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Brownian snails with removal: epidemics in diffusing populations*

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Abstract

Two stochastic models of susceptible/infected/removed (SIR) type are introduced for the spread of infection through a spatially-distributed population. Individuals are initially distributed at random in space, and they move continuously according to independent diffusion processes. The disease may pass from an infected individual to an uninfected individual when they are sufficiently close. Infected individuals are permanently removed at some given rate α. Such processes are reminiscent of so-called frog models, but differ through the action of removal, as well as the fact that frogs jump whereas snails slither.

Two models are studied here, termed the 'delayed diffusion' and the 'diffusion' models. In the first, individuals are stationary until they are infected, at which time they begin to move; in the second, all individuals start to move at the initial time 0. Using a perturbative argument, conditions are established under which the disease infects a.s. only finitely many individuals. It is proved for the delayed diffusion model that there exists a critical value $\alpha_c \in (0, \infty)$ for the survival of the epidemic.

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1 Introduction

1.1 Outline of the models

Numerous mathematical models have been introduced to describe the spread of a disease around a population. Such models may be deterministic or stochastic, or a mixture of each; they may incorporate a range of factors including susceptibility,

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infectivity, recovery, and removal; the population members (termed 'particles') may be distributed about some given space; and so on. We propose two models in which (i) the particles move randomly about the space that they inhabit, (ii) infection may be passed between particles that are sufficiently close to one another, and (iii) after the elapse of a random time since infection, a particle is removed from the process. These models differ from that of Beckman, Dinan, Durrett, Huo, and Junge [3] through the introduction of the permanent 'removal' of particles, and this new feature brings a significant new difficulty to the analysis.

We shall concentrate mostly on the case in which the particles inhabit $\mathbb{R}^d$ where $d \geq 2$. Here is a concrete example of the processes studied here.

(a) Particles are initially distributed in $\mathbb{R}^d$ in the manner of a rate-λ Poisson process conditioned to contain a point at the origin 0.

(b) Particles move randomly within $\mathbb{R}^d$ according to independent Brownian motions with variance-parameter σ^2.

(c) At time 0 the particle at the origin (the initial 'infective') suffers from an infectious disease, which may be passed to others when sufficiently close.

(d) When two particles, labelled P and P', are within a given distance δ, and P is already infected, then particle P' becomes infected.

(e) Each particle is infected for a total period of time having the exponential distribution with parameter $\alpha \in [0, \infty)$, and is then permanently removed.

The fundamental question is to determine for which vectors $(\lambda, \delta, \sigma, \alpha)$ it is the case that (with strictly positive probability) infinitely many particles become infected. For simplicity, we shall assume henceforth that

$$\delta = \sigma = 1. \tag{1.1}$$

We shall generally assume $\alpha > 0$. In the special case $\alpha = 0$, (studied in [3]) a particle once infected remains infected forever, and the subsequent analysis is greatly facilitated by a property of monotonicity that is absent in the more challenging case $\alpha > 0$ considered in the current work.

Two protocols for movement feature in this article.

A. *Delayed diffusion model.* The initial infective starts to move at time 0, and all other particles remain stationary until they are infected, at which times they begin to move.

B. *Diffusion model.* All particles begin to move at time 0.

The main difficulty in studying these models arises from the fact that particles are permanently removed after a (random) period of infectivity. This introduces a potential non-monotonicity into the model, namely that the presence of infected particles may hinder the growth of the process through the creation of islands of 'removed' particles that can act as barriers to the further spread of infection. A related situation (but without the movement of particles) was considered by Kuulasma [21] in a discrete setting, and the methods derived there are useful in our Section 3.7 (see also Alves et al. [1, p. 4]). This issue may be overcome for the delayed diffusion model, but remains problematic in the case of the diffusion process.

Let I denote the set of particles that are ever infected, and

$$\theta(\lambda, \alpha) := \mathbb{P}_{\lambda,\alpha}(|I| = \infty). \tag{1.2}$$

We say the process

$$\begin{aligned} \text{becomes extinct} \quad & \text{if } \theta(\lambda,\alpha) = 0, \\ \text{survives} \quad & \text{if } \theta(\lambda,\alpha) > 0. \end{aligned}$$

Let λ_c denote the critical value of λ for the disk (or 'Boolean') percolation model with radius 1 on $\mathbb{R}^d$ (see, for example, [25]). It is immediate for both models above that

$$\theta(\lambda,\alpha) > 0 \qquad \text{if } \lambda > \lambda_c \text{ and } \alpha \geq 0, \tag{1.3}$$

since in that case the disease spreads instantaneously to the percolation cluster C containing the initial infective, and in addition we have $\mathbb{P}_{\lambda,\alpha}(|C| = \infty) > 0$.

1.2 Two exemplars of results

We write θ_d (respectively, θ_{dd}) for the function θ of (1.2) in the case of the diffusion model (respectively, delayed diffusion model). The following two theorems are proved in Sections 3 and 4 as special cases of results for more general epidemic models than those given above.

Theorem 1.1 (Brownian delayed diffusion model)**.** *Let $d \geq 2$. There exists a non-decreasing function $\alpha_c : (0,\infty) \to (0,\infty]$ such that*

$$\theta_{dd}(\lambda,\alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_c(\lambda), \\ > 0 & \text{if } \alpha < \alpha_c(\lambda). \end{cases} \tag{1.4}$$

Furthermore, $\alpha_c(\lambda) = \infty$ when $\lambda > \lambda_c$, and there exists $\underline{\lambda} \in (0, \lambda_c]$ such that $\alpha_c(\lambda) < \infty$ when $0 < \lambda < \underline{\lambda}$.

The delayed diffusion model has no phase transition when $d = 1$; see Theorems 3.2 and 3.6.

Theorem 1.2 (Brownian diffusion model)**.** *Let $d \geq 1$. There exists $\underline{\lambda} \in (0, \lambda_c]$ and a non-decreasing function $\underline{\alpha} : (0, \underline{\lambda}) \to (0,\infty)$ such that $\theta_d(\lambda,\alpha) = 0$ when $\alpha > \underline{\alpha}(\lambda)$ and $0 < \lambda < \underline{\lambda}$.*

For the diffusion model, we have no proof of survival for $d \geq 2$ and small positive α (that is, that $\theta_d(\lambda,\alpha) > 0$ for some $\lambda < \lambda_c$ and $\alpha > 0$), and neither does the current work answer the question of whether or not survival *ever* occurs when $d = 1$. See Section 4.3. The above theorems are proved using a perturbative argument, and thus fall short of the assertion that $\underline{\lambda} = \lambda_c$.

The methods of proof may be made quantitative, leading to bounds for the numerical values of the critical points α_c. Such bounds are far from precise, and therefore we do not explore them here. Our basic estimates for the growth of infection hold if the intensity λ of the Poisson process is non-constant so long as it is bounded uniformly between two strictly positive constants. The existence of the subcritical phase may be proved for more general diffusions than Brownian motion.

1.3 Literature and notation

The related literature is somewhat ramified, and a spread of related problems have been studied by various teams. We mention a selection of papers but do not attempt a full review, and we concentrate on works associated with $\mathbb{R}^d$ rather than with trees or complete graphs.

The delayed diffusion model may be viewed as a continuous-time version of the 'frog' random walk process studied in Alves et al. [1, 2], Ramirez and Sidoravicius [29], Fontes et al. [7], Benjamini et al. [4], and Hoffman, Johnson, and Junge [15, 16]. See Popov [28]

for an early review. Kesten and Sidoravicius [17, 18, 19] considered a variant of the frog model as a model for infection, both with and without recuperation (that is, when infected frogs recover and become available for reinfection—see also Section 4.3 of the current work). The paper of Beckman et al. [3] is devoted to the delayed diffusion model without removal (that is, with $\alpha = 0$). Peres et al. [27] studied three geometric properties of a Poissonian/Brownian cloud of particles, in work inspired in part by the dynamic Boolean percolation model of van den Berg et al. [5]. Related work has appeared in Gracar and Stauffer [9].

A number of authors have considered the frog model with recuperation under the title 'activated random walks'. The reader is referred to the review by Rolla [30], and for recent work to Stauffer and Taggi [33] and Rolla et al. [31].

We write $\mathbb{Z}_0 = \{0, 1, 2, \dots\}$ and 1_A (or $1(A)$) for the indicator function of an event or set A. Let $S(r)$ denote the closed r-ball of $\mathbb{R}^d$ with centre at the origin, and $S = S(1)$. The d-dimensional Lebesgue measure of a set A is written $|A|_d$, and the Euclidean norm $\|\cdot\|_d$. The *radius* of $M \subseteq \mathbb{R}^d$ is defined by

$$\mathrm{rad}(M) := \sup\{\|m\|_d : m \in M\}.$$

We abbreviate $\mathbb{P}_{\lambda,\alpha}$ (respectively, $\mathbb{E}_{\lambda,\alpha}$) to the generic notation $\mathbb{P}$ (respectively, $\mathbb{E}$).

The contents of this paper are as follows. The two models are defined in Section 2 with a degree of generality that includes general diffusions and a more general process of infection. The delayed diffusion model is studied in Section 3, and the diffusion model in Section 4. Theorem 1.1 (respectively, Theorem 1.2) is contained within Theorem 3.1 (respectively, Theorem 4.1).

1.4 Open problems

This introduction closes with a short account of some of the principal remaining open problems. For concreteness, we restrict ourselves to the Brownian models of Section 1.2 without further reference to the random-walk versions of these models, and the general models of Section 2.1. This section is positioned here despite the fact that it refers sometimes to versions of the models that have not yet been fully introduced (see Section 2).

A. For the Brownian delayed diffusion model, show that the critical value $\alpha_{\mathrm{c}}(\lambda)$ of Theorem 1.1 satisfies $\alpha_{\mathrm{c}}(\lambda) < \infty$ whenever $\lambda < \lambda_{\mathrm{c}}$.

B. When $d \geq 2$, prove survival in the Brownian diffusion model for some $\lambda < \lambda_{\mathrm{c}}$ and small $\alpha > 0$. More specifically, show that $\theta_{\mathrm{d}}(\lambda, \alpha) > 0$ for suitable λ and α.

C. Having resolved problem B, show the existence of a critical value $\alpha_{\mathrm{c}} = \alpha_{\mathrm{c}}(\lambda)$ for the Brownian diffusion model such that survival occurs when $\alpha < \alpha_{\mathrm{c}}$ and not when $\alpha > \alpha_{\mathrm{c}}$. Furthermore, identify the set of λ such that $\alpha_{\mathrm{c}}(\lambda) < \infty$.

D. Decide whether or not survival can ever occur for the Brownian diffusion model in one dimension.

E. In either model, prove a shape theorem for the set of particles that are either infected or removed at time t.

2 General models

2.1 The general set-up

Let $d \geq 1$. A *diffusion process* in $\mathbb{R}^d$ is a solution ζ to the stochastic differential equation

$$d\zeta(t) = a(\zeta(t))\, dt + \sigma(\zeta(t))\, dW_t, \tag{2.1}$$

where W is a standard Brownian motion in $\mathbb{R}^d$. (We may write either W_t or $W(t)$.) For definiteness, we shall assume that: $\zeta(0) = 0$; ζ has continuous sample paths; the instantaneous drift vector a and variance matrix σ are locally Lipschitz continuous. We do not allow a, σ to be time-dependent. We call the process 'Brownian' if ζ is a standard Brownian motion, which is to say that a is the zero vector and σ is the identity matrix.

Let ζ be such a diffusion, and let $(\zeta_i : i \in \mathbb{Z}_0)$ be independent copies of ζ. Let $\alpha \in (0, \infty)$, $\rho \in [0, \infty)$, and let $\mu : \mathbb{R}^d \to [0, \infty)$ be integrable with

$$\mathrm{Int}(\mu) := \int_{\mathbb{R}^d} \mu(x)\, dx \in (0, \infty). \tag{2.2}$$

We call μ *radially decreasing* if

$$\mu(rx) \le \mu(x) \qquad x \in \mathbb{R}^d,\, r \in [1, \infty). \tag{2.3}$$

Let $\Pi = (X_0 = 0, X_1, X_2, \dots)$ be a Poisson process on $\mathbb{R}^d$ (conditioned to possess a point at the origin 0) with constant intensity $\lambda \in (0, \infty)$. At time 0, particles with label-set $\mathcal{P} = \{P_0, P_1, P_2, \dots\}$ are placed at the respective points $X_0 = 0, X_1, X_2, \dots$. We may refer to a particle P_i by either its index i or its initial position X_i.

We describe the process of infection in a somewhat informal manner (see also Section 2.4). For $i \in \mathbb{Z}_0$, at any given time t particle P_i is in one of three states S (susceptible), I (infected), and R (removed). Thus the state space is $\Omega = \Pi \times \{\mathrm{S}, \mathrm{I}, \mathrm{R}\}^{\mathbb{Z}_0}$, and we write $\omega(t) = (\omega_i(t) : i \in \mathbb{Z}_0)$ for the state of the process at time t. Let S_t (respectively, I_t, R_t) be the set of particles in state S (respectively, I, R) at time t. We take

$$\omega_i(0) = \begin{cases} \mathrm{I} & \text{if } i = 0, \\ \mathrm{S} & \text{otherwise,} \end{cases}$$

so that $I_0 = \{P_0\}$ and $S_0 = \mathcal{P} \setminus \{P_0\}$. The only particle-transitions that may occur are S $\to$ I and I $\to$ R. The transitions S $\to$ I occur at rates that depend on the locations of the currently infected particles.

We shall refer to the above (in conjunction with the specific infection assumptions of Sections 2.2 or 2.3) as the *general model*. When $a \equiv 0$ and $\sigma \equiv 1$ in (2.1) (or, more generally, σ is constant), we shall refer to it as the *Brownian model*. We shall prove the existence of a subcritical phase (characterized by the absence of survival) for the general model subject to weak conditions. Our proof of survival for the delayed diffusion model is for the Brownian model alone. Estimates for the volume of the sausage generated by ζ play roles in the calculations, and it may be that, in this regard or another, the behaviour of a general model is richer than that of its Brownian version.

2.2 Delayed diffusion model

Each particle P_j is stationary if and only if it is in state S. If it becomes infected (at some time B_j, see (2.5)), henceforth it follows the diffusion $X_j + \zeta_j$. We write

$$\pi_j(t) = \begin{cases} X_j & \text{if } t \le B_j, \\ X_j + \zeta_j(t - B_j) & \text{if } t > B_j, \end{cases}$$

for the position of P_j at time t.

We describe next the rate at which a given particle P infects another particle P'. The function μ, given above, encapsulates the spatial aspect of the infection process, and a parameter $\rho \in (0, \infty)$ represents its intensity,

(S → I) Let $t > 0$, and let P_j be a particle that is in state S at all times $s < t$. Each $P_i \in I_t$ (with $i \neq j$) infects P_j at rate $\rho\mu(X_j - \pi_i(t))$. The aggregate rate at which P_j becomes infected is

$$\sum_{i \in I_t,\, i \neq j} \rho\mu(X_j - \pi_i(t)). \tag{2.4}$$

(I → R) An infected particle is removed at rate α.

