Algebr Represent Theor
https://doi.org/10.1007/s10468-018-9791-7

On the p'-Subgraph of the Young Graph

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Received: 21 August 2017 / Accepted: 23 April 2018

Abstract Let p be a prime number. In this article we study the restriction to $\mathfrak{S}_{n-1}$ of irreducible characters of degree coprime to p of $\mathfrak{S}_n$. In particular, we study the combinatorial properties of the subgraph $\mathbb{Y}_{p'}$ of the Young graph $\mathbb{Y}$. This is an extension to odd primes of the work done in Ayyer et al. (2016) for $p = 2$.

Keywords Young graph · Characters of symmetric groups · Abacus combinatorics

Mathematics Subject Classification (2010) 20C30

1 Introduction

Let $\mathcal{P}$ denote the set of partitions of natural numbers. For λ a partition of n, also written $\lambda \vdash n$, and $\mu \in \mathcal{P}$ we let $(\lambda, \mu) \in \mathcal{E}$ if and only if χ^μ is an irreducible constituent of the multiplicity-free character $(\chi^\lambda)_{\mathfrak{S}_{n-1}}$. Here we denoted by χ^λ the ordinary irreducible character of the symmetric group $\mathfrak{S}_n$ naturally labelled by λ (this notation will be kept throughout the article). The Young graph $\mathbb{Y}$ has $\mathcal{P}$ as its set of vertices and $\mathcal{E}$ as its set of edges. We refer the reader to Section 2.2 for a more precise description of $\mathbb{Y}$.

Presented by: Radha Kessar

The first author's research was funded by Trinity Hall, University of Cambridge.

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We remark that the Young graph is a well-understood and intensively studied combinatorial object, deeply connected to the representation theory of symmetric groups. It is somewhat surprising that only very recently in [1], the following remarkable fact was shown to hold.

Theorem 1 (Unique Parent Theorem in [1]) *Let $n \in \mathbb{N}$ and let χ be an irreducible character of odd degree of $\mathfrak{S}_n$. Then the restriction $\chi_{\mathfrak{S}_{n-1}}$ has a unique irreducible constituent of odd degree.*

Let p be a prime number. For a partition $\lambda \vdash n$, we write $\lambda \vdash_{p'} n$ if the corresponding character χ^λ labelled by λ has p'-degree, that is, degree coprime to p. Let $\mathbb{Y}_{p'}$ be the subgraph of $\mathbb{Y}$ induced by the subset of vertices (partitions) labelling irreducible characters of p'-degree. Theorem 1 shows that $\mathbb{Y}_{2'}$ is a rooted tree. Starting from this beautiful observation, the rest of [1] is devoted to describing the combinatorial structure of $\mathbb{Y}_{2'}$. We remark that the relevance of [1] transcends the study of the Young graph. In fact, Theorem 1 was recently used to construct several types of character correspondences (see [3–5] and [6]).

The main aim of this paper is to study the combinatorial structure of $\mathbb{Y}_{p'}$ for any odd prime p. As remarked in [1, Section 7], $\mathbb{Y}_{3'}$ is not a tree. Indeed, for every odd prime p, there exists an irreducible character χ of p'-degree of some $\mathfrak{S}_n$ whose restriction $\chi_{\mathfrak{S}_{n-1}}$ has more than one irreducible constituent of p'-degree. Yet notably, given *any* prime p and any irreducible character χ of p'-degree of $\mathfrak{S}_n$, Theorem A (below) describes the number of irreducible constituents of p'-degree of $\chi_{\mathfrak{S}_{n-1}}$. In particular, this is a generalisation of Theorem 1 to all primes.

Given a partition $\lambda \vdash n$ we denote by $\lambda^-_{p'}$ the set consisting of all partitions $\mu \vdash_{p'} n-1$ such that χ^μ is an irreducible constituent of $(\chi^\lambda)_{\mathfrak{S}_{n-1}}$. Moreover we define $\mathcal{E}_n$ to be the set

$$\mathcal{E}_n = \left\{ |\lambda^-_{p'}| \ : \ \lambda \vdash_{p'} n \right\},$$

and we let $br(n)$ be the maximal value in $\mathcal{E}_n$ ($br(n)$ is well-defined, as we remark in Section 2.2). Our first result describes $\mathcal{E}_n$ and gives a recursive formula for the exact value of $br(n)$.

Theorem A *Let $n \in \mathbb{N}$ and let p be a prime. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be the p-adic expansion of n, for some $0 \leq n_1 < n_2 < \cdots < n_t$ with $a_j > 0$ for all $1 \leq j \leq t$. Then $\mathcal{E}_n = \{1, 2, \ldots, br(n)-1, br(n)\}$ and*

$$br(n) = br(a_1 p^{n_1}) + \sum_{j=2}^{t} \Phi(a_j, br(m_j))$$

where $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$, and where Φ is the function described explicitly in Definition 2.1 below.

In Section 5 we determine $br(ap^k)$ exactly for any prime p, any $k \in \mathbb{N}_0$ and any $a \in \{1, \ldots, p-1\}$. The following result serves as the base case for computing $br(n)$ for any natural number n, using the recursive expression given in Theorem A.

Theorem B *Let p be an odd prime, $k \in \mathbb{N}_0$ and $a \in \{1, \dots, p-1\}$. Then*

$$br(ap^k) = \begin{cases} f(2a) & \text{if } k = 0, \\ p - 1 + 2\lfloor \frac{2a-(p-1)}{6} \rfloor & \text{if } k = 1 \text{ and } \frac{p}{2} < a < p, \\ 2a & \text{otherwise.} \end{cases}$$

Here $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$.

We remark that by [1], if $p = 2$ then $br(2^k) = 1$ for all $k \in \mathbb{N}_0$.

Theorems A and B provide us with a recursive formula for $br(n)$, the maximal number of downward edges from a vertex on level n of $\mathbb{Y}_{p'}$ to level $n-1$. In the second part of our article we show that the slightly involved expression for the value of $br(n)$ described in Theorem A can be bounded from above by a much nicer function of the p-adic digits of n.

Theorem C *Let $n \in \mathbb{N}$ and let p be a prime. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be the p-adic expansion of n, for some $0 \leq n_1 < n_2 < \cdots < n_t$ with $a_j > 0$ for all $1 \leq j \leq t$. Then $1 \leq br(n) \leq \mathcal{B}_n$, where*

$$\mathcal{B}_n = br(a_1 p^{n_1}) + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor \leq 2a_1 + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor.$$

Theorem C has some interesting direct applications (see Section 4). For instance, in Remark 4.2 below, we observe that when $p \in \{2, 3\}$ then $\mathcal{B}_n = br(n)$. In particular our result is a generalization of Theorem 1. Moreover, for any prime p we observe that the upper bound $\mathcal{B}_n$ is attained for every n having all of its p-adic digits lying in $\{0, 1, 2, 3\}$.

We further show that the upper bound $\mathcal{B}_n$ given in Theorem C is indeed a good approximation of $br(n)$. In fact, the following result shows that the difference $\varepsilon_n := \mathcal{B}_n - br(n)$ can be bounded by a constant depending only on the prime p, and not on $n \in \mathbb{N}$.

Theorem D *For any $n \in \mathbb{N}$, we have $\varepsilon_n < \frac{p}{2}\log_2(p)$.*

A consequence of Theorem D is that for any odd prime p we have $\sup\{br(n) : n \in \mathbb{N}\} = \infty$. This is false when $p = 2$, by Theorem 1.

2 Notation and Preliminaries

In this section we fix the notation that will be used throughout the article and recall some basic facts in the representation theory of symmetric groups (we refer the reader to [8] and [10] for detailed accounts of the theory). We begin by introducing the technical notation necessary to state and prove Theorem A.

Definition 2.1 For $a \in \mathbb{N}_0$ and $L \in \mathbb{N}$, define

$$\Phi(a, L) := \max \left\{ \sum_{i=1}^{L} f(a_i) \,\middle|\, a_1 + \cdots + a_L \leq a \text{ and } a_i \in \mathbb{N}_0 \ \forall\, 1 \leq i \leq L \right\},$$

where $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$.

We now record some properties of this function Φ which will be useful for later proofs.

Lemma 2.2 *Let $a \in \mathbb{N}_0$ and $L \in \mathbb{N}$. Then $\Phi(a, L) \leq \lfloor a/2 \rfloor$. In particular, if $L \geq \lfloor a/2 \rfloor$ then $\Phi(a, L) = \lfloor a/2 \rfloor$.*

Proof Suppose $\Phi(a, L) = f(a_1) + \cdots + f(a_L)$ such that $a_i \in \mathbb{N}_0$ and $a_1 + \cdots + a_L \leq a$. Observe that for all integers $x \geq 2$, we have $f(x) \leq f(2) + f(x-2)$. Hence

$$f(a_i) \leq \lfloor a_i/2 \rfloor \cdot f(2) + f(\delta_i)$$

for all $1 \leq i \leq L$, where $\delta_i = a_i - 2\lfloor a_i/2 \rfloor \in \{0, 1\}$. Thus

$$\Phi(a, L) \leq \sum_{i=1}^{L} (\lfloor a_i/2 \rfloor \cdot f(2) + f(\delta_i)) = \sum_{i=1}^{L} \lfloor a_i/2 \rfloor \leq \lfloor a/2 \rfloor,$$

where the middle equality follows from the fact that $f(2) = 1$ and $f(1) = f(0) = 0$.

