Projective twists and the Hopf correspondence

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Declaration.

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

Projective twists and the Hopf correpondence.

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Abstract.

This dissertation is the fruit of a research project on a class of symplectic automorphisms called projective twists.

In the first part of the thesis (Chapters 3, 4) we use Picard–Lefschetz theory to introduce a new local model for the planar projective twists $\tau_{\mathbb{AP}^2} \in \mathrm{Symp}_{ct}(T^*\mathbb{AP}^2)$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. In each case, we construct an exact Lefschetz fibration $\pi \colon T^*\mathbb{AP}^2 \to \mathbb{C}$ with three singular fibres, and define a compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{AP}^2)$ on the total space. Given two disjoint Lefschetz thimbles $\Delta_\alpha, \Delta_\beta \subset T^*\mathbb{AP}^2$, we compute the Floer cohomology groups $\mathrm{HF}(\varphi^k(\Delta_\alpha), \Delta_\beta; \mathbb{Z}/2\mathbb{Z})$ and verify (partially for $\mathbb{CP}^2$) that φ is indeed isotopic to (a power of) the standard local projective twist.

The constructions we present are governed by *generalised lantern relations*, which provide an isotopy between the global monodromy of a Lefschetz fibration and a fibred twist along an S^1-fibred coisotropic submanifold of the smooth fibre. We also use these relations to study two classes of monotone Lagrangian submanifolds of $(T^*\mathbb{CP}^2, d\lambda_{T^*\mathbb{CP}^2})$.

In the second part of the thesis, starting from Chapter 5, we investigate the properties of projective twists within the symplectic mapping class group of Liouville/Stein manifolds. We define the *Hopf correspondence*, a Lagrangian correspondence (in the sense of [WW09]) aimed at assigning Lagrangian spheres $L_1, \dots, L_m$ of a Liouville manifold (Y, Ω) to given Lagrangian (real, complex) projective spaces $K_1, \dots, K_m$ of a Liouville manifold (W, ω). When this correspondence can be established (according to a cohomology condition on a class $\alpha \in H^*(W)$), it intertwines the (real, complex) projective twists $\tau_{K_i} \in \pi_0(\mathrm{Symp}_{ct}(W))$ (and the induced autoequivalences of the compact Fukaya category $\mathcal{F}\mathrm{uk}(W)$) with the Dehn twists $\tau_{L_i} \in \pi_0(\mathrm{Symp}_{ct}(Y))$ (and the corresponding autoequivalences of $\mathcal{F}\mathrm{uk}(Y)$), for $i = 1, \dots m$. Using the Hopf correspondence, we obtain a free generation result for projective twists in a clean plumbing of projective spaces and a result about products of positive powers of real projective twists in Liouville manifolds. The same techniques are also used to show that in infinitely many dimensions n, the Hamiltonian class of the local projective twist in $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$ does depend on a choice of *framing*, i.e a choice of smooth parametrisation of the Lagrangian projective space used to define the twist. Another application of the Hopf correspondence delivers smooth homotopy complex projective spaces $K \simeq \mathbb{CP}^n$, that do not admit Lagrangian embeddings into $(T^*\mathbb{CP}^n, d\lambda_{T^*\mathbb{CP}^n})$, for $n = 4, 7$.

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Ailsa Keating first suggested that the model $\mathbb{RP}^2$ twist could be constructed using a Lefschetz fibration as we present in Chapter 4. I am very thankful to her for sharing the ideas which provided the momentum to realise that part of the thesis, and for helpful conversations in the initial stage of my research.

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Contents

Chapter 1

Introduction

Fogo eterno prá afugentar
O inferno prá outro lugar
Fogo eterno prá consumir
O inferno, fora daqui!

Gilberto Gil, *Palco*

A Stein manifold is a complex manifold which can be properly embedded as a complex submanifold of some Euclidean space $\mathbb{C}^N$. Traditionally, Stein manifolds have been studied from a complex geometrical perspective, but the work of Eliashberg and Cieliebak [CE12] demonstrated how the topology of these manifolds is governed by symplectic geometry. A Stein manifold admits a symplectic structure (canonical up to isomorphism) obtained from the complex structure and a plurisubharmonic function.

One way of studying the topology of Stein manifolds in a setting able to render both their flexible and rigidity features is via *Lefschetz fibrations*; following a result by Giroux–Pardon [GP17], every Stein manifold admits the structure of a Lefschetz fibration, whose smooth fibres are also Stein manifolds.

A Lefschetz fibration on a symplectic manifold (E, ω_E) is a map $\pi\colon E \to \mathbb{C}$ that is a submersion away from a finite set, at which there are local complex coordinates with respect to which the map has a quadratic singularity, and for which the smooth fibres are symplectic smooth manifolds. Seidel initiated ([Sei03, Sei08a]) pseudoholomorphic curve analysis on Lefschetz fibrations to study their symplectic invariants from a Floer theoretical and Fukaya categorical point of view (giving results that include, among others, the exact triangle (1.2) below).

A fundamental property of a Lefschetz fibration is that its (local) monodromies give rise to one of the most important classes of symplectomorphisms: Dehn twists. Dehn twists are symplectic automorphisms that can be defined on symplectic manifolds admitting Lagrangian embeddings of spheres, and symplectic Picard–Lefschetz theory has been a very productive tool used to study them. Unfortunately, not every symplectic manifold admits embeddings of Lagrangian spheres, in absence of which there is in general no systematic way of constructing an element of the symplectic mapping class group. However, if the ambient manifold happens to contain Lagrangian projective spaces, then the symplectic mapping class group does contain another interesting class of symplectic automorphisms; projective twists. Projective twists were first defined by Seidel ([Sei00]) using the periodicity of the geodesic flow on $\mathbb{CP}^n$. These symplectomorphisms can be thought of as belonging to a class of symplectomorphisms called *fibred twists* formally introduced by Perutz in [Per07] (see Section 2.2.5). Fibred twists arise as monodromies of symplectic fibrations with Morse–Bott singularities, called *Morse–Bott–Lefschetz* fibrations.

These fibrations (under some exactness or monotonicity conditions for the smooth fibres) also admit a pseudoholomorphic curve analysis ([Per07, WW16]), that is often more complicated than that for symplectic Lefschetz fibrations.

In this thesis we study projective twists in the mapping class group of certain Stein and Liouville manifolds (the latter are a class of symplectic manifolds including Stein manifolds, see Definition 2.2.1). The modus operandi of most of this investigation is to transpose statements about projective twists to statements involving Dehn twists (in an auxiliary symplectic manifold), to eventually harness these questions with the current literature on the properties of Dehn twists.

The contents of the thesis are taken from the author's papers [Tor20a] (Chapter 4) and [Tor20b] (Chapters 5 - 8).

1.1 Dehn twists

Given a symplectic manifold (M,ω) with contact boundary, a natural object of study is the group $\mathrm{Symp}_{ct}(M)$ of compactly supported symplectomorphisms that are the identity in a neighbourhood of the boundary. Its quotient $\pi_0(\mathrm{Symp}_{ct}(M))$ by the relation of symplectic isotopy is the *symplectic mapping class group*, and is already a highly non-trivial object. When $H^1(M;\mathbb{R})=0$, a symplectic isotopy is automatically Hamiltonian, and $\pi_0(\mathrm{Symp}_{ct}(M))$ coincides with the quotient $\mathrm{Symp}_{ct}(M)/\mathrm{Ham}_{ct}(M)$ by the subgroup $\mathrm{Ham}_{ct}(M)\subset\mathrm{Symp}_{ct}(M)$ of (compactly supported) Hamiltonian symplectomorphisms (namely time-1 maps of compactly supported Hamiltonian flows).

The symplectic mapping class group carries a (forgetful) comparison map

$$c:\pi_0(\mathrm{Symp}_{ct}(M))\longrightarrow\pi_0(\mathrm{Diff}^+_{ct}(M)) \tag{1.1}$$

to the (compactly supported and orientation-preserving) smooth mapping class group of M. In general, the map is neither injective nor surjective. Its kernel is of particular interest as it captures phenomena which are exclusively symplectic and not visible in the smooth mapping class group. The question of whether a symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(M)$ is a non-trivial element of the kernel of c (i.e is smoothly isotopic to the identity but not symplectically so) is called the *symplectic isotopy problem*.

In dimension two, the kernel of c is always trivial, and the symplectic mapping class group is isomorphic to the smooth mapping class group $\pi_0(\mathrm{Diff}^+_{ct}(M))$; this is a consequence of *Moser's argument* ([Mos65]). The main known constructions of symplectomorphisms can be grouped in two categories:

1. Automorphisms obtained as isometries of Kähler manifolds (induced by Lie group actions).
2. Automorphisms obtained as monodromy maps of algebraic families. In particular, if X is a smooth projective variety varying in a smooth family $\chi \to M$, parallel transport yields a monodromy representation $\pi_1(M) \to \mathrm{Symp}(X)$.

One of the many outlets of applications of the theory of pseudoholomorphic curves pioneered by Gromov ([Gro85]) was to compute the symplectomorphism groups of $(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2})$ and $(\mathbb{CP}^2, \omega_{FS})$, whose elements belong to the first category above. Gromov proved that $\mathrm{Symp}(\mathbb{CP}^2)$ is homotopy equivalent to the isometry group $\mathbb{P}U(3)$ and used that to show that $\mathrm{Symp}_{ct}(\mathbb{C}^2)$ is contractible. On the other hand, he showed that $\mathrm{Symp}(S^2 \times S^2)$ retracts onto the subgroup $(\mathrm{SO}(3) \times \mathrm{SO}(3)) \rtimes \mathbb{Z}/2\mathbb{Z}$ of Kähler isometries. In contrast, little is known about the smooth mapping class group of the cited manifolds.

In the cases above, the symplectomorphism group retracts to the isometry group K of the manifold, and therefore its elements are not the best candidates to study the symplectic isotopy question; namely, if K is connected then all of its elements are connected to the identity.

Dehn twists are the simplest examples of symplectomorphisms of the second type, and were first discussed by Arnol'd in [Arn]. Given a sphere L (and a choice of parametrisation, called *framing*, see Definition 2.1.7), the periodicity of the (co)geodesic flow can be used to construct a compactly supported symplectomorphism of the cotangent bundle $\tau_L \in \mathrm{Symp}_{ct}(T^*L)$ (see Definition 2.1.8), called a standard Dehn twist.

The standard Dehn twist has infinite symplectic order, i.e infinite order in $\pi_0(\mathrm{Symp}_{ct}(T^*S^n))$ ([Sei00]), and for $n = 2$, it generates the entire mapping class group $\pi_0(\mathrm{Symp}_{ct}(T^*S^2))$ ([Sei98]).

Given a general symplectic manifold (M, ω) and a Lagrangian sphere $L \subset M$, the local construction of the standard Dehn twist can be implanted in a neighbourhood of L via Weinstein's neighbourhood theorem, to yield a compactly supported symplectomorphism $\tau_L \in \mathrm{Symp}_{ct}(M)$. When $\dim(L)$ is even, the Dehn twist has finite order in $\mathrm{Diff}^+_{ct}(M)$ but often has infinite order in $\mathrm{Symp}_{ct}(M)$. Seidel's early investigations generated the first global examples of (symplectically) non-trivial Dehn twists, in particular non-trivial elements of the kernel of the comparison map (1.1). For example, for a $K3$-surface (M,ω) containing two disjoint Lagrangian spheres $L_1, L_2 \subset M$, the class of τ_{L_1} has infinite order in $\pi_0(\mathrm{Symp}_{ct}(M))$, and hence in that case c has infinite kernel ([Sei00]). Other important examples in which the kernel of c is large comprise Dehn twists in Milnor fibres of any isolated hypersurface singularity ([Kea13]) and Dehn twists in projective hypersurfaces of degree $d > 2$ (and other more general divisors, [Ton16]).

One of the methodologies widely used in these investigations is symplectic Picard–Lefschetz theory. In this context, Dehn twists are regarded as the class of symplectomorphisms that encodes symplectic monodromy maps associated to nodal degenerations, i.e monodromies of Lefschetz fibrations ([Sei03, Sei08a]). For an exact symplectic manifold (M,ω), any Dehn twist τ_L along a Lagrangian sphere $L \subset (M,\omega)$ can be realised as the local monodromy of an exact Lefschetz fibration (with exact smooth fibre (M,ω) and exact base). The global monodromy of such Lefschetz fibrations can never be isotopic to the identity in the symplectic mapping class group ([BGZ19]), so Dehn twists represent an important source of (symplectically) non-trivial symplectic automorphisms of exact symplectic manifolds.

A central finding of symplectic Picard–Lefschetz theory is the celebrated *Seidel long exact sequence* ([Sei03]), which characterises the action of a Dehn twist on a Floer theoretical level. Let (M,ω) be an exact symplectic manifold and $L \subset M$ a Lagrangian sphere generating a Dehn twist τ_L, then for every Lagrangians $L_0, L_1 \subset M$, there is an exact triangle

$$\begin{array}{ccc} \mathrm{HF}^*(L_0,L_1) & \longrightarrow & \mathrm{HF}^*(L_0,\tau_L(L_1)) \\ & \nwarrow \qquad \swarrow & \\ & \mathrm{HF}^*(L,L_1)\otimes\mathrm{HF}^*(L_0,L) & \end{array} \tag{1.2}$$

where $\mathrm{HF}^*(L_i,L_j)$ is the *Lagrangian Floer cohomology* of the pair (L_i,L_j), a symplectic invariant obtained as the cohomology of the cochain complex $(\mathrm{CF}^*(L_i,L_j),\partial)$ generated by the intersections of the Lagrangians, with a differential ∂ counting the pseudoholomorphic curves connecting them (see Section 4.3.1).

This exact sequence is a Floer theoretical version of the standard Picard–Lefschetz formula (the latter can be obtained by taking the Euler characteristics of (1.2)) and enables to capture symplectic phenomena that cannot be detected by the classical description of the monodromy action on half-dimensional homology classes.

Any Dehn twist in $\mathrm{Symp}_{ct}(M)$ generates an autoequivalence (a functor that is invertible up to quasi-isomorphism) of the *Fukaya category* $\mathcal{F}\mathrm{uk}(M)$, an A_∞-category whose objects are Lagrangians (with some additional algebraic structures) and morphisms are Floer cochains (see Section 2.3), and there is a refined version of the exact sequence (1.2) at the level of functors ([Sei08a]).

The inquiries that engendered this research are grounded in the study of relations of Dehn twists within the symplectic mapping class group of an exact manifold. More precisely, given an exact symplectic manifold containing a set of Lagrangian spheres, one of the points of departure of this project is the question of whether products of Dehn twists along the given spheres can generate non-trivial symplectic automorphisms.

In the case in which such products are restricted to admit *positive powers* of Dehn twists (we call such a product a *positive word*), the following result is a relevant motivation for our investigations.

Theorem 1 ([BGZ19, Theorem 1.4])**.** *Let (M,ω) be a Liouville manifold, and let $L_1,\dots,L_m \subset M$ be Lagrangian spheres. Let $\phi = \prod_{i=1}^{k} \tau_{L_{j_i}} \in \mathrm{Symp}_{ct}(M)$, $j_i \in \{1,\dots,m\}$ be a positive word of Dehn twists. Then ϕ is not compactly supported isotopic to the identity in $\mathrm{Symp}_{ct}(M)$.*

In a more general setting, we can consider both positive as well as negative powers of Dehn twists. In this case, it becomes necessary to have a way of measuring the intersections of the Lagrangians generating the twists. For example, if L, L' are two Lagrangian spheres of a Liouville manifold (M,ω) which intersect in a single point, then the corresponding twists $\tau_L, \tau_{L'} \in \mathrm{Symp}_{ct}(M)$ satisfy the *braid relation* $\tau_L\tau_{L'}\tau_L \simeq \tau_{L'}\tau_L\tau_{L'}$ ([Sei99, ST01]). In a general situation, Keating shows that the suitable quantifier that obstructs the possibility of a non-trivial relation between the twists $\tau_L, \tau_{L'}$ is the rank of the Floer cohomology group $\mathrm{HF}(L,L')$, as follows.

Theorem 1.1.1 ([Kea13, Theorem 1.1 and 1.2])**.** *Let (M,ω) be a Liouville manifold of dimension greater than 2, and $L, L' \subset M$ be two Lagrangian spheres satisfying $\operatorname{rank}\mathrm{HF}(L,L') \geqslant 2$, and such that L, L' are not quasi-isomorphic in the Fukaya category. Then the Dehn twists $\tau_L, \tau_{L'}$ generate a free subgroup of $\pi_0(\mathrm{Symp}_{ct}(M))$, and the associated functors T_L, T_L' generate a free subgroup of $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(M))$.*

Note that the two-dimensional case holds via a result due to Ishida ([Ish96]). Keating's result hinges on a classification of A_∞-modules over the formal A_∞-algebra $H^*(S^n)$, that delivers an iterated version of Seidel's (categorical) long exact sequence, which is then used to control the rank of Floer cohomology groups.

1.2 Projective twists: questions

In [Sei00], Seidel introduced a class of symplectomorphisms defined from Lagrangian submanifolds with periodic geodesic flow. This type of Lagrangian includes spheres—in which case the symplectomorphisms are squared Dehn twists—and projective spaces. This thesis focuses on the latter class of symplectomorphisms, that we call *projective twists* (the appellation *Dehn* will be associated exclusively to Dehn twists along spheres). The complex projective analogues of Dehn twists are *always* contained in the kernel of the comparison map (1.1) ([Sei00, Proposition 4.6]), thereby provide examples of symplectomorphisms which are never detectable by the smooth structure.

Unlike their spherical counterparts, projective twists have not been in the spotlight of research in symplectic toplogy, and this is for a number of reasons. The definition of projective twist requires the existence of a Lagrangian embedding of a projective space, which can result in strong topological restrictions to the ambient manifold. Moreover, the symplectic Picard–Lefschetz theory of [Sei08a] does not have such immediate applications as for Dehn twists.

Nevertheless, a series of recent results indicates that projective twists do have interesting properties of the caliber of Dehn twists: Evans [Eva11], Harris [Har11], Mak–Wu [MW18a]. In particular, [MW18a] brought the necessary understanding of the autoequivalences of the Fukaya category induced by projective twists, by showing that they also fit in an exact triangle. In this case, the induced functor has the shape of a double cone (see [MW18a, Theorem 6.10]).

Projective twists can be viewed as examples of *fibred twists*, a class of symplectomorphisms introduced in the first instance by Biran–Giroux and later in more generality by Perutz ([Per07]). Fibred twists are defined as compactly supported symplectomorphisms that act as Dehn twists on the fibres of a spherically fibred coisotropic manifold (see Definition 2.2.11). In this presentation, projective twists arise as monodromies of Morse–Bott degenerations. The relevant fibration-like structures admitting such singularities are called *Morse–Bott–Lefschetz* (MBL) fibrations (see Section 2.2.2), which were utilised in [WW16] to obtain a generalisation of Seidel's long exact sequence for fibred twists. MBL fibrations deliver the current geometric local model for projective twists (which we illustrate in Section 2.2.5). In the first part of this thesis (Chapters 3 and 4), we use Picard–Lefschetz theory to establish a new local model for the standard projective twists $\tau_{\mathbb{RP}^2}, \tau_{\mathbb{CP}^2}$ in the symplectic mapping class group of their respective cotangent bundles.

We then investigate the properties of projective twists from the viewpoint of the following motivating questions.

Questions 1. Let (M, ω) be a Liouville manifold.

(a) Can a reduced word of projective twists be symplectically isotopic to the identity (i.e are there twists satisfying any non-trivial relations) in $\mathrm{Symp}_{ct}(M)$?

(b) Can a reduced *positive* word (i.e a product of positive powers) of projective twists be symplectically isotopic to the identity in $\mathrm{Symp}_{ct}(M)$?

The recent advances in the study of Fukaya categories and the mirror symmetry conjecture have made these questions relevant also on a categorical level. When possible, we also ask to which extent the functors induced by these symplectic automorphisms (on suitable versions of the Fukaya category) behave like their geometrical counterparts, and consider the categorical version of the questions above.

1.3 Model planar projective twists

The first part of this thesis (which occupies Chapters 3 and 4) presents new local models for the planar projective twists $\tau_{\mathbb{RP}^2} \in \mathrm{Symp}_{ct}(T^*\mathbb{RP}^2)$ and $\tau_{\mathbb{CP}^2} \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$ that enhance the geometric understanding of these symplectomorphisms. As discussed above, the current available local model of a projective twist is issued from its presentation as the local monodromy of a MBL fibration (we discuss these models in Section 2.2.5). In the new models we present in this thesis, the planar projective twists are built as symplectomorphisms of the total space of two Lefschetz fibrations $\pi\colon E_{\mathbb{AP}^2} \to \mathbb{C}$ (with compact exact fibres) with total spaces exact symplectomorphic to $(T^*\mathbb{AP}^2, d\lambda_{T^*\mathbb{AP}^2})$, $\mathbb{A} \in \{\mathbb{R},\mathbb{C}\}$.

Lefschetz fibrations were first thought of as fibration like structures arising from families of hyperplane sections (with nodal singularities) on complex projective varieties ([Lam81, Lef24]), so called *Lefschetz pencils* (see Definition 3.2.1). A smooth projective variety (X,ω_X) with an ample line bundle $\mathcal{L} \to X$ can be viewed as a Kähler manifold with integral symplectic class $c_1(\mathcal{L}) = [\omega] \in H^2(X;\mathbb{Z})$. Then the zero locus of a section $s_\infty \in H^0(\mathcal{L})$ that vanishes transversely is a smooth hypersurface $\Sigma = s_\infty^{-1}(0)$ Poincaré dual to $[\omega]$. Given another generic section $s_0 \in H^0(\mathcal{L})$, the family $\{\lambda s_0 + \mu s_\infty = 0\}_{[\lambda:\mu]\in\mathbb{CP}^1}$ parametrised by $\mathbb{CP}^1$ determines a Lefschetz pencil; $\Sigma_\infty \subset X$ is a smooth fibre in this family.

(i) On one hand, the (closure of) the complement $X \setminus \Sigma_\infty$ is a Stein domain with plurisubharmonic function $f\colon X\setminus\Sigma_\infty \to \mathbb{R}$ given by $f := -log\|s_\infty\|^2$, on which the ratio s_0/s_∞ induces the structure of an exact Lefschetz fibration (see [Sei08a, (19b)]).

(ii) On the other hand, if $f\colon X\setminus\Sigma_\infty \to \mathbb{R}$ is Morse, its unstable manifolds are isotropic. Let $\Xi \subset X$ be the union of these isotropic submanifolds. A theorem of Biran ([Bir01]) proves that $X \setminus \Xi$ is symplectomorphic to an open disc normal bundle to Σ_∞ (see Theorem 2.2.15).

In Chapter 2, we consider the following applications of Biran's decomposition, that we use in combination with observation (i) in order to produce the desired Lefschetz fibrations.

Example A ([Bir01, 3.1.2]). Let $X = (\mathbb{CP}^n, \omega_{FS})$ with the ample line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{CP}^n}(2) \to \mathbb{CP}^n$. A generic section $s \in H^0(\mathcal{L})$ is given by a homogeneous polynomial of degree two in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{CP}^n$. Consider the section $s := \sum_{i=0}^n z_i^2$. The divisor arising as the zero locus of s is the smooth quadric $\Sigma := s^{-1}(0) = \{z_0^2 + \cdots + z_n^2 = 0\} \cong Q^{n-1} \subset \mathbb{CP}^n$, which is a Kähler submanifold with the integral symplectic structure $\omega_\Sigma = \omega_{FS}|_\Sigma$. The smooth function

$$\rho\colon \mathbb{CP}^n \longrightarrow \mathbb{R},\ \rho([z_0:\cdots:z_n]) = \frac{|\sum_{i=0}^n z_i^2|^2}{(\sum_{i=0}^n |z_i|^2)^2},$$

is a reparametrisation of $\|s\|^2$, and is Morse–Bott on $X\setminus\Sigma$. To study the unstable submanifolds of $f := -\log(\rho)\colon X\setminus\Sigma \to \mathbb{R}$ it is enough to understand the critical locus

$$\mathrm{Crit}(\rho|_{X\setminus\Sigma}) = \mathrm{Crit}(\rho)\setminus\Sigma = \{[z_0:\cdots:z_n]\in\mathbb{CP}^n | z_j\in\mathbb{R} \text{ for all } 0\leqslant j\leqslant n\} \simeq \mathbb{RP}^n \subset \mathbb{CP}^n,$$

which is embedded as a Lagrangian in $(\mathbb{CP}^n, \omega_{FS})$. The positive gradient vector field points out of $\mathrm{Crit}(\rho)\setminus\Sigma$, so for the union of unstable manifolds Ξ we have $\Xi = \mathrm{Crit}(\rho)\setminus\Sigma \simeq \mathbb{RP}^n \subset \mathbb{CP}^n$.

The applications of Biran's theorem we discuss in Chapter 2 (see 2.2.3, 2.2.4) imply that the complement $X\setminus\Sigma$ is symplectomorphic to an open disc subbundle $\mathring{D}_\varepsilon T^*\mathbb{RP}^n \subset T^*\mathbb{RP}^n$:

$$\mathbb{CP}^n \cong \mathring{D}_\varepsilon T^*\mathbb{RP}^n \cup \Sigma,\ \varepsilon > 0. \tag{1.3}$$

Example B ([Bir01, 3.3])**.** Let $X = (\mathbb{CP}^n\times\mathbb{CP}^n, \omega_{FS}\oplus\omega_{FS})$ with the ample line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{CP}^n\times\mathbb{CP}^n}(1,1) := pr_1^*(\mathcal{O}_{\mathbb{CP}^n}(1))\otimes pr_2^*(\mathcal{O}_{\mathbb{CP}^n}(1))$, where $pr_i\colon \mathbb{CP}^n\times\mathbb{CP}^n \to \mathbb{CP}^n$ is the projection to the i-th factor, $i = 1,2$. A generic section $s\in H^0(\mathcal{L})$ is a homogeneous polynomial of degree $(1,1)$, in the homogeneous coordinates $(\underline{x},\underline{y}) = ([x_0:\cdots:x_n],[y_0:\cdots:y_n])$ of $\mathbb{CP}^n\times\mathbb{CP}^n$. Let $s := \sum_{i=0}^n x_i y_i$; the divisor obtained as the zero set of s is given as $\Sigma := \{\sum_{i=0}^n x_i y_i = 0\} \subset \mathbb{CP}^n\times\mathbb{CP}^n$.

The function

$$\rho\colon \mathbb{CP}^n\times\mathbb{CP}^n \to \mathbb{R},\ \rho(\underline{x},\underline{y}) = \frac{|\sum_{i=0}^n x_i y_i|^2}{(\sum_{i=0}^n |x_i|^2)^2(\sum_{i=0}^n |y_i|^2)^2}$$

is a smooth reparametrisation of $\|s\|^2$, and is Morse–Bott on $X\setminus\Sigma$ with critical locus given by

$$\mathrm{Crit}(\rho|_{X\setminus\Sigma}) = \mathrm{Crit}(\rho)\setminus\Sigma = \{([x_0:\cdots:x_n],[\overline{x_0}:\cdots:\overline{x_n}]),\ [x_0\cdots:x_n]\in\mathbb{CP}^n\} \simeq \mathbb{CP}^n \subset \mathbb{CP}^n\times\mathbb{CP}^n,$$

which is embedded as a Lagrangian in $(\mathbb{CP}^n\times\mathbb{CP}^n, \omega_{FS}\oplus\omega_{FS})$. Following the same reasoning as in the previous example, the union of unstable manifolds Ξ is given by $\Xi = \mathrm{Crit}(\rho)\setminus\Sigma \simeq \mathbb{CP}^n$. The arguments of Chapter 2 (see 2.2.3, 2.2.4) then yield a decomposition

$$\mathbb{CP}^n\times\mathbb{CP}^n \cong \mathring{D}_\varepsilon T^*\mathbb{CP}^n \cup \Sigma,\ \varepsilon > 0. \tag{1.4}$$

Combining the decompositions (1.3) and (1.4) with the study of a pencil of quadrics on $(\mathbb{CP}^2, \omega_{FS})$ and a pencil of $(1,1)$-divisors on $(\mathbb{CP}^2\times\mathbb{CP}^2, \omega_{FS}\oplus\omega_{FS})$ respectively, we obtain the following Lefschetz fibrations.

Proposition 1.3.1. *1. There is an exact Lefschetz fibration $\pi\colon E_{\mathbb{RP}^2} \to \mathbb{C}$ with smooth fibre exact symplectomorphic to a sphere with four boundary components, three singular fibres and such that the completion of the total space $E_{\mathbb{RP}^2}$ is exact symplectomorphic to $(T^*\mathbb{RP}^2, d\lambda_{T^*\mathbb{RP}^2})$ (Section 3.3).*

2. There is an exact Lefschetz fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ with smooth fibre exact symplectomorphic to a plumbing of disc cotangent bundles $D_\varepsilon(T^*S^3 \#_{S^1} T^*S^3)$ along a circle and three singular fibres (see Section 6.1 for a definition of plumbing), such that the completion of the total space $E_{\mathbb{CP}^2}$ is exact symplectomorphic to $(T^*\mathbb{CP}^2, d\lambda_{T^*\mathbb{CP}^2})$ (Section 3.4).

As they arise from Lefschetz pencils of hypersurfaces on projective varieties, these two Lefschetz fibrations have the property that for any vanishing cycle in the smooth fibre, $V \subset M$, the global monodromy of the fibration $\phi \in \mathrm{Symp}_{ct}(M)$ satisfies

$$\phi(V) \simeq V. \tag{1.5}$$

This is a consequence of the *general lantern relation*, a relation in the mapping class group of the smooth fibre (M, ω), see Lemma 3.2.6. This special feature of the monodromy enables us, in the two cases, to apply a construction of [Sei15], which takes as input a symplectomorphism of the fibre of a Lefschetz fibration and outputs a compactly supported symplectomorphism on the total space (Section 4.2). This construction represents a fundamentally new method of building a symplectomorphism that differs from the two currently known categories of Section 1.1. In Chapter 4 we illustrate this method, so far rather unexplored, for the fibrations above to obtain non-trivial elements of the symplectic mapping class groups $\pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{AP}^2))$ that we compare with the standard planar projective twists (Sections 4.4 and 4.5 respectively).

1.4 Positive products of Dehn twists

In Chapter 7, we begin the investigation about positive relations of twists. In a first instance, we analyse Question 1 (a) for Dehn twists, and we show that in a Liouville manifold (M, ω) there can never be a product $\phi \in \mathrm{Symp}_{ct}(M)$ of (positive powers of) Dehn twists that is isotopic to the identity, via a compactly supported symplectic isotopy. We thereby provide an alternative proof of Theorem 1, which was originally proved (by Barthes–Geiges–Zehmisch in [BGZ19]) via techniques involving open book decompositions. Our proof is implemented using Picard–Lefschetz theory. The idea is to build a Lefschetz fibration $\pi\colon E \to \mathbb{C}$ with smooth fibre the Liouville manifold (M, ω) and vanishing cycles the given Lagrangian spheres involved in the product $\phi \in \mathrm{Symp}_{ct}(M)$. In that way, the monodromy of π is given by ϕ.

Assuming that there exists an isotopy $\phi \simeq Id$ as in the statement, we extend π to a fibration over $\mathbb{CP}^1$, and, by analysing the moduli space of pseudoholomorphic sections (following [Sei03]), obtain a contradictory statement.

Consider a punctured torus $M := T^2 \setminus \{*\}$, and the two (Lagrangian) circles a and b, representatives of the homological generators. The product $\phi := (\tau_a \tau_b)^6 \in \mathrm{Symp}_{ct}(M)$ satisfies the well-known relation $\phi \simeq \tau_d$, where τ_d is the Dehn twist along the boundary curve d encircling the puncture (this is a consequence of the *chain relation*, see [FM11, Proposition 4.12]). In particular, ϕ is non-trivial (Theorem 1 validates this fact). However, ϕ does act trivially on any exact compact circle in M (which is necessarily disjoint from the support of τ_d). If, on the other hand, we consider an open Lagrangian $T \subset M$ with two boundary components at the puncture, then ϕ operates a twist on T, so the latter cannot be preserved by this product.

Going back to a general Liouville manifold (M, ω), it can happen that a product of Dehn twists, despite being necessarily not (compactly supported symplectically) isotopic to the identity, preserves some Lagrangian submanifolds of M. The question arises, whether one can find a Lagrangian $T \subset M$ such that there can be no compactly supported symplectic isotopy $\phi(T) \simeq T$. The existence of such a Lagrangian would result in a stronger version of Theorem 1.

In the second part of Chapter 7, we address this question. We find one possible candidate Lagrangian $T \subset M$ with the above properties, but unfortunately cannot prove that such a Lagrangian always exist.

A Lagrangian $T \subset M$ is called conical if it is an exact, properly embedded Lagrangian that is preserved by the Liouville flow over the cylindrical ends of M.

Theorem 2. *Let (M^{2n}, ω) be a Liouville manifold containing embedded Lagrangian spheres $L_1, \ldots, L_m$ and a conical Lagrangian disc T intersecting one of the spheres L_j transversely in a point. Let $\phi := \prod_{i=1}^{k} \tau_{L_{j_i}} \in \mathrm{Symp}_{ct}(M)$, $j_i \in \{1, \ldots, m\}$ be a positive word of Dehn twists involving τ_{L_j}. Then the Lagrangians T and $\phi(T)$ are not isotopic via a compactly supported Lagrangian isotopy.*

Example 1.4.1. For $m > 0$, consider an iterated transverse plumbing $M := T^*S^m \#_{pt} T^*S^m \#_{pt} T^*S^m \cdots \#_{pt} T^*S^m$ (see Section 6.1 for the definition of plumbing). Let $\phi \in \mathrm{Symp}_{ct}(M)$ be a product of Dehn twists along the Lagrangian spheres of M, such that ϕ contains the Dehn twist along the j-th sphere. In this case, the conical Lagrangian disc $T \subset M$ of Theorem 2 is given by any cotangent fibre of the j-th summand. $\diamondsuit$

The arguments we use in the proof of Theorem 2 are centred around the same principles of the method used for Theorem 1, with some necessary adjustments due to the non-compactness of the Lagrangian $T \subset M$.

1.5 From projective spaces to spheres: the Hopf correspondence

How can we study projective twists? Because much of the scholarship that emerged from the study of Dehn twists is the result of successful applications of symplectic Picard–Lefschetz theory, a first intuitive move is to approach the study of projective twists by means of their presentation as monodromies of Morse–Bott–Lefschetz fibrations. In a preliminary phase of this research, we attempted to adapt some of the arguments originally tailored for Dehn twists to a more general Picard–Morse–Bott–Lefschetz theory as developed in [WW16]. One of the challenges we encountered in this setting was that the total space of Morse–Bott–Lefschetz fibations is in general not exact, and the singular locus (a smooth manifold of the singular fibre) often admits rational curves. This fact induced many complications related to loss of compactness of the moduli spaces of pseudoholomorphic curves of these fibrations.

To examine the properties of these symplectomorphisms, we eventually adopted a strategy that allowed to reduce the study of projective twists to that of Dehn twists in an auxiliary Liouville manifold; this was made possible via the theoretical device of *Lagrangian correspondences*.

A Lagrangian correspondence between two symplectic manifolds (W,ω) and (Y,Ω) is a Lagrangian submanifold of the product $W^- \times Y := (W \times Y, -\omega \oplus \Omega)$. By the work of [WW09, WW10a, WW10b, Gao17a, Gao17b], under suitable conditions, a Lagrangian correspondence induces a functor which associates a Lagrangian in Y to a Lagrangian in W.

In a first stage, we define an appropriate Lagrangian correspondence that relates a set of Lagrangian projective spaces in a given Liouville manifold (W,ω) to a set of Lagrangian spheres in an auxiliary manifold (Y,Ω) expressly built under some cohomological conditions. Fix a tuple $(\mathbb{A},k,*,R) \in \{(\mathbb{R},0,1,\mathbb{Z}/2\mathbb{Z}),(\mathbb{C},1,2,\mathbb{Z})\}$. Assume there are Lagrangian projective spaces $\mathbb{AP}^n \cong K_1,\dots K_m \subset W$ and a non-trivial class $\alpha \in H^*(W;R)$ such that $\alpha|_{K_i}$ generates $H^*(\mathbb{AP}^n;R)$. Then there is a Liouville manifold (Y,Ω), realised as a T^*S^k-bundle $q\colon Y \to W$, so that over each Lagrangian projective space K_i, the bundle restricts a Lagrangian sphere L_i. The total space (Y,Ω) contains an S^k-fibred coisotropic submanifold $V \to W$, which defines a Lagrangian correspondence $\Gamma := \{(q(y),y),\ y \in V\} \subset W^- \times Y$ in the sense of [Per08]. Over each projective Lagrangian $K_i \subset W$, the correspondence yields a Lagrangian sphere $L_i \subset Y$, for $i = 1,\dots,m$. We name Γ the *Hopf correspondence*.

Once the Hopf correspondence constructed, we use Ma'u–Wehrheim–Woodward theory to show that it induces a functor $\Theta_\Gamma\colon \mathcal{F}\mathrm{uk}(W) \to \mathcal{F}\mathrm{uk}(Y)$ between the compact Fukaya categories (see Section 5.2.1). We then prove the existence of a commuting diagram (Section 5.4)

$$\begin{array}{ccc} \mathcal{F}\mathrm{uk}(Y) & \xrightarrow{\quad T_{L_i} \quad} & \mathcal{F}\mathrm{uk}(Y) \\ {\scriptstyle \Theta_\Gamma}\uparrow & & {\scriptstyle \Theta_\Gamma}\uparrow \\ \mathcal{F}\mathrm{uk}(W) & \xrightarrow{\quad T_{K_i} \quad} & \mathcal{F}\mathrm{uk}(W) \end{array} \tag{1.6}$$

where $T_{K_i} \in Auteq(\mathcal{F}\mathrm{uk}(W))$ and $T_{L_i} \in Auteq(\mathcal{F}\mathrm{uk}(Y))$ are the twists-functors induced by the graphs of the respective twists $\tau_{K_i} \in \mathrm{Symp}_{ct}(W), \tau_{L_i} \in \mathrm{Symp}_{ct}(Y)$.

1.6 Products of projective twists

In Chapter 6 we consider Question 1(b) and give a first answer to it. We consider a *clean plumbing* (see Definition 6.1.1) of Lagrangian projective spaces; a symplectic construction in which two copies of cotangent bundles $T^*\mathbb{AP}^n$ are glued along a common submanifold of the zero sections, and prove the following.

Theorem 3. *Let $W = T^*\mathbb{AP}^n \#_{\mathbb{AP}^\ell} T^*\mathbb{AP}^n$ be a clean plumbing of (real, complex) projective spaces along a linearly embedded sub-projective space $\mathbb{AP}^\ell \subset W$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. Let $K_1, K_2 \subset W$ denote the Lagrangian core components of the plumbing. Then the projective twists τ_{K_1} and τ_{K_2} generate a free group inside $\pi_0(\mathrm{Symp}_{ct}(W))$, and the associated functors T_{K_1}, T_{K_2} generate a free subgroup of $Auteq(\mathcal{F}\mathrm{uk}(W))$.*

In the complex case, this theorem yields a new criterion for projective twists to generate a free subgroup of the kernel of the comparison map (1.1). We prove Theorem 3 using the Hopf correspondence to relate the functors $T_{K_1}, T_{K_2} \in Auteq(\mathcal{F}\mathrm{uk}(W))$ to functors $T_{L_1}, T_{L_2} \in Auteq(\mathcal{F}\mathrm{uk}(Y))$ induced by Dehn twists in a Liouville manifold (Y, Ω) constructed as a T^*S^k-bundle over W, $k \in \{0, 1, 3\}$. This is made possible via the commuting diagram (1.6).

We can then apply Keating's result (Theorem 1.1.1) to our setting to obtain a free generation result for T_{L_1}, T_{L_2}, that we translate into a free generation result for T_{K_1}, T_{K_2} via the Hopf correspondence.

Remark 1.6.1. *The case $W := T^*\mathbb{CP}^1_1 \#_{pt} T^*\mathbb{CP}^1_2$ can be obtained with the current literature ([Sei99], [ST01], [KS02]), by considering W as an A_2-configuration and using the isotopies $\tau_{\mathbb{CP}^1_i} \simeq \tau^2_{S^2_i}$ (see Remark 6.2.1).* //

In Section 7.3, we restrict our attention to products of positive powers of projective twists, i.e Question 1(b), and prove a result that can be considered the (real) projective counterpart to Theorem 1.

Theorem 4. *Let (W^{2n}, ω) be a Liouville manifold containing Lagrangian real projective spaces $K_1, \dots K_m$, $K_i \cong \mathbb{RP}^n$. Suppose that there is a class $\alpha \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ such that for every $i = 1, \dots, m$,*

$\alpha|_{K_i}$ generates $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. Let $\varphi \in \mathrm{Symp}_{ct}(W)$ be a positive word in the subset of projective twists $\{\tau_{K_i}\}_{i\in\{1,\dots,m\}}$. Then φ is not isotopic to the identity in $\pi_0(\mathrm{Symp}_{ct}(M))$.

Using the cohomological assumption of the theorem, we establish the Hopf correspondence and prove the theorem by contradiction. The idea is that in these circumstances, there exists a product of Dehn twists $\phi \in \mathrm{Symp}_{ct}(\widetilde{W})$ in the symplectic double cover $q\colon (\widetilde{W},\widetilde{\omega}) \longrightarrow (W,\omega)$, such that $q\circ\phi = \varphi\circ q$. Then, an isotopy $\varphi \simeq Id$ in $\mathrm{Symp}_{ct}(W)$ can be lifted to an isotopy $\phi \simeq Id$ in $\mathrm{Symp}_{ct}(\widetilde{W})$, contradicting Theorem 1.

Unfortunately, the same techniques do not yield a result for complex projective twists; in that case, the auxiliary manifold (Y,Ω) defines a $\mathbb{C}^*$-bundle $q\colon Y \to W$ and a compactly supported isotopy on W does not lift to a compactly supported isotopy on Y, so that the arguments used for Theorem 4 do not apply here.

1.7 Non-standard parametrised projective twists

The last chapter uses the Hopf correspondence to examine the ways in which the symplectic structure interferes with the underlying topological structures, such as diffeomorphism and homeomorphism class, of Lagrangian homotopy projective spaces. In this thesis, a manifold that is homeomorphic but not diffeomorphic to a standard (real or complex) projective space is called an AD projective space. A manifold that is homotopy equivalent but not homeomorphic to a standard (real or complex) projective space is called an AT projective space. Similarly, an AD sphere is a sphere that is homeomorphic but not diffeomorphic to the standard sphere (we decide to drop the usual epithet *exotic*, see Definition 8.1.6).

A notorious conjecture, known under the name of *nearby Lagrangian conjecture*, states that given a closed smooth manifold Q, any closed exact Lagrangian submanifold of $(T^*Q, d\lambda_Q)$ is Hamiltonian isotopic to the zero section. This conjecture has generated a lot of interest in the symplectic community, but its statement is currently confirmed only up to simple homotopy equivalence ([Abo12b], [Kra13], [AK18]; in Section 8.1 we summarise the state of the art of this conjecture). For a homotopy sphere L, it is known that the choice of smooth structure can be an obstruction to the existence of a Lagrangian embedding $L \hookrightarrow T^*S^n$. Namely, for $n > 4$ odd, AD spheres which do not bound parallelisable manifolds admit no Lagrangian embedding into T^*S^n ([Abo12a, EKS16]).

Using the existing literature about S^1-actions on AD spheres ([Bre67], [Jam80], [Kas16]), we find, in Section 8.1, examples of non-standard homotopy complex projective spaces which do not admit Lagrangian embeddings into $T^*\mathbb{CP}^n$. These results are compatible with the predictions derived from the nearby Lagrangian conjecture.

Theorem 5 (Theorems 8.1.11, 8.1.13)**.** *(i) There is a manifold P homotopy equivalent to $\mathbb{CP}^4$*

and with the same first Pontryagin class such that neither P nor $P\#\Sigma^8$ admit an exact Lagrangian embedding into $T^*\mathbb{CP}^4$.

(ii) There is an element Σ^{14} in the group of homotopy 14*-spheres Θ_{14} such that $\mathbb{CP}^7\#\Sigma^{14}$ does not admit an exact Lagrangian embedding into $T^*\mathbb{CP}^7$.*

On the other hand, in Section 8.2, we present new results which prove that in general, the Hamiltonian isotopy class of projective twists does depend on a choice of framing, i.e a choice of smooth parametrisation $f\colon \mathbb{CP}^n \to L$ (see Definition 2.1.7). It was proved by Dimitroglou Rizell and Evans ([DRE15]) that a non-standard parametrisation $S^n \to L$ of a Lagrangian sphere can give rise to a Dehn twist that is not isotopic to the standard Dehn twist τ_{S^n}.

We use classical homotopy theory and the Hopf correspondence to transpose the existence of non-standard parametrisations of Dehn twists of [DRE15] into instances of projective twists depending on their framing.

Theorem 6 (Corollary 8.2.13)**.** *The $\mathbb{CP}^n$-twist depends on the framing when $n = 19, 23, 25, 29$.*

This shows that in general, $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$ is not generated by the standard projective twist along the zero section $\tau_{\mathbb{CP}^n}$ (see Corollary 8.2.16). Moreover, we notice that the use of advanced topological technology (*topological modular forms*) can prove the existence of infinitely many non-standardly framed (complex) projective twists (Remark 8.2.14).

Chapter 2

Twists

This chapter provides the necessary background for studying Dehn twists and projective twists in symplectic topology; it can be skipped by the expert reader. We summarise the geometric construction of these maps from the geodesic flow perspective (Section 2.1) and the interpretation as monodromies of topological fibrations (in Section 2.2).

2.1 Twists from geodesic flow

Given a connected closed Riemannian manifold (K,g) admitting a periodic (co-) geodesic flow, Seidel ([Sei00]) constructs a symplectomorphism in $\mathrm{Symp}_{ct}(T^*K)$. We review the construction (the main references are [Sei03, 1.2], [Sei00, 4.b]) of this class of symplectomorphisms, which we call twists. More specifically, for $K \cong S^n$ and g the round metric, this is the well-known symplectic *Dehn twist*, and in the cases $K \in \{\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n\}$, and g the standard submersion metric, the construction yields what we call a *projective twist*.

Let (K,g) be a closed connected Riemannian manifold admitting a periodic cogeodesic flow $\Phi_K^t\colon T^*K \to T^*K$ on its cotangent bundle $(T^*K, d\lambda_{T^*K})$, such that each geodesic of length 2π is closed (so that the shortest period of a unit-speed geodesic is 2π).

Let $\|\cdot\|_K$ be the norm associated to the given Riemannian metric g. The normalised cogeodesic flow satisfies $\Phi_K^{2\pi} = Id$ and can be extended to a Hamiltonian S^1-action σ_t^H on $T^*K \setminus K$, with moment map $H : T^*K \setminus K \to \mathbb{R}$, $H(v) = \|v\|_K$. Note that the cogeodesic flow coincides with the Reeb flow of λ_{T^*K}.

Definition 2.1.1. Let K be diffeomorphic to S^n. For $\varepsilon > 0$, define an auxiliary smooth cut-off function $r_\varepsilon\colon \mathbb{R}^+ \to \mathbb{R}$ such that $0 < r_\varepsilon(t) < \pi$ for all $t < \varepsilon$ and

$$r_\varepsilon(t) = \begin{cases} \pi - t & t \ll \varepsilon \\ 0 & t \geqslant \varepsilon \end{cases} \tag{2.1}$$

The model Dehn twist $\tau_K^{loc}: T^*K \to T^*K$ is defined as

$$\tau_K^{loc}(\xi) = \begin{cases} \sigma^H_{r_\varepsilon(\|\xi\|_K)}(\xi) & \xi \notin K \\ -\xi & \xi \in K. \end{cases} \tag{2.2}$$

♢

Definition 2.1.2. Let K be diffeomorphic to $\mathbb{AP}^n$ for $\mathbb{A} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$. For $\varepsilon > 0$, let $r_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ be a smooth cut-off function such that $0 < r_\varepsilon(t) < 2\pi$ for all $t < \varepsilon$ and

$$r_\varepsilon(t) = \begin{cases} 2\pi - t & t \ll \varepsilon \\ 0 & t \geqslant \varepsilon \end{cases} \tag{2.3}$$

The model projective twist $\tau_K^{loc}: T^*K \to T^*K$ is defined as

$$\tau_K^{loc}(\xi) = \begin{cases} \sigma^H_{r_\varepsilon(\|\xi\|_K)}(\xi) & \xi \notin K \\ \xi & \xi \in K. \end{cases} \tag{2.4}$$

♢

Remark 2.1.3. *Our choice of cut-off functions r_ε follows [MW18a, 2.1], but the construction is independent of such choices, up to suitable isotopy ([Sei00]).* //

Theorem 2.1.4 ([Sei00, Corollary 4.5])**.** *Let (K,g) be a Riemannian manifold admitting a periodic (co-)geodesic flow and satisfying $H^1(K;\mathbb{R}) = 0$. Then the symplectomorphisms τ_K^{loc} have infinite order in $\pi_0(\mathrm{Symp}_{ct}(T^*K))$.*

Theorem 2.1.5 ([Sei00, Proposition 4.6])**.** *The symplectomorphism $\tau^{loc}_{\mathbb{CP}^n}$ of Definition 2.1.1 is isotopic to the identity in $\mathrm{Diff}_{ct}(T^*\mathbb{CP}^n)$.*

Remark 2.1.6. *We will often denote the standard twists by $\tau_{S^n} := \tau^{loc}_{S^n}$ or, for $\mathbb{A} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$, $\tau_{\mathbb{AP}^n} := \tau^{loc}_{\mathbb{AP}^n}$. With the conventions* (2.1) *and* (2.3)*, the isomorphisms $S^1 \cong \mathbb{RP}^1$, $S^2 \cong \mathbb{CP}^1$ and $S^4 \cong \mathbb{HP}^1$ induce identifications $\tau^2_{S^1} \simeq \tau_{\mathbb{RP}^1}$, $\tau^2_{S^2} \simeq \tau_{\mathbb{CP}^1}$ and $\tau^2_{S^4} \simeq \tau_{\mathbb{HP}^1}$ respectively (see [Sei00], [Har11]).* //

Now suppose (L,g) is a Riemannian manifold admitting a Lagrangian embedding $L \subset M$ into a general symplectic manifold (M,ω).