Transitions of other types are not permitted. We take the sample path $\omega = (\omega(t) : t \geq 0)$ to be pointwise right-continuous, which is to say that, for $i \in \mathbb{Z}_0$, the function $\omega_i(\cdot)$ is right-continuous. The *infection time* B_j of particle P_j is given by

$$B_j = \inf\{t \geq 0 : P_j \in I_t\}. \tag{2.5}$$

The infection rates $\rho\mu(X_j - \pi_i(t))$ of (2.4) are finite, and hence infections take place at a.s. distinct times. We may thus speak of P_j as being 'directly infected' by P_i. We speak of a point $z \in \Pi$ as being *directly infected* by a point $y \in \Pi$ when the associated particles have that property. If P_j is infected directly by P_i, we call P_j a *child* of P_i, and P_i the *parent* of P_j.

Following its infection, particle P_i remains infected for a further random time T_i, called the *lifetime* of P_i, and is then removed. The times T_i are random variables with the exponential distribution with parameter $\alpha > 0$, and are independent of one another and of the X_j and ζ_j.

In the above version of the delayed diffusion model, ρ is assumed finite. When $\rho = \infty$, we shall consider only situations in which

$$\rho = \infty, \qquad \mu = 1_M \text{ where } M \subseteq \mathbb{R}^d \text{ is compact.} \tag{2.6}$$

In this situation, a susceptible particle P_j becomes infected at the earliest instant that it belongs to $\pi_i(t) + M$ for some $P_i \in I_t$, $i \neq j$. This happens when either (i) P_i infects P_j as P_i diffuses around $\mathbb{R}^d$ post-infection, or (ii) at the moment B_i of infection of P_i, particle P_j is infected instantaneously by virtue of the fact that $X_j \in \pi_i(B_i) + M$. These two situations are investigated slightly more fully in the following definition of 'direct infection'.

The role of the Boolean model of continuum percolation becomes clear when $\rho = \infty$, and we illustrate this, subject to the simplifying assumption that M is symmetric in the sense that $x \in M$ if and only if $-x \in M$. Let $\Pi = (X_i : i \in \mathbb{Z}_0)$ be a Poisson process in $\mathbb{R}^d$ with constant intensity λ, and declare two points X_i, X_j to be *adjacent* if and only if $X_j - X_i \in M$. This adjacency relation generates a graph G with vertex-set Π. In the delayed diffusion process on the set Π, entire clusters of the percolation process are infected simultaneously.

Since there can be many (even infinitely many) simultaneous infections at the same time instant when $\rho = \infty$, the notion of 'direct infection' requires amplification. For $j \neq 0$, we say that P_j is *directly infected* by P_0 if P_j is in state S at all times $s < Y_j$, where $Y_j = \inf\{t \geq 0 : X_j \in \zeta_0(t) + M\}$, and in addition $Y_j < T_0$. We make a similar definition, as follows, for direct infections by P_i with $i \neq 0$. Let $i \neq 0$ and $j \neq 0, i$.

(a) We say that P_j is *dynamically infected* by P_i if the following holds. Particle P_i (respectively, P_j) is in state I (respectively, state S) at all times $Y_{i,j} - \epsilon$ for $\epsilon \in (0, \epsilon_0)$ and some $\epsilon_0 > 0$, where $Y_{i,j} = \inf\{t \geq B_i : X_j \in X_i + \zeta_i(t) + M\}$, and in addition $Y_{i,j} - B_i \leq T_i$.

(b) We say that P_j is *instantaneously infected* by P_i if the following holds. There exist $n \geq 1$ and $k, i_1, i_2, \ldots, i_n = i, i_{n+1} = j$ such that i_1 is dynamically infected by k (at time B_{i_1}) and

$$X_{i_{m+1}} \in T_m \setminus T_{m-1}, \qquad m = 1, 2, \ldots, n,$$

where

$$T_0 := \varnothing, \quad T_m = \bigcup_{r=1}^{m} (X_r + M).$$

Condition (a) corresponds to infection through movement of P_i, and (b) corresponds to instantaneous infection at the moment of infection of P_i. We say that P_j is *directly infected* by P_i if it is infected by P_i either dynamically or instantaneously.

Certain events of probability 0 are overlooked in the above informal description including, for example, the event of being dynamically infected by two or more particles, and the event of being instantaneously infected by an infinite chain (i_r) but by no finite chain.

In either case $\rho < \infty$ or $\rho = \infty$, we write $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ for the probability that infinitely many particles are infected. For concreteness, we note our special interest in the case in which:

(a) ζ is a standard Brownian motion,

(b) $\mu = 1_S$ with S the closed unit ball of $\mathbb{R}^d$.

2.3 Diffusion model

The diffusion model differs from the delayed diffusion model of Section 2.2 in that all particles begin to move at time $t = 0$. The location of P_j at time t is $X_j + \zeta_j(t)$, and the transition rates are given as follows. First, suppose $\rho \in (0, \infty)$.

(S $\to$ I) Let $t > 0$, and let P_j be susceptible at all times $s < t$. Each $P_i \in I_t$ (with $i \neq j$) infects P_j at rate $\rho\mu(X_j + \zeta_j(t) - X_i - \zeta_i(t))$. The aggregate rate at which P_j becomes infected is

$$\sum_{i \in I_t,\, i \neq j} \rho\mu\big(X_j + \zeta_j(t) - X_i - \zeta_i(t)\big). \tag{2.7}$$

(I $\to$ R) An infected particle is removed at rate α.

As in Section 2.2, we may allow $\rho = \infty$ and $\mu = 1_M$ with M compact. In either case $\rho < \infty$ or $\rho = \infty$ we write $\theta_{\mathrm{d}}(\lambda, \rho, \alpha)$ for the probability that infinitely many particles are infected.

2.4 Construction

We shall not investigate the formal construction of the above processes as strong Markov processes with right-continuous sample paths. The interested reader may refer to the related model involving random walks on $\mathbb{Z}^d$ (rather than diffusions or Brownian motions on $\mathbb{R}^d$) with $\alpha = 0$, as considered in some depth by Kesten and Sidoravicius in [17] and developed for the process with 'recuperation' in their sequel [18].

Instead, we sketch briefly how such processes may be built around a triple $(\Pi, \boldsymbol{\zeta}, \boldsymbol{\delta})$, where Π is a rate-λ Poisson process of initial positions, $\boldsymbol{\zeta} = (\zeta_i : i \in \mathbb{Z}_0)$ is a family of independent copies of the diffusion ζ, and $\boldsymbol{\delta} = (\delta_i : i \in \mathbb{Z}_0)$ is a family of independent rate-α Poisson processes on $(0, \infty)$, that are independent of the pair $(\Pi, \boldsymbol{\zeta})$. We place particles P_i at the points of Π, and P_i deviates from its initial point according to ζ_i. An initial infected particle P_0 is placed at the origin at time 0, and it diffuses according to

ζ_0. The infection is communicated according to the appropriate rules (either delayed or not, and either (i) via the pair (ρ, μ) with $\rho < \infty$, or (ii) with $\rho = \infty$ and $\mu = 1_S$). After infection, P_i is removed at the next occurrence of the Poisson process δ_i.

The above construction is straightforward so long as there exist, at any given time a.s., only finitely many simultaneous infections. In the two models with $\rho < \infty$, simultaneous infections can occur only after the earliest time T_∞ at which there exist infinitely many infected particles. It is a consequence of Proposition 3.8(c) that $\mathbb{P}(T_\infty < \infty) = 0$ for the general delayed diffusion model with $\rho < \infty$.

The issue is slightly more complex when $\rho = \infty$ and $\mu = 1_S$, since infinitely many simultaneous infections may take place in the supercritical phase of the percolation process of moving disks. In this case, we assume invariably that $\lambda < \lambda_c$, so that there is no percolation of disks at any fixed time, and indeed it was shown in [5] that, a.s., there is no percolation at all times. Therefore, there exist, a.s., only finitely many simultaneous infections at any given instant. Bounds for the growth of generation sizes are found at (4.4) and (4.21).

3 The delayed diffusion model

3.1 Main results

We consider the Brownian delayed diffusion model of Section 2.2, and we adopt the notation of that section. Recall the critical point λ_c of the Boolean continuum percolation on $\mathbb{R}^d$ in which a closed unit ball is placed at each point of a rate-λ Poisson process. Let $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ be the probability that the process survives.

Theorem 3.1. *Consider the Brownian delayed diffusion model on $\mathbb{R}^d$ where $d \geq 2$.*

(a) *Let $\rho \in (0, \infty)$. There exists a function $\alpha_c : (0, \infty)^2 \to (0, \infty)$ such that*

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_c(\lambda, \rho), \\ > 0 & \text{if } \alpha < \alpha_c(\lambda, \rho). \end{cases} \tag{3.1}$$

The function $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ is non-increasing in α and non-decreasing in ρ. Therefore, $\alpha_c = \alpha_c(\lambda, \rho)$ is non-decreasing in ρ.

(b) *Let $\rho = \infty$ and $\mu = 1_S$ where S is the closed unit ball in $\mathbb{R}^d$. There exists a non-decreasing function $\alpha_c : (0, \infty) \to (0, \infty]$ such that, for $0 < \lambda < \lambda_c$,*

$$\theta_{\mathrm{dd}}(\lambda, \infty, \alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_c(\lambda), \\ > 0 & \text{if } \alpha < \alpha_c(\lambda). \end{cases} \tag{3.2}$$

Furthermore, there exists $\underline{\lambda} \in (0, \lambda_c]$ such that

$$\alpha_c(\lambda) \begin{cases} < \infty & \text{if } 0 < \lambda < \underline{\lambda}, \\ = \infty & \text{if } \lambda > \lambda_c. \end{cases}$$

Moreover, the function $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha)$ is non-increasing in α.

This theorem extends Theorem 1.1. Its proof is found in Sections 3.2–3.7.

The situation is different in one dimension, where it turns out that the Brownian model has no phase transition. The proof of the following theorem, in a version valid for the general delayed diffusion model, may be found in Section 3.3.

Theorem 3.2. *Consider the Brownian delayed diffusion model on $\mathbb{R}$ with $\mu = 1_S$. We have that $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = 0$ for all $\lambda, \alpha > 0$ and $\rho \in (0, \infty]$.*

Theorems 3.1 and 3.2 are stated for the case of a single initial infective. The proofs are valid also with a finite number of initial infectives distributed at the points of some arbitrary subset I_0 of $\mathbb{R}^d$. By the proof of the forthcoming Proposition 3.4, the set I of ultimately infected particles is stochastically increasing in I_0.

3.2 Percolation representation of the delayed diffusion model

Consider the delayed diffusion model with $d \geq 2$. Suppose that either $\rho \in (0, \infty)$ with μ as in (2.2), or $\rho = \infty$ and

$$\mu(x) = 1_S(x), \qquad x \in \mathbb{R}^2, \tag{3.3}$$

where S is the closed unit ball with centre at the origin. It turns out that the set of infected particles may be considered as a type of percolation model on the random set Π. This observation will be useful in exploring the phases of the former model.

The proof of the main result of this section, Proposition 3.3, is motivated in part by work of Kuulasmaa [21] where a certain epidemic model was studied via a related percolation process (a similar argument is implicit in [1, p. 4]). Recall the initial placements $\Pi = (X_0 = 0, X_1, X_2, \dots)$ of particles P_i, with law denoted P (and corresponding expectation E); we condition on Π.

Fix $i \geq 0$, and consider the following infection process. The particle P_i is the *unique* initially infected particle, and it diffuses according to ζ_i and has lifetime T_i. All other particles P_j, $j \neq i$, are kept stationary for all time at their respective locations X_j. As P_i moves around $\mathbb{R}^d$, it infects other particles in the usual way; newly infected particles are permitted neither to move nor to infect others. Let J_i be the (random) set of particles infected by P_i in this process.

Let $\tau_{i,j} \in (0, \infty]$ be the time of the first infection by P_i of P_j, *assuming that P_i is never removed*. Write $i \to j$ if $\tau_{i,j} < T_i$, which is to say that this infection takes place before P_i is removed. Thus,

$$J_i = \{j : i \to j\}. \tag{3.4}$$

Suppose first that $\rho < \infty$. Given (Π, ζ_i, T_i), the vector $\vec{\tau}_i = (\tau_{i,j} : j \neq i)$ contains conditionally independent random variables with respective distribution functions

$$F_{i,j}(t) = 1 - \exp\left(-\int_0^t \rho\mu(X_j - X_i - \zeta_i(s))\, ds\right), \qquad t \geq 0, \tag{3.5}$$

and

$$\mathbb{P}(i \to j \mid \Pi, \zeta_i, T_i) = F_{i,j}(T_i). \tag{3.6}$$

When $\rho = \infty$, we have that

$$\tau_{i,j} = \inf\{t > 0 : X_j \in X_i + \zeta_i(t) + S\}, \tag{3.7}$$

the first hitting time of $X_j - X_i$ by the radius-1 sausage of ζ_i. As above, we write $i \to j$ if $\tau_{i,j} < T_i$, with J_i and $\vec{\tau}_i$ given accordingly.

One may thus construct sets J_i for all $i \geq 0$; given Π, the set J_i depends only on (ζ_i, T_i), and therefore the J_i are conditionally independent given Π. The sets $\{J_i : i \geq 0\}$ generate a directed graph $\vec{G} = \vec{G}_\Pi$ with vertex-set $\mathbb{Z}_0$ and directed edge-set $\vec{E} = \{[i, j\rangle : i \to j\}$. Write $\vec{I}$ for the set of vertices k of $\vec{G}$ such that there exists a directed path of $\vec{G}$ from 0 to k. To the edges of $\vec{G}$ we attach random labels, with edge $[i, j\rangle$ receiving the label $\tau_{i,j}$.

From the vector $(\vec{\tau}_i, T_i : i \geq 0)$, we can construct a copy of the general delayed diffusion process by allowing an infection by P_i of P_j whenever $i \to j$ and in addition P_j has not been infected previously by another particle. Let I denote the set of ultimately infected particles in this coupled process.

Proposition 3.3. *For $\rho \in (0, \infty]$, we have $I = \vec{I}$.*

We turn our attention to the Brownian case. By rescaling in space/time, we obtain the following. The full parameter-set of the process is $\{\lambda, \rho, \alpha, \mu, \sigma\}$, where σ is the standard-deviation parameter of the Brownian motion, and we shall sometimes write $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha, \mu, \sigma)$ accordingly.

Proposition 3.4. *Consider the Brownian delayed diffusion model. Let $\rho \in (0, \infty]$.*

(a) *For given $\lambda \in (0, \infty)$, the function $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ is non-decreasing in ρ and non-increasing in α.*

(b) *We have that*

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha, \mu, 1) = \theta_{\mathrm{dd}}(\lambda/r^d, \rho/r^2, \alpha/r^2, \mu_r, 1), \qquad r \geq 1, \tag{3.8}$$

where $\mu_r(x) := \mu(x/r)$.

