi_R^2\big)(v)\Big]K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\int_{\mathbb{R}^d}\big(f^2H\varphi_R^2\big)(v)\left(\int_{\mathbb{R}^d}\big(K_f(v,w)-K_f(w,v)\big)\,\mathrm{d}w\right)\mathrm{d}v\\ &\quad+\int_{B_{2R}}\int_{\mathbb{R}^d\setminus B_{2R}}\big[f(w)-f(v)\big]\big(fH\varphi_R^2\big)(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\le \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}} f^2(v)\Big[\big(H\varphi_R^2\big)(w)-\big(H\varphi_R^2\big)(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}}\big(f^2H\varphi_R^2\big)(v)\big[K_f(w,v)-K_f(v,w)\big]\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\int_{\mathbb{R}^d}\big(f^2H\varphi_R^2\big)(v)\left(\int_{\mathbb{R}^d}\big(K_f(v,w)-K_f(w,v)\big)\,\mathrm{d}w\right)\mathrm{d}v\\ &\quad+\int_{B_{2R}}\int_{\mathbb{R}^d\setminus B_{2R}}\big[f(w)-f(v)\big]\big(fH\varphi_R^2\big)(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &=: \mathcal{I}_{loc}^{H}+\mathcal{I}_{loc}^{skew}+\mathcal{I}_{nonloc}^{skew}+\mathcal{I}_{tail}.\end{aligned}\tag{6.43}$$

First we note that the last integral in (6.43) tends to zero as $R\to\infty$:

$$\begin{aligned}\mathcal{I}_{tail}&=\int_{B_{2R}}\int_{\mathbb{R}^d\setminus B_{2R}}\big[f(w)-f(v)\big]\big(fH\varphi_R^2\big)(v)K_f(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\le C\Lambda R^{-2s}\left\|H\varphi_R^2\right\|_{L^\infty}\|f\|_{L^1(B_R)}\|f\|_{L^\infty(\mathbb{R}^d\setminus B_{2R})}\xrightarrow[R\to\infty]{}0.\end{aligned}\tag{6.44}$$

Second, we distinguish the singular from the non-singular part to bound $\mathcal{I}_{loc}^{H}$. We have

$$\begin{aligned}\mathcal{I}_{loc}^{H}&=\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}} f^2(v)\Big[\big(H\varphi_R^2\big)(w)-\big(H\varphi_R^2\big)(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v\\ &=\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}\cap B_\rho(v)}\ldots\,\mathrm{d}w\,\mathrm{d}v+\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}\setminus B_\rho(v)}\ldots\,\mathrm{d}w\,\mathrm{d}v=:\mathcal{I}_{loc}^{H,s}+\mathcal{I}_{loc}^{H,ns}.\end{aligned}\tag{6.45}$$

Then we bound the non-singular part using (1.32)

$$\begin{aligned}\mathcal{I}_{loc}^{H,ns}&=\frac{1}{2}\int_{B_{2R}}\int_{B_{2R}\setminus B_\rho(v)} f^2(v)\Big[\big(H\varphi_R^2\big)(w)-\big(H\varphi_R^2\big)(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v\\ &\le C\rho^{-2s}\left\|H\varphi_R\right\|_{L^\infty(B_{2R})}\|f\|_{L^2(B_{2R})}^2.\end{aligned}\tag{6.46}$$

For the singular part, we observe that $\varphi_R(v)=\varphi_R(w)$ for any $w\in B_\rho(v)$ and $v\in B_{\frac{R-1}{2}}$ if

$R \geq 1 + 2\rho$, so that we further split

$$\begin{aligned}\mathcal{I}^{H,s}_{loc} &= \frac{1}{2}\int_{B_{2R}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\Big[\big(H\varphi_R^2\big)(w) - \big(H\varphi_R^2\big)(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v \\ &= \frac{1}{2}\int_{B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\varphi_R^2(v)\Big[H(w) - H(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + \frac{1}{2}\int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\Big[\big(H\varphi_R^2\big)(w) - \big(H\varphi_R^2\big)(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v.\end{aligned} \tag{6.47}$$

Then note that as $R \to \infty$, the last integral tends to zero as the integrand is bounded: Taylor expanding $H\varphi_R$, using (1.32) and (1.30)

$$\begin{aligned}&\frac{1}{2}\int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\Big[\big(H\varphi_R^2\big)(w) - \big(H\varphi_R^2\big)(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad= \frac{1}{2}\int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v) D_v\big(H\varphi_R^2\big)(v)\cdot\big(w-v\big)K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v \\ &\qquad + C\left\|D_v^2\big(H\varphi_R^2\big)\right\|_{L^\infty(B_{2R}\setminus B_{\frac{R-1}{2}})}\int_{B_{2R}\setminus B_{\frac{R-1}{2}}}\int_{B_{2R}\cap B_\rho(v)} f^2(v)\,|w-v|^2\,K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad\leq C\rho^{1-2s}\int_{B_{2R}\setminus B_{\frac{R-1}{2}}} f^2(v) D_v\big(H\varphi_R^2\big)(v)\,\mathrm{d}v \\ &\qquad + C\rho^{2-2s}\left\|D_v^2\big(H\varphi_R^2\big)\right\|_{L^\infty(B_{2R}\setminus B_{\frac{R-1}{2}})}\int_{B_{2R}\setminus B_{\frac{R-1}{2}}} f^2(v)\,\mathrm{d}v \\ &\quad\xrightarrow[R\to\infty]{} 0.\end{aligned}$$

We combine (6.45), (6.46), (6.47) and let $R \to \infty$, so that

$$\begin{aligned}\mathcal{I}^H_{loc} &\leq C\rho^{-2s}\,\|H\|_{L^\infty(\mathbb{R}^d)}\,\|f\|_{L^2(\mathbb{R}^d)} \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{B_\rho(v)} f^2(v)\Big[H(w) - H(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v.\end{aligned} \tag{6.48}$$

Thus, by letting $R \to \infty$, we conclude from (6.43), (6.44), (6.48),

$$\begin{aligned}\int_{\mathbb{R}^d}\big(\tilde{\mathcal{L}}f\big)fH\,\mathrm{d}v &\leq \frac{1}{2}\int_{\mathbb{R}^d} f^2(v)\int_{B_\rho(v)}\Big[H(w) - H(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d}(f^2H)(v)\int_{\mathbb{R}^d}\big[K_f(v,w) - K_f(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + C\rho^{-2s}\,\|H\|_{L^\infty(\mathbb{R}^d)}\,\|f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{6.49}$$

Step 3: Conclusion.

We assemble the pieces. Equation (6.42) implies

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_1,x,v)H(\tau_1,x,v)\varphi_R^2\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad\le \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\varphi_R^2\,\mathrm{d}x\,\mathrm{d}v + CR^{-2s}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)H(z)\varphi_R(x,v)\,\mathrm{d}z \\
&\qquad\quad + \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)\varphi_R^2(x,v)\mathcal{T}H(z)\,\mathrm{d}z + \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} \big(\tilde{\mathcal{L}}f\big)(z)f(z)H(z)\varphi_R^2(x,v)\,\mathrm{d}z,
\end{aligned}$$

which as $R\to\infty$ yields together with (6.49)

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_1,x,v)H(\tau_1,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\quad\le \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v + \int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)\mathcal{T}H(z)\,\mathrm{d}z \\
&\qquad + \frac{1}{2}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)\int_{B_\rho(v)}\Big[H(w)-H(v)\Big]K_f(w,v)\,\mathrm{d}w\,\mathrm{d}z \\
&\qquad + \frac{1}{2}\int_{[\tau_0,\tau_1]\times\mathbb{R}^{2d}} f^2(z)H(z)\int_{\mathbb{R}^d}\big[K_f(v,w)-K_f(w,v)\big]\,\mathrm{d}w\,\mathrm{d}v \\
&\qquad + C\rho^{-2s}\left\|H\right\|_{L^\infty}\left\|f\right\|^2_{L^2([\tau_0,\tau_1]\times\mathbb{R}^{2d})} \\
&\quad\le \frac{1}{2}\int_{\mathbb{R}^{2d}} f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v + C\rho^{-2s}\left\|H\right\|_{L^\infty}\left\|f\right\|^2_{L^2([\tau_0,\tau_1]\times\mathbb{R}^{2d})},
\end{aligned}$$

since by construction H satisfies (6.41). This concludes the proof of Proposition 6.2. □

It now only remains to show that we can construct a function H satisfying (6.41).

Lemma 6.2. *Let $y_0, w_0 \in \mathbb{R}^d$. Let $0<\rho$ and $0\le\tau_0<\sigma$. Let $k\ge 1$ and $\alpha\ge 0$ be such that $\sigma-\tau_0\le\frac{\rho^{2s}}{4k}$. For $(t,x,v)\in[\tau_0,\sigma]\times\mathbb{R}^{2d}$, we define $\delta(t):=2(\sigma-\tau_0)-(t-\tau_0)$ and*

$$H(t,x,v) := e^{-\max\left\{1,\frac{1}{3\rho}\max\left(|v-w_0|,|x-y_0-(\sigma+t-2\tau_0)w_0|^{\frac{1}{1+2s}}\right)\right\}\log\left(\frac{\rho^{2s}}{k\delta(t)}\right)} e^{\alpha\frac{(\sigma-t)}{\rho^{2s}}}. \tag{6.50}$$

Then there exist constants $C_1, C_2 > 0$ depending only on s, d, Λ_0 such that, if $k > C_1$ and $\alpha > C_2$, then H satisfies (6.41)*, where K_f is a non-negative kernel satisfying* (1.32)*,* (1.29)*,* (1.30)*.*

To check that H satisfies (6.41), we note that due to (1.29), we choose α such that

$$-\frac{\alpha}{\rho^{2s}}H(v) + \frac{1}{2}H(v)\int_{\mathbb{R}^d}\left[K_f(v,w)-K_f(w,v)\right]\mathrm{d}w \le -\alpha H + \Lambda_0 H \le 0. \tag{6.51}$$

Then, for ease of notation, we write

$$H(t,x,v) = H_0(t,x,v)e^{\alpha\frac{(\sigma-t)}{\rho^{2s}}},$$

so that if H_0 satisfies

$$\mathcal{T}H_0 + \frac{1}{2}\int_{B_\rho(v)} \Big[H_0(w) - H_0(v)\Big] K_f(w,v)\,\mathrm{d}w \le 0, \tag{6.52}$$

then in particular H satisfies (6.41) due to (6.51). One can verify (6.52) by a case distinction, similar to Lemma 6.1.

Note that the remainder of the proof of Theorem 6.5 and Theorem 6.6 carries over almost verbally, upon replacing Proposition 6.1 and Lemma 6.1 by Proposition 6.2, and Lemma 6.2, respectively. In particular, the Theorem 1.5 follows similarly to Theorem 6.1.

6.4.3 On the exponential lower bound

The only part where we use the divergence form symmetry of the non-local operator in the derivation of the exponential lower bound on the fundamental solution is in Section 6.3.3. Instead, we argue as follows.

Fix $(\tau_0, y_0, w_0) \in [0,T] \times \mathbb{R}^{2d}$ and let $0 < \tau_0 < \sigma < T$. Consider for $\tau_1 < \sigma$ and for some α large enough the function

$$f(\tau_1, y_1, w_1) = M \int_{\max\left\{|x-y_0-(\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}}, |v-w_0|^{2s}\right\} < \alpha(\sigma-\tau_0)} J(\sigma, x, v; \tau_1, y_1, w_1)\,\mathrm{d}x\,\mathrm{d}v,$$

where $M \ge 1$ is some constant such that $f(\tau_1, y_1, w_1) \le M$ for $(\tau_1, y_1, w_1) \in [0,\sigma] \times \mathbb{R}^{2d}$. Define $g(\tau_1, y_1, w_1) := f(\tau_1, y_1, w_1) e^{-c_0(\tau_1 - \sigma)}$ for $\tau_1 < \sigma$. Then g is a super-solution to the adjoint of (1.16) in $(0,\sigma) \times \mathbb{R}^{2d}$, with initial values

$$\begin{cases} g(\sigma, y_1, w_1) = M, & \text{if } \max\left\{|y_1 - y_0 - (\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}}, |w_1 - w_0|^{2s}\right\} < \alpha(\sigma-\tau_0), \\ g(\sigma, y_1, w_1) = 0, & \text{if } \max\left\{|y_1 - y_0 - (\sigma-\tau_0)w_0|^{\frac{2s}{1+2s}}, |w_1 - w_0|^{2s}\right\} > \alpha(\sigma-\tau_0). \end{cases}$$

since

$$\mathcal{L}^* g(v) = \int_{\mathbb{R}^d} \big(g(w) - g(v)\big) K_f(v,w)\,\mathrm{d}w + \left(\int_{\mathbb{R}^d} K_f(w,v) - K_f(v,w)\,\mathrm{d}w\right) g(v),$$

so that using (1.29) and choosing $c_0 > 0$ large enough, we get

$$\mathcal{T}^* g(v) - \mathcal{L}^* g(v) = -e^{-c_0(\tau_1-\sigma)}\mathcal{T}f - e^{-c_0(\tau_1-\sigma)}\mathcal{L}f - \Lambda_0 g + c_0 g \ge 0.$$

If we set

$$\tilde{g}(\tau_1, y_1, w_1) = \begin{cases} g(\tau_1, y_1, w_1), & \text{if } \tau_1 < \sigma, \\ M e^{-c_0(\tau_1-\sigma)}, & \text{if } \tau_1 > \sigma, \end{cases}$$

then $\tilde{g}$ is a non-negative super-solution of the adjoint of (1.16) in $(0,\infty)\times\mathbb{R}^{2d}$ since $\tilde{g}\le Me^{-c_0(\tau_1-\sigma)}$ for $(\tau_1,y_1,w_1)\in[0,\infty)\times\mathbb{R}^{2d}$, so that for $\tau_1>\sigma$

$$\begin{aligned}\mathcal{T}^*\tilde{g}-\mathcal{L}^*\tilde{g}&=c_0Me^{-c_0(\tau_1-\sigma)}-\mathcal{L}\tilde{g}-\Lambda_0\tilde{g}\\&=Me^{-c_0(\tau_1-\sigma)}(c_0-\Lambda_0)+\int\left[Me^{-c_0(\tau_1-\sigma)}-\tilde{g}(w)\right]K_f(v,w)\,\mathrm{d}w\ge 0.\end{aligned}$$

Thus by the weak Harnack inequality [115, Theorem 1.6], we get using $-c_0(\tau_1-\sigma)\ge 0$ for $c_0\ge 0$ and the fact that $\tilde{f}(\sigma,y,w)=M$ for some $\delta>0$ sufficiently small

$$\begin{aligned}\tilde{g}(\tau_0,y_0,w_0)&\ge\left(\int_{\sigma-\delta}^{\sigma+\delta}\int_{\max\left\{|y'-y_0-(\tau'-\tau_0)w_0|^{\frac{2s}{1+2s}},|w'-w_0|^{2s}\right\}<\alpha(\sigma-\tau_0)}\tilde{g}^{\zeta}(\tau',y',w')\,\mathrm{d}y'\,\mathrm{d}w'\,\mathrm{d}\tau'\right)^{\frac{1}{\zeta}}\\&\ge\left(\int_{\sigma-\delta}^{\sigma+\delta}\int_{\max\left\{|y'-y_0-(\tau'-\tau_0)w_0|^{\frac{2s}{1+2s}},|w'-w_0|^{2s}\right\}<\alpha(\sigma-\tau_0)}\tilde{f}^{\zeta}(\tau',y',w')\,\mathrm{d}y'\,\mathrm{d}w'\,\mathrm{d}\tau'\right)^{\frac{1}{\zeta}}\\&\ge c>0,\end{aligned}$$

so that

$$f(\tau_0,y_0,w_0)\ge ce^{c_0(\tau_0-\tau_1)}.$$

We conclude the proof of Theorem 1.6 with [153, Theorem 7.2].

6.5 Appendix: Upper bound in the parabolic case

In this subsection, we discuss how the method outlined above to derive the polynomial decay of Theorem 6.1 can be improved for kernels that satisfy a pointwise upper bound (6.7) in case that we are in the parabolic setting: J is constant in x. The most significant sub-optimality stems from the way we treated the non-singular part in Aronson's method in Proposition 6.1. For pointwisely bouded kernels, instead of (6.15), we obtain:

$$\begin{aligned}&\sup_{t\in(\tau_0,\sigma)}\int_{\mathbb{R}^{2d}}f^2(t,x,v)H(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\\&\qquad\le\int_{\mathbb{R}^{2d}}f^2(\tau_0,x,v)H(\tau_0,x,v)\,\mathrm{d}x\,\mathrm{d}v\\&\qquad\quad+C\rho^{-(d+2s)}\left\|H\right\|_{L^\infty([\tau_0,\sigma]\times\mathbb{R}^{2d})}\left\|f\right\|^2_{L^2_t([\tau_0,\sigma];L^1_v(\mathbb{R}^d))}.\end{aligned}\tag{6.53}$$

This follows by replacing the estimate on the non-singular part of (6.20) by

$$\begin{aligned}\mathcal{I}_{loc}^{ns} &= \int_{B_{2R}}\int_{B_{2R}} \big[f(w)-f(v)\big]\Big[\big(fH\varphi_R^2\big)(v)-\big(fH\varphi_R^2\big)(w)\Big]K(v,w)\chi_{|v-w|>\rho}\,\mathrm{d}w\,\mathrm{d}v\\ &\leq \int_{B_{2R}}\int_{B_{2R}} \Big[f(w)\big(fH\varphi_R^2\big)(v)+f(v)\big(fH\varphi_R^2\big)(w)\Big]K(v,w)\chi_{|v-w|>\rho}\,\mathrm{d}w\,\mathrm{d}v\\ &\leq C\rho^{-2s-d}\,\|H\|_{L^\infty}\int_{B_{2R}} f(w)\,\mathrm{d}w\int_{B_{2R}} f(v)\,\mathrm{d}v.\end{aligned}$$

The construction of H remains the same. Therefore, this improvement leads in the proof of Theorem 6.6 to the replacement of (6.31) by

$$\sup_{t\in(\tau_0,\sigma)}\int_{B^0_{2\rho}(t,w_0)} f^2(t,v)\,\mathrm{d}v \leq \left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{1-\frac{\gamma}{3\rho}}\|f_0\|^2_{L^2(\mathbb{R}^d)}+\rho^{-(d+2s)}(\sigma-\tau_0)\,\|f_0\|^2_{L^1(\mathbb{R}^d)},$$

which subsequently implies an improved estimate in Theorem 6.6:

$$\begin{aligned}|f(\sigma,w_0)|^2 \leq & C(\sigma-\tau_0)^{-\frac{d}{2s}}\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{1-\frac{\gamma}{3\rho}}\|f_0\|^2_{L^2(\mathbb{R}^{2d})}\\ &+\rho^{-(d+2s)}(\sigma-\tau_0)^{1-\frac{d}{2s}}\,\|f_0\|^2_{L^1(\mathbb{R}^d)}.\end{aligned}\tag{6.54}$$

Using (6.54) in the proof of Theorem 6.5, yields the following improved estimate in Theorem 6.5:

$$J(\sigma,v;\tau_0,w_0)\leq C\left((\sigma-\tau_0)^{-\frac{d}{2s}}\left(\frac{\rho^{2s}}{k(\sigma-\tau_0)}\right)^{\frac{1}{4}-\frac{\gamma}{12\rho}}+\rho^{-\frac{d+2s}{4}}(\sigma-\tau_0)^{\frac{2s-3d}{8s}}\right),$$

so that eventually, choosing ρ as in (6.32), we conclude for $(\sigma-\tau_0)\leq\frac{\rho^{2s}}{4}$

$$J(\sigma,v;\tau_0,w_0)\lesssim(\sigma-\tau_0)\,|v-w_0|^{-(d+2s)}+(\sigma-\tau_0)^{\frac{2s-3d}{8s}}\,|v-w_0|^{-\frac{d+2s}{4}}.$$

Note that in the parabolic case, the optimal decay in velocity is $d+2s$, which suggests that our method - which does not rely on Meyer's decomposition - is sub-optimal by a factor of $\frac{1}{4}$.

Part II

Pointwise decay for Boltzmann

CHAPTER 7

Boltzmann on the boundary

This chapter is concerned with the generation of decay estimates in the velocity variable for solutions of the space-inhomogeneous Boltzmann equation without cutoff on a bounded spatial domain for hard and moderately soft potentials. It is based on the article [113] with Cyril Imbert. The result holds for general non-negative suitable weak sub-solutions, provided that mass, energy and entropy density functions are under control. The we treat physically meaningful boundary conditions: in-flow, bounce-back, specular reflection, diffuse reflection and Maxwell reflection. The generation of arbitrary polynomial decay in the velocity variable for the Boltzmann equation without cutoff is new even for classical solutions.

7.1 Introduction

7.1.1 Boltzmann on the boundary

We consider the Boltzmann equation (1.2) posed in bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^d$ (with $d \geq 2$), which we recall here

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f), \qquad t \in (0, T), x \in \Omega, v \in \mathbb{R}^d,$$

where the unknown function $f = f(t, x, v)$ is non-negative; $Q(f, f)$ denotes the Boltzmann collision operator given in (1.3). We are concerned with the case when $\gamma + 2s > 0$ corresponding to so-called moderately soft potentials.

The equation is supplemented with an initial condition $f(0, x, v) = f_{\text{in}}(x, v)$ for $(x, v) \in \Omega \times \mathbb{R}^d$ and a boundary condition. In order to describe the latter, we let n denote the outward unit normal vector on the boundary and we denote by Γ the domain that considers boundary points in space, that is $\Gamma := \partial\Omega \times \mathbb{R}^d$, by Γ_- the incoming part of the boundary,

and by Γ_+ the outcoming part of the boundary, that is

$$\Gamma_- := \{(x,v) \in \Gamma : v \cdot n(x) < 0\},$$

and

$$\Gamma_+ := \{(x,v) \in \Gamma : v \cdot n(x) > 0\}.$$

The Boltzmann equation (1.2) is then supplemented with boundary conditions that are commonly considered in the literature,

(i). In-flow: $f(t,x,v)|_{\Gamma_-} = f_b(t,x,v)$ for a given function f_b.

(ii). Bounce-back: $f(t,x,v) = f(t,x,-v)$ on Γ.

(iii). Specular reflection: $f(t,x,v) = f(t,x,\mathcal{R}_x v)$ with $\mathcal{R}_x v = v - 2(v \cdot n)n$.

(iv). Diffuse reflection: $f(t,x,v)|_{\Gamma_-} = c_\mu \mu(v) \int_{\Gamma_+} f(t,x,v')(v' \cdot n)\,\mathrm{d}v'$, where $\mu(v) = e^{-|v|^2}$ is the Maxwellian, and $c_\mu = c_\mu(x)$ is a normalisation factor, such that $c_\mu \int_{\Gamma_+} \mu(v') v' \cdot n\,\mathrm{d}v' = 1$.

(v). Maxwell reflection: $f(t,x,v)|_{\Gamma_-} = (1-\iota) f(t,x,\mathcal{R}_x v) + \iota c_\mu \mu(v) \int_{\Gamma_+} f(t,x,v')(v' \cdot n)\,\mathrm{d}v'$ where $\iota : \partial\Omega \to [0,1]$ is the accommodation coefficient.

We recall that we work in a conditional regime: we assume that there exist positive constants m_0, M_0, E_0, H_0 such that for almost every $(t,x) \in (0,T) \times \Omega$, the function $f(v) = f(t,x,v)$ satisfies (1.20).

7.1.2 Truncated convex inequalities

In order to present the truncated convex inequalities satisfied by our weak solutions, it is necessary to recall the formulation in (1.16),

$$Q(f,f)(v) = \int_{\mathbb{R}^d} \left[f(v') K_f(v,v') - f(v) K_f(v',v) \right] \mathrm{d}v',$$

where we denote here and in the sequel $f' = f(v')$ and $f = f(v)$. We derive decay estimates by studying the evolution along time of some L^{q_0}-Lebesgue norm of the function $(f-A)_+$. To this end, we consider a general function $\varphi = \varphi(t,x,v,r)$ that is convex in r instead of $\varphi_0((f-A)_+) = (f-A)_+^{q_0}$. For such a φ, we recall the formal computation in (1.38), where we add and subtract

$$\mathcal{D}_\varphi(f,f') := \varphi(t,x,v',f') - \varphi(t,x,v,f) - \partial_r \varphi(t,x,v,f)\,(f'-f)$$

as an integrand, so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(v,f)\,\mathrm{d}v\,\mathrm{d}x = &-\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &+\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\Big[\partial_r\varphi(v,f)f-\varphi(v,f)\\ &\qquad-\partial_r\varphi(v',f')f'+\varphi(v',f')\Big]K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &+\iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(v,f)\,\mathrm{d}v\,\mathrm{d}x-\iint_\Gamma\varphi(f)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x).\end{aligned}$$

If we denote by

$$\Phi(f):=\partial_r\varphi(t,x,v,f)f-\varphi(t,x,v,f),$$

then we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(v,f)\,\mathrm{d}v\,\mathrm{d}x = &-\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &+\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\left[\Phi(f)-\Phi(f')\right]K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &+\iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(v,f)\,\mathrm{d}v\,\mathrm{d}x-\iint_\Gamma\varphi(f)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x).\end{aligned}$$

Due to the cancellation lemma, see (1.21), we are thus lead to,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(v,f)\,\mathrm{d}v\,\mathrm{d}x = &-\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &+c_b\iint_{\mathbb{R}^d\times\Omega}\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x\\ &+\iint_{\mathbb{R}^d\times\Omega}[\partial_t\varphi](v,f)\,\mathrm{d}v\,\mathrm{d}x-\iint_\Gamma\varphi(f)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x).\end{aligned}$$

We can now use the special structure of the function φ that we are going to use. More precisely,

$$\varphi(t,v,r)=\bar\varphi((r-A(t,v))_+)$$

for some convex function $\bar\varphi$ vanishing at 0 together with its derivative $\dot{\bar\varphi}$. Then we compute,

$$\Phi(f)=\bar\Phi((f-A)_+)$$

and

$$\mathcal{D}_\varphi(f,f')=d_{\bar\varphi}((f-A)_+,(f'-A')_+)+\dot{\bar\varphi}((f-A)_+)(A'-f')_+-\dot{\bar\varphi}((f-A)_+)(A'-A),\quad(7.1)$$

where

$$
\begin{aligned}
&d_{\bar\varphi}((f-A)_+,(f'-A')_+)\\
&\quad=\bar\varphi((f'-A')_+)-\bar\varphi((f-A)_+)-\dot{\bar\varphi}((f-A)_+)\big((f'-A')_+-(f-A)_+\big).
\end{aligned}
\tag{7.2}
$$

We remark that the first two terms in $\mathcal{D}_\varphi$ are non-negative, while the third one is an error that we will have to handle.

7.1.3 Suitable weak sub-solutions

In order to define suitable weak solutions, we consider a family of convex functions $\varphi_a(r)=(r-a)_+$ associated with $a>0$. These functions are commonly used in the theory of entropy solutions for scalar conservation laws and are known as *Kruzhkov's semi-entropies*, see for instance [20, 119]. In this case,

$$
d_{\varphi_a}(r,s)=\begin{cases}(s-a)_+ & \text{if } r\le a\\ (a-s)_+ & \text{if } r>a\end{cases}
\tag{7.3}
$$

and $\Phi_a(r)=a\mathbf{1}_{\{f>a\}}$.

The upper bound on the mass density – see condition (1.20) – and the bound on the time-integrated entropy production (see Theorem 7.1) suggest to consider solutions that are L^{q_0} in all variables with $q_0=1+\frac{2s}{d}$ – see Lemma 7.2.

In accordance with the formal computation that we performed above, we introduce the following notion of weak sub-solutions for the inhomogeneous Boltzmann equation.

Definition 7.1 (Suitable weak sub-solutions)**.** Let $T\in(0,+\infty]$ and let Ω be a C^1 domain of $\mathbb{R}^d$. A non-negative function $f\in L^1((0,T)\times\Omega\times\mathbb{R}^d)$ is a *suitable weak sub-solution* of (1.2) if

(i). IN-FLOW: given $f_b\colon(0,T)\times\Gamma_-\to[0,+\infty)$, for any real number $a>0$ and any function $A\colon(0,T)\times\mathbb{R}^d\to(0,+\infty)$ such that $\partial_t A$ and $\partial^2_{v_i,v_j}A$ exist and are bounded

continuous in $(0,T)\times\mathbb{R}^d$, there holds in $\mathcal{D}'((0,T))$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\iint_{\mathbb{R}^d\times\Omega}\varphi_a(f-A)\,\mathrm{d}v\,\mathrm{d}x \\ \quad+\displaystyle\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega} d_{\varphi_a}((f-A)_+,(f'-A')_+)K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathrm{d}x \\ \quad+\displaystyle\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\dot\varphi_a(f-A)(A'-f')_+K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathrm{d}x \\ \quad\le\displaystyle\iint_{\mathbb{R}^d\times\Omega} c_b\Big(\Phi_a(f-A)+A\dot\varphi_a(f-A)\Big)(f*_v|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x \\ \quad-\displaystyle\iint_{\mathbb{R}^d\times\Omega}\dot\varphi_a(f-A)\partial_t A\,\mathrm{d}v\,\mathrm{d}x \\ \quad+\displaystyle\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\dot\varphi_a(f-A)(A'-A)K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathrm{d}x \\ \quad-\displaystyle\iint_{\Gamma_-}\varphi_a(f_b-A)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x) \end{cases} \tag{7.4}$$

with $\varphi_a(r)=(r-a)_+$, $\Phi_a(r)=a\mathbf{1}_{\{f>a\}}$ and d_{φ_a} given by (7.3).

(ii). BOUNCE-BACK: it satisfies the (7.4) with $f_b=0$.

(iii). SPECULAR REFLECTION: it satisfies (7.4) with $f_b=0$.

(iv). DIFFUSE REFLECTION: it satisfies (7.4) with $f_b(t,x,v):=c_\mu\mu(v)\int_{\Gamma_+}f(t,x,v')(v'\cdot n)\,\mathrm{d}v'$.

(v). MAXWELL REFLECTION : it satisfies (7.4) with $f_b(t,x,v):=\iota c_\mu\mu(v)\int_{\Gamma_+}f(t,x,v')(v'\cdot n)\,\mathrm{d}v'$.

Remark 7.1 (General convex functions). The family of inequalities (7.4) is only imposed for elementary (non-decreasing non-negative) convex functions $\varphi_a(r)=(r-a)_+$. But this implies that such inequalities hold true for general Lipschitz convex functions φ such that $\varphi(0)=\dot\varphi(0)=0$ by simply integrating in the parameter a. As already mentioned in [88], a general C^2 convex function φ satisfies,

$$\varphi(r)=\varphi(0)+\dot\varphi(0)r+\int_0^{+\infty}\ddot\varphi(a)(r-a)_+\,\mathrm{d}a,$$

and we can easily check that

$$\Phi(r)=\int_0^{+\infty}\ddot\varphi(a)\Phi_a(r)\,\mathrm{d}a\quad\text{ and }\quad d_\varphi(r,\rho)=\int_0^{+\infty}\ddot\varphi(a)d_{\varphi_a}(r,\rho)\,\mathrm{d}a.$$

We will follow this idea through a truncation procedure (see Section 7.3).

Remark 7.2 (Link with renormalised solutions). R. J. DiPerna and P.-L. Lions [68] constructed weak solutions of the Boltzmann equation in the cutoff case by considering the (Lipschitz) concave non-linear change of variables $\varphi(f)$ with $\varphi(r) = \ln(1+r)$. We consider here (Lipschitz) convex functions φ.

Remark 7.3 (Positive terms). The first positive term in the left hand side of (7.4) corresponds to the classical entropy production term. The second one is reminiscent of the "good extra term" first exhibited by L. Caffarelli, C. H. Chan and A. F. Vasseur in [35] for parabolic equations, and by us in Chapter 3 for kinetic equations. It also plays a crucial role in the recent work by Z. Ouyang and L. Silvestre [169].

Remark 7.4 (Error terms). Right-hand sides of time differential inequalities are well-defined and integrable in time. Indeed, $\dot{\varphi}_a$ and Φ_a are bounded and $f *_v |\cdot|^\gamma$ is integrable in (t,x,v). The term involving the difference $A' - A$ comes from the dependence of the barrier function A on the velocity variable. It did not appear in the work of Q. Ouyang and L. Silvestre [169]. It is reminiscent of the "bad terms" in [121].

7.1.4 Main result

The main result is stated in **Theorem 1.7**: we show the generation of pointwise decay on f at *any polynomial rate* in velocity.

7.1.5 Comments

This work forms part of a large body of literature dealing with velocity decay of solutions to kinetic equations. Before reviewing the literature, we make several comments about our main theorem and its proof.

Conditional decay estimates. The conditional decay estimates that we obtain for the space-inhomogeneous Boltzmann equation in the non-cutoff case are to be compared with the ones obtained by C. Imbert, C. Mouhot and L. Silvestre (2020). They consider classical solutions: using a barrier argument and a maximum principle they show that f decays at a certain algebraic rate in velocity. We extend their result in several directions. First, the Boltzmann equation is posed in a domain supplemented with physically meaningful boundary conditions. Second, we consider weak (sub-)solutions instead of classical ones. But most importantly, we generate *any polynomial velocity decay* even for moderately soft potentials, which was not contained in any previous literature result.

We already mentioned that deriving decay estimates, such as pointwise or moment bounds, is a classical theme in kinetic theory; it is key if the conditional regularity programme by Imbert-Silvestre (see [118] and also [116]) is to be extended to the case of bounded domains.

Conditional boundedness. Our methods of proofs are developed after the article by Z. Ouyang and L. Silvestre [169] about conditional pointwise bounds of weak solutions. They do not establish a velocity decay estimate, but show conditional boundedness of the solutions. Their technique is closer to De Giorgi's truncation argument (revisited for instance by B. Perthame and A. F. Vasseur for evolution equations [170]): they study the evolution of the L^2-norm of the positive part of $(f - a)$ for some time-dependent function $a = a(t)$ along the flow of the equation.

Truncated Convex Inequalities. In this work, we consider solutions with an integrability that stems from the control of the mass density and the entropy production (see Remark 1.4). We also choose the framework of weak solutions satisfying a family of inequalities associated with convex functions. Such weak solutions were recently introduced by C. Imbert, F. Golse and L. Silvestre in the homogeneous case for very soft potentials [88]. It turns out that the classical notion of weak solutions constructed in [186] termed H-solutions fit into our framework. We thus call them suitable weak sub-solutions. We emphasise that we do not use the equation but only the family of inequalities described above. It is reminiscent of the notion of De Giorgi classes from classical elliptic regularity; we refer the interested reader to [140] or also to [146] for a modern presentation.

Coercive terms. In the inequalities (7.4), there are coercive terms, that is to say positive terms on the left hand side; and error terms, corresponding to all terms appearing on the right hand side (some of which are positive, others have *a priori* no sign). Amongst the coercive terms, we distinguish two types:

Some are counterparts to the $\dot{H}^1$-norm in De Giorgi's original article. They ensure that the sub-solution enjoys higher order integrability. These terms were exploited in [88] with techniques developed in [47, 115]. They are non-local in nature, but also linear. They rely on properties of the kernel K that are derived from the hydrodynamical bounds – see Subsection 1.6.2.

The second type of coercive term is what we call the "good extra term", in reference to [35], where an additional "coercive" term was exhibited in the non-local setting. This term was exploited for kinetic equations for the first time in [152], see Chapter 3 and Chapter 5, where the cross terms play a crucial role. This additional term is the most important in the proof. It is a purely non-local effect.

Error terms. There are also two types of error terms:

First, there is the error coming from the dependence of the barrier A on the velocity variable. Even if it resembles certain error terms in [121], here we use a truncation

argument in contrast to the pointwise contact barrier argument employed by [121], and we split the collision operator differently.

Second, all other terms use the cancellation lemma. They are non-linear in nature: they make appear a product with the convolution of f with $|\cdot|^\gamma$.

Generation of pointwise decay. Our main result asserts that, as long as mass, energy and entropy densities are under control, suitable sub-solutions satisfy a polynomial decay in the velocity variable of any order q. The result is first proved for some exponent q_{nsl} (for not-so-large), by using the estimates from below for coercive terms and from above for error terms. We then use this first decay to improve the exponent q, measuring the polynomial decay rate, iteratively. This iteration procedure is the main novelty of our article.

Main ideas. The strategy is as follows. We start with exploiting the coercivity of the signed terms that stems from the boundedness of the macroscopic quantities in the conditional regime (1.20). As explained above, this yields two different terms: the first is linear and can be associated to the energy from the equation; the second is the "good extra term", which occurs from the non-locality of the collision operator (1.3). We then bound the error terms, that is all the terms on the right hand side of (7.4), by these two types of coercive terms. This is done mainly through interpolation, weighted Hölder estimates, and a careful study of the non-linearity of (1.3). This results in the monotonicity of the L^{q_0}-norm of $(f-A)_+$, where q_0 is the integrability of our weak sub-solutions, and where $A = A(t,v) = a(t)\langle v\rangle^{-q}$ encodes the temporal and velocity decay. In this first step, we have to impose the restriction $q \in [0, q_{\mathrm{nsl}}]$ with q_{nsl} given in (vii). This extends the result of [121] to weak solutions on the boundary. But in a next step, we exploit this generation of moments in order to improve the "good extra term" to a "very good extra term": we can improve the coercivity of this second term which in turn loosens the restriction on the decay exponent q. We roughly get a pointwise decay for $q \approx q_{\mathrm{nsl}} + 2s$. This process can be iteratively continued with the end result stated in Theorem 1.7. Showing the monotonicity of a Lebesgue norm on a level set function $(f-A)_+$ is reminiscent of De Giorgi's method, but the usual iteration argument of De Giorgi is replaced by solving an ODE on the temporal decay.

7.1.6 Review of literature

The study of velocity moments of solutions to kinetic equations plays a central role since Boltzmann's (and Landau's) collision operator integrates the velocity variable in the open space $\mathbb{R}^d$. It has a long history, mostly in the space-homogeneous setting that goes back to

the article by T. Carleman [42]. It is precisely described in the introduction of [121]. Let us give references, summarise the review of literature from [121] and review the literature written since then. The reader is also referred to references contained in the articles that are quoted in this subsection.

VELOCITY MOMENTS. We first review contributions to the study of moments of solutions in the velocity variable. Maxwell potentials are treated in [112, 182]. In the case of hard potentials and angular cutoff, Povzner's inequalities [173] are commonly used: see works by T. Elmroth [71], L. Desvillettes [57], S. Mischler and B. Wennberg [163], X. Lu [155] and B. Wennberg (in the non-cutoff case) [191]. A. Bobylev considered exponential moments in [26], see also [76], and in particular the work [5] in this direction, where they establish the creation and the propagation of exponential moments to the spatially homogeneous Boltzmann equation for hard potentials. The case of moderately soft potentials and angular cutoff is addressed by L. Desvillettes [57], see also [59, 188] in this direction. We finally mention the work by M. Gualdani, S. Mischler and C. Mouhot [95] that focusses on hard spheres and makes assumptions on hydrodynamical quantities that are similar to what is assumed in this work. More recently, C. Cao, L. B. He and J. Ji [41] studied the propagation of exponential moments in L^2 for (very) soft potentials in a perturbative regime.

POINTWISE DECAY. There are fewer results about pointwise decay. It starts with works by T. Carleman [42, 43] and extended by L. Arkeryd [11]. Exponential pointwise upper bounds were obtained in [76], see also [77].

More recently, the work by C. Imbert, C. Mouhot and L. Silvestre [121] addressed the question of appearance and propagation of polynomial decay in the velocity variable under condition (1.20) on hydrodynamical quantities. In this vein, S. Cameron and S. Snelson [37] established similar results in the case $\gamma+2s>2$. The study of polynomial decay is also central in [108] in which the authors are able to deal with very soft potentials ($\gamma+2s<0$). See also [107] for results dealing with the Landau equation. In [104], C. Henderson and W. Wang are interested in very soft potentials and short time existence. They work in the class of solutions with polynomial decay.

Open Problems 7.1. 1. Solutions of the inhomogeneous Boltzmann equation converge to Maxwellians for large times [61]. We recall that under the condition (1.20), C. Imbert, C. Mouhot and L. Silvestre proved in [120] that solutions stay above a Maxwellian. It is thus natural to ask ourselves if they can be bounded from above by another Maxwellian. Unfortunately, our proof does not yield neither such an (optimal) upper bound nor any exponential decay (in v). The first open question is thus to show the propagation of pointwise Gaussian bounds.

2. Another natural open question is the possibility of bridging the gap between generating *any* polynomial pointwise bound to the generation of pointwise Gaussian bounds. Is it possible to show that solutions to the non-cutoff Boltzmann equation with moderately soft potentials generate Gaussian bounds of the form $e^{-a(t)\langle v\rangle^{\gamma+2s}}$ for some function $a = a(t)$?

3. The critical cases $\gamma + 2s = 0$ and $\gamma + 2s = 1$ are left open. Finally, the case of very soft potentials is also another natural open question.

7.1.7 Notation

For $p \in [1, \infty]$ and $k \in \mathbb{R}$, the weighted Lebesgue space L^p_k is given by

$$L^p_k(\mathbb{R}^d) := \left\{ f \colon \mathbb{R}^d \to \mathbb{R} \ \text{ s.t. } \int_{\mathbb{R}^d} f^p(v)\langle v\rangle^{kp} \, \mathrm{d}v < +\infty \right\}, \tag{7.5}$$

where $\langle v\rangle := (1 + |v|^2)^{\frac{1}{2}}$. When $k = 0$, we simply write L^p.

7.1.8 Constants

We gather here parameters and constants that are used repeatedly in statements and proofs.

(i). The dimension of the x and v variables is denoted by d. It is always larger than or equal to 2.

(ii). The parameters γ and s from the kernel satisfy: $\gamma \in (-d, 1]$ and $s \in (0, 1)$ and $0 < \gamma + 2s < 1$.

(iii). The Lebesgue exponent $p_0 > 1$,

$$\frac{1}{p_0} = 1 - \frac{2s}{d}$$

comes from the entropy production estimate.

(iv). The Lebesgue exponent $q_0 > 1$,

$$q_0 = 1 + \frac{2s}{d}$$

is related to integrability of solutions.

(v). The moment exponent

$$k_0 = (\gamma + 2s) - \frac{2s}{d}$$

appears in the entropy production estimate.

(vi). To any decay exponent $q \geq 1$ is associated a moment exponent

$$l_q = (\gamma + 2s) + \frac{2s}{d} - q\left(1 + \frac{2s}{d}\right).$$

(vii). The decay exponent $q_{\rm nsl}$ (for not-so-large) is given by

$$q_{\rm nsl} = \begin{cases} d + 1 + \frac{d}{2s}\min(\gamma, 0), & \text{if } -\frac{2sd}{d+2s} \leq \gamma \leq 1, \\ d + 1 - \frac{d}{d+2s}(2 - \gamma), & \text{if } -2s < \gamma < -\frac{2sd}{d+2s}. \end{cases}$$

It corresponds to the first decay that is generated by coercivity and error estimates.

7.2 Preliminaries

We gather in this section known results and technical lemmas that will be used in next sections.

7.2.1 Integrability of solutions

7.2.1.1 Weighted Hölder's inequality

The following lemma is a consequence of Hölder's inequality and a proof is given in [88].

Lemma 7.1 (Hölder's inequality with weights)**.** *Let $p, q, r \in [1, +\infty]$ and $p \leq r \leq q$. Then,*

$$\|f\|_{L^r_{k_r}} \leq \|f\|^{\alpha}_{L^p_{k_p}} \|f\|^{1-\alpha}_{L^q_{k_q}}$$

with $\alpha \in (0,1)$ such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ and $k_r = \alpha k_p + (1-\alpha)k_q$.

We can use the previous lemma to derive the following one.

Lemma 7.2 (Integrability of solutions)**.** *If $f \in L^\infty((0,T)\times\bar{\Omega}; L^1_2(\mathbb{R}^d))$ and $f \in L^1((0,T)\times\bar{\Omega}; L^{p_0}_{k_0}(\mathbb{R}^d))$ with k_0 given by* (v)*, then $f \in L^{q_0}_{k_0^*}((0,T)\times\bar{\Omega}\times\mathbb{R}^d)$, where q_0 and k_0^* are given by*

$$q_0 = 2 - \frac{1}{p_0} = 1 + \frac{2s}{d} \in (1,2), \qquad k_0^* = \frac{2s + d(\gamma + 2s)}{d + 2s} > 0.$$

7.2.1.2 Entropy production estimate

The entropy production of a function f is given by the following formula,

$$\begin{aligned} D(f) &= -\langle Q(f,f), \ln f\rangle \\ &= \frac{1}{2}\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} [f(v_*)f(v) - f(v'_*)f(v')] \ln\frac{f(v_*)f(v)}{f(v'_*)f(v')} B(|v-v_*|,\cos\theta)\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v'. \end{aligned} \tag{7.6}$$

Theorem 7.1 (Entropy production estimate – [47, Theorem 0.1])**.** *Let $\gamma \in (-d, 1]$ and $s \in (0,1)$. Then, for any non-negative solution f of (1.2) verifying the bounds (1.20), the entropy production $D(f)$ satisfies*

$$\|f\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} \leq C_0 D(f),$$

where p_0 is such that $\frac{1}{p_0} = 1 - \frac{2s}{d}$ and $k_0 = \gamma + 2s - \frac{2s}{d}$. The constant C_0 only depends on mass, energy and entropy of f.

In particular, as a consequence of the entropy production estimate and Lemma 7.2, we note that for any solution of (1.2) satisfying the hydrodynamical condition (1.20) there holds $f \in L^{q_0}_{k_0^*}((0,T)\times\bar{\Omega}\times\mathbb{R}^d)$.

7.2.2 Interpolation estimates

This subsection is devoted to the proofs of interpolation estimates that will be used in the proof of the main theorem.

Hölder's inequality with weights (see Lemma 7.1) applied to $p = r = q = 1$ yields the following result.

Lemma 7.3 (First weighted L^1 estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have a finite 2-moment. Then there holds,*

$$\forall k \in \mathbb{R},\ \forall \alpha_0 \in (0,1), \qquad \int_{\mathbb{R}^d} f(v)\langle v\rangle^k\,\mathrm{d}v \leq E_0^{\alpha_0}\left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{\frac{k-2\alpha_0}{1-\alpha_0}}\,\mathrm{d}v\right)^{1-\alpha_0}.$$

Second, we use that $q_0 \in (1,2)$ to get $1 \in [q_0 - 1, q_0]$ and interpolate $f \in L^1$ between $f \in L^{q_0}$ and $f \in L^{q_0-1}$ (with weights). Since $q_0 - 1 < 1$, the interpolation is applied to $f^{(q_0-1)^{-1}}$.

Lemma 7.4 (Second weighted L^1 estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have a finite 2-moment. Let $q_0 = \frac{d+2s}{d} \in (1,2)$, $\frac{1}{p_0} = 1 - \frac{2s}{d}$, $k_0 = \gamma + 2s - \frac{2s}{d}$. Then there holds,*

$$\forall k_2 \in \mathbb{R}, \qquad \int_{\mathbb{R}^d} f(v)\langle v\rangle^{k_2}\,\mathrm{d}v \leq \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\frac{d(d-2s)}{4s^2+d^2}}\left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{\frac{4sd}{4s^2+d^2}},$$

with

$$m_2 = \frac{4s^2+d^2}{4sd}\left[k_2 - \frac{d(d-2s)}{4s^2+d^2}k_0\right].$$

Proof. We interpolate

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{k_2}\,\mathrm{d}v \le \left(\int_{\mathbb{R}^d} f^{q_0p_0}(v)\langle v\rangle^{k_0p_0}\,\mathrm{d}v\right)^{\frac{\alpha_3}{p_0}}\left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{\alpha_4},$$

where we need α_3, α_4 and m_2 to satisfy,

$$\frac{\alpha_3}{p_0} + \alpha_4 = 1, \qquad \alpha_3 q_0 + \alpha_4(q_0-1) = 1, \qquad \alpha_3 k_0 + \alpha_4 m_2 = k_2.$$

This yields

$$\alpha_3 = \frac{p(2-q_0)}{q_0p_0+1-q_0} = \frac{d(d-2s)}{4s^2+d^2}, \qquad \alpha_4 = \frac{q_0p_0-1}{q_0p_0+1-q_0} = \frac{4sd}{4s^2+d^2},$$
$$m_2 = \frac{1}{\alpha_4}(k_2-\alpha_3k_0) = \frac{4s^2+d^2}{4sd}\left[k_2 - \frac{d(d-2s)}{4s^2+d^2}\left(\gamma+2s-\frac{2s}{d}\right)\right]. \qquad \square$$

We can then apply Lemma 7.1 with $p=1$ and $q = p_0q_0$.

Lemma 7.5 (General weighted L^{p_1} estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$. Then for all $p_1 \in [1, p_0q_0]$ and all $k_1 \in \mathbb{R}$,*

$$\|f\|_{L^{p_1}_{k_1/p_1}} \le \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\frac{d(p_1-1)}{p_1 4s}}\left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1}\,\mathrm{d}v\right)^{\frac{d+2s-p_1(d-2s)}{p_1 4s}},$$

with

$$m_1 = \frac{4s}{d+2s-p_1(d-2s)}\left[k_1 - \frac{d(p_1-1)}{4s}k_0\right].$$

Proof. We interpolate

$$\int_{\mathbb{R}^d} f^{p_1}(v)\langle v\rangle^{k_1}\,\mathrm{d}v \le \left(\int_{\mathbb{R}^d} f^{q_0p_0}(v)\langle v\rangle^{k_0p_0}\,\mathrm{d}v\right)^{\frac{\alpha_1}{p_0}}\left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1}\,\mathrm{d}v\right)^{\alpha_2},$$

where we need α_1, α_2 and m_1 to satisfy,

$$\frac{\alpha_1}{p_0} + \alpha_2 = 1, \qquad \alpha_1 q_0 + \alpha_2 = p_1, \qquad \alpha_1 k_0 + \alpha_2 m_1 = k_1.$$

This yields

$$\alpha_1 = \frac{(p_1-1)p_0}{q_0p_0-1} = \frac{d(p_1-1)}{4s}, \qquad \alpha_2 = \frac{q_0p_0-p_1}{q_0p_0-1} = \frac{d+2s-p_1(d-2s)}{4s},$$
$$m_1 = \frac{1}{\alpha_2}(k_1-\alpha_1k_0) = \frac{4s}{d+2s-p_1(d-2s)}\left[k_1 - \frac{d(p_1-1)}{4s}\left(\gamma+2s-\frac{2s}{d}\right)\right]. \qquad \square$$

Finally, as a consequence of Lemmas 7.3, 7.4 and 7.5, there holds the following estimate.

Lemma 7.6 (Weighted L^{q_0} estimate)**.** *Let $f : \mathbb{R}^d \to [0, \infty)$ have a finite 2-moment. Then there holds for any $k_3 \in \mathbb{R}$ and any $\varepsilon \in (0, 1)$*

$$\int_{\mathbb{R}^d} f^{q_0}(v)\langle v\rangle^{k_3} \,\mathrm{d}v \leq \varepsilon \left\| f^{q_0} \right\|_{L^{p_0}_{k_0}} + C(\varepsilon, E_0) \left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_3} \,\mathrm{d}v \right),$$

with

$$m_3 = \left(1 + \frac{d}{2s}\right) k_3 - \frac{d}{2s} k_0 - 2,$$

and $C(\varepsilon, E_0)$ only depends on E_0, ε, s and d.

Proof. We first interpolate with Lemma 7.5 for $p_1 = q_0$ and $k_1 = k_3$,

$$\int_{\mathbb{R}^d} f^{q_0}(v)\langle v\rangle^{k_3} \,\mathrm{d}v \leq \left\| f^{q_0} \right\|^{\alpha_1}_{L^{p_0}_{k_0}} \left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1} \,\mathrm{d}v \right)^{\alpha_2},$$

where α_1, α_2, m_1 are given by

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{d + 2s}{2d}, \qquad m_1 = \frac{2d}{d + 2s} \left[k_3 - \frac{1}{2} \left(\gamma + 2s - \frac{2s}{d} \right) \right].$$

Then apply Lemma 7.3 for any $\alpha_0 \in (0, 1)$ and with $k = m_1$.

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1} \,\mathrm{d}v \leq E_0^{\alpha_0} \left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{\frac{m_1 - 2\alpha_0}{1-\alpha_0}} \,\mathrm{d}v \right)^{1-\alpha_0}.$$

Next we interpolate with Lemma 7.4 for $k_2 = \frac{m_1 - 2\alpha_0}{1-\alpha_0}$,

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{\frac{m_1 - 2\alpha_0}{1-\alpha_0}} \,\mathrm{d}v \leq \left\| f^{q_0} \right\|^{\alpha_3}_{L^{p_0}_{k_0}} \left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2} \,\mathrm{d}v \right)^{\alpha_4},$$

where

$$\alpha_3 = \frac{d(d - 2s)}{4s^2 + d^2}, \qquad \alpha_4 = \frac{4sd}{4s^2 + d^2},$$
$$m_2 = \frac{4s^2 + d^2}{4sd} \left[\frac{m_1 - 2\alpha_0}{1 - \alpha_0} - \frac{d(d - 2s)}{4s^2 + d^2} \left(\gamma + 2s - \frac{2s}{d} \right) \right].$$

Finally we combine the three previous inequalities

$$\int f^{q_0}(v)\langle v\rangle^{k_3} \,\mathrm{d}v \leq E_0^{\alpha_2\alpha_0} \left\| f^{q_0} \right\|^{\alpha_1 + (1-\alpha_0)\alpha_2\alpha_3}_{L^{p_0}_{k_0}} \left(\int f^{q_0-1}\langle v\rangle^{m_2} \,\mathrm{d}v \right)^{(1-\alpha_0)\alpha_2\alpha_4}$$

we use Young's inequality for some $\theta > 1$ and for any $\varepsilon \in (0,1)$,

$$\leq \varepsilon \, \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\theta(\alpha_1+(1-\alpha_0)\alpha_2\alpha_3)} + C(\varepsilon, E_0) \left(\int_{\mathbb{R}^d} f^{q_0-1}(v) \langle v \rangle^{m_2} \, \mathrm{d}v \right)^{\frac{\theta(1-\alpha_0)\alpha_2\alpha_4}{\theta-1}}$$

with $C(\varepsilon, E_0) = E_0^{\frac{\alpha_2\alpha_0\theta}{\theta-1}} \varepsilon^{-\frac{1}{\theta-1}}$. We choose θ and α_0 such that

$$\theta(\alpha_1 + (1-\alpha_0)\alpha_2\alpha_3) = 1, \qquad \frac{\theta}{\theta-1}(1-\alpha_0)\alpha_2\alpha_4 = 1.$$

Then we showed

$$\int_{\mathbb{R}^d} f^{q_0}(v) \langle v \rangle^{k_3} \, \mathrm{d}v \leq \varepsilon \, \|f^{q_0}\|_{L^{p_0}_{k_0}} + C(\varepsilon, E_0) \left(\int f^{q_0-1} \langle v \rangle^{m_2} \, \mathrm{d}v \right).$$

Note that we find

$$\theta = \frac{\alpha_3 + \alpha_4}{\alpha_3 + \alpha_1\alpha_4} = \frac{d+2s}{d}, \qquad \alpha_0 = \frac{\alpha_1 - 1 + \alpha_2(\alpha_3+\alpha_4)}{\alpha_2(\alpha_3+\alpha_4)} = \frac{4sd}{(d+2s)^2},$$

(in particular $\theta > 1$), so that

$$m_3 = m_2 = \frac{(d+2s)k_0 - d\left(\gamma + 2s - \frac{2s}{d}\right)}{2s} - 2.$$

This yields the result for $\iota = 1$. To get it for any ι, apply the result to ε/ι and recall the definition of $C(\varepsilon, E_0)$. We get $\iota^{1+\frac{1}{\theta-1}}$ (with $\theta - 1 = \frac{2s}{d}$) in front of the second term. □

7.3 Truncated Lebesgue norms

In order to get the decay estimates from Theorem 1.7, we aim at proving that the following quantity equals zero,

$$\iint_{\mathbb{R}^d \times \Omega} (f(t,x,v) - A(t,v))_+^{q_0} \, \mathrm{d}v \, \mathrm{d}x$$

with $A(t,v) = \mathfrak{a}(t)\langle v \rangle^{-q}$ for some well chosen C^1 function $\mathfrak{a} > 0$ and some exponent $q > 0$. We recall that $q_0 = 1 + \frac{2s}{d}$ (see Lemma 7.2) and that r_+ denotes $\max(0, r)$ for any real number r. In order to prove that this functional vanishes when applied to a solution of the space-inhomogeneous Boltzmann equation, we investigate how it evolves with time.

7.3.1 Truncated convex inequalities

Keeping in mind that $q_0 > 1$, we thus consider the convex function $\varphi_0(r) = r_+^{q_0}$ and we would like to apply the definition of suitable weak sub-solutions – see (7.4) – and integrate

against $\ddot{\varphi}_0$ (see Remark 7.1). Unfortunately, we are not sure that the right hand side of (7.4) is integrable with respect to a against $\ddot{\varphi}_0(a)$. For this reason, we follow [88] and approximate $\varphi_0(r)$ with $\varphi_{0,\kappa}(r)$ defined for $\kappa > 0$ by

$$\varphi_{0,\kappa}(r) = \big(r_+ \wedge \kappa\big)^{q_0} + q_0 \kappa^{q_0 - 1}(r - \kappa)_+. \tag{7.7}$$

In particular $\varphi_{0,\kappa}(0) = (\dot{\varphi_{0,\kappa}})(0) = 0$. We can now apply the definition of suitable weak sub-solutions and integrate with respect to $\ddot{\varphi_{0,\kappa}}(a)$ for $a \in [0, \kappa]$. What we obtain is expressed in the following statement.

Lemma 7.7 (Truncated Convex Inequalities)**.** *Let f be a suitable weak sub-solution of the Boltzmann equation with either in-flow, bounce-back, specular/ diffuse / Maxwell reflection boundary condition. Then,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega\times\mathbb{R}^d} \varphi_{0,\kappa}\big(f - A\big)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + \iiint_{\Omega\times\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}((f-A)_+, (f'-A')_+) K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + \iiiint_{\Omega\times\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\
&\quad \le c_b \iint_{\Omega\times\mathbb{R}^d} \Phi\big(f\big)\,(f * |\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + \iiint_{\Omega\times\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-A)\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad - \iint_{\Omega\times\mathbb{R}^d} \dot{\varphi_{0,\kappa}}(f-A)\partial_t A\,\mathrm{d}v\,\mathrm{d}x - \iint_{\partial\Omega\times\mathbb{R}^d} \varphi_{0,\kappa}(f-A)(v\cdot n(x))\,\mathrm{d}v\,\mathrm{d}S(x),
\end{aligned} \tag{7.8}$$

with Φ given by,

$$\Phi(f) := \dot{\varphi_{0,\kappa}}(f-A)f - \varphi_{0,\kappa}(f-A) = \Phi_{0,\kappa}(f-A) + A\dot{\varphi_{0,\kappa}}(f-A). \tag{7.9}$$

7.3.2 Useful properties associated with $\varphi_{0,\kappa}$

The derivatives of the approximate convex function $\varphi_{0,\kappa}$ are given by the following formulas,

$$\dot{\varphi_{0,\kappa}}(r) = q_0\big(r_+ \wedge \kappa\big)^{q_0-1} \quad \text{and} \quad \ddot{\varphi_{0,\kappa}}(r) = q_0(q_0-1)r^{q_0-2}\mathbf{1}_{\{0\le r\le\kappa\}}.$$

Moreover, we compute for $r \ge 0$,

$$\Phi_{0,\kappa}(r) = (\dot{\varphi_{0,\kappa}})(r)r - \varphi_{0,\kappa}(r) = (q_0-1)\big(r\wedge\kappa\big)^{q_0},$$

and for $r, s \ge 0$,

$$d_{\varphi_{0,\kappa}}(r,s) = \varphi_{0,\kappa}(s) - \varphi_{0,\kappa}(r) - \dot{\varphi_{0,\kappa}}(r)(s-r), \tag{7.10}$$

so that, if $0 \le r, s \le \kappa$,

$$d_{\varphi_{0,\kappa}}(r,s) = s^{q_0} - r^{q_0} - q_0 r^{q_0-1}(s-r).$$

Note that if $r \le k$, then for $q_0 > 1$

$$d_{\varphi_{0,\kappa}}\Big(r, \frac{r}{2}\Big) = c_{q_0} r^{q_0} \quad \text{with} \quad c_{q_0} = 2^{-q_0} - 1 + \frac{q_0}{2} > 0.$$

In particular,

$$d_{\varphi_0}(r,s) \ge c_{q_0} r^{q_0} \quad \text{in} \quad \{0 \le s < r/2\} \tag{7.11}$$

with $c_{q_0} = d_{\varphi_0}(1, 1/2) = 2^{-q_0} - 1 + q_0/2 > 0$. We also have for $r \ge 0$,

$$d_{\varphi_{0,\kappa}}\Big(r, \frac{r \wedge \kappa}{2}\Big) = c_{q_0}\Big(r \wedge \kappa\Big)^{q_0}. \tag{7.12}$$

7.4 Coercivity estimates

This section is devoted to the study of positive terms appearing in the left hand side of the family of convex inequalities associated with the approximate L^{q_0}-norm, see (7.8). The term in which $(A' - f')_+$ appears is referred to as the "good extra term" (see the introduction).

7.4.1 First coercivity estimate

In order to get our first coercivity estimate, we follow the method introduced in [47] and used in [169] and [88]. Precisely, the proof of the following proposition is adapted from the proof of [88, Lemma 3.1] in which $k_0 < 0$. We recall that $k_0 = (\gamma + 2s) - \frac{2s}{d}$.

Proposition 7.1 (Coercivity estimate for $k_0 \ge 0$)**.** *Assume f satisfies* (1.20)*. If $k_0 \ge 0$, then there exists a constant $C_{\mathrm{cor}} > 0$, depending on d, γ, s and constants in* (1.20)*, such that for all $f, g \colon \mathbb{R}^d \to [0, +\infty)$,*

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g') K_f(v, v') \, \mathrm{d}v \, \mathrm{d}v' \ge C_{\mathrm{cor}} \left\| \big(g \wedge \kappa\big)^{q_0} \right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}. \tag{7.13}$$

Proof. For $R > 0$ (to be chosen), we define

$$T_R = \begin{cases} B_{R/2}(v) & \text{if } |v| \ge R \\ B_{3R}(v) \setminus B_{2R}(v) & \text{if } |v| \le R. \end{cases}$$

In particular, for $v' - v \in T_R$, we have $|v| \le 2|v'|$, which implies $\langle v \rangle \le 2\langle v' \rangle$.

Then we use the non-degeneracy cone $\Xi(v)$ associated with the kernel K_f and get from (1.23) and (7.11)

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\quad\geq c_{q_0}\int_{\mathbb{R}^d}\int_{\left\{g'<\frac{g\wedge\kappa}{2}\right\}\cap\{v'-v\in\Xi(v)\cap T_R\}} \big(g\wedge\kappa\big)^{q_0}(v)K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v \\
&\quad\geq c\int_{\mathbb{R}^d}\int_{\{g'<g/2\}\cap\{v'-v\in\Xi(v)\cap T_R\}} \big(g\wedge\kappa\big)^{q_0}(v)\langle v\rangle^{\gamma+2s+1}\,|v-v'|^{-d-2s}\,\mathrm{d}v'\,\mathrm{d}v \\
&\quad\geq c\int_{\mathbb{R}^d} \big(g\wedge\kappa\big)^{q_0}(v)R^{-d-2s}\langle v\rangle^{\gamma+2s+1}\left|\left\{g'<\frac{g\wedge\kappa}{2}\right\}\cap\{v+\Xi(v)\cap T_R\}\right|\,\mathrm{d}v,
\end{aligned}$$

where $c=c_{q_0}\lambda_0$. Moreover, for any $v\in\mathbb{R}^d$, we use either (1.25) or (1.24) and Chebyshev's inequality to estimate the Lebesgue norm of the sub-level set as follows,

$$\begin{aligned}
&\left|\left\{g'<\frac{g\wedge\kappa}{2}\right\}\cap\{v+\Xi(v)\cap T_R\}\right| \\
&\quad=|\{v+\Xi(v)\cap T_R\}|-\left|\left\{g'>\frac{g\wedge\kappa}{2}\right\}\cap\{v+\Xi(v)\cap T_R\}\right| \\
&\quad\geq c_0R^d\langle v\rangle^{-1}-\frac{2^{p_0(q_0-k_0)}}{\langle v\rangle^{k_0p_0}\big(g\wedge\kappa\big)^{q_0p_0}(v)}\int_{\mathbb{R}^d}(g'\wedge\kappa)^{q_0p_0}\langle v'\rangle^{k_0p_0}\,\mathrm{d}v'.
\end{aligned}$$

The last line uses that $k_0\geq 0$. Then, we choose R such that

$$R^d=c_1\,\|g^{q_0}\|_{L^{p_0}_{k_0}}^{p_0}\,\langle v\rangle^{-k_0p_0+1}\big(g\wedge\kappa\big)^{-q_0p_0}(v), \tag{7.14}$$

with $c_1=\frac{2^{p_0(q_0-k_0)+1}}{c_0}$ and get

$$\left|\left\{g'<\frac{g\wedge\kappa}{2}\right\}\cap\{v+\Xi(v)\cap T_R\}\right|\geq c_0R^d\langle v\rangle^{-1}-\frac{c_0}{2}R^d\langle v\rangle^{-1}=\frac{c_0}{2}R^d\langle v\rangle^{-1}.$$

Therefore, with such a choice of R,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\quad\geq c\frac{c_0}{2}\int_{\mathbb{R}^d}\big(g\wedge\kappa\big)^{q_0}(v)R^{-2s}\langle v\rangle^{\gamma+2s}\,\mathrm{d}v \\
&\quad= c\frac{c_0}{2}c_1^{-\frac{2s}{d}}\,\|g^{q_0}\|_{L^{p_0}_{k_0}}^{-\frac{2sp_0}{d}}\int_{\mathbb{R}^d}\langle v\rangle^{\gamma+2s+\frac{2s(k_0p_0-1)}{d}}\big(g\wedge\kappa\big)^{q_0\left(1+\frac{2sp_0}{d}\right)}(v)\,\mathrm{d}v,
\end{aligned}$$

and remarking that p_0 and k_0 are such that $p_0=1+\frac{2sp_0}{d}$ and $k_0p_0=\gamma+2s+\frac{2s(k_0p_0-1)}{d}$,

$$\geq c\frac{c_0}{2}c_1^{-\frac{2s}{d}}\,\|g^{q_0}\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}^{1-p_0}\,\big\|\big(g\wedge\kappa\big)^{q_0}\big\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}^{p_0}.$$

We reach the desired conclusion with $C_{cor} = c\frac{c_0}{2}c_1^{-\frac{2s}{d}}$. □

Proposition 7.2 (Coercivity estimate for $k_0 < 0$)**.** *Assume $k_0 < 0$. Then there exists $C_{\mathrm{cor}} > 0$, depending on s, d, M_0, E_0, such that for all $f, g\colon \mathbb{R}^d \to [0, +\infty)$,*

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g')K_f(v, v')\,\mathrm{d}v\,\mathrm{d}v' \\ \geq C_{\mathrm{cor}} \left\|\left(g \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} - C_{\mathrm{cor}} \int_{\mathbb{R}^d} (g \wedge \kappa)^{q_0}\langle v\rangle^{\gamma}\,\mathrm{d}v.\end{aligned} \tag{7.15}$$

In particular,

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g')K_f(v, v')\,\mathrm{d}v\,\mathrm{d}v' \\ \geq C_{\mathrm{cor}} \left\|\left(g \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} - C_{\mathrm{cor}} \left\|\left(g \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{\gamma-d-1}(\mathbb{R}^d)}.\end{aligned} \tag{7.16}$$

Proof. The estimate (7.15) follows the same lines as the proof of Proposition 7.1 and is proved in [88, Lemma 3.1].

We can obtain (7.16) by applying Lemma 7.6 to $g \wedge \kappa$ and $\varphi = 1$ and $k_3 = \gamma$ and get for all $\varepsilon \in (0, 1)$,

$$\int_{\mathbb{R}^d} \left(g \wedge \kappa\right)^{q_0}\langle v\rangle^{\gamma}\,\mathrm{d}v \leq \varepsilon \left\|\left(g \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} + C(\varepsilon, E_0) \left\|\left(g \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{m_3}}$$

with $m_3 = \gamma - d - 1$. □

We now prove another coercivity estimate for $\gamma < 0$.

Proposition 7.3 (Second coercivity estimate)**.** *There exists two positive constants $C_{\mathrm{cor}'}$ and κ_0, only depending on s, d, γ and hydrodynamical bounds from* (1.20)*, such that for all $\kappa \geq \kappa_0$ and all $f, g\colon \mathbb{R}^{2d} \to [0, +\infty)$,*

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g')K_f(v, v')\,\mathrm{d}v\,\mathrm{d}v' \geq C_{\mathrm{cor}'}\,\kappa^{\frac{2s}{d}q_0}N^{-\frac{2s}{d}} \int_{\mathbb{R}^d} (g(v) - \kappa)_+\langle v\rangle^{\gamma}\,\mathrm{d}v. \tag{7.17}$$

with $N = \int_{\mathbb{R}^d}(g \wedge \kappa)^{q_0-1}\langle v\rangle^{-(d-1)}\,\mathrm{d}v$.

Proof. We explain how to adapt the proof of [88, Lemma 3.2]. The only difference lies in the choice of R. In our case, we choose $R = R(v')$ such that

$$\frac{R^d}{\langle v'\rangle} = C_0\frac{N}{\kappa^{q_0-1}\langle v'\rangle^{-(d-1)}}.$$

We can argue as in [88] by remarking that $N \leq \kappa^{q_0-2}M_0$. We thus choose C_0 and κ_0 such

that

$$C_0 \frac{M_0}{\varepsilon_0^d \kappa_0} \leq 1$$

to ensure that $R \leq \varepsilon_0 \langle v' \rangle$, which allows us to use [88, Lemma 2.6]. The idea is to consider the cone of non-degeneracy for fixed v'

$$S(v') = \left\{ v \in \mathbb{R}^d : K(v, v') \geq \mu_0 \langle v' \rangle^{\gamma+2s+1} |v - v'|^{-(d+2s)} \right\}.$$

Then Lemma 2.6 in [88] proves the existence of $\mu_0 > 0$ depending only on the hydrodynamical quantities such that

$$|S(v') \cap B_R(v')| \geq \mu_0 R^d \langle v \rangle^{-1}.$$

The weight in N implies that the estimate of the sub-level set is modified as follows:

$$\begin{aligned} |\{g \wedge \kappa > \kappa/2\} \cap S(v') \cap B_R(v')| &\leq \frac{2^{q_0-1}}{\kappa^{q_0-1}} \int_{S(v') \cap B_R(v')} (g(v) \wedge \kappa)^{q_0-1} \frac{\langle v \rangle^{d-1}}{\langle v \rangle^{d-1}} \, \mathrm{d}v \\ &\leq \frac{2^{q_0+d-2}}{\kappa^{q_0-1}} \langle v' \rangle^{d-1} \int_{S(v') \cap B_R(v')} (g(v) \wedge \kappa)^{q_0-1} \langle v \rangle^{-(d-1)} \, \mathrm{d}v. \end{aligned}$$

We used that for $v \in B_R(v')$ and R such that $R \leq \varepsilon_0 \langle v' \rangle$, we have $\langle v \rangle \leq 2 \langle v' \rangle$ (since $\varepsilon_0 \leq 1$).

Following the proof of [88, Lemma 3.2] and keeping in mind that $\varphi_{0,\kappa}(g') = q_0 \kappa^{q_0-1}(g' - \kappa)$ in $\{g' > \kappa\}$, we get

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g') K_f(v, v') \, \mathrm{d}v \, \mathrm{d}v' &\geq C_{2,q_0} \frac{\mu_0^2}{2} \int_{\{g' > \kappa\}} \varphi_{0,\kappa}(g') \langle v' \rangle^{\gamma+2s} R^{-2s} \, \mathrm{d}v' \\ &\geq C_{2,q_0} \frac{\mu_0^2}{2} C_0^{-\frac{2s}{d}} N^{-\frac{2s}{d}} q_0 \kappa^{q_0-1} \kappa^{(q_0-1)\frac{2s}{d}} \int_{\mathbb{R}^d} (g' - \kappa)_+ \langle v' \rangle^{\gamma} \, \mathrm{d}v'. \end{aligned}$$

In view of the definition of k_0 and q_0, we obtain the announced lower bound for some constant $C_{\mathrm{cor}'}$ equal to $C_{2,q_0} \frac{\mu_0^2}{2} C_0^{-\frac{2s}{d}} q_0$. □

7.4.2 The good extra term

We exhibit a third coercivity term. It controls the weighted L^{q_0-1}-norm of $(f - A)_+$. This lemma is similar to [88, Lemma 5.3], but more general since A depends on v. In contrast with the previous coercivity estimates, it is nonlinear in nature, it cannot be stated for two general functions f and g.

Proposition 7.4 (The good extra term for not-too-large q's)**.** *Let $q \in [0, d+1]$. Then*

there exists a constant $C_{\rm get} > 0$ *depending on* m_0, M_0, E_0, H_0, d, *such that if* $\mathfrak{a}$ *satisfies*

$$\mathfrak{a}(t) \geq a_{\rm get} \qquad \textit{with} \qquad a_{\rm get} := 2^{d+1}(M_0 + E_0)c_0^{-1} \tag{7.18}$$

with c_0 *from* (1.24), *then for any* f *satisfying* (1.20), *there holds,*

$$\begin{aligned} &\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f - A)\,(A' - f')_+\,K_f(v, v')\,\mathrm{d}v\,\mathrm{d}v' \\ &\qquad \geq C_{\rm get}\,\mathfrak{a}^{1+\frac{2s}{d}} \int_{\mathbb{R}^d} \Big((f - A)_+ \wedge \kappa\Big)^{q_0 - 1} \langle v\rangle^{l_q}\,\mathrm{d}v. \end{aligned} \tag{7.19}$$

Proof. Let $R > 0$ (to be chosen). We denote the "good" set of velocities by $G := \{v : \langle v\rangle \geq 2R\}$. For $v' \in B_R(v)$ and $v \in G$, we have $\langle v'\rangle \leq 2\langle v\rangle$ and $\langle v\rangle \leq 2\langle v'\rangle$. In particular, $A(v') \geq 2^{-q}A(v)$ for such v's and v''s. We now use the non-degeneracy cone $\Xi(v)$, see in particular (1.23), in order to write,

$$\begin{aligned} &\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f - A)\,(A' - f')_+\,K_f(v, v')\,\mathrm{d}v\,\mathrm{d}v' \\ &\quad \geq \frac{q_0}{2} \int_G \int_{\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}} \Big((f - A)_+ \wedge \kappa\Big)^{q_0 - 1}(v) A' K_f(v, v')\,\mathrm{d}v'\,\mathrm{d}v \\ &\quad \geq \frac{q_0 \lambda_0}{2^{q+1}} \int_G R^{-d-2s}\langle v\rangle^{\gamma+2s+1} \Big((f - A)_+ \wedge \kappa\Big)^{q_0 - 1} A \left|\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right|\,\mathrm{d}v. \end{aligned}$$

We then use (1.24) and Chebyshev's inequality to bound the sublevel set from below for $v \in G$ as follows,

$$\begin{aligned} \left|\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| &= |\{v + \Xi(v) \cap B_R\}| - \left|\left\{f' \geq \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| \\ &\geq c_0 R^d \langle v\rangle^{-1} - \int_{v + \Xi(v) \cap B_R} \frac{2}{A'} f'\,\mathrm{d}v' \\ &\geq c_0 R^d \langle v\rangle^{-1} - \frac{2}{\mathfrak{a}} \int_{v + \Xi(v) \cap B_R} \langle v'\rangle^{q-2} f(v') \langle v'\rangle^2\,\mathrm{d}v' \end{aligned}$$

we use again that $\langle v'\rangle \leq 2\langle v\rangle$ if $q \geq 2$ and $\langle v\rangle \leq 2\langle v'\rangle$ if $q \leq 2$ to get,

$$\begin{aligned} &\geq c_0 R^d \langle v\rangle^{-1} - \frac{2^{1+|q-2|}}{A(v)\langle v\rangle^2} \int_{\mathbb{R}^d} f(v') \langle v'\rangle^2\,\mathrm{d}v' \\ &\geq c_0 R^d \langle v\rangle^{-1} - \frac{2^d}{A(v)\langle v\rangle^2}(M_0 + E_0). \end{aligned}$$

We choose next $R = R(v)$ such that

$$R^d = \frac{2^{d+1}(M_0 + E_0)}{c_0 A(v)\langle v\rangle}.$$

Then

$$\left|\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| \geq \frac{c_0}{2} R^d \langle v\rangle^{-1}.$$

With such an estimate of the sublevel set at hand, we can finish to justify the estimate from below of the good extra term,

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f - A)\,(A' - f')_+ K_f(v, v')\,\mathrm{d}v'\,\mathrm{d}v \\ &\geq \frac{q_0 \lambda_0 c_0}{2^{q+2}} \int_G R^{-2s} \langle v\rangle^{\gamma+2s} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1}(v) A(v)\,\mathrm{d}v \\ &= C_{\mathrm{get}} \int_G \langle v\rangle^{\gamma+2s+\frac{2s}{d}} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1}(v) A^{1+\frac{2s}{d}}(v)\,\mathrm{d}v\end{aligned}$$

with $C_{\mathrm{get}} = \frac{q_0 \lambda_0 c_0}{2^{q+2}} \left(2^{d+1} c_0^{-1} (M_0 + E_0)\right)^{-\frac{2s}{d}}$.

Finally, since $A(v) = \mathfrak{a}\langle v\rangle^{-q}$, we have

$$G = \{v : \langle v\rangle \geq 2R\} = \left\{v : \langle v\rangle^{d+1-q} \geq \frac{2^{d+1}(M_0 + E_0)}{c_0 \mathfrak{a}}\right\}.$$

If $q \leq d + 1$ and if $\mathfrak{a}$ is such that $\mathfrak{a} \geq 2^{d+1}(M_0 + E_0) c_0^{-1}$ then $G = \mathbb{R}^d$. □

We conclude this section devoted to coercivity estimates by exhibiting an improved lower bound for the good extra term in the case of large decay exponents q (in contrast with assumptions of Proposition 7.4) by taking advantage of the fact that the function is already known to decay at some rate.

Proposition 7.5 (The very good extra term)**.** *Let $q \geq 0$. Let $f \colon \mathbb{R}^d \to [0, +\infty)$ and $A(v) := \mathfrak{a}\langle v\rangle^{-q}$ for some $\mathfrak{a} \geq 0$ and assume there is $\mathfrak{b}_* \geq 0$ depending on $m_0, M_0, E_0, s, d, \gamma$ (and time) such that*

$$f(v) \leq \mathfrak{b}_* \langle v\rangle^{-q_*}.$$

for some $q_* \geq 0$. If $\mathfrak{a}$ satisfies

$$\mathfrak{a} \geq 2^{2+q-q_*} \mathfrak{b}_*,$$

then

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f - A)(v)\,(A' - f')_+ K_f(v, v')\,\mathrm{d}v\,\mathrm{d}v' \\ \geq \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}} \mathfrak{a}^{1+2s} \int_{\mathbb{R}^d} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1}(v) \langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\end{aligned} \tag{7.20}$$

for some a constant $C_{\mathrm{vget}} > 0$ depending on m_0, M_0, E_0, H_0, d, s.

