Received: 31 January 2024 | Revised: 5 June 2024 | Accepted: 19 June 2024

DOI: 10.1111/sapm.12734

SPECIAL ISSUE ARTICLE

STUDIES IN APPLIED MATHEMATICS WILEY

Modelling Wave Propagation: Mathematical Theory and Numerical Analysis, in Memory of V. Dougalis

The Saffman–Taylor problem and several sets of remarkable integral identities

A. S. Fokas[1,2,3] | K. Kalimeris[3]

[1]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, UK

[2]Viterbi School of Engineering, University of Southern California, Los Angeles, Los Angeles, California, USA

[3]Mathematics Research Center, Academy of Athens, Athens, Greece

Correspondence
K. Kalimeris, Mathematics Research Center, Academy of Athens, Greece.
Email: kkalimeris@academyofathens.gr

Thanasis Fokas dedicates this paper to Vassilis, a deep scholar and a great friend.

Funding information
Research Committee of Academy of Athens

Abstract

The methodology based on the so-called global relation, introduced by the first author, has recently led to the derivation of a novel nonlinear integral-differential equation characterizing the classical problem of the Saffman–Taylor fingers with nonzero surface tension. In the particular case of zero surface tension, this equation is satisfied by the explicit solution obtained by Saffman and Taylor. Here, first, for the case of zero surface tension, we present a new nonlinear integrodifferential equation characterizing the Saffman–Taylor fingers. Then, by using the explicit Saffman–Taylor solution valid for the particular case of zero surface tension, we show that the above equations give rise to sets of remarkable integral trigonometric identities.

KEYWORDS
global relation, integral identities, Saffman–Taylor finger

1 | INTRODUCTION

The penetration of a gas into a viscous liquid provides a classical paradigm of the dynamic formation of patterns. In the simplified approach proposed by Saffman and Taylor (1958),[1] the flow of the two fluids is considered to be two-dimensional and the interface is a one-dimensional curve. The pressure of the gas is assumed to be uniform, and the motion of the viscous fluid is assumed to

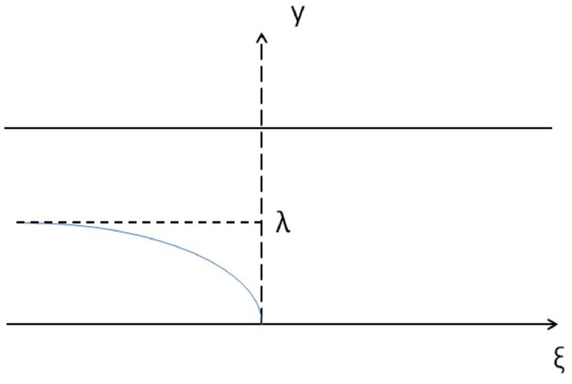

FIGURE 1 The Saffman–Taylor finger. The blue curve depicts the function $y = \Lambda(\xi),\ \xi < 0$.

obey Darcy's law. Thus, all the nonlinearities of the system originate in the boundary conditions at the interface.

The most striking observation of the experimental analysis of Saffman and Taylor was that, for a large value of the interface speed, U, the system evolves toward a state where a single finger occupies approximately half of the channel width. Under the assumption of zero surface tension, γ, Saffman and Taylor, by using a hodograph transformation and conformal mappings, were able to derive a continuous family of steady state solutions. These exact solutions involve a dimensionless parameter, $\lambda,\ 0 < \lambda < 1$, which represents the relative size of the finger compared to the width of the channel. For the particular choice of $\lambda = 1/2$, the exact solution agrees very well with the observed shape of the finger in the regime of high capillary numbers, $Ca = \mu U/\gamma$, where μ denotes the viscosity of the fluid.

The problem was later reconsidered by McLean and Saffman.[2] Their numerical study led to the value of $\lambda = 1/2$, which was in good agreement with the experiments. The question of deciphering the mechanism that selects the particular solutions that is observed experimentally remains open. The equations obtained by McLean–Saffman have more than one solution. In particular, Romero[3] computed numerically a second set of solutions to these equations, and it is shown in Ref. [4] that it exists a discrete family of solutions to the McLean–Saffman equations. However, only one solution is observed in the experiments.[5]

The Saffman–Taylor problem was revisited in Ref. [6] using the methodology of the "global relation" introduced by the first author[7] and employed in the Fokas Method.[8] The global relation has also been used in other fluid mechanics problems, see, for example, Refs. [9–12]. In the current work, this methodology gives rise to a specific nonlinear integrodifferential equation characterizing the interface, which is denoted by Λ, see Equation (5).

In the particular case of zero surface tension, the above nonlinear equations possess the analytical solution obtained by Saffman and Taylor (Figure 1). This implies that, for $\gamma = 0$, the nonlinear integrodifferential equation characterizing Λ gives rise to families of remarkable integral trigonometric identities. We have derived an alternative nonlinear integrodifferential equation characterizing the Saffman–Taylor problem. Although this equation is more complicated than the one derived in Ref. [6], it can also be used for the derivation of a set of trigonometric identities. Thus, for brevity of presentation, we derive only this alternative equation for the case of $\gamma = 0$, see (6). It is worth noting that for the particular case of $\lambda = 1/2$, these identities can be easily verified directly. However, for a general value of λ, we have not been able to verify directly these identities. Numerical calculations are consistent with our proof that they are indeed valid.