Finally, if $L \geq \lfloor a/2 \rfloor$ then we see that $\Phi(a, L) = \lfloor a/2 \rfloor$ by considering

$$a_1 = a_2 = \cdots = a_{\lfloor a/2 \rfloor} = 2 \text{ and } a_{\lfloor a/2 \rfloor + 1} = \cdots = a_L = 0,$$

which satisfy $\sum_{i=1}^{L} a_i = 2 \cdot \lfloor a/2 \rfloor \leq a$ and $\sum_{i=1}^{L} f(a_i) = \lfloor a/2 \rfloor$. □

Lemma 2.3 *Let $k \in \mathbb{N}$. Then $2^{k-1} \leq \Phi(2^k + 2, 2^{k-1}) \leq 2^{k-1} + 1$.*

Proof When $k = 1$, we note that $\Phi(4, 1) = 1$. Now assume $k \geq 2$. The upper bound follows from Lemma 2.2. The lower bound follows from the fact that $2^k + 2 = 6 + 2 \cdot (2^{k-1} - 2) + 0$, and $f(6) + f(2) \cdot (2^{k-1} - 2) + f(0) = 2^{k-1}$. □

2.1 Combinatorics of Partitions

Let n be a natural number. We denote by $\mathcal{P}(n)$ the set of partitions of n and we let

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}(n).$$

Hence the notation $\lambda \vdash n$ is sometimes replaced by $\lambda \in \mathcal{P}(n)$. For any natural number e, we denote by $C_e(\lambda)$ and $Q_e(\lambda) = (\lambda_0, \lambda_1, \ldots, \lambda_{e-1})$ the e-core and the e-quotient of λ respectively (see [10, Chapter I] for the precise definitions). The e-weight of λ is the natural number $w_e(\lambda)$ defined by $w_e(\lambda) = |\lambda_0| + |\lambda_1| + \cdots + |\lambda_{e-1}|$. We remark that given a partition λ of n, the e-quotient $Q_e(\lambda)$ is uniquely determined up to a cyclic shift of its components. Moreover, it is well-known that (up to the above mentioned shift) any partition is uniquely determined by its e-core and e-quotient (we refer the reader to [10] for a detailed discussion on the topic).

Let $\mathcal{H}(\lambda)$ be the set of hooks of λ and denote by $\mathcal{H}_e(\lambda)$ the subset of $\mathcal{H}(\lambda)$ consisting of those hooks of λ having length divisible by e. We also let $\mathcal{H}(Q_e(\lambda)) = \cup_{i=0}^{e-1} \mathcal{H}(\lambda_i)$. As explained in [10, Theorem 3.3], there is a bijection between $\mathcal{H}_e(\lambda)$ and $\mathcal{H}(Q_e(\lambda))$ mapping hooks in λ of length ex to hooks in the quotient of length x. Moreover the bijection respects the process of hook removal. Namely, the partition μ obtained by removing a ex-hook from λ is such that $C_e(\mu) = C_e(\lambda)$ and the e-quotient of μ is obtained by removing a x-hook from one of the e partitions involved in the e-quotient of λ. The other fundamental result we need to recall is [10, Proposition 3.6], which can be stated as follows.

Proposition 2.4 *Let $\lambda \in \mathcal{P}(n)$. The number of e-hooks that must be removed from λ to obtain $C_e(\lambda)$ is $w_e(\lambda)$. Moreover ìw$_e(\lambda) = |\mathcal{H}_e(\lambda)| = (|\lambda| - |C_e(\lambda)|)/e$.*

James' Abacus All of the operations on partitions concerning addition and removal of e-hooks described above are best illustrated on James' abacus. We give here a brief description of this important object, and introduce some pieces of notation that will be used extensively throughout. We refer the reader to [8, Chapter 2] for a complete account of the combinatorial properties of James' abacus, and to the beautiful article [2] for further properties of hooks in Young diagrams.

Let λ be a partition of n and let A be an e-abacus configuration for λ. Denote by A_0, $A_1, \dots, A_{e-1}$ the runners in A from left to right and label the rows by integers such that the row numbers increase downwards. As is customary, all abaci contain finitely many rows and hence finitely many beads, but in all instances enough to perform all of the necessary operations. For $j \in \{0, \dots, e-1\}$, denote by $|A_j|$ the number of beads on runner j. Moreover, we denote by $A^{\uparrow}$ the e-abacus obtained from A by sliding all beads on each runner as high as possible. Extending the notation just introduced, we denote by $A_0^{\uparrow}, \dots, A_{e-1}^{\uparrow}$ the runners of $A^{\uparrow}$. As explained in [8, Chapter 2], $A^{\uparrow}$ is an e-abacus for $C_e(\lambda)$. Let the operation of sliding any single bead down (resp. up) one row on its runner be called a *down-move* (resp. *up-move*). Of course, such a move is only possible for a bead in position (i, j) (that is, in row i on runner A_j) if the respective position $(i \pm 1, j)$ was empty initially. Sometimes we call an empty position a *gap*. We say that position (x, y) is the *first* gap in A if there are beads in positions (i, j) for all $i < x$ and all j, and in positions (x, j) for all $j < y$.

On the level of partitions, performing a down- or up-move corresponds to adding or removing an e-hook, respectively. In analogy with the notation used for partitions, we denote by $w(A)$ the total number of up-moves needed to obtain $A^{\uparrow}$ from A. Similarly, for $i \in \{0, \dots, e-1\}$ we let $w(A_i)$ be the number of those up-moves that were performed on runner i in the transition from A to $A^{\uparrow}$. It is easy to see that $w_e(\lambda) = w(A) = w(A_0) + \dots + w(A_{e-1})$.

Suppose that c is a bead in position (i, j) of A. We say that c is a *removable bead* if $j \neq 0$ and there is no bead in $(i, j-1)$, or if $j = 0$ and there is no bead in $(i-1, e-1)$. Denote by $A^{\leftarrow c}$ the abacus configuration obtained by sliding c into position $(i, j-1)$ (respectively $(i-1, e-1)$).

Let λ^- be the subset of $\mathcal{P}(n-1)$ consisting of all partitions whose Young diagram can be obtained from that of λ by removing a node. Clearly $A^{\leftarrow c}$ is an abacus configuration for a partition $\mu \in \lambda^-$, and conversely any $\mu \in \lambda^-$ can be represented by $A^{\leftarrow c}$ for some such c, since removable beads in an abacus of λ correspond to removable nodes in the Young diagram of λ.

Finally, for $j \in \{0, \dots, e-1\}$ we denote by $\mathrm{Rem}(A_j)$ the number of removable beads in A lying on runner A_j. In particular, we have that $|\lambda^-| = \mathrm{Rem}(A_0) + \dots + \mathrm{Rem}(A_{e-1})$.

Lemma 2.5 *Let $e \in \mathbb{N}$. Let λ be a partition of any natural number, and denote by A an e-abacus configuration for λ. Suppose c is a removable bead on runner A_j and let $\mu \vdash n-1$ be the partition represented by $A^{\leftarrow c}$. Then*

$$w_e(\mu) = w_e(\lambda) + \begin{cases} |A_j| - |A_{j-1}| - 1 & \text{if } j \neq 0 \\ |A_0| - |A_{e-1}| - 2 & \text{if } j = 0. \end{cases}$$

Proof First suppose $j \neq 0$. Without loss of generality we can relabel the rows of the e-abacus A such that all rows labelled by negative integers do not have empty positions. To ease the notation we let $B := A^{\leftarrow c}$. Clearly $w(A_i) = w(B_i)$ for all $i \neq j-1, j$ in $\{0, \dots, e-1\}$. Hence

$$w_e(\mu) - w_e(\lambda) = w(B_{j-1}) + w(B_j) - w(A_{j-1}) - w(A_j).$$

Let s and t be the numbers of beads lying in rows labelled by non-negative integers in runners A_{j-1} and A_j respectively. Suppose that the s beads on A_{j-1} lie in rows $0 \leq x_1 < \cdots < x_s$ and that the t beads on A_j lie in rows $0 \leq y_1 < \cdots < y_t$. Then

$$w(A_{j-1})+w(A_j) = \sum_{i=1}^{s}(x_i-(i-1))+\sum_{i=1}^{t}(y_i-(i-1)) = \sum_{i=1}^{s}x_i+\sum_{i=1}^{t}y_i-\tfrac{s(s-1)}{2}-\tfrac{t(t-1)}{2}.$$

Suppose that the bead c lies in row y_l for some $l \in \{1,\dots,t\}$. Since c is removable, $y_l \neq x_i$ for all $i \in \{1,\dots,s\}$. Thus the beads on B_{j-1} lie in rows $0 \leq x'_1 < \cdots < x'_{s+1}$ with $\{x'_1,\dots,x'_{s+1}\} = \{x_1,\dots,x_s,y_l\}$ and the beads on B_j lie in rows $0 \leq y'_1 < \cdots < y'_{t-1}$ with $\{y'_1,\dots,y'_{t-1}\} = \{y_1,\dots,y_{l-1},y_{l+1},\dots,y_t\}$. Hence

$$w(B_{j-1})+w(B_j) = \sum_{i=1}^{s+1}(x'_i-(i-1))+\sum_{i=1}^{t-1}(y'_1-(i-1)) = \sum_{i=1}^{s}x_i+\sum_{i=1}^{t}y_i-\tfrac{s(s+1)}{2}-\tfrac{(t-1)(t-2)}{2}$$

and we conclude that $w_e(\mu) - w_e(\lambda) = t - s - 1 = |A_j| - |A_{j-1}| - 1$.

The case when $j = 0$ is similar. □

Remark 2.6 In this note, given a partition λ and a fixed e-abacus configuration A for λ we let λ_i be the partition corresponding to the runner A_i, considered as a 1-abacus. The resulting e-quotient $(\lambda_0, \lambda_1, \dots, \lambda_{e-1})$ depends on the choice of the abacus A (a different choice of the e-abacus may induce a cyclic shift on the components of the e-quotient). Nevertheless, all of the results presented in Section 2 onwards hold independently of this observation. For instance, the e-weight $w_e(\lambda)$ introduced at the beginning of Section 2.1 does not depend on the choice of the e-abacus; the same discussion holds for Theorem 2.8 below.