Proof. The proof follows the same lines as for Proposition 7.4. We find

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\qquad \geq \frac{q_0\lambda_0}{2^{q+1}} \int_G R^{-d-2s}\langle v\rangle^{\gamma+2s+1}\big((f-A)_+\wedge\kappa\big)^{q_0-1} A \left|\left\{f' < \frac{A'}{2}\right\} \cap \{v+\Xi(v)\cap B_R\}\right| \,\mathrm{d}v,
\end{aligned}$$

where $G := \{v : \langle v\rangle \geq 2R\}$ for some $R \in (0,1)$ to be chosen. Then we use Chebyshev's inequality, so that for $v \in G$,

$$\begin{aligned}
\left|\left\{f' < \frac{A'}{2}\right\} \cap \{v+\Xi(v)\cap B_R\}\right| &= |\{v+\Xi(v)\cap B_R\}| - \left|\left\{f' \geq \frac{A'}{2}\right\} \cap \{v+\Xi(v)\cap B_R\}\right| \\
&\geq |\{v+\Xi(v)\cap B_R\}| - \frac{2}{\mathfrak{a}} \int_{v+\Xi(v)\cap B_R} \langle v'\rangle^{q-q_*} f(v')\langle v'\rangle^{q_*}\,\mathrm{d}v' \\
&\geq |\{v+\Xi(v)\cap B_R\}| - \frac{2^{1+(q-q_*)}}{\mathfrak{a}}\mathfrak{b}_* \,|\{v+\Xi(v)\cap B_R\}|\,.
\end{aligned}$$

We pick $\mathfrak{a}$ sufficiently large, so that

$$1 - \frac{2^{1+(q-q_*)}}{\mathfrak{a}}\mathfrak{b}_* \geq \frac{1}{2}.$$

Then, using (1.24), we bound

$$\left|\left\{f' < \frac{A'}{2}\right\} \cap \{v+\Xi(v)\cap B_R\}\right| \geq \frac{1}{2} c_0 R^d \langle v\rangle^{-1}.$$

Now pick R as follows,

$$R = \frac{2^{3+(q-q_*)}}{\mathfrak{a}}\mathfrak{b}_* \leq \frac{1}{2}.$$

The fact that $2R \leq 1$ implies in particular that the good set G is the whole space,

$$G = \{v : \langle v\rangle \geq 2R\} = \mathbb{R}^d.$$

We finally get

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\qquad \geq \frac{c_0 q_0\lambda_0}{2^{q+2}} \int_G R^{-2s}\langle v\rangle^{\gamma+2s}\big((f-A)_+\wedge\kappa\big)^{q_0-1} A\,\mathrm{d}v \\
&\qquad = \frac{c_0 q_0\lambda_0}{2^{q+2}2^{2s(3+(q-q_*))}}\mathfrak{b}_*^{-2s}\mathfrak{a}^{1+2s} \int_{\mathbb{R}^d} \big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v. \qquad \square
\end{aligned}$$

7.5 Error term due to the velocity dependence of the barrier

This section is devoted to the estimate of the first error term appearing in the right hand side of the Truncated Convex Inequalities, see Lemma 7.7. We recall that it has the following form,

$$\mathcal{E}_1 := \iint_{\mathbb{R}^{2d}} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\,(A'-A)\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v.$$

In the next paragraph, we use ideas from [121].

7.5.1 Not-so-large decay exponents

Lemma 7.8 (Error term in the velocity variable)**.** *We consider a function f satisfying* (1.20)*. Let $q \in [0, d+1]$ and $A(v) := \mathfrak{a}\langle v\rangle^{-q}$ for some $\mathfrak{a} > 0$. There exists C_1 depending on d, m_0, M_0, E_0, H_0 such that*

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\big(A(v')-A(v)\big)K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v \\ \leq C_1\mathfrak{a}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{l_q}\,\mathrm{d}v,\end{aligned} \tag{7.21}$$

with l_q given by (vi)*.*

Recalling that the kernel K_f is given by,

$$K_f(v, v+z) = \frac{1}{|z|^{d+2s}}\int_{v'_*\in v+z^\perp} f(v'_*)\,|v-v'_*|^{\gamma+2s+1}\,\tilde{b}(\cos\theta)\,\mathrm{d}v'_*,$$

we aim at estimating

$$\mathcal{E}_1 = \int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\mathcal{I}(v)\,\mathrm{d}v$$

with

$$\begin{aligned}\mathcal{I}(v) &:= \int_{\mathbb{R}^d}(A(v')-A(v))\,K_f(v,v')\,\mathrm{d}v' \\ &= \int_{\mathbb{R}^d}\frac{(A(v+z)-A(v))}{|z|^{d+2s}}\left\{\int_{v'_*\in v+z^\perp} f(v'_*)\,|v-v'_*|^{\gamma+2s+1}\,\tilde{b}(\cos\theta)\,\mathrm{d}v'_*\right\}\mathrm{d}z \\ &= \int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s}\left\{\int_{z\in(v-v'_*)^\perp}\frac{(A(v+z)-A(v))}{|z|^{d+2s-1}}\tilde{b}(\cos\theta)\,\mathrm{d}z\right\}\mathrm{d}v'_*,\end{aligned}$$

where we changed the order of integration in the last equality.

We distinguish the singular part from the non-singular one,

$$\mathcal{I}(v) = \mathcal{I}_1(v) + \mathcal{I}_2(v)$$

with

$$\begin{aligned}
\mathcal{I}_1(v) &= \int_{\mathbb{R}^d} f(v'_*)\,|v - v'_*|^{\gamma+2s} \left\{ \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z \right\} \mathrm{d}v'_* \\
\mathcal{I}_2(v) &= \int_{\mathbb{R}^d} f(v'_*)\,|v - v'_*|^{\gamma+2s} \left\{ \int_{z\in(v-v'_*)^\perp} \chi_{|z|<\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z \right\} \mathrm{d}v'_*.
\end{aligned} \tag{7.22}$$

We now estimate each part separately.

Lemma 7.9 (Estimate of the non-singular part)**.**

$$\mathcal{I}_1(v) \le \begin{cases} CE_0\mathfrak{a}\langle v\rangle^{\gamma-2-\min(q,d-1)}, & \text{if } q \neq d-1, \\ CE_0\mathfrak{a}\langle v\rangle^{\gamma-d}, & \text{if } q = d-1. \end{cases} \tag{7.23}$$

In particular, if $q \le d+1$, then

$$\mathcal{I}_1(v) \le C\mathfrak{a}\langle v\rangle^{l_q}, \tag{7.24}$$

where l_q is given by (vi).

Proof. For the non-singular part we further distinguish two cases: $|v'_*| < \frac{1}{8}|v|$ and $\frac{1}{8}|v| < |v'_*|$,

$$\mathcal{I}_1(v) = \mathcal{I}_{1,1}(v) + \mathcal{I}_{1,2}(v)$$

with

$$\begin{cases} \mathcal{I}_{1,1}(v) = & \int_{|v'_*|<\frac{1}{8}|v|} f(v'_*)\,|v - v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_* \\ \mathcal{I}_{1,2}(v) = & \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v - v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_*. \end{cases}$$

The idea for $\mathcal{I}_{1,1}$ is to show that $|v+z| > |v|$ by exploiting the orthogonality $v + z \perp v'_*$ and the smallness $|v'_*| < \frac{1}{8}|v|$. In particular, $\mathcal{I}_{1,1}$ has a good sign. For the second term $\mathcal{I}_{1,2}$, we exploit the 2-moment of f.

We start with $\mathcal{I}_{1,1}$. Since and $z \perp (v - v'_*)$ and $|v'_*| < \frac{1}{8}|v|$ and $|z| > \frac{1}{2}\langle v\rangle > \frac{1}{2}|v|$, we

find

$$
\begin{aligned}
|v+z|^2 &= |v|^2 + |z|^2 + 2v\cdot z = |v|^2 + |z|^2 + 2v_*'\cdot z \\
&\geq |v|^2 + |z|\left(|z| - 2\,|v_*'|\right) > |v|^2 + |z|\left(|z| - \frac{1}{4}\,|v|\right) \\
&> |v|^2 + \frac{1}{4}\,|z|\,|v| > \left(1+\frac{1}{8}\right)|v|^2 .
\end{aligned} \tag{7.25}
$$

Since $q \geq 0$, $A(v+z) - A(v) = \mathfrak{a}\left[\langle v+z\rangle^{-q} - \langle v\rangle^{-q}\right] \leq 0$, we conclude that $\mathcal{I}_{1,1}(v) < 0$.

For $\mathcal{I}_{1,2}$ we find

$$
\begin{aligned}
&\mathcal{I}_{1,2}(v) \\
&= \int_{\frac{1}{8}|v|<|v_*'|} f(v_*')\,|v-v_*'|^{\gamma+2s} \int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v_*'
\end{aligned}
$$

we use that $\tilde{b}(\cos\theta) \leq \tilde{b}_+$ and $A(v) \geq 0$,

$$
\begin{aligned}
&\leq \tilde{b}_+\mathfrak{a} \int_{\frac{1}{8}|v|<|v_*'|} f(v_*')\,|v-v_*'|^{\gamma+2s} \int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z\,\mathrm{d}v_*' \\
&\leq \tilde{b}_+\mathfrak{a} \int_{\frac{1}{8}|v|<|v_*'|} f(v_*') \frac{(9\,|v_*'|)^{\gamma+2s}\langle v_*'\rangle^2}{\langle v_*'\rangle^2} \int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z\,\mathrm{d}v_*' \\
&\leq \left(9^{\gamma+2s}8^{2-\gamma-2s}\tilde{b}_+\right)\mathfrak{a}\langle v\rangle^{\gamma+2s-2} \int_{\frac{1}{8}|v|<|v_*'|} f(v_*')\langle v_*'\rangle^2 \int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z\,\mathrm{d}v_*'
\end{aligned}
$$

since $2 \geq \gamma + 2s$.

Finally, we compute,

$$
\begin{aligned}
&\int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \\
&\quad= \int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle}\chi_{|v+z|>|v|} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \\
&\qquad+ \int_{z\in(v-v_*')^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle}\chi_{|v+z|<|v|} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z
\end{aligned}
$$

we let ω_{d-2} denote the volume of the unit sphere of dimension $d-2$,

$$
\begin{aligned}
&\leq \omega_{d-2}\langle v\rangle^{-q} \int_{\frac{1}{2}\langle v\rangle}^\infty r^{-d+1-2s+d-2}\,\mathrm{d}r \\
&\qquad+ (\langle v\rangle/2)^{-d-2s+1} \int_{z\in(v-v_*')^\perp} \chi_{|v+z|<|v|}\langle v+z\rangle^{-q}\,\mathrm{d}z \\
&\leq C\langle v\rangle^{-q-2s} + C\langle v\rangle^{-(d-1)-2s} \int_0^{|v|} \frac{r^{d-2}}{(1+r^2)^{q/2}}\,\mathrm{d}r,
\end{aligned}
$$

for some positive constant C only depending on d and s only.

We now distinguish cases. If $q > d-1$, then $\frac{r^{d-2}}{(1+r^2)^{q/2}} \in L^1(\mathbb{R})$ and

$$\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \leq C_{q,1}\langle v\rangle^{-(d-1)-2s},$$

with

$$C_{q,1} = C + C\int_0^{+\infty} \frac{r^{d-2}}{(1+r^2)^{q/2}}\,\mathrm{d}r \leq C + C(1+2^{-q/2}(q-(d-1))^{-1}).$$

(split the domain integration between $r \leq 1$ and $r \geq 1$).

If $q < d-1$, then we simply write,

$$\int_0^{|v|} \frac{r^{d-2}}{(1+r^2)^{q/2}}\,\mathrm{d}r \leq \int_0^{|v|} r^{d-2-q}\,\mathrm{d}r = \frac{|v|^{d-1-q}}{d-1-q} \leq \frac{\langle v\rangle^{d-1-q}}{d-1-q}.$$

This implies that

$$\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \leq C\langle v\rangle^{-q-2s}.$$

If $q = d-1$, then

$$\int_0^{|v|} \frac{r^{d-2}}{(1+r^2)^{(d-1)/2}}\,\mathrm{d}r \leq \int_0^{|v|} r^{-1}\,\mathrm{d}r = \ln|v|.$$

This implies in this case that

$$\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \leq C\langle v\rangle^{-(d-1)-2s}\ln\langle v\rangle.$$

We thus proved that,

$$\mathcal{I}_{1,2}(v) \leq \begin{cases} CE_0\mathfrak{a}\langle v\rangle^{\gamma-2-\min(q,d-1)}, & \text{if } q \neq d-1, \\ CE_0\mathfrak{a}\langle v\rangle^{\gamma-d}, & \text{if } q = d-1. \end{cases} \tag{7.26}$$

Using $\mathcal{I}_{1,1} < 0$ and (7.26) we obtain (7.23) and (7.24).

We can check that

$$\gamma - 2 - \min(q, d-1) \leq (\gamma+2s) + \frac{2s}{d} - q\frac{d+2s}{d} = l_q,$$

for $q \neq d-1$, and

$$\gamma - d \leq (\gamma+2s) + \frac{2s}{d} - (d-1)\frac{d+2s}{d} = l_{d-1}.$$

The proof is now complete. □

Lemma 7.10 (Estimate of the singular part)**.**

$$\mathcal{I}_2(v) \le C_{q,2}\mathfrak{a}\langle v\rangle^{\gamma-q}, \tag{7.27}$$

with $C_{q,2} = \omega_{d-2}\frac{q(q+4)}{2}2^{\frac{3}{2}(q+2)}$.

In particular, for $q \le d+1$ *there holds* $\gamma - q \le l_q$ *where* l_q *is given in* (vi).

Proof. For the singular part, we do a Taylor expansion: there exists $\theta \in (0,1)$ such that

$$A(v+z) - A(v) - \nabla_v A(v+\theta z)\cdot z = \frac{1}{2}\mathrm{D}_v^2 A(v+\theta z)z\cdot z,$$

for some $\theta \in (0,1)$. Moreover, $A(w) = \mathfrak{a}\mathcal{A}(|w|)$ with $\mathcal{A}(r) = (1+r^2)^{-q/2}$. In particular,

$$\frac{1}{\mathfrak{a}}D_v^2 A(w) = \mathcal{A}''(|w|)\frac{w}{|w|}\otimes\frac{w}{|w|} + \frac{\mathcal{A}'(|w|)}{|w|}\left(I - \frac{w}{|w|}\otimes\frac{w}{|w|}\right),$$

where I denotes the identity matrix. As far as derivatives of $\mathcal{A}$ are concerned, we have

$$\frac{|\mathcal{A}'(r)|}{r} \le q(1+r^2)^{-\frac{q+2}{2}} \quad \text{and} \quad |\mathcal{A}''(r)| \le q(q+3)(1+r^2)^{-\frac{q+2}{2}}.$$

This implies that for all $w, \xi \in \mathbb{R}^d$, we have

$$D_v^2 A(w)\xi\cdot\xi \le q(q+4)\mathfrak{a}\langle w\rangle^{-q-2}|\xi|^2.$$

In particular,

$$|A(v+z) - A(v) - \nabla_v A(v+\theta z)\cdot z| \le \frac{q(q+4)}{2}\mathfrak{a}\langle v+\theta z\rangle^{-q-2}|z|^2.$$

We further note for $|z| \le \frac{1}{2}\langle v\rangle$,

$$\langle v+\theta z\rangle \ge \frac{1}{2\sqrt{2}}\langle v\rangle. \tag{7.28}$$

Indeed,

$$\langle v+\theta z\rangle \ge \frac{1}{\sqrt{2}}(1+|v+\theta z|) \ge \frac{1}{\sqrt{2}}(1+|v|-|z|) \ge \frac{1}{\sqrt{2}}(1+|v|-\frac{1}{2}\langle v\rangle) \ge \frac{1}{2\sqrt{2}}\langle v\rangle.$$

We thus have,

$$|A(v+z) - A(v) - \nabla_v A(v+\theta z)\cdot z| \le C_{q,2}\mathfrak{a}\langle v\rangle^{-q-2}|z|^2,$$

with $C_{q,2} = \frac{q(q+4)}{2}2^{\frac{3}{2}(q+2)}$. Due to the symmetry (1.28) and the fact that $\gamma + 2s > 0$, we

thus obtain

$$\begin{aligned}\mathcal{I}_2(v) &= \int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|<\frac{1}{2}\langle v\rangle}\,\frac{(A(v+z)-A(v))}{|z|^{d+2s-1}}\tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_* \\ &\le C_{q,2}\tilde{b}_+\mathfrak{a}\langle v\rangle^{-q-2}\int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s}\int_{z\in(v-v'_*)^\perp}\chi_{|z|<\frac{1}{2}\langle v\rangle}\,|z|^{2-d-2s+1}\,\mathrm{d}z\,\mathrm{d}v'_* \\ &\le \omega_{d-2}C_{q,2}\tilde{b}_+\mathfrak{a}\langle v\rangle^{-q-2}\left\{\int_{\mathbb{R}^d} f(v'_*)\left(|v|^{\gamma+2s}+|v'_*|^{\gamma+2s}\right)\mathrm{d}v'_*\right\}\left\{\int_0^{\frac{1}{2}\langle v\rangle} r^{3-d-2s+d-2}\,\mathrm{d}r\right\} \\ &\le \omega_{d-2}C_{q,2}\mathfrak{a}\langle v\rangle^{-q-2}\langle v\rangle^{\gamma+2s}\langle v\rangle^{-2s+2}.\end{aligned}$$

This implies (7.27). □

Proof of Lemma 7.8. Therefore, if we now denote by $\mathcal{E}_1$ the left hand side of (7.21), then (7.24) and (7.27) imply

$$\begin{aligned}\mathcal{E}_1 &= \int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\mathcal{I}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\left[\mathcal{I}_1(v)+\mathcal{I}_2(v)\right]\mathrm{d}v \\ &\le C\mathfrak{a}\int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\langle v\rangle^{l_q}\,\mathrm{d}v.\end{aligned}$$

This concludes the proof. □

7.5.2 Large decay exponents

For large q's, we need a better estimate than the one obtained in Lemma 7.8 above in order to establish the propagation of decay.

Lemma 7.11 (Error term in the velocity variable for large q's)**.** *Let $q > d+1$ and $A := \mathfrak{a}\langle v\rangle^{-q}$ for some $\mathfrak{a}\ge 0$. We consider a function f satisfying* (1.20)*. Assume there is an exponent $q_* > d+\gamma+2s$ and a function on time $\mathfrak{b}_* = \mathfrak{b}_*(t) > 0$ depending only on $m_0, M_0, E_0, s, d, \gamma$ and q_* such that*

$$f(v) \le \mathfrak{b}_*\langle v\rangle^{-q_*}. \tag{7.29}$$

Then there exists C_{1,q_*} depending on $\gamma, s, d, q_*, m_0, M_0, E_0, H_0$ such that

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\,(A(v')-A(v))\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v \\ &\qquad\le C_{1,q_*}\mathfrak{a}\Bigg(\mathfrak{b}_*\int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v \\ &\qquad\qquad+\int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v\Bigg).\end{aligned} \tag{7.30}$$

Remark 7.5 (Growth of C_{1,q_*} with q_*). We will see that the constant C_{1,q_*} grows like 8^{q_*} (up to a multiplicative constant).

The proof of this lemma follows the same steps as the one of Lemma 7.8 above. We bound the left hand side of (7.30) by splitting the inner integral in v' into the non-singular and the singular parts as we did above in (7.22). For the singular part, we can use Lemma 7.10. For the non-singular part, we need to improve Lemma 7.9. We recall from (7.22) that the non-singular part $\mathcal{I}_1$ is given by

$$\mathcal{I}_1(v) = \int_{\mathbb{R}^d} f(v'_*)\,|v - v'_*|^{\gamma+2s} \left\{ \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z \right\} \mathrm{d}v'_*,$$

for which the following estimate holds.

Lemma 7.12 (Estimate of the non-singular part for large q's)**.** *Let $q > d+1$ and $A(t,v) = \mathfrak{a}(t)\langle v\rangle^{-q}$. Assume there is an exponent $q_* > d+\gamma+2s$ and a function $\mathfrak{b}_* = \mathfrak{b}_*(t) > 0$ depending only on $m_0, M_0, E_0, s, d, \gamma$ and q_* such that*

$$f(v) \le \mathfrak{b}_* \langle v\rangle^{-q_*}.$$

Then there is a constant C depending on γ, s, d, q_* such that

$$\mathcal{I}_1(v) \le C\mathfrak{a}\mathfrak{b}_* \langle v\rangle^{\gamma-q_*}. \tag{7.31}$$

Proof. To bound $\mathcal{I}_1$, we distinguish the cases $\frac{1}{8}|v| > |v'_*|$ and $\frac{1}{8}|v| < |v'_*|$, and we denote these parts by $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$, respectively. We note as above that $\mathcal{I}_{1,1} < 0$, see (7.25). For $\mathcal{I}_{1,2}$ we find

$$\mathcal{I}_{1,2}(v) = \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v-v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_*$$

we change the order of integration

$$\begin{aligned} &= \int_{z\in\mathbb{R}^d} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s}} \\ &\quad \times \left\{ \int_{v'_*\in v+z^\perp} \chi_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v-v'_*|^{\gamma+2s+1}\,\tilde{b}(\cos\theta)\,\mathrm{d}v'_* \right\} \mathrm{d}z \end{aligned}$$

we use that $\tilde{b}(\cos\theta) \le \tilde{b}_+$ and $A(v) \ge 0$, $|z| > \langle v\rangle/2$, $|v| \le 8\,|v'_*|$ and $f(v'_*) \le \mathfrak{b}_*\langle v'_*\rangle^{-q_*}$

$$
\begin{aligned}
&\le 2^{d+2s}\tilde{b}_+\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{-d-2s}\int_{z\in\mathbb{R}^d}\chi_{|z|>\frac{1}{2}\langle v\rangle}\langle v+z\rangle^{-q}\\
&\qquad\times\int_{v'_*\in v+z^\perp}\chi_{\frac{1}{8}|v|<|v'_*|}\langle v'_*\rangle^{-q_*}(9\langle v'_*\rangle)^{\gamma+2s+1}\,\mathrm{d}v'_*\,\mathrm{d}z\\
&\le 2^{d+2s}\omega_{d-2}9^{\gamma+2s+1}\tilde{b}_+\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{-d-2s}\int_{z\in\mathbb{R}^d}\chi_{|z|>\frac{1}{2}\langle v\rangle}\langle v+z\rangle^{-q}\int_{\frac{1}{8}\langle v\rangle}^{\infty} r^{\gamma+2s+1-q_*+d-2}\,\mathrm{d}r\,\mathrm{d}z\\
&\le C\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{-d-2s}\langle v\rangle^{\gamma+2s+d-q_*}\int_{z\in\mathbb{R}^d}\langle v+z\rangle^{-q}\,\mathrm{d}z,
\end{aligned}
$$

with $C = 2^{d+2s}\omega_{d-2}9^{\gamma+2s+1}\tilde{b}_+8^{q_*-\gamma-2s-d}$, provided that $q_* > d+\gamma+2s$. Note that $\langle\cdot\rangle^{-q} \in L^1(\mathbb{R}^d)$ for $q > d$. Thus

$$
\mathcal{I}_{1,2}(v) \le C\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{\gamma-q_*}. \qquad \square
$$

We use this to prove Lemma 7.11.

Proof of Lemma 7.11. We use Lemma 7.12 for the non-singular part $\mathcal{I}_1$, and Lemma 7.10 for the singular part $\mathcal{I}_2$, and find

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2d}}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\,(A(v')-A(v))\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\\
&\qquad=\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\big[\mathcal{I}_1(v)+\mathcal{I}_2(v)\big]\,\mathrm{d}v\\
&\qquad\le C\mathfrak{a}\mathfrak{b}_*\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\\
&\qquad\quad+C\mathfrak{a}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v. \qquad \square
\end{aligned}
$$

7.6 Remaining error terms for hard potentials

In the case of hard potentials, that is to say in the case where $\gamma \ge 0$, we can treat error terms appearing in the right hand side of (7.8) all together. More precisely, we can estimate,

$$
\begin{aligned}
&\int\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\\
&\quad=\int\Big((q_0-1)\Big((f-A)_+\wedge\kappa\Big)^{q_0}(v)+q_0A\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\Big)(v)\,(f*|\cdot|^\gamma)\,\mathrm{d}v,
\end{aligned}
$$

with Φ from (7.9). Note that the case $\gamma \ge 0$ is significantly simpler than $\gamma < 0$.

7.6.1 Generation of decay

We start with an estimate that is used to prove the generation of moments in the case of hard potentials.

Lemma 7.13 (Error terms for hard potentials)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have finite mass M_0 and energy E_0 and let $q \in [0, d+1]$. There exists a constant $C_{\mathrm{hard}} > 0$ depending on M_0, E_0, d such that for any $\varepsilon \in (0,1)$ there holds*

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi(f)\,(f * |\cdot|^\gamma)\,\mathrm{d}v \leq & \varepsilon \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0} \right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} \\ & + C_{\mathrm{hard}}\,(\mathfrak{a} + C(\varepsilon)) \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \right\|_{L^1_{\gamma-q}(\mathbb{R}^d)}.\end{aligned} \tag{7.32}$$

Proof. We first remark that

$$\int_{\mathbb{R}^d} f(w)\,|v-w|^\gamma\,\mathrm{d}w \leq C_\gamma \int_{\mathbb{R}^d} f(w)(\langle v\rangle^\gamma + |w|^2)\,\mathrm{d}w,$$

for some constant C_γ only depending on γ. We used that for $\gamma \in [0,1]$, $(a+b)^\gamma \leq C_\gamma((a^2+1)^{\frac{\gamma}{2}} + b^2)$. Let $C_{\gamma,0}$ denote $C_\gamma(M_0+E_0)$. We have,

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi(f)\,(f * |\cdot|^\gamma)\,\mathrm{d}v &\leq C_{\gamma,0} \int_{\mathbb{R}^d} \Phi(f)\langle v\rangle^\gamma\,\mathrm{d}v \\ &\leq q_0 C_{\gamma,0} \int_{\mathbb{R}^d} \Big(\mathfrak{a}\big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q} + \big((f-A)_+ \wedge \kappa\big)^{q_0}\langle v\rangle^\gamma \Big)\,\mathrm{d}v.\end{aligned}$$

Using the interpolation from Lemma 7.6 with $k_3 = \gamma$ and $\iota = 1$, there holds for $m_3 = \gamma - (d+1) \leq \gamma - q$, (recall $q \leq d+1$) and any $\varepsilon \in (0,1)$

$$\begin{aligned}\int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0}\langle v\rangle^\gamma\,\mathrm{d}v \leq & \varepsilon \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0} \right\|_{L^{p_0}_{k_0}} \\ & + C(\varepsilon, E_0) \left(\int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v \right).\end{aligned}$$

This yields the claim (7.32). □

7.6.2 Error terms for large exponents

For large q's, the following estimate is used instead of Lemma 7.13 to establish the decay estimate.

Lemma 7.14 (Error terms for hard potentials and large q's)**.** *Let $\gamma > 0$. Let $f : \mathbb{R}^d \to [0,\infty)$ satisfy* (1.20)*. Assume*

$$f \leq \mathfrak{b}_* \langle v\rangle^{-q_*}$$

for some $q_* \geq 0$ and some $\mathfrak{b}_* = \mathfrak{b}_*(t) \geq 0$. We also suppose that $\mathfrak{a} \geq \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \mathfrak{b}_*$. Then, for any $q \geq 0$, there holds

$$\begin{aligned}\int_{\mathbb{R}^d} \Phi(f)\,(f * |\cdot|^\gamma)\,\mathrm{d}v \leq & \frac{1}{2}\frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v \\ & + q_0 C_{\gamma,0}\mathfrak{b}_*\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v.\end{aligned} \tag{7.33}$$

Proof. As in the proof of Lemma 7.13, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v \\ &\quad\leq q_0 C_{\gamma,0}\int_{\mathbb{R}^d}\Big(\mathfrak{a}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q} + \big((f-A)_+\wedge\kappa\big)^{q_0}\langle v\rangle^{\gamma}\Big)\,\mathrm{d}v.\end{aligned}$$

Then we remark first that $\gamma - q < \gamma + 2s - q$, and we thus bound for $\mathfrak{a} \geq \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \mathfrak{b}_*$,

$$\begin{aligned}&q_0 C_{\gamma,0}\mathfrak{a}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q}\,\mathrm{d}v \\ &\qquad\leq \frac{C_{\mathrm{vget}}}{2}\frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v.\end{aligned}$$

For the second term we use the decay assumption (7.29) and we find

$$q_0 C_{\gamma,0}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0}\langle v\rangle^{\gamma}\,\mathrm{d}v \leq q_0 C_{\gamma,0}\mathfrak{b}_*\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v. \qquad \square$$

7.7 Main error terms for soft potentials

We next explain how to estimate the following term in the case of soft potentials, *i.e.* when $\gamma < 0$,

$$\begin{aligned}&\iint_{\mathbb{R}^d\times\Omega}\Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x \\ &= \iint_{\mathbb{R}^d\times\Omega}\left((q_0-1)\Big((f-A)_+\wedge\kappa\Big)^{q_0}(v) + q_0 A\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\right)(v)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

We will use the following auxiliary results.

7.7.1 Auxiliaries

Lemma 7.15 (Convolution product)**.** *Let $\gamma < 0$ and $f : \mathbb{R}^d \to [0,\infty)$ have finite mass M_0. Let $A(t,v) = \mathfrak{a}\langle v\rangle^{-q}$ for some $q \geq 0$ and suppose, for all $t \geq 0$, $\mathfrak{a}(t) \geq 2^d \frac{M_0}{C_{q,3}}$ with*

$C_{q,3} = \frac{\omega_{d-1}(2\sqrt{2})^q}{(d+\gamma)}$. *Then there exists* $C_{\mathrm{conv}} > 0$ *depending on* d, γ, q *and* M_0, *such that*

$$(f \wedge A) * |\cdot|^\gamma \leq C_{\mathrm{conv}}\, \mathfrak{a}^{-\frac{\gamma}{d}} \langle v \rangle^{\gamma \min\left(1, \frac{q}{d}\right)}.$$

Proof. For $R > 0$ such that $R < \langle v \rangle / 2$, we write

$$\begin{aligned}\int_{\mathbb{R}^d} (f \wedge A)(v-w)\, |w|^\gamma \, \mathrm{d}w &= \int_{B_R} (f \wedge A)(v-w)\, |w|^\gamma \, \mathrm{d}w + \int_{\mathbb{R}^d \setminus B_R} (f \wedge A)(v-w)\, |w|^\gamma \, \mathrm{d}w \\ &\leq \mathfrak{a} \int_{B_R} \langle v-w \rangle^{-q} |w|^\gamma \, \mathrm{d}w + R^\gamma M_0\end{aligned}$$

and for $R < \langle v \rangle / 2$ and $w \in B_R$, we have $\langle v - w \rangle \geq \frac{1}{2\sqrt{2}} \langle v \rangle$ (argue as in (7.28)),

$$\begin{aligned}&\leq \mathfrak{a}\, \omega_{d-1} (2\sqrt{2})^q \langle v \rangle^{-q} \frac{R^{d+\gamma}}{d+\gamma} + R^\gamma M_0 \\ &\leq C_{q,3} a \langle v \rangle^{-q} R^{d+\gamma} + M_0 R^\gamma,\end{aligned}$$

where

$$C_{q,3} = \frac{\omega_{d-1}(2\sqrt{2})^q}{(d+\gamma)}. \tag{7.34}$$

Choosing $R = \left(\frac{M_0}{C_{q,3}\mathfrak{a}}\right)^{\frac{1}{d}} \langle v \rangle^{\min(1, \frac{q}{d})} \leq \frac{1}{2} \langle v \rangle$ (since $\mathfrak{a} \geq 2^d \frac{M_0}{C_{q,3}}$) yields

$$\begin{aligned}\int_{\mathbb{R}^d} (f \wedge A)(v-w)\, |w|^\gamma \, \mathrm{d}w &\leq (C_{q,3})^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}} \mathfrak{a}^{-\frac{\gamma}{d}} \langle v \rangle^{-q + (d+\gamma)\min(1, \frac{q}{d})} \\ &\quad + (C_{q,3})^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}} \mathfrak{a}^{-\frac{\gamma}{d}} \langle v \rangle^{\gamma \min(1, \frac{q}{d})} \\ &\leq 2 (C_{q,3})^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}} \mathfrak{a}^{-\frac{\gamma}{d}} \langle v \rangle^{\gamma \min(1, \frac{q}{d})}.\end{aligned}$$

□

We next collect auxiliary computations in the following lemma.

Lemma 7.16 (Auxiliary technical result)**.** *Let* $\gamma \in (-2s, 0)$ *and*

$$1 \leq q \leq 2 + \frac{d}{2s} k_0, \qquad l_1 = \min(2, q).$$

If we pick

$$\theta = \frac{d+2s}{d-2s}, \qquad \frac{1}{\beta} = \frac{d^2+4s^2}{(d+2s)^2}, \qquad \frac{1}{\beta'} = \frac{4sd}{(d+2s)^2}, \tag{7.35}$$

then

$$\frac{1}{\beta} + \frac{1}{\beta'} = 1, \qquad \frac{1}{\theta} + \frac{1}{\theta'} = 1, \qquad \gamma_2 \theta = 1, \qquad \gamma_3 \theta' = 1, \tag{7.36}$$

where

$$\gamma_1 := \frac{8s^2(d+\gamma)}{d(4sd+\gamma(d+2s))}, \qquad \gamma_2 := \frac{1}{\beta}\frac{(d-2s)(d+2s)}{d^2+4s^2}, \qquad \gamma_3 := \frac{1}{\beta}\frac{4s(d+2s)}{d^2+4s^2}. \quad (7.37)$$

Moreover,

$$\gamma_1\theta' < 1 + \frac{2s}{d}, \quad (7.38)$$

and there holds

$$m_1 \leq \frac{2}{\beta'} + \frac{m_3}{\beta}, \quad (7.39)$$

and

$$m_4 \leq \frac{2}{\beta'} + \frac{m_3}{\beta}, \quad (7.40)$$

with

$$\begin{aligned} m_1 &= -\frac{4sd^2}{(d+2s)(4sd+\gamma(d+2s))}l_1 + \frac{\gamma d}{4sd+\gamma(d+2s)}k_0, \\ m_3 &= \frac{4sd}{d^2+4s^2}l_q + \frac{d(d-2s)}{d^2+4s^2}k_0, \\ m_4 &= -2\frac{4sd}{(d+2s)(\gamma+2s)} - \frac{(2s-\gamma)d}{(d+2s)(\gamma+2s)}k_0. \end{aligned}$$

Proof. The relations in (7.36) are verified by a straightforward computation. We then rewrite (7.38) as

$$\frac{2s(d+2s)(d+\gamma)}{d(4sd+\gamma(d+2s))} < 1 + \frac{2s}{d},$$

which holds for $\gamma + 2s > 0$.

It remains to check that (7.39) and (7.40) are satisfied. We rewrite (7.39) as

$$-\frac{4sd^2}{4sd+\gamma(d+2s)}l_1 + \frac{\gamma d(d+2s)}{4sd+\gamma(d+2s)}k_0 \leq 2\frac{4sd}{(d+2s)} + \frac{4sd}{(d+2s)}l_q + \frac{d(d-2s)}{(d+2s)}k_0.$$

Then using that $l_q = k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}$ we get

$$-\frac{4sd^2}{(4sd+\gamma(d+2s))}l_1 + \frac{\gamma d(d+2s)}{4sd+\gamma(d+2s)}k_0 \leq 8s - 4sq + dk_0,$$

or equivalently,

$$-d^2 l_1 \leq (2-q)(4sd+\gamma(d+2s)) + d^2 k_0.$$

If $2 \geq q$ and $k_0 \geq 0$ then (7.39) is satisfied since $l_1 \geq 0$.

If $2 \leq q$ and $k_0 \geq 0$ then $l_1 = 2$, and since $q \leq 2 + \frac{d}{2s}k_0$, (7.39) is satisfied if

$$-2d^2 \leq -\frac{d}{2s}k_0(4sd+\gamma(d+2s)) + d^2k_0 = -d^2k_0 - \frac{d}{2s}\gamma(d+2s)k_0,$$

which holds if

$$\left(\gamma + 2s - \frac{2s}{d}\right) = k_0 \leq 2,$$

which is true since $\gamma + 2s \leq 2$.

If $k_0 < 0$ then $q \leq 2 + \frac{d}{2s}k_0 < 2$ and in particular $l_1 = q$, so that (7.39) is satisfied if

$$0 \leq (2-q)(4sd + \gamma(d+2s)) + d^2 k_0 + d^2 q.$$

We remark that $4sd + \gamma(d+2s) = (\gamma+2s)d + 2s(\gamma+d) \geq 0$. Since $2 - q \geq \frac{d}{2s}(-k_0) \geq 0$ and $q \geq 1$, we have to check that

$$\left(\frac{\gamma}{d} + \frac{\gamma+2s}{2s}\right) k_0 \leq 1,$$

is satisfied. If $\frac{\gamma}{d} + \frac{\gamma+2s}{2s} \geq 0$ then (7.39) is satisfied. If $\frac{\gamma}{d} + \frac{\gamma+2s}{2s} \leq 0$, then we see that (7.39) holds as soon as

$$\gamma k_0 = \gamma\left(\gamma + 2s - \frac{2s}{d}\right) \leq d,$$

which is satisfied. Indeed, $0 < -\gamma < d$ and $s < 1$ ensures that $-\gamma\frac{2s}{d} \leq 2$. We conclude by recalling that $d \geq 2$. Thus in any case we see that (7.39) is satisfied for $1 \leq q \leq 2 + \frac{d}{2s}k_0$.

Finally, we rewrite (7.40) as

$$\begin{aligned}
&-2\frac{4sd}{(d+2s)(\gamma+2s)} - \frac{(2s-\gamma)d}{(d+2s)(\gamma+2s)}k_0 \\
&\qquad \leq 2\frac{4sd}{(d+2s)^2} + \frac{d^2+4s^2}{(d+2s)^2}\left(\frac{4sd}{d^2+4s^2}l_q + \frac{d(d-2s)}{d^2+4s^2}k_0\right), \\
\Leftrightarrow\ &-2\frac{4sd}{(\gamma+2s)} - \frac{(2s-\gamma)d}{(\gamma+2s)}k_0 \\
&\qquad \leq 2\frac{4sd}{(d+2s)} + \frac{1}{(d+2s)}\left(4sd\left[k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}\right] + d(d-2s)k_0\right), \\
\Leftrightarrow\ &4sq \leq 8s + 2\frac{4sd}{(\gamma+2s)} + dk_0 + \frac{(2s-\gamma)d}{(\gamma+2s)}k_0.
\end{aligned}$$

We thus see, that for $q \leq 2 + \frac{d}{2s}k_0$, (7.40) is satisfied if

$$\gamma k_0 = \gamma\left(\gamma + 2s - \frac{2s}{d}\right) \leq 4s.$$

This is satisfied since $-\gamma \leq d$. □

7.7.2 Splitting

We split the error term involving $\Phi(f)$ by writing

$$f = (f \wedge A) + (f - A)_+ \wedge \kappa + (f - A - \kappa)_+,$$

so that

$$\int_{\mathbb{R}^d} \Phi\big(f\big)\,(f * |\cdot|^\gamma)\,\mathrm{d}v = \mathcal{E}_2 + \mathcal{E}_4 + \mathcal{E}_3 + \mathcal{E}_5,$$

with

$$\begin{cases} \mathcal{E}_2 &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)^{q_0-1}\big((f\wedge A) * |\cdot|^\gamma\big)\,\mathrm{d}v \\ \mathcal{E}_3 &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)_+^{q_0-1}\big((f-A)_+ \wedge \kappa * |\cdot|^\gamma\big)\,\mathrm{d}v \\ \mathcal{E}_4 &= (q_0-1) c_b \int \Big((f-A)_+ \wedge \kappa\Big)^{q_0}\,(f * |\cdot|^\gamma)\,\mathrm{d}v \\ \mathcal{E}_5 &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)_+^{q_0-1}\big((f-A-\kappa)_+ * |\cdot|^\gamma\big)\,\mathrm{d}v. \end{cases} \tag{7.41}$$

7.7.3 The term $\mathcal{E}_2$

Lemma 7.17 (Estimate of $\mathcal{E}_2$)**.** *Let $\gamma \in (-d, 0]$ and $\gamma + 2s > 0$ and $q \in [0, d+1]$. Then*

$$\mathcal{E}_2 \le \frac{1}{2} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \right\|_{L^1_{l_q}}, \tag{7.42}$$

as soon as $\mathfrak{a}$ satisfies for all $t \ge 0$

$$\mathfrak{a}(t) \ge \underline{a}^{(2)} = \max\left(\frac{2^d M_0}{C_{q,3}}, \left(\frac{2}{C_{\mathrm{get}}}\right)^{\frac{d}{\gamma+2s}}, a_{get} \right), \tag{7.43}$$

(with $a_{get}, C_{\mathrm{get}}$ from Proposition 7.4).

Proof. By Lemma 7.15 (see the assumption on $\mathfrak{a}$) there holds

$$\mathcal{E}_2 \le C_{\mathrm{conv}}\, q_0 c_b \mathfrak{a}^{-\frac{\gamma}{d}} \int_{\mathbb{R}^d} A\Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma \min(1, \frac{q}{d})}\,\mathrm{d}v.$$

It suffices to check that $1 - \frac{\gamma}{d} < 1 + \frac{2s}{d}$, which is true since $\gamma + 2s > 0$, and that $\gamma \min(1, \frac{q}{d}) - q \le l_q$, which again is true since $q \le d+1$. Thus

$$\mathfrak{a}^{1-\frac{\gamma}{d}} \int \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma\min(1,\frac{q}{d})-q}\,\mathrm{d}v \le \frac{1}{2} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \right\|_{L^1_{l_q}},$$

where C_{get} is the constant from Proposition 7.4. We used the assumption on $\mathfrak{a}$ from the statement. □

7.7.4 The term $\mathcal{E}_3$

We consider next $\mathcal{E}_3$ defined in (7.41).

Lemma 7.18 (Estimate of $\mathcal{E}_3$)**.** *Let $\gamma < 0$ and $q \in [0, d+1+\frac{d}{2s}\gamma]$. Then there exists a constant $\underline{a}^{(3)} > 0$, only depending on the parameters $d, \gamma, s, 2 \wedge q$ and the constants c_b, a_{get}, $C_{\rm cor}$ and $C_{\rm get}$, such that*

$$\mathcal{E}_3 \leq \frac{1}{2} C_{\rm cor} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{4} C_{\rm get}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \right\|_{L^1_{l_q}}, \tag{7.44}$$

(with $C_{\rm cor}$, a_{get} and $C_{\rm get}$ from Propositions 7.2 and 7.4) as soon as $a \geq \underline{a}^{(3)}$.

To prove this lemma, we show that we can split $\mathcal{E}_3$ as follows.

Lemma 7.19 (Splitting $\mathcal{E}_3$)**.** *We have $\mathcal{E}_3 \leq \mathcal{E}_3^{(i)} + \mathcal{E}_3^{(ii)}$ with*

$$\mathcal{E}_3^{(i)} = C_3^{(i)} \left\| (f \wedge A) \langle \cdot \rangle^{l_1} \right\|_{L^{p_1}} \left\| \big((f-A)_+ \wedge \kappa \big)^{q_0-1} \langle \cdot \rangle^{-\frac{2sl_1}{d+2s}} \right\|_{L^{p_2}} \left\| \big((f-A)_+ \wedge \kappa \big) \langle \cdot \rangle^{-\frac{dl_1}{d+2s}} \right\|_{L^{q_3}},$$
$$\mathcal{E}_3^{(ii)} = q_0 c_b M_0 \mathfrak{a} \left\| \big((f-A)_+ \wedge \kappa \big)^{q_0-1} \right\|_{L^1_{-q}},$$

where $l_1 = \min(q, 2)$, and $p_1, p_2, p_3, q_3 \in [1, +\infty]$ satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \qquad 1 + \frac{1}{p_3} = \frac{1}{q_3} + \frac{-\gamma}{d}. \tag{7.45}$$

The constant $C_3^{(i)}$ is given by $2^{\frac{dl_1}{2(d+2s)}} q_0 c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}}$.

Proof. We use that for $w \in B_1$ and $v \in \mathbb{R}^d$, we have,

$$\langle v - w \rangle^2 \leq 1 + |v|^2 + |w|^2 \leq 2 + |v|^2 \leq 2 \langle v \rangle^2.$$

We now distinguish two domains of integration: $|w| \leq 1$ and $|w| \geq 1$. We use the fact that

$\gamma < 0$ to get $|w|^\gamma \le 1$ for $|w| \ge 1$ and we write for $l_1 = \min(q, 2)$,

$$
\begin{aligned}
\mathcal{E}_3 &= q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\big((f-A)_+ \wedge \kappa\big)^{q_0-1} \left(\int_{B_1} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\quad + q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\big((f-A)_+ \wedge \kappa\big)^{q_0-1} \left(\int_{\mathbb{R}^d \setminus B_1} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\le q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\langle v\rangle^{l_1}\langle v\rangle^{-l_1}\big((f-A)_+ \wedge \kappa\big)^{q_0-1} \\
&\qquad\qquad \times \left(\int_{B_1} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v \\
&\le 2^{\frac{dl_1}{2(d+2s)}} q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\langle v\rangle^{l_1}\big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-\frac{2sl_1}{d+2s}} \\
&\qquad\qquad \times \left(\int_{B_1} \langle v-w\rangle^{-\frac{dl_1}{d+2s}}\big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v
\end{aligned}
$$

we use weak Young's inequality,

$$
\begin{aligned}
&\le 2^{\frac{dl_1}{2(d+2s)}} q_0 c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}} \left\| (f \wedge A)\langle\cdot\rangle^{l_1} \right\|_{L^{p_1}} \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle\cdot\rangle^{-\frac{2sl_1}{d+2s}} \right\|_{L^{p_2}} \\
&\quad \times \left\| \big((f-A)_+ \wedge \kappa\big)\langle\cdot\rangle^{-\frac{dl_1}{d+2s}} \right\|_{L^{q_3}} \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v.
\end{aligned}
$$

□

Then we estimate each piece separately.

Lemma 7.20 (Estimate of $\mathcal{E}_3^{(ii)}$)**.**

$$
\mathcal{E}_3^{(ii)} \le \frac{1}{8} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \right\|_{L^1_{l_q}} \tag{7.46}
$$

as soon as $\mathfrak{a} \ge \underline{a}^{(3,ii)} := \left(\frac{8 q_0 c_b M_0}{C_{\mathrm{get}}} \right)^{\frac{d}{2s}}$.

Proof. We note that $-q \le l_q$ for $q \le 2 + \frac{d}{2s} k_0$, and $1 < 1 + \frac{2s}{d}$, so that the conclusion holds true. □

Lemma 7.21 (Estimate of $\mathcal{E}_3^{(i)}$)**.** *There exists* $\underline{a}^{(3,i)}$*, that only depends on* d, γ, s, E_0, l_1*,* c_b*,* C_{cor} *and* C_{get} *(see Propositions 7.1 and 7.4), such that*

$$
\mathcal{E}_3^{(i)} \le \frac{1}{2} C_{\mathrm{cor}} \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{8} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \right\|_{L^1_{l_q}}
$$

as soon as $\mathfrak{a} \geq \underline{a}^{(3,i)}$.

Proof. For $\mathcal{E}_3^{(i)}$, we first notice that,

$$\left\|(f\wedge A)\langle\cdot\rangle^{l_1}\right\|_{L^{p_1}} \leq \left\|(f\wedge A)\langle\cdot\rangle^{l_1}\right\|_{L^1}^{\frac{1}{p_1}} \left\|(f\wedge A)\langle\cdot\rangle^{l_1}\right\|_{L^\infty}^{1-\frac{1}{p_1}} \leq E_0^{\frac{1}{p_1}}\mathfrak{a}^{1-\frac{1}{p_1}}.$$

Next we choose $p_3=\infty$ and $p_2=\frac{q_3}{q_0-1}\geq 1$ in (7.45), which implies

$$\frac{1}{q_3}=\frac{d+\gamma}{d},\qquad \frac{1}{p_2}=\frac{2s(d+\gamma)}{d^2},\qquad 1-\frac{1}{p_1}=\frac{2s(d+\gamma)}{d^2}.$$

Thus

$$\begin{aligned}\mathcal{E}_3^{(i)} &= C_3^{(i)}\left\|(f\wedge A)\langle\cdot\rangle^{l_1}\right\|_{L^{p_1}}\left\|\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle\cdot\rangle^{-\frac{2sl_1}{d+2s}}\right\|_{L^{p_2}}\left\|\big((f-A)_+\wedge\kappa\big)\langle\cdot\rangle^{-\frac{dl_1}{d+2s}}\right\|_{L^{q_3}}\\ &\leq C_3^{(i)}E_0^{\frac{1}{p_1}}\mathfrak{a}^{\frac{2s(d+\gamma)}{d^2}}\left(\int\big((f-A)_+\wedge\kappa\big)^{\frac{d}{d+\gamma}}\langle v\rangle^{-\frac{d^2l_1}{(d+2s)(d+\gamma)}}\,\mathrm{d}v\right)^{\frac{(d+\gamma)(d+2s)}{d^2}}.\end{aligned}$$

Step 1: First interpolation. Then we interpolate with Lemma 7.5 for $k_1=-\frac{d^2l_1}{(d+2s)(d+\gamma)}$

$$\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{\frac{d}{d+\gamma}}\langle v\rangle^{k_1}\,\mathrm{d}v \leq \left\|\big((f-A)_+\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\alpha_1}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\alpha_2},$$

where

$$\begin{aligned}\alpha_1&=\frac{-\gamma d}{4s(d+\gamma)},\qquad \alpha_2=\frac{4sd+\gamma(d+2s)}{4s(d+\gamma)},\\ m_1&=-\frac{4sd^2}{(d+2s)(4sd+\gamma(d+2s))}l_1+\frac{\gamma d}{4sd+\gamma(d+2s)}k_0.\end{aligned}$$

We can check that $\frac{-\gamma(d+2s)}{4sd}$ since $-\gamma\in[0,d]$ and $\gamma+2s\geq 0$. Together with Young's inequality, this implies that there exists $C=C(C_{\mathrm{cor}},c_b,E_0,l_1,\gamma,d,s)$ such that,

$$\begin{aligned}\mathcal{E}_3^{(i)} &\leq C_3^{(i)}E_0^{\frac{1}{p_1}}\mathfrak{a}^{\frac{2s(d+\gamma)}{d^2}}\left\|\big((f-A)_+\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\frac{-\gamma(d+2s)}{4sd}}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\frac{(d+2s)(4sd+\gamma(d+2s))}{4sd^2}}\\ &\leq \frac{1}{4}C_{\mathrm{cor}}\left\|\big((f-A)_+\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}}+C\mathfrak{a}^{\gamma_1}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\frac{d+2s}{d}}\end{aligned}$$

where γ_1 is given in Lemma 7.16, see (7.37).

Step 2: Second interpolation. Now we pick $\beta,\beta'\geq 1$ as in (7.35), so that (7.39) implies together with Hölder's inequality

$$\begin{aligned}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}} &\leq \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{\frac{2}{\beta_1'}+\frac{m_3}{\beta_1}}} \leq \left\|(f-A)_+\wedge\kappa\right\|_{L^1_2}^{\frac{1}{\beta_1'}}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_3}}^{\frac{1}{\beta_1}}\\ &\leq E_0^{\frac{1}{\beta_1'}}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_3}}^{\frac{1}{\beta_1}}.\end{aligned}$$

To bound the right hand side of the previous inequality, we interpolate again, now with Lemma 7.4 and $k_2 = m_3$,

$$\|(f-A)_+ \wedge \kappa\|_{L^1_{m_3}} \leq \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\alpha_3} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{m_2}}^{\alpha_4},$$

where

$$\alpha_3 = \frac{d(d-2s)}{d^2+4s^2}, \qquad \alpha_4 = \frac{4sd}{d^2+4s^2}, \qquad m_2 = \frac{4s^2+d^2}{4sd}\left[m_3 - \frac{d(d-2s)}{d^2+4s^2}k_0\right].$$

Note that the choice of m_3 in Lemma 7.16 is such that $m_2 = l_q$, that is $m_3 = \frac{4sd}{d^2+4s^2}l_q + \frac{d(d-2s)}{d^2+4s^2}k_0$. In particular, we find

$$\begin{aligned}\|(f-A)_+ \wedge \kappa\|_{L^1_{m_1}}^{\frac{d+2s}{d}} &\leq E_0^{\frac{1}{\beta'}\frac{d+2s}{d}} \|(f-A)_+ \wedge \kappa\|_{L^1_{m_3}}^{\frac{1}{\beta}\frac{d+2s}{d}} \\ &\leq E_0^{\frac{1}{\beta'}\frac{d+2s}{d}} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{l_q}}^{\gamma_3},\end{aligned}$$

where γ_2 and γ_3 also come from Lemma 7.16, see (7.37).

Step 3: Young's inequality. We combine Step 1 and Step 2 with Young's inequality, so that for $\theta \geq 1$ and any $\varepsilon \in (0,1)$

$$\begin{aligned}\mathcal{E}_3^{(i)} &\leq \frac{1}{4}C_{\mathrm{cor}} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}} + C\mathfrak{a}^{\gamma_1} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{l_q}}^{\gamma_3} \\ &\leq \frac{1}{4}C_{\mathrm{cor}} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}} + \frac{1}{4}C_{\mathrm{cor}} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}} \\ &\quad + \bar{C}\mathfrak{a}^{\gamma_1\theta'} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{l_q}}^{\gamma_3\theta'},\end{aligned}$$

with $\bar{C} = \bar{C}(C_{\mathrm{cor}}, c_b, E_0, l_1, \gamma, d, s)$ and $\theta' = \frac{\theta}{1-\theta}$, see Lemma 7.16 again. We used that $\gamma_2\theta = 1$, see (7.36). To complete the proof of Lemma 7.21, we use (7.38) from Lemma 7.16. □

Proof of Lemma 7.18. Combine Lemmas 7.19, 7.20 and 7.21. □

7.7.5 The term $\mathcal{E}_4$

The estimate of $\mathcal{E}_4$ defined in (7.41) follows with similar methods.

Lemma 7.22 (Estimate of $\mathcal{E}_4$)**.** *Let $f : [0,\infty) \to \mathbb{R}$ have finite mass M_0 and finite 2-moment E_0, and let $q \in [0, \frac{d}{2s}k_0]$.*

Then there exists a constant $\underline{a}^{(4)} > 0$, only depending on the parameters $d, \gamma, s, \min(2,q)$

and the constants c_b, $C_{\rm cor}$ and $C_{\rm get}$ (see Propositions 7.1 and 7.4), such that

$$\mathcal{E}_4 \le \frac{1}{4} C_{\rm cor} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{8} C_{\rm get}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \right\|_{L^1_{l_q}}, \tag{7.47}$$

(with $C_{\rm cor}$ and $C_{\rm get}$ from Propositions 7.2 and 7.4) as soon as $a \ge \underline{a}^{(4)}$.

To prove this lemma, we split $\mathcal{E}_4$ as follows.

Lemma 7.23 (Splitting $\mathcal{E}_4$)**.** *Let $p_4, p_5, q_5 \in [1, +\infty]$ satisfy*

$$\frac{1}{p_4} + \frac{1}{p_5} = 1, \qquad 1 + \frac{1}{p_5} = \frac{1}{q_5} + \frac{-\gamma}{d}. \tag{7.48}$$

We have $\mathcal{E}_4 \le \mathcal{E}_4{}^{(i)} + \mathcal{E}_4{}^{(ii)}$ with

$$\begin{aligned} \mathcal{E}_4{}^{(i)} &= C_4^{(i)} \left\| \big((f-A)_+ \wedge \kappa \big)^{q_0} \langle \cdot \rangle^{-2} \right\|_{L^{p_4}} \left\| (f-A)_+ \langle \cdot \rangle^2 \right\|_{L^{q_5}}, \\ \mathcal{E}_4{}^{(ii)} &= M_0 (q_0 - 1) c_b \left\| (f-A)_+ \wedge \kappa \right\|_{L^{q_0}}^{q_0}, \end{aligned}$$

with $C_4^{(i)} = 4(q_0-1)c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}}$.

Proof. We use for $w \in B_1$ and $v \in \mathbb{R}^d$

$$|v| \le |v-w| + |w| \le |v-w| + 1,$$

which implies $\langle v \rangle \le 2 \langle v - w \rangle$, so that distinguishing two domains of integration, $|w| \le 1$ and $|w| \ge 1$, we find

$$\begin{aligned} \mathcal{E}_4 &= (q_0-1) c_b \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa \big)^{q_0} \langle v \rangle^{-2} \langle v \rangle^2 \left(\int_{B_1} f(v-w) \, |w|^\gamma \, \mathrm{d}w \right) \mathrm{d}v \\ &\quad + (q_0-1) c_b \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa \big)^{q_0} \left(\int_{\mathbb{R}^d \setminus B_1} f(v-w) \, |w|^\gamma \, \mathrm{d}w \right) \mathrm{d}v \\ &\le 4(q_0-1) c_b \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa \big)^{q_0} \langle v \rangle^{-2} \left(\int_{B_1} \langle v-w \rangle^2 f(v-w) \, |w|^\gamma \, \mathrm{d}w \right) \mathrm{d}v \\ &\quad + M_0 (q_0-1) c_b \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa \big)^{q_0} \, \mathrm{d}v \\ &\le 4(q_0-1) c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}} \left\| \big((f-A)_+ \wedge \kappa \big)^{q_0} \langle \cdot \rangle^{-2} \right\|_{L^{p_4}} \left\| f \langle \cdot \rangle^2 \right\|_{L^{q_5}} \\ &\quad + M_0 (q_0-1) c_b \left\| (f-A)_+ \wedge \kappa \right\|_{L^{q_0}}^{q_0}. \end{aligned}$$

□

Again we estimate each piece separately.

Lemma 7.24 (Estimate of $\mathcal{E}_4{}^{(ii)}$)**.** *There holds*

$$\mathcal{E}_4{}^{(ii)} \le \frac{1}{8} C_{\rm cor} \left\| \big((f-A)_+ \wedge \kappa \big)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{16} C_{\rm get}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \right\|_{L^1_{l_q}},$$

(with $C_{\rm cor}$ and $C_{\rm get}$ from Propositions 7.1 and 7.4) as soon as $\mathfrak{a} \geq \underline{a}^{(4,ii)}$ with $\underline{a}^{(4,ii)} > 0$ only depending on $C_{\rm cor}, c_b, q_0, C_{\rm get}, d, \gamma, s, M_0$.

Proof. We use Lemma 7.6 with $k_3 = 0$ to get for some constant $C = C(C_{\rm cor}, q_0, M_0, c_b, d, \gamma, s)$,

$$\mathcal{E}_4{}^{(ii)} \leq \frac{1}{8} C_{\rm cor} \left\| \left((f - A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + C \left\| \left((f - A)_+ \wedge \kappa \right)^{q_0 - 1} \right\|_{L^1_{m_3}},$$

with $m_3 = -\frac{d}{2s} k_0 - 2$. For $q \leq 2 + \frac{d}{2s} k_0$, the exponent m_3 satisfies $m_3 \leq l_q$. □

Lemma 7.25 (Estimate of $\mathcal{E}_4{}^{(i)}$)**.** *There holds*

$$\mathcal{E}_4{}^{(i)} \leq \frac{1}{8} C_{\rm cor} \left\| \left((f - A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{16} C_{\rm get}\, \mathfrak{a}^{1 + \frac{2s}{d}} \left\| \left((f - A)_+ \wedge \kappa \right)^{q_0 - 1} \right\|_{L^1_{l_q}},$$

as soon as $\mathfrak{a} \geq \underline{a}^{(4,i)}$ with $\underline{a}^{(4,i)} > 0$ only depending on $C_{\rm cor}, c_b, C_{\rm get}, d, \gamma, s, M_0$ (with $C_{\rm cor}$ and $C_{\rm get}$ from Propositions 7.1 and 7.4).

Proof. We choose $q_5 = 1$ in (7.48). Then

$$\frac{1}{p_5} = \frac{-\gamma}{d} \in (0, 1), \qquad \frac{1}{p_4} = 1 + \frac{\gamma}{d} = \frac{d + \gamma}{d} \in (0, 1),$$

and thus

$$\begin{aligned} \mathcal{E}_4{}^{(i)} &\leq 4(q_0 - 1) c_b \left\| \left((f - A)_+ \wedge \kappa \right)^{q_0} \langle \cdot \rangle^{-2} \right\|_{L^{p_4}} \left\| f \langle \cdot \rangle^2 \right\|_{L^{q_5}} \\ &\leq 4(q_0 - 1) c_b E_0 \left(\int \left((f - A)_+ \wedge \kappa \right)^{\frac{d+2s}{d+\gamma}} \langle v \rangle^{-2 \frac{d}{d+\gamma}} \, \mathrm{d}v \right)^{\frac{d+\gamma}{d}}. \end{aligned}$$

Step 1: First interpolation. We then interpolate thanks to Lemma 7.5 with $p_1 = \frac{d+2s}{d+\gamma} \in [1, p_0 q_0]$,

$$\begin{aligned} &\left(\int \left((f - A)_+ \wedge \kappa \right)^{\frac{d+2s}{d+\gamma}} \langle v \rangle^{-2 \frac{d}{d+\gamma}} \, \mathrm{d}v \right)^{\frac{d+\gamma}{d}} \\ &\qquad \leq \left(\int \left((f - A)_+ \wedge \kappa \right)^{q_0 p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{\alpha_5}{p_0}} \left(\int \left((f - A)_+ \wedge \kappa \right) \langle v \rangle^{m_4} \, \mathrm{d}v \right)^{\alpha_6}, \end{aligned}$$

where α_5, α_6, m_4 are given by

$$\begin{aligned} \alpha_5 &= \frac{2s - \gamma}{4s}, \qquad \alpha_6 = \frac{(d + 2s)(\gamma + 2s)}{4sd}, \\ m_4 &= -2 \frac{4sd}{(d + 2s)(\gamma + 2s)} - \frac{(2s - \gamma) d}{(d + 2s)(\gamma + 2s)} k_0. \end{aligned}$$

We remark that $\alpha_5 \in (0,1)$. Then Young's inequality yields for any $\varepsilon_1 \in (0,1)$

$$\mathcal{E}_4{}^{(i)} \leq \frac{1}{16} C_{\mathrm{cor}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + C(C_{\mathrm{cor}}, M_0, E_0, c_b) \left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{m_4}}^{\frac{d+2s}{d}} .$$

Step 2: Second interpolation. We pick $\beta, \beta' \geq 1$ as in (7.35). In particular, (7.40) allows us to use Hölder's inequality in order to get,

$$\left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{m_4}} \leq \left\| (f-A)_+ \wedge \kappa \right\|_{L^1_2}^{\frac{1}{\beta'}} \left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{m_3}}^{\frac{1}{\beta}} \leq E_0^{\frac{1}{\beta'}} \left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{m_3}}^{\frac{1}{\beta}} .$$

If we now apply the interpolation Lemma 7.4 for $k_2 = m_3$, we find

$$\begin{aligned} &\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right) \langle v \rangle^{m_3} \, \mathrm{d}v \\ &\qquad \leq \left(\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0 p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{\alpha_3}{p_0}} \left(\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0 - 1} \langle v \rangle^{m_2} \, \mathrm{d}v \right)^{\alpha_4} , \end{aligned}$$

where

$$\alpha_3 = \frac{d(d-2s)}{4s^s + d^2}, \qquad \alpha_4 = \frac{4sd}{4s^2 + d^2}, \qquad m_2 = \frac{4s^2 + d^2}{4sd} \left[m_3 - \frac{d(d-2s)}{4s^2 + d^2} k_0 \right] .$$

Again the choice of m_3 in Lemma 7.16 is such that $m_2 = l_q$. In particular, there holds

$$\left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{m_4}}^{\frac{d+2s}{d}} \leq E_0^{\frac{1}{\beta'} \frac{d+2s}{d}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{l_q}}^{\gamma_3} ,$$

with γ_2, γ_3 given by (7.37).

Step 3: Young's inequality. We combine Step 1 and Step 2 with Young's inequality, so that for $\theta \geq 1$,

$$\begin{aligned} \mathcal{E}_4{}^{(i)} &\leq \frac{1}{16} C_{\mathrm{cor}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} \\ &\quad + C(C_{\mathrm{cor}}) \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0 - 1} \right\|_{L^1_{l_q}}^{\gamma_3} \\ &\leq \frac{1}{16} C_{\mathrm{cor}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} \\ &\quad + \frac{1}{16} C_{\mathrm{cor}} \left\| (f-A)_+^{q_0} \right\|_{L^{p_0}_{k_0}}^{\gamma_2 \theta} + \bar{C}(C_{\mathrm{cor}}) \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0 - 1} \right\|_{L^1_{l_q}}^{\gamma_3 \theta'} . \end{aligned}$$

If we pick θ as in (7.35), then Lemma 7.16 implies (7.36) and we conclude by choosing $\underline{a}^{(4,i)} = \underline{a}$ such that $\frac{1}{16} C_{\mathrm{get}} \underline{a}^{1 + \frac{2s}{d}} \geq \bar{C}(C_{\mathrm{cor}})$. □

Proof of Lemma 7.22. The result follows as a consequence of Lemmas 7.23, 7.24 and 7.25. □

7.7.6 The error term due to truncation for soft potentials

Finally it remains to estimate $\mathcal{E}_5$ in (7.41).

Lemma 7.26 (Estimate of $\mathcal{E}_5$)**.** *Let $\gamma < 0$ and $q \le q_{\mathrm{nsl}}$ where q_{nsl} is given in* (vii)*. We also assume that $q \ge d - 1 - 2s\frac{d-2}{d+2s}$ if $\gamma < -\frac{2sd}{d+2s}$. There holds*

$$\begin{aligned}\mathcal{E}_5 \le & \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{\frac{d+2s}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{l_q}\,\mathrm{d}v \\ & + \frac{1}{2} C_{\mathrm{cor}'}\, \kappa^{\frac{2s}{d} q_0} N^{-\frac{2s}{d}} \int_{\mathbb{R}^d} (f-A-\kappa)_+ \langle v\rangle^{\gamma}\,\mathrm{d}v \\ & + C_5 \bigg(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v\rangle^{k_0 p_0}\,\mathrm{d}v \bigg)^{\frac{1}{p_0}},\end{aligned}$$

with $N = \int_{\mathbb{R}^d} ((f-A)_+ \wedge \kappa)^{q_0-1} \langle v\rangle^{-(d-1)}\,\mathrm{d}v$. The positive constant C_5 only depends on parameters d, γ, s and constants $c_b, E_0, M_0, C_{\mathrm{get}}$ and $C_{\mathrm{cor}'}$ (see Propositions 7.3 and 7.4).

Proof. To estimate $\mathcal{E}_5$, we first recall its definition,

$$\mathcal{E}_5 = q_0 c_b \int_{\mathbb{R}^d} A(v) \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \big((f-A-\kappa)_+ * |\cdot|^{\gamma} \big)\,\mathrm{d}v.$$

Step 1: Splitting. We split the integration domain as before and we find for a constant m to be chosen later,

$$\begin{aligned}\mathcal{E}_5 &= q_0 c_b \int_{\mathbb{R}^d} A(v) \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{m} \langle v\rangle^{-m} \int_{B_1} (f-A-\kappa)_+(v-w)\,|w|^{\gamma}\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + q_0 c_b \int_{\mathbb{R}^d} A(v) \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \int_{\mathbb{R}^d \setminus B_1} (f-A-\kappa)_+(v-w)\,|w|^{\gamma}\,\mathrm{d}w\,\mathrm{d}v \\ &\le 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \int_{\mathbb{R}^d} \langle v\rangle^{-q-m} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \int_{B_1} \langle v-w\rangle^{m} (f-A-\kappa)_+(v-w)\,|w|^{\gamma}\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{-q}\,\mathrm{d}v \\ &=: \mathcal{E}_5^{(i)} + \mathcal{E}_5^{(ii)}.\end{aligned}$$

Note that in case that $m \ge 0$, we used $\langle v\rangle \le \langle v-w\rangle$ for $w \in B_1$ and $v \in \mathbb{R}^d$. In case that $m < 0$, we used $\langle v-w\rangle \le \sqrt{2}\langle v\rangle$ for $w \in B_1, v \in \mathbb{R}^d$.

The term $\mathcal{E}_5^{(ii)}$ is treated by checking that $-q \le l_q$ holds true for any $q \le d + 1 + \frac{d\gamma}{2s}$.

We use again Hölder's and Young's inequalities to write,

$$\begin{aligned}\mathcal{E}_5^{(i)} &\leq 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle \cdot \rangle^{-(q+m)} \right\|_{L^{\frac{d}{d+\gamma}}} \|(f-A-\kappa)_+\|_{L^1_m} \\ &\leq 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \kappa^{-\gamma \frac{2s}{d^2}} \left(\int_{\mathbb{R}^d} ((f-A)_+ \wedge \kappa)^{q_0-1} \langle v \rangle^{-(q+m)\frac{d}{d+\gamma}} \, \mathrm{d}v \right)^{\frac{d+\gamma}{d}} \|(f-A-\kappa)_+\|_{L^1_m} .\end{aligned}$$

It is now convenient to introduce some notation. We consider for any $\ell \in \mathbb{R}$,

$$N_\ell = \int_{\mathbb{R}^d} ((f-A)_+ \wedge \kappa)^{q_0-1} \langle v \rangle^\ell \, \mathrm{d}v \quad \text{and} \quad M_\ell = \int_{\mathbb{R}^d} (f-A-\kappa)_+ \langle v \rangle^\ell \, \mathrm{d}v.$$

We also let k_m denote $-(q+m)\frac{d}{d+\gamma}$ and we write the previous inequality under the following compact form,

$$\mathcal{E}_5^{(i)} \leq 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \kappa^{-\gamma \frac{2s}{d^2}} N_{k_m}^{\frac{d+\gamma}{d}} M_m. \tag{7.49}$$

We distinguish two cases: $\frac{-2sd}{d+2s} \leq \gamma < 0$ and $-2s < \gamma < \frac{-2sd}{d+2s}$.

Step 2: estimate of $\mathcal{E}_5^{(i)}$ for not too negative γ's. We start by assuming $\frac{-2sd}{d+2s} \leq \gamma < 0$. We use Young's inequality, so that

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{\frac{d}{d+\gamma}} N_{k_m} + \left(\frac{1}{16} C_{\mathrm{get}} \right)^{\frac{d+\gamma}{\gamma}} \left(2^{\frac{m}{2}} q_0 c_b \right)^{-\frac{d}{\gamma}} \kappa^{\frac{2s}{d}} M_m^{\frac{d}{-\gamma}}.$$

We pick $m = k_0 = \gamma + 2s - \frac{2s}{d}$. We then observe that for any $\gamma \geq -\frac{2sd}{d+2s}$, there holds $\frac{d}{d+\gamma} \leq 1 + \frac{2s}{d}$, and moreover $k_m \leq l_q$, provided that $q \leq q_{\mathrm{nsl}} = d + 1 + \frac{\gamma d}{2s}$, $\gamma \geq -\frac{2sd}{d+2s}$ and $d \geq 2$, see Lemma 7.27. This shows

$$\frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{\frac{d}{d+\gamma}} N_{k_m} \leq \frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{1+\frac{2s}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{l_q} \, \mathrm{d}v.$$

Moreover, we use Hölder's and Chebyshev's inequalities to bound

$$\begin{aligned}\kappa^{\frac{2s}{d}} M_{k_0}^{\frac{d}{-\gamma}} &= \kappa^{\frac{2s}{d}} M_{k_0}^{\frac{d}{-\gamma}-1} M_{k_0} \leq \kappa^{\frac{2s}{d}} M_{k_0}^{\frac{d+\gamma}{-\gamma}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}} |\{f-A > \kappa\}|^{\frac{2s}{d}} \\ &\leq \kappa^{\frac{2s}{d}} \kappa^{-\frac{2s}{d}} E_0^{\frac{d+\gamma}{-\gamma}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}} \left(\int_{\mathbb{R}^d} (f-A)_+ \, \mathrm{d}v \right)^{\frac{2s}{d}} \\ &= E_0^{\frac{d+\gamma}{-\gamma}} M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}}.\end{aligned}$$

We used that $k_0 < 2$ to bound M_{k_0} from above by E_0.

Gathering the estimates, we get in this first case,

$$\begin{aligned}\mathcal{E}_5^{(i)} \leq & \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{l_q} \,\mathrm{d}v \\ & + \left(\frac{1}{16} C_{\mathrm{get}} \right)^{\frac{d+\gamma}{\gamma}} \left(2^{\frac{m}{2}} q_0 c_b \right)^{-\frac{d}{\gamma}} E_0^{\frac{d+\gamma}{-\gamma}} M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f - A - \kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \,\mathrm{d}v \right)^{\frac{1}{p_0}}. \end{aligned} \tag{7.50}$$

Step 3: estimate of $\mathcal{E}_5^{(i)}$ for very negative γ's. We now assume $-2s < \gamma < \frac{-2sd}{d+2s}$. It remains to bound $\mathcal{E}_5^{(i)}$ for this range of γ. We start from (7.49) and we recall $k_m = -(q+m)\frac{d}{d+\gamma}$ for some $m \in \mathbb{R}$ to be determined,

$$\mathcal{E}_5^{(i)} \leq 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} N_{k_m}^{\frac{d+\gamma}{d}} M_m = 2^{\frac{m}{2}} q_0 c_b \kappa^{\frac{-\gamma 2s}{d^2}} \mathfrak{a} N_{k_m}^{\frac{d}{d+2s}} N_{k_m}^{\frac{d+\gamma}{d} - \frac{d}{d+2s}} M_m.$$

Note that

$$\frac{d+\gamma}{d} - \frac{d}{d+2s} < 0,$$

for $\gamma < -\frac{2sd}{d+2s}$.

We use Young's inequality

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{\frac{d+2s}{d}} N_{k_m} + C_5^{(i)} \kappa^{-\frac{\gamma(d+2s)}{d^2}} N_{k_m}^{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}} M_m^{\frac{d+2s}{2s}},$$

where $C_5^{(i)} = \left(\frac{1}{16} C_{\mathrm{get}}\right)^{-\frac{d}{2s}} \left(2^{\frac{m}{2}} q_0 c_b\right)^{\frac{d+2s}{2s}}$. We further bound for any $\alpha_0 \in [0,1)$

$$M_m \leq E_0^{\alpha_0} M_{m_{\alpha_0}}^{1-\alpha_0} \quad \text{with} \quad m_{\alpha_0} := \frac{m - 2\alpha_0}{1-\alpha_0},$$

thanks to Lemma 7.3. We found,

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{\frac{d+2s}{d}} N_{k_m} + C_5^{(i)} E_0^{\alpha_0 \frac{d+2s}{2s}} \kappa^{-\frac{\gamma(d+2s)}{d^2}} N_{k_m}^{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}} M_{m_{\alpha_0}}^{(1-\alpha_0)\frac{d+2s}{2s}}. \tag{7.51}$$

Then, in order to absorb the first term in the good extra term (Proposition 7.4), we need

$$k_m := -(q+m)\frac{d}{d+\gamma} \leq l_q = k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}.$$

For the second term, we want to use the second coercivity estimate from Proposition 7.3. To this end, we require

$$k_m := -(q+m)\frac{d}{d+\gamma} \geq -(d-1),$$

and

$$m_{\alpha_0} := \frac{m - 2\alpha_0}{1-\alpha_0} \leq k_0 + \frac{2s}{d} k_m.$$

In order to satisfy the first two constraints, q has to satisfy,

$$-(d-1) \leq l_q,$$

which holds true for $q \leq q_{\text{nsl}}$.

We pick m such that

$$k_m = -(q+m)\frac{d}{d+\gamma} = -(d-1),$$

and we use Proposition 7.3 with $k_m = -(d-1)$ for which we have

$$N := \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0 - 1} \langle v \rangle^{-(d-1)} \, \mathrm{d}v = N_{-(d-1)}.$$

Thus we consider

$$\bar{\theta} = -\frac{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}}{\frac{2s}{d}} = \frac{-\gamma(d+2s) - 2sd}{4s^2},$$

and we find

$$\begin{aligned}
\kappa^{-\frac{\gamma(d+2s)}{d^2}} N_{k_m}^{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}} M_{m_{\alpha_0}}^{\frac{d+2s}{2s}(1-\alpha_0)} &= \kappa^{\frac{2s}{d}q_0\bar{\theta}} N^{-\frac{2s}{d}\bar{\theta}} M_{m_{\alpha_0}}^{\bar{\theta}} \kappa^{-\frac{\gamma(d+2s)}{d^2} - \frac{2s}{d}q_0\bar{\theta}} M_{m_{\alpha_0}}^{\frac{(d+2s)(1-\alpha_0)}{2s} - \bar{\theta}} \\
&= \kappa^{\frac{2s}{d}q_0\bar{\theta}} N^{-\frac{2s}{d}\bar{\theta}} M_{m_{\alpha_0}}^{\bar{\theta}} \kappa^{\frac{(\gamma+2s)(d+2s)}{2sd}} M_{m_{\alpha_0}}^{\frac{(d+2s)(\gamma+2s)}{4s^2} + \frac{d-\alpha_0(d+2s)}{2s}}.
\end{aligned}$$

We plug this into (7.51) and use Young's inequality, to get

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\text{get}} \mathfrak{a}^{\frac{d+2s}{d}} N_{k_m} + \frac{1}{8} C_{cor'} \kappa^{\frac{2s}{d}q_0} N^{-\frac{2s}{d}} M_{m_{\alpha_0}} + C_5 \kappa^{\frac{2s}{d}} M_{m_{\alpha_0}}^{1 + \frac{2sd}{(\gamma+2s)(d+2s)} - \alpha_0 \frac{2s}{\gamma+2s}}. \tag{7.52}$$

where $C_5 := \left(\frac{1}{8}C_{cor'}\right)^{\frac{\gamma(d+2s)+2sd}{(\gamma+2s)(d+2s)}} \left(C_5^{(i)} E_0^{\alpha_0 \frac{d+2s}{2s}}\right)^{\frac{4s^2}{(\gamma+2s)(d+2s)}}$. We used that

$$\frac{1}{1-\theta} = \frac{4s^2}{(\gamma+2s)(d+2s)}.$$

Note that the estimate (7.52) holds for any $0 \leq \alpha_0 < 1$, but we want to pick α_0, such that

$$1 + \frac{2sd}{(\gamma+2s)(d+2s)} - \alpha_0 \frac{2s}{\gamma+2s} = 1, \tag{7.53}$$

but at the same time we need to make sure that

$$m_{\alpha_0} = \frac{m - 2\alpha_0}{1-\alpha_0} \leq k_0 + \frac{2s}{d} k_m = \gamma, \tag{7.54}$$

where we recall that we picked m and k_m as

$$m = -q + \frac{(d-1)(d+\gamma)}{d}, \qquad k_m = -(d-1).$$

We verify (7.54) in Lemma 7.28 below.