(c) *If μ is radially decreasing (see* (2.3)*), then*

$$\alpha_{\mathrm{c}}(\lambda, \rho) \geq r^2 \alpha_{\mathrm{c}}(\lambda/r^d, \rho/r^2), \qquad r \geq 1.$$

(d) *If $\rho = \infty$ and μ is radially decreasing, then $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha)$ and $\alpha_{\mathrm{c}}(\lambda, \infty)$ are non-decreasing in λ.*

Proof of Proposition 3.3. This is a deterministic claim. Assume Π is given. If $i \in I$, there exists a chain of direct infection from 0 to i, and this chain generates a directed path of $\vec{G}$ from 0 to i. Suppose, conversely, that $k \in \vec{I}$. Let $\mathcal{P}_k$ be the set of directed paths of $\vec{G}$ from 0 to k. Let $\pi \in \mathcal{P}_k$ be a shortest such path (where the length of an edge $[i, j\rangle$ is taken to be the label $\tau_{i,j}$ of that edge). We may assume that the $\tau_{i,j}$, for $i \to j$, are distinct; no essential difficulty emerges on the complementary null set. Then the path π is a geodesic, in that every sub-path is the shortest directed path joining its endvertices. Therefore, when infection is initially introduced at P_0, it will be transmitted directly along π to P_k. □

Proof of Proposition 3.4. (a) By Proposition 3.3, if the parameters are changed in such a way that each J_i is stochastically increased (respectively, decreased), then the set I is also stochastically increased (respectively, decreased). The claims follow by (3.5)–(3.6) when $\rho < \infty$, and by (3.7) when $\rho = \infty$.

(b) We shall show that the probabilities of infections are the same for the two sets of parameter-values in (3.8). Let $r \geq 1$, and consider the effect of dilating space by the ratio r. The resulting stretched Poisson process $r\Pi$ has intensity λ/r^d, the resulting Brownian motion $r\zeta_i(t)$ is distributed as $\zeta_i(r^2 t)$, and μ is replaced by μ_r. Therefore,

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha, \mu, 1) = \theta_{\mathrm{dd}}(\lambda/r^d, \rho, \alpha, \mu_r, r). \tag{3.9}$$

Next, we use the construction of the process in terms of the J_i given above Proposition 3.3. If $\rho < \infty$ then, by (3.6) and the change of variables $u = r^2 s$,

$$\begin{aligned}
\mathbb{P}(i \to j \mid \Pi, \zeta_i, T_i) &= 1 - \exp\left(- \int_0^{T_i} \rho \mu(X_j - X_i - \zeta_i(s))\, ds \right) \\
&\stackrel{\mathrm{d}}{=} 1 - \exp\left(- \int_0^{T_i} \rho \mu_r(rX_j - rX_i - \zeta_i(r^2 s))\, ds \right) \\
&\stackrel{\mathrm{d}}{=} 1 - \exp\left(- \int_0^{r^2 T_i} \rho \mu_r(rX_j - rX_i - \zeta_i(u))\, \frac{du}{r^2} \right),
\end{aligned}$$

where $\stackrel{\mathrm{d}}{=}$ means equality in distribution. Since $r^2 T_i$ is exponentially distributed with parameter α/r^2, the right side of (3.9) equals $\theta_{\mathrm{dd}}(\lambda/r^d, \rho/r^2, \alpha/r^2, \mu_r, 1)$, as claimed. The same conclusion is valid for $\rho = \infty$, by (3.7).

(c) Since $\mu_r \geq \mu$ by assumption, the J_i are stochastically monotone in μ, it follows by (3.8) that

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha, \mu, 1) \geq \theta_{\mathrm{dd}}(\lambda/r^d, \rho/r^2, \alpha/r^2, \mu, 1), \qquad r \geq 1.$$

By the monotonicity of θ_{dd} in α, if $\alpha > \alpha_{\mathrm{c}}(\lambda, \rho)$ then $\alpha/r^2 \geq \alpha_{\mathrm{c}}(\lambda/r^d, \rho/r^2)$ as claimed.

(d) This holds as in part (c). □

Remark 3.5. In the forthcoming proof of Section 3.7.4 we shall make use of a consequence of Proposition 3.3, namely that

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = \mathrm{E}\big(\mathbb{Q}_\Pi(|\vec{I}| = \infty)\big), \tag{3.10}$$

where $\mathbb{Q}_\Pi$ is the conditional law of $\vec{G}$ given Π, and E is expectation with respect to Π. In proving survival, it therefore suffices to prove the right side of (3.10) is strictly positive.

3.3 No survival in one dimension

It was stated in Theorem 3.2 that the Brownian model with $\mu = 1_S$ never survives in one dimension. We state and prove a version of this 'no survival' theorem for the general delayed diffusion model of Section 2.2, subject to a weak condition which includes the Brownian model.

Throughout this section, T_α denotes a random variable having the exponential distribution with parameter α, assumed to be independent of all other random variables involved in the models. A typical diffusion is denoted ζ, and we write $M_t = \sup\{|\zeta(s)| : s \in [0, t]\}$.

Theorem 3.6. *Consider the general delayed diffusion model on $\mathbb{R}$ with infection parameters (ρ, μ).*

(a) *Let $\rho < \infty$. Assume that α is such that $\mathbb{E}|\zeta(T_\alpha)| < \infty$, and in addition that $\int_{\mathbb{R}} |y|\mu(y)\, dy < \infty$. Then $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = 0$ for all $\lambda > 0$.*

(b) *Let $\rho = \infty$ and μ have bounded support. Assume that α is such that $\mathbb{E}(M_{T_\alpha}) < \infty$. Then $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha) = 0$ for all $\lambda > 0$.*

It follows that. for all α, there is no survival if

either: $\rho < \infty$ and, for all α, we have $\mathbb{E}|\zeta(T_\alpha)| < \infty$,

or: $\rho = \infty$, μ has bounded support, and, for all α, we have $\mathbb{E}(M_{T_\alpha}) < \infty$.

These two conditions (that $\mathbb{E}|\zeta(T_\alpha)| < \infty$ and $\mathbb{E}(M_{T_\alpha}) < \infty$) are equivalent when ζ is Brownian motion (see, for example, [12, Thm 13.4.6]), and indeed they hold for all α in the Brownian case. This implies Theorem 3.2. The proof of Theorem 3.6 has some similarity to that of [1, Thm 1.1].

Proof. (a) Assume the required conditions. Let $A > 0$; later we will take A to be large. Let Π' be a Poisson process on $\mathbb{R}$ with intensity λ, and let $L = \Pi' \cap (-\infty, 0]$ and $R = \Pi' \cap [A, \infty)$. Write $\{S_1 \to S_2\}$ for the event that, in the percolation representation of the last section, some particle in S_1 infects some particle in S_2. We prove first that

$$\mathbb{P}(L \to R) \to 0 \qquad \text{as } A \to \infty. \tag{3.11}$$

Note that, for suitable functions f,

$$\mathbb{E}\left(\int_0^{T_\alpha} f(\zeta(s))\,ds\right) = \int_0^\infty \mathbb{E}(f(\zeta(s))e^{-\alpha s}\,ds = \frac{1}{\alpha}\mathbb{E}(f(\zeta(T_\alpha)). \tag{3.12}$$

Since $\mathbb{P}(L \to R)$ is no larger than the mean number of infections from L into R, we have by the Campbell–Hardy Theorem for Poisson processes (see [12, Exer. 6.13.2]), Fubini's Theorem, and (3.12) that

$$\begin{aligned}\mathbb{P}(L \to R) &\le \lambda^2 \int_{-\infty}^0 du \int_A^\infty dv\, \mathbb{E}\left(\rho \int_0^{T_\alpha} \mu(v-u-\zeta_u(s))\,ds\right)\\ &= \frac{\lambda^2\rho}{\alpha}\mathbb{E}\left(\int_{-\infty}^0 du \int_A^\infty dv\,\mu(v-u-\zeta(T_\alpha))\right)\\ &= \frac{\lambda^2\rho}{\alpha}\mathbb{E}\left(\int_A^\infty dv\, I(v-\zeta(T_\alpha))\right) = \frac{\lambda^2\rho}{\alpha}\mathbb{E}(Z_A),\end{aligned} \tag{3.13}$$

where

$$I(x) := \int_x^\infty \mu(y)\,dy, \qquad Z_A := \int_{A-\zeta(T_\alpha)}^\infty I(v)\,dv.$$

Since $I(x) \le I(-\infty) < \infty$, we have

$$\mathbb{E}(Z_A) \le I(-\infty)\mathbb{E}|\zeta(T_\alpha)| + \int_A^\infty I(v)\,dv.$$

Furthermore, Z_A is integrable since

$$\int_A^\infty I(v)\,dv \le \int_0^\infty I(v)\,dv = \int_0^\infty y\mu(y)\,dy < \infty.$$

Since $Z_A \to 0$ a.s. as $A \to \infty$, we have by monotone convergence that $\mathbb{E}(Z_A) \to 0$ also. Equation (3.11) now follows by (3.13). By a similar argument, $\mathbb{P}(R \to L) \to 0$ as $A \to \infty$.

We pick A sufficiently large that

$$\mathbb{P}(L \to R) \le \tfrac{1}{4}, \qquad \mathbb{P}(R \to L) \le \tfrac{1}{4}.$$

On the event $E_A := \{L \not\to R\} \cap \{R \not\to L\} \cap \{\Pi' \cap (0, A) = \varnothing\}$, there can be no chain of infection between particles in L and particles in R. Note that

$$\mathbb{P}(E_A) \ge \tfrac{1}{2}e^{-\lambda A} > 0. \tag{3.14}$$

Let $B_0 = E_A$ and, for $k \in \mathbb{Z}$, let B_k be the event defined similarly to E_A but with the interval $(0, A)$ replaced by $(kA, (k+1)A)$. By (3.14),

$$\mathbb{P}(B_k) = \mathbb{P}(E_A) \ge \tfrac{1}{2}e^{-\lambda A} > 0. \tag{3.15}$$

By the ergodic theorem, the limit

$$\Lambda := \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} 1_{B_k}$$

exists a.s. and has mean at least $\frac{1}{2}e^{-\lambda A}$. Since Λ is translation-invariant and the underlying probability measure is a product measure, we have $\Lambda \ge \frac{1}{3}e^{-\lambda A}$ a.s. Therefore, B_k occurs infinitely often a.s. This would imply the claim were it not for the extra particle at

0 and its associated Brownian motion ζ_0. However, the range of ζ_0, up to time T_0, is a.s. bounded, and this completes the proof.

(b) Let $\rho = \infty$ and, for simplicity, take $\mu = 1_S$ (the proof for general μ with bounded support is essentially the same). The proof is close to that of part (a).

Let Π' be a Poisson process on $\mathbb{R}$ with intensity λ, let $(\zeta_X : X \in \Pi')$ be independent copies of ζ, and let $(T_X : X \in \Pi')$ be independent copies of T_α. Write $F_x = \inf\{t \geq 0 : \zeta(t) = x\}$ for the first-passage time of ζ to the point $x \in \mathbb{R}$.

Let

$$A = \bigcap_{X \in \Pi' \cap (-\infty, 0)} \{G_X > T_X\},$$

where G_X is the first-passage time to 0 of $X + \zeta_X$. Then

$$\mathbb{P}(A) \geq \mathbb{P}\big(\Pi' \cap [-1, 0] = \varnothing\big)\mathbb{E}\left(\prod_{X \in \Pi' \cap (-\infty, -1)} (1 - p_X)\right), \tag{3.16}$$

where

$$p_x = \mathbb{P}(G_x \leq T_x) = \mathbb{P}(F_{-x} \leq T_\alpha) \leq \mathbb{P}(M_{T_\alpha} \geq |x|). \tag{3.17}$$

There exists $\epsilon > 0$ such that $p_x < 1 - \epsilon$ for $x \geq 1$, and therefore there exists $c = c(\epsilon) \in (0, \infty)$ such that $1 - p_x \geq e^{-cp_x}$ for $x \geq 1$. By (3.16) and Jensen's inequality,

$$\mathbb{P}(A) \geq e^{-\lambda} \exp\left(-c\mathbb{E}\Big(\sum_{X \in \Pi' \cap (-\infty, -1)} p_X\Big)\right). \tag{3.18}$$

By the Campbell–Hardy Theorem,

$$\mathbb{E}\Big(\sum_{X \in \Pi' \cap (-\infty, -1)} p_X\Big) = \lambda \int_{-\infty}^{-1} dx \int_0^\infty dt\, \alpha e^{-\alpha t} \mathbb{P}(F_{-x} \leq t). \tag{3.19}$$

By (3.17), we obtain after interchanging the order of integration that

$$\mathbb{E}\Big(\sum_{X \in \Pi' \cap (-\infty, -1)} p_X\Big) \leq \lambda \mathbb{E}(M_{T_\alpha}). \tag{3.20}$$

By (3.18)–(3.20),

$$\mathbb{P}(A) \geq \exp\big(-\lambda - c\lambda \mathbb{E}(M_{T_\alpha})\big) > 0. \tag{3.21}$$

For $k \in \mathbb{Z}$, let B_k be the event that, for all $X \in \Pi'$, the diffusion $X + \zeta_X$ hits the interval $[k-1, k+1]$ only after time T_X. We repeat the argument of part (a) with (3.21) in place of (3.14), and thereby obtain the claim. □

3.4 A condition for subcriticality when $\rho < \infty$

Consider the general delayed diffusion model of Section 2.2, and assume first that $\rho \in (0, \infty)$. Let $I_0 = \{0\}$. We call $y \in \Pi$ a *first generation infected point up to time* t if y is directly infected by P_0 at or before time t. Let $I_{1,t}$ be the set of all first generation infected points up to time t. For $n \geq 2$, we call $z \in \Pi$ an *nth generation infected point up to time* t if, at or before time t, z is directly infected by some $y \in I_{n-1,t}$, and we define $I_{n,t}$ accordingly. Write $I_n = \lim_{t \to \infty} I_{n,t}$, the set of all nth generation infected points, and let $I = \bigcup_n I_n$ be the set of points that are ever infected.

In the following, we shall sometimes use the coupling of the delayed diffusion model with the percolation-type system of the Section 3.2, and we shall use the notation of that section. In particular, we have that $I_1 \subseteq J_0$, and by Proposition 3.3 that $I = \vec{I}$ (note that I_1 is a strict subset of J_0 if there exist $i, j \in J_0$ such that, in the notation of (3.4), we have $0 \to i \to j$).

Proposition 3.7. *Consider the general delayed diffusion model. Let $\rho \in (0,\infty)$ and*

$$L_t(x) = \mathbb{E}\left(1 - \exp\left(-\int_0^t \rho\mu(x-\zeta(s))\,ds\right)\right). \tag{3.22}$$

We have that $\mathbb{E}|I_{1,t}| \le R_t$ and $\mathbb{E}|I_1| \le R$, where

$$R_t = \lambda \int_{\mathbb{R}^d} \left[\int_0^t L_s(x)\alpha e^{-\alpha s}\,ds + L_t(x)e^{-\alpha t}\right] dx, \tag{3.23}$$

$$R = \lim_{t\to\infty} R_t = \lambda \int_{\mathbb{R}^d}\int_0^\infty L_s(x)\alpha e^{-\alpha s}\,ds\,dx. \tag{3.24}$$

The constant R in (3.24) is an upper bound for the so-called *reproductive rate* of the process. In the notation of Section 3.2, we have $R = \mathbb{E}|J_0|$.