Definition 2.1.7. Let $K \in \{S^n,\mathbb{RP}^n,\mathbb{CP}^n,\mathbb{HP}^n\}$. A *framed Lagrangian* sphere/projective space is a Lagrangian submanifold $L \subset M$ together with an equivalence class $[f]$ of diffeomorphisms $f: K \to L$; $f_1 \sim f_2$ iff $f_2^{-1}f_1$ is isotopic, in Diff(K), to an element of the isometry group Iso(K,g). An equivalence class $[f]$ as above is called a *framing*. ♢

Definition 2.1.8. Let $(L,[f])$ be a framed Lagrangian sphere/projective space in (M,ω). Using Weinstein's neighbourhood theorem, extend a framing representative $f\colon K \to L$ to a symplectic embedding $\iota\colon D_sT^*K \to M$, $s > 0$, where $D_sT^*K := \{v \in T^*K, \|v\|_K < s\}$, $s > 0$. There is a model twist τ_K^{loc}, supported in the interior of D_s^*K, and define

$$\tau_L \cong \begin{cases} \iota \circ \tau_K^{loc} \circ \iota^{-1} & \text{on } \mathrm{Im}(\iota) \\ \mathrm{Id} & \text{elsewhere} \end{cases}$$

In the case where L is a sphere, the map τ_L is the standard symplectic *Dehn* twist. When L is a projective space, the resulting map is called a *projective twist*. In this thesis, the term *Dehn* twist is exclusively reserved for twists that are constructed from a Lagrangian sphere. ♢

Remark 2.1.9.

1. *A Dehn twist along an exact Lagrangian sphere, or a projective twist along an exact projective Lagrangian in an exact symplectic manifold are exact symplectomorphisms in the sense of Definition 2.2.2. The same holds for products of such twists. This follows by construction (for direct computations see for example [BGZ19, Lemma 4.4], [CDVK16, Lemma 2.1]).*

2. *Theorem 2.1.5 implies that given a symplectic manifold (M,ω), any Lagrangian $L \cong \mathbb{CP}^n \subset M$ will define an element $\tau_L \in \mathrm{Symp}_{ct}(M)$ that is isotopic to the identity in $\mathrm{Diff}_{ct}(M)$.*

//

As shown by Dimitroglou Rizell and Evans in [DRE15], the choice of framing does play a role in determining the symplectic isotopy class of a spherical Dehn twist. In Section 8, we prove that this is also the case for projective twists. Before then, any given Lagrangian submanifold involved in the construction of a twist is assumed to be endowed with a choice of framing and we omit mentioning this datum as the results of Sections 2-7.3 are independent of such choices. This is because the autoequivalence of the Fukaya category induced by a Dehn twist (see Section 2.3) is independent of a choice of framing (as a consequence of the shape of the functor, see [Sei08a, Corollary 17.17]). The same is true for the functor induced by the projective twist ([MW18a, Theorem 6.10]).

2.2 Twists as monodromies

This section approaches twists from a different perspective, one that presents these symplectomorphisms as monodromy maps of fibration-like structures.

A celebrated feature that has made the study of Dehn twists particularly productive is the fact that Dehn twists occur as (local) monodromies of Lefschetz fibrations (we review this in Section 2.2.1). On the other hand, it is a lesser known fact that projective twists can be modelled as local monodromies of *Morse–Bott–Lefschetz* fibrations, another class of fibrations admitting more degenerate singularities (this is reviewed in Section 2.2.2). Known by the experts, such a model is not explicitly discussed in the existing literature; an illustration can be found in Sections 2.2.3-2.2.5.

2.2.1 Dehn twists and Lefschetz fibrations

This section is a review of Lefschetz fibrations (mainly following [Sei08a, MS10]) aimed at setting the notation for future chapters, and recalling the well-known *Picard–Lefschetz theorem*.

Definition 2.2.1. A Liouville manifold of finite type is an exact symplectic manifold $(W, \omega = d\lambda_W)$, where $\lambda_W \in \Omega^1(W)$ is called the Liouville form, such that there exists a proper function $h_W \colon W \to [0,\infty)$ and $c_0 > 0$ with the following property. For all $x \in (c_0,\infty)$ and $y \in h^{-1}(x)$, the vector field Z_W dual to λ_W, called the Liouville vector field, satisfies $dh_W(Z_W)(y) > 0$.

For a regular value c of h_W, a closed sublevel set $M := h_W^{-1}([0,c])$ of a Liouville manifold $(W, d\lambda_W)$ is a compact symplectic manifold with contact type boundary $(\Sigma := h_W^{-1}(c), \lambda_W|_\Sigma)$, and it is called a Liouville domain. ♢

Definition 2.2.2. An exact symplectomorphism between two Liouville manifolds $(W_1, d\lambda_1), (W_2, d\lambda_2)$ is a diffeomorphism $\psi\colon W_1 \to W_2$ satisfying $\psi^*\lambda_2 - \lambda_1 = df$, for a compactly supported function $f\colon W_1 \to \mathbb{R}$. ♢

Definition 2.2.3. Let now $(M, d\lambda)$ be a Liouville domain with contact boundary $(\Sigma = \partial M, \alpha = \lambda|_\Sigma)$. The negative Liouville flow identifies a collar neighbourhood $C(\Sigma)$ of the boundary with $(-\varepsilon, 0] \times \partial M$, such that $\lambda|_{C(\Sigma)} = e^t\alpha$. An almost complex structure J of *contact type near the boundary* is one that satisfies $de^t \circ J = -\lambda$. ♢

Definition 2.2.4. Given a Liouville domain $(M, d\lambda)$ as above, we can use the identification of the collar neighbourhood $C(\Sigma)$ to glue an infinite cone and define the symplectic completion of M:

$$(W, \omega_W) := (M \cup [0,\infty) \times \partial M, d(e^t\alpha)), \tag{2.5}$$

where t is the coordinate on $(0,\infty)$, such that the Liouville flow extends to Z_W with $Z_W|_{[0,\infty)\times\partial M} = \partial_t$

An almost complex structure J of contact type extends to an almost complex structure J_W on the completion satisfying

- $J_W(\frac{\partial}{\partial t}) = R_\alpha$, where R_α is the Reeb vector field associated to α,
- J_W is invariant under translations in the t-direction,
- $J_W|_M = J$.

This kind of almost complex structure will be called cylindrical. ♢

We will only consider Liouville manifolds that are complete (i.e with complete Liouville vector field) and of finite type, which we can identify as the union of a Liouville domain with a cylindrical non-compact end, equipped with an almost complex structure cylindrical at infinity.

Let $(E^{2n+2}, \Omega_E, \lambda_E)$ be a Liouville manifold, with a compatible almost complex structure J_E, and consider the complex plane with its standard symplectic form and complex structure $j_{\mathbb{C}}$. Let $\pi\colon E \to \mathbb{C}$ be a map with finitely many critical points, which are all non-degenerate, and contained in a compact set of E. Denote by $\mathrm{Crit}(\pi) := \{x \in E,\ D_x\pi = 0\}$ the set of critical points, and by $\mathrm{Crit}v(\pi) := \pi(\mathrm{Crit}(\pi))$ the set of critical values.

Definition 2.2.5. A Lefschetz fibration on (the Liouville manifold) E is a $(J_E, j_{\mathbb{C}})$-holomorphic map π, i.e $D\pi \circ J_E = j_{\mathbb{C}} \circ D\pi$, with the above properties and the following additional features.

1. For all $x \in E \setminus \mathrm{Crit}(\pi)$, $\ker(D_x\pi) \subset T_xE$ is symplectic.

2. Every smooth fibre is symplectomorphic to the completion of a Liouville domain $(M, d\lambda_M)$.

3. There is an open neighbourhood $U^h \subset E$ such that $\pi\colon E \setminus U^h \longrightarrow \mathbb{C}$ is proper and $\pi|_{U^h}$ can be trivialised via an isomorphism $f\colon U^h \cong \mathbb{C} \times ([0,\infty) \times \partial M)$ such that

$$f^*(\lambda_E) = \lambda_{\mathbb{C}} + e^t \lambda_M. \tag{2.6}$$

$\diamondsuit$

For more details about how this fibration is modeled outside of a neighbourhood of the critical points, see [MS10, (2.1)].

By the first point above, there is a symplectic splitting

$$T_xE = \ker(D_x\pi) \oplus T_xE^h, \tag{2.7}$$

where T_xE^h is the symplectic complement of $\ker(D_x\pi)$ with respect to Ω_E. The decomposition in (2.7) defines a canonical connection over $\mathbb{C} \setminus \mathrm{Crit}v(\pi)$. By the triviality condition 3., for every path $\gamma\colon [0,1] \to D \setminus \mathrm{Crit}(\pi)$, there are well-defined parallel transport maps $h_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ which yield symplectomorphisms between smooth fibres.

Definition 2.2.6. A pair $(J_E, j_{\mathbb{C}})$ is said to be *compatible* with π if the following holds.

- $D\pi \circ J_E = j_{\mathbb{C}} \circ D\pi$.

- There is a local Kähler structure J_0 such that $J_E = J_0$ in a neighbourhood of $\mathrm{Crit}(\pi)$.

- On the neighbourhood U^h, J_E is a product, $f^*(J_E) = (j_{\mathbb{C}}, J^{vv})$, where J^{vv} is a cylindrical almost complex structure compatible with $d(e^t\lambda_M)$.

- $\Omega_E(\cdot, J_E\cdot)$ is symmetric and positive definite.

◇

Remark 2.2.7. *This choice of almost complex structure is not generic. However, the space of compatible almost complex structures on the total space of an exact Lefschetz fibration is contractible ([Sei03, Section 2.1]), and the moduli spaces we will consider still meet the usual regularity requirements ([Sei03, Section 2.2]).* //

For a Lefschetz fibration on a Liouville manifold (E, Ω_E), the proper fibration obtained as $E \setminus U^h \longrightarrow \mathbb{C}$, for an open neighbourhood $U^h \subset E$ as above, carries the same symplectic information as π with the difference that its fibres are Liouville domains, and as result the total space admits a non-trivial *horizontal boundary*, given by the union of the boundaries of all fibres.

In most of the thesis we will employ this latter type of Lefschetz fibration (for notational simplicity), and unless specified, an *exact* Lefschetz fibration will denote a fibration obtained in this way.

Let now $\pi\colon E \to \mathbb{C}$ be an exact Lefschetz fibration, with smooth fibre given by the Liouville domain $(M, d\lambda)$. By the triviality assumption of Definition 2.2.5, there is a neighbourhood of $U^\partial \subset E$ of the horizontal boundary $\partial^h E$ that is isomorphic to an open neighbourhood of the trivial bundle $\mathbb{C} \times \partial M$:

$$U^\partial \cong \mathbb{C} \times M^{out} \subset \mathbb{C} \times M \tag{2.8}$$

where $M^{out} \subset M$ is a collar neighbourhood of ∂M. The isomorphism is compatible with the Liouville forms and the almost complex structures.

Let $\pi\colon E \to \mathbb{C}$ be a Lefschetz fibration with exact compact fibre (M, ω) and distinct critical values $\mathrm{Critv}(\pi) = \{w_0, \ldots, w_m\} \subset D_R$, where $D_R \subset \mathbb{C}$ is a disc of radius R. Fix a base point $z_* \in \mathbb{R}$, such that $z_* \gg R$, and an identification $\pi^{-1}(z_*) \cong M$. In what follows we will frequently use the fact that via parallel transport, any fibre $\pi^{-1}(z)$ for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > R$ can be symplectically identified with the smooth fixed fibre M via parallel transport.

Definition 2.2.8.

1. A vanishing path associated to a critical value $w_i \in \mathrm{Critv}(\pi)$ is a properly embedded path $\gamma_i\colon \mathbb{R}^+ \to \mathbb{C}$ with $\gamma_i^{-1}(\mathrm{Critv}(\pi)) = \{0\}, \gamma_i(0) = w_i$ and $\lim_{t\to\infty} \mathrm{Re}(\gamma(t)) = \infty$ such that outside of a compact set containing the critical values, the image of γ_i is a horizontal half ray at height $a_i \in \mathbb{R}$:

$$\exists T > 0 \text{ such that } \forall t > T,\ \mathrm{Re}(\gamma_i(t)) > R,\ \mathrm{Im}(\gamma_i(t)) = a_i. \tag{2.9}$$

2. A distinguished basis of vanishing paths for π is a collection of $m+1$ disjoint paths $(\gamma_0, \ldots \gamma_m) \subset \mathbb{C}$ defined as above, with pairwise distinct heights satisfying $a_0 < a_1 < \cdots < a_m$.

3. The corresponding basis of Lefschetz thimbles is the unique set of Lagrangian discs $(\Delta_{\gamma_0}, \ldots, \Delta_{\gamma_m}) \subset E$ where Δ_{γ_i} is defined as the set of points which under the limit $t \to 0$ of the parallel transport maps over γ_i are mapped to the critical point in $\pi^1(w_i)$ (the proof of

uniqueness can be found in [Sei08a, (16b)]). Given a general Lefschetz thimble $\mathcal{L}$, define its height $a(\mathcal{L})$ as the value defined in (2.9). For a pair of thimbles $(\mathcal{L}_0, \mathcal{L}_1)$ set $\mathcal{L}_0 > \mathcal{L}_1$ if $a(\mathcal{L}_0) > a(\mathcal{L}_1)$.

4. There is an associated basis of vanishing cycles $(V_0, \dots, V_m)$ where for all $i = 0, \dots, m$,

$$V_i = \partial \Delta_{\gamma_i} = \Delta_{\gamma_i} \cap M \subset M$$

(using the above identification for smooth fibres). Every vanishing cycle $V_i \subset M$ is an exact Lagrangian sphere which comes with an equivalence class in of diffeomorphisms $S^n \to V_i$ defined up to the action of $\mathrm{O}(n+1)$ (called a *framing*). This is induced by the restriction of a diffeomorphism $D^{n+1} \longrightarrow \Delta_i$ (which is canonical, see [Sei03, Lemma 1.14]).

♢

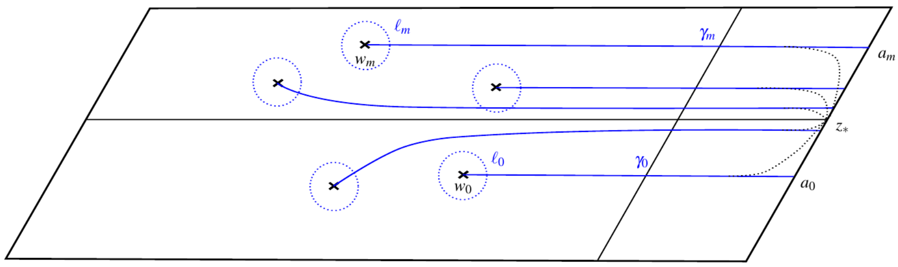

Figure 2.1: A distinguished basis of vanishing paths $(\gamma_0, \dots, \gamma_m)$.

Definition 2.2.9. The global monodromy is the symplectomorphism $\phi \in \mathrm{Symp}_{ct}(M)$ whose Hamiltonian isotopy class is defined by the anticlockwise parallel transport map around a loop through the base point z_* encircling all the critical values of the fibration. (Typically, this loop is defined as the smoothing of the concatenation of the loops centered at z_* going around a single critical value as in Figure 2.1.) ♢

The symplectic Picard-Lefschetz theorem ([Arn]) states that the global monodromy ϕ is isotopic to the product of the Dehn twists along the vanishing cycles $(V_0, \dots, V_m)$,

$$\phi \simeq \tau_{V_0} \cdots \tau_{V_m} \in \mathrm{Symp}_{ct}(M), \tag{2.10}$$

and the Hamiltonian isotopy class is independent of the choice of basis of vanishing paths.

On the other hand, given the data $(M, (V_0, \dots, V_m))$, there is an exact Lefschetz fibration $\pi\colon E \to \mathbb{C}$ with fibre (M, ω), and vanishing cycles $(V_0, \dots, V_m) \subset M$, unique up to exact symplectomorphism ([Sei08a, (16e)]).

2.2.2 Projective twists and MBL fibrations

Lefschetz fibrations can be viewed as a special case of *Morse–Bott–Lefschetz* (abbreviated MBL) fibrations, a class of fibrations which allows non-isolated singularities. The monodromies of such fibrations are symplectomorphisms called *fibred twists* ([Per07]), which naturally generalise Dehn twists. The relevance of fibred twists in this discussion is Lemma 2.2.14, which shows that projective twists are just a special type of fibred twists, thereby providing a first local model for these symplectomorphisms (obtained as monodromy of a suitable MBL fibration).

In this section we briefly discuss MBL fibrations and (S^1-)fibred twists, to be able to state Lemma 2.2.14, which we'll prove in Section 2.2.5. Sections 2.2.3 and 2.2.4 are auxiliary to this proof.

Let D be a disc with standard almost complex structure j, and (E,Ω_E) a $(2n+2)$-manifold with a closed 2-form Ω_E and an almost complex structure J_E. A MBL fibration $\pi\colon E\to D$ is a proper (J_E,j)-holomorphic map such that Ω_E is non-degenerate on $\ker(D\pi)$, and whose singular fibres admit Morse–Bott singularities (the definition is more involved, see [Per07, Definition 2.1]). This means that the critical locus $\mathrm{Crit}(\pi)\subset E$ can be a set of smooth closed symplectic submanifolds $(Q_i,\Omega_E|_{Q_i})$ of (real) dimension $2n-2k_i$, $k_i\geqslant 0$ ([Per07, Definition 2.1]), and that the Hessian of π is non-degenerate only along the normal bundle to $\mathrm{Crit}(\pi)$. For this reason, in general the two form on the total space cannot be made exact.

Every critical point in a component $Q_i\in E^{crit}$ of the singular locus has a neighbourhood with local complex coordinates $(x_0,\dots x_n)$ such that in that neighbourhood,

$$\pi(x_0,\dots x_n)=\sum_{j=0}^{k_i}x_j^2. \tag{2.11}$$

A MBL fibration is characterised by its smooth fibre (M,ω) and its "fibred vanishing cycles", a set of submanifolds of the smooth fibre playing the analogue role of vanishing cycles of a Lefschetz fibration. Fibred vanishing cycles are defined as the sets of points in the smooth fibre that are mapped into the critical locus via the parallel transport maps. For any component Q_i of the critical locus, the associated fibred vanishing cycle is a coisotropic submanifold of (M,ω) which is an S^{k_i}-bundle $V_i\to Q_i$. The analogy with vanishing cycles is that the image of V_i under the limit of the parallel transport into Q_i collapses the spherical fibres of V_i ([Per07, Section 2.3]).

The monodromy of a MBL fibration with smooth fibre (M,ω), one singularity with (compact) critical locus Q and associated fibred coisotropic $p\colon V\to Q$ is given by the *fibred twist* (see Definition 2.2.11) $\tau_V\in\mathrm{Symp}_{ct}(M)$ around V ([Per07, Theorem 2.16]). On the other hand, any fibred twist can be realised as (local) monodromy of such a MBL fibration ([Per07, Section 2.4.1]).

Instead of giving the general definition of fibred twists, we focus on a special case of interest, that of S^1-fibred twists along a coisotropic $V \to Q$ which has the structure of a *prequantisation bundle*.

Definition 2.2.10. Let $p\colon V \to Q$ be a principal S^1-bundle over a closed symplectic manifold (Q, ω_Q). We call $p\colon V \to Q$ a *prequantisation bundle* if ω_Q is an integral symplectic form (meaning $[\omega_Q]$ lies in the image of the map $H^2(Q;\mathbb{Z}) \to H^2(Q;\mathbb{R})$), and the Euler class satisfies $e(p) = -k[\omega_Q]$, for some $k > 0$. In that case, there is a connection 1-form $\alpha \in \Omega^1(V;\mathbb{R})$ (regarded as a regular real valued 1-form) which is a contact form whose Reeb vector field generates the S^1-action on V, and $d\alpha = 2k\pi p^*(\omega_Q)$ (see e.g [Gei08, Theorem 7.2.4]). $\diamondsuit$

Let (W, ω_W) be a symplectic manifold, and $V \subset W$ a coisotropic submanifold admitting the structure of a prequantisation bundle $p\colon V \to Q$ over a closed symplectic manifold (Q, ω_Q).

By the coisotropic neighbourhood theorem (see [Per07, Section 2.4.1], [WW16, Section 2.3]), a neighbourhood U of $V \subset W$ is isomorphic to an associated symplectic bundle $(V \times_{S^1} T^*S^1, p^*(\omega_Q) + d\lambda_{T^*S^1} + d\langle \mu, \alpha \rangle)$, where the quotient is taken with respect to the S^1-action

$$(x, (\theta, t)) \cdot \varphi = (x \cdot \varphi, (\theta - \varphi, t)),\ \varphi \in S^1,\ (\theta, t) \in T^*S^1. \tag{2.12}$$

The two-form given above is invariant under the S^1-action (2.12) so it descends to a well-defined two form on the quotient. Here $\lambda_{T^*S^1} = d(t\theta)$ is the standard Liouville form on T^*S^1 and $\mu : T^*S^1 \to \mathbb{R}$ is the moment map of the standard S^1-action on T^*S^1 (which is the distance function), and $\langle \cdot, \cdot \rangle$ is the Lie algebra pairing (see (2.17) for the concrete description of this two-form).

Definition 2.2.11. 1. **Local fibred twist.** Let $\tau_{S^1} : T^*S^1 \to T^*S^1$ the standard S^1-Dehn twist of Definition 2.1.1. On the product space $V \times T^*S^1$, define a diffeomorphism

$$\bar{\tau}\colon V \times T^*S^1 \longrightarrow V \times T^*S^1 \tag{2.13}$$

$$(x, (\theta, t)) \mapsto (x, \tau_{S^1}(\theta, t)) \tag{2.14}$$

Since $\bar{\tau}$ commutes with the diagonal S^1 action, it descends to a well-defined compactly supported diffeomorphism τ_{loc} of the quotient $V \times_{S^1} T^*S^1$ with the properties that

(a) τ_{loc} is a symplectomorphism,

(b) τ_{loc} covers the identity on Q,

(c) τ_{loc} acts as a standard Dehn twist on each fibre of $V \times_{S^1} T^*S^1$

see ([Per07, Lemma 2.3], [WW16, Definition 2.7])

2. **Fibred twist along V.** For $\varepsilon > 0$, let $\psi\colon V \times_{S^1} D_\varepsilon T^*S^1 \to U$ be the symplectomorphism of

neighbourhoods as above. Then

$$\tau_V := \begin{cases} \psi \circ \tau_{loc} \circ \psi^{-1} & \text{on } U \\ Id & \text{on } W \setminus U \end{cases}$$

defines a compactly supported symplectomorphism in $\mathrm{Symp}_{ct}(W)$.

♢

We now describe another construction from the literature, formalised by Chiang–Ding–Koert, and we show it corresponds to the special instance of fibred twist above.

Let $p\colon (V,\alpha) \to (\Sigma, \omega_\Sigma)$ be a prequantisation bundle. The flow $\sigma_t^{R\alpha}$ of the Reeb vector field R_α, given its periodicity, gives rise to a loop of contactomorphisms on V. It is then possible to *suspend* this loop of contactomorphisms to a symplectomorphism of the symplectisation of V (see [CKSC18, Proposition 3.1] for a general statement about such a construction).

Definition 2.2.12 ([CDVK16, Definition 2.7, Lemma 2.1])**.** Let (W, ω_W) be a Liouville domain such that $\partial W = V$, $\omega_W|_V = d\alpha$. Consider a neighbourhood of $V \subset W$, i.e a piece of symplectisation

$$(V \times [0,1], d(e^t\alpha)). \tag{2.15}$$

Fix a smooth function $f\colon [0,1] \to \mathbb{R}$ with $f(0) = 2\pi$ and $f(1) = 0$, define

$$\begin{aligned} \tau_V'\colon V \times [0,1] &\longrightarrow V \times [0,1] \\ (x,t) &\mapsto (\sigma_{f(t)}^{R_\alpha}(x), t) \end{aligned} \tag{2.16}$$

and extend to $W \setminus (V \times [0,1])$ via the identity (τ_V' is the identity near the boundary of $V \times [0,1]$). Then τ_V' defines a symplectomorphism of W. ♢

Lemma 2.2.13. *The construction of the fibred twist in Definition 2.2.11 is equivalent to a suspension of a loop of contactomorphisms in the sense of Definition 2.2.12.*

Proof. As a first step, we explain how the neighbourhoods of $V \subset W$ used to define τ_V and τ_V' are symplectomorphic (abstractly, this is clear by the coisotropic neighbourhood theorem). In Definition 2.2.11, we used an associated symplectic T^*S^1-bundle

$$(V \times_{S^1} D_\varepsilon T^*S^1, p^*(\omega_Q) + d\lambda_{T^*S^1} + d\langle \mu, \alpha \rangle = \frac{d\alpha}{2\pi} + d(t\theta) + d(t\alpha)). \tag{2.17}$$

For $\varepsilon = \frac{1}{2}$ we can use the identification $D_\varepsilon T^*S^1 \cong S^1 \times [0,1]$ and get $V \times_{S^1} D_\varepsilon T^*S^1 \cong V \times [0,1]$, so that the Liouville form $\lambda_{T^*S^1}$ gets identified with $d(t\alpha)$ and (2.17) is symplectomorphic to a collar neigbourhood $(V \times [0,1], d((2t + \frac{1}{2\pi})\alpha))$. The correct symplectic form is obtained after a change of variable.

It is clear from the definition that τ_V' covers the identity on Σ and after the above identification, it acts as a standard S^1-Dehn twist on the fibres of $V \times [0,1]$ (note that we can linearly interpolate between the choices of cut-off functions in the two definitions).

Therefore, the symplectomorphism $\tau_{V'}$ agrees with the fibred twist we have defined in Section 2.2.2. □

The projective twists $\tau_{\mathbb{A}\mathbb{P}^n}, \mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$ are S^1-fibred twists along the unit cotangent bundle of $(T^*\mathbb{A}\mathbb{P}^n, d\lambda_{T^*\mathbb{A}\mathbb{P}^n})$.

Lemma 2.2.14. *Let $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. The projective twist $\tau_{\mathbb{A}\mathbb{P}^n} \in \mathrm{Symp}_{ct}(T^*\mathbb{A}\mathbb{P}^n)$ (Definition 2.1.1) is isotopic to the S^1-fibred Dehn twist (Definitions 2.2.11 and 2.2.12) along the coisotropic*

$$(V, \alpha) = (ST^*\mathbb{A}\mathbb{P}^n, \lambda_{T^*\mathbb{A}\mathbb{P}^n}|_{ST^*\mathbb{A}\mathbb{P}^n}) \to (\Sigma, \omega_\Sigma) \tag{2.18}$$

where

1. *For $\mathbb{A} = \mathbb{R}$, $\Sigma := \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathbb{P}^n,\ \sum_{i=0}^n z_i^2 = 0\} \subset \mathbb{C}\mathbb{P}^n$ is the $(n-1)$-complex quadric with the restriction of the Fubini–Study form ω_{FS};*
2. *For $\mathbb{A} = \mathbb{C}$, $\Sigma := \{([x_0 : \cdots : x_n], [y_0 : \ldots y_n]) \in \mathbb{C}\mathbb{P}^n \times \mathbb{C}\mathbb{P}^n,\ \sum_{i=0}^n x_i y_i = 0\} \subset \mathbb{C}\mathbb{P}^n \times \mathbb{C}\mathbb{P}^n$ with the restriction of $\omega_{FS} \oplus \omega_{FS}$*

In particular, a projective twist is the local monodromy of a MBL fibration with fibred vanishing cycle $V \to \Sigma$.

The proof will be given in Section 2.2.5, after the excursus of Sections 2.2.3 and 2.2.4, which is aimed at clarifying what the coisotropic submanifolds $(V, \alpha) \to (\Sigma, \omega_\Sigma)$ are in the two examples.

2.2.3 Symplectic decompositions of complex projective varieties

To understand the structure of the fibred coisotropics (2.18), it is necessary to provide an alternative description of the cotangent bundle of the real and complex projective spaces.

This section contains the necessary theoretical tools to do so, by viewing these (disc) cotangent bundles as the complement of a divisor $\Sigma \subset X$ in a complex projective variety X (the concrete examples are carried out in Section 2.2.4). The main reference for this section is [Bir01, Section 2.1].

Let X^{2n} be a smooth complex projective variety with an ample line bundle $\mathcal{L} \to X$; we view X as a Kähler manifold with an integral symplectic form ω_X with the property that $c_1(\mathcal{L}) = [\omega_X]$. Assume there is a section $s \in H^0(\mathcal{L})$ such that $\Sigma := s^{-1}(0) \subset X$ is a smooth projective subvariety (transversely cut out). Then $(\Sigma, \omega_\Sigma = \omega_X|_\Sigma)$ is a Kähler submanifold of X, with fundamental class $[\Sigma] \in H_{2n-2}(X; \mathbb{Z})$ Poincaré dual to $[\omega_X]$.

The (closure of the) complement $X \setminus \Sigma$ is a Stein domain, with the plurisubharmonic function $\varphi := -\frac{1}{4\pi} log(\rho) = -\frac{1}{4\pi} log(\|s\|^2)$ and Liouville vector field $Z = grad(\varphi)$. The restriction $\omega := \omega_X|_{X \setminus \Sigma}$ is exact and coincides with $-dd^{\mathbb{C}}\varphi$. By the symplectic neighbourhood theorem ([MS17, Theorem

3.4.10]), a neighbourhood of $\Sigma \subset X$ is symplectomorphic to the *symplectic normal bundle* to Σ, that we define below in (2.22). On an algebro-geometric level, the normal bundle to $\Sigma \subset X$ is modelled on $\mathcal{L}|_\Sigma$. Define $P := \{v \in \mathcal{L}^*|_\Sigma, \|v\| = 1\}$, for a hermitian metric on the dual bundle induced by one on $\mathcal{L}$ (we make this choice of orientation in order for P to be viewed as a convex boundary, positively oriented with respect to the Liouville vector field Z above).

The bundle $p\colon P \to \Sigma$ is a principal S^1-bundle with Euler class $e(p) = -[\omega_X]|_\Sigma = -[\omega_\Sigma]$. Let $\alpha_P \in \Omega^1(P;\mathbb{R})$ be a connection 1-form as in Definition 2.2.10, with curvature $d\alpha_P = 2\pi p^*(\omega_\Sigma)$ and such that it is a contact form whose associated Reeb vector field R_{α_P} generates the S^1-action on P. It gives P the structure of prequantisation bundle

$$p\colon (P,\alpha_P) \longrightarrow (\Sigma,\omega_\Sigma). \tag{2.19}$$

Consider the following disc bundle associated to P,

$$(P \times_{S^1} D(1), p^*(\omega_\Sigma) + d(r^2 d\theta) - d(r^2\alpha_P)) \tag{2.20}$$

where $D(1) \subset \mathbb{C}$ is the open unit disc with radial coordinate r, and $\theta \in S^1$ acts on $(x,z) \in P \times D(1)$ as

$$(x,z)\cdot\theta = (x\cdot\theta, e^{2\pi i\theta}z). \tag{2.21}$$

The form $p^*(\omega_\Sigma)+d(r^2d\theta)-d(r^2\alpha_P) = d((\frac{1}{2\pi}-r^2)(\alpha_P-d\theta)) =: d\lambda_\nu \in \Omega^1(P\times D(1))$ is invariant under the S^1-action and therefore induces a well-defined one form on the quotient $P\times_{S^1} D(1)$. Note that this bundle is positive, with first Chern class given by $+[\omega_\Sigma] = -e(p)$.

By the symplectic neighbourhood theorem, there is a symplectic embedding

$$\nu\colon (P\times_{S^1} D(1), d\lambda_\nu) \longrightarrow (X,\omega_X), \tag{2.22}$$

and the symplectic disc normal bundle is the image $N_{\Sigma/X} := \nu(P\times_{S^1} D(1))$. The zero section of the associated bundle on the left hand side is mapped to Σ, and on its complement there is a vector field $X_\nu := -\frac{1-r^2}{2r}\partial_r$, such that $\nu_*(X_\nu)$ is a multiple of the Liouville vector field Z on $X\setminus\Sigma$ (see [Oba20, 2.2]).

Theorem 2.2.15 ([Bir01, Theorem 1.A])**.** *Let (X,ω_X) be a smooth projective variety as above, with an ample line bundle $\mathcal{L}\to X$ defining a submanifold $\Sigma\subset X$ as the smooth zero locus of a non-trivial section $s\in H^0(\mathcal{L})$. Assume that the function $\rho\colon X\to\mathbb{R}$, $\rho(x) = \|s(x)\|^2$ (with respect to the same hermitian metric as above) is Morse on $X\setminus\Sigma$ and let*

$$\Xi := \bigcup_{p\in\mathrm{Crit}(\rho)\setminus\Sigma} W^u_p(\rho,g), \tag{2.23}$$

where $W^u_p(\rho, g)$ *is the unstable manifold of a critical point* $p \in \mathrm{Crit}(\rho)$ *with respect to the (positive) gradient flow, defined via the Kähler metric* g *on* X *compatible with* ω_X*. Then* $\Xi \subset X$ *is an isotropic CW-complex whose complement, the open dense subset* $(X \setminus \Xi, \omega_X)$*, is symplectomorphic to the symplectic open disc normal bundle* $N_{\Sigma/X}$*.*

In Chapter 4, we will use this decomposition in the situation in which the isotropic CW complex is a smooth Lagrangian submanifold of the given Kähler manifold.

Lemma 2.2.16. *Assume* X, ω, ρ *and* $\varphi := -\frac{1}{4\pi} log(\rho)$ *as above. Suppose that the isotropic CW complex* $\Xi \subset X$ *defined in* (2.23) *is a smooth Lagrangian submanifold which coincides with* $\mathrm{Crit}(\varphi) = \mathrm{Crit}(\rho) \setminus \Sigma$*. Let* $U_{\Sigma/X} \supset \Sigma$ *be an open disc subbundle of the normal bundle* $N_{\Sigma/X}$ *given in* (2.22)*. Then the complement* $X \setminus U_{\Sigma/X}$ *is symplectomorphic to a disc subbundle (of some radius* $\varepsilon > 0$*) of the cotangent bundle* $(T^*\Xi, d\lambda_{T^*\Xi})$*:*

$$X \cong D_\varepsilon T^*\Xi \cup U_{\Sigma/X}. \tag{2.24}$$

Proof. This lemma follows as a corollary to Biran's decomposition theorem, see [Bir01, Section 7].

Let $U_{\Xi/X} \supset \Xi$ be an open neighbourhood of $\Xi \subset X$, and $U_{\Sigma/X} \supset \Sigma$ an open neighbourhood of Σ, the subbundle of $N_{\Sigma/X}$ as in the statement.

Consider the Liouville vector field $Z = grad(-\frac{1}{4\pi} log\|s\|^2) = grad(\varphi)$ on $X \setminus \Sigma$ with $\mathcal{L}_Z \omega_X = -dd^{\mathbb{C}} \varphi = \omega_X|_{X \setminus \Sigma}$ for the plurisubharmonic function φ. For any $t > 0$, the flow Z^t of Z acts by conformal symplectomorphisms. Then, there is a symplectic cobordism between the boundaries of the closures of $U_{\Sigma/X}$ and $U_{\Xi/X}$, and since Ξ is the only critical set of φ in $X \setminus \Sigma$, the flow lines of the vector field Z define a trivialisation of the cobordism (so the two boundaries are contactomorphic). By the Weinstein's neighbourhood theorem, $U_{\Xi/X}$ is a disc cotangent bundle, and there is $C > 0$ and $t \in [-C, C]$ such that $X \cong Z^t(U_{\Xi/X}) \cup_{ST^*\Xi} U_{\Sigma/X}$. ☐

2.2.4 Examples: disc cotangent bundles of projective spaces

Consider a compact Riemannian manifold (Ξ, g) with periodic geodesic flow, and a circle bundle of radius $r \in (0,1)$ inside $(T^*\Xi, d\lambda_{T^*\Xi})$, denoted by $ST^*\Xi = \{v \in T^*\Xi;\ \|v\|_g = r\}$. The latter is a contact hypersurface of the cotangent bundle, with contact form $\lambda_{T^*\Xi}|_{ST^*\Xi}$. The Hamiltonian vector field of the distance function $\mu\colon T^*\Xi \to \mathbb{R}\ \mu(v) = \|v\|_g$ is (a multiple of) the Reeb vector field of the contact form on $ST^*\Xi$, whose flow coincides with the (co-) geodesic flow on Ξ. The latter is periodic, so the Reeb vector field generates a free S^1-action on $(ST^*\Xi, \lambda_{T^*\Xi}|_{ST^*\Xi})$, which is a regular level set of μ. So the unit cotangent bundle is a principal S^1-bundle over the space of geodesics.

In this case, a result stronger than Lemma 2.2.16 holds.

Proposition 2.2.17 ([Aud07, Proposition 4.1, 4.3]). *Let $\Xi \in \{S^n, \mathbb{RP}^n, \mathbb{CP}^n\}$. Then there is projective variety (X, ω_X) where $\Xi \subset X$ is embedded as Lagrangian submanifold, and such that there is an open disc bundle $\mathring{D}_\varepsilon T^*\Xi \subset T^*\Xi$ with a symplectic decomposition*

$$X \cong \mathring{D}_\varepsilon T^*\Xi \cup \Sigma \tag{2.25}$$

where $\Sigma := ST^*\Xi/S^1$ is the space of (unparametrised, oriented) geodesics of Ξ.

The decomposition (2.25) is a special instance of (2.24); there is an ample line bundle $\mathcal{L} \to X$ and a holomorphic section $s \in H^0(\mathcal{L})$ such that $s^{-1}(0) \cong \Sigma$. The restriction $\omega_X|_{X\setminus\Sigma}$ is a multiple of the standard symplectic form on the cotangent bundle $d\lambda_{T^*\Xi}$, and the coisotropic $(ST^*\Xi, \lambda_{T^*\Xi}|_{ST^*\Xi}) \to (\Sigma, \omega|_\Sigma)$ is a prequantisation bundle contactomorphic to (P, α_P) in (2.19).

We illustrate two examples in which we can use a decomposition of the type (2.24) to present a disc subbundle of $T^*\mathbb{AP}^n$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$ as the complement of an ample divisor in a projective variety.

Example 2.2.18 ([Bir01, 3.1.2]). Let $X = (\mathbb{CP}^n, \omega_{FS})$ with the ample line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{CP}^n}(2) \to \mathbb{CP}^n$. A generic section $s \in H^0(\mathcal{L})$ is given by a homogeneous polynomial of degree two in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{CP}^n$. Consider the section $s := \sum_{i=0}^n z_i^2$. The divisor arising as the zero locus of s is the smooth quadric $\Sigma := s^{-1}(0) = \{z_0^2 + \cdots + z_n^2 = 0\} \cong Q^{n-1} \subset \mathbb{CP}^n$, which is a Kähler submanifold with the integral symplectic structure $\omega_\Sigma = \omega_{FS}|_\Sigma$. The smooth function

$$\rho \colon \mathbb{CP}^n \longrightarrow \mathbb{R},\ \rho([z_0 : \cdots : z_n]) = \frac{|\sum_{i=0}^n z_i^2|^2}{(\sum_{i=0}^n |z_i|^2)^2},$$

is a reparametrisation of $\|s\|^2$, and is Morse–Bott on $X \setminus \Sigma$ with unique critical locus given by

$$\mathrm{Crit}(\rho|_{X\setminus\Sigma}) = \mathrm{Crit}(\rho) \setminus \Sigma = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n | z_j \in \mathbb{R} \text{ for all } 0 \leqslant j \leqslant n\} \simeq \mathbb{RP}^n \subset \mathbb{CP}^n,$$

which is embedded as a Lagrangian in $(\mathbb{CP}^n, \omega_{FS})$. The positive gradient vector field points out of $\mathrm{Crit}(\rho) \setminus \Sigma$, so the union of unstable manifolds coincides with the critical locus of $\rho|_{X\setminus\Sigma}$: $\Xi = \mathrm{Crit}(\rho) \setminus \Sigma \simeq \mathbb{RP}^n \subset \mathbb{CP}^n$.

Therefore, by Lemma 2.2.16 and Proposition 2.2.17, the complement $X \setminus \Sigma$ is symplectomorphic to an open disc cotangent bundle of Ξ:

$$\mathbb{CP}^n \cong \mathring{D}_\varepsilon T^*\mathbb{RP}^n \cup \Sigma,\ \varepsilon > 0. \tag{2.26}$$

The unit cotangent bundle is a prequantisation bundle $(ST^*\mathbb{RP}^n, d\lambda_{T^*\mathbb{RP}^n}) \longrightarrow (\Sigma, \omega_\Sigma)$ over the quadric $\Sigma \subset \mathbb{CP}^n$. ♢

Example 2.2.19 ([Bir01, 3.3]). Let $X = (\mathbb{CP}^n \times \mathbb{CP}^n, \omega_{FS} \oplus \omega_{FS})$ with the ample line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{CP}^n\times\mathbb{CP}^n}(1,1) := pr_1^*(\mathcal{O}_{\mathbb{CP}^n}(1)) \otimes pr_2^*(\mathcal{O}_{\mathbb{CP}^n}(1))$, where $pr_i \colon \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{CP}^n$ is the projection to the i-th factor, $i = 1, 2$. A generic section $s \in H^0(\mathcal{L})$ is a homogeneous polynomial of degree $(1,1)$, in the homogeneous coordinates $(\underline{x},\underline{y}) = ([x_0 : \cdots : x_n],[y_0 : \cdots : y_n])$ of $\mathbb{CP}^n \times \mathbb{CP}^n$. Let $s := \sum_{i=0}^n x_i y_i$; the divisor obtained as the zero set of s is given as

$$\Sigma := \left\{ \sum_{i=0}^n x_i y_i = 0 \right\} \subset \mathbb{CP}^n \times \mathbb{CP}^n. \tag{2.27}$$

The function

$$\rho\colon \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{R},\ \rho(\underline{x},\underline{y}) = \frac{|\sum_{i=0}^n x_i y_i|^2}{(\sum_{i=0}^n |x_i|^2)^2 (\sum_{i=0}^n |y_i|^2)^2}$$

is a smooth reparametrisation of $\|s\|^2$, and is Morse–Bott on $X \setminus \Sigma$ with critical locus given by

$$\mathrm{Crit}(\rho|_{X\setminus\Sigma}) = \mathrm{Crit}(\rho) \setminus \Sigma = \{([x_0 : \cdots : x_n],[\overline{x_0} : \cdots : \overline{x_n}]),\ [x_0 \cdots : x_n] \in \mathbb{CP}^n\} \simeq \mathbb{CP}^n \subset \mathbb{CP}^n \times \mathbb{CP}^n,$$

which is embedded as a Lagrangian in $(\mathbb{CP}^n \times \mathbb{CP}^n, \omega_{FS} \oplus \omega_{FS})$. Following the same reasoning as in the previous example, the union of unstable manifolds Ξ is given by $\Xi = \mathrm{Crit}(\rho) \setminus \Sigma \simeq \mathbb{CP}^n$.

Using Lemma 2.2.16 and Proposition 2.2.17, we obtain a decomposition

$$\mathbb{CP}^n \times \mathbb{CP}^n \cong \mathring{D}_\varepsilon T^*\mathbb{CP}^n \cup \Sigma,\ \varepsilon > 0. \tag{2.28}$$

As we explain below in Remark 2.2.20, the unit cotangent bundle is a prequantisation bundle $(ST^*\mathbb{CP}^n, d\lambda_{T^*\mathbb{CP}^n}) \longrightarrow (\Sigma, \omega_\Sigma)$ over the projectivisation of the holomorphic cotangent bundle; $\Sigma \cong \mathbb{P}(T^*_{hol}\mathbb{CP}^n)$, which, for $n = 2$, is isomorphic to the Flag 3-fold. ♢

Remark 2.2.20. *Consider*

$$\Sigma' := \{(\ell, H),\ \ell \textit{ is a line in } \mathbb{C}^{n+1},\ H \textit{ is a hyperplane in } \mathbb{C}^{n+1} \textit{with } \ell \subset H\} \subset \mathbb{CP}^n \times (\mathbb{CP}^n)^*.$$

This admits a projection

$$\begin{aligned} \Sigma' &\longrightarrow \mathbb{CP}^n \\ (\ell, H) &\longmapsto \ell. \end{aligned} \tag{2.29}$$

that realises Σ' as the projectivised cotangent bundle $\Sigma' \cong \mathbb{P}(T^*_{hol}\mathbb{CP}^n)$ (see for example [EH13, Proposition 10.22]). [The fibre over $\ell \in \mathbb{P}(\mathbb{C}^{n+1})$ of (2.29) *is given by $\{w \in \mathbb{P}((\mathbb{C}^{n+1})^*), w(\ell) = 0\} = \mathbb{P}(\mathbb{C}^{n+1}/\ell)$.]*

Consider the diffeomorphism

$$\begin{aligned} f\colon \mathbb{CP}^n &\to (\mathbb{CP}^n)^* \\ \ell &\mapsto \ell^\perp \end{aligned} \tag{2.30}$$

sending a line $\ell \in \mathbb{CP}^n$ to its orthogonal complement (with respect to a fixed hermitian metric), i.e the hyperplane whose normal vector is ℓ. Then $\mathbb{CP}^n \times \mathbb{CP}^n$ can be identified with $\mathbb{CP}^n \times (\mathbb{CP}^n)^*$ using f. Under this identification, we can identify Σ from (2.27) *with Σ'.* //

In the two dimensional case, $\Sigma \subset \mathbb{CP}^2 \times \mathbb{CP}^2$ is a Flag 3-fold embedded in $\mathbb{CP}^2 \times \mathbb{CP}^2$. Write the Flag 3-fold (the manifold of complete flags in $\mathbb{C}^3$) as

$$Fl_3 := \{(\ell, H),\ \ell \subset \mathbb{C}^3 \text{ is line through } 0\ , H \subset \mathbb{C}^3 \text{ is a plane such that } \ell \subset H\}, \tag{2.31}$$

then lines ℓ can be parametrised by the first $\mathbb{CP}^2$ factor, and planes H by the second $\mathbb{CP}^2$ factor so that in coordinates we obtain the identification $Fl_3 \cong \left\{\sum_{i=0}^2 x_i y_i = 0\right\} = \Sigma$.

Consider the Flag 3-fold as the set of pairs (ℓ, H) as in (2.31). Then the projection (2.29) gives $Fl_3 \cong \mathbb{P}(T^*_{hol}\mathbb{CP}^2)$.

2.2.5 The projective twist as a fibred twist: proof of Lemma 2.2.14

Let $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. We use the decompositions of the previous section to prove that the projective twists $\tau_{\mathbb{AP}^n} \in \operatorname{Symp}_{ct}(T^*\mathbb{AP}^n)$, defined via the geodesic flow in Definition 2.1.1, can also be viewed as S^1-fibred twists along the unit cotangent bundle $ST^*\mathbb{AP}^n \subset T^*\mathbb{AP}^n$.

Proof of Lemma 2.2.14. The second part of the statement about the shape of the coisotropic manifolds is clear by Examples 2.2.18 and 2.2.19.

As we have shown (in Lemma 2.2.13) the equivalence of Definitions 2.2.11 and 2.2.12, it suffices to prove that the projective twist coincides with the fibred twist of Definition 2.2.12.

Let $j\colon ST^*\mathbb{AP}^n \times [0,1] \hookrightarrow T^*\mathbb{AP}^n$ be the inclusion with $j^*(\omega) = d(e^t\alpha)$. Then the projective twist, defined as in (2.4), pulls back under the inclusion j to a symplectomorphism of the shape of (2.16) (see [Sei00, 4.a.]), which acts on the symplectisation as the Reeb flow in one component and the identity in the other.

Namely, the map ϕ_t of (2.4) is obtained as an extension of the Reeb flow on $ST^*\mathbb{AP}^n$ and hence preserves hypersurfaces of fixed distance t from the zero section (level sets of the distance function), and acts on the fibres of such hypersurfaces via the Reeb flow $\sigma^{R_\alpha}_{g(t)}$, for a smooth function $g\colon [0,1] \to \mathbb{R}$ with $g(t) = r_\varepsilon(t/\varepsilon)$; $g(0) = 2\pi$ and $g(1) = 0$. As we have done here, one can interpolate between different choices of cut-off functions, so the resulting isotopy class of symplectomorphism is independent of such choice.