Finally, we conclude the proof just as in the case for larger γ above: we use that $\frac{m-2\alpha_0}{1-\alpha_0} \leq k_0$ and find with Chebyshev's inequality

$$\begin{aligned}
\kappa^{\frac{2s}{d}} M_{m_{\alpha_0}} &\leq \kappa^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f - A - \kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}} |\{f - A > \kappa\}|^{\frac{2s}{d}} \\
&\leq \kappa^{\frac{2s}{d}} \kappa^{-\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f - A - \kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}} \left(\int_{\mathbb{R}^d} (f - A)_+ \, \mathrm{d}v \right)^{\frac{2s}{d}} \\
&\leq M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f - A - \kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}}.
\end{aligned}$$

We plug this into (7.52) and use (7.53) to find

$$\begin{aligned}
\mathcal{E}_5^{(i)} &\leq \frac{1}{16} C_{\mathrm{get}} \mathfrak{a}^{\frac{d+2s}{d}} N + \frac{1}{8} C_{cor'} \kappa^{\frac{2s}{d} q_0} N^{-\frac{2s}{d}} M_\gamma + C_5 \kappa^{\frac{2s}{d}} M_{m_{\alpha_0}} \\
&\leq \frac{1}{16} C_{\mathrm{get}} \mathfrak{a}^{\frac{d+2s}{d}} N + \frac{1}{8} C_{cor'} \kappa^{\frac{2s}{d} q_0} N^{-\frac{2s}{d}} M_\gamma \\
&\quad + C_5 M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f - A - \kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}}.
\end{aligned} \tag{7.55}$$

We conclude the proof for $\gamma \in \big[-\frac{2sd}{d+2s}, 0 \big)$ using (7.50), and for $\gamma \in \big(-2s, -\frac{2sd}{d+2s} \big)$ with (7.55). □

Lemma 7.27. *Let $d \geq 2$, $s \in (0,1)$, $\gamma < 0$ and $q \leq d + 1 + \frac{d\gamma}{2s}$. Then*

$$-(q + k_0) \frac{d}{d+\gamma} \leq l_q = k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}. \tag{7.56}$$

Proof. We rewrite (7.56) as

$$\left(\frac{d+2s}{d} - \frac{d}{d+\gamma} \right) q \leq \left(1 + \frac{d}{d+\gamma} \right) k_0 + 2\frac{2s}{d}.$$

or also

$$(\gamma(d+2s) + 2sd)\, q \leq d\,(2d+\gamma)\, k_0 + 4s(d+\gamma).$$

If we now use $q \leq \frac{d}{2s}k_0 + 2$, we find

$$(\gamma(d+2s)+2sd)\left(\frac{d}{2s}k_0+2\right) \leq d\,(2d+\gamma)\,k_0 + 4s(d+\gamma).$$

or equivalently,

$$\left(\gamma\frac{d^2}{2s}+d^2\right)k_0 \leq 2d^2k_0 - 2\gamma d,$$

which is satisfied if

$$0 \leq \left(d-\gamma\frac{d}{2s}\right)k_0 - 2\gamma.$$

Now we use that $k_0 = \gamma + 2s - \frac{2s}{d}$. We find

$$0 \leq \left(d-\gamma\frac{d}{2s}\right)\left(\gamma+2s-\frac{2s}{d}\right) - 2\gamma = d\left(\frac{2s-\gamma}{2s}\right)(\gamma+2s) - (2s+\gamma).$$

We divide by $\gamma + 2s > 0$ to find

$$0 \leq \frac{d}{2s}(2s-\gamma) - 1,$$

which is true for any $d \geq 1$ and $\gamma < 0$. □

Lemma 7.28. *Let $d \geq 2$, $s \in (0,1)$, $-2s < \gamma < -\frac{2sd}{d+2s}$ and $q \geq d - 1 - 2s\frac{d-2}{d+2s}$. There exists $\alpha_0 \in (0,1)$ such that* (7.53) *and*(7.54) *hold true.*

Proof. In view of (7.53), we pick

$$\alpha_0 = \frac{d}{d+2s} \in (0,1).$$

We rewrite the constraint (7.54) as

$$\frac{d^2-d-\gamma-dq}{d(2-\gamma)} \leq \alpha_0$$

and check that (7.54) is satisfied. It is equivalent to,

$$(d+2s)(d^2-\gamma) - d(d+2s)(q+1) \leq d^2(2-\gamma).$$

We now remark that for $q \geq d - 1 - 2s\frac{d-2}{d+2s}$, we have

$$(d+2s)(q+1) \geq d^2 + 4s.$$

We are thus left with checking that

$$(d+2s)(d^2-\gamma)-d(d^2+4s)\leq d^2(2-\gamma).$$

We rearrange terms and get,

$$(-\gamma)(d+2s-d^2)-4sd-(2-2s)d^2\leq 0.$$

We conclude by remarking that for $d\geq 2$ and $s\in(0,1)$, we have $d+2s-d^2\leq 0$. □

7.7.7 Final Estimate

Combining Lemmas 7.17, 7.18, 7.22 and 7.26 implies the following statement.

Lemma 7.29 (Error terms for soft potentials)**.** *Let $\gamma<0$. Let $0\leq q\leq q_{\mathrm{nsl}}$ where q_{nsl} is given by* (vii) *and $q\geq d-1-2s\frac{d-2}{d+2s}$ if $\gamma<-\frac{2sd}{d+2s}$. There exists a positive constant $\underline{a}$ only depending on d,γ,s, M_0, E_0, c_b, C_{get}, C_{cor} (see Propositions 7.1 and 7.4) and $C_{q,3}$ (see formula* (7.34)*) such that*

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\Omega}\Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad\leq\frac{3}{4}C_{\mathrm{cor}}\,\Big\|\big((f-A)_+\wedge\kappa\big)^{q_0}\Big\|_{L^{p_0}_{k_0}}+\frac{15}{16}C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\,\Big\|\big((f-A)_+\wedge\kappa\big)^{q_0-1}\Big\|_{L^1_{l_q}}\\
&\qquad+C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}}\,\mathrm{d}x\\
&\qquad+\frac{1}{2}C_{\mathrm{cor}'}\,\kappa^{\frac{2s}{d}q_0}\int_\Omega\left[\Big\|\big((f-A)_+\wedge\kappa\big)^{q_0-1}\Big\|^{-\frac{2s}{d}}_{L^1_{-(d-1)}}\int_{\mathbb{R}^d}(f-A-\kappa)_+\langle v\rangle^\gamma\,\mathrm{d}v\right]\mathrm{d}x
\end{aligned}\tag{7.57}$$

as soon as $\mathfrak{a}\geq\underline{a}$.

Remark 7.6. The constants C_{cor}, $C_{\mathrm{cor}'}$, C_{get} and C_5 appear in Propositions 7.2, 7.3, 7.4 and Lemma 7.26 respectively.

Proof. Estimate (7.57) is a consequence of (7.41) and Lemmas 7.17, 7.18, 7.22, and 7.26. □

7.8 Estimates for large exponents

In this section, we establish estimates that we will use to generate more decay.

7.8.1 Convolution product

Lemma 7.30 (Convolution product for decaying functions and soft potentials)**.** *Let $\gamma < 0$ and $f : [0,T] \times \mathbb{R}^d \to [0,\infty)$ have finite mass M_0. Assume that*

$$f(t,v) \le \mathfrak{b}_* \langle v \rangle^{-q_*}, \qquad \forall (t,v) \in [0,T] \times \mathbb{R}^d,$$

for some $q_* > 0$ and some $\mathfrak{b}_* = \mathfrak{b}_*(t)$ such that $\mathfrak{b}_* \ge 2^d \frac{M_0}{c_{q_*}}$ with $c_{q_*} = \frac{|\partial B_1|(2\sqrt{2})^{q_*}}{(d+\gamma)}$. Then

$$f * |\cdot|^\gamma \le C_{\mathrm{conv}'}\, \mathfrak{b}_*^{-\frac{\gamma}{d}} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d})},$$

with $C_{\mathrm{conv}'} = 2 c_{q_*}^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}}$.

Proof. We proceed as in the proof of Lemma 7.15, exploiting the assumed pointwise decay. For $R > 0$ such that $R < \langle v \rangle / 2$, we write

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(v-w)\,|w|^\gamma \,\mathrm{d}w \\
&\quad = \int_{B_R} f(v-w)\,|w|^\gamma \,\mathrm{d}w + \int_{\mathbb{R}^d \setminus B_R} f(v-w)\,|w|^\gamma \,\mathrm{d}w \\
&\quad \le \mathfrak{b}_* \int_{B_R} \langle v-w \rangle^{-q_*} |w|^\gamma \,\mathrm{d}w + R^\gamma M_0 \le \mathfrak{b}_* |\partial B_1| (2\sqrt{2})^{q_*} \langle v \rangle^{-q_*} \frac{R^{d+\gamma}}{d+\gamma} + R^\gamma M_0 \\
&\quad \le c_{q_*} \mathfrak{b}_* \langle v \rangle^{-q_*} R^{d+\gamma} + M_0 R^\gamma
\end{aligned}$$

with $c_{q_*} = \frac{|\partial B_1|(2\sqrt{2})^{q_*}}{(d+\gamma)}$. Then we pick $R = \left(\frac{M_0}{c_{q_*}\mathfrak{b}_*}\right)^{\frac{1}{d}} \langle v \rangle^{\min(1, \frac{q_*}{d})} \le \frac{1}{2}\langle v \rangle$ (since $\mathfrak{b}_* \ge 2^d \frac{M_0}{c_{q_*}}$) so that

$$\begin{aligned}
\int_{\mathbb{R}^d} f(v-w)\,|w|^\gamma \,\mathrm{d}w &\le c_{q_*}^{-\frac{\gamma}{d}} \mathfrak{b}_*^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}} \left(\langle v \rangle^{-q_* + (d+\gamma)\min(1, \frac{q_*}{d})} + \langle v \rangle^{\gamma \min(1, \frac{q_*}{d})} \right) \\
&\le 2 c_{q_*}^{-\frac{\gamma}{d}} \mathfrak{b}_*^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d})}.
\end{aligned}$$

We thus get the desired estimate. □

7.8.2 Estimate of the remaining error term

Lemma 7.31 (Error terms for soft potentials and decaying functions)**.** *Let $\gamma < 0$. Assume $f : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ has finite mass M_0. If for a.e. $(t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^d$,*

$$f(t,x,v) \le \mathfrak{b}_*(t) \langle v \rangle^{-q_*}$$

for some $q_* \geq 0$ *and some* $\mathfrak{b}_* \geq 2^d \frac{M_0}{c_{q_*}}$ *with* $c_{q_*} = \frac{\omega_{d-1}(2\sqrt{2})^{q_*}}{(d+\gamma)}$*, then*

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\Omega} \Phi\big(f\big)\,(f * |\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad \leq \frac{1}{2} C_{\mathrm{vget}} \frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}} \iint_{\mathbb{R}^d\times\Omega} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad\quad + \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}} \iint_{\mathbb{R}^d\times\Omega} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q_*}\,\mathrm{d}v\,\mathrm{d}x,
\end{aligned} \tag{7.58}$$

as soon as $\mathfrak{a}$ *satisfies* $\mathfrak{a} \geq \bar{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}}$ *with* $\bar{C} = 16 q_0 c_b \frac{C_{\mathrm{conv}'}}{C_{\mathrm{vget}}}$ *and* $\underline{C} = C_{\mathrm{conv}'}(q_0-1)c_b$*.*

Proof. In order to prove the lemma, we split the error term involving $\Phi(f)$ naturally into two pieces,

$$\int_{\mathbb{R}^d} \Phi\big(f\big)\,(f * |\cdot|^\gamma)\,\mathrm{d}v = \bar{\mathcal{E}} + \underline{\mathcal{E}}$$

with

$$\begin{cases} \bar{\mathcal{E}} &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)^{q_0-1}\big(f * |\cdot|^\gamma\big)\,\mathrm{d}v, \\ \underline{\mathcal{E}} &= (q_0-1)c_b \int \Big((f-A)_+ \wedge \kappa\Big)^{q_0}\,(f * |\cdot|^\gamma)\,\mathrm{d}v. \end{cases} \tag{7.59}$$

We now estimate each part.

Estimate of $\bar{\mathcal{E}}$. We first claim that

$$\bar{\mathcal{E}} \leq \frac{1}{2} C_{\mathrm{vget}} \frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q}\,\mathrm{d}v, \tag{7.60}$$

as soon as the constant $\mathfrak{a}$ satisfies

$$\mathfrak{a} \geq \bar{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}}$$

for some constant $\bar{C} = 2q_0 c_b \frac{C_{\mathrm{conv}'}}{C_{\mathrm{vget}}}$ with C_{vget} from Proposition 7.5 and $C_{\mathrm{conv}'}$ from Lemma 7.30. Indeed, by Lemma 7.30 there holds

$$\begin{aligned}
\bar{\mathcal{E}} &\leq C_{\mathrm{conv}'}\, q_0 c_b \mathfrak{b}_*^{-\frac{\gamma}{d}} \int_{\mathbb{R}^d} A\Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma\min(1,\frac{q_*}{d})}\,\mathrm{d}v \\
&\leq \frac{1}{2} C_{\mathrm{vget}} \frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v.
\end{aligned}$$

The last inequality follows from the condition imposed on a.

Estimate of $\underline{\mathcal{E}}$. We next claim there exists a constant $\underline{C} > 0$ depending on s, d, γ, c_b, q_* such that

$$\underline{\mathcal{E}} \leq \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q_*}\,\mathrm{d}v. \tag{7.61}$$

Indeed, by Lemma 7.30 there holds

$$\begin{aligned}\underline{\mathcal{E}} &\leq C_{\mathrm{conv}'}\,(q_0-1)c_b\mathfrak{b}_*^{-\frac{\gamma}{d}}\int_{\mathbb{R}^d}\Big((f-A)_+\wedge\kappa\Big)^{q_0}\langle v\rangle^{\gamma\min(1,\frac{q_*}{d})}\,\mathrm{d}v\\ &\leq C_{\mathrm{conv}'}\,(q_0-1)c_b\mathfrak{b}_*^{1-\frac{\gamma}{d}}\int_{\mathbb{R}^d}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q_*}\,\mathrm{d}v.\end{aligned}$$

We used that $(f-A)_+\leq f\leq \mathfrak{b}_*\langle v\rangle^{-q_*}$ to get the second line.

Conclusion. Estimate (7.58) is derived from the Truncated Convex Inequalities (7.8) after using (7.59) together with 7.60 and 7.61. □

7.9 Proof of the decay estimate

7.9.1 Monotonicity for generation

The next lemma is obtained by combining Lemma 7.7 with (coercivity and error) estimates from previous sections.

Lemma 7.32 (Monotonicity)**.** *Let f be a suitable weak sub-solution of the Boltzmann equation with either in-flow, bounce-back, specular / diffuse / Maxwell reflection boundary condition. Let*

$$A(t,v):=\mathfrak{a}(t)\langle v\rangle^{-q_{\mathrm{nsl}}}\qquad\text{with}\qquad\mathfrak{a}(t):=a_*\left(1+t^{-\frac{d}{2s}}\right),$$

with q_{nsl} from (vii) *and for some constant $a_*>0$ large enough depending on d,γ,s,c_b,M_0,E_0 and C_b. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\leq\mathcal{I}_\kappa(t)\qquad\text{in }\mathcal{D}'((0,T)),\tag{7.62}$$

with $\mathcal{I}_\kappa(t)$ such that for any $T>0$, we have

$$\int_0^T\mathcal{I}_\kappa(t)\,\mathrm{d}t\xrightarrow[\kappa\to\infty]{}0.$$

Remark 7.7. The constant C_b comes from the assumption on the boundary data f_b in the statement of the main result, see Theorem 1.7.

Proof. We write q instead of q_{nsl} for clarity. We assume that a_* satisfies

$$a_*\geq a_{\mathrm{get}}\qquad\text{and}\qquad a_*\geq C_b\quad(\text{and } a_*\geq\underline{a}\text{ if }\gamma<0),\tag{7.63}$$

with a_{get} given in (7.18) and C_b in the statement of Theorem 1.7 (it is related to boundary values) and $\underline{a}$ comes from Lemma 7.29. The first condition ensures that we can use Proposition 7.4 about the good extra term while the second one ensures that boundary terms will not appear in the Truncated Convex Inequalities, see Lemma 7.7.

We distinguish the cases of hard and soft potentials.

THE CASE OF HARD POTENTIALS. In this case, we apply Lemma 7.7 and collect estimates (7.16) with $g = (f-A)_+$, (7.19), (7.21), (7.32) to get that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le -C_{\text{cor}}\int_\Omega \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\mathrm{d}x + C_{\text{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{\gamma-d-1}(\mathbb{R}^d)}\mathrm{d}x\\
&\qquad - C_{\text{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\mathrm{d}x\\
&\qquad + (C_1+C_{\text{hard}})\mathfrak{a}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\mathrm{d}x\\
&\qquad + \varepsilon\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^p_{k_0}(\mathbb{R}^d)}\mathrm{d}x + C_{\text{hard}}\,C(\varepsilon)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\mathrm{d}x\\
&\qquad - \dot{\mathfrak{a}}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{-q}(\mathbb{R}^d)}\mathrm{d}x,
\end{aligned}$$

we then pick $\varepsilon = C_{\text{cor}}/2$ and get,

$$\begin{aligned}
&\le -\frac{C_{\text{cor}}}{2}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\mathrm{d}x\\
&\qquad + \bar{C}\left(-\bar{c}\mathfrak{a}^{1+\frac{2s}{d}}+1+\mathfrak{a}(t)-\dot{\mathfrak{a}}(t)\right)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\mathrm{d}x
\end{aligned}$$

for some positive constants $\bar{C}$ and $\bar{c}$ depending on C_{cor}, C_{get}, C_{hard}, $C(C_{\text{cor}})$, C_1, d and s. The last inequality follows from the fact that $\gamma - d - 1 \le l_q$ and $-q \le l_q$ (recall that $\dot{\mathfrak{a}} < 0$ and l_q is given in (vi)) for $q \le d+1$ and $\gamma \ge 0$. Finally, we choose a_* large enough (depending on $\bar{c}$, d and s) such that

$$-\bar{c}\,\mathfrak{a}^{1+\frac{2s}{d}}(t) + 1 + \mathfrak{a}(t) - \dot{\mathfrak{a}}(t) \le 0. \tag{7.64}$$

In particular, this yields, $\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big((f-A)_+\big)\,\mathrm{d}v\,\mathrm{d}x \le 0$ and concludes the proof for hard potentials ($\gamma \ge 0$).

THE CASE OF SOFT POTENTIALS. In this case, we apply Lemma 7.7 and collect estimates (7.16) with $g = (f-A)_+$, (7.19), (7.21), (7.57) and use Lemma 7.7 to deduce,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le -C_{\mathrm{cor}}\int_\Omega \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x + C_{\mathrm{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad - C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + C_1\mathfrak{a}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + \frac{3}{4}C_{\mathrm{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + \frac{15}{16}C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x - \dot{\mathfrak{a}}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{-q}(\mathbb{R}^d)}\,\mathrm{d}x.
\end{aligned}$$

Using that $-q\le l_q$ for $q\le d+1+\frac{\gamma d}{2s}$ (and in particular for $q=q_{\mathrm{nsl}}$), we get,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le \bar{C}\left(-\bar{c}\mathfrak{a}^{1+\frac{2s}{d}}(t)+\mathfrak{a}(t)-\dot{\mathfrak{a}}(t)\right)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x.
\end{aligned}$$

Finally, we choose a_* such that both (7.63) and (7.64) (for a new $\bar{c}$) hold true and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big((f-A)_+\big)\,\mathrm{d}v\,\mathrm{d}x\le\mathcal{I}_\kappa(t),$$

with

$$\mathcal{I}_\kappa(t)=C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x.$$

Finally we notice that $f\in L^1([0,T]\times\Omega;L^{p_0}_{k_0}(\mathbb{R}^d))$ ensures that,

$$\int_0^T\mathcal{I}_\kappa(t)\,\mathrm{d}t\to 0\qquad\text{as }\kappa\to\infty.$$

This concludes the proof in the case of soft potentials. □

7.9.2 Proof of generation for not so large exponents

We now state and prove a result that will be used several times.

Lemma 7.33 (Monotonicity implies decrease)**.** *Let f be a suitable weak sub-solution of the Boltzmann equation with either in-flow, bounce-back, specular / diffuse / Maxwell reflection boundary condition. Let*

$$A(t,v) := \mathfrak{a}(t)\langle v\rangle^{-q}$$

for some smooth bounded function $\mathfrak{a}\colon (0,T) \to (0,+\infty)$ and for some $q \geq 0$. Assume that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x \leq \mathcal{I}_\kappa(t) \qquad \text{in } \mathcal{D}'((0,T)), \tag{7.65}$$

for some $\mathcal{I}_\kappa(t)$ such that for any $T>0$, we have

$$\int_0^T \mathcal{I}_\kappa(t)\,\mathrm{d}t \xrightarrow[\kappa\to\infty]{} 0.$$

Then $f(t,x,v) \leq A(t,v)$ almost everywhere in $(0,T)\times\Omega\times\mathbb{R}^d$.

Proof. For all $t \in (0,T)$, we define

$$m_\kappa(t) := \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}(f-A)(t,x,v)\,\mathrm{d}v\,\mathrm{d}x. \tag{7.66}$$

Since $\varphi_{0,\kappa}(f-A) \in L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, there holds $m_\kappa \in L^1((0,T))$.

Step 1. Lebesgue points. We now take two Lebesgue points $t_1, t_2 \in (0,T)$ such that $t_1 < t_2$, and we consider for any $\varepsilon > 0$ the following cutoff $\eta_\varepsilon \in C_c^\infty(\mathbb{R})$ in time given by

$$\eta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}(t_2 - t), & \text{if } t \in [t_2-\varepsilon, t_2), \\ 1, & \text{if } t \in (t_1+\varepsilon, t_2-\varepsilon), \\ \frac{1}{\varepsilon}(t - t_1), & \text{if } t \in (t_1, t_1+\varepsilon], \\ 0, & \text{else.} \end{cases}$$

Then we test (7.62) with η_ε and integrate over $(0,T)$, so that

$$\int_0^T m_\kappa'(t)\eta_\varepsilon(t)\,\mathrm{d}t \leq \int_0^T \mathcal{I}_\kappa(t)\eta_\varepsilon(t)\,\mathrm{d}t.$$

The left hand side yields

$$\int_0^T m'_\kappa(t)\eta_\varepsilon(t)\,\mathrm{d}t = -\int_0^T m(t)\eta'_\varepsilon(t)\,\mathrm{d}t = \frac{1}{\varepsilon}\int_{t_2-\varepsilon}^{t_2} m(t)\,\mathrm{d}t - \frac{1}{\varepsilon}\int_{t_1}^{t_1+\varepsilon} m(t)\,\mathrm{d}t,$$

so that we deduce after taking $\varepsilon \to 0$, we deduce that for almost every $t_1, t_2 \in (0,T)$ with $t_1 < t_2$,

$$m_\kappa(t_2) - m_\kappa(t_1) \le \int_{t_1}^{t_2} \mathcal{I}_\kappa(t)\,\mathrm{d}t. \tag{7.67}$$

Note that the right hand side converges to 0 as $\kappa \to \infty$. It also converges to 0 as $t_2 \to t_1$ or as $t_1 \to t_2$, so that $m_\kappa(t)$ is (coincides a.e. with a) càdlàg (function) and the values of $m_\kappa(t)$ are well-defined for all $t \in (0,T)$, see [169, Corollary 4.9.1]. We deduce from (7.67) that m_∞ is non-increasing in $(0,T)$.

Step 2. Shifting the barrier in time. To finish the proof of Theorem 1.7, it is sufficient to show that $m_\infty(t) = 0$ for a.e. $t \in (0,T)$. Indeed this in turn implies $f(t,x,v) \le \mathfrak{a}(t)\langle v\rangle^{-q}$ for almost every $(t,x,v) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

Even if m_∞ is monotonically decreasing, the difficulty is that $m_\infty(t)$ is not defined at $t = 0$. The following argument is taken from [169, Proof of Theorem 1.1]. We define a shifted version of the function m defined in (7.66),

$$m_{\kappa,t_1}(t) := \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}\big(f(t,x,v) - A(t-t_1,v)\big)\,\mathrm{d}v\,\mathrm{d}x.$$

After updating $\mathfrak{a}$, we know from Step 1 that $m_{\infty,t_1}(t)$ is monotonically decreasing for $t \in (t_1,T)$. Since $\mathfrak{a}(t) \to \infty$ as $t \to 0$, we note

$$\lim_{t_1\to t_2} m_{\infty,t_1}(t_2) = 0.$$

Thus for any $\varepsilon \in (0,1)$ there exists $t_1 \in (0,t_2)$ such that $m_{\infty,t_1}(t_2) < \varepsilon$.

Moreover, there holds for a.e. $t \in (t_2,T)$

$$m_{\infty,t_2}(t) \le m_{\infty,t_1}(t) \le m_{\infty,t_1}(t_2) < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we deduce $m_{\infty,t_2}(t) = 0$ for a.e. $t \in (t_2,T)$. This implies that for a.e. $t_2 \in (0,T)$ and $t \in (t_2,T)$, we have $(x,v) \in \Omega\times\mathbb{R}^d$, $f(t,x,v) \le A(t-t_2,v)$. We can now fix $t_0 > 0$ and consider $t_2 < t_0 < t$ and outside the sets of null measure. By letting $t_2 \to 0$, this implies $f(t,x,v) \le A(t,v)$ for $t > t_0$. Since t_0 is arbitrary, the proof is complete. □

Proof of Theorem 1.7 for not so large q's. It is enough to deal with the case $q = q_{\mathrm{nsl}}$. In

this case, the conclusion of the theorem is a consequence of Lemmas 7.32 and 7.33. □

7.9.3 Improving decay

In order to prove the appearance of decay for large values of the exponent q, we improve it iteratively. We start from Lemma 7.7 and write,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_{\mathbb{R}^d}\varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x+\int_\Omega\iint_{\mathbb{R}^{2d}}d_{\varphi_{0,\kappa}}((f-A)_+,(f'-A')_+)K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad+\int_\Omega\iint_{\mathbb{R}^{2d}}\dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le c_b\int_\Omega\int_{\mathbb{R}^d}\Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x+\int_\Omega\iint_{\mathbb{R}^{2d}}\dot{\varphi_{0,\kappa}}(f-A)\,(A'-A)\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad-\int_\Omega\int_{\mathbb{R}^d}\dot{\varphi_{0,\kappa}}(f-A)\partial_tA\,\mathrm{d}v\,\mathrm{d}x-\int_{\partial\Omega}\int_{\mathbb{R}^d}\varphi_{0,\kappa}(f-A)(v\cdot n(x))\,\mathrm{d}v\,\mathrm{d}S(x).
\end{aligned}$$

We recall that we assume that, depending on the boundary condition that we impose, $f\le f_b\le\mathfrak{a}(t)\langle v\rangle^{-q}$ on Γ_-, so that the boundary term vanishes.

When q is lower or equal to $d+1$, we can use the lower bound from the good extra term from Proposition 7.4 and estimate the error term due to the v-dependence of A thanks to Lemma 7.8 and the convolution term thanks to Lemma 7.13.

When q is greater than $d+1$, for the term on the second line we use the coercivity estimate from Proposition 7.5. Finally, for the error terms and q large, we use Lemma 7.11 to estimate $\mathcal{E}_1$ and either Lemma 7.14 (in case of hard potentials) or Lemma 7.31 (in case of soft potentials) to estimate the term involving the convolution.

7.9.3.1 Hard potentials

Lemma 7.34 (Improving decay – hard potentials)**.** *Let $\gamma\ge0$ be such that $0<\gamma+2s<1$ and $q_*\ge d+1$ and $q=q_*+2s$. If*

$$f\le C_{q_*}(1+t^{-\beta_{q_*}})\langle v\rangle^{-q_*},\qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

then

$$f\le C_q(1+t^{-\beta_q})\langle v\rangle^{-q},\qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d$$

with $\beta_q=\frac{1+2s}{2s}\beta_{q_*}$ and $C_q=a_*C_{q_*}^{1+\frac{1}{2s}}$ with a_* only depending on $d,\gamma,s,C_{\mathrm{vget}},C_{\gamma,0},C_b$, β_{q_*} and C_{1,q_*} from Lemma 7.11.

Proof. We consider the barrier $A(t,v)=\mathfrak{a}(t)\langle v\rangle^{-q}$ and the function $\mathfrak{a}(t)=C_q(1+t^{-\beta_q})$. It is convenient to write

$$\mathfrak{b}_*(t)=C_{q_*}(1+t^{-\beta_{q_*}}).\tag{7.68}$$

Truncated Convex Inequalities. Without loss of generality, we can assume that $C_{q_*} \geq 1$, so that $\mathfrak{b}_*(t) \geq 1$. We impose that

$$\mathfrak{a} \geq \underline{c}\, \mathfrak{b}_* \quad \text{with} \quad \underline{c} = \max\left(2^{2+2s}, \left(\frac{2q_0 C_{\gamma,0}}{C_{\text{vget}}}\right)^{\frac{1}{2s}}, C_b\right) \tag{7.69}$$

so that we can use Proposition 7.5, Lemma 7.11 (since $q > d+1$ and $q_* > d+\gamma+2s$), Lemma 7.14 (since $q_* \geq 0$) in order to write

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad \leq C_{1,q_*}\mathfrak{a}(t)\mathfrak{b}_*(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + C_{1,q_*}\mathfrak{a}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + q_0 C_{\gamma,0}\mathfrak{b}_*(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad - \dot{\mathfrak{a}}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad - \frac{1}{2}\frac{C_{\text{vget}}}{\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Using the fact that $q = q_* + 2s$ and $-2s \leq \gamma$, we get,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad \leq C_{1,q_*}\left(\mathfrak{b}_*(t)+1\right)\mathfrak{a}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + q_0 C_{\gamma,0}\mathfrak{b}_*(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad - \dot{\mathfrak{a}}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad - \frac{1}{2}\frac{C_{\text{vget}}}{\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We also impose to $\mathfrak{a}$ to satisfy,

$$-\dot{\mathfrak{a}}(t) + C_{1,q_*}\big(\mathfrak{b}_*(t)+1\big)\mathfrak{a}(t) + q_0 C_{\gamma,0}\mathfrak{b}_*(t) \leq \frac{C_{\text{vget}}}{2\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t). \tag{7.70}$$

Construction of the time barrier. We now consider

$$\mathfrak{a}(t) = a_* \mathfrak{b}_*^{1+\frac{1}{2s}}$$

and we choose $a_* \geq 1$ large enough so that $\mathfrak{a}$ satisfies both (7.69) and (7.70). In order to satisfy (7.69), we remark that $\mathfrak{b}_* \geq C_{q_*}$ and in particular $\mathfrak{a} \geq a_* C_{q_*}^{\frac{1}{2s}} \mathfrak{b}_*$, so that (7.69) holds true if we impose

$$a_* \geq C_{q_*}^{-\frac{1}{2s}} \underline{c} = C_{q_*}^{-\frac{1}{2s}} \max\left(2^{2+2s}, \left(\frac{2q_0 C_{\gamma,0}}{C_{\text{vget}}}\right)^{\frac{1}{2s}}, C_b\right). \tag{7.71}$$

As far as (7.70) is concerned, we plug $\mathfrak{a} = a_* \mathfrak{b}_*^{1+\frac{1}{2s}}$ into the ODE and get (after dividing by $a_* \mathfrak{b}_*^{\frac{1}{2s}}$),

$$\frac{1+2s}{2s}(-\dot{\mathfrak{b}_*})(t) + C_{1,q_*}\mathfrak{b}_*^2(t) + (C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0})\mathfrak{b}_*(t) \leq \frac{C_{\text{vget}}}{2} a_*^{2s} \mathfrak{b}_*^2(t)$$

(we used that $a_* \geq C_{q_*}^{-\frac{1}{2s}} \underline{c} \geq \mathfrak{b}_*^{-\frac{1}{2s}} \underline{c}$). Since we have $\mathfrak{b}_*(t) \geq 1$, this amounts to check that

$$\frac{1+2s}{2s}(-\dot{\mathfrak{b}_*})(t) + \left(2C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0}\right)\mathfrak{b}_*^2(t) \leq \frac{C_{\text{vget}}}{2} a_*^{2s} \mathfrak{b}_*^2(t).$$

We now pick a_* such that

$$a_* \geq \left(2C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0}\right)^{\frac{1}{2s}} \left(\frac{4}{C_{\text{vget}}}\right)^{\frac{1}{2s}}, \tag{7.72}$$

and we are left with verifying that

$$(-\dot{\mathfrak{b}_*})(t) \leq \frac{2s C_{\text{vget}}}{4(1+2s)} a_*^{2s} \mathfrak{b}_*^2(t).$$

Recalling that $\mathfrak{b}_*$ is given by (7.68), we have to check that for all $t > 0$,

$$C_{q_*}^{-1} \beta_{q_*} t^{-\beta_{q_*}-1} \leq \frac{2s C_{\text{vget}}}{4(1+2s)} a_*^{2s} (1 + t^{-\beta_{q_*}})^2.$$

It is enough to pick a_* such that

$$a_* \geq C_{q_*}^{-\frac{1}{2s}} \left(\frac{4(1+2s)\beta_{q_*}}{2s C_{\text{vget}}}\right)^{\frac{1}{2s}}, \tag{7.73}$$

(we used that $t^{-\beta_{q_*}-1} \leq (1 + t^{-\beta_{q_*}})^2$ for all $t > 0$ since $\beta_{q_*} > 1$).

Conclusion. Thanks to Lemma 7.33, we thus conclude for $\beta_q = \big(1+\frac{1}{2s}\big)\beta_{q_*}$

$$f \le a_* C_{q_*}^{1+\frac{1}{2s}} (1+t^{-\beta_{q_*}})^{1+\frac{1}{2s}} \langle v\rangle^{-q} \le a_* C_{q_*}^{1+\frac{1}{2s}} 2^{\frac{1}{2s}} (1+t^{-\beta_q})\langle v\rangle^{-q}.$$

(we used the convexity of $r \mapsto r^{1+\frac{1}{2s}}$) as soon as $a_* \ge 1$ satisfies (7.71), (7.72) and (7.73).

Since we assume $C_{q_*} \ge 1$, it is enough to impose,

$$a_* \ge \max\left(\underline{c}, \left(2C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0}\right)^{\frac{1}{2s}} \left(\frac{4}{C_{\text{vget}}}\right)^{\frac{1}{2s}}, \left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\text{vget}}}\right)^{\frac{1}{2s}}\right),$$

with $\underline{c} = \max\left(2^{2+2s}, \left(\frac{2q_0 C_{\gamma,0}}{C_{\text{vget}}}\right)^{\frac{1}{2s}}, C_b\right)$. □

7.9.3.2 Soft potentials

Lemma 7.35 (Improving decay – soft potentials)**.** *Let $\gamma < 0$ be such that $0 < \gamma + 2s < 1$ and q_{nsl} be given by* (vii)*. If for some $q_* \ge q_{\text{nsl}}$,*

$$f \le C_{q_*}(1+t^{-\beta_{q_*}})\langle v\rangle^{-q_*}, \qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

then for $q = q_* + \delta_*$, there holds

$$f \le C_q(1+t^{-\beta_q})\langle v\rangle^{-q}, \qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

with

$$\delta_* = \begin{cases} \min\left(\gamma+2s-\gamma\frac{q_*}{d}, d+1-q_*\right) & \text{if } q_* < d, \\ \min(2s, d+1-q_*) & \text{if } d \le q_* < d+1, \\ 2s & \text{if } q_* \ge d+1, \end{cases} \tag{7.74}$$

and $\beta_q = \frac{1+2s}{2s}\beta_{q_*}$ and $C_q = a_* C_{q_*}^{1+\frac{1}{2s}}$ with a_* only depending on $d, \gamma, s, C_{\text{vget}}, C_{\gamma,0}, C_b$ and q_*, β_{q_*} and C_{q_*}.

Remark 7.8. See (7.76) and (7.78) in the proof of the lemma for conditions on a_*.

Proof. We consider again the barrier $A(t,v) = \mathfrak{a}(t)\langle v\rangle^{-q}$ with $q = q_* + \delta_*$ (with δ_* to be chosen) and the function $\mathfrak{a}$ such that for all $t \in [0,T)$, $\mathfrak{a}(t) \ge C_b$. We also write $\mathfrak{b}_*(t)$ for $C_{q_*}(1+t^{-\beta_{q_*}})$ and we can assume without loss of generality that

$$C_{q_*} \ge 2^d \frac{M_0}{c_{q_*}} \quad \text{and} \quad C_{q_*} \ge 1,$$

(see the statement of Lemma 7.31 for the definition of c_{q_*}).

Let $q_* < d+1$. We remark that in this case, δ_* is chosen so that $q = q_* + \delta_* \le d+1$. We impose that

$$\mathfrak{a} \ge C_b \quad \text{and} \quad \mathfrak{a} \ge 2^{2+\delta_*}\mathfrak{b}_* \quad \text{and} \quad \mathfrak{a} \ge \overline{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}}, \tag{7.75}$$

so that we can apply Proposition 7.5, Lemma 7.8 (since $q \le d+1$), and Lemma 7.31, and get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x + \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x \\
\le \frac{1}{2}C_{\mathrm{vget}}\frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}}\iint_{\mathbb{R}^d\times\Omega}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x \\
+ \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}}\iint_{\mathbb{R}^d\times\Omega}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
+ C_1\mathfrak{a}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{l_q}\,\mathrm{d}v\,\mathrm{d}x \\
- \dot{\mathfrak{a}}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We remark that $l_q \le \gamma+2s-q$ since $q > q_{\mathrm{nsl}} \ge 1$. We also have $\gamma\min(1,\frac{q_*}{d})-q_* \le \gamma+2s-q$ by the choice of δ_* in (7.74). We look for $\mathfrak{a}$ such that

$$-\dot{\mathfrak{a}} + C_1\mathfrak{a} + \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}} \le \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}.$$

We plug next $\mathfrak{a} = a_*\mathfrak{b}_*^{1+\frac{1}{2s}}$ and get (after dividing by $a_*\mathfrak{b}_*^{\frac{1}{2s}}$),

$$-\frac{1+2s}{2s}\dot{\mathfrak{b}_*} + C_1\mathfrak{b}_* + \underline{C}a_*^{-1}\mathfrak{b}_*^{1-\frac{\gamma}{d}-\frac{1}{2s}} \le \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2.$$

We remark next that since $\mathfrak{b}_* \ge 1$ and $a_* \ge 1$ and $1-\frac{\gamma}{d}-\frac{1}{2s} \le 2$, it is sufficient to check that $\mathfrak{b}_*$ and a_* satisfy

$$-\frac{1+2s}{2s}\dot{\mathfrak{b}_*} + (C_1+\underline{C})\mathfrak{b}_*^2 \le \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2.$$

We thus pick a_* such that all three conditions below are verified:

$$\begin{aligned}
&a_* \ge \max(C_b, 2^{2+\delta_*}, 1, \overline{C}) \\
&a_*^{2s} \ge \frac{4(C_1+\underline{C})}{C_{\mathrm{vget}}} \\
&a_* \ge C_{q_*}^{-\frac{1}{2s}}\left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\mathrm{vget}}}\right)^{\frac{1}{2s}},
\end{aligned} \tag{7.76}$$

(see (7.73) above for the last condition) so that (7.75) holds true together with the differential inequality. We can conclude as in the case of hard potentials (thanks to Lemma 7.33) that

$$f \le a_* C_{q_*}^{1+\frac{1}{2s}} (1+t^{-\beta_{q_*}})^{1+\frac{1}{2s}} \langle v\rangle^{-q} \le a_* C_{q_*}^{1+\frac{1}{2s}} 2^{\frac{1}{2s}} (1+t^{-\beta_q}) \langle v\rangle^{-q},$$

where $\beta_q = \beta_{q_*}\big(1+\frac{1}{2s}\big)$ and a_* satisfies (7.76).

Assume now that $q_* \ge d+1 > d+\gamma+2s$, we pick $\delta_* = 2s$ and $q = q_* + \delta_* \ge d+1+2s$. We impose that

$$\mathfrak{a} \ge C_b \quad \text{and} \quad \mathfrak{a} \ge 2^{2+2s}\mathfrak{b}_* \quad \text{and} \quad \mathfrak{a} \ge \overline{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}}, \tag{7.77}$$

so that we can use Proposition 7.5, Lemma 7.11 (since $q > d+1$ and $q_* > d+\gamma+2s$) and Lemma 7.31 and find,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x + \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x,\\
\le C_{1,q_*}\mathfrak{a}(t)\mathfrak{b}_*(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
+ C_{1,q_*}\mathfrak{a}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v\,\mathrm{d}x\\
+ \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x\\
+ \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
- \dot{\mathfrak{a}}(t) \iint_{\mathbb{R}^d\times\Omega} \big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We remark that $\gamma - q_* = \gamma+2s-q$, that $\gamma - q \le \gamma+2s-q$ and that $-q \le \gamma+2s-q$. We thus look for $\mathfrak{a}$ such that

$$-\dot{\mathfrak{a}} + 2C_{1,q_*}\mathfrak{a}\mathfrak{b}_* + \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}} \le \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s},$$

(we used that $\mathfrak{b}_* \ge 1$). We plug once again $\mathfrak{a} = a_*\mathfrak{b}_*^{1+\frac{1}{2s}}$ and get (after dividing by $a_*\mathfrak{b}_*^{\frac{1}{2s}}$),

$$-\frac{1+2s}{2s}\dot{\mathfrak{b}_*} + (2C_{1,q_*}+\underline{C})\mathfrak{b}_*^2 \le \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2.$$

We thus pick a_* such that all three conditions are verified:

$$\begin{aligned} a_* &\geq \max(C_b, 2^{2+2s}, 1, \overline{C}) \\ a_*^{2s} &\geq \frac{4(2C_{1,q_*} + \underline{C})}{C_{\mathrm{vget}}} \\ a_* &\geq C_{q_*}^{-\frac{1}{2s}} \left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\mathrm{vget}}} \right)^{\frac{1}{2s}} \end{aligned} \tag{7.78}$$

(see (7.73) above for the last condition) so that (7.77) holds true together with the differential inequality. We can conclude as in the case of hard potentials (thanks to Lemma 7.33) that

$$f \leq a_* C_{q_*}^{1+\frac{1}{2s}} (1 + t^{-\beta_{q_*}})^{1+\frac{1}{2s}} \langle v \rangle^{-q} \leq a_* C_{q_*}^{1+\frac{1}{2s}} 2^{\frac{1}{2s}} (1 + t^{-\beta_q}) \langle v \rangle^{-q},$$

where $\beta_q = \beta_{q_*}\big(1 + \frac{1}{2s}\big)$ and a_* satisfies (7.78). □

7.9.4 Proof of generation for large exponents

Proof of Theorem 1.7 for large q's. The idea is to iterate the gain of decay from Lemma 7.34 and Lemma 7.35 for hard and soft potentials respectively. We have shown in Subsection 7.9.2 that

$$f \leq C_{q_*}(1 + t^{-\beta_{q_*}}) \langle v \rangle^{-q_*}, \qquad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^d,$$

with

$$q_* = q_{\mathrm{nsl}} =: q^{(0)}, \qquad \beta_{q_*} = \frac{d}{2s} =: \beta^{(0)}.$$

where we recall that q_{nsl} is given by (vii).

Hard potentials Lemma 7.34 then implies that there is $C_q = a_0 C_{q_*}$ for some $a_0 > 0$ such that

$$f \leq C_q(1 + t^{-\beta_q}) \langle v \rangle^{-q}, \qquad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^d,$$

with

$$q = q_* + 2s =: q^{(1)}, \qquad \beta_q = \frac{1+2s}{2s} \beta_{q_*} =: \beta^{(1)}.$$

We apply Lemma 7.34 again, now with $q^{(1)}$ and with $\beta^{(1)}$. We find

$$f \leq C_q(1 + t^{-\beta^{(2)}}) \langle v \rangle^{-q^{(2)}}, \qquad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^d$$

with

$$q^{(2)} = q^{(1)} + 2s = q_* + 2 \cdot 2s = d + 1 + 4s, \qquad \beta^{(2)} = \left(\frac{1+2s}{2s} \right)^2 \beta_{q_*} = \left(\frac{1+2s}{2s} \right)^2 \frac{d}{2s}.$$

We iterate this process and conclude for any $n \geq 0$

$$f \leq C_q(1 + t^{-\beta^{(n)}})\langle v\rangle^{-q^{(n)}}, \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d$$

with

$$q^{(n)} = d + 1 + n \cdot 2s, \qquad \beta^{(n)} = \left(\frac{1+2s}{2s}\right)^n \frac{d}{2s}.$$

Soft potentials We follow the same reasoning in the case of soft potentials. We apply Lemma 7.35 with $q_* = q_{\text{nsl}}$ and $\beta_{q_*} = \frac{d}{2s}$: then there is $C_q = a_0 C_{q_*}$ with $a_0 > 0$ such that

$$f \leq C_q(1 + t^{-\beta_q})\langle v\rangle^{-q}, \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

with

$$q = q_* + \delta_* =: q^{(1)}, \qquad \beta_q = \frac{1+2s}{2s}\beta_{q_*} =: \beta^{(1)},$$

where $\delta_* > 0$ depends on q_*. We iterate this process and we conclude for any $n \geq 0$

$$f \leq C_q\left(1 + t^{-\beta_q^{(n)}}\right)\langle v\rangle^{-q^{(n)}}, \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

with

$$q^{(n)} = q^{(n-1)} + \delta_*^{(n-1)}, \qquad \beta^{(n)} = \left(\frac{1+2s}{2s}\right)^n \frac{d}{2s}.$$

Writing $q^{(0)} = q_{\text{nsl}}$, we remark that there exists $n_0 \in \mathbb{N}$ such that

$$q^{(n_0+1)} = d + 1.$$

More precisely, we remark that

$$q^{(n+1)} = \begin{cases} \min\left(q^{(n)} + (\gamma+2s) + (-\gamma)\frac{q^{(n)}}{d}, d+1\right) & \text{if } q^{(n)} < d \\ \min\left(q^{(n)} + 2s, d+1\right) & \text{if } d \leq q^{(n)} < d+1 \\ q^{(n)} + 2s & \text{if } q^{(n)} \geq d+1. \end{cases}$$

In particular, as long as $q^{(n)} < d+1$, we have $q^{(n+1)} \geq q^{(n)} + (\gamma + 2s)$ and since $\gamma + 2s > 0$, this cannot hold for all $n \geq 1$.

We thus have for $n \geq n_0 + 1$,

$$q^{(n)} = d + 1 + (n - n_0 - 1)2s,$$

ensuring in particular $q^{(n)} \to \infty$ as $n \to \infty$. □

a-priori

Part III

Well-posedness for Landau-Coulomb

CHAPTER 8

Sub-critical Landau

In this final part, we consider the Landau equation with Coulomb potential in the spatially homogeneous case. We establish a well-posedness theory for rough initial data. We start in Chapter 8 with the sub-critical regime, where we first show that small initial data, in the sense of data that is close to equilibrium in sub-critical Lebesgue norms, leads to a global-in-time smooth solution. We then show that a global-in-time result in a perturbative regime is in a certain sense equivalent to a local-in-time result for general initial data in sub-critical Lebesgue spaces. In Chapter 9, we move to the critical regime: we show sufficiently strong a-priori estimates to construct a local-in-time smooth solution for initial data in the critical Lebesgue space $L^{\frac{3}{2}}$, where the coefficients fail to be bounded. Due to the breakthrough result of Guillen-Silvestre [100], we manage to show that our solution actually exists for all times. Finally, in Chapter 10, we show that a solution with initial data in $L^1 \cap L \log L$ has in fact a bounded Fisher information, and thus exists for all times. The results in this part are born out of a collaboration with W. Golding, M. Gualdani, and L. Desvillettes.

In this first chapter, we show short time propagation of smallness in L^p norms for $p > 3/2$ and instantaneous regularisation in Sobolev spaces. This yields new short time quantitative a-priori estimates that are unconditional near equilibrium. We combine these estimates with existing literature on global well-posedness for regular data to extend the well-posedness theory to small L^p data with p arbitrarily close to $3/2$. This is based on our article with W. Golding and M. Gualdani [83].

We then show with similar methods that we can also construct a local-in-time solution for large initial data, away from the perturbative regime. This was a collaboration with W. Golding [81].

8.1 Introduction

The Landau-Coulomb equation is a fundamental model in plasma physics. Despite its widespread use, the mathematical understanding of (1.41), has at the time of writing the articles [81, 83], been limited, particularly with regard to the existence of global-in-time smooth solutions.

8.1.1 Literature

Complete existence theory for solutions to (1.39) has been established only for weak solutions. In [186], Villani constructed global-in-time very weak solutions (often referred to as H-solutions) to (1.39) in the spatially homogeneous case for general initial datum. These H-solutions conserve mass, momentum and energy, and have decreasing entropy. Subsequently, in [58], Desvillettes used propagation of L^1 moments to clarify that Villani's solutions satisfy (1.39) in a standard weak sense. The moment estimates were then improved in [45] by showing that the solutions propagate L^1 moments of all orders and stretched exponential moments and used to provide quantitative rates of decay to equilibrium in L^1. Notably, despite recent progress on the eventual regularity and partial regularity of H-solutions [62, 86, 87], the question of uniqueness of these solutions remains open, even for regular initial datum.

If one starts with more regular, or more integrable initial datum, the situation is reversed. The uniqueness of weak solutions with bounded mass, momentum, energy, and entropy which further lie in $L^1(0,T;L^\infty)$ was established by Fournier in [75] using probabilistic methods. Subsequently, the authors of [52] relaxed the boundedness assumption to $L^\infty(0,T;L^p)$ for $p > 3/2$, assuming the initial datum decays sufficiently fast. However, these results do not necessarily apply to the global solutions constructed by Villani, as it is currently unknown whether general weak solutions belong to either integrability class. Let us also remark that by applying the parabolic De Giorgi-Nash-Moser theorem and parabolic Schauder estimates appropriately, it is known that $L^\infty_{loc}(0,T;L^\infty(\mathbb{R}^3))$ strong solutions are instantaneously smooth. However, the existence theory of strong solutions is much less developed.

Few unconditional regularity results exist for general weak solutions of (1.41). The authors of [86] showed partial regularity in time through the use of a De Giorgi-type iteration, which takes advantage of the control of entropy to determine a dimensional bound on the set of singular times. The results were then extended in [87] to provide a bound on the dimension of the set of singular points in space-time.

While unconditional regularity results remain elusive, various conditional results have been established based on classical elliptic-parabolic iteration methods that highlight the smoothing effects of the diffusion in (1.41). In [93], the second author and Guillen used

barrier arguments to obtain pointwise bounds for radially symmetric solutions, conditional to the $L^\infty(0,T;L^p)$ norm with $p > 3/2$. This is a strong enough estimate to build global-in-time solutions to a related model, the isotropic Landau equation introduced in [136] (see also [90]), but not for (1.41). In [178], Silvestre treated (1.41) as a non-divergence form parabolic equation and applied arguments inspired by Krylov-Safanov to deduce a-priori pointwise estimates for smooth solutions to (1.41). This provides unconditional bounds in the case of moderately soft potentials, but only provides bounds conditional on a weighted $L^\infty(0,T;L^p)$-norm for $p > 3/2$ of the solution for Coulomb potential. In a subsequent work, [94], the authors used the theory of A_p weights and a Moser-type iteration to obtain pointwise bounds for weak solutions, provided solutions belong to $L^\infty(0,T;L^p)$-norm for $p > 3/2$.

For large initial data[1], until recently, the only result was that of [15]: the authors show local-in-time existence of a strong solution for initial data in L^∞. The L^∞ norm is rather amenable to estimation using maximum principle type arguments, which neglect the helpful diffusive term in (1.41). By contrast, it is significantly more difficult to close energy estimates in lower L^p norms. Recently, in [62], the authors were able to prove a corresponding result for initial datum in a polynomially weighted $\dot{H}^1$ space. We mention also that existence of strong solutions for radial initial data belonging to L^p with $p > 3/2$ is established in [93]. Finally, there are additional works in the rough data, inhomogeneous Landau-Coulomb setting, i.e. [103, 106, 107] and the references therein, but the class of initial data considered is always more stringent than just L^∞, which was addressed in [15] for homogeneous initial data.

For small initial data, i.e. near equilibrium, there are more results. First, in [101], the author constructs global strong solutions are constructed for data close to equilibrium in weighted H^k for some $k \geq 8$. Second, in [62], they demonstrated the existence of a monotone functional that leads to the eventual smoothness of general weak solutions. If the initial value is close to the equilibrium in weighted H^1 space, the authors were able to show global-in-time well-posedness. Third, the article [44] discusses a global-in-time existence result for the inhomogeneous Landau equation, assuming closeness to the equilibrium in a weighted $H^1_x L^2_v$-space. The proof in [44] uses spectral methods, which are significantly more complicated for general L^p spaces when $p \neq 2$. We would also like to mention the related result in [132]: there, the authors proved that smooth solutions for inhomogeneous Landau-Coulomb equation exist globally in time for initial data close to equilibrium in a weighted L^∞ sense. The method relies upon pointwise estimates using a De Giorgi iteration to close their estimates.

[1] by which we mean far from the unique corresponding Maxwellian steady state

8.1.2 Aims

8.1.2.1 Small data - long time

In the first part of this chapter, we aim to show unconditional regularisation effects and pointwise bounds when initial data are close to equilibrium in L^p with p arbitrarily close to $3/2$. This part relies on [83]. Such unconditional, quantitative estimates are strong enough to prove existence of global in time classical solutions. This provides the first proof of global well-posedness near equilibrium for initial data in low L^p spaces and without smallness in weighted norms. Previous results require initial data near equilibrium in a weighted L^2 space [44], a weighted H^1 space [62], or a weighted L^∞ space [132]. The restriction $p > 3/2$ is essential to our arguments, which hinge upon short-time smoothing estimates. Our smoothing estimates hold when the diffusion dominates on short time scales. This occurs in *sub-critical* norms; for our equation, all $L^p(\mathbb{R}^3)$ norms with p strictly greater than $3/2$ are sub-critical, in the sense that, for $g \geq 0$,

$$\frac{\int_{\mathbb{R}^3} g^{p+1} \, \mathrm{d}v}{\int_{\mathbb{R}^3} \left|\nabla g^{p/2}\right|^2 \, \mathrm{d}v} \lesssim \left(\int_{\mathbb{R}^3} g^{3/2} \, \mathrm{d}v \right)^{2/3}.$$

We will derive an inequality similar to the preceding one that involves the Landau diffusion term (instead of the Laplacian) and use it to quantitatively track the time evolution of the L^p norm for a short time. An iteration argument inspired by the De Giorgi method will yield smoothing estimates. Qualitatively, this implies that if the initial data is small in some L^p spaces ($p > 3/2$), the corresponding solution will be regular and small (in all Sobolev norms) after some time passes. Uniqueness will allow us to piece together the local-in-time solution constructed here with the global-in time solution for small smooth data constructed in [62], or alternatively in [44] or [132]. This yields a single global in time solution for small L^p data with p arbitrarily close to $3/2$. Let us also stress that we require smallness only for f, not for moments of f.

8.1.2.2 Large data - short time

The second part of this chapter aims to adapt the energy methods and smoothing estimates in the first part to show local existence of strong solutions for initial data merely in L^p, for any $p > 3/2$. Moreover, we attempt to unify the smoothing estimates that exist within the literature, which are essential to our method. Accordingly, we deduce a uniqueness result for solutions in unweighted mixed space-time Lebesgue spaces, reminiscent of the classical Prodi-Serrin results in incompressible fluid mechanics. This part is based on [81].

It is rather natural to expect smoothing estimates for solutions of (1.41), given the parabolic nature of the equation. This perspective has yielded several conditional results,

beginning with [93], which shows radial solutions regularise from $L^\infty_{loc}(\mathbb{R}^+; L^{\frac{3}{2}+})$ to $L^\infty_{t,v}$ and become instantaneously smooth, and [178], which extended this to non-radial solutions, at the expense of weights in velocity. This was reproved via new techniques in [94], which interestingly suggests that the nonlinear diffusive effects of (1.41) are far stronger than the corresponding diffusive effects of the heat equation. Additionally, recent work in [24] suggests some of these estimates can be localized in velocity. Finally, a framework of $L^\infty_{loc}(\mathbb{R}^+; L^{\frac{3}{2}+})$ plus some energy control appeared in [7] to obtain pointwise control for the closely related Landau-Fermi-Dirac equation. Subsequently, this technique was modified for the use of the Landau-Coulomb equation in [83] and reappeared in [8], generalised to the full range of soft potentials and dimensions. We attempt to unite these smoothing results in Proposition 8.4 below, which is then used to build strong solutions.

8.1.3 Notation

Before presenting our main result, we first establish some notation and revisit the some basic properties of (1.39). The notation introduced in this section is valid for all remaining chapters, Chapter 8, Chapter 9, and Chapter 10. We use the bracket convention $\langle v\rangle := (1+|v|^2)^{\frac{1}{2}}$ and define the L^p_m norm for $p \in [1,\infty)$ and $m \in \mathbb{R}$, as follows:

$$\|f\|^p_{L^p_m(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^m \,\mathrm{d}v, \qquad \|f\|_{L^\infty_m(\mathbb{R}^3)} := \operatorname*{ess\,sup}_{v\in\mathbb{R}^3} \langle v\rangle^m |f(v)|\,.$$

We would like to make the reader aware of the different convention employed in Chapter 7 to introduce weighted Lebesgue norms, see (7.5). We define the weighted Sobolev norm $H^k_m(\mathbb{R}^3)$ as

$$\|f\|^2_{H^k_m(\mathbb{R}^3)} := \sum_{|\alpha|\le k} \int_{\mathbb{R}^3} \left|\partial^\alpha\left(\langle v\rangle^{\frac{m}{2}} f\right)\right|^2 \mathrm{d}v.$$

8.1.4 Results

8.1.4.1 Small data, long time

We now present the main result in a perturbative regime.

Theorem 8.1 (Small data, long time)**.** *Fix $p > \frac{3}{2}$ and let m be any positive number greater than*

$$\max\left(\frac{9}{2}\frac{p-1}{p-\frac{3}{2}}\max\left(1, \frac{p(p-3/2)}{p^2-2p+3/2}\right), 55\right).$$

For any given M, H and $f_{in} \in L^1_m \cap L^p(\mathbb{R}^3)$ satisfying

$$\|f_{in}\|_{L^1_m} \le M \qquad \text{and} \qquad \int_{\mathbb{R}^3} f_{in}\,|\log(f_{in})|\;dv \le H, \tag{8.1}$$

there exists $\delta = \delta(p, m, M, H) > 0$ *such that if*

$$\int_{\mathbb{R}^3} |f_{in} - \mu|^p \leq \delta,$$

equation (1.41) *admits a unique global classical solution on* $(0, \infty) \times \mathbb{R}^3$ *with initial datum* f_{in}*. This solution satisfies, for* $t > 0$*, the smoothing estimate*

$$\|f(t)\|_{L^\infty} \leq C(p, m, M, H) \left(1 + t^{-\frac{1}{1+\gamma}}\right), \tag{8.2}$$

for some $\gamma > 0$ *that only depends on* p *and* m*:*

$$\gamma := \frac{2(p - 3/2)}{3m} \left[m - \frac{9}{2} \frac{p-1}{p-3/2}\right] \in \left(0, \frac{2}{3}\left(p - \frac{3}{2}\right)\right).$$

Theorem 8.1 provides further evidence that all L^p norms for $p > 3/2$ are sub-critical in the sense that diffusive effects dominate. Combined with the conditional regularity results in [93, 94, 178] mentioned above, this emphasizes the role of the $L^{3/2}$ norm as a sort of critical norm for the equation. If one relies on the analogy to semi-linear parabolic problems, one could expect blow-up in L^p norms for $1 < p < 3/2$, which seems unrealistic for (1.41), since the L^1 norm is conserved. Theorem 8.1 also clarifies when the analogy of (1.41) to semi-linear equations remains valid and provides useful intuition, which is when $p > 3/2$. On the other hand, we believe that Theorem 8.1 should be true also for p arbitrarily close to one, but the proof seems at the moment out of reach.

Since we are in the sub-critical regime, heuristically, the solution obtained in Theorem 8.1 is expected to behave similarly to solutions to the heat equation, with a correction that takes into account the degeneracy of the diffusion coefficient $A[f]$ at large velocities. In fact, ignoring the m dependence of the constant C and letting $m \to \infty$ in the exponent γ, the smoothing estimate in (8.2) approaches the well-known $L^p - L^\infty$ estimate for the heat equation on $\mathbb{R}^3$:

$$\left\|e^{t\Delta} f_{in}\right\|_{L^\infty} \leq C(p) t^{-\frac{3}{2p}} \|f_{in}\|_{L^p}.$$

The nonlinear dependence of the small parameter δ from the size of the initial data in L^1_m and $L \log L$ should be interpreted as follows: if a sequence of initial data f^k_{in} converges to μ in L^p and remains equi-bounded in L^1_m and $L \log L$, then for sufficiently large k, these initial data will admit global classical solutions according to Theorem 8.1.

Unlike the results of [62], [132] and [44], Theorem 8.1 does not require smallness for the moments of the initial data. This distinction is subtle but non-trivial. Our method uses weighted spaces only to quantify the strength of the diffusion in our energy estimates, not to bound our energy functional. This is possible thanks to several weighted Sobolev inequalities combined with moment estimates.

The following corollary is immediate from Theorem 8.1 and [62, Proposition 1.2].

Corollary 8.1. *Let f_{in} be an initial datum satisfying the assumptions of Theorem 8.1 and let f be the corresponding unique solution. This solution f satisfies the long-time decay estimate*

$$\|\nabla(f-\mu)\|_{L^2_2} + \|f-\mu\|_{L^2_1} \leq \frac{C(m,H,M,p,\delta)}{(1+t)^{15/4}}, \quad t >> 1,$$

where μ is the Maxwellian with the same mass, momentum and energy as f_{in}.

Let us state the precise definition of solution found in Theorem 8.1: the function $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^+$ is a global solution to (1.41) with initial datum f_{in} in the following sense:

- For each $T > 0$, $f \in C(0,T; L^1 \cap L^p) \cap L^\infty(0,T; L^1_m) \cap L^1(0,T; L^\infty)$;
- For each $0 < t < T$, $f \in L^\infty(t,T; L^\infty \cap H^1_2)$;
- This f solves (1.41) in the sense of distributions on $[0,\infty) \times \mathbb{R}^3$, i.e. for each $\varphi \in C^\infty_c([0,\infty) \times \mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} f_{in}\varphi(0)\,\mathrm{d}v - \int_0^\infty \int_{\mathbb{R}^3} f\partial_t\varphi\,\mathrm{d}v\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^3} \nabla^2\varphi : A[f]f + 2\nabla\varphi\cdot\nabla a[f]f\,\mathrm{d}v\,\mathrm{d}t. \tag{8.3}$$

- This f is a classical solution of (1.41) on $(0,\infty)\times\mathbb{R}^3$ and satisfies the short time smoothing estimate (8.2) with

$$\|f(t)\|_{L^\infty} \leq C(p,m,M,H)\left(1 + t^{-\frac{1}{1+\gamma}}\right),$$

where

$$\gamma := \frac{2(p-3/2)}{3m}\left[m - \frac{9}{2}\frac{p-1}{p-3/2}\right] \in \left(0, \frac{2}{3}\left(p - \frac{3}{2}\right)\right).$$

The nonlinear nature of (1.41) and the low regularity of the initial data f_{in} cause some minor difficulties in stating precisely what notion of solution holds up to time 0. To illustrate the main ideas of Theorem 8.1 and provide the main step of its proof, we isolate the result of Theorem 8.1 for Schwartz class initial data in the following proposition:

Proposition 8.1. *Fix p, m, H, M, and f_{in} as in Theorem 8.1 and suppose additionally that $f_{in} \in \mathcal{S}(\mathbb{R}^3)$. Then, there is an $\delta = \delta(p,m,H,M) > 0$ such that $\|f_{in} - \mu\|_{L^p} < \delta$ implies there is a unique global-in-time solution $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^+$ to (1.41) satisfying $f \in C(0,\infty; L^p)$. Furthermore, f is regular in the sense that $f \in C^\infty(\mathbb{R}^+; \mathcal{S}(\mathbb{R}^3))$.*

For $h := f - \mu$ a Schwartz class solution, we show that $h(t,v)$ is small in L^∞ after some time t_0, where t_0 is explicitly computed. The method used here is a variation of an

ODE argument introduced by Corrias and Perthame in [53] for the Keller-Segel model. Higher regularity estimates then imply the smallness of $h(t)$ in a weighted Sobolev norm for $t \in (t_0, 3t_0)$. This quantitative control over higher regularity of h enables us to apply a result from [62] (summarized in Theorem 8.5 below) to conclude that there is a global solution to (1.41) with initial datum $f(2t_0)$. The uniqueness result of [75] (or [52]) allows us to piece together the local-in time solution constructed in the time interval $(0, 2t_0)$ with the global solution on $(2t_0, +\infty)$ to obtain a single global-in-time solution with initial datum f_{in}. Finally, we remove the Schwartz class assumption using a compactness argument and obtain Theorem 8.1.

8.1.4.2 Large data, short time

Here we now present the main result for general initial data, but only for a short amount of time.

Theorem 8.2. *Fix $p > \frac{3}{2}$ and let $m > \frac{9}{2}\frac{p-1}{p-\frac{3}{2}}$. For any given M, H and $f_{in} \in L^1_m \cap L^p(\mathbb{R}^3)$ satisfying*

$$\|f_{in}\|_{L^1_m} \leq M \qquad \text{and} \qquad \int_{\mathbb{R}^3} f_{in} \left|\log(f_{in})\right| \, dv \leq H, \tag{8.4}$$

there exists $0 < T = T(p, m, M, H)$ such that equation (1.41) admits a smooth solution on $(0, T) \times \mathbb{R}^3$ with initial datum f_{in} in the sense that $\lim_{t\to 0^+} \|f(t) - f_{in}\|_{L^p} = 0$. This solution satisfies, for $t \in (0, T)$, the smoothing estimate

$$\|f(t)\|_{L^\infty} \leq C(p, m, M, H) \left(1 + t^{-\beta^*}\right), \tag{8.5}$$

where $\beta^* = \beta^*(p, m)$ is explicitly computable and satisfies $\frac{3}{2p} < \beta^* < 1$. Thus, f is the unique solution to (1.41) in $L^1(0, T; L^\infty)$ with initial data f_{in}.

As a consequence, we deduce the following continuation criterion, which has appeared previously in [93] for radial solutions.

Corollary 8.2. *Fix $p > \frac{3}{2}$ and let $m > \frac{9}{2}\frac{p-1}{p-\frac{3}{2}}$ and f_{in} an initial data satisfying the assumptions of Theorem 8.2. Let f be the corresponding solution constructed in Theorem 8.2. Then, the maximal time T^* for which f can be uniquely continued as a smooth solution is characterized by*

$$\lim_{t \nearrow T^*} \|f(t)\|_{L^p} = +\infty. \tag{8.6}$$

desvillettesHsols The proof of Theorem 8.2 is based on the widely used lower bound $A[f] \gtrsim \langle v \rangle^{-3}$ that suggests that (1.41) should behave like the semi-linear heat equation

$$\partial_t f = \Delta f + f^2.$$

Indeed, the smoothing estimate (8.5) is consistent with the well-known $L^p(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ heat regularisation estimate,

$$\left\| e^{t\Delta} f_{in} \right\|_{L^\infty} \lesssim t^{-\frac{3}{2p}} \left\| f_{in} \right\|_{L^p},$$

up to a correction accounting for the degenerate diffusion at large velocities encoded in the parameter m. Because the semi-linear equation is only locally well-posed for arbitrary initial data in L^p for $p \geq 3/2$, it does not seem that one can go beyond the limitation $p > 3/2$ using solely the lower bound $A[f] \gtrsim \langle v \rangle^{-3}$ and not relying on the full strength of the diffusion. Indeed, [94] suggests that the heat smoothing estimate is suboptimal for Landau, but we cannot see this difference using methods relying on the lower bound $A[f] \gtrsim \langle v \rangle^{-3}$.

As a final consequence of our methods, we are able to adapt our a-priori estimates to apply to rather general weak solutions to (1.41) satisfying a Prodi-Serrin type condition in *unweighted* Lebesgue spaces. We prove a quantitative conditional regularity result which implies a more natural uniqueness class for the solutions constructed in Theorem 8.2, namely $L^\infty(0,T;L^p)$, and implies the stability of these solutions under weak $L^r(0,T;L^p)$ convergence:

Theorem 8.3. *Suppose $g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ is an H-solution[2] to (1.41) such that*

- *$g \in L^r(0,T;L^p(\mathbb{R}^3))$ for some $r > \frac{2p}{2p-3}$ and $p > 3/2$;*
- *$g \in W^{1,\infty}(\Omega)$ for each Ω compactly contained in $(0,T] \times \mathbb{R}^3$;*
- *and $\lim_{t \to 0^+} g(t) = g_{in}$ in the sense of distributions, where $g_{in} \in L^1_m \cap L \log L$ for some $m > m^*(p,r)$.*

Then, $g \in L^1(0,T;L^\infty)$ and g is smooth. Consequently, g is the unique smooth solution to (1.41) in $L^r(0,T;L^p)$ with initial data g_{in}.

The role of the qualitative regularity of g, namely the locally Lipschitz assumption, is merely to guarantee that truncations of g are sufficiently regular to be used as a test functions in a weak formulation of (1.41). This assumption can be replaced by an appropriate notion of suitable weak solution, as in the seminal work of Caffarelli, Kohn, and Nirenberg on the partial regularity for the incompressible Navier-Stokes equations. However, we expect that in the $L^r(0,T;L^p)$ setting, this assumption is unnecessary and could potentially be removed by using a mollified version of g as a test function and sending the mollification parameter to 0^+ using the DiPerna-Lions commutator estimates,

[2] More precisely, g is a solution in the sense of distributions to the collisional form of the homogeneous Landau equation with constant mass, momentum, and energy, decreasing entropy, and bounded entropy production. As a consequence of [58], the bounded entropy production implies $\nabla \sqrt{g} \in L^2(0,T;L^2_{-3})$.

another tool frequently used in fluid mechanics. We do not investigate these directions further.

One can explicitly compute that $m^* = \frac{9}{2}\frac{p-1}{p-3/2}$ when $r = \infty$. As a result, we find that the solution constructed in Theorem 8.2 is the unique locally Lipschitz solution belonging to $L^\infty(0,T;L^p)$. This uniqueness class is the same one considered by Chern and Gualdani in [52], up to our additional qualitative regularity assumption. Although the estimates presented in [52] are sound as a-priori estimates, it remains unclear whether all low regularity solutions satisfy the same estimates in the absence of additional regularity hypotheses. Our assumption that g is locally Lipschitz is used in the proof of Theorem 8.3 to rigorously justify the uniqueness result of [52], which is especially intricate for $p \in (3/2, 2)$.

The term Prodi-Serrin condition originates in [8], where the authors show the following conditional regularity result, analogous to the classical Prodi-Serrin result for the three-dimensional incompressible Navier-Stokes equation: If f is a suitable solution to (1.41) with initial data in L^1_m for some sufficiently large m and

$$\langle\cdot\rangle^3 f \in L^r((0,T);L^p(\mathbb{R}^3)), \quad \frac{2}{r}+\frac{3}{p}=2, \quad 1 \le r,\, p \le \infty, \quad p \neq \frac{3}{2}, \tag{8.7}$$

then $f \in L^\infty(t,T;L^\infty)$ for each $t > 0$ and, consequently, f is instantaneously smooth. The core of their argument is a novel proof of an ε-Poincaré inequality, which we use in Section 8.4.4. Note that except in the case $r = 1$, $p = \infty$, it remains unclear whether assumption (8.7) is enough to guarantee uniqueness. The main contribution of Theorem 8.3 is that for the full range sub-critical range of parameters, namely $\frac{2}{r} + \frac{3}{p} < 2$, one can rigorously prove that uniqueness holds as a consequence of smoothing estimates. We expect that uniqueness in the critical case, i.e. under assumption (8.7), requires a significantly different proof, because by analogy to the semi-linear heat equation, one expects only the estimate

$$\|f(t)\|_{L^\infty} \le \frac{C}{t},$$

which does not imply $f \in L^1(0,T;L^\infty)$.

8.1.5 Outline

This chapter is organised as follows: In Section 8.2, we list known results that are central to our proof of both, the perturbative result and the short time result.

Then we focus first on the case of small initial data for which we construct a global-in-time solution. In Section 8.3.1, we analyse the evolution of $f - \mu$ in L^p and show that the L^p norm *remains small* for a short time. In Section 8.3.2, we use the De Giorgi method to prove short time smoothing estimates. Qualitatively, this implies that if the initial

datum has small L^p norm, the solution will have small L^∞ norm after some time passes. In Section 8.3.3, we prove Proposition 8.1, which concludes the argument for Schwartz class initial data. In Section 8.3.4, we deduce Theorem 8.1 from Proposition 8.1 using a compactness argument to remove the Schwartz class assumption on the initial data.

Then, we move to the proof of Theorem 8.2 for large data. We derive two a-priori estimates: In Section 8.4.1, we use an ODE argument to show propagation of L^p norms of the initial datum for $p > 3/2$ for some quantitative short time. In Section 8.4.2, we use a new De Giorgi type argument to show quantitative L^∞ regularisation, for sufficiently integrable solutions. A combination of these estimates yields the quantitative short-time smoothing estimates claimed in (8.5). Then, in Section 8.4.3, we use a compactness argument and the estimates (8.5) to build our smooth solutions and conclude the proof of Theorem 8.2 and Corollary 8.2. Finally, in Section 8.4.4, we show how to use our a-priori estimates to obtain the result in Theorem 8.3.

8.2 Preliminaries

We collect here known results that are used without proof.

8.2.1 Existence results

8.2.1.1 Local

We begin with a local-in-time well-posedness result of Henderson, Snelson, and Tarfulea [107]. We present the result in a simplified form, suitable for our purposes:

Theorem 8.4 (From [107, Theorem 1.2 and Theorem 1.3])**.** *Suppose $f_{in} \in \mathcal{S}(\mathbb{R}^3)$ is non-negative. Then, there is a time $0 < T^* \leq \infty$ and a function $f : [0, T^*) \times \mathbb{R}^3 \to \mathbb{R}^+$ with $f \in C^\infty(0, T^*; \mathcal{S}(\mathbb{R}^3))$ such that $[0, T^*)$ is the maximal time interval on which f is the unique classical solution to* (1.41) *with initial datum f_{in}. Furthermore, if $T^* < \infty$, then f satisfies*

$$\lim_{t \nearrow T^*} \|f(t)\|_{L^\infty} = +\infty.$$

8.2.1.2 Global

We continue with a global-in-time existence result for small data shown by Desvillettes, He and Jiang in [62]:

Theorem 8.5 ([62, Proposition 1.2])**.** *Suppose $\|f_{in}\|_{L \log L} + \|f_{in}\|_{L^1_{55}} \leq K$ and $f_{in} \in H^1_2$ is a non-negative initial datum. Let μ be the corresponding Maxwellian. Then, there is an*

$\varepsilon_1 = \varepsilon_1(K) > 0$ *and* $C = C(K) > 0$ *such that if*

$$\|\nabla f_{in} - \nabla\mu\|_{L^2_2} + \|f_{in} - \mu\|_{L^2_1} < \varepsilon_1,$$

then there is a global solution $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^+$ *to* (1.41)*, satisfying*

$$\|f(t) - \mu\|_{H^1_2} \leq \frac{C}{(1+t)^{15/4}}.$$

8.2.2 Uniqueness

We conclude this section by recalling a uniqueness result due to Fournier:

Theorem 8.6 (Uniqueness [75, Theorem 2])**.** *Suppose* $f,\ g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ *are solutions to* (1.41) *in the sense of distributions with the same initial datum* $f_{in} \in L^1_2$ *and both belong to* $L^\infty(0,T;L^1_2) \cap L^1(0,T;L^\infty)$*. Then,* $f(t) = g(t)$ *for all* $0 < t < T$*.*

8.2.3 Moments

We will frequently use that (1.41) propagates L^1-moments of any order. More precisely, L^1-moments of any order $s > 2$ grow at most linearly in time.

Lemma 8.1 (From [45, Lemma 2.1])**.** *Let* $k > 2$*. Fix a non-negative initial datum* $f_{in} \in L^1_k \cap L\log L$ *satisfying the normalisation* (1.46)*. Suppose* $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^+$ *is any weak solution of* (1.41) *with initial datum* f_{in}*. Then, for each* $t \geq 0$*,*

$$\int_{\mathbb{R}^3} f(t,v) \langle v\rangle^k \, \mathrm{d}v \leq C(1+t). \tag{8.8}$$

We also use a moment bound on $h = f - \mu$.

Lemma 8.2 ([62, Lemma 2.1])**.** *Let* $l > \frac{19}{2}$*. Fix a non-negative initial datum* $f_{in} \in L^1_l \cap L\log L$ *such that* $\int f_{in}\,\mathrm{d}v = 1$*,* $\int f_{in}\,|v|^2\,\mathrm{d}v = 3$*. Suppose* $f : \mathbb{R}^3 \to \mathbb{R}$ *is any weak solution of* (1.41) *and let* $h := f - \mu$*. Then for all* $\theta \in [0,l]$ *and* $q < q_{l,\theta}$ *with*

$$q_{l,\theta} = -\frac{2l^2 - 25l + 57}{18(l-2)}\Big(1 - \frac{\theta}{l}\Big) + \frac{\theta}{l}, \tag{8.9}$$

there exists a constant $C_3 = C_3(\theta, l, K)$*, where* K *is such that* $\|f_{in}\|_{L^1_l} + \|f_{in}\|_{L\log L} \leq K$*, such that*

$$\forall t \geq 0, \qquad \|h(t,\cdot)\|_{L^1_\theta} \leq C_3(1+t)^q. \tag{8.10}$$

8.2.4 Coefficients

8.2.4.1 Lower bound

We generally require mass, energy and entropy of f to be bounded uniformly. If we know that $f \in L^p$ for some $p > 1$, then the entropy bound follows naturally from the L^p bound.

Lemma 8.3. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}^+$ satisfies for any $p > 1$*

$$0 < m_0 \leq \int_{\mathbb{R}^3} f \, \mathrm{d}v \leq M_0, \qquad \int_{\mathbb{R}^3} |v|^2 f \, \mathrm{d}v \leq E_0, \qquad \int_{\mathbb{R}^3} f^p \, \mathrm{d}v \leq K.$$

Then,

$$\int_{\mathbb{R}^3} f \left|\log(f)\right| \, \mathrm{d}v \leq C(m_0, M_0, E_0, K, p).$$

We then state the uniform lower bound satisfied by the diffusion coefficient. Note that by the previous Lemma, we can relax the entropy bound by the L^p bound on f, if $p > 1$. This ellipticity result is by now standard in the literature.

Lemma 8.4. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}^+$ satisfies*

$$0 < m_0 \leq \int_{\mathbb{R}^3} f \ \mathrm{d}v \leq M_0, \qquad \int_{\mathbb{R}^3} |v|^2 f \ \mathrm{d}v \leq E_0, \qquad \int_{\mathbb{R}^3} f \left|\log(f)\right| \ \mathrm{d}v \leq H_0.$$

Then, there is a constant $c_0 = c_0(m_0, M_0, E_0, H_0)$ such that

$$A[f](v) \geq \frac{c_0}{\langle v \rangle^3} I.$$

8.2.4.2 Upper bounds

The following lemma controls the degenerate diffusion associated to (1.41) and quantifies the parabolic nature associated to the Landau equation when written in the form (1.41). Each of these coefficient bounds stated here are now standard and can be found in many places in the literature, [23, Lemma 3.1], [82, Lemma 2.1], [178, Lemma 3.2], [44, Lemma 4.2].