Proposition 3.8. *Consider the general delayed diffusion model. Let $\rho \in (0,\infty)$.*

(a) *We have that $\mathbb{E}|I_n| \le R^n$ for $n \ge 0$, where R is given in* (3.24).

(b) *If $R < 1$, then $\mathbb{E}|I| \le 1/(1-R)$, and hence $\theta_{\mathrm{dd}}(\lambda,\rho,\alpha) = 0$.*

(c) *We have that $R \le \lambda\rho\,\mathrm{Int}(\mu)/\alpha$.*

Note that parts (b) and (c) imply that

$$\theta_{\mathrm{dd}}(\lambda,\rho,\alpha) = 0 \quad \text{if} \quad \alpha > \lambda\rho\,\mathrm{Int}(\mu). \tag{3.25}$$

Proof of Proposition 3.7. Let $\mathcal{F}_0(t)$ be the σ-field generated by $(\zeta_0(s) : 0 \le s \le t)$. Conditional on $\mathcal{F}_0(t)$, for $i \ge 1$, let $A_i = (A_i^k : k \ge 0)$ be a Poisson process on $[0,\infty)$ with rate function

$$r_{X_i}(s) := \rho\mu(X_i - \zeta_0(s)), \qquad s \in [0,\infty).$$

Assume the A_i are independent conditional on $\mathcal{F}_0(t)$, and write $N_i = |\{k : A_i^k \le t\}|$. We say that P_0 'contacts' P_i at the times $\{A_i^k : k \ge 1\}$. Let $U_t = \{X_i : i \ge 1,\ N_i \ge 1\}$ be the set of points in Π that P_0 contacts up to time t. Note that $I_{1,t}$ is dominated stochastically by U_t. The domination is strict since there may exist $X_i \in U_t$ such that P_i is infected before time t by some previously infected $P_j \ne P_0$.

Consider a particle, labelled P_j say, with initial position $x \in \mathbb{R}^d$. Conditional on $\mathcal{F}_0(t)$, P_0 contacts P_j prior to time t with probability

$$1 - \exp\left(-\int_0^t r_x(s)\,ds\right).$$

Therefore,

$$\mathbb{P}\big(X_j \in I_{1,t} \,\big|\, X_j = x,\, \mathcal{F}_0(t)\big) \le \mathbb{E}\left(1 - \exp\left(-\int_0^t r_x(s)\,ds\right) \,\middle|\, \mathcal{F}_0(t)\right). \tag{3.26}$$

By the colouring theorem for Poisson processes (see, for example, [12, Thm 6.13.14]), conditional on $\mathcal{F}_0(t)$, U_t is a Poisson process with inhomogeneous intensity function given by

$$\Lambda_{t,\zeta_0}(x) = \lambda\mathbb{E}\left(1 - \exp\left(-\int_0^t r_x(s)\,ds\right) \,\middle|\, \mathcal{F}_0(t)\right).$$

By Fubini's theorem,

$$\begin{aligned}\mathbb{E}|I_{1,t}| &\le \mathbb{E}\big(\mathbb{E}(|U_t| \,\big|\, T_0)\big)\\ &= \int_{\mathbb{R}^d}\left[\lambda\int_0^t L_s(x)\alpha e^{-\alpha s}\,ds + L_t(x)\mathbb{P}(T_0 > t)\right] dx,\end{aligned} \tag{3.27}$$

and (3.23) follows. Equation (3.24) follows as $t \to \infty$ by the monotone and bounded convergence theorems. □

Proof of Proposition 3.8. (a) This may be proved directly, but it is more informative to use the percolation representation of Section 3.2. Let $\vec{G} = \vec{G}_\Pi$ be as defined there, and note that, in the given coupling, we have $I_n = \{i \in \mathbb{Z}_0 : \delta(0,i) = n\}$ where δ denotes graph-theoretic distance on $\vec{G}$.

We write

$$|I_n| \le \sum_{i \in \mathbb{Z}_0} |J_i| 1(i \in I_{n-1}).$$

By the independence of J_i and the event $\{i \in I_{n-1}\}$,

$$\mathbb{E}|I_n| \le \sum_{i \in \mathbb{Z}_0} \mathbb{E}|J_i| \, \mathbb{P}(i \in I_{n-1}) \le R \, \mathbb{E}|I_{n-1}|,$$

and the claim follows.

(b) By part (a) and the assumption $R < 1$,

$$\mathbb{E}|I| = \sum_{n=0}^{\infty} \mathbb{E}|I_n| \le \frac{1}{1-R} < \infty.$$

Therefore, $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = \mathbb{P}(|I| = \infty) = 0$.

(c) Since $1 - e^{-z} \le z$ for $z \ge 0$, by (3.22) and Fubini's theorem,

$$\int_{\mathbb{R}^d} L_t(x) \, dx \le \rho t \, \mathrm{Int}(\mu).$$

By (3.24),

$$R \le \lambda \rho \, \mathrm{Int}(\mu) \int_0^\infty s \alpha e^{-\alpha s} \, ds = \frac{\lambda \rho}{\alpha} \, \mathrm{Int}(\mu),$$

as claimed. □

3.5 Infection with compact support

Suppose $\mu = 1_M$ with M compact. By (3.22) and (3.24), $R = R(\rho)$ is given by

$$R(\rho) = \lambda \int_{\mathbb{R}^d} \int_0^\infty L_s(x) \alpha e^{-\alpha s} \, ds \, dx, \tag{3.28}$$

where

$$L_t(x) = \mathbb{E}\big(1 - \exp\left(-\rho Q_t(x)\right)\big), \tag{3.29}$$

and

$$Q_t(x) = \big|\{s \in [0,t] : x \in \zeta(s) + M\}\big|_1.$$

We denote by Σ_t the M-sausage of ζ, that is,

$$\Sigma_t := \bigcup_{s \in [0,t]} \big[\zeta(s) + M\big], \qquad t \ge 0. \tag{3.30}$$

Consider the limit $\rho \to \infty$. By (3.28) and dominated convergence,

$$R(\rho) \uparrow \overline{R} := \lambda \int_{\mathbb{R}^d} \int_0^\infty \overline{L}_s(x) \alpha e^{-\alpha s} \, ds \, dx, \tag{3.31}$$

where

$$\overline{L}_t(x) = \mathbb{P}(Q_t(x) > 0) = \mathbb{P}(x \in \Sigma_t).$$

Therefore,

$$\overline{R} = \lambda \int_0^\infty \mathbb{E}|\Sigma_s|_d \, \alpha e^{-\alpha s} \, ds, \tag{3.32}$$

where the integral is the mean volume of the sausage Σ up to time T_0. This formula is easily obtained from first principles applied to the $\rho = \infty$ delayed diffusion process (see Section 3.6).

Example 3.9 (Bounded motion)**.** If, in addition to the assumptions above, each particle is confined within some given distance $\Delta < \infty$ of its initial location, then $\Sigma_t \subseteq S(\Delta + \mathrm{rad}(M))$. Therefore, by (3.31)–(3.32),

$$R(\rho) \le \overline{R} \le \lambda \big| S(\Delta + \mathrm{rad}(M)) \big|_d. \tag{3.33}$$

If the right side of (3.33) is strictly less than 1, then $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = 0$ for $\rho \in (0, \infty)$ by Proposition 3.8. This is an improvement over (3.25) for large ρ.

3.6 A condition for subcriticality when $\rho = \infty$

Let $d \ge 2$, $\rho = \infty$, and $\mu = 1_M$ with M compact. The argument of Sections 3.4–3.5 is easily adapted subject to a condition on the volume of the sausage Σ of (3.30), namely

$$\mathrm{C}_{\gamma,\sigma}\text{: for } t \ge 0,\ \mathbb{E}|\Sigma_t|_d \le \gamma e^{\sigma t}, \tag{3.34}$$

for some $\gamma, \sigma \in [0, \infty)$. Let

$$R(\infty) = \lambda \int_0^\infty \mathbb{E}|\Sigma_s|_d \, \alpha e^{-\alpha s} \, ds, \tag{3.35}$$

in agreement with (3.31)–(3.32). Note that $R(\infty)$ equals the mean number of points of the Poisson process $\Pi \setminus \{0\}$ lying in the sausage Σ_T, where T is independent of Σ and is exponentially distributed with parameter α.

Theorem 3.10.

(a) *If $R(\infty) < 1$ then $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha) = 0$.*

(b) *Assume condition $\mathrm{C}_{\gamma,\sigma}$ of* (3.34) *holds, and $\lambda < \underline{\lambda} := 1/\gamma$. If $\alpha > \underline{\alpha} := \sigma/(1 - \lambda\gamma)$, then $R(\infty) < 1$ for $\alpha > \underline{\alpha}$.*

Proof. (a) This holds by the argument of Proposition 3.8 adapted to the case $\rho = \infty$.

(b) Subject to condition (3.34) with $\lambda\gamma < 1$,

$$R(\infty) \le \lambda \int_0^\infty \alpha\gamma e^{-(\alpha-\sigma)s} \, ds = \frac{\lambda\alpha\gamma}{\alpha - \sigma}, \qquad \alpha > \sigma, \tag{3.36}$$

and the second claim follows. □

Example 3.11 (Brownian motion with $d = 2$)**.** Suppose $d = 2$, ζ is a standard Brownian motion, and $M = S$. By (3.35) and the results of Spitzer [32, p. 117],

$$\begin{aligned} R(\infty) &= \lambda|S|_2 + \lambda \int_0^\infty \alpha e^{-\alpha s} \int_{\mathbb{R}^2 \setminus S} \mathbb{P}(x \in \Sigma_s) \, dx \, ds \\ &= \lambda\pi + \lambda \int_{\mathbb{R}^2 \setminus S} \frac{K_0(\|x\|_2 \sqrt{2\alpha})}{K_0(\sqrt{2\alpha})} \, dx = \lambda Z_\alpha, \end{aligned}$$

where

$$Z_\alpha = \pi + \frac{2\pi}{\sqrt{\alpha}} \frac{K_1(\sqrt{2\alpha})}{K_0(\sqrt{2\alpha})} = \pi + \frac{2\pi}{\sqrt{\alpha}} + \mathrm{o}(\alpha^{-\frac{1}{2}}) \qquad \text{as } \alpha \to \infty. \tag{3.37}$$

Here, K_1 (respectively, K_0) is the modified Bessel function of the second kind of order 1 (respectively, order 0) given by

$$K_0(x) = \int_0^\infty e^{-x \cosh s} \, ds, \qquad K_1(x) = \int_0^\infty e^{-x \cosh s} \cosh s \, ds.$$

Therefore, if $\lambda < \underline{\lambda} := 1/\pi$, there exists $\underline{\alpha} \in (0, \infty)$ such that $R(\infty) < 1$ when $\alpha > \underline{\alpha}$.

Example 3.12 (Brownian motion with $d \geq 5$)**.** Suppose $d \geq 5$, ζ is a standard Brownian motion, and $M = S$. Getoor [8, Thm 2] has shown an explicit constant C such that

$$\mathbb{E}|\Sigma_t|_d - tc_d \uparrow C \qquad \text{as } t \to \infty,$$

where c_d is the Newtonian capacity of the closed unit ball S of $\mathbb{R}^d$. By (3.35),

$$R(\infty) \leq \lambda \left(\frac{c_d}{\alpha} + C \right).$$

Therefore, if $\lambda < \underline{\lambda} := 1/C$, there exists $\underline{\alpha} \in (0, \infty)$ such that $R(\infty) < 1$ when $\alpha > \underline{\alpha}$. Related estimates are in principle valid for $d = 3, 4$, though the behaviour of $\mathbb{E}|\Sigma_t|_d - tc_d$ is more complicated (see [8]).

Example 3.13 (Brownian motion with constant drift)**.** Let $d \geq 2$, $M = S$, with ζ a Brownian motion with constant drift. It is standard (with a simple proof using subadditivity) that the limit $\gamma := \mathbb{E}|\Sigma_t|_d/t$ exists and in addition is strictly positive when the drift is non-zero. Thus, for $\epsilon > 0$, there exists C_ϵ such that

$$\mathbb{E}|\Sigma_t|_d \leq C_\epsilon + (1 + \epsilon)\gamma t, \qquad t \geq 0.$$

As in Example 3.12, if $\lambda < \underline{\lambda} := 1/C_\epsilon$, there exists $\underline{\alpha} \in (0, \infty)$ such that $R(\infty) < 1$ when $\alpha > \underline{\alpha}$. See also [13, 14].

Example 3.14 (Ornstein–Uhlenbeck process)**.** Let $M = S$ and consider the Ornstein–Uhlenbeck process on $\mathbb{R}^d$ satisfying

$$d\zeta(t) = A\zeta(t)\, dt + dW_t$$

where W is standard Brownian motion, A is a $d \times d$ real matrix, and $\zeta(0) = 0$. The solution to this stochastic differential equation is

$$\zeta(t) = \int_0^t e^{A(t-s)}\, dW_s,$$

so that $\|\zeta(t)\|_d \leq e^{|A|t}\|X_t\|_d$ where

$$X_t = \int_0^t e^{-As}\, dW_s$$

defines a martingale, with $|A|$ denoting operator norm. By the Burkholder–Davis–Gundy inequality applied to X (see, for example, [23, Thm 1.1]), the function $M_t = \sup\{\|\zeta(s)\|_d : s \in [0, t]\}$ satisfies

$$\mathbb{E}(M_t^d) \leq c e^{d|A|t} \left(\int_0^t e^{2|A|s}\, ds \right)^{d/2},$$

for some $c < \infty$. Now,

$$|\Sigma_t|_d \leq 2^d (1 + M_t)^d,$$

whence Condition $\mathrm{C}_{\gamma,\sigma}$ holds for suitable $\gamma, \sigma < \infty$.

3.7 Proof of Theorem 3.1

This is proved in several stages, as described in the next subsections.

§3.7.1 The existence and some basic properties of $\alpha_{\mathrm{c}}(\lambda, \rho)$ are proved.

§3.7.2 Let $d = 2$. Suppose $\mu = 1_S$. The remaining properties of α_{c} are established in the respective cases $\rho = \infty$ and $\rho \in (0, \infty)$.

§3.7.3 The previous results are proved for general μ and $\rho \in (0, \infty)$.

§3.7.4 Corresponding statements are proved for $d \geq 3$.

3.7.1 Existence of $\alpha_{\mathbf{c}}$

Consider the Brownian delayed diffusion model with $d \geq 2$, $\rho \in (0, \infty]$. When $\rho = \infty$, we assume in addition that

$$\mu(x) = 1_S(x), \qquad x \in \mathbb{R}^2, \tag{3.38}$$

where S is the closed unit ball with centre at the origin; note in this case that μ is radially decreasing.