2 | AN ALTERNATIVE NONLINEAR INTEGRAL EQUATION FOR $\Lambda(x)$

Let the functions $F(k,\lambda)$ and $G(k,\lambda)$ be defined as follows:

$$F(k,\lambda)=\int_{-\infty}^{0} e^{-ik\xi}\Lambda'(\xi)\cosh[k(\Lambda(\xi)-1)]d\xi, \qquad \mathrm{Im}k\geq 0,\ 0<\lambda<1, \tag{1}$$

$$G(k,\lambda)=\int_{-\infty}^{0} e^{-ik\xi}g'(\xi)\sinh[k(\Lambda(\xi)-1)]d\xi, \qquad \mathrm{Im}k\geq 0,\ 0<\lambda<1, \tag{2}$$

where prime denotes differentiation and the function $g(\xi)$ is defined by

$$g(\xi)=\frac{\gamma\Lambda''(\xi)}{\left(1+(\Lambda'(\xi))^2\right)^{3/2}}. \tag{3}$$

By employing the "global relation," it is shown in Ref. [6] that the Saffman–Taylor problem is completely characterized by the equation

$$\begin{aligned}&\frac{\lambda}{k}+\frac{2}{e^k-e^{-k}}\left[F(k,\lambda)+\frac{i}{M-1}G(k,\lambda)\right]\\&\quad=\sum_{n=1}^{\infty}\frac{1}{k-n\pi i}\left[F(n\pi i,\lambda)e^{n\pi i}+\frac{i}{M-1}G(n\pi i,\lambda)e^{n\pi i}\right],\ \ \mathrm{Im}k\geq 0,\end{aligned} \tag{4}$$

where M is the ratio of the viscosity of the displaced fluid to that of the displacing fluid.

Starting with this equation, and using the inverse Fourier transform, the following nonlinear integrodifferential equation for $\Lambda(x)$ is derived in Ref. [6]:

$$\int_{-\infty}^{0}\left(1+\frac{g'(x)}{M-1}\right)\frac{\sin(\pi\Lambda(x))}{\cosh(\pi(x-\xi))-\cos(\pi\Lambda(x))}dx=2(1-\lambda), \quad \xi<0,\ \ 0<\lambda<1. \tag{5}$$

In what follows, for the case $\gamma=0$, starting with (4), we derive an alternative integral equation for $\Lambda(\xi)$.

Proposition 1. *Let* $\xi<0$, $M>1$, *and* $\gamma=0$. *The function* $\Lambda(\xi)$ *characterizing the Saffman–Taylor fingers satisfies the equation*

$$\begin{aligned}&\int_{-\infty}^{\xi}\left\{2\pi x-\ln\left[1+e^{-2\pi(\xi-x)}-2e^{-\pi(\xi-x)}\cos(\pi\Lambda(x))\right]\right\}\Lambda'(x)dx\\&\quad-\int_{\xi}^{0}\ln\left[1+e^{2\pi(\xi-x)}-2e^{\pi(\xi-x)}\cos(\pi\Lambda(x))\right]\Lambda'(x)dx=2\pi\xi(\Lambda(\xi)-\lambda).\end{aligned} \tag{6}$$

This equation can also be written in the form

$$\begin{aligned}&\int_{\lambda}^{\Lambda(\xi)}2\pi x(\Lambda)-\ln\left[1+e^{-2\pi(\xi-x(\Lambda))}-2e^{-\pi(\xi-x(\Lambda))}\cos(\pi\Lambda)\right]d\Lambda\\&\quad-\int_{\Lambda(\xi)}^{0}\ln\left[1+e^{2\pi(\xi-x(\Lambda))}-2e^{\pi(\xi-x(\Lambda))}\cos(\pi\Lambda)\right]d\Lambda=2\pi\xi(\Lambda(\xi)-\lambda).\end{aligned} \tag{7}$$

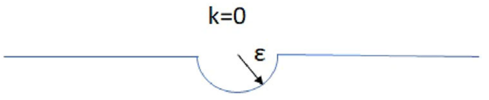

FIGURE 2 The contour L.

Proof. We divide (4) by k, multiply by $e^{ik\xi}$, $\xi < 0$, and integrate along the contour L of the k-complex plane, which is the real line deformed by a semicircular arc on the lower complex half plane, around $k = 0$ with small radius $\epsilon > 0$; see Figure 2.

Since $\xi < 0$ and the poles $k_n = n\pi i$, $n > 0$ occur on $\mathbb{C}^+$, the integral of the RHS of the equation obtained from the integral along L of the RHS of (4) vanishes by closing in $\mathbb{C}^-$. Hence, for $\gamma = 0$, we obtain

$$\int_L \left[\frac{\lambda}{k} + 2\frac{F(k,\lambda)}{e^k - e^{-k}}\right]\frac{e^{ik\xi}}{k}dk = 0,\ \xi < 0. \tag{8}$$

We split the integral appearing in F as an integral along $(-\infty, \xi)$ and an integral along $(\xi, 0)$. Then, in the first and second cases, we compute the integral along L by closing in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. In the first case, we have the poles $k = 0$ and $k = n\pi i$, $n = 1, 2, \ldots$, whereas in the second case, we have only the poles $k = -n\pi i$, $n = 1, 2, \ldots$.

Observe that the term $\frac{\lambda}{k}\frac{e^{ik\xi}}{k}$ is bounded at the lower half plane, thus closing at $\mathbb{C}^-$, the relevant integral yields zero contribution. Hence, the residue of the pole $k = 0$ is given by the term involving F, namely

$$\int_{-\infty}^{\xi} \operatorname*{Res}_{k=0}\left\{2\frac{e^{-ikx}\Lambda'(x)\cosh[k(\Lambda(x)-1)]}{e^k - e^{-k}}\frac{e^{ik\xi}}{k}\right\}dx = -i\int_{-\infty}^{\xi} x\Lambda'(x)dx + i\int_{-\infty}^{\xi} \xi\Lambda'(x)dx.$$