2.2 Characters of $\mathfrak{S}_n$

For each $n \in \mathbb{N}$, the elements of the set $\mathrm{Irr}(\mathfrak{S}_n)$ of irreducible characters of $\mathfrak{S}_n$ are naturally labelled by partitions of n. For $\lambda \in \mathcal{P}(n)$, the corresponding irreducible character is denoted by χ^λ. In this article we will often identify the labelling partition with the corresponding irreducible character, and hence write $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ to denote at once the partition λ of n and the irreducible character χ^λ. The meaning of this notation will always be clear from the context. We recall the *Branching rule* (see [7, Chapter 9]) which tells us that

$$(\chi^\lambda)_{\mathfrak{S}_{n-1}} = \sum_{\mu\in\lambda^-} \chi^\mu.$$

By convention we let $\mathfrak{S}_0$ be the trivial 1-element group and $\mathcal{P}(0) = \{\emptyset\}$.

From now on let p be a prime. We denote by $\mathrm{Irr}_{p'}(\mathfrak{S}_n)$ the set of irreducible characters of $\mathfrak{S}_n$ of degree coprime to p. Thus $\lambda \vdash_{p'} n$ is equivalent to $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$, and in this case we also say that λ is a p'-partition of n (and sometimes simply write $\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$). We remark here that $br(n) = \max \mathcal{E}_n$ is well-defined. Indeed, for any $n \in \mathbb{N}$, if $\lambda \vdash_{p'} n$ then $|\lambda^-_{p'}| \geq 1$, so $\mathcal{E}_n$ is non-empty and $br(n) \geq 1$.

Irreducible characters of $\mathfrak{S}_n$ of p'-degree were completely described in [9]. We restate this result in language that will be particularly convenient for our purposes.

Theorem 2.7 *Let n be a natural number and let $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$. Let $a \in \{1,\dots,p-1\}$ and $k \in \mathbb{N}_0$ be such that $ap^k \leq n < (a+1)p^k$. Then $\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$ if and only if $C_{p^k}(\lambda) \in \mathrm{Irr}_{p'}(\mathfrak{S}_{n-ap^k})$.*

Theorem 2.7 says that λ is a p'-partition if and only if $w_{p^k}(\lambda) = a$ and the partition $C_{p^k}(\lambda)$ obtained from λ by successively removing all possible p^k-hooks is a p'-partition of $n - ap^k$. It will sometimes be useful to use the following equivalent version of Theorem 2.7.

Theorem 2.8 *Let $n = \sum_{j=0}^{k} a_j p^j$ be the p-adic expansion of $n \in \mathbb{N}$. Let $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ and let $Q_p(\lambda) = (\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$. Then $\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$ if and only if*

(i) $C_p(\lambda) \vdash a_0$, *and*
(ii) *for all $t \in \{0, 1, \ldots, p-1\}$ there exists $b_{1t}, b_{2t}, \ldots, b_{kt} \in \mathbb{N}_0$ such that*

$$\sum_{t=0}^{p-1} b_{jt} = a_j \text{ for all } j \in \{1, \ldots k\}, \text{ and such that } \lambda_t \vdash_{p'} \sum_{j=1}^{k} b_{jt} p^{j-1}.$$

Proof This characterization of p'-partitions of $n \in \mathbb{N}$ can be easily proved using the *p-core tower* associated to any partition of n. We refer the reader to [10, Chapters I and II] for the precise description of this combinatorial object. □

3 The Core Map and the Proof of Theorem A

In this section we state some combinatorial results that will play a fundamental role in the proofs of all of our main theorems. As a consequence of these observations, we are able to give a proof of Theorem A. We warn the reader that the key Theorem 3.13 will be assumed to be true in this section. Its rather technical proof is postponed to Section 5. As appropriately remarked later in this section, the proofs of Theorems B and C are postponed to Section 5 to improve readability.

Notation 3.1 Unless otherwise stated, in this section we fix $n \in \mathbb{N}$ such that $n = ap^k + m$ for some $k \geq 1$, $a \in \{1, \ldots, p-1\}$ and $0 < m < p^k$. To be precise this will be the standing assumption from Lemma 3.2 to Proposition 3.11.

Lemma 3.2 *Let $\lambda \vdash n$ be such that $w_{p^k}(\lambda) = w \leq a$ and denote by A a p^k-abacus configuration for λ. Suppose c is a removable bead on runner A_j and let $\mu \vdash n-1$ be the partition represented by $A^{\leftarrow c}$. Then $w_{p^k}(\mu) = w$ if and only if*

$$|A_j| = \begin{cases} 1 + |A_{j-1}| & \text{if } j \neq 0, \\ 2 + |A_{p^k-1}| & \text{if } j = 0. \end{cases}$$

Proof This is immediate by Lemma 2.5. □

The following result, which we believe is of independent interest, is one of the key steps in proving Theorem A.

Theorem 3.3 *Let $\lambda \vdash_{p'} n$ and let $\alpha \in \lambda_{p'}^{-}$. Then $C_{p^k}(\alpha) \in \mu_{p'}^{-}$, where $\mu := C_{p^k}(\lambda)$. In particular we deduce that the map*

$$C_{p^k} : \lambda_{p'}^{-} \longrightarrow \mu_{p'}^{-},$$

is well-defined. Moreover, it is surjective.

Proof Let A be the p^k-abacus configuration for μ having first gap in position $(0, 0)$. It is easy to see that rows $i \geq 1$ must be empty, since $|\mu| = m < p^k$. (We will not need rows i with $|i| > a$, so we may assume row $-a$ is the top row of the abacus and $+a$ the bottom row.) So $|A_0| = a$ and $a \leq |A_j| \leq a + 1$, for all $j \in \{0, \ldots, p^k - 1\}$. Let B be the p^k-abacus configuration for λ such that $B^{\uparrow} = A$. By Proposition 2.4, we have $w_{p^k}(\lambda) = a$ and we see that B is obtained from A after performing exactly a down-moves.

Let c be the bead in B such that $B^{\leftarrow c}$ is an abacus configuration for α, and suppose c lies on runner B_j. Since α is a p'-partition of $n - 1 = ap^k + (m - 1) \geq ap^k$ we deduce from Theorem 2.7 that $w_{p^k}(\alpha) = a$. Hence by Lemma 3.2 we have $|B_j| = 1 + |B_{j-1}|$ (j cannot be 0 because $|B_l| = |A_l| \in \{a, a+1\}$ for all $l \in \{0, \ldots, p^k - 1\}$). It follows that there exists a bead d in position $(0, j)$ of A and that position $(0, j - 1)$ of A is empty. Hence $A^{\leftarrow d}$ is a p^k-abacus configuration for $C_{p^k}(\alpha)$, which by Theorem 2.7 must be a p'-partition. Thus $C_{p^k}(\alpha) \in \mu^-_{p'}$ and the map $C_{p^k} : \lambda^-_{p'} \longrightarrow \mu^-_{p'}$ is well-defined.

To show that the map is surjective we proceed as follows. Let A be the p^k-abacus configuration for μ as described above. For any $\beta \in \mu^-_{p'}$ there exists a bead d in A such that $A^{\leftarrow d}$ is a p^k-abacus configuration for β. Let $j \in \{1, \ldots, p^k - 1\}$ be such that d is in position $(0, j)$ in A and such that position $(0, j - 1)$ is empty. Let B be the p^k-abacus for λ described above. Clearly we have that $|B_j| = |A_j| = 1 + |A_{j-1}| = 1 + |B_{j-1}|$. Hence there exists a row $y \in \{-a, \ldots, a\}$ such that position $(y, j - 1)$ of B is empty and such that there is a bead (say e) in position (y, j). Let α be the partition corresponding to the p^k-abacus $B^{\leftarrow e}$. By Lemma 3.2 we deduce that $w_{p^k}(\alpha) = a$. Moreover it is clear that $C_{p^k}(\alpha) = \beta \in \mathrm{Irr}_{p'}(\mathfrak{S}_{n-ap^k})$. By Theorem 2.7 we deduce that $\alpha \in \lambda^-_{p'}$ and therefore C_{p^k} is surjective. □

Corollary 3.4 *Let* $\lambda \vdash_{p'} n$. *Then* $|C_{p^k}(\lambda)^-_{p'}| \leq |\lambda^-_{p'}|$.

Keeping $n = ap^k + m$ as in Notation 3.1, we now introduce the following notation. Given $\gamma \vdash_{p'} m$, define

$$br(n, \gamma) := \max\{|\lambda^-_{p'}| \ : \ \lambda \vdash_{p'} n \ \text{ and } \ C_{p^k}(\lambda) = \gamma\}.$$

Clearly $br(n)$, the main object of our study, is equal to the maximal $br(n, \gamma)$ where γ is any p'-partition of m. Corollary 3.4 allows us to give the following definition.

Definition 3.5 Let $n = ap^k + m$ be as in Notation 3.1, and let $\gamma \vdash_{p'} m$. We define $N(a, p^k, \gamma) \in \mathbb{N}_0$ to be such that $|\gamma^-_{p'}| + N(a, p^k, \gamma) = br(n, \gamma)$.

One of the main goals of the present section is to prove the following fact.

Proposition 3.6 *Let* $\gamma \vdash_{p'} m$ *and let* $L = |\gamma^-_{p'}|$. *Then* $N(a, p^k, \gamma) = \Phi(a, L)$, *where* Φ *is as described in Definition 2.1.*

In order to prove Proposition 3.6, we need to introduce the following combinatorial concepts.

Definition 3.7 Let $n = ap^k + m$ be as in Notation 3.1, and let $\gamma \vdash_{p'} m$. Denote by A_γ the p^k-abacus configuration for γ having first gap in position $(0, 0)$. Define $\mathcal{R}_{A_\gamma}$ to be the subset of $\{0, 1, \ldots, p^k - 1\}$ such that $j \in \mathcal{R}_{A_\gamma}$ if and only if there is a removable bead c on runner j of A_γ such that the partition corresponding to the p^k-abacus $A_\gamma^{\leftarrow c}$ is a p'-partition of $m-1$.

Since A_γ has first gap in position $(0,0)$ and since $|\gamma| = m < p^k$ we deduce that all removable beads in A_γ lie in row 0. Hence $|\mathcal{R}_{A_\gamma}| = |\gamma_{p'}^-|$. By definition of removable bead, we have in particular that $0 \notin \mathcal{R}_{A_\gamma}$, and for $1 \leq j \leq p^k - 2$ we have that if $j \in \mathcal{R}_{A_\gamma}$ then $j+1 \notin \mathcal{R}_{A_\gamma}$.