By Proposition 3.4(a, d), $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ is non-decreasing in ρ, and non-increasing in α, and is moreover non-decreasing in λ if $\rho = \infty$ (the radial monotonicity of μ has been used in this case). With

$$\alpha_{\mathrm{c}}(\lambda, \rho) := \inf\bigl\{\alpha : \theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = 0\bigr\},$$

we have that

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) \begin{cases} > 0 & \text{if } \alpha < \alpha_{\mathrm{c}}(\lambda, \rho), \\ = 0 & \text{if } \alpha > \alpha_{\mathrm{c}}(\lambda, \rho), \end{cases}$$

and, furthermore, α_{c} is non-decreasing in ρ.

In case (a) of the theorem, by Proposition 3.8, $\alpha_{\mathrm{c}}(\lambda, \rho) < \infty$ for all λ, ρ. In case (b), by Theorem 3.10 and Example 3.13, there exists $\underline{\lambda} \in (0, \lambda_{\mathrm{c}}]$ such that $\alpha_{\mathrm{c}}(\lambda, \infty) < \infty$ when $\lambda \in (0, \underline{\lambda})$. As remarked after (1.2), $\alpha_{\mathrm{c}}(\lambda, \infty) = \infty$ when $\lambda > \lambda_{\mathrm{c}}$.

It remains to show that $\alpha_{\mathrm{c}}(\lambda, \rho) > 0$ for all $\lambda \in (0, \infty)$, $\rho \in (0, \infty]$, and the rest of this proof is devoted to that. This will be achieved by comparison with a directed site percolation model on $\mathbb{Z}_0^2$ viewed as a directed graph with edges directed away from the origin. When $d = 2$, the key fact is the *recurrence* of Brownian motion, which permits a static block argument. This fails when $d \geq 3$, in which case we employ a dynamic block argument and the *transience* of Brownian motion.

3.7.2 The case $d = 2$ with $\mu = 1_S$

Assume first that $d = 2$, for which we use a static block argument. Let $\epsilon > 0$. We choose $a > 0$ such that

$$\mathbb{P}(\Pi' \cap aS \neq \varnothing) > 1 - \epsilon, \tag{3.39}$$

where $\Pi' = \Pi \setminus \{0\}$. For $\mathbf{x} \in \mathbb{Z}^2$, let $S_{\mathbf{x}} = 3a\mathbf{x} + aS$ be the ball with radius a and centre at $3a\mathbf{x}$. We declare $\mathbf{x}$ *occupied* if $\Pi \cap S_{\mathbf{x}} \neq \varnothing$, and *vacant* otherwise; thus, the origin 0 is invariably occupied. Note that the occupied/vacant states of different $\mathbf{x}$ are independent. If a given $\mathbf{x} \neq 0$ is occupied, we let $Q_{\mathbf{x}} \in \Pi \cap S_{\mathbf{x}}$ be the least such point in the lexicographic ordering, and we set $Q_0 = 0$. If $\mathbf{x}$ is occupied, we denote by $\zeta_{\mathbf{x}}$ the diffusion associated with the particle at $Q_{\mathbf{x}}$, and $T_{\mathbf{x}}$ for the lifetime of this particle.

Let ζ be a standard Brownian motion on $\mathbb{R}^2$ with $\zeta(0) = 0$, and let

$$\mathrm{ws}_t(\zeta) := \bigcup_{s \in [0,t]} \bigl[\zeta(s) + S\bigr], \qquad t \in [0, \infty), \tag{3.40}$$

be the corresponding Wiener sausage.

Suppose for now that $\rho = \infty$; later we explain how to handle the case $\rho < \infty$. First we explain what it means to say that the origin 0 is *open*. Let

$$F(\zeta, z) = \inf\{t : z \in \mathrm{ws}_t(\zeta)\}, \qquad z \in \mathbb{R}^2,$$

be the first hitting time of z by $\mathrm{ws}(\zeta)$.

For $\mathbf{y} \in \mathbb{Z}^2$, we define the event

$$K(\zeta_0, \mathbf{y}) = \bigcap_{z \in S_{\mathbf{y}}} \{F(\zeta_0, z) < T_0\},$$

and

$$K(\zeta_0) = \bigcap_{\mathbf{y}\in N} K(\zeta_0, \mathbf{y}),$$

where $N = \{(0,1),(1,0)\}$ is the neighbour set of 0 in the directed graph on $\mathbb{Z}_0^2$. By the recurrence of ζ_0, we may choose $\alpha > 0$ sufficiently small that

$$p_\alpha(0) := \mathbb{P}(K(\zeta_0)) \quad \text{satisfies} \quad p_\alpha(0) > 1 - \epsilon. \tag{3.41}$$

We call 0 *open* if the event $K(\zeta_0)$ occurs. If 0 is not open, it is called *closed*. (Recall that 0 is automatically occupied.)

We now explain what is meant by declaring $\mathbf{x} \in \mathbb{Z}^2 \setminus \{0\}$ to be open. Assume $\mathbf{x}$ is occupied and pick $Q_\mathbf{x}$ as above. For $\mathbf{y} \in \mathbf{x} + N$, we define the event

$$K(\zeta_\mathbf{x}, \mathbf{y}) = \bigcap_{z\in S_\mathbf{y}} \{F(Q_\mathbf{x} + \zeta_\mathbf{x}, z) < T_\mathbf{x}\}, \tag{3.42}$$

and

$$K(\zeta_\mathbf{x}) = \bigcap_{\mathbf{y}\in N} K(\zeta_\mathbf{x}, \mathbf{y}).$$

By the recurrence of ζ, we may choose α such that

$$p_\alpha(\mathbf{x}) := \mathbb{P}\big(K(\zeta_\mathbf{x}) \,\big|\, \mathbf{x} \text{ is occupied}\big) \quad \text{satisfies} \quad p_\alpha(\mathbf{x}) > 1 - \epsilon. \tag{3.43}$$

We declare $\mathbf{x} \in \mathbb{Z}^2$ *open* if $\mathbf{x}$ is occupied, and in addition the event $K(\zeta_\mathbf{x})$ occurs. A vertex of $\mathbb{Z}^2$ which is not open is called *closed*. Conditional on the set of occupied vertices, the open/closed states are independent.

The open/closed state of a vertex $\mathbf{x} \in \mathbb{Z}^2$ depends only on the existence of $Q_\mathbf{x}$ and on the diffusion $\zeta_\mathbf{x}$, whence the open/closed states of different $\mathbf{x} \in \mathbb{Z}^2$ are independent. By (3.39)–(3.41), the configuration of open/closed vertices forms a family of independent Bernoulli random variables with density at least $(1-\epsilon)^2$. Choose $\epsilon > 0$ such that $(1-\epsilon)^2$ exceeds the critical probability of directed site percolation on $\mathbb{Z}_0^2$ (cf. [11, Thm 3.30]). With strictly positive probability, the origin is the root of an infinite directed cluster of the latter process. Using the definition of the state 'open' for the delayed diffusion model, we conclude that the graph $\vec{G}$ (of Section 3.2) contains an infinite directed path from the origin with strictly positive probability. The corresponding claim of Theorem 3.1(b) follows by Lemma 3.3.

Suppose now that $\rho \in (0,\infty)$. We adapt the above argument by redefining the times $F(\zeta, z)$ and the events $K(\zeta)$ as follows. Consider first the case of the origin. Let

$$E(\zeta, z, t) = \big|\{s \in [0,t] : z \in \zeta(s) + S\}\big|_1, \qquad z \in \mathbb{R}^2. \tag{3.44}$$

Pick $F > 0$ such that $e^{-\rho F} < \epsilon$, and write

$$\overline{K}(\zeta_0, t) = \bigcap_{\mathbf{y}\in N,\, z\in S_\mathbf{y}} \{E(\zeta_0, z, t) > F\}.$$

In words, $\overline{K}(\zeta_0, t)$ is the event that the Wiener sausage, started at 0 and run for time t, contains every $z \in S_{(0,1)} \cup S_{(1,0)}$ for an aggregate time exceeding F. It follows that, given that $Q_\mathbf{y} \in \Pi \cap S_\mathbf{y}$ for some $\mathbf{y} \in N$, then P_0 infects $Q_\mathbf{y}$ with probability at least $1 - e^{-\rho F} > 1 - \epsilon$.

By elementary properties of a recurrent Brownian motion, we may pick t and then $\alpha = \alpha(t)$ such that (cf. (3.41))

$$p_\alpha(0) := \mathbb{P}\big(\overline{K}(\zeta_0, t) \cap \{t < T_0\}\big) \quad \text{satisfies} \quad p_\alpha(0) > 1 - \epsilon. \tag{3.45}$$

Turning to general $\mathbf{x} \in \mathbb{Z}^2 \setminus \{0\}$, a similar construction is valid for an event $\overline{K}(\zeta_\mathbf{x}, t)$ as in (3.45), and we replicate the above comparison with directed percolation with $(1-\epsilon)^2$ replaced by $(1-\epsilon)^3$.

3.7.3 The case $d = 2$ with general μ and $\rho < \infty$

We consider next the Brownian delayed diffusion process in two dimensions with infections governed by the pair (ρ, μ), as described in Section 2.2. Assume that $\rho \in (0, \infty)$ and $\mathrm{Int}(\mu) \in (0, \infty)$. The basic method is to adapt the arguments of Section 3.7.2. The new ingredient is a proof of a statement corresponding to (3.45), as follows.

Let $\mathbf{y} \in N$ and write $S_{\mathbf{y}} = 3a\mathbf{y} + aS$ as before. For $\epsilon > 0$, pick a such that $\mathbb{P}(\Pi \cap S_{\mathbf{y}} \neq \varnothing) > 1 - \epsilon$. Suppose that $\Pi \cap S_{\mathbf{y}} \neq \varnothing$, and write $Q := Q_{\mathbf{y}}$ for the least point in the lexicographic ordering of $\Pi \cap S_{\mathbf{y}}$. Consider Q henceforth as given. The following concerns only two particles, namely P_0 and the particle P at Q. Consider the process in which P_0 diffuses forever according to $\zeta := \zeta_0$, and P remains stationary. Given ζ and Q, let A be a Poisson process of times $(A_k : k = 1, 2, \dots)$ with rate function $r(s) = \rho\mu(Q - \zeta(s))$. We say that P_0 'contacts' P at the times of A, and we claim that

$$\mathbb{P}(A_1 < \infty) = 1. \tag{3.46}$$

This implies that, for $\epsilon > 0$ there exists t such that $\mathbb{P}(A_1 < t) > 1 - \epsilon$, and we may then pick $\alpha > 0$ sufficiently small that $\mathbb{P}(A_1 < T_0) > 1 - \epsilon$, where T_0 is the lifetime of P_0. Therefore, subject to (3.46), P_0 infects P with probability at least $1 - 2\epsilon$. This is enough to allow the argument of Section 3.7.2 to proceed, and we turn to the proof of (3.46).

Fix $\mathbf{z} \in \mathbb{R}^2$ to be chosen soon, and write T_b for the disk $Q - \mathbf{z} + bS$. By the Lebesgue density theorem (see, for example, [24, Cor. 2.14]), we may pick $\mathbf{z} \in \mathbb{R}^2$ and $\eta > 0$ such that

$$\int_{T_2} \mu(Q - \mathbf{u})\, d\mathbf{u} = \int_{\mathbf{z}+2S} \mu(\mathbf{v})\, d\mathbf{v} \geq \eta\mu(\mathbf{z}) > 0. \tag{3.47}$$

We shall suppose without loss of generality that $0 \notin T_1$. Let H be the hitting time (by ζ) of the disk T_1 and let $H' > H$ be the subsequent exit time of the disk T_3. The probability that P_0 contacts P during the time-interval (H, H') is

$$p := 1 - \mathbb{E}\left[\exp\left(-\rho \int_H^{H'} \mu(Q - \zeta(t))\, dt\right)\right]. \tag{3.48}$$

By spherical symmetry and [26, Thm 3.31],

$$\mathbb{E}\int_H^{H'} \mu(Q - \zeta(t))\, dt = \int_{T_3} \mu(Q - \mathbf{u})G(\mathbf{x}, \mathbf{u})\, d\mathbf{u}$$

for any given fixed $\mathbf{x} \in \partial T_1$, where G is the appropriate Green's function of [26, Lem. 3.36]. There exists $c > 0$ such that $G(\mathbf{x}, \mathbf{u}) \geq c$ for $\mathbf{u} \in T_2$, so that

$$\mathbb{E}\int_H^{H'} \mu(Q - \zeta(t))\, dt \geq \int_{T_2} \mu(Q - \mathbf{u})G(\mathbf{x}, \mathbf{u})\, d\mathbf{u} \geq c\eta\mu(\mathbf{z}) > 0,$$

by (3.47). By (3.48), we have $p > 0$.

We now iterate the above. Each time ζ revisits T_1, having earlier departed from T_3, there is probability p of such a contact. These contact events are independent, and, by recurrence, a.s. some such contact occurs ultimately. Equation (3.46) is proved.

3.7.4 The case $d \geq 3$

Let $d = 3$; the case $d \geq 4$ is handled similarly. This time we use a *dynamic* block argument, combined with Remark 3.5. The idea is the following. Let ζ_0 be the diffusion of particle P_0. We track the projection of ζ_0, denoted $\overline{\zeta}_0$, on the plane $\mathbb{R}^2 \times \{0\}$. By the

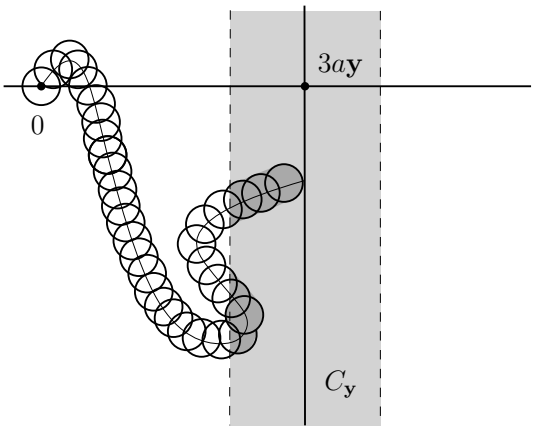

Figure 3.1: The Wiener sausage $\mathrm{ws}(\zeta_0)$ stopped when it hits the line $(3a\mathbf{y}) \times \mathbb{R}$. The dark shaded areas constitute the region $L(\zeta_0, \mathbf{y})$.

recurrence of $\overline{\zeta}_0$, the Wiener sausage $\mathrm{ws}(\zeta_0)$ a.s. visits every line $\mathbf{z} \times \mathbb{R}$ infinitely often, for $\mathbf{z} \in \mathbb{R}^2$ (such $\mathbf{z}$ will be chosen later). At such a visit, we may choose a point $Q'_{\mathbf{z}}$ of Π lying in $\mathrm{ws}(\zeta_0)$ 'near to' the line $\mathbf{z} \times \mathbb{R}$. The construction is then iterated with $Q'_{\mathbf{z}}$ as the starting particle. We build this process in each of two independent directions, and may choose the parameter values such that it dominates the cluster at 0 of a supercritical directed site percolation process.