Thus, computing the the residue contribution of (8) yields

$$\begin{aligned}
&\int_{-\infty}^{\xi} 2\pi x\Lambda'(x)dx - \int_{-\infty}^{\xi} 2\pi\xi\Lambda'(x)dx \\
&+ \int_{-\infty}^{\xi}\sum_{n=1}^{\infty}\left\{\frac{\Lambda'(x)}{n}\left[e^{-n\pi(\xi-x)+in\pi\Lambda(x)} + e^{-n\pi(\xi-x)-in\pi\Lambda(x)}\right]dx\right. \\
&+ \int_{\xi}^{0}\sum_{n=1}^{\infty}\left\{\frac{\Lambda'(x)}{n}\left[e^{n\pi(\xi-x)-in\pi\Lambda(x)} + e^{n\pi(\xi-x)+in\pi\Lambda(x)}\right]dx = 0.\right.
\end{aligned} \tag{9}$$

Employing the identity

$$\sum_{n=1}^{\infty}\frac{e^{n\alpha}}{n} = -\ln(1 - e^{\alpha}), \qquad \alpha < 0, \tag{10}$$

Equation (9) becomes

$$0 = \int_{-\infty}^{\xi} 2\pi x\Lambda'(x)dx - 2\pi\xi(\Lambda(\xi) - \lambda)$$

$$-\int_{-\infty}^{\xi} \ln\left[\left(1-e^{-\pi(\xi-x)+i\pi\Lambda(x)}\right)\left(1-e^{-\pi(\xi-x)-i\pi\Lambda(x)}\right)\right]\Lambda'(x)dx$$
$$-\int_{\xi}^{0} \ln\left[\left(1-e^{\pi(\xi-x)-i\pi\Lambda(x)}\right)\left(1-e^{\pi(\xi-x)+i\pi\Lambda(x)}\right)\right]\Lambda'(x)dx$$

Simplifying this equation, we find (6).

Noting that $\Lambda(x)$ is a monotonic function, it follows that $\Lambda(x)$ can be used as the independent variable. Thus, (6) yields (7). □

Remark 1. Letting $\xi = 0$ in (6), we find

$$\int_{-\infty}^{0} \left\{2\pi x - \ln\left[1+e^{2\pi x} - 2e^{\pi x}\cos(\pi\Lambda(x))\right]\right\}\Lambda'(x)dx = 0. \tag{11}$$

This equation also follows from the limit of (4) as $k \to 0$, using the small k expansion of $\frac{2F(k,\lambda)}{e^k - e^{-k}}$, namely

$$\frac{2F(k,\lambda)}{e^k-e^{-k}} + \frac{\lambda}{k} = -i\int_{-\infty}^{0} x\Lambda'(x)dx + O(k), \qquad k \to 0. \tag{12}$$

$$0 = -i\int_{-\infty}^{0} x\Lambda'(x)dx + \sum_{n=1}^{\infty}\frac{1}{n\pi i}F(n\pi i,\lambda)e^{n\pi i}. \tag{13}$$

Using in this equation, the identities (which follow from the definitions of F and G),

$$F(n\pi i,\lambda)e^{n\pi i} = \frac{1}{2}\int_{-\infty}^{0} e^{2n\pi x}\left[e^{in\pi\Lambda(x)} + e^{-in\pi\Lambda(x)}\right]\Lambda'(x)dx, \quad n = 1,2,3\ldots,$$

and then using (10), Equation (13) becomes (11).

3 | THE CASE OF ZERO SURFACE TENSION

In the case of $\gamma = 0$, using Λ as independent variable and recalling that $\Lambda(0) = 0,\ \Lambda(-\infty) = \lambda$, Equation (5) becomes

$$\int_{-\infty}^{0} \frac{\sin(\pi\Lambda)}{\cosh(\pi(x-\xi)) - \cos(\pi\Lambda)}\frac{d\Lambda}{\Lambda'(x)} = 2(1-\lambda), \quad \xi < 0,\ \ 0<\lambda<1, \tag{14}$$

where x is a function of Λ. In order to have an integral defined in the canonical interval $[0,\pi/2]$, we use the new variable θ instead of Λ, where

$$\theta = \frac{\pi\Lambda}{2\lambda}. \tag{15}$$

Then, (14) becomes

$$\int_{0}^{\frac{\pi}{2}} \frac{\sin(2\lambda\theta)}{\left[e^{\pi(x-\xi)} + e^{-\pi(x-\xi)} - 2\cos(2\lambda\theta)\right]}\frac{d\theta}{\theta'(x)} = \lambda - 1. \tag{16}$$

We represent the unknown function $e^{\pi x}$ in the form

$$e^{\pi x} = \Psi(\theta(x)), \tag{17}$$

where $\Psi(\theta)$ must be determined from Equation (16). Differentiating (17) with respect to x, we find

$$\theta'(x) = \frac{\pi\Psi}{\Psi_\theta}. \tag{18}$$

Hence, (16) becomes

$$e^{\pi\xi}\int_0^{\frac{\pi}{2}} \frac{\Psi_\theta \sin(2\lambda\theta)\, d\theta}{e^{2\pi\xi} - 2\cos(2\lambda\theta)\Psi e^{\pi\xi} + \Psi^2} = \pi(\lambda - 1), \quad -\infty < \xi < 0, \;\; 0 < \lambda < 1. \tag{19}$$

Proposition 2. *Let θ and x be defined in terms of Λ and θ, respectively, by the equations*

$$\theta = \frac{\pi\Lambda}{2\lambda}, \qquad e^{\pi x} = \Psi(\theta). \tag{20}$$

The Saffman–Taylor fingers for the case of zero surface tension are characterized by the equation

$$Re\left\{ \int_0^{\frac{\pi}{2}} \frac{d\theta}{\Psi(\theta)\, e^{-\pi\xi - 2i\lambda\theta} - 1} \right\} = 0, \qquad \xi < 0, \;\; 0 < \lambda < 1. \tag{21}$$