The result is a symplectomorphism that is obtained as suspension of a loop of contactomorphisms induced by the Reeb flow on level-set hypersurfaces; this is exactly as τ'_V was defined in (2.16). Since we proved that τ'_V coincided with the fibred twist τ_V of Definition 2.2.11, this completes the proof.

□

2.3 Functor twists

This section only contains the notation (and the general notions involved) that we will use to denote the functors of the Fukaya category that are induced by twists.

Let (M,ω) be a Liouville manifold and let k be a field of characteristic 2. Very simply put, the compact Fukaya category, $\mathcal{F}\text{uk}(M)$, is an A_∞-category whose objects are closed exact Lagrangian *branes*, which are Lagrangian submanifolds with additional algebraic data, and morphisms the Floer cochain groups between transversely intersecting Lagrangians ([Sei08a, (9j),(12g)]). This category encodes intersection data associated to all its objects, including the Floer differential $\partial = \mu^1$, the Floer cup product μ^2 and higher order products μ^k (see e.g [Sei08a, (9j, (12g))]). It is well-defined for any Liouville manifold (see [Sei08a]).

Two Lagrangians that are Hamiltonian isotopic are quasi-isomorphic objects in the Fukaya category, which means they are isomorphic objects of the associated cohomological category, that we denote by $H(\mathcal{F}\text{uk}(M))$. We denote the automorphisms of $H(\mathcal{F}\text{uk}(M))$ (i.e the automorphisms of the Fukaya category up to quasi-isomorphism) by $\text{Auteq}(\mathcal{F}\text{uk}(M))$.

Let $Tw(\mathcal{F}\text{uk}(M))$ be the category of twisted complexes in $\mathcal{F}\text{uk}(M)$ (see [Sei08a, (3k)]), and $D^\flat\mathcal{F}\text{uk}(M) := H(Tw\mathcal{F}\text{uk}(M))$ the cohomology category of $Tw(\mathcal{F}\text{uk}(M))$.

There is a map

$$\Phi : \text{Symp}_{ct}(M) \to \text{Auteq}(D^\flat\mathcal{F}\text{uk}(M)) \tag{2.32}$$

to the group of auto-equivalences of the Fukaya category (modulo quasi-isomorphism), such that given $\phi \in \text{Symp}_{ct}(M)$, $\Phi(\phi)$ sends a Lagrangian $L \subset M$ to another Lagrangian $\phi(L) \subset M$ (we avoid discussing a graded situation in this context). The map factors through the quotient by the subgroup $\text{Ham}_{ct}(M) \subset \text{Symp}_{ct}(M)$ of compactly supported Hamiltonian diffeomorphisms, so given an exact Lagrangian sphere/projective space L and its associated twist τ_L, $\Phi(\tau_L)$ defines a well-defined element of $\text{Aut}(D^\flat\mathcal{F}\text{uk}(M))$ that we denote by T_L.

In [Sei03], Seidel showed that for a Dehn twist τ_L, the induced functor $T_L \in \text{Aut}(D^\flat\mathcal{F}\text{uk}(M))$ fits into an exact triangle (see [Sei08a, (17j)]).

Recently, there have been generalisations of Seidel's triangle for a wider class of symplectomorphisms, achieved through a range of different techniques. Wehrheim–Woodward ([WW16]) proved the existence of an exact triangle for fibred twists using quilt theory adapted to Morse–Bott Lefschetz fibrations.

Mak and Wu ([MW18a]) treated the case of projective twists, using Lagrangian cobordism theory as developed in [BC13] and [BC14]. They proved that the autoequivalence induced by a (real, complex, quaternionic) projective twist is isomorphic to a double cone of functors in $\mathrm{Aut}(Tw\mathcal{F}\mathrm{uk}(M))$ ([MW18a, Theorem 6.10]).

Under the appropriate circumstances, the mirror symmetry conjecture gives conjectural descriptions of such functors. If a symplectic manifold (M,ω) has a mirror complex manifold (X,J), there are autoequivalences of the Fukaya category of M that are induced by autoequivalences of the derived category of coherent sheaves of X (we call such autoequivalences *algebraic twist functors*, and will only refer to them in Remark 5.4.3).

Chapter 3

Lefschetz fibrations on $T^*\mathbb{A}\mathbb{P}^2$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$

3.1 Introduction

In Chapter 2 (Section 2.2.4), we have used Biran's symplectic decomposition ([Bir01]) to understand the complement of a smooth hypersurface of two complex projective varieties $(\mathbb{CP}^2, \omega_{FS})$ and $(\mathbb{CP}^2 \times \mathbb{CP}^2, \omega_{FS} \oplus \omega_{FS})$. In this chapter, we look at these projective varieties as carrying the structure of a Lefschetz pencil.

A smooth projective variety (X, ω_X) with an ample line bundle $\mathcal{L} \to X$ can be viewed as a Kähler manifold with integral symplectic class $[\omega] \in H^2(X;\mathbb{Z})$. Then the zero locus of a section $s_\infty \in H^0(\mathcal{L})$ that vanishes transversely is a smooth hypersurface $\Sigma = s_\infty^{-1}(0)$ Poincaré dual to $[\omega]$. Given another generic section $s_0 \in H^0(\mathcal{L})$, the family $\{\lambda s_0 + \mu s_\infty = 0\}_{[\lambda:\mu]\in\mathbb{CP}^1}$ parametrised by $\mathbb{CP}^1$ determines a Lefschetz pencil; $\Sigma_\infty \subset X$ is a smooth fibre in this family. The important feature of a Lefschetz pencil is that it induces a Lefschetz fibration (Section 3.2.4).

We employ the decompositions (2.26), (2.28) to study a Lefschetz pencil of hyperplane sections on $(\mathbb{CP}^2, \omega_{FS})$ and $(\mathbb{CP}^2 \times \mathbb{CP}^2, \omega_{FS} \oplus \omega_{FS})$ respectively, to build Lefschetz fibrations $\pi\colon T^*\mathbb{AP}^2 \to \mathbb{C}$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$ on cotangent bundles of projective spaces.

These fibrations will be a key component in the construction of the model planar projective twists of Chapter 4. At the end of this chapter (Section 3.5), we find an additional application, in a slightly different context. We study Lagrangian submanifolds of $(T^*\mathbb{CP}^2, d\lambda_{T^*\mathbb{CP}^2})$ that are diffeomorphic to $S^1 \times S^3$. By viewing such manifolds as Lagrangians of the total space of the fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$, we show that they can be classified in (at least) two Lagrangian isotopy classes (Lemma 3.5.7).

3.2 Geometry of Lefschetz pencils

The Lefschetz fibrations we build in this chapter are obtained from pencils of hyperplane sections. For this reason, we include a short review of classical material about Lefschetz pencils ([Lam81]) and the properties, on a symplectic level, of Lefschetz fibrations arising from this context ([Aur03, Sei08a, Oba21]).

Let (X, ω_X) be an $(n+1)$-dimensional smooth complex projective variety, and let $\mathcal{L} \to X$ be a holomorphic ample line bundle. Let $s_0, s_\infty \in H^0(\mathcal{L})$ be linearly independent holomorphic sections of $\mathcal{L}$. Then

$$\Sigma_{[\lambda:\mu]} := \{\lambda s_0 + \mu s_\infty = 0\}_{[\lambda:\mu] \in \mathbb{CP}^1} \subset X \tag{3.1}$$

defines a family of (projective) hypersurfaces. There is a rational function

$$X \dashrightarrow \mathbb{CP}^1,\ z \mapsto [s_0(z) : s_\infty(z)] \tag{3.2}$$

defined away from the base locus

$$B := \bigcap_{[\lambda:\mu] \in \mathbb{CP}^1} \Sigma_{[\lambda:\mu]} = \{s_0 = s_\infty = 0\}. \tag{3.3}$$

Definition 3.2.1. Let $\Sigma_\infty := \Sigma_{[0:1]} = s_\infty^{-1}(0)$ be smooth. The family (3.1) is called a (algebraic) Lefschetz pencil if

1. The base locus B is a smooth submanifold of X (of real codimension 4).

2. The map $p_X : X \setminus B \to \mathbb{CP}^1$ induced by (3.2) admits a (finite) set of non-degenerate critical points $\mathrm{Crit}(p_X)$, and is a submersion away from $\mathrm{Crit}(p_X)$. Denote the set of critical values by $\mathrm{Crit}v(p_X)$.

♢

The two above properties imply that:

3. In a neighbourhood of $p \in B$ there are local coordinates $(z_0, \dots, z_n)$ such that $B = \{z_0 = z_1 = 0\}$ and $p_X(z_0, \dots, z_n) = \frac{z_1}{z_0}$.

4. In a neighbourhood of $p \in \mathrm{Crit}(p_X)$ there are local coordinates $(z_0, \dots, z_n)$ such that $p_X(z_0, \dots, z_n) = z_0^2 + \dots + z_n^2$.

Remark 3.2.2. *Pencils exist on any projective variety (see for example [Voi03, Lemma 2.10]), and Donaldson ([Don96, Don99]) proved the existence of a Lefschetz pencil on any closed integral symplectic manifold (X, ω).* //

3.2.1 From pencil to fibration

A natural way of turning a Lefschetz pencil into a Lefschetz fibration is to perform a blow-up of X at the base locus B. Let $\widetilde{X} := Bl_B X = \{(z,y) \in \mathbb{CP}^1 \times X :\ s_0(y) = z s_\infty(y)\}$. The projection map $\widetilde{p}\colon \widetilde{X} \to \mathbb{CP}^1$ defines a Lefschetz fibration over $\mathbb{CP}^1$ with closed fibres, each of which contains a copy of the base locus embedded as a smooth hypersurface.

Note that the geometry of the blown-up space yields a well known formula relating the Euler characteristics of the various components of a Lefschetz pencil, which will be helpful for later computations.

Lemma 3.2.3. *Let $X \subset \mathbb{CP}^N$ be a smooth projective variety of complex dimension $n+1$. Let $(\Sigma_{[\lambda:\mu]})_{[\lambda:\mu]\in\mathbb{CP}^1}$ be a Lefschetz pencil on X with base locus B, generic fibre Σ and $r+1$ critical points, then the Euler characteristics of X, Σ and B are related as follows:*

$$\chi(X) = 2\chi(\Sigma) - \chi(B) + (-1)^{n+1}(r+1). \tag{3.4}$$

□

Another strategy to produce a Lefschetz fibration from the projection map of a Lefschetz pencil is to remove the smooth generic fibre $\Sigma_\infty = s_\infty^{-1}(0)$ from the family (3.1). In the process, the base locus is removed, and the restriction $p_X|_{X\setminus\Sigma_\infty}$ gives rise to a well-defined map

$$p\colon X \setminus \Sigma_\infty \longrightarrow \mathbb{C}. \tag{3.5}$$

The total space $X \setminus \Sigma_\infty$ is affine, and as noted in Section 2.2.3, $\varphi := -\frac{1}{4\pi} log(\|s\|^2)$ is a plurisubharmonic function on it. Similarly, a generic smooth fibre of (3.5) is given by the complement of the base locus $\Sigma \setminus B$, which is also affine.

The "affine" fibration p can be turned into a an Lefschetz fibration with exact compact fibres as follows. Consider the space $E := X \setminus (\Sigma_\infty \cup U_{B/X})$, where $U_{B/X}$ is a disc subbundle of the normal bundle $N_{B/X}$. The latter can be obtained as in (2.22), with the difference that in this case, the normal bundle to $B \subset X$ is modelled on $\mathcal{L} \oplus \mathcal{L}$. Equivalently, E is obtained as the complement $X \setminus U_{\Sigma_\infty/X}$ of an open subbundle $U_{\Sigma_\infty/X} \subset N_{\Sigma_\infty/X}$ of the normal bundle to the smooth fibre in X. The smooth fibres of the restriction $p|_E$ are Stein domains $M := \Sigma \setminus U_{B/\Sigma}$, where $U_{B/\Sigma} \subset N_{B/\Sigma}$ is an open subbundle of the normal bundle to $B \subset \Sigma$.

Proposition 3.2.4 ([Oba21, Proposition 2.6])**.** *Let (X, ω_X) be a smooth projective variety with a Lefschetz pencil induced by a holomorphic line bundle $\mathcal{L} \to X$, with base locus B and generic smooth fibre Σ_∞. Then, for a disc bundle $U_{B/X} \supset B$ as above there is a closed two form Ω on $E := X \setminus U_{\Sigma_\infty/X}$ such that*

1. *$\pi = p|_E \colon (E,\Omega) \longrightarrow \mathbb{C}$ is an exact Lefschetz fibration with smooth fibre (M,ω),*

2. *For every $z \in \mathbb{C}$, $\omega_X|_{\pi^{-1}(z)} = \Omega|_{\pi^{-1}(z)}$,*

3. *Ω coincides with ω_X outside a collar neighbourhood of the horizontal boundary of E.*

The total space (E,Ω) can be (symplectically) completed to a Stein manifold.

Remark 3.2.5. *By construction, the boundary of the smooth fibre (M,ω) has the structure of a prequantisation bundle $P := \{\nu \in \mathcal{L}^*|_B,\ \|\nu\| = 1\}$ over the base locus B (see Section 2.2.3).* //

The symplectic Picard–Lefschetz theorem expresses the global monodromy of a Lefschetz fibration as the product of Dehn twists along the Lagrangian vanishing cycles of the singularities. The global monodromy of an exact Lefschetz fibration induced by a Lefschetz pencil, like that of Theorem 3.2.4, has an additional property, that we call the *generalised lantern relation* (the isotopy (3.6)).

Lemma 3.2.6 ([Aur03, Section 2.4], [Gom04, Section 3] [Oba21, Theorem 2.7])**.** *Consider a Lefschetz pencil on the projective variety (X,ω_X), with generic smooth fibre Σ and base locus B. Let $\pi\colon E \to \mathbb{C}$ be the exact Lefschetz fibration induced by the pencil as in Theorem 3.2.4, with smooth fibre (M,ω), and global monodromy $\phi \in \mathrm{Symp}_{ct}(M)$. Then there is a symplectic isotopy*

$$\phi \simeq \tau_V, \tag{3.6}$$

where the map $\tau_V \in \mathrm{Symp}_{ct}(M)$ is a fibred *twist along the S^1-fibred coisotropic submanifold $V = \partial M \to B$, obtained as in Remark 3.2.5.*

3.2.2 The monodromy as a graded symplectomorphism

In some cases, it is possibe to define a notion of $\mathbb{Z}$ grading, for Lagrangian submanifolds, Floer cohomology groups and symplectomorphisms. In this section we study how the global monodromy of a Lefschetz fibration that is built as in Section 3.2.4 acts (as a graded symplectomorphism) on vanishing cycles (as graded Lagrangians). To illustrate this, we follow [Sei00, 2] and [Sei08a, 11].

For now, let (M^{2n},ω) be a general symplectic manifold satifying $2c_1(M) = 0$. Since $2c_1(M) = 2c_1(TM) = c_1((\bigwedge^n_{\mathbb{C}} TM)^{\otimes 2})$, $2c_1(M) = 0$ implies the bundle $\mathcal{K}^2_M := (\bigwedge^n TM)^{\otimes -2}$ is trivial so it admits a nowhere vanishing section, i.e a quadratic complex volume form η_M.

Let $LGr(n)$ be the Grassmannian of Lagrangian planes in $(\mathbb{R}^{2n},\omega_{std})$, and $LGr(TM)$ the bundle of Lagrangian Grassmannians in the tangent bundle of M, which is a $LGr(n)$-bundle over M.

Definition 3.2.7. The squared phase map $\alpha_M \colon LGr(TM) \to S^1$, is defined as

$$\alpha_M \colon LGr(TM) \to S^1,\ \alpha_M(V) = \frac{\eta_M((v_1 \wedge \cdots \wedge v_n)^2)}{|\eta_M((v_1 \wedge \cdots \wedge v_n)^2)|} = \arg(\eta_M|_V),$$

where $v_1,\dots,v_n$ is any basis of V.

$\diamondsuit$

Let $L \subset M$ be a closed connected Lagrangian submanifold of M. There is a section $s_L\colon L \to LGr(TM)$ sending $p \in L$ to the corresponding tangent plane $T_pL \in LGr(T_pM)$. Consider the composition $\alpha_L = \alpha_M \circ s_L\colon L \to S^1$.

Definition 3.2.8. The Maslov class of L, denoted by $\mu_L \in \mathrm{Hom}(\pi_1(L),\mathbb{Z}) = H^1(L;\mathbb{Z})$ is the homotopy class of the map $[\alpha_L] \in [L,S^1] = H^1(L;\mathbb{Z})$. In other words, μ_L is the pullback of the angle form $d\theta \in H^1(S^1;\mathbb{Z})$ under the map α_L.

Given a pseudoholomorphic (with respect to some given compatible almost complex structure) disc $u\colon (D,\partial D) \to (M,L)$, its Maslov index is given by $\mu(u) = \int_\gamma \mu_L$ where γ is the loop in $LGr(n)$ given by the (symplectic) trivialisation of $(u|_{\partial D})^*(TL)$.

The minimal Maslov index of L is defined as

$$N_L := \min\{\mu([u]) > 0,\ u\colon (D,\partial D) \to (M,L) \text{ pseudoholomorphic}\}.$$

$\diamondsuit$

Definition 3.2.9. A graded Lagrangian is a pair $(L,\tilde{\alpha}_L)$ consisting of a Lagrangian submanifold $L \subset M$ with a lift of α_L, i.e a map $\tilde{\alpha}_L\colon L \to \mathbb{R}$ such that $\exp(2\pi i\tilde{\alpha}_L(x)) = \alpha_L(TL_x)$. If $\mu_L = 0$, such a lift exists. $\diamondsuit$

We shall focus on grading shifts that graded exact Lagrangian submanifolds undergo under a symplectomorphism.

Definition 3.2.10. A graded symplectomorphism is a pair (ϕ,α_ϕ) consisting of a symplectomorphism $\phi \in \mathrm{Symp}(M)$ and a map $\alpha_\phi\colon LGr(TM) \to \mathbb{R}$ satisfying $\exp(2\pi i\alpha_\phi(V)) = \frac{\alpha_M(D\phi(V))}{\alpha_M(V)}$. $\diamondsuit$

An important example of graded symplectomorphism is the *shift* operator ([Sei08a, (11k)]), which can be considered as the pair $(id,-k)$, $k \in \mathbb{Z}$.

On a graded Lagrangian $(L,\tilde{\alpha}_L)$, the shift sends $\tilde{\alpha}_L$ to $\tilde{\alpha}_L - k$. We will simply denote the action of shift on the Lagrangian as $L[k]$.

Definition 3.2.11. Given a pair (L_0,L_1) of graded exact closed Lagrangian submanifolds of M, the degree of an intersection point $x \in L_0 \cap L_1$ is defined as

$$\deg(x) = \tilde{L}_1(x) - \tilde{L}_2(x), \tag{3.7}$$

where $\tilde{L}_i(x) := \tilde{\alpha}_{L_i}(x)$. $\diamondsuit$

The Floer cohomology group $\mathrm{HF}(L_0, L_1)$ can be endowed with a grading ([Sei00, 2.f]), and we have the following formula for shifts ([Sei00, Section 2f])

$$\mathrm{HF}^*(L_0[j], L_1) = \mathrm{HF}^{*-j}(L_0, L_1) = \mathrm{HF}^*(L_0, L_1[-j]),\ j \in \mathbb{Z}.$$

For the remainder of the section, assume that $\pi \colon E^{2n+2} \to \mathbb{C}$ is an exact Lefschetz fibration with smooth fibre (M, ω), that is induced by a Lefschetz pencil on a projective variety (X, ω_X) as in Lemma 3.2.4. Let $\mathcal{L} \to X$ be the ample line bundle that generates the pencil. The generalised lantern relation (Lemma 3.2.6) ensures that the (ungraded) global monodromy of π preserves the vanishing cycles. We introduce an extra assumption on X that yields a graded version of this statement (Lemma 3.2.12).

Assumption A. The canonical bundle of X, $\mathcal{K}_X$, satisfies:

$$\mathcal{K}_X^2 \cong \mathcal{L}^{\otimes -d},\ d \in \mathbb{Z}^{>0} \tag{3.8}$$

Below we explain how, under Assumption (A), the symplectic parallel transport maps and in particular the global monodromy can be treated as graded symplectomorphisms.

Let $s_0, s_\infty \in H^0(\mathcal{L})$ be linearly independent sections of $\mathcal{L}$. The preimage of $s_\infty^{\otimes -d}$ (which is a section of $\mathcal{L}^{\otimes -d}$) under (3.8) is the square of a $(n+1)$-form, which has a zero ($d < 0$) or a pole ($d > 0$) along $\Sigma_\infty = s_\infty^{-1}(0)$. That gives a trivialisation of $\mathcal{K}_X^2$, the bundle of quadratic forms, over the complement $W = X \setminus \Sigma_\infty$, where it restricts to a quadratic form η_X, which induces a quadratic volume form on the total space E of the Lefschetz fibration. There is an associated relative quadratic form η_X / dz^2 (here dz is the volume form on $\mathbb{C}$), which is a section of $\mathcal{K}_{X/\mathbb{C}}^2 = \pi^*(\mathcal{K}_\mathbb{C}^{-2}) \otimes \mathcal{K}_X^2$, on the complement of Σ_∞. This defines a quadratic complex volume form on the smooth fibre (M, ω) of E.

Lemma 3.2.12 ([Sei08a, (19b)])**.** *The global monodromy $\phi \in \mathrm{Symp}_{ct}(M)$ of the exact Lefschetz fibration $\pi \colon E \to \mathbb{C}$ above (obtained from a Lefschetz pencil on a projective variety X satisfying Assumption* (A)*) acts as a shift on every graded closed exact Lagrangian vanishing cycle $V \subset M$. In other words, it acts as the identity, equipped with a constant grading, and in the notation of the discussion above we have*

$$\phi(V) = V[4-d]. \tag{3.9}$$

Proof. By Lemma 3.2.6, the global monodromy $\phi \in \mathrm{Symp}_{ct}(M)$ of a fibration as in the statement is isotopic to a fibred twist whose support is disjoint from the vanishing cycles. Therefore, ϕ preserves setwise the vanishing cycles, and will act on each of them as a shift. This shift depends on the pole order of η_X/dz^2. Under the assumption (A), the relative quadratic form η_X/dz^2 does not extend smoothly over infinity—where the section s_∞ has a pole.

By (3.8), η_X has a pole of order d at infinity, while a standard argument shows that dz^2 has a pole of order 4. Therefore, $\eta' := z^{4-d}(\eta_X/dz^2)$ is holomorphic, so that the phase function of η' is preserved under parallel transport around a large circle through z_* (that can be homotoped to be away at infinity). This implies that, on the other hand, the phase function of $\eta_X/dz^2 = z^{d-4}\eta'$ must shift by $d-4$ when doing parallel transport 'around infinity' (since the phase function of η' must remain constant). Then, by the convention of [Sei08a, (11k)], the grading of the graded Lagrangian vanishing cycles V_i is shifted as $\phi(V_i) = V_i[-(d-4)] = V_i[4-d]$. □

3.3 Lefschetz fibration on $T^*\mathbb{RP}^2$

From a pencil of quadrics on $(\mathbb{CP}^2, \omega_{FS})$, we define an exact Lefschetz fibration $\pi\colon E_{\mathbb{RP}^2} \to \mathbb{C}$, with smooth fibre a 2-sphere with four punctures, three singular fibres, and whose total space admits a symplectic completion to a Stein manifold exact symplectomorphic to $(T^*\mathbb{RP}^2, d\lambda_{T^*\mathbb{RP}^2})$. This is a well-known case study in symplectic toplogy (see for example [Aur03, Section 3.1]).

Recall Example 2.2.18. Consider a Lefschetz pencil $C_{[\lambda:\mu]} := \{\lambda s_0 + \mu s_\infty = 0\}_{[\lambda:\mu]\in\mathbb{CP}^1}$ on $\mathbb{CP}^2$, generated by sections of $\mathcal{O}_{\mathbb{CP}^2}(2)$ denoted by s_0 and s_∞, which are homogeneous polynomials in degree two in the coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{CP}^2$. In a generic family of conics, the (complex) curves intersect in a common set of four points (two degree two complex curves generically intersect in four points), the base locus $B = \{s_0 = 0 = s_\infty\}$. Every fibre contains these four points, and there are three degenerate cases, in which the fibre is made up of two curves (each containing two of the base points) intersecting at a point. These are the three singular fibres, any of which has one singularity (at the aforementioned intersection point) which is at most an ordinary double point. This might be best illustrated with an example; take $s_0 := z_0^2 - z_2^2$, $s_\infty := z_0^2 + z_1^2 + z_2^2$. Then $B = \{[1 : i\sqrt{2} : 1], [1 : -i\sqrt{2} : 1], [1 : i\sqrt{2} : -1], [1 : -i\sqrt{2} : -1]\}$, and there are three singular fibres at $[\lambda : \mu] = \{[0 : 1], [1 : 1], [1 : -1]\}$.

One can check that each singular conic is composed by two lines in $\mathbb{CP}^2$ containing two of the base points each; three such configurations can arise (the number of critical fibres can also be checked with the formula (3.4)). For a basis of vanishing paths, the three associated vanishing cycles are the classes of circles which collapse to the intersection of the two lines in each singular conic. Hence, each vanishing cycle in a smooth fibre is represented by a loop encircling two points of the base locus (see Figure 3.1).

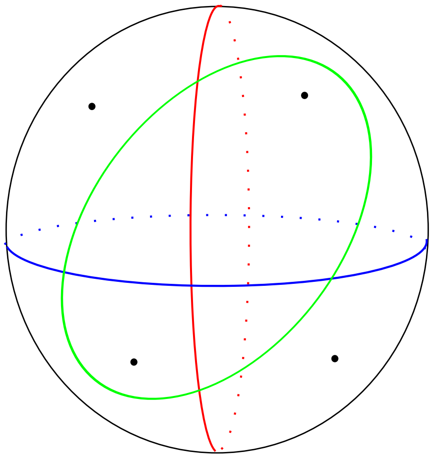

Figure 3.1: A smooth fibre of the pencil, and the three configurations of vanishing cycles.

Let $[z_0 : z_1 : z_2]$ be homogeneous coordinates on $\mathbb{CP}^2$. There is a rational map

$$\mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1, \ \underline{z} = [z_0 : z_1 : z_2] \mapsto [s_0(\underline{z}) : s_\infty(\underline{z})]$$

sending each point to the hyperplane in $\mathbb{CP}^2$ containing it. By removing one of the smooth fibres, for example $\Sigma_\infty := s_\infty^{-1}(0)$, we obtain a well defined map

$$p := \frac{s_0}{s_\infty} : \mathbb{CP}^2 \setminus \Sigma_\infty \longrightarrow \mathbb{CP}^1 \setminus \{\infty\} \cong \mathbb{C}. \tag{3.10}$$

This defines a Lefschetz fibration whose fibres are 2-spheres with four punctures (the base locus B has been removed in the process) and base $\mathbb{C}$.

Lemma 3.3.1. *The total space $\mathbb{CP}^2 \setminus \Sigma_\infty$ of p is exact symplectomorphic to an open disc subbundle of $(T^*\mathbb{RP}^2, d\lambda_{T^*\mathbb{RP}^2})$.*

Proof. Follows from Example 2.2.18. □

The fibration (3.10) can be adjusted (Lemma 3.2.4) to become an exact Lefschetz fibration

$$\pi \colon E_{\mathbb{RP}^2} \to \mathbb{C} \tag{3.11}$$

with smooth fibre a 2-sphere with four boundary components, and whose total space admit a symplectic completion to $(T^*\mathbb{RP}^2, d\lambda_{T^*\mathbb{RP}^2})$.

The vanishing cycles $V_0, V_1, V_2 \subset M$ are exact Lagrangian circles which partition the four boundary components in three possible configurations of pairs. There is only one Hamiltonian class for each vanishing cycle (each homotopy class has only one exact Lagrangian representative, by Stokes's theorem) and since $\mathrm{Diff}(S^1) \simeq O(2)$, there is a unique choice of framing.

3.3.1 Monodromy

In this section we study the properties of the monodromy of the Lefschetz fibration $\pi\colon E_{\mathbb{RP}^2} \to \mathbb{C}$. Let $\mathrm{Critv}(\pi) = \{w_0, w_1, w_2\}$ be the set of critical values of π. Fix a base-point $z_* \in \mathbb{C}$, smooth fibre $\pi^{-1}(z_*) \cong M$ (a four punctured sphere) and basis of vanishing paths $(\gamma_0, \gamma_1, \gamma_2)$. Let $(\Delta_0, \Delta_1, \Delta_2)$ and $(V_0, V_1, V_2) \subset M$ be the associated bases of Lagrangian thimbles and vanishing cycles respectively.

Lemma 3.3.2. *Let $(d_1, d_2, d_3, d_4) \subset M$ be simple closed curves with $d_i \cap V_j = \emptyset$ ($i = 1,2,3,4$ and $j = 0,1,2$), such that each d_i encircles a distinct boundary component of the fibre. The global monodromy $\phi \in \mathrm{Symp}_{ct}(T^*\mathbb{RP}^2)$ of π satisfies*

$$\phi := \tau_{V_0}\tau_{V_1}\tau_{V_2} \simeq \prod_{i=1}^{4} \tau_{d_i}. \tag{3.12}$$

In particular, ϕ commutes with each individual twist, and preserves each vanishing cycle.

Proof. The expression (3.12) is *the* lantern relation (see for example [FM11, Proposition 5.1]), a special instance of the generalised lantern relation of Lemma 3.2.6, according to which the global monodromy is isotopic to the fibred twist in the circle bundle of the normal bundle to the base locus $B \subset \mathbb{CP}^1$. In this low-dimensional case, the unit normal bundle is simply the union of the four boundary circles. □

Remark 3.3.3. *The isotopy* (3.12) *also implies that any cyclic permutation of $\tau_{V_0}\tau_{V_1}\tau_{V_2}$ defines the same element of the mapping class group.* //

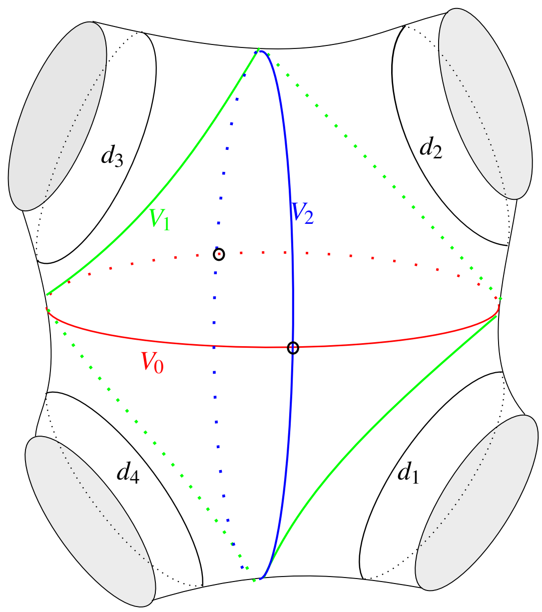

Figure 3.2: The boundary circles d_i around the boundary components of the smooth fibre are disjoint and therefore the twists in the composition $\Pi_{i=1}^4 \tau_{d_i}$ commute.

The lantern relation forces the (ungraded) monodromy to act trivially on vanishing cycles. However, if we consider ϕ as a graded symplectomorphism, it acts on vanishing cycles by a shift determined as in Lemma 3.2.12. The line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^2}(2)$ defining the pencil, and the canonical bundle $\mathcal{K}_{\mathbb{CP}^2} = \mathcal{O}(-3)$ satisfy Assumption (A), with $d = 3$:

$$\mathcal{O}(-3)^{\otimes 2} \cong \mathcal{O}(2)^{\otimes -3}. \tag{3.13}$$

On graded Lagrangian vanishing cycles $V_i \subset M$, we therefore have the shift $\phi(V_i) = V_i[4-3] = V_i[1]$.

3.4 Lefschetz fibration on $T^*\mathbb{CP}^2$

We find an exact Lefschetz fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ with smooth fibre a clean plumbing of disc cotangent bundles of 3-spheres $D_\varepsilon(T^*S^3 \#_{S^1} T^*S^3)$ along a circle (see Section 6.1 for the definition of clean Lagrangian plumbing), three singular fibres and whose total space admits a symplectic completion to a Stein manifold exact symplectomorphic to $(T^*\mathbb{CP}^2, d\lambda_{T^*\mathbb{CP}^2})$. The three vanishing cycles appear as the two core components of the plumbing, and the surgery thereof.

Recall example 2.2.19. Let $(\mathbb{CP}^2 \times \mathbb{CP}^2, \omega_{FS} \oplus \omega_{FS})$ with homogeneous coordinates $(\underline{x}, \underline{y}) := ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ on the first and second factor respectively. Fix a hypersurface

$$\Sigma := \left\{ \sum_{i=0}^{2} x_i y_i = 0 \right\} \subset \mathbb{CP}^2 \times \mathbb{CP}^2, \tag{3.14}$$

obtained as the image of the embedding of the Flag variety $Fl_3 \hookrightarrow \mathbb{CP}^2 \times \mathbb{CP}^2$. Let s_0, s_∞ be sections of $\mathcal{O}_{\mathbb{CP}^2 \times \mathbb{CP}^2}(1,1)$. Consider a pencil

$$\Sigma_{[\lambda:\mu]} := \{\lambda s_0(\underline{x}, \underline{y}) + \mu s_\infty(\underline{x}, \underline{y}) = 0\}_{[\lambda:\mu] \in \mathbb{CP}^1} \subset \mathbb{CP}^2 \times \mathbb{CP}^2. \tag{3.15}$$

A generic fibre of such a pencil is isomorphic to the three-fold Flag variety $Fl_3 \subset \mathbb{C}^3$.

Lemma 3.4.1. *The base locus B is diffeomorphic to the 3-point blow up of $\mathbb{CP}^2$, $B \cong \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$, equipped with its monotone symplectic form.*

Proof. The base locus is a symplectic manifold of dimension 4, and since it is obtained as the intersection of hyperplane sections of $\mathbb{CP}^2 \times \mathbb{CP}^2$ of bidegree $(1,1)$, its Chern class is a positive class by the adjunction formula. Hence B is a monotone Fano (or del Pezzo) surface, i.e a non-singular projective algebraic surface with ample anticanonical divisor. Thus B is either isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ or to the blow up $\mathbb{CP}^2 \# r\overline{\mathbb{CP}^2}$ at r points in general position, for $0 \leqslant r \leqslant 8$ (see for example [Dem80]).

Embed $\mathbb{CP}^2 \times \mathbb{CP}^2$ in $\mathbb{CP}^8$ via the Segre embedding, and by Lemma 3.2.3, we know

$$\chi(B) = 2\chi(Fl_3) - \chi(\mathbb{CP}^2 \times \mathbb{CP}^2) + 3. \tag{3.16}$$

Now $\chi(\mathbb{CP}^2\times\mathbb{CP}^2)=\chi(\mathbb{CP}^2)^2=9$ and the flag manifold Fl_3 is known to have Euler characteristic $\chi(Fl_3)=3!=6$, so that $\chi(B)=6$, and hence $B\cong\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. □

Given a generic pencil of $(1,1)$ divisors (3.15), whose smooth fibre "at infinity" is denoted by $\Sigma_\infty := s_\infty^{-1}(0)\cong Fl_3$ consider the map

$$p=\frac{s_0(x,y)}{s_\infty(x,y)}\colon \mathbb{CP}^2\times\mathbb{CP}^2\setminus\Sigma_\infty\longrightarrow\mathbb{C}. \tag{3.17}$$

Lemma 3.4.2 (see Lemma 3.4.5)**.** *The fibre of* (3.17)*, which is the affine* 3*-fold $Fl_3\setminus\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is the interior of a Stein domain whose symplectic completion is exact symplectomorphic to a clean plumbing $T^*S^3\#_{S^1}T^*S^3$.*

Lemma 3.4.3. *The total space $\mathbb{CP}^2\times\mathbb{CP}^2\setminus\Sigma_\infty$ of p is exact symplectomorphic to an open disc subbundle of $(T^*\mathbb{CP}^2,d\lambda_{T^*\mathbb{CP}^2})$.*

Proof. Follows from Example 2.2.19. □

The fibration (3.10) can be adjusted (Lemma 3.2.4) to become an exact Lefschetz fibration

$$\pi\colon E_{\mathbb{RP}^2}\longrightarrow\mathbb{C} \tag{3.18}$$

whose total space admit a symplectic completion to $(T^*\mathbb{CP}^2,d\lambda_{T^*\mathbb{CP}^2})$. In the next section, we study the fibres of π.

3.4.1 Topology of the fibre: A MBL fibration on the affine Flag 3-fold

In this section we enhance our understanding of the topology of the fibres of the Lefschetz fibration $\pi\colon E_{\mathbb{CP}^2}\to\mathbb{C}$ via a construction due to Jonny Evans [Eva]. We will show why Lemma 3.4.2 is true.

Let $Fl_3=\{x_0y_0+x_1y_1+x_2y_2=0\}\subset\mathbb{CP}^2\times\mathbb{CP}^2$ be the Flag 3-fold as in Example 2.2.19 (where we use the same choice of coordinates on $\mathbb{CP}^2\times\mathbb{CP}^2$ as before). Consider a pencil of divisors $Y_{[\lambda:\mu]}:=\{\lambda(x_0y_0-x_2y_2)+\mu x_1y_1=0\}_{[\lambda:\mu]\in\mathbb{CP}^1}\subset Fl_3$. By the definition of the pencil, generic fibre of $Y_{[\lambda:\mu]}$ can be understood as the intersection of two hyperplane sections of $\mathbb{CP}^2\times\mathbb{CP}^2$ of bidegree $(1,1)$. Therefore, by Lemma 3.4.1, the generic fibre is a copy of the del Pezzo surface $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. There are three critical fibres at $[\lambda:\mu]\in\{[1:0],[1:1],[-1:1]\}$. The base locus of this pencil is the set $\Theta:=\{x_0y_0=x_1y_1=x_2y_2=0\}$, composed of six lines $L'_{ijk}=\{x_i=x_j=y_k=0\}$, $L'_{ijk}=\{y_i=y_j=x_k\}$ where ijk is a permutation of 012. For all i, let $z_i=x_iy_i$. Remove a smooth fibre Y_∞ and define a Morse–Bott Lefschetz fibration (see the proof of [SW20, Lemma 4.5])

$$p_1\colon Fl_3\setminus Y_\infty\to\mathbb{C},\ ([x_0:x_1:x_2],[y_0:y_1:y_2])\mapsto\frac{x_1y_1}{x_2y_2-x_0y_0}. \tag{3.19}$$

The general fibre is isomorphic to $(\mathbb{C}^*)^2$ and there are three singular fibres over $\{0, \pm 1\}$ isomorphic to $\mathbb{C}^* \times (\mathbb{C} \vee_0 \mathbb{C})$, so that in every singular fibre, there is a copy of $\mathbb{C}^*$ that is the singular locus of that fibre.

Lemma 3.4.4 ([Eva]). *The fibration p_1 admits three 3-dimensional matching spheres of the total space, which pairwise intersect cleanly along a circle.*

Proof. The symplectic structure on $Y \setminus Y_\infty$ yields a preferred choice of symplectic connection (the symplectic orthogonal to each fibre of p_1) which gives local parallel transport maps.

We first explain why these parallel transport maps are globally defined on the total space of the fibration p_1. There is a Hamiltonian T^2-action on $\mathbb{CP}^2 \times \mathbb{CP}^2$ given by

$$(\theta, \phi) \cdot ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = ([e^{i\theta} x_0 : x_1 : e^{i\phi} x_2], [e^{-i\theta} y_0 : y_1 : e^{-i\phi} y_2]), \tag{3.20}$$

which preserves the fibres of p_1. The (restricted) moment map $H \colon Fl_3 \setminus Y_\infty \to \mathbb{R}^2$ of (3.20) has the shape

$$H([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = \frac{1}{2} \left(\frac{|x_0|^2}{|a|^2} - \frac{|y_0|^2}{|b|^2}, \frac{|x_2|^2}{|a|^2} - \frac{|y_2|^2}{|b|^2} \right) \tag{3.21}$$

and its level sets are compact. These level sets are preserved by the parallel transport maps (see [SW20, Lemma 4.1] for a proof). Therfore, parallel transport maps are defined globally via the action (3.20).

Let $\gamma_\pm(t) = \pm t$, $t \in [0, 1]$ be the path in $\mathbb{C}$ connecting the critical value 0 to ± 1. Over $z \notin \{-1, 0, 1\}$, the restriction $T_z := H^{-1}(0) \cap p_1^{-1}(z)$ is a smooth torus in the smooth fibre $(\mathbb{C}^*)^2$. Since parallel transport preserves the level sets of H, set $S_\pm := H^{-1}(0) \cap p_1^{-1}(\gamma_\pm([0, 1]))$ obtained by parallel transporting such a torus along the path $\gamma_\pm$, defines a matching cycle. To understand $S_\pm$, we look at how the torus T_z degenerates in the critical fibres over $z \in \{-1, 0, 1\}$.

The action (3.20) exhibits three circles given by $\{\phi = 0\}, \{\theta = 0\}$ and $\{\phi - \theta = 0\}$. We can set the generators of $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ to be the θ- and ϕ-circles. We explain below that each circle collapses in one of the singular fibres.

The critical locus over $z = -1$ is given by the equations $\{x_0 = y_0 = 0,\ x_1 y_1 + x_2 y_2 = 0\}$ so it is made of pairs $([0 : x_1 : x_2], [0 : -x_2 : x_1])$. Therefore, as $t \to -1$ on γ_-, the $\{\theta = 0\}$ circle must collapse to a point.

The critical locus over 0 is given by the equations $\{x_1 = y_1 = 0,\ x_0 y_0 + x_2 y_2 = 0$, so it is made of pairs $([x_0 : 0 : x_2], [-x_2 : 0 : x_0])$. The T^2 action becomes trivial if $\theta = \phi$, which means that at $t \to 0$ on $\gamma_\pm$, the circle $\{\theta = \phi\}$ collapses to a point.

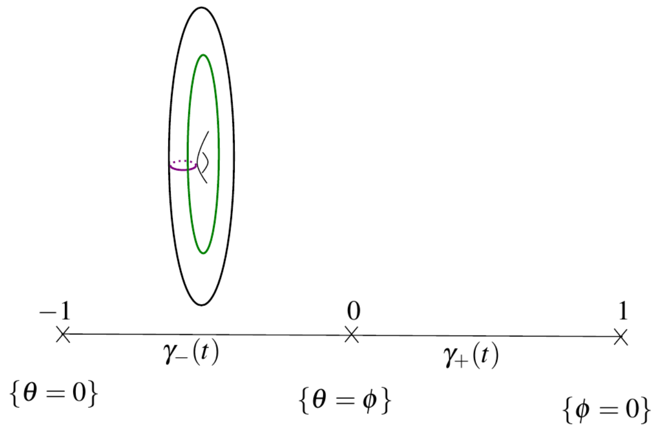

Figure 3.3: Depiction of the restriction of the smooth fibre $T_z = H^{-1}(0) \cap p_1^{-1}(z)$, such that over each singular value one of the circles $\{\theta = 0\}$, $\{\theta = \phi\}$, $\{\phi = 0\}$ collapses.

The critical locus over 1 is given by $\{x_2 = y_2 = 0,\ x_0y_0 = -x_1y_1\}$ so it is made of pairs $([x_0 : x_1 : 0],[x_1 : x_0 : 0])$. In this case, as $t \to 1$ on γ_+, the $\{\phi = 0\}$ circle collapses to a point.

From these observations, one can see that the matching cycles $S_\pm$ are unions of solid tori $S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$ glued along the torus $S^1 \times S^1 \subset \mathbb{C}^*$ in the smooth fibre; see also [SW20, Example 4.2]. This union has to be a three sphere, because the circles $\{\theta = 0\}$ and $\{\phi = 0\}$ become trivial homology classes in $S_\pm$.

By this description we also understand that the matching spheres $S_\pm$ intersect in $p_1^{-1}(0)$ cleanly in a circle (see also discussion of [SW20, Lemma 4.5], these intersection loci are the real part of the three components of the critical locus). A third matching sphere can be obtained as the Morse-Bott surgery of the other two (there are two possible such surgeries).

□

Lemma 3.4.5 ([Eva]). *The union of $S_\pm$ form a Lagrangian skeleton for the clean Lagrangian plumbing $T^*S^3 \#_{S^1} T^*S^3$ (see Definition 6.1.1). In particular, the symplectic completion of $M := Fl_3 \setminus Y_\infty$ is symplectomorphic to this plumbing.*

The proof of this lemma is beyond the scope of this section. The idea in [Eva] is to define a plurisubharmonic function $Fl_3 \setminus Y \to \mathbb{R}$ with skeleton formed by the pair of Lagrangians 3-spheres over $[-1,0]$ and $[0,1]$, meeting cleanly in a circle over 0. A clean plumbing of spheres $T^*S^3 \#_{S^1} T^*S^3$ is determined by the topology of the surgery of the Lagrangian cores ([SW20]), and the plumbing of the statement is such that the surgery of the core spheres is another sphere.

Remark 3.4.6. *The plumbing of Lemma 3.4.5 does not coincide with the plumbing we use in the applications of Chapter 6. The difference lies in the choice of trivialisation of the normal bundle of*

S^1 inside the two spheres, which in this case also determines the gluing of the two core components of the plumbing. In this case, the framing is such that the surgery of the two spheres along S^1 is again sphere.

//

3.4.2 Topology of the fibre: a bifibration

Consider $(1,1)$-divisors on $(\mathbb{CP}^2 \times \mathbb{CP}^2, \omega_{FS} \oplus \omega_{FS})$ given by a linear combination of monomials x_0y_0, x_1y_1, x_2y_2. This gives a rational map

$$\mathbb{CP}^2 \times \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2 \tag{3.22}$$

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \longmapsto [z_0 = x_0y_0 : z_1 = x_1y_1 : z_2 = x_2y_2]. \tag{3.23}$$

The base locus of this map is the set Θ as in Section 3.4.1, so consider the well-defined restriction

$$q\colon \mathbb{CP}^2 \times \mathbb{CP}^2 \setminus \Theta \to \mathbb{CP}^2.$$

A generic fibre over $z = [z_0 : z_1 : z_2]$ is determined by the equations $\{x_iy_i = z_i\}$. The smooth fibre is $(\mathbb{C}^*)^3$. However, it reduces to $(\mathbb{C}^*)^2$ after quotienting by the $\mathbb{C}^*$ action given by $(x_i, y_i) \mapsto (\zeta x_i, \zeta^{-1}y_i)$ for all $i \in \{0,1,2\}$, $\zeta \in \mathbb{C}^*$.

Over the coordinate lines $\{z_i = x_iy_i = 0\} \setminus \{[1:0:0],[0:1:0],[0:0:1]\}$ of $\mathbb{CP}^2$, the fibration is singular with singular fibres $\mathbb{C}^* \times (\mathbb{C} \vee_0 \mathbb{C})$. There are three such degenerations, one for each factor of $(\mathbb{C}^*)^3$.

Lemma 3.4.7. *There is a Lefschetz fibration over $\mathbb{C}$, with total space exact symplectomorphic to a disc cotangent bundle of $T^*\mathbb{CP}^n$ and fibres exact symplectomorphic to the affine Flag 3-fold (see Lemma 3.4.2) with the following property. The three vanishing cycles (associated to a basis of vanishing paths) of this fibration are three-spheres which pairwise intersect cleanly in a circle.*

Proof. To show the claim, we consider a 1-parameter family of lines in (3.22). For every generic line $\ell := \{b_0z_0 + b_1z_1 + b_2z_2 = 0,\ b_i \neq 0\}$ in the base $\mathbb{CP}^2$, the restriction $q|_\ell$ gives a $(\mathbb{C}^*)^2$-fibration of an open subset of the Flag 3-fold in $\mathbb{CP}^2 \times \mathbb{CP}^2$. There are three singular fibres, over the points in which ℓ intersects the coordinate axes.

A deformation of ℓ in a family $\ell_{[\alpha:\beta]}$ of lines in $\mathbb{CP}^2$ corresponds to a pencil of flag 3-folds in $\mathbb{CP}^2 \times \mathbb{CP}^2$, and the restriction of q over $\ell_{[\alpha:\beta]}$, $[\alpha : \beta] \in \mathbb{CP}^1$ defines a Lefschetz fibration π as in Section 3.4.

In this family, consider a path $\delta(t)$ joining a generic line $\delta(0) := \ell_{gen}$ to a line $\delta(1) = \ell_{crit}$ passing through either of $[1:0:0],[0:1:0],[0:0:1]$; in the total space this traces a family of Morse–Bott–Lefschetz fibrations, each with a configuration of three singular fibres. We can choose ℓ_{gen} such that $q|_{\ell_{gen}} = p_1$ from the previous section. As the path reaches $\ell_{crit} = \delta(1)$, two of the critical loci of this configuration will come together over the singular restriction $q|_{\ell_{crit}}$ and this produces a matching sphere of the fibration p_1, as described in Lemma 3.4.4. Namely, for any $w \in \{[1:0:0],[0:1:0],[0:0:1]\}$, $q^{-1}(w) = (\mathbb{C}\vee\mathbb{C})\times(\mathbb{C}\vee\mathbb{C})$: as two of the singular points of q come together, *two* of the original factors in $(\mathbb{C}^*)^3$ collapse, and after quotienting by the $\mathbb{C}^*$-action, the degenerations are recognisable as ordinary double points. As the restriction $q|_{\ell_{crit}}$ defines a singular fibre for a fibration of the type of π, the matching spheres of p_1 (Lemma 3.4.4) coincide with the vanishing cycles of π.