For $A \subseteq \mathbb{R}^3$, we write $\overline{A}$ for its projection onto the first two coordinates. We abuse notation by identifying $\mathbf{x} = (x_1, x_2, 0, \dots, 0) \in \overline{\mathbb{R}^3}$ (respectively, $\overline{\mathbb{Z}^3}$, etc) with the 2-vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ (respectively, $\mathbb{Z}^2$, etc). Thus, $\overline{\mathbb{R}^3} = \mathbb{R}^2 \times \{0\}$ is the plane of the first two coordinates, and similarly $\overline{\mathbb{Z}^3} = \mathbb{Z}^2 \times \{0\}$, $\overline{\mathbb{Z}^3_0} = \mathbb{Z}^2_0 \times \{0\}$, and $\overline{S} = S \cap \overline{\mathbb{R}^3}$.

For $\mathbf{x} \in \overline{\mathbb{Z}^3}$, let $\overline{S}_{\mathbf{x}} = 3a\mathbf{x} + a\overline{S}$ be the two-dimensional ball with radius $a > 1$ and centre at $3a\mathbf{x}$, and let $C_{\mathbf{x}} = \overline{S}_{\mathbf{x}} \times \mathbb{R}$ be the *cylinder* generated by $\mathbf{x}$. We explain later how a is chosen. Let $\zeta = (\zeta^{(i)} : i = 1, 2, 3)$ be a standard Brownian motion in $\mathbb{R}^3$ with $\zeta(0) = 0$ and coordinate processes $\zeta^{(i)}$, and let $\overline{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, 0)$ be its projection onto the first two coordinates. Note that $\overline{\zeta}$ is a recurrent process on $\overline{\mathbb{R}^3}$.

We declare the particle at 0 to be *open*, and let $\mathbf{y} \in N := \{(1, 0), (0, 1)\}$. We shall see that, with a probability to be bounded below, there exists a (random) particle at some $Q_{\mathbf{y}} \in C_{\mathbf{y}}$ such that P_0 infects this particle. If this occurs, we declare $\mathbf{y}$ to be open. On the event that $\mathbf{y}$ is open, we may iterate the construction starting at $Q_{\mathbf{y}}$, to find a number of further random vertices of $\vec{G}$. By a comparison with a supercritical directed site percolation model, we shall show (for large α) that $\vec{G}$ contains an infinite directed cluster with root 0. The claim then follows by Proposition 3.3 and Remark 3.5.

Suppose for now that $\rho = \infty$. Let $\epsilon > 0$. With ζ a standard Brownian motion on $\mathbb{R}^3$ with $\zeta(0) = 0$, let $\mathrm{ws}_t(\zeta)$ be the corresponding Wiener sausage (3.40). We explain next the state open/closed for a vertex $\mathbf{y} \in N$. Let

$$F(\zeta_0, \mathbf{y}) = \inf\big\{t : ((3a\mathbf{y}) \times \mathbb{R}) \cap \mathrm{ws}_t(\zeta_0) \neq \varnothing\big\}. \tag{3.49}$$

Since $\overline{\zeta}_0$ is recurrent, we have $F(\zeta_0, \mathbf{y}) < \infty$ a.s. Let T_0 be the lifetime of P_0, and define the event

$$K(\zeta_0, \mathbf{y}) = \big\{F(\zeta_0, \mathbf{y}) < T_0\big\}. \tag{3.50}$$

We explain next how a is chosen (see Figure 3.1). Let $a > 1$ and, for $\mathbf{y} \in N$, consider the intersection

$$L(\zeta_0, \mathbf{y}) := \mathrm{ws}_{F(\zeta_0, \mathbf{y})}(\zeta_0) \cap C_{\mathbf{y}}.$$

Lemma 3.15. *There exists $c > 0$ such that the volume of $L(\zeta_0, \mathbf{y})$ satisfies*

$$|L(\zeta_0, \mathbf{y})|_3 \geq ca.$$

Proof. The set $L(\zeta_0, \mathbf{y})$ is the union of disjoint subsets of the Wiener sausage, exactly one of which, denoted L', touches the line $(3a\mathbf{y}) \times \mathbb{R}$. The volume of L' is bounded below by the volume of the union of a cylinder with radius 1 and length $a - 1$, and a half-sphere with radius 1. Thus,

$$|L(\zeta_0, \mathbf{y})|_3 \geq (a-1)\pi + \tfrac{2}{3}\pi \geq \tfrac{2}{3}\pi a,$$

whence the lemma holds with $c = \frac{2}{3}\pi$. □

By Lemma 3.15, we may pick $a > 1$ sufficiently large that

$$\mathbb{P}\big(N_{\mathbf{y}} \mid K(\zeta_0, \mathbf{y})\big) > 1 - \epsilon \qquad \text{where} \qquad N_{\mathbf{y}} := \{\Pi \cap L(\zeta_0, \mathbf{y}) \neq \varnothing\}.$$

If $\Pi \cap L(\zeta_0, \mathbf{y}) \neq \varnothing$, we pick the least point in the intersection (in lexicographic order) and denote it $Q_{\mathbf{y}}$, and we say that $Q_{\mathbf{y}}$ has been *occupied from* 0. We call $\mathbf{y}$ *open* if $K(\zeta_0, \mathbf{y}) \cap N_{\mathbf{y}}$ occurs, and *closed* otherwise.

By the recurrence of $\overline{\zeta}$, we may choose $\alpha > 0$ such that, for $\mathbf{y} \in N$,

$$p_\alpha(\mathbf{y}) := \mathbb{P}(\mathbf{y} \text{ is open}) \quad \text{satisfies} \quad p_\alpha(\mathbf{y}) > 1 - \epsilon. \tag{3.51}$$

In order to define the open/closed states of other $\mathbf{x} \in \overline{\mathbb{Z}^3}$, it is necessary to generalize the above slightly, and we do this next. Instead of considering a Brownian motion ζ starting at $\zeta(0) = 0$, we move the starting point to some $q \in \overline{\mathbb{R}^3}$. Thus ζ becomes $q + \zeta$, and (3.49)–(3.50) become

$$\begin{aligned} F(\zeta, q, \mathbf{y}) &= \inf\big\{t : ((3a\mathbf{y}) \times \mathbb{R}) \cap (q + \mathsf{ws}_t(\zeta)) \neq \varnothing\big\}, \\ K(\zeta, q, \mathbf{y}, T) &= \{F(\zeta, q, \mathbf{y}) < T\}. \end{aligned}$$

By the recurrence of $\overline{\zeta}$, we may choose α such that

$$\overline{p}_\alpha(\mathbf{y}) := \inf\big\{\mathbb{P}(K(\zeta_0, q, \mathbf{y}, T_0)) : q \in \overline{S}\big\} \quad \text{satisfies} \quad \overline{p}_\alpha(\mathbf{y}) > 1 - \epsilon. \tag{3.52}$$

The extra notation introduced above will be used at the next stage.

We construct a non-decreasing sequence pair (V_n, W_n) of disjoint subsets of $\overline{\mathbb{Z}^3_0}$ in the following way. The set V_n is the set of vertices known to be open at stage n of the construction, and W_n is the set known to be closed. Our target is to show that the V_n dominate some supercritical percolation process.

The vertices of $\overline{\mathbb{Z}^3_0}$ are ordered in L^1 order: for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, we declare

$$\mathbf{x} < \mathbf{y} \qquad \text{if} \qquad \text{either } x_1 + x_2 < y_1 + y_2, \quad \text{or } x_1 + x_2 = y_1 + y_2 \text{ and } x_1 < y_1.$$

Let $G_n = \{(x_1, x_2) \in \mathbb{Z}^2_0 : x_1 + x_2 = n\}$, and call G_n the nth generation of $\mathbb{Z}^2_0$.

First, let

$$V_0 = \{0\}, \qquad W_0 = \varnothing.$$

We choose the least $\mathbf{y} \in N$, and set:

$$\begin{aligned} \text{if } \mathbf{y} \text{ is open:} \quad & V_1 = V_0 \cup \{\mathbf{y}\},\ W_1 = W_0, \\ \text{otherwise:} \quad & V_1 = V_0,\ W_1 = W_0 \cup \{\mathbf{y}\}. \end{aligned}$$

In the first case, we say that '$\mathbf{y}$ is occupied from 0'.

For $A \subset \overline{\mathbb{Z}^3_0}$, let ΔA be the set of vertices $b \in \overline{\mathbb{Z}^3_0} \setminus A$ such that b has some neighbour $a \in A$ with $a < b$. Suppose (V_k, W_k) have been defined for $k = 1, 2, \ldots, n$, and define

(V_{n+1}, W_{n+1}) as follows. Select the least $\mathbf{z} \in \Delta V_n \setminus W_n$. If such $\mathbf{z}$ exists, find the least $\mathbf{x} \in V_n$ such that $\mathbf{z} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{y} \in N$. Thus $\mathbf{x}$ is known to be open, and there exists a vertex of $\vec{G}$ at the point $Q_{\mathbf{x}} \in C_{\mathbf{x}}$.

As above,

$$L(\zeta_{\mathbf{x}}, Q_{\mathbf{x}}, \mathbf{z}) := \mathrm{ws}_{F(\zeta_{\mathbf{x}}, Q_{\mathbf{x}}, \mathbf{y})}(Q_{\mathbf{x}} + \zeta_{\mathbf{x}}) \cap C_{\mathbf{z}},$$
$$N_{\mathbf{z}} := \{\Pi \cap L(\zeta_{\mathbf{x}}, Q_{\mathbf{x}}, \mathbf{y}) \neq \varnothing\}.$$

If $K(\zeta_{\mathbf{x}}, Q_{\mathbf{x}}, \mathbf{z}, T_{\mathbf{x}}) \cap N_{\mathbf{z}}$ occurs we call $\mathbf{z}$ *open*, and we say that $\mathbf{z}$ is occupied from $\mathbf{x}$; otherwise we say that $\mathbf{z}$ is *closed*.

$$\begin{aligned} \text{If } \mathbf{z} \text{ is open:} \quad & V_{n+1} = V_n \cup \{\mathbf{z}\},\ W_{n+1} = W_n, \\ \text{otherwise:} \quad & V_{n+1} = V_n,\ W_{n+1} = W_n \cup \{\mathbf{z}\}. \end{aligned}$$

By (3.51)–(3.52), the vertex $\mathbf{z}$ under current scrutiny is open with conditional probability at least $(1-\epsilon)^2$.

This process is iterated until the earliest stage at which no such $\mathbf{z}$ exists. If this occurs for some $n < \infty$, we declare $V_m = V_n$ for $m \geq n$, and in any case we set $V_\infty = \lim_{m\to\infty} V_m$.

The resulting set V_∞ is the cluster at the origin of a type of dependent directed site percolation process which is built by generation-number. Having discovered the open vertices $\mathbf{z}$ in generation n together with the associated points $Q_{\mathbf{z}}$, the law of the next generation is (conditionally) independent of the past and is 1-dependent.

We now apply a stochastic-domination argument. Such methods have been used since at least [6], and the following core lemma was systematized by Liggett, Schonmann, and Stacey [22, Thm 0.0] (see also [10, Thm 7.65] and the references therein). Let $\delta \in (0,1)$, and let $\mathbf{X} = (X_x : \mathbf{x} \in \mathbb{Z}_0^2)$ be a 1-dependent family of Bernoulli random variables such that $\mathbb{E}(X_{\mathbf{x}}) > 1 - \delta$ for all $\mathbf{x}$. There exists $\eta(\delta) > 0$, satisfying $\eta(\delta) \to 0$ as $\delta \to 0$, such that $\mathbf{X}$ dominates stochastically a family $\mathbf{Y} = (Y_x : \mathbf{x} \in \mathbb{Z}_0^2)$ of independent Bernoulli variables with parameter $1 - \eta(\delta)$. We choose $\delta > 0$ such that $1 - \eta(\delta)$ exceeds the critical probability of directed site percolation on $\mathbb{Z}_0^2$. By the above, for sufficiently small $\delta > 0$, there is strictly positive probability of an infinite directed path on $\mathbb{Z}_0^2$ comprising vertices $\mathbf{x}$ with $X_{\mathbf{x}} = 1$.

With δ chosen thus and $\epsilon = \delta/2$, we deduce as required that $\mathbb{P}(|V_\infty| = \infty) > 0$. By a consideration of the geometry of the above construction, and the definition of the local states open/occupied, by (3.10) this entails $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha) > 0$.

A minor extra complication arises at the last stage, in that the events $\{\mathbf{x} \text{ is open}\}$ are not 1-dependent, but only 1-dependent within a given generation conditional on earlier generations. This may be viewed as follows. Begin with a family $\mathbf{Y} = (Y_x : \mathbf{x} \in \mathbb{Z}_0^2)$ of Bernoulli variables with parameter $1 - \eta(\delta)$. Having constructed the subsequence $(V_0, V_1, \dots, V_{n-1})$, the set V_n (or more precisely the set of its indicator functions) dominates stochastically the nth generation of $\mathbf{Y}$. This holds inductively for all n, and the claim follows.

When $\rho \in (0, \infty)$, we extend the earlier argument (around (3.50) and later). Rather than presenting all the required details, we consider the special case of (3.50); the general case is similar. Let $\mathbf{y} \in N$ and $X_t := \mathrm{ws}_t(\zeta_0) \cap C_{\mathbf{y}}$. We develop the previous reference to the first hitting time $F(\zeta_0, \mathbf{y})$ with a consideration of the limit set $X_\infty = \lim_{t\to\infty} X_t$. Since $\overline{\zeta}_0$ is recurrent and ζ_0 is transient, there exists a deterministic $\eta > 0$ such that:

(a) a.s., X_∞ contains infinitely many disjoint closed connected regions $R_1, R_2, \dots$, each with volume exceeding $\frac{1}{2}ca$, and

(b) every point $\mathbf{x} \in \bigcup_i R_i$ is such that

$$|\{t \geq 0 : \mathbf{x} \in \mathrm{ws}_t(\zeta_0)\}|_1 \geq \eta. \tag{3.53}$$

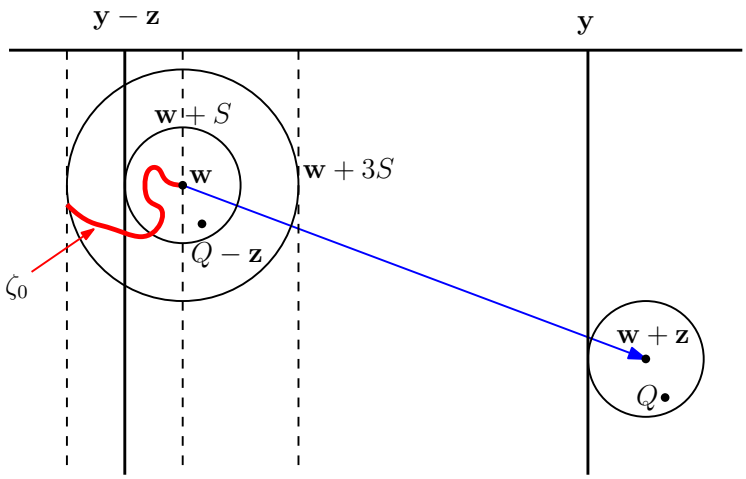

Figure 3.2: An illustration of the proof that, with strictly positive probability (and, therefore, by iteration, with probability 1) P_0 infects some particle in $C_{\mathbf{y}}$. The arrow denotes the vector $\mathbf{z}$.