Lemma 3.8 *Let $\gamma \vdash_{p'} m$. Let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$ and let B be the p^k-abacus for λ such that $B^\uparrow = A_\gamma$. Let c be a removable bead on runner j of B and let μ be the partition of $n-1$ corresponding to $B^{\leftarrow c}$. Then μ is a p'-partition if and only if $j \in \mathcal{R}_{A_\gamma}$.*

Proof Let $A := A_\gamma$. First suppose $j \in \mathcal{R}_A$. In particular, $j \neq 0$. Then

$$|B_j| = |A_j| = |A_{j-1}| + 1 = |B_{j-1}| + 1,$$

so $w_{p^k}(\mu) = a$ by Lemma 3.2. We also have that $(B^{\leftarrow c})^\uparrow$ is an abacus configuration for $C_{p^k}(\mu)$. Moreover if d is the bead in position $(0, j)$ of A then $(B^{\leftarrow c})^\uparrow = A^{\leftarrow d}$. Therefore we deduce that $C_{p^k}(\mu) \in \gamma_{p'}^-$ and hence that $\mu \vdash_{p'} n-1$, by Theorem 2.7.

Now suppose that $j \notin \mathcal{R}_A$. If $j = 0$ then $|B_0| = |A_0| \neq |A_{p^k-1}| + 2 = |B_{p^k-1}| + 2$. Hence $w_{p^k}(\mu) \neq a$ by Lemma 3.2 and therefore μ is not a p'-partition, by Theorem 2.7. Now we may assume that $j \neq 0$. If $|A_j| \neq |A_{j-1}| + 1$ then $|B_j| \neq |B_{j-1}| + 1$ and hence $w_{p^k}(\mu) \neq a$, by Lemma 3.2. In particular μ is not a p'-partition, by Theorem 2.7. If $|A_j| = |A_{j-1}| + 1$, then $C_{p^k}(\mu) \in \gamma^-$ is represented by the p^k-abacus $(B^{\leftarrow c})^\uparrow$. Again we have that $(B^{\leftarrow c})^\uparrow = A^{\leftarrow d}$, where d is the bead in position $(0, j)$ of A. Since $j \notin \mathcal{R}_A$ we deduce that $C_{p^k}(\mu)$ is not a p'-partition. It follows that $\mu \vdash n-1$ is not a p'-partition, by Theorem 2.7. □

Corollary 3.9 *Let $\gamma \vdash_{p'} m$ and let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$. Let B be the p^k-abacus for λ such that $B^\uparrow = A_\gamma$. Then*

$$|\lambda_{p'}^-| = \sum_{j \in \mathcal{R}_{A_\gamma}} \mathrm{Rem}(B_j).$$

Recall from Definition 2.1 that $f(x) = \max\{y \in \mathbb{N}_0 \mid y(y+1) \leq x\}$. The following lemma describes the key relationship between this function f and certain removable beads, which will be necessary for the proof of Proposition 3.6 (below).

Lemma 3.10 *Let $\lambda \in \{\emptyset, (1)\}$ and let T_λ denote the 2-abacus configuration of λ having first gap in position $(0,0)$. Let $x \in \mathbb{N}_0$ and let $\mathcal{T}_\lambda(x)$ be the set of all 2-abaci U such that $w(U) = x$ and $U^\uparrow = T_\lambda$. Then*

$$\max\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}_\lambda(x)\} = \begin{cases} f(x)+1 & \text{if } \lambda = (1), \\ \lfloor \sqrt{x} \rfloor & \text{if } \lambda = \emptyset. \end{cases}$$

Proof This is clear if $x = 0$ or $x = 1$, so we may assume now that $x \geq 2$ (and hence $f(x) > 0$). We first fix $\lambda = (1)$; this is the case that we will need to use in the proof of Proposition 3.6 below. Since λ is now fixed, we ease the notation by letting $T_{(1)} = T$ and $\mathcal{T}_{(1)}(x) = \mathcal{T}(x)$, for all $x \in \mathbb{N}_0$. Moreover, let $F(x) := \max\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}(x)\}$. We first show that there exists $A \in \mathcal{T}(x)$ such that $w(A_0) = 0$ (equivalently $w(A_1) = x$) and $\mathrm{Rem}(A_1) = F(x)$.

Let $U \in \mathcal{T}(x)$ be such that $w(U_0) = \ell$ and $\mathrm{Rem}(U_1) = r$ for some $l \in \{1, 2, \dots, x\}$ and some $r \in \{0, 1, \dots, F(x)\}$. Then there exists a 2-abacus $V \in \mathcal{T}(y)$ for some $y \leq x$ such that $w(V_0) < \ell$ and $\mathrm{Rem}(V_1) \geq r$. This follows from the following observation. Since $\ell \geq 1$ there exists $i \in \mathbb{Z}$ such that there is a bead in position $(i, 0)$ of U and such that position $(i-1, 0)$ of U is empty. Denoting beads by X and gaps by O, consider the four possibilities for rows $i-1$ and i of U:

$i-1$	OO	OO	OX	OX
i	XX	XO	XX	XO

In the first three instances, we can move the bead in $(i, 0)$ to $(i-1, 0)$ to obtain the desired abacus configuration V. In the fourth case, we need to additionally move the bead in $(i-1, 1)$ to $(i, 1)$. Hence, if $B \in \mathcal{T}(x)$ is such that $\mathrm{Rem}(B_1) = F(x)$ then there exists $y \leq x$ and $A' \in \mathcal{T}(y)$ such that $\mathrm{Rem}(A'_1) = F(x)$, $w(A'_0) = 0$ and $w(A'_1) = y$. Let $(i, 1)$ be the lowest position occupied by a bead (say d) in A'. Moving d to position $(i + (x - y), 1)$ we obtain a 2-abacus configuration $A \in \mathcal{T}(x)$ such that $\mathrm{Rem}(A_1) = \mathrm{Rem}(A'_1) = F(x)$, $w(A_0) = 0$ and $w(A_1) = x$, as desired.

We want to prove that $F(x) = f(x) + 1$. First suppose for a contradiction that $F(x) \geq f(x)+2$, and let $A \in \mathcal{T}(x)$ be such that $\mathrm{Rem}(A_1) = F(x)$ and $w(A_0) = 0$. By construction there exists integers $0 \leq j_1 < j_2 < \cdots < j_{f(x)+2}$ such that there is a bead in position $(j_k, 1)$ of A for all $k \in \{1, \dots, f(x)+2\}$. This implies that $w(A) = w(A_1) \geq (f(x)+1)(f(x)+2) > x$, a contradiction. Hence $F(x) \leq f(x)+1$.

Now let $y := f(x)(f(x)+1) \leq x$. Let B be the 2-abacus configuration obtained from T by first sliding down the bead in position $(0, 1)$ to position $(f(x)+x-y, 1)$ and then sliding down the bead in position $(i, 1)$ to position $(i + f(x), 1)$ for $i = -1, -2, \dots, -f(x)$. Clearly $B \in \mathcal{T}(x)$ and $\mathrm{Rem}(B_1) = f(x)+1$. We conclude that $F(x) = f(x)+1$, as desired.

The case $\lambda = \emptyset$ is similar. □

Proof of Proposition 3.6 Let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$ and $|\lambda^-_{p'}| = br(n, \gamma)$. Let B be the p^k-abacus for λ such that $B^{\uparrow} = A_\gamma$. In particular, B is obtained from A_γ by performing a down-moves. Let $\mathcal{R}_{A_\gamma} = \{j_1, \dots, j_L\}$. Then by Corollary 3.9, we have

$$L + N(a, p^k, \gamma) = br(n, \gamma) = |\lambda^-_{p'}| = \sum_{i=1}^{L} \mathrm{Rem}(B_{j_i}).$$

Let $a_i = w(B_{j_i-1}) + w(B_{j_i})$ for $i \in \{1, 2, \dots, L\}$, so $a_1 + \cdots + a_L \leq a$. Since no two numbers in $\mathcal{R}_{A_\gamma}$ are consecutive (as remarked after Definition 3.7), we can regard the pairs of runners of $(B_{j_1-1}, B_{j_1}), (B_{j_2-1}, B_{j_2}), \dots, (B_{j_L-1}, B_{j_L})$ as L disjoint 2-abaci, whose 2-cores are all equal to the 2-abacus $T_{(1)}$ considered in Lemma 3.10. It is easy to see that the 2-abacus identified by the pair (B_{j_i-1}, B_{j_i}) lies in $\mathcal{T}_{(1)}(a_i)$ for all $i \in \{1, \dots, L\}$. Lemma 3.10, together with the maximality of $|\lambda^-_{p'}|$ among all the p'-partitions of n with p^k-core equal to γ, allows us to deduce that $\mathrm{Rem}(B_{j_i}) = f(a_i) + 1$, for all $i \in \{1, \dots, L\}$. Hence we obtain

$$N(a, p^k, \gamma) = \sum_{i=1}^{L} \mathrm{Rem}(B_{j_i}) - L = \sum_{i=1}^{L} f(a_i).$$

We conclude the proof by showing that

$$N(a, p^k, \gamma) = \max \left\{ \sum_{i=1}^{L} f(a'_i) \mid a'_1 + \cdots + a'_L \leq a,\ a'_i \in \mathbb{N}_0\ \forall\, i \right\} = \Phi(a, L).$$