□

Remark 3.4.8. *Any smooth fibre $q^{-1}(z)$, $z = [z_0 : z_1 : z_2]$ with $z_i \neq 0$ compactifies to $q^{-1}(z) \cup \Theta = \mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. Note that this coincides with the base locus of the pencil in Lemma 3.4.1. For the closure of the singular fibres, see [Tyu11, Section 2].*

//

3.4.3 Monodromy

Let $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ be the exact Lefschetz fibration discussed in the previous subsections, with $\mathrm{Crit}v = \{w_0, w_2, w_4\}$. Let $z_* \in \mathbb{C}$ be a basepoint, which fixes the smooth fibre $\pi^{-1}(z_*) \cong M$, which admits a symplectic completion to the plumbing $T^*S^3\#_{S^1}T^*S^3$. Choose a distinguished basis of vanishing paths $(\gamma_0, \gamma_2, \gamma_4)$ and associated vanishing thimbles $(\Delta_0, \Delta_2, \Delta_4)$ (so any two of them, Δ_j, Δ_k are disjoint and satisfy $h(\Delta_j) > h(\Delta_k)$ if $j > k$). Let $(V_0, V_2, V_4) \subset M$ be the basis of vanishing cycles associated to the three singular points, from which we define the Dehn twists $\tau_{V_0}, \tau_{V_2}, \tau_{V_4} \in \mathrm{Symp}_{ct}(M)$. The global monodromy of the fibration π is isotopic to $\tau_{V_0}\tau_{V_2}\tau_{V_4}$.

Lemma 3.4.9. *The global monodromy satisfies the generalised lantern relation*

$$\tau_{V_0}\tau_{V_2}\tau_{V_4} \simeq \tau_V \in \pi_0(\mathrm{Symp}_{ct}(M)) \tag{3.24}$$

where τ_V is the fibred twist in the unit normal bundle $V \to B$ to base locus $B = \mathbb{CP}^2\#3\overline{\mathbb{CP}^2} \subset \mathbb{CP}^2 \times \mathbb{CP}^2$.

Proof. This follows from Lemma 3.2.6, for the pencil defined in Section 3.4. □

Corollary 3.4.10. *The global monodromy commutes with every single twist τ_{V_i} for $i = 0, 2, 4$.*

Proof. Follows from Lemma 3.4.9. □

Recall that the monodromy can be made into a graded symplectomorphism, and that for the pencil we consider, the canonical bundle $\mathcal{K}_{\mathbb{CP}^2\times\mathbb{CP}^2} \cong \mathcal{O}_{\mathbb{CP}^2\times\mathbb{CP}^2}(-3,-3)$ satisfies Assumption (A) with $d = 6$:

$$\mathcal{K}^2_{\mathbb{CP}^2\times\mathbb{CP}^2} = \mathcal{L}^{\otimes -d} \tag{3.25}$$

for the line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{CP}^2\times\mathbb{CP}^2}(1,1)$ generating the pencil. So if $V_i \subset M$ is a graded vanishing cycle, we have

$$\phi(V_i) = V_i[-2].$$

3.5 Monotone Lagrangian submanifolds of $T^*\mathbb{CP}^2$

This final section is a short digression in which we include an application derived from the construction of Section 3.4 to discuss monotone Lagrangian submanifolds of $(T^*\mathbb{CP}^2, \lambda_{T^*\mathbb{CP}^2})$ diffeomorphic to $S^1 \times S^3$. We show that there are at least two distinct Lagrangian isotopy classes of such Lagrangians (3.5.7).

The Lagrangians we study in this section can be obtained by parallel transport of the vanishing cycles of the exact Lefschetz fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ defined in Section 3.4. The construction of such Lagrangians imitates that of two well known families of Lagrangians in T^*S^2; the Chekanov and Clifford tori. These tori can be defined in various ways (see below), but in particular they do admit a presentation as Lagrangian submanifolds of the Lefschetz fibration on $(T^*S^2, d\lambda_{T^*S^2})$, a fibration obtained from a Lefschetz pencil of conics on $(\mathbb{CP}^1\times\mathbb{CP}^1, \omega_{FS}\oplus\omega_{FS})$. These two tori are not Hamiltonian isotopic to each other, and the motivation behind this section is to disclose whether the Lagrangians $S^1 \times S^3 \subset T^*\mathbb{CP}^2$ obtained in the analogous way (in the total space of $E_{\mathbb{CP}^2} \to \mathbb{C}$) can be divided into two (or more) distinct Lagrangian/Hamiltonian isotopy classes.

The Chekanov torus is an important type of monotone torus originally studied in its Lagrangian embedding in $(\mathbb{C}^n, \omega_{std})$ ([Che96] and later ([CS]) in complex projective spaces $\mathbb{CP}^n$ and products of spheres $\times_n\mathbb{CP}^1$ (in particular in $(\mathbb{CP}^2, \omega_{FS})$ and $(\mathbb{CP}^1\times\mathbb{CP}^1, \omega_{FS}\oplus\omega_{FS})$). In all such examples, this kind of torus stands out as being not Hamiltonian isotopic to the Clifford torus, which is the “standard” product of circles $\times_n S^1(r)$.

Consider the decompositions of the Kähler manifolds

$$\mathbb{CP}^2 \cong D_\varepsilon T^*\mathbb{RP}^2 \cup C_\infty \tag{3.26}$$

$$\mathbb{CP}^1\times\mathbb{CP}^1 \cong D_\varepsilon T^*S^2 \cup \Delta \tag{3.27}$$

(see Section 2.2.5, the second is a special case of Example 2.2.19 that can also be found in [Bir01, Aud07] as referred above) where $C_\infty \subset \mathbb{CP}^2$ is the quadric at infinity and $\Delta \subset \mathbb{CP}^1\times\mathbb{CP}^1$

is the holomorphic diagonal. In [Gad13], Gadbled proved that the Chekanov tori in $(\mathbb{CP}^2, \omega_{FS})$ and $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{FS} \oplus \omega_{FS})$ can be constructed as circle bundles over C_∞ and Δ respectively (this was proved using the *circle bundle construction* of [Bir06]). Therefore, these tori must be preserved in the complement of the divisors in the decompositions above, giving rise to Lagrangian tori in the cotangent bundles $(T^*\mathbb{RP}^2, d\lambda_{T^*\mathbb{RP}^2})$ and $(T^*S^2, d\lambda_{T^*S^2})$.

The well-studied Lefschetz fibration on $(T^*S^2, d\lambda_{T^*S^2})$ with smooth fibre $(T^*S^1, d\lambda_{T^*S^1})$, has two critical fibres, one vanishing cycle V_0 (the zero section of the smooth fibre) and a matching sphere corresponding to the zero section. This fibration arises from a pencil of conics on $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{FS} \oplus \omega_{FS})$ (whose base locus is a pair of points), by removing the conic at infinity, the holomorphic diagonal $\mathbb{CP}^1 \cong \Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ (see [Bir01, Example 3.2.1] and [Aud07, Section 4.3]).

It is known that the Chekanov torus in T^*S^2 admits a presentation as Lagrangian submanifold of the total space of the Lefschetz fibration $T^*S^2 \to \mathbb{C}$ above (see [Aur07, 5.1]). In this picture, the torus emerges from flowing the vanishing cycle V_0 by parallel transport over a loop that does not encircle any of the critical values. Then, the Chekanov torus is fibred by V_0 over this loop.

Recall the decomposition

$$\mathbb{CP}^2 \times \mathbb{CP}^2 \cong D_\varepsilon T^*\mathbb{CP}^2 \cup \Sigma_\infty \tag{3.28}$$

where the divisor at infinity is the Flag 3-fold.

Question 1. Can we use 3.28 to get an interesting Lagrangian $S^1 \times S^3 \subset T^*\mathbb{CP}^2$, comparable (in its construction) to the Chekanov torus in T^*S^2?

We could attempt to answer Question 1 by applying Biran's circle bundle construction ([Bir06]) over a three-sphere contained in Σ_∞, to generalise the methods used for the Chekanov tori. However, there is no distinguished choice of sphere in the divisor Σ_∞, contrasting the Chekanov torus construction in which the circle bundle is taken over the equator in $S^2 \cong \Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. We discard this approach, and privilege another, which is to consider the products $S^1 \times S^3$ as Lagrangians in the total space of the Lefschetz fibration $\pi \colon E_{\mathbb{CP}^2} \to \mathbb{C}$.

3.5.1 Lagrangian in the total space of the standard fibration on $\mathbb{C}^4$

We begin by studying monotone Lagrangian submanifolds of $\mathbb{C}^4$ obtained by parallel transport in the total space of the "standard fibration"

$$\begin{aligned} q\colon \mathbb{C}^4 &\longrightarrow \mathbb{C} \\ z &\longmapsto z_1^2 + z_2^2 + z_3^2 + z_4^2 \end{aligned} \tag{3.29}$$

with smooth fibres $q^{-1}(z) = \{(x,y) \in \mathbb{C}^4,\ |x|^2 - |y|^2 = 1,\ \langle x,y \rangle = 1\} \cong T^*S^3$ for $z \neq 0$ and a singular fibre over 0, in which the zero section of T^*S^3 collapses to a point. The monodromy around a loop in the base $t \mapsto e^{2\pi i t}$ is then given by the Dehn twist along this vanishing cycle, i.e $\tau_{S^3} \in \mathrm{Symp}_{ct}(T^*S^3)$. For $z \in \mathbb{C}^*$, let $V_z \cong S^3 \subset q^{-1}(z)$ be the zero section of the fibre over z (a representative of the vanishing cycle).

We construct two (distinct) families of Lagrangian $S^1 \times S^3 \subset \mathbb{C}^4$ as follows. Let $\sigma\colon [0,1] \to \mathbb{C}$ be a loop in the base and let

$$T_\sigma := \bigcup_{z \in \mathrm{Im}(\sigma)} V_z \subset \mathbb{C}^4. \tag{3.30}$$

This is a Lagrangian submanifold of the total space that can be obtained by flowing the vanishing cycle by parallel transport over the loop σ. If $0 \in \mathrm{Im}(\sigma)$ then T_σ is an immersed four-sphere (with a nodal singularity at 0). If, on the other hand, $0 \notin \mathrm{Im}(\sigma)$, we call $T_\sigma \cong S^3 \times S^1$ of type 1 if σ encloses the origin, and of type 2 otherwise.

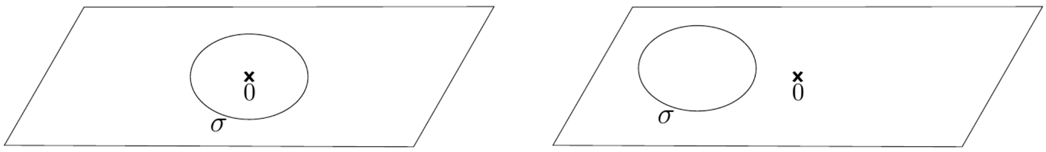

Figure 3.4: Type 1 (left) and Type 2 (right) Lagrangians.

For simplicity, consider the exact Lefschetz fibration $f\colon E \to \mathbb{C}$ obtained by cutting the the fibres into Liouville domains exact symplectomorphic to a disc cotangent bundles $D_rT^*S^3$, $r > 0$, and call the new fibration $f\colon E \to \mathbb{C}$ (see [Sei03, Section 1.2] for the precise description of the exact local model).

Lemma 3.5.1. *Let* $0 \notin \mathrm{Im}(\sigma)$. *The Lagrangian* T_σ *is monotone.*

Proof. Recall that a Lagrangian $L \subset (X,\omega)$ is monotone if the the area homomorphism $\omega\colon \pi_2(X,L) \to \mathbb{R}$ and the Maslov (index) homomorphism $\mu\colon \pi_2(X,L) \to \mathbb{Z}$ are proportional, i.e

$$\forall u \in \pi_2(X,L),\ \omega(u) = c\mu(u). \tag{3.31}$$

By the homotopy long exact sequence for pairs, we have $\pi_2(\mathbb{C}^4, T_\sigma) \cong \pi_1(T_\sigma) \cong \mathbb{Z}$, for $T_\sigma \cong S^3 \times S^1 \subset \mathbb{C}^4$. Then, the two homomorphisms are forced to be proportional. □

The characterisation of pseudoholomorphic discs we present below revolves around the use of a section-count invariant associated to Lefschetz fibrations as defined in [Sei03, Section 2.1]. Let (J, j) be a pair of almost complex structures on $\mathbb{C}^4$ and $\mathbb{C}$ compatible with π (as in Definition 2.2.6).

Let $D \subset \mathbb{C}$ be the disc in the base bounded by the loop σ, such that $\partial D = \mathrm{Im}(\sigma)$, and consider pseudoholomorphic sections of the restriction $f|_D$, the curves $s\colon D \to E$ satisfying

1. $\forall z \in D$: $f(s(z)) = z$,

2. $s(\partial D) = s(\sigma) \subset T_\sigma$ (i.e the section s has a boundary condition defined by T_σ),

3. $Ds(z) + J(s) \circ Ds(z) \circ j = 0$.

Let $\mathcal{M}_{E/D} := \{s\colon D \to E \text{ satisfies } (1),(2),(3)\}$ be the moduli space of (J, j)-holomorphic sections described above. A Fredholm analysis can be adapted to this situation to show that $\mathcal{M}_{E/D}$ is a smooth manifold (see [Sei03, Lemma 2.5], [Sei08a, p.237]). The choice of almost complex structures forces J-holomorphic discs in the total space to project to j-holomorphic discs in the base, so by the open mapping theorem, the moduli space $\mathcal{M}_{E/D}$ corresponds to the moduli space of J-holomorphic discs $u\colon (D,\partial D) \to (E, T_\sigma)$. Moreover, if $0 \in int(D)$, the moduli space $\mathcal{M}_{E/D}$ can be identified with the unit cotangent bundle ST^*S^3 (see [Sei03, p.1033]).

Lemma 3.5.2. *Assume the loop $\sigma\colon S^1 \to \mathbb{C}$ bounding $D \subset \mathbb{C}$ is such that $0 \in int(D)$. Then the Lagrangian T_σ has minimal Maslov index* 4.

Proof. Let N_{T_σ} be the minimal Maslov index associated to $T_\sigma \subset \mathbb{C}^4$ (see Definition 3.2.8). The moduli space $\mathcal{M}_{E/D}$ of J-holomorphic discs $s\colon (D,\partial D) \to (X, T_\sigma)$ with boundary in the homotopy class of the generating loop of $\pi_1(L)$ has expected (virtual) dimension $\dim(\mathcal{M}_{E/D}) = 4 + N_{T_\sigma} - 3$ (this is a standard result that derives from the study of the Fredholm operator involved in the equations defining pseudo-holomorphic discs, see for example [Oh93]). By the observation above, we know $\dim(\mathcal{M}_{E/D}) = \dim(ST^*S^3) = 5$, so that $N_{T_\sigma} = 4$. □

Lemma 3.5.3. *Assume the loop $\sigma\colon S^1 \to \mathbb{C}$ bounding $D \subset \mathbb{C}$ is chosen such that $0 \notin D$. Then the Lagrangian T_σ has minimal Maslov index* 2.

Proof. Since $0 \notin D$, the restriction $f|_D\colon E|_D \to D$ can be trivialised so that $E|_D = f^{-1}(D) \cong D_r T^*S^3 \times D$. The only non-trivial discs are generated by the base loop σ onto which T_σ projects to (there is no contribution from the smooth fibre since $T_\sigma|_z := T_\sigma \cap f^{-1}(z) \subset f^{-1}(z) \cong T^*S^3$ are both simply connected). Any disc generated this way has Maslov index 2.

□

Remark 3.5.4. *Lemma 3.5.2 can be somewhat surprising at first. Namely, one could be mislead by the two-dimensional situation and expect the index of those discs to be two. The minimal Maslov index of any non-trivial disc with boundary conditions on any Lagrangian torus in $(\mathbb{C}^2, \omega_{std})$ is two ([Vit90]), so in particular that is the case for the Clifford torus (which would be a type-one torus* *in the total space of the model Lefschetz fibration on* $\mathbb{C}^2$*). This is also the result obtained from the expected dimension of the moduli space of such discs, which is* $\dim(ST^*S^2) = 3$ *(so that the minimal Maslov index is indeed* $3-2+1=2$*). But this only holds in dimension two; it is incorrect to think that this would generalise to the discs of Lemma 3.5.2.* //

Corollary 3.5.5. *The two Lagrangians are distinguished by their Maslov index, hence they are neither Lagrangian isotopic, nor Hamiltonian isotopic.*

□

3.5.2 Monotone Lagrangian $S^3 \times S^1 \subset T^*\mathbb{CP}^2$

Let $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ the exact Lefschetz fibration (3.18), with critical values $\mathrm{Critv}(\pi) = \{w_0, w_2, w_4\}$ and vanishing cycles V_0, V_2, V_4. For $i = 0, 2, 4$, set $V_{z,i}$ to be the representative of V_i in the fibre $\pi^{-1}(z)$.

For each critical value $w_i \in \mathrm{Critv}(\pi)$ there is a pair of Lagrangians $S^3 \times S^1 \subset T^*\mathbb{CP}^2$ as follows.

Definition 3.5.6. For $i \in \{0,2,4\}$, fix $w_i \in \mathrm{Critv}(\pi)$. Let $\sigma\colon S^1 \to \mathbb{C}$ be a loop in the base of the fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$, bounding a disc D with $\partial D = \mathrm{Im}(\sigma)$, $\mathrm{Critv}(\pi) \cap \mathrm{Im}(\sigma) = \emptyset$ and $w_j \notin D$ if $j \neq i$.

There is a monotone Lagrangian

$$T_{\sigma,w_i} := \bigcup_{z \in \mathrm{Im}(\sigma)} V_{z,i} \subset E_{\mathbb{CP}^2}, \tag{3.32}$$

that we can view as Lagrangian of $(T^*\mathbb{CP}^2, d\lambda_{T^*\mathbb{CP}^2})$, obtained by flowing the vanishing cycle V_i under parallel transport around the loop σ.

We say T_{σ,w_i} is

1. of Type 1 if $w_i \in int(D)$,
2. of Type 2 if $w_i \notin D$.

♢

Lemma 3.5.7. *Fix* $w_i \in \mathrm{Critv}(\pi)$*. Let* $\sigma_1, \sigma_2\colon S^1 \to \mathbb{C}$ *be two loops as in Definition 3.5.6, and assume that the Lagrangian* T_{σ_1,w_i} *is of type* 1*, and* T_{σ_2,w_i} *is of type* 2*, both associated to the same critical value. Then the Lagrangians* T_{σ_1,w_i} *and* T_{σ_2,w_i} *are not Lagrangian isotopic.*

Proof. Consider the homotopy long exact sequence for the pair $(X,L) = (T^*\mathbb{CP}^2, S^3 \times S^1)$. Exactness of X and monotonicity of L imply that the Maslov homomorphism $\mu\colon \pi_2(X,L) \to \mathbb{Z}$ descends to a map $\mu\colon \pi_1(L) \to \mathbb{Z}$, which represents an element of $H^1(L;\mathbb{Z})$. Therefore, we can use the considerations of the previous section to prove the claim. □

Chapter 4

Model planar projective twists

4.1 Introduction

This chapter marks the beginning of the investigations on projective twists.

As showed in Chapter 2 (Section 2.2.5), projective twists can be identified with S^1-fibred twists, in a local model in which the coisotropic submanifolds (defining these twists) are given by the unit cotangent bundles $ST^*\mathbb{AP}^n$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. In this chapter, we use Picard–Lefschetz theory and the constructions of Chapter 3 to introduce an alternative local model for real and complex projective twists in dimension two.

Consider the exact Lefschetz fibrations $\pi \colon E_{\mathbb{AP}^2} \to \mathbb{C}$ of Sections 3.3 ($\mathbb{A} = \mathbb{R}$) and 3.4 ($\mathbb{A} = \mathbb{C}$) respectively. The generalised lantern relations (3.12) and (3.24) force the total monodromy of these fibrations to preserve every vanishing cycle. We use this property, together with a construction from [Sei15], to build compactly supported symplectomorphisms $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{AP}^2)$. The construction involves lifting a Dehn twist along an annulus in the base $\mathbb{C}$ to a symplectomorphism of the total space, that we adjust to a compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{AP}^2)$.

We then measure the Floer theoretical action of φ on a Lefschetz thimble by computing the Floer cohomology groups $\mathrm{HF}(\varphi^k(\Delta_\alpha), \Delta_\beta; \mathbb{Z}/2\mathbb{Z})$ for elements $\Delta_\alpha, \Delta_\beta$ in a distinguished basis of Lefschetz thimbles. In the real case, we can combine these computations with the knowledge of the mapping class group $\pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{RP}^2))$ (computed in [Eva11]) to obtain:

Theorem 7. *The symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{RP}^2)$ is isotopic to a power of the projective twist $\tau^k_{\mathbb{RP}^2}$, $k \in \mathbb{Z}^*$.*

However, in the complex case, very little is known about $\pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{CP}^2))$. By the results of Chapter 8, we know that for $n \geqslant 19$, there are cases in which a non-standard framing of the projective twist produces a symplectomorphism that is not Hamiltonian isotopic to the standard $\tau_{\mathbb{CP}^n} \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$, but that doesn't say anything about $n = 2$. As a consequence, we only obtain a partial result that is based on our Floer cohomological computations.

Theorem 8. *The symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$ is of (symplectic) infinite order.*

We can only conjecture that φ is isotopic to the complex planar projective twist. However, there is strong evidence that our constructions correspond to the projective twists. Namely, for the two examples, the current literature seems to confirm these symplectomorphisms are indeed projective twists (this is shown in Sections 4.4.2 and 4.5.2).

4.2 A compactly supported symplectomorphism on the total space of a Lefschetz fibration

This section outlines the construction (after [Sei15]) of a compactly supported symplectomorphism on the total space of a class of Lefschetz fibrations (satisfying the Assumption (B) below). This is the general recipe that we use in building the models of the real and complex projective twists (in Sections 4.4, 4.5 respectively) starting from the exact Lefschetz fibrations $\pi\colon E_{\mathbb{AP}^2} \to \mathbb{C}$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$.

Let $\pi\colon E^{2n+2} \to \mathbb{C}$ be an exact Lefschetz fibration with smooth fibre a Liouville domain (M^{2n}, ω), and denote the set of critical values by $\mathrm{Critv}(\pi) = \{w_0, \dots, w_r\} \subset D_R$, where $D_R \subset \mathbb{C}$ is a disc of radius $R > 0$. Fix a base point $z_* \in \mathbb{C}$, with $\mathrm{Re}(z_*) \gg R$, and use it to fix a representative of the symplectomorphism class of the smooth fibre $\pi^{-1}(z_*) \cong M$.

Fix a basis of vanishing paths $(\gamma_0, \dots, \gamma_r)$, the corresponding set of framed Lagrangian vanishing cycles $(V_0, \dots, V_r)$ in M (following the conventions of Section 2.2.1) and denote by $\phi \in \mathrm{Symp}_{ct}(M)$ the total monodromy.

Assumption B. There is a symplectomorphism $\phi' \in \mathrm{Symp}_{ct}(M)$ with the following property. There is a symplectic isotopy $\phi \simeq \phi'$ such that

$$\forall i = 0, \dots, r, \;\; \phi'(V_i) = V_i \;\text{ and } \phi'|_{V_i} \text{ is homotopic to an element of } \mathrm{O}(n+1). \tag{4.1}$$

To simplify the notation, in what follows the perturbed map will also be denoted by ϕ.

Note that the second condition in (4.1) ensures that for all $i = 0, \dots, r$, the homotopy class of the framing of the vanishing cycle V_i and that of the framing of its image under ϕ coincide.

Lemma 4.2.1. *Under Assumption* (B), $\phi \in \mathrm{Symp}_{ct}(M)$ *can be extended to an element of* $\mathrm{Symp}(E)$ *which acts fibrewise as* ϕ.

Proof. For $i = 0, \ldots, r$, let $f_i \colon S^n \to V_i$ be the framings of the vanishing cycles of the Lefschetz fibration $\pi \colon E \to \mathbb{C}$. Then the data $(M, \phi \circ f \colon S^n \to V_i)$ define a new Lefschetz fibration $\pi' \colon E' \to \mathbb{C}$, which is obtained by pulling back all data in the construction of E (as in [Sei08a, (16e)]) by ϕ. This construction yields a symplectomorphism $q \colon E \to E'$. Since the framings of the vanishing cycles of E and E' are in the same homotopy class, there is a 1-parameter family of exact Lefschetz fibrations $(E_t, \Omega_{E_t})_{t \in [0,1]}$ that interpolates between E $(t = 0)$ and E' $(t = 1)$ ([Sei08a, (16e)]). By exactness, Moser's argument yields another symplectomorphism $\tilde{q} \colon E \to E'$. The map $\tilde{q}^{-1} \circ q \colon E \to E$ is then a symplectomorphism of the total space which by construction acts (up to isotopy) fibrewise as ϕ. □

Lemma 4.2.1 applies to the total monodromy of E but also to its inverse, $\phi^{-1} \in \mathrm{Symp}_{ct}(M)$.

Let $D_{R-2\varepsilon} \subset D_{R-\varepsilon} \subset \mathbb{C}$ be two discs centered at the origin, of radii $R - 2\varepsilon$ and $R - \varepsilon$ respectively, such that $\mathrm{Critv}(\pi) \subset D_{R-2\varepsilon}$. Consider an anticlockwise rotational vector field on the base, supported on the annulus $A_{2\varepsilon} := D_R \setminus D_{R-2\varepsilon}$, defined as follows. Let $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that

$$\begin{array}{ll} \psi(r) = 0 & r \leqslant R - 2\varepsilon \\ \psi'(r) = 1 & r > R - \varepsilon. \end{array} \tag{4.2}$$

Define a Hamiltonian function $H \colon \mathbb{C} \to [0, \infty)$, $H(z) = \psi(|z|)$ and let (b_t) be its associated Hamiltonian flow. The 2π-flow $(b_{2\pi})$ defines a Dehn twist in $A_{2\varepsilon}$, which acts as the identity on $D_{R-2\varepsilon}$, as an anticlockwise 2π rotation on $\mathbb{C} \setminus int(D_R)$ and interpolates between the two on $A_{2\varepsilon}$.

For every $t \in [0, 2\pi]$, (b_t) can be lifted via parallel transport to a family of symplectomorphisms (Φ_t) of the total space (see also [May09, Observation 6.4]). The element $\Phi_{2\pi}$ covers the base twist $b_{2\pi}$, and therefore is fibre preserving over $\mathbb{C} \setminus int(A_{2\varepsilon})$, namely over $D_{R-2\varepsilon}$ and over $\mathbb{C} \setminus int(D_R)$. In particular:

- For $z \in D_{R-2\varepsilon}$ the map acts fibrewise as $\Phi_{2\pi}|_{\pi^{-1}(z)} = Id$,
- By the Picard–Lefschetz theorem, $\Phi_{2\pi}|_{\pi^{-1}(z_*)} \simeq \phi$ in $\mathrm{Symp}_{ct}(M)$, since $z_* \in \mathbb{C} \setminus int(D_R)$.

By Lemma 4.2.1, $\phi^{-1} \in \mathrm{Symp}_{ct}(M)$ can be extended to a symplectomorphism of the total space $\tilde{\phi}^{-1} \in \mathrm{Symp}(E)$. Let $\tilde{\varphi} := \tilde{\phi}^{-1} \circ \Phi_{2\pi}$. This is a symplectomorphism $\tilde{\varphi} \in \mathrm{Symp}(E)$ which defines another lift of the base twist $b_{2\pi}$, but is supported over the compact region D_R. So $\tilde{\varphi}$ is compactly supported in the horizontal direction, however not so in the vertical direction. Namely, $\tilde{\varphi}$ acts non-trivially on $\pi^{-1}(A_{2\varepsilon}) \cap \partial^h E$.

In what follows we adjust the map by finding an isotopy $\tilde{\varphi} \simeq \varphi$ to a compactly supported symplectomorphim $\varphi \in \mathrm{Symp}_{ct}(E)$.

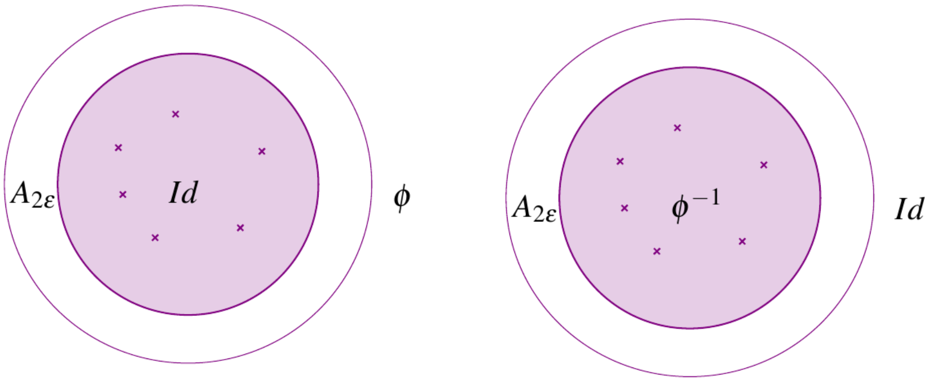

Figure 4.1: The fibrewise action of $\Phi_{2\pi}$ (left) and that of $\widetilde{\varphi}$ (right) over the disc $D_{R-2\varepsilon}$ and over the complement $\mathbb{C} \setminus D_R$.

The horizontal boundary $\partial^h E$ can be trivialised as in (2.8): an open neighbourhoood of $\partial^h E$ is isomorphic to $U^\partial \cong \mathbb{C} \times M^{out} \subset \mathbb{C} \times M$, where $M^{out} \subset M$ is an open neighbourhood of ∂M.

The above trivialisation induces a decomposition $\widetilde{\varphi}|_{U^\partial} \cong \tau_{\mathbb{C}} \times Id_M$, where $\tau_{\mathbb{C}}$ is the Dehn twist in a circle in $\mathbb{C}$. The latter admits a symplectic isotopy with the identity, $(\alpha_t)_{t\in[0,1]}$, $\alpha_0 = \tau_{\mathbb{C}}, \alpha_1 = Id_{\mathbb{C}}$. Since the symplectic form is product-like in the region U^h, there is a symplectic isotopy $(\alpha_t \times Id)_{t\in[0,1]}$ connecting $\tau_{\mathbb{C}} \times Id_M$ to $Id_{\mathbb{C}} \times Id_M$, supported in U^∂ and fixing ∂U^∂. This isotopy does not interfere with the support of $\tilde{\varphi}$ in the compact region, so after applying the isotopy, we obtain a compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(E)$ whose support is $U^c := D_R \times (M \setminus M^{out})$, and $\varphi|_{U^c} = \tilde{\varphi}|_{U^c}$.

4.3 Floer cohomology computations

For a given Lefschetz fibration $\pi \colon E \to \mathbb{C}$ admitting a compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(E)$ as in the previous section, the question arises on whether φ is a non-trivial element in the mapping class group of E, and if so, how to study its properties.

One possibility is to compute the Floer theoretical action of φ. This section focuses on the computation of Floer cohomology groups $\mathrm{HF}^*(\varphi^k(\Delta_\alpha), \Delta_\beta; \mathbb{Z}/2\mathbb{Z})$, where $\Delta_\alpha, \Delta_\beta \subset E$ are (disjoint) Lefschetz thimbles of π. The image $\varphi^k(\Delta_\alpha)$ is another thimble, which intersects Δ_β over k regular values $z_i \in \mathbb{C} \setminus \mathrm{Critv}(\pi)$, $i = 0, \dots, k-1$, and the components of the intersection are given by $\varphi^k(\Delta_\alpha) \cap \Delta_\beta = \bigcup_{i=0,\dots,k-1} \{V_{z_i,\alpha} \cap V_{z_i,\beta}\}$, where $V_{z_i,\alpha} := \varphi^k(\Delta_\alpha) \cap \pi^{-1}(z_i)$, $V_{z_i,\beta} := \Delta_\beta \cap \pi^{-1}(z_i)$.

The main computation tool is a spectral sequence adapted from [MS10] (Section 4.3.3), whose first page is given by $\bigoplus_{i=0}^{k-1} \mathrm{HF}^*(V_{z_i,\alpha}, V_{z_i,\beta}; \mathbb{Z}/2\mathbb{Z})$, and which converges to $\mathrm{HF}(\varphi^k(\Delta_\alpha), \Delta_\beta; \mathbb{Z}/2\mathbb{Z})$.

The computations take place in Sections 4.3.2, 4.3.3 and 4.3.4. Before that, we set up, in Section

4.3.1, the conventions for ungraded Floer cohomology of Lagrangians in the total space and in the smooth fibre of a Lefschetz fibration.

4.3.1 Floer cohomology conventions

In what follows the ground field for all the Floer cohomology groups will always be assumed to be $\mathbb{Z}/2\mathbb{Z}$; this avoids the use of spin structures.

Let $\pi\colon E \to \mathbb{C}$ be an exact Lefschetz fibration. Let $\mathcal{J}(E,\pi,j)$ be the set of almost complex structures compatible with π in the sense of Definition 2.2.6. Such a choice of almost complex structure is not generic, but one can always choose a generic element $J_E \in \mathcal{J}(E,\pi,j)$ (as in [Sei03, Section 2.1]) making the elements of the moduli spaces below regular. Fix such a $J_E \in \mathcal{J}(E,\pi,j)$.

Let $\mathcal{L}_0,\mathcal{L}_1 \subset E$ be two Lagrangian thimbles with $h(\mathcal{L}_0) > h(\mathcal{L}_1)$, and define $\mathscr{C}(\mathcal{L}_0,\mathcal{L}_1)$ to be the set of intersection points of the pair $(\mathcal{L}_0,\mathcal{L}_1)$ after a suitable Hamiltonian perturbation to make the intersection transverse.

Given $\xi_\pm \in \mathscr{C}(\mathcal{L}_0,\mathcal{L}_1)$, let $u\colon \mathbb{R}\times[0,1] \to E$ be a solution to the (possibly perturbed) Floer equation with the properties

$$\begin{gathered} \partial_s u(s,t) + J_E(u)\partial_t u(s,t) = 0, \\ \forall s \in \mathbb{R}:\ \ u(s,0) \in \mathcal{L}_0,\ u(s,1) \in \mathcal{L}_1 \\ \lim_{s\to-\infty} u(s,t) = \xi_-,\ \lim_{s\to+\infty} u(s,t) = \xi_+. \end{gathered} \tag{4.3}$$

Let $\mathcal{M}(\xi_-,\xi_+,[u];J_E)$ be the moduli space of unparametrised J_E-holomorphic curves in the class $[u]$, satisfying (4.3). Then $\mathcal{M}(\xi_-,\xi_+,[u];J_E)$ is a smooth manifold of dimension $\mu([u])-1$ ([Oh93]).

The Floer complex $\mathrm{CF}(\mathcal{L}_0,\mathcal{L}_1)$ is generated, as a $\mathbb{Z}/2\mathbb{Z}$-vector space, by the elements of $\mathscr{C}(\mathcal{L}_0,\mathcal{L}_1)$, and the Floer differential of an element $\xi_+ \in \mathscr{C}(\mathcal{L}_0,\mathcal{L}_1)$ is given by

$$\partial\xi_+ = \sum_{\substack{\xi_-\in\mathscr{C}(\mathcal{L}_0,\mathcal{L}_1)\\ \mu([u])=1}} (\sharp_{\mathbb{Z}/2\mathbb{Z}}\mathcal{M}(\xi_-,\xi_+,[u];J_E))\langle\xi_-\rangle. \tag{4.4}$$

The moduli spaces of curves $\mathcal{M}(\xi_-,\xi_+,[u];J_E)$ are compact (we explain this below), so that the above expression is well-defined.

We show that the elements of these moduli spaces satisfy compactness properties in both "vertical" and "horizontal" directions (see also [May09, Section 6]).

For compactness in the fibre (vertical) direction, let $U^\partial \subset E$ be an open neighbourhoood of $\partial^h E$. As before, there is an isomorphism $U^\partial \cong \mathbb{C}\times M^{out} \subset \mathbb{C}\times M$ for an open neighbourhood $M^{out} \subset M$

of ∂M, under which both the Liouville form and the almost complex structure split in a product-like fashion. Assume a pseudoholomorphic curve u as in (4.3) was to enter the neighbourhood $\partial^h U$. Then the projection of u to the second factor would be a pseudoholomorphic curve in M, with interior points in M^{out} and boundary conditions in the compact part away from M^{out}, a contradiction to the maximum principle.

On the other hand, since π is $(J_E, j_{\mathbb{C}})$-holomorphic, the projection of a non-trivial J_E-holomorphic curve u to the base $\mathbb{C}$ is a holomorphic strip with boundary condition on $(\gamma_0(\mathbb{R}^+), \gamma_1(\mathbb{R}^+))$, and by the maximum principle the strip cannot escape a compact neighbourhood in $\mathbb{C}$ containing the intersection points of the paths. Therefore, u cannot go arbitrarily far in the horizontal direction. .

By exactness, the action functional only depends on the endpoints $\xi_\pm$ and therefore gives a common upper bound for the energy of the curves in these moduli spaces. So by the Gromov compactness theorem (for a statement of this theorem see [MS94]) together with $\dim(\mathcal{M}(\xi_-, \xi_+, [u]; J_E)) = 0$, the only remaining issue are bubble phenomena. But these cannot occur by exactness.

Now consider two Lagrangian thimbles $\mathcal{L}_0, \mathcal{L}_1 \subset E$ with associated vanishing paths $\gamma_0, \gamma_1 : \mathbb{R}^+ \to \mathbb{C}$ and assume $h(\mathcal{L}_0) \leqslant h(\mathcal{L}_1)$. For a point in the base $z \in \mathbb{C}$, let $x := \mathrm{Re}(z)$, $y := \mathrm{Im}(z)$.

Definition 4.3.1. For $\varepsilon > h(\mathcal{L}_1) - h(\mathcal{L}_0)$, define a map $H_\varepsilon \in C^\infty(\mathbb{C}, \mathbb{R})$ satisfying

$$\begin{array}{ll} H_\varepsilon(x) = 0 & x < c \\ H'_\varepsilon(x) = \varepsilon & x > c + K. \end{array} \tag{4.5}$$

for $c > R$ and $K > 0$. Define the Hamiltonian vector field $Y_\varepsilon = H'_\varepsilon(x)\partial_y$, call its time-1 flow χ_ε and let $\widetilde{\chi_\varepsilon}$ be a lift of χ_ε to the total space. If $\mathcal{L}_0$ is a Lefschetz thimble, then $\widetilde{\chi_\varepsilon}(\mathcal{L}_0)$ is a another Lefschetz thimble isotopic to $\mathcal{L}_0$ and $\pi(\widetilde{\chi_\varepsilon}(\mathcal{L}_0)) = \chi_\varepsilon(\gamma_0)$. ♢

In the case $h(\mathcal{L}_0) \leqslant h(\mathcal{L}_1)$, the Floer complex $\mathrm{CF}(\mathcal{L}_0, \mathcal{L}_1)$ is defined as above, but considering the modified pair $(\chi_\varepsilon(\mathcal{L}_0), \mathcal{L}_1)$.

For Lagrangian vanishing cycles, the situation is fairly standard. For a pair $(V_0, V_1) \subset (M, \omega)$ of closed exact Lagrangian submanifolds of the fibre, the Floer cohomology groups $\mathrm{HF}(V_0, V_1; \mathbb{Z}/2\mathbb{Z})$ are well defined (see for example [Sei08a, 8,9]).

4.3.2 The Floer complex $\mathrm{CF}(\varphi^k(\Delta_a), \Delta_c)$

Let $\pi\colon E \to \mathbb{C}$ be an exact Lefschetz fibration with smooth fibre a Liouville domain $(M, d\lambda)$, $r+1$ critical fibres and associated critical values $\mathrm{Crit}v = \{w_0, \dots, w_r\} \subset \mathbb{C}$. Let $z_* \in \mathbb{C}$ be a base-point with $\mathrm{Re}(z_*) \gg 0$, fixing the smooth fibre $M := \pi^{-1}(z_*)$. Fix a distinguished basis of vanishing paths $(\gamma_0, \dots, \gamma_r)$ for the critical values, and the associated basis of Lagrangian vanishing thimbles

$(\Delta_0, \ldots, \Delta_r)$. Call $(V_0, \ldots, V_r) \subset M$ the resulting basis of vanishing cycles in the fixed fibre. Choose two elements of the basis $\Delta_\alpha, \Delta_\beta$, $\alpha, \beta \in \{0, \ldots, r\}$ with the property that $h(\Delta_\alpha) > h(\Delta_\beta)$.

For the entire section, assume that $\phi := \tau_{V_0} \cdots \tau_{V_r}$ satisfies Assumption (B), which ensures the existence of a symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(E)$ as constructed in Section 4.2. Note that we don't use Assumption (A) until Section 4.3.4.

Let $R > 0$, and $D_{R-2\varepsilon} \subset D_R \subset \mathbb{C}$ two discs in the base (of radii $R - 2\varepsilon$ and R respectively, for $\varepsilon > 0$) containing the critical values. Let $b_{2\pi}$ be a Dehn twist in the annulus $A_{2\varepsilon} = D_R \setminus D_{R-2\varepsilon}$ as defined in Section 4.2. Note that the image $\varphi(\Delta_\gamma)$ of the vanishing thimble associated to a vanishing path γ is Lagrangian isotopic to the vanishing thimble $\Delta_{b_{2\pi}(\gamma)}$ associated to the twisted path $b_{2\pi}(\gamma)$.

For $k \in \mathbb{Z}$, set $\gamma_\alpha^k := b_{2\pi}^k(\gamma_\alpha)$ and $\Delta_\alpha^k := \Delta_{\gamma_\alpha^k} = \varphi^k(\Delta_\alpha)$.

Call the set of intersection points $I := \{\gamma_\alpha^k \cap \gamma_\beta\} = \{z_0, \ldots, z_{k-1}\} \subset A_{2\varepsilon}$, where z_0 is the innermost and z_{k-1} the outermost intersection point.

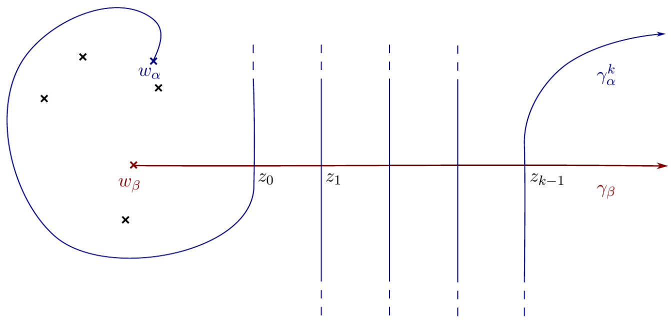

Figure 4.2: The intersection pattern of the pair of paths $(\gamma_\alpha^k, \gamma_\beta)$

Each component of the intersection locus $\{\Delta_\alpha^k \cap \Delta_\beta\}$ lying over $z_j \in I$ is determined by the intersection of the pair $(V_{z_j,\alpha}, V_{z_j,\beta}) \subset E_{z_j} := \pi^{-1}(z_j)$, where $V_{z_j,\alpha} := E_{z_j} \cap \Delta_\alpha^k$, $V_{z_j,\beta} := E_{z_j} \cap \Delta_\beta$.

Moreover, by the discussion of Section 3.2.2, the global monodromy $\phi \in \mathrm{Symp}_{ct}(M)$ of $\pi\colon E \to \mathbb{C}$ preserves the vanishing cycles up to a shift in their gradings. As a consequence, the intersection $\Delta_\alpha^k \cap \Delta_\beta \cap E_{z_j}$ is the same at any $z_j \in I$. Therefore, we can apply identical (local, compactly supported) Hamiltonian perturbations in each fibre E_{z_j} in order to turn all intersections into a configuration of points $\{\xi_{z_j,1}, \ldots, \xi_{z_j,\ell}\}$ for a fixed $\ell \in \mathbb{N}$, and $j \in \{0, \ldots, k-1\}$. Fix a generic element $J_E \in \mathcal{J}(E, \pi, j)$ as in Section 4.3.1. For all $j \in \{0, \ldots, k-1\}$, the (ungraded) Floer cohomologies $\mathrm{HF}(\Delta_\alpha^k, \Delta_\beta; \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{HF}(V_{z_j,\alpha}, V_{z_j,\beta}; \mathbb{Z}/2\mathbb{Z})$ are well defined by the previous section (4.3.1).

Recall that Δ_α and Δ_β are disjoint and $h(\Delta_\alpha) > h(\Delta_\beta)$ so $\mathrm{HF}(\Delta_\alpha^0, \Delta_\beta; \mathbb{Z}/2\mathbb{Z}) = \mathrm{HF}(\Delta_\alpha, \Delta_\beta; \mathbb{Z}/2\mathbb{Z}) = 0$ (by the conventions of Section 4.3.1).

Lemma 4.3.2. *The first power of φ satisfies*

$$\mathrm{HF}(\Delta_\alpha^1,\Delta_\beta) = \mathrm{HF}(\varphi(\Delta_\alpha),\Delta_\beta) \cong \mathrm{HF}(V_{z_0,\alpha},V_{z_0,\beta}). \tag{4.6}$$

Proof. If we consider the first power of the map φ, the vanishing paths associated to the two thimbles intersect in a single point z_0. Then there can be no "horizontal" Floer differential. Namely, all generators are contained in $\pi^{-1}(z_0)$, so any pseudoholomorphic curve $u \in \mathcal{M}(\Delta_\alpha^1,\Delta_\beta)$ connecting these generators must be confined in that fibre, since π is $(J_E, j_{\mathbb{C}})$-holomorphic and the open mapping theorem applies to the image $\pi(u)$. Therefore, we have $\mathrm{HF}(\varphi(\Delta_\alpha),\Delta_\beta) \cong \mathrm{HF}(V_{z_0,\alpha},V_{z_0,\beta})$. $\square$

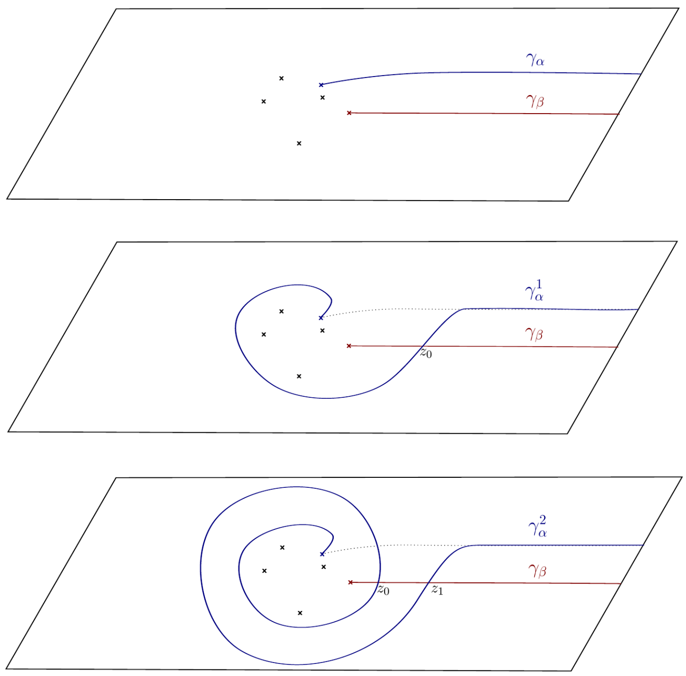

Figure 4.3: Top: the initial vanishing paths $(\gamma_\alpha,\gamma_\beta)$, middle: the first power $(\gamma_\alpha^1,\gamma_\beta)$, bottom: the second power $(\gamma_\alpha^2,\gamma_\beta)$.

To determine the groups $\mathrm{HF}(\varphi^k(\Delta_\alpha),\Delta_\beta)$ for $k > 1$, we will need to introduce additional computational tools. In the next section, we opt for a spectral sequence that was expressly developed to compute the Floer cohomology of Lefschetz thimbles in [MS10].

4.3.3 A spectral sequence

In this section we complete the computation of the Floer cohomology groups $\mathrm{HF}^*(\varphi^k(\Delta_\alpha),\Delta_\beta;\mathbb{Z}/2\mathbb{Z})$ of the Lefschetz thimbles $\varphi^k(\Delta_\alpha)$ and Δ_β with the help of a spectral sequence orginally defined in [MS10].

Let $\pi\colon E \to \mathbb{C}$ be the Lefschetz fibration we have considered in the previous subsection, and assign gradings for the the Lefschetz thimbles and the vanishing cycles.

Consider the holomorphic strips in the base formed by the pair of arcs $(\gamma_\alpha^k, \gamma_\beta)$, and connecting the intersection points of $I = \{\gamma_\alpha^k \cap \gamma_\beta\} = \{z_0, \dots, z_{k-1}\}$. We assign indices to these points as follows. Fix $\mathrm{ind}(z_0) = 0$ and for $j = 1, \dots, k-1$, set

$$\mathrm{ind}(z_j) = j \cdot (-2). \tag{4.7}$$

For any $z \in I$, $\mathrm{ind}(z)$ is an even negative value, $-2(k-1) \leqslant \mathrm{ind}(z) \leqslant 0$. Geometrically, $\mathrm{ind}(z_j)$ corresponds to the Maslov index of the holomorphic strip with endpoints z_{j-1}, z_j and boundary condition on $(\gamma_\alpha^k, \gamma_\beta)$.