Proof. Using the identity

$$\frac{e^{\pi\xi}\sin(2\lambda\theta)}{e^{2\pi\xi} - 2\cos(2\lambda\theta)e^{\pi\xi}\Psi + \Psi^2} = \frac{1}{2i}\left[\frac{1}{\Psi - e^{\pi\xi + 2i\lambda\theta}} - \frac{1}{\Psi - e^{\pi\xi - 2i\lambda\theta}}\right], \tag{22}$$

Equation (19) becomes

$$\mathrm{Im}\left\{ \int_0^{\frac{\pi}{2}} \frac{\Psi_\theta\, e^{-\pi\xi}\, d\theta}{\Psi\, e^{-\pi\xi} - e^{-2i\lambda\theta}} \right\} = \pi(\lambda - 1), \qquad \xi < 0, \;\; 0 < \lambda < 1. \tag{23}$$

By adding and subtracting the term $2i\lambda e^{2i\lambda\theta}$ in the numerator of the integrand of (23), we find

$$\mathrm{Im}\left\{ \ln\left[\Psi e^{-\pi\xi} - e^{2i\lambda\theta}\right]\Big|_{\theta=0}^{\theta=\pi/2} \right\} + \mathrm{Im}\left\{ \int_0^{\frac{\pi}{2}} \frac{2i\lambda\, e^{2i\lambda\theta - \pi\xi}\, d\theta}{\Psi\, e^{-\pi\xi} - e^{2i\lambda\theta}} \right\} = \pi(\lambda - 1). \tag{24}$$

Remarkably, the first term of the LHS of (24) cancels the RHS of (24). Indeed, $\theta = 0$ corresponds to $\Lambda = 0$ and hence to $x = 0$, that is, to $\Psi = 1$. Also, $\theta = \pi/2$ corresponds to $\Lambda = \lambda$ and hence to $x \to -\infty$, that is, to $\Psi = 0$. Thus,

$$\mathrm{Im}\left\{ \ln\left[\Psi e^{-\pi\xi} - e^{2i\lambda\theta}\right]\Big|_{\theta=0}^{\theta=\pi/2} \right\} = \mathrm{Im}\left\{\ln\left(-e^{-i\lambda\pi}\right) - \ln\left(e^{-\pi\xi} - 1\right)\right\} = \pi(\lambda - 1). \tag{25}$$

Hence, (23) becomes (21). □

The famous Saffman–Taylor solution is given by

$$\Psi(\theta) = (\cos\theta)^{2(1-\lambda)}. \tag{26}$$

Actually, it will be verified below that the slightly more general expression obtained from (26) by multiplying with constant α, with $|\alpha| < 1$, satisfies (21). This constant is fixed to be 1 from the requirement that $\Psi(0) = 1$.

In order to prove this result, we insert in the LHS of (21) the expression (26) multiplied by α and obtain the integral,

$$I(\xi,\lambda) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{A(\xi)e^{-2i\lambda\theta}\left(e^{i\theta}+e^{-i\theta}\right)^{2(1-\lambda)} - 1}, \qquad A(\xi) = \alpha\, 2^{2(\lambda-1)}\, e^{-\pi\xi}. \tag{27}$$

Letting

$$z = e^{i\theta}, \qquad d\theta = \frac{dz}{iz},$$

we find

$$I(\xi,\lambda) = \frac{1}{i}\int_C \frac{dz}{Az^{1-2\lambda}\left(z+\frac{1}{z}\right)^{2(1-\lambda)} - z}, \tag{28}$$

where C denotes the circular arch of the unit circle from $\theta = 0$ to $\theta = \pi/2$.

The integrand of I has branch points at $z + \frac{1}{z} = 0$ or $z = \pm i$. This motivates the branch cut defined by the finite interval connecting these two branch points. Cauchy's theorem implies that

$$I = -I_x + I_y, \tag{29}$$

where I_x and I_y denote the associated integrals along the real axis, $z = x,\ 0 < x < 1$ and the imaginary axis $z = iy,\ 0 < y < 1$, respectively:

$$I_x = \frac{1}{i}\int_0^1 \frac{dx}{A(\xi)x^{1-2\lambda}\left(x+\frac{1}{x}\right)^{2(1-\lambda)} - x}, \tag{30}$$

$$I_y = i\int_0^1 \frac{dy}{A(\xi)y^{1-2\lambda}\left(\frac{1}{y}-y\right)^{2(1-\lambda)} + y}. \tag{31}$$

Noting that $0 < y < 1$, that is, $\frac{1}{y} - y > 0$, we observe that both integrals are purely imaginary, thus $\mathrm{Re}\{I\} = 0$.

4 | SEVERAL SETS OF REMARKABLE INTEGRAL IDENTITIES

Using into (19) the expression of $\Psi(\theta)$ defined by (26), we obtain an identity, valid for all ξ and λ. Equation (19) is not uniformly valid, thus we cannot differentiate with respect to λ to obtain additional identities. However, the large k expansion of the basic Equation (4) provides additional identities. Similar considerations apply to (6) with $g' = 0$. We summarize these results in the form of a proposition.

Proposition 3. *For any $\lambda \in (0,1)$ and $A \geq 1$, the following sets remarkable identities are valid:*

$$\int_0^{\frac{\pi}{2}} \frac{\sin\theta\ \sin(2\lambda\theta)\,d\theta}{A(\cos\theta)^{3-2\lambda} + A^{-1}(\cos\theta)^{2\lambda-1} - 2\cos\theta\cos(2\lambda\theta)} = \frac{\pi}{2}. \tag{32}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin\theta\sin(2\lambda\theta)\left[(\cos\theta)^{2(\lambda-1)} - (\cos\theta)^{2(1-\lambda)}\right]d\theta}{\cos\theta\left[(\cos\theta)^{2(1-\lambda)} + (\cos\theta)^{2(\lambda-1)} - 2\cos(2\lambda\theta)\right]^2} = \frac{\pi}{8\lambda^2}. \tag{33}$$

$$\int_{\frac{\pi}{2}}^{P(A)} \left\{4(1-\lambda)\ln(\cos\theta) - \ln\left[1 + A^2(\cos\theta)^{4(1-\lambda)} - 2A(\cos\theta)^{2(1-\lambda)}\cos(2\lambda\theta)\right]\right\}d\theta \tag{34}$$

$$-\int_{P(A)}^{0} \ln\left[1 + A^{-2}(\cos\theta)^{4(\lambda-1)} - 2A^{-1}(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta = 2\left[\frac{\pi}{2} - P(A)\right]\ln A,$$

with $\cos P(A) = A^{\frac{1}{2(\lambda-1)}}$.