Suppose for a contradiction that there exists a natural number $y \leq a$ and $(a'_1, \ldots, a'_L)$ a composition of y such that $\sum_{i=1}^{L} f(a'_i) > N(a, p^k, \gamma)$. Since f is a non-decreasing function, without loss of generality we can assume that $y = a$. Then by using constructions analogous to those in the proof of Lemma 3.10, we can construct a partition $\tilde{\lambda} \vdash_{p'} n$ with $C_{p^k}(\tilde{\lambda}) = \gamma$, $w_{p^k}(\tilde{\lambda}) = a$ and p^k-abacus configuration $\tilde{B}$ satisfying $\tilde{B}^{\uparrow} = A_\gamma$ such that $w(\tilde{B}_{j_i}) = a'_i$ and $\mathrm{Rem}(\tilde{B}_{j_i}) = f(a'_i) + 1$ for all $i \in \{1, \ldots, L\}$. This implies that

$$br(n, \gamma) \geq |\tilde{\lambda}^-_{p'}| = L + \sum_{i=1}^{L} f(a'_i) > L + N(a, p^k, \gamma) = |\lambda^-_{p'}| = br(n, \gamma),$$

which is a contradiction. Hence $N(a, p^k, \gamma) = \Phi(a, L)$. □

Proposition 3.11 *Let $\gamma \vdash_{p'} m$. Then $br(n) = br(n, \gamma)$ if and only if $|\gamma^-_{p'}| = br(m)$. In particular, $br(n) = br(m) + \Phi(a, br(m))$.*

Proof First suppose that $br(n) = br(n, \gamma)$. Let $\lambda \vdash_{p'} n$ be such that $C_{p^k}(\lambda) = \gamma$ and $|\lambda^-_{p'}| = br(n)$, so that $br(n) = |\gamma^-_{p'}| + \Phi(a, |\gamma^-_{p'}|)$ by Proposition 3.6. Let $\delta \vdash_{p'} m$ be such that $|\delta^-_{p'}| = br(m)$. Then, since $\Phi(X, Y)$ is non-decreasing in each argument (when the other argument is fixed), we have

$$br(n) \geq br(n, \delta) = |\delta^-_{p'}| + \Phi(a, |\delta^-_{p'}|) = br(m) + \Phi(a, br(m)) \geq |\gamma^-_{p'}| + \Phi(a, |\gamma^-_{p'}|) = br(n),$$

whence equalities hold in the above. This proves all three statements: $br(m) = |\gamma^-_{p'}|$ gives the only if direction; $br(n) = br(n, \delta)$ gives the if direction (with δ in place of γ); and the final assertion is clear. □

Corollary 3.12 *Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be the p-adic expansion of n, for some $0 \leq n_1 < \cdots < n_t$. Let $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$, then*

$$br(n) = br(a_1 p^{n_1}) + \sum_{j=2}^{t} \Phi(a_j, br(m_j)).$$

Thus we have shown that the second statement of Theorem A holds. In the last part of this section we aim to complete the proof of Theorem A by studying the set $\mathcal{E}_n = \{|\lambda^-_{p'}| : \lambda \vdash n \text{ and } p \nmid \chi^\lambda(1)\}$. We first state and assume the following theorem.

Theorem 3.13 *Let p be a prime, $k \in \mathbb{N}_0$ and $a \in \{1, 2, \ldots, p-1\}$. Then we have that $\mathcal{E}_{ap^k} = \{1, 2, \ldots, br(ap^k)\}$.*

The proof of Theorem 3.13 is rather more technical and so has been postponed to Section 5. More precisely, Theorem 3.13 follows from Propositions 5.1, 5.4 and 5.14, which are proved in Section 5 below.

The next statement extends the observations already made in Lemma 3.10, and is crucial to completing the description of the set $\mathcal{E}_n$.

Lemma 3.14 *Let $B = T_{(1)}$ denote the 2-abacus configuration of the partition* (1) *having first gap in position* $(0, 0)$. *Let $x \in \mathbb{N}_0$ and let $\mathcal{T}(x)$ be the set consisting of all 2-abaci U such that $w(U) = x$ and $U^{\uparrow} = B$. Then* $\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}(x)\} = \{1, 2, \ldots, f(x) + 1\}$.

Proof From Lemma 3.10 we know that $f(x) + 1$ is the maximal value in $\{\mathrm{Rem}(U_1) \mid U \in \mathcal{T}(x)\}$. For any $r \in \{0, 1, \ldots, f(x)\}$, let $U(r)$ be the 2-abacus configuration obtained from B by first sliding down the bead in position $(0, 1)$ to position $(x - r(r + 1), 1)$ and then (if $r > 0$) sliding down the bead in position $(i, 1)$ to position $(i + r, 1)$ for $i = -1, -2, \ldots, -r$. Clearly $U(r) \in \mathcal{T}(x)$ and $\mathrm{Rem}(U(r)_1) = r + 1$. □

Theorem 3.15 *Let $n \in \mathbb{N}$ and let p be a prime. Let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be the p-adic expansion of n, for some $0 \leq n_1 < n_2 < \cdots < n_t$ with $a_j > 0$ for all $1 \leq j \leq t$. Then* $\mathcal{E}_n = \{1, 2, \ldots, br(n)\}$.

Proof We prove the assertion by induction on t, the p-adic length of n. If $t = 1$ then the statement follows from Theorem 3.13.

Assume that $t \geq 2$. Let $m = \sum_{j=1}^{t-1} a_j p^{n_j}$ and let γ be a p'-partition of m such that $|\gamma_{p'}^{-}| = br(m)$. For convenience, let $L = br(m)$ and $k = n_t$. As in Definition 3.7 let $A := A_\gamma$ be the p^k-abacus configuration for γ having first gap in position $(0, 0)$. Moreover, let $\mathcal{R}_A = \{j_1, \ldots, j_L\}$.

Applying Lemma 3.14 to the L pairs of runners $(A_{j_i - 1}, A_{j_i})$ of A, we see that for each $r \in \{0, 1, \ldots, \Phi(a_t, L)\}$, there exists a sequence of a_t down-moves that can be performed on A to produce a p^k-abacus B^r such that

$$\sum_{j \in \mathcal{R}_A} \mathrm{Rem}(B_j^r) = L + r.$$

Let $\lambda(r)$ be the partition of n corresponding to B^r. Clearly $C_{p^k}(\lambda(r)) = \gamma$ and by Theorem 2.7 we deduce that $\lambda(r) \vdash_{p'} n$. Moreover, $|\lambda(r)_{p'}^{-}| = L + r$ by Corollary 3.9. Hence $L + r \in \mathcal{E}_n$, and thus $\{L, L + 1, \ldots, br(n)\} \subseteq \mathcal{E}_n$, noting that $L + \Phi(a_t, L) = br(n, \gamma) = br(n)$ by Proposition 3.11.

If $L = 1$ then the proof is complete; otherwise, using the inductive hypothesis we have that for any $i \in \{1, 2, \ldots, L - 1\}$, there exists $\gamma(i) \vdash_{p'} m$ such that $|\gamma(i)_{p'}^{-}| = i$. Taking $r = 0$ and replacing γ by $\gamma(i)$ in the above construction, we construct $\beta(i) \vdash_{p'} n$ such that $C_{p^k}(\beta(i)) = \gamma(i)$ and $|\beta(i)_{p'}^{-}| = i + 0$. Hence $\{1, 2, \ldots, L - 1\} \subseteq \mathcal{E}_n$, and we conclude that $\mathcal{E}_n = \{1, 2, \ldots, br(n)\}$. □

Proof of Theorem A This follows directly from Corollary 3.12 and Theorem 3.15. □

4 The Upper Bound $\mathcal{B}_n$

In this section we prove Theorem D. Let $n \in \mathbb{N}$ and let $n = \sum_{j=1}^{t} a_j p^{n_j}$ be the p-adic expansion of n, for some $0 \leq n_1 < \cdots < n_t$ with $a_j > 0$ for all $1 \leq j \leq t$. Recall that

$$\mathcal{B}_n = br(a_1 p^{n_1}) + \sum_{j=2}^{t} \left\lfloor \frac{a_j}{2} \right\rfloor.$$

From Lemma 2.2 and Corollary 3.12, we see that $br(n) \leq \mathcal{B}_n$, and the difference $\varepsilon_n = \mathcal{B}_n - br(n)$ can be written as

$$\varepsilon_n = \sum_{j=2}^{t} \left(\lfloor a_j/2 \rfloor - \Phi(a_j, br(m_j)) \right)$$

where $m_j = \sum_{i=1}^{j-1} a_i p^{n_i}$. The following statement will be useful in the proof of Theorem D, below.

Lemma 4.1 *Let $s, t \in \mathbb{N}_0$ with $s \leq t$. Let $b_0, b_1, \ldots, b_t \in \{0, 1, \ldots, p-1\}$ with $b_0, b_1, \ldots, b_s$ not all zero. Then $br\left(\sum_{j=0}^{s} b_j p^j\right) \leq br\left(\sum_{j=0}^{t} b_j p^j\right)$.*

Proof This follows directly from Proposition 3.11. □

Proof of Theorem D Fix $n \in \mathbb{N}$ and its p-adic expansion as above. Let $\varepsilon(j) = \lfloor a_j/2 \rfloor - \Phi(a_j, br(m_j))$. If $a_j \leq 3$ then $\varepsilon(j) = 0$ by Lemma 2.2, since $br(m_j) \geq 1$. Hence if $a_j \leq 3$ for all $j \geq 2$, then in fact $\varepsilon_n = 0$. Thus if $p \leq 3$ then $\varepsilon_n = 0$, so from now on we may assume $p \geq 5$ and that there exists $i \in \{2, \ldots, t\}$ such that $a_i \geq 4$. In particular, there exists a unique $k \in \{1, \ldots, t\}$ and integers $1 = i_0 < i_1 < i_2 < \cdots < i_k \leq t$ such that for all $j \in \{1, \ldots, k\}$,

$$i_j := \min \left\{ x \in \{i_{j-1} + 1, \ldots, t-1, t\} \mid a_x \geq 2^j + 2 \right\},$$

and such that $\{x \in \{i_k + 1, \ldots, t-1, t\} \mid a_x \geq 2^{k+1} + 2\} = \emptyset$. Note that k must satisfy $2^k < p$, because if $2^k \geq p$ then $a_{i_k} \geq 2^k + 2 > p - 1$, contradicting the fact that a_{i_k} is a p-adic digit.