We define an order on the set I as follows: $z_-, z_+ \in I$ satisfy $z_+ > z_-$ if there is a sequence $v_1, \dots, v_m \colon \mathbb{R} \times [0,1] \to \mathbb{C}$ of holomorphic curves with bounded energy, of the form

$$\begin{aligned} v_i(\mathbb{R} \times \{0\}) \subset \gamma_\alpha^k(\mathbb{R}^+),\ v_i(\mathbb{R} \times \{1\}) \subset \gamma_\beta(\mathbb{R}^+), \\ \lim_{s \to +\infty} v_i(s, \cdot) = \lim_{s \to -\infty} v_{i+1}(s, \cdot), \\ \lim_{s \to -\infty} v_1(s, \cdot) = z_-,\ \lim_{s \to +\infty} v_m(s, \cdot) = z_+, \end{aligned}$$

By the open mapping theorem, $z_0 > \dots > z_{k-1}$ and note that for $z_-, z_+ \in I$, $\mathrm{ind}(z_+) < \mathrm{ind}(z_-) \iff z_+ > z_-$.

Proposition 4.3.3. *There is a cohomological spectral sequence with bigraded differentials* $d_c\colon E_c^{p,q} \to E_c^{p+c, q-c+1}$*, converging to* $HF^*(\Delta_\alpha^k, \Delta_\beta; \mathbb{Z}/2\mathbb{Z})$*. The starting page is generated by*

$$E_1^{pq} = \begin{cases} \mathrm{HF}^{p+q}(V_{z_{(-p)},\alpha}, V_{z_{(-p)},\beta}; \mathbb{Z}/2\mathbb{Z}) & -(k-1) \leqslant p \leqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Proof. From now onwards, we will omit the coefficient field $\mathbb{Z}/2\mathbb{Z}$ in the notation. The spectral sequence is obtained as a special instance of [MS10, Proposition 4.1], whose proof is adapted below. Let $(J_t)_{t\in[0,1]}$ be a family of almost complex structures on E such that for each $t \in [0,1]$, the tuple (J_t, j) is compatible with the fibration π in the sense of Definition 2.2.6. Then, for a pair of intersection points $(\xi_-, \xi_+) \in \Delta_\alpha^k \cap \Delta_\beta$ we can define a relation $\xi_+ > \xi_-$ if there is a sequence $u_1, \dots, u_m \colon \mathbb{R} \times [0,1] \to E$ of pseudoholomorphic curves of bounded energy such that for all

$i = 1, \dots, m$

$$\partial_s u_i(s,t) + J_t \partial_t u_i(s,t) = 0,$$
$$u_i(\mathbb{R} \times \{0\}) \subset \Delta_\alpha^k,\ u_i(\mathbb{R} \times \{1\}) \subset \Delta_\beta,$$
$$\lim_{s\to+\infty} u_i(s,\cdot) = \lim_{s\to-\infty} u_{i+1}(s,\cdot),$$
$$\lim_{s\to-\infty} u_1(s,\cdot) = \xi_-,\ \lim_{s\to+\infty} u_m(s,\cdot) = \xi_+.$$

If $\xi_+ > \xi_-$, then $\xi_\pm$ must either lie in the same fibre or their projection satisfy $\pi(\xi_+) > \pi(\xi_-)$ in the ordering defined for intersection points on the base. This means that $\xi_+ > \xi_-$ implies $\operatorname{ind}(\pi(\xi_+)) \leqslant \operatorname{ind}(\pi(\xi_-))$.

As noted in Section 2.2.1, the choice of almost complex structures is not generic, but by applying a small perturbation to the family $(J_t)_{t\in[0,1]}$, these curves meet the usual regularity and compactness requirements.

Let $I_p := \{z \in I,\ \operatorname{ind}(z) \geqslant 2p\} \subset I$, so that I_p is non-empty for $-(k-1) \leqslant p \leqslant 0$. The complex $\mathrm{CF}(\Delta_\alpha^k, \Delta_\beta)$ admits a filtration F^*, where each term F^p is generated by the intersection points in $\pi^{-1}(I_p)$. By definition of the order, the Floer differential preserves F^p so that there is an induced differential on F^p/F^{p+1}. The latter is therefore a cochain complex, generated by the intersection points in $\pi^{-1}(z)$, for the unique element $z = z_{-p} \in I_p \setminus I_{p+1}$ This gives rise to a spectral sequence, whose first page is given by the cohomology $H^*(F^p/F^{p+1}) = \mathrm{HF}(V_{z,\alpha}, V_{z,\beta})$. The last equality holds because regularity for J_t-holomorphic strips in the total space is equivalent to regularity of $J_t|_{\pi^{-1}(z)}$-holomorphic strip in the fibre obtained by restriction (see the original proof of [MS10] for details). □

4.3.4 The special case of a Lefschetz fibration obtained by a Lefschetz pencil

The spectral sequence above becomes computationally relevant when there is a concrete understanding of the grading shift that the monodromy causes on vanishing cycles. This is the case for Lefschetz fibrations that are induced by Lefschetz pencils satisfying Assumption (A); we now restrict to this specific situation to obtain a more enlightening version of (4.8).

We have seen in Lemma 3.2.6 that in the case of a Lefschetz fibration $\pi\colon E \to \mathbb{C}$ induced by a Lefschetz pencil, the total monodromy is isotopic to a symplectomorphism (a fibred twist) whose restriction to the vanishing cycles is the identity (because the fibred twist is supported away from the cycles). In particular, (a stronger version of) Assumption (B) holds, and one can build a symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(E)$ as in Section 4.2. When moreover we assume the pencil satisfies Assumption (A), we can observe interesting patterns in the gradings of vanishing cycles and their images under iterations of φ.

We keep all the notation as in the previous subsection, and again assume the Lagrangians Δ_α^k, Δ_β have been perturbed by a Hamiltonian isotopy to make their intersection transverse (a set of ℓ points) over each $z_j \in I$, $j \in \{0,\dots,k-1\}$. This section studies the grading of intersection points in $\{\xi_{j,1},\dots,\xi_{j,\ell}\}$ obtained after the perturbations applied to $\{\Delta_\alpha^k \cap \Delta_\beta\}$.

For $j \in \{0,\dots,k-1\}$ and $i \in \{1,\dots,\ell\}$, choose $\xi_{z_j,i} \in \Delta_\alpha^k \cap \Delta_\beta \cap E_{z_j}$ and let $\xi_{z_{j+1},i} \in \Delta_\alpha^k \cap \Delta_\beta \cap E_{z_{j+1}}$, the corresponding point in the next fibre.

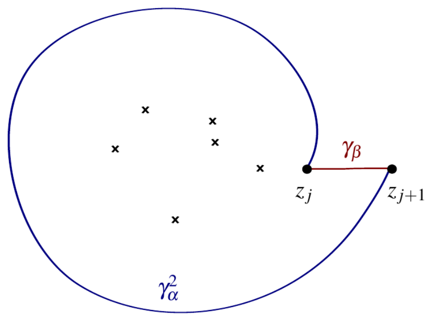

Figure 4.4: The projection in $\mathbb{C}$ of a pseudoholomorphic strip connecting ξ_j to ξ_{j+1}.

Lemma 4.3.4. *The degree shift is given by*

$$s_\varphi := \deg(\xi_{z_{j+1},i}) - \deg(\xi_{z_j,i}) = d-2. \tag{4.9}$$

Proof. We compute the index of a pseudoholomorphic strip $u\colon \mathbb{R}\times[0,1] \to E$ with boundary conditions

$$\forall s \in \mathbb{R}:\ u(s,0) \in \Delta_\alpha^k,\ u(s,1) \in \Delta_\beta,$$
$$\lim_{s\to-\infty} u(s,t) = \xi_{z_j,i},\ \lim_{s\to+\infty} u(s,t) = \xi_{z_{j+1},i}.$$

This will give the shift in grading between a fibre and the next, s_φ, which is independent of the homotopy class of u.

After fixing a trivialisation of $u^*|_{TM}$, the maps

$$\delta_\alpha := u^*|_{\mathbb{R}\times\{0\}} T\Delta_\alpha^k,\ \delta_\beta := u^*|_{\mathbb{R}\times\{1\}} T\Delta_\beta$$

can be viewed as paths $\delta_{\alpha,\beta}(s)\colon [-\infty,+\infty] \to LGr(n)$ connecting $T_{\xi_{z_j,i}}\Delta_\alpha^k$ to $T_{\xi_{z_{j+1},i}}\Delta_\alpha^k$ and $T_{\xi_{z_j,i}}\Delta_\beta$ to $T_{\xi_{z_{j+1},i}}\Delta_\beta^k$ respectively.

The Maslov index of u can be computed as ([RS93, Section 3], see also [Aur14, 1.3]) the number of times (counted with multiplicities) at which $\delta_\alpha(s)$ and $\delta_\beta(s)$ are not transverse to each other. Since for every $x \notin \mathrm{Crit}(\pi)$, the tangent space T_xE admits symplectic splitting (2.7) $T_xE \cong T_x^vE \oplus T_x^hE =$

$ker(D_x\pi) \oplus \pi^*(T_x\mathbb{C})$, and the Maslov index is additive under such decomposition ([RS93, Theorem 2.3]), we can break the computation of the index into horizontal (base) and vertical (fibre) direction.

In the base direction, it suffices to look at the paths γ_α^k and γ_β to see that the contribution is $+2$ (see Figure 4.4). In the vertical direction, the winding of δ_α relative to δ_β is given by the change in phase of the relative quadratic form η_X/dz^2 under a 2π-rotation, which, as in Lemma 3.2.12, is given by $d-4$.

We therefore obtain a total degree shift of

$$s_\varphi := (d-4)+2 = d-2. \tag{4.10}$$

□

This value indicates the shift in the grading of the vanishing cycles from one intersection fibre to the next. Namely, after parallel transporting $(V_{z_j,\alpha}, V_{z_j,\beta})$ and $(V_{z_{j+1},\alpha}, V_{z_{j+1},\beta})$ into the smooth fibre $M = \pi^{-1}(z_*)$ (to be able to compare them), we can write $(V_{z_{j+1},\alpha}, V_{z_{j+1},\beta}) = (V_{z_j,\alpha}[-s_\varphi], V_{z_j,\beta})$ and therefore, for $j = 1, \ldots, k-1$

$$(V_{z_j,\alpha}, V_{z_j,\beta}) = (V_{z_0,\alpha}[-j\cdot s_\varphi], V_{z_j,\beta}) = (V_{z_0,\alpha}[j(2-d)], V_{z_j,\beta}). \tag{4.11}$$

The formula (4.10) indicates how this shift affects the Floer cohomology of the cycles; for $j = 1, \ldots k-1$, we have

$$\mathrm{HF}^*(V_{z_j,\alpha}, V_{z_j,\beta}; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{HF}^*(V_{z_0,\alpha}[j\cdot(2-d)], V_{z_0,\beta}; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{HF}^{*-j\cdot(2-d)}(V_{z_0,\alpha}, V_{z_0,\beta}; \mathbb{Z}/2\mathbb{Z}). \tag{4.12}$$

Now recall the subsets $I_p = \{z \in I,\ \mathrm{ind}(z) \geqslant 2p\} \subset I$ used in the definition of the spectral sequence of Proposition 4.3.3, $-(k-1) \leqslant p \leqslant 0$. Using (4.12), we can see that the nontrivial (p,q)-entries of the first page of the spectral sequence (4.8) are given by (recall $-(k-1) \leqslant p \leqslant 0$)

$$E_1^{pq} = \mathrm{HF}^{p+q}(V_{z_{(-p)},\alpha}, V_{z_{(-p)},\beta}) \cong \mathrm{HF}^{p+q}\left(V_{z_0,\alpha}\left[-p(2-d)\right], V_{z_0,\beta}\right) \cong \mathrm{HF}^{p+q-p(d-2)}(V_{z_0,\alpha}, V_{z_0,\beta}). \tag{4.13}$$

4.4 $\mathbb{RP}^2$ twist

We define a non-trivial, compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{RP}^2)$ on the total space of the Lefschetz fibration $\pi\colon E_{\mathbb{RP}^2} \to \mathbb{C}$ built in Section 3.3, by applying the construction of Section 4.2 to this fibration.

Using the Floer theoretical computations for Lefschetz thimbles of Section 4.3.3, we prove that φ is isotopic to a power of the projective twist $\tau_{\mathbb{RP}^2}^k \in \pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{RP}^2))$, $k \in \mathbb{Z}$.

4.4.1 The real projective twist

Let $\pi\colon E_{\mathbb{RP}^2} \to \mathbb{C}$ be the Lefschetz fibration of Section 3.3, with base point $z_* \in \mathbb{C}$, and smooth fibre $\pi^{-1}(z_*)$ exact symplectomorphic to a 2-sphere with four boundary components. Consider the vanishing paths $(\gamma_0,\gamma_1,\gamma_2)$, Lefschetz thimbles $(\Delta_0,\Delta_1,\Delta_2)$ and vanishing cycles (V_0,V_1,V_2) as in Section 3.3.1.

According to the conventions of Section 4.3.3, for $j,k \in \{0,1,2\}$, $j \neq k$ any two thimbles Δ_j, Δ_k are disjoint and satisfy $h(\Delta_j) > h(\Delta_k)$ if $j > k$. Any element in the basis of vanishing cycles satisfies Condition (B) by Lemma 3.3.2, so Section 4.2 yields a well-defined compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{RP}^2)$ on the total space of the Lefschetz fibration $\pi\colon E_{\mathbb{RP}^2} \to \mathbb{C}$.

Let $\gamma_2^k = b_{2\pi}^k(\gamma_2)$ and $\Delta_2^k := \Delta_{\gamma_2^k} \cong \varphi^k(\Delta_2)$ and $I := \gamma_2^k(\mathbb{R}^+) \cap \gamma_0(\mathbb{R}^+) = \{z_0,\dots,z_{k-1}\}$. Over each intersection point $z \in I$, the thimbles Δ_2^k and Δ_0 intersect in their vanishing cycles $V_{z,2}$ and $V_{z,0}$ respectively, which are Lagrangian circles meeting at two points (see Figure 3.2)

$$V_{z,0} \cap V_{z,2} := \{\xi_z^-,\xi_z^+\} \subset \pi^{-1}(z). \tag{4.14}$$

For each $z \in I$, the Lagrangians $(V_{z,0},V_{z,2})$ trace four punctured strips between ξ_z^- and ξ_z^+, so there cannot be any non-trivial differentials and hence the pairs (ξ_z^-,ξ_z^+) generate the Floer cohomology $\mathrm{HF}^*(V_{z,2},V_{z,0};\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Together with Proposition 4.3.2, we obtain:

Proposition 4.4.1. *The symplectomorphism $\varphi \in \pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{RP}^2))$ is a non-trivial element of the symplectic mapping class group.* □

Endow the Lagrangian vanishing cycles in the smooth fibres and the Lagrangian thimbles with a $\mathbb{Z}$-grading as in Section 3.2.2. Set $\mathrm{ind}(z_0) = 0$ and $\mathrm{ind}(z_j) = -2j$ for $j = 1,\dots,k-1$ as in 4.3.3.

Let (ξ_0^-,ξ_0^+) be the pair of generators of $\mathrm{CF}^*(V_{z_0,0},V_{z_0,2})$.

Lemma 4.4.2. *For every $z \in I$, the degree of both generators (ξ_z^-,ξ_z^+) is the same, and (ξ_0^-,ξ_0^+) can be set in degree 0.*

Therefore, we fix

$$\mathrm{HF}^*(V_{z_0,2},V_{z_0,0};\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \text{ for } * = 0. \tag{4.15}$$

Proof. The relation $\mathcal{K}_{\mathbb{CP}^2}^{\otimes 2} \cong \mathcal{O}(2)^{\otimes -3}$ for the canonical bundle $\mathcal{K}_{\mathbb{CP}^2} \cong \mathcal{O}(-3)$ induces a quadratic volume form η_X with four poles at the base locus points, no zeroes. As discussed in Section 3.2.2, in turn this yields a relative quadratic form on the smooth fibres, that is holomorphic except at the four base points where it has poles of single order (this can be seen with a computation in local coordinates).

Since $\mathcal{K}_{\mathbb{CP}^1}^{\otimes 2} \cong \mathcal{O}_{\mathbb{CP}^1}(-4)$, $\mathcal{K}_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}(D) = \mathcal{O}$ for a divisor $D \subset \mathbb{CP}^1$ of four points, the space of sections of $\mathcal{K}_{\mathbb{CP}^1}^2$ with four single poles is the space of sections of the trivial bundle. Therefore, on a smooth fibre there is a unique (up to scale) quadratic form as above. Fix one such form, which will be of the shape $\eta_M := dz^2 \cdot r(z)$, for a rational function $r(z)$ with four simple poles.

Consider now the double cover of the sphere, branched at four points

$$p\colon T^2 \longrightarrow S^2 \tag{4.16}$$

obtined by quotienting by the covering involution $w \mapsto -w$ (where w is a coordinate on $\mathbb{C}/\mathbb{Z}^2$).

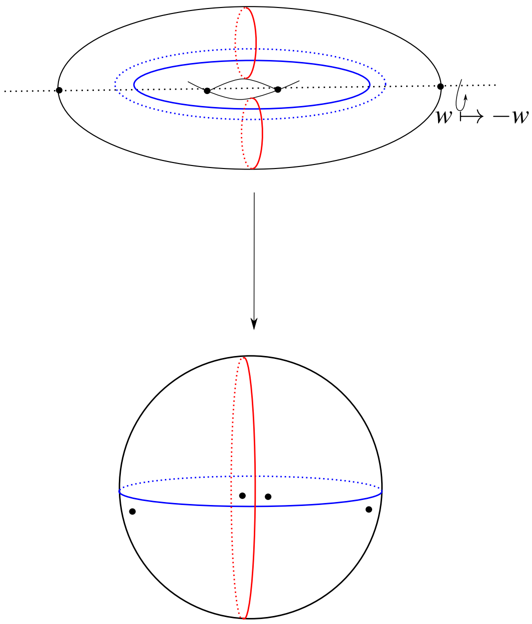

Figure 4.5: The branched double cover of the sphere.

The pull-back η_M under (4.16) is a holomorphic form (the pullback of dz^2/z is holomorphic), so it must be a multiple of the square of the standard form on the torus, dw^2. This form is translation invariant.

The vanishing cycles in the sphere lift to the double cover, so the degree of their intersection points can be computed by computing the degree of the the intersections of their lifts on the torus, using dw^2. The lift of any of the vanishing cycles has two components (two disjoint curves, two longitudes and two meridians, see Figure 4.5). Namely, every vanishing cycle encircles two of the base points in the sphere, and since p is a double cover, the monodromy of p around two of the branched points is the identity. On the torus, the lifts of the vanishing cycles intersect in four points pairwise related by the covering involution $w \mapsto -w$. Within one of these pairs, the two points are related by translation. As w is translation invariant, all intersection points must have the same degree.

□

With the gradings of 4.4.2, the non-trivial entries of the spectral sequence (4.13), generated by the Floer cohomology groups of the vanishing cycles in the intersection fibres, are given by (see (4.13))

$$E_1^{pq} = \mathrm{HF}^{p+q}\left(V_{z_0,2}[p(d-2)], V_{z_0,0}\right) \cong \mathrm{HF}^{p+q}\left(V_{z_0,2}[p], V_{z_0,0}\right) \cong \mathrm{HF}^{q}(V_{z_0,2}, V_{z_0,0}), \tag{4.17}$$

where $d = 3$ comes from (3.13).

						q
	0	0	0	0	0	$\vdots$
	0	0	0	0	0	1
	$\mathbb{Z}/2\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}^2$	0
p	-4	-3	-2	-1	0	

Equation 4.17: The shape of the spectral sequence for $k = 5$.

Note that there is room for a nontrivial differential $d_1 \colon E^{(p,q)} \to E^{(p+1,q)}$, so it is not possible to directly compute the Floer cohomology groups $\mathrm{HF}^*(\varphi^k(\Delta_2), \Delta_0)$. However, using [Eva11], we can infer the following.

Corollary 4.4.3. *The symplectomorphism* $\varphi \in \pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{RP}^2))$ *is isotopic to a power of the projective twists* $\tau^k_{\mathbb{RP}^2}$, $k \in \mathbb{Z}^*$.

Proof. By Proposition 4.4.1, φ is a non-trivial compactly supported symplectomorphism of $T^*\mathbb{RP}^2$. The symplectic mapping class group $\mathrm{Symp}_{ct}(T^*\mathbb{RP}^2)$ is known to be generated by $\tau_{\mathbb{RP}^2}$ ([Eva11]) so φ is isotopic to a power $\tau^k_{\mathbb{RP}^2}$, $k \in \mathbb{Z}$. □

We expect the rank of $\mathrm{HF}(\varphi^k(\Delta_2), \Delta_0)$ to increase linearly with k (see below); if that was indeed the case, φ would indeed be isotopic to the projective twist, or its inverse.

4.4.2 Expected results

Let $\mathbb{N}_k := \{0, 1 \ldots, k-1\}$. The Floer cohomology group are expected to be of the shape

$$\mathrm{HF}^*(\varphi^k(\Delta_2),\Delta_0;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & * = -\mathbb{N}_k \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Note that if all the differentials of the spectral sequence (4.17) vanish, (4.18) is the limit to which the sequence converges to. The nature of the above prediction rests in a series of claims, some of which are still in conjectural state in the literature. We sketch the reasoning below.

Consider the symplectomorphism of the total space $\Phi_{2\pi} \in \mathrm{Symp}(T^*\mathbb{RP}^2)$ we defined in Section 4.2.

Assume that Δ_0, Δ_2 have been suitably deformed into Lagrangians with Legendrian boundary at infinity, to become objects of the wrapped Fukaya category $\mathcal{W}(T^*\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$.

Then the following is expected (see [MS10, Remark 3.1], [BEE12, Appendix], [GPS20, Lemma 3.37], [AG])

$$\varinjlim_k \mathrm{HF}^*(\Phi_{2\pi}^k(\Delta_2),\Delta_0;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{HW}^*(\Delta_2,\Delta_0;\mathbb{Z}/2\mathbb{Z}). \tag{4.19}$$

Moreover, if we identify the thimbles with cotangent fibres $T^*_{q_1}, T^*_{q_2} \in T^*\mathbb{CP}^2$, we obtain

$$\varinjlim_k \mathrm{HF}^*(\Phi_{2\pi}^k(\Delta_2),\Delta_0) = \mathrm{HW}^*(\Delta_2,\Delta_0) \cong \mathrm{HW}^*(T^*_{q_1},T^*_{q_2}) \cong \mathrm{HW}^*(T^*_{q_1},T^*_{q_1}). \tag{4.20}$$

where the last isomorphism is induced by invariance under Hamiltonian isotopies of wrapped Floer cohomology ([AS10, Section 3.1]).

Let $\Omega\mathbb{RP}^2$ be the based loop space of $\mathbb{RP}^2$. By [AS06], there is an isomorphism $\mathrm{HW}^*(T^*_{q_1},T^*_{q_1};\mathbb{Z}/2\mathbb{Z}) \cong H_{-*}(\Omega\mathbb{RP}^2)$; which, combined with (4.20) yields the conjectural relation

$$\varinjlim_k \mathrm{HF}^*(\Phi_{2\pi}^k(\Delta_2),\Delta_0) \cong H_{-*}(\Omega\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}). \tag{4.21}$$

Lemma 4.4.4. *The homology of the based loop space $\Omega\mathbb{RP}^2$ is given by*

$$\forall * \in \mathbb{N},\ \ H_*(\Omega\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \cong H_*(\Omega S^2)\oplus H_*(\Omega S^2) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}. \tag{4.22}$$

Proof. Consider the fibration $\mathbb{Z}/2\mathbb{Z} \to S^2 \to \mathbb{RP}^2$, and let $P\mathbb{RP}^2$ the space of based paths on $\mathbb{RP}^2$. Pulling back the path-loop fibration $\Omega\mathbb{RP}^2 \to P\mathbb{RP}^2 \to \mathbb{RP}^n$ to $S^2 \to \mathbb{RP}^2$ yield another fibration $\Omega\mathbb{RP}^2 \to \mathbb{Z}/2\mathbb{Z} \to S^2$, see for example [May99, Section 5] (alternatively, this can be deduced from the homotopy lifting property). Iterating this process we obtain $\Omega S^2 \to \Omega\mathbb{RP}^2 \to \mathbb{Z}/2\mathbb{Z}$.

In this situation, despite having a disconnected base $\mathbb{Z}/2\mathbb{Z}$ (and $\Omega\mathbb{RP}^2$ has two components), the fibration does have homotopy equivalent fibres, an explicit homotopy equivalence being the following. Let γ be a fixed based loop in $\mathbb{RP}^2$ representing the non-trivial homology class. Then, concatenation with γ defines a map from a component of $\Omega\mathbb{RP}^2$ to the other; concatenating twice gives a map from each component to itself. Therefore, $H_*(\Omega\mathbb{RP}^2) \cong H_*(\Omega S^2) \oplus H_*(\Omega S^2)$, and $H_*(\Omega S^2) \cong k[x]$, for x in degree 1. (The homology of based loop spaces of spheres is known ([FHT12, p.235]), and can be obtained by applying the Serre spectral sequence). ☐

Remark 4.4.5. *The expected result* (4.18) *matches the computation of [FS05, Theorem 1.3].* //

4.5 $\mathbb{CP}^2$-twist

Following the instructions of Section 4.2, this time for the Lefschetz fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ built in 3.4, we define a symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$.

Using the techniques of Section 4.3.3, we then compute the Floer cohomology of two vanishing thimbles, one of which acted upon by a power φ^k. We find that the rank of the Floer cohomology is given by $2k$, which implies that φ has infinite order in $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$. Comparing this outcome with results of [FS05] and predictions from wrapped Floer cohomology computations, we make the following conjecture.

Conjecture 4.5.1. *The symplectomorphism φ is isotopic to the $\mathbb{CP}^2$-twist.*

4.5.1 A symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$ of infinite order.

Let $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$ be the Lefschetz fibration of Section 3.4. Recall that the smooth fibres are Stein domains whose symplectic completion is exact symplectomorphic to the plumbing of spheres $T^*S^3\#_{S^1}T^*S^3$. Let $z_* \in \mathbb{C}$ be the base point and consider the vanishing paths $(\gamma_0,\gamma_2,\gamma_4)$, Lefschetz thimbles $(\Delta_0,\Delta_2,\Delta_4)$ and vanishing cycles (V_0,V_2,V_4) as in Section 3.4.3. By Lemma 3.4.9, the ungraded monodromy preserves the vanishing cycles, and the graded version shifts their grading as $\phi(V_i) = V_i[-2]$. Then Corollary 3.4.10 grants the well-definedness of the construction of Section 4.2, that we use to build a compactly supported symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$ of the total space of the Lefschetz fibration $\pi\colon E_{\mathbb{CP}^2} \to \mathbb{C}$.

To compute the Floer cohomology group $\mathrm{HF}(\varphi^k(\Delta_4),\Delta_0;\mathbb{Z}/2\mathbb{Z})$, we utilise the spectral sequence of Proposition 4.3.3, and much of the discussion for the $\mathbb{RP}^2$ case (Section 4.4.1) applies.

Over each intersection point $z \in I$, the intersection of the thimbles is determined by the intersection of their associated vanishing 3-spheres $(V_{z,4},V_{z,0}) = (\Delta_4 \cap \pi^{-1}(z), \Delta_0 \cap \pi^{-1}(z))$. The latter intersect cleanly in a circle $V_{z,4} \cap V_{z,0} \cong S^1$. In a neighbourhood of the intersection locus, one can apply a perturbation by a Morse function with two critical points (one minimum and one maximum), so that after perturbing, the two vanishing cycles intersect in two points $\{\xi_z^-,\xi_z^+\} \subset \pi^{-1}(z)$, and the latter generate the Floer complex $\mathrm{CF}(V_{z,4},V_{z,0})$ in each fibre $\pi^{-1}(z)$, $z \in I$. The generators $\{\xi_z^-,\xi_z^+\}$

represent non-trivial cocycles in $\mathrm{CF}(V_{z,4}, V_{z,0})$, since the clean intersection $V_{z,4} \cap V_{z,0}$ has one component, and by exactness, the results of [Poź94, Theorem 3.4.11] apply (see also [Sei99, Theorem 3.1]). It follows that the Floer cohomology of the vanishing cycles is the standard cohomology (up to a grading shift $l \in \mathbb{Z}$ discussed below)

$$\forall z \in I,\ \mathrm{HF}^{*+l}(V_{z,4}, V_{z,0}; \mathbb{Z}/2\mathbb{Z}) \cong H^*(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \tag{4.23}$$

By Lemma 4.3.2, we know that $\mathrm{HF}^*(\varphi(\Delta_4), \Delta_0; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\{\xi_0^-, \xi_0^+\}$. On the other hand, $\mathrm{HF}(\Delta_4, \Delta_0; \mathbb{Z}/2\mathbb{Z}) = 0$ so we obtain:

Proposition 4.5.2. *The symplectomorphism $\varphi \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^2)$ is a non-trivial element of the symplectic mapping class group.* □

Let $\gamma_4^k = b_{2\pi}^k(\gamma_4)$ and $\Delta_4^k := \Delta_{\gamma_4^k} \cong \varphi^k(\Delta_4)$ and $I := \gamma_4^k(\mathbb{R}^+) \cap \gamma_0(\mathbb{R}^+) = \{z_0, \dots, z_{k-1}\}$. Equip the Lagrangian vanishing cycles in the smooth fibres and the Lagrangian thimbles and with a $\mathbb{Z}$-grading as in Section 3.2.2.

Set $\mathrm{ind}(z_0) = 0$ and $\mathrm{ind}(z_j) = -2j$ for $z_j \in I$, as in Section 4.3.3. The elements $\{\xi_j^-, \xi_j^+\} \subset \pi^{-1}(z)$ of any generating pair of the complex $\mathrm{CF}^*(V_{z,4}, V_{z,0}; \mathbb{Z}/2\mathbb{Z})$ must be shifted in their degrees as $\deg(\xi_j^+) = \deg(\xi_j^-) \pm 1$. Namely, this shift is established by the Morse perturbation applied above (see [Sei99, (4.4)]). Set $\deg(\xi_0^-) = -1$, $\deg(\xi_0^+) = 0$. This fixes

$$\mathrm{HF}^*(V_{z_0,4}, V_{z_0,0}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},\ * = -1, 0$$

This choice determines the grading of the Floer cohomology groups $\mathrm{HF}^*(V_{z,4}, V_{z,0}; \mathbb{Z}/2\mathbb{Z})$, for every $z \in I$. Inserting $d = 6$ (from (3.25)) in the formula (4.13), we obtain, for $-(k-1) \leqslant p \leqslant 0$,

$$E_1^{pq} = \mathrm{HF}^{p+q}(V_{z_{(-p)},4}, V_{z_{(-p)},0}) \cong \mathrm{HF}^{p+q-p(d-2)}(V_{z_0,4}, V_{z_0,0}) \cong \mathrm{HF}^{q-3p}(V_{z_0,4}, V_{z_0,0}). \tag{4.24}$$

The difference in total degree between generators of different fibres is always greater than one: by Lemma 4.3.4, we know it is at least $s_\varphi = d - 2 = 6 - 2 = 4$. Therefore, there is no non-trivial "horizontal" differential (counting pseudoholomorphic curves between different fibres). This can also be noticed by the position of the generators in the first page of the sequence, in Diagram (4.25) (the total degree of a generator is obtained by adding p and q). With (4.23) it follows that all differentials are trivial, and the sequence collapses at the first page, and we have the ungraded result $\mathrm{HF}(\Delta_4^k, \Delta_0) \cong \oplus_{j=0}^{k-1} \mathrm{HF}(V_{z_j,4}, V_{z_j,0})$. Together with the grading computations above, this implies the following.

Theorem 4.5.3. *Let $\mathbb{N}_k := \{0, 1 \dots, k-1\}$. The graded Floer cohomology groups are given by*

$$\mathrm{HF}^*(\varphi^k(\Delta_4), \Delta_0; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & * = -4\mathbb{N}_k, -4\mathbb{N}_k - 1 \\ 0 & . \end{cases} \tag{4.25}$$

p	-3	-2	-1	0	q
	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	-1
	0	0	0	0	-2
	0	0	$\mathbb{Z}/2\mathbb{Z}$	0	-3
	0	0	$\mathbb{Z}/2\mathbb{Z}$	0	-4
	0	0	0	0	-5
	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	-6
	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	-7

(4.25)

The shape of the spectral sequence for $k = 3$.

In particular, omitting the gradings, $\mathrm{HF}(\varphi^k(\Delta_4), \Delta_0; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^k$.

□

Corollary 4.5.4. *The symplectomorphism* φ *has infinite order in* $\pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{CP}^2))$.

Remark 4.5.5. *The symplectic mapping class group* $\pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{CP}^2))$ *has not yet been computed. In particular, it is not known whether it is solely generated by the standard projective twist* $\tau_{\mathbb{CP}^2}$*. For example, there could be other potential generators such as* $\mathbb{CP}^2$*-twists which are not Hamiltonianly isotopic to* $\tau_{\mathbb{CP}^2}$*. However, as we shall see in Chapter 8, such phenomena have only been observed in dimensions* $n \geqslant 18$*, so we cannot draw further conclusions.* //

4.5.2 Expected results

As for the real projective twists, we compare our results with the literature. First note that if we identify the Lefschetz thimble with cotangent fibres of the cotangent bundle $T^*\mathbb{CP}^2$, Theorem 4.5.3

matches with the computations of [FS05].

The Floer cohomology of a cotangent fibre $T^*_{q_1}$ of $T^*\mathbb{CP}^2$ twisted by a projective twist and another (untwisted) cotangent fibre $T^*_{q_2} \subset T^*\mathbb{CP}^2$ was computed in [FS05, Theorem 2.13]. Let $q_1, q_2 \in \mathbb{CP}^2$ be two distinct points and $T^*_{q_1}, T^*_{q_2} \in T^*\mathbb{CP}^2$ their associated cotangent fibres. Then, according to [FS05],

$$\mathrm{HF}(\tau^k_{\mathbb{CP}^2}(T^*_{q_1}), T^*_{q_2}; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^k. \tag{4.26}$$

We can also apply the observations of Section 4.18 to get

$$\begin{aligned} \varinjlim_k \mathrm{HF}^*(\Phi^k_{2\pi}(\Delta_4), \Delta_0; \mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{HW}^*(\Delta_4, \Delta_0; \mathbb{Z}/2\mathbb{Z}) \cong \\ &\cong \mathrm{HW}^*(T^*_{q_1}, T^*_{q_2}; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{HW}^*(T^*_{q_1}, T^*_{q_1}; \mathbb{Z}/2\mathbb{Z}). \end{aligned} \tag{4.27}$$

In this case, $\mathbb{CP}^2$ is not a Spin manifold, but since we work over a field of coefficient two, the isomorphism $\mathrm{HW}^*(T^*_{q_1}, T^*_{q_1}; \mathbb{Z}/2\mathbb{Z}) \cong H_{-*}(\Omega\mathbb{CP}^2; \mathbb{Z}/2\mathbb{Z})$ from [AS06] still holds. We therefore obtain the expected isomorphism

$$\varinjlim_k \mathrm{HF}^*(\Phi^k_{2\pi}(\Delta_4), \Delta_0; \mathbb{Z}/2\mathbb{Z}) \cong H_{-*}(\Omega\mathbb{CP}^2; \mathbb{Z}/2\mathbb{Z}).$$

The homology of the loop space $\Omega\mathbb{CP}^2$ can be computed by considering the fibration $\Omega S^5 \to \Omega\mathbb{CP}^2 \to S^1$ obtained by "looping twice" the standard fibration $S^1 \to S^5 \to \mathbb{CP}^2$ and (as we did in Section 4.4.2) and applying the Serre spectral sequence (see [FHT12, Chapter 16 (a)]).

Lemma 4.5.6. *For a ground field k of characteristic* 2*,* $H_*(\Omega\mathbb{CP}^2) \cong H_*(S^1) \otimes H_*(\Omega S^5)$ *with generators x in degree* $+4$ *and y in degree* $+1$*, and* $y^2 = 0$*.*

□

Chapter 5

The Hopf correspondence

In this section we discuss the main theoretical device in action in the rest of the thesis; the *Hopf correspondence*. Given a real/complex projective Lagrangian $K \subset W$ in a Liouville manifold (W, ω) satisfying Assumptions (D)/ (C) (see next section), the Hopf correspondence is a Lagrangian correspondence that associates to K a Lagrangian sphere $L \subset Y$ in an auxiliary Liouville manifold (Y, Ω). This tool will be central in the arguments we present in Sections 6, 7 and 8.

Then, given the projective twist $\tau_K \in \mathrm{Symp}_{ct}(W)$, there is a Dehn twist $\tau_L \in \mathrm{Symp}_{ct}(Y)$ along the sphere associated to K. Let $T_K \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ and $T_L \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ be the twist functors induced by τ_K, τ_L. The key use of the Hopf correspondence is aimed at proving the existence of a commuting diagram (Section 5.4)

$$\begin{array}{ccc} \mathcal{F}\mathrm{uk}(Y) & \xrightarrow{T_L} & \mathcal{F}\mathrm{uk}(Y) \\ \Theta_\Gamma \uparrow & & \Theta_\Gamma \uparrow \\ \mathcal{F}\mathrm{uk}(W) & \xrightarrow{T_K} & \mathcal{F}\mathrm{uk}(W). \end{array} \tag{5.1}$$

where Θ_Γ is a functor induced by the correspondence (whose existence is also non-trivial, see 5.2.1).

We first prove the geometric statements necessary to obtain this diagram at the level of spaces, after which we review the main concepts from Wehrheim–Woodward's Lagrangian correspondence theory (Section 5.2) and introduce the Hopf correspondence (Section 5.3). We then show that the functors of $\mathcal{F}\mathrm{uk}(W)$ induced by projective twists are entwined, via the correspondence, with the functors of $\mathcal{F}\mathrm{uk}(Y)$ induced by the Dehn twists.

In Section 5.5, we show that the Hopf correspondence can be used to build a symplectic Gysin sequence as established in [Per08]. These results will be used in Chapter 6 to prove the free generation result of Theorem 3.

5.1 Commuting diagrams of twists

In this section we introduce the geometric ideas underpinning the philosophy of the Hopf correspondence. We prove a criterion for relating projective twists in a Liouville manifold (W,ω) to Dehn twists in another Liouville manifold (Y,Ω).

5.1.1 Complex projective Lagrangians

We begin by considering Lagrangian complex projective spaces. Fix the round metric on S^{2n+1}, with norm $\|\cdot\|_S$ and consider the free S^1-action on S^{2n+1} by complex multiplication. The orbits of the action are great circles ("Hopf circles"), hence geodesics, and the action is isometric.

Consider the quotient map $h\colon S^{2n+1} \to \mathbb{CP}^n$, which is the (generalised) Hopf fibration. It is a Riemannian submersion that uniquely defines the Fubini-Study metric g_P on $\mathbb{CP}^n$. Identify the tangent bundles with their corresponding cotangent bundles $TS^{2n+1} \cong T^*S^{2n+1}$, $T\mathbb{CP}^n \cong T^*\mathbb{CP}^n$ via the the canonical isomorphism induced by the metrics.

The Hopf action on S^{2n+1} lifts to a Hamiltonian S^1-action on the cotangent bundle $(T^*S^{2n+1}, \omega_{T^*S^n})$ ([GS90]). Let $\mu\colon T^*S^{2n+1} \to \mathbb{R}$ be the moment map of this action. Assume 0 is a regular value of μ and consider the level set $\mu^{-1}(0) \subset T^*S^{2n+1}$, which has the structure of a principal S^1-bundle $p\colon \mu^{-1}(0) \to T^*\mathbb{CP}^n$ over the symplectic quotient $T^*S^{2n+1} /\!\!/ S^1 := \mu^{-1}(0)/S^1 \cong T^*\mathbb{CP}^n$. Note that the (co)vectors in $\mu^{-1}(0)$ are horizontal because $\mu^{-1}(0) = \{(x,\xi) \in T^*S^{2n+1},\ \xi(v) = 0\}$ where $v \in T_xS^{2n+1}$ is the vector field obtained by the infinitesimal action induced by the Hopf action at $x \in S^{2n+1}$.

Lemma 5.1.1. *Let $\tau_{S^{2n+1}} \in \mathrm{Symp}_{ct}(T^*S^{2n+1})$, $\tau_{\mathbb{CP}^n} \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$ be the model Dehn and projective twists respectively. Let $p\colon V := \mu^{-1}(0) \to T^*\mathbb{CP}^n$ be the symplectic quotient map as above. There is a commuting diagram*

$$\begin{array}{ccc} V & \xrightarrow{\ \tau_{S^{2n+1}}|_V\ } & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ T^*\mathbb{CP}^n & \xrightarrow{\ \tau_{\mathbb{CP}^n}\ } & T^*\mathbb{CP}^n. \end{array} \quad . \tag{5.2}$$

Proof. The Hopf action is isometric, i.e for any $g \in S^1$, the induced map $\psi_g \in \mathrm{Diff}(S^{2n+1})$ is an isometry. This implies that the differential maps on the tangent bundles $D_p\psi_g\colon T_pS^{2n+1} \to T_{\psi_g(p)}S^{2n+1}$ commute with the geodesic flow.

The co-geodesic flow Φ_H^t on T^*S^{2n+1} is induced by the Hamiltonian function

$$\begin{aligned} \widetilde{H}\colon T^*S^{2n+1} &\longrightarrow \mathbb{R} \\ (p,\xi) &\longmapsto \|\xi\|_S. \end{aligned} \tag{5.3}$$

This is S^1-invariant, so that there is a Hamiltonian function $H\colon T^*\mathbb{CP}^n \to \mathbb{R}$ defined on the quotient, with respect to the submersion metric g_P, which induces the (co-)geodesic flow on $T^*\mathbb{CP}^n$. Since p is induced by a Riemannian submersion, we have the relation $p\circ\Phi^t_{\widetilde{H}}|_V = \Phi^t_H \circ p|_V$,

and for any choice of cut-off function r_ε as in Section 2.1,

$$p\circ\sigma^{\widetilde{H}}_{r_\varepsilon(\|\xi\|_S)}(\xi) = \sigma^{H}_{r_\varepsilon(\|p(\xi)\|_P)}\circ p(\xi),\ \xi\in V\subset T^*S^{2n+1} \tag{5.4}$$

where $\sigma^H_t, \sigma^{\widetilde{H}}_t$ are the Hamiltonian S^1-actions induced by H and $\widetilde{H}$ respectively, as in (2.1.1).

Any geodesic connecting a point on S^{2n+1} to its antipode projects to a closed geodesic of minimal period on $\mathbb{CP}^n$, so the definitions of the twists in Section 2.1 imply that $p\circ\tau_{S^{2n+1}}|_V = \tau_{\mathbb{CP}^n}\circ p$. □

We now extend the above discussion to a more global situation; in order to do that it is necessary to set the following assumption.

Assumption C. Let (W,ω) be a $4n$-dimensional Liouville manifold with a homology class $\alpha\in H^2(W;\mathbb{Z})$ and Lagrangian complex projective spaces $K_1,\ldots,K_m\subset W$ such that

$$\forall i\colon\ \alpha|_{K_i} = x\in H^2(\mathbb{CP}^n;\mathbb{Z}),$$

where $x = c_1(\mathcal{O}_{\mathbb{CP}^n}(-1))$ is the generator of the cohomology ring $H^*(\mathbb{CP}^n;\mathbb{Z})\cong\mathbb{Z}[x]/x^{n+1}$.

Proposition 5.1.2. *Let (W,ω) be a $4n$-dimensional Liouville manifold containing embedded Lagrangian complex projective spaces $K_1,\ldots,K_m\subset W$. Assume there exists a class $\alpha\in H^2(W;\mathbb{Z})$ satisfying Assumption C. Then there is a $(4n+2)$-dimensional Liouville manifold (Y,Ω) with Lagrangian spheres $L_1,\ldots,L_m\subset Y$, a coisotropic submanifold $V\subset Y$ with the structure of an S^1-fibre bundle $p\colon V\to W$ such that for each $i\in\{1,\ldots,m\}$, $L_i\subset V$ and there is a commuting diagram*

$$\begin{array}{ccc} V & \xrightarrow{\ \tau_{L_i}|_V\ } & V \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ W & \xrightarrow{\ \tau_{K_i}\ } & W. \end{array} \qquad . \tag{5.5}$$

The class $\alpha\in H^2(W;\mathbb{Z})$ restricts to a generator $x\in H^2(\mathbb{CP}^n;\mathbb{Z})$ on each Lagrangian K_i, so there is a complex line bundle $\mathcal{L}\to W$ satisfying $c_1(\mathcal{L})=\alpha$ which is modelled on the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ over K_i, for $i=1,\ldots m$. Fix a metric $\|\cdot\|_{\mathcal{L}}$ on $\mathcal{L}$, and for $u\in\mathcal{L}$ define a function $r(u):=\|u\|_{\mathcal{L}}$. Set $V:=\{u\in\mathcal{L},\ r(u)=1\}$. Over K_i, V defines a sphere $L_i:=V|_{K_i}$.

Lemma 5.1.3. *The $\mathbb{C}^*$-bundle associated to $\mathcal{L}$ is a Liouville domain where the spheres L_i are embedded as Lagrangian submanifolds.*

Proof. Denote this bundle by $q\colon Y\to W$. Following [Rit14, 7.2], we build a symplectic form Ω on Y, making the spheres L_i Lagrangian, and find the appropriate vector field which will be Liouville with respect to Ω.

The metric induces a connection one form γ^∇ on $\mathcal{L}\setminus 0$ satisfying

$$\begin{aligned}\gamma^\nabla|_{H_u^\nabla} = 0,\ \gamma^\nabla|_{T_u^v\mathcal{L}} = \gamma\ \forall u\in\mathcal{L}\setminus 0\\ [d\gamma^\nabla] = -q^*(c_1(\mathcal{L})) = -q^*(\alpha).\end{aligned}\tag{5.6}$$

where $H_u^\nabla\mathcal{L}$ is the horizontal distribution associated to the connection ∇ at u, $T_u^v\mathcal{L}$ the vertical distribution, and γ the fibrewise angular form defined by the metric. Let $\Omega := q^*\omega + d(f(r)\gamma^\nabla)$, for a function $f\in C^\infty(\mathbb{R})$ with

$$\begin{aligned}&f(1) = 0\\ &f'(r) > 0 \quad \text{for all } r\in\mathbb{R}.\end{aligned}$$

Then Ω defines a symplectic form in a neighbourhood of $\{r=1\}$, and L_i is Lagrangian with respect to Ω. Let λ be the Liouville 1-form on W with $d\lambda = \omega$. Define $\lambda_Y := q^*\lambda + f(r)\gamma^\nabla$ so that $d(\lambda_Y) = \Omega$. Then (λ_Y,Ω) defines a Liouville structure near $\{r=1\}$ (the symplectic dual to λ_Y points outwards along a small neighbourhood of $\{r=1\}$). Therefore, a symplectic completion along this neighbourhood yields a Liouville manifold that is diffeomorphic to Y, containing the Lagrangian spheres $L_1,\dots,L_m$. □

Proof of Proposition 5.1.2. Let $\mathcal{L}\to W$ be the complex line bundle we have constructed above with $c_1(\mathcal{L}) = \alpha$. For each Lagrangian projective space $K_i\subset W$, the restriction of the bundle $\mathcal{L}|_{K_i}$ is modeled on the tautological line bundle, which implies that $L_i\to K_i$ is modelled on the Hopf quotient map $h\colon S^{2n+1}\to\mathbb{CP}^n$. The commutativity of (5.5) follows by the local commuting diagram of cotangent bundles (5.2). □

Example 5.1.4. Without assumption (C), Proposition 5.1.2 is in general not true, as the following example illustrates. Consider the manifold W obtained by attaching a 3-handle to the contact boundary of $DT^*\mathbb{CP}^2$ such that $H^2(W;\mathbb{Z}) = 0$. On one hand, W contains a non-trivial Lagrangian $K=\mathbb{CP}^2\subset W$ coming from the zero section (which is preserved by the handle attachment, since it is disjoint from the boundary. Note that the handle attachment is subcritical, so in fact the whole wrapped Fukaya category is preserved, see [GPS18]). However, as there is no non-trivial 2-cohomology class on W, there is no non-trivial S^1-bundle over W that can be used to build a sphere over K. ♢

5.1.2 Real projective Lagrangians

A similar procedure can be applied to a Liouville manifold containing real projective Lagrangians with an appropriate cohomology criterion. First recall the following.

Let $S^0 \cong \mathbb{Z}/2\mathbb{Z}$ act on the sphere S^n by the antipodal map. The quotient map $h : S^n \longrightarrow \mathbb{RP}^n$ is in this case a covering map, and induces a symplectic double cover $q\colon T^*S^n \to T^*\mathbb{RP}^n$ with $q^*\omega_{T^*\mathbb{RP}^n} = \omega_{T^*S^n}$.

Lemma 5.1.5 ([MW18a, Lemma 2.4])**.** *Let $\tau_{\mathbb{RP}^n} \in \mathrm{Symp}_{ct}(T^*\mathbb{RP}^n)$ be the $\mathbb{RP}^n$-twist defined as in Section 2.1. Then the diagram*

$$\begin{array}{ccc} T^*S^n & \xrightarrow{\tau_{S^n}} & T^*S^n \\ \downarrow q & & \downarrow q \\ T^*\mathbb{RP}^n & \xrightarrow{\tau_{\mathbb{RP}^n}} & T^*\mathbb{RP}^n \end{array} . \tag{5.7}$$

commutes.