Equation (34) is equivalent to the equation

$$\int_0^{\frac{\pi}{2}} \ln\left[1 + A^{-2}(\cos\theta)^{4(\lambda-1)} - 2A^{-1}(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta = 0. \tag{35}$$

Proof. Replacing in (19) Ψ and Ψ_θ, respectively, by

$$\Psi(\theta) = (\cos\theta)^{2(1-\lambda)}, \qquad \Psi_\theta = -2(1-\lambda)(\cos\theta)^{1-2\lambda}\sin\theta,$$

we find Equation (32), by using the notation $A = e^{-\pi\xi} \geq 1,\ \xi \leq 0$.

Equating the terms $O\left(1/k^2\right)$ of (19) and letting $\gamma = 0$, we find

$$i\pi\sum_{n=1}^{\infty} F(in\pi,\lambda)e^{in\pi} = -i\frac{\Lambda''(0)}{\Lambda'(0)^3}. \tag{36}$$

Using the identity

$$\sum_{n=1}^{\infty} ne^{\alpha n} = \frac{e^{-\alpha}}{\left(e^{-\alpha}-1\right)^2} = \frac{1}{\left(e^{\alpha/2} - e^{-\alpha/2}\right)^2}, \qquad \alpha < 0, \tag{37}$$

Equation (36) yields

$$\int_{-\infty}^{0}\left\{\frac{1}{\left[e^{\frac{\pi}{2}(x+i\Lambda(\xi))} - e^{-\frac{\pi}{2}(x+i\Lambda(\xi))}\right]^2} - \frac{1}{\left[e^{\frac{\pi}{2}(x-i\Lambda(\xi))} - e^{-\frac{\pi}{2}(x-i\Lambda(\xi))}\right]^2}\right\}dx = \frac{2i}{\pi}\frac{\Lambda''(0)}{\Lambda'(0)^3}, \tag{38}$$

or

$$\int_{-\infty}^{0} \frac{\sin(\pi\Lambda(x))\sinh(\pi x)}{\left[\cosh(\pi x) - \cos(\pi\Lambda(x))\right]^2}dx = -\frac{1}{\pi}\frac{\Lambda''(0)}{\Lambda'(0)^3}. \tag{39}$$

Differentiating the Saffman–Taylor solution, we obtain

$$\Lambda'(x) \sim -\frac{2\lambda^2}{\pi(1-\lambda)}\frac{1}{\Lambda(x)}, \qquad x \to 0. \tag{40}$$

Hence,

$$\Lambda''(x) \sim -\frac{4\lambda^4}{\pi^2(1-\lambda)^2}\frac{1}{\Lambda(x)^3}, \qquad x \to 0. \tag{41}$$

Thus,

$$\frac{\Lambda''(x)}{\Lambda'(x)} = \frac{\pi}{2}\frac{1-\lambda}{\lambda^2}, \tag{42}$$

and (39) becomes

$$\int_{-\infty}^{0} \frac{\sin(\pi\Lambda(x))\sinh(\pi x)}{[\cosh(\pi x)-\cos(\pi\Lambda(x))]^2}dx = \frac{1-\lambda}{2\lambda^2}. \tag{43}$$

We employ the change of variables $\Lambda = 2\lambda\theta/\pi$, along with the transformation

$$e^{\pi x} = (\cos\theta)^{2(1-\lambda)}, \tag{44}$$

which implies that $x = 0, -\infty$, correspond to $\theta = \pi/2,\ 0$, respectively. Using θ as an independent variable, Equation (43) becomes (33).

Employing the change of variables $\Lambda = 2\lambda\theta/\pi$ in (6), we find

$$\int_{-\infty}^{\xi} \left\{2\pi x\theta' - \ln\left[1+e^{2\pi x}e^{-2\pi\xi} - 2e^{\pi x}e^{-\pi\xi}\cos(2\lambda\theta)\right]\theta'\right\} dx$$
$$- \int_{\xi}^{0} \ln\left[1+e^{-2\pi x}e^{2\pi\xi} - 2e^{-\pi x}e^{\pi\xi}\cos(2\lambda\theta)\right]\theta'\, dx = 2\pi\xi\left(\theta(\xi)-\frac{\pi}{2}\right).$$

Using (44), this equation becomes

$$\int_{\frac{\pi}{2}}^{Q(\xi)} \left\{4(1-\lambda)\ln(\cos\theta) - \ln\left[1+e^{-2\pi\xi}(\cos\theta)^{4(1-\lambda)} - 2e^{-\pi\xi}(\cos\theta)^{2(1-\lambda)}\cos(2\lambda\theta)\right]\right\} d\theta$$
$$- \int_{Q(\xi)}^{0} \ln\left[1+e^{2\pi\xi}(\cos\theta)^{4(\lambda-1)} - 2e^{\pi\xi}(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta = 2\pi\xi\left(Q(\xi)-\frac{\pi}{2}\right), \tag{45}$$

with $\cos Q(\xi) = e^{\frac{\pi\xi}{2(1-\lambda)}}$. Using the notation $A = e^{-\pi\xi} \geq 1,\ \xi \leq 0$, we obtain (34).