We first show that $br(m_{i_j}) \geq 2^{j-1}$ for all $j \in \mathbb{N}$ by induction. This is clear for $j = 1$. For $j \in \{2, \ldots, t\}$, we have

$$br(m_{i_j}) \geq br(m_{i_{j-1}+1}) = br(m_{i_{j-1}}) + \Phi(a_{i_{j-1}}, br(m_{i_{j-1}})) \geq 2^{j-2} + \Phi(2^{j-1}+2, 2^{j-2}) \geq 2^{j-1}.$$

The inequalities above hold by Lemma 4.1, the fact that Φ is non-decreasing in each argument, the inductive hypothesis and Lemma 2.3, while the equality follows from Proposition 3.11. Thus for all $x \geq i_j + 1$ we have

$$br(m_x) \geq br(m_{i_j+1}) = br(m_{i_j}) + \Phi(a_{i_j}, br(m_{i_j})) \geq 2^{j-1} + \Phi(2^j + 2, 2^{j-1}) \geq 2^j.$$

Now let $x \in \{2, \ldots, t\}$ be such that $i_j < x < i_{j+1}$ for some $j \in \{1, \ldots, k\}$. Since $x < i_{j+1}$, we have $a_x \leq 2^{j+1} + 1$, and since $x > i_j$, we have by the above discussion that $br(m_x) \geq 2^j$. Therefore $br(m_x) \geq \lfloor a_x/2 \rfloor$ and hence $\varepsilon(x) = 0$ by Lemma 2.2. Similarly if $x < i_1$ then $a_x \leq 3$ and so $\varepsilon(x) = 0$, while if $x > i_k$ then $br(m_x) \geq 2^k \geq \lfloor a_x/2 \rfloor$ and thus $\varepsilon(x) = 0$ also. Hence

$$\varepsilon_n = \sum_{j=1}^{k} \varepsilon(i_j).$$

Finally, for each $j \in \{1, \ldots, k\}$, we have by Lemma 2.3 that

$$\varepsilon(i_j) = \lfloor a_{i_j}/2 \rfloor - \Phi(a_{i_j}, br(m_{i_j})) \leq \tfrac{p-1}{2} - \Phi(2^j + 2, 2^{j-1}) \leq \tfrac{p-1}{2} - 2^{j-1}.$$

Hence

$$\varepsilon_n = \sum_{j=1}^{k} \varepsilon(i_j) \leq \sum_{i=0}^{k-1} \left(\frac{p-1}{2} - 2^i\right) = k \cdot \frac{p-1}{2} - (2^k - 1) < k \cdot \frac{p}{2} < \frac{p}{2} \log_2 p.$$

□

Remark 4.2 Theorem D shows that the difference between the upper bound B_n and the actual value of $br(n)$ is relatively small, and can be bounded independently of n. If $p \in \{2, 3\}$ then $\varepsilon_n = 0$, as observed in the first part of the proof of Theorem D above. In particular, fixing $p = 2$ we recover [1, Theorem 1]. As already mentioned in the introduction, the proof of Theorem D also shows for any prime p, we have $\mathcal{B}_n = br(n)$ whenever all of the p-adic digits of n are at most 3.

5 The Value of $br(ap^k)$ and the Set $\mathcal{E}_{ap^k}$

The main goals in this section are to prove Theorem B (i.e. determining the value of $br(ap^k)$), Theorem C and Theorem 3.13 (i.e. showing that $\mathcal{E}_{ap^k} = \{1, 2, \dots, br(ap^k)\}$). As already remarked in the introduction, these two results play the role of base cases for Theorem A.

From now on, let p be an odd prime. The case when $k = 0$ is straightforward and is described in the following proposition.

Proposition 5.1 *Let* $a \in \{1, 2, \dots, p-1\}$. *Then* $br(a) = f(2a)$ *and* $\mathcal{E}_a = \{1, 2, \dots, br(a)\}$.

Proof Every partition of $a - 1$ is a p'-partition, and we can always construct a partition λ of a such that $|\lambda^-| = m$ for any $1 \leq m \leq f(2a)$, since $f(2a)$ is the maximum number of parts of distinct size achieved by a partition of a. □

In the following proposition we provide a naive upper bound for $br(ap^k)$, for all $k \in \mathbb{N}$ and $a \in \{1, \dots, p-1\}$. As we will show in the rest of this section, this bound turns out to be tight for almost all values of a and k.

Proposition 5.2 *Let* $a \in \{1, 2, \dots, p-1\}$ *and let* $k \in \mathbb{N}$. *Then* $br(ap^k) \leq 2a$.

Proof Let C and D be p^k-abacus configurations such that D is obtained from C by performing a single down-move. It is easy to see that the number of removable beads in D is at most the number of removable beads in C plus two. Hence if λ is a partition such that $C_{p^k}(\lambda) = \emptyset$ then $|\lambda^-| \leq 2w_{p^k}(\lambda)$. Now let $n = ap^k$ and let $\lambda \vdash_{p'} n$ be such that $|\lambda^-_{p'}| = br(n)$. From Theorem 2.7 we know that $C_{p^k}(\lambda) = \emptyset$ and $w_{p^k}(\lambda) = a$. The result follows. □

We have now all the ingredients to prove Theorem C.

Proof of Theorem C This is a straightforward consequence of Lemma 2.2, Corollary 3.12 and Proposition 5.2. □

To complete the proof of Theorem B, it will be convenient to split the remainder of this section into two parts. In each part we will appropriately fix the natural numbers a and k according to the statement of Theorem B.

5.1 Part I

In this first part, we consider the case $k = 1$ and $a < \frac{p}{2}$, and the case $k \geq 2$.

Proposition 5.3 *Let $a \in \{1, 2, \dots, p-1\}$ and let $k \in \mathbb{N}$. If $k = 1$ and $a < \frac{p}{2}$ or if $k \geq 2$, then $br(ap^k) = 2a$.*

Proof It is enough to construct $\lambda \vdash_{p'} ap^k$ such that $|\lambda^-_{p'}| = 2a$, by Proposition 5.2.

(i) First suppose that $k = 1$ and $a < \frac{p}{2}$. Let λ be the partition of ap defined by

$$\lambda = (p-1, p-2, \dots, p-a, a, a-1, \dots, 2, 1).$$

The following diagram is the p-abacus configuration for λ having first gap in position $(0, 0)$, where we have indicated the row numbers on the left and the runner numbers above each column:

	0	1	2	3	$\cdots$	$2a-2$	$2a-1$	$2a$	$\cdots$	$p-1$
-1	×	×	×	×	$\cdots$	×	×	×	$\cdots$	×
0	∘	×	∘	×	$\cdots$	∘	×	∘	$\cdots$	∘
1	×	∘	×	∘	$\cdots$	×	∘	∘	$\cdots$	∘

Since $C_p(\lambda) = \emptyset$ we have that $\lambda \vdash_{p'} ap$ by Theorem 2.7. Moreover, we observe that $w_p(\mu) = w_p(\lambda) - 1 = a - 1$ for each $\mu \in \lambda^-$, by Lemma 2.5, and so $C_p(\mu) \vdash p-1$ by Proposition 2.4. But every partition of $p-1$ is of p'-degree, so by Theorem 2.7 we have that $\mu \vdash_{p'} ap - 1$ for every $\mu \in \lambda^-$, whence $\lambda^-_{p'} = \lambda^-$ and so $|\lambda^-_{p'}| = 2a$.

(ii) Suppose now that $k \geq 2$. Let $r = p^{k-1} - a > 0$ and let $\lambda^j = a + p - 2 + rp + (a - j)(p-1) = p^k - (j-1)(p-1) - 1$, for each $j \in \{1, 2, \dots, a\}$. Let λ be the partition of ap^k defined by

$$\lambda = \left(\lambda^1, \lambda^2, \dots, \lambda^a, a, (a-1)^{p-1}, (a-2)^{p-1}, \dots, 2^{p-1}, 1^{p-1}\right).$$

The best way to verify that λ has the required properties is to look at it on James' abacus. We describe and depict below a p-abacus configuration A corresponding to λ:

- The first gap is in position $(1, 0)$;
- Rows $1 \leq i \leq a-1$ have a gap only in position $(i, 0)$;
- Row a has a bead only in position $(a, 1)$;
- Rows $a+1$ to $a+r$ are all empty;

- Rows $a+1+r \leq i \leq 2a+r$ have a bead only in position $(i, 0)$;
- There is a gap in position (x, y) for all $x > 2a + r$.

$$
\begin{array}{lccccc}
 & 0 & 1 & 2 & \cdots & p-1 \\
1 & \circ & \times & \times & \cdots & \times \\
\vdots & \vdots & & & & \vdots \\
a-1 & \circ & \times & \times & \cdots & \times \\
a & \circ & \times & \circ & \cdots & \circ \\
a+1 & \circ & \circ & \circ & \cdots & \circ \\
\vdots & \vdots & & & & \vdots \\
a+r & \circ & \circ & \circ & \cdots & \circ \\
a+1+r & \times & \circ & \circ & \cdots & \circ \\
\vdots & \vdots & & & & \vdots \\
2a+r & \times & \circ & \circ & \cdots & \circ
\end{array}
$$

From the structure of A we observe that $Q_p(\lambda) = (\lambda_0, \emptyset, \ldots, \emptyset)$, where

$$\lambda_0 = (\underbrace{p^{k-1}, \ldots, p^{k-1}}_{a \text{ times}}).$$

From the discussion in Section 2.1 (or [10, Theorem 3.3]), we deduce that $w_{p^k}(\lambda) = w_{p^{k-1}}(\lambda_0) = a$ and $C_{p^k}(\lambda) = \emptyset$. Thus $\lambda \vdash_{p'} ap^k$, by Theorem 2.7.