Lemma 8.5 (Coefficient bounds)**.** *For each $1 \leq p < 3/2 < q \leq \infty$,*

$$\|A[f]\|_{L^\infty} \lesssim \|f\|_{L^1}^{\frac{2q-3}{3q-3}} \|f\|_{L^q}^{\frac{q}{3q-3}} \qquad \text{and} \qquad \|A[f]\|_{L^\infty} \lesssim \|f\|_{L^p}^{\frac{2p}{3}} \|f\|_{L^\infty}^{\frac{3-2p}{3}}.$$

Furthermore, for each $1 \leq p < 3 < q \leq \infty$,

$$\|\nabla a[f]\|_{L^\infty} \lesssim \|f\|_{L^1}^{\frac{q-3}{3q-3}} \|f\|_{L^q}^{\frac{2q}{3q-3}} \qquad \textit{and} \qquad \|\nabla a[f]\|_{L^\infty} \lesssim \|f\|_{L^p}^{\frac{p}{3}} \|f\|_{L^\infty}^{\frac{3-p}{3}}.$$

Furthermore, if $3/2 < q \leq \infty$ *then*

$$\|\nabla a[f]\|_{L^3} \leq C\|f\|_{L^1}^{\frac{2}{3}\frac{q-\frac{3}{2}}{q-1}} \|f\|_{L^q}^{\frac{1}{3}\frac{q}{q-1}}.$$

Furthermore, if $1 < q < \infty$ *then*

$$\left\|\nabla^2 a[f]\right\|_{L^q} \leq C\|f\|_{L^q}.$$

Finally, for any $3 < p \leq \infty$ *and* $m > 3\,(p-1)$,

$$|\nabla a[f](v)| + |\nabla A[f](v)| \lesssim \langle v\rangle^{-2} \|f\|_{L^p_m}.$$

8.2.5 Interpolation, Sobolev and Sawyer-Wheeden

Very frequently we will use the following weighted Lebesgue interpolation result, which is a simple consequence of Hölder's inequality.

Lemma 8.6. *Suppose* $\theta \in (0,1)$, $1 \leq q, p_1, p_2 \leq \infty$, *and* $\beta, \alpha_1, \alpha_2 \in \mathbb{R}$ *satisfy the relations*

$$\frac{1}{q} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \textit{and} \quad \beta = \theta\alpha_1 + (1-\theta)\alpha_2.$$

Then, for any $f \in L^{p_1}_{\alpha_1} \cap L^{p_2}_{\alpha_2}$, $f \in L^q_\beta$ *and*

$$\left\| \langle v\rangle^{\beta} f\right\|_{L^q} \leq \left\| \langle v\rangle^{\alpha_1} f\right\|_{L^{p_1}}^{\theta} \left\| \langle v\rangle^{\alpha_2} f\right\|_{L^{p_2}}^{1-\theta}.$$

Finally, we repeatedly use weighted interpolation and weighted Sobolev estimates. For the convenience of the reader, we collect these estimates in the following lemma. Its proof may be found in the Appendix 8.5, adapting the extensive literature on weighted Sobolev inequalities (see for instance [175]).

Lemma 8.7 (Sobolev and Interpolation Inequalities)**.** *Let* $1 \leq s \leq 6$. *Then, there are universal constants* $C_1(s)$ *and* $C_2(s)$ *such that for all* $f \in \mathcal{S}(\mathbb{R}^3)$

$$\left(\int_{\mathbb{R}^3} |f|^6 \langle v\rangle^{-9}\, dv\right)^{1/3} \leq C_1 \int_{\mathbb{R}^3} |\nabla f(v)|^2 \langle v\rangle^{-3}\, \mathrm{d}v + C_2 \left(\int_{\mathbb{R}^3} |f|^s\, \mathrm{d}v\right)^{2/s}. \tag{8.11}$$

Moreover, for each $k \geq 3$, *there are* $C_1 = C_1(k) > 0$ *and* $C_2 = C_2(k) > 0$ *so that for all* $f \in \mathcal{S}(\mathbb{R}^3)$,

$$\left(\int_{\mathbb{R}^3} \langle v\rangle^{3k-9} f^6\, \mathrm{d}v\right)^{\frac{1}{3}} + C_1 \int_{\mathbb{R}^3} \langle v\rangle^{k-5} f^2\, \mathrm{d}v \leq C_2 \int_{\mathbb{R}^3} \langle v\rangle^{k-3} |\nabla f|^2\, \mathrm{d}v. \tag{8.12}$$

Finally, let $1 < p < q < 3p < \infty$. *Then, for any* $f \in \mathcal{S}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \langle v \rangle^k f^q \leq C(k,p) \left(\int_{\mathbb{R}^3} \langle v \rangle^m f^p \, \mathrm{d}v \right)^{\frac{3p-q}{2p}} \left(\int_{\mathbb{R}^3} \langle v \rangle^{k-3} \left| \nabla f^{p/2} \right|^2 \, \mathrm{d}v \right)^{\frac{3(q-p)}{2p}}, \tag{8.13}$$

where

$$m = \frac{2kp - (k-3)(3q-3p)}{3p-q}.$$

8.3 Small data, long time

8.3.1 Local-in-Time Propagation of L^p Norms

In this section, we study the evolution of L^p-norms for $\frac{3}{2} < p < \infty$ of $h = f - \mu$, where f is a solution to (1.41) and μ is the corresponding Maxwellian. We show that the L^p-norms of initial datum h_{in} are propagated by smooth solutions of (1.41), at least for some short time interval $[0, T_0]$. Since we work with smooth solutions, the calculations in this section are rigorous estimates. We note that for $p = 2$ a version of the following estimate has previously appeared as an a-priori bound in [4].

Lemma 8.8. *Fix* $\frac{3}{2} < p < \infty$, $m > \frac{9}{2} \frac{p-1}{p-\frac{3}{2}}$, $H \in \mathbb{R}$ *and* $M \in \mathbb{R}^+$. *Suppose* f_{in} *is a non-negative Schwartz class function, satisfying the normalisation* (1.46) *and the bounds*

$$\|f_{in}\|_{L^1_m} \leq M \qquad \text{and} \qquad \int_{\mathbb{R}^3} f_{in} \left| \log(f_{in}) \right| \, dv \leq H. \tag{8.14}$$

Let $f : [0, T^*) \to \mathbb{R}^+$ *be the unique Schwartz class solution to* (1.41) *with initial datum* f_{in} *on its maximal interval of existence* $[0, T^*)$. *Then, there is an* $\varepsilon_0(m, p, H, M)$ *sufficiently small such that for any* $0 < \varepsilon < \varepsilon_0$, *there are* $\delta = \delta(\varepsilon, m, p, H, M)$ *and* $T_0 = T_0(\varepsilon, m, p, H, M) \in (0, 1]$ *such that for* $h := f - \mu$, *if* $\|h_{in}\|_{L^p}^p < \delta$ *then*

$$\sup_{0 < t < \min(T_0, T^*)} \left(\|h(t)\|_{L^p}^p + \frac{c_0}{2} \int_0^t \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left| \nabla h^{p/2} \right|^2 \, dv ds \right) \leq \varepsilon, \tag{8.15}$$

where $c_0 = c_0(M, H)$ *is the constant from Lemma 8.4. Moreover,* T_0 *may be chosen such that*

$$C(m, p, H, M) \varepsilon^{1-\alpha} \leq T_0,$$

where $\alpha = \left(1 - \frac{1}{p} + \frac{1}{3(p-1)} \right) \in (0, 1)$.

Remark 8.1. In the proof of Lemma 8.8, we study the evolution of the L^p-norm of h via an ODE method. The L^p norm satisfies a differential inequality of the form,

$$y' \leq y + y^\alpha, \qquad \alpha \in (0, 1).$$

For small data, the sublinear evolution dominates and restricts our estimate to a small time interval (of order $\varepsilon^{1-\alpha}$).

Lemma 8.8 provides a quantitative estimate on how (1.41) preserves L^p-smallness for a short amount of time. This is the first step towards a global-in-time existence result for small L^p initial data. One might hope that this argument could extend to large initial data, at least to show local-in-time well-posedness. However, the ODE method used in the proof of Lemma 8.8 requires smallness of the initial data and does not seem to yield a corresponding result for large data.

Proof. We start by noting that μ is a steady state solution to (1.41), so that h satisfies

$$\partial_t h = \nabla \cdot (A[f]\nabla h - \nabla a[f] h) + A[h] : \nabla^2 \mu + h\mu. \tag{8.16}$$

Testing (8.16) with h^{p-1}, integrating by parts, and rearranging yields

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}\|h(t)\|_{L^p}^p}{\mathrm{d}t} &+ \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3} A[f]\nabla h^{p/2}\cdot\nabla h^{p/2}\,\mathrm{d}v \\ &= \frac{p-1}{p}\int_{\mathbb{R}^3} h^{p+1}\,\mathrm{d}v + \frac{2p-1}{p}\int_{\mathbb{R}^3} h^p\mu\,\mathrm{d}v + \int_{\mathbb{R}^3} h^{p-1}\left(A[h]:\nabla^2\mu\right)\,\mathrm{d}v \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{8.17}$$

Beginning with I_1, we use Hölder and a weighted Sobolev inequality to get:

$$\begin{aligned} &\int_{\mathbb{R}^3} h^{p+1}\,\mathrm{d}v \\ &\quad= \int_{\mathbb{R}^3}\left(\frac{h^{p/2}}{\langle v\rangle^{3/2}}\right)^2 \langle v\rangle^3 h\,\mathrm{d}v \\ &\quad\le \left(\int_{\mathbb{R}^3}\left(h^{p/2}\right)^6\langle v\rangle^{-9}\,\mathrm{d}v\right)^{1/3}\left(\int_{\mathbb{R}^3} h^{3/2}\langle v\rangle^{9/2}\,\mathrm{d}v\right)^{2/3} \\ &\quad\le \left[C_1\int_{\mathbb{R}^3}\left|\nabla h^{p/2}\right|^2\langle v\rangle^{-3}\,\mathrm{d}v + C_2\left(\int_{\mathbb{R}^3} h^{3p/4}\,\mathrm{d}v\right)^{4/3}\right]\left(\int_{\mathbb{R}^3} h^{3/2}\langle v\rangle^{9/2}\,\mathrm{d}v\right)^{2/3}, \end{aligned} \tag{8.18}$$

using (8.11) of Lemma 8.7 with $s = 3/2$. Hölder's inequality and that $p > 3/2$ yield:

$$\left(\int_{\mathbb{R}^3} h^{3/2}\langle v\rangle^{9/2}\,\mathrm{d}v\right)^{2/3} \le \left(\int_{\mathbb{R}^3} h^p\,\mathrm{d}v\right)^{\frac{1}{3(p-1)}}\left(\int_{\mathbb{R}^3} h\langle v\rangle^{\frac{9}{2}\frac{p-1}{p-3/2}}\,\mathrm{d}v\right)^{\frac{2}{3}\frac{p-3/2}{p-1}}$$

$$\left(\int_{\mathbb{R}^3} h^{3p/4}\,\mathrm{d}v\right)^{4/3} \le \left(\int_{\mathbb{R}^3} h^p\,\mathrm{d}v\right)^{\frac{p-4/3}{(p-1)}}\left(\int_{\mathbb{R}^3} h\,\mathrm{d}v\right)^{\frac{p}{3(p-1)}}$$

Summarising, the term I_1 is bounded as

$$I_1 \leq C_1 \left\| \langle v \rangle^{-3/2} \nabla h^{p/2} \right\|_{L^2}^2 \|h\|_{L^p}^{\frac{p}{3(p-1)}} \|h \langle v \rangle^m\|_{L^1}^{\frac{2}{3}\frac{p-\frac{3}{2}}{p-1}} + C_2 \|h\|_{L^p}^p \|h \langle v \rangle^m\|_{L^1}.$$

The term I_2 is bounded via

$$I_2 \leq 2 \|h\|_{L^p}^p \|\mu\|_{L^\infty}.$$

The term I_3 is bound using Hölder's inequality, Lemma 8.5, and conservation of mass, as

$$I_3 \leq \|A[h]\|_{L^\infty} \|h\|_{L^p}^{p-1} \|\mu\|_{W^{2,p}} \leq \|\mu\|_{W^{2,p}} \|h\|_{L^p}^{p-1} \|h\|_{L^1}^{1-\frac{p}{3(p-1)}} \|h\|_{L^p}^{\frac{p}{3(p-1)}} \leq C(p) \|h\|_{L^p}^{p\left(1-\frac{1}{p}+\frac{1}{3(p-1)}\right)}.$$

In summary, we have shown

$$\begin{aligned} \frac{\mathrm{d}\|h(t)\|_{L^p}^p}{\mathrm{d}t} + \int_{\mathbb{R}^3} A[f] \nabla h^{p/2} \cdot \nabla h^{p/2} \, \mathrm{d}v \\ \leq C(p) \Bigg(\left\| \langle v \rangle^{-3/2} \nabla h^{p/2} \right\|_{L^2}^2 \|h\|_{L^p}^{\frac{p}{3(p-1)}} \|h \langle v \rangle^m\|_{L^1}^{\frac{2}{3}\frac{p-\frac{3}{2}}{p-1}} \\ + \|h\|_{L^p}^p \|h \langle v \rangle^m\|_{L^1} + \|h\|_{L^p}^p + \|h\|_{L^p}^{p\alpha} \Bigg), \end{aligned}$$

where $\alpha = \left(1 - \frac{1}{p} + \frac{1}{3(p-1)}\right)$. The coercivity estimate from Lemma 8.4 and the moment estimates from Lemma 8.2 then imply that there are constants $\overline{M} = \overline{M}(M, m, H)$ and $c_0 = c_0(H)$ for which

$$\frac{c_0}{\langle v \rangle^3} \leq A[f] \qquad \text{and} \qquad \sup_{0<t<\min(T^*,1)} \|h(t)\|_{L^1_m} \leq \overline{M},$$

and thus for a fixed constant $\tilde{C} = \tilde{C}(p)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|h\|_{L^p}^p \leq -\left(\int_{\mathbb{R}^3} |\nabla h^{p/2}|^2 \langle v \rangle^{-3} \, \mathrm{d}v \right) \left(c_0 - \tilde{C} \|h\|_{L^p}^{\frac{p}{3(p-1)}} \overline{M}^{\frac{2}{3}\frac{p-\frac{3}{2}}{p-1}} \right) \\ + \tilde{C}(\overline{M}+1)\|h\|_{L^p}^p + \tilde{C}\|h\|_{L^p}^{p\alpha}, \end{aligned} \tag{8.19}$$

for $0 \leq t < \min(1, T^*)$. Introducing y, G and N as

$$y(t) := \|h(t)\|_{L^p}^p \quad \text{and} \quad G(t) := \left(\int_{\mathbb{R}^3} |\nabla h^{p/2}|^2 \langle v \rangle^{-3} \, \mathrm{d}v \right),$$

and

$$N(y) := \left(c_0 - \tilde{C} y^{\frac{1}{3(p-1)}} \overline{M}^{\frac{2}{3}\frac{p-\frac{3}{2}}{p-1}} \right),$$

we have shown in (8.19) that y satisfies the differential inequality

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq -G(t)N(y) + \tilde{C}(\overline{M}+1)y + \tilde{C}y^{\alpha}. \tag{8.20}$$

We prefer to study (8.20) in integral form: Integrating from 0 to t, we find for any $0 < t < \min(T^*, 1)$,

$$y(t) \leq y_0 + \tilde{C}\int_0^t (\overline{M}+1)y(s) + y(s)^{\alpha}\, ds - \int_0^t G(s)N(y(s))\, ds. \tag{8.21}$$

We are now ready to fix our choice of parameters ε_0, T_0, and δ. First, we choose ε_0 as

$$\varepsilon_0 = \left(\frac{c_0^{3(p-1)}}{(2\tilde{C})^{3(p-1)}\overline{M}^{2p-3}}\right), \qquad \text{so that} \qquad N(y) > \frac{c_0}{2} \quad \text{if and only if} \quad y < \varepsilon_0.$$

Second, for any $\varepsilon \in (0, \varepsilon_0)$, we pick T_0 and δ as

$$T_0 = (3\tilde{C})^{-1}\min\left(\left(\overline{M}+1\right)^{-1}, \varepsilon^{1-\alpha}\right) \qquad \text{and} \qquad \delta < \varepsilon/3. \tag{8.22}$$

The choice of δ guarantees $y_0 < \varepsilon$ and consequently $N(y_0) > c_0/2$. Therefore, if $y_0 < \delta$, by continuity of y, either $y(t) < \varepsilon$ for all $0 \leq t \leq \min(T^*, 1)$ or there is a first time $t_0 \in (0, \min(T^*, 1))$ such that $y(t_0) = \varepsilon$. By definition of t_0, we then have $y(t) < \varepsilon$ and consequently $N(y(t)) > c_0/2$ for $0 \leq t < t_0$. So, for any $0 < t < \min(t_0, T_0)$, we combine (8.21) with the choice of T_0 in (8.22) and the non-negativity of G to find

$$y(t) < \varepsilon/3 + \tilde{C}(\overline{M}+1)\varepsilon t + \tilde{C}\varepsilon^{\alpha}t - \frac{c_0}{2}\int_0^t G(s)\, dt < \varepsilon. \tag{8.23}$$

We find $t_0 \geq T_0$ and rearranging (8.23) yields the desired estimate

$$\sup_{0<t<\min(T_0,T^*)} y(t) + \frac{c_0}{2}\int_0^{\min(T_0,T^*)} G(s)\ \mathrm{d}t < \varepsilon.$$

The explicit choice of T_0 in (8.22) yields the claimed asymptotic behavior. □

8.3.2 Quantitative L^{∞} Regularisation

In this section, we prove a quantitative regularisation estimate for the L^{∞} norm of smooth solutions to (1.41). More precisely, we show that for smooth solutions to (1.41), the L^{∞} norm is controlled by the energy functional that appears in (8.15). Our ultimate goal is to construct solutions to (1.41) for initial data that are close to equilibrium in L^p, even for rough profiles. Hence, we ensure that the estimate depends only on L^1 moments and entropy, making the estimate independent of the solution's smoothness. The proof,

inspired by [7], is a modification of the iteration procedure initially introduced by De Giorgi in [56] for the study of elliptic equations. A similar version of our estimates has already been obtained by Silvestre in [178]. One could possibly replace the forthcoming estimates with Silvestre's estimates to obtain a similar result to Theorem (8.1), where smallness is instead required in a weighted L^p space. We prefer to assume smallness in the strictly weaker unweighted L^p norm and so prove our own regularisation estimates:

Proposition 8.2. *Fix $p > \frac{3}{2}$, $m > \frac{9}{2}\frac{p-1}{p-\frac{3}{2}}$, $H \in \mathbb{R}$ and $M \in \mathbb{R}^+$. Suppose f_{in} is a non-negative Schwartz class function, satisfying the normalisation* (1.46) *and the bounds*

$$\|f_{in}\|_{L^1_m} \leq M \qquad \text{and} \qquad \int_{\mathbb{R}^3} f_{in} \left|\log(f_{in})\right| \, dv \leq H.$$

Let $f : [0, T^*) \to \mathbb{R}^+$ be the unique Schwartz class solution to (1.41) *with initial datum f_{in} and $T^* > 0$ its maximal time of existence. Then, for γ, β_0, β_1 and β_2, defined in terms of m and p via*

$$\beta_0 = \frac{1}{3(p-1)}, \qquad \beta_1 = \frac{2}{3} - \frac{3}{m}, \qquad \beta_2 = \frac{3}{m}, \qquad \gamma = \frac{2p-3}{3m}\left[m - \frac{9(p-1)}{2p-3}\right], \quad (8.24)$$

there is a constant $C = C(p, m, H, M)$ such that for any $0 < t < T^*$ there holds

$$\|h(t)\|_{L^\infty(\mathbb{R}^3)} \leq C(1+t)^{\frac{1+\beta_2}{\gamma}} \left(E_0^{\frac{\beta_1}{\gamma}} + E_0^{\frac{\beta_1}{1+\gamma}} + E_0^{\frac{\beta_1}{2+\gamma}} + E_0^{\frac{\beta_1+\beta_0}{2+\gamma}} + \left(E_0^{\beta_1} t^{-1}\right)^{\frac{1}{\gamma+1}} \right),$$

where E_0 is defined via

$$E_0 := \sup_{0<t<T^*} \|h(t)\|_{L^p}^p + \int_0^{T^*} \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left|\nabla h^{p/2}\right|^2 \, \mathrm{d}v \, \mathrm{d}t.$$

Remark 8.2. It is not clear whether the short-time smoothing rates controlled by γ and β_1 in Proposition 8.2 are optimal, but we have attempted to sharpen this gain as much as possible within our method. By contrast, the exponent β_2, which governs the long-time decay rate in Proposition 8.2, is quite far from optimal. For instance, rather than using the more complicated decay of L^1 moments of h found in Lemma 8.2 above, we frequently choose to bound the L^1 norm via the far simpler estimate

$$\|h(t)\|_{L^1_m} \leq \|\mu\|_{L^1_m} + \|f(t)\|_{L^1_m} \leq C(m, M, H)(1+t). \quad (8.25)$$

Despite our stated goal of global-in-time existence for small data, our approach relies upon quantitative short-time smoothing estimates, for which (8.25) is sufficient.

Step 1: Energy estimate for level set functions (Differential Form)

For any fixed $\ell \in \mathbb{R}^+$, we consider the part of h above ℓ, denoted by

$$h_\ell(t,v) := h(t,v) - \ell, \quad h_\ell^+(t,v) = \max\left(h_\ell(t,v), 0\right).$$

Our first goal is to derive an energy estimate on these level set functions, which is contained in the following lemma:

Lemma 8.9. *Suppose m, p, M, H, and h are as in Proposition 8.2. Then, there is a constant $C = C(p,m)$ such that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(h_\ell^+)^p\,\mathrm{d}v + c_0\int_{\mathbb{R}^3}\left|\nabla(h_\ell^+)^{\frac{p}{2}}\right|^2\langle v\rangle^{-3}\,\mathrm{d}v \\ \leq C\left[\frac{1}{(\ell-k)^\gamma} + \frac{1+\ell}{(\ell-k)^{1+\gamma}} + \frac{1+\ell+E^{\beta_0}(t)+\ell^2}{(\ell-k)^{2+\gamma}}\right] \\ \times\left[\left(\left\|\langle\cdot\rangle^{-\frac{3}{2}}\nabla(h_k^+)^{\frac{p}{2}}\right\|_{L^2}^2 + \left\|h_k^+\right\|_{L^p}^p\right)\left\|h_k^+\right\|_{L^p}^{p\left(\frac{q-p-\frac{2}{3}}{p-1}\right)}\left\|h_k^+\right\|_{L^1_m}^{\frac{3}{2m}}\right],\end{aligned} \tag{8.26}$$

where $c_0 = c_0(M,H)$ is the constant from Lemma 8.4 and

$$q = \frac{5}{3}p - \frac{3(p-1)}{m}, \qquad \gamma = q - (p+1), \qquad \beta_0 = \frac{1}{3(p-1)}, \qquad E(t) = \|h(t)\|_{L^p}^p.$$

For the proof of this lemma, we test the weak formulation of (8.16) with $(h_\ell^+)^{p-1}$. Using that $\partial_t h_\ell^+ = \partial_t h \chi_{\{h\geq\ell\}}$ and $\nabla_v h_\ell^+ = \nabla_v h \chi_{\{h\geq\ell\}}$, we rearrange terms as in the proof of Lemma 8.8 to obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(h_\ell^+)^p\,\mathrm{d}v + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}A[h]\left|\nabla(h_\ell^+)^{\frac{p}{2}}\right|^2 \\ \leq (p-1)\int_{\mathbb{R}^3}\nabla h_\ell^+ h\nabla a[f](h_\ell^+)^{p-2}\,\mathrm{d}v \\ + \int_{\mathbb{R}^3}(h_\ell^+)^{p-1}\left(A[h]:\nabla^2\mu\right)\,\mathrm{d}v + \int_{\mathbb{R}^3}h\mu(h_\ell^+)^{p-1}\,\mathrm{d}v.\end{aligned} \tag{8.27}$$

For the first term on the right hand side, we integrate by parts and continue adding and

subtracting to remove the h and f, until we have more terms, each involving only h_ℓ^+:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla h_\ell^+ h \nabla a[f](h_\ell^+)^{p-2}\,\mathrm{d}v \\
&\quad= \int_{\mathbb{R}^3} \nabla h_\ell^+ \nabla a[f](h_\ell^+)^{p-1}\,\mathrm{d}v + \ell \int_{\mathbb{R}^3} \nabla h_\ell^+ \nabla a[f](h_\ell^+)^{p-2}\,\mathrm{d}v \\
&\quad= \frac{1}{p}\int_{\mathbb{R}^3} f(h_\ell^+)^p\,\mathrm{d}v + \frac{\ell}{p-1}\int_{\mathbb{R}^3} f(h_\ell^+)^{p-1}\,\mathrm{d} \\
&\quad= \frac{1}{p}\int_{\mathbb{R}^3} (h_\ell^+)^{p+1}\,\mathrm{d}v + \frac{\ell}{p-1}\int_{\mathbb{R}^3} (h_\ell^+)^p\,\mathrm{d}v \\
&\qquad+ \frac{1}{p}\int_{\mathbb{R}^3} (\mu+\ell)(h_\ell^+)^p\,\mathrm{d}v + \frac{\ell}{p-1}\int_{\mathbb{R}^3} (\mu+\ell)(h_\ell^+)^{p-1}\,\mathrm{d}v \\
&\quad\le C(p)\int_{\mathbb{R}^3} (h_\ell^+)^{p+1}\,\mathrm{d}v + C(p)\,(\ell+1)\int_{\mathbb{R}^3} (h_\ell^+)^p\,\mathrm{d}v \\
&\qquad+ C(p)\left(\ell+\ell^2\right)\int_{\mathbb{R}^3} (h_\ell^+)^{p-1}\,\mathrm{d}v.
\end{aligned}$$

Bounding the remaining (linear) terms on the right hand side of (8.27), we find

$$\begin{aligned}
&\int_{\mathbb{R}^3} (h_\ell^+)^{p-1}\left(A[h] : \nabla^2\mu\right)\,\mathrm{d}v + \int_{\mathbb{R}^3} h\mu(h_\ell^+)^{p-1}\,\mathrm{d}v \\
&\qquad\le C\left\|A[h]\right\|_{L^\infty}\int (h_\ell^+)^{p-1}\,\mathrm{d}v + \int (h_\ell^+)^p\,\mathrm{d}v + \ell\int (h_\ell^+)^{p-1}\,\mathrm{d}v \\
&\qquad\le \left(1+CE(t)^{\beta_0}\right)\int (h_\ell^+)^{p-1}\,\mathrm{d}v + \int (h_\ell^+)^p\,\mathrm{d}v + \ell\int (h_\ell^+)^{p-1}\,\mathrm{d}v,
\end{aligned}$$

where $E(t) = \|h(t)\|_{L^p}^p$ and $\beta_0 = \frac{1}{3(p-1)}$ come from the upper bound for A in Lemma 8.5 and the constant C depends only on p as the mass of f is fixed.

Due to the lower bound on $A[f]$ in Lemma 8.4, we conclude for some universal constant C depending only on p,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (h_\ell^+)^p\,\mathrm{d}v + c_0\int_{\mathbb{R}^3} \left|\nabla (h_\ell^+)^{\frac{p}{2}}\right|^2 \langle v\rangle^{-3}\,\mathrm{d}v \\
&\qquad\le C\int_{\mathbb{R}^3} (h_\ell^+)^{p+1}\,\mathrm{d}v + C(1+\ell)\int_{\mathbb{R}^3} (h_\ell^+)^p\,\mathrm{d}v \\
&\qquad\quad+ C\left(1+E^{\beta_0}+\ell+\ell^2\right)\int_{\mathbb{R}^3} (h_\ell^+)^{p-1}\,\mathrm{d}v.
\end{aligned} \tag{8.28}$$

After obtaining an energy estimate like (8.28), the standard way of applying the De Giorgi approach to parabolic equations is to first integrate over time and use a Sobolev embedding in the variable v to obtain control of the norms $L_t^\infty L_v^p$ and $L_t^p L_v^{p^*}$, where p^* is the Sobolev conjugate of p. Next, one interpolates to gain control of $L_{t,v}^q$ for some $q > p$. In the case of the heat equation, this results in $q = 10/3$, providing improved integrability. The De Giorgi method then repeats this process iteratively, ultimately concluding a quantitative $L_{t,v}^\infty$

bound. However, the presence of the weight $\langle v\rangle^{-3}$ in (8.28) prevents the direct application of a Sobolev inequality, but, similar to [8], a modified De Giorgi procedure can still be used to gain integrability.

Indeed, by passing from a lower level set to a higher lever set, we observe that for any $0 \le k < \ell$,

$$0 \le h_\ell^+ \le h_k^+,$$

and if $h_\ell \ge 0$, then $h \ge \ell > k$. In this case, $h_k = h_\ell + \ell - k$ and $\frac{h_k}{\ell-k} = \frac{h_\ell}{\ell-k} + 1 \ge 1$. Therefore,

$$\mathbb{1}_{\{h_\ell \ge 0\}} \le \frac{h_k^+}{\ell - k}.$$

In particular for any $\alpha > 0$

$$\mathbb{1}_{\{h_\ell \ge 0\}} \le \left(\frac{h_k^+}{\ell - k}\right)^\alpha.$$

We deduce

$$h_\ell^+ \le (\ell - k)^{-\alpha} (h_k^+)^{1+\alpha} \quad \text{for any } \alpha > 0. \tag{8.29}$$

Inequality (8.29) is crucial for the proof of the next lemma, which explains how to estimate the terms on the right hand side of (8.28).

Lemma 8.10. *Let p, m, and h be as in Proposition 8.2. Then, for each $0 \le k < \ell$ and for $i \in \{0, 1, 2\}$ there holds*

$$\int_{\mathbb{R}^3} (h_\ell^+)^{p+1-i} \le \frac{C}{(\ell - k)^{i+\gamma}} \left[\left(\left\| \langle\cdot\rangle^{-\frac{3}{2}} \nabla (h_k^+)^{\frac{p}{2}} \right\|_{L^2}^2 + \left\| h_k^+ \right\|_{L^p}^p \right) \left\| h_k^+ \right\|_{L^p}^{p\left(\frac{2}{3} - \frac{3}{m}\right)} \left\| h_k^+ \right\|_{L^1_m}^{\frac{3}{m}} \right],$$

where

$$\gamma = \frac{2}{3}\left(p - \frac{3}{2}\right) - \frac{3}{m}(p-1).$$

Proof. First, let us derive a useful bound on $\|g\|_{L^r}^r$, where $g : \mathbb{R}^3 \to \mathbb{R}$ is arbitrary. Interpolating, we find

$$\|g\|_{L^r} \le \left\| \langle\cdot\rangle^{-\frac{3}{p}} g \right\|_{L^{3p}}^{\theta_1} \|g\|_{L^p}^{\theta_2} \|\langle\cdot\rangle^\alpha g\|_{L^1}^{\theta_3},$$

provided $\theta_1,\ \theta_2,\ \theta_3 \in (0,1)$ and the following relations are satisfied:

$$\theta_1 + \theta_2 + \theta_3 = 1, \qquad \frac{\theta_1}{3p} + \frac{\theta_2}{p} + \theta_3 = \frac{1}{r}, \qquad \frac{-3}{p}\theta_1 + \alpha\theta_3 = 0.$$

Imposing further that $r\theta_1 = p$, we find

$$r\theta_2 = \frac{p\left[r - p - \frac{2}{3}\right]}{p-1}, \qquad r\theta_3 = \frac{5p - 3r}{3(p-1)}, \qquad \alpha = \frac{9(p-1)}{5p-3r}, \qquad r \in \left(p + \frac{2}{3}, \frac{5p}{3}\right).$$

The weighted Sobolev inequality (8.11) of Lemma 8.7 with $s = 2$ then implies for a constant

depending only on p, r and α

$$\|g\|_{L^r}^r \leq C\left(\left\|\langle\cdot\rangle^{-\frac{3}{2}}\nabla\big(g^{\frac{p}{2}}\big)\right\|_{L^2}^2 + \|g\|_{L^p}^p\right)\|g\|_{L^p}^{p\left(\frac{r-p-\frac{2}{3}}{p-1}\right)}\|\langle\cdot\rangle^{\alpha}g\|_{L^1}^{\frac{5p-3r}{3(p-1)}}. \tag{8.30}$$

Note that the admissible range for r, i.e. $(p+2/3, 5p/3)$ is non-degenerate if and only if $p>1$ and, moreover, $p+1$ is within this range if and only if $p>3/2$. With the estimate (8.30) in hand for general g and r, let us return to our specific setting. For simplicity, let us only prove the case when $i=0$. Let us fix q by setting $\alpha = m$, or, in other words, define q via the relation

$$m = \frac{9(p-1)}{5p-3q} \qquad \text{or, equivalently,} \qquad q = \frac{5p}{3} - \frac{3(p-1)}{m}.$$

With this definition of q, we see that q is in the admissible range $(p+2/3, 5p/3)$ provided $m > 9/2$. However, since we need to bound the L^{p+1} norm, we see that $q > p+1$ if and only if

$$m > \frac{9}{2}\frac{p-1}{p-\frac{3}{2}}.$$

Since by assumption we have enough moments, we set $\gamma = q-(p+1)$ for γ positive. Finally, we combine (8.29) with (8.30) to obtain for $0 \leq k < \ell$,

$$\begin{aligned}\int_{\mathbb{R}^3}(h_\ell^+)^{p+1}\,\mathrm{d}v &\leq \frac{1}{(\ell-k)^\gamma}\int (h_k^+)^{p+1+\gamma}\,\mathrm{d}v\\ &\leq \frac{1}{(\ell-k)^\gamma}\int (h_k^+)^{q}\,\mathrm{d}v\\ &\leq \frac{C}{(\ell-k)^\gamma}\left[\left(\left\|\langle\cdot\rangle^{-\frac{3}{2}}\nabla(h_k^+)^{\frac{p}{2}}\right\|_{L^2}^2 + \left\|h_k^+\right\|_{L^p}^p\right)\left\|h_k^+\right\|_{L^p}^{p\left(\frac{q-p-\frac{2}{3}}{p-1}\right)}\left\|h_k^+\right\|_{L^1_m}^{\frac{5p-3q}{3(p-1)}}\right],\end{aligned}$$

where both q and γ have been fixed in terms of m and p. □

Lemma 8.10 bounds the terms on the right hand side of (8.28) using only the quantities appearing on the left hand side, with the exception of L^1-moments, which we know stay

bounded by Lemma 8.2. This yields the final differential form of our energy estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(h_\ell^+)^p\,\mathrm{d}v + c_0\int_{\mathbb{R}^3}\left|\nabla(h_\ell^+)^{\frac{p}{2}}\right|^2\langle v\rangle^{-3}\,\mathrm{d}v \\ \leq C\left[\frac{1}{(\ell-k)^\gamma}+\frac{1+\ell}{(\ell-k)^{1+\gamma}}+\frac{1+E^{\beta_0}+\ell+\ell^2}{(\ell-k)^{2+\gamma}}\right] \\ \times\left[\left(\left\|\langle\cdot\rangle^{-\frac{3}{2}}\nabla(h_k^+)^{\frac{p}{2}}\right\|_{L^2}^2+\left\|h_k^+\right\|_{L^p}^p\right)\left\|h_k^+\right\|_{L^p}^{p\left(\frac{q-p-\frac{2}{3}}{p-1}\right)}\left\|h_k^+\right\|_{L^1_m}^{\frac{5p-3q}{3(p-1)}}\right],\end{aligned}$$

and concludes the proof of Lemma 8.9.

Step 2: Energy estimate for level set functions (Integral Form)

The energy estimate (8.26) gives us control between different level sets, however, we will find this more useful in integral form. This leads to the definition of our energy functional $\mathcal{E}$ for any $\ell \geq 0$ and $0 \leq T_1 \leq T_2 \leq T^*$ as

$$\mathcal{E}_\ell(T_1,T_2) := \sup_{t\in[T_1,T_2]}\left\|h_\ell^+(t)\right\|_{L^p_v}^p + c_0\int_{T_1}^{T_2}\left\|\langle\cdot\rangle^{-\frac{3}{2}}\nabla(h_\ell^+)^{\frac{p}{2}}\right\|_{L^2_v}^2\,\mathrm{d}t,$$

where c_0 is the constant from Lemma 8.4. We will now integrate (8.26) to obtain an estimate purely in terms of this energy:

Lemma 8.11. *Let m, p, H, M, and h be as in Proposition 8.2. Further, let β_0, γ, and q be the exponents from Lemma 8.9. Then, there is a constant $C = C(m,p,H,M)$ such that for any $0 \leq T_1 \leq T_2 \leq T_3 \leq T^*$ and $0 \leq k \leq \ell$, there holds*

$$\begin{aligned}\mathcal{E}_\ell(T_2,T_3) \leq C(1+T_3)^{1+\beta_2}\mathcal{E}_k(T_1,T_3)^{1+\beta_1} \\ \times\left[\frac{1}{(T_2-T_1)(\ell-k)^{1+\gamma}}+\frac{1}{(\ell-k)^\gamma}+\frac{1+\ell}{(\ell-k)^{1+\gamma}}+\frac{1+\ell+\ell^2+\mathcal{E}_0^{\beta_0}(T_1,T_3)}{(\ell-k)^{2+\gamma}}\right],\end{aligned}$$

where

$$\beta_1 = \frac{2}{3}-\frac{3}{m} \qquad \text{and} \qquad \beta_2 = \frac{3}{m}.$$

Proof. Fix $0 \leq T_1 < T_2 \leq T_3 \leq T^*$. We integrate (8.26) over (t_1,t_2) and denoting the right hand side of (8.26) by $RHS(t)$, we obtain

$$\int_{\mathbb{R}^3}(h_\ell^+)^p(t_2)\,\mathrm{d}v + c_0\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left|\nabla(h_\ell^+)^{\frac{p}{2}}\right|^2\langle v\rangle^{-3}\,\mathrm{d}v\,\mathrm{d}t \leq \int_{\mathbb{R}^3}(h_\ell^+)^p(t_1)\,\mathrm{d}v + \int_{t_1}^{t_2}RHS(t)\,\mathrm{d}t.$$

Since $RHS(t)$ is positive, this implies for $T_1 \leq t_1 \leq T_2 \leq t_2 \leq T_3$,

$$\int_{\mathbb{R}^3}(h_\ell^+)^p(t_2)\,\mathrm{d}v + c_0\int_{T_2}^{t_2}\int_{\mathbb{R}^3}\left|\nabla(h_\ell^+)^{\frac{p}{2}}\right|^2\langle v\rangle^{-3}\,\mathrm{d}v\,\mathrm{d}t \leq \int_{\mathbb{R}^3}(h_\ell^+)^p(t_1)\,\mathrm{d}v + \int_{T_1}^{t_2}RHS(t)\,\mathrm{d}t.$$

Next, taking the the supremum over $t_2 \in [T_2, T_3]$ and then averaging over $t_1 \in [T_1, T_2]$, we find

$$\sup_{t\in[T_2,T_3]} \int_{\mathbb{R}^3} (h_\ell^+)^p(t)\,\mathrm{d}v + c_0 \int_{T_2}^{T_3} \int_{\mathbb{R}^3} \left|\nabla (h_\ell^+)^{\frac{p}{2}}\right|^2 \langle v\rangle^{-3}\,\mathrm{d}v\,\mathrm{d}t \\ \leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{\mathbb{R}^3} (h_\ell^+)^p(t)\,\mathrm{d}v\,\mathrm{d}t + \int_{T_1}^{T_3} RHS(t)\,\mathrm{d}t.$$

The left hand side is $\mathcal{E}_\ell(T_2, T_3)$. It remains to relate the right hand side back to the energy functional $\mathcal{E}_k(T_1, T_3)$. Indeed, we have the obvious estimate

$$\int_{T_1}^{T_3} \left[\left(\left\| \langle\cdot\rangle^{-\frac{3}{2}} \nabla (h_k^+)^{\frac{p}{2}} \right\|_{L^2}^2 + \left\| h_k^+ \right\|_{L^p}^p \right) \left\| h_k^+ \right\|_{L^p}^{p\left(\frac{2}{3} - \frac{3}{m}\right)} \left\| h_k^+ \right\|_{L^1_m}^{\frac{3}{m}} \right] \mathrm{d}t \\ \leq \mathcal{M}^{\beta_2} \left(c_0^{-1} + (T_3 - T_1) \right) \mathcal{E}_k(T_1, T_3)^{1+\beta_1},$$

where

$$\beta_1 = \frac{2}{3} - \frac{3}{m}, \qquad \beta_2 = \frac{3}{m}, \qquad \text{and} \qquad \mathcal{M} = \sup_{T_1 < t < T_3} \|h(t)\|_{L^1_m}.$$

Also, recalling the definition of $E(t)$, we see

$$\sup_{T_1<t<T_3} E(t)^{\beta_0} = \sup_{T_1<t<T_3} \|h(t)\|_{L^p}^{p\beta_0} \leq \mathcal{E}_0(T_1, T_3)^{\beta_0}.$$

Therefore, we see

$$\int_{T_1}^{T_3} RHS(t)\,\mathrm{d}t \\ \leq C\mathcal{M}^{\beta_2} \left(c_0^{-1} + (T_3 - T_1) \right) \mathcal{E}_k(T_1, T_3)^{1+\beta_1} \\ \times \left[\frac{1}{(\ell - k)^\gamma} + \frac{1+\ell}{(\ell-k)^{1+\gamma}} + \frac{1 + \ell + \ell^2 + \mathcal{E}_0(T_1, T_3)^{\beta_0}}{(\ell - k)^{2+\gamma}} \right],$$

where the constant $C = C(p, m)$ is still independent of h, the level sets l and k, and the times T_1, T_2, and T_3. Finally, using Lemma 8.10 once more, we obtain

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{\mathbb{R}^3} (h_\ell^+)^p\,\mathrm{d}v\,\mathrm{d}t \leq \frac{C\mathcal{M}^{\beta_2} \left(c_0^{-1} + (T_2 - T_1) \right) \mathcal{E}_k(T_1, T_3)^{1+\beta_1}}{(T_2 - T_1)(\ell - k)^{1+\gamma}}.$$

By Lemma 8.2, $\mathcal{M}$ grows at most linearly in T_3, and thus we conclude

$$\mathcal{E}_\ell(T_2, T_3) \leq C(H, M, p, m)(1 + T_3)^{1+\beta_2} \mathcal{E}_k(T_1, T_3)^{1+\beta_1} \\ \times \left[\frac{1}{(T_2 - T_1)(\ell - k)^{1+\gamma}} + \frac{1}{(\ell - k)^\gamma} + \frac{1+\ell}{(\ell-k)^{1+\gamma}} + \frac{1 + \ell + \ell^2 + \mathcal{E}_0(T_1, T_3)^{\beta_0}}{(\ell - k)^{2+\gamma}} \right],$$

as desired. This concludes the proof. □

Step 3: De Giorgi iteration

We now finish the proof of Proposition 8.2 using an iteration procedure. Let $0 < t < T^*$ be the fixed times in the statement of Proposition 8.2 and take $t < T \leq T^*$. We consider for $n \in \mathbb{N}$

$$\ell_n = K\left(1 - 2^{-n}\right), \quad t_n = t\left(1 - 2^{-n}\right), \quad \text{and} \quad E_n = \mathcal{E}_{\ell_n}(t_n, T),$$

where $K > 0$ is a parameter to be chosen appropriately. Indeed, our main goal is to find a value of K for which $\lim_{n\to\infty} E_n = 0$. By the definition of the energy and the choice of iteration parameters, this would imply $h(\tau, v) \leq K$ for almost every $t < \tau < T$ and $v \in \mathbb{R}^3$.

To this end, we apply Lemma 8.11 with $T_1 = t_n$, $T_2 = t_{n+1}$, and $T_3 = T$ and deduce the following recurrence relation for E_n

$$\begin{aligned}
&E_{n+1} \\
&\leq C(1+T)^{1+\beta_2} E_n^{1+\beta_1} \left[\frac{1}{(t_{n+1}-t_n)(\ell_{n+1}-\ell_n)^{\gamma+1}} + \frac{1}{(\ell_{n+1}-\ell_n)^{\gamma}} \right. \\
&\qquad \left. + \frac{1+\ell_{n+1}}{(\ell_{n+1}-\ell_n)^{\gamma+1}} + \frac{1+E_0^{\beta_0}+\ell_{n+1}+\ell_{n+1}^2}{(\ell_{n+1}-\ell_n)^{\gamma+2}} \right] \\
&= C(1+T)^{1+\beta_2} E_n^{1+\beta_1} \left[\frac{2^{(\gamma+1)(n+1)}}{tK^{\gamma+1}} + \frac{2^{(\gamma+1)(n+1)} + 2^{(\gamma+2)(n+1)}(1-2^{-(n+1)})}{K^{\gamma+1}} \right. \\
&\qquad + \frac{2^{(\gamma+2)(n+1)}(1+E_0^{\beta_0})}{K^{\gamma+2}} \\
&\qquad \left. + \frac{2^{\gamma(n+1)} + 2^{(\gamma+1)(n+1)}\left(1-2^{-(n+1)}\right) + 2^{(\gamma+2)(n+1)}\left(1-2^{-(n+1)}\right)^2}{K^{\gamma}} \right] \\
&\leq C(1+T)^{1+\beta_2} E_n^{1+\beta_1} \left[\frac{2^{n(\gamma+1)}}{tK^{\gamma+1}} + \frac{2^{n(\gamma+2)}\left(1+E_0^{\beta_0}\right)}{K^{\gamma+2}} + \frac{2^{n(\gamma+2)}}{K^{\gamma+1}} + \frac{2^{n(\gamma+2)}}{K^{\gamma}} \right].
\end{aligned} \tag{8.31}$$

Now, because the power β_1 is positive, E_n should decay, even though the coefficients in the recurrence are geometric (in n). To prove this, we use a trick from [8] to find a quantitative estimate for K despite differing homogeneity in K. In essence, we expect E_n to decay exponentially, so we search for a choice of parameters K and $Q > 0$ such that the sequence E_n^* defined as

$$E_n^* := E_0 Q^{-n}, \quad n \in \mathbb{N}$$

satisfies (8.31) with the reversed inequality, and so by induction will remain larger than

E_n. Therefore, we want

$$E^*_{n+1} \gtrsim (1+T)^{1+\beta_2}(E^*_n)^{1+\beta_1}\left[\frac{2^{n(\gamma+1)}}{tK^{\gamma+1}} + \frac{2^{n(\gamma+2)}\left(1+E_0^{\beta_0}\right)}{K^{\gamma+2}} + \frac{2^{n(\gamma+2)}}{K^{\gamma+1}} + \frac{2^{n(\gamma+2)}}{K^{\gamma}}\right]. \quad (8.32)$$

Using the definition of E^*_n, (8.32) holds if

$$1 \geq C(1+T)^{1+\beta_2}E_0^{\beta_1}Q\left(2^{\gamma+2}Q^{-\beta_1}\right)^n\left[\frac{1}{tK^{\gamma+1}} + \frac{1+E_0^{\beta_0}}{K^{\gamma+2}} + \frac{1}{K^{\gamma+1}} + \frac{1}{K^{\gamma}}\right].$$

We now choose Q such that

$$2^{\gamma+2}Q^{-\beta_1} \leq 1 \qquad \text{or equivalently,} \qquad Q \geq 2^{\frac{(\gamma+2)}{\beta_1}}.$$

Then, for (8.32) to hold we need

$$1 \geq C(1+T)^{1+\beta_2}E_0^{\beta_1}Q\left[\frac{1}{tK^{\gamma+1}} + \frac{1+E_0^{\beta_0}}{K^{\gamma+2}} + \frac{1}{K^{\gamma+1}} + \frac{1}{K^{\gamma}}\right].$$

Next, choosing K so each of the terms is smaller than a quarter, the recurrence (8.32) holds. More precisely, we choose K as

$$K = C\max\left\{E_0^{\frac{\beta_1}{\gamma}}; E_0^{\frac{\beta_1}{1+\gamma}}; E_0^{\frac{\beta_1}{2+\gamma}}; E_0^{\frac{\beta_0+\beta_1}{2+\gamma}}; E_0^{\frac{\beta_1}{1+\gamma}}t^{-\frac{1}{1+\gamma}}\right\}(1+T)^{\frac{1+\beta_2}{\gamma}}. \quad (8.33)$$

Then, as discussed above, since $E_0 = E^*_0$ it follows by induction that $E_n \leq E^*_n$ for $n \in \mathbb{N}$. Moreover, since $Q > 1$ we deduce

$$\lim_{n\to\infty} E_n \leq \lim_{n\to\infty} E^*_n = 0.$$

Therefore, the definition of our energy functional E_n implies

$$\sup_{t<\tau<T}\left\|h^+_K(\tau)\right\|^p_{L^p_v} = 0, \qquad \text{or equivalently,} \qquad \|h\|_{L^\infty((t,T)\times\mathbb{R}^3)} \leq K.$$

Finally, from the choice of K, we have for all $v \in \mathbb{R}^3$ and $\tau \in (t,T)$

$$h(\tau,v) \leq K \leq C\left(E_0^{\frac{\beta_1}{\gamma}} + E_0^{\frac{\beta_1}{1+\gamma}} + E_0^{\frac{\beta_1}{2+\gamma}} + E_0^{\frac{\beta_0+\beta_1}{2+\gamma}} + E_0^{\frac{\beta_1}{1+\gamma}}t^{-\frac{1}{1+\gamma}}\right)(1+T)^{\frac{1+\beta_2}{\gamma}}.$$

Taking $T = 2t$ yields the desired estimate. Finally, recalling the definitions of the exponents β_0, β_1, β_2, and γ, and performing some basic algebraic manipulations gives the simplified expressions found in (8.24) and completes the proof of Proposition 8.2.

8.3.3 Global existence for smooth initial data

In this section, we prove Proposition 8.1 by demonstrating global-in-time existence of solutions for initial data that are very smooth and rapidly decaying. Let us define the class of initial data considered in the proposition as:

$$\mathcal{A}(p,m,H,M,\delta) = \Bigg\{ h_{in} \in \mathcal{S}(\mathbb{R}^3) \mid f_{in} = h_{in} + \mu \text{ satisfies (1.46)}, \ 0 \leq f_{in}, \\ \|f_{in}\|_{L^1_m} \leq M, \ \int_{\mathbb{R}^3} f_{in} |\log(f_{in})| \, \mathrm{d}v \leq H, \ \|h_{in}\|_{L^p}^p \leq \delta \Bigg\}.$$

We now fix p, m, H, M such that $p > 3/2$ and

$$m > \frac{9}{2} \frac{p-1}{p-\frac{3}{2}} \max\left(1, \frac{p\left(p-\frac{3}{2}\right)}{p^2 - 2p + 3/2}\right),$$

and attempt to find a corresponding δ so that for each $h_{in} \in \mathcal{A}(\delta)$, there exists a unique global-in-time Schwartz class solution to (1.41) with initial datum $f_{in} = h_{in} + \mu$.

By Theorem 8.4, for any $h_{in} \in \mathcal{S}(\mathbb{R}^3)$, there is a time $T^*(h_{in}) > 0$ and function $h : [0, T^*) \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(t) = h(t) + \mu$ is the unique Schwartz class solution to (1.41) on $[0, T^*) \times \mathbb{R}^3$ with initial datum $f_{in} = h_{in} + \mu$ with maximal time of existence T^*. Moreover, T^* is characterized via the blow up criterion

$$\lim_{t \nearrow T^*} \|f(t)\|_{L^\infty} = \infty. \tag{8.34}$$

Our first goal is to show that T^* is uniformly large for all $h_{in} \in \mathcal{A}(\delta)$. By Lemma 8.8, for any $\varepsilon > 0$ sufficiently small, there are $T_0(\varepsilon) \leq 1$ and $\delta_0(\varepsilon) > 0$ such that for any $h_{in} \in \mathcal{A}(\delta_0)$, the following estimate holds:

$$\sup_{0 < t < \min(T_0, T^*)} \|h\|_{L^p}^p + c_0(H) \int_0^{\min(T_0,T^*)} \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left| \nabla h^{p/2} \right|^2 \, \mathrm{d}v \, \mathrm{d}t \leq \varepsilon.$$

By Proposition 8.2, we find for $0 < t < \min(T_0, T^*)$ and for ε sufficiently small

$$\|h(t)\|_{L^\infty} \leq C(p,m,H,M) \left(\varepsilon^{\min\left(\frac{\beta_1}{\gamma}, \frac{\beta_1}{1+\gamma}, \frac{\beta_1}{2+\gamma}, \frac{\beta_0+\beta_1}{2+\gamma}\right)} + \left(\varepsilon^{\beta_1} t^{-1}\right)^{\frac{1}{1+\gamma}} \right) < \infty, \tag{8.35}$$

where the exponents β_0, β_1, and γ are defined in Proposition 8.2. Using the characterization of T^* in (8.34), we must have $T^* > T_0$ for every $h_{in} \in \mathcal{A}(\delta_0(\varepsilon))$. Moreover, Lemma 8.8

guarantees $T_0(\varepsilon) \geq C\varepsilon^{1-\alpha}$ for $\alpha = 1 - \frac{1}{p} + \frac{1}{3(p-1)}$, so that (8.35) implies

$$\lim_{\varepsilon \to 0^+} \sup_{h_{in} \in \mathcal{A}(\delta_0(\varepsilon))} \left[\sup_{T_0(\varepsilon)/4 < t < T_0(\varepsilon)} \|h(t)\|_{L^\infty} + \int_0^{T_0(\varepsilon)} \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left| \nabla h^{p/2} \right|^2 \, \mathrm{d}v \, \mathrm{d}t \right] = 0, \quad (8.36)$$

provided $\alpha + \beta_1 > 1$. We note that simplifying the expressions for α and β_1,

$$\alpha + \beta_1 - 1 = 0 \qquad \text{if and only if} \qquad m = \frac{9}{2} \frac{p-1}{p - \frac{3}{2}} \left(\frac{p\left(p - \frac{3}{2}\right)}{p^2 - 2p + 3/2} \right).$$

Therefore, our constraint on m exactly guarantees that $\alpha + \beta_1 - 1 > 0$ and so (8.36) holds.

We would like to apply the global-in-time existence result Theorem 8.5 to conclude that there is a δ sufficiently small such that for each $h_{in} \in \mathcal{A}(\delta)$, we have $T^*(h_{in}) = \infty$. Consequently, f_{in} admits a global-in-time Schwartz class solution if f_{in} is suitably close to the Maxwellian μ in L^p. However, to apply Theorem 8.5 to the profile $h(T_0/2)$, which is well-defined for $h_{in} \in \mathcal{A}(\delta_0(\varepsilon))$, we must show the condition:

$$\lim_{\varepsilon \to 0^+} \sup_{h_{in} \in \mathcal{A}(\delta_0(\varepsilon))} \left(\|h(T_0/2)\|_{L^2_1} + \|\nabla h(T_0/2)\|_{L^2_2} \right) < \varepsilon_1, \quad (8.37)$$

where ε_1 is the fixed smallness parameter in Theorem 8.5. Supposing for the moment that (8.37) holds, we will show how to deduce Proposition 8.1.

Use (8.37) to pick ε_0 sufficiently small so that

$$\sup_{h_{in} \in \mathcal{A}(\delta_0(\varepsilon_0))} \left(\|h(T_0/2)\|_{L^2_1} + \|\nabla h(T_0/2)\|_{L^2_2} \right) < \varepsilon_1.$$

Then, for each $h_{in} \in \mathcal{A}(\delta_0(\varepsilon_0))$, it follows that one may apply Theorem 8.5 to $h(T_0/2)$ and obtain a solution $\tilde{f} = \tilde{h} + \mu$ to (1.41) on $[T_0/2, \infty)$, where $\tilde{f}(T_0/2) = f(T_0/2)$, which further satisfies $\sup_{t > T_0/2} \|\tilde{f}\|_{L^2} < \infty$. Note that restricting f to $[T_0/2, T_0]$ yields another solution to (1.41) with the same profile at time $T_0/2$. Since both f and $\tilde{f}$ lie within $L^1(T_0/2, T_0; L^\infty)$ by Proposition 8.2, applying Theorem 8.6, we conclude f and $\tilde{f}$ agree on $[T_0/2, T_0]$. Therefore,

$$f(t) = \begin{cases} f(t) & \text{if } t \leq T_0/2, \\ \tilde{f}(t) & \text{if } t \geq T_0/2, \end{cases}$$

defines a global-in-time solution to (1.41) with initial datum f_{in}. Finally, since $\tilde{f}$ is constructed by Theorem 8.5, we obtain the long-time bound

$$\sup_{T_0/2 < t < \infty} \|h(t)\|_{H^1} \leq C(p, m, H, M).$$

Applying Proposition 8.2 once more yields that $\|h(t)\|_{L^\infty}$ remains finite, and by (8.34) we

conclude $T^* = \infty$. The constructed solution f remains Schwartz class, which concludes the proof of Proposition 8.1.

Proof of Claim 8.37

We now show condition (8.37). The first term in (8.37) is easily controlled by using the moment bound from Lemma 8.2 and (8.36). Indeed, for any $h_{in} \in \mathcal{A}(\delta_0(\varepsilon))$

$$\int_{\mathbb{R}^3} |h(T_0/2)|^2 \langle v \rangle^2 \, \mathrm{d}v \leq \left(\sup_{T_0/4<t<T_0} \|h(t)\|_{L^\infty} \right) \left(\sup_{0<t<T_0} \|h(t)\|_{L^1_2} \right),$$

where the right hand side decays as $\varepsilon \to 0^+$.

The second term in (8.37) requires more care. We know by (8.36) the L^∞ norm of $h(T_0/2)$ is uniformly small for $h_{in} \in \mathcal{A}(\delta_0(\varepsilon))$. We need to show that this implies the gradient $\nabla h(T_0/2)$ is also small (uniformly in h_{in}). Note that because $T_0(\varepsilon) \sim \varepsilon^{1-\alpha}$ for $\alpha \in (0,1)$, the H^1 estimates in the forthcoming Lemma 8.12 imply for $m \geq 8$,

$$\text{For } h_{in} \in \mathcal{A}(\delta_0(\varepsilon)), \qquad \|\nabla h(T_0/2)\|_{L^2_2} \leq C(H,M)\varepsilon^\alpha,$$

which concludes the proof of (8.37). Now, let us prove the necessary higher regularity estimates:

Lemma 8.12 (Uniform H^1 Regularity)**.** *Let f be a smooth, rapidly decaying solution to* (1.41) *on $[0,T] \times \mathbb{R}^3$ for some $0 < T < 1$, satisfying the normalisation* (1.46) *and with entropy H. Suppose further that $h = f - \mu$ satisfies $\|h\|_{L^\infty([0,T]\times\mathbb{R}^3)} < \varepsilon \leq 1$. Then, for any $k \geq 0$, we have the following weighted H^1 estimates: For any $0 < t < T$,*

$$\sup_{t<s<T} \int_{\mathbb{R}^3} \langle v \rangle^k |\nabla h(s)|^2 \, \mathrm{d}v + c_0 \int_t^T \langle v \rangle^{k-3} \left|\nabla^2 h(s)\right|^2 \, \mathrm{d}v \, \mathrm{d}s \leq C\left(k, H, \|h_{in}\|_{L^1_{k+6}}\right) \varepsilon \left(1 + \frac{1}{t}\right).$$

Proof. As a first step towards H^1 regularity, we perform a weighted L^2 estimate by testing (8.16) with $\langle v \rangle^k h$ and integrating by parts.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^k h^2 + \int_{\mathbb{R}^3} \langle v \rangle^k A[f] \nabla h \cdot \nabla h \\
&\quad \leq -\frac{1}{2} \int_{\mathbb{R}^3} \nabla \langle v \rangle^k \cdot A[f] \nabla h^2 \, \mathrm{d}v + \int_{\mathbb{R}^3} h^2 \nabla \langle v \rangle^k \cdot \nabla a[f] \, \mathrm{d}v \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}^3} \langle v \rangle^k \nabla a[f] \cdot \nabla h^2 \, \mathrm{d}v + \int_{\mathbb{R}^3} \langle v \rangle^k h (A[h] : \nabla^2 \mu + h\mu) \, \mathrm{d}v \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Recall, by conservation of mass, $\|f\|_{L^\infty_t L^1_v} = 1$ and since $f = h + \mu$, $\|f\|_{L^\infty_{t,v}} \leq \varepsilon + \|\mu\|_{L^\infty} \leq 2$. It follows from Lemma 8.5 that there is a universal constant such that

$$\|A[f]\|_{L^\infty} \leq \|f\|_{L^1}^{2/3} \|f\|_{L^\infty}^{1/3} \leq C \qquad \text{and} \qquad \|\nabla a[f]\|_{L^\infty} \leq \|f\|_{L^1}^{1/3} \|f\|_{L^\infty}^{2/3} \leq C. \tag{8.38}$$

Note that the estimates do not depend on the sign of f and therefore analogous estimates hold for h as well. From (8.38), we immediately bound I_2 as

$$|I_2| \leq C(k) \int_{\mathbb{R}^3} h^2 \nabla \langle v \rangle^k \, \mathrm{d}v \leq C(k)\varepsilon \|h\|_{L^1_k}.$$

For I_3, we integrate by parts and similarly obtain from (8.38),

$$|I_3| \leq C(k) \int_{\mathbb{R}^3} h^2 \left| \nabla \langle v \rangle^k \cdot \nabla a[f] + \langle v \rangle^k f \right| \mathrm{d}v \leq C(k)\varepsilon \left\| h \right\|_{L^1_k}.$$

For I_1, we integrate by parts and use (8.38) again to obtain

$$\begin{aligned} I_1 &= C \int_{\mathbb{R}^3} \nabla \langle v \rangle^k \cdot \left[\nabla \cdot \left(A[f]h^2 \right) - \nabla a[f] h^2 \right] \mathrm{d}v \\ &\leq C(k) \int_{\mathbb{R}^3} \left[\langle v \rangle^{k-2} + \langle v \rangle^{k-1} \right] h^2 \, \mathrm{d}v \leq C(k)\varepsilon \left\| h \right\|_{L^1_k}. \end{aligned}$$

Finally, for I_4, we use (8.38) for h and find

$$\begin{aligned} |I_4| &\leq C(k) \|A[h]\|_{L^\infty} \int_{\mathbb{R}^3} \langle v \rangle^k |h| \left| \nabla^2 \mu \right| \mathrm{d}v + C(k) \int_{\mathbb{R}^3} h^2 \langle v \rangle^k \mu \, \mathrm{d}v \\ &\leq C(k)\varepsilon^{4/3} + C(k)\varepsilon \left\| h \right\|_{L^1_k}. \end{aligned}$$

Summarising, we have shown using the lower bound on $A[f]$ from Lemma 8.4

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^k |h(t)|^2 \, \mathrm{d}v + c_0(H) \int_{\mathbb{R}^3} \langle v \rangle^{k-3} |\nabla h|^2 \, \mathrm{d}v \leq C(k)\varepsilon \int_{\mathbb{R}^3} \langle v \rangle^k |h| \, \mathrm{d}v.$$

Therefore, integrating in time and using the linear-in-time growth bound on L^1-moments of h from Lemma 8.2, we obtain

$$\begin{aligned} \sup_{0<t<T} \|h\|^2_{L^2_k} + c_0(H) \int_0^T \|\nabla h(t)\|^2_{L^2_{k-3}} \, \mathrm{d}t &\leq C(k)\varepsilon \left\| h \right\|_{L^\infty_t L^1_k} T + \|h_{in}\|^2_{L^2_k} \\ &\leq C(\|h_{in}\|_{L^1_k}, H)\varepsilon. \end{aligned} \tag{8.39}$$

As a second step towards higher regularity, we perform weighted H^1_k estimates by testing (8.16) with

$\partial_{v_i}\left(\langle v\rangle^k \partial_{v_i} h\right)$ for $i \in \{1,2,3\}$ and integrating by parts to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^k (\partial_{v_i}h)^2 + \int_{\mathbb{R}^3}\langle v\rangle^k A[f]\nabla\partial_{v_i}h\cdot\nabla\partial_{v_i}h \\
&\quad\le -\int_{\mathbb{R}^3}\nabla(\langle v\rangle^k\partial_{v_i}h)\cdot\partial_{v_i}A[f]\nabla h\,\mathrm{d}v - \int A[f]\partial_{v_i}h\nabla\partial_{v_i}h\nabla\langle v\rangle^k \\
&\qquad + \int_{\mathbb{R}^3}\nabla(\langle v\rangle^k\partial_{v_i}h)\cdot\partial_{v_i}(\nabla a[f]h)\,\mathrm{d}v + \int_{\mathbb{R}^3}\partial_{v_i}\left(\langle v\rangle^k\partial_{v_i}h\right)\left(A[h]:\nabla^2\mu + h\mu\right)\,\mathrm{d}v \\
&\quad= J_1+J_2+J_3+J_4.
\end{aligned}$$

Before we estimate $J_i, i\in\{1,\dots,4\}$, we observe that Lemma 8.5 gives the following estimates on the coefficients $A[f]$ and $\nabla a[f]$

$$\|\nabla A[f]\|_{L^\infty_v}\le\|f\|_{L^1}^{1/3}\|f\|_{L^\infty}^{2/3}\le C,\qquad \|\nabla^2 a[f]\|_{L^r_v}\le\|f\|_{L^r}\le C\quad\text{for } 1<r<\infty. \tag{8.40}$$

Since these estimates use only the order of decay of the kernel and do not rely upon the sign of f, analogous estimates hold for $A[h]$ and $\nabla a[h]$, where the smallness of h gives an improvement (in powers of ε) over the universal constant C. Now, we split J_1 as

$$J_1=\int_{\mathbb{R}^3}\langle v\rangle^k\nabla\partial_{v_i}h\cdot\partial_{v_i}A[f]\nabla h\,\mathrm{d}v+\int_{\mathbb{R}^3}\partial_{v_i}h\nabla\langle v\rangle^k\partial_{v_i}A[f]\nabla h = J_{11}+J_{12}.$$

For J_{11}, we use Young's inequality and (8.40) to conclude

$$|J_{11}|\le\frac{c_0}{4}\int_{\mathbb{R}^3}\langle v\rangle^{k-3}|\nabla\partial_{v_i}h|^2\,\mathrm{d}v + C(k,H)\int_{\mathbb{R}^3}|\nabla h|^2\langle v\rangle^{k+3}\,\mathrm{d}v.$$

For J_{12}, we simply use (8.40) to obtain

$$|J_{12}|\le C(k)\int_{\mathbb{R}^3}|\nabla h|^2\langle v\rangle^{k-1}\,\mathrm{d}v.$$

For J_2 we use (8.38) and Young's inequality to get

$$J_2\le\frac{c_0}{4}\int_{\mathbb{R}^3}|\nabla\partial_{v_i}h|^2\langle v\rangle^{k-3}\,\mathrm{d}v + C(k,H)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}|\nabla h|^2\,\mathrm{d}v.$$

Next, we split J_3 as

$$J_3=\int_{\mathbb{R}^3}\langle v\rangle^k\nabla\partial_{v_i}h\cdot\partial_{v_i}(\nabla a[f]h)\,\mathrm{d}v+\int_{\mathbb{R}^3}\partial_{v_i}h\nabla\langle v\rangle^k\cdot\partial_{v_i}(\nabla a[f]h)\,\mathrm{d}v = J_{31}+J_{32}.$$

For J_{31}, we use Young's inequality, (8.38), Hölder's inequality, and (8.40) with $r=\frac{2(k+6)}{3}$

to obtain

$$|J_{31}| \leq \frac{c_0}{4}\int_{\mathbb{R}^3} |\nabla\partial_{v_i}h|^2 \langle v\rangle^{k-3}\,\mathrm{d}v + C(k,H)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}\left(|\nabla\partial_{v_i}a[f]|^2|h|^2 + |\nabla a[f]|^2|\partial_{v_i}h|^2\right)$$
$$\leq \frac{c_0}{4}\int_{\mathbb{R}^3} |\nabla\partial_{v_i}h|^2 \langle v\rangle^{k-3}\,\mathrm{d}v + C(k,H)\varepsilon^{\frac{k+9}{k+6}}\|h\|^{\frac{k+3}{k+6}}_{L^1_{k+6}} + C(k,H)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}|\nabla h|^2\,\mathrm{d}v.$$

For J_{32}, we use Young's inequality, (8.38), Hölder's inequality, and (8.40) with $r = \frac{k+6}{3}$ to obtain

$$|J_{32}| \leq C(k)\int_{\mathbb{R}^3}\langle v\rangle^{k-1}\left(|\nabla a[f]||\partial_{v_i}h|^2 + |\nabla\partial_{v_i}a[f]||h\partial_{v_i}h|\right)$$
$$\leq C(k)\int_{\mathbb{R}^3}\langle v\rangle^{k-1}|\nabla h|^2\,\mathrm{d}v + C(k)\int_{\mathbb{R}^3}|\nabla h|^2\langle v\rangle^{k+3}\,\mathrm{d}v + C(k)\varepsilon^{\frac{k+12}{k+6}}\|h\|^{\frac{k}{k+6}}_{L^1_{k+6}}.$$

Lastly, we split the linear terms in J_4 as

$$J_4 = \int_{\mathbb{R}^3}\langle v\rangle^k\partial_{v_iv_i}h\left(A[h]:\nabla^2\mu + \mu h\right)\mathrm{d}v + \int_{\mathbb{R}^3}\partial_{v_i}\langle v\rangle^k\partial_{v_i}h\left(A[h]:\nabla^2\mu + h\mu\right)\mathrm{d}v = J_{41}+J_{42}.$$

For J_{41}, we use Young's inequality, Hölder's inequality, (8.38) for h, and the embedding $L^2_{3+}\hookrightarrow L^1$ so that

$$|J_{41}| \leq \frac{c_0}{4}\int_{\mathbb{R}^3}\langle v\rangle^{k-3}|\nabla\partial_{v_i}h|^2\,\mathrm{d}v + C(k,H)\int_{\mathbb{R}^3}|A[h]|^2\langle v\rangle^{k+3}\left|\nabla^2\mu\right|^2\mathrm{d}v$$
$$+ C(k,H)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}|\mu|^2h^2\,\mathrm{d}v$$
$$\leq \frac{c_0}{4}\int_{\mathbb{R}^3}\langle v\rangle^{k-3}|\nabla\partial_{v_i}h|^2\,\mathrm{d}v + C(k,H)\varepsilon^{4/3}\|h\|^{2/3}_{L^1_{3+}} + C(k,H)\varepsilon^2.$$

For J_{42}, we use Young's inequality, (8.38) for h, and $L^2_{3+}\hookrightarrow L^1$ to obtain

$$|J_{42}| \leq C(k)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}|\nabla h|^2\,\mathrm{d}v + C(k)\int_{\mathbb{R}^3}\langle v\rangle^k|A[h]|^2\left|\nabla^2\mu\right|^2\mathrm{d}v + C(k)\int_{\mathbb{R}^3}\langle v\rangle^k\mu^2h^2\,\mathrm{d}v$$
$$\leq C(k)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}|\nabla h|^2\,\mathrm{d}v + C(k)\varepsilon^{4/3}\|h\|^{2/3}_{L^1_{3+}} + C(k)\varepsilon^2.$$

To summarise, using the lower bound for $A[f]$ from Lemma 8.4 and absorbing the highest order terms without a sign, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^k|\partial_{v_i}h|^2\,\mathrm{d}v + \frac{c_0(H)}{4}\int_{\mathbb{R}^3}\langle v\rangle^{k-3}|\nabla\partial_{v_i}h|\,\mathrm{d}v$$
$$\leq C(k,H)\int_{\mathbb{R}^3}\langle v\rangle^{k+3}|\nabla h|^2\,\mathrm{d}v + C(k,H)\varepsilon\left(1+\int_{\mathbb{R}^3}\langle v\rangle^{k+6}|h|\,\mathrm{d}v\right)^{\max\left(\frac{2}{3},\frac{k}{k+6},\frac{k+3}{k+6}\right)}.$$

After integrating over (t_1,t_2) for $0<t_1<t<t_2<T<1$ and using the linear-in-time

growth bound on L^1 moments of h from Lemma 8.2, we have

$$\int_{\mathbb{R}^3} \langle v\rangle^k |\nabla h(t_2)|^2 \,\mathrm{d}v + \frac{c_0(H)}{4}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{k-3} \left|\nabla^2 h\right| \,\mathrm{d}v \\ \leq \int_{\mathbb{R}^3} \langle v\rangle^k |\nabla h(t_1)|^2 \,\mathrm{d}v + C(k,H)\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{k+3} |\nabla h|^2 \,\mathrm{d}v\,\mathrm{d}s + C\left(k,H,\|h\|_{L^1_{k+6}}\right)\varepsilon.$$

Therefore, taking a supremum over $t_2 \in (t,T)$, averaging over $t_1 \in (0,t)$, appealing to (8.39) with k replaced by $k+3$, we find

$$\sup_{t<s<T}\int_{\mathbb{R}^3} \langle v\rangle^k |\nabla h|^2 \,\mathrm{d}v + \frac{c_0(H)}{4}\int_t^T\int_{\mathbb{R}^3} \langle v\rangle^{k-3} \left|\nabla^2 h\right| \,\mathrm{d}v\,\mathrm{d}s \\ \leq C\left(k,H,\|h_{in}\|_{L^1_{k+6}}\right)\varepsilon\left(1+\frac{1}{t}\right),$$

which is the claimed estimate. □

8.3.4 Global existence for rough initial data: Proof of Theorem 8.1

In this section, we deduce Theorem 8.1 from Proposition 8.1 via a compactness argument.

Step 1: Construction of approximating sequence

Let us fix p, m, H, and M as in the statement of Theorem 8.1. Then, we fix $\delta_0 = \delta_0(p,m,H,M)$ as defined in Proposition 8.1. For simplicity, dependence of constants on these fixed parameters will be suppressed.

Now, we fix our initial datum $f_{in} : \mathbb{R}^3 \to \mathbb{R}^+$ as any profile $f_{in} \in L^1_m \cap L^p$ satisfying the normalisation (1.46) (recall this fixes the Maxwellian as μ) and further satisfying the bounds

$$\int_{\mathbb{R}^3} \langle v\rangle^m |f_{in}(v) - \mu(v)| \,\mathrm{d}v \leq M \qquad \text{and} \int_{\mathbb{R}^3} f_{in}|\log f_{in}| \leq H. \tag{8.41}$$

Let us pick a sequence of Schwartz class functions $f^n_{in} \in \mathcal{S}(\mathbb{R}^3)$ that approximate f_{in} and satisfy

- Convergence in $L^1_m \cap L^p$,

$$\lim_{n\to\infty} \|f^n_{in} - f_{in}\|_{L^1_m} + \|f^n_{in} - f_{in}\|_{L^p} = 0;$$

- Normalisation (1.46);
- and uniform control of L^1-moments, entropy and L^p-distance from equilibrium

$$\int_{\mathbb{R}^3} \langle v\rangle^m |f^n_{in}(v) - \mu(v)| \,\mathrm{d}v \leq M, \quad \int_{\mathbb{R}^3} f^n_{in}|\log f^n_{in}| \leq H, \quad \int_{\mathbb{R}^3} |f^n_{in} - \mu|^p \,\mathrm{d}v \leq \delta_0.$$

From Proposition 8.1, we obtain a corresponding sequence of global-in-time Schwartz class solutions to (1.41), namely $f_n : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^+$.

Step 2: Uniform-in-n estimates

We recall that the ODE argument in Lemma 8.8, the gain of L^∞-regularity in Proposition 8.2, and the H^1-regularity estimates in Lemma 8.12 imply the existence of a short time interval $[0, T_0]$ on which we have the following uniform-in-n control:

$$\begin{aligned} &\sup_{0<t<T_0} \left(t^{\frac{1}{1+\gamma}} \|f_n(t)\|_{L^\infty(\mathbb{R}^3)} + \|f_n(t)\|_{L^1_m} + \|f_n(t)\|_{L^p} \right) \leq C, \\ &\sup_{s<t<T_0} \|f_n\|_{H^1_k} \leq C, \qquad \text{for each } s \in (0, T_0]. \end{aligned} \tag{8.42}$$

Moreover, the long time behavior from Theorem 8.5 combined with the gain of regularity in Proposition 8.2 yields the estimates: for each $T_0 < s < \infty$,

$$\sup_{T_0<t<s} \|f_n(t)\|_{H^1_2} \leq C(s) \quad \text{and} \quad \sup_{T_0<t<s} \|f_n(t)\|_{L^\infty} \leq C(s) \tag{8.43}$$

In order to apply the Aubin-Lions lemma and conclude strong compactness of the family $\{f_n\}$, we note that the above control implies corresponding control on the time derivative via standard duality arguments. More precisely, for any $0 < s < t < \infty$ and any for $\Phi \in L^1(s,t,H^1)$,

$$\begin{aligned} \left| \int_s^t \int_{\mathbb{R}^3} \Phi(\tau) \partial_t f_n(\tau) \, \mathrm{d}v \, \mathrm{d}\tau \right| &\leq \left| \int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot (A[f_n] \nabla f_n - \nabla a[f_n] f_n) \, \mathrm{d}v \, \mathrm{d}\tau \right| \\ &\leq \|\nabla\Phi\|_{L^1(s,t;L^2)} \left(\|A[f_n]\|_{L^\infty_{t,x}} \|\nabla f_n\|_{L^\infty(s,t;L^2)} + \|\nabla a[f_n]\|_{L^\infty_{t,x}} \|f_n\|_{L^\infty(s,t;L^2)} \right) \\ &\leq C(s,t) \|\Phi\|_{L^1(s,t;H^1)}, \end{aligned} \tag{8.44}$$

where we have used the coefficient bounds in Lemma 8.5. Therefore,

$$\|\partial_t f_n\|_{L^\infty(s,t;H^{-1})} = \sup_{\|\Phi\|_{L^1(s,t;H^1)}=1} \int_s^t \int_{\mathbb{R}^3} \Phi(\tau) \partial_t f_n(\tau) \, \mathrm{d}v \, \mathrm{d}\tau \leq C(s,t). \tag{8.45}$$

On the other hand, to find a bound on $\partial_t f_n$ which holds all the way up to time 0, we integrate by parts and obtain for any $\Phi \in L^1(0, T_0; W^{2,p'})$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \partial_t f_n \Phi \, \mathrm{d}v \, \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left(\nabla^2\Phi : A[f_n] f_n \right) + 2\nabla\Phi \cdot \nabla a[f_n] f_n \, \mathrm{d}v \, \mathrm{d}t. \tag{8.46}$$

Therefore, using Hölder's inequality, $A[f_n] \in L^\infty(0, T_0; L^\infty)$, $\nabla a[f_n] \in L^\infty(0, T_0; L^3)$, and

the Sobolev embedding $\dot{W}^{1,r}(\mathbb{R}^3) \hookrightarrow L^{\frac{3r}{3-r}}(\mathbb{R}^3)$ for $1 \leq r < 3$, we find

$$\begin{aligned}\int_0^{T_0} \int_{\mathbb{R}^3} \partial_t f_n \Phi \, \mathrm{d}v \, \mathrm{d}t &\leq \|A[f_n]\|_{L^\infty_{t,v}} \|\nabla^2 \Phi\|_{L^1_t L^{p'}_v} \|f\|_{L^\infty_t L^p_v} \\ &\quad + 2\|\nabla \Phi\|_{L^1 L^{\frac{3}{2}\frac{p}{p-3/2}}} \|\nabla a[f_n]\|_{L^\infty_t L^3_v} \|f_n\|_{L^\infty_t L^p_v} \\ &\leq C\|\nabla^2 \Phi\|_{L^1_t L^{\frac{p}{p-1}}_v}.\end{aligned} \tag{8.47}$$

By duality, this gives the bound

$$\|\partial_t f_n\|_{L^\infty(0,T_0;W^{-2,p})} = \sup_{\|\Phi\|_{L^1(0,T_0;W^{2,p'})}=1} \int_0^{T_0} \Phi \partial_t f_n \, \mathrm{d}t \leq C. \tag{8.48}$$

Finally, from the short time smoothing estimates in (8.42), we see $\{f_n\}$ is bounded in $L^{(1+\gamma)-}(0,T_0;L^\infty)$. Therefore, interpolation with the $L^\infty(0,T_0;L^1)$ bound yields

$$\int_0^{T_0} \|f_n(t)\|^2_{L^2} \, \mathrm{d}t \leq C. \tag{8.49}$$

Step 3: Construction of the limit

First, note that the Banach-Alaoglu theorem in $L^\infty_{loc}(\mathbb{R}^+;H^1_k)$ and $L^2(0,T_0;L^2)$ implies the existence of a weak-star limit $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ of the sequence $\{f_n\}$. Now, we use a diagonalization argument to combine the uniform-in-estimates from (8.42), (8.43), (8.45), (8.48), and (8.49) with the Aubin-Lions lemma, Banach-Alaoglu theorem, or Riesz's theorem to pick a subsequence (still denoted f_n) converging to f in each of the following senses:

- Strongly in $C(s,t;L^2)$ for each $0 < s < t < \infty$;
- weak-starly in $L^\infty(s,t;H^1_k) \cap W^{1,\infty}(s,t;H^{-1})$, for each $0 < s < t < \infty$;
- weak-starly in $L^\infty(0,T_0;L^p) \cap W^{1,\infty}(0,T_0;W^{-2,p})$;
- strongly in $C(0,T_0;W^{-1,p})$;
- weakly in $L^2([0,T_0] \times \mathbb{R}^3)$;
- and pointwise almost everywhere in $[0,\infty) \times \mathbb{R}^3$.