Each such region contains a point of Π with probability at least $1 - e^{-\frac{1}{2}\lambda ca}$. Each such point is infected by P_0 with probability at least $1 - e^{-\rho\eta}$. Pick N such that, in N independent trials each with probability of success $1 - e^{-\frac{1}{2}\lambda ca} - e^{-\rho\eta}$, there exists at least one success with probability exceeding $1 - \epsilon$. Finally, pick the deterministic time τ such that there is probability at least $1 - \epsilon$ that X_τ contains at least N disjoint closed connected regions R_j, each with volume exceeding $\frac{1}{2}ca$, and such that, for every j and every $\mathbf{x} \in R_j$, inequality (3.53) holds.

Finally, we pick α such that

$$\mathbb{P}(T_0 > \tau) \geq 1 - \epsilon.$$

With these choices, the probability that X_τ contains some particle that is infected from 0 is at least $(1 - \epsilon)^3$. The required argument proceeds henceforth as before.

We turn finally to the case of general μ and $\rho \in (0, \infty)$, and we indicate briefly how the method of Section 3.7.3 may be applied in the current context. First, let $\mathbf{y} \in N$ and $a > 3$. It suffices as above to show that, with probability near 1, P_0 infects some particle in $C_{\mathbf{y}} := \overline{S}_{\mathbf{y}} \times \mathbb{R}$ where, as usual, $S_{\mathbf{y}} = 3a\mathbf{y} + aS$. The following argument is illustrated in Figure 3.2.

Pick $\mathbf{z} \in \mathbb{R}^3$ and $\eta > 0$ such that the $d = 3$ version of (3.47) holds, namely,

$$\int_{\mathbf{z}+S} \mu(\mathbf{v})\, d\mathbf{v} \geq \eta\mu(\mathbf{z}) > 0. \tag{3.54}$$

By recurrence, the projected diffusion $\overline{\zeta}_0$ visits the disk $\overline{S}_{\mathbf{y}-\mathbf{z}}$ infinitely often, a.s., and therefore ζ_0 visits the tube $T := \overline{S}_{\mathbf{y}-\mathbf{z}} \times \mathbb{R}$ similarly. By the transitivity of ζ_0, its entry points into T are a.s. unbounded. Following each such entry to T, at the point $\mathbf{w} \in \mathbb{R}^3$ say, there is an exit from the ball $\mathbf{w} + 3S$. Let H be the time of the first such entry and H' the time of the subsequent such exit.

Let $\zeta > 0$ denote the volume of the ball S, so that

$$\mathbb{P}\big([\mathbf{w} + \mathbf{z} + S] \cap \Pi \neq \varnothing\big) \geq 1 - e^{-\lambda\zeta}. \tag{3.55}$$

On the event that $[\mathbf{w} + \mathbf{z} + S] \cap \Pi \neq \varnothing$, let Q be the least point in that intersection, so that $Q \in C_{\mathbf{y}}$. Conditional on Q, the probability that P_0 infects the particle at Q during

the time-interval (H, H') is

$$p := 1 - \mathbb{E}\left[\exp\left(-\rho \int_H^{H'} \mu(Q - \zeta(t))\, dt\right)\right]. \tag{3.56}$$

By spherical symmetry and [26, Thm 3.31], conditional on $\mathbf{w}$,

$$\mathbb{E}\int_H^{H'} \mu(Q - \zeta(t))\, dt = \int_{\mathbf{w}+3S} \mu(Q - \mathbf{u}) G(\mathbf{w}, \mathbf{u})\, d\mathbf{u},$$

where G is the appropriate Green's function of [26, Lem. 3.32]. There exists $c > 0$ such that $G(\mathbf{w}, \mathbf{u}) \geq c$ for $\mathbf{u} \in \mathbf{w} + 2S$. We make the change of variable $\mathbf{u} = Q - \mathbf{x} + \mathbf{v}$, and note that $Q - \mathbf{z} + S \subseteq \mathbf{w} + 2S$, to deduce that

$$\mathbb{E}\int_H^{H'} \mu(Q - \zeta(t))\, dt \geq c \int_{\mathbf{z}+S} \mu(\mathbf{v})\, d\mathbf{v} \geq c\eta\mu(\mathbf{z}) > 0, \tag{3.57}$$

by (3.54). By (3.56), we have $p > 0$.

On combining (3.55) and (3.57), we deduce that there exists $\delta > 0$ such that

$$\mathbb{P}\big(\exists Q \in \Pi \cap C_{\mathbf{y}}, \text{ and } P_0 \text{ infects } Q \text{ between times } H \text{ and } H'\big) \geq \delta. \tag{3.58}$$

The proof is completed by using the iterative argument around (3.53).

4 The diffusion model

4.1 A condition for subcriticality

We consider the diffusion model in the general form of Sections 2.1 and 2.3, and we adopt the notation of those sections. Recall the critical point λ_{c} of the Boolean continuum percolation on $\mathbb{R}^d$ in which a closed unit ball is centred at each point of a rate-λ Poisson process on $\mathbb{R}^d$. We shall prove the existence of a subcritical phase.

Condition (3.34) is now replaced as follows. Let ζ' be an independent copy of ζ, and define the sausage

$$\Sigma'_t := \bigcup_{s\in[0,t]} \big[\zeta(s) - \zeta'(s) + S\big], \qquad s \geq 0. \tag{4.1}$$

We shall assume

$$\mathrm{C}'_{\gamma,\sigma}\text{: for } t \geq 0,\ \mathbb{E}|\Sigma'_t|_d \leq \gamma e^{\sigma t}, \tag{4.2}$$

for some $\gamma, \sigma \in [0, \infty)$, and we make a note about this condition in Remark 4.3.

Let $\theta_{\mathrm{d}}(\lambda, \rho, \alpha)$ be the probability that the diffusion process survives.

Theorem 4.1. *Consider the general diffusion model on $\mathbb{R}^d$ where $d \geq 1$.*

(a) *Let $\rho \in (0, \infty)$ and $\underline{\alpha}(\lambda, \rho) = \lambda\rho\,\mathrm{Int}(\mu)$. Then $\theta_{\mathrm{d}}(\lambda, \rho, \alpha) = 0$ if $\alpha > \underline{\alpha}(\lambda, \rho)$.*

(b) *Let $\rho = \infty$ and $\mu = 1_S$. Assume in addition that condition $\mathrm{C}'_{\gamma,\sigma}$ of (4.2) holds. Let $\underline{\alpha}(\lambda) = \sigma/(1 - \lambda\gamma)$ and $\underline{\lambda} = 1/\gamma$. Then $\theta_{\mathrm{d}}(\lambda, \infty, \alpha) = 0$ if $\alpha > \underline{\alpha}(\lambda)$ and $0 < \lambda < \underline{\lambda}$.*

This theorem extends Theorem 1.2. Its proof is related to that given in Section 3.4 for the delayed diffusion model.

Proof. (a) Let $\lambda \in (0, \infty)$, and suppose that $\rho < \infty$. We shall enhance the probability space on which the diffusion model is defined. Let $(T_i : i \in \mathbb{Z}_0)$ be random variables with the exponential distribution with parameter α; these are independent of one another and of all other random variables so far. We call T_i the 'lifetime' of P_i, and it is the length of the period between infection and removal of P_i.

For $i \neq j$, we introduce Poisson processes $A_{i,j}$ of points in $[0,\infty)$, and we say that P_i 'contacts' P_j at the times of $A_{i,j}$. The intensity functions of the $A_{i,j}$ depend as follows on the positions of P_i and P_j. Conditional on Π and the diffusions $(\zeta_r : r \in \mathbb{Z}_0)$, let $(A_{i,j} : i,j \in \mathbb{Z}_0,\, i \neq j)$ be independent Poisson processes on $[0,\infty)$ with respective rate functions

$$r_{i,j}(s) := \rho\mu\big(X_j + \zeta_j(s) - X_i - \zeta_i(s)\big), \qquad s \geq 0.$$

The points of $A_{i,j}$ are denoted $(A_{i,j}^k : k \in \mathbb{Z}_0)$. Let

$$\underline{A}_{i,j}(t) := \inf\{A_{i,j} : k \in \mathbb{Z}_0,\, A_{i,j}^k > t\}, \qquad t > 0,$$

and let $B_{i,j}(t)$ be the event that $\underline{A}_{i,j}(t) - t < T_i$ and P_j is susceptible at all times $\underline{A}_{i,j}(t) - \epsilon$ for $\epsilon > 0$. Suppose that P_i becomes infected at time τ. The first contact by P_i of P_j after time τ results in an infection if and only the event $B_{i,j}(\tau)$ occurs (in which case we say that P_i infects P_j *directly*). Write $\underline{A}_{i,j} = \underline{A}_{i,j}(0)$ and $B_{i,j} = B_{i,j}(0)$.

Proposition 3.7 holds with the same proof but with $L_t(x)$ replaced by

$$\widetilde{L}_t(x) = \mathbb{E}\left(1 - \exp\left(-\int_0^t \rho\mu(x + \zeta(s) - \zeta'(s))\, ds\right)\right), \tag{4.3}$$

where ζ' is an independent copy of ζ. By the Poisson colouring theorem, $\widetilde{L}_t(x)$ equals the probability that P_0 contacts a particle started at $x \in \mathbb{R}$ during the time interval $(0,t]$. With this new $\widetilde{L}_t(x)$, the new bound $R = R(\rho)$ now satisfies

$$R(\rho) = \lambda \int_{\mathbb{R}^d} \int_0^\infty \widetilde{L}_s(x) \alpha e^{-\alpha s}\, ds\, dx \leq \frac{\lambda\rho}{\alpha}\, \mathrm{Int}(\mu). \tag{4.4}$$

In other words, $R(\rho)$ is the mean number of particles that P_0 contacts during its lifetime (it is *not* the mean total number of contacts by P_0, since P_0 may contact any given particle many times).

By an inductive definition as before, we define the nth generation I_n of infected particles from 0. We claim that

$$\mathbb{E}|I_n| \leq R(\rho)^n, \qquad n \geq 1. \tag{4.5}$$

By (4.5), $\mathbb{E}|I| < \infty$ whenever $R(\rho) < 1$, and the claim of part (a) follows by (4.4) as in the proof of Proposition 3.8(b, c). We turn therefore to the proof of (4.5), which we prove first with $n = 1$.

Recall that each label $i \in \mathbb{Z}_0$ corresponds to a point $X_i \in \Pi$, an associated diffusion ζ_i, and a lifetime T_i. The lifetime T_i is the residual time to removal of P_i after its first infection.

We have that

$$|I_1| = \sum_{j \in \mathbb{Z}_0 \setminus \{0\}} 1(B_{0,j}) \leq \sum_{j \in \mathbb{Z}_0 \setminus \{0\}} 1(\underline{A}_{0,j} < T_0), \tag{4.6}$$

whence, by the remark after (4.4),

$$\mathbb{E}|I_1| \leq \sum_{j \in \mathbb{Z}_0 \setminus \{0\}} \mathbb{P}(\underline{A}_{0,j} < T_0) = R(\rho), \tag{4.7}$$

as claimed.

Suppose next that $n \geq 2$. We introduce some further notation. Let $i_0 = 0$, and let $\vec{\imath} = (i_1, i_2, \dots, i_n)$ be an ordered vector of distinct members of $\mathbb{Z}_0 \setminus \{0\}$; we shall consider $\vec{\imath}$ as both a vector and a set. Define the increasing sequence $\tau(\vec{\imath}) = (\tau_j : 0 \leq j \leq n)$ of times by

$$\tau_0 = 0, \quad \tau_1 = \underline{A}_{i_0,i_1}, \quad \tau_2 = \underline{A}_{i_1,i_2}(\tau_1), \quad \dots, \quad \tau_{j+1} = \underline{A}_{i_j,i_{j+1}}(\tau_j). \tag{4.8}$$

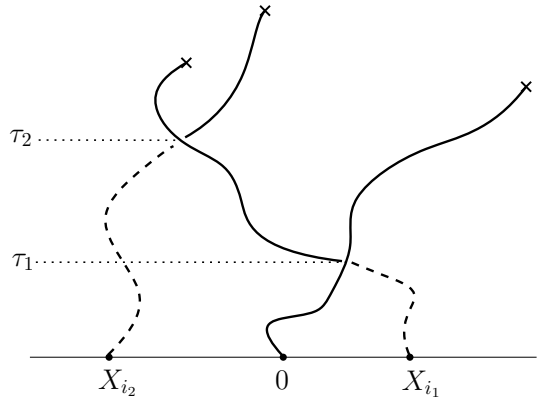

Figure 4.1: The horizontal axis represents one-dimensional space $\mathbb{R}$, and the vertical axis represents time. This is an illustration of the summand $f(0, i_1, i_2)$ in (4.10) when $d = 1$. In this conceptual view, infections occur where pairs of diffusions intersect, and times of removal are marked by crosses.

By iterating the argument leading to (4.6), we obtain

$$|I_n| \le W_n, \tag{4.9}$$

where

$$W_n = \sum_{\vec{\imath}} f(\vec{\imath}), \tag{4.10}$$

and

$$f(\vec{\imath}) = 1(\tau_1 < T_{i_0})1(\tau_2 - \tau_1 < T_{i_1}) \cdots 1(\tau_n - \tau_{n-1} < T_{i_{n-1}}). \tag{4.11}$$

Equations (4.9)–(4.10) are implied by the following observation: if $P_{i_n} \in I_n$, then there exists a sequence $i_0 = 0, i_1, \ldots, i_{n-1}$ such that, for $0 \le j < n$, P_{i_j} infects $P_{i_{j+1}}$ directly at the time τ_{j+1}. See Figure 4.1.

By (4.10),

$$\mathbb{E}(W_n) \le \sum_{\vec{\imath}} \mathbb{E}\big[\mathbb{P}(C_1 \cap C_2 \cap \cdots \cap C_n \mid \mathcal{G}(\vec{\imath}))\big],$$

where $C_j = \{\tau_j - \tau_{j-1} < T_{i_{j-1}}\}$ and $\mathcal{G}(\vec{\imath})$ is the σ-field generated by the random variables

$$(X_{i_j}, \zeta_{i_j}, T_{i_j}) \text{ for } 0 \le j < n-1, \quad X_{i_{n-1}},\ \tau_{n-1},\ (\zeta_{i_{n-1}}(s) : s \in [0, \tau_{n-1}]).$$

Note that $C_1, C_2, \ldots, C_{n-1}$ are $\mathcal{G}(\vec{\imath})$-measurable, so that

$$\mathbb{E}(W_n) \le \sum_{\vec{\imath}} \mathbb{E}\big[1(C_1 \cap \cdots \cap C_{n-1})\mathbb{P}(C_n \mid \mathcal{G}(\vec{\imath}))\big].$$

Therefore,

$$\mathbb{E}(W_n) \le \mathbb{E}\left[\sum_{i_1,\ldots,i_{n-1}} 1(C_1 \cap \cdots \cap C_{n-1}) \sum_{i_n} \mathbb{P}(C_n \mid \mathcal{G}(\vec{\imath}))\right] \tag{4.12}$$

where the summations are over distinct $i_1, \ldots, i_n \ne 0$.