Notice that λ has exactly $2a$ removable nodes, corresponding to the $2a$ removable beads in A lying in positions $(i, 1)$ and $(a + r + i, 0)$ for $i \in \{1, \ldots, a\}$. Let c be a removable bead in position $(i, 1)$ of A, for some $i \in \{1, \ldots, a\}$. Then $A^{\leftarrow c}$ corresponds to the partition $\mu \vdash ap^k - 1$ such that $C_p(\mu) = (p - 1) \vdash p - 1$ and $Q_p(\mu) = (\mu_0, \mu_1, \emptyset, \ldots, \emptyset)$, where

$$\mu_0 = (\underbrace{p^{k-1} - 1, \ldots, p^{k-1} - 1}_{a \text{ times}}, i - 1) \text{ and } \mu_1 = (1^{a-i}).$$

We observe that $\mu_0 \vdash_{p'} (a - 1)p^{k-1} + m$, where $m := p^{k-1} - a + (i - 1)$. This follows from Theorem 2.7, since $w_{p^{k-1}}(\mu_0) = a - 1$ and $C_{p^{k-1}}(\mu_0) = (m) \vdash_{p'} m$. Moreover, we clearly have that $\mu_1 \vdash_{p'} a - i$. We can now use Theorem 2.8 to deduce that $\mu \vdash_{p'} ap^k - 1$ and therefore $\mu \in \lambda^-_{p'}$.

A similar argument shows that for every $j \in \{1, \ldots, a\}$ the p-abacus configuration $A^{\leftarrow d}$ obtained from A by sliding the bead d in position $(a + r + j, 0)$ to position $(a + r + j - 1, p - 1)$, corresponds to a p'-partition μ of $ap^k - 1$, that is, $\mu \in \lambda^-_{p'}$. Thus $|\lambda^-_{p'}| = 2a$. □

Proposition 5.4 *Let $a \in \{1, 2, \ldots, p - 1\}$ and let $k \in \mathbb{N}$. If $k = 1$ and $a < \frac{p}{2}$ or if $k \geq 2$, then $\mathcal{E}_{ap^k} = \{1, 2, \ldots br(ap^k)\}$.*

Proof It is enough to construct $\lambda \vdash_{p'} ap^k$ such that $|\lambda^-_{p'}| = m$ for each $m \in \{1, 2, \ldots, 2a - 1\}$, by Proposition 5.2.

(i) First suppose that $k = 1$ and $a < \frac{p}{2}$. We first exhibit $\lambda(j) \vdash_{p'} ap$ such that $|\lambda(j)^-_{p'}| = 2j$ for each $j \in \{1, 2, \ldots, a - 1\}$:

- Let $\lambda(1) = (ap - 1, 1)$;
- For each fixed $j \in \{2, \ldots, a - 1\}$, let $\lambda(j) = (\lambda_1, \lambda_2, \ldots, \lambda_{2j})$ where

- $\lambda_1 = (a - j + 1)p - 2j + 1$,
- $\lambda_x = p + 2 - x$ for $x \in \{2, \dots, j\}$, and
- $\lambda_y = 2j + 1 - y$ for $y \in \{j + 1, \dots, 2j\}$.

For convenience, we depict the p-abacus of $\lambda(j)$ having first gap in position $(0, 0)$:

$$\begin{array}{l|cccccccccc}
 & 0 & 1 & 2 & 3 & \cdots & 2j-2 & 2j-1 & 2j & \cdots & p-1 \\
-1 & \times & \times & \times & \times & \cdots & \times & \times & \times & \cdots & \times \\
0 & \circ & \times & \circ & \times & \cdots & \circ & \times & \circ & \cdots & \circ \\
1 & \circ & \circ & \times & \circ & \cdots & \times & \circ & \circ & \cdots & \circ \\
2 & \circ & \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ \\
\vdots & \vdots & & & & & & & & & \vdots \\
a-j+1 & \times & \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ
\end{array}$$

By Theorem 2.7, we have $\lambda(j) \vdash_{p'} ap$. Moreover, we observe that $w_p(\mu) = w_p(\lambda(j)) - 1 = a - 1$ for each $\mu \in \lambda(j)^-$ by Lemma 2.5, and so $C_p(\mu) \vdash p - 1$ by Proposition 2.4. But then $C_p(\mu) \vdash_{p'} p - 1$ and so by Theorem 2.7 we have that $\mu \vdash_{p'} ap - 1$ for each $\mu \in \lambda(j)^-$, whence $\lambda(j)^-_{p'} = \lambda(j)^-$ and so $|\lambda(j)^-_{p'}| = 2j$. Hence $\{2, 4, \dots, 2a-2\} \subseteq \mathcal{E}_{ap}$.

Next we exhibit $\beta(j) \vdash_{p'} ap$ such that $|\beta(j)^-_{p'}| = 2j - 1$ for each $j \in \{1, 2, \dots, a\}$:

- Let $\beta(1) = ((a - 1)p + 1, 1^{p-1})$;
- Let $\beta(a) = (2a - 1, 2a - 2, \dots, a + 1, a^{p-2a+2}, a - 1, \dots, 2, 1)$;
- For each fixed $j \in \{2, \dots, a - 1\}$, let $\beta(j) = (\beta_1, \dots, \beta_p)$ where
  - $\beta_1 = (a - j)p + 1$,
  - $\beta_x = 2j + 2 - x$ for $x \in \{2, \dots, j\}$,
  - $\beta_y = j$ for $y \in \{j + 1, \dots, p - j + 1\}$, and
  - $\beta_z = p + 1 - z$ for $z \in \{p - j + 2, \dots, p\}$.

For convenience, we depict the p-abacus of $\beta(j)$ having first gap in position $(0, 0)$:

$$\begin{array}{l|cccccccccc}
 & 0 & 1 & 2 & 3 & \cdots & 2j-2 & 2j-1 & 2j & \cdots & p-1 \\
-1 & \times & \times & \times & \times & \cdots & \times & \times & \times & \cdots & \times \\
0 & \circ & \times & \circ & \times & \cdots & \circ & \times & \times & \cdots & \times \\
1 & \circ & \circ & \times & \circ & \cdots & \times & \circ & \circ & \cdots & \circ \\
2 & \circ & \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ \\
\vdots & \vdots & & & & & & & & & \vdots \\
a-j+1 & \times & \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ
\end{array}$$

Again by Theorem 2.7 we have $\beta(j) \vdash_{p'} ap$. By Lemma 2.5, if $j \neq a$ then $|\beta(j)^-| = 2j$ and $|\beta(j)^- \setminus \beta(j)^-_{p'}| = 1$, while if $j = a$ then $|\beta(j)^-_{p'}| = |\beta(j)^-| = 2a - 1$. In both cases we have $|\beta(j)^-_{p'}| = 2j - 1$, giving $\{1, 3, \dots, 2a - 1\} \subseteq \mathcal{E}_{ap}$. Thus $\mathcal{E}_{ap} = \{1, 2, \dots, 2a\}$ as claimed.

(ii) Suppose now that $k \geq 2$. We first construct a partition $\lambda(j) \vdash_{p'} ap^k$ such that $|\lambda(j)^-_{p'}| = 2a - j$, for all $j \in \{1, 2, \dots, a - 1\}$. Let $r = p^{k-1} - a > 0$ and let

$$\lambda(j) := \left(\eta_{a-1}, \dots, \eta_j, \theta_j, \dots, \theta_1, a, (a-1)^{p-1}, \dots, (j+1)^{p-1}, j^{p-2}, (j-1)^{p-1}, \dots, 1^{p-1}\right),$$

where $\theta_t = a + pr + t(p - 1)$ and $\eta_t = \theta_t + (p - 2)$ for $t \in \{1, \dots, a - 1\}$. As usual, it is useful to look at $\lambda(j)$ on James' abacus. We describe and depict below a p-abacus A^j of $\lambda(j)$:

- The first gap is in position $(1, 1)$;
- Rows $1 \leq x \leq j$ have a gap only in position $(x, 1)$;
- Rows $j + 1 \leq x \leq a - 1$ have a gap only in position $(x, 0)$;
- Row a has a bead only in position $(a, 1)$;
- Rows $a + 1$ to $a + r$ are all empty;
- Rows $a + r + 1 \leq x \leq a + r + j$ have a bead only in position $(x, 1)$;
- Rows $a + r + j + 1 \leq x \leq 2a + r$ have a bead only in position $(x, 0)$;
- There is a gap in position (x, y) for all $x > 2a + r$.

$$\begin{array}{l cccccc}
 & 0 & 1 & 2 & \cdots & p-1 \\
1 & \times & \circ & \times & \cdots & \times \\
\vdots & \vdots & & & & \vdots \\
j & \times & \circ & \times & \cdots & \times \\
j+1 & \circ & \times & \times & \cdots & \times \\
\vdots & \vdots & & & & \vdots \\
a-1 & \circ & \times & \times & \cdots & \times \\
a & \circ & \times & \circ & \cdots & \circ \\
a+1 & \circ & \circ & \circ & \cdots & \circ \\
\vdots & \vdots & & & & \vdots \\
a+r & \circ & \circ & \circ & \cdots & \circ \\
a+r+1 & \circ & \times & \circ & \cdots & \circ \\
\vdots & \vdots & & & & \vdots \\
a+r+j & \circ & \times & \circ & \cdots & \circ \\
a+r+j+1 & \times & \circ & \circ & \cdots & \circ \\
\vdots & \vdots & & & & \vdots \\
2a+r & \times & \circ & \circ & \cdots & \circ
\end{array}$$

Since j is fixed, we will denote $\lambda(j)$ by λ and A^j by A from now on. Arguing as in the proof of Proposition 5.2, we deduce that $\lambda \vdash_{p'} ap^k$. Moreover, it is clear that $|\lambda^-| = 2a$. Let $x \in \{1, \dots j\}$ and let c be the bead lying in position $(x, 2)$ of A. Let μ^x be the partition of $ap^k - 1$ corresponding to the p-abacus $A^{\leftarrow c}$. Then $C_p(\mu^x) = (p, 1^{p-1})$. Therefore μ^x is not a p'-partition, by Theorem 2.8. It follows that $|\lambda^-_{p'}| \leq 2a - j$.