Further simplifying the first integrand in (34), we obtain the identity

$$- \int_{\frac{\pi}{2}}^{P(A)} \ln\left[A^2 + (\cos\theta)^{4(\lambda-1)} - 2A(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta$$
$$- \int_{P(A)}^{0} \ln\left[1+A^{-2}(\cos\theta)^{4(\lambda-1)} - 2A^{-1}(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta = 2\left(\frac{\pi}{2}-P(A)\right)\ln A,$$

which becomes

$$-\int_{\frac{\pi}{2}}^{P(A)} \ln A^2 d\theta - \int_{\frac{\pi}{2}}^{P(A)} \ln\left[1 + A^{-2}(\cos\theta)^{4(\lambda-1)} - 2A^{-1}(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta$$

$$-\int_{P(A)}^{0} \ln\left[1 + A^{-2}(\cos\theta)^{4(\lambda-1)} - 2A^{-1}(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta = 2\left(\frac{\pi}{2} - P(A)\right)\ln A.$$

Observing that

$$-\int_{\frac{\pi}{2}}^{P(A)} \ln A^2 d\theta = 2\ln A\int_{P(A)}^{\frac{\pi}{2}} d\theta = 2\left(\frac{\pi}{2} - P(A)\right)\ln A,$$

we obtain (35). □

Remark 2. Evaluating (35) at $A = 1$ yields the set of identities

$$\int_0^{\frac{\pi}{2}} \ln\left[1 + (\cos\theta)^{4(\lambda-1)} - 2(\cos\theta)^{2(\lambda-1)}\cos(2\lambda\theta)\right]d\theta = 0, \qquad 0 < \lambda < 1. \tag{46}$$

This set can also be derived by letting in Equation (6) $\xi = 0$, namely

$$\int_{-\infty}^{0} \left\{2\pi x - \ln\left[1 + e^{2\pi x} - 2e^{\pi x}\cos(\pi\Lambda(x))\right]\right\}\Lambda'(x)dx = 0. \tag{47}$$

Employing the change of variables $\Lambda = 2\lambda\theta/\pi$, we find

$$\int_{-\infty}^{0} \left\{2\pi x - \ln\left[1 + e^{2\pi x} - 2e^{\pi x}\cos(2\lambda\theta)\right]\right\}\theta' dx = 0. \tag{48}$$

Employing the transformation (44) and using θ as an independent variable, Equation (48) becomes

$$\int_0^{\frac{\pi}{2}} \left\{\ln\left[(\cos\theta)^{4(1-\lambda)}\right] - \ln\left[1 + (\cos\theta)^{4(1-\lambda)} - 2(\cos\theta)^{2(1-\lambda)}\cos(2\lambda\theta)\right]\right\} d\theta = 0, \tag{49}$$

which simplifies to (46).

We note that (49) can also be derived by evaluating (45) at $\xi = 0$, and using that $Q(0) = 0$.

Remark 3. For $\lambda = \frac{1}{2}$, Equation (32) can be derived directly. Indeed, letting

$$T = \tan\theta, \qquad dT = \left(T^2 + 1\right)d\theta,$$

(32) becomes

$$A\int_0^{\infty} \frac{T^2\, dT}{(T^2+1)(1+B+T^2)} = \frac{\pi}{2}, \qquad B = A(A-2).$$

Hence,

$$\frac{1}{i\pi}\frac{A}{B}\int_0^\infty \left\{\frac{1}{1+iT}+\frac{1}{1-iT}-(A-1)\left(\frac{i}{A-1+iT}+\frac{i}{A-1-iT}\right)\right\}dT=1,$$

where we have used $\sqrt{B+1}=A-1$.

Hence,

$$\frac{i}{\pi}\frac{A}{B}\left[\ln\left(\frac{1+iT}{1-iT}\right)-(A-1)\ln\left(\frac{A-1+iT}{A-1-iT}\right)\right]\Bigg|_{T=0}^{T=\infty}=1,$$

or

$$\frac{A(A-2)}{B}=1,$$

which is valid.

Similarly, for $\lambda=\frac{1}{2}$, (33) can be verified by a direct computation:

$$\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\left[(\cos\theta)^{-1}-\cos\theta\right]}{\cos\theta[\cos\theta+(\cos\theta)^{-1}-2\cos\theta]^2}d\theta=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\left[(\cos\theta)^{-1}-\cos\theta\right]}{\cos\theta[(\cos\theta)^{-1}-\cos\theta]^2}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\cos\theta[(\cos\theta)^{-1}-\cos\theta]}d\theta=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{1-\cos^2\theta}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\sin^2\theta}d\theta=\int_0^{\frac{\pi}{2}}d\theta=\frac{\pi}{2}.$$

Finally, for $\lambda=\frac{1}{2}$ and $A=1$, the identity (35) can also be proven directly:

$$\int_0^{\frac{\pi}{2}}\left\{2\ln(\cos\theta)-\ln\left[1+(\cos\theta)^2-2(\cos\theta)^2\right]\right\}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\ln\left[(\cos\theta)^2\right]d\theta-\int_0^{\frac{\pi}{2}}\ln\left[(\sin\theta)^2\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\ln\left[(\cos\theta)^2\right]d\theta-\int_0^{\frac{\pi}{2}}\ln\left[\left(\cos\left(\frac{\pi}{2}-\theta\right)\right)^2\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\ln\left[(\cos\theta)^2\right]d\theta+\int_{\frac{\pi}{2}}^{0}\ln\left[(\cos\phi)^2\right]d\phi=0,$$

where we made the change of variables $\phi=\frac{\pi}{2}-\theta$.