Assumption D. Let (W,ω) be a $2n$-dimensional Liouville manifold with a homology class $\alpha \in H^1(W;\mathbb{Z}/2\mathbb{Z})$ and Lagrangian real projective spaces $K_1,\dots K_m \subset W$ such that

$$\forall i:\ \alpha|_{K_i} = x \in H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$$

where $x = e(\gamma_{\mathbb{R}}^{1,n+1})$ is the the Euler class of the real tautological bundle $\gamma_{\mathbb{R}}^{1,n+1} \to \mathbb{RP}^n$, and generator of the cohomology ring $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/x^{n+1}$.

Proposition 5.1.6. *Let (W,ω) be a $2n$-dimensional Liouville manifold containing embedded Lagrangian real projective spaces $K_1,\dots K_m \subset W$. Assume there is a class $\alpha \in H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ satisfying Assumption D. Then, there is a $2n$-dimensional Liouville manifold $(\widetilde{W},\tilde{\omega})$ containing Lagrangian spheres $L_1,\dots,L_m \subset \widetilde{W}$ and a commuting diagram*

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\tau_{L_i}} & \widetilde{W} \\ \downarrow q & & \downarrow q \\ W & \xrightarrow{\tau_{K_i}} & W. \end{array} . \tag{5.8}$$

Proof. In this case, the class $\alpha \in H^1(W;\mathbb{Z}/2\mathbb{Z})$ defines a symplectic double cover $q\colon (\widetilde{W},\widetilde{\omega}) \to (W,\omega)$. Each Lagrangian $K_i \cong \mathbb{RP}^n$ then lifts to its double cover L_i, which is a sphere $S^n \subset \widetilde{W}$. Let λ be the Liouville form on W. As q is symplectic, $\widetilde{\omega} = q^*(\omega) = q^*(d\lambda) = d(q^*(\lambda))$, and $\widetilde{\lambda} := q^*(\lambda)$ defines a Liouville form on $\widetilde{W}$, which gives $\widetilde{W}$ the structure of a Liouville manifold. Then the result follows by the local case illustrated by Lemma 5.1.5.

Remark 5.1.7. *It is possible to obtain an analogous diagram for the quaternionic twist as follows. Consider the free $S^3 \simeq Sp(1)$-action on S^{4n+3} inducing the quotient map $h\colon S^{4n+3} \to \mathbb{HP}^n$. This is* *a submersion as in the complex case, and the same arguments (with the natural metrics) yield the local commuting diagram*

$$\begin{array}{ccc} \mu^{-1}(0) & \xrightarrow{\tau_{S^{4n+3}}|_{\mu^{-1}(0)}} & \mu^{-1}(0) \\ \downarrow p & & \downarrow p \\ T^*\mathbb{HP}^n & \xrightarrow{\tau_{\mathbb{HP}^n}} & T^*\mathbb{HP}^n \end{array} \quad . \tag{5.9}$$

where $p\colon \mu^{-1}(0) \longrightarrow T^*\mathbb{HP}^n$ *is the* S^3*-fibre bundle induced given by the symplectic quotient map of the Hamiltonian action induced on* T^*S^{4n+3}.

Given an $8n$*-dimensional symplectic manifold* (W,ω) *containing quaternionic projective Lagrangians, one would hope to find a cohomological condition to ensure the existence of a symplectic* $(8n+6)$*-dimensional manifold* (Y,Ω) *with corresponding Lagrangian spheres, as we did for the real and complex cases. However, homotopy classes of maps* $W \to \mathbb{HP}^\infty \cong Sp(1)$ *do not classify quaternionic line bundles over* W*, so there is no analogue of Assumptions C, D to ensure the existence of such a manifold, and a commuting diagram of the form of* (5.5).

//

5.2 Lagrangian correspondences

We summarise the basic definitions and results associated to Lagrangian correspondences in the setting of [WW09, WW10a, WW10b, MWW16]. For the entire section we let k be a coefficient field of characteristic two.

Definition 5.2.1 ([WW10b])**.** A *Lagrangian correspondence* between two symplectic manifolds (M_k,ω_k) and (M_{k+1},ω_{k+1}) ("from M_k to M_{k+1}") is a Lagrangian submanifold $L_{k,k+1} \subset (M_k^- \times M_{k+1}) := (M_k \times M_{k+1}, -\omega_k \oplus \omega_{k+1})$. A *cycle of Lagrangian correspondences* of length $r \geqslant 1$ is a sequence of symplectic manifolds $(M_0,\dots,M_{r+1} = M_0)$ together with a sequence of Lagrangian correspondences $\underline{L} := (L_{01}, L_{12}, \dots, L_{r-1,r}, L_{r,0})$ such that $L_{k,k+1} \subset M_k^- \times M_{k+1}$ for $k = 0,\dots,r$. ♢

For example, a Lagrangian submanifold L of a symplectic manifold (M,ω) is a trivial example of Lagrangian correspondence, seen as $L \subset \{pt\}^- \times M = M$ (see other examples below).

Definition 5.2.2 ([WW10a, Definition 2.0.4])**.** Let $(M_i,\omega_i), i = 0,1,2$ be symplectic manifolds and $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ be Lagrangian correspondences.

1. The correspondence transpose to L_{01} is defined as $L_{01}^t := \{(m_1,m_0) | (m_0,m_1) \in L_{01}\} \subset M_1^- \times M_0$. Note that for a simple Lagrangian $L \subset M$ of a single symplectic manifold M, we won't distinguish L from its conjugate.

2. The composition of L_{01} and L_{12} is defined as

$$L_{01} \circ L_{12} := \left\{ (m_0, m_2) \in M_0^- \times M_2 \middle| \, \exists m_1 \in M_1 : \begin{array}{c} (m_0, m_1) \in L_{01} \\ (m_1, m_2) \in L_{12} \end{array} \right\} \subset M_0^- \times M_2 \tag{5.10}$$

and it is called embedded if $L_{01} \circ L_{12}$ defines an embedded Lagrangian submanifold of $M_0^- \times M_2$.

♢

Example 5.2.3 ([Per08, 1.1])**.** Let (M^{2n}, ω_M) a symplectic manifold with a coisotropic embedding $\iota \colon V \hookrightarrow M$. If the foliation defined by the integrable distribution TV^ω is a fibration $p \colon V \to B$ then the leaf space is a symplectic manifold (B, ω_B) satisfying $p^*\omega_B = \iota^*(\omega_M)$. The (transpose) graph of p,

$$\Gamma := \{(p(v), v), \; v \in V\} \subset (B \times M, -\omega_B \oplus \omega_M)$$

is a Lagrangian correspondence.

♢

A special case of Example 5.2.3 is when the coiosotropic submanifold is obtained as a level set of a moment map induced by a Hamiltonian action:

Example 5.2.4 ([WW10b, Example 2.0.2 (e)])**.** Let (M, ω_M) be a symplectic manifold. Let G be a compact Lie group acting on M Hamiltonianly with moment map $\mu \colon M \to \mathfrak{g}^*$. If G acts freely on $\mu^{-1}(0)$, the latter is a smooth $G-$fibred coisotropic over the symplectic quotient $W := M /\!\!/ G = \mu^{-1}(0)/G$. W is a symplectic manifold with symplectic structure $\omega_{M/\!\!/G}$ given by the Marsden-Weinstein theorem (see for example [MS17, Section 5.4]). The graph of the quotient map $p \colon \mu^{-1}(0) \to W$ is a Lagrangian submanifold of $(M \times W, -\omega_M \oplus \omega_W)$ and defines a Lagrangian correspondence, relating Lagrangians of M with Lagrangians of its symplectic quotient. ♢

5.2.1 Induced functors

In [WW10a] and [WW10b], Wehrheim and Woodward introduced a Floer cohomology theory adapted to cycles of closed Lagrangian correspondences $\underline{L} := (L_{01}, \ldots, L_{r0})$, called *quilted Floer cohomology* and denoted by $\mathrm{HF}(\underline{L}; k)$. Pseudo-holomorphic quilts are a generalisation of the usual pseudoholomorphic strips used in standard Lagrangian Floer theory, and the quilted invariant is defined by counting pseudoholomorphic quilts with boundary constraints defined by the Lagrangian correspondences ([WW10b, Section 5]). It can be viewed as a Floer theory in product symplectic manifolds (we refer to [WW10b, Section 4.3] for definitions).

One of the main features is that given a cycle $\underline{L}$ of Lagrangian correspondences, quilted Floer cohomology is invariant under embedded composition (as in Definition 5.2.2) of subsequent Lagrangians in $\underline{L}$.

Theorem 5.2.5 ([WW10b, Theorem 5.4.1])**.** *Let $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$ be a cyclic sequence of closed, exact embedded and oriented Lagrangian correspondences between Liouville manifolds $(M_0, \ldots, M_{r+1} = M_0)$ such that $\forall i, L_{(i-1)i} \circ L_{i(i+1)}$ is embedded. Then, for $\underline{L'} := (L_{01}, \ldots, L_{(j-1)j} \circ L_{j(j+1)}, \ldots, L_{r(r+1)})$, there is an isomorphism $HF(\underline{L};k) \cong HF(\underline{L'};k)$.*

In [MWW16] the same authors and Ma'u proved that under certain assumptions, a Lagrangian correspondence L_{01} between given symplectic manifolds (M_0, ω_0) and (M_1, ω_1), defines an A_∞-functor Γ_{01} between $\mathcal{F}\mathrm{uk}(M_0)$ and the dg-category of A_∞-modules over $\mathcal{F}\mathrm{uk}(M_1)$. The functor is realised as the geometric composition $(\cdot) \circ \Gamma$ of Lagrangians submanifolds of M_0 with the correspondence, and this important result relies on the invariance Theorem 5.2.5. If for every Lagrangian in M_0 the composition outputs an embedded Lagrangian of M_1, the induced functor is between Fukaya categories.

Theorem 5.2.6 ([MWW16, Theorem 1.1])**.** *Assume M_0, M_1 are Liouville manifolds, and let $\Gamma_{01} \subset M_0^- \times M_1$ be a closed, exact and embedded correspondence such that for any closed embedded Lagrangian $K_0 \subset M_0$, the geometric composition*

$$L_1 := K_0 \circ \Gamma_{01} = \{m_1 \in M_1, \; \exists m_0 \in K_0 \text{ such that } (m_0, m_1) \in \Gamma_{01}\} \subset M_1 \tag{5.11}$$

is a closed embedded Lagrangian in M_1. This assignement defines an A_∞-functor

$$\begin{aligned} \Theta_{01} : &\mathcal{F}uk(M_0) \longrightarrow \mathcal{F}uk(M_1), \\ &\Theta_{01}(K_0) = L_1. \end{aligned} \tag{5.12}$$

In the above theorem, the correspondences are required to be closed, exact (or satify suitable monotonicity conditions) and embedded. Gao ([Gao17a, Gao17b]) developed non-compact generalisations of Theorem 5.2.6, including non-compact Lagrangian correspondences, in the setting of wrapped Fukaya categories.

In both cases, the main theoretical device at work behind a result such as Theorem 5.2.6 (or Gao's equivalent) is quilted Floer theory, which, in [Gao17b], was adapted to a version suitable for non-compact correspondences.

In this work we focus on a Lagrangian correspondence in a setting that features some properties of both theories. Before introducing our setting (see below), we review the types of Lagrangians that are admitted in a Gao's setting.

Let (M_0,ω_0), (M_1,ω_1) be Liouville manifolds with cylindrical almost complex structures J_0, J_1 and Liouville flows Z_0,Z_1 respectively. The product manifold $(M_0\times M_1,-\omega_0\times\omega_1)$ is a Liouville manifold with respect to the product almost complex structure $J_{01} := -J_0\times J_1$ and Liouville flow $Z_{01} := \pi_0^*(Z_0)+\pi_1^*(Z_1)$, for the projections $\pi_i\colon M_0\times M_1 \longrightarrow M_i$, $i=1,2$.

Let $\Sigma\subset M_0\times M_1$ be the contact hypersurface given in [Gao17b, 2.2], such that we can fix a choice of cylindrical end that is compatible with the choices above. In other words, there is a compact set $U\subset M_0\times M_1$ bounded by Σ, such that there is a diffeomorphism $M_0\times M_1\setminus U\cong[0,\infty)\times\Sigma$ ([Gao17b, (2.5)]).

Definition 5.2.7. A Lagrangian submanifold is said to be *conical* if it is an exact, properly embedded Lagrangian which is preserved by the Liouville vector field over the cylindrical ends. ♢

Definition 5.2.8 ([Gao17b, Definition 3.9])**.** A Lagrangian submanifold $\Gamma_{01}\subset M_0^-\times M_1$ is called admissible if it is

1. Either a product of conical Lagrangian submanifolds of M_0^- and M_1;
2. Or a Lagrangian that is conical with respect to the cylindrical end $\Sigma\times[0,+\infty)$.

♢

Gao defines geometric composition for this type of Lagrangian correspondences and proves the analogue of Theorem 5.2.5 ([Gao17b, Theorem 1.5]). Moreover, he shows the open analogue of Theorem 5.2.6, namely that such a Lagrangian correspondence induces a functor of wrapped Fukaya categories ([Gao17a, Theorem 1.2]).

Below we focus on the type of correspondences we consider in this thesis, which arises as a special case of Example 5.2.3 for a non-compact coisotropic. It is a class of exact, embedded, but not closed correspondences between Liouville manifolds.

Setting. Let (M_0,ω_0) and (M_1,ω_1) be Liouville manifolds such that there is a fibration $q\colon M_1\longrightarrow M_0$ with Liouville fibres.

Let $\Gamma\subset M_0^-\times M_1$ be a Lagrangian correspondence obtained as the (transpose) graph of a proper fibration $p\colon V\to M_0$, where $V\subset M_1$ is a fibred coisotropic as in Example 5.2.3, and $q|_V=p$.

On $M_0^-\times M_1$ set the product almost complex structure $J_{01}=-J_0\times J_1$ for cylindrical almost complex structures on M_0 and M_1, such that the fibration $(id,q)\colon M_0^-\times M_1\longrightarrow M_0^-\times M_0$ is (J_{01},J_{00})-holomorphic, for $J_{00}=-J_0\times J_0$.

Then $\Gamma=\{(p(v),v),\ v\in V\}$ is properly fibred over the diagonal $\Delta_{M_0}:=\{(p(v),p(v)),\ v\in V\}=\subset$

$M_0^- \times M_0$ which is a conical Lagrangian correspondence in $M_0^- \times M_0$. However, the original correspondence Γ is not conical, or more generally admissible in the sense of Definition 5.2.8.

Consequently, the above setting doesn't exactly fit either the compact nor the open quilted theories, but is located between the two: it represents a class of non-compact correspondences which nevertheless induces a functor of compact Fukaya categories.

Axiom 1. The type of Lagrangian correspondence $\Gamma \subset M_0^- \times M_1$ defined in the above setting induces a functor

$$\begin{aligned} \Theta_{01} :&\mathcal{F}uk(M_0) \longrightarrow \mathcal{F}uk(M_1), \\ &\Theta_{01}(K_0) = L_1. \end{aligned} \tag{5.13}$$

Experts will recognise the validity of the above statement that we have set as an axiom. Proving it as a theorem would require a lengthy digression necessary to fill in all details covered in [WW10a, Gao17b, Gao17a]. In Lemma 5.2.9, we restrict to proving the invariance of quilted Floer cohomology under Lagrangian correspondences. Given invariance, the results of [WW10a] yield a functor on the cohomological category. The extension to an A_∞-functor, which would turn Axiom 1 into a theorem, can then be obtained by considering higher A_∞-products, which we omit here.

Lemma 5.2.9. *Let $K \subset M_0$, $L' \subset M_1$ be closed exact Lagrangians and consider the cycle of correspondences $(K,\Gamma,L') \subset (pt,M_0,M_1)$. Then the quilted Floer cohomology group $\mathrm{HF}(K,\Gamma,L')$ is well-defined and satisfies the invariance property*

$$\mathrm{HF}(K,\Gamma,L') \cong \mathrm{HF}(K \circ \Gamma, L') = \mathrm{HF}(L,L'). \tag{5.14}$$

Proof. By definition (see [WW10b, Section 4.3]), the generators of the cochain complex $\mathrm{CF}(K,\Gamma,L')$ are given by the generators of $\mathrm{CF}(K \times L,\Gamma)$. These intersection points must be contained in a compact region, since $K \subset M_0$ and $L' \subset M_1$ are closed Lagrangians. By [WW09, Proposition 2.2.1] the cochain groups $\mathrm{CF}(K \circ \Gamma, L') = \mathrm{CF}(L,L')$ and $\mathrm{CF}(K \times L,\Gamma)$ are isomorphic.

We now analyse the Floer trajectories involved in the computation of $\mathrm{HF}(K,\Gamma,L')$. By the maximum principle, the only non-compactness phenomenon that could occur would be a J_{01}-holomorphic curve escaping a compact set on the non-compact boundary condition Γ. However, all such curves, and any Floer trajectory of interest, are contained in a compact set, as we now explain.

By assumption, J_{01}-holomorphic curves with boundary conditions on (K,Γ,L') project under (id,q) to J_{00}-holomorphic curves involved in the complex for the tuple (K,Δ_{M_0},K'), where $K' \circ \Gamma = L'$ and $(id,q)(\Gamma) = \Delta_{M_0}$.

The (quilted) Floer cohomology group for the cycle of Lagrangian correspondences $(K,\Delta_{M_0},K') \subset (pt,M_0,M_0)$ can be defined as the Floer cohomology group $\mathrm{HF}(K,\Delta_{M_0},K') := \mathrm{HF}^*(K \times K',\Delta_{M_0})$

([Gao17b, Lemma 4.8]). Moreover by [Gao17a, Theorem 1.2], Δ_{M_0} induces the identity functor, so clearly all the J_{00}-holomorphic strips involved in the complex $\mathrm{CF}(K,\Delta_{M_0},K')$ are well-behaved, and moreover we have $\mathrm{HF}(K,\Delta_{M_0},K') \cong \mathrm{HF}(K,K')$.

Because of properness of $(q,id)|_\Gamma\colon \Gamma_{01} \to M_0 \times M_0$, if there was any J_{01}-holomorphic curve escaping to infinity at the boundary condition Γ, then it would project to a J_{00}-holomorphic curve escaping to infinity at the boundary condition on Δ_{M_0}, which cannot happen. □

Remark 5.2.10. *Let $K,K' \subset (M_0,\omega_0)$ be closed exact Lagrangians. Note that for any conical correspondence (not just the diagonal) $\Gamma_{00} \subset M_0^- \times M_0$, compactness of moduli spaces of curves involved in the quilted complex $\mathrm{CF}(K,\Gamma_{00},K')$ (for compact Lagrangians $K,K' \subset M_0$) is preserved. Namely, all intersection points lie in a compact region, so by exactness both energy and symplectic area are bounded. We can apply a reverse isoperimetric inequality according to which the length of the boundary of such a curve is bounded by a quantity proportional to its area ([GS14, Theorem 1.4]). This ensures that the boundary of all pseudoholomorphic curves is contained in a compact set, which can then be determined by using a monotonicity lemma like [SS05, Lemma 13]. Again, by exactness there is no bubbling so the moduli spaces of such curves are compact.*

//

5.3 Hopf correspondence

We can finally introduce the correspondence of interest; the *Hopf correspondence*. This a Lagrangian correspondence obtained as the graph of a spherically fibred coisotropic submanifold as in Example 5.2.3.

We use the discussions of Sections 5.1.1, 5.1.2 to explain how, for each type of Lagrangian projective space $K \cong \mathbb{AP}^n \subset W$, $\mathbb{A} \in \{\mathbb{R},\mathbb{C}\}$ in a Liouville manifold (W,ω) satisfying the appropriate cohomology Assumption (D), (C), there is a Lagrangian correspondence relating K to a Lagrangian sphere L in an auxiliary Liouville manifold (Y,Ω).

5.3.1 Lagrangian $\mathbb{CP}^n$

Let (W^{4n},ω) be a Liouville manifold admitting Lagrangian submanifolds $K_i \cong \mathbb{CP}^n \hookrightarrow W$, $i = 1,\dots,m$. Assume there is a class $\alpha \in H^2(W;\mathbb{Z})$ satisfying Assumption (C). The discussion of Section 5.1.1 delivers a $\mathbb{C}^*$-bundle $q\colon Y \longrightarrow W$ (associated to the complex line bundle $\mathcal{L} \to W$ with $c_1(\mathcal{L}) = \alpha$), whose total space is a Liouville manifold (Y,Ω) (proof of Proposition 5.1.2). Set $V := Y|_{\{r=1\}}$, the unit length bundle (determined by the metric on Y induced by a choice of hermitian metric on $\mathcal{L}$). If $\iota\colon V \longhookrightarrow Y$ is the inclusion, then by construction $\iota^*\Omega = q^*\omega|_V$, so the symplectic reduction of V by S^1 is given by (W,ω), and V is a fibred coisotropic submanifold of (Y,Ω) with S^1-fibre bundle structure $p = q|_V : V \to W$.

For any Lagrangian projective space $K_i \subset W$, the restriction $V|_{K_i} \to K_i$ is a Lagrangian sphere $L_i \cong S^{2n+1} \subset Y$.

Definition 5.3.1. The (transpose) graph

$$\Gamma := \{(p(y),y), y \in V\} \subset W^- \times Y \tag{5.15}$$

defines a Lagrangian correspondence ([Per08, Proposition 1.1]), which we call the *Hopf correspondence*. By construction, for $K_i \cong \mathbb{CP}^n \subset \{pt\} \times W$, the correspondence maps K_i to the embedded Lagrangian sphere $L_i := K_i \circ \Gamma \cong S^{2n+1} \subset \{pt\} \times Y$, $i = 1,\dots,m$ via geometric composition. ♢

Remark 5.3.2. *This Lagrangian correspondence can equivalently be thought of as a correspondence of the type of Example 5.2.4, where the coisotropic V is a regular level set of a Hamiltonian S^1 "Hopf"-action, and (W,ω) its symplectic quotient (note that the local models* (5.2), (5.7), (5.9) *are obtained from this perspective). This explains the choice of name for the correspondence.*

//

5.3.2 Lagrangian $\mathbb{RP}^n$

Let (W^{2n},ω) be a Liouville manifold admitting Lagrangian embeddings $K_i \cong \mathbb{RP}^n \hookrightarrow W$, $i = 1,\dots m$. Assume there is a cohomology class $\alpha \in H^1(W;\mathbb{Z}/2\mathbb{Z})$ satisfying Assumption D. Then, there is a Liouville manifold $(Y,\Omega) = (\widetilde{W}^{2n},\widetilde{\omega})$ obtained as symplectic double cover of W, and containing Lagrangian spheres $L_1,\dots,L_m \subset \widetilde{W}$. The double cover $q\colon \widetilde{W} \longrightarrow W$ defines an S^0-fibration over W, and in this case the "coisotropic submanifold" is the total space itself. As above, we define the Hopf correspondence as $\Gamma := \{(q(y),y), y \in \widetilde{W}\} \subset W^- \times \widetilde{W}$.

5.4 Commuting diagrams of twist functors

Let (W,ω) and (Y,Ω) be Liouville manifolds and $K_1,\dots K_m \subset W$ be real/complex projective Lagrangians satisfying Assumptions (D)/(C). Let $q\colon Y \longrightarrow W$ be the fibration we constructed in the previous subsections, and $\Gamma \subset W^- \times Y$ be the Hopf correspondence, obtained as the graph $\Gamma = \{(p(v),v),\ v \in V\}$ of the spherically fibred coisotropic $p = q|_V : V \to W$. This correspondence is properly fibred over the diagonal $\Delta_W = \{(p(v),p(v)),\ v \in V)\} \subset W^- \times W$, via $(id,q)\colon W \times Y \longrightarrow W \times W$ and satisfies the conditions of Axiom 1. Therefore, there is a well-defined functor

$$\Theta_\Gamma : \mathcal{F}\text{uk}(W) \to \mathcal{F}\text{uk}(Y),\ \Theta_\Gamma(K) = K \circ \Gamma =: L. \tag{5.16}$$

Let $L_1, \ldots, L_m \subset Y$ the Lagrangian spheres associated to $K_1, \ldots, K_m$ through the correspondence. For each $i = 1, \ldots m$, let $T_{K_i} \in Auteq(\mathcal{F}\mathrm{uk}(W))$ and $T_{L_i} \in Auteq(\mathcal{F}\mathrm{uk}(Y))$ be the (geometric) twists functors induced by the graphs of the respective twists $\tau_{K_i} \in \mathrm{Symp}_{ct}(W), \tau_{L_i} \in \mathrm{Symp}_{ct}(Y)$.

Corollary 5.4.1. *There is a commuting diagram at the level of compact Fukaya categories*

$$\begin{array}{ccc} \mathcal{F}\mathrm{uk}(Y) & \xrightarrow{T_{L_i}} & \mathcal{F}\mathrm{uk}(Y) \\ {\scriptstyle \Theta_\Gamma}\uparrow & & \uparrow{\scriptstyle \Theta_\Gamma} \\ \mathcal{F}\mathrm{uk}(W) & \xrightarrow{T_{K_i}} & \mathcal{F}\mathrm{uk}(W). \end{array} \tag{5.17}$$

In particular, iterative applications of this diagram yield

$$\Theta_\Gamma \circ \prod T_{K_i}^{k_i} = \prod T_{L_i}^{k_i} \circ \Theta_\Gamma. \tag{5.18}$$

Proof. Consider the functors T_{K_i} and T_{L_i} as correspondences induced by the graphs of the respective twists $\tau_{K_i} \in \mathrm{Symp}_{ct}(W), \tau_{L_i} \in \mathrm{Symp}_{ct}(Y)$. Then we have to check that the compositions of correspondences $\Theta_\Gamma \circ T_{K_i} = T_{L_i} \circ \Theta_\Gamma$, as Lagrangians in $W^- \times Y$, coincide. By construction, this equality amounts to the commutativity of diagrams (5.5) or (5.8), respectively. □

Remark 5.4.2. *Note that for a coefficient field of characteristic zero, the functor associated to the real projective twist has a different shape which produces a different diagram ([MW18b, Corollary 1.3]).* //

Remark 5.4.3. *Given a hypothetical mirror pair (X, M) for a symplectic manifold (M, ω) with $c_1(M) = 0$ and complex manifold (X, J) we can make the following observation.*

In [HT06], Huybrechts and Thomas conjectured that the functors induced by projective twists on the derived Fukaya category $D^\flat(\mathcal{F}\mathrm{uk}(M))$ should be mirror to a class of autoequivalences of $D^\flat(X)$, induced by so called $\mathbb{P}$-objects (see [HT06, Definition 1.1]). This is the analogue of the statement proved by Seidel that autoequivalences of $D^\flat(\mathcal{F}\mathrm{uk}(M))$ induced by Dehn twists should be mirror to autoequivalences of $D^\flat(X)$ induced by "spherical objects" (see [ST01, Definition 1.1]).

From this perspective, we can view the diagram (5.4.1) *as a conjectural mirror to the following situation.*

By [HT06, Proposition 1.4], a $\mathbb{P}$-object $\mathcal{P} \in D^\flat(X)$ in the central fibre of an algebraic deformation $j \colon X \hookrightarrow \mathcal{X}$ and satisfying $0 \neq A(\mathcal{P}) \cdot \kappa(\mathcal{X}) \in \mathrm{Ext}^2(\mathcal{P}, \mathcal{P})$ has an associated spherical object given by $j_*(\mathcal{P}) \in D^\flat(\mathcal{X})$. Here, $A(\mathcal{P}) \in \mathrm{Ext}^1(\mathcal{P}, \mathcal{P} \otimes \Omega^1_X)$ is the Atiyah class of $\mathcal{P}$ and $\kappa(\mathcal{X}) \in H^1(X, \mathcal{T}_X)$ the Kodaira-Spencer class of the family $\mathcal{X}$. Furthermore, the autoequivalences associated to each object *(also called "twists"), $T_{\mathcal{P}}$ and $T_{j_*\mathcal{P}}$, are related by a commutative diagram ([HT06, Proposition 2.7])*

$$\begin{array}{ccc} D^\flat(X) & \xrightarrow{j_*} & D^\flat(\mathcal{X}) \\ \downarrow T_{\mathcal{P}} & & \downarrow T_{j_*\mathcal{P}} \\ D^\flat(X) & \xrightarrow{j_*} & D^\flat(\mathcal{X}) \end{array} \tag{5.19}$$

//

5.5 Lagrangian Gysin sequence

Let $\Gamma \subset W^- \times Y$ be the Hopf correspondence. Given projective Lagrangian submanifolds $K, K' \subset W$ and their corresponding spherical lifts $L, L' \subset Y$ through the functor Θ_Γ, a version of Perutz's Gysin sequence of [Per08] can be used to establish a relationship between the ranks of the Floer cohomology groups $\mathrm{HF}(K,K')$ and $\mathrm{HF}(L,L')$. We will need this relation in the next chapter, for the proof of Theorem 3.

Let $V \to W$ be the S^k-fibred coisotropic defining the correspondence, $k \in \{0,1\}$, with Euler class $\alpha \in H^{k+1}(W;R)$, $R \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\}$ and Lagrangian projective spaces $K, K' \subset W$ satisfying Assumptions (D)/(C) respectively. Let $L = \Theta_\Gamma(K) = K \circ \Gamma \subset Y$, $L' = \Theta_\Gamma(K') = K' \circ \Gamma \subset Y$ be the associated Lagrangian sphere given by the correspondence.

Lemma 5.5.1. *There is an exact triangle of the shape*

$$\begin{array}{ccc} \mathrm{HF}^*(K,K') & \xrightarrow{\alpha\,\cup\,\cdot} & \mathrm{HF}^{*+k+1}(K,K') \\ & \nwarrow \qquad \swarrow_{\Gamma_*} & \\ & \mathrm{HF}^{*+k+1}(L,L') & \end{array} \tag{5.20}$$

Proof. This exact sequence follows from the Gysin triangle proved by Perutz in [Per08, Theorem 1], which has the more general form

$$\ldots \longrightarrow \mathrm{HF}^*(K,K') \xrightarrow{e(V)\cup\,\cdot} \mathrm{HF}^{*+k+1}(K,K') \xrightarrow{\Gamma_*} \mathrm{HF}^{*+k+1}(K,\Gamma^t,\Gamma,K') \longrightarrow \ldots \tag{5.21}$$

where the last group is the quilted Floer cohomology group of the cycle of Lagrangian correspondences $\underline{L} := (K,\Gamma,\Gamma^t,K') \subset (pt,W,Y,W)$, satisfying $\mathrm{HF}(K,\Gamma,\Gamma^t,K') \cong \mathrm{HF}^*(K\circ\Gamma, K'\circ\Gamma) \cong \mathrm{HF}(L,L')$. The isomorphism follows from Axiom 1 (in particular Lemma (5.2.9 applied to a sequence of four Lagrangian correspondences). The compositions $L = K\circ\Gamma$, $L' = \Gamma^t \circ K' = K'\circ\Gamma$ are embedded, and coincide with the spheres (which are Lagrangian in W) in the sphere bundle V over K, respectively K'.

The first map in the original exact sequence (5.20) is the quantum cup product with the Euler class $e(V) \in QH^*(W)$. In this case the exactness assumptions on the ambient symplectic manifold W

ensure the well-definedness of the operation and as $QH^*(W) \cong H^*(W)$ is a ring isomorphism, there is no quantum deformation involved and we obviously have $e(V) = \alpha \in H^*(W)$. The second map, Γ_*, is induced by the Lagrangian correspondence, and needs to be understood in the context of quilted Floer theory. We refer the reader to [Per08, Section 4.1] for a more refined description of the maps (in the setting of Hamiltonian Floer theory). □

Corollary 5.5.2. *The Gysin sequence produces the following rank inequality*

$$hf(L,L') := \operatorname{rank}\mathrm{HF}(L,L') \leqslant 2\operatorname{rank}\mathrm{HF}(K,K'). \tag{5.22}$$

In Chapter 6, we will need to compare functors induced by (projective and Dehn) twists to the identity functor. In particular, it will be necessary to distinguish objects of $\mathcal{F}\mathrm{uk}(W)$ with their image under the twist functors. The following lemma gives a helpful criterion.

Lemma 5.5.3. *Let* $K', \overline{K}'$ *be quasi-isomorphic objects in* $\mathcal{F}\mathrm{uk}(W)$. *Then the maps* $f_1\colon \mathrm{CF}^*(K,K') \xrightarrow{\alpha\cup\cdot} \mathrm{CF}^{*+k+1}(K,K')$ *and* $f_2\colon \mathrm{CF}^*(K,\overline{K}') \xrightarrow{\alpha\cup\cdot} \mathrm{CF}^{*+k+1}(K,\overline{K}')$ *have quasi-isomorphic mapping cones.*

Proof. Consider the long exact sequences associated to the mapping cones of the cup product maps $f_1\colon \mathrm{CF}^*(K,K') \longrightarrow \mathrm{CF}^{*+k+1}(K,K')$, $f_2\colon \mathrm{CF}^*(K,\overline{K}') \longrightarrow \mathrm{CF}^{*+k+1}(K,\overline{K}')$.

These sequences fit in a diagram of the shape

$$\begin{array}{ccccccc}
\mathrm{CF}^*(K,K') & \xrightarrow{f_1=\alpha\cup\cdot} & \mathrm{CF}^{*+k+1}(K,K') & \longrightarrow & Cone(f_1) & \longrightarrow & \mathrm{CF}^{*+k+1}(K,K') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{CF}^*(K,\overline{K}') & \xrightarrow{f_2=\alpha\cup\cdot} & \mathrm{CF}^{*+k+1}(K,\overline{K}') & \longrightarrow & Cone(f_2) & \longrightarrow & \mathrm{CF}^{*+k+1}(K,\overline{K}')
\end{array}$$

Since $K', \overline{K}'$ are quasi-isomorphic objects in $\mathcal{F}\mathrm{uk}(W)$, there is a characteristic element $\eta \in \mathrm{HF}(K',\overline{K}')$ which induces isomorphism $\mathrm{HF}(K,K') \to \mathrm{HF}(K,\overline{K}')$ (the Floer product with η, see [Sei08a, (8k)]). Therefore, the vertical maps $\mathrm{CF}^*(K,K') \longrightarrow \mathrm{CF}^*(K,\overline{K}')$ are well-defined, and they are quasi-isomorphisms. By the five lemma, it follows that the mapping cones $Cone(f_1)$ and $Cone(f_2)$ are also quasi-isomorphic.

□

Chapter 6

Free generation in plumbing of projective spaces

In this chapter we apply the Hopf correspondence to prove our first result about products of projective twists.

Consider a transverse plumbing $W := T^*\mathbb{AP}^n \#_{pt} T^*\mathbb{AP}^n$ of cotangent bundles of projective spaces, for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. Then, the main result of this chapter (Theorem 3) shows that the Lagrangian cores of the plumbing define two projective twists which generate a free subgroup of $\pi_0(\mathrm{Symp}_{ct}(W))$. In fact, Theorem 3 is a stronger statement that holds not only for transverse plumbings, but also more generally for *clean* plumbings along sub-projective spaces (see Definition 6.1.1).

For the proof, we use the Hopf correspondence to reduce the statement of Theorem 3 into a statement about Dehn twists, and apply *Keating's free generation result* (Theorem 6.2.2) for Dehn twists ([Kea13]).

As a corollary, we show that there are infinitely many Lagrangian isotopy classes of embeddings $\mathbb{CP}^n \hookrightarrow W$ which are smoothly isotopic, but pairwise not Lagrangian isotopic.

6.1 Clean Lagrangian plumbing

We first recall a construction from [Abo11, Appendix A] of clean *Lagrangian plumbing* of two Riemannian manifolds Q_1, Q_2 along a submanifold $B \subset Q_i$, $i = 1, 2$. Fix three closed smooth manifolds B, Q_1, Q_2, for each $i = 1, 2$ an embedding $B \hookrightarrow Q_i$ and an isomorphism $\rho : \nu_{B/Q_1} \to \nu^*_{B/Q_2}$ from the normal bundle ν_{B/Q_1} to the conormal bundle ν^*_{B/Q_2}.

Pick a Riemannian metric on B, an inner product and a connection on $\nu_{B/Q_1} \cong \nu^*_{B/Q_2}$ (which induces an inner product and connection on $\nu_{B/Q_2} \cong \nu^*_{B/Q_1}$). This data induces a metric on the total spaces

ν_{B/Q_i}, and a neighbourhood U_i of $B \subset Q_i$ can be identified with a disc subbundle $D_\varepsilon \nu_{B/Q_i}$ of radius $\varepsilon > 0$. With this identification we write $x \in U_i$ as $x = (a,b)$ for $b \in B$, $a \in D_\varepsilon(\nu_{B/Q_i})_b$ (the fibre over b).

For each $x = (a,b) \in U_i$, the connection gives a decomposition of the fibres $T^*_x Q_i \cong T^*_b B \oplus (\nu^*_{B/Q_i})_b$. We get an identification of a neighbourhood of $B \subset T^*Q_i$ as

$$D_\varepsilon \nu_{B/Q_i} \oplus D_\varepsilon T^*B \oplus D_\varepsilon \nu^*_{B/Q_i}. \tag{6.1}$$

Let V_i be a neighbourhood of $Q_i \subset T^*Q_i$ which in (6.1) coincides with $D_\varepsilon T^*B \oplus D_\varepsilon \nu^*_{B/Q_i}$ over $U_i \cong D_\varepsilon \nu_{B/Q_i}$.

Definition 6.1.1. 1. As a smooth manifold, the clean plumbing of Q_1, Q_2 along B, denoted by $M := D_\varepsilon(T^*Q_1 \#_B T^*Q_2)$ is defined by gluing V_1 to V_2 along $D_\varepsilon \nu_{B/Q_1} \oplus D_\varepsilon T^*B \oplus D_\varepsilon \nu^*_{B/Q_1} \subset V_1$ identified with $D_\varepsilon \nu^*_{B/Q_2} \oplus D_\varepsilon T^*B \oplus D_\varepsilon \nu_{B/Q_2}$ via $(\rho, id_{T^*B}, -\rho^*)$. Its Liouville completion will be denoted by $T^*Q_1 \#_B T^*Q_2$.

2. The plumbing construction inherits an exact symplectic structure, since the identification maps of 1. preserve the canonical structures on D^*Q_i. Let Z_i be the standard radial Liouville vector field on V_i. We define a Liouville vector field Z on the plumbing by letting $Z = \rho_1 Z_1 + \rho_2 Z_2$, for smooth functions $\rho_i : M \to [0,1]$ supported on V_i such that $\rho_1 + \rho_2 = 1$. This endows M with the structure of an exact symplectic manifold.

$\diamondsuit$

In the next sections, we will apply this plumbing construction to cotangent bundles of projective spaces and spheres. We will work with (ungraded) Floer cohomology groups $\mathrm{HF}(Q_1, Q_2, k)$ where k is a coefficient field of characteristic two. Note that by exactness of the Lagrangians and the manifold W, the Floer differentials in $\mathrm{CF}(Q_i, Q_i)$ vanish, and as Q_1 and Q_2 intersect cleanly along B, there is an isomorphism $\mathrm{HF}(Q_1, Q_2) \cong H^*(B)$ ([Poź94]).

6.2 Theorem 3

We now prove the main theorem of this chapter.

Theorem 3. *Let $W = T^*\mathbb{AP}^n \#_{\mathbb{AP}^\ell} T^*\mathbb{AP}^n$ be a clean plumbing of (real, complex) projective spaces along a linearly embedded sub-projective space $\mathbb{AP}^\ell \subset W$, $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}\}$. Let $K_1, K_2 \subset W$ denote the Lagrangian core components of the plumbing. Then the projective twists τ_{K_1} and τ_{K_2} generate a free group inside $\pi_0(\mathrm{Symp}_{ct}(W))$, and the associated functors T_{K_1}, T_{K_2} generate a free subgroup of $Auteq(\mathcal{F}\mathrm{uk}(W))$.*

Remark 6.2.1. *The case* $W := T^*\mathbb{CP}^1_1 \#_{pt} T^*\mathbb{CP}^1_2$ *can be deduced from the existing literature, by considering* X *as an* A_2*-configuration and the isotopies* $\tau_{\mathbb{CP}^1_i} \simeq \tau^2_{S^2_i}$ *of Section 2. There is a homomorphism ([Sei99, Proposition 8.4])* $\rho : Br_3 \to \pi_0(\mathrm{Symp}_{ct}(W))$ *sending the generators of the braid group* σ_i *to* $\rho(\sigma_i) = \tau_{S^2_i}$, $i = 1,2$. *The associated homomorphism* $\hat{\rho} : Br_3 \to \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ *fits in the diagram*

$$\begin{array}{ccc} Br_3 & \xrightarrow{\ \rho\ } & \pi_0(\mathrm{Symp}_{ct}(W)) \\ & \searrow^{\hat{\rho}} & \downarrow \\ & & Auteq(\mathcal{F}\mathrm{uk}(W)) \end{array} \quad . \tag{6.2}$$

and its injectivity ([ST01]) implies the injectivity of ρ. *Then, as* $\langle \sigma_1^2, \sigma_2^2 \rangle \cong Free_2$, *it follows* $\langle \tau_{K_1}, \tau_{K_2} \rangle \cong Free_2$.
Also note that ρ *is in fact an isomorphism ([Wu14]), so* $\pi_0(\mathrm{Symp}_{ct}(T^*S^2 \#_{pt} T^*S^2)) = Br_3$. //

Theorem 3 takes inspiration from Keating's free generation result for Dehn twists in Liouville manifolds.

Theorem 6.2.2. *[Kea13, Theorem 1.1 and 1.2] Let* (Y, Ω) *be a Liouville manifold of dimension greater than* 2, *and* $L, L' \subset Y$ *be two Lagrangian spheres satisfying* $\mathrm{rank}\,\mathrm{HF}(L, L') \geqslant 2$, *and such that* L, L' *are not quasi-isomorphic in* $\mathcal{F}\mathrm{uk}(Y)$. *Then the Dehn twists* $\tau_L, \tau_{L'}$ *generate a free subgroup of* $\pi_0(\mathrm{Symp}_{ct}(Y))$, *and the associated functors* $T_L, T_{L'} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ *generate a free subgroup of* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$.

Keating proves the geometric part of Theorem 3 by making a categorical detour, first proving that the associated functors $T_L, T_{L'} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ induced by the Dehn twists generate a free subgroup of $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ so that the composition

$$Free_2 \to \pi_0 \mathrm{Symp}_{ct}(Y) \to \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y)) \tag{6.3}$$

is injective.

By identifying a Dehn twist with its associated functor, Keating exploits the algebraic properties of the latter to arrive at the following rank inequalities (which are central in her final proof).

Lemma 6.2.3. *[Kea13, Lemma 8.1] Let* $\tilde{L}, L, L' \subset Y$ *be Lagrangians such that* $\tilde{L}$ *is a sphere,* $\tilde{L} \ncong L$ *in the Fukaya category, and* $hf(\tilde{L}, L) := \mathrm{rank}(\mathrm{HF}(\tilde{L}, L')) \geqslant 2$. *Then, for all* $n \neq 0$*:*

$$hf(\tilde{L}, L') > hf(L, L') \Rightarrow hf(\tilde{L}, \tau^n_{\tilde{L}}(L')) < hf(L, \tau^n_{\tilde{L}}(L')). \tag{6.4}$$

Lemma 6.2.4. *[Kea13, Claim 8.2] Let* $L, L' \subset Y$ *be two Lagrangian spheres in an exact symplectic manifold as in Theorem 6.2.2 satisfying* $hf(L, L') = 2$. *Then for all* $m \neq 0$ *we have*

$$hf(L', L) = hf(L', \tau^m_{L'} L) < hf(L, \tau^m_{L'} L). \tag{6.5}$$

We will apply these inequalities to Lagrangian spheres obtained from the Hopf correspondence, to produce similar results for projective twists and prove Theorem 3.

6.2.1 Strategy

The plumbing (W,ω) and its real/complex projective Lagrangian cores $K_1, K_2 \subset W$ satisfy the cohomological conditions (D)/(C).

In the case in which W is a transverse plumbing (which retracts to the wedge sum of the two spheres) there is a ring isomorphism ($\mathbb{A} \in \{\mathbb{R},\mathbb{C}\}$, $R \in \{\mathbb{Z}/2\mathbb{Z},\mathbb{Z}\}$)

$$\tilde{H}^*(W;R) \cong \tilde{H}^*(K_1;R) \oplus \tilde{H}^*(K_2;R) \cong \tilde{H}^*(\mathbb{AP}^n;R) \oplus \tilde{H}^*(\mathbb{AP}^n;R),$$

so that it is immediate to see the existence of a class $\alpha = (\alpha_1,\alpha_2) \in H^k(W;R)$ restricting to K_1 and K_2 to the generator of $H^*(\mathbb{AP}^n;R)$, for $k \in \{1,2\}$. For a clean plumbing along a linearly embedded sub-projective space $\mathbb{AP}^\ell$, this restriction property still holds because the 'difference' map of the Mayer-Vietoris sequence is always zero.

By Propositions 5.1.2, 5.1.6, the cohomological conditions ensure the existence of a Liouville manifold $(Y,\Omega) \longrightarrow (W,\omega)$ and a Hopf correspondence $\Gamma \subset W^- \times Y$ that gives rise to associated Lagrangian spheres $S^m \cong L_i = K_i \circ \Gamma \subset Y$, $i = 1,2$ and commuting diagrams of twist functors (5.17). Then, given a product (a word in τ_{K_1}, τ_{K_2}) $\varphi \in \mathrm{Symp}_{ct}(W)$ of projective twists, the Hopf correspondence yields a corresponding product of Dehn twists (a word in τ_{L_1}, τ_{L_2}) $\phi \in \mathrm{Symp}_{ct}(Y)$.

Remark 6.2.5. *In the real projective case, the geometric statement of Theorem 3 can be obtained by an isotopy-lifting argument using the geometric diagrams of Section 5.1 (the strategy adopted in Section 7.3). Assuming the projective twists do satisfy a relation, this procedure lifts the isotopy to* $\mathrm{Symp}_{ct}(Y)$*, producing a relation between Dehn twists, which cannot hold by Keating's theorem. However, this geometric argument does not give a statement at the level of Fukaya categories, for which the use of the Hopf correspondence at the level of functors in* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ *is necessary (Sections 5.4, 5.5).*

//

The spheres $L_1, L_2 \cong S^m$ intersect cleanly along a sub-sphere S^r, for the tuple $(\mathbb{A},m,r) \in \{(\mathbb{R},n,\ell),(\mathbb{C},2n+1,2\ell+1)\}$, and as noted before, $\mathrm{HF}(L_1,L_2;k) \cong H^*(S^r;k)$. Since $L_i \subset Y$ are exact spheres $\mathrm{HF}(L_i,L_i;k) \cong H^*(S^m;k)$ ([Flo88]) and therefore

$$\operatorname{rank}\mathrm{HF}(L_1,L_2) = \operatorname{rank}\mathrm{HF}(L_i,L_i) = 2,\ i = 1,2. \tag{6.6}$$

In the following sections we will study the ranks of the Floer cohomology groups $\mathrm{HF}(\cdot,\varphi(\cdot))$ and show that there is always a Lagrangian $\hat{K} \subset W$ such that

$$\mathrm{HF}(\hat{K},\hat{K}) \ncong \mathrm{HF}(\hat{K},\varphi(\hat{K})). \tag{6.7}$$

As a result, $\hat{K}$ and $\varphi(\hat{K})$ are not quasi-isomorphic objects in $\mathcal{F}\mathrm{uk}(W)$, and therefore the functor induced by φ cannot be isomorphic to the identity in $\mathrm{Auteq}\,\mathcal{F}\mathrm{uk}(W)$. This will also rule out the possibility of φ being isotopic to the identity in $\mathrm{Symp}_{ct}(W)$.

We prove (6.7) by applying the rank inequalities (6.4), (6.5) to Lagrangian spheres in (Y,Ω) obtained via the correspondence Γ, in combination with the symplectic Gysin sequence associated to Γ (Corollary 5.5.2).

In the following section we reiterate a part of Keating's proof of Theorem 6.2.2 using the rank inequalities (6.4), (6.5) that hold for the word of Dehn twists $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle$ associated to φ via the Hopf correspondence. This will clarify the methods used in proving the analogous statement for projective twists (namely, Theorem 3) in Section 6.2.3.

6.2.2 Associated word of Dehn twists.

Let $\varphi \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{ct}(W)$ be a word of projective twists as in the statement of Theorem 3. Consider the Hopf correspondence $\Gamma \subset W^- \times Y$ and the associated word of Dehn twists $\phi \in \mathrm{Symp}_{ct}(Y)$ as above. In this section we replicate the last steps in Keating's proof of the injectivity of the homomorphism

$$Free_2 \longrightarrow \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W)).$$

We first make the following observation about a word of twists and its conjugates.