It is tempting to argue as follows. The diffusions $(\zeta_k : k \notin \{i_0, \ldots, i_{n-1}\})$ are independent of $\mathcal{G}(\vec{\imath})$, and τ_{n-1} is $\mathcal{G}(\vec{\imath})$-measurable. By the Poisson displacement theorem (see [20, Sec. 5.2]), the positions $\Pi' = (X_k + \zeta_k(\tau_{n-1}) : k \notin \{i_0, \ldots, i_{n-1}\})$ are a subset of a rate-λ Poisson process. It follows that

$$\sum_{i_n} \mathbb{P}(C_n \mid \mathcal{G}(\vec{\imath})) \le R(\rho). \tag{4.13}$$

By (4.10)–(4.13),

$$\mathbb{E}(W_n) \le \mathbb{E}(W_{n-1})R(\rho). \tag{4.14}$$

Inequality (4.5) follows by iteration and (4.9) There is a subtlety in the argument leading to (4.13), namely that the distribution of the subset $(X_k : k \notin \{i_1, \dots, i_{n-1}\})$ of Π will generally depend on the choice of $i_1, \dots, i_{n-1}$. This may be overcome as follows.

We decouple the indices of particles and their starting positions in a classical way (see [12, Thm 6.13.11]) by giving a more prescriptive recipe for the construction of the Poisson process Π. Let m be a positive integer and let $\Lambda_m = [-m, m]^d \subset \mathbb{R}^d$; later we shall take the limit as $m \to \infty$. Let M have the Poisson distribution with parameter $\lambda(2m)^d$. Conditional on M, let $X_1, X_2, \dots, X_M$ be independent random variables with the uniform distribution on Λ_m. Thus, points in $\Pi \cap \Lambda_m$ are indexed $\{0\} \cup J$ where $J = \{1, 2, \dots, M\}$, with P_0 retaining the index 0.

Let

$$W_n(m) = \sum_{\vec{\imath} \subseteq J} f(\vec{\imath}), \tag{4.15}$$

so that $W_n(m) \to W_n$ as $m \to \infty$, and furthermore,

$$\mathbb{E}(W_n(m)) \to \mathbb{E}(W_n) \qquad \text{as } m \to \infty, \tag{4.16}$$

by the monotone convergence theorem. The sum $W_n(m)$ may be represented in terms of the average of $f(\vec{S}_n)$ where $\vec{S}_n$ is a random ordered n-subset of indices in J, namely,

$$W_n(m) = \mathbb{E}\left(\frac{M!}{(M-n)!} f(\vec{S}_n)\right). \tag{4.17}$$

The term $f(\vec{S}_n)$ is intepreted as 0 if $n > M$. With $\vec{S}_n = (s_1, s_2, \dots, s_n)$ and $\vec{S}_{n-1} = (s_1, s_2, \dots, s_{n-1})$, we have as in (4.12) that

$$W_n(m) = \mathbb{E}\left(\frac{M!}{(M-n)!} f(\vec{S}_{n-1}) Z_n\right) \tag{4.18}$$

where

$$Z_n = 1(\tau_n - \tau_{n-1} < T_{s_{n-1}}), \tag{4.19}$$

and $\tau(\vec{S}_n) = (\tau_0, \tau_1, \dots, \tau_n)$.

For an ordered $(n-1)$-subset $\vec{\imath}$ of J, let $\overline{R}(\vec{\imath})$ be the supremum over $s \in \Lambda_m$ of the mean number of particles infected by a given initial particle located at s, in the subset of the rate-λ Poisson process obtained from Π by deleting $(X_i : i \in \{0\} \cup \vec{\imath})$. Note that

$$\overline{R}(\vec{\imath}) \le R(\rho). \tag{4.20}$$

By (4.18)–(4.19),

$$\begin{aligned}\mathbb{E}(Z_n \mid \mathcal{G}(\vec{S}_{n-1}), M) &= \frac{1}{M-n+1} \sum_{s \in J \setminus \vec{S}_{n-1}} \mathbb{P}\big(\tau_n - \tau_{n-1} < T_{s_{n-1}} \,\big|\, \mathcal{G}(\vec{S}_n), M\big) \\ &\le \frac{1}{M-n+1} \overline{R}(\vec{S}_{n-1}).\end{aligned}$$

By (4.18) and (4.20),

$$\begin{aligned}W_n(m) &\le \mathbb{E}\left(\frac{M!}{(M-n+1)!} f(\vec{S}_{n-1}) \overline{R}(\vec{S}_{n-1})\right) \\ &= \sum_{\vec{\imath} \subseteq J} f(\vec{\imath}) \overline{R}(\vec{\imath}) \le W_n(m-1) R(\rho).\end{aligned}$$

By (4.16), on letting $m \to \infty$, we deduce inequality (4.14), and the proof is completed as before.

(b) Let $\rho = \infty$. We repeat the argument in the proof of part (a) (cf. Section 3.6) with $R(\infty)$ defined as the mean number of particles P_j for which there exists $t < T_0$ with $X_j + \zeta_j(t) \in \zeta_0(t) + S$. That is, with ζ' an independent copy of ζ,

$$\begin{aligned} R(\infty) &= \int_{\mathbb{R}^d} \lambda\, dx\, \mathbb{P}\big(x + \zeta'(t) - \zeta(t) \in S \text{ for some } t < T_0\big) \\ &= \int_{\mathbb{R}^d} \lambda\, dx \int_0^\infty \mathbb{P}(x \in \Sigma'_s) \alpha e^{-\alpha s}\, ds \\ &= \lambda \int_0^t \mathbb{E}|\Sigma'_s|_d\, \alpha e^{-\alpha s}\, ds, \end{aligned} \tag{4.21}$$

where Σ'_s is given in (4.1). As in Theorem 3.10(b) adapted to the diffusion model, we have by $\mathrm{C}'_{\gamma,\sigma}$ that $R(\infty) < 1$ if $\lambda < \underline{\lambda} := 1/\gamma$ and $\alpha > \underline{\alpha}(\lambda) := \sigma/(1 - \lambda\gamma)$. By the argument of the proof of part (a), $\theta_{\mathrm{d}}(\lambda, \rho, \alpha) = 0$ for $\lambda \in (0, \underline{\lambda})$ and $\alpha > \underline{\alpha}(\lambda)$. □

Example 4.2 (Bounded motion)**.** Let $\rho = \infty$ and $\mu = 1_M$ as above, and suppose in addition that each particle is confined within some given distance $\Delta < \infty$ of its initial location. By (4.21),

$$R(\infty) \le \lambda \big| S\big(2(\Delta + \mathrm{rad}(M))\big) \big|_d.$$

If the right side is strictly less than 1, then $\theta_{\mathrm{d}}(\lambda, \infty, \alpha) = 0$ by Proposition 3.8(b) adapted to the current context.

Remark 4.3 (Condition $\mathrm{C}'_{\gamma,\sigma}$)**.** Let $M_t = \sup\{\|\zeta(s)\|_d : s \in [0,t]\}$, the maximum displacement of ζ up to time t, and let M'_t be given similarly in terms of ζ'. By Minkowski's inequality,

$$\mathbb{E}|\Sigma'_t|_d \le \mathbb{E}\big([M_t + M'_t + 1]^d\big) \le \big(2\|M_t\| + 1\big)^d,$$

Here, $\|\cdot\|$ denotes the L^d norm. Therefore, $\mathrm{C}'_{\gamma,\sigma}$ holds for some γ, σ if $\|M_t\| \le \gamma' e^{\sigma' t}$ for suitable γ', σ'.

4.2 The Brownian diffusion model

Suppose that $\rho \in (0, \infty]$, $\mu = 1_S$, and ζ is a standard Brownian motion (one may allow it to have constant non-zero drift, but for simplicity we set the drift to 0). Since $(\zeta - \zeta')/\sqrt{2}$ is a standard Brownian motion, it is easily seen that $\mathbb{E}|\Sigma'_s|_d = \mathbb{E}|W_{2s}|_d$ where W is the usual radius-1 Wiener sausage. Therefore,

$$R(\infty) = \lambda \int_0^\infty \mathbb{E}|W_{2s}|_d\, \alpha e^{-\alpha s}\, ds = \lambda \int_0^\infty \mathbb{E}|W_s|_d\, (\alpha/2) e^{-\alpha s/2}\, ds.$$

Hence, $\underline{\alpha}(\lambda) = 2\underline{\alpha}_{\mathrm{dd}}(\lambda)$ where $\underline{\alpha}_{\mathrm{dd}}(\lambda)$ is the corresponding quantity $\underline{\alpha}$ of Example 3.13 for the delayed diffusion model.

4.3 Survival

We close with some remarks on the missing 'survival' parts of Theorems 1.2 and 4.1. An iterative construction similar to that of Section 3.7 may be explored for the diffusion model. However, Proposition 3.3 is not easily extended or adapted when the particles are *permanently removed* following infection.

The situation is different when either there is no removal (that is, $\alpha = 0$, see [3]), or 'recuperation' occurs in that particles become susceptible again post-infection. A model of the latter type, but involving random walks rather than Brownian motions, has been studied by Kesten and Sidoravicius in their lengthy and complex work [18]. Each of

these variants has structure not shared with our diffusion model, including the property that the set of infectives increases when the set of initially infected particles increases. Heavy use is made of this property in [18]. Unlike the delayed diffusion model (see the end of Section 3.1 and Proposition 3.4), neither the diffusion model nor its random-walk version has this property, in contradiction of the claim of Remark 4 of [18].

References

[1] O. S. M. Alves, F. P. Machado, and S. Yu. Popov, *Phase transition for the frog model*, Electron. J. Probab. **7** (2002), paper no. 16, 21 pp. MR1943889

[2] O. S. M. Alves, F. P. Machado, and S. Yu. Popov, *The shape theorem for the frog model*, Ann. Appl. Probab. **12** (2002), 533–546. MR1910638

[3] E. Beckman, E. Dinan, R. Durrett, R. Huo, and M. Junge, *Asymptotic behavior of the Brownian frog model*, Electron. J. Probab. **23** (2018), paper no. 104, 19 pp. MR3870447

[4] I. Benjamini, L. R. Fontes, J. Hermon, and F. P. Machado, *On an epidemic model on finite graphs*, Ann. Appl. Probab. **30** (2020), 208–258. MR4068310

[5] J. van den Berg, R. Meester, and D. G. White, *Dynamic Boolean models*, Stoch. Proc. Appl. **69** (1997), 247–257. MR1472953

[6] J.-D. Deuschel and A. Pisztora, *Surface order large deviations for high-density percolation*, Probab. Theory Related Fields **104** (1996), 467–482. MR1384041

[7] L. R. Fontes, F. P. Machado, and A. Sarkar, *The critical probability for the frog model is not a monotonic function of the graph*, J. Appl. Probab. **41** (2004), 292–298. MR2036292

[8] R. K. Getoor, *Some asymptotic formulas involving capacity*, Z. Wahrsch'theorie und verwandte Gebiete **4** (1965), 248–252. MR0190988

[9] P. Gracar and A. Stauffer, *Percolation of Lipschitz surface and tight bounds on the spread of information among mobile agents*, Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, LIPIcs, Leibniz Int. Proc. Inform., vol. 116, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, paper 39. MR3857277

[10] G. R. Grimmett, *Percolation*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 321, Springer-Verlag, Berlin, 1999. MR1707339

[11] G. R. Grimmett, *Probability on Graphs*, 2nd ed., Cambridge University Press, Cambridge, 2018. MR3751350

[12] G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*, 4th ed., Oxford University Press, Oxford, 2020. MR4229142

[13] Y. Hamana and H. Matsumoto, *A formula for the expected volume of the Wiener sausage with constant drift*, Forum Math. **29** (2017), 369–381. MR3619119

[14] P. H. Haynes, V. H. Hoang, J. R. Norris, and K. C. Zygalakis, *Homogenization for advection–diffusion in a perforated domain*, Probability and Mathematical Genetics, vol. 37, Cambridge University Press, 2010, pp. 397–415. MR2744249

[15] C. Hoffman, T. Johnson, and M. Junge, *Cover time for the frog model on trees*, Forum Math. Sigma **7** (2019), Paper No. e41, 49 pp. MR4031108

[16] T. Johnson and M. Junge, *Stochastic orders and the frog model*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), 1013–1030. MR3795075

[17] H. Kesten and V. Sidoravicius, *The spread of a rumor or infection in a moving population*, Ann. Probab. **33** (2005), 2402–2462. MR2184100

[18] H. Kesten and V. Sidoravicius, *A phase transition in a model for the spread of an infection*, Illinois J. Math. **50** (2006), 547–634. MR2247840

[19] H. Kesten and V. Sidoravicius, *A shape theorem for the spread of an infection*, Ann. of Math. (2) **167** (2008), 701–766. MR2415386

[20] J. F C. Kingman, *Poisson Processes*, Oxford Studies in Probability, vol. 3, Oxford University Press, Oxford, 1993. MR1207584

[21] K. Kuulasmaa, *The spatial general epidemic and locally dependent random graphs*, J. Appl. Probab. **19** (1982), 745–758. MR675138

[22] T. M. Liggett, R. H. Schonmann, and A. M. Stacey, *Domination by product measures*, Ann. Probab. **25** (1997), 71–95. MR1428500

[23] C. Marinelli and M. Röckner, *On the maximal inequalities of Burkholder, Davis and Gundy*, Exposit. Mathem. **34** (2016), 1–26. MR3463679

[24] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995. MR1333890

[25] R. Meester and R. Roy, *Continuum Percolation*, Cambridge Tracts in Mathematics, vol. 119, Cambridge University Press, Cambridge, 1996. MR1409145

[26] P. Mörters and Y. Peres, *Brownian Motion*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 30, Cambridge University Press, Cambridge, Cambridge, 2010. MR2604525

[27] Y. Peres, A. Sinclair, P. Sousi, and A. Stauffer, *Mobile geometric graphs: Detection, coverage and percolation*, Probab. Theory Related Fields **156** (2013), 273–305. MR3055260

[28] S. Yu. Popov, *Frogs and some other interacting random walks models*, Discrete Random Walks, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2003, pp. 277–288. MR1819703

[29] A. F. Ramírez and V. Sidoravicius, *Asymptotic behavior of a stochastic growth process associated with a system of interacting branching random walks*, C. R. Math. Acad. Sci. Paris **335** (2002), 821–826. MR1947707

[30] L. T. Rolla, *Activated random walks on* $\mathbb{Z}^d$, Probab. Surveys **17** (2020), 478–544. MR4152668

[31] L. T. Rolla, V. Sidoravicius, and O. Zindy, *Universality and sharpness in activated random walks*, Ann. Henri Poincaré, Theor. & Math. Phys. **20** (2019), 1823–1835. MR3956161

[32] F. Spitzer, *Electrostatic capacity, heat flow and Brownian Motion*, Z. Wahrsch'theorie und verwandte Geb. **3** (1964), 110–121. MR172343

[33] A. Stauffer and L. Taggi, *Critical density of activated random walks on transitive graphs*, Ann. Probab. **46** (2018), 2190–2220. MR3813989

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