In fact, the identity (35), for $\lambda=\frac{1}{2}$, can be also verified directly for any $A\geq 1$, by employing the integral identity

$$\int_0^{\frac{\pi}{2}}\ln\left[c\,(\cos\theta)^2+b^2\right]d\theta=\pi\ln\left(\frac{b+\sqrt{b^2+c}}{2}\right),\qquad b^2+c\geq 0. \tag{50}$$

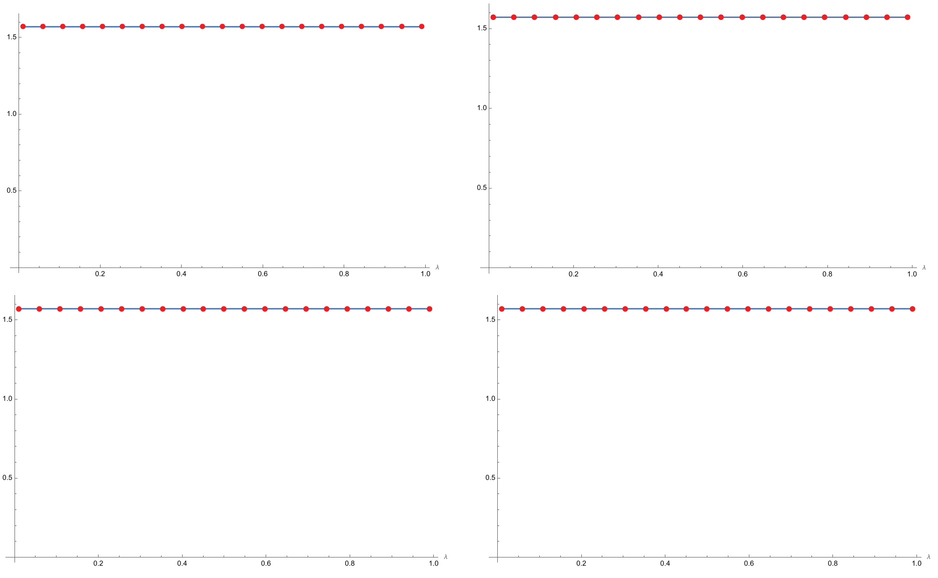

FIGURE 3 Numerical evidence for the validity of (32) for four different values of A, namely, 1, 2, 57/7, $\pi\sqrt{5}$. The blue horizontal line depicts the RHS; the red dots depict the values of the integral for different values of λ.

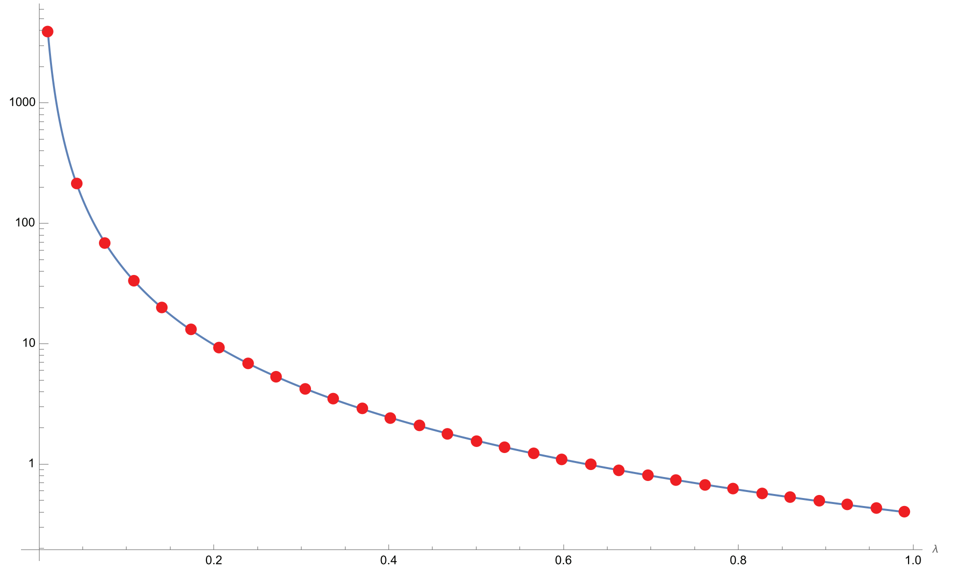

FIGURE 4 Numerical evidence for the validity of (33). The blue curve depicts the RHS; the red dots depict the values of the integral for different values of λ.

Indeed, (35), for $\lambda = \frac{1}{2}$, reads

$$\int_0^{\frac{\pi}{2}} \ln\left[1 + A^{-2}(\cos\theta)^{-2} - 2A^{-1}\right] d\theta = 0,$$

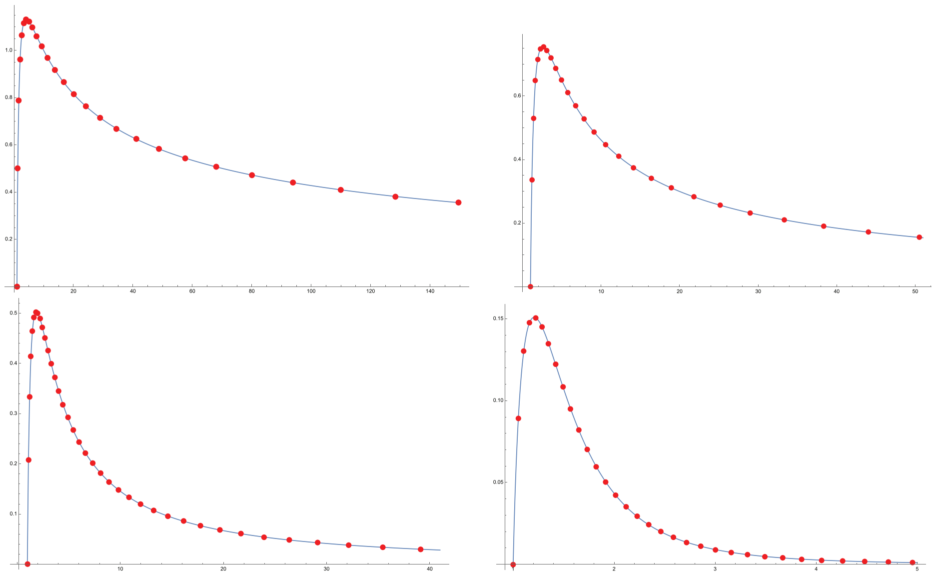

FIGURE 5 Numerical evidence for the validity of (34) for four different values of λ, namely, 1/4, 1/2, 2/3, 9/10. The blue horizontal line depicts the RHS; the red dots depict the values of the integral for different values of A.