Lemma 6.2.6. *Let $\phi \in \mathrm{Symp}_{ct}(Y)$ be a symplectomorphism which has the shape of global conjugate, i.e $\phi = \psi^{-1}\phi'\psi$, for $\psi, \phi' \in \mathrm{Symp}_{ct}(Y)$ not isotopic to the identity. Then there is a closed Lagrangian $\tilde{L} \subset Y$ such that $\mathrm{HF}(\tilde{L},\phi(\tilde{L})) \ncong \mathrm{HF}(\tilde{L},\tilde{L})$ if and only if ϕ' satisfies $\mathrm{HF}(\hat{L},\phi'(\hat{L})) \ncong \mathrm{HF}(\hat{L},\hat{L})$ for some closed Lagrangian $\hat{L} \subset Y$.*

Proof. Assume there is a Lagrangian $\hat{L} \subset Y$ such that $\mathrm{HF}(\hat{L},\phi'(\hat{L})) \ncong \mathrm{HF}(\hat{L},\hat{L})$. Then by invariance of Floer cohomology under symplectomorphisms $\mathrm{HF}(\psi^{-1}\hat{L},\psi^{-1}\phi'\psi(\psi^{-1}\hat{L})) \cong \mathrm{HF}(\psi^{-1}\hat{L},\psi^{-1}(\phi'\hat{L}) \cong \mathrm{HF}(\hat{L},\phi'\hat{L}) \ncong \mathrm{HF}(\hat{L},\hat{L})$, so for $\tilde{L} := \psi^{-1}(\hat{L})$ we have $\mathrm{HF}(\tilde{L},\phi(\tilde{L})) \ncong \mathrm{HF}(\tilde{L},\tilde{L})$. The other direction is similar. □

We will apply the above lemma to a word of Dehn twists $\phi \in \mathrm{Symp}_{ct}(Y)$ which in its reduced has the shape of a global conjugate, i.e a word $\phi = \psi^{-1}\phi'\psi$, for two reduced words $\psi, \phi' \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{ct}(Y)$ not isotopic to the identity. Then the lemma shows that it is always possible to switch between ϕ and its conjugate, as the correct choice of Lagrangian keeps track of the Floer cohomological action of the original word.

Without loss of generality, we can therefore use this conjugation argument to restrict the focus on reduced words $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle$ which are either (for $i, j \in \{1,2\}$):

1. A power of a single Dehn twist, i.e $\phi = \tau_{L_i}^s$, $s \in \mathbb{Z}^*$ (Lemma 6.2.7).

2. A word starting with a power of τ_{L_i} and ending in a power of τ_{L_j}, $i \neq j$ (Lemma 6.2.9).

Lemma 6.2.7. *Let $\phi = \tau_{L_i}^s \in \mathrm{Symp}_{ct}(Y)$ be a reduced word of Dehn twists which is a power of a single Dehn twist, $i, j \in \{1,2\}$, $s \in \mathbb{Z}^*$. The associated functor is not isomorphic to the identity in* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$*, so in particular, ϕ cannot be isotopic to the identity in* $\mathrm{Symp}_{ct}(Y)$.

Proof. We show that there exists a closed Lagrangian $\hat{L} \subset Y$ such that

$$\mathrm{HF}(\hat{L}, \phi(\hat{L})) \ncong \mathrm{HF}(\hat{L}, \hat{L}).$$

For $\phi = \tau_{L_i}^s$, a possible candidate is given by $\hat{L} = L_j$, $i, j \in \{1,2\}$, $i \neq j$.

Namely, the rank inequality stated by Lemma 6.2.4, gives

$$2 = hf(L_j, L_j) = hf(L_i, L_j) = hf(L_i, \tau_{L_i}^s L_j) < hf(L_j, \tau_{L_i}^s L_j).$$

□

Remark 6.2.8. *The geometric result of the above lemma can also be proven independently from Keating's results, as a corollary to Theorem 1.* //

Lemma 6.2.9. *Let $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{ct}(Y)$ be a reduced word of Dehn twists around the Lagragian (spherical) cores which is a product where the first and last factors are powers of distinct Dehn twists. Then the functor associated to ϕ is not isomorphic to the identity in* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$ *so in particular ϕ cannot be isotopic to the identity in* $\mathrm{Symp}_{ct}(Y)$.

Proof. We show that there is a closed Lagrangian $\hat{L} \subset Y$ such that $\mathrm{HF}(\hat{L}, \phi\hat{L}) \ncong \mathrm{HF}(\hat{L}, \hat{L})$.

We can assume w.l.o.g that the first factor of ϕ to be a power of τ_{L_2} and the last is a power of τ_{L_1} (otherwise consider ϕ^{-1}), so that we have a word of the following shape

$$\phi = \tau_{L_2}^{b_k} \tau_{L_1}^{a_k} \cdots \tau_{L_2}^{b_1} \tau_{L_1}^{a_1},\ a_i, b_i \in \mathbb{Z}^* \text{ for } 1 \leqslant i \leqslant k. \tag{6.8}$$

In the case we are considering, we have $hf(L_1, L_2) = 2$. Apply Lemma 6.2.4 to get

$$2 = hf(L_1, L_1) = hf(L_2, L_1) = hf(L_2, \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1) < hf(L_1, \tau_{L_2}^{b_1} \tau_{L_1}^{a_1} L_1).$$

Now apply Lemma 6.2.3 (with $n = a_2$, $\tilde{L} = L_1, L = L_2$ and $L' = \tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1$) and get

$$hf(L_1, \tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1) = hf(L_1, \tau_{L_1}^{a_2}\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1) < hf(L_2, \tau_{L_1}^{a_2}\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1).$$

Apply Lemma 6.2.3 again (with $n = b_2$, $\tilde{L} = L_2, L = L_1$ and $L' = \tau_{L_1}^{a_2}\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1$)

$$hf(L_2, \tau_{L_1}^{a_2}\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1) = hf(L_2, \tau_{L_2}^{b_2}\tau_{L_1}^{a_2}\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1) < hf(L_1, \tau_{L_2}^{b_2}\tau_{L_1}^{a_2}\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1).$$

Continue to apply Lemma 6.2.3 iteratively until the final step

$$hf(L_2, \tau_{L_2}^{b_k}\tau_{L_1}^{a_k}\cdots\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1) < hf(L_1, \tau_{L_2}^{b_k}\tau_{L_1}^{a_k}\cdots\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1).$$

Then

$$hf(L_1, \tau_{L_2}^{b_k}\tau_{L_1}^{a_k}\cdots\tau_{L_2}^{b_1}\tau_{L_1}^{a_1}L_1) > 2 + 2k - 1 = 2k + 1.$$

So setting $\hat{L} = L_1$ we have $\mathrm{HF}(\hat{L}, \phi(\hat{L})) \ncong \mathrm{HF}(\hat{L}, \hat{L})$. □

Corollary 6.2.10. *Let $\phi \in \langle \tau_{L_1}, \tau_{L_2} \rangle \subset \mathrm{Symp}_{ct}(Y)$ be a word of Dehn twists that is a product of the shape* (6.8)*. Then there is a Lagrangian $\hat{L} \subset Y$ such that*

$$\lim_{s\to\infty} \mathrm{rank}\, HF^*(\hat{L}, \phi^s(\hat{L})) = \infty.$$

Proof. Let ϕ be of the shape (6.8). Then

$$\phi^s = (\tau_{L_2}^{b_k}\tau_{L_1}^{a_k}\cdots\tau_{L_2}^{b_1}\tau_{L_1}^{a_1})(\cdots)(\cdots)(\tau_{L_2}^{b_k}\tau_{L_1}^{a_k}\cdots\tau_{L_2}^{b_1}\tau_{L_1}^{a_1})$$

has 'factor length' $k \cdot s$ (in the sense of (6.8)). By the proof of Lemma 6.2.9, the rank of $\mathrm{HF}(L_1, \phi(L_1))$ depends on the number $k \in \mathbb{N}$ appearing in the factor decomposition of ϕ. Therefore

$$hf(L_1, \phi^s(L_1)) > 2ks + 1$$

so we can set $\hat{L} := L_1$. □

6.2.3 Proof

We now go back to the original word $\varphi \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{ct}(W)$ of projective twists, and we show that it cannot induce the identity functor in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$. Lemma 6.2.6 holds for any symplectomorphism, so by the same conjugation argument explained before, we can focus the attention on words that are either (for $i, j \in \{1, 2\}$)

1. A power of a single twist $\varphi = \tau_{K_i}^s$, $s \in \mathbb{Z}^*$.
2. A mixed product of the shape $\varphi := \tau_{K_i}^{b_k}\tau_{K_j}^{a_k}\cdots\tau_{K_i}^{b_1}\tau_{K_j}^{a_1} \in \mathrm{Symp}_{ct}(W)$, $i \neq j$, $a_m, b_m \in \mathbb{Z}^*$, $1 \leqslant m \leqslant k$.

Proposition 6.2.11. *Let $\varphi = \tau_{K_i}^s \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{ct}(W)$ be a reduced word of projective twists which is a power of a single twist, $i \in \{1,2\}$, $s \in \mathbb{Z}^*$. Then the functor induced by φ is not isomorphic to the identity in* $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$*, and in particular φ cannot be isotopic to the identity in* $\mathrm{Symp}_{ct}(W)$.

Proof. Let $\varphi = \tau_{K_i}^s \in \mathrm{Symp}_{ct}(W)$, $s \in \mathbb{Z}^*$. Assume by contradiction that the functor induced by φ (still denoted by φ) is isomorphic to the identity, so that any Lagrangian $\hat{K} \subset W$ is quasi-isomorphic, as an object of $\mathcal{F}\mathrm{uk}(W)$, to $\varphi(\hat{K})$.

By Lemma 5.5.3, there is a quasi-isomorphism of the mapping cones of the cup product maps $f_1\colon \mathrm{CF}^*(\hat{K},\hat{K}) \longrightarrow \mathrm{CF}^{*+k+1}(\hat{K},\hat{K})$ and $f_2\colon \mathrm{CF}^*(\hat{K},\varphi(\hat{K})) \longrightarrow \mathrm{CF}^{*+k+1}(\hat{K},\varphi(\hat{K}))$ (we are considering ungraded Floer cohomology groups so technically the degrees are irrelevant here). Therefore, by the exact triangle of Lemma 5.5.1, if $\hat{L} \subset Y$ is the Lagrangian lift of $\hat{K}$ through the correspondence Γ, and $\phi \in \mathrm{Symp}_{ct}(Y)$ the symplectomorphism associated to φ, then $\mathrm{HF}(\hat{L},\hat{L}) \cong \mathrm{HF}(\hat{L},\phi(\hat{L}))$.

So if we set $\hat{K} := K_j$, $j \neq i$, by assumption we have $\mathrm{HF}(K_j,\varphi(K_j)) \cong \mathrm{HF}(K_j,K_j)$ and the above argument yields $\mathrm{HF}(L_j,\phi(L_j)) = \mathrm{HF}(L_j,\tau_i^s(L_j)) \cong \mathrm{HF}(L_j,L_j)$ which is clearly in contradiction to (the proof of) Lemma 6.2.7 (according to which these two groups have distinct ranks). Hence, φ cannot be isomorphic to the identity functor in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$. $\square$

Proposition 6.2.12. *Let $\varphi \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{ct}(W)$ be a reduced word of projective twists around the Lagragian cores which is a product where the first and last factors are powers of distinct projective twists. Then the functor induced by φ is not isomorphic to the identity in* $\mathrm{Auteq}\mathcal{F}\mathrm{uk}(W)$*, so in particular φ is not isotopic to the identity in* $\mathrm{Symp}_{ct}(W)$.

Proof. By the analogous discussion as in the proof of Lemma 6.2.9, it is enough to prove the statement for a word whose reduced form is of the shape

$$\varphi := \tau_{K_2}^{b_k}\tau_{K_1}^{a_k}\cdots\tau_{K_2}^{b_1}\tau_{K_1}^{a_1} \in \mathrm{Symp}_{ct}(W),\ a_m,b_m \in \mathbb{Z}^*,\ 1 \leqslant m \leqslant k. \tag{6.9}$$

Denote the product of twist functors induced by (6.9) also by $\varphi \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$. By iteratively using commutativity of the functors in diagram (5.17), one can define the corresponding composition of twist functors $\phi := T_{L_2}^{b_k}T_{L_1}^{a_k}\cdots T_{L_2}^{b_1}T_{L_1}^{a_1} \in \mathrm{Auteq}(\mathcal{F}\mathrm{uk}(Y))$, which by Theorem 6.2.2 cannot be isomorphic to the identity functor.

Moreover, Corollary 6.2.10 shows that not only is $HF(L_1,\phi(L_1))$ not only non-isomorphic to $HF(L_1,L_1)$, but also that $\lim_{s\to\infty} hf(L_1,\phi^s(L_1)) \to \infty$.

The Lagrangian $L_1 = \Gamma \circ K_1$ is the Lagrangian associated to K_1 via the Hopf correspondence, and the symplectic Gysin exact sequence (Corollary 5.5.2) applied to the Hopf correspondence gives the inequality

$$hf(L_1, \phi(L_1)) \leqslant 2hf(K_1, \varphi(K_1)), \tag{6.10}$$

which implies that the rank $hf(K_1, \varphi^s(K_1))$ also grows at least linearly with s. □

Corollary 6.2.13. *Let $\varphi \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{ct}(W)$ be a word of projective twists of the shape* (6.9). *Then there is a Lagrangian $\hat{K} \subset W$ such that*

$$\lim_{s \to \infty} \operatorname{rank} HF^*(\hat{K}, \varphi^s(\hat{K})) = \infty.$$

□

Finally, we can summarise the proof of Theorem 3.

Proof of Theorem 3. Let $\varphi \in \langle \tau_{K_1}, \tau_{K_2} \rangle \subset \mathrm{Symp}_{ct}(W)$ a word in the projective twists along the Lagrangian cores of W.

1. If the word has the shape $\varphi = \tau^s_{K_i} \in \mathrm{Symp}_{ct}(W)$, $i \in \{1,2\}$, $s \in \mathbb{Z}^*$, then its induced functor is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ by Proposition 6.2.11.
2. If the word has the shape $\varphi := \tau^{b_k}_{K_i} \tau^{a_k}_{K_j} \cdots \tau^{b_1}_{K_i} \tau^{a_1}_{K_j} \in \mathrm{Symp}_{ct}(W)$, $i, j \in \{1,2\}$ $i \neq j$, $a_m, b_m \in \mathbb{Z}^*$, $1 \leqslant m \leqslant k$, then its induced functor is not isomorphic to the identity in $\mathrm{Auteq}(\mathcal{F}\mathrm{uk}(W))$ by Proposition 6.2.12.
3. If φ has any other form, then it must be a conjugate of a word of shape 1. or 2. and hence the induced functor is not isomorphic to the identity by Corollary 6.2.6.

□

6.3 Knotted Lagrangian projective spaces

The phenomenon that a single (smooth) isotopy class of submanifolds contains infinitely many Lagrangian isotopy classes is called Lagrangian “knottedness” ([Sei99**?** , Hin12, LW12, Wu14]). Often, the quest for knottedness is intimately related to the study of isotopy classes of Dehn twists.

In the plumbing of spheres $L_i \cong S^2$, $Y := T^*L_1 \#_{pt} T^*L_2$, we know that for any $r \in \mathbb{Z}$, $\tau^{2r}_{L_2}(L_1)$ is smoothly isotopic to the identity, but not symplectically; as first shown by Seidel ([Sei99, Theorem 1.1]), none of the powers $\tau^{2r}_{L_2}(L_1)$ are Hamiltonian isotopic. Our results yield the analogue for plumbing of complex projective spaces (of any dimension).

Corollary 6.3.1. *Let $W := T^*\mathbb{CP}^n \#_{\mathbb{CP}^\ell} T^*\mathbb{CP}^n$ be a clean plumbing along a projective subspace $\mathbb{CP}^\ell \subset \mathbb{CP}^n$. Each Lagrangian core $K_i \cong \mathbb{CP}^n$ of W defines a smooth isotopy class which contains infinitely many symplectic isotopy classes of Lagrangian projective spaces.*

Proof. Let $K_1, K_2 \cong \mathbb{CP}^n \subset W$ be the two Lagrangian cores of the plumbing. For $i, j \in \{1,2\}$, $i \neq j$, define the element $\varphi := \tau_{K_i}\tau_{K_j}$. Then by Proposition 6.2.12 we have

$$\lim_{s\to\infty} \operatorname{rank} \mathrm{HF}(K_j, \varphi^s(K_j)) = \infty \tag{6.11}$$

which in particular means that $\varphi^{s_a}(K_j)$ is not Lagrangian isotopic to $\varphi^{s_b}(K_j)$ for any $s_a \neq s_b$, despite being smoothly isotopic (by Theorem 2.1.5). □

Remark 6.3.2.

1. *The low dimensional case $n = 1$ corresponds to a transverse plumbing of spheres $W := T^*L_1 \#_{pt} T^*L_2$, $L_i \cong S^2$. In that case, the symplectic mapping class group $\pi_0(\mathrm{Symp}_{ct}(W))$ is generated by the Dehn twists τ_{L_1} and τ_{L_2} (see Remark 6.2.1). Moreover, Hind ([Hin12]) proved that for any Lagrangian sphere $L \subset W$, there is a word $\tau \in \langle \tau_{L_1}, \tau_{L_2} \rangle$ such that $\tau(L)$ is isotopic to one of the cores L_1 or L_2.*

2. *For higher dimensional spheres, it is known ([DRE15]) that the two twists cannot always generate the whole symplectic mapping class group, because there could be Dehn twists not Hamiltonian isotopic to τ_{L_i} (coming from a non-standard choice of framing, see Section 8.2). On the other hand, it is not known wheter Hind's result generalises for higher dimensional spheres.*

3. *In the general projective case for $n > 1$, we know by Chapter 8 of this thesis that the homomorphism $Free_2 \hookrightarrow \pi_0(\mathrm{Symp}_{ct}(W))$ is in general not surjective, due to the presence of projective twists with non-standard framing. As for high dimensional spheres, it is still open whether there exist Lagrangian projective spaces in $T^*\mathbb{CP}^n_1 \#_{pt} T^*\mathbb{CP}^n_2$ that are not isotopic to an image of one of the cores under a composition of (standard) projective twists $\tau_{\mathbb{CP}^n_1}, \tau_{\mathbb{CP}^n_2}$.*

//

Chapter 7

Positive products of twists in Liouville manifolds

The present chapter covers our results about products of positive powers of Dehn and projective twists.

In the first part, Section 7.1, we analyse products of (positive powers of) Dehn twists. We re–prove a theorem by Barth–Geiges–Zehmisch (Theorem 1) asserting that for a Liouville manifold (M,ω), no product $\phi \in \mathrm{Symp}_{ct}(M)$ of positive powers of Dehn twists can be symplectically isotopic to the identity. We provide an alternative proof that was suggested by Paul Seidel. Based on symplectic Picard–Lefschetz theory, the argument for the proof relies on a count of pseudoholomorphic sections of a Lefschetz fibration constructed from the data given by ϕ and the Lagrangian spheres associated to the Dehn twists.

Using similar tools, we then prove Theorem 2 (Section 7.2), which can be interpreted as a relative version of Theorem 1. This states that a Liouville manifold (M,ω) containing Lagrangian spheres and a conical Lagrangian disc T (Definition 5.2.7) intersecting one of the spheres transversely at a point cannot admit a positive product of Dehn twists preserving the Lagrangian up to compactly supported symplectic isotopy.

In Section 7.3, we explore the analogous questions for projective twists, by means of the tools developed in Chapter 5. After setting the necessary conditions to ensure the existence of the Hopf correspondence, we use Theorem 1 to prove a comparable result for real projective twists.

7.1 Product of Dehn twists

In this section we focus on the properties of *positive words*—i.e products of positive powers—of Dehn twists in Liouville manifolds. We identify such symplectomorphisms with monodromies of

Lefschetz fibrations and question whether these can be symplectically isotopic to the identity. This approach leads to an alternative proof to the following theorem which currently can be found in [BGZ19].

Theorem 1 ([BGZ19, Theorem 1.4])**.** *Let (M,ω) be a Liouville manifold, and let $L_1,\dots,L_m \subset M$ be Lagrangian spheres. Let $\phi = \prod_{i=1}^{k} \tau_{L_{j_i}} \in \mathrm{Symp}_{ct}(M)$, $j_i \in \{1,\dots,m\}$ be a positive word of Dehn twists. Then ϕ is not compactly supported isotopic to the identity in* $\mathrm{Symp}_{ct}(M)$.

Example 7.1.1. The exactness condition of Theorem 1 is necessary, as the following examples show.

(a) Consider the 2-torus $M := T^2$, and let $a,b \subset M$ represent the longitude and meridian of M. Then the associated Dehn twists satisfy $(\tau_a\tau_b)^6 = Id$ in $\pi_0(\mathrm{Symp}_{ct}(M))$. This is a classical result, see for example [FM11] (see [Aur03, 3.1] for the same example in a symplectic setting).

(b) Let $(M := S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2})$, and consider the antidiagonal $\overline{\Delta} := \{(x,y) \in S^2 \times S^2 | x+y = 0\} \subset M$. Then the Dehn twist $\tau_{\overline{\Delta}}$ is symplectically isotopic to an involution $(x,y) \mapsto (y,x)$, which implies $\tau_{\overline{\Delta}}^2 = Id$ in $\pi_0(\mathrm{Symp}_{ct}(M))$ (see [Sei08b, Example 2.9]).

♢

Remark 7.1.2. *1. The two dimensional case of Theorem 1 (for a product of Dehn twists in a Riemann surface) is a consequence of [Smi01, Theorem 1.3].*

2. The outcome of Theorem 1 is strictly geometric, and may not hold for the compact Fukaya category; we are not able to obtain information about the functors associated to the Dehn twists. Consider a punctured torus $M := T^2 \setminus \{*\}$ (the same applies to a punctured genus g surface), and the two (Lagrangian) circles a and b, representatives of the homological generators. In the closed case, the composition $(\tau_a\tau_b)^6$ is isotopic to the identity by the example above. In the punctured torus, there is an isotopy $(\tau_a\tau_b)^6 \simeq \tau_d$, where τ_d is the Dehn twist along the boundary curve d encircling the puncture (this is a consequence of the chain relation*, see [FM11, Proposition 4.12]). But since the support of τ_d is disjoint from any exact compact circle in M, the product $(\tau_a\tau_b)^6$ still acts as the identity on objects of the* compact *Fukaya category* $\mathcal{F}\mathrm{uk}(M)$.

//

7.1.1 Proof of Theorem 1

The original proof of Theorem 1 in [BGZ19] relies on the theory of open book decompositions, whereas the proof below uses Picard–Lefschetz theory. To simplify notation we prove the version of the theorem where $(M,\omega = d\lambda_M)$ is a Liouville domain.

Let $\phi = \prod_{i=1}^{k} \tau_{L_{j_i}} \in \mathrm{Symp}_{ct}(M)$, $j_i \in \{1, \dots, m\}$ be the word in positive powers of Dehn twists in the given collection, and assume by contradiction that this product is compactly supported Hamiltonianly isotopic to the identity (recall ϕ is an exact symplectomorphism).

Build an exact Lefschetz fibration $\pi\colon (E, \Omega_E, \lambda_E) \longrightarrow (\mathbb{C}, \lambda_{\mathbb{C}})$ with smooth fibre the Liouville domain (M, λ_M) identified over base point $z_* \in \mathbb{C}$, vanishing cycles the given Lagrangian spheres $(L_{j_1}, \dots, L_{j_k})$, and monodromy given by the product $\phi \in \mathrm{Symp}_{ct}(M)$ (see Section 2.2.1). Let $j_{\mathbb{C}}$ be the standard complex structure on $\mathbb{C}$.

By assumption, the fibration built from the above data has monodromy isotopic to the identity via a compactly supported Hamiltonian isotopy with $\phi_0 = \phi$ and $\phi_1 = Id$. Then π can be extended to a fibration $\hat{\pi}\colon \hat{E} \longrightarrow \mathbb{CP}^1$ as follows. Let $D_R \subset \mathbb{C}$ be a large circle of radius $R > 0$ passing through z_* and containing all the critical values. Define a fibration $E' \to D_{R+1}$ by extending $E|_{D_R}$ to a larger disc D_{R+1} such that for $t \in [0,1]$, the monodromy around D_{R+t} is $\phi_t \in \mathrm{Symp}_{ct}(M)$. Then $\hat{E}$ is obtained after gluing E' to a trivial fibration with fibre (M, ω) over a disc neighbourhood of the point at "infinity", $\hat{z} \in \mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$.

Moreover, as the symplectic connection around the fibre $\hat{\pi}^{-1}(\hat{z})$ is trivial, $\hat{\pi}\colon \hat{E} \longrightarrow \mathbb{CP}^1$ has the following properties.

1. There is a closed (possibly degenerate) two form $\hat{\Omega}_{\hat{E}}$ on $\hat{E}$ satisfying $\hat{\Omega}_{\hat{E}}|_{\hat{\pi}^{-1}(z)} = \Omega_E|_{\pi^{-1}(z)}$ for all $z \in \mathbb{CP}^1 \setminus \hat{z}$,

2. A neighbourhood of the horizontal boundary $V \supset \partial^h \hat{E}$ can be trivialised as $V \cong \mathbb{CP}^1 \times M^{out}$, where $M^{out} \subset M$ is an open neighbourhood of the boundary of the smooth fibre.

Definition 7.1.3. The set of almost complex structures compatible with $\hat{\pi}$, denoted by $\mathcal{J}(\hat{E}, \hat{\pi}, j)$, is defined as follows. An element $\hat{J} \in \mathcal{J}(\hat{E}, \hat{\pi}, j)$ satisfies

- $D\hat{\pi} \circ \hat{J} = j \circ D\hat{\pi}$ where j is the standard complex structure on $\mathbb{CP}^1$,

- There is an integrable almost complex structure J_0 such that $\hat{J} = J_0$ in a neighbourhood of $\mathrm{Crit}(\hat{\pi})$,

- For all $z \in \mathbb{CP}^1$, the restriction $J^{vv} := \hat{J}|_{\hat{\pi}^{-1}(z)}$ is an almost complex structure of contact type compatible with the Liouville form λ_M, and its restriction to V is isomorphic to a product $j \times J^{vv}$,

- $\hat{\Omega}_{\hat{E}}(\cdot, \hat{J}\cdot)$ is symmetric and positive definite.

◇

The form $\hat{\Omega}_{\hat{E}}$ can be modified to a symplectic form $\hat{\Omega} := \hat{\Omega}_E + \hat{\pi}^*(\beta)$ that tames $\hat{J}$, for $\beta \in \Omega^2(\mathbb{CP}^1)$ (similar to [Sei03, Lemma 2.1], [MS17, Theorem 6.1.4]).

From now onwards, we fix a generic element $\hat{J} \in \mathcal{J}(\hat{E}, \hat{\pi}, j)$, so that, by the same arguments of [Sei03, Lemma 2.4], all the moduli spaces we encounter satisfy the necessary regularity conditions.

Consider the moduli space of closed $(\hat{J}, j)$-holomorphic sections

$$\mathcal{M}_{\hat{J}} = \left\{ u \colon \mathbb{CP}^1 \to \hat{E},\ \hat{\pi} \circ u = id_{\mathbb{CP}^1},\ \hat{J} \circ Du = Du \circ j \right\}. \tag{7.1}$$

The moduli space has a non-empty boundary, but as we explain below this does not cause compactness issues, as the only sections reaching the boundary must be trivial sections.

Lemma 7.1.4. *The space $\mathcal{M}_{\hat{J}}$ is not empty. Moreover, there is a compact subset $K \subset \hat{E} \setminus \partial^h \hat{E}$ such that for all $u \in \mathcal{M}_{\hat{J}}$, either* $\mathrm{Im}(u) \subset K$*, or u is a trivial section.*

Proof. Let $q \in V$ a point in a neighbourhood $\partial^h \hat{E} \subset V$ of the horizontal boundary as in 2. above. Via the trivialisation of this neighbourhood, one obtains a trivial section $s : \mathbb{CP}^1 \to \hat{E}$ with $s(z) = q$ for all $z \in \mathbb{CP}^1$, which is a regular $(\hat{J}, j)$-holomorphic section, so $\mathcal{M}_{\hat{J}}$ is not empty. The rest of the proof follows from a maximum principle as in [Sei03, Lemma 2.2]. □

We can adapt the argument of [Sei03, Lemma 2.3] to the case of closed curves to show that for our choice of almost complex structure $\hat{J}$, the moduli space $\mathcal{M}_{\hat{J}}$ is a compact smooth manifold with boundary. The only issue that could possibly occur is a loss of compactness for the component containing sections outside the compact part K, which, by Lemma 7.1.4, can only be "constant" sections. These elements have bounded energy, as they are all in the same homology class. By the Gromov compactness theorem, it follows that the only non-compact phenomenon that can occur in this case is sphere bubbling. The next lemma shows how to discard bubbles.

Lemma 7.1.5. *Let u_∞ be the limit of a (sub)sequence of pseudoholomorphic sections $(u_n)_{n \in \mathbb{N}}$ of the Lefschetz fibration $\hat{\pi}$. A component of u_∞ is either an element in the class $[u_i]$ (for $i \in \mathbb{N}$) or is contained in a single fibre. In the latter case, the component is a bubble.*

Proof. Let $v_1, v_2, \dots v_k$ be the components of u_∞. The limiting curve u_∞ is assumed to be $(\hat{J}, j)$-holomorphic and nonconstant, so it has to have degree one, as $\sum_{j=1}^k [\pi \circ v_i] = [\pi \circ u_\infty] = [\mathbb{CP}^1]$. It follows that the degrees of its components sum up to one. All degrees are non-negative, so there is only one component with degree one. If in addition there was a bubble, it would be represented in a degree zero component and therefore would have to be entirely contained in a fibre (note that by positivity of intersections, the bubble cannot intersect other fibres). □

Since the fibres are exact, there can be no bubbling of the type of Lemma 7.1.5, so the moduli space $\mathcal{M}_{\hat{J}}$ is compact.

Lemma 7.1.6. *Through each point of the smooth fibre M there is at least one holomorphic section* $s \in \mathcal{M}_{\hat{J}}$.

Proof. As in the proof of Lemma 7.1.4, we consider a neighbourhood of the horizontal boundary $V \supset \partial^h \hat{E}$, $q \in V$ such that $\hat{\pi}(q) =: z_{gen} \in \mathbb{CP}^1 \setminus \mathrm{Crit}v(\hat{\pi})$ and the trivial section through q, $s\colon \mathbb{CP}^1 \to \hat{E}$. Consider

$$\mathcal{M}(\hat{J}, q) := \{u \in \mathcal{M}_{\hat{J}},\ q \in \mathrm{Im}(u)\} \subset \mathcal{M}_{\hat{J}}.$$

It is a smooth compact manifold (by the same arguments as for $\mathcal{M}_{\hat{J}}$). Moreover, by Lemma 7.1.4, the only element in $\mathcal{M}(\hat{J}, q)$ is the trivial $\hat{J}$-holomorphic section $s\colon \mathbb{CP}^1 \to \hat{E}$ through q.

Let $p \in \hat{\pi}^{-1}(z_{gen})$ be any other point in the fibre of q, and consider a path $\alpha : [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$. For every point $\alpha(t)$, $t \in [0,1]$, define $\mathcal{M}(\hat{J}, \alpha(t), [s]) := \{u \in \mathcal{M}_{\hat{J}},\ \alpha(t) \in \mathrm{Im}(u),\ \text{and } [u] = [s]\}$. Clearly, $\mathcal{M}(\hat{J}, \alpha(1), [s]) = \mathcal{M}(\hat{J}, q)$.

Consider

$$\mathcal{M}_{cob} := \bigcup_{t \in [0,1]} \mathcal{M}(\hat{J}, \alpha(t), [s]) \subset \mathcal{M}_{\hat{J}}. \tag{7.2}$$

The boundary components of (7.2) are given by $\partial \mathcal{M}_{cob} = \mathcal{M}(\hat{J}, p, [s]) \sqcup \mathcal{M}(\hat{J}, q)$. We want to show that the space $\mathcal{M}_{cob}$ is compact, so that it defines a one-dimensional cobordism between $\mathcal{M}(\hat{J}, p, [s])$ and $\mathcal{M}(\hat{J}, q)$. As before, since $\hat{J} \in \mathcal{J}(\hat{E}, \hat{\pi}, j)$ is chosen to be generic, $\mathcal{M}_{cob}$ is a smooth manifold.

To show that $\mathcal{M}_{cob}$ is compact, the same strategy applies as in the case of $\mathcal{M}_{\hat{J}}$. In particular, consider a sequence $t_i \subset [0,1]$ and for each i a section $u_i \in \mathcal{M}(\hat{J}, \alpha(t_i), [s]) \subset \mathcal{M}_{cob}$.

All the sections of the sequence we are considering belong to the same homology class, by definition. In particular they have the same area, so that Gromov's theorem applies. Consequently, as t_i tends to a limit value t_∞, the sequence u_i converges to a stable map u_∞. As before (in the proof of Lemma 7.1.5), if sphere bubbling occured, bubbles would have to be "vertical" (meaning entirely contained in the fibres), which is impossible by the exactness of the fibres.

It follows that $\mathcal{M}_{cob}$ is compact and hence $\mathcal{M}(\hat{J}, p, [s])$ and $\mathcal{M}(\hat{J}, q)$ are indeed cobordant. Since the signed count of the boundary components of a 1-dimensional compact manifold is zero, it follows that the 0-dimensional components of the two spaces have the same cardinality. In particular, for any $p \in \hat{\pi}^{-1}(z_{gen})$, $\mathcal{M}(\hat{J}, p, [s])$ is not empty which means there is at least one element in $\mathcal{M}_{\hat{J}}$ that passes through p.

□

Corollary 7.1.7. *The map induced by the evaluation map*

$$\begin{aligned} ev\colon \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 &\longrightarrow \hat{E} \\ (u,z) &\longmapsto ev_z(u) = u(z). \end{aligned} \tag{7.3}$$

is surjective.

Proof. By Lemma 7.1.6, the image of the map (7.3) is dense, since each point on a smooth fibre has a preimage. As $\mathcal{M}_{\hat{J}}$ is compact and the mapping is continuous, the result extends to all points of $\hat{E}$ and hence (7.3) is surjective. □

Proof of Theorem 1. Assume by contradiction that the product $\phi = \tau_{L_{j_1}} \dots \tau_{L_{j_k}}$ is isotopic to the identity, and build the fibration $\hat{\pi}\colon \hat{E} \longrightarrow \mathbb{CP}^1$ and the moduli space $\mathcal{M}_{\hat{J}}$ as above. By Corollary 7.1.7, the evaluation map $ev\colon \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 \longrightarrow \hat{E}$ is surjective. Consider the commuting diagram

$$\begin{array}{ccc} \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 & \xrightarrow{ev} & \hat{E} \\ & \searrow^{pr_2} & \downarrow \hat{\pi} \\ & & \mathbb{CP}^1 \end{array}$$

where $pr_2\colon \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1 \to \mathbb{CP}^1$ is the projection to the second factor.

Let $x \in \mathrm{Crit}(\hat{\pi}) \subset \hat{E}$ be any point in the critical set. By the surjectivity of ev, there is a pair $(u,w) \in \mathcal{M}_{\hat{J}} \times \mathbb{CP}^1$ such that $u(w) = x$, so that $w \in \mathbb{CP}^1$ is the critical value associated to x. From the diagram we can write

$$\begin{aligned} D_{(u,w)}(pr_2) &= D_{(u,w)}(\hat{\pi} \circ ev) \\ D_{(u,w)}(pr_2) &= D_x\hat{\pi} D_{(u,w)}(ev). \end{aligned}$$

As x is a critical point, $D_x\hat{\pi} = 0$, which forces $D_{(u,w)}(pr_2)$ to be the zero map. But this is in contradiction with $D_{(u,w)}(pr_2)$ being surjective. □

Corollary 7.1.8. *There is no exact Lefschetz fibration with global monodromy symplectically isotopic to the identity, except for the trivial fibration.*

□

7.2 A relative version

Let

$$M := T^*S^m \#_{pt} T^*S^m \#_{pt} T^*S^m \cdots \#_{pt} T^*S^m \tag{7.4}$$

be a "multi-plumbing" of m spheres (an iterated construction of transverse plumbing of spheres, see Section 6.1 for the definition of plumbing). By Theorem 1, we know that no product $\phi \in \mathrm{Symp}_{ct}(M)$ of Dehn twist along the core spheres can be compactly supported symplectically isotopic to the identity. However, the theorem, a priori, doesn't prevent such a product to act trivially on some Lagrangian submanifolds of M. Is it possible to tell whether there are Lagrangians that detect the non-triviality of ϕ? Let T be a cotangent fibre of the j-th T^*S^n-summand, for $j \in \{1, \dots, m\}$. The theorem we prove in this section shows that any product of positive Dehn twists along Lagrangian cores and involving the j-th sphere does not preserve T up to compactly supported symplectic isotopy.

Theorem 2. *Let (M^{2n}, ω) be a Liouville manifold containing embedded Lagrangian spheres $L_1, \dots, L_m$ and a conical Lagrangian disc T intersecting one of the spheres L_j transversely in a point. Let $\phi := \prod_{i=1}^k \tau_{L_{j_i}} \in \mathrm{Symp}_{ct}(M)$, $j_i \in \{1, \dots, m\}$ be a positive word of Dehn twists involving τ_{L_j}. Then the Lagrangians T and $\phi(T)$ are not isotopic via a compactly supported Lagrangian isotopy.*

We prove the statement of Theorem 2 in the equivalent version where $(M, \omega = d\lambda_M)$ is a Liouville domain and $T \subset M$ is a Lagrangian disc preserved by the Liouville flow near the boundary ∂M (so that $\partial T \subset \partial M$). This is only a choice so that the Lefschetz fibrations involved have exact compact fibres.

As in the statement, write $\phi = \prod_{i=1}^k \tau_{L_{j_i}}$, $j_i \in \{1, \dots, m\}$. By assumption, there is at least one index $\ell \in \{1, \dots, k\}$ such that $j_\ell = j$. Assuming $\phi(T) \simeq T$ via a compactly supported isotopy, we arrive at the contradictory statement $j \notin \{j_1, \dots, j_k\}$.

From the data $(M, (L_{j_1}, \dots, L_{j_\ell}, \dots, L_{j_k}))$, build an exact Lefschetz fibration $\pi' \colon (E', \Omega_{E'}, \lambda_{E'}) \to (\mathbb{C}, \lambda_{\mathbb{C}})$ with smooth fibre the Liouville domain $(M, d\lambda)$, base point $z_* \in \mathbb{R}$, $z_* \gg 0$, such that the k critical values $\mathrm{Crit}v(\pi') = \{w_{j_1}, \dots, w_{j_\ell}, \dots, w_{j_k}\}$ are ordered vertically on the imaginary line, $\mathrm{Crit}v(\pi) \subset i\mathbb{R}$ with a basis of vanishing paths $(\gamma_{j_1}, \dots \gamma_{j_k})$ ([Sei08a, 16e]).

Let $(\Delta_{\gamma_{j_1}}, \dots, \Delta_{j_k})$ be the corresponding basis of Lefschetz thimbles and for every $i = 1, \dots, k$, $V_{j_i} := \pi^{-1}(z_*) \cap \Delta_{\gamma_{j_i}}$ the associated vanishing cycles, which, under the identification $\pi^{-1}(z_*) = M$ correspond to L_{j_i}. Let $\sigma \colon S^1 \to \mathbb{C}$ be a loop encircling all critical values.

Build a new exact fibration $\pi \colon (E, \Omega_E, \lambda_E) \to (\mathbb{C}, \lambda_{\mathbb{C}})$ associated to the data

$(M,(V_{j_1},\ldots,V_{j_\ell},\ldots,V_{j_k},V_{j_\ell}))$, with base point $z_*\in\mathbb{C}$, an extra critical value $w_{j+1}\in\mathrm{Critv}(\pi)\subset i\mathbb{R}$ and an extra vanishing path $\gamma_{j_{k+1}}$ such that $\mathrm{Im}(\gamma_{j_{k+1}})\cap\mathrm{Im}(\sigma)=\emptyset$ (all the other choices are the same as for π').

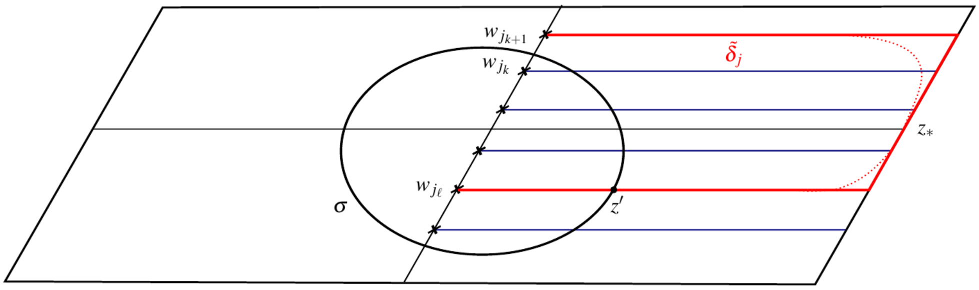

Figure 7.1: The new fibration π has an extra critical value $w_{j_{k+1}}$ and a matching sphere that fibres over the smoothing δ_j of the red arc $\tilde{\delta}_j$.

Compared to π', there are now two critical points $w_{j_\ell},w_{j_{k+1}}$ associated to the same vanishing cycle V_{j_ℓ}. Therefore, there is a matching path $\delta_j\colon[0,1]\to\mathbb{C}$ with $\delta_j(0)=w_{j_\ell},\ \delta(\frac{1}{2})=z_*,\ \delta_j(1)=w_{j_{k+1}}$, whose parallel transport is a Lagrangian matching sphere $S_j\cong S^{n+1}\subset E$ (Section 2.2.1, [Sei08a, (16g)]) fibred by Lagrangians isomorphic to L_j (see Figure 7.1). Let $z'\in\mathrm{Im}(\delta_j)\cap\mathrm{Im}(\sigma)$, and via parallel transport identify $T\subset\pi^{-1}(z_*)$ with a copy of the Lagrangian in $\pi^{-1}(z')$.

By construction, the monodromy around σ is given by the product ϕ. By assumption there is an isotopy $\phi(T)\simeq T$, so parallel transport of T along σ produces a well-defined Lagrangian $P_\sigma\subset E$. For $z\in\mathrm{Im}(\sigma)$, let $T_z\subset\pi^{-1}(z)$ be the exact fibres of P_σ. Then $\Omega_E|_{P_\sigma}=df_\sigma+\pi^*(\kappa_\sigma)$ for a function $f_\sigma\in C^\infty(P_\sigma,\mathbb{R})$ such that for every $z\in\mathrm{Im}(\sigma)$, $f_\sigma|_{\pi^{-1}(z)}$ makes T_z exact and $\kappa_\sigma\in\Omega^1(\mathrm{Im}(\sigma))$ ([Sei03, Lemma 1.3]).

Lemma 7.2.1. *The Lagrangian P_σ defines a nontrivial class in $H_{n+1}(E,\partial E;\mathbb{Z})$.*

Proof. The matching sphere S_j and the disc bundle P_σ are properly embedded Lagrangian submanifolds meeting transversely at the point $y\in L_j$ lying over the intersection between σ and the matching path associated to S_j. Their homological intersection, which is the image of a non-degenerate pairing

$$H_{n+1}(E;\mathbb{Z})\times H_{n+1}(E,\partial E;\mathbb{Z})\to\mathbb{Z}$$

is one, so in particular P_σ represents a non trivial homology class in $H_{n+1}(E,\partial E;\mathbb{Z})$. □

7.2.1 Proof of Theorem 2

Let $D\subseteq\mathbb{C}$ be the disc bounded by the loop σ in the base of π. The idea for the proof of Theorem 2 is based on a section-count which follows the same principles of Section 7.1. In this context however, we consider pseudoholomorphic sections definining boundary conditions for $E|_D$ on P_σ.

Let $\mathcal{J}(\pi, E, j_{\mathbb{C}})$ be the set of almost complex structures compatible with π (see Definition 2.2.6), where $j_{\mathbb{C}}$ is the standard complex structure on $\mathbb{C}$. For a generic element $J \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$,

$$\mathcal{M}(J, P_\sigma) := \{u\colon (D, \partial D) \longrightarrow (E, P_\sigma),\ \pi \circ u = id_D,\ J \circ Du = Du \circ j_{\mathbb{C}}|_D\} \tag{7.5}$$

be the moduli space of pseudoholomorphic sections with boundary condition on P_σ.

The Lagrangian P_σ is fibred by copies of the exact Lagrangian $T \subset M$, and therefore $P_\sigma \cap \partial^h E \neq \emptyset$, it is not disjoint from the horizontal boundary. As a result, the moduli space $\mathcal{M}(J, P_\sigma)$ is not compact, but fortunately its non-compact ends are very well-behaved.

Below, we show that for a generic almost complex structure in $\mathcal{J}(\pi, E, j_{\mathbb{C}})$, the "non-compact" elements (those sections reaching the horizontal boundary) of the moduli space (7.5) are regular. We do this by showing that such sections must be trivial—and the trivial section can be made regular, as the almost complex structure is product-like near $\partial^h E$. For all the other holomorphic sections, which are entirely contained in the compact region, the same regularity arguments as in [Sei03] apply.

Lemma 7.2.2. *There is $\hat{J} \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ with the following property. There is no $\hat{J}$-holomorphic section $v\colon (D, \partial D) \to (E, P_\sigma)$ with boundary condition on P_σ, such that there are $z_1, z_2 \in \partial D$ with $v(z_1) \in P_\sigma \setminus (P_\sigma \cap \partial^h E)$ and $v(z_2) \in P_\sigma \cap \partial^h E$.*

Proof. We show that any generic element $J \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ can be deformed to an almost complex structure $\hat{J} \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ as in the statement. To do that we use a *reverse isoperimetric inequality* from [GS14] that applies to the Liouville completion of E.

Identify a collar neighbourhood of $\partial^h E$ with $C(\partial^h E) := \mathbb{C} \times ((-\varepsilon, 0] \times \partial M))$, and consider the Liouville completion of E, $(\overline{E}, \omega_{\overline{E}})$, obtained by gluing a cylindrical end $U^h := \mathbb{C} \times ([0, \infty) \times \partial M)$ along a collar neighbourhood of the horizontal boundary $\partial^h E$, such that $\omega_{\overline{E}}|_{U^h} = d(\lambda_{\mathbb{C}} + e^t \lambda_M|_{\partial M})$, for the coordinate t on $[0, \infty)$.

Let $(\overline{M}, \overline{\omega})$ be the generic smooth fibre of $\overline{E}$, and $\overline{T} \subset \overline{M}$ the Lagrangian obtained from T by gluing a conical end at the boundary. Accordingly, let $\overline{P}_\sigma \subset \overline{E}$ be the "completion" of $P_\sigma \subset E$ in $\overline{E}$. This Lagrangian can be trivialised outside of a compact set as $\partial D \times U^\infty \subset \mathbb{C} \times U^\infty \subset \overline{E}$, where $U^\infty \subset \overline{M}$ is a neighbourhood of the cylindrical end of $\overline{T}$. Extend J to a cylindrical almost complex structure $\overline{J}$ on $\overline{E}$ (see Definition 2.2.3).

By [GPS20, Lemma 2.43], $(\overline{E}, \omega_{\overline{E}})$ has *bounded geometry* in the sense of [GPS20, Definition 2.42], which is equivalent to the notion of bounded geometry of [GS14, 1.4], see [GPS20, p.104]. The same holds for the Lagrangian $\overline{P}_\sigma \subset \overline{E}$, as it is compact in the base direction, and conical in the fibre direction (see the proof of [GPS20, Lemma 2.43]). Bounded geometry implies that for any $\overline{J}$–holomorphic section $u\colon (D, \partial D) \to (\overline{E}, \overline{P}_\sigma)$ there is a *reverse isoperimetric inequality* ([GS14, Theorem 1.4])

$$\ell(u|_{\partial D}) \leqslant a(u) \cdot C, \tag{7.6}$$

where ℓ is the length function associated to a $\overline{J}$-compatible metric $g_{\overline{J}}$, $C > 0$ is a constant depending on $\overline{E}$, and $a(u)$ is the area of the curve.

Let $A := \int_{\partial D} \kappa_\sigma$ (for $\kappa_\sigma \in \Omega^1(\mathrm{Im}(\sigma))$ as above), and set $R := A \cdot C$. For $R > 0$, consider a piece of symplectisation $(E_{R+1} := E \cup \mathbb{C} \times ([0, R+1] \times \partial M), \omega_{R+1})$ with $\omega_{R+1}|_E = \omega_E$ and $\omega_{R+1}|_{\mathbb{C}\times([0,R+1]\times\partial M)} = d(\lambda_{\mathbb{C}} + e^t \lambda_M)$, and a compatible almost complex structure $J_{R+1} = \overline{J}|_{E_{R+1}}$ of contact type. Clearly $E \subset E_{R+1} \subset \overline{E}$, and there is a diffeomorphism $\psi\colon E_{R+1} \to E$, that is the identity on $E \setminus C(\partial^h E)$ and compresses $\mathbb{C} \times ((-\varepsilon, R+1] \times \partial M)$ to $\mathbb{C} \times ((-\varepsilon, 0] \times \partial M)$ via the negative Liouville flow.

Every $\overline{J}$-holomorphic curve $u\colon (D, \partial D) \to (\overline{E}, \overline{P}_\sigma)$ such that there are $z_1, z_2 \in \partial D$ with $u(z_1) \in Int(E)$ and $u(z_2) \in \overline{E} \setminus E_{R+1}$ satisfies $d(u(z_1), u(z_2)) > A \cdot C$ and the inequality (7.6)

Now set $\hat{J} := \psi_*(J_{R+1})$. This satisfies the requirements of the Lemma. Namely, let $v\colon (D, \partial D) \to (E, P_\sigma)$ be a $\hat{J}$-holomorphic section as in the statement, i.e such that there are $z_1, z_2 \in \partial D$ with $v(z_1) \in P_\sigma \setminus (P_\sigma \cap \partial^h E)$ and $v(z_2) \in P_\sigma \cap \partial^h E$.