The pointwise convergence first ensures that $f \geq 0$ almost everywhere. Moreover, combined with Fatou's lemma, $f \in L^\infty(0,\infty;L^1_2) \cap L^\infty_{loc}(0,\infty;L^1_m)$ and f has bounded entropy. Now,

for every time $0 \leq s \leq t < \infty$, we find

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_n(s) = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_n(t). \tag{8.50}$$

Now, recall $f_n \to f$ pointwise almost everywhere and for almost every t, $\|f_n(t)\|_{L^1_m \cap L^2} \leq C(t)$. So, for almost every t, the fixed time profiles $f_n(t)$ are tight and uniformly integrable and converge pointwise to $f(t)$. The Vitali convergence theorem implies (8.50) holds for f as well, i.e. f conserves mass, momentum, and energy. Moreover, the same argument implies f has decreasing entropy and satisfies the normalisation (1.46). Finally, note that the pointwise convergence implies f satisfies the same short-time smoothing estimates as f_n.

Step 4: The limit is instantaneously a strong solution to Landau

We now show that the limit f satisfies the Landau equation (1.41) in the appropriate sense. Because we have strong compactness in $L^\infty_{loc}(\mathbb{R}^+; L^2)$, passing to the limit in the nonlinear, nonlocal terms is simple. Indeed, for any $\Phi \in L^2(s,t;H^1)$, since f_n solves (1.41), we find

$$\int_s^t \int_{\mathbb{R}^3} \Phi \partial_t f_n \, \mathrm{d}v \, \mathrm{d}\tau = -\int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot (A[f_n]\nabla f_n - \nabla a[f_n] f_n) \, \mathrm{d}v \, \mathrm{d}\tau.$$

Analyzing the convergence in the top order terms, we have

$$\begin{aligned} &\left| \int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot (A[f_n]\nabla f_n - A[f]\nabla f) \, \mathrm{d}v \, \mathrm{d}\tau \right| \\ &\quad \leq \left| \int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot A[f_n - f]\nabla f_n \, \mathrm{d}v \, \mathrm{d}\tau \right| \\ &\qquad + \left| \int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot A[f] \left(\nabla f_n - \nabla f\right) \, \mathrm{d}v \, \mathrm{d}\tau \right|. \end{aligned} \tag{8.51}$$

For the first term on the right hand side of (8.51), we find

$$\begin{aligned} \left| \int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot A[f_n - f]\nabla f_n \, \mathrm{d}v \, \mathrm{d}\tau \right| &\leq \|\nabla\Phi\|_{L^2_{t,x}} \|\nabla f_n\|_{L^2_{t,x}} \|A[f - f_n]\|_{L^\infty_{t,x}} \\ &\leq C(s,t) \|f - f_n\|^{\frac{1}{3}}_{L^\infty(s,t;L^1)} \|f - f_n\|^{\frac{2}{3}}_{L^\infty(s,t;L^2)}, \end{aligned}$$

which converges to 0 by conservation of mass and strong compactness in $L^\infty(s,t;L^2)$. The second term on the right hand side of (8.51) converges to 0 because $f_n \to f$ weakly in $L^2(s,t;H^1)$ and $A[f]\nabla\Phi \in L^2(s,t;H^1)$.

Next, we analyse the lower order terms, namely,

$$\begin{aligned}
&\left| \int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot (f_n \nabla a[f_n] - f\nabla a[f]) \, \mathrm{d}v \, \mathrm{d}\tau \right| \\
&\leq \int_s^t \int_{\mathbb{R}^3} |\nabla\Phi \cdot (f_n - f)\nabla a[f_n]| \, \mathrm{d}v \, \mathrm{d}\tau + \int_s^t \int_{\mathbb{R}^3} |\nabla\Phi \cdot f \nabla a[f_n - f]| \, \mathrm{d}v \, \mathrm{d}\tau \\
&\leq \|\Phi\|_{L^2(s,t;H^1)} \left(\|f_n - f\|_{L^2(t,s;L^2)} \|\nabla a[f_n]\|_{L^\infty_{t,x}} + \|f\|_{L^2_{t,x}} \|\nabla a[f_n - f]\|_{L^\infty_{t,x}} \right)
\end{aligned} \tag{8.52}$$

and the right hand side converges to 0 as $n \to \infty$. Finally, using $\partial_t f_n \rightharpoonup \partial_t f$ in $L^2(s,t;H^{-1})$, we find f satisfies Landau in the sense that for any $0 < s < t < \infty$ and any $\Phi \in L^2(s,t;H^1)$,

$$\int_s^t \int_{\mathbb{R}^3} \Phi \partial_t f \, \mathrm{d}v \, \mathrm{d}\tau = -\int_s^t \int_{\mathbb{R}^3} \nabla\Phi \cdot (A[f]\nabla f - \nabla a[f] f) \, \mathrm{d}v \, \mathrm{d}\tau. \tag{8.53}$$

We have shown f is a weak solution to (1.41) on $(0,\infty) \times \mathbb{R}^3$ in the sense of (8.53). It remains to show that $f \in C^1(0,\infty; C^2_v)$, so that f is a classical solution for positive times.

However, improving $f \in L^\infty_{t,v}$ to classical regularity for f is relatively standard in the literature. For the sake of completeness, we provide a sketch of one possible argument following [82] and refer the reader to [82, Lemma 4.2 and Lemma 4.3] to fill in the omitted details. We note that f belongs to $L^\infty(s,t;L^\infty)$ and the weak formulation (8.53) is sufficiently strong to allow us to use f as a test function. This allows us to perform weighted H^k estimates and obtain for $k \in \mathbb{R}$, $\ell \in \mathbb{N}$, and $0 < s < t < \infty$:

$$\begin{aligned}
&\sup_{s<\tau<t} \|\nabla^\ell f\|^2_{L^2_k} + \int_s^t \|\nabla^{\ell+1} f\|_{L^2_{k-3}} \, \mathrm{d}\tau \\
&\qquad \leq C\left(t, H, \|f_{in}\|_{L^1_{k+6\ell}}, \|f\|_{L^\infty(s,t;L^\infty)}\right) \left(1 + \frac{1}{s}\right)^\ell.
\end{aligned} \tag{8.54}$$

The estimates for $\ell = 0$ and $\ell = 1$ follow from the proof of Lemma 8.12, while the estimates for $\ell \geq 2$ are explained thoroughly in [82, Lemma 4.2]. From (8.54) with $\ell = 2$ and $k = 0$ and the Sobolev embedding $H^2 \hookrightarrow W^{1,6}$, we find

$$\|f\|_{L^\infty(s,t;W^{1,6})} \leq C\left(s,t,\|f_{in}\|_{L^1_{12}}\right), \qquad \|\partial_t f\|_{L^\infty(s,t;W^{-1,6})} \leq C\left(s,t,\|f_{in}\|_{L^1_{12}}\right), \tag{8.55}$$

after using a duality argument to bound the time derivative like in (8.48). Real interpolation of (vector-valued) Sobolev spaces implies

$$\|f\|_{W^{\theta,6}(s,t;W^{(1-2\theta)-,6})} \leq C(s,t), \qquad \text{for any } \theta \in (0,1). \tag{8.56}$$

Picking $1/6 < \theta < 1/4$, Morrey's inequality implies $f \in C^{0,\alpha/2}(s,t;C^{0,\alpha})$ for some $\alpha \in (0,1)$. Because f is globally (in velocity) Hölder with exponent α, the non-local coefficients $A[f]$ and $\nabla a[f]$ lie in the same Hölder space as f. Thus, (1.41) is now a divergence form

parabolic equation with Hölder continuous coefficients. Since the lower bound for $A[f]$ still degenerates like $\langle v\rangle^{-3}$ for large velocities, performing classical parabolic Schauder estimates on compact subsets of $(s,t)\times\mathbb{R}^3$ yields $f\in C^{1,\alpha/2}(s,t;C^{2,\alpha}_{loc})$, which implies f is a classical solution to (1.41).

Step 5: Behavior at Initial Time

It remains only to address sense in which f satisfies (1.41) up to time $t=0$ and the sense in which the initial datum f_{in} is obtained. For $\varphi\in C_c^\infty([0,\infty)\times\mathbb{R}^3)$, we integrate by parts in (1.41) for f_n and use $\nabla\cdot A=\nabla a$, to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} f_n(0)\varphi(0)\,\mathrm{d}v-\int_{\mathbb{R}}\int_{\mathbb{R}^3} f_n\partial_t\varphi\,\mathrm{d}v\,\mathrm{d}t\\ =\int_{\mathbb{R}}\int_{\mathbb{R}^3}\left(\nabla^2\varphi:A[f_n]f_n\right)+2\nabla\varphi\cdot\nabla a[f_n]f_n\,\mathrm{d}v\,\mathrm{d}t.\end{aligned}\tag{8.57}$$

We use $f_n(0)\to f_{in}$ in L^p by construction and $f_n\rightharpoonup f$ in $L^2(0,T;L^2)$ to conclude that the left hand side of (8.57) converges to

$$\int_{R^3} f_{in}\varphi(0)\,\mathrm{d}v-\int_{\mathbb{R}}\int_{\mathbb{R}^3} f\partial_t\varphi\,\mathrm{d}v\,\mathrm{d}t.$$

On the other hand, say φ is supported in the time interval $[0,t]$. Then, splitting the time integral on the right hand side of (8.57) into $[0,s]$ and $[s,t]$, and using f is a classical solution to Landau on $[s,t]$, we find that for any $s>0$,

$$\begin{aligned}\int_{\mathbb{R}^3} f_{in}\varphi(0)\,\mathrm{d}v-\int_{\mathbb{R}}\int_{\mathbb{R}^3} f\partial_t\varphi\,\mathrm{d}v\,\mathrm{d}t\\ =\int_s^\infty\int_{\mathbb{R}^3}\left(\nabla^2\varphi:A[f]f\right)+2\nabla\varphi\cdot\nabla a[f]f\,\mathrm{d}v\,\mathrm{d}t\\ +\lim_{n\to\infty}\int_0^s\int_{\mathbb{R}^3}\left(\nabla^2\varphi:A[f_n]f_n\right)+2\nabla\varphi\cdot\nabla a[f_n]f_n\,\mathrm{d}v\,\mathrm{d}t.\end{aligned}$$

Taking $s\to 0^+$, it suffices to show that

$$\lim_{s\to0^+}\sup_n\int_0^s\int_{\mathbb{R}^3}\left(\nabla^2\varphi:A[f_n]f_n\right)+2\nabla\varphi\cdot\nabla a[f_n]f_n\,\mathrm{d}v\,\mathrm{d}t=0.$$

First, using Lemma 8.5 and $f_n\rightharpoonup f$ in $L^2(0,T_0;L^2)$, we have

$$\begin{aligned}\int_0^s\nabla^2\varphi:A[f_n]f_n&\le s^{1/2}\|\nabla^2\varphi_{L^\infty(0,T_0;L^2)}\|A[f_n]\|_{L^\infty(0,T_0;L^\infty)}\|f_n\|_{L^2(0,T_0;L^2)}\\ &\le Cs^{1/2}\|\varphi\|_{L^\infty(0,T_0;H^2)},\end{aligned}$$

which evidently converges to 0 as $s\to0^+$, uniformly in n. Second, using Lemma 8.5 and

$f_n \rightharpoonup f$ in $L^2(0,T_0;L^2)$ again,

$$\int_0^s \|\nabla\varphi\cdot\nabla a[f_n]f_n\|_{L^1}\,\mathrm{d}t \le \int_0^s \|\nabla\varphi\|_{L^6}\|\nabla a[f_n]\|_{L^3}\|f_n\|_{L^2}\,\mathrm{d}t \le Cs^{1/2}\|\varphi\|_{L^\infty(0,T_0;H^2)},$$

which evidently converges to 0 as $s\to 0^+$. Thus, by the Lebesgue dominated convergence theorem, f satisfies Landau in the sense that for each $\varphi\in C_c^\infty([0,\infty)\times\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} f_{in}\varphi(0)\,\mathrm{d}v - \int_{\mathbb{R}}\int_{\mathbb{R}^3} f\partial_t\varphi\,\mathrm{d}v\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}^3}\left(\nabla^2\varphi : A[f]f\right) + 2\nabla\varphi\cdot\nabla a[f]f\,\mathrm{d}v\,\mathrm{d}t. \quad (8.58)$$

It remains only to show that the initial datum is obtained strongly in $L^1\cap L^p$. First, we note that $f\in C(0,T_0;W^{-1,p})$ and so f is certainly continuous in the sense of distributions. Now, taking test functions in (8.58) of the form $\psi(t)\varphi(v)$ for ψ a smooth approximation of $\chi_{[0,s]}(t)$ and $\varphi\in C_c^\infty(\mathbb{R}^3)$, and using the monotone convergence theorem we find for any $0<s$ and $\varphi\in C_c^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} f_{in}\varphi\,\mathrm{d}v - \int_{\mathbb{R}^3} f(s)\varphi\,\mathrm{d}v = \int_0^s\int_{\mathbb{R}^3}\left(\nabla^2\varphi : A[f]f\right) + 2\nabla\varphi\cdot\nabla a[f]f\,\mathrm{d}v\,\mathrm{d}t. \quad (8.59)$$

Since the right hand side of (8.59) converges to 0 as $s\to 0$ for any $\varphi\in C_c^\infty$, we conclude that

$$\lim_{s\to 0^+}\int_{\mathbb{R}^3}\varphi f(s)\,\mathrm{d}v = \int_{\mathbb{R}^3}\varphi f_{in}\,\mathrm{d}v, \qquad \text{for each } \varphi\in C_c^\infty(\mathbb{R}^3).$$

Because f is continuous in the sense of distributions, $f(0)=f_{in}$, and the initial datum is obtained in the sense of distributions. By a density argument using $f\in L^\infty(0,T_0;L^p)$, the initial datum f_{in} is obtained weakly in L^p so that by weak lower semi-continuity of norms, we obtain

$$\|f_{in}\|_{L^p} \le \liminf_{s\to 0^+}\|f(s)\|_{L^p}.$$

Next, analyzing the evolution of the L^p norms of the smooth solutions f_n, we find the simple estimate,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int f_n^p &= -\int \nabla f_n^{p-1}\cdot\left(A[f_n]\nabla f_n - \nabla a[f_n]f_n\right)\,\mathrm{d}v\\ &\le -C(p)\int \nabla f_n^{p/2}\cdot A[f_n]\nabla f_n^{p/2}\,\mathrm{d}v + C(p)\int_{\mathbb{R}^3} f_n^{p+1}\,\mathrm{d}v\\ &\le \|f_n\|_{L^p}^p\|f_n\|_{L^\infty}.\end{aligned}$$

Integrating in time, for each $0\le t\le T_0$,

$$\|f_n(t)\|_{L^p}^p \le \|f_n(0)\|_{L^p}^p + \|f_n\|_{L^1(0,t;L^\infty)}\|f_n\|_{L^\infty(0,T_0;L^p)}^p. \quad (8.60)$$

Now taking the limit as $n \to \infty$ in (8.60), we find for each $0 < t < T_0$,

$$\|f(t)\|_{L^p}^p \leq \|f_{in}\|_{L^p}^p + \|f\|_{L^1(0,t;L^\infty)}\|f\|_{L^\infty(0,T_0;L^p)}^p.$$

Since $f \in L^1(0,T_0;L^\infty)$, taking $t \to 0^+$, we find

$$\limsup_{t\to 0^+} \|f(t)\|_{L^p} \leq \|f_{in}\|_{L^p}.$$

Combining $f(t) \rightharpoonup f_{in}$ in L^p with $\|f(t)\|_{L^p} \to \|f_{in}\|_{L^p}$ implies strong convergence in L^p. Finally, $f \in C(0,T_0;L^p)$ and $f \in L^\infty(0,T_0;L^1_m)$ for $m > 2$ guarantees $f \in C(0,T_0;L^1_2 \cap L^p)$.

8.4 Large data, short time

We now move to initial data that is not assumed to be sufficiently close the steady-state, the Maxwellian μ. We construct a unique solution that exists for some amount of time. We adapt the ODE methods and conditional regularity results above to deduce new short time $L^p \to L^\infty$ smoothing estimates. These estimates enable us to construct local-in-time smooth solutions for large L^p initial data, and allow us to show directly conditional regularity results for solutions verifying *unweighted* Prodi-Serrin type conditions. As a consequence, we obtain additional stability and uniqueness results for the solutions we construct.

8.4.1 Propagation of L^p norms

In this section, we present an ODE argument that provides a quantitative bound on the time that a smooth solution to the homogeneous Landau equation remains in L^p, in terms of norms of the initial datum. This is a variant of the ODE argument used in [83] to quantify how long solutions with small initial data remain small. We note that a version of this a-priori estimate specialized to the case $p = 2$ first appeared in [4].

Proposition 8.3. *Fix $p > \frac{3}{2}$, $m > \frac{9}{2}\frac{p-1}{p-3/2}$, and $M \in \mathbb{R}^+$. Let $f : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ be any smooth, rapidly decaying solution to* (1.41) *satisfying* (1.46) *and having initial data $f_{in} \in L^p$ with*

$$\int_{\mathbb{R}^3} \langle v \rangle^m f_{in} \, \mathrm{d}v + \int_{\mathbb{R}^3} f_{in} \left|\log(f_{in})\right| \, \mathrm{d}v \leq M. \tag{8.61}$$

Then, there is a time $T_0 > 0$ depending only on m, p, M, and $\|f_{in}\|_{L^p}$ such that

$$\sup_{t\in(0,\min(T,T_0))} \|f(t)\|_{L^p}^p + \int_0^{\min(T,T_0)} \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left|\nabla f^{p/2}\right|^2 \, \mathrm{d}v \, \mathrm{d}t \leq C, \tag{8.62}$$

where C depends only on m, p, M, and $\|f_{in}\|_{L^p}$.

Proof. Multiplying (1.41) with f^{p-1} and integrating in v, we integrate by parts and rearrange terms to obtain:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f^p \,\mathrm{d}v + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3} A[f]\nabla f^{p/2}\cdot\nabla f^{p/2}\,\mathrm{d}v = \frac{p-1}{p}\int_{\mathbb{R}^3} f^{p+1}\,\mathrm{d}v.$$

Using the lower bound on the diffusion coefficient from Lemma 8.4, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f^p \,\mathrm{d}v + c_0\int_{\mathbb{R}^3}\left|\nabla f^{\frac{p}{2}}\right|^2\langle v\rangle^{-3}\,\mathrm{d}v \le C(p)\int_{\mathbb{R}^3} f^{p+1}\,\mathrm{d}v. \tag{8.63}$$

To bound the right hand side, we first interpolate between L^{3p}_{-9}, L^p, and L^1_m to find

$$\|f\|_{L^{p+1}} \le \left\|\langle\cdot\rangle^{-\frac{3}{p}} f\right\|^{\theta_1}_{L^{3p}} \|f\|^{\theta_2}_{L^p}\,\|\langle\cdot\rangle^m f\|^{\theta_3}_{L^1}, \tag{8.64}$$

provided $\theta_1,\ \theta_2,\ \theta_3 \in (0,1)$ and the following relations are satisfied:

$$\theta_1+\theta_2+\theta_3 = 1, \qquad \frac{\theta_1}{3p}+\frac{\theta_2}{p}+\theta_3 = \frac{1}{p+1}, \qquad \frac{-3}{p}\theta_1 + m\theta_3 = 0.$$

We solve this system of constraints in terms of m and p as:

$$\begin{aligned}\theta_1 &= \frac{3mp}{(p+1)(2pm-9p+9)}, \qquad \theta_2 = \frac{2p^2m-9p^2-mp}{(p+1)(2pm-9p+9)},\\ \theta_3 &= \frac{9}{(p+1)(2pm-9p+9)},\end{aligned} \tag{8.65}$$

where we note that θ_1, θ_2, and θ_3 belong to $(0,1)$ for $p>\frac{3}{2}$ and $m>\frac{9p}{2p-1}$. Applying the weighted Sobolev inequality (8.11) from Lemma 8.7 gives

$$\|f\|^{p+1}_{L^{p+1}} \le \left(\int_{\mathbb{R}^3}\langle v\rangle^{-3}\left|\nabla f^{\frac{p}{2}}\right|^2\mathrm{d}v + \int_{\mathbb{R}^3} f^p\,\mathrm{d}v\right)^{\frac{(p+1)\theta_1}{p}} \|f\|^{(p+1)\theta_2}_{L^p}\,\|\langle\cdot\rangle^m f\|^{(p+1)\theta_3}_{L^1},$$

where $\frac{(p+1)\theta_1}{p}<1$, using the restriction that $m>\frac{9}{2}\frac{p-1}{p-3/2}>\frac{9p}{2p-1}$. Therefore, an application of Young's inequality yields

$$\begin{aligned}\|f\|^{p+1}_{L^{p+1}} \le& \frac{c_0}{2}\left(\int_{\mathbb{R}^3}\langle v\rangle^{-3}\left|\nabla f^{\frac{p}{2}}\right|^2\mathrm{d}v + \int_{\mathbb{R}^3} f^p\,\mathrm{d}v\right)\\ &+ C(c_0,m,p)\left(\|f\|^{\theta_2}_{L^p}\,\|\langle\cdot\rangle^m f\|^{\theta_3}_{L^1}\right)^{\frac{p(p+1)}{p-(p+1)\theta_1}}.\end{aligned} \tag{8.66}$$

Therefore, setting $y(t)=\|f(t)\|^p_{L^p}$, we arrive at the differential inequality

$$\frac{dy}{dt} + \frac{c_0}{2}\int_{\mathbb{R}^3}\langle v\rangle^{-3}\left|\nabla f^{p/2}\right|^2 \le C(p,c_0)y(t) + C(p,c_0,m)y^{\alpha_1}(t)\left(\int_{\mathbb{R}^3} f\langle v\rangle^m\,\mathrm{d}v\right)^{\alpha_2}, \tag{8.67}$$

where $\alpha_1 = \frac{(p+1)\theta_2}{p-(p+1)\theta_1} > 1$ and $\alpha_2 = \frac{p(p+1)\theta_3}{p-(p+1)\theta_1}$. Since $\|f(t)\|_{L^1_m}$ grows at most linearly in time by Lemma 8.1, this simplifies to

$$\frac{dy}{dt} \leq C(p, c_0)y + C(p, c_0, m)y^{\alpha_1}(1+t)^{\alpha_2} =: C_1 y + g(t)y^{\alpha_1}.$$

In the case of equality, this is a Bernoulli type ODE, motivating the substitution $v = \frac{y^{1-\alpha_1}}{1-\alpha_1}$, which gives

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leq C_1(1-\alpha_1)v + g(t).$$

Performing a second substitution of $w = e^{C_1(\alpha_1-1)t}v$ yields

$$\frac{\mathrm{d}w}{\mathrm{d}t} \leq e^{C_1(\alpha_1-1)t}g(t).$$

Thus, integrating in time,

$$e^{C_1(\alpha_1-1)t}v(t) = w(t) \leq w(0) + \int_0^t e^{C_1(\alpha_1-1)\tau}g(\tau)\,\mathrm{d}\tau = v(0) + \int_0^t e^{C_1(\alpha_1-1)\tau}g(\tau)\,\mathrm{d}\tau. \quad (8.68)$$

We conclude that for $0 < t < T$, y satisfies:

$$y(t) \leq e^{C_1 t}\left[y_0^{1-\alpha_1} + (1-\alpha_1)\int_0^t e^{-C_1(1-\alpha_1)\tau}g(\tau)\,\mathrm{d}\tau\right]^{1/(1-\alpha_1)}.$$

Choosing $2T_0$ sufficiently small, depending only on M, m, and p, we can enforce

$$\sup_{0<t<\min(T,2T_0)} y(t) \leq Cy_0, \quad (8.69)$$

for some constant $C > 0$ depending only on m, p, and M. Plugging this bound for $y(t)$ back into the right hand side of (8.67), we find the desired bound until the time T_0. □

8.4.2 Quantitative short time smoothing

In this subsection, we present a quantitative regularisation estimate for the L^∞ norm of smooth solutions to (1.41). More precisely, we show that for smooth solutions to (1.41), the L^∞ norm is controlled by the $L^{p+1}_{t,v}$ norm for $p > 3/2$. For solutions of (1.41), this norm is controlled by the energy functional appearing in (8.62) using interpolation and (8.11) from Lemma 8.7. The proof relies on a De Giorgi type iteration, introduced in [56] to study the regularity of weak solutions to elliptic equations. The specific version here is adapted from [83], which followed [8] and used the energy functional $\mathcal{E}$ defined below in (8.71) as a control quantity. Here we refine the method and use certain space-time Lebesgue norms as control quantities.

Proposition 8.4. *Fix $p > \frac{3}{2}$, $m > \frac{9}{2}\frac{p-1}{p-\frac{3}{2}}$, $M \in \mathbb{R}^+$. Suppose f_{in} is a non-negative Schwartz class function, satisfying the normalisation* (1.46) *and the bounds*

$$\|f_{in}\|_{L^1_m} \leq M \qquad \text{and} \qquad \int_{\mathbb{R}^3} f_{in} \left|\log(f_{in})\right| \, dv \leq M.$$

Let $f : [0, T^*) \times \mathbb{R}^3 \to \mathbb{R}^+$ be the unique Schwartz class solution to (1.41) *with initial datum f_{in} and $0 < T^* \leq \infty$ its maximal time of existence. Then, for γ and β, defined in terms of m and p via*

$$\beta = \frac{2}{3} - \frac{3}{m} \qquad \text{and} \qquad \gamma = \frac{2(p-3/2)}{3m}\left[m - \frac{9(p-1)}{2(p-3/2)}\right], \tag{8.70}$$

there is a constant $C = C(M, m, p)$ such that for any $0 < t < \min(1, T^*)$, f satisfies

$$\|f(t)\|_{L^\infty(\mathbb{R}^3)} \leq C\left(A_0^{\frac{\beta}{\gamma}} + A_0^{\frac{\beta}{1+\gamma+\beta}} t^{-\frac{1+\beta}{1+\gamma+\beta}}\right), \qquad \text{where} \qquad A_0 := \int_0^t \int_{\mathbb{R}^3} f^{p+1} \, \mathrm{d}v \, \mathrm{d}t.$$

Step 1: Introduction of Level Set Functions and Functionals

For any fixed $\ell \in \mathbb{R}^+$, we consider the level set function, denoted by

$$f_\ell(t, v) := f(t, v) - \ell, \quad f_\ell^+(t, v) = \max\left(f_\ell(t, v), 0\right).$$

For any $\ell \geq 0$ and $0 \leq T_1 \leq T_2 \leq \min(1, T^*)$, we define two functionals of f_ℓ^+:

$$\begin{aligned} \mathcal{E}_\ell(T_1, T_2) &= \sup_{T_1 < t < T_2} \int_{\mathbb{R}^3} (f_\ell^+)^p \, \mathrm{d}v + \int_{T_1}^{T_2} \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left|\nabla (f_\ell^+)^{p/2}\right|^2 \, \mathrm{d}v \, \mathrm{d}t, \\ \mathcal{A}_\ell(T_1, T_2) &= \int_{T_1}^{T_2} \int_{\mathbb{R}^3} (f_\ell^+)^{p+1} \, \mathrm{d}v \, \mathrm{d}t. \end{aligned} \tag{8.71}$$

The functional $\mathcal{E}_\ell$ is an energy functional associated to (1.41), which appears when one attempts standard L^p energy estimates. Note that we unconditionally control $\mathcal{E}(0, T_0)$ from the ODE argument in Proposition 8.3. The functional $\mathcal{A}_\ell$ is the "control quantity" appearing in the statement of Proposition 8.4, which will be the iterated functional below. Our goal now is to show the following two inequalities relating the two functionals: for any $0 \leq T_1 < T_2 \leq T_3 \leq \min(1, T^*)$ and any $0 < k < \ell$,

$$\mathcal{A}_\ell(T_1, T_3) \leq \frac{C(m, p, M)}{(\ell - k)^\gamma} \mathcal{E}_k(T_1, T_3)^{1+\beta}, \tag{8.72}$$

$$\mathcal{E}_\ell(T_2, T_3) \leq C(m, p, M)\left[\frac{1}{(T_2 - T_1)(\ell - k)} + 1 + \frac{\ell}{(\ell - k)} + \frac{\ell^2}{(\ell - k)^2}\right] \mathcal{A}_k(T_1, T_3), \tag{8.73}$$

where β and γ are as in the statement of Proposition 8.4.

Step 2: Proof of (8.72)

Let us show (8.72). This step does **NOT** use that f satisfies the Landau equation, but rather relies on functional inequalities. We start by noting that the weighted Sobolev inequality (8.11) in Lemma 8.7 gives for any function $g : \mathbb{R}^3 \to \mathbb{R}$

$$\|g\|_{L^r} \leq \left\| \langle\cdot\rangle^{-\frac{3}{p}} g \right\|_{L^{3p}}^{\theta_1} \|g\|_{L^p}^{\theta_2} \|\langle\cdot\rangle^{\alpha} g\|_{L^1}^{\theta_3},$$

provided $0 < \theta_1,\ \theta_2,\ \theta_3 < 1$ and the following relations are satisfied:

$$\theta_1 + \theta_2 + \theta_3 = 1, \qquad \frac{\theta_1}{3p} + \frac{\theta_2}{p} + \theta_3 = \frac{1}{r}, \qquad \frac{-3}{p}\theta_1 + \alpha\theta_3 = 0.$$

Imposing further that $r\theta_1 = p$, we find

$$r\theta_2 = \frac{p\left[r - p - \frac{2}{3}\right]}{p-1}, \qquad r\theta_3 = \frac{5p - 3r}{3(p-1)}, \qquad \alpha = \frac{9(p-1)}{5p-3r}, \qquad r \in \left(p + \frac{2}{3}, \frac{5p}{3}\right).$$

Next we want to use the interpolation estimate in (8.30). Note that the admissible range for r, i.e. $(p + 2/3, 5p/3)$ is non-degenerate if and only if $p > 1$ and, moreover, $p + 1$ is within this range if and only if $p > 3/2$. With the estimate (8.30) at hand for general g and r, let us return to our specific setting: Let us fix r by setting $\alpha = m$, or, in other words, define r via the relation

$$m = \frac{9(p-1)}{5p-3r} \qquad \text{or equivalently,} \qquad r = \frac{5p}{3} - \frac{3(p-1)}{m}.$$

With this definition of r, we see that r is in the admissible range $(p + 2/3, 5p/3)$ provided $m > 9/2$. However, since we need to bound the L^{p+1} norm, we see that $r > p + 1$ if and only if

$$m > \frac{9}{2}\frac{p-1}{p-\frac{3}{2}}.$$

Since by assumption we have enough moments, we set $\gamma = r - (p+1) > 0$. Therefore, applying (8.30) to f_k^+, we find the pointwise-in-time estimate

$$\left\|f_k^+(t)\right\|_{L^{p+1+\gamma}}^{p+1+\gamma} \leq C\left(\int_{\mathbb{R}^3} \langle v\rangle^{-3} \left|\nabla (f_k^+)^{\frac{p}{2}}\right|^2 \,\mathrm{d}v + \left\|f_k^+\right\|_{L^p}^p\right) \left\|f_k^+\right\|_{L^p}^{p\left(\frac{\gamma+1/3}{p-1}\right)} \left\|\langle\cdot\rangle^{\alpha} f_k^+\right\|_{L^1}^{\frac{5p-3r}{3(p-1)}}.$$

A simple exponent computation shows $(r - p - 2/3)/(p-1) = 2/3 - 3/m = \beta$. So, integrating over $T_1 < t < T_3 \leq \min(T^*, 1)$ and using Lemma 8.1 to bound the L^1 moments

of f, we obtain

$$\begin{aligned}
&\left\|f_k^+\right\|_{L^{p+1+\gamma}([T_1,T_3]\times\mathbb{R}^3)}^{p+1+\gamma} \\
&\quad\le C(M)\left(\int_{T_1}^{T_3}\int_{\mathbb{R}^3}\langle v\rangle^{-3}\,\nabla(f_k^+)^{\frac{p}{2}}\,\mathrm{d}v\,\mathrm{d}t\right)\left\|f_k^+\right\|_{L^\infty(T_1,T_3;L^p)}^{p\beta} \\
&\qquad + C(M)\left\|f_k^+\right\|_{L^\infty(T_1,T_3;L^p)}^{p(1+\beta)} \\
&\quad\le C(M)\mathcal{E}_k(T_1,T_3)^{1+\beta}.
\end{aligned} \tag{8.74}$$

Finally, we pass from the lower level set k to a higher lever set ℓ as we did before in (8.29): combining (8.74) with (8.29) where $\alpha=\gamma$, we obtain for any $0\le k<\ell$,

$$\mathcal{A}_\ell(T_1,T_3)\le\frac{1}{(\ell-k)^\gamma}\int_{T_1}^{T_3}\int_{\mathbb{R}^3}(f_k^+)^{p+\gamma+1}\,\mathrm{d}v\,\mathrm{d}t\le\frac{C(M)}{(\ell-k)^\gamma}\mathcal{E}_k(T_1,T_3)^{1+\beta},$$

which completes the proof of (8.72).

Step 3: Proof of (8.73)

Let us now show (8.73). Contrary to the proof of (8.72), the proof of (8.73) relies essentially on f solving the Landau equation. We begin by testing the weak formulation of (1.41) with $(f_\ell^+)^{p-1}$. Since $\partial_t f_\ell^+=\partial_t f\chi_{\{f\ge\ell\}}$ and $\nabla_v f_\ell^+=\nabla_v f\chi_{\{f\ge\ell\}}$, we rearrange terms as in the proof of Proposition 8.3 to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(f_\ell^+)^p\,\mathrm{d}v+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}A[f]\nabla(f_\ell^+)^{\frac{p}{2}}\cdot\nabla(f_\ell^+)^{\frac{p}{2}}\,\mathrm{d}v \\
&\qquad=\frac{(p-1)}{p}\int_{\mathbb{R}^3}(f_\ell^+)^{p+1}\,\mathrm{d}v+\frac{(2p-1)\ell}{p}\int_{\mathbb{R}^3}(f_\ell^+)^p\,\mathrm{d}v+\ell^2\int_{\mathbb{R}^3}(f_\ell^+)^{p-1}\,\mathrm{d}v.
\end{aligned}$$

Using the lower bound on the coefficient $A[f]$ from Lemma 8.4 and discarding constants that depend only on p and M yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(f_\ell^+)^p\,\mathrm{d}v+\int_{\mathbb{R}^3}\langle v\rangle^{-3}\left|\nabla(f_\ell^+)^{\frac{p}{2}}\right|^2\,\mathrm{d}v \\
&\qquad\lesssim_{p,M}\int_{\mathbb{R}^3}(f_\ell^+)^{p+1}\,\mathrm{d}v+\ell\int_{\mathbb{R}^3}(f_\ell^+)^p\,\mathrm{d}v+\ell^2\int_{\mathbb{R}^3}(f_\ell^+)^{p-1}\,\mathrm{d}v.
\end{aligned} \tag{8.75}$$

Then, integrating (8.75) over (t_1,t_2) yields

$$\begin{aligned}
&\int_{\mathbb{R}^3}(f_\ell^+(t_2))^p\,\mathrm{d}v+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\langle v\rangle^{-3}\left|\nabla(f_\ell^+)^{\frac{p}{2}}\right|^2\,\mathrm{d}v \\
&\qquad\lesssim_{p,M}\int_{\mathbb{R}^3}(f_\ell^+)^p(t_1)\,\mathrm{d}v+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(f_\ell^+)^{p+1}\,\mathrm{d}v\,\mathrm{d}t \\
&\qquad\quad+\ell\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(f_\ell^+)^p\,\mathrm{d}v\,\mathrm{d}t+\ell^2\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(f_\ell^+)^{p-1}\,\mathrm{d}v\,\mathrm{d}t.
\end{aligned}$$

Finally, taking the the supremum over $t_2 \in [T_2, T_3]$, averaging over $t_1 \in [T_1, T_2]$, and using (8.29) with $\alpha \in \{0, 1, 2\}$, we find

$$\begin{aligned}
\mathcal{E}_\ell(T_2, T_3) &= \sup_{s\in(T_2,T_3)} \int_{\mathbb{R}^3} (f_\ell^+(s))^p \,\mathrm{d}v + \int_{T_2}^{T_3} \int_{\mathbb{R}^3} \langle v\rangle^{-3} \left|\nabla (f_\ell^+)^{\frac{p}{2}}\right|^2 \,\mathrm{d}v \\
&\lesssim_{M,p} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{\mathbb{R}^3} (f_\ell^+)^p(t) \,\mathrm{d}v \,\mathrm{d}t \\
&\quad + \int_{T_1}^{T_3} \int_{\mathbb{R}^3} (f_\ell^+)^{p+1} \,\mathrm{d}v \,\mathrm{d}t + \ell \int_{T_1}^{T_3} \int_{\mathbb{R}^3} (f_\ell^+)^p \,\mathrm{d}v \,\mathrm{d}t \\
&\quad + \ell^2 \int_{T_1}^{T_3} \int_{\mathbb{R}^3} (f_\ell^+)^{p-1} \,\mathrm{d}v \,\mathrm{d}t \\
&\lesssim_{M,p} \left[\frac{1}{(T_2 - T_1)(\ell - k)} + 1 + \frac{\ell}{(\ell - k)} + \frac{\ell^2}{(\ell - k)^2}\right] \mathcal{A}_k(T_1, T_3),
\end{aligned} \tag{8.76}$$

which completes the proof of (8.73).

Step 4: De Giorgi iteration

We now finish the proof of Proposition 8.4 using an iteration procedure inspired by that of De Giorgi, introduced in [56]. In particular, we will interweave the inequalities (8.72) and (8.73) passing to smaller dyadic space-time domains and higher level sets. To this end, we define the following iteration quantities: Let $0 < t < T = \min(T^*, 1)$ be the fixed time in the statement of Proposition 8.4 and consider for $n \in \mathbb{N}$,

$$\ell_n = K\left(1 - 2^{-n}\right), \quad t_n = t\left(1 - 2^{-n}\right), \quad \text{and} \quad A_n = \mathcal{A}_{\ell_n}(t_n, T), \tag{8.77}$$

where $K > 0$ is a parameter to be chosen appropriately. Indeed, the main remaining task is to find a value of K for which $\lim_{n\to\infty} A_n = 0$. By the definition of the control quantity and the choice of iteration parameters, this would imply $f(\tau, v) \le K$ for almost every $t < \tau < T$ and $v \in \mathbb{R}^3$.

To this end, we first use (8.72) to obtain

$$A_{n+2} \le \frac{C}{(\ell_{n+2} - \ell_{n+1})^\gamma} \mathcal{E}_{\ell_{n+1}}(t_{n+2}, T)^{1+\beta}.$$

Next, we use (8.73) to obtain the recurrence

$$\begin{aligned}
A_{n+2} &\le \frac{C}{(\ell_{n+2} - \ell_{n+1})^\gamma} \left[\frac{1}{(\ell_{n+1} - \ell_n)(t_{n+2} - t_n)} + 1 + \frac{\ell_{n+1}}{(\ell_{n+1} - \ell_n)} + \frac{\ell_{n+1}^2}{(\ell_{n+1} - \ell_n)^2}\right]^{1+\beta} A_n^{1+\beta} \\
&\le \frac{C_1 2^{n(1+\gamma+\beta)} A_n^{1+\beta}}{K^\gamma} \left[\frac{1}{Kt} + 1\right]^{1+\beta},
\end{aligned} \tag{8.78}$$

where C_1 is now a fixed constant. Because β is positive, A_n is super-geometric in n and A_n should decay fast enough to beat the geometrically-growing (in n) coefficients in the recurrence. To prove this, we employ a discrete barrier argument to find a quantitative estimate for K. We expect A_n to decay exponentially, so we search for a choice of parameters K and $Q > 0$ such that the sequence A_n^* defined as

$$A_n^* := A_0 Q^{-n}, \quad \text{for } n \in \mathbb{N}$$

satisfies (8.78) with the reversed inequality, and so by induction remains larger than A_n. More precisely, we want to find a choice of K and A so that A_n^* satisfies

$$A_{n+2}^* \geq \frac{C_1 2^{n(\gamma+1+\beta)} (A_n^*)^{1+\beta}}{K^\gamma} \left[\frac{1}{tK} + 1\right]^{1+\beta}. \tag{8.79}$$

Inserting our ansatz for A_n^*, (8.79) holds if

$$1 \geq \frac{C_1 A_0^\beta Q^2 \left(2^{\gamma+1+\beta} Q^{-\beta}\right)^n}{K^\gamma} \left[\frac{1}{tK} + 1\right]^{1+\beta}.$$

We now choose Q such that

$$2^{\gamma+1+\beta} Q^{-\beta} \leq 1 \qquad \text{or equivalently,} \qquad Q \geq 2^{\frac{\gamma+1+\beta}{\beta}}.$$

Then, for (8.79) to hold we need

$$1 \geq \left[\frac{C_1 A_0^\beta Q^2}{K^\gamma}\right]^{\frac{1}{1+\beta}} \left[\frac{1}{tK} + 1\right].$$

Next, choosing K so each of the terms is smaller than a half, the recurrence (8.79) holds. More precisely, we choose K as

$$K = \max\left(\left(2^{1+\beta} C_1 Q^2\right)^{\frac{1}{\gamma}}, \left(2^{1+\beta} C_1 Q^2\right)^{\frac{1}{1+\beta+\gamma}}\right) \max\left\{A_0^{\frac{\beta}{\gamma}}; A_0^{\frac{\beta}{\gamma+1+\beta}} t^{-\frac{1+\beta}{1+\gamma+\beta}}\right\}. \tag{8.80}$$

With the above choices of K and Q, if $A_n \leq A_n^*$, by construction

$$A_{n+2} \leq \frac{C_1 2^{n(\gamma+1+\beta)} A_n^{1+\beta}}{K^\gamma} \left[\frac{1}{tK} + 1\right]^{1+\beta} \leq \frac{C_1 2^{n(\gamma+1+\beta)} (A_n^*)^{1+\beta}}{K^\gamma} \left[\frac{1}{tK} + 1\right]^{1+\beta} \leq A_{n+2}^*.$$

Since $A_0 = A_0^*$, it follows by induction that $A_n \leq A_n^*$ for n even. Moreover, since $Q > 1$ we deduce

$$\lim_{n\to\infty} A_{2n} \leq \lim_{n\to\infty} A_{2n}^* = 0.$$

Recalling the definition of A_n in (8.77), the monotone convergence theorem implies

$$\left\|f_K^+\right\|_{L^{p+1}}^{p+1} = 0, \qquad \text{or equivalently,} \qquad \|f\|_{L^\infty((t,T)\times\mathbb{R}^3)} \leq K.$$

Finally, from the choice of K, we have for all $v \in \mathbb{R}^3$ and $\tau \in (t,T)$

$$f(\tau,v) \leq K \leq C\left(A_0^{\frac{\beta}{\gamma}} + A_0^{\frac{\beta}{1+\gamma+\beta}} t^{-\frac{1+\beta}{1+\gamma+\beta}}\right).$$

Recalling the definitions of the exponents β and γ and performing some basic algebraic manipulations gives the simplified expressions found in (8.70), completing the proof of Proposition 8.4.

8.4.3 Short time existence

8.4.3.1 Proof of Theorem 8.2

Step 1: Approximation procedure

Fix $p > 3/2$ and $m > \frac{9}{2}\frac{p-1}{p-3/2}$ as in the statement Theorem 8.2. Take $f_{in} \geq 0$ any initial data satisfying the hypotheses of Theorem 8.2, namely that f_{in} is normalised according to (1.46) and f_{in} satisfies

$$\int_{\mathbb{R}^3} f_{in}|\log f_{in}| + f_{in}^p + \langle v\rangle^m f_{in}\,\mathrm{d}v \leq K < \infty.$$

Now, we approximate f_{in} by a family of Schwartz class initial datum $\{f_{in}^\varepsilon\}_{\varepsilon>0}$, satisfying

- **Regularity:** For each $\varepsilon > 0$, $f_{in}^\varepsilon \in \mathcal{S}(\mathbb{R}^3)$;
- **Normalisation:** For each $\varepsilon > 0$, $f_{in}^\varepsilon \geq 0$ and f_{in}^ε satisfies (1.46).
- **Uniform bounds:** For each $\varepsilon > 0$,

$$\int_{\mathbb{R}^3} f_{in}^\varepsilon|\log f_{in}^\varepsilon| + |f_{in}^\varepsilon|^p + \langle v\rangle^m f_{in}^\varepsilon\,\mathrm{d}v \leq 2K$$

- **Convergence:** The family $\{f_{in}\}_{\varepsilon>0}$ converges to f_{in} as $\varepsilon \to 0^+$ strongly in $L^p \cap L^1_m$ and pointwise almost everywhere.

Consequently, by Theorem 8.4, there exist local-in-time Schwartz class solutions $f_\varepsilon : [0,T_\varepsilon)\times\mathbb{R}^3 \to \mathbb{R}^+$ to (1.41), with initial data f_{in}^ε where T_ε, the maximal time of existence, is either infinite or satisfies

$$\lim_{t\nearrow T_\varepsilon} \|f_\varepsilon(t)\|_{L^\infty} = \infty. \tag{8.81}$$

For the remainder of the proof, the parameters p and m, the constant K, and the family f_ε are fixed.

Step 2: Uniform estimates

We first find a uniform lower bound on T_ε. To this end, we apply Proposition 8.3 and 8.4 to find uniform-in-ε estimates on the family $\{f_\varepsilon\}$. Appealing to Proposition 8.3 with $M = 2K$, we find a uniform time $0 < T_0 \leq 1$ such that

$$\sup_{0<t<\min(T_0,T^*)} \|f_\varepsilon(t)\|_{L^p}^p + c_0 \int_0^{\min(T_0,T_*)} \int_{\mathbb{R}^3} \langle v\rangle^{-3} |\nabla f_\varepsilon^p| \, \mathrm{d}v \, \mathrm{d}t \leq C \|f_{in}^\varepsilon\|_{L^p}^p \lesssim_K 1, \quad (8.82)$$

where both C and c_0 are uniform in ε. Second, note that using the same method as in the proof of Proposition 8.4, i.e. the estimate (8.74) and (8.82) yields

$$\|f_\varepsilon\|_{L^{p+1+\gamma}([0,t]\times\mathbb{R}^3)} \lesssim_K 1.$$

Therefore, we use Proposition 8.4 with $\tilde{p} = p + \gamma$, which implies for each $\varepsilon > 0$,

$$\|f_\varepsilon(t)\|_{L^\infty} \leq Ct^{-\frac{1+\beta}{1+\beta+\tilde{\gamma}}}, \qquad \tilde{\gamma} = \frac{2(p+\gamma-3/2)}{3m}\left[m - \frac{9(p+\gamma-1)}{2(p+\gamma-3/2)}\right]. \quad (8.83)$$

We define $\beta^*(p,m) = \beta^* = \frac{1+\beta}{1+\beta+\tilde{\gamma}}$, which clearly satisfies $\beta^* \in (0,1)$. Therefore,

$$\|f_\varepsilon(t)\|_{L^\infty} \leq Ct^{-\beta^*}, \qquad 0 < t < \min(T_0, T_\varepsilon), \quad (8.84)$$

where again C is uniform in ε. We note that if $T_\varepsilon < T_0$ for some $\varepsilon > 0$, then

$$\lim_{t \nearrow T_\varepsilon} \|f_\varepsilon(t)\|_{L^\infty} \leq CT_\varepsilon^{-\beta^*} < \infty,$$

which contradicts the continuation criteria (8.81). So, we have found our uniform lower bound $T_0 \leq T_\varepsilon$. We now want to construct our strong solution $f(t)$ on $[0, T_0]$. To this end, we would like uniform-in-ε Hölder regularity estimates to gain strong compactness. We begin with Hölder estimates on the coefficients.

The following estimate is taken from [93, Proposition 4.7], where it is stated in a restricted setting. Nevertheless, the same proof implies the localized estimate we need. Before we state the result, let us mention that the notion of Hölder continuity that we work with respects the parabolic scaling of (1.41): we define the distance between two points $(t,v), (s,w) \in \mathbb{R}_+ \times \mathbb{R}^3$ as

$$d\left((t,v),(s,w)\right) := \left(|t-s| + |v-w|^2\right)^{\frac{1}{2}}.$$

Then, Hölder spaces are defined using this notion of distance, that is $f \in C^{0,\alpha}(\mathbb{R}_+ \times \mathbb{R}^3)$ for some $\alpha \in (0,1)$ if for any $(t,v),(s,w) \in \mathbb{R}_+ \times \mathbb{R}^3$ there holds

$$|f(t,v) - f(s,w)| \leq C d\left((t,v),(s,w)\right)^\alpha .$$

The constant C is the Hölder seminorm $C = [f]_{C^{0,\alpha}(\mathbb{R}_+ \times \mathbb{R}^3)}$. Higher order Hölder spaces are then introduced as usual. For more details, see [139, p.7].

Lemma 8.13. *Suppose $g : [0,T] \times \mathbb{R}^3$ is a solution to (1.41) satisfying*

$$\|g\|_{L^\infty([0,T]\times\mathbb{R}^3)} + \|g_{in}\|_{L\log(L)} + \|g_{in}\|_{L^1_2} \leq L$$

and $g \in C^{k,\alpha}((0,T) \times B_R)$ for some $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $R > 0$. Then, $A[g], \nabla a[g] \in C^{k,\alpha}((0,T) \times B_{R/2})$, depending only on L, k, R.

Proof. We prove first the case $k = 0$ for $A[g]$. That is, assume $g \in C^{0,\alpha}([0,T] \times B_R)$. Our goal is to show the same regularity for $A[g]$ on a smaller domain.

Recall that $A[g]$ is defined as convolution with the matrix-valued kernel $K(v) = \frac{\Pi(v)}{8\pi|v|}$. Let $0 \leq \varphi \leq 1$ a smooth function which is identically 1 on $B_{R/4}$ and vanishes outside $B_{R/2}$. We decompose the kernel $K = K_1 + K_2$ where $K_1 = \varphi K$ is supported in B_R and $K_2 = (1-\varphi)K \in C^\infty(\mathbb{R}^3)$. Note that $K_1 \in L^1(\mathbb{R}^3)$ and $K_2 \in W^{2,\infty}(\mathbb{R}^3)$. Let us first estimate the localized part of the kernel. For $0 < t_1 < t_2 < T$ and $v, w \in B_{R/2}$,

$$\begin{aligned} |K_1 * g(t_1,v) - K_1 * g(t_2,w)| &\leq \int_{\mathbb{R}^3} K_1(v^*) \left|g(t_1, v - v^*) - g(t_2, w - v^*)\right| \,\mathrm{d}v^* \\ &\leq \|g\|_{C^\alpha((0,T)\times B_R)} \left(|t_1 - t_2| + |w - v|^2\right)^{\alpha/2} \int_{B_R} K_1(v^*) \,\mathrm{d}v^*. \end{aligned}$$

We conclude $K_1 * g \in C^{0,\alpha}((0,T) \times B_{R/2})$. Next, we look at the smooth, non-localized part of the kernel:

$$\|K_2 * g\|_{W^{1,\infty}_v} \leq \|\nabla K_2 * g\|_{L^\infty} \leq \|\nabla K_2\|_{L^\infty} \|g\|_{L^1_v} ,$$

which implies $K_2 * g$ is spatially Hölder continuous on $(0,T) \times \mathbb{R}^3$. For temporal regularity, we differentiate in time, use the equation, and integrate by parts:

$$\begin{aligned} \partial_t (K_2 * g)(t,v) &= \int_{\mathbb{R}^3} K_2(v - v^*) \partial_t g(t,v^*) \,\mathrm{d}v^* \\ &= -\int_{\mathbb{R}^3} \nabla_{v^*} K_2(v - v^*) \cdot \left[A[g]\nabla_{v^*} g - \nabla_{v^*} a[g] g\right](v^*) \, dv^*. \end{aligned}$$

Integrating by parts once more, and using that $\sum_j \partial_j A_{ij} = -\partial_i a$, we have

$$\partial_t (K_2 * g)(t,v) = \int_{\mathbb{R}^3} \nabla_{v^*}^2 K_2(v-v^*) : A[g]g - 2\nabla_{v^*} K_2(v-v^*) \cdot \nabla_{v^*} a[g] g \, \mathrm{d}v^*.$$

Therefore, using Lemma 8.5, conservation of mass, and $g \in L^\infty([0,T]\times\mathbb{R}^3)$, we estimate

$$\begin{aligned}
|\partial_t (K_2 * g)| &\leq \left\|\nabla^2 K_2\right\|_{L^\infty(\mathbb{R}^3)} \|A[g]\|_{L^\infty_{t,v}} \|g\|_{L^\infty(0,T;L^1(\mathbb{R}^3))} \\
&\quad + 2\|\nabla K_2\|_{L^\infty(\mathbb{R}^3)} \|\nabla a[g]\|_{L^\infty_{t,v}} \|g\|_{L^\infty(0,T;L^1(\mathbb{R}^3))} \\
&\leq C\|K_2\|_{W^{2,\infty}(\mathbb{R}^3)} \left(\|g\|^{5/3}_{L^\infty(0,T;L^1)} \|g\|^{1/3}_{L^\infty_{t,v}} + \|g\|^{4/3}_{L^\infty(0,T;L^1)} \|g\|^{2/3}_{L^\infty_{t,v}} \right) < \infty
\end{aligned}$$

Combining our estimates, we conclude $K_2 * g \in W^{1,\infty}([0,T]\times\mathbb{R}^3)$. It follows that $A[g] \in C^\alpha_{loc}((0,T)\times B_{R/2})$, as desired. The same technique of splitting the kernel into a localized L^1 kernel and a non-localized $W^{2,\infty}$ kernel yields the corresponding result for $\nabla a[g]$. To prove the higher order estimates, i.e. when $k > 0$, we note that both $A[g]$ and $\nabla a[g]$ are convolution-type operators. Therefore, we may put an integral number of derivatives on g and use the $k = 0$ estimate to conclude. □

As a direct consequence of these local Hölder estimates on the coefficients, we may bootstrap the regularity of f_ε, uniformly in ε, using Schauder estimates.

Lemma 8.14 (Higher Regularity)**.** *The family $f_\varepsilon : [0,T_0]\times\mathbb{R}^3 \to \mathbb{R}^+$ constructed above in Step 1 satisfies for each $R > 0$, $0 < t < T_0$, $k \in \mathbb{N}$,*

$$\|f_\varepsilon\|_{C^k((t,T_0)\times B_R)} \leq C(K,R,t,k),$$

uniformly in ε.

Proof. We will prove this by a bootstrap argument. Fix some t and $R > 0$. Then, by Lemma 8.5, the coefficients $A[f_\varepsilon]$ and $\nabla a[f_\varepsilon]$ satisfy the global upper bounds

$$\begin{aligned}
|A[f_\varepsilon]| &\leq C\|f_\varepsilon\|^{2/3}_{L^\infty(0,T;L^1)} \|f_\varepsilon\|^{1/3}_{L^\infty(0,T\times\mathbb{R}^3)} \leq C(K,T), \\
|\nabla a[f_\varepsilon]| &\leq C\|f_\varepsilon\|^{1/3}_{L^\infty(0,T;L^1)} \|f_\varepsilon\|^{2/3}_{L^\infty(0,T\times\mathbb{R}^3)} \leq C(K,T),
\end{aligned}$$

as well as the locally uniform coercivity estimate:

$$A[f_\varepsilon] \geq \frac{c_0}{\langle v\rangle^3} \geq c(K,R).$$

Therefore, we conclude (1.41) is a divergence-form parabolic equation with bounded measurable coefficients. By the parabolic version of the De Giorgi-Nash-Moser theorem

(see [139, Theorem 10.1]), for each $0 < t \leq T$ and $R > 0$, there is an $\alpha \in (0,1)$ such that

$$f_\varepsilon \in C^\alpha((t,T) \times B_{4R}).$$

Therefore, by Lemma 8.13, $A[f_\varepsilon]$, $\nabla a[f_\varepsilon] \in C^\alpha((t,T) \times B_{2R})$. Thus, by parabolic Schauder estimates (see [139, Theorem 12.1]), we conclude $f_\varepsilon \in C^{1,\alpha}((t,T) \times B_R)$, which proves the claim with $k = 1$. Iterating this procedure, gaining one derivative at a time, we conclude $f_\varepsilon \in C^k((t,T) \times B_R)$ for any $k > 0$. □

Step 3: Compactness and properties of the limit

As a consequence of the uniform estimates, (8.84), Lemma 8.14 and standard compactness results, we have a limit $f : [0,T_0] \times \mathbb{R}^3 \to \mathbb{R}$ such that $f_\varepsilon \to f$ in the following senses:

- Strongly in $C^k((t,T_0) \times B_R)$, for each $k \in \mathbb{N}$, $R > 0$ and each $0 < t < T_0$;
- Pointwise almost everywhere in $[0,T_0] \times \mathbb{R}^3$;
- Weak-starly in $L^\infty([0,T_0]; L^p)$;
- Strongly in $L^1([0,T_0] \times \mathbb{R}^3)$,

where the last strong convergence follows from uniform integrability, pointwise convergence, and tightness. Note that the convergence immediately implies we retain $f \geq 0$, f satisfies the normalisation (1.46), and f has decreasing entropy. Moreover, we can assume that $t^{-\beta^*} f_\varepsilon \to t^{-\beta^*} f$ weak-starly in $L^\infty([0,T_0] \times \mathbb{R}^3)$. Therefore, by lower semi-continuity of the norm, we retain the desired short time smoothing estimates, (8.84) and, in particular, $f \in L^1(0,T_0; L^\infty)$. We retain also that $f \in C^\infty((t,T_0) \times \mathbb{R}^3)$ for each $0 < t < T_0$, i.e. that f becomes instantaneously smooth.

Step 4: Passing to the limit

We now pass to the limit and show f constructed in the previous step satisfies the Landau equation (1.41) on $(0,T_0) \times \mathbb{R}^3$. Since f is smooth on this domain, the notion of solution is unambiguous and, in particular, it suffices to show f is a distributional solution to (1.41). We first show convergence of the non-local coefficients: By Lemma 8.5,

$$\left\| A[f - f_\varepsilon] \right\|_{L^1(0,T_0;L^\infty(\mathbb{R}^3))} \leq C \left\| f - f_\varepsilon \right\|^{2/3}_{L^1(0,T_0;L^1(\mathbb{R}^3))} \left\| f - f_\varepsilon \right\|^{1/3}_{L^1(0,T_0;L^\infty(\mathbb{R}^3))},$$

where the right hand side tends to 0 as $\varepsilon \to 0^+$, using the short-time smoothing estimates (8.84) and strong convergence in $L^1([0,T_0] \times \mathbb{R}^3)$. Using Lemma 8.5 again, we have the analogous estimate for $\nabla a[f - f_\varepsilon]$:

$$\left\| \nabla a[f - f_\varepsilon] \right\|_{L^1(0,T_0;L^\infty(\mathbb{R}^3))} \leq C \left\| f - f_\varepsilon \right\|^{1/3}_{L^1(0,T_0;L^1(\mathbb{R}^3))} \left\| f - f_\varepsilon \right\|^{2/3}_{L^1(0,T_0;L^\infty(\mathbb{R}^3))},$$

which again converges to zero as $\varepsilon \to 0^+$. Now, integrating by parts, f_ε satisfies the following distributional formulation of (1.41): For any $\varphi \in C_c^\infty((0,T_0)\times\mathbb{R}^3)$,

$$\int_0^{T_0}\int_{\mathbb{R}^3} \varphi \partial_t f_\varepsilon \,\mathrm{d}v\,\mathrm{d}t = -\int_0^{T_0}\int_{\mathbb{R}^3} \nabla\varphi\cdot(A[f_\varepsilon]\nabla f_\varepsilon - \nabla a[f_\varepsilon]f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}t.$$

Therefore, global convergence of the non-local coefficients and the local compactness in C^1 is sufficient to show that f satisfies (1.41) in the following distributional sense: for each $\varphi \in C_c^\infty((0,T_0)\times\mathbb{R}^3)$,

$$\int_0^{T_0}\int_{\mathbb{R}^3} \partial_t f \varphi \,\mathrm{d}v = -\int_0^{T_0}\int_{\mathbb{R}^3} \nabla\varphi\cdot(A[f]\nabla f - \nabla a[f]f)\,\mathrm{d}v\,\mathrm{d}t.$$

Because f is smooth, we may integrate by parts and conclude f is a classical solution to the Landau equation, i.e. f solves (1.41) pointwise.

Step 5: Behavior at initial time

It only remains to show that the initial data is obtained in the appropriate strong sense. To this end, we recall that each f_ε solves the following distributional form of (1.41): For any $\varphi \in C_c^\infty([0,T_0)\times\mathbb{R}^3)$,

$$\int_0^{T_0}\int_{\mathbb{R}^3} \partial_t f_\varepsilon \varphi \,\mathrm{d}v = \int_0^{T_0}\int_{\mathbb{R}^3} \nabla^2\varphi : A[f_\varepsilon]f_\varepsilon + 2\nabla\varphi\cdot\nabla a[f_\varepsilon]f_\varepsilon \,\mathrm{d}v\,\mathrm{d}t.$$

We estimate the first term on the right hand side using Hölder's inequality and Lemma 8.5 as

$$\begin{aligned}
\int_0^{T_0}\nabla^2\varphi : A[f_\varepsilon]f_\varepsilon \,\mathrm{d}v\,\mathrm{d}t &\le \|\varphi\|_{L^1\left(0,T_0;W^{2,\frac{p}{p-1}}\right)} \|f_\varepsilon\|_{L^\infty(0,T_0;L^p)} \|A[f_\varepsilon]\|_{L^\infty([0,T_0]\times\mathbb{R}^3)} \\
&\le \|\varphi\|_{L^1\left(0,T_0;W^{2,\frac{p}{p-1}}\right)} \|f_\varepsilon\|^{1+\frac{1}{3}\frac{p}{p-1}}_{L^\infty(0,T_0;L^p)} \|f_\varepsilon\|^{\frac{2}{3}\frac{p-3/2}{p-1}}_{L^\infty(0,T_0;L^1)} \\
&\le C(K)\,\|\varphi\|_{L^1\left(0,T_0;W^{2,\frac{p}{p-1}}\right)}.
\end{aligned}$$

We then estimate the second term using Hölder's inequality, the Sobolev embedding $\dot{W}^{1,\frac{p}{p-1}}(\mathbb{R}^3) \hookrightarrow L^{\frac{2p-3}{3p}}$ for $p > 3/2$, Lemma 8.5, and the Hardy-Littlewood-Sobolev inequality as

$$\begin{aligned}
\int_0^{T_0}\nabla\varphi\cdot\nabla a[f_\varepsilon]f_\varepsilon \,\mathrm{d}v\,\mathrm{d}t &\le \|\nabla\varphi\|_{L^1\left(0,T_0;L^{\frac{3p}{2p-3}}\right)} \|\nabla a[f_\varepsilon]\|_{L^\infty(0,T_0;L^3)} \|f_\varepsilon\|_{L^\infty(0,T_0;L^p)} \\
&\le \left\|\nabla^2\varphi\right\|_{L^1\left(0,T_0;L^{\frac{p}{p-1}}\right)} \|f_\varepsilon\|_{L^\infty(0,T_0;L^{3/2})} \|f_\varepsilon\|_{L^\infty(0,T_0;L^p)} \\
&\le C(K)\left\|\nabla^2\varphi\right\|_{L^1\left(0,T_0;L^{\frac{p}{p-1}}\right)}.
\end{aligned}$$

Therefore, by a duality argument, we conclude that

$$\|\partial_t f_\varepsilon\|_{L^\infty(0,T_0;W^{-2,p})} \leq C(K).$$

Combined with $\|f_\varepsilon\|_{L^\infty(0,T_0;L^p)} \leq C(K)$, the Aubin-Lions Lemma yields compactness of f_ε in $C(0,T_0;W^{-1,p})$. Therefore, up to a further subsequences we may assume $f_\varepsilon \to f$ in $C(0,T_0;W^{-1,p})$. In particular, $f_\varepsilon(0) = f_{in}^\varepsilon$ implies f is continuous in time and $f(0) = f_{in}$ in the sense of distributions. Using $f \in L^\infty(0,T_0;L^p)$ once more, f is weakly continuous in time taking values in L^p and $f(0) = f_{in}$ almost everywhere. By lower-semicontinuity of the norm in the weak topology, we also have

$$\|f_{in}\|_{L^p} \leq \liminf_{t \searrow 0^+} \|f(t)\|_{L^p}.$$

Finally, performing a standard L^p energy estimate on f_ε and throwing away the coercive term, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} f_\varepsilon^p(t)\,\mathrm{d}v \leq C(p) \int_{\mathbb{R}^3} f_\varepsilon^{p+1}\,\mathrm{d}v \leq C(p)\,\|f_\varepsilon(t)\|_{L^\infty}\,\|f_\varepsilon\|_{L^p}^p \leq Ct^{-\beta^*},$$

where the constant C is uniform in ε. Therefore, integrating in time, for any $0 < t < T_0$,

$$\sup_{0<s<t} \|f_\varepsilon(s)\|_{L^p_v} \leq \|f_{in}^\varepsilon\|_{L^p_v} + Ct^{1-\beta^*}.$$

We take $\varepsilon \to 0^+$ and use that $f_{in}^\varepsilon \to f_{in}$ strongly in L^p and $f_\varepsilon \to f$ weak starly in $L^\infty(0,T;L^p)$ to obtain

$$\sup_{0<s<t} \|f(s)\|_{L^p} \leq \|f_{in}\|_{L^p} + Ct^{1-\beta^*}, \qquad \text{for each } 0 < t < T_0.$$

Since f is weakly continuous taking values in L^p, for every $0 < t < T_0$,

$$\|f(t)\|_{L^p} \leq \|f_{in}\|_{L^p} + Ct^{1-\beta^*}.$$

Finally, taking $t \to 0^+$ and using $\beta^* \in (0,1)$, we obtain

$$\limsup_{t \searrow 0^+} \|f(t)\|_{L^p} \leq \|f_{in}\|_{L^p}.$$

We have shown $\lim_{t\searrow 0^+} \|f(t)\|_{L^p} = \|f_{in}\|_{L^p}$. Combined with $f(t) \rightharpoonup f_{in}$ in L^p, we deduce that $f(t) \to f_{in}$ strongly in L^p, which concludes the proof of Theorem 8.2.

8.4.3.2 Continuation criterion

In this part, we prove the continuation criterion contained in Theorem 8.2. To this end, assume $f : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^+$ is a solution constructed in the proof of Theorem 8.2. We need only show that if $\limsup_{t \nearrow T} \|f(t)\|_{L^p} < \infty$, then the solution f may be continued smoothly past time T. Indeed, for each $0 < \varepsilon < T$, set $g^\varepsilon_{in} = f(T-\varepsilon)$. Then, the propagation of moments from Lemma 8.1 and the decay of entropy imply the a uniform-in-ε bound:

$$\int_{\mathbb{R}^3} g^\varepsilon_{in} \left|\log(g^\varepsilon_{in})\right| + |g^\varepsilon_{in}|^p + \langle v \rangle^m g^\varepsilon_{in} \, \mathrm{d}v \leq K.$$

Therefore, Theorem 8.2 and the invariance of (1.41) under translations in time implies there is a uniform time $2T_0$, such that g^ε_{in} gives rise to a smooth solution g_ε to (1.41) on $[T-\varepsilon, T+2T_0-\varepsilon]$ with $g^\varepsilon(T-\varepsilon) = g^\varepsilon_{in}$. Pick $\varepsilon = T_0$ and set $\tilde{f}$ to be the concatenation of g_ε and f:

$$\tilde{f}(t) = \begin{cases} f(t) & 0 \leq t \leq T - T_0 \\ g_\varepsilon(t) & T - T_0 \leq t \leq T + T_0. \end{cases}$$

Note that by construction $f(T-T_0) = g_\varepsilon(T-T_0)$ and both f and g are solutions to (1.41) on $[T-T_0, T)$ that lie in $L^1(T-T_0, T; L^\infty)$ by the short time smoothing estimates of Theorem 8.2. Therefore, by the uniqueness result of Fournier in [75], $f(t) = g(t)$ on $[T-T_0, T)$ and $\tilde{f}$ is a smooth solution to (1.41) on $[0, T+T_0)$, as desired.

8.4.4 Uniqueness

In this section, we prove Theorem 8.3 by extending the a-priori estimates of the preceding sections to a more general class of solutions to (1.41). In Section 8.4.1 and Section 8.4.2, we showed that solutions to (1.41) which are smooth, rapidly decay, and belong to $L^\infty(0,T;L^p)$ for some $p > 3/2$ are quantitatively bounded in $L^1(0,T;L^\infty)$. In this section, we extend this $L^1(0,T;L^\infty)$ bound to locally Lipschitz solutions $g \in L^r(0,T;L^p)$ for some $p > 3/2$ and $r > \frac{2p}{2p-3}$. The proof is divided into two steps:

In step one, mimicking Section 8.4.1, we use an ODE argument to quantitatively bound the potential growth of of the L^p norm. More precisely, we show the estimate:

$$\sup_{t<\tau<T} \|g(t)\|^p_{L^p} + \int_t^T \int_{\mathbb{R}^3} \langle v \rangle^{-3} \left|\nabla g^{p/2}\right|^2 \, \mathrm{d}v \, \mathrm{d}\tau \leq \frac{C}{t^{p/r}}, \qquad \text{for each } 0 < t < T. \tag{8.85}$$

This gives a quantitative estimate on the $L^\infty(t,T;L^p)$ norm of g. We give first a formal proof of (8.85), which is rigorous for rapidly decaying solutions. In step two, we use the methods from Section 8.4.2 to obtain an estimate on the potential growth of the L^∞ norm of g. We show $\|g(t)\|_{L^\infty} \leq Ct^{-1+\delta}$ for some small $\delta > 0$, which implies $g \in L^1(0,T;L^\infty)$. We conclude with a rigorous proof of (8.85) for merely locally Lipschitz solutions to (1.41),

which depends crucially on the so called ε-Poincaré inequalities used to propagate L^p norms for solutions to (1.41) (see [7, 8, 94]). The particular version used was recently introduced as [8, Proposition 1.7].

Step 1: Formal proof of (8.85)

Let $g \in L^r(0,T;L^p) \cap W^{1,\infty}(\Omega)$ for each Ω compactly contained in $(0,T)\times\mathbb{R}^3$) be a solution to (1.41) with initial datum g_{in}, attained in the sense of distributions. We assume that g_{in} satisfies the bounds

$$\int_{\mathbb{R}^3} \langle v\rangle^m \, g_{in} \, \mathrm{d}v \le M \qquad \text{and} \qquad \int_{\mathbb{R}^3} g_{in} \, |\log(g_{in})| \, \mathrm{d}v \le M,$$

for some $p > 3/2$, $r > \frac{2p}{2p-3}$, and $m > \frac{9}{2}\frac{p-1}{p-3/2}$. Following the a-priori estimates in Section 8.4.1, we use g^{p-1} as a test function[3] and derive a differential inequality for $y(t) := \|g(t)\|_{L^p}^p$ of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \int_{\mathbb{R}^3} \langle v\rangle^{-3} \left|\nabla g^{p/2}\right|^2 \mathrm{d}v \le Cy + Cy^{\alpha_1}, \qquad \text{for each } 0 < t < T, \tag{8.86}$$

where the constant depends on g only through M, T, m, and p. The exponent α_1 is the same as in (8.67), which we recall is

$$\alpha_1 := \frac{(2p^2-p)m - 9p^2}{(2p^2-3p)m - 9p^2 + 9p} \qquad \text{and} \qquad p\alpha_1 - p = \frac{2mp - 9p}{m(2p-3) - 9p + 9}.$$

To avoid the significant regularity issues at time $t = 0$, we integrate over $[s,t]$, for $0 < s < t < T$, which yields

$$\begin{aligned} y(t) + \int_s^t \int_{\mathbb{R}^3} \langle v\rangle^{-3} \left|\nabla g^{p/2}\right|^2 \mathrm{d}v \, \mathrm{d}\tau \\ \le y(s) + C\int_s^t y(\tau)\,\mathrm{d}\tau + C\int_s^t \left(\int_{\mathbb{R}^3} g^p(\tau,v)\,\mathrm{d}v\right)^{\alpha_1 - 1} y(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{8.87}$$

By Grönwall's inequality, we obtain the quantitative bound

$$\sup_{s<t<T} y(t) \le y(s) \exp\left(C(T-s) + C\int_s^T \|g(\tau)\|_{L^p}^{p\alpha_1 - p} \right).$$

[3]The local regularity of g is not sufficient to justify using g^{p-1} as a test function, which is the reason why this argument is only formal. One should instead use $\varphi_R g^{p-1}$ as a test function, where φ_R is a smooth cutoff to B_R. Integrating in time, one obtains an integral equation for the truncated L^p norm. Using $g \in L^r(0,T;L^p)$, one obtains a uniform-in-R estimate on the truncated L^p norm analagous to (8.87), from which (8.88) and (8.85) follow by the monotone convergence theorem.

A straightforward computation implies

$$p\alpha_1 - p = \frac{2mp - 9p}{m(2p-3) - 9p + 9} \qquad \text{and} \qquad \lim_{m\to\infty} p\alpha_1 - p = \frac{2p}{2p-3}.$$

Since is $p\alpha_1 - p$ is decreasing as a function of m provided $m > \frac{9}{2}\frac{p-1}{p-3/2}$, we may take m even larger, depending only on p and r, so that $\frac{2p}{2p-3} < p\alpha_1 - p < r$. For this choice of m,

$$y(t) \leq Cy(s) \exp\left(\int_0^T \|g(\tau)\|_{L^p}^r \, \mathrm{d}\tau\right) \leq Cy(s) \qquad \text{for each } 0 < s < t < T.$$

Raising to the power r/p and averaging over $s \in [0,t]$, we obtain a quantitative bound on the appearance of the L^p norm of g:

$$\|g(t)\|_{L^p}^r \leq \frac{C}{t}\int_0^t \|g(s)\|_{L^p}^r \, \mathrm{d}s \leq \frac{C}{t}. \tag{8.88}$$

Consequently, using (8.88) to control the right hand side of (8.87), we deduce (8.85):

$$\sup_{t<\tau<T} \|g(t)\|_{L^p}^p + \int_t^T \int_{\mathbb{R}^3} \langle v\rangle^{-3} \left|\nabla g^{p/2}\right|^2 \, \mathrm{d}v \, \mathrm{d}\tau \leq \frac{C}{t^{p/r}}, \qquad \text{for each } 0 < t < T. \tag{8.89}$$

Step 2: $L^\infty(t,T;L^p) \to L^1(0,T;L^\infty)$ regularisation estimates

Our goal now is to use Proposition 8.4 to obtain a quantitative bound on the L^∞ norm of g. To this end, we use interpolation as in (8.30) to bound higher $L^q([0,T]\times\mathbb{R}^3)$ norms of g:

$$\|g\|_{L^q}^q \leq C\left(\left\|\langle\cdot\rangle^{-\frac{3}{2}} \nabla g^{\frac{p}{2}}\right\|_{L^2}^2 + \|g\|_{L^p}^p\right) \|g\|_{L^p}^{p\left(\frac{q-p-\frac{2}{3}}{p-1}\right)} \|\langle\cdot\rangle^\alpha g\|_{L^1}^{\frac{5p-3q}{3(p-1)}},$$

where $\alpha = \frac{9(p-1)}{5p-3q}$. Integrating in time and using (8.85), we find for each $q \in \left(p, \frac{5p}{3}\right)$, for $m > \alpha(p,q)$,

$$A_q(t) := \int_t^T \int_{\mathbb{R}^3} g^q \, \mathrm{d}v \, \mathrm{d}\tau \leq Ct^{-\frac{p}{r}\frac{q-5/3}{p-1}}$$

Therefore, using the $L^q(0,T;L^q) \to L^\infty(t,T;L^\infty)$ smoothing estimate[4] in Proposition 8.4, we obtain

$$\begin{aligned} \|f(t)\|_{L^\infty} &\leq CA_q(t/2)^{\frac{\beta}{\gamma}} + \left(A_q(t/2)^\beta t^{-1-\beta}\right)^{\frac{1}{1+\beta+\gamma}} \\ &\leq Ct^{-\frac{p\beta}{r\gamma}\frac{q-5/3}{p-1}} + Ct^{-\frac{p}{r}\frac{q-5/3}{p-1}\frac{\beta}{1+\beta+\gamma} - \frac{1+\beta}{1+\beta+\gamma}} \\ &:= Ct^{\omega_1} + Ct^{\omega_2}. \end{aligned} \tag{8.90}$$

[4]Technically, Proposition 8.4 is stated for smooth, rapidly decaying solutions. Nevertheless, we claim that the estimate applies to g without change. The qualitative regularity and decay is solely used to ensure that $(f-k)_+^{p-1}$ for constant $k \in \mathbb{R}^+$ can be used as a test function in (1.41) in the proof of the energy estimate (8.73). Crucially, (8.73) can be easily justified for g because $g \in W^{1,\infty}_{loc}$ and g satisfies (8.85).

where $\beta = \beta(m)$ and $\gamma = \gamma(q, m)$ are exponents given explicitly by the formulas:

$$\beta = \frac{2}{3} - \frac{3}{m} \qquad \text{and} \qquad \gamma = \frac{2(q-1)-3}{3m}\left[m - \frac{9(q-2)}{2(q-1)-3}\right].$$

To conclude $g \in L^1(0,T;L^\infty)$, it remains only to show that there is a choice of q and m for which the exponents ω_1 and ω_2 in (8.90) are strictly greater than -1. Instead of computing directly, we take the limit as $q \to 5p/3$ and $m \to \infty$ in the exponents. In particular,

$$\lim_{m\to\infty} \beta = \frac{2}{3} \qquad \text{and} \qquad \lim_{m\to\infty,\, q\to 5p/3} \gamma = \frac{10p-15}{9}$$

and, consequently, since $r > \frac{2p}{2p-3}$,

$$\begin{aligned}
\lim_{m\to\infty,\, q\to 5p/3} \omega_1 &= -\frac{10p}{(10p-15)r} > -1 \\
\lim_{m\to\infty,\, q\to 5p/3} \omega_2 &= -\frac{\frac{10p}{9r} + \frac{5}{3}}{1 + \frac{2}{3} + \frac{10p-15}{9}} = -\frac{10p+15r}{10pr} > -1.
\end{aligned}$$

We conclude that for m sufficiently large (depending only on p and r) we can use the $L^q(0,T;L^q) \to L^\infty(t,T;L^\infty)$ smoothing estimate from Proposition 8.4 with q close enough to $5p/3$ (depending on r) so that $\omega_1 > -1$ and $\omega_2 > -1$. Thus, we conclude for $g_{in} \in L^1_m$ for m sufficiently large, $g \in L^1(0,T;L^\infty)$. By the uniqueness result of Fournier [75], we conclude any two locally Lipschitz solutions with the same initial data g_{in} that both also belong to $L^r(0,T;L^p)$ are equal.