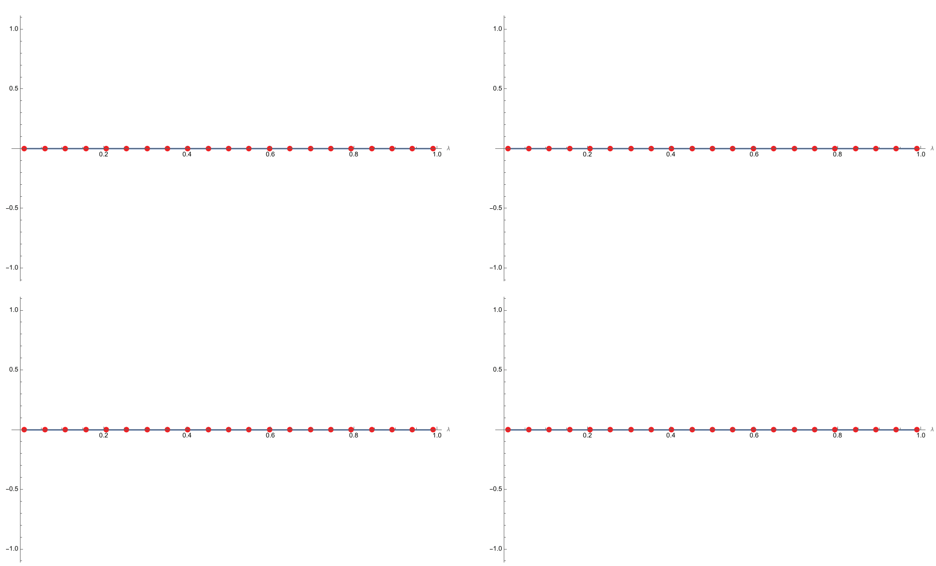

FIGURE 6 Numerical evidence for the validity of (35) for four different values of A, namely, 1, 2, 57/7, $\pi\sqrt{5}$. The blue horizontal line depicts the RHS; the red dots depict the values of the integral for different values of λ.

which takes the form

$$\int_0^{\frac{\pi}{2}} \ln\left[\left(1-\frac{2}{A}\right)(\cos\theta)^2+\frac{1}{A^2}\right]d\theta = \int_0^{\frac{\pi}{2}} \ln\left[(\cos\theta)^2\right]d\theta.$$

For the evaluation of the LHS, we employ (50) for $b = \frac{1}{A},\ c = 1 - \frac{2}{A}$. For the evaluation of the RHS, we employ (50) for $b = 0,\ c = 1$. Both yield the contribution $-\pi \ln 2$.

However, for other values of λ, we have not been able to give a direct analytical proof for (32)–(35). Numerical computations suggest that these identities are indeed valid, see Figures 3–6.

ACKNOWLEDGMENTS

This work is a part of a project initiated by the first author and Yannis Yortsos. The second author later joined this project and made important contributions. KK was partially supported by the research program 200/984 of the Research Committee of the Academy of Athens.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

REFERENCES

1. Saffman PG, Taylor GI. The penetration of a fluid into a porous medium or Hele–Shaw cell containing a more viscous liquid. *Proc R Soc London, A*. 1958;245(1242):312-329.
2. McLean JW, Saffman PG. The effect of surface tension on the shape of fingers in a Hele Shaw cell. *J Fluid Mech*. 1981;102:455-469.
3. Romero LA. *I. Similarity solutions of the equations of three phase flow through porous media. II. The fingering problem in a Hele-Shaw cell*. Dissertation. California Institute of Technology; 1982.
4. Van den-Broeck JM. Fingers in Hele–Shaw cells with surface tension. *Phys Fluids*. 1983;26:2033.
5. Tabeling P, Zocchi G, Libchaber A. An experimental study of the Saffman–Taylor instability. *J Fluid Mech*. 1987;177:67-82.
6. Fokas AS, Kalimeris K, Yortsos Y. In preparation. A new approach to the Saffman–Taylor problem.
7. Fokas AS. On the integrability of linear and nonlinear partial differential equations. *J Math Phys*. 2000;41(6):4188-4237.
8. Fokas AS. A unified transform method for solving linear and certain nonlinear PDEs. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*. 1997;453(1962):1411-1443.
9. Ablowitz MJ, Fokas AS, Musslimani ZH. On a new non-local formulation of water waves. *J Fluid Mech*. 2006;562:313-343.
10. Deconinck B, Oliveras K. The instability of periodic surface gravity waves. *J Fluid Mech*. 2011;675:141-167.
11. Fokas AS, Kalimeris K. Water waves with moving boundaries. *J Fluid Mech*. 2017;832:641-665.
12. Fokas AS, Nachbin A. Water waves over a variable bottom: a non-local formulation and conformal mappings. *J Fluid Mech*. 2012;695:288-309.

How to cite this article: Fokas AS, Kalimeris K. The Saffman–Taylor problem and several sets of remarkable integral identities. *Stud Appl Math*. 2024;e12734.
https://doi.org/10.1111/sapm.12734