Then we certainly have $d(v(z_1), v(z_2)) < \ell(v|_{\partial D})$ for the distance function d and the length ℓ associated to a compatible metric $g_{\hat{J}}$. On the other hand, the area of v is bounded by a fixed upper bound since $a(v) = \int_D v^*\Omega_E = \int_D d(v^*\lambda_E) = \int_{\partial D} v^*(\lambda_E) = \int_{\partial D} \kappa_\sigma = A$ by exactness of Ω_E and fibrewise exactness of P_σ.

By stretching the neck in a neighbourhood of the boundary of E to E_{R+1}, the pullback $\psi^*(v)$ produces a contradiction, since $d(\psi^*(v(z_1)), \psi^*(v(z_2)) < \ell(\psi^*(v|_{\partial D})) < a(\psi^*(v)) \cdot C = A \cdot C$, but also $d(\psi^*(v(z_1)), \psi^*(v(z_2)) > A \cdot C$ by construction of E_{R+1}. This concludes the proof.

□

From now onwards, fix an almost complex structure $\hat{J} \in \mathcal{J}(\pi, E, j_{\mathbb{C}})$ as in Lemma 7.2.2. The above results imply that the only possible scenario left to consider in the case of a section with boundary condition on P_σ intersecting $\partial^h E$, is to be entirely contained in the horizontal boundary of the fibration.

Lemma 7.2.3. *Let $u\colon D \to E$ be a $\hat{J}$-holomorphic section such that $\mathrm{Im}(u) \subset \partial^h E$. Then u is a constant section.*

Proof. Assume there is a non-constant section $u\colon D \to E$ such that $\mathrm{Im}(u) \subset \partial^h E$. Identify (via a trivialisation as in (2.6)) a neighbourhood of $\partial^h E$ as $U^\partial \cong \mathbb{C} \times M^{out} \subset \mathbb{C} \times M$ for an open neighbourhood

$M^{out} \subset M$ of ∂M. Then the projection of $\mathrm{Im}(u)$ to M defines a non–constant $\hat{J}|_M$–holomorphic disc $u\colon (D,\partial D) \to (M,T)$, which, by the exactness assumptions on M, cannot exist. Therefore, u must be a constant section. □

We now prove that there are no compactness issues. The moduli space $\mathcal{M}(\hat{J},P_\sigma)$ has a non-compact end, but by the regularity discussion above, the only sections reaching it are the constant ones, and all elements of $\mathcal{M}(\hat{J},P_\sigma)$ have bounded energy so that the Gromov compactness theorem applies. The bubbles components in the Gromov limit of a sequence of $(\hat{J}, j_{\mathbb{C}})$-holomorphic sections in $\mathcal{M}(\hat{J},P_\sigma)$ are either spheres in the fibres over D, or discs in the fibres $\pi^{-1}(z)$, for $z \in \mathrm{Im}(\sigma)$, with boundary condition on T_z. Both options can be discarded by exactness of E and fibrewise exactness of P_σ.

Lemma 7.2.4. *The evaluation map*

$$ev : \mathcal{M}(\hat{J},P_\sigma) \times D \longrightarrow E \qquad (7.7)$$
$$(u,z) \longmapsto u(z)$$

(i) is proper;

(ii) restricts to a surjective map $\mathcal{M}(\hat{J},P_\sigma) \times \partial D \longrightarrow P_\sigma$ of degree one.

Proof. (i) To prove this property is enough to show that every sequence of sections $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}(\hat{J},P_\sigma)$ whose image under ev lies in a relatively compact set of E has a convergent subsequence. Consider such a sequence. If its image under (7.7) lie in a compact set then by exactness there is an upper bound to the energy of all elements in the sequence (which is bounded by a finite value determined by the maximum among all areas of the curves). Then, by the Gromov compactness theorem, $\{u_k\}_k$ admits a subsequence converging to a stable map, which, in the absence of bubbles, can only be another section.

(ii) To prove the second point, we show that the algebraic count of sections through every point of P_σ is one. Let $U^\partial \supset \partial^h E$ be a neighbourhood of the horizontal boundary as in the proof of the previous lemma and $q \in U^\partial \cap \pi^{-1}(\sigma)$. Since ϕ is compactly supported in a neighbourhood of the vanishing cycles, the monodromy around σ preserves q. By lemmata 7.2.2, 7.2.3, the moduli space $\mathcal{M}(\hat{J},q) := \{u \in \mathcal{M}(\hat{J},P_\sigma), q \in \mathrm{Im}(u)\} \subset \mathcal{M}(\hat{J},P_\sigma)$ is compact and only contains the constant section $s\colon D \to E$ through q.

Given another point $p \in P_\sigma$, consider the path $\alpha\colon [0,1] \to M$ with $\alpha(0) = p, \alpha(1) = q$ and define

$$\mathcal{M}(\hat{J},P_\sigma,\alpha(t),[s]) := \{u \in \mathcal{M}(\hat{J},P_\sigma),\ \alpha(t) \in \mathrm{Im}(u) \text{ and } [u] = [s]\}.$$

Clearly, $\mathcal{M}(\hat{J},P_\sigma,\alpha(1),[s]) = \mathcal{M}(\hat{J},q)$.

Consider

$$\mathcal{M}_{cob} := \bigcup_{t\in[0,1]} \mathcal{M}(\hat{J}, P_\sigma, \alpha(t), [s]) \subset \mathcal{M}(\hat{J}, P_\sigma). \tag{7.8}$$

All elements in $\mathcal{M}_{cob}$ are in the same homology class so that the same compactness arguments apply as above. Compactness implies that for every $p \in P_\sigma$, the moduli space $\mathcal{M}(\hat{J}, P_\sigma, p, [s])$ is cobordant to the moduli space $\mathcal{M}(\hat{J}, q)$. Therefore, by the same reasoning as in the proof of Lemma 7.1.6, through each point of P_σ there is algebraically a unique section in $\mathcal{M}(\hat{J}, P_\sigma)$, so that the restriction $\mathcal{M}(\hat{J}, P_\sigma) \times \partial D \longrightarrow P_\sigma$ is surjective and of degree one. □

Proof of Theorem 2. Under the assumption that $\phi(T) \simeq T$, we have proved that P_σ represents a non-trivial class in $H_{n+1}(E, \partial E)$ (Lemma 7.2.1). The same assumption however also yields Lemma 7.2.4, which in particular implies that $ev_*(\mathcal{M}_{\hat{J}}(E, P_\sigma) \times \partial D) = [P_\sigma] \in H_{n+1}(E, \partial E)$ is realised as the boundary of the chain $ev_*(\mathcal{M}_{\hat{J}}(E, P_\sigma) \times D) \in C_{n+2}(E, \partial E)$. This is a contradiction, which concludes the proof of the theorem. □

7.3 Products of projective twists

We continue the investigation on positive products of twists in Liouville manifolds, this time focussing on projective twists. Ideally, one would try to generalise as many results from Section 7.1 to this situation.

The previous section heavily relied on the link between Dehn twists and Lefschetz fibrations, and many constructions we used depended on section count invariants of Lefschetz fibrations.

Perutz showed in [Per07] that any *fibred twist* admits a representation as the local monodromy of a Morse–Bott–Lefschetz (MBL) fibration (see also Section 2.2.2). As explained in Section 2.2.5, projective twists can be thought of as an example of S^1-fibred twists, so we could envisage extending the mechanisms behind the proof for the spherical case to the setting of MBL fibrations (following [Per07] and [WW16]) to show the analogous statement for projective twists.

Question 2. Let $\varphi \in \mathrm{Symp}_{ct}(W)$ be a non-empty composition of positive powers of projective twists on a Liouville manifold (W, ω) of dimension at least four. Can φ be isotopic to the identity in $\mathrm{Symp}_{ct}(W)$?

Unfortunately, the section-count strategy presents a route filled with obstacles; the central problem being the lack of compactness of moduli spaces of sections of MBL fibrations. The critical locus $\mathrm{Crit}(\pi)$ of such a fibration is a compact symplectic submanifold of the total space, and in general contains rational curves. The total space of a MBL fibration $\pi\colon E \longrightarrow \mathbb{C}$ associated to a projective twist cannot be made into an exact symplectic manifold, so bubbling phenomena can become an issue when considering moduli spaces of pseudoholomorphic sections.

Instead, the idea remains, as in Chapter 6, to use the Hopf correspondence to translate a situation involving projective twists into one involving Dehn twists.

Theorem 4. *Let (W^{2n}, ω) be a Liouville manifold containing Lagrangian real projective spaces $K_1, \dots K_m$, $K_i \cong \mathbb{RP}^n$. Suppose that there is a class $\alpha \in H^1(W;\mathbb{Z}/2\mathbb{Z})$ such that for every $i = 1,\dots,m$, $\alpha|_{K_i}$ generates $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. Let $\varphi \in \mathrm{Symp}_{ct}(W)$ be a positive word in the subset of projective twists $\{\tau_{K_i}\}_{i\in\{1,\dots,m\}}$. Then φ is not isotopic to the identity in $\pi_0(\mathrm{Symp}_{ct}(M))$.*

Proof. As in Section 5.3.2, let $q\colon (\tilde{W}, \tilde{\omega}) \to (W, \omega)$ be the symplectic double cover given by the class α and $L_1, \dots, L_m \subset \tilde{W}$ Lagrangian spheres obtained as double cover of $K_1, \dots, K_m \subset W$. The composition of projective twists $\varphi \in \mathrm{Symp}_{ct}(W)$ lifts to a composition of spherical Dehn twists $\phi \in \mathrm{Symp}_{ct}(\tilde{W})$. Assume there is an isotopy $(\varphi_t)_{0\leqslant t\leqslant 1}$ connecting the composition of projective twists $\varphi_0 = \varphi$ to the identity $\varphi_1 = Id$. The isotopy lifts to a family of compactly supported maps $(\phi_t)_{0\leqslant t\leqslant 1}$ in the double cover $\tilde{W}$, where $\phi_0 = \phi$ is the lift of φ. Then, ϕ_1 covers the identity and can therefore only be the identity or a deck transformation. The latter type would define a non-compactly supported symplectomorphism, hence $\phi \in \mathrm{Symp}_{ct}(\tilde{W})$ is a composition of Dehn twists in a Liouville domain which is isotopic to the identity, contradicting Theorem 1. □

Remark 7.3.1. *A similar argument fails when applied to complex projective twists. Let (W^{4n}, ω) be a symplectic manifold with complex projective Lagrangians $K_1, \dots, K_m$ satisfying Assumption* (C)*. The fibration $(Y, \Omega) \to (W, \omega)$ constructed from the cohomological condition is not proper, so an isotopy in* $\mathrm{Symp}_{ct}(W)$ *cannot be lifted to an isotopy in* $\mathrm{Symp}_{ct}(Y)$*.* //

Chapter 8

Epilogue: framings of projective twists, homotopy projective Lagrangians

As a last application of the Hopf correspondence, we examine homotopy projective Lagrangians. We prove two non-embedding results for Lagrangian projective spaces in non-standard homeomorphism/diffeomorphism classes (Theorems 8.1.11 and 8.1.13), and for $n \geqslant 19$, the existence of projective twists obtained from a non-standard choice of framing, that are not Hamiltonian isotopic to the standard $\tau_{\mathbb{CP}^n} \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$:

Theorem 6 (Corollary 8.2.13)**.** *The $\mathbb{CP}^n$-twist depends on the framing when $n = 19, 23, 25, 29$.*

Embedding theorems are obtained in Section 8.1 using homotopy theory results combined with the existing state of the art of the nearby Lagrangian conjecture, and the use of the Hopf correspondence.

We subsequently investigate the question of framings for projective twists in Section 8.2. For that purpose, we utilise the current literature on framing of Dehn twists, a pairing constructed by Bredon, and the Hopf correspondence. This enables us to obtain instances in which the (Hamiltonian isotopy class of the) local projective twist does depend on a choice of framing of the associated Lagrangian projective space. With the additional use of topological modular forms, we explain why there should be infinitely many such examples.

8.1 Lagrangian non-embeddings of projective spaces

The *nearby Lagrangian conjecture* states that given a closed smooth manifold Q, any closed exact Lagrangian submanifold of $(T^*Q, d\lambda_Q)$ is Hamiltonian isotopic to the zero section. If this conjecture was true, the existence of another closed exact Lagrangian embedding $L \hookrightarrow T^*Q$ would yield a diffeomorphism, $L \cong Q$. By Weinstein's neighbourhood theorem, the latter version of the statement can also be read as: if $(T^*L, d\lambda_{T^*L})$ is symplectomorphic to $(T^*Q, d\lambda_{T^*Q})$, then L is diffeomorphic to Q.

The conjecture has been verified for some specific examples ($T^*S^2, T^*\mathbb{RP}^2$ by Hind [Hin12] and Li–Wu [LW12], T^*T^2 by Dimitroglou Rizell–Goodman–Ivrii [RGI16]), and weaker versions of it have been proved. Currently, the most general feature one can deduce from an exact Lagrangian embedding in $(T^*Q, d\lambda_{T^*Q})$ is (simple) homotopy equivalence.

Theorem 8.1.1 ([Abo12b], [Kra13], [AK18]). *If $L \subset T^*Q$ is a closed exact Lagrangian embedding, then the projection $L \subset T^*Q \xrightarrow{p} Q$ is a (simple) homotopy equivalence.*

Remark 8.1.2. *Note that if $L \subset T^*Q \xrightarrow{p} Q$ is a Lagrangian as in the above statement, then $TL \otimes \mathbb{C} \cong p^*(TQ \otimes \mathbb{C})$. The Pontryagin classes $p_i \in H^{4k}(\cdot)$ satisfy $2p_i(L) = 2p_i(Q)$. Moreover, the (rational) Pontryagin classes p_i are homeomorphism invariants ([Nov65]).* //

Equipped with the connected sum operation, the set of h-cobordism classes of homotopy m-spheres Θ_m has an abelian group structure (where the standard sphere plays the role of neutral element). We will always assume $m > 5$, in which case the elements of Θ_m correspond to diffeomorphism classes of m-spheres.

The group Θ_m fits in an exact sequence ([KM63])

$$0 \longrightarrow bP_{m+1} \longrightarrow \Theta_m \xrightarrow{\psi} \operatorname{coker}(J_m) \longrightarrow bP_m. \tag{8.1}$$

Here $bP_{m+1} = ker(\psi) \subset \Theta_m$ denotes the subgroup of homotopy m-spheres bounding an $(m+1)$-dimensional parallelisable manifold, and $J_m \colon \pi_m(O) \to \pi_m(S)$ is a map from the m-th stable homotopy group $\pi_m(O) = \lim_{\ell \to \infty} \pi_m(SO(\ell))$ to the m-th stable homotopy group of spheres $\pi_m(S) := \lim_{\ell \to \infty} \pi_{m+\ell}(S^\ell)$ (see e.g [Lev85, Section 3]). This group is also called the *m-th stable stem*.

Throughout the chapter, we will repeatedly use the following fact about the sequence (8.1).

Theorem 8.1.3 ([KM63, Theorem 5.1]). *If m is an odd integer, $bP_m = 0$. Consequently, for any odd m, $\psi \colon \Theta_m \to \operatorname{coker}(J_m)$ is surjective.*

In the symplectic setting, homotopy spheres are good candidates to test the nearby Lagrangian conjecture.

Theorem 8.1.4 ([Abo12a], extended by [EKS16]). *Let $m > 4$ odd. If $\Sigma, \Sigma' \in \Theta_m$ and $T^*\Sigma$ is symplectomorphic to $T^*\Sigma'$, then $[\Sigma] = \pm[\Sigma'] \in \Theta_m / bP_{m+1}$.*

It will be practical to paraphrase the above theorem as follows.

Corollary 8.1.5. *If $m > 4$ is odd and $\Sigma \in \Theta_m \setminus bP_{m+1}$, then Σ does not admit a Lagrangian embedding into T^*S^m.*

Definition 8.1.6. By personal choice of the author, we depart from the classic terminology of *exotic* manifolds. Instead, we will call a smooth manifold that is homeomorphic, but not diffeomorphic, to the standard sphere an *AD* sphere (AD stands for Alternative Differentiable structure). Correspondingly, a smooth manifold that is homeomorphic, but not diffeomorphic, to the standard $\mathbb{CP}^n$ will be called an *AD* projective space. Finally, a smooth manifold that is homotopy equivalent, but not homeomorphic, to the standard projective space will be called an *AT* projective space (where AT stands for Alternative Topological structure). ♢

8.1.1 Results

The results of this section hinge on the existence of homotopy projective spaces that are obtained as the reduced space of a circle action on an AD sphere. It is not always possible to relate an n-dimensional AD/AT projective space to a $(2n+1)$-dimensional AD sphere in this way. Below, we start by exploring a few facts about AD/AT projective spaces, after which we can discuss three interesting examples where the desired phenomenon is observed (the spaces of Theorems 8.1.11, 8.1.13).

Definition 8.1.7 ([Kaw69])**.** The inertia group $I(M)$ of an oriented closed smooth manifold M is the subgroup of Θ_m consisting of homotopy spheres $S \in \Theta_m$ such that the connected sum $M \# S$ is in the same diffeomorphism class as M. ♢

If $I(\mathbb{CP}^n) = 0$ and Θ_{2n} is non-trivial, one can build an AD projective space as follows. Given an AD sphere $\Sigma \in \Theta_{2n}$ the connected sum $\mathbb{CP}^n \# \Sigma$ (a 0-dimensional surgery) is another manifold homeomorphic to $\mathbb{CP}^n$ but not diffeomorphic to it. For $n \geqslant 8$, there are examples for which the inertia group $I(\mathbb{CP}^n)$ is non-trivial (see [Kaw69]); in those cases the smooth structure of the resulting manifold is not automatically distinct from the standard smooth structure on $\mathbb{CP}^n$. In dimension four, we know:

Theorem 8.1.8 ([Kas16])**.** *There are two possible distinct smooth structures on a manifold homeomorphic to $\mathbb{CP}^4$: the standard $\mathbb{CP}^4$-structure, and the one on $\mathbb{CP}^4 \# \Sigma^8$, where $\Sigma^8 \in \Theta_8$ is the unique AD 8-sphere.*

In contrast, it is known that there is an abundance of AT projective spaces; for even integers $n \geqslant 4$, there are infinitely many AT projective spaces, distinguished by the first Pontryagin class $p_1 \in H^4(\mathbb{CP}^n;\mathbb{Z})$ ([Hsi66]).

Is there a way to associate an AD sphere to an AD/AT projective space? Given an AD/AT projective space K, the unit bundle of the line bundle $\mathcal{L} \to K$ satisfying $c_1(\mathcal{L}) = \alpha_K$ (where $\alpha_K \in H^2(K;\mathbb{Z})$ is the cohomology generator) could still be diffeomorphic to a standard sphere. Note that in the special case where the projective space is a surgery of the form $K = \mathbb{CP}^n \# \Sigma$, for an AD sphere $\Sigma \in \Theta_{2n}$, the $(2n+1)$-sphere obtained as the unit bundle of $\mathcal{L} \to K$ is given by $stab(\Sigma) \in \Theta_{2n+1}$ (where $stab$ is the map constructed in Section 8.2, see Remark 8.2.6).

On the other hand, one could examine S^1-quotients of AD spheres $\tilde{S} \in \Theta_{2n+1}$. A priori this is not always a successful strategy, as not all homotopy spheres admit a smooth free circle action. But if such an action exists, then the quotient $P := \tilde{S}/S^1$ resulting from it is an AD or AT projective space. Namely, this reduced space is necessarily homotopy equivalent to a projective space ([Hsi66]), but it is at least not diffeomorphic to the standard $\mathbb{CP}^n$ (since circle bundles over P are classified by elements of $H^2(P;\mathbb{Z})$, and if P was the standard projective space, then the total space of the line bundle would have to be a standard sphere).

Theorem 8.1.9 ([Jam80, Sections 2-3]). *There is a homotopy* 9*-sphere* $\widetilde{S}$ *such that*

(i) $\widetilde{S} \notin bP_{10} \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) $\widetilde{S}$ *admits a free action of* S^1.

(iii) The quotient $P := \widetilde{S}/S^1$ *is not homeomorphic to* $\mathbb{CP}^4$.

(iv) P *and the standard* $\mathbb{CP}^4$ *have the same tangent bundles.*

Remark 8.1.10. *In [Jam80, Section 3], James notes that there is another* S^1*-action on* $\widetilde{S}$ *with quotient space* $P\#\Sigma^8$*. The latter is an AT projective space that is not diffeomorphic to* P. //

We now have enough material to state and prove the results of this section.

Theorem 8.1.11. *There is a manifold* P *homotopy equivalent to* $\mathbb{CP}^4$ *and with the same first Pontryagin class such that neither* P *nor* $P\#\Sigma^8$ *admit an exact Lagrangian embedding into* $T^*\mathbb{CP}^4$.

Proof. Consider the homotopy 9-sphere $\widetilde{S}$ admitting a free S^1-action of Theorem 8.1.9. The quotient $P = \widetilde{S}/S^1$ is homotopy equivalent to $\mathbb{CP}^4$, but by Theorem 8.1.9 (iii), it is not homeomorphic to it. The first Pontryagin classes of P and $\mathbb{CP}^4$ coincide by Theorem 8.1.9 (iv). Assume there is a Lagrangian embedding $P \hookrightarrow T^*\mathbb{CP}^4$. The Hopf correspondence (see Section 5.1.1) lifts P to $\widetilde{S}$, giving an exact Lagrangian embedding $\widetilde{S} \hookrightarrow T^*S^9$. However, by Theorem 8.1.9, $\widetilde{S} \in \Theta_9 \setminus bP_{10}$, so the existence of the Lagrangian embedding contradicts Corollary 8.1.5.
The same argument applies to prove that $P\#\Sigma^8$ does not embed as Lagrangian into $T^*\mathbb{CP}^4$. Namely,

the Hopf correspondence would, in that case too, lift (via the S^1-action of Remark 8.1.10) $P\#\Sigma^8$ to $\widetilde{S}$ ([Jam80, Section 3]). □

Remark 8.1.12. *Our techniques do not allow to prove whether the AD projective space* $\mathbb{CP}^4\#\Sigma^8$ *of Theorem 8.1.8 does admit a Lagrangian embedding into* $T^*\mathbb{CP}^4$ *or not.* //

Theorem 8.1.13. *Let* $\Sigma^{14} \in \Theta_{14}$ *be the unique AD* 14*-sphere. Then* $\mathbb{CP}^7\#\Sigma^{14}$ *does not admit an exact Lagrangian embedding into* $T^*\mathbb{CP}^7$.

Proof. First note that $\Theta_{14} \cong \mathbb{Z}/2\mathbb{Z}$ and $bP_{15} = 0$ ([KM63]), so there is a unique AD 14-sphere, denoted by Σ^{14} as in the statement. By [Bre67, Theorem 4.6] there is an AD sphere $\Sigma^{15} \in \Theta_{15} \setminus bP_{16}$ admitting a free S^1-action, with quotient $P := \mathbb{CP}^7\#\Sigma^{14}$. If P admitted a Lagrangian embedding in $T^*\mathbb{CP}^7$, the Hopf correspondence would yield a Lagrangian embedding $\Sigma^{15} \hookrightarrow T^*S^{15}$. But $\Sigma^{15} \notin bP_{16}$, which would contradict Corollary 8.1.5 (for the same reasons as in the proof of Theorem 8.1.11). □

8.2 Framing of projective twists

The background material that we use to examine the question of framing of projective twists is based on [DRE15], in which Dimitroglou Rizell and Evans proved that the Hamiltonian isotopy class of a Dehn twist does in general depend on a choice of framing.
Let (M,ω) be a symplectic manifold. Given a framing of a Lagrangian sphere $L \subset M$, i.e a diffeomorphism $S^n \longrightarrow L$ (see Section 2.1), the precomposition with an element $F \in \mathrm{Diff}(S^m)$ yields another framing.

Consider the symplectomorphism $F^*\colon T^*S^m \to T^*S^m$ induced by the lift of F to the cotangent bundle T^*S^m. The standard model twist $\tau^{loc}_{S^m} \in \mathrm{Symp}_{ct}(T^*S^m)$ can be replaced by $F^* \circ \tau^{loc}_{S^m} \circ (F^{-1})^*$, and the latter can be implanted in a Weinstein neighbourhood as in Definition 2.1.8 to produce a new element in $\mathrm{Symp}_{ct}(M)$. To study framings of twists, we can then restrict to these parametrisations of the standard model twist $\tau_{S^m} := \tau^{loc}_{S^m} \in \mathrm{Symp}_{ct}(T^*S^m)$.

A core fact for the study of parametrisations of twists is the isomorphism $\pi_0(\mathrm{Diff}^+(S^m)) \cong \Theta_{m+1}$ ([KM63], [Cer70]). In particular, given a non-trivial diffeomorphism $F \in \mathrm{Diff}^+(S^m)$, there is a $(m+1)$-dimensional AD sphere constructed as follows.

Definition 8.2.1. Let $F \in \mathrm{Diff}(S^m)$ not isotopic to the identity. Then $\Sigma_F := D^{m+1} \cup_F D^{m+1} \in \Sigma_{m+1}$ is an $(m+1)$-dimensional homotopy sphere obtained by gluing two $(m+1)$-discs along their boundary S^m twisted by F. In the notation of [DRE15, Definition 1.4] (which is more apt to visualise the Lagrangian suspension we utilise in Section 8.2.2), this is equivalent to

$$\Sigma_F := (D^{m+1} \times S^0) \cup_\Phi S^m \times [0,1]$$

glued along $S^m \times S^0$ via $\Phi\colon S^m \times S^0 \to S^m \times S^0$, $\Phi(x,y) = (F(x),y)$. ♢

Also recall that there is an isomorphism $\pi_0(\mathrm{Diff}^+(S^m)) \cong \pi_0(\mathrm{Diff}^+_{ct}(D^m))$ induced by a map $\mathrm{Diff}^+_{ct}(D^m) \to \mathrm{Diff}^+(S^m)$ which extends all elements of $\mathrm{Diff}^+_{ct}(D^n)$ over a capping disc.

Dimitroglou Rizell and Evans proved the existence of Dehn twists, whose Hamiltonian isotopy class depends on the choice of framing.

Definition 8.2.2 ([DRE15, Definition 1.1])**.** Fix a cotangent fibre $\Lambda \subset T^*S^m$ and let $\mathcal{L}_m \subset \Theta_m$ be the subset of homotopy spheres which admit a Lagrangian embedding into T^*S^m with the additional requirement that the embedding intersects Λ transversely in exactly one point. ♢

Theorem 8.2.3 ([DRE15, Theorem A])**.** *Let $F \in \mathrm{Diff}^+(S^m)$ be such that $\Sigma_F \notin \mathcal{L}_{m+1}$. Then $\tau_{S^m}^{-1} \circ (F^* \circ \tau_{S^m} \circ (F^{-1})^*)$ is not trivial in $\pi_0(\mathrm{Symp}_{ct}(T^*S^m))$.*

In the rest of the chapter, we analyse the analogous problem for reparametrisations $f \in \mathrm{Diff}(\mathbb{CP}^n)$ of projective twists. We prove that there exist $n \in \mathbb{N}$ such that the twist $\tau_f := f^* \circ \tau_{\mathbb{CP}^n} \circ (f^{-1})^*$ is not isotopic to the standard projective twist in $\pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n))$, where $f^*\colon T^*\mathbb{CP}^n \to T^*\mathbb{CP}^n$ is the symplectomorphism induced by the lift of f to the cotangent bundle. We will not directly use Theorem 8.2.3 but an intermediary result (Proposition 8.2.4 below) that Dimitroglou Rizell and Evans proved (using [Abo12a, AK18, ES14]) to support their arguments.

Proposition 8.2.4 ([DRE15, Proposition 1.2])**.** *There is an inclusion $\mathcal{L}_m \subset bP_{m+1}$.*

Remark 8.2.5. *There is a slight abuse of terminology in the entirety of the chapter. A* framing *will be employed (as in the rest of the thesis) in the non-standard sense à la Seidel to signify a smooth parametrisation of a sphere. The classical topological notion of framing (as a trivialisation of the normal bundle) is also needed in this section, and in order to avoid a conflict of nomenclature, we call the latter a* normal framing. //

8.2.1 Bredon's pairing

We begin by introducing an essential component of the arguments of this section; a map

$$stab\colon \Theta_m \longrightarrow \Theta_{m+1} \tag{8.2}$$

obtained as a special case of a homomorphism $\Theta_m \otimes \pi_\ell(S) \longrightarrow \Theta_{m+\ell}$ studied in [Bre67].

Consider the linear action of $\mathrm{SO}(2) \simeq S^1$ on $S^{m+1} \subset \mathbb{R}^{m+2}$ via the representation

$$\mathrm{SO}(2) \longrightarrow \mathrm{SO}(m+2),\, A \longmapsto \varphi(A) = \begin{bmatrix} A & & \\ & A & \\ & & \ddots \end{bmatrix}$$

with 1 in the right hand bottom corner if m is odd. This is the linear S^1-action on S^{m+1} which is free if m even (in which case it is the standard Hopf action), and whose fixed point set is S^0 if m odd.

For $\Sigma \in \Theta_m$, Bredon's construction ([Bre67, Sections 1, 4]) yields a homotopy $(m+1)$-sphere as follows. Let $V \subset \Sigma$ be an open neighbourhood of a point $p \in \Sigma$, and $g\colon (V,p) \to (\mathbb{R}^m, 0)$ an orientation reversing diffeomorphism. Let $B := g^{-1}(D^m) \subset V \subset \Sigma$, where $D^m \subset \mathbb{R}^m$ is the unit disc.

Let $\mathcal{C} \cong S^1 \subset S^{m+1}$ be a principal orbit of the SO(2)-action on S^{m+1}, equipped with a normal framing $\mathcal{F}\colon \mathcal{C} \times \mathbb{R}^m \to S^{m+1}$.

Define

$$stab(\Sigma) := S^{m+1} \setminus (\mathcal{F}(\mathcal{C} \times D^m)) \,\cup\, \mathcal{C} \times (\Sigma \setminus B) \tag{8.3}$$

where the two pieces are glued along their boundaries, which can be identified via a diffeomorphism $\mathcal{F}(\mathcal{C} \times (\mathbb{R}^m \setminus \{0\})) \cong \mathcal{C} \times (V \setminus \{p\})$ as in [Bre67, p. 435].

The normally framed orbit $(\mathcal{C}, \mathcal{F})$ represents an element $\gamma \in \pi_{m+1}(S^m) \cong \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ via the Thom–Pontryagin construction (see [Mil65, 87]). With this identification in mind, the map *stab* is derived from a pairing $\Theta_m \times \pi_{m+1}(S^m) \to \Theta_{m+1},\ (\Sigma, \gamma) \mapsto stab(\Sigma) = \langle \Sigma, \gamma \rangle$ (see [Bre67, (1)]). The latter induces a homomorphism ([Bre67, (2)])

$$\Theta_m \otimes \pi_1(S) \longrightarrow \Theta_{m+1}. \tag{8.4}$$

To determine the class γ, we follow [Bre67, (4.1)] and find that $\gamma = \eta^j$, where $\eta \in \pi_1(S) := \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is the non-trivial element in the stable stem $\pi_1(S)$, and

$$j = \begin{cases} \frac{m}{2} & \text{If } m \textbf{ even} \text{ (i.e the action on } S^{m+1} \text{ is free)} \\ \frac{m-1}{2} & \text{If } m \textbf{ odd} \text{ (i.e the action } S^{m+1} \text{ is not free).} \end{cases}$$

Intuitively, if a normally framed Hopf circle in S^3 represents the class $\eta \in \pi_4(S^3)$, then γ is determined by the number of times (mod 2) that this normal framing fits in the normal bundle to $\mathcal{C} \subset S^{m+1}$.

For $m+1 = 2n+1$, and $m+1 = 2n+2$, we have $j = n$ and

$$\gamma = \eta^n = \begin{cases} \eta & \text{If } n \textbf{ odd} \\ 0 & \text{If } n \textbf{ even}. \end{cases} \tag{8.5}$$

Remark 8.2.6. *For an even dimensional homotopy sphere $\Sigma \in \Theta_{2n}$, the image $stab(\Sigma)$ can also be described as follows (this remark is relevant for Section 8.1). Consider the surgery $\mathbb{CP}^n \# \Sigma$ and the complex line bundle $\mathcal{L} \to \mathbb{CP}^n \# \Sigma$ associated to the generator of $H^2(\mathbb{CP}^n \# \Sigma; \mathbb{Z})$. Then, $stab(\Sigma)$ is the homotopy sphere obtained as the unit circle bundle of $\mathcal{L}$.* //

We now focus on the case $m+1 = 2n+2$.

Lemma 8.2.7. *The map $\Theta_{2n+1} \longrightarrow \Theta_{2n+2}/bP_{2n+3}$ is non-trivial for $n = 19, 23, 25, 29$.*

Proof. There is a commuting diagram (see [Bre67, Corollary 2.2]) obtained from the exact sequence (8.1),

$$\begin{array}{ccc} \Theta_{2n+1} & \xrightarrow{stab} & \Theta_{2n+2} \\ \downarrow \psi & & \downarrow \psi \\ \operatorname{coker}(J_{2n+1}) & \xrightarrow{(-)\cdot\eta^n} & \operatorname{coker}(J_{2n+2}) \end{array} \quad . \tag{8.6}$$

where $(-)\cdot\eta^n\colon \operatorname{coker}(J_{2n+1}) \to \operatorname{coker}(J_{2n+2})$, is a map descending from the multiplication $\pi_{2n+1}(S) \times \pi_1(S) \to \pi_{2n+2}(S)$ with the class $\eta \in \pi_1(S) \cong \mathbb{Z}/2\mathbb{Z}$, which is well defined since for $\ell + 1 < m$, $\operatorname{Im}(J_m) \cdot \operatorname{Im}(J_\ell) \subseteq \operatorname{Im}(J_{m+\ell})$, the image of the J-homomorphism is preserved under multiplication with elements of the stable stems.

By (8.5), we know that a necessary requirement for the map *stab* to be non-trivial is to have $n = 2k+1$ for some $k \in \mathbb{N}$ so that $\eta^n = \eta$ is non-trivial. In that case, we get

$$\begin{array}{ccc} \Theta_{4k+3} & \xrightarrow{stab} & \Theta_{4(k+1)} \\ \downarrow \psi & & \downarrow \psi \\ \operatorname{coker}(J_{4k+3}) & \xrightarrow{(-)\cdot\eta} & \operatorname{coker}(J_{4(k+1)}) \end{array} \quad . \tag{8.7}$$

The vertical maps are both surjective since ψ is always surjective in odd dimensions, and when $m \equiv 0 \mod 4$ (see [Lev85, Theorem 5.4]).

The exact sequence (8.1) implies that $\operatorname{coker}(J_{4k+4}) \cong \Theta_{4k+4}/\ker(\psi) \cong \Theta_{4k+4}$, and the non-triviality of the composition $\psi \circ stab\colon :\Theta_{4k+3} \to \Theta_{4k+4}$ is equivalent to the non-triviality of the multiplication $(-)\cdot\eta\colon \operatorname{coker}(J_{4k+3}) \to \operatorname{coker}(J_{4k+4})$. This amounts to looking for elements in the stable stems whose η-multiples are not in the image of J. As η is of order two, this information can be found in the "two-primary part" of the stable stems, the subgroups obtained after quotienting all elements of odd order. These are tabulated in a diagram in [Hat01, p.385], where the elements of interest appear to be in degrees $2n+1 \in \{39, 47, 51, 59\}$, which means that $n \in \{19, 23, 25, 29\}$. □

The rest of the section is dedicated to explain how to relate a parametrisation $f \in \operatorname{Diff}^+(\mathbb{CP}^n)$ for the projective twist to a parametrisation $F \in \operatorname{Diff}^+(S^{2n+1})$ of the Dehn twist.

Lemma 8.2.8. *Let n be an odd integer and $f \in \operatorname{Diff}^+(\mathbb{CP}^n)$ an orientation preserving diffeomorphism. There exists a diffeomorphism $F \in \operatorname{Diff}^+(S^{2n+1})$ satisfying $h \circ F = f \circ h$, i.e F is the lift of f by the Hopf bundle map.*

Proof. Let $h\colon S^{2n+1} \to \mathbb{CP}^n$ the Hopf bundle map. A diffeomorphism $f\colon \mathbb{CP}^n \to \mathbb{CP}^n$ induces a continuous function $F\colon f^*(S^{2n+1}) \to S^{2n+1}$ covering f such that the following diagram commutes

$$\begin{array}{ccc} f^*(S^{2n+1}) & \xrightarrow{\quad F \quad} & S^{2n+1} \\ \Big\downarrow h' & & \Big\downarrow h \\ \mathbb{CP}^n & \xrightarrow{\quad f \quad} & \mathbb{CP}^n \end{array} \quad . \tag{8.8}$$

where $h'\colon f^*(S^{2n+1}) \to \mathbb{CP}^n$ is the pullback bundle of h by f. The map induced by f on the second cohomology $\overline{f}\colon H^2(\mathbb{CP}^n) \to H^2(\mathbb{CP}^n)$ is $\pm Id$. Therefore, the Euler classes of $h\colon S^{2n+1} \to \mathbb{CP}^n$ and $h'\colon f^*(S^{2n+1}) \to \mathbb{CP}^n$ coincide up to sign so that these principal S^1-bundles must have diffeomorphic total spaces. It follows that $F\colon f^*(S^{2n+1}) \to S^{2n+1}$ is in fact a diffeomorphism $F\colon S^{2n+1} \to S^{2n+1}$ covering f satisfying $h \circ F = f \circ h$.

□

Lemma 8.2.9. *Let n be an odd integer and $f \in \mathrm{Diff}^+(\mathbb{CP}^n)$ be an orientation preserving diffeomorphism supported in an open chart, i.e f is induced by an element of $\mathrm{Diff}^+_{ct}(D^{2n})$, and let $\Sigma_f \in \Theta_{2n+1}$ be the homotopy $(2n+1)$-sphere associated to f.*

Let $F \in \mathrm{Diff}^+(S^{2n+1})$ be the S^1-equivariant lift of f of Lemma 8.2.8, and $\Sigma_F \in \Theta_{2n+2}$ the corresponding homotopy $(2n+2)$-sphere. Then $stab(\Sigma_f) = \Sigma_F$.

Proof. The lift $F \in \mathrm{Diff}^+(S^{2n+1})$ is supported in a tubular neighbourhood of a Hopf circle in S^{2n+1}. To build Σ_F, identify S^{2n+1} with an equator in S^{2n+2}, and consider the above circle as a normally framed circle $\mathcal{C} \subset S^{2n+2}$. That requires a choice of trivialisation of the normal bundle to a Hopf circle, a normal framing $\mathcal{F}\colon \mathcal{C} \times \mathbb{R}^{2n+1} \to S^{2n+2}$ that defines the support of the gluing map for the construction of Σ_F: on $\mathcal{F}(\mathcal{C} \times D^{2n+1})$, F acts as $id \times f$. By the same arguments as in the beginning of this section, the normal framing of this Hopf circle corresponds to the class $\eta^n \in \pi_1(S)$, so that $\Sigma_F = stab(\Sigma_f)$ by the construction (8.3). □

8.2.2 Results

We first mention an auxiliary result from [DRE15] we will need in the proof of Theorem 6.

Lemma 8.2.10 ([DRE15, Proposition 2.5])**.** *Consider $(T^*S^{2n+1}, d\lambda_{T^*S^{2n+1}})$ with the well-known structure of Lefschetz fibration $T^*S^{2n+1} \to \mathbb{C}$ with smooth fibre $(T^*S^{2n}, d\lambda_{T^*S^{2n}})$ and two singular fibres. Let $L \subset T^*S^{2n+1}$ be the standard Lagrangian embedding of the zero section. There is an open symplectic embedding*

$$e\colon T^*S^{2n+1} \times T^*[0,1] \longrightarrow T^*S^{2n+2} \tag{8.9}$$

such that

- *$L\times[0,1]$ is sent to a subset of the zero section $S^{2n+2}\subset T^*S^{2n+2}$ (the matching sphere);*

- *The image of the embedding is disjoint from a particular cotangent fibre $\Lambda\subset T^*S^{2n+2}$ (a Lefschetz thimble).*

Proposition 8.2.11. *If the map $stab:\Theta_{2n+1}\to\Theta_{2n+2}$ is non-trivial and n is odd, then the $\mathbb{CP}^n$-twist depends on a choice of framing.*

Proof. Choose a framing $f\in \mathrm{Diff}^+(\mathbb{CP}^n)$, coming from an element of $\mathrm{Diff}^+_{ct}(D^{2n})$ extended by the identity on the projective space. Let $F\in\mathrm{Diff}^+(S^{2n+1})$ be the S^1-equivariant lift of f as in Lemma 8.2.8, supported in a tubular neighbourhood of a Hopf circle $\mathcal{F}\colon\ \cong S^1\times D^{2n}\subset S^{2n+1}$. Let $\Sigma_f\in\Theta_{2n+1}$ be the sphere associated to f, and $\Sigma_F\in\Theta_{2n+2}$ that associated to F. By Lemma 8.2.9, $\Sigma_F=stab(\Sigma_f)=\langle\Sigma_f,\eta^n\rangle\in\Theta_{2n+2}$. Since n is odd, $\eta^n=\eta$ and $\Sigma_F=stab(\Sigma_f)\in\Theta_{2n+2}$ is non-trivial.

The map f^* induced by f on the cotangent bundle is not compactly supported, but can be used to define the compactly supported conjugation

$$\tau_f:=f^*\circ\tau_{\mathbb{CP}^n}\circ(f^{-1})^*\colon T^*\mathbb{CP}^n\longrightarrow T^*\mathbb{CP}^n \tag{8.10}$$

of the projective twist $\tau_{\mathbb{CP}^n}\in\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$.

We show in Lemma 8.2.12 below that τ_f belongs to a Hamiltonian class distinct from that of the standard projective twist. □

Lemma 8.2.12. *The twist τ_f is not isotopic to the standard twist $\tau_{\mathbb{CP}^n}\in\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$.*

Proof. Assume by contradiction that $\tau_{\mathbb{CP}^n}^{-1}\circ\tau_f$ is (Hamiltonian) isotopic to the identity in $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$. Let $(\phi_t)_{t\in[0,1]}$ be an isotopy connecting the two symplectomorphisms in $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$ such that there are $s'>s\in(0,1)$ with

$$\phi_t=\begin{cases}\tau_{\mathbb{CP}^n}^{-1}\circ\tau_f & \text{If } t\leqslant s\\ Id & \text{If } t\geqslant s'.\end{cases} \tag{8.11}$$

Let $H\colon T^*\mathbb{CP}^n\times[0,1]\to\mathbb{R}$ be the generating Hamiltonian function. Define the Lagrangian embedding

$$\begin{aligned}\psi'\colon K\times[0,1]&\longrightarrow T^*\mathbb{CP}^n\times T^*[0,1]\\ (x,t)&\longmapsto(\phi_t(x),t,-H(\phi_t(x),t)),\end{aligned} \tag{8.12}$$

where $K\subset T^*\mathbb{CP}^n$ is the standard Lagrangian embedding of the zero section. By construction, near the ends of the interval, $K\times[0,1]$ is preserved by (8.12) in the sense that $\psi'(K\times[0,s])\subset K\times[0,s]$ and $\psi'(K\times[s',1])\subset K\times[s',1]$.

On each fibre $T^*\mathbb{CP}^n$ of $T^*\mathbb{CP}^n \times T^*[0,1]$, apply the Hopf correspondence to lift the image of (8.12) to a Lagrangian embedding

$$\Psi \colon L \times [0,1] \to T^*S^{2n+1} \times T^*[0,1] \tag{8.13}$$

(where $L \subset T^*S^{2n+1}$ is the standard Lagrangian embedding of the zero section) such that $L \times I$ is preserved by Ψ near the ends of the interval.

By Lemma 8.2.10 we can replace $e(L \times [0,1]) \subset T^*S^{2n+2}$ by the Lagrangian suspension $\Psi(L \times [0,1])$, so that the ends of $\Psi(L \times [0,1])$ are "capped" into a $(2n+2)$-dimensional sphere diffeomorphic to $\Sigma_F \in \Theta_{2n+2}$ (see [DRE15, 3.3]) which intersects a cotangent fibre once transversely and is therefore contained in $\mathcal{L}_{2n+2}$. By Lemma 8.2.4, $\mathcal{L}_{2n+2} \subset bP_{2n+3}$ and since $bP_{2n+3} = 0$ (this holds for all odd integers, see [KM63]), Σ_F has to be the standard sphere. However, as we have seen above, $\Sigma_F = \langle \eta^n, \Sigma_f \rangle \in \Theta_{2n+2}$ is non-trivial as n is odd. This is a contradiction, which proves Proposition 8.2.12; τ_f cannot be isotopic to the standard projective twist in $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$. □

Combining Lemma 8.2.7 with Proposition 8.2.11 yields the result we want.

Corollary 8.2.13. *The $\mathbb{CP}^n$-twist depends on the framing when $n = 19, 23, 25, 29$.*

□

Remark 8.2.14. *The $\mathbb{CP}^n$-twist depends on the choice of framing for infinitely many dimensions n.* //

Proof. One way to obtain infinite families of nontrivial multiples of η which are not contained in the image of J is by detecting them in topological modular forms, denoted tmf (we refer to [Hen14] for a survey on the subject). There is a "Hurewicz homomorphism" $\pi_*(\mathbb{S}) \to \pi_*(tmf)$ between the ring of stable homotopy groups of spheres and the homotopy ring of tmf, and the two primary components of the ring of homotopy groups have a certain kind of periodicity of degree 192. Therefore, if we can identify an element in one of the homotopy groups $\pi_{4k+3}(tmf)$ that is also in the image of the Hurewicz homomorphism and arises as a product of η, we obtain a periodic family of elements to which the argument of Lemma 8.2.7 applies.

A (partially conjectural) diagram depicting the two-primary components can be found in [Hen14] and it is helpful to first identify a potential candidate. Degree $39 = 4 \cdot 9 + 3$ presents an element which has been confirmed to be the image of a non-trivial multiple of η (see [HM14, Corollary 11.2], there the element in question is called u and arises as image of a product of $\overline{\kappa}, \nu, \eta$ and κ; all of these are standard names of generators of stable homotopy groups stems). It follows that in every dimension $m \equiv 39 \pmod{192}$ there is an element for which the map $(-) \cdot \eta \colon \mathrm{coker}(J_m) \to \mathrm{coker}(J_{m+1})$ and hence $stab \colon \Theta_m \to \Theta_{m+1}$ are not trivial. Recall that $m = 4k+3 = 2n+1$ so that by Proposition

8.2.11, the projective twist depends on the framing for $n \equiv 19 \pmod{96}$. Further scrutiny of the literature would provide other such elements—e.g for $m = 59$ $(n = 29)$. □

Remark 8.2.15. *It is very likely that a version of Corollary 8.2.14 holds for $\mathbb{HP}^n$-twists as well. Bredon computes (see [Bre67, p.446]) the class that would be associated to a framing of $S^3 \subset S^{4n+3}$, which is a power of $\nu \in \pi_3(S) = \lim_m \pi_{m+3}(S^m) \cong \pi_8(S^5) \cong \mathbb{Z}_{24}$. Non-triviality results for the map stab in this case would not only be depending on the parity of n, so a non-vanishing criterion would be harder to obtain. But such a criterion could then be combined with existence of smooth semi free actions of S^3 on homotopy $(4k+3)$-spheres explicitly computed in [Bre67, Theorem 4.4, 4.7] (also note that there are infinitely many inequivalent free S^3-actions on homotopy S^{4k+3}-spheres, by [Hsi66, Theorem 3]). Then, the above strategy could be applied to obtain infinitely many dimensions in which the $\mathbb{HP}^n$-twist would depend on the framing.*

//

Corollary 8.2.16. *In the above dimensions,* $\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n) \not\simeq \mathbb{Z}$.

Proof. If $\tau_{\mathbb{CP}^n} \in \pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n))$ is the standardly framed twist along the zero section, then we claim that $\mathbb{Z}\langle\tau_{\mathbb{CP}^n}\rangle \subsetneq \pi_0(\mathrm{Symp}_{ct}(T^*\mathbb{CP}^n))$. Let $f \in \mathrm{Diff}^+_{ct}(\mathbb{CP}^n)$ be a framing such that the projective twist $\tau_f \in \mathrm{Symp}_{ct}(T^*\mathbb{CP}^n)$ defined using f is not isotopic to $\tau_{\mathbb{CP}^n}$, as in Corollary 8.2.13. Then, $\tau_f^{-1} \circ \tau_{\mathbb{CP}^n}$ cannot be isotopic to any power $\tau^k_{\mathbb{CP}^n}$, for any $k \in \mathbb{Z}$. This is because $\tau_{\mathbb{CP}^n}$, viewed as a graded symplectomorphism, acts non-trivially on the grading of the zero section, viewed as a graded Lagrangian (see [Sei00, Lemma 5.7]), whereas $\tau_f^{-1} \circ \tau_{\mathbb{CP}^n}$ acts trivially on the grading (see also [DRE15, Remark 1.5]). □

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