We conclude the manuscript with a rigorous proof of (8.85).

Rigorous Proof of (8.85)

We take g a locally Lipschitz weak solution to (1.41) as in the statement of Theorem 8.3. Combining $g \in L^r(0,T;L^p)$ for $p > 3/2$ with Lemma 8.5 and interpolation of Lebesgue spaces, we conclude that the nonlocal coefficients satisfy the pointwise in time bounds:

$$\|A[f]\|_{L^\infty} \leq \|g\|_{L^1}^{\frac{2p-3}{3p-3}} \|g\|_{L^p}^{\frac{p}{3p-3}} \quad \text{and} \quad \|\nabla a[f]\|_{L^3} \leq \|f\|_{L^{3/2}} \leq \|f\|_{L^1}^{\frac{2p-3}{3p-3}} \|f\|_{L^p}^{\frac{p}{3(p-1)}}.$$

Consequently,

$$A[g] \in L^{\frac{3r(p-1)}{p}}(0,T;L^\infty) \quad \text{and} \quad \nabla a[g] \in L^{\frac{3r(p-1)}{p}}(0,T;L^3).$$

Since $g \in W^{1,\infty}(\Omega)$ for each Ω compactly contained in $(0,T)\times\mathbb{R}^3$, a rather straightforward density argument implies for any $\varphi \in L^\infty(s,t;W^{1,\infty}(\mathbb{R}^3))$ with compact support Ω,

$$\int_0^T \int_{\mathbb{R}^3} \varphi \partial_t g \, \mathrm{d}v \, \mathrm{d}\tau = -\int_0^T \int_{\mathbb{R}^3} \nabla\varphi \cdot (A[g]\nabla g - \nabla a[g] g) \, \mathrm{d}v \, \mathrm{d}\tau. \tag{8.91}$$

For $R > 1$, we take φ_R to be the cutoff function:

$$\varphi_R(v) = \begin{cases} 1 - R^{-(p+1)}(|v| - R)_+^{p+1} & \text{if } |v| < 3R/2 \\ R^{-(p+1)}(2R - |v|)_+^{p+1} & \text{otherwise.} \end{cases}$$

We note that $\varphi_R \in W^{1,\infty}(\mathbb{R}^3)$ satisfies the usual cutoff properties, namely, $0 \leq \varphi_R \leq 1$, $\varphi_R = 1$ on B_R, and $\varphi_R = 0$ on $\mathbb{R}^3 \setminus B_{2R}$. However, φ_R satisfies the additional useful pointwise inequality:

$$|\nabla \varphi_R|^2 \leq C R^{-2(p+1)} \varphi_R^{\frac{2p}{p+1}}. \tag{8.92}$$

Since g is locally Lipshitz and φ_R is compactly supported, $g^{p-1}\varphi_R^2 \chi_{[s,t]}$ an admissible test function in (8.91). Therefore, the fundamental theorem of calculus for Lipschitz functions guarantees

$$\int_{\mathbb{R}^3} \varphi_R^2 g^p(t)\,\mathrm{d}v - \int_{\mathbb{R}^3} \varphi_R^2 g^p(s)\,\mathrm{d}v = -p\int_0^T \int_{\mathbb{R}^3} \nabla\left(\varphi_R^2 g^{p-1}\right) \cdot \left(A[g]\nabla g - \nabla a[g] g\right)\,\mathrm{d}v\,\mathrm{d}\tau.$$

We rewrite the right hand side by rearranging powers of g and integrating by parts as

$$\begin{aligned} RHS = &-\frac{4(p-1)}{p}\int_s^t \int_{\mathbb{R}^3} \varphi_R^2 \nabla g^{p/2} \cdot A[g]\nabla g^{p/2}\,\mathrm{d}v\,\mathrm{d}\tau \\ &+ 2\int_s^t \int_{\mathbb{R}^3} g^{p/2}\varphi_R \nabla \varphi_R \cdot A[g]\nabla g^{p/2}\,\mathrm{d}v\,\mathrm{d}\tau \\ &+ \int_s^t \int_{\mathbb{R}^3} g^p \nabla \varphi_R^2 \cdot \nabla a[g]\,\mathrm{d}v\,\mathrm{d}\tau + (p-1)\int_s^t \int_{\mathbb{R}^3} \varphi_R^2 g^{p+1}\,\mathrm{d}v\,\mathrm{d}\tau := \sum_{i=1}^4 I_i. \end{aligned}$$

The term I_1 is the coercive term. The error terms I_2 and I_3 coming from the cutoff φ_R are bounded together using the identity $\nabla \cdot A[g] = \nabla a[g]$ as

$$\begin{aligned} I_3 &= \int_s^t \int_{\mathbb{R}^3} \nabla \varphi_R^2 \cdot \nabla \cdot (A[g]g^p) - A[g]\nabla g^p\,\mathrm{d}v\,\mathrm{d}\tau = -\int_s^t \int_{\mathbb{R}^3} \nabla^2 \varphi_R^2 : A[g]g^p\,\mathrm{d}v\,\mathrm{d}\tau - I_2 \\ &\leq -I_2 + \frac{1}{R^2}\int_s^t \|A[g(\tau)]\|_{L^\infty} \left\|\varphi_R g(\tau)^{p/2}\right\|_{L^2}^2\,\mathrm{d}\tau. \end{aligned}$$

Lastly, the main term I_4 presents a difficulty not seen in the global bounds anywhere else in the manuscript. The cutoff functions must be distributed correctly in order to close the argument. To this end, we bound using the ε-Poincare lemma found as Proposition 1.7 in [8]. For the reader's convenience, we reproduce the proof here: To bound I_4, we use Hölder's inequality, the fractional Sobolev embedding $\dot{H}^s(\mathbb{R}^3) \hookrightarrow L^{\frac{2q}{q-1}}(\mathbb{R}^3)$, the Gagliardo-Nirenberg-Sobolev inequality, and Young's inequality: for any $q > 3/2$ and any

$\varepsilon > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^3} \varphi_R^2 g^{p+1} &\le \left\| \langle v \rangle^3 g \right\|_{L^q(B_{2R})} \left\| \varphi_R g^{p/2} \langle v \rangle^{-3/2} \right\|^2_{L^{\frac{2q}{q-1}}} \\
&\le \left\| \langle v \rangle^3 g \right\|_{L^q(B_{2R})} \left\| \varphi_R g^{p/2} \langle v \rangle^{-3/2} \right\|^2_{\dot{H}^s} \\
&\le \left\| \langle v \rangle^3 g \right\|_{L^q(B_{2R})} \left\| \varphi_R g^{p/2} \langle v \rangle^{-3/2} \right\|^{2s}_{\dot{H}^1} \left\| g^{p/2} \langle v \rangle^{-3/2} \right\|^{2-2s}_{L^2} \\
&\le \varepsilon \left\| \varphi_R g^{p/2} \langle v \rangle^{-3/2} \right\|^2_{\dot{H}^1} + C(\varepsilon) \left\| \langle v \rangle^3 g \right\|^{\frac{1}{1-s}}_{L^q(B_{2R})} \left\| \varphi_R g^{p/2} \langle v \rangle^{-3/2} \right\|^2_{L^2},
\end{aligned}$$

where the relation $\frac{q-1}{2q} = \frac{1}{2} + \frac{s}{3}$ implies

$$\begin{aligned}
\int_{\mathbb{R}^3} \varphi_R^2 g^{p+1} \, \mathrm{d}v \le \varepsilon \int_{\mathbb{R}^3} \left| \nabla (\varphi_R g^{p/2} \langle v \rangle^{-3/2}) \right|^2 \, \mathrm{d}v \\
+ C(\varepsilon) \left\| \langle v \rangle^3 g \right\|^{\frac{2q}{2q-3}}_{L^q(B_{2R})} \int_{\mathbb{R}^3} \varphi_R^2 g^p \langle v \rangle^{-3} \, \mathrm{d}v.
\end{aligned}$$

By explicit computation, the pointwise estimate (8.92), and Young's inequality,

$$\begin{aligned}
\left| \nabla \left(\varphi_R g^{p/2} \langle v \rangle^{-3/2} \right) \right|^2 &\le C \langle v \rangle^{-3} \varphi_R^2 \left| \nabla g^{p/2} \right|^2 + C \langle v \rangle^{-3} g^p \left| \nabla \varphi_R \right|^2 + C \langle v \rangle^{-5} \varphi_R^2 g^p \\
&\le C \langle v \rangle^{-3} \varphi_R^2 \left| \nabla g^{p/2} \right|^2 + \varphi_R^2 g^{p+1} + C \varphi_R^2 g^p + C R^{-2(p+1)^2} \langle v \rangle^{-3(p+1)}.
\end{aligned}$$

Therefore, picking ε sufficiently small, and using the lower bound for $A[g]$ guaranteed by Lemma 8.4, for any $s < t < T$ and for any $q > 3/2$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi_R^2 g^p(t) \, \mathrm{d}v + c_0 \int_s^t \int_{\mathbb{R}^3} \langle v \rangle^{-3} \varphi_R^2 \left| \nabla g^{\frac{p}{2}} \right|^2 \, \mathrm{d}v \, \mathrm{d}\tau \\
&\quad \le \int_{\mathbb{R}^3} \varphi_R^2 g^p(s) \, \mathrm{d}v \\
&\qquad + C \int_s^t \left(R^{-2(p+1)^2} + R^{-2} \left\| A[g] \right\|_{L^\infty} + \left\| g \langle v \rangle^3 \right\|^{\frac{2q}{2q-3}}_{L^q(B_{2R})} \right) \left(\int_{\mathbb{R}^3} \varphi_R^2 g^p \, \mathrm{d}v \right) \mathrm{d}\tau.
\end{aligned} \tag{8.93}$$

By Grönwall's lemma applied to the function $\tau \mapsto \int_{\mathbb{R}^3} \varphi_R^2 g^p(\tau) \, \mathrm{d}v$, we conclude the explicit bound: for any $0 < s < t < T$ and any $q > 3/2$,

$$\begin{aligned}
\int_{\mathbb{R}^3} \varphi_R^2 g(t)^p \, \mathrm{d}v \le \left(\int_{\mathbb{R}^3} \varphi_R^2 g^p(s) \, \mathrm{d}v \right) \\
\times \exp \left(C \int_0^T R^{-2(p+1)^2} + R^{-2} \left\| A[g] \right\|_{L^\infty} + \left\| g \langle v \rangle^3 \right\|^{\frac{2q}{2q-3}}_{L^q(B_{2R})} \, \mathrm{d}\tau \right).
\end{aligned} \tag{8.94}$$

Since we have already shown $A[g] \in L^1(0, T; L^\infty)$, the monotone convergence theorem implies

$$\int_{\mathbb{R}^3} g(\tau)^p \, \mathrm{d}v \le \left(\int_{\mathbb{R}^3} g^p(s) \, \mathrm{d}v \right) \exp \left(C \int_0^T \left\| g \langle v \rangle^3 \right\|^{\frac{2q}{2q-3}}_{L^q} \, \mathrm{d}\tau \right). \tag{8.95}$$

To conclude as in the formal proof given in Step 1, we need only show that $\langle v\rangle^3 g \in L^{\frac{2q}{2q-3}}(0,T;L^q)$ for some $q > 3/2$. Because (r,p) is a sub-critical exponent pair–meaning $\frac{2}{r}+\frac{3}{p} < 2$–and $(\infty,1)$ is a super-critical pair–meaning $\frac{2}{\infty}+\frac{3}{1} > 2$–we can interpolate between $L^r(0,T;L^p)$ and $L^\infty(0,T;L^1_m)$ to find $(\frac{2q}{2q-3},q)$, a critical exponent pair. Interpolating between L^p and L^1_m, for $3/2 < q < p$,

$$\left\|\langle v\rangle^3 g\right\|_{L^q} \leq \left\|\langle v\rangle^{\frac{3}{\theta}} g\right\|_{L^1}^{\theta} \|g\|_{L^p}^{1-\theta}, \quad \text{where} \quad \frac{\theta}{1}+\frac{1-\theta}{p}=\frac{1}{q}.$$

Solving for θ, we find $\theta = \frac{p-q}{q(p-1)}$ and $1-\theta = \frac{p(q-1)}{q(p-1)}$, so that

$$\begin{aligned}
&\int_0^T \left(\int_{\mathbb{R}^3} \langle v\rangle^{3q} g^q \,\mathrm{d}v\right)^{\frac{2}{2q-3}} \,\mathrm{d}\tau \\
&\quad\leq \int_0^T \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{3q(p-1)}{p-q}} g \,\mathrm{d}v\right)^{\frac{2(p-q)}{(p-1)(2q-3)}} \left(\int_{\mathbb{R}^3} g^p \,\mathrm{d}v\right)^{\frac{2q-2}{(p-1)(2q-3)}} \,\mathrm{d}\tau \\
&\quad\leq C(T,p,q,r)\, \|g\|_{L^\infty(0,T;L^1_m)}^{\frac{2(p-q)}{(p-1)(2q-3)}} \|g\|_{L^r(0,T;L^p)}^{r},
\end{aligned}$$

provided q is chosen so that $r > \frac{2(q-1)p}{(p-1)(2q-3)}$. If $r=\infty$ this constraint is satisfied for any $3/2 < q < p$. So, taking q arbitrarily close to $3/2$, we recover the constraint on the number of moments, $m > \frac{9}{2}\frac{p-1}{p-3/2}$ as claimed. On the other hand, if $r < \infty$, taking the limit $q \to p^-$ the constraint becomes $r > \frac{2p}{2p-3}$. So, for $r < \infty$, we can always pick q sufficiently close to p and $m > \frac{3q(p-1)}{p-q}$ so that the constraint is satisfied. Returning to (8.95), we obtain: for each $0 < s < t < T$,

$$\int_{\mathbb{R}^3} g(t)^p \,\mathrm{d}v \leq C(p,r,T,m,M,\|f\|_{L^r(0,T;L^p)}) \left(\int_{\mathbb{R}^3} g^p(s) \,\mathrm{d}v\right).$$

As in Step 1, taking an average over $s \in (0,t)$, we obtain $\|g(t)\|_{L^p}^p \leq Ct^{-p/r}$. Returning to (8.93) and taking the limit as $R \to \infty$, we conclude (8.85).

8.5 Appendix: weighted Sobolev inequalities

In this section, we provide a brief proof of the Sobolev inequalities claimed in Lemma 8.7. We start with the proof of (8.11). Weighted Sobolev and Poincare inequalities of the form

$$\left(\int_Q |\phi|^q w_1 \, dv\right)^{1/q} \leq C \left(\int_Q |\nabla\phi|^p \, w_2 dv\right)^{1/p},$$

for $q \geq p \geq 1$, and ϕ either compactly supported in Q or with zero average over Q, are guaranteed to hold if certain averages involving w_1 and w_2 are bounded over all cubes contained in a cube slightly larger than Q. For $p=2$ and $2 \leq q < \infty$, $r > 1$, a cube Q,

weights w_1 and w_2, we define

$$\sigma_{q,2,r}(Q, w_1, w_2) := |Q|^{\frac{1}{d}-\frac{1}{2}+\frac{1}{q}} \left(\fint_Q w_1^r \, dv \right)^{\frac{1}{qr}} \left(\fint_Q w_2^{-r} \, dv \right)^{\frac{1}{2r}}.$$

The precise statement is summarized in the following theorem. For its proof, we refer to [175, Theorem 1].

Theorem 8.7. *Consider two nonnegative weights w_1, w_2, a cube Q, $2 \le q < \infty$ and $r > 1$. Then, for any Lipschitz function ϕ which has compact support in Q or is such that $\int_Q \phi \, dv = 0$, we have*

$$\left(\int_Q |\phi|^q w_1 \, dv \right)^{1/q} \le \mathcal{C}_{q,2,r}(Q, w_1, w_2) \left(\int_Q |\nabla \phi|^2 \, w_2 \, dv \right)^{1/2}, \tag{8.96}$$

where for some constant $C(d, r, q)$,

$$\mathcal{C}_{q,2,r}(Q, w_1, w_2) := C(d, r, q) \sup_{Q' \subset 8Q} \sigma_{q,2,r}(Q', w_1, w_2).$$

We are now prepared to show (8.11) in Lemma 8.7 by combining Theorem 8.7 with a covering argument.

Proof of (8.11) **in Lemma 8.7**

Let $Q = Q(v_0)$ be a cube of side length 1, centred at v_0. For any $\alpha, \ m > 0$, we compute

$$\frac{1}{|Q|} \int_Q (1 + |v|)^m \ \mathrm{d}v \le \begin{cases} 2^m (1 + |v_0|)^m, & v_0 \in B_2(0)^c, \\ c, & v_0 \in B_2(0), \end{cases}$$

and

$$\frac{1}{|Q|} \int_Q \frac{1}{(1 + |v|)^\alpha} \ \mathrm{d}v \le \begin{cases} \frac{2}{(1+|v_0|)^\alpha}, & v_0 \in B_2(0)^c, \\ c, & v_0 \in B_2(0). \end{cases}$$

Therefore,

$$\sigma_{6,2,r}(Q, \langle v \rangle^{-9}, \langle v \rangle^{-3}) = \left(\fint_Q \langle v \rangle^{-9r} \ \mathrm{d}v \right)^{\frac{1}{6r}} \left(\fint_Q \langle v \rangle^{3r} \ \mathrm{d}v \right)^{\frac{1}{2r}} \le C,$$

where C is independent on the center of Q. Applying Theorem 8.7, we find (8.96) holds

for $w_1 = \langle v\rangle^{-9}$ and $w_2 = \langle v\rangle^{-3}$:

$$\left(\int_Q |\phi - (\phi)_Q|^6 \langle v\rangle^{-9}\ \mathrm{d}v\right)^{1/3} \leq C \int_Q |\nabla\phi|^2\ \langle v\rangle^{-3}\ \mathrm{d}v. \tag{8.97}$$

Here and below $(\phi)_Q$ denotes the average of ϕ in Q.

Next, we consider a family $\mathcal{Q}$ of non-overlapping cubes of side length 1 which cover $\mathbb{R}^3$. Decomposing the integral over $\mathbb{R}^3$ using this partition and Young's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\phi|^6\ \langle v\rangle^{-9}\ dv = \sum_{Q\in\mathcal{Q}} \int_Q |\phi|^6\ \langle v\rangle^{-9}\ dv &\leq c_1 \sum_{Q\in\mathcal{Q}} \int_Q |\phi - (\phi)_Q|^6\ \langle v\rangle^{-9}\ \mathrm{d}v \\ &\quad + c_2 \sum_{Q\in\mathcal{Q}} |(\phi)_Q|^6 \int_Q \langle v\rangle^{-9}\ \mathrm{d}v.\end{aligned}$$

For each $Q \in \mathcal{Q}$, we apply (8.97) and find

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} \int_Q |\phi - (\phi)_Q|^6\ \langle v\rangle^{-9}\ \mathrm{d}v &\leq \sum_{Q\in\mathcal{Q}} \int_Q |\phi - (\phi)_Q|^6\ \langle v\rangle^{-9}\ \mathrm{d}v \\ &\leq C \sum_{Q\in\mathcal{Q}} \left(\int_Q |\nabla\phi|^2\ \langle v\rangle^{-3}\ \mathrm{d}v\right)^3 \leq C \left(\int_{\mathbb{R}^3} |\nabla\phi|^2\ \langle v\rangle^{-3}\ \mathrm{d}v\right)^3\end{aligned}$$

Additionally, since $\langle v\rangle^{-9} \leq 1$ and $|Q| = 1$, for any $1 \leq s \leq 6$, Jensen's inequality implies

$$\sum_{Q\in\mathcal{Q}} |(\phi)_Q|^6 \int_Q \langle v\rangle^{-9}\ \mathrm{d}v \leq \sum_{Q\in\mathcal{Q}} |(\phi)_Q|^6 \leq \sum_{Q\in\mathcal{Q}} \left(\int_Q |\phi|^s\ \mathrm{d}v\right)^{6/s} \leq \left(\int_{\mathbb{R}^3} |\phi|^s\ \mathrm{d}v\right)^{6/s}.$$

Summarizing, we obtain for any $1 \leq s \leq 6$,

$$\int_{\mathbb{R}^3} |\phi|^6\ \langle v\rangle^{-9}\ \mathrm{d}v \leq C_1 \left(\int_{\mathbb{R}^3} |\nabla\phi|^2\ \langle v\rangle^{-3}\ \mathrm{d}v\right)^3 + C_2 \left(\int_{\mathbb{R}^3} |\phi|^s\ \mathrm{d}v\right)^{6/s},$$

as claimed.

Proof of the weighted Sobolev inequality (8.12)

Proof. The case $k = 3$ follows from the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and Hardy's inequality.

For $k > 3$, we first compute derivatives of our weight $\langle v\rangle^m$ for $m \in \mathbb{R}$ arbitrary.

$$\nabla \langle v\rangle^m = m \langle v\rangle^{m-2} v \quad \text{and} \quad \Delta \langle v\rangle^m = 3m \langle v\rangle^{m-2} + m(m-2) \langle v\rangle^{m-4} |v|^2$$

Now, we apply the Sobolev embedding to the function $\varphi = \langle v\rangle^m f$ and find for $C_1 > 0$ the

Sobolev constant,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3} \langle v\rangle^{6m} f^6\right)^{1/3} \\
&\qquad \le C_1 \int_{\mathbb{R}^3} |\nabla \langle v\rangle^m f|^2 \\
&\qquad = C_1 \int_{\mathbb{R}^3} (f\nabla \langle v\rangle^m + \langle v\rangle^m \nabla f)^2 \\
&\qquad = C_1 \int_{\mathbb{R}^3} \langle v\rangle^{2m} |\nabla f|^2 \,\mathrm{d}v + \int_{\mathbb{R}^3} m^2 f^2 \langle v\rangle^{2(m-2)} |v|^2 + 2mf \langle v\rangle^{2m-2} \nabla f \cdot v \\
&\qquad = C_1 \int_{\mathbb{R}^3} \langle v\rangle^{2m} |\nabla f|^2 \,\mathrm{d}v + C_1 \int_{\mathbb{R}^3} m^2 f^2 \langle v\rangle^{2(m-2)} |v|^2 + \frac{1}{2}\nabla f^2 \cdot \nabla \langle v\rangle^{2m} \\
&\qquad = C_1 \int_{\mathbb{R}^3} \langle v\rangle^{2m} |\nabla f|^2 \,\mathrm{d}v + C_1 \int_{\mathbb{R}^3} m^2 f^2 \langle v\rangle^{2(m-2)} |v|^2 - \frac{1}{2} f^2 \Delta \langle v\rangle^{2m} \\
&\qquad \le C_1 \int_{\mathbb{R}^3} \langle v\rangle^{2m} |\nabla f|^2 + C_2 \int_{\mathbb{R}^3} f^2 \langle v\rangle^{2m-2}.
\end{aligned}$$

For $m > 0$, we use $|v|^2 \le \langle v\rangle^2$ to compute C_2 as

$$C_2 = C_1(m^2 - \frac{1}{2}(6m + 2m(2m-2))) = C_1(m^2 - 3m + 2m^2 + 2m) = -C_1(m^2 + m) < 0.$$

Finally, setting $2m = k - 3$, for $k > 3$, we conclude

$$\left(\int_{\mathbb{R}^3} \langle v\rangle^{3k-9} f^6\right)^{1/3} \le C_1 \int_{\mathbb{R}^3} \langle v\rangle^{k-3} |\nabla f|^2 - \frac{C_1}{4}(k-3)(k-1) \int_{\mathbb{R}^3} f^2 \langle v\rangle^{k-5}.$$

□

Proof of the interpolation estimate (8.13)

Proof. We use interpolation of weighted Lebesgue spaces (i.e. Hölder's inequality) to bound $\|f\|_{L^q_k}$ as

$$\left\|\langle v\rangle^{\frac{k}{q}} f\right\|_{L^q} \le \left\|\langle v\rangle^{\frac{m}{p}} f\right\|_{L^p}^{\theta} \left\|\langle v\rangle^{\frac{k-3}{p}} f\right\|_{L^{3p}}^{1-\theta},$$

provided $\theta \in (0,1)$ and $m \in \mathbb{R}$ satisfy the relations

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{3p} \qquad \text{and} \qquad \frac{k}{q} = \frac{m\theta}{p} + \frac{(k-3)(1-\theta)}{p}.$$

Solving this system of constraints for θ and m, we obtain

$$\theta = \frac{3p-q}{2q}, \qquad 1-\theta = \frac{3q-3p}{2q}, \qquad \text{and} \qquad m = \frac{2kp - (k-3)(3q-3p)}{3p-q}.$$

Therefore, since $k \geq 3$, we use the weighted Sobolev inequality (8.12) to obtain

$$\begin{aligned}\|f\|_{L^q_k}^q &\leq \|f\|_{L^p_m}^{\frac{3p-q}{2}} \left\|\langle v\rangle^{\frac{k-3}{p}} f\right\|_{L^{3p}}^{\frac{3}{2}(q-p)} \\ &\leq \|f\|_{L^p_m}^{\frac{3p-q}{2}} \left\|f^{p/2}\right\|_{L^6_{3k-9}}^{\frac{3}{p}(q-p)} \leq C_{k,p}\,\|f\|_{L^p_m}^{\frac{3p-q}{2}} \left\|\nabla f^{p/2}\right\|_{L^2_{k-3}}^{\frac{3}{p}(q-p)},\end{aligned}$$

which completes the proof. □

a-priori

CHAPTER 9

Critical Landau

This chapter discusses the well-posedness of the homogeneous Landau equation on $\mathbb{R}^3$ with Coulomb potential for rough initial data in the critical Lebesgue space $L^{\frac{3}{2}}$. It also shows that the solutions constructed in Chapter 8 exist for all times. To this end we exploit the breakthrough result of Guillen-Silvestre [100], which appeared during the preparation of the manuscript of [84]. Our principal aim was to construct local solutions in the critical Lebesgue space. Due to the global well-posedness result for rapidly decaying smooth data in [100], we managed to show that our solutions are in fact global solutions.

9.1 Introduction

We establish the global-in-time existence of smooth solutions to (1.41) for slowly decaying initial data in $L^{3/2}$. We show that such solutions satisfy the quantitative short time regularisation estimates:

$$\sup_{0<t<1} \|f(t)\|_{L^\infty} \leq \frac{C}{t} \qquad \text{and} \qquad \sup_{0<t<1} i(f(t)) \leq \frac{C}{t}.$$

Since the Fisher information becomes instantaneously finite, we adapt the main result of [100] to our setting and continue the local-in-time solutions to global-in-time solutions. We also prove existence of smooth solutions with initial data in L^p for any $p > 3/2$. In this case, we additionally show uniqueness.

Previous existence results for smooth solutions considered L^∞ data [15, 107, 132] , L^2 data [44], $\dot{H}^1$ data [62], and recently L^p data with $p > 3/2$, see Chapter 8 based on [81, 83].[1] Each of these works considers either global-in-time solutions for initial data close to equilibrium or local-in-time solutions for general initial data. Furthermore, each of these

[1] Technically, each of these results with the exception of Arsenev-Peskov [15] requires a weighted a version of the space or additional L^1 moments.

results requires the initial data to belong in a space which embeds into L^p for some $p > 3/2$. Recall that the value $p = 3/2$ is a threshold for the boundedness of the coefficients ($A[f]$ can be unbounded for $f \in L^{3/2}$) and also a threshold for time integrability of $\|f(t)\|_{L^\infty}$. Consequently, $p = 3/2$ is the threshold at which most known analytical techniques break down and no longer apply.

9.1.1 Results

We denote by L^p_m the space of measurable functions such that

$$\|f\|_{L^p_m} := \left(\int_{\mathbb{R}^3} f^p \langle v \rangle^m \, dv \right)^{1/p} < +\infty.$$

Our main results are summarised in the following theorems:

Theorem 9.1. *(Local-in-Time well-posedness) Let $\frac{3}{2} \leq p < +\infty$. If the initial data f_{in} is such that $f_{in}\langle v \rangle^{\frac{9}{2p}} \in L^p(\mathbb{R}^3)$, then there exists a smooth solution $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ to (1.41) such that*

$$\|f\|_{C([0,T];L^p_{9/2})} + \sup_{0<t\leq T} t^{\frac{3}{2p}} \|f(t)\|_{L^\infty} \lesssim C(f_{in}). \tag{9.1}$$

Moreover, if $p > 3/2$, this solution is unique among all smooth solutions[2] $g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ to (1.41) which obtain the initial data in the following sense:

(1) $\lim_{t\to 0^+} g(t) = f_{in}$ *in the sense of distributions*

(2) $g \in L^r(0,T;L^q_{9/2})$ *for some pair* (r,q), $\quad 3/2 < q \leq \infty, \quad \dfrac{2q}{2q-3} < r \leq \infty.$

The $L^{3/2} \to L^\infty$ regularisation estimate in (9.1) agrees with the regularisation rate observed for the heat equation. However, we conjecture that the $L^p \to L^\infty$ regularisation rate of the heat equation is sub-obtimal for the Landau diffusion operator, when $p < 3/2$, as predicted in [31, 94]. The enhanced regularisation rate is expected because the diffusion operator $\nabla \cdot (A[f]\nabla f)$ behaves more nonlinearly if $p < 3/2$. Let us comment further on the meaning of the threshold $p = 3/2$. First, we recall that for any function f, the diffusion matrix is uniformly bounded below as

$$A[f](v,t) \geq c_0 \langle v \rangle^{-3} I,$$

[2]More precisely, following the previous Chapter 8, we assume g is a locally bounded H-solution: we assume g is a distributional solution; we assume the entropy dissipation of g belongs to $L^1(0,T)$; and we assume g satisfies the additional regularity assumption $g \in L^\infty(s,T;L^\infty(\Omega))$ for each $s > 0$ and $\Omega \subset \mathbb{R}^3$ compact. It is well-known that such solutions are necessarily qualitatively smooth. This technical regularity assumption is solely to ensure that truncations of g are admissible as test functions for (1.41). We expect this assumption is unnecessary and can be removed.

where c_0 only depends on the mass, energy, and entropy of f. One is then led to the toy problem:

$$u_t = \Delta u + u^2. \tag{9.2}$$

and the associated 1-parameter scaling $u_\lambda = \lambda^2 u(\lambda^2 t, \lambda x)$. This scaling preserves (9.2) and leaves the $L^\infty(0,T,L^{3/2})$ norm invariant. In this sense, the $L^{3/2}$ norm is critical with respect to the scaling of this toy problem. This analogy is, however, simultaneously both helpful and misleading. This comparison is helpful in identifying the $L^{3/2}$ as an important norm for Landau, but not critical in the usual scaling sense. This is the motivation for the ε-regularity criterion, presented in Section 9.3, which shows that if a solution to the Landau equation (1.41) is small in a (critical) weighted $L^{3/2}$ norm, then the solution is smooth. Because the smallness required by the ε-regularity criterion cannot be directly removed, we are not able to show regularisation estimates of the form $L^\infty(0,T,L^{3/2}) \to L^\infty(t,T,L^\infty)$, nor are we able to propagate $L^{3/2}$ directly. Instead, we can replace the smallness by estimates of the form $C(0,T,L^{3/2}) \to L^\infty(t,T,L^\infty)$ and quantitatively propagate continuity in $L^{3/2}$ by solving a system of coupled differential inequalities. While the comparison to (9.2) provides inspiration for the smoothing estimates, the standard techniques for the semilinear heat equation and for the incompressible three-dimensional Navier-Stokes equation do not apply here, due to the nonlinear and nonlocal nature of (1.41) and the presence of the unbounded coefficients.

For all positive times, the solution constructed in Theorem 9.1 is smooth and the notion of solution is unambiguous. However, the initial data is rough, and the sense in which the initial data is obtained is a delicate point, especially when considering uniqueness. For $p = 3/2$, the solution f does not belong to $L^1(0,T,L^\infty)$, and the uniqueness result by Fournier [75] does not apply. Nevertheless, we expect this solution to be unique among smooth solutions in the class $C([0,T];L^{3/2}_{9/2})$, but the proof is still open. For readers interested in investigating uniqueness, we note that the solutions constructed here in the $p = 3/2$ case additionally satisfy:

$$\begin{aligned} &f \in L^{\frac{2q}{2q-3}}(0,T;L^q_{9/2}) \quad \text{if } q \in [3/2,\infty), \\ &\nabla f^{3/4} \in L^2(0,T;L^2_{3/2}), \quad \text{and} \quad \lim_{t\to 0^+} t\,\|f(t)\|_{L^\infty} = 0. \end{aligned}$$

Note that the unweighted $p = \infty$ result of Arsenev-Peskov [15] fits naturally in our framework. Our decay assumptions are less restrictive in terms of weights than the ones in [8] used to show conditional regularity. This is a result of a new (weighted) ε-Poincaré inequality used to rigorously show uniqueness (see Lemma 9.11). The proof of Lemma 9.11 follows from the Caffarelli-Kohn-Nirenberg inequality and is located in Appendix 9.9. The integrability condition $f \in L^1_2$ and the normalisation (1.46) for f are used solely in the lower bound for $A[f]$.

To obtain global-in-time solutions from Theorem 9.1 we use the monotonicity of the Fisher information proved by Guillen and Silvestre in [100]. Their framework, as stated, requires C^1 initial data with Maxwellian tails. We cannot expect that the solutions constructed in Theorem 9.1 satisfy these assumptions for generic initial data. Instead, we use the following adaptation of their result:

Theorem 9.2. *Suppose $g_{in} \in L^1_m$ for some $m > 6$ and $g_{in} \in W^{1,1}_{loc}$ has finite Fisher information[3]. Then, there is a unique global-in-time smooth solution to* (1.41) *with initial datum g_{in} and decreasing Fisher information.*

The proof of Theorem 9.2 uses an approximation argument that ensures the Guillen-Silvestre a-priori estimates can be applied here. Consequently, applying Theorem 9.2, we obtain the global-in-time existence result in **Theorem 1.8**, which we recall here:

Theorem 9.3. *(Global-in-time Existence) Fix $\frac{3}{2} \leq p < \infty$, $m > 6$, and $m_0 \geq 9/2$, and initial data f_{in} such that*

$$f_{in} \in L^p_{m_0} \cap L^1_m \quad \text{and} \quad f_{in} \geq \frac{a}{\langle v \rangle^k}, \quad \text{for some } a > 0 \text{ and } k \leq \frac{m_0 - 3}{p-1}.$$

Then, (1.41) *admits a global-in-time smooth solution $f : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^+$ with initial data f_{in}. Moreover, f has decreasing Fisher information, which satisfies the short-time smoothing estimate:*

$$\int_{\mathbb{R}^3} \frac{|\nabla f|^2}{f} \, \mathrm{d}v \leq \frac{C(a, p, k, m_0, f_{in})}{t} \qquad \text{for} \qquad 0 < t < 1.$$

Remark 9.1. There is an implicit compatibility condition hidden within assumption (1.47). Namely, the lower bound $\langle v \rangle^{-k}$ must belong to $L^p_{m_0} \cap L^1_m$. This holds if

$$\frac{m_0 - 3}{p-1} \geq k > \max\left(\frac{m_0 + 3}{p}, m + 3 \right).$$

In particular, this compatibility condition is satisfied for initial data f_{in} satisfying

$$f_{in} \in L^{3/2}(B_R), \qquad f_{in} \gtrsim 1 \quad \text{on } B_R, \qquad f_{in} \sim |v|^{-k} \quad \text{for } |v| \geq R,$$

provided $k > 9$.

Theorem 1.8 is *NOT* proved by iterating Theorem 9.1. Instead, Theorem 9.1 is used to show that solutions to (1.41) with initial data satisfying (1.47) satisfy the regularisation

[3]There are two reasonable definitions for the Fisher information of a general nonnegative function $g \in L^1(\mathbb{R}^3)$ that will be used frequently, namely $\left\| \sqrt{g} \right\|^2_{\dot{H}^1}$ and $\left\| g^{-1} |\nabla g|^2 \right\|_{L^1}$, with the convention that the integrand is 0 on the set $\{g = 0\}$. We remind the reader that under the mild assumption that $g \in W^{1,1}_{loc}$ these definitions are equivalent up to a factor of four.

estimates (9.1) and (1.48) for a short time. We then apply Theorem 9.2 with $g_{in} = f(t)$ for any small time $t > 0$ to obtain a global solution. Global existence fundamentally relies upon the L^1 moments and the Fisher information, which are the only quantities besides the entropy we know how to propagate for all time.

Let us comment briefly on (1.48). In fact, (1.48) is a consequence of the decay of the Fisher information $i(f)$ and the bound $i(f) \in L^1_{loc}(\mathbb{R}^+)$. The L^1 bound on the Fisher information is obtained by combining pointwise lower bounds for f with the weighted L^2 estimates for ∇f. Because the Landau equation does not generate polynomial lower bounds, we need to propagate a pointwise lower bound from the initial data. We note that this assumption has the undesirable side effect of precluding vacuum regions in the initial data. We expect that the pointwise lower can be weakened to a lower bound in an L^p sense, but we do not pursue this here. Moreover, to overcome the technical difficulties created by $A[f]$ being unbounded at time 0 for $p = 3/2$, we take a Stampacchia approach to the maximum principle. Additionally, one could conceivably obtain a version of Theorem 1.8 for initial data with Gaussian or stretched exponential tails, using the extensive study of weighted L^2 estimates in [44].

It is natural to ask whether the regularisation estimates found in Theorem 1.1 hold for all time. Since we are not iterating Theorem 1.1 to produce a global solution, it is unclear in what form the estimates fond in Theorem 1.1 hold for arbitrarily large times. Using known regularisation techniques, one can show that for $t > 0$ the function f can be bounded as

$$\|f(t+1)\|_{L^\infty} \leq C(\|f(t)\|_{L^1_{6+}}, \|f(t)\|_{L^3}) \leq C(\|f(t)\|_{L^1_{6+}}, i(f(t))). \tag{9.3}$$

While the Fisher information decays, the L^1_m moments may grow at most linearly in time, raising the possibility of infinite time blow-up. However, taking $m > 16$ in (1.47), one can show that the L^1_{6+} norm of $f(t)$ is bounded uniformly in time (for a precise reference see [62, Lemma 2.1]). As expected, this rules out infinite time blow-up for initial data satisfying (1.47). Previously, the results in [62] quantitatively ruled out infinite time blow up for H^1 initial datum using a novel monotonicity formula connecting the $\dot{H}^1$ norm to the relative entropy. Alternatively, [86] qualitatively ruled out infinite time blow up via estimates on the Hausdorff dimension of the singular set of suitable H-solutions. See also [87] for related partial regularity results in space-time.

While (9.3) shows non-degeneracy for large time, more precise estimates can be obtained to study the convergence to equilibrium. Using the estimates in Section 9.6 with $p = 2$ modified for $h = f - \mu$ (instead of f), where μ is the Maxwellian corresponding to the normalisation of f (1.46), we extend known convergence to equilibrium results in L^1 (see [45, Theorem 2]) and L^2 (see [44, Corollary 1.3]) to L^∞:

Corollary 9.1. *For any smooth, rapidly decaying f that solves Landau, and any $m \geq 9/2$,*

$$\|f(T) - \mu\|_{L^\infty} \lesssim_m \sup_{T-1<t<T} \|f(t) - \mu\|_{L^2_m}^4 + \|f(t) - \mu\|_{L^2_m}^{4/7}, \tag{9.4}$$

where μ is the Maxwellian associated to the normalisation (1.46)*. If f_{in} has finite Fisher information and $f_{in} \in L^2(\exp(\lambda \langle v \rangle^s))$ for some $s \in (0, 1/2)$ and $\lambda > 0$, then*

$$\|f(t) - \mu\|_{L^\infty} \lesssim_{s,\lambda,f_{in}} \exp\left(-\lambda_0 t^{\frac{s}{3}}\right), \tag{9.5}$$

for some $\lambda_0 = \lambda_0(s, \lambda, f_{in})$.

The regularisation estimate (9.4) is proved using a De Giorgi iteration that has essentially appeared in the authors' previous work as Proposition 8.2.

The asymptotic behavior in (9.5) then follows combining known results on convergence to equilibrium in weighted L^1 (specifically [45, Theorem 2]), weighted L^2 (specifically [44, Theorem 1.1]), and the monotonicity of the Fisher information for polynomial localized solutions proved in Theorem 9.2) above.

The construction of the global solution in Theorem 1.8 relied on propagating the polynomial lower bound that is assumed on the initial data for some amount of time. Interestingly, compared to the statement in Corollary 9.1, Maxwellian behaviour only forms at larger times.

9.1.2 Open problems

Given the rapid recent development of the theory for the homogeneous Landau equation, we collect here some interesting open problems. We emphasise problems whose resolution, in our opinion, would advance the understanding of the collision operator in a manner that can be transferred to the physical inhomogeneous Landau equation.

- regularisation estimates for smooth solutions using only the mass, energy, and entropy. That is, do suitably nice solutions to (1.41) satisfy $L^\infty(0, T; L^1_2 \cap L \log L) \to L^\infty(t, T; L^\infty)$ estimates? If so, can one verify the conjectured regularisation rate of $\frac{1}{t}$?

- A study of L^∞ solutions, with only locally finite mass: Such solutions appear naturally in the study of local regularity as the limit of suitable rescaling procedures (zooming in about a single point). Additionally, self-similar solutions necessarily have infinite mass. See [179] for a more in depth discussion.

- Existence of additional Lypanov functionals: For example, are there monotone weighted quantities? Are there higher order functionals, such as the derivatives of the Fisher information, which are monotone?

9.1.3 Outline

In Section 9.2, we recall short time well posedness results and prove Theorem 9.2. The rest of the paper is then devoted to proofs of Theorem 9.1 and Theorem 1.8. The proofs split into two distinct cases, $p = 3/2$ and $p > 3/2$. Section 9.3, Section 9.4, and Section 9.5 are devoted to the case $p = 3/2$. In Section 9.3, we prove novel $L^{3/2} \to L^\infty$ regularisation estimates for smooth solutions using the De Giorgi method. In Section 9.4, we use an ODE argument to propagate the $L^{3/2}$ norm for a short time. Combined with the L^∞ regularisation estimates, additional higher regularity estimates, and a compactness argument we deduce the $p = 3/2$ case of Theorem 9.1. In Section 9.5, we propagate the lower bound of the initial data using a barrier argument and perform additional weighted estimates to show that under the hypotheses of Theorem 1.8, the Fisher information becomes instantaneously finite, which concludes the proof of Theorem 1.8 for the $p = 3/2$ case. In Section 9.6, we show the analogous regularisation estimates for $p > 3/2$ and show Theorem 9.1 for the $p > 3/2$ case. In Section 9.7, we show Theorem 1.8 for the $p > 3/2$ case.

9.2 Prior well-posedness results and proof of Theorem 9.2

This work will build on the local-in-time existence result in sub-critical norms from Chapter 8. In particular, we refer to Theorem 8.2 and its Corollary 8.2.

We also need a rather recent result by Guillen-Silvestre [100] on the global well-posedness for C^1 initial datum with sufficient decay. The proof is based on showing that the Fisher information $i(f)$ is monotone decreasing as a function of time:

Theorem 9.4 (From [100, Theorem 1.2])**.** *Let $f_0 : \mathbb{R}^3 \to [0, \infty)$ be an initial datum that is C^1 and satisfies $f_0(v) \leq C_0 \exp\left(-\beta |v|^2\right)$ for some positive parameters C_0 and β. Assume also that f_0 has finite Fisher information. Then there is a unique C^1 solution $f : [0, \infty) \times \mathbb{R}^3 \to [0, \infty)$ to the Landau equation (1.41) with initial datum $f(0, v) = f_0(v)$. For any positive time, this function f is strictly positive, in the Schwartz space, and bounded above by a Maxwellian. The Fisher information $i(f)$ is non-increasing.*

The proof of Theorem 9.2 relies on approximating rough slowly, decaying initial data by initial data satisfying the hypotheses of Theorem 9.4. However, because the Fisher information is a nonlinear quantity, we will need to use a particular approximation. The following lemma will be used to construct the approximation:

Lemma 9.1. *For any $R > 0$, there is a radial, monotone decreasing $\eta \in C_c^\infty(\mathbb{R}^3)$ that satisfies:*

$$0 \le \eta \le 1, \quad \eta(x) = \begin{cases} 1 & \text{if } x \in B_R(0) \\ 0 & \text{if } x \notin B_{2R}(0), \end{cases} \qquad |\nabla \eta| \le \frac{C\sqrt{\eta}}{R}, \quad \text{and} \quad |\nabla \eta| \le \frac{C\sqrt{1-\eta}}{R},$$

where C is a universal constant.

The proof is based on the standard construction of a smooth partition of unity in differential geometry. For completeness, we provide a proof in Appendix 9.8. We are now ready to prove an analogue of Theorem 9.4 for polynomially decaying initial data:

Proof of Theorem 9.2. We prove the theorem first for $g_{in} \in C^\infty$. For $R > 0$, fix η_R as in Lemma 9.1 and approximate g_{in} by

$$g_{in,R} := g_{in}(v)\eta_R(v) + (1 - \eta_R(v))e^{-|v|^2}.$$

Note that $g_{in,R} \in C^\infty$, $g_{in,R}$ has Gaussian tails, and $g_{in,R} \to g_{in}$ pointwise as $R \to \infty$. Also,

$$\int_{\mathbb{R}^3} \langle v \rangle^m |g_{in,R} - g_{in}| \, \mathrm{d}v \le \int_{\mathbb{R}^3 \setminus B_R(0)} \langle v \rangle^m \left(g_{in}(v) + e^{-|v|^2} \right) \mathrm{d}v, \tag{9.6}$$

which converges to 0 since $g_{in} \in L^1_m$. A similar argument gives $g_{in,R} \to g_{in}$ in L^3 as a consequence of the finite Fisher information and Sobolev's embedding. We show next that the Fisher information converges. For $|v| \le R$, we have $g_{in,R} = g_{in}$. On the other hand, for $|v| \ge 2R$, $g_{in,R} = e^{-|v|^2}$, so we have

$$\int_{\mathbb{R}^3 \setminus B_{2R}} \frac{|\nabla g_{in,R}|^2}{g_{in,R}} \, \mathrm{d}v \le 4 \int_{\mathbb{R}^3 \setminus B_R} |v|^2 e^{-|v|^2} \, \mathrm{d}v,$$

which evidently converges to 0 as $R \to \infty$. It remains to bound terms on the intermediate annulus A_R given by $R \le |v| \le 2R$. We start by computing

$$\nabla g_{in,R} = \nabla g_{in}(v)\eta_R(v) + g_{in}(v)\nabla \eta_R(v) + \nabla \left[(1 - \eta_R(v))e^{-|v|^2} \right]$$

and so by Cauchy Schwarz inequality,

$$\begin{aligned} &\int_{A_R} \frac{|\nabla g_{in,R}|^2}{g_{in,R}} \, \mathrm{d}v \\ &\quad \le \int_{A_R} \frac{|\nabla g_{in}(v)|^2 \eta_R(v)^2 + g_{in}(v)^2 |\nabla \eta_R(v)|^2 + \left| \nabla \left[(1 - \eta_R(v))e^{-|v|^2} \right] \right|^2}{g_{in,R}} \, \mathrm{d}v \\ &\quad =: I_1 + I_2 + I_3. \end{aligned}$$

The first term we control as

$$I_1 \leq \int_{A_R} \frac{|\nabla g_{in}(v)|^2 \eta_R(v)^2}{g_{in}\eta_R(v)} \, \mathrm{d}v \leq \int_{A_R} \frac{|\nabla g_{in}(v)|^2}{g_{in}} \, \mathrm{d}v.$$

Because $i(g_{in}) < \infty$, the Lebesgue dominated convergence theorem implies I_1 converges to 0 as $R \to \infty$. We control the second term using the specific form of η_R:

$$I_2 \leq \int_{A_R} \frac{g_{in}(v)^2 |\nabla \eta_R(v)|^2}{g_{in}\eta_R(v)} \, \mathrm{d}v \leq CR^{-2} \int_{A_R} g_{in}(v) \, \mathrm{d}v \leq CR^{-2},$$

which converges to 0 as $R \to \infty$. We control the third term similarly:

$$\begin{aligned} I_3 &\leq \int_{A_R} \frac{|\nabla \left[(1-\eta_R(v)) \exp(-|v|^2)\right]|^2}{(1-\eta_R(v)) \exp(-|v|^2)} \, \mathrm{d}v \\ &\leq 4 \int_{A_R} |v|^2 e^{-|v|^2} \, \mathrm{d}v + \int_{A_R} \exp(-|v|^2) \frac{|\nabla \eta_R(v)|^2}{1-\eta_R(v)} \, \mathrm{d}v. \end{aligned}$$

The first term converges to 0 again as $R \to \infty$, since the exponential dominates any polynomial. The second term also converges to 0 as $R \to \infty$, where we use the specific form of $1 - \eta_R$ to conclude. Thus, we we have shown

$$\lim_{R \to \infty} i(g_{in,R}) = i(g_{in}).$$

We now use Theorem 9.4 to deduce that for any $R > 1$, there exists $g_R : [0, \infty) \times \mathbb{R}^3 \to [0, \infty)$ solving (1.41) with initial datum $g_{in,R}$, and whose Fisher information is monotonically decreasing. The Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ implies that $\|g_R(t)\|_{L^3} \leq Ci(g_{in,R})$ for each $t \geq 0$ and $R > 1$. Using the propagation of L^1 moments in Lemma 8.1, we apply Theorem 8.2 with initial data $g_R(t)$ for any $t \in [0, T]$. By uniqueness of $L^1_{loc}(\mathbb{R}^+; L^\infty)$ weak solutions, we conclude that g_R satisfies the smoothing estimate (8.5): for any $T > 0$ and any $t \in (0, T)$,

$$\|g_R(t)\|_{L^\infty} \leq C \left(\sup_{0<t<T} \|g_R(t)\|_{L^3}, \sup_{0<t<T} \|g_R(t)\|_{L^1_m}, m \right) \left(1 + t^{-\beta^*}\right),$$

for some explicit $\beta^* \in (0, 1)$. Therefore, there is an $g : [0, \infty) \times \mathbb{R}^3 \to [0, \infty)$ such that as $R \to \infty$, $g_R \to g$ weak starly in $L^\infty(0, T; L^3)$, and weakly in $L^1_{loc}(\mathbb{R}^+; L^\infty)$. By well-known higher regularity estimates (recall Lemma 8.14), $g_R \to g$ in $C^k(\Omega)$ for any $k \in \mathbb{N}$ and any $\Omega \subset (0, \infty) \times \mathbb{R}^3$ compact. This is certainly sufficient to conclude g is a smooth solution to (1.41) with initial data g_{in}.

Further, by the Banach-Alaoglu theorem and monotonicity of the $i(g_R(t))$, $\sqrt{g} \in$

$L^\infty(\mathbb{R}^+;\dot{H}^1)$ and

$$\sup_{t\in\mathbb{R}^+} i(g(t)) = \sup_{t\in\mathbb{R}^+} \|\sqrt{g}\|_{\dot{H}^1}^2 \leq \liminf_{R\to\infty} \sup_{t\in\mathbb{R}^+} \left\|\sqrt{g_R(t)}\right\|_{\dot{H}^1}^2 \leq \liminf_{R\to\infty} i(g_{in,R}) = i(g_{in}).$$

To summarize, we have shown for any smooth g_{in} with finite Fisher information, there is a unique global-in-time smooth solution g to Landau with initial datum g_{in}. This smooth solution satisfies the bounds

$$i(g(t)) \leq i(g_{in}),$$

and

$$\sup_{0<t<T} t^{\beta^*} \|g(t)\|_{L^\infty} \leq C(\|g_{in}\|_{L^1_m}, i(g_{in}), m, T) \quad \text{for each } T > 0.$$

Finally, using uniqueness, we can show that g has decreasing Fisher information. In particular, fix $0 \leq s \leq t$ and apply this result to $g(s)$ to construct another solution $\tilde{g}$ with initial data $\tilde{g}_{in} = g(s)$. By construction, $i(\tilde{g}(t)) \leq i(\tilde{g}_{in}) = i(g(s))$. By the uniqueness from Theorem 8.6, $\tilde{g}(\tau) = g(s+\tau)$ for all $\tau \geq 0$. Taking $\tau = t - s$, we conclude $i(g(t)) = i(\tilde{g}(t)) \leq i(g(s))$.

This concludes the proof, if g_{in} is qualitatively smooth. If $g_{in} \in W^{1,1}_{loc}$ is not smooth, then one can approximate g_{in} by

$$g_{in,\delta} = \left(\varphi_\delta * \sqrt{g_{in}(\cdot)}\right)^2,$$

where φ_δ is a standard mollifier. The seemingly strange choice of approximation is exactly to easily imply $i(g_{in,\delta}) \to i(g_{in})$ as $\delta \to 0^+$. Sending $\delta \to 0^+$ one then concludes in the same manner as when taking $R \to \infty$. □

9.3 An ε-regularity criterion for Landau-Coulomb

In this section, we derive a smoothing estimate based on a De Giorgi argument for smooth functions, similar to Sections 8.3.2 and 8.4.2 above. Variants of the De Giorgi method have been used previously in [7, 8, 81, 83, 86, 87] to obtain L^∞ regularisation estimates for homogeneous kinetic equations. The novelty of the method presented here is that we use an $L^{3/2}$-based control quantity and use crucially the translation invariance of the estimates in the level set parameter. The main result of this section is the following ε-regularity criterion:

Proposition 9.1. *Let $c_0 > 0$ arbitrary and let $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ a smooth, rapidly decaying solution to* (1.41) *satisfying* (1.46) *with $A[f] \geq c_0 \langle v\rangle^{-3} I$. Then, there is a*

constant $\varepsilon_0 > 0$, depending only on c_0, such that if f satisfies

$$\varepsilon(f) := \sup_{0<\tau<T} \int_{\mathbb{R}^3} \langle v\rangle^{9/2} (f-K)_+^{3/2} \, \mathrm{d}v + \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{3/2} \left|\nabla (f-K)_+^{3/4}\right|^2 \, \mathrm{d}v < \varepsilon_0,$$

for some threshold $K \geq 0$, then

$$f(\tau, v) \leq C^*(K+1) + C^* \varepsilon^{2/3} t^{-1} \qquad \text{for each } (\tau, v) \in [t, T] \times \mathbb{R}^3,$$

where $C^* > 0$ depends only on c_0.

We start by noting that for any smooth, rapidly decaying solution to (1.41), Lemma 8.4 and Lemma 8.3 imply that there is some $c_0 > 0$ such that

$$A[f](v) \geq \frac{c_0}{\langle v\rangle^3} I.$$

Step 1: Introduction of Level Sets and Weighted Energy Functional

For f a solution to (1.41) and for any $\ell \in \mathbb{R}^+$, we denote the level set function

$$f_\ell(t, v) := (f(t, v) - \ell)_+ = \max(f(t, v) - \ell, 0).$$

We now define our main weighted energy functional associated to f: for $0 \leq T_1 < T_2 \leq T$ and $\ell \in \mathbb{R}^+$,

$$\mathcal{E}_\ell(T_1, T_2) := \sup_{T_1<t<T_2} \int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{3/2} \, \mathrm{d}v + \int_{T_1}^{T_2} \int_{\mathbb{R}^3} \langle v\rangle^{3/2} \left|\nabla f_\ell^{3/4}\right|^2 \, \mathrm{d}v.$$

Our first main goal is showing the following nonlinear inequality: for any $\ell > k \geq 0$ and $0 \leq T_1 < T_2 < T_3 \leq T$,

$$\begin{aligned} &\mathcal{E}_\ell(T_2, T_3) \\ &\leq C \left[1 + \frac{1}{(\ell-k)(T_2-T_1)} + \frac{\mathcal{E}_k^{1/2}(T_1, T_3)}{(\ell-k)^{3/4}} + \frac{1+\ell}{(\ell-k)} + \frac{\ell^2}{(\ell-k)^2}\right] \mathcal{E}_k^{5/3}(T_1, T_3), \end{aligned} \tag{9.7}$$

where C depends on c_0. To prove (9.7), we will show a slightly more precise bound. It will be convenient at times to work with each half of $\mathcal{E}$ separately, therefore we introduce

the following auxiliary functionals:

$$\begin{aligned}\mathcal{A}_\ell(T_1,T_2) &:= \sup_{T_1<t<T_2}\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\ell^{3/2}\,\mathrm{d}v\\ \mathcal{B}_\ell(T_1,T_2) &:= \int_{T_1}^{T_2}\int_{\mathbb{R}^3}\langle v\rangle^{3/2}\left|\nabla f_\ell^{3/4}\right|^2\,\mathrm{d}v\,\mathrm{d}t.\end{aligned} \tag{9.8}$$

Step 2: Differential Inequality

We begin with the computation of the L^p norm of the level set function f_ℓ. This computation will be used frequently throughout the remainder of the paper.

Lemma 9.2. *For $f:[0,T]\times\mathbb{R}^3\to\mathbb{R}^+$ a solution to (1.41), the following differential inequality holds for each $\ell\geq 0$, for each $1<p<\infty$ and $m\geq 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v &+ \int_{\mathbb{R}^3}\langle v\rangle^{m-3}\left|\nabla f_\ell^{p/2}\right|^2\,\mathrm{d}v\\ &\leq C\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p+1}\,\mathrm{d}v + C(\|A[f]\|_{L^\infty}+\ell)\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v\\ &\quad + C\ell^2\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v,\end{aligned} \tag{9.9}$$

where the constant C depends only on c_0, p, and m.

Testing (1.41) by $\langle v\rangle^m f_\ell^{p-1}$, and using $\nabla f_\ell = \chi_{\{f>\ell\}}\nabla f$ and $\partial_t f_\ell = \chi_{\{f>\ell\}}\partial_t f$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p &+ \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}\langle v\rangle^m A[f]\nabla f_\ell^{p/2}\cdot\nabla f_\ell^{p/2}\,\mathrm{d}v\\ &\leq -\int_{\mathbb{R}^3} f_\ell^{p-1}(\nabla\langle v\rangle^m)\cdot A[f]\nabla f_\ell\,\mathrm{d}v\\ &\quad + \int_{\mathbb{R}^3}\langle v\rangle^m\nabla f_\ell^{p-1}\cdot\nabla a[f]f\,\mathrm{d}v\\ &\quad + \int_{R^3} f_\ell^{p-1}\nabla(\langle v\rangle^m)\cdot\nabla a[f]f\,\mathrm{d}v\\ &=: I_1+I_2+I_3\end{aligned}$$

We bound I_1 using Young's inequality as

$$\begin{aligned}I_1 = -\frac{2}{p}\int_{\mathbb{R}^3} f_\ell^{p/2}(\nabla\langle v\rangle^m)\cdot A[f]\nabla f_\ell^{p/2}\,\mathrm{d}v &\leq \frac{p-1}{2p^2}\int_{\mathbb{R}^3}\langle v\rangle^m A[f]\nabla f_\ell^{p/2}\cdot\nabla f_\ell^{p/2}\,\mathrm{d}v\\ &\quad + C(p)\,\|A[f]\|_{L^\infty}\int_{\mathbb{R}^3}\langle v\rangle^{m-2} f_\ell^p\,\mathrm{d}v.\end{aligned}$$

Next, by integrating by parts, we rewrite I_2, which contains the highest order terms:

$$\begin{aligned}
I_2 &= -I_3 + \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p-1} f^2 \,\mathrm{d}v - \frac{1}{p}\int_{\mathbb{R}^3} \langle v\rangle^m \nabla a[f]\cdot\nabla f_\ell^p \,\mathrm{d}v \\
&= -I_3 + \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p+1} \,\mathrm{d}v + 2\ell\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p \,\mathrm{d}v + \ell^2\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p f \,\mathrm{d}v + \frac{1}{p}\int_{\mathbb{R}^3} f_\ell^p(\nabla\langle v\rangle^m)\cdot\nabla a[f]\,\mathrm{d}v \qquad (9.10)\\
&= -I_3 + \frac{p-1}{p}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p+1}\,\mathrm{d}v + \frac{(2p-1)\ell}{p}\int_{R^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v \\
&\quad + \ell^2\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v + \frac{1}{p}\int_{\mathbb{R}^3}(\nabla\langle v\rangle^m)\cdot[\nabla\cdot(A[f]f_\ell^p) - A[f]\nabla f_\ell^p]\,\mathrm{d}v
\end{aligned}$$

Therefore, we bound I_2 and I_3 together as

$$\begin{aligned}
I_2 + I_3 &\le C\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p+1}\,\mathrm{d}v + C\ell\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v + C\ell^2\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} f_\ell^p A[f] : (\nabla^2\langle v\rangle^m)\,\mathrm{d}v - \frac{2}{p}\int_{\mathbb{R}^3}(\nabla\langle v\rangle^m) f_\ell^{p/2}\cdot A[f]\nabla f_\ell^{p/2}\,\mathrm{d}v \\
&\le C\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p+1}\,\mathrm{d}v + C(\|A[f]\|_{L^\infty}+\ell)\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v \\
&\quad + C\ell^2\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v + \frac{p-1}{2p^2}\int_{\mathbb{R}^3}\langle v\rangle^m A[f]\nabla f_\ell^{p/2}\cdot\nabla f_\ell^{p/2}\,\mathrm{d}v,
\end{aligned}$$

where the last inequality follows from Young's, as in (9.10). After reabsorbing terms and using the lower bound for $A[f]$, we find the desired inequality:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v + \int_{\mathbb{R}^3}\langle v\rangle^{m-3}\left|\nabla f_\ell^{p/2}\right|^2\,\mathrm{d}v \\
&\qquad \le C\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p+1}\,\mathrm{d}v + C(\|A[f]\|_{L^\infty}+\ell)\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^p\,\mathrm{d}v + C\ell^2\int_{\mathbb{R}^3}\langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v.
\end{aligned}$$

This concludes the proof of Lemma 9.2.

Step 3: Gain of Integrability

In this step, we use Lemma 9.2 to obtain a slightly more precise form of the nonlinear inequality (9.7) relating the energy functionals defined in (9.8).

Lemma 9.3. *Under the assumptions of Proposition 9.1 and notation introduced in* (9.8) *of Step 1, the energy functional $\mathcal{E}$ satisfies for each $0 \le T_1 < T_2 < T_3 \le T$, each $0 \le k < \ell$,*

$$\begin{aligned}
&\mathcal{E}_\ell(T_2,T_3) \\
&\le C\left[1 + \frac{\mathcal{A}_k^{1/2}(T_1,T_3)}{(\ell-k)^{3/4}} + \frac{1}{(T_2-T_1)(\ell-k)} + \frac{1+\ell}{(\ell-k)} + \frac{\ell^2}{(\ell-k)^2}\right]\mathcal{A}_k^{2/3}(T_1,T_3)\mathcal{B}_k(T_1,T_3),
\end{aligned}$$

where the constant C depends only on c_0, the coercivity constant of $A[f]$.

We begin by integrating (9.9) (with $m = 9/2$, $p = 3/2$) over $\tau \in [t_1, t_2] \subset [0, T]$, to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{3/2}(t_2)\,\mathrm{d}v + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{3/2} \left|\nabla f_\ell^{3/4}\right|^2 \,\mathrm{d}v\,\mathrm{d}\tau \\ \leq \int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{3/2}(t_1)\,\mathrm{d}v + C\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{5/2}\,\mathrm{d}v\,\mathrm{d}\tau \\ + C\ell\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{3/2}\,\mathrm{d}v + C\ell^2\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{1/2}\,\mathrm{d}v\,\mathrm{d}\tau \\ + C\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} \left\|A[f(\tau)]\right\|_{L^\infty} f_\ell^{3/2}\,\mathrm{d}v\,\mathrm{d}\tau.\end{aligned}$$

Taking a supremum over $t_2 \in [T_2, T_3]$, then averaging over $t_1 \in [T_1, T_2]$, and recalling the definition of $\mathcal{E}_\ell$ in (9.8), we find

$$\begin{aligned}\mathcal{E}_\ell(T_2, T_3) \leq \frac{C}{T_2 - T_1}\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{3/2}\,\mathrm{d}v\,\mathrm{d}\tau + C\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{5/2}\,\mathrm{d}v\,\mathrm{d}\tau \\ + C\ell\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{3/2}\,\mathrm{d}v\,\mathrm{d}\tau + C\ell^2\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{1/2}\,\mathrm{d}v\,\mathrm{d}\tau \\ + C\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} \left\|A[f(\tau)]\right\|_{L^\infty} f_\ell^{3/2}\,\mathrm{d}v\,\mathrm{d}\tau.\end{aligned} \tag{9.11}$$

Now, we note a simple consequence of the definition of f_ℓ: if $0 \leq k < \ell$, then

$$\{f_\ell > 0\} = \{f_k > \ell - k\}.$$

Therefore, using Hölder's inequality, Chebychev's inequality for the measure $\mathrm{d}\nu(v) = \langle v\rangle^{9/2}\,\mathrm{d}v$, for any $0 < \alpha_1 < \alpha_2$, and the fact that $f_\ell \leq f_k$ for any $k \leq \ell$,

$$\begin{aligned}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^{\alpha_1}\,\mathrm{d}v &\leq \nu\left(\{f_\ell > 0\}\right)^{1-\frac{\alpha_1}{\alpha_2}} \left(\int_{\mathbb{R}^3} f_\ell^{\alpha_2}\,\mathrm{d}\nu(v)\right)^{\frac{\alpha_1}{\alpha_2}} \\ &\leq \nu\left(\{f_k^{\alpha_2} > (\ell - k)^{\alpha_2}\}\right)^{1-\frac{\alpha_1}{\alpha_2}} \left(\int_{\mathbb{R}^3} f_\ell^{\alpha_2}\,\mathrm{d}\nu(v)\right)^{\frac{\alpha_1}{\alpha_2}} \\ &\leq (\ell - k)^{\alpha_1 - \alpha_2}\left(\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_k^{\alpha_2}\,\mathrm{d}v\right).\end{aligned} \tag{9.12}$$

Combining (9.11) and (9.12) (with $\alpha_1 \in \{1/2, 3/2\}$ and $\alpha_2 = 5/2$ and integrated over

$\tau \in [T_1, T_3]$), we have shown for $0 \le k < \ell$

$$\begin{aligned}
&\mathcal{E}_\ell(T_2, T_3) \\
&\le C\left[1 + \frac{1}{(T_2 - T_1)(\ell - k)} + \frac{\ell}{(\ell - k)} + \frac{\ell^2}{(\ell - k)^2}\right] \int_{T_1}^{T_3} \int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{5/2} \, \mathrm{d}v \, \mathrm{d}\tau \\
&\quad + C \int_{T_1}^{T_3} \int_{\mathbb{R}^3} \langle v \rangle^{9/2} \left\| A[f(\tau)] \right\|_{L^\infty} f_\ell^{3/2} \, \mathrm{d}v \, \mathrm{d}\tau.
\end{aligned} \tag{9.13}$$

We now bound the last term, using the identity $f = f_\ell + \min(f, \ell)$, the linearity of A, the coefficient bounds in Lemma 8.5, and the embedding $L^2_m \hookrightarrow L^1$ for $m > 1$, and (9.12) (with $\alpha_1 \in \{3/2, 2\}$ and $\alpha_2 \in \{13/6, 5/2\}$):

$$\begin{aligned}
&\left\| A[f] \right\|_{L^\infty} \left\| f_\ell \right\|_{L^{3/2}_{9/2}}^{3/2} \\
&\le \left(\left\| A[f_\ell] \right\|_{L^\infty} + \left\| A[\min(f, \ell)] \right\|_{L^\infty} \right) \left\| f_\ell \right\|_{L^{3/2}_{9/2}}^{3/2} \\
&\le C \left(\left\| f_\ell \right\|_{L^1}^{1/3} \left\| f_\ell \right\|_{L^2}^{2/3} + \left\| \min(f, \ell) \right\|_{L^1}^{2/3} \left\| \min(f, \ell) \right\|_{L^\infty}^{1/3} \right) \left\| f_\ell \right\|_{L^{3/2}_{9/2}}^{3/2} \\
&\le C \left\| f_\ell \right\|_{L^2_{9/2}} \left\| f_\ell \right\|_{L^{3/2}_{9/2}}^{3/2} + C\ell^{1/3} \left\| f_\ell \right\|_{L^{3/2}_{9/2}}^{3/2} \\
&\le \frac{C}{(\ell - k)^{3/4}} \left(\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{13/6} \, \mathrm{d}v \right)^{\frac{3}{2}} + \frac{C\ell^{1/3}}{(\ell - k)} \left(\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{5/2} \, \mathrm{d}v \right)
\end{aligned} \tag{9.14}$$

Next, we use Lemma 8.7 with $q \in \{5/2, 13/6\}$, $p = 3/2$, and $9/2$ moments:

$$\begin{aligned}
\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{5/2} \, \mathrm{d}v &\le \left(\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{3/2} \, \mathrm{d}v \right)^{2/3} \left(\int_{\mathbb{R}^3} \langle v \rangle^{3/2} \left| \nabla f_k^{3/4} \right|^2 \, \mathrm{d}v \right), \\
\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{13/6} \, \mathrm{d}v &\le \left(\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_\ell^{3/2} \, \mathrm{d}v \right)^{7/9} \left(\int_{\mathbb{R}^3} \langle v \rangle^{3/2} \left| \nabla f_k^{3/4} \right|^2 \, \mathrm{d}v \right)^{2/3}.
\end{aligned}$$

Integrating in time and recalling the definitions of $\mathcal{A}_k$ and $\mathcal{B}_k$ in (9.8), we find

$$\begin{aligned}
\int_{T_1}^{T_3} \int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{5/2} \, \mathrm{d}v \, \mathrm{d}\tau &\le \mathcal{A}_k^{2/3}(T_1, T_3) \mathcal{B}_k(T_1, T_3), \\
\int_{T_1}^{T_3} \left(\int_{\mathbb{R}^3} \langle v \rangle^{9/2} f_k^{13/6} \, \mathrm{d}v \right)^{3/2} \mathrm{d}\tau &\le \mathcal{A}_k^{7/6}(T_1, T_3) \mathcal{B}_k(T_1, T_3).
\end{aligned} \tag{9.15}$$

Combining (9.13), (9.14), and (9.15), and using $\ell^{1/3} \le C(1 + \ell)$ we estimate $\mathcal{E}_\ell$ as

$$\begin{aligned}
\mathcal{E}_\ell(T_2, T_3) \le C \Bigg[1 + \frac{\mathcal{A}_k^{1/2}(T_1, T_3)}{(\ell - k)^{3/4}} + \frac{1}{(T_2 - T_1)(\ell - k)} + \frac{1 + \ell}{(\ell - k)} + \frac{\ell^2}{(\ell - k)^2} \Bigg] \\
\times \mathcal{A}_k^{2/3}(T_1, T_3) \mathcal{B}_k(T_1, T_3).
\end{aligned}$$

This concludes the proof of Lemma 9.3.

Step 4: De Giorgi Iteration

We now conclude the proof of Proposition 9.1 by a De Giorgi iteration. We fix $0 < t < T$ the given times and $K > 0$ the given threshold from the statement of Proposition 9.1. Then, for $\eta > 0$ arbitrary, we define our iteration quantities:

$$\ell_n = K + (1 - 2^{-n})\eta, \qquad t_n = t(1 - 2^{-n}), \qquad \text{and} \qquad E_n = \mathcal{E}_{\ell_n}(t_n, T).$$

Then, using Lemma 9.3, monotonicity in n, and we have the following recurrence:

$$\begin{aligned} E_{n+1} &= \mathcal{E}_{\ell_{n+1}}(t_{n+1}, T) \\ &\lesssim \left(1 + \frac{\mathcal{A}_{\ell_n}(t_n, T)^{1/2}}{(\ell_{n+1} - \ell_n)^{3/4}} + \frac{1}{(t_{n+1} - t_n)(\ell_{n+1} - \ell_n)} + \frac{1 + \ell_n}{(\ell_{n+1} - \ell_n)} + \frac{\ell_n^2}{(\ell_{n+1} - \ell_n)^2}\right) \\ &\qquad \times \mathcal{A}_{\ell_n}^{2/3}(t_n, T)\mathcal{B}_{\ell_n}(t_n, T) \\ &\leq C_1 E_n^{5/3} + C_2 2^{2n}\left(1 + \frac{1}{\eta t} + \frac{E_0^{1/2}}{\eta^{3/4}} + \frac{K}{\eta} + \frac{K^2}{\eta^2}\right) E_n^{5/3}, \end{aligned}$$

where C_1, $C_2 > 1$ are now fixed constants depending only on c_0. We attempt to find a barrier sequence of the form $B_n := B^{-n}E_0$ for some $B > 1$. That is, we wish to find values of η and B for which

$$B_{n+1} \geq C_1 B_n^{5/3} + C_2 2^{2n}\left(1 + \frac{1}{\eta t} + \frac{E_0^{1/2}}{\eta^{3/4}} + \frac{K}{\eta} + \frac{K^2}{\eta^2}\right) B_n^{5/3}.$$

Inserting the definition of B_n, this is implied by

$$1 \geq C_1 B^{\frac{-2n+3}{3}} E_0^{2/3} + C_2 B^{\frac{-2n+3}{3}} 2^{2n} E_0^{2/3}\left(1 + \frac{1}{\eta t} + \frac{E_0^{1/2}}{\eta^{3/4}} + \frac{K}{\eta} + \frac{K^2}{\eta^2}\right).$$

We then pick $B = 8$ so that $B^{\frac{2}{3}} = 4$ and we need only show

$$1 \geq 8(C_1 + C_2)E_0^{2/3} + 8C_2 E_0^{2/3}\left(\frac{1}{\eta t} + \frac{E_0^{1/2}}{\eta^{3/4}} + \frac{K}{\eta} + \frac{K^2}{\eta^2}\right).$$

Next, we pick ε_0 so that the first term is less than $1/2$ if $E_0 \leq \varepsilon_0$. More precisely, we pick

$$\varepsilon_0 = \max\left(1, \frac{1}{64(C_1 + C_2)^{3/2}}\right).$$

We finally pick η so that each remaining term is less than $1/8$:

$$\begin{aligned}\eta &= \max\left(64C_2, 256C_2^{4/3}, 8C_2^{1/2}\right)\max\left(E_0^{2/3}t^{-1}, E_0^{14/9}, E_0^{2/3}K, E_0^{1/3}K\right)\\ &\lesssim_{C_2} 1 + K + E_0^{2/3}t^{-1},\end{aligned}$$

where the implicit constant depends only on C_2. With these choices of B, η, and ε_0, we find that if $E_0 \le \varepsilon_0$, then $B_n \ge E_n$ for all n. More precisely, $B_0 = E_0$, and by our choice of parameters B_n satisfies the reverse inequality so that by induction,

$$\begin{aligned}E_{n+1} &\le C_1 E_n^{5/3} + C_2 2^{2n}\left(1 + \frac{1}{\eta t} + \frac{E_0^{1/2}}{\eta^{3/4}} + \frac{1+K}{\eta} + \frac{K^2}{\eta^2}\right)E_n^{5/3}\\ &\le C_1 B_n^{5/3} + C_2 2^{2n}\left(1 + \frac{1}{\eta t} + \frac{E_0^{1/2}}{\eta^{3/4}} + \frac{1+K}{\eta} + \frac{K^2}{\eta^2}\right)B_n^{5/3} \le B_{n+1}.\end{aligned}$$

Therefore,

$$\left\|f\chi_{\{f\ge K+\eta\}}\right\|_{L^\infty(t,T;L^{3/2}_{9/2})} \le \lim_{n\to\infty} E_n \le \lim_{n\to\infty} B_n = 0,$$

and we conclude $f(\tau, v) \le K + \eta$ pointwise a.e. on $[t,T]\times\mathbb{R}^3$. By the choice of η, and noting $\varepsilon = E_0 < \varepsilon_0 \le 1$, we have shown the desired pointwise estimate:

$$f(\tau, v) \le C^*(K+1) + C^*\varepsilon^{2/3}t^{-1} \qquad \text{for almost all } \tau, v \in [t,T]\times\mathbb{R}^3.$$

9.4 The local in time $L^{3/2}$ theory

9.4.1 Propagation of $L^{3/2}_m$ for smooth data

We now wish to propagate the $L^{3/2}_m$ norm of a solution f to (1.41) for a time depending only on the initial data f_{in}. This will enable us to use Proposition 9.1 to gain unconditional smoothing estimates, which can be used to construct solutions to (1.41). The main result of this section is in the following proposition:

Proposition 9.2. *Suppose $f_{in} \in L^{3/2}_m$ is Schwartz class and satisfies*

$$\int_{\mathbb{R}^3}\langle v\rangle^m f_{in}^{3/2}\,\mathrm{d}v \le H \qquad \text{and} \qquad \int_{\mathbb{R}^3}\langle v\rangle^m (f_{in}-K)_+^{3/2}\,\mathrm{d}v = \delta,$$

for some $m \ge 9/2$ and $K > 0$. Then, there is an constant $\delta_1 = \delta_1(H,m) > 0$ such that if $\delta < \delta_1$ and $f : [0,T^*)\times\mathbb{R}^3 \to \mathbb{R}_+$ is any smooth, rapidly decaying solution to (1.41) with

initial datum f_{in}, there is $T = T(H, m, K) \in (0, 1]$ such that

$$\begin{aligned}\sup_{0<\tau<\min(T,T^*)} &\int_{\mathbb{R}^3} \langle v\rangle^m (f-K)_+^{3/2}\,\mathrm{d}v \\ &+ \int_0^{\min(T,T^*)} \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla (f-K)_+^{3/4}\right|^2 \,\mathrm{d}v\,\mathrm{d}\tau < 4\delta,\end{aligned} \tag{9.16}$$

and, moreover,

$$\begin{aligned}\sup_{0<\tau<\min(T,T^*)} &\int_{\mathbb{R}^3} \langle v\rangle^m f^{3/2}\,\mathrm{d}v \\ &+ \int_0^{\min(T,T^*)} \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f^{3/4}\right|^2 \,\mathrm{d}v\,\mathrm{d}\tau \le C(m, K, H).\end{aligned} \tag{9.17}$$

The proof of Proposition 9.2 is inspired by our preceding results in Section 8.3.1, but our current setting presents significant complications. First, to use Proposition 9.1, we need more than $f \in L^\infty(0, T; L^{3/2}_m)$; we need additional regularity in time. For example, $f \in C(0, T; L^{3/2}_m)$ or $f \in BV(0, T; L^{3/2}_m)$ would be sufficient. To this end, we introduce the following quantities:

$$f_K := (f-K)_+, \quad y(t) := \int_{\mathbb{R}^3} \langle v\rangle^m f_K^{3/2}\,\mathrm{d}v, \quad F(t) := \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f_K^{3/4}\right|^2 \,\mathrm{d}v. \tag{9.18}$$

Since K will be fixed, the dependence of y on K is suppressed. Control of $y(t)$ uniformly in time encodes uniform integrability of the profiles $\{f(t)\}_{t\ge 0}$. Our goal now is to propagate smallness of y.

This leads, however, to another complication. We are not able to obtain an equation for the evolution of $y(t)$ using just that f solves (1.41). Instead, we must consider the following auxiliary "bulk" quantities:

$$g_{2K} := \min(f, 2K), \quad z(t) := \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v, \quad G(t) := \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla g_{2K}^{3/4}\right|^2 \,\mathrm{d}v. \tag{9.19}$$

Note that f and its derivatives decompose as

$$f = f_K + g_K \quad \text{and} \quad \partial_{t,v_i} f = \partial_{t,v_i} f_K + \partial_{t,v_i} g_K.$$

Although their derivatives have essentially disjoint supports, f_K and g_K do *NOT* have disjoint support. Therefore, to study the evolution of y, the nonlinear reaction term in (1.41) becomes

$$f^2 = f_K^2 + 2 f_K g_K + g_K^2.$$

We begin by finding the evolution f_K, i.e. show that $y(t)$ satisfies a differential inequality, which essentially appeared already in the proof of Proposition 9.1:

Lemma 9.4. *Suppose $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ is a smooth, rapidly decaying solution to* (1.41) *with $A[f] \geq c_0 \langle v \rangle^{-3}$. Let $m \geq 9/2$. Then, for any $K > 0$, the corresponding quantities defined in* (9.18) *and* (9.19) *satisfy the differential inequality:*

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + F(t) \lesssim_m F(t) y^{2/3}(t) + (1+K) y(t) + y^{7/5}(t) + K z(t), \tag{9.20}$$

where the implicit constant C depends only on m and c_0.

Proof. Applying Lemma 9.2 (with $p = 3/2$, $\ell = K$, and $m \geq 9/2$ fixed), we have the energy estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^m f_K^{3/2} \,\mathrm{d}v + \frac{c_0}{2} \int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f_K^{3/4} \right|^2 \,\mathrm{d}v \\ \leq C \int_{\mathbb{R}^3} \langle v \rangle^m f_K^{5/2} \,\mathrm{d}v + C(\|A[f]\|_{L^\infty} + K) \int_{\mathbb{R}^3} \langle v \rangle^m f_K^{3/2} \,\mathrm{d}v \\ + CK^2 \int_{\mathbb{R}^3} \langle v \rangle^m f_K^{1/2} \,\mathrm{d}v.\end{aligned}$$

where the constant C depends only on c_0 and m. Bounding the highest order term first, we find by Lemma 8.7 (with $p = 3/2$, $k = 9/2$, and $q = 5/2$):

$$\int_{\mathbb{R}^3} \langle v \rangle^m f_K^{5/2} \,\mathrm{d}v \lesssim \left(\int_{\mathbb{R}^3} \langle v \rangle^m f_K^{3/2} \,\mathrm{d}v \right)^{2/3} \left(\int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f_K^{3/4} \right|^2 \,\mathrm{d}v \right) \lesssim F(t) y^{2/3}.$$

Next, we bound the term with $\|A[f]\|_{L^\infty}$ using Lemma 8.5 (with $q = 9/2$), $\|f_K\|_{L^1} \leq 1$, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and Young's inequality:

$$\begin{aligned}\|A[f]\|_{L^\infty} \int_{\mathbb{R}^3} \langle v \rangle^m f_K^{3/2} \,\mathrm{d}v &\leq (\|A[f_K]\|_{L^\infty} + \|A[g_K]\|_{L^\infty}) \, y(t) \\ &\leq C \left(\|f_K\|_{L^1}^{\frac{4}{7}} \|f_K\|_{L^{9/2}}^{\frac{3}{7}} + \|g_K\|_{L^1}^{2/3} \|g_K\|_{L^\infty}^{1/3} \right) y(t) \\ &\leq C \left(\int_{\mathbb{R}^3} \left| \nabla f_K^{3/4} \right|^2 \right)^{\frac{2}{7}} y + CK^{1/3} y \\ &\leq \delta F(t) + C\delta^{-\frac{2}{5}} y^{\frac{7}{5}} + CK^{1/3} y.\end{aligned}$$

Lastly, we bound the lowest order term using Hölder's inequality and Chebychev's inequality

for the measure $\mathrm{d}\nu = \langle v\rangle^m \,\mathrm{d}v$:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle v\rangle^m f_K^{1/2}\,\mathrm{d}v \\
&\quad= \int_{\{f\geq 2K\}} f_K^{1/2}\,\mathrm{d}\nu + \int_{\{K\leq f<2K\}} f_K^{1/2}\,\mathrm{d}\nu \\
&\quad\leq \nu\left(\{f_K \geq K\}\right)^{2/3}\left(\int_{\mathbb{R}^3} f_K^{3/2}\,\mathrm{d}\nu\right)^{1/3} + \nu\left(\{g_{2K}\geq K\}\right)^{2/3}\left(\int_{\mathbb{R}^3} g_K^{3/2}\,\mathrm{d}\nu\right)^{1/3} \\
&\quad\leq K^{-1}\int_{\mathbb{R}^3}\langle v\rangle^m f_K^{3/2}\,\mathrm{d}v + K^{-1}\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v = K^{-1}(y+z).
\end{aligned}$$

Picking δ sufficiently small depending only on c_0 and m and combining our estimates, we have shown that y satisfies the differential inequality

$$\frac{\mathrm{d}y}{\mathrm{d}t} + F(t) \lesssim_{m,c_0} F(t)y^{2/3} + (1+K)y + y^{7/5} + Kz,$$

where the implicit constant C depends only on m and c_0. □

Next, we show that the bulk portion of f evolves according to a relatively simple integral inequality. More precisely, in the following lemma, we show that $z(t)$ solves a *constant coefficient, linear equation* with a forcing term, which is nonlinear in $y(t)$. Nonetheless, Grönwall's lemma enables us to bound $z(t)$ in terms of $y(t)$ and reduce (9.20) to a scalar, albeit nonlinear, integral inequality.

Lemma 9.5. *Under the same notations and hypotheses as Lemma 9.4, there is a constant $C>0$ depending only on m and c_0 such that*

$$\begin{aligned}
z(t) + \int_0^t G(s)\,\mathrm{d}s \leq& z(0) + 2y(0) + C\,(K+1)\int_0^t z(s)\,\mathrm{d}s + CK^{3/2}\int_0^t y(s)^{\beta_1}\,\mathrm{d}s \\
&+ C\int_0^t \left(F^{4/7}(s) + K^{1/3} + K\right) y(s)\,\mathrm{d}s,
\end{aligned} \tag{9.21}$$

where $\beta_1 \geq 1$ is explicitly computed and depends only on m. For $m = 9/2$, $\beta_1 = 1$.

Proof. Using that $\partial_t g_{2K} = \chi_{\{f<2K\}}\partial_t f = \partial_t f - \partial_t f_{2K}$ and $g_{2K} = 2K$ on the support of f_{2K}, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= \frac{3}{2}\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{1/2}\partial_t g_{2K}\,\mathrm{d}v \\
&= \frac{3}{2}\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{1/2}\partial_t f\,\mathrm{d}v - \frac{3(2K)^{1/2}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^m f_{2K}\,\mathrm{d}v.
\end{aligned} \tag{9.22}$$

Our main task will be to use that f solves (1.41) to bound the first term on the right hand side of (9.22), which is the result of taking $\langle v\rangle^m g_{2K}^{1/2}$ as a test function in the weak

formulation of (1.41): we use $\nabla g_{2K} = \chi_{\{f<2K\}}\nabla f$ and find

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2}\partial_t f \\
&\quad = -\int_{\mathbb{R}^3} \nabla\left(\langle v\rangle^m g_{2K}^{1/2}\right)\cdot\left(A[f]\nabla f - \nabla a[f] f\right)\,\mathrm{d}v \\
&\quad = -\int_{\mathbb{R}^3} \langle v\rangle^m \nabla g_{2K}^{1/2}\cdot A[f]\nabla f\,\mathrm{d}v - \int_{\mathbb{R}^3} g_{2K}^{1/2}\nabla\langle v\rangle^m \cdot A[f]\nabla f\,\mathrm{d}v \\
&\qquad + \int_{\mathbb{R}^3} \langle v\rangle^m \nabla g_{2K}^{1/2}\cdot \nabla a[f] f\,\mathrm{d}v + \int_{\mathbb{R}^3} g_{2K}^{1/2}\nabla\langle v\rangle^m \cdot \nabla a[f] f\,\mathrm{d}v \\
&\quad = \sum_{j=1}^4 I_j.
\end{aligned} \tag{9.23}$$

The first term, I_1, is our usual coercive term, which we rewrite as

$$I_1 = \frac{-8}{9}\int_{\mathbb{R}^3}\langle v\rangle^m A[f]\nabla g_{2K}^{3/4}\cdot\nabla g_{2K}^{3/4}\,\mathrm{d}v.$$

We expand the second term using the identity $f = f_{2K} + g_{2K}$ and that f_{2K} and ∇g_{2K} have disjoint support

$$\begin{aligned}
I_2 &= -\frac{4}{3}\int_{\mathbb{R}^3} g_{2K}^{3/4}\nabla\langle v\rangle^m\cdot A[f]\nabla g_{2K}^{3/4}\,\mathrm{d}v - \int_{\mathbb{R}^3} g_{2K}^{1/2}\nabla\langle v\rangle^m\cdot A[f]\nabla f_{2K}\,\mathrm{d}v \\
&= I_{2,1} - \int_{\mathbb{R}^3} g_{2K}^{1/2}\nabla\langle v\rangle^m\cdot\left(\nabla\cdot(A[f]f_{2K}) - \nabla a[f] f_{2K}\right)\,\mathrm{d}v \\
&= I_{2,1} + \int_{\mathbb{R}^3} g_{2K}^{1/2} f_{2K} A[f] : \nabla^2\langle v\rangle^m\,\mathrm{d}v + \int_{\mathbb{R}^3} g_{2K}^{1/2} f_{2K}\nabla\langle v\rangle^m\cdot\nabla a[f]\,\mathrm{d}v \\
&= I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned} \tag{9.24}$$

We bound $I_{2,1}$ using Young's inequality: for any $\delta > 0$,

$$\begin{aligned}
|I_{2,1}| &\le \delta\int_{\mathbb{R}^3}\langle v\rangle^m A[f]\nabla g_{2K}^{3/4}\cdot g_{2K}^{3/4}\,\mathrm{d}v + C\delta^{-1}\int_{\mathbb{R}^3} g_{2K}^{3/2} A[f]\nabla\langle v\rangle^m\cdot\nabla\langle v\rangle^m\,\mathrm{d}v \\
&\le \delta\int_{\mathbb{R}^3}\langle v\rangle^m A[f]\nabla g_{2K}^{3/4}\cdot g_{2K}^{3/4}\,\mathrm{d}v + C\delta^{-1}\int_{\mathbb{R}^3}|A[f]|\langle v\rangle^{m-2} g_{2K}^{3/2}\,\mathrm{d}v
\end{aligned}$$

We bound $I_{2,2}$ and $I_{2,3}$ using Hölder's inequality and Chebychev's inequality as in (9.12) (with $l = 2K$, $k = K$, $\alpha_1 = 1$, and $\alpha_2 = 3/2$):

$$\begin{aligned}
|I_{2,2}| + |I_{2,3}| &\le CK^{1/2}\left(\|A[f]\|_{L^\infty} + \|\nabla a[f]\|_{L^\infty}\right)\int_{\mathbb{R}^3}\langle v\rangle^m f_{2K}\,\mathrm{d}v \\
&\le C\left(\|A[f]\|_{L^\infty} + \|\nabla a[f]\|_{L^\infty}\right)\int_{\mathbb{R}^3}\langle v\rangle^m f_K^{3/2}\,\mathrm{d}v.
\end{aligned}$$

Next, we further decompose the terms I_3 and I_4. Integrating by parts and using that

$f < 2K$ on $\{f < 2K\}$, so that $g_{2K} f_{2K} = 2K f_{2K}$ on the support of f_{2K}, we find

$$\begin{aligned}
I_3 + I_4 &= \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2} f^2 \,\mathrm{d}v - \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2} \nabla a[f] \cdot \nabla f \,\mathrm{d}v \\
&= \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2} f^2 \,\mathrm{d}v - \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2} \nabla a[f] \cdot \nabla g_{2K} \,\mathrm{d}v \\
&\quad - \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2} \nabla a[f] \cdot \nabla f_{2K} \,\mathrm{d}v \\
&= \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2} \left(g_{2K} + f_{2K}\right) f \,\mathrm{d}v - \frac{2}{3} \int_{\mathbb{R}^3} \langle v\rangle^m f g_{2K}^{3/2} \,\mathrm{d}v \\
&\quad + \frac{2}{3} \int_{\mathbb{R}^3} g_{2K}^{3/2} \nabla \langle v\rangle^m \cdot \nabla a[f] \,\mathrm{d}v + \int_{\mathbb{R}^3} \langle v\rangle^m f_{2K} \nabla g_{2K} \cdot \nabla a[f] \,\mathrm{d}v \\
&\quad - \int_{\mathbb{R}^3} \langle v\rangle^m f f_{2K} g_{2K}^{1/2} + \int_{\mathbb{R}^3} g_{2K}^{1/2} f_{2K} \nabla \langle v\rangle^m \cdot \nabla a[f] \,\mathrm{d}v \\
&= \frac{1}{3} \int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{3/2} f \,\mathrm{d}v + \frac{2}{3} \int_{\mathbb{R}^3} g_{2K}^{3/2} \nabla \langle v\rangle^m \cdot \nabla a[f] \,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^3} g_{2K}^{1/2} f_{2K} \nabla \langle v\rangle^m \cdot \nabla a[f] \,\mathrm{d}v \\
&= I_{3,1} + I_{3,2} + I_{3,3},
\end{aligned} \tag{9.25}$$

Note that $I_{3,3} = I_{2,3}$ has already been bounded. Now, we bound $I_{3,1}$ using Hölder's inequality and Chebychev's inequality as in (9.12) (with $l = 2K$, $k = K$, $\alpha_1 = 1$, and $\alpha_2 = 3/2$):

$$\begin{aligned}
\int_{\mathbb{R}^3} \langle v\rangle^m f_{2K} g_K^{3/2} \,\mathrm{d}v &\le (2K)^{3/2} \int_{\mathbb{R}^3} \langle v\rangle^m f_{2K} \,\mathrm{d}v \le (2K)^{3/2} K^{-1/2} \int_{\mathbb{R}^3} \langle v\rangle^m f_K^{3/2} \,\mathrm{d}v \\
&\le CK \int_{\mathbb{R}^3} \langle v\rangle^m f_K^{3/2} \,\mathrm{d}v.
\end{aligned}$$

We bound $I_{3,2}$ by using $\nabla a[f] = \nabla \cdot A[f]$, integrating by parts, and applying Young's inequality to obtain for any $\delta > 0$,

$$\begin{aligned}
I_{3,2} &= \frac{2}{3} \int_{\mathbb{R}^3} \nabla \langle v\rangle^m \cdot \left[\nabla \cdot \left(A[f] g_{2K}^{3/2}\right) - A[f] \nabla g_{2K}^{3/2}\right] \,\mathrm{d}v \\
&= -\frac{2}{3} \int_{\mathbb{R}^3} g_{2K}^{3/2} A[f] : \nabla^2 \langle v\rangle^m \,\mathrm{d}v - \frac{8}{9} \int_{\mathbb{R}^3} g_{2K}^{3/4} \nabla \langle v\rangle^m \cdot A[f] \nabla g_{2K}^{3/4} \,\mathrm{d}v \\
&\le \delta \int_{\mathbb{R}^3} \langle v\rangle^m A[f] \nabla g_{2K}^{3/4} \cdot \nabla g_{2K}^{3/4} \,\mathrm{d}v + C(\delta^{-1} + 1) \int_{\mathbb{R}^3} |A[f]| \langle v\rangle^{m-2} g_{2K}^{3/2} \,\mathrm{d}v.
\end{aligned}$$

Collecting these estimates, we have shown that upon choosing δ sufficiently small

$$\begin{aligned}\int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{1/2}\partial_t f + \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla g_{2K}^{3/4}\right|^2 \mathrm{d}v \\ \leq CK\int_{\mathbb{R}^3} \langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v + C\int_{\mathbb{R}^3} |A[f]|\,\langle v\rangle^{m-2} g_{2K}^{3/2}\,\mathrm{d}v \\ + C\left(K + \|A[f]\|_{L^\infty} + \|\nabla a[f]\|_{L^\infty}\right)\int_{\mathbb{R}^3}\langle v\rangle^m f_K^{3/2}\,\mathrm{d}v.\end{aligned}$$

Set $q = \min\left(\frac{4}{3}, \frac{m}{m-2}\right)$ so that $\frac{q}{q-1} \geq 4$. Then, by linearity of A and Hölder's inequality, we find

$$\begin{aligned}&\int_{\mathbb{R}^3} |A[f]|\,\langle v\rangle^{m-2} g_{2K}^{3/2}\,\mathrm{d}v \\ &\quad= \int_{\mathbb{R}^3} |A[f_K]|\,\langle v\rangle^{m-2} g_{2K}^{3/2}\,\mathrm{d}v + \int_{\mathbb{R}^3} |A[g_K]|\,\langle v\rangle^{m-2} g_{2K}^{3/2}\,\mathrm{d}v \\ &\quad\leq \|A[f_K]\|_{L^{\frac{q}{q-1}}}\left(\int_{\mathbb{R}^3}\langle v\rangle^{q(m-2)} g_{2K}^{3q/2}\,\mathrm{d}v\right)^{1/q} + \|A[g_K]\|_{L^\infty}\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v.\end{aligned}$$

Then we use Hardy-Littlewood-Sobolev's inequality, see [145], (recalling that the kernel of A belongs to the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$), to bound

$$\|A[f_K]\|_{L^{\frac{q}{q-1}}} \leq \|f_K\|_{L^{q'}},$$

with

$$\frac{1}{q'} = 1 + \frac{q-1}{q} - \frac{1}{3} = \frac{5q-3}{3q}.$$

Combined with the fact that $g_{2K} \leq 2K$, an interpolation of Lebesgue spaces, and Young's inequality we arrive at

$$\begin{aligned}&\int_{\mathbb{R}^3} |A[f]|\,\langle v\rangle^{m-2} g_{2K}^{3/2}\,\mathrm{d}v \\ &\quad\leq CK^{\frac{3(q-1)}{2q}}\|f_K\|_{L^{\frac{3q}{5q-3}}}\left(\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v\right)^{1/q} + C\|g_K\|_{L^1}^{2/3}\|g_K\|_{L^\infty}^{1/3}\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v \\ &\quad\leq CK^{\frac{3(q-1)}{2q}}\|f_K\|_{L^1}^{\frac{3q-3}{q}}\|f_K\|_{L^{3/2}}^{\frac{3-2q}{q}}\left(\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v\right)^{1/q} + CK^{1/3}\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v \\ &\quad\leq CK^{\frac{3}{2}}\|f_K\|_{L^{3/2}}^{\frac{3-2q}{q-1}} + C(1+K)\int_{\mathbb{R}^3}\langle v\rangle^m g_{2K}^{3/2}\,\mathrm{d}v,\end{aligned}$$

or in terms of F and z, we may equivalently write

$$\begin{aligned}&\int_{\mathbb{R}^3} g_{2K}^{1/2}\partial_t f\,\mathrm{d}v + G(t) \\ &\quad\leq C(1+K)\,z(t) + C\left(\|A[f]\|_{L^\infty} + \|\nabla a[f]\|_{L^\infty} + K\right)y(t) + CK^{3/2}y(t)^{\beta_1},\end{aligned} \tag{9.26}$$

where $\beta_1 = \frac{3-2q}{q-1}$ for $q = \min\left(\frac{4}{3}, \frac{m}{m-2}\right)$. We now estimate $\|A[f]\|_{L^\infty}$ with Lemma 8.5, $\|(f-2K)_+\|_{L^1} \leq 1$, and Lemma 8.7:

$$\begin{aligned}\|A[f]\|_{L^\infty} &= \|A[f_K]\|_{L^\infty} + \|A[g_K]\|_{L^\infty} \\ &\leq C\,\|f_K\|_{L^1}^{4/7}\,\|f_K\|_{L^{9/2}}^{3/7} + C\,\|g_K\|_{L^1}^{2/3}\,\|g_K\|_{L^\infty}^{1/3} \\ &\leq C\left(\int_{\mathbb{R}^3}\left|\nabla f_K^{3/4}\right|^2 \mathrm{d}v\right)^{2/7} + CK^{1/3}.\end{aligned} \tag{9.27}$$

Similarly, we bound $\|\nabla a[f]\|_{L^\infty}$ as

$$\begin{aligned}\|\nabla a[f]\|_{L^\infty} &\leq \|\nabla a[f_K]\|_{L^\infty} + \|\nabla a[g_K]\|_{L^\infty} \\ &\leq C\,\|f_K\|_{L^1}^{1/7}\,\|f_K\|_{L^{9/2}}^{6/7} + C\,\|g_K\|_{L^1}^{1/3}\,\|g_K\|_{L^\infty}^{2/3} \\ &\leq C\left(\int_{\mathbb{R}^3}\left|\nabla f_K^{3/4}\right|^2 \mathrm{d}v\right)^{4/7} + CK^{2/3}.\end{aligned} \tag{9.28}$$

Combining (9.26) with the coefficient bounds (9.27) and (9.28), we obtain the following differential inequality:

$$\int_{\mathbb{R}^3} g_{2K}^{1/2}\partial_t f\,\mathrm{d}v + G(t) \lesssim_{m,c_0} (1+K)\,z(t) + \left(F^{4/7}(t) + K^{1/3} + K\right)y(t) + K^{3/2}y(t)^{\beta_1}.$$

Using the identity (9.22), we have shown

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}z(t) + (2K)^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^m f_{2K}\,\mathrm{d}v + G(t) \\ &\qquad\leq C\,(1+K)\,z(t) + C\left(F^{4/7}(t) + K^{1/3} + K\right)y(t) + CK^{3/2}y(t)^{\beta_1}.\end{aligned}$$

Integrating in time, we find the integral inequality:

$$\begin{aligned}z(t) + \int_0^t G(s)\,\mathrm{d}s \leq z(0) + 2K^{1/2}\int_{\mathbb{R}^3}\langle v\rangle^m f_{2K}(0)\,\mathrm{d}v + C\,(1+K)\int_0^t z(s)\,\mathrm{d}s \\ + C\int_0^t \left(F^{4/7}(s) + K^{1/3} + K\right)y(s)\,\mathrm{d}s + CK^{3/2}\int_0^t y(s)^{\beta_1}\,\mathrm{d}s.\end{aligned}$$

Finally, using Hölder's inequality and Chebychev's inequality as in (9.12) (with $l = 2K$, $k = K$, $\alpha_1 = 1$, and $\alpha_2 = 3/2$):

$$K^{1/2}\int_{\mathbb{R}^3}\langle v\rangle^m f_{2K}(0)\,\mathrm{d}v \leq \int_{\mathbb{R}^3}\langle v\rangle^m f_K^{3/2}(0)\,\mathrm{d}v = y(0),$$

which concludes the proof of (9.21). □

From the previous two lemmas, we obtain a system of differential inequalities for the evolution of $y(t)$ and $z(t)$. We now show that for a quantifiable time, the system propagates

smallness of $y(t)$ and boundedness of z, the bulk of the $L^{3/2}$ norm of f.

Proof of Proposition 9.2

Fix initial data f_{in}, a weight m, and bound H. Recall from Lemma 8.4 that there is a $c_0 = c_0(H)$ such that for any smooth solution f to (1.41) with initial datum f_{in}, $c_0 \langle v \rangle^{-3} \leq A[f]$. For $K > 0$ arbitrary and $f : [0, T^*) \times \mathbb{R}^3 \to R^+$ the unique Schwartz class solution to (1.41) with data f_{in} define $y(t)$, $F(t)$, $z(t)$, and $G(t)$ via (9.18) and (9.19). From Lemma 9.4, for any $K > 0$, there is a constant $\alpha = \alpha(m, c_0) > 0$ such that y solves

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{1}{2}F(t) \lesssim_{c_0,m} -F(t)\left(1 - \alpha y^{2/3}\right) + (1+K)y(t) + y^{7/5}(t) + Kz(t). \tag{9.29}$$

We now pick $\delta_1 = \delta_1(m, H)$ as the solution to

$$1 - \alpha\left(2\delta_1\right)^{2/3} = 0 \qquad \text{or equivalently} \qquad \delta_1 = \frac{1}{2\alpha^{3/2}}.$$

Finally, fix any $\delta < \delta_1$ and $K > 0$ such that

$$y(0) = \int_{\mathbb{R}^3} \langle v \rangle^{9/2} \left(f_{in} - K\right)_+^{3/2} \mathrm{d}v = \delta,$$

which completely fixes y, z, F, and G. From here on, inessential constants are allowed to depend on m, H, and also K. Now, integrating (9.21), and noting that $z(0) \leq \|f_{in}\|_{L^{3/2}_m} = H$, z solves the integral inequality,

$$z(t) + \int_0^t G(s)\,\mathrm{d}s \leq 2H + C\int_0^t F^{4/7}(s)y(s) + y(s) + y(s)^{\beta_1}\,\mathrm{d}s + C\int_0^t z(s)\,\mathrm{d}s,$$

where $\beta_1 = \beta_1(m) \geq 1$ is defined in Lemma 9.5. Since F and y are nonnegative, the first integral on the right hand side is increasing in t and the integral form of Grönwall's inequality implies that for each $0 \leq t \leq 1$,

$$z(t) \leq C + C\int_0^t F^{4/7}(s)y(s) + y(s) + y(s)^{\beta_1}\,\mathrm{d}s. \tag{9.30}$$

Substituting this explicit inequality for z into the right hand side of (9.29), we obtain

$$\begin{aligned}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{1}{2}F(t) \leq -CF(t)\left(1 - \alpha y^{2/3}\right) + Cy(t) + Cy^{7/5}(t) \\ + C\left(\int_0^t F^{4/7}(s)y(s) + y(s) + y(s)^{\beta_1}\,\mathrm{d}s + C\right).\end{aligned}$$

Integrating in time, and using monotonicity to remove the double integrals in time, for

each $0 \le t \le 1$,

$$\begin{aligned} y(t) + \frac{1}{2}\int_0^t F(s)\,\mathrm{d}s &\le \delta - C\int_0^t F(s)\left(1-\alpha y^{2/3}(s)\right)\,\mathrm{d}s + C\int_0^t y(s) + y^{7/5}(s)\,\mathrm{d}s \\ &\quad + C\int_0^t \int_0^s F^{4/7}(\tau)y(\tau) + y(\tau) + y(\tau)^{\beta_1}\,\mathrm{d}\tau\,\mathrm{d}s + Ct \\ &\le \delta - C\int_0^t F(s)\left(1-\alpha y^{2/3}(s)\right)\,\mathrm{d}s + C\int_0^t y(s) + y^{7/5}(s)\,\mathrm{d}s \\ &\quad + C\int_0^t F^{4/7}(s)y(s) + y(s) + y(s)^{\beta_1}\,\mathrm{d}s + Ct. \end{aligned}$$

Using Young's inequality, we arrive at the final form of our differential inequality for y:

$$\begin{aligned} y(t) + \frac{1}{4}\int_0^t F(s)\,\mathrm{d}s \le& \delta + C_1 t - C_1\int_0^t F(s)\left(1-\alpha y^{2/3}(s)\right)\,\mathrm{d}s \\ &+ C_1\int_0^t y(s) + y^{\max(\beta_1,7/3)}(s)\,\mathrm{d}s, \end{aligned} \tag{9.31}$$

where $C_1 > 0$ is now a fixed constant. We then fix $T = T(m,H,K)$ as the minimum of 1 and the unique solution of

$$2\delta = \delta + C_1 T\left(1 + 2\delta + (2\delta)^{\max(\beta_1,7/3)}\right),$$

We claim that $y(t) \le 2\delta$ for each $0 < t < \min(T,T^*)$. By assumption $y(0) = \delta < 2\delta$. Thus, by continuity of y, either there holds $y(t) < 2\delta$ for any $t \in (0,\min(T,T^*))$ or there is $t_0 \in (0,\min(T,T^*))$ such that $y(t_0) = 2\delta$.

Suppose for the sake of contradiction that $t_0 < \min(T,T^*)$. Then, for any $t \in [0,t_0]$ there holds $y(t) \le 2\delta < 2\delta_1$. By the definition of δ_1, we conclude

$$-C\int_0^t F(s)\left(1-\alpha y^{2/3}(s)\right)\,\mathrm{d}s \le 0, \qquad \text{for each } 0 \le t \le t_0.$$

Therefore, (9.31) implies that y satisfies

$$y(t) \le \delta + C_1 t + C_1\int_0^t y(s) + y^{\max(\beta_1,7/3)}(s)\,\mathrm{d}s \le \delta + C_1 t\left(1 + 2\delta + (2\delta)^{\max(\beta_1,7/3)}\right),$$

for each $0 \le t \le t_0$. Therefore, evaluating at $t = t_0$,

$$2\delta \le \delta + C_1 t_0\left(1 + 2\delta + (2\delta)^{\max(\beta_1,7/3)}\right) < \delta + C_1 T\left(1 + 2\delta + (2\delta)^{\max(\beta_1,7/3)}\right) \le 2\delta,$$

a contradiction. It follows that $t_0 \ge \min(T,T^*)$, i.e. $y(t) \le 2\delta$ for $t \in [0,\min(T,T_0))$.

From (9.31), we conclude that for $0 < t < \min(T, T^*)$, there holds

$$\begin{aligned}\frac{1}{4}\int_0^t F(s)\,\mathrm{d}s &\leq \delta + C_1 t + C_1 \int_0^t y(s) + y^{\max(\beta_1,7/3)}(s)\,\mathrm{d}s \\ &\leq \delta + C_1 T\left(1 + 2\delta + (2\delta)^{\max(\beta_1,7/3)}\right) \leq 2\delta,\end{aligned}$$

which implies (9.16). Therefore, returning to (9.30) and (9.21), we use the bounds on $y(t)$ and $F(t)$ to obtain

$$z(t) + \int_0^t G(s)\,\mathrm{d}s \leq C(K, H, m).$$

Finally, using the bounds on y and z we propagate the $L^{3/2}_{9/2}$ and using the bounds on F and G we show the creation of bounds on the energy:

$$\|f\|^{3/2}_{L^\infty(0,T;L^{3/2}_m)} \leq y(t) + z(t) \leq C(K, H, m),$$

and

$$\left\|\nabla f^{3/4}\right\|_{L^2(0,T;L^2_{m-3})} \leq \int_0^t F(s) + G(s)\,\mathrm{d}s \leq C(K, H, m).$$

9.4.2 Construction of solution

In this section, we prove Theorem 9.1 (with $p = 3/2$) using an approximation argument similar to Section 3.5.3. A similar argument will also be used in Section 9.6 to construct solutions for $p > 3/2$.

Approximation

Take $f_{in} \in L^{3/2}_{9/2}(\mathbb{R}^3)$ satisfying (1.46) and such that

$$\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_{in}^{3/2}\,\mathrm{d}v \leq H.$$

Let $\{f_{in,\varepsilon}\}_{\varepsilon>0}$ be a family of Schwartz class initial datum such that

i. for each $\varepsilon > 0$, $f_{in,\varepsilon} \in \mathcal{S}(\mathbb{R}^3)$,

ii. for each $\varepsilon > 0$, $f_{in,\varepsilon}$ is normalised through (1.46),

iii. for each $\varepsilon > 0$,
$$\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_{in,\varepsilon}^{3/2}\,\mathrm{d}v \leq 2H,$$

iv. $\{f_{in,\varepsilon}\}_{\varepsilon>0} \to f_{in}$ as $\varepsilon \to 0^+$ strongly in $L^{3/2}_{9/2} \cap L^1_2$ and pointwise almost everywhere.

Then, by Theorem 8.2 and Corollary 8.2, there is a family of local-in-time Schwartz class solutions $f_\varepsilon : [0, T_\varepsilon) \times \mathbb{R}^3 \to \mathbb{R}_+$ for (1.41) with initial data $f_{in,\varepsilon}$, existing up to time T_ε, which is either infinite, or else

$$\lim_{t \nearrow T_\varepsilon} \|f_\varepsilon(t)\|_{L^\infty} = +\infty. \tag{9.32}$$

Uniform lower bound on time

We first show that the approximations exist on a uniform time interval, i.e. the maximal time of existence T_ε is bounded below, uniformly in ε. Fix $\delta_1 = \delta_1(H)$ as in Proposition 9.2 and $\varepsilon_0 = \varepsilon_0(H)$ as in Proposition 9.1, which are uniform in ε. Then, because $\langle v \rangle^{9/2} f_{in,\varepsilon}^{3/2} \to \langle v \rangle^{9/2} f_{in}^{3/2}$ in $L^1(\mathbb{R}^3)$, the family is uniformly integrable, allowing us to choose K independent of ε so that

$$\int_{\mathbb{R}^3} \langle v \rangle^{\frac{9}{2}} (f_{in,\varepsilon} - K)_+^{\frac{3}{2}} \, \mathrm{d}v < \frac{\varepsilon_0}{6}. \tag{9.33}$$

Applying Proposition 9.2, there is a $T = T(K, H) \in (0, 1]$ independent of ε such that

$$\sup_{0<\tau<\min(T,T_\varepsilon)} \int_{\mathbb{R}^3} \langle v \rangle^{9/2} (f_\varepsilon - K)_+^{3/2} \, \mathrm{d}v + \int_0^{\min(T,T_\varepsilon)} \int_{\mathbb{R}^3} \langle v \rangle^{3/2} \left| \nabla (f_\varepsilon - K)_+^{3/4} \right|^2 \, \mathrm{d}v < \varepsilon_0.$$

By Proposition 9.1, there is $C = C(K, H) > 0$, independent of ε, such that

$$\|f_\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \left(1 + t^{-1}\right), \qquad 0 < t < \min(T, T_\varepsilon). \tag{9.34}$$

Finally, the continuation criterion (9.32) for f_ε, implies $T_\varepsilon > T$ for each $\varepsilon > 0$, providing a uniform in ε lower bound on the time of existence of f_ε. We will construct solutions on this uniform time interval $[0, T]$.

Higher regularity estimates

To construct a smooth solution f on $[0, T]$ we need strong compactness, which follows from uniform in ε higher regularity estimates in Lemma 8.14.

Passing to the limit

As a consequence of the uniform estimates in (9.34), Lemma 8.14, Proposition 9.2, the Sobolev embedding (8.12), and standard compactness results, we have a limit $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}_+$, such that $f_\varepsilon \to f$ in the following topologies:

i. strongly in $C^k((t, T) \times B_R)$ for each $k \in \mathbb{N}$, for each $R > 0$ and for $0 < t < T$;

ii. pointwise almost everywhere in $[0, T] \times \mathbb{R}^3$;

iii. strongly in $L^1([0,T]\times\mathbb{R}^3)$;[4]

iv. weak-starly in $L^\infty(0,T;L^{3/2}_{9/2}(\mathbb{R}^3))\cap L^{5/2}(0,T;L^{5/2}_{9/2})$;

v. $\nabla f_\varepsilon^{3/4}\to\nabla f^{3/4}$ weakly in $L^2(0,T;L^2_{3/2})$;

vi. and $t^{-1}f_\varepsilon\to t^{-1}f$ weak-starly in $L^\infty((0,T)\times\mathbb{R}^3)$.

In particular, the limit function f satisfies $f\geq 0$ and by Fatou's lemma $f\in L^\infty(0,T;L^1_2)$. Therefore, once we show that the limit satisfies (1.41), f will automatically satisfy the normalisation (1.46). Moreover, from iv. and v., we see that f satisfies the desired estimates:

$$\sup_{0<t<T}\|f(t)\|_{L^{3/2}_{9/2}}+\sup_{0<t<T}t\,\|f(t)\|_{L^\infty}\leq C(H,K)$$

Finally, f also becomes instantaneously smooth, that is $f\in C^\infty((t,T)\times\mathbb{R}^3)$ for each $0<t<T$.

Limit equation

It remains to show that the limit f satisfies (1.41) on $[0,T]\times\mathbb{R}^3$ with initial data f_{in} and that $f\in C([0,T];L^{3/2}_{9/2})$. First, we show that f is a classical solution to (1.41) on $(0,T)\times\mathbb{R}^3$. Because f is smooth, it suffices to show that the equation is satisfied in a weak sense. Since f_ε solves (1.41), there holds for any $\varphi\in C_c^\infty((t,T)\times\mathbb{R}^3)$,

$$\int_t^T\int_{\mathbb{R}^3}\varphi\partial_t f_\varepsilon\,\mathrm{d}v\,\mathrm{d}t=-\int_t^T\int_{\mathbb{R}^3}\nabla\varphi\cdot\left(A[f_\varepsilon]\nabla f_\varepsilon-\nabla a[f_\varepsilon]f_\varepsilon\right)\,\mathrm{d}v\,\mathrm{d}\tau. \tag{9.35}$$

The diffusion coefficient converges to the limit diffusion coefficient, since by Lemma 8.5 and Hölder's inequality, we have

$$\|A[f_\varepsilon-f]\|_{L^1(t,T;L^\infty(\mathbb{R}^3))}\leq C\,\|f_\varepsilon-f\|^{\frac{2}{3}}_{L^1(t,T;L^1(\mathbb{R}^3))}\,\|f_\varepsilon-f\|^{\frac{1}{3}}_{L^1(t,T;L^\infty(\mathbb{R}^3))},$$

which converges to zero as $\varepsilon\to 0^+$ by iv. and v. above. Similarly, for ∇a, again by Lemma 8.5 and Hölder's inequality, we have the bound

$$\|\nabla a[f_\varepsilon-f]\|_{L^1(t,T;L^\infty(\mathbb{R}^3))}\leq C\,\|f_\varepsilon-f\|^{\frac{1}{3}}_{L^1(t,T;L^1(\mathbb{R}^3))}\,\|f_\varepsilon-f\|^{\frac{2}{3}}_{L^1(t,T;L^\infty(\mathbb{R}^3))},$$

which also converges to zero as $\varepsilon\to 0^+$ by iv. and v. above. The global convergence of the coefficients together with the local compactness in C^1 given by i. implies in particular for any $\varphi\in C_c^\infty((t,T)\times\mathbb{R}^3)$

$$\int_t^T\int_{\mathbb{R}^3}\varphi\partial_t f\,\mathrm{d}v\,\mathrm{d}\tau=-\int_t^T\int_{\mathbb{R}^3}\nabla\varphi\cdot\left(A[f]\nabla f-\nabla a[f]f\right)\,\mathrm{d}v\,\mathrm{d}\tau.$$

[4]The decay of the entropy implies and uniform bound on the energy implies strong compactness in L^1.

Thus, f solves (1.41) in a distributional sense, and consequently f solves (1.41) in a classical sense on $(0,T)\times\mathbb{R}^3$.

Convergence to the initial datum

To show that the initial datum is obtained in a suitable strong sense, we integrate by parts in (9.35) to obtain that f_ε satisfies for any $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^3)$

$$\int_0^T\int_{\mathbb{R}^3}\varphi\partial_t f_\varepsilon\,\mathrm{d}v\,\mathrm{d}t=\int_0^T\int_{\mathbb{R}^3}\nabla^2\varphi:A[f_\varepsilon]f_\varepsilon+2\nabla\varphi\cdot\nabla a[f_\varepsilon]f_\varepsilon\,\mathrm{d}v\,\mathrm{d}t.$$

Then we bound with Hölder's inequality, $f_\varepsilon\in L^\infty(0,T;L^{3/2}(\mathbb{R}^3))$, Lemma 8.5, and the smoothing estimate (9.34)

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}\nabla^2\varphi:A[f_\varepsilon]f_\varepsilon\,\mathrm{d}v\,\mathrm{d}t\\
&\quad\le\|f_\varepsilon\|_{L^\infty(0,T;L^{3/2}(\mathbb{R}^3))}\int_0^T\left(\int_{\mathbb{R}^3}\left|\nabla^2\varphi\right|^3\,\mathrm{d}v\right)^{2/3}\|A[f_\varepsilon](t)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}t\\
&\quad\le C(H,K)\left(\int_0^T\int_{\mathbb{R}^3}\left|\nabla^2\varphi\right|^3\,\mathrm{d}v\,\mathrm{d}t\right)^{1/3}\left(\int_0^T\|f_\varepsilon(t)\|_{L^1(\mathbb{R}^3)}^{2/3}\|f_\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)}^{1/3}\,\mathrm{d}t\right)^{\frac{2}{3}}\\
&\quad\le C(H,K)\left\|\nabla^2\varphi\right\|_{L^3((0,T)\times\mathbb{R}^3)}\left(\int_0^T t^{-\frac{1}{2}}\,\mathrm{d}t\right)^{\frac{2}{3}}\\
&\quad\le C(H,K)T^{\frac{1}{3}}\|\varphi\|_{L^3(0,T;W^{2,3}(\mathbb{R}^3))}.
\end{aligned}$$

Similarly, we bound with Hölder's inequality, Lemma 8.5 and the smoothing estimate (9.34)

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}\nabla\varphi\cdot\nabla a[f_\varepsilon]f_\varepsilon\,\mathrm{d}v\,\mathrm{d}t\\
&\qquad\le\|f_\varepsilon\|_{L^\infty(0,T;L^{3/2}(\mathbb{R}^3))}\int_0^T\|\nabla\varphi(t)\|_{L^3(\mathbb{R}^3)}\|\nabla a[f_\varepsilon](t)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}t\\
&\qquad\le C(H,K)\|\varphi\|_{L^3(0,T;W^{2,3}(\mathbb{R}^3))}\int_0^T\|f_\varepsilon(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{1}{2}}\|f_\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}}\,\mathrm{d}t\\
&\qquad\le C(H,K)\|\varphi\|_{L^3(0,T;W^{2,3}(\mathbb{R}^3))}\|f_\varepsilon\|_{L^\infty(0,T;L^{3/2}(\mathbb{R}^3))}^{\frac{1}{2}}\left(\int_0^T t^{-\frac{3}{4}}\,\mathrm{d}t\right)^{\frac{2}{3}}\\
&\qquad\le C(H,K)T^{\frac{1}{6}}\|\varphi\|_{L^3(0,T;W^{2,3}(\mathbb{R}^3))}.
\end{aligned}$$

By a simple density and duality argument, we have shown that

$$\|\partial_t f_\varepsilon\|_{L^{3/2}(0,T;W^{-2,3/2}(\mathbb{R}^3))}\le C(H,K).$$

Combined with $\|f_\varepsilon\|_{L^\infty(0,T;L^{3/2})}\le C(H,K)$, the Aubin-Lions Lemma gives strong com-

pactness of f_ε in $C(0,T;W^{-1,3/2})$. Thus, up to extracting a subsequence, f_ε converges to f in $C(0,T;W^{-1,3/2}(\mathbb{R}^3))$. Together with the fact that $f_\varepsilon(0)=f_{in,\varepsilon}$, for each $\varepsilon>0$, f is continuous up to time 0 and $f(0)=f_{in}$ in the sense of distributions. But then, using that $f\in L^\infty(0,T;L^{3/2}_{9/2}(\mathbb{R}^3))$, we conclude that f is weakly continuous in time with values in $L^{3/2}_{9/2}$ and $f(0)=f_{in}$ almost everywhere. Moreover, by lower semicontinuity of the norm in the weak topology, we have

$$\|f_{in}\|_{L^{3/2}_{9/2}(\mathbb{R}^3)} \le \liminf_{t\searrow 0^+} \|f(t)\|_{L^{3/2}_{9/2}(\mathbb{R}^3)}\,. \tag{9.36}$$

To conclude that $\lim_{t\to 0^+} f(t)=f_{in}$ strongly in $L^{3/2}_{9/2}$, it remains to show that

$$\limsup_{t\searrow 0^+} \|f(t)\|_{L^{3/2}_{9/2}(\mathbb{R}^3)} \le \|f_{in}\|_{L^{3/2}_{9/2}(\mathbb{R}^3)}\,. \tag{9.37}$$

To this end, we recall the energy estimate from Lemma 9.4 (with $\ell=0$) implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{3/2}\,\mathrm{d}v + \int_{\mathbb{R}^3}\langle v\rangle^{3/2}\left|\nabla f_\varepsilon^{3/4}\right|^2\,\mathrm{d}v \\ \le C\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{5/2}\,\mathrm{d}v + C\,\|A[f_\varepsilon]\|_{L^\infty}\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{3/2}\,\mathrm{d}v.\end{aligned}$$

We now bound the first term using Lemma 8.7:

$$\begin{aligned}&\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{5/2}\,\mathrm{d}v\\ &\quad\le C\int_{\mathbb{R}^3}\langle v\rangle^{9/2}(f_\varepsilon-K)_+^{5/2}\,\mathrm{d}v + CK\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{3/2}\,\mathrm{d}v\\ &\quad\le C\left(\int_{\mathbb{R}^3}\langle v\rangle^{9/2}(f_\varepsilon-K)_+^{3/2}\,\mathrm{d}v\right)^{2/3}\left(\int_{\mathbb{R}^3}\langle v\rangle^{3/2}\left|\nabla(f_\varepsilon-K)_+^{3/4}\right|^2\,\mathrm{d}v\right) + C(K,H).\end{aligned}$$

As in (9.33), relying on Proposition 9.2, we pick K uniform in ε so that

$$\left(\int_{\mathbb{R}^3}\langle v\rangle^{9/2}(f_\varepsilon(t)-K)_+^{3/2}\,\mathrm{d}v\right)^{2/3} << 1 \qquad \text{for } 0<t<T(K,H).$$

combined with Lemma 8.5 and (9.34), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{3/2}\,\mathrm{d}v \le C(1+t^{-1/3}), \quad \text{for } 0<t<T(K,H), \tag{9.38}$$

where the constant is independent of ε. Integrating in time, we find

$$\sup_{0<s<t}\left(\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f_\varepsilon^{3/2}(s)\,\mathrm{d}v\right) \le \|f_{in,\varepsilon}\|_{L^{3/2}_{9/2}} + C(H,K)t^{2/3}.$$

By $f_\varepsilon \overset{*}{\rightharpoonup} f$ in $L^\infty(0,T;L^{3/2}_{9/2})$, which implies the same convergence on smaller time domains, and lower semi-continuity of the norm, with respect to the weak star topology,

$$\sup_{0<s<t}\left(\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f^{3/2}(s)\,\mathrm{d}v\right) \leq \|f_{in}\|_{L^{3/2}_{9/2}} + C(H,K)t^{2/3}.$$

Since f is weakly continuous taking values in $L^{3/2}_{9/2}$, we conclude by lower semi-continuity again that for every $0<t<T$,

$$\left(\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f^{3/2}(t)\,\mathrm{d}v\right) \leq \|f_{in}\|_{L^{3/2}_{9/2}} + C(H,K)t^{2/3}.$$

Taking $t\to 0^+$, we conclude that $f\to f_{in}$ in $L^{3/2}_{9/2}$ using (9.36). It remains now to show strong continuity of f at any later time $t>0$.

Continuity

To show f is continuous in the strong topology on $L^{3/2}_{9/2}$, we show that the function

$$t\mapsto \int_{\mathbb{R}^3}\langle v\rangle^{9/2} f(t)^{3/2}\,\mathrm{d}v$$

is absolutely continuous as a function of time. To rigorously justify the following computations, we work with a truncation of the norm. For any $R>0$, take $\varphi_R\in C^\infty_c(\mathbb{R}^3)$ a cutoff function that satisfies:

$$0\leq\varphi_R\leq 1,\quad \varphi_R(v)=\begin{cases}1 & \text{if } v\in B_R(0)\\ 0 & \text{if } v\notin B_{2R}(0),\end{cases}\qquad |\nabla\varphi_R|\leq\frac{C}{R},\quad\text{and}\quad \left|\nabla^2\varphi_R\right|\leq\frac{C}{R^2},$$

Because $f\in C^\infty$ and f, $A[f]$, $\nabla a[f]\in L^\infty([s,T]\times\mathbb{R}^3)$ for each $s>0$, we can use $\varphi_R^2 f^{1/2}\langle v\rangle^{9/2}$ as a test function for (1.41) and integrate in time for $\tau\in[s,t]$. For positive times, integrating by parts several times and using $\nabla\cdot A[f]=\nabla a[f]$, the same computations

as in Lemma 9.2 imply that for each $0 < s < t < T$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi_R^2 \langle v\rangle^{9/2} f^{3/2}(t)\,\mathrm{d}v - \int_{\mathbb{R}^3} \varphi_R^2 \langle v\rangle^{9/2} f^{3/2}(s)\,\mathrm{d}v \\
&\quad = -\frac{3}{2}\int_s^t \int_{\mathbb{R}^3} \nabla\left(\varphi_R^2 \langle v\rangle^{9/2} f^{1/2}\right)\cdot\left[A[f]\nabla f - \nabla a[f] f\right]\,\mathrm{d}v\,\mathrm{d}\tau \\
&\quad = -\frac{4}{3}\int_s^t \int_{\mathbb{R}^3} \varphi_R^2 \langle v\rangle^{9/2} A[f]\nabla f^{3/4}\cdot\nabla f^{3/4}\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^3} \varphi_R^2 \langle v\rangle^{9/2} f^{5/2}\,\mathrm{d}v \\
&\qquad - \int_{\mathbb{R}^3}\left(\nabla^2\left(\varphi_R^2\langle v\rangle^{9/2}\right) : A[f]\right) f^{3/2}\,\mathrm{d}v \\
&\qquad - 4\int_{\mathbb{R}^3} f^{3/4}\nabla\left(\varphi_R^2\langle v\rangle^{9/2}\right)\cdot A[f]\nabla f^{3/4}\,\mathrm{d}v \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{9.39}$$

Because $f \in L^{5/2}(0,T;L^{5/2}_{9/2})$, the term I_2 is bounded uniformly in R. Similarly, $f \in L^\infty(0,T;L^{3/2}_{9/2})$ and $A[f] \in L^\infty(s,t;L^\infty)$ together imply I_3 is bounded uniformly in R. By Young's inequality, we bound I_4 as

$$\begin{aligned}
I_4 &= -4\int_{\mathbb{R}^3} f^{3/4}\left[\varphi_R\langle v\rangle^{9/2}\nabla\varphi_R + \varphi_R^2\nabla\langle v\rangle^{9/2}\right]\cdot A[f]\nabla f^{3/4}\,\mathrm{d}v \\
&\le -\frac{1}{2}I_1 + \frac{C}{R^2}\int_{\mathbb{R}^3} \|A[f]\|_{L^\infty}\langle v\rangle^{9/2} f^{3/2}\,\mathrm{d}v \\
&\qquad + C\int_{\mathbb{R}^3}\langle v\rangle^{-9/2}\varphi_R^2 f^{3/2} A[f]\nabla\langle v\rangle^{9/2}\cdot\nabla\langle v\rangle^{9/2}\,\mathrm{d}v \\
&= -\frac{1}{2}I_1 + I_5 + I_6,
\end{aligned}$$

where I_5 and I_6 are bounded uniformly in R. Thus, (9.39) and $f \in L^\infty(0,T;L^{3/2}_{9/2})$ imply I_1 is bounded uniformly in R. Therefore, using the monotone convergence theorem and the uniform in R bounds, taking the limit as $R\to\infty$ in (9.39), for every $0 < s < t < T$,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f^{3/2}(t)\,\mathrm{d}v - \int_{\mathbb{R}^3}\langle v\rangle^{9/2} f^{3/2}(s)\,\mathrm{d}v \\
&\quad = -\frac{4}{3}\int_s^t\int_{\mathbb{R}^3}\langle v\rangle^{9/2} A[f]\nabla f^{3/4}\cdot\nabla f^{3/4}\,\mathrm{d}v + \frac{1}{2}\int_s^t\int_{\mathbb{R}^3}\langle v\rangle^{9/2} f^{5/2}\,\mathrm{d}v \\
&\qquad - \int_s^t\int_{\mathbb{R}^3}\left(\nabla^2\langle v\rangle^{9/2} : A[f]\right) f^{3/2}\,\mathrm{d}v - 4\int_s^t\int_{\mathbb{R}^3} f^{3/4}\nabla\langle v\rangle^{9/2}\cdot A[f]\nabla f^{3/4}\,\mathrm{d}v.
\end{aligned}$$

It follows that $t \mapsto \int_{\mathbb{R}^3}\langle v\rangle^{9/2} f^{3/2}(t)\,\mathrm{d}v$ is the integral of an L^1 function and thus is absolutely continuous. Recalling $f \in C_w(0,T;L^{3/2}_{9/2})$, this finishes the proof of continuity and Theorem 9.1 for $p = 3/2$.

9.5 The global in time $L^{3/2}$ theory

9.5.1 Pointwise lower bounds

Here we show the solution constructed in Theorem 9.1 propagates lower bounds, which will be used to control the Fisher information. The basic idea is that the function $\psi(t,v) = a\exp(-\eta t^{2/3})\langle v\rangle^{-k}$ satisfies

$$\partial_t \psi \leq A[f] : \nabla^2 \psi + f\psi,$$

provided η is chosen suitably large, because by combining Lemma 8.5 and our a-priori estimate $\|f\|_{L^\infty} \lesssim t^{-1}$, we see $\|A[f]\|_{L^\infty} \lesssim t^{-1/3}$. Therefore, we expect that if the initial datum is above $\langle v\rangle^{-k}$, the solution f should remain above ψ. However, we are not studying a parabolic equation with bounded coefficients, meaning that the pointwise approach to the maximum principle fails. Instead, to fully justify that $f \geq \psi$, we take the Stampacchia approach here. We begin with a general weighted estimate:

Lemma 9.6. *Suppose $f : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^+$ is a smooth, rapidly decaying solution to (1.41). For any smooth function $\ell : (0,T) \to \mathbb{R}^+$ and for any $k > 0$, define $g(t,v) = \langle v\rangle^k f(t,v)$ and $g_\ell(t,v) = (\ell(t) - g(t,v))_+$. Then, for $n < -3$ sufficiently large, g_ℓ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \langle v\rangle^n g_\ell^{3/2}\,\mathrm{d}v \leq \left(\ell'(t) + C\,\|A[f(t)]\|_{L^\infty}\,\ell(t)\right)\left(\int_{\mathbb{R}^3}\langle v\rangle^{n-2} g_\ell^{1/2}\,\mathrm{d}v\right),$$

where $C > 0$ depends only on n and k.

Remark 9.2. Although the assumption that f is rapidly decaying implies that g is rapidly decaying, g_ℓ is *NOT* rapidly decaying. The role of the negative weights, i.e. $\langle v\rangle^n$ with $n < -3$ is exactly to ensure that all integrals are finite. Note that in the proof we will frequently integrate by parts, and assume that the boundary terms vanish without comment, because, in each instance, one can use the rapid decay of g, ∇g_ℓ, or $\langle v\rangle^n$ for large enough n as a justification.

Proof. We begin by using $\partial_t g_\ell = (\ell'(t) - \partial_t g)\chi_{\{g<\ell\}}$ to note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{3/2}\,\mathrm{d}v = \frac{3}{2}\ell'(t)\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{1/2}\,\mathrm{d}v - \frac{3}{2}\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{1/2}\partial_t g\,\mathrm{d}v.$$

The first term is already in the correct form, while for the second we need to use that f solves (1.41). Recalling that $g = \langle v\rangle^k f$, we write down an equation for g:

$$\begin{aligned}\partial_t g &= \langle v\rangle^k \nabla\cdot(A[f]\nabla f - \nabla a[f] f)\\ &= \langle v\rangle^k \nabla\cdot\left(\langle v\rangle^{-k} A[f]\nabla g + gA[f]\nabla\langle v\rangle^{-k} - \langle v\rangle^{-k}\nabla a[f] g\right).\end{aligned}$$

Multiplying by $-\langle v\rangle^n g_\ell^{1/2}$, we find

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{1/2}\partial_t g\,\mathrm{d}v\\
&\quad=\int_{\mathbb{R}^3}\nabla\left(\langle v\rangle^{n+k} g_\ell^{1/2}\right)\cdot\left(\langle v\rangle^{-k} A[f]\nabla g+gA[f]\nabla\langle v\rangle^{-k}-\langle v\rangle^{-k}\nabla a[f]g\right)\mathrm{d}v\\
&\quad=I_1+I_2+I_3.
\end{aligned}$$

We further subdivide I_1 as

$$I_1=\int_{\mathbb{R}^3}\langle v\rangle^n\nabla g_\ell^{1/2}\cdot A[f]\nabla g\,\mathrm{d}v+\int_{\mathbb{R}^3}\langle v\rangle^{-k}\left(\nabla\langle v\rangle^{n+k}\right)g_\ell^{1/2}\cdot A[f]\nabla g\,\mathrm{d}v=I_{1,1}+I_{1,2}.$$

Using that $\nabla g_\ell=-\chi_{\{g<\ell\}}\nabla g$, the term $I_{1,1}$ is our usual coercive term:

$$I_{1,1}=-\frac{8}{9}\int_{\mathbb{R}^3}\langle v\rangle^n A[f]\nabla g_\ell^{3/4}\cdot\nabla g_\ell^{3/4}\,\mathrm{d}v.$$

Using that $\nabla\langle v\rangle^k=k\langle v\rangle^{k-2}v$, the term $I_{1,2}$ simplifies as:

$$I_{1,2}=-\frac{4(n+k)}{3n}\int_{\mathbb{R}^3}g_\ell^{3/4}\nabla\langle v\rangle^n\cdot A[f]\nabla g_\ell^{3/4}\,\mathrm{d}v.$$

For I_2, we integrate by parts and further subdivide as

$$\begin{aligned}
I_2=&-\int_{\mathbb{R}^3}\langle v\rangle^{n+k} g_\ell^{1/2}\nabla g\cdot A[f]\nabla\langle v\rangle^{-k}\,\mathrm{d}v-\int_{\mathbb{R}^3}\langle v\rangle^{n+k} g_\ell^{1/2} g\left(A[f]:\nabla^2\langle v\rangle^{-k}\right)\mathrm{d}v\\
&-\int_{\mathbb{R}^3}\langle v\rangle^{n+k} g_\ell^{1/2} g\nabla a[f]\cdot\nabla\langle v\rangle^{-k}\,\mathrm{d}v=I_{2,1}+I_{2,2}+I_{2,3}.
\end{aligned}$$

Since we will use some cancellations between I_1, I_2 and I_3, before bounding terms, we integrate by parts and further subdivide I_3 as

$$\begin{aligned}
I_3&=\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{1/2}\nabla g\cdot\nabla a[f]\,\mathrm{d}v-\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{1/2} gf\,\mathrm{d}v\\
&\quad+\int_{\mathbb{R}^3}\langle v\rangle^{n+k} g_\ell^{1/2} g\nabla a[f]\cdot\nabla\langle v\rangle^{-k}\,\mathrm{d}v\\
&=I_{3,1}+I_{3,2}+I_{3,3}.
\end{aligned}$$

Note that $I_{2,3}=-I_{3,3}$ and $I_{3,2}$, which contains the most singular terms, has a favorable sign, $I_{3,2}\le 0$. Next, we expand $I_{2,1}$ as

$$I_{2,1}=\frac{-4k}{3n}\int_{\mathbb{R}^3}g_\ell^{3/4}\nabla\langle v\rangle^n\cdot A[f]\nabla g_\ell^{3/4}\,\mathrm{d}v.$$

Then, for $I_{2,2}$, we compute explicitly

$$\begin{aligned} I_{2,2} &= -k(k+2)\int_{\mathbb{R}^3}\langle v\rangle^{n-4} g_\ell^{1/2} g\left(A[f]:v\otimes v\right)\,\mathrm{d}v + k\int_{\mathbb{R}^3}\langle v\rangle^{n-2} g g_\ell^{1/2}\mathrm{tr}\left(A[f]\right)\,\mathrm{d}v \\ &\le k\int_{\mathbb{R}^3}\langle v\rangle^{n-2}\left(\ell(t) g_\ell^{1/2} - g_\ell^{3/2}\right)\mathrm{tr}\left(A[f]\right)\,\mathrm{d}v \\ &\le k\ell(t)\int_{\mathbb{R}^3}\langle v\rangle^{n-2} g_\ell^{1/2}\mathrm{tr}\left(A[f]\right)\,\mathrm{d}v, \end{aligned}$$

where only one term does not have a favorable sign. We rewrite $I_{3,1}$ as

$$\begin{aligned} I_{3,1} &= -\frac{2}{3}\int_{\mathbb{R}^3}\langle v\rangle^n \nabla g_\ell^{3/2}\cdot\nabla a[f]\,\mathrm{d}v = \frac{2}{3}\int_{\mathbb{R}^3} g_\ell^{3/2}\nabla\langle v\rangle^n\cdot\nabla a[f]\,\mathrm{d}v - \frac{2}{3}\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{3/2} f\,\mathrm{d}v \\ &\le \frac{2}{3}\int_{\mathbb{R}^3} g_\ell^{3/4}\nabla\langle v\rangle^n\cdot\left(\nabla\cdot(A[f]g_\ell^{3/4}) - A[f]\nabla g_\ell^{3/4}\right)\,\mathrm{d}v \\ &= -\frac{4}{3}\int_{\mathbb{R}^3} g_\ell^{3/4}\nabla\langle v\rangle^n\cdot A[f]\nabla g_\ell^{3/4} - \frac{2}{3}\int_{\mathbb{R}^3} g_\ell^{3/2}\left(A[f]:\nabla^2\langle v\rangle^n\right)\,\mathrm{d}v, \end{aligned}$$

where again the most singular term has a favorable sign. Expanding the last term further and using $g\le\ell(t)$ on the support of g_ℓ

$$\begin{aligned} -\frac{2}{3}\int_{\mathbb{R}^3} g_\ell^{3/2}\left(A[f]:\nabla^2\langle v\rangle^n\right)\,\mathrm{d}v &= -\frac{2n(n-2)}{3}\int_{\mathbb{R}^3}\langle v\rangle^{n-4} g_\ell^{3/2}\left(A[f]:v\otimes v\right)\,\mathrm{d}v \\ &\quad - \frac{2n}{3}\int_{\mathbb{R}^3}\langle v\rangle^{n-2} g_\ell^{3/2}\left(\mathrm{tr}A[f]\right)\,\mathrm{d}v \\ &\le -\frac{2n}{3}\ell(t)\int_{\mathbb{R}^3}\langle v\rangle^{n-2} g_\ell^{1/2}\mathrm{tr}\left(A[f]\right)\,\mathrm{d}v, \end{aligned}$$

where the first term has a favorable sign. Therefore, summing over $I_{i,j}$, we summarize our estimates as:

$$\begin{aligned} -\int_{\mathbb{R}^3}\langle v\rangle^n g_\ell^{1/2}\partial_t g\,\mathrm{d}v + \frac{8}{9}\int_{\mathbb{R}^3}\langle v\rangle^n A[f]\nabla g_\ell^{3/4}\cdot\nabla g_\ell^{3/4}\,\mathrm{d}v \\ \le -\frac{8(n+k)}{3n}\int_{\mathbb{R}^3} g_\ell^{3/4} A[f]\nabla\langle v\rangle^n\cdot\nabla g_\ell^{3/4}\,\mathrm{d}v \\ + \frac{3k-2n}{3}\ell(t)\int_{\mathbb{R}^3}\langle v\rangle^{n-2} g_\ell^{1/2}\mathrm{tr}\left(A[f]\right)\,\mathrm{d}v. \end{aligned}$$

Now, we bound the terms on the right hand side. Using $g\le\ell(t)$ on the support of g_ℓ, for

any $\delta > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} g_\ell^{3/4} A[f] \nabla \langle v \rangle^n \cdot \nabla g_\ell^{3/4} \, \mathrm{d}v \\
&\quad \leq \delta \int_{\mathbb{R}^3} \langle v \rangle^n A[f] \nabla g_\ell^{3/4} \cdot \nabla g_\ell^{3/4} \, \mathrm{d}v + \frac{1}{4\delta} \int_{\mathbb{R}^3} g_\ell^{3/2} \langle v \rangle^{-n} A[f] \nabla \langle v \rangle^n \cdot \nabla \langle v \rangle^n \, \mathrm{d}v \\
&\quad \leq \delta \int_{\mathbb{R}^3} \langle v \rangle^n A[f] \nabla g_\ell^{3/4} \cdot \nabla g_\ell^{3/4} \, \mathrm{d}v + \frac{n^2}{4\delta} \int_{\mathbb{R}^3} \langle v \rangle^{n-4} g_\ell^{3/2} A[f] v \cdot v \, \mathrm{d}v \\
&\quad \leq \delta \int_{\mathbb{R}^3} \langle v \rangle^n A[f] \nabla g_\ell^{3/4} \cdot \nabla g_\ell^{3/4} \, \mathrm{d}v + \frac{n^2 \left\| A[f(t)] \right\|_{L^\infty} \ell(t)}{4\delta} \int_{\mathbb{R}^3} \langle v \rangle^{n-2} g_\ell^{1/2} \, \mathrm{d}v.
\end{aligned}$$

Taking δ sufficiently small depending only on n and k, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^n g_\ell^{3/2} \, \mathrm{d}v \leq \left(\ell'(t) + C(n,k) \left\| A[f(t)] \right\|_{L^\infty} \ell(t) \right) \left(\int_{\mathbb{R}^3} \langle v \rangle^{n-2} g_\ell^{1/2} \, \mathrm{d}v \right),$$

as desired. □

Lemma 9.7. *Suppose $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ is a smooth, rapidly decaying solution to (1.41) such that $\left\| A[f(t)] \right\|_{L^\infty} \leq M t^{-1/3}$ for $t \in (0,T]$. Then, for each $a > 0$, $k > 0$, and $n < -3$ sufficiently negative, there is an $\eta = \eta(n,k,M) > 0$ sufficiently large so that f satisfies on $(0,T)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^n \left(a \exp\left(-\eta t^{2/3}\right) - f \langle v \rangle^k \right)_+^{3/2} \, \mathrm{d}v \leq 0.$$

Proof. Fix $k > 0$ and $n < -3$ arbitrary. Then, set $\ell(t) = a \exp(-\eta t^{2/3})$ for $\eta > 0$ to be determined, $g = \langle v \rangle^k f$, and $g_\ell = (\ell - g)_+$. We compute

$$\ell'(t) = -\frac{2a\eta}{3t^{1/3}} \exp(-\eta t^{2/3}) = -\frac{2\eta}{3t^{1/3}} \ell(t).$$

Therefore, by Lemma 9.6, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^n g_\ell^{3/2} \, \mathrm{d}v &\leq \left(\ell'(t) + C(n,k) \left\| A[f(t)] \right\|_{L^\infty} \ell(t) \right) \left(\int_{\mathbb{R}^3} \langle v \rangle^{n-2} g_\ell^{1/2} \, \mathrm{d}v \right) \\
&\leq \left(C(n,k) M - 2\eta/3 \right) \frac{\ell(t)}{t^{1/3}} \left(\int_{\mathbb{R}^3} \langle v \rangle^{n-2} g_\ell^{1/2} \, \mathrm{d}v \right).
\end{aligned}$$

Picking $\eta \geq 3C(n,k)M/2$, we conclude that the right hand side is negative. □

9.5.2 Bound on the Fisher information and proof of Theorem 1.8

Since we do not have uniqueness, we follow the proof of Theorem 9.1. Suppose that the initial datum $f_{in} \in L^1_m \cap L^{3/2}_{m_0}$ for some $m > 6$ and $m_0 > 6$ and $f_{in} \geq a \langle v \rangle^{-k}$ for some $a > 0$ and $k > 0$ as in the statement of Theorem 1.8. Then, let $f_{in,\varepsilon} \in \mathcal{S}(\mathbb{R}^3)$ be such that $f_{in,\varepsilon} \to f_{in}$ in $L^1_m \cap L^{3/2}_{m_0}$. By Theorem 9.1, $f_{in,\varepsilon}$ give rise to $f_\varepsilon : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$, Schwartz class solutions to (1.41) on $[0,T]$, for some $0 < T \leq 1$ independent of ε.

From the proof of Theorem 9.1, we have the uniform bounds

$$\begin{aligned} \sup_{0<t<T} \|f_\varepsilon(t)\|^{3/2}_{L^{3/2}_{m_0}} + \int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{m_0-3} \left|\nabla f_\varepsilon^{3/4}\right|^2 \,\mathrm{d}v \,\mathrm{d}t \\ + \sup_{0<t<T} t \,\|f_\varepsilon(t)\|_{L^\infty} \leq C(f_{in}, m_0, T). \end{aligned} \tag{9.40}$$

Moreover, since $\|f_\varepsilon(t)\|_{L^\infty} \lesssim t^{-1}$, by Lemma 8.5, $\|A[f_\varepsilon(t)]\|_{L^\infty} \lesssim t^{-1/3}$. So, for $n < -3$ sufficiently negative, we can apply Lemma 9.7 and conclude for some $\eta = \eta(n, k, f_{in}, T) > 0$, independent of ε:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v \rangle^n \left(a \exp\left(-\eta t^{2/3}\right) - f_\varepsilon \langle v \rangle^k\right)_+^{3/2} \,\mathrm{d}v \leq 0, \qquad \text{on } (0,T).$$

Integrating in time, we see that for each $0 < t < T$,

$$\int_{\mathbb{R}^3} \langle v \rangle^n \left(a \exp\left(-\eta t^{2/3}\right) - f_\varepsilon(t,v) \langle v \rangle^k\right)_+^{3/2} \,\mathrm{d}v \leq \int_{\mathbb{R}^3} \langle v \rangle^n \left(a - f_{in,\varepsilon} \langle v \rangle^k\right)_+^{3/2} \,\mathrm{d}v.$$

Noting that $a \leq f_{in} \langle v \rangle^k$ and $f_{in,\varepsilon} \to f_{in}$ in L^1, the right hand side converges to 0 as $\varepsilon \to 0^+$. Since $f_\varepsilon \to f$ in $L^1([0,T] \times \mathbb{R}^3)$ as $\varepsilon \to 0^+$, we find for almost every $t \in (0,T)$,

$$\int_{\mathbb{R}^3} \langle v \rangle^n \left(a \exp\left(-\eta t^{2/3}\right) - f(t,v) \langle v \rangle^k\right)_+^{3/2} \,\mathrm{d}v \leq 0.$$

Since f is smooth, $f \langle v \rangle^k \geq a \exp\left(-\eta t^{2/3}\right)$ for every $t \in (0,T)$. Combined with (9.40), we have shown the following upper bound on the Fisher information:

$$\begin{aligned} \int_0^T i(f(t) \,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla f(t)|^2}{\sqrt{f}(t)\sqrt{f}(t)} \,\mathrm{d}v \,\mathrm{d}t \\ &\leq C(a,T,k) \int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{\frac{k}{2}} \left|\nabla f^{3/4}(t)\right|^2 \,\mathrm{d}v \,\mathrm{d}t \leq C(a,T,k,f_{in},m_0), \end{aligned}$$

provided $k \leq 2m_0 - 6$. Thus $i(f)$ is bounded almost everywhere in $[0,T]$. Finally, since L^1 moments are propagated by Lemma 8.1, $f(t) \in L^1_m$, where $m > 6$. Thus, applying Theorem 9.2 with initial data given by the profile $f(s)$, for each $0 < s < T$, there is unique

global-in-time smooth solution $g_s : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ with decreasing Fisher information and $\|g_s(t)\|_{L^\infty} < \infty$ for each t. Denote by $\tilde{f}$ the concatenation:

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } 0 < t < T/2, \\ g_{T/2}(t - T/2) & \text{otherwise.} \end{cases}$$

By Fournier's uniqueness result from Theorem 8.6, we conclude $g_s(t) = f(t - s)$ for each $0 < s < t < T$. Consequently, $\tilde{f}$ is the desired global-in-time smooth solution. Moreover, $\tilde{f}$ has decreasing Fisher information and L^1 in time: for each $0 < t < T$

$$i(f)(t) \leq C \left(1 + t^{-1}\right).$$

9.6 The sub-critical local in time theory

This section is devoted to the proof of Theorem 9.1 in the sub-critical case, i.e. when $p > 3/2$. The general strategy of proof is similar to the critical $p = 3/2$ case, but the proofs are comparatively simpler partially due to uniform bounds on the diffusion coefficient $A[f]$: the analogue of the ε-regularity criterion in Proposition 9.1 is a standard $L^p \to L^\infty$ regularisation estimate enabling a significant simplification in of the propagation of weighted norms in Proposition 9.2. We remark that local well-posedness for initial datum in $L^p \cap L^1_m$ for $p > 3/2$ was considered in Chapter 8, but the methods break down as $p \to 3/2^+$ in the framework of Chapter 8. The alternative weighted L^p framework offers the advantage of simplicity and does not break down as p tends to either ∞ or $3/2$. Due to these considerations, we show full proofs of necessary a-priori estimates and omit substantial details of the construction of the solution, which mimics Section 9.4 here, or Section 3.5.3 in Chapter 8.

By contrast with the $p = 3/2$ case, when $p > 3/2$ we can derive a uniqueness result based on the $L^1(0, T; L^\infty)$ uniqueness result of Fournier [75]. The uniqueness follows the strategy in Section 8.4.4, which was based on the Prodi-Serrin criteria and the novel proof of an ε-Poincaré inequality by Alonso et al. in [8].

9.6.1 Sub-critical De Giorgi

Proposition 9.3. *Suppose $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^+$ is a smooth, rapidly decaying solution to (1.41) where $\lambda \langle v \rangle^{-3} \leq A[f] \leq \Lambda$. For any $m \geq 9/2$ and $3/2 < p < \infty$, set*

$$E_0 := \left(\sup_{0<t<T} \int_{\mathbb{R}^3} \langle v \rangle^m f^p \right)^{2/5} \left(\int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f^{p/2} \right|^2 \mathrm{d}v \, \mathrm{d}\tau \right)^{3/5}.$$

Then, we have the estimate

$$\|f\|_{L^\infty([t,T]\times\mathbb{R}^3)} \le C \max\left(E_0^{\frac{1}{p}} \left(1 + t^{-\frac{3}{2p}}\right), E_0^{\frac{2}{2p-3}} \right),$$

where C depends only on p, m, λ, and Λ.

Step 1: Introduction of Level Sets and Weighted Energy Functionals

For f a solution to (1.41) and for any $\ell \in \mathbb{R}^+$, we denote the level set function

$$f_\ell(t,v) := (f(t,v) - \ell)_+ = \max(f(t,v) - \ell, 0).$$

For $m \ge 9/2$ fixed and $p > 3/2$, we define three weighted energy functionals associated to f: for $0 \le T_1 < T_2 \le T$ and $0 \le \ell < \infty$,

$$\begin{aligned}
\mathcal{A}_\ell(T_1,T_2) &:= \sup_{T_1<t<T_2} \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p \,\mathrm{d}v \\
\mathcal{B}_\ell(T_1,T_2) &:= \int_{T_1}^{T_2} \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f_\ell^{p/2}\right| \,\mathrm{d}v \\
\mathcal{E}_\ell(T_1,T_2) &:= \sup_{T_1<t<T_2} \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p/2} + \int_{T_1}^{T_2} \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f_\ell^{p/2}\right|^2 \,\mathrm{d}v\,\mathrm{d}t
\end{aligned}$$

Our first goal is showing the following nonlinear inequality: for any $0 \le k < \ell$ and $0 \le T_1 < T_2 < T_3 \le T$,

$$\begin{aligned}
\mathcal{E}_\ell(T_2,T_3) \le C \left[\frac{1}{(\ell-k)^{2p/3}(T_2-T_1)} + \frac{1}{(\ell-k)^{2p/3-1}} + \frac{1+\ell}{(\ell-k)^{2p/3}} + \frac{\ell^2}{(\ell-k)^{2p/3+1}} \right] \\
\times \mathcal{A}_k^{2/3}(T_1,T_3)\mathcal{B}_k(T_1,T_3),
\end{aligned}$$

where C depends on λ, Λ, p, and m.

Step 2: Differential Inequality

Lemma 9.8. *Under the assumptions of Proposition 9.3 and notation introduced in Step 1, the following level set inequality holds for each $\ell > 0$,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p \,\mathrm{d}v + \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f_\ell^{p/2}\right|^2 \,\mathrm{d}v \\
&\quad \le C \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p+1} \,\mathrm{d}v + C(1+\ell) \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p \,\mathrm{d}v + C\ell^2 \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p-1} \,\mathrm{d}v,
\end{aligned}$$

where the constant C depends on p, m, λ, and Λ.

The proof follows from Lemma 9.2, noting that $\|A[f]\|_{L^\infty} \le \Lambda$.

Step 3: Gain of Integrability

In this step, we use the inequality derived in Lemma 9.8 to obtain a nonlinear inequality for our energy functional $\mathcal{E}$ defined in Step 1.

Lemma 9.9. *Under the assumptions of Proposition 9.3 and notation introduced in Step 1, the energy functional $\mathcal{E}$ satisfies for each $0 \le T_1 < T_2 < T_3 \le T$ and each $0 \le k < \ell$,*

$$\mathcal{E}_\ell(T_2,T_3) \le C\left[\frac{1}{(\ell-k)^{2p/3}(T_2-T_1)} + \frac{1}{(\ell-k)^{2p/3-1}} + \frac{1+\ell}{(\ell-k)^{2p/3}} + \frac{\ell^2}{(\ell-k)^{2p/3+1}}\right] \times \mathcal{A}_k^{2/3}(T_1,T_3)\mathcal{B}_k(T_1,T_3),$$

where the constant C depends only on p, λ, Λ, and m.

Following the proof of Proposition 9.1, we integrate the inequality from Lemma 9.8 over $\tau \in [t_1,t_2] \subset [0,T]$, take a supremum over $t_2 \in [T_2,T_3]$, and average over $t_1 \in [T_1,T_2]$ to find

$$\begin{aligned}
&\sup_{\tau\in[T_2,T_3]} \int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p(\tau)\,\mathrm{d}v + \int_{T_2}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f_\ell^{p/2}\right|^2 \mathrm{d}v\,\mathrm{d}\tau \\
&\qquad \le \frac{1}{T_2-T_1}\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p\,\mathrm{d}v\,\mathrm{d}\tau + C\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p+1}\,\mathrm{d}v\,\mathrm{d}\tau \\
&\qquad\qquad + C(1+\ell)\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^p\,\mathrm{d}v\,\mathrm{d}\tau + C\ell^2\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{p-1}\,\mathrm{d}v\,\mathrm{d}\tau.
\end{aligned} \tag{9.41}$$

Recall that as a consequence of the definition of the level set functions f_ℓ and Chebychev's inequality for the measure $\mathrm{d}\nu(v) = \langle v\rangle^m\,\mathrm{d}v$, we have from (9.12), for any $0 \le k < \ell$ and $0 < \alpha_1 < \alpha_2$,

$$\int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{\alpha_1}\,\mathrm{d}v\,\mathrm{d}\tau \le (\ell-k)^{\alpha_1-\alpha_2}\int_{T_1}^{T_3}\int_{\mathbb{R}^3} f_k^{\alpha_2}\,\mathrm{d}v\,\mathrm{d}\tau. \tag{9.42}$$

Combining (9.41) and (9.42) (with $\alpha_1 \in \{p-1,\, p,\, p+1\}$ and $\alpha_2 = 5p/3$), we have shown

$$\begin{aligned}
&\mathcal{E}_\ell(T_2,T_3) \\
&\le C\left[\frac{1}{(T_2-T_1)(\ell-k)^{2p/3}} + \frac{1}{(\ell-k)^{(2p-3)/3}} + \frac{1+\ell}{(\ell-k)^{2p/3}} + \frac{\ell^2}{(\ell-k)^{(2p+3)/3}}\right] \\
&\qquad \times \int_{T_1}^{T_3}\int_{\mathbb{R}^3} \langle v\rangle^m f_k^{5p/3}\,\mathrm{d}v\,\mathrm{d}\tau.
\end{aligned}$$

By interpolation and the weighted Sobolev inequality from Lemma 8.7,

$$\int_{\mathbb{R}^3} \langle v\rangle^m f_\ell^{5p/3}\,\mathrm{d}v \le C\left(\int_{\mathbb{R}^3} \langle v\rangle^{9/2} f_\ell^p\,\mathrm{d}v\right)^{2/3}\left(\int_{\mathbb{R}^3} \langle v\rangle^{m-3}\left|\nabla f_\ell^{p/2}\right|\,\mathrm{d}v\right),$$

Since $m \geq 9/2$, integrating in time we find:

$$\begin{aligned}\int_{T_1}^{T_3} \int_{\mathbb{R}^3} \langle v \rangle^m f_\ell^{\frac{5p}{3}} \, \mathrm{d}v \, \mathrm{d}\tau &\leq C \int_{T_1}^{T_3} \left(\int_{\mathbb{R}^3} \langle v \rangle^m f_\ell^p \, \mathrm{d}v \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f_\ell^{p/2} \right| \, \mathrm{d}v \right) \mathrm{d}\tau \\ &\leq C \mathcal{A}_\ell^{2/3}(T_1, T_3) \mathcal{B}_\ell(T_1, T_3).\end{aligned}$$

This concludes the proof of Lemma 9.9.

Step 4: De Giorgi Iteration

We now finish the proof of Proposition 9.3 using a De Giorgi iteration. We set $0 < t < T$ the times from the statement of Proposition 9.3. Then, we define our iteration quantities:

$$\ell_n = K(1 - 2^{-n}) \qquad t_n = t(1 - 2^{-n}) \qquad E_n = \mathcal{A}_{\ell_n}^{2/5}(t_n, T) \mathcal{B}_{\ell_n}(t_n, T)^{3/5}.$$

Then, using Young's inequality and Lemma 9.9, we have the following recurrence for E_n:

$$\begin{aligned}E_{n+1} &\leq C \left(\mathcal{A}_{\ell_{n+1}}(t_{n+1}, T) + \mathcal{B}_{\ell_{n+1}}(t_{n+1}, T) \right) = C \mathcal{E}_{\ell_{n+1}}(t_{n+1}, T) \\ &\leq C \Bigg(\frac{1}{(t_{n+1} - t_n)(\ell_{n+1} - \ell_n)^{2p/3}} + \frac{1}{(\ell_{n+1} - \ell_n)^{2p/3 - 1}} \\ &\qquad + \frac{1 + \ell_{n+1}}{(\ell_{n+1} - \ell_n)^{2p/3}} + \frac{\ell_{n+1}^2}{(\ell_{n+1} - \ell_n)^{2p/3+1}} \Bigg) E_n^{5/3} \\ &\leq C 2^{n\kappa} \left(\frac{1}{K^{2p/3} t} + \frac{1}{K^{2p/3-1}} + \frac{1 + K}{K^{2p/3}} + \frac{K^2}{K^{2p/3+1}} \right) E_n^{5/3} \\ &\leq C^* 2^{n\kappa} \left(\frac{1}{K^{\frac{2p}{3}} t} + \frac{1}{K^{\frac{2p}{3}}} + \frac{1}{K^{\frac{2p-3}{3}}} \right) E_n^{5/3},\end{aligned}$$

where $\kappa = 2p/3 + 1$ and $C^* > 1$ is now a fixed constant. We attempt to find a barrier sequence of the form $B_n := B^n E_0$ for some $B > 1$. That is, we seek values of K and B for which

$$B_{n+1} \geq C^* 2^{n\kappa} \left(\frac{1}{K^{\frac{2p}{3}} t} + \frac{1}{K^{\frac{2p}{3}}} + \frac{1}{K^{\frac{2p-3}{3}}} \right) B_n^{5/3}.$$

Inserting the definition of B_n, this is implied by

$$1 \geq C^* B^{\frac{2n-3}{3}} 2^{n\kappa} E_0^{2/3} \left(\frac{1}{K^{\frac{2p}{3}} t} + \frac{1}{K^{\frac{2p}{3}}} + \frac{1}{K^{\frac{2p-3}{3}}} \right).$$

We first pick B sufficiently large so that $B^{\frac{2}{3}} \geq 2^\kappa$. Second, we pick K so that each remaining term is less than $1/3$:

$$K = \widetilde{C}(C^*, B, p) \max \left(E_0^{\frac{1}{p}} t^{-\frac{3}{2p}}, E_0^{\frac{1}{p}}, E_0^{\frac{2}{2p-3}} \right)$$

With these choices of K and B, since $B_0 = E_0$, we find inductively:

$$\begin{aligned} E_{n+1} &\leq C^* 2^{n\kappa} \left(\frac{1}{K^{\frac{2p}{3}} t} + \frac{1}{K^{\frac{2p}{3}}} + \frac{1}{K^{\frac{2p-3}{3}}} \right) E_n^{5/3} \\ &\leq C^* 2^{n\kappa} \left(\frac{1}{K^{\frac{2p}{3}} t} + \frac{1}{K^{\frac{2p}{3}}} + \frac{1}{K^{\frac{2p-3}{3}}} \right) B_n^{5/3} \leq B_{n+1}. \end{aligned}$$

Therefore,

$$\left\| f \chi_{\{f \geq K\}} \right\|_{L^\infty(t,T;L^p_m)} \leq \lim_{n\to\infty} E_n \leq \lim_{n\to\infty} B_n = 0,$$

and we conclude $f(\tau, v) \leq K$ pointwise a.e. on $[t, T] \times \mathbb{R}^3$. By the choice of K, we have shown the estimate:

$$f(\tau, v) \leq C \max\left(E_0^{\frac{1}{p}} \left(1 + t^{-\frac{3}{2p}} \right), E_0^{\frac{2}{2p-3}} \right) \qquad \text{for almost all } \tau, v \in [t, T] \times \mathbb{R}^3.$$

This concludes the proof of Proposition 9.3.

9.6.2 Propagation of L^p_m for smooth data

Lemma 9.10. *Suppose $p > 3/2$ and $m \geq 9/2$, $f_{in} \in \mathcal{S}(\mathbb{R}^3)$, and $f : [0, T^*) \to \mathbb{R}^+$ is a smooth rapidly decaying solution to (1.41) with initial data f_{in}. Then, there is a time $T = T(\|f_{in}\|_{L^p_m}, m, p) > 0$ such that f satisfies*

$$\sup_{0<t<\min(T,T^*)} \|f(t)\|^p_{L^p_m} + \int_0^{\min(T,T^*)} \int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f^{p/2} \right|^2 \, \mathrm{d}v \, \mathrm{d}\tau \leq 4 \|f_{in}\|^p_{L^p_m}.$$

Proof. From Lemma 9.8 with $\ell = 0$, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathbb{R}^3} \langle v \rangle^m f^p \, \mathrm{d}v + \int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f^{p/2} \right| \, \mathrm{d}v \\ \leq C \int_{\mathbb{R}^3} \langle v \rangle^m f^{p+1} \, \mathrm{d}v + C \int_{\mathbb{R}^3} f^p \langle v \rangle^m \, \mathrm{d}v. \end{aligned} \tag{9.43}$$

By the interpolation estimate in Lemma 8.7 with $q = p + 1$ (since $m \geq \frac{9}{2}$) and Young's inequality (since $3/2p < 1$ by assumption), we find for any $\delta > 0$,

$$\begin{aligned} \int_{\mathbb{R}^3} \langle v \rangle^m f^{p+1} \, \mathrm{d}v &\leq C \left(\int_{\mathbb{R}^3} \langle v \rangle^m f^p \, \mathrm{d}v \right)^{1-1/2p} \left(\int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f^{p/2} \right|^2 \, \mathrm{d}v \right)^{3/2p} \\ &\leq \delta \left(\int_{\mathbb{R}^3} \langle v \rangle^{m-3} \left| \nabla f^{p/2} \right|^2 \, \mathrm{d}v \right) + C\delta^{-\frac{2p}{2p-3}} \left(\int_{\mathbb{R}^3} \langle v \rangle^m f^p \, \mathrm{d}v \right)^{\frac{2p-1}{2p-3}}. \end{aligned}$$

Taking δ sufficiently small, depending only on λ, Λ, p, and m, and setting $\alpha = \frac{2p-1}{2p-3} > 1$,

$y(t) = \|f(t)\|_{L^p_m}^p$, and $F(t) = \left\|\nabla f(t)^{p/2}\right\|_{L^2_m}^2$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + F(t) \leq C y^\alpha(t) + C y(t).$$

This is a Bernoulli-type differential inequality for y, which can be explicitly solved by a suitable change of coordinates: Setting $v = e^{-C(1-\alpha)t} y^{1-\alpha}$, we find v satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} v = (1-\alpha) e^{-C(1-\alpha)t} y^{-\alpha} y'(t) - C(1-\alpha) v \geq C(1-\alpha) e^{-C(1-\alpha)t}.$$

Integrating then gives that

$$v(t) \geq v(0) - \left[\exp(C(\alpha-1)t) - 1\right].$$

This implies the following bound on $y(t)$:

$$y(t) \leq \exp(Ct) \left[y_0^{1-\alpha} + 1 - \exp(C(\alpha-1)t)\right]^{\frac{1}{1-\alpha}}$$

Since $\alpha > 1$, we find a $T = T(y_0, \alpha, C)$ sufficiently small such that $y(t) \leq 2y_0$ on $[0, \min(T, T^*)]$. Consequently, we integrate to find

$$\int_0^{\min(T,T^*)} F(t)\,\mathrm{d}t \leq y_0 + C(y_0^\alpha + y_0)T \leq 2y_0,$$

where the last inequality follows from picking T possibly smaller. □

9.6.3 Construction of solution

We now construct local-in-time solutions. For $f_{in} \in L^p_m \cap L^1_2$, we take $f_{in,\varepsilon} \in \mathcal{S}(\mathbb{R}^3)$ so that $f_{in,\varepsilon}$ converges to f_{in} in $L^1_2 \cap L^p_m$. Using local-in-time well-posedness for $f_{in,\varepsilon}$, there are corresponding solutions $f_\varepsilon : [0, T^*_\varepsilon) \times \mathbb{R}^3 \to \mathbb{R}^+$. From Lemma 9.10, there is a uniform-in-ε time T such that for $0 < t < \min(T, T^*_\varepsilon)$, we have the uniform estimate

$$\left(\sup_{0<\tau<t} \int_{\mathbb{R}^3} \langle v\rangle^m f_\varepsilon^p \,\mathrm{d}v\right)^{2/5} \left(\int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{m-3} \left|\nabla f_\varepsilon^{p/2}\right|^2 \,\mathrm{d}v\,\mathrm{d}\tau\right)^{3/5} \lesssim 1.$$

Using Proposition 9.3, we find that $T^*_\varepsilon > T$ for each $\varepsilon > 0$ and there holds the uniform-in-ε estimate:

$$\|f_\varepsilon(t)\|_{L^\infty} \lesssim \left(1 + \frac{1}{t^{3/2p}}\right) \qquad \text{for each } 0 < t < T.$$

Passing to the limit, $f_\varepsilon \to f$, where these bounds guarantee $f \in C(0, T; L^p_m)$ is smooth solution for positive time and satisfies the claimed smoothing estimates. Moreover, by uniqueness result of Fournier, Theorem 8.6, the limit f is the unique $L^1(0, T; L^\infty)$ solution

with initial data f_{in}.

9.6.4 Uniqueness

Let $g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ be a weak solution of (1.41) with initial datum $g_{in} \in L^1_2 \cap L \log L$, attained in the sense of distributions. We suppose further that $g \in L^r(0,T; L^p_{9/2}(\mathbb{R}^3)) \cap L^\infty(s,T; W^{1,\infty}(\Omega))$ for each $0 < s < T$ and each open, bounded $\Omega \subset \mathbb{R}^3$ and some pair (r,p) satisfying $p > 3/2$ and $r > \frac{2p}{2p-3}$.

Our goal is to derive a differential inequality for $y(t) := \|g(t)\|^p_{L^p_{9/2}(\mathbb{R}^3)}$ much as in Lemma 9.10 to obtain weighted $L^r(0,T;L^p) \to L^\infty(t,T;L^p)$ smoothing estimates. Formally, the idea is to test (1.41) with $\langle v\rangle^{9/2} g^{p-1}$ and use the Sobolev embedding in Lemma 8.7 to control the highest order term. Formally testing (1.41) with $g^{p-1}\langle v\rangle^{9/2}$, Lemma 9.8 (with $\ell = 0$ and $m = 9/2$) implies the following differential inequality:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \langle v\rangle^{9/2} g^p \,\mathrm{d}v + \int_{\mathbb{R}^3} \langle v\rangle^{3/2} \left|\nabla g^{p/2}\right|^2 \,\mathrm{d}v \\ \leq C\int_{\mathbb{R}^3} \langle v\rangle^{9/2} g^{p+1}\,\mathrm{d}v + C\int_{\mathbb{R}^3}\langle v\rangle^{9/2} g^p \,\mathrm{d}v. \end{aligned} \tag{9.44}$$

Then, we interpolate and use the Sobolev embedding from Lemma 8.7 and apply Young's inequality to obtain: for any $\varepsilon \in (0,1)$

$$\begin{aligned} \int_{\mathbb{R}^3} \langle v\rangle^{9/2} g^{p+1}\,\mathrm{d}v &\leq \|g\|^{p-\frac12}_{L^p_{9/2}} \|g\|^{\frac32}_{L^{3p}_{9/2}} \leq \|g\|^{p-\frac12}_{L^p_{9/2}} \left\|g^{p/2}\right\|^{\frac3p}_{L^6_{9/2}} \\ &\leq C\|g\|^{p-\frac12}_{L^p_{9/2}} \left(\int_{\mathbb{R}^3}\langle v\rangle^{3/2}\left|\nabla g^{p/2}\right|^2\,\mathrm{d}v\right)^{\frac{3}{2p}} \\ &\leq C(\varepsilon)\|g\|^{\left(p-\frac12\right)\frac{2p}{2p-3}}_{L^p_{9/2}} + \varepsilon\left(\int_{\mathbb{R}^3}\langle v\rangle^{3/2}\left|\nabla g^{p/2}\right|^2\,\mathrm{d}v\right). \end{aligned}$$

Picking ε sufficiently small, the second term can absorbed by left hand side of (9.44). Denoting by $y(t) := \|g(t)\|^p_{L^p_m}$, for $t \in (0,T]$ we find:

$$\frac{d}{dt}y + \left\|\nabla g^{p/2}(t)\right\|^2_{L^2_{3/2}} \leq C(\varepsilon) y^{1+\frac{2}{2p-3}}(t) + y(t).$$

Since y is only absolutely continuous for positive times, we integrate over $[s,t]$ for $0 < s < t < T$ to obtain:

$$y(t) + \int_s^t \left\|\nabla g^{p/2}(\tau)\right\|^2_{L^2_{3/2}}\,\mathrm{d}\tau \leq y(s) + C(\varepsilon)\int_s^t y^{1+\frac{2}{2p-3}}(\tau)\,\mathrm{d}\tau + \int_s^t y(\tau)\,\mathrm{d}\tau. \tag{9.45}$$

Grönwall's inequality and $\frac{2p}{2p-3} \leq r$ then implies the explicit bound:

$$\sup_{s\leq t\leq T} y(t) \leq y(s) \exp\left(C(T-s) + C\int_s^T y^{\frac{2}{2p-3}}(\tau)\,\mathrm{d}\tau \right) \leq Cy(s).$$

Rearranging powers and averaging over $s \in [0,t]$, we find

$$\|g(t)\|_{L^p_{9/2}}^r \leq Ct^{-1}\int_0^t \|g(s)\|_{L^p_{9/2}}^r \,\mathrm{d}s \leq Ct^{-1}.$$

We plug this estimate back into (9.45), to obtain for each $0 < t < T$,

$$\begin{aligned} &\sup_{t<\tau<T} \|g(\tau)\|_{L^p_{9/2}}^p + \int_t^T \left\|\nabla g^{p/2}(\tau)\right\|_{L^2_{3/2}}^2 \,\mathrm{d}\tau \\ &\qquad \leq \|g(t)\|_{L^p_{9/2}}^p + C(\varepsilon)\int_t^T \|g(\tau)\|_{L^p_{9/2}}^{p+r}\,\mathrm{d}\tau + \int_t^T \|g(\tau)\|_{L^p_{9/2}}^p\,\mathrm{d}\tau \\ &\qquad \leq Ct^{-\frac{p}{r}}. \end{aligned} \tag{9.46}$$

In a second step, we use smoothing estimates to show that g belongs to Fournier's uniqueness class, i.e. $g \in L^1(0,T;L^\infty)$. Using the smoothing estimate of Proposition 9.3 - which can be rigorously justified since $g \in W^{1,\infty}_{loc}$ satisfies the global (in v) bound (9.46) - we obtain

$$\|g\|_{L^\infty([t,T]\times\mathbb{R}^3)} \leq C\max\left\{ t^{-\frac{1}{r}}\left(1+t^{-\frac{3}{2p}}\right), t^{-\frac{2p}{r(2p-3)}} \right\} \leq Ct^{-\left(\frac{1}{r}+\frac{3}{2p}\right)} + Ct^{-\frac{2p}{r(2p-3)}}.$$

Since $r > \frac{2p}{2p-3}$, we see that $g \in L^1(0,T;L^\infty(\mathbb{R}^3))$. Since this holds for any locally Lipschitz $g \in L^r(0,T;L^p_{9/2})$, by Fournier's uniqueness result in Theorem 8.6, we conclude that any two such solutions g with coinciding initial data are equal.

We conclude this section by noting that while the above computations are merely formal, they can be made rigorous. Although g is locally Lipschitz, g does not have sufficient (global) regularity to merit its use as a test function in (1.41). To rigorously justify the formal argument presented here, one should truncate the norm, by using $\langle v\rangle^{9/2}\varphi^2 g^{p-1}$ as a test function, where φ is a Lipschitz cutoff to B_R. To ensure that the formal arguments presented transfer to the local truncations (uniformly in R), we need a more precise version of the interpolation from Lemma 8.7 used above:

Lemma 9.11. *Suppose g, φ are any Schwartz class functions. Then, for any $3/2 < q < \infty$, there is a universal constant $C = C(q)$ such that for any $\varepsilon > 0$, there holds:*

$$\begin{aligned} &\int_{\mathbb{R}^3} \langle v\rangle^{9/2}\varphi^2 g^{p+1}\,\mathrm{d}v \\ &\qquad \leq \varepsilon\int_{\mathbb{R}^3}\langle v\rangle^{3/2}\left|\nabla\left(\varphi g^{p/2}\right)\right|^2\,\mathrm{d}v + C\left(\varepsilon^{-3}\|g\|_{L^q_{9/2}}^{2q}\right)^{\frac{1}{2q-3}}\int_{\mathbb{R}^3}\langle v\rangle^{9/2}\varphi^2 g^p\,\mathrm{d}v. \end{aligned}$$

Such estimates were first applied to the study of the homogeneous Landau equation in the work of the second author and Guillen [94], where they were termed ε-Poincaré inequalities. More recently, a new ε-Poincaré inequality appeared in the work of Alonso et al. [8], enabling the authors to obtain a condition for propagation and creation of L^p norms. The ε-Poincaré inequality of [8] was the crucial tool in Section 8.4.4 for turning a-priori smoothing arguments into a rigorous uniqueness proof. We postpone a proof of Lemma 9.11 to Appendix 9.9 and refer the reader to Section 8.4.4 for details on how to use Lemma 9.11 to provide a rigorous justification of the formal argument above.

9.7 The sub-critical global in time theory

9.7.1 Pointwise lower bounds

Here we show the solution constructed in Theorem 9.1 propagates lower bounds, which will be used to control the Fisher information. When $p > 3/2$, $\|A[f]\|_{L^\infty} \lesssim 1$ so that we can show that the function $\psi(t,v) = a\exp(-\eta t)\langle v\rangle^{-k}$ satisfies

$$\partial_t \psi \leq A[f] : \nabla^2 \psi + f\psi,$$

provided η is chosen suitably large. We still take Stampacchia approach to show $f_{in} \geq \langle v\rangle^{-k}$ implies $f \geq \psi$, but take less care in justifying computations, particularly when using integration by parts. Because we have uniqueness in the $p > 3/2$ case, a-priori estimates can easily be extended to rough data. We begin by proving that ψ as above is indeed a subsolution to the linearized equation:

Lemma 9.12. *Suppose $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ is a smooth solution to (1.41) with*

$$C_1 \langle v\rangle^{-3} \leq A[f] \leq C_2.$$

Then, for each $k > 5$ and $a > 0$, there is $\eta = \eta(C_1, C_2, k)$ sufficiently large so that $\psi(t,v) = a\exp(-\eta t)\langle v\rangle^{-k}$ satisfies

$$\partial_t \psi \leq A[f] : \nabla^2 \psi + f\psi.$$

Proof. Define $\psi(t,v) = a\exp(-\eta t)\langle v\rangle^{-k}$. Then, we compute derivatives of ψ.

$$\begin{aligned}
&\partial_t \psi = -\eta\psi, \qquad \nabla\psi = a\exp(-\eta t)\left(-k\langle v\rangle^{-k-2} v\right), \\
&\nabla^2\psi = a\exp(-\eta t)\left[k(k+2)\langle v\rangle^{-k-4} v\otimes v - k\langle v\rangle^{-k-2} Id\right] \\
&\qquad = \left[k(k+2)\frac{v\otimes v}{\langle v\rangle^4} - k\frac{Id}{\langle v\rangle^2}\right]\psi.
\end{aligned}$$

So, using $f \geq 0$, $\operatorname{tr}(A[f]) \leq C_1$, $A[f] \geq C_2 \langle v \rangle^{-3}$, by Young's inequality

$$\begin{aligned}
\partial_t \psi - A[f] : \nabla^2 \psi - f\psi &\leq \left[\frac{C_1 k}{\langle v \rangle^2} - \eta - \frac{C_2 k(k+2)|v|^2}{\langle v \rangle^7} \right] \psi \\
&\leq \left[\frac{\tilde{C}_1}{\langle v \rangle^2} - \eta - \frac{\tilde{C}_2}{\langle v \rangle^5} \right] \psi \\
&\leq \left[\tilde{C}_1 \left(\frac{5}{2\delta} \right)^{10/3} + \tilde{C}_1 \delta \langle v \rangle^{-5} - \eta - \tilde{C}_2 \langle v \rangle^{-5} \right] \psi.
\end{aligned}$$

Therefore, picking $\delta = \frac{\tilde{C}_2}{\tilde{C}_1}$ and then picking $\eta > \tilde{C}_1 (\frac{5}{2\delta})^{10/3}$, concludes the proof. □

Lemma 9.13. *Suppose $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ is a smooth solution to (1.41) with initial data $f_{in} \in L^p_{9/2}$ for $p > 3/2$ and $f_{in} \geq a \langle v \rangle^{-k}$ for $k > 5$ and $a > 0$. Suppose further that $f \in L^\infty(0,T; L^p) \cap L^1(0,T; L^\infty)$. Then, there is an $\eta = \eta(p, \|f_{in}\|_{L^p}) > 0$ sufficiently large such that*

$$f(t) \geq a \exp(-\eta t) \langle v \rangle^{-k} \qquad \text{for } 0 \leq t \leq T.$$

Proof. Let ψ be as in Lemma 9.12, so that $f_{in}(v) \geq \psi(0,v)$ and

$$\partial_t \psi \leq A[f] : \nabla^2 \psi + \psi f.$$

Set $\varphi = \psi - f$ so that by linearity,

$$\partial_t \varphi \leq A[f] : \nabla^2 \varphi + \varphi f.$$

Then, performing an L^p estimate, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varphi_+^p &= - \int_{\mathbb{R}^3} \nabla \varphi_+^{p-1} \cdot (A[f] \nabla \varphi - \nabla a[f] \varphi) \, \mathrm{d}v \\
&= - \frac{4(p-1)}{p^2} \int_{\mathbb{R}^3} \nabla \varphi_+^{p/2} \cdot A[f] \nabla \varphi_+^{p/2} \, \mathrm{d}v + \frac{p-1}{p} \int_{\mathbb{R}^3} \nabla \varphi_+^p \cdot \nabla a[f] \, \mathrm{d}v \\
&\leq \int_{\mathbb{R}^3} \varphi_+^p f \, \mathrm{d}v \leq \int_{\mathbb{R}^3} \varphi_+^p \psi \, \mathrm{d}v \leq \|\psi(t)\|_{L^\infty} \int_{\mathbb{R}^3} \varphi_+^p \, \mathrm{d}v.
\end{aligned}$$

Since $\psi \in L^1(0,T; L^\infty)$ and $\varphi_+(0) = 0$, by Grönwall's inequality, $\varphi_+ = 0$ on $[0,T] \times \mathbb{R}^3$, which completes the proof. □

9.7.2 Bound on the Fisher information and proof of Theorem 1.8

Fix initial datum $f_{in} \in L^1_m \cap L^p_{m_0}$ for some $p \in (3/2, +\infty)$, with m and m_0 as stated in the theorem. From Theorem 9.1, we have a solution $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^+$ satisfying the

bounds

$$\sup_{0<t<T} \|f(t)\|_{L^p_{m_0}}^p + \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{m_0-3} \left|\nabla f^{p/2}\right|^2 \,\mathrm{d}v\,\mathrm{d}t + \sup_{0<t<T} t^{3/2p} \|f(t)\|_{L^\infty} \leq C(f_{in}, m_0, p, T). \tag{9.47}$$

Since $\|f(t)\|_{L^p} \lesssim 1$, by Lemma 8.5, $\|A[f(t)]\|_{L^\infty} \lesssim 1$. So, by Lemma 9.13, for some $\eta = \eta(k, p, \|f_{in}\|_{L^p}, T) > 0$ there holds:

$$f(t,v) \geq a \exp(-\eta t) \langle v\rangle^{-k} \qquad \text{for each } (t,v) \in [0,T] \times \mathbb{R}^3.$$

Combined with (9.47), we have shown the following upper bound on the Fisher information:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^3} f^{p-2} \frac{|\nabla f|^2}{f^{p-1}} \,\mathrm{d}v\,\mathrm{d}t &\leq C(a,T,k) \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{k(p-1)} \left|\nabla f^{p/2}\right|^2 \,\mathrm{d}v\,\mathrm{d}t \\ &\leq C(a,T,k,f_{in},m_0,p)\end{aligned}$$

provided $k(p-1) \leq m_0 - 3$. We conclude as in the $p = 3/2$ case of Theorem 9.2 that f can be continued to a global-in-time smooth solution with decreasing Fisher information.

9.8 Appendix: Proof of Lemma 9.1

In this section, we construct the smooth cutoff used in the proof of Theorem 9.2.

Proof. We define first an auxiliary function $\tilde{\varphi}$ as

$$\tilde{\varphi}(x) = \begin{cases} \exp(-x^{-1}) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

From $\tilde{\varphi}$, we translate and rescale $x \mapsto \frac{\tilde{\varphi}(x)}{\tilde{\varphi}(x)+\tilde{\varphi}(1-x)}$ and define the radial profile of the cutoff:

$$\begin{aligned}\varphi(x) &= \left(\frac{\tilde{\varphi}\left(2 - x/R\right)}{\tilde{\varphi}\left(2 - x/R\right) + \tilde{\varphi}\left(1 - (2 - x/R)\right)}\right)^2 \\ &= \begin{cases} 1, & \text{if } x \leq R, \\ \left(\frac{\exp\left(-(2-x/R)^{-1}\right)}{\exp(-(2-x/R)^{-1}) + \exp(-(x/R-1)^{-1})}\right)^2, & \text{if } R < x < 2R, \\ 0 & \text{if } 2R \leq x. \end{cases}\end{aligned}$$

The cutoff we seek is $\eta(v) = \varphi(|v|)$. Note that because $\tilde{\varphi} \in C^\infty$, $\eta \in C^\infty$ and we can easily verify $0 \leq \eta \leq 1$, $\eta = 1$ on B_R, and $\eta = 0$ on $\mathbb{R}^3 \setminus B_R$. It remains only to verify the differential inequalities, which are nontrivial only when $R < |v| < 2R$.

Because η is radial, $\nabla\eta(v) = \frac{v}{|v|}\varphi'(|v|)$, so that $|\nabla\eta(v)| = |\varphi'(|v|)|$. For $R < |v| < 2R$, by explicit computation,

$$\begin{aligned}\varphi'(x) &= 2\sqrt{\varphi(x)}\left(\frac{\tilde{\varphi}(2-x/R)}{\tilde{\varphi}(2-x/R)+\tilde{\varphi}(1-(2-x/R))}\right)' \\ &= -\frac{2\sqrt{\varphi(x)}}{R}\left(\frac{\tilde{\varphi}'(2-x/R)\tilde{\varphi}(1-(2-x/R))+\tilde{\varphi}(2-x/R)\tilde{\varphi}'(1-(2-x/R))}{\left[\tilde{\varphi}(2-x/R)+\tilde{\varphi}(1-(2-x/R))\right]^2}\right).\end{aligned}$$

We bound the term in parentheses by computing it explicitly and realizing that for $R < x < 2R$ the denominator is minimized at $x = 3R/2$, and that the numerator is bounded since $x \mapsto e^{-x^{-1}}x^{-2}$ is bounded, as the exponential dominates:

$$0 \leq \frac{e^{-(2-x/R)^{-1}}e^{-(1-(2-x/R))^{-1}}\left[(2-x/R)^{-2}+(1-(2-x/R))^{-2}\right]}{\left[e^{-(2-x/R)^{-1}}+e^{-(1-(2-x/R))^{-1}}\right]^2} \leq C.$$

Therefore, φ is decreasing and

$$|\nabla\eta(v)| = |\varphi'(|v|)| = -\varphi'(|v|) \leq \frac{2C\sqrt{\varphi(|v|)}}{R} = \frac{2C\sqrt{\eta}}{R}. \tag{9.48}$$

From (9.48), we deduce one differential equality for η. On the other hand, we see for $R < x < 2R$,

$$\begin{aligned}1-\varphi(x) &= \frac{\tilde{\varphi}\left(1-(2-x/R)\right)\left(\varphi(1-(2-x/R))+2\tilde{\varphi}(2-x/R)\right)}{\left[\tilde{\varphi}\left(2-x/R\right)+\tilde{\varphi}\left(1-(2-x/R)\right)\right]^2} \\ &= \frac{e^{-(1-(2-x/R))^{-1}}\left[e^{-(1-(2-x/R))^{-1}}+2e^{-(2-x/R)^{-1}}\right]}{\left(e^{-(2-x/R)^{-1}}+e^{-(1-(2-x/R))^{-1}}\right)^2},\end{aligned}$$

so that the explicit expressions for φ yield

$$\begin{aligned}\frac{|\varphi'(x)|^2}{1-\varphi(x)} &= \frac{4R^{-2}e^{-(2-x/R)^{-1}}}{\left[e^{-(1-(2-x/R))^{-1}}+2e^{-(2-x/R)^{-1}}\right]} \\ &\qquad \times\left[\frac{e^{-(2-x/R)^{-1}}e^{-1/2(1-(2-x/R))^{-1}}\left[(2-x/R)^{-2}+(1-(2-x/R))^{-2}\right]}{\left[e^{-(2-x/R)^{-1}}+e^{-(1-(2-x/R))^{-1}}\right]^2}\right]^2 \\ &\leq \frac{C}{R^2},\end{aligned}$$

where we bounded the term in brackets as before, minimizing the denominator and again using that the exponential dominates any polynomial. We conclude

$$|\nabla\eta(v)| \leq \frac{C\sqrt{1-\eta(v)}}{R}, \tag{9.49}$$

and (9.49) is the second differential inequality claimed. □

9.9 Appendix: Proof of Lemma 9.11

In this section, we provide a proof of the ε-Poincaré inequality in Lemma 9.11:

Proof. We prove first an unweighted version of the desired inequality. Using Holder's inequality and the Gagliardo-Nirenberg-Sobolev inequality, we obtain:

$$\begin{aligned}\int_{\mathbb{R}^3} \varphi^2 g^{p+1}\,\mathrm{d}v &\le \left(\int_{\mathbb{R}^3} g^q\,\mathrm{d}v\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^3} \left(\varphi^2 g^p\right)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} = \|g\|_{L^q} \left\|\varphi g^{p/2}\right\|^2_{L^{\frac{2q}{q-1}}} \\ &\le C\,\|g\|_{L^q} \left\|\nabla(\varphi g^{p/2})\right\|^{2\theta}_{L^2} \left\|\varphi g^{p/2}\right\|^{2(1-\theta)}_{L^2} \\ &\le \varepsilon \left(\int_{\mathbb{R}^3} \left|\nabla(\varphi g^{p/2})\right|^2\,\mathrm{d}v\right) + C\varepsilon^{-\frac{\theta}{1-\theta}}\,\|g\|^{\frac{1}{1-\theta}}_{L^q} \left(\int_{\mathbb{R}^3} \varphi^2 g^p\,\mathrm{d}v\right),\end{aligned}$$

where $\frac{q-1}{2q} = \frac{1}{2} - \frac{\theta}{3}$. Solving for θ, we find $\theta = \frac{3}{2q} \in (0,1)$ for $q > 3/2$. Inserting this value of θ, we find

$$\int_{\mathbb{R}^3} \varphi^2 g^{p+1}\,\mathrm{d}v \le \varepsilon \left(\int_{\mathbb{R}^3} \left|\nabla(\varphi g^{p/2})\right|^2\,\mathrm{d}v\right) + C\varepsilon^{-\frac{3}{2q-3}}\,\|g\|^{\frac{2q}{2q-3}}_{L^q} \left(\int_{\mathbb{R}^3} \varphi^2 g^p\,\mathrm{d}v\right). \tag{9.50}$$

Repeating the above argument with homogeneous weights of the form $|v|^m$ and using the Caffarelli-Kohn-Nirenberg inequality (see [34] or later improvements in [70, 148]) in place of the Gagliardo-Nirenberg-Sobolev inequality, we find:

$$\begin{aligned}&\int_{\mathbb{R}^3} |v|^{9/2}\,\varphi^2 g^{p+1}\,\mathrm{d}v \\ &\quad\le \left(\int_{\mathbb{R}^3} |v|^{9/2}\,g^q\,\mathrm{d}v\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^3} |v|^{9/2}\left(\varphi^2 g^p\right)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \\ &\quad= \left\||v|^{\frac{9}{2q}}\,g\right\|_{L^q} \left\||v|^{\frac{9q}{4(q-1)}}\,\varphi g^{p/2}\right\|^2_{L^{\frac{2q}{q-1}}} \\ &\quad\le C\left\||v|^{\frac{9}{2q}}\,g\right\|_{L^q} \left\||v|^{3/4}\,\nabla(\varphi g^{p/2})\right\|^{2\theta}_{L^2} \left\||v|^{9/4}\,\varphi g^{p/2}\right\|^{2(1-\theta)}_{L^2} \\ &\quad\le \varepsilon \left(\int_{\mathbb{R}^3} |v|^{3/2}\left|\nabla(\varphi g^{p/2})\right|^2\,\mathrm{d}v\right) + C\varepsilon^{-\frac{\theta}{1-\theta}} \left\||v|^{\frac{9}{2q}}\,g\right\|^{\frac{1}{1-\theta}}_{L^q} \left(\int_{\mathbb{R}^3} |v|^{9/2}\,\varphi^2 g^p\,\mathrm{d}v\right),\end{aligned}$$

where θ is now given by

$$\frac{q-1}{2q} + \frac{\frac{9(q-1)}{4q}}{3} = \theta\left(\frac{1}{2} - \frac{1-3/4}{3}\right) + (1-\theta)\left(\frac{1}{2} + \frac{9/4}{3}\right).$$

Solving for θ, we once again find $\theta = \frac{3}{2q} \in (0,1)$ for $q > 3/2$. Consequently, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |v|^{9/2} \varphi^2 g^{p+1} \,\mathrm{d}v \leq & \varepsilon \left(\int_{\mathbb{R}^3} |v|^{3/2} \left| \nabla (\varphi g^{p/2}) \right|^2 \,\mathrm{d}v \right) \\ & + C\varepsilon^{-\frac{3}{2q-3}} \left\| |v|^{\frac{9}{2q}} g \right\|_{L^q}^{\frac{2q}{2q-3}} \left(\int_{\mathbb{R}^3} |v|^{9/2} \varphi^2 g^p \,\mathrm{d}v \right).\end{aligned} \tag{9.51}$$

Summing (9.50) and (9.51), we obtain the desired bound. □

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CHAPTER 10

Super-critical Landau

In our final chapter, we consider the Landau-Coulomb equation for initial data with bounded mass, finite numbers of moments, and entropy. We show the existence of a global weak solution that has bounded Fisher information for positive times. This solution is therefore a global strong solution away from the initial time. We propose an alternative approach, based on already existing estimates, to the study of the appearance of Fisher information recently performed by Ji in [123].

Recent work in the groundbreaking paper [100] has established the existence of solutions that remain bounded over arbitrarily long times for smooth initial data, along with the proof that the Fisher information is monotonically decaying in time. Furthermore, in the previous Chapter 9 we have extended these results to potentially singular initial data, considering initial data in $L^p(\mathbb{R}^3)$ for any $p \geq 3/2$. This result was very recently improved by Sehyun Ji [123], who showed that the Fisher information appears for any strictly positive time as soon as the initial datum has finite entropy and 5 moments in L^1, thanks to a careful study of the evolution of the Fisher information itself.

In the present chapter, we propose a similar (though somewhat weaker, in terms of moments and rate of regularisation) result, which has the specificity of combining computations from previous papers, allowing for a very concise presentation. Specifically, we draw on the weighted $L^1(L^3)$ estimate from [58], the propagation of L^2 moments on short time interval from Section 8.4.1 in Chapter 8, the existence of smooth solutions for less regular initial data from Chapter 9, estimates for the evolution of a weighted entropy from [6, 7], and the decay of the Fisher information from Theorem 9.2. We use this to prove **Theorem 1.9**, a global existence result away for strictly positive times in the super-critical regime.

Our result pushes the global well-posedness of the Landau-Coulomb equation (1.39) with (1.40) to initial data in the natural function space $L^1 \cap L \log L$, which is associated with the mass and the entropy of the particles. The strength of our approach is that it is

a combination and modification of results that were already present in the literature. It is thus a concise completion of the global existence results for the Landau equation with Coulomb potentials.

The proof is a combination of the monotonicity of the Fisher information, the entropy dissipation estimates, and the L^2 existence theory. Using more precise entropy and L^p estimates (instead of L^2), a straightforward adaptation of our proof can reduce the number of moments and slightly improve the regularisation rate in Theorem 1.9.

10.1 Proof of the main result, Theorem 1.9

10.1.1 Estimates on approximated coefficients

We build our solution using an approximation procedure. Let f_{in} be as in Theorem 1.9. We first introduce the sequence of approximated initial data:

$$f_{in,n} := (f_{in}\, 1_{|\cdot|\leq n}) * \chi_n + \frac{\mathcal{M}}{n}, \tag{10.1}$$

where $\mathcal{M}(v) := e^{-|v|^2}$, and χ_n is a standard mollifier. Note that

$$\int f_{in,n}(1, |v|^2, \ln f_{in,n})\, dv \leq C(f_{in}) + O\left(\frac{1}{n}\right),$$

(see for example beginning of Section 3.2 in [86]). We also introduce the approximate Landau-Coulomb kernel, defined as

$$K_n(|x|) = |x|^{-1}, \qquad \text{when} \qquad |x| \geq \frac{1}{n},$$

and $K_n(|x|) = -n^3 \frac{|x|^2}{2} + \frac{3}{2}n$ when $|x| \leq \frac{1}{n}$, in such a way that $K_n \in C^1(\mathbb{R}_+)$, and $K_n(|x|) \leq |x|^{-1}$ on $\mathbb{R}_+$. For any $g : \mathbb{R}^3 \to \mathbb{R}$ such that the convolution makes sense, and any $i, j = 1, 2, 3$, we define

$$A_n[g]_{ij} := K_n \Pi_{ij} * g, \qquad b_n[g]_i := \sum_j \partial_j (K_n \Pi_{ij}) * g,$$

$$c_n[g] := \sum_i \sum_j \partial_{ij} (K_n \Pi_{ij}) * g.$$

In the following we will denote by $\|\cdot\|_{L^p}$ or $\|\cdot\|_p$ the classical L^p norm on $\mathbb{R}^3$, and by $\|\cdot\|_{L^p_k}$ the norm defined by

$$\|g\|^p_{L^p_k} := \int_{\mathbb{R}^3} |g(v)|^p \langle v \rangle^{pk}\, dv,$$

for any $p \in [1, \infty[$, $k \in \mathbb{R}$, and $\langle v \rangle^2 := 1 + |v|^2$. We also define the entropy of a function g as the quantity

$$H(g) := \int_{\mathbb{R}^3} g(v) \ln g(v) \, dv.$$

The quantities $A_n[g]$, $b_n[g]$, $c_n[g]$ satisfy the following properties, uniformly with respect to n:

Proposition 10.1. *For any $n \in \mathbb{N} \setminus \{0\}$, $i, j \in \{1, 2, 3\}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ such that the convolutions are defined, one has*

$$|A_n[g]_{ij}(v)| \leq C \left(|g| * |\cdot|^{-1} \right)(v), \qquad \|A_n[g]_{ij}\|_\infty \leq C \left(\|g\|_{L^1} + \|g\|_{L^2} \right), \tag{10.2}$$

$$|b_n[g]_i(v)| \leq C \left(|g| * |\cdot|^{-2} \right)(v),$$

$$|c_n[g](v)| \leq C \, |g(v)| \, ,$$

where $C > 0$ is an absolute constant.

Proof. The proofs for all the inequalities use the fact that $K_n(|x|) \leq |x|^{-1}$. For the second inequality in (10.2) we recall (see for example Lemma 8.5) that

$$\|A_n[g]_{ij}\|_\infty \leq \|A[g]_{ij}\|_\infty \lesssim \|g\|_1^{1/3} \|g\|_2^{2/3}.$$

□

The following lower bounds also hold uniformly with respect to n:

Proposition 10.2. *For any $n \in \mathbb{N} \setminus \{0\}$, $g : \mathbb{R}^3 \to \mathbb{R}_+$ lying in $L^1_2(\mathbb{R}^3)$ and such that $\int g \, |\ln g| \, dv < +\infty$, we have*

$$A_n[g](v) : \xi \otimes \xi \geq c_0 \langle v \rangle^{-3} \, |\xi|^2, \quad \text{for any } v, \xi \in \mathbb{R}^3 \tag{10.3}$$

$$D_n(g) + 1 \geq c_1 \int_{\mathbb{R}^3} \frac{|\nabla g|^2}{g}(v) \langle v \rangle^{-3} \, dv, \tag{10.4}$$

where $D_n(g)$ is the entropy dissipation

$$D_n(g) := \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_n(|v - w|) \, g(v) \, g(w) \, \left(\frac{\nabla g}{g}(v) - \frac{\nabla g}{g}(w) \right)^T \times \Pi(v - w) \Big(\frac{\nabla g}{g}(v) - \frac{\nabla g}{g}(w) \Big) \, dv dw,$$

and c_0, c_1 are constants that depend only on (upper and lower bounds of) the mass and energy of g, and on an upper bound of $\int g \ln g \, dv$.

Proof. The lower bound for A_n can be computed using the calculations in Lemma 3.3. and 3.4 in [178]. We report the modified steps in the Appendix 10.2. The second estimate follows from Theorem 3 in [58]. □

10.1.2 The approximated solution: existence and properties

We consider the approximated equation

$$\partial_t f = \nabla \cdot \left[A_n[f] \nabla f - b_n[f] f \right] + \frac{1}{n} \Delta f = A_n[f] : \nabla \nabla f - c_n[f] f + \frac{1}{n} \Delta f, \tag{10.5}$$

with initial data as in (10.1). We address existence and uniqueness of smooth solutions to the approximated Landau equation in the following proposition. These solutions also satisfy a bound on the entropy and a propagation of moments.

Proposition 10.3. *For any $n \in \mathbb{N} \setminus \{0\}$, there exists a unique global smooth solution $0 < f_n := f_n(t,v) \in C^\infty(\mathbb{R}_+; \mathcal{S}(\mathbb{R}^3))$ to equation (10.5) satisfying the initial condition (10.1):*

$$f_n(0, \cdot) = f_{in,n},$$

and such that (for any $T > 0$)

$$\forall t \in [0,T], \qquad f_n(t,v) \geq C_{n,T}\, e^{-\frac{|v|^2}{2}}, \tag{10.6}$$

for some constant $C_{n,T} > 0$, depending on n and T.

Moreover, this solution satisfies the conservation of mass momentum and energy

$$\forall t \geq 0, \qquad \int_{\mathbb{R}^3} f_n(t,v)\, \psi(v)\, dv = \int_{\mathbb{R}^3} f_{in,n}(v)\, \psi(v)\, dv,$$

where $\psi = 1, v_i, \frac{|v|^2}{2}$, the entropy identity

$$\forall\, t \geq s \geq 0, \qquad H(f_n(t,\cdot)) + \int_s^t D_n(f_n(\sigma,\cdot))\, d\sigma = H(f_n(s,\cdot)), \tag{10.7}$$

and it has a monotonically decreasing in time Fisher information.

Finally, for any $k \geq 0$ and $T > 0$, there exists a constant $C > 0$ depending only on the mass and the energy of f_{in} and on $H(f_{in})$, and a constant $C_{k,T} > 0$ depending additionally on k, T, $\|f_{in}\|_{L^1_k(\mathbb{R}^3)}$, such that

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3} f_n(t,v) \langle v \rangle^k\, dv \leq C_{k,T}, \tag{10.8}$$

$$\sup_{t \in [0,T]} H(f_n(t,\cdot)) \leq C\,(1 + H(f_{in})). \tag{10.9}$$

Proof. The arguments in [186, Section 6] can be used to show existence, uniqueness and smoothness of f_n. Maximum principle applies, see [14], giving us the lower bound (10.6), so that

$$|\ln f_n(v)| \leq C_{n,T}\left(1+|v|^2\right),$$

which allows to make rigorous all formal manipulations involving the entropy. The proof of (10.8) and (10.9) follows along the lines of Proposition 4 in [58], using the uniform upper bound (10.2), and the fact that the moments of $f_{in,n}$ are bounded uniformly in n by the moments of f_{in}. □

10.1.3 Main uniform estimates

The next four propositions contain the crucial arguments. We will show that the function f_n has uniformly-in-n bounded (weighted) L^3 norms and Fisher information for all times $t > 0$. These bounds will yield the construction of a global in time solution which is smooth for all times away from zero.

The first step is to control a weighted L^2 norm of f_n at a certain time t_0. This time t_0 might depend on n. The bound is provided by the entropy inequality.

Proposition 10.4. *Let f_{in} be as in Theorem 1.9. Let $k > 2$. Fix also a time $0 < t \leq 1$. There exists a constant $C_k^* > 0$ depending only on k, $\|f_{in}\|_{L^1_{9+4k}(\mathbb{R}^3)}$ and $H(f_{in})$, the mass and energy of f_{in}, and a time $t_0 \in [0, \frac{t}{2}]$ (t_0 might depend on n) such that the solution f_n to (10.5), (10.1) constructed in Proposition 10.3 satisfies the estimate*

$$\|f_n(t_0,\cdot)\|^2_{L^2_k} = \int_{\mathbb{R}^3} f_n(t_0,v)^2\langle v\rangle^{2k}\,dv \leq \frac{C_k^*}{t^{3/2}}. \tag{10.10}$$

Proof. We start with the entropy dissipation estimate (remember that $t \leq 1$, estimate (10.7), and the elementary inequality (10.16))

$$\int_0^t D_n(f_n(s,\cdot))\,ds \leq C.$$

Estimate (10.4) implies

$$\int_0^t \int_{\mathbb{R}^3} \frac{|\nabla f_n(s,v)|^2}{f_n(s,v)}\langle v\rangle^{-3}\,dv\,ds \leq C.$$

Using the Sobolev estimate

$$\|g\|_{L^6(\mathbb{R}^3)} \leq C_{Sob}\,\|\nabla g\|_{L^2(\mathbb{R}^3)}$$

for $g := \sqrt{f_n}\langle v\rangle^{3/2}$ and the mass conservation, we end up with

$$\int_0^t \|f_n(s,\cdot)\|_{L^3_{-3}}\, ds \le C.$$

We now multiply both sides by $t/2$ (and restrict the integral to the interval $[0,t/2]$)

$$\frac{1}{t/2}\int_0^{t/2} \|f_n(s,\cdot)\|_{L^3_{-3}}\, ds \le \frac{2C}{t}.$$

The mean value theorem implies the existence of a time $t_0 \in [0,\frac{t}{2}]$ (depending on n) such that

$$\|f_n(t_0,\cdot)\|_{L^3_{-3}} \le \frac{2C}{t}.$$

Therefore, the interpolation inequality

$$\|g\|_{L^2_k} \le \|g\|^{3/4}_{L^3_{-3}}\, \|g\|^{1/4}_{L^1_{9+4k}},$$

and estimate (10.8) of Proposition 10.3 yield our bound

$$\|f_n(t_0,\cdot)\|_{L^2_k}\, ds \le \frac{C^*_k}{t^{3/4}},$$

provided that $\|f_{in}\|_{L^1_{9+4k}} < \infty$. □

The previous proposition shows that given $t>0$, any solution f_n to (10.5) has uniformly in n L^2-bounds at a certain time $0 < t_0 \le \frac{t}{2}$ (t_0 depends on n). The next proposition shows that this uniform bound can be propagated in a time interval $[t_0,t_1]$ with t_1 controlled.

Proposition 10.5. *Let $k \ge 9/4$, and f_n be the solution to* (10.5), (10.1) *constructed in Proposition 10.3 for an initial datum f_{in} with finite mass, entropy, and L^1_{9+4k} moment. There exists a constant $C'_k > 0$ depending only on k, $\|f_{in}\|_{L^1_{9+4k}(\mathbb{R}^3)}$, $H(f_{in})$ and the mass and energy of f_{in}, such that*

$$\sup_{\sigma\in[t_0,t_1]} \|f_n(\sigma,\cdot)\|^2_{L^2_k} \le \sqrt{2}\, \|f_n(t_0,\cdot)\|^2_{L^2_k}, \tag{10.11}$$

where $0 < t_0 \le 1/2$ and

$$t_1 := t_0 + (2C_k)^{-1} \ln\left(1 + \frac{1}{1+2\,\|f_n(t_0,\cdot)\|^4_{L^2_k}}\right) < 1,$$

where $C_k > 1$ is a constant depending only on k and c_0.

Proof. We proceed as in Lemma 4.1 in [8] (cf. also Chapter 9 and Section 8.4.1 in Chapter

8) , and get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{f_n^2}{2}\langle v\rangle^{2k}\,dv\\
&\quad=-\int_{\mathbb{R}^3}(\nabla f_n)^T\,A_n[f_n]\,\nabla f_n\langle v\rangle^{2k}\,dv+\,k\int_{\mathbb{R}^3}\nabla\otimes(v\langle v\rangle^{2k-2}):A_n[f_n]\,f_n^2\\
&\qquad+2k\int_{\mathbb{R}^3}b_n[f_n]\cdot v\langle v\rangle^{2k-2}\,f_n^2-\frac{1}{2}\int_{\mathbb{R}^3}c_n[f_n]\langle v\rangle^{2k}\,f_n^2\\
&\qquad-\frac{1}{n}\int_{\mathbb{R}^3}|\nabla f_n|^2\,\langle v\rangle^{2k}\,dv+\frac{k}{n}\int_{\mathbb{R}^3}f_n^2\nabla\cdot(v\langle v\rangle^{2k-2})\,dv\\
&\qquad\le-c_0\int_{\mathbb{R}^3}|\nabla f_n|^2\langle v\rangle^{2k-3}\,dv\\
&\qquad\quad+\,C_k\int_{\mathbb{R}^3}f_n^2\left[|A_n[f_n]|\langle v\rangle^{2k-2}+|b_n[f_n]|\,\langle v\rangle^{2k-1}+f_n\langle v\rangle^{2k}\right]dv\\
&\qquad\quad+\frac{C_k}{n}\int_{\mathbb{R}^3}f_n^2\langle v\rangle^{2k-2}\,dv\\
&\le-\frac{c_0}{2}\int_{\mathbb{R}^3}|\nabla(f_n\,\langle v\rangle^{k-\frac{3}{2}})|^2\,dv+\,C_k\left(\,\|A_n[f_n]\|_\infty+1\right)\int_{\mathbb{R}^3}f_n^2\langle v\rangle^{2k-2}\,dv\\
&\quad+C_k\int_{\mathbb{R}^3}f_n^2\left|f_n*|\cdot|^{-2}1_{|\cdot|\le1}\right|\,\langle v\rangle^{2k-1}dv\\
&\quad+\,C_k\int_{\mathbb{R}^3}f_n^2\left|f_n*|\cdot|^{-2}1_{|\cdot|\ge1}\right|\langle v\rangle^{2k-1}dv+C_k\,\|f_n\|^3_{L^3_{\frac{2}{3}k}}\\
&\le-\frac{c_0}{2C_{Sob}^2}\,\|f_n\|^2_{L^6_{k-\frac{3}{2}}}+C_k\left(\,\|f_n\|_{L^1}+\|f_n\|_{L^2}+1\right)\|f_n\|^2_{L^2_{k-1}}\\
&\quad+\,C_k\,\left\|f_n^2\langle v\rangle^{2k-1}\right\|_{L^{\frac{5}{4}}}\left\|f_n*|\cdot|^{-2}1_{|\cdot|\le1}\right\|_{L^5}+C_k\,\|f_n\|_{L^1}\,\|f_n\|^2_{L^2_{k-\frac{1}{2}}}\\
&\quad+C_k\,\|f_n\|^{3/2}_{L^6_{k-\frac{3}{2}}}\,\|f_n\|^{3/2}_{L^2_k}\,,
\end{aligned}
$$

using the interpolation

$$\|g\|_{L^3_{k-\frac{3}{4}}}\le\|g\|^{1/2}_{L^6_{k-\frac{3}{2}}}\,\|g\|^{1/2}_{L^2_k}\,,$$

and the inequality $k-\frac{3}{4}\ge\frac{2}{3}k$ when $k\ge\frac{9}{4}$. Then we use Young's inequality to get

$$\|f_n\|^{3/2}_{L^6_{k-\frac{3}{2}}}\,\|f_n\|^{3/2}_{L^2_k}\le\frac{c_0}{4C_{Sob}^2}\,\|f_n\|^2_{L^6_{k-\frac{3}{2}}}+C_k\,\|f_n\|^6_{L^2_k}\,,$$

and Young's inequality for convolutions and conservation of mass, so that

$$
\begin{aligned}
\frac{1}{2}\,\frac{d}{dt}\,\|f_n\|^2_{L^2_k}\le&-\frac{c_0}{4C_{Sob}^2}\,\|f_n\|^2_{L^6_{k-\frac{3}{2}}}+C_k\left(1+\|f_n\|_{L^2_k}\right)\,\|f_n\|^2_{L^2_k}\\
&+\,C_k\,\|f_n\|^2_{L^{\frac{5}{2}}_{k-\frac{1}{2}}}\,\|f_n\|_{L^2}+C_k\,\|f_n\|^2_{L^2_{k-\frac{1}{2}}}+C_k\,\|f_n\|^6_{L^2_k}\,.
\end{aligned}
$$

At this point, we use the interpolation

$$\|g\|_{L^{\frac{5}{2}}_{k-\frac{1}{2}}} \leq \|g\|^{3/10}_{L^6_{k-\frac{5}{3}}} \, \|g\|^{7/10}_{L^2_k},$$

and observe that $k - \frac{5}{3} \leq k - \frac{3}{2}$, so that using Young's inequality,

$$\|f_n\|^2_{L^{\frac{5}{2}}_{k-\frac{1}{2}}} \, \|f_n\|_{L^2} \leq C_k \, \|f_n\|^{3/5}_{L^6_{k-\frac{3}{2}}} \, \|f_n\|^{12/5}_{L^2_k} \leq \frac{c_0}{4C^2_{Sob}} \, \|f_n\|^2_{L^6_{k-\frac{3}{2}}} + C_k \, \|f_n\|^{24/7}_{L^2_k},$$

and denoting $Y := \|f_n\|^2_{L^2_k}$, we end up with the differential inequality

$$Y' \leq C_k \left(Y + Y^{3/2} + Y^{12/7} + Y^3 \right) \leq C_k (Y + Y^3).$$

To solve the above differential inequality, we use a transformation $g := -\frac{1}{2} Y^{-2}$ so that g solves

$$g'(\sigma) \leq C_k \, (-2g(\sigma) + 1), \qquad \sigma > t_0,$$

and is bounded by

$$g(\sigma) \leq \frac{1}{2} + e^{-2C_k(\sigma - t_0)} \left(g(t_0) - \frac{1}{2} \right), \qquad \sigma > t_0.$$

Going back with the transformation from g to Y, we recover

$$Y^2(\sigma) \leq \frac{1}{e^{-2C_k(\sigma - t_0)} \left(\frac{1}{Y^2(t_0)} + 1 \right) - 1}.$$

The estimate (10.11) holds as long as $\sigma - t_0 \leq (2C_k)^{-1} \ln \left(1 + \frac{1}{1 + 2\|f_n(t_0, \cdot)\|^4_{L^2_k}} \right)$. □

We now introduce an estimate for the derivative of a weighted entropy. Thanks to the weight, the new entropy production controls the classical Fisher information, up to lower order terms.

Proposition 10.6. *Let f_n be the solution to* (10.5), (10.1) *built in Poposition 10.3, for f_{in} with finite mass, entropy and L^1_6 moment. Let $H_3(f_n|\mathcal{M})$ be defined as*

$$H_3(f_n|\mathcal{M}) := \int_{\mathbb{R}^3} \Big(f_n \, \ln f_n - f_n + f_n \, |v|^2 + \mathcal{M}(v) \Big) \langle v \rangle^3 \, dv.$$

Then there exists a constant $K > 0$ depending only on the mass and energy of f_{in}, on

$\|f_{in}\|_{L^1_6(\mathbb{R}^3)}$, *and on* $H(f_{in})$, *such that when* $0 \le s_1 < s_2 \le 1$*:*

$$\begin{aligned} H_3(f_n|\mathcal{M})(s_2) + c_0 \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \frac{|\nabla f_n(\sigma,v)|^2}{f_n(\sigma,v)}\, dv d\sigma \\ \le H_3(f_n|\mathcal{M})(s_1) + K \int_{s_1}^{s_2} \Big(1 + \|f_n(\sigma,\cdot)\|_{L^2_2}\Big)^3\, d\sigma. \end{aligned} \tag{10.12}$$

Proof. We first observe that (similarly to the computation of [7], beginning of the proof of Prop. 5.5, cf. also [6])

$$\begin{aligned} \frac{d}{dt} H_3(f_n|\mathcal{M}) = &- \int_{\mathbb{R}^3} \langle v\rangle^3 \, (\nabla f_n)^T A_n[f_n] \frac{\nabla f_n}{f_n}\, dv - \int_{\mathbb{R}^3} \langle v\rangle^3 \, c_n[f_n]\, f_n\, dv \\ &+ 3 \int_{\mathbb{R}^3} A_n[f_n] : \nabla \otimes (v\langle v\rangle)\, (f_n \ln f_n - f_n)\, dv \\ &+ 6 \int_{\mathbb{R}^3} b_n[f_n] \cdot v\langle v\rangle\, (f_n \ln f_n - f_n)\, dv \\ &+ 2 \int_{\mathbb{R}^3} f_n\, b_n[f_n] \cdot \nabla(\langle v\rangle^3 |v|^2)\, dv \\ &+ \int_{\mathbb{R}^3} f_n\, A_n[f_n] : \nabla \otimes \nabla(\langle v\rangle^3 |v|^2)\, dv \\ &+ \frac{1}{n} \int_{\mathbb{R}^3} \Delta f_n \left(\ln f_n + |v|^2\right) \langle v\rangle^3\, dv. \end{aligned}$$

Thanks to the coercivity estimate (10.3), we know that

$$\int_{\mathbb{R}^3} \langle v\rangle^3 (\nabla f_n)^T A_n[f_n] \frac{\nabla f_n}{f_n}\, dv \ge c_0 \int_{\mathbb{R}^3} \frac{|\nabla f_n|^2}{f_n}\, dv. \tag{10.13}$$

Then systematically using the estimates of Proposition 10.1, we get first

$$\left| \int_{\mathbb{R}^3} \langle v\rangle^3 \, c_n[f_n]\, f_n\, dv \right| \le C \int_{\mathbb{R}^3} \langle v\rangle^3 \, f_n^2\, dv \le C\, \|f_n\|^2_{L^2_{3/2}}; \tag{10.14}$$

second,

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} A_n[f_n] : \nabla \otimes (v\langle v\rangle)\, (f_n \ln f_n - f_n)\, dv \right| \\ &\quad \le C\, \|A_n[f_n]\|_\infty \int_{\mathbb{R}^3} \langle v\rangle \Big[f_n (\ln f_n)^+ + f_n \langle v\rangle^2 + e^{-1} \mathcal{M}(v) \langle v\rangle^2 + f_n \Big]\, dv \\ &\quad \le C\, (\|f_n\|_{L^1} + \|f_n\|_{L^2}) \big(\|f_n\|^2_{L^2_{1/2}} + \|f_n\|_{L^1_3} + 1 \big), \end{aligned} \tag{10.15}$$

where we used the elementary inequalities (for $x > 0, v \in \mathbb{R}^3$)

$$x |\ln x| \le x\, (\ln x)^+ + x\langle v\rangle^2 + e^{-1} \mathcal{M}(v) \langle v\rangle^2, \qquad (\ln x)^+ \le x; \tag{10.16}$$

third, we get

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^3} b_n[f_n] \cdot v\langle v\rangle \left(f_n \ln f_n - f_n\right) dv \right| \\
&\quad \le C \int_{\mathbb{R}^3} \langle v\rangle^2 \left(f_n * |\cdot|^{-2} 1_{|\cdot|\le 1}\right) \left(f_n |\ln f_n| + f_n\right) dv \\
&\qquad + C \int_{\mathbb{R}^3} \langle v\rangle^2 \left(f_n * |\cdot|^{-2} 1_{|\cdot|\ge 1}\right) \left(f_n |\ln f_n| + f_n\right) dv \\
&\quad \le C \left\| f_n * |\cdot|^{-2} 1_{|\cdot|\le 1} \right\|_{L^4} \left\| \left[f_n (\ln f_n)^+ + f_n \langle v\rangle^2 + e^{-1} M(v) \langle v\rangle^2 + f_n \right] \langle v\rangle^2 \right\|_{L^{4/3}} \\
&\qquad + C \, \|f_n\|_{L^1} \int_{\mathbb{R}^3} \langle v\rangle^2 \left(f_n |\ln f_n| + f_n\right) dv \\
&\quad \le C \, \|f_n\|_{L^2} \left(\left\| [f_n^{4/3} [(\ln f_n)^+]^{4/3} \langle v\rangle^{8/3} + f_n^{4/3} \langle v\rangle^{16/3} \right\|_{L^1}^{3/4} + 1 \right) \\
&\qquad + C \, \|f_n\|_{L^1} \left\| [f_n (\ln f_n)^+ + f_n \langle v\rangle^2 + e^{-1} \mathcal{M}(v) \langle v\rangle^2 + f_n] \langle v\rangle^2 \right\|_{L^1} \\
&\quad \le C \, \|f_n\|_{L^2} \left(\|f_n\|_{L^2_{4/3}}^{3/2} + \|f_n\|_{L^2_2}^{1/2} \, \|f_n\|_{L^1_6}^{1/2} + 1 \right) \\
&\qquad + C \, \|f_n\|_{L^1} \left(\|f_n\|_{L^2_1}^2 + \|f_n\|_{L^1_4} + 1 \right),
\end{aligned}
\tag{10.17}
$$

where we used Young's inequality for convolutions, the elementary inequalities (10.16) and $(\ln x)^+ \le C x^{1/2}$, together with the interpolation inequality

$$
\|g\|_{L^{4/3}_4} \le \|g\|_{L^2_2}^{1/2} \, \|g\|_{L^1_6}^{1/2} ; \tag{10.18}
$$

fourth, we obtain

$$
\begin{aligned}
\left| \int_{\mathbb{R}^3} f_n \, A_n[f_n] : \nabla \otimes \nabla (\langle v\rangle^3 |v|^2) \, dv \right| &\le C \int_{\mathbb{R}^3} f_n \, |A_n[f_n]| \, \langle v\rangle^3 \, dv \\
&\le C \big(\|f_n\|_{L^1} + \|f_n\|_{L^2} \big) \, \|f_n\|_{L^1_3} ;
\end{aligned}
\tag{10.19}
$$

and fifth (using again interpolation (10.18) in the last inequality)

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^3} f_n \, b_n[f_n] \cdot \nabla (\langle v\rangle^3 |v|^2) \, dv \right| \\
&\qquad \le C \int_{\mathbb{R}^3} f_n \, |b_n[f_n]| \langle v\rangle^4 \, dv \\
&\qquad \le C \int_{\mathbb{R}^3} \langle v\rangle^4 \left(f_n * |\cdot|^{-2} 1_{|\cdot|\le 1}\right) f_n \, dv + C \int_{\mathbb{R}^3} \langle v\rangle^4 \left(f_n * |\cdot|^{-2} 1_{|\cdot|\ge 1}\right) f_n \, dv \\
&\qquad \le C \left\| f_n * |\cdot|^{-2} 1_{|\cdot|\le 1} \right\|_{L^4} \left\| f_n \langle v\rangle^4 \right\|_{L^{4/3}} + C \, \|f_n\|_{L^1} \, \|f_n\|_{L^1_4} \\
&\qquad \le C \, \|f_n\|_{L^2} \, \|f_n\|_{L^1_6}^{1/2} \, \|f_n\|_{L^2_2}^{1/2} + C \, \|f_n\|_{L^1} \, \|f_n\|_{L^1_4} .
\end{aligned}
\tag{10.20}
$$

Finally, the last term is bounded by

$$\begin{aligned}
&\frac{1}{n}\int_{\mathbb{R}^3}\Delta f_n\left(\ln f_n+|v|^2\right)\langle v\rangle^3\,dv\\
&\qquad\le\frac{1}{n}\int_{\mathbb{R}^3}\left[|f_n\ln f_n-f_n|\,|\Delta\langle v\rangle^3|+f_n|\Delta\left(\langle v\rangle^3|v|^2\right)|\right]\,dv\\
&\qquad\le\frac{1}{n}\left(\|f_n\|^2_{L^2_{1/2}}+\|f_n\|_{L^1_3}\right).
\end{aligned}\tag{10.21}$$

Regrouping estimates (10.13) to (10.21), using estimate (10.8) in Proposition 10.3 and integrating in (s_1,s_2), we get estimate (10.12). □

Thanks to Theorem 1.1 in [100], we know that the Fisher information for f_n is bounded (more explicitly monotone decreasing in time). This bound, however, depends on the Fisher information of the initial data and will be lost in the limit $n\to+\infty$. The next proposition shows that at time $\min(t_1,t)$ (depending on n), the Fisher information is finite and not depending on n. Thanks to its monotonicity, this uniform bound will hold for all times larger than $\min(t_1,t)$, and therefore for all times larger than t.

Proposition 10.7. *Let $f_{in}:=f_{in}(v)\ge 0$ be as in Theorem 1.9. For any $t\in]0,1]$, the solution f_n to the approximated Landau-Coulomb equation* (10.5), (10.1) *built in Proposition 10.3 satisfies*

$$\sup_{\sigma\in[t,+\infty)}\int_{\mathbb{R}^3}\frac{|\nabla f_n(\sigma,v)|^2}{f_n(\sigma,v)}\,dv\le C'\,t^{-9/2},\tag{10.22}$$

where C' depend only on the mass, energy and entropy of f_{in}, and on $||f_{in}||_{L^1_{18}}$.

Proof. We first observe that (for any $g\ge 0$ such that the quantity exists)

$$H_3(g|\mathcal{M})=\int_{\mathbb{R}^3}\left[\frac{g}{\mathcal{M}}\ln\left(\frac{g}{\mathcal{M}}\right)-\frac{g}{\mathcal{M}}+1\right]\mathcal{M}\langle v\rangle^3\,dv\ge 0,\tag{10.23}$$

and (using both inequalities of (10.16))

$$\begin{aligned}
H_3(g|\mathcal{M})&\le\int_{\mathbb{R}^3}[\,g|\ln g|+g\langle v\rangle^2+\mathcal{M}(v)]\langle v\rangle^3\,dv\\
&\le C\int_{\mathbb{R}^3}[g(\ln g)^+ +g\langle v\rangle^2+\mathcal{M}(v)\langle v\rangle^2]\langle v\rangle^3\,dv\\
&\le C\,(\|g\|^2_{L^2_{3/2}}+\|g\|_{L^1_5}+1).
\end{aligned}\tag{10.24}$$

Let t_0 and t_1 be as in Propositions 10.4 and 10.5 with $k=9/4$. We use (10.12) with

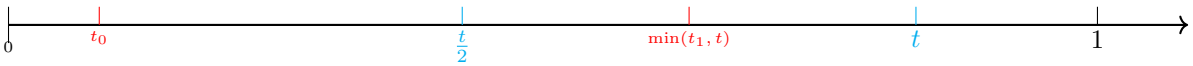


Figure 10.1: The timeline. Note that t is independent of n, whereas t_0 and t_1 depend on n.

$s_1 = t_0$ and $s_2 = \min(t_1, t)$. We get

$$H_3(f_n|M)(\min(t_1,t)) + c_0 \int_{t_0}^{\min(t_1,t)} \int_{\mathbb{R}^3} \frac{|\nabla f_n(s,v)|^2}{f_n(s,v)}\, dvds \\ \leq H_3(f_n|M)(t_0) + K\left(\min(t_1,t) - t_0\right) \sup_{s\in[t_0,\min(t_1,t)]} \left(1 + \|f_n(s,\cdot)\|_{L^2_2}\right)^3.$$

Thanks to the monotonicity of the Fisher information [100, Theorem 1.1], using estimates (10.23) and (10.24), as well as (10.8) from Proposition 10.3, we see that

$$\begin{aligned} c_0 \int_{\mathbb{R}^3} \frac{|\nabla f_n(\tau,v)|^2}{f_n(\tau,v)}\, dvds \\ &\leq \frac{C}{\min(t_1,t) - t_0} \left(1 + \|f_n(t_0,\cdot)\|^2_{L^2_{3/2}}\right) + K \sup_{s\in[t_0,\min(t_1,t)]} \left(1 + \|f_n(s,\cdot)\|_{L^2_2}\right)^3 \\ &\leq \frac{C}{\min(t_1,t) - t_0} \left(1 + \|f_n(t_0,\cdot)\|^2_{L^2_{9/4}}\right) + K \sup_{s\in[t_0,\min(t_1,t)]} \left(1 + \|f_n(s,\cdot)\|_{L^2_{9/4}}\right)^3, \end{aligned}$$

for $\tau = \min(t_1, t)$. We now use Proposition 10.4 (with $k = 9/4$, so that $4k + 9 = 18$), and get

$$\|f_n(t_0,\cdot)\|_{L^2_{9/4}} \leq C^*_{9/4}\, t^{-3/4}.$$

Then Proposition 10.5 (applied with $k = 9/4$) yields

$$\sup_{s\in[t_0,\min(t_1,t)]} \|f_n(s,\cdot)\|_{L^2_{9/4}} \leq 2^{1/4}\, \|f_n(t_0,\cdot)\|_{L^2_{9/4}} \leq 2^{1/4}\, C^*_{9/4}\, t^{-3/4}, \tag{10.25}$$

and we find

$$\int_{\mathbb{R}^3} \frac{|\nabla f_n(\tau,v)|^2}{f_n(\tau,v)}\, dv \leq \frac{1}{\min(t_1,t) - t_0} \int_{t_0}^{\min(t_1,t)} \int_{\mathbb{R}^3} \frac{|\nabla f_n(s,v)|^2}{f_n(s,v)}\, dvds \\ \leq \frac{C}{\min(t_1,t) - t_0} \left(1 + (C^*_{9/4}\, t^{-3/4})^2\right) + C\left(1 + 2^{1/4}\, C^*_{9/4}\, t^{-3/4}\right)^3.$$

We note that (remember that $t \in]0, 1]$)

$$t_1 - t_0 = (2C_{9/4})^{-1} \ln \left(1 + \frac{1}{1 + 2 \left\| f_n(t_0, \cdot) \right\|_{L^2_{9/4}}^4} \right)$$
$$\geq \left\{ (2C_{9/4})^{-1} \ln \left[1 + \frac{1}{1 + 2 \left(C^*_{9/4} \, t^{-3/4} \right)^4} \right] \right\} \geq D^{-1} \ln \left(1 + \frac{t^3}{t^3 + C} \right)$$
$$\geq D^{-1} \ln \left(1 + \frac{t^3}{1 + C} \right) \geq D^{-1} \, t^3,$$

(where C, D are constants regrouping previous constants), and (since $t - t_0 \geq \frac{t}{2}$)

$$\min(t_1, t) - t_0 \geq \min(D^{-1} \, t^3, \frac{t}{2}) \geq D^{-1} \, t^3,$$

so that

$$\int_{\mathbb{R}^3} \frac{|\nabla f_n(\tau, v)|^2}{f_n(\tau, v)} \, dv \leq \frac{K}{t^3} (1 + K \, t^{-3/2}) + K \, (1 + K \, t^{-3/4})^3 \leq K \, t^{-9/2}.$$

Thanks to the monotonicity of the Fisher information [100, Theorem 1.1], we then conclude that

$$\sup_{\sigma \in [\min(t_1, t), +\infty)} \int_{\mathbb{R}^3} \frac{|\nabla f_n(\sigma, v)|^2}{f_n(\sigma, v)} \, dv \leq K \, t^{-9/2}, \tag{10.26}$$

which implies (10.22).

□

10.1.4 The limit

We now finish the proof of Theorem 1.9 by taking the limit $n \to +\infty$. We restate here Theorem 1.9 in a slightly more precise way, presenting the

Theorem 10.1. *Let $f_{in} := f_{in}(v) \geq 0$ lying in $L^1_{18}(\mathbb{R}^3)$ be such that $H(f_{in})$ is finite. Then, there exists a (global in time) H-solution (also weak solution in the sense given by [58]) f to the Landau-Coulomb equation* (1.39) *with* (1.40) *with initial datum f_{in}. Moreover this solution is strong, and belongs to $C^\infty \cap L^1_{18}(\mathbb{R}^3))$ on any time interval $(0, +\infty)$. Finally, f satisfies for all $t > 0$ the bound:*

$$\sup_{\sigma \in [t, +\infty)} \int_{\mathbb{R}^3} \frac{|\nabla f(\sigma, v)|^2}{f(\sigma, v)} \, dv \leq C' \, t^{-9/2},$$

where C' depends only on the mass and energy of f_{in}, $H(f_{in})$ and $\|f_{in}\|_{L^1_{18}}$.

Proof. Up to subsequences, the sequence of solutions f_n found in Proposition 10.3 converges, as $n \to +\infty$ to a H-solution f to the Landau-Coulomb equation (see [186] Section 6 for

details), which satisfies the following weak formulation

$$\int f_{in}\varphi(\cdot,0)\,dv + \int_0^T \int f\partial_t\varphi\,dvds$$
$$= \int_0^T \iint \left(\frac{f(v)f(w)}{|v-w|}\right)^{1/2} (\nabla_v\varphi - \nabla_w\varphi)\cdot\Pi(v-w)(\nabla_v - \nabla_w)\left(\frac{f(v)f(w)}{|v-w|}\right)^{1/2} dvdwds,$$

for all test functions $\varphi \in C_0^2([0,T)\times\mathbb{R}^3)$. This H-solution is also a weak solution in the sense of Corollary 11 in [58]. Moreover $f_n \to f$ almost everywhere in time and velocity thanks to Aubin-Lions lemma, $f \in C_w([0,T], L^1(\mathbb{R}^3))$, and, thanks to Fatou lemma, using (10.22) (and the link between $\frac{|\nabla f_n|^2}{f_n}$ and $|\nabla\sqrt{f_n}|^2$), that is (for $t>0$)

$$\sup_{[t,+\infty)} \int_{\mathbb{R}^3} |\nabla\sqrt{f_n}|^2\,dv \le C'\,t^{-9/2},$$

we see that

$$\sup_{[t,+\infty)} \int_{\mathbb{R}^3} |\nabla\sqrt{f}|^2\,dv \le C'\,t^{-9/2}.$$

At this point, classical arguments imply that f is bounded for all times greater than t and therefore smooth [8, 94, 178] (and satisfies the equation in a strong sense).

□

10.2 Appendix: Coercivity for the truncated kernel

Proof of (10.3): Let f be a function with bounded mass, second moment and entropy. Following the same computations as in Lemma 3.1 in [178], we can show that there exists $\ell,\ R,\ \mu$ not depending on n such that the measure of the set

$$B := \{v \in B_R \mid f \ge \ell\}$$

is smaller (or equal) than μ. Let us now show (10.3); since $K_n \le K_{n+1}$, it is enough to show the lower bound for K_1. Let e be an unitary vector. We have

$$A_1[g](v) : e\otimes e = \int f(v-w)K_1(w)(1-\cos^2\theta)dw,$$

where θ is the angle between w and e. Hence

$$A_1[g](v) : e\otimes e \ge \ell \int_{\{w-v\in B_R\}\cap\{f(v-w)\ge\ell\}} K_1(w)(1-\cos^2\theta)dw,$$

Since $(1-\cos^2\theta) \sim 0$ when w is parallel to e, and since w belongs to a subset of B_R+v, the smallest value for $A_1[g](v) : e \otimes e$ is obtained when $e = \frac{v}{|v|}$. Let v be in a neighborhood of the origin. Then one can find a cone with center at the origin and aperture ε in direction $e = \frac{v}{|v|}$ such that the intersection of this cone with the set $\{B+v\}$ has measure smaller than $\mu/4$. In this set $(1-\cos^2\theta) \geq \varepsilon^2$ and

$$A_1[g](v) : e \otimes e \geq \ell \int_{\{w-v \in B_R\} \cap \{f(v-w) \geq \ell\} \cap \text{cone}_\varepsilon} K_1(w)(1-\cos^2\theta)dw \geq c(\ell, R, \mu).$$

If v is away from the origin, then $K(w) = \frac{1}{|w|}$ in $\{B+v\}$ and

$$A_1[g](v) : e \otimes e \geq \ell \int_{\{w-v \in B_R\} \cap \{f(v-w) \geq \ell\} \cap \text{cone}_\varepsilon} \frac{1}{|w|}(1-\cos^2\theta)dw \geq c(\ell, R, \mu) \geq \frac{c_0}{(1+|v|^3)},$$

as in the classical case.

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