We will now show that all of the other $2a - j$ removable beads in A correspond to p'-partitions of $ap^k - 1$. Let $x \in \{j + 1, j + 2, \dots, a\}$ and let c be the bead in position $(x, 1)$ of A. Let μ^x be the partition of $ap^k - 1$ corresponding to the p-abacus $A^{\leftarrow c}$. Then $C_p(\mu^x) = (p - 1) \vdash_{p'} p - 1$ and $Q_p(\mu^x) = (\mu_0, \mu_1, \emptyset, \dots, \emptyset)$, where

$$\mu_0 = \big((p^{k-1} - 1)^{a-j}, x - j - 1\big) \text{ and } \mu_1 = \big((r + j + 1)^j, (j + 1)^{a-x}, j^{x-j-1}\big).$$

By Theorem 2.7, we have that $\mu_0 \vdash_{p'} |\mu_0|$ and $\mu_1 \vdash_{p'} |\mu_1|$, where

$$|\mu_0| = (a - j - 1)p^{k-1} + (p - 1)\sum_{i=1}^{k-2} p^i + [(p - 1) - (a - x)]$$

and

$$|\mu_1| = jp^{k-1} + (a - x).$$

This implies $\mu^x \vdash_{p'} ap^k - 1$, by Theorem 2.8.

Now let c be the bead in position $(a + r + x, 1)$ for some $x \in \{1, \dots, j\}$, and let μ^x be the partition corresponding to the p-abacus $A^{\leftarrow c}$. Arguing as before, we deduce from Theorem 2.8 that $\mu^x \vdash_{p'} ap^k - 1$.

Finally, let c be the bead in position $(a+r+x, 0)$ for some $x \in \{j+1, \dots, a\}$ and let μ^x be the partition corresponding to $A^{\leftarrow c}$. First, we observe that $C_p(\mu^x) = (p-2, 1) \vdash_{p'} p-1$. Moreover, $Q_p(\mu^x) = (\mu_0, \mu_1, \emptyset, \dots, \emptyset, \mu_{p-1})$, where

$$\mu_0 = \left((p^{k-1}+1)^{a-x}, (p^{k-1})^{x-j-1}\right), \quad \mu_1 = \left((r+j)^j, j^{a-j}\right), \quad \text{and} \quad \mu_{p-1} = (r+x-1).$$

Again, $\mu^x \vdash_{p'} ap^k - 1$ by Theorem 2.8, and so $|\lambda_{p'}^-| = 2a - j$. Thus $\{a+1, a+2, \dots, 2a-1\} \subseteq \mathcal{E}_{ap^k}$.

Finally, we construct a partition $\beta(j) \vdash_{p'} ap^k$ such that $|\beta(j)_{p'}^-| = a - j$, for all $j \in \{0, 1, \dots, a-1\}$. Let B^j be the p-abacus configuration obtained from the p-abacus A^j described above by replacing the bead in position $(a, 1)$ with a gap so that row a is now empty. Let $\beta(j)$ be the partition of ap^k corresponding to the p-abacus B^j. Again, since j is fixed we will now denote B^j by B and $\beta(j)$ by β.

It is clear that $\beta \vdash_{p'} ap^k$ and $|\beta^-| = 2a - j - 1$. Moreover, if c is one of the $a - 1$ removable beads lying on runner 1 of B and μ is the partition of $ap^k - 1$ corresponding to the p-abacus $B^{\leftarrow c}$, then $C_p(\mu) = (p, 1^{p-1})$ and therefore μ is not a p'-partition by Theorem 2.8. Hence $|\beta_{p'}^-| \leq a - j$. Arguing as before, the partition corresponding to the p-abacus $B^{\leftarrow c}$ for any removable bead c lying on runner 0 of B is a p'-partition of $ap^k - 1$. Hence $|\beta_{p'}^-| = a - j$, and so $\{1, 2, \dots, a\} \subseteq \mathcal{E}_{ap^k}$. Thus $\mathcal{E}_{ap^k} = \{1, 2, \dots, 2a\}$ as claimed. □

5.2 Part II

In this second part of Section 5, we fix $k = 1$ and $a \in \mathbb{N}$ such that $\frac{p}{2} < a < p$. The main aim in Part II is to prove the following fact.

Proposition 5.5 *Let $a \in \mathbb{N}$ be such that $\frac{p}{2} < a < p$. Then $br(ap) = p - 1 + 2\lfloor \frac{2a-(p-1)}{6} \rfloor$.*

The proof of Proposition 5.3 is split into a series of technical lemmas. We start by fixing some notation which will be kept throughout Part II.

Notation 5.6 Let $a \in \mathbb{N}$ be such that $\frac{p}{2} < a < p$. We let $x := a - \frac{p-1}{2}$, and we write $x = 3q + \delta$ for some $q \in \mathbb{N}_0$ and $\delta \in \{0, 1, 2\}$. In particular we have $q = \lfloor \frac{x}{3} \rfloor = \lfloor \frac{2a-(p-1)}{6} \rfloor$.

Definition 5.7 Denote by $A_\emptyset$ the p-abacus configuration for the empty partition $\emptyset$ such that $A_\emptyset$ has first gap in position $(0, 0)$. We then define $\mathcal{Z}(a)$ to be the set of p-abaci B such that $w(B) = a$ and $B^\uparrow = A_\emptyset$. It is clear by Theorem 2.7 that $\mathcal{Z}(a)$ is naturally in bijection with $\mathrm{Irr}_{p'}(\mathfrak{S}_{ap})$.

Lemma 5.8 *Let $\lambda \vdash_{p'} ap$ and let $B \in \mathcal{Z}(a)$ be the p-abacus corresponding to λ. Then*

$$|\lambda_{p'}^-| = \sum_{i=1}^{p-1} \mathrm{Rem}(B_i) \quad \text{and} \quad br(ap) = \max_{B \in \mathcal{Z}(a)} \sum_{i=1}^{p-1} \mathrm{Rem}(B_i).$$

Proof The statement follows directly from Lemma 2.5 and Theorem 2.7. □

Lemma 5.9 *For $a \in \mathbb{N}$ such that $\frac{p}{2} < a < p$, we have $br(ap) \geq p - 1 + 2q$.*

Proof We exhibit a partition $\beta \vdash_{p'} ap$ such that $|\beta^-_{p'}| = p - 1 + 2q$. If $\delta = 0$ then let β be the following partition of ap:

$$(p + 2q, p + 2q - 1, \dots, p + q + 1, p + q - 1, \dots, q + 1, q^{p-2q+1}, q - 1, \dots, 2, 1),$$

while if $\delta \neq 0$ then let β be the following partition of ap:

$$\big(p(\delta+1)+2, p+2q+1, p+2q, \dots, p+q+3, p+q-1, \dots, q+1, q^{p-2q+1}, q-1, \dots, 1\big).$$

We describe and depict below a p-abacus $B_\beta \in \mathcal{Z}(a)$ of β:

- For $j \in \{0, 2, \dots, p - 3, p - 1\}$, runner j has beads in positions (x, j) for all $x \leq -1$;
- Runner 1 has beads in positions $(0, 1)$, $(1 + \delta, 1)$ and $(y, 1)$ for all $y \leq -3$;
- For $j \in \{3, 5, \dots, 2q - 1\}$, runner j has beads in positions $(0, j)$, $(1, j)$ and (y, j) for all $y \leq -3$;
- For $j \in \{2q + 1, 2q + 3, \dots, p - 2\}$, runner j has beads in positions $(0, j)$ and (y, j) for all ≤ -2.

B_β :	0	1	2	3	4	$\cdots$	$2q-2$	$2q-1$	$2q$	$2q+1$	$2q+2$	$\cdots$	$p-3$	$p-2$	$p-1$
-3	×	×	×	×	×	$\cdots$	×	×	×	×	×	$\cdots$	×	×	×
-2	×	○	×	○	×	$\cdots$	×	○	×	×	×	$\cdots$	×	×	×
-1	×	○	×	○	×	$\cdots$	×	○	×	○	×	$\cdots$	×	○	×
0	○	×	○	×	○	$\cdots$	○	×	○	×	○	$\cdots$	○	×	○
1	○	○	○	×	○	$\cdots$	○	×	○	○	○	$\cdots$	○	○	○
2	○	○	○	○	○	$\cdots$	○	○	○	○	○	$\cdots$	○	○	○
$\vdots$	$\vdots$														$\vdots$
$1+\delta$	○	×	○	○	○	$\cdots$	○	○	○	○	○	$\cdots$	○	○	○

Observe that $C_p(\beta) = \emptyset$ and $w_p(\beta) = a$, whence $\beta \vdash_{p'} ap$ by Theorem 2.7. Moreover by Lemma 5.8 we have $\beta^- = \beta^-_{p'}$. Hence $br(ap) \geq |\beta^-_{p'}| = p - 1 + 2q$. □

Thus it remains to show that $|\lambda^-_{p'}| \leq p - 1 + 2q$ for all $\lambda \vdash_{p'} ap$. In order to do this we introduce a new combinatorial object.

Definition 5.10 Let $T_\emptyset$ be the 2-abacus configuration for the empty partition $\emptyset$ having first gap in position $(0, 0)$. Let $U^{(0)}, U^{(1)} \dots, U^{(p-1)}$ be 2-abaci such that $(U^{(i)})^\uparrow = T_\emptyset$ for all $i \in \{0, 1, \dots, p - 1\}$. If $w(U^{(0)}) + w(U^{(1)}) + \cdots + w(U^{(p-1)}) = w \in \mathbb{N}_0$ then we call the sequence $\underline{U} = (U^{(0)}, U^{(1)}, \dots, U^{(p-1)})$ a *doubled p-abacus of weight w* (we write $w(\underline{U}) = w$ in this case). Moreover we denote by $\mathcal{D}(w)$ the set of doubled p-abaci of weight w.

Finally, given any $w \in \mathbb{N}_0$ we let $M(w) = \max\big\{\rho(\underline{U}) \mid \underline{U} \in \mathcal{D}(w)\big\}$, where for any $\underline{U} \in \mathcal{D}(w)$ we define $\rho(\underline{U})$ as

$$\rho(\underline{U}) = \sum_{i=1}^{p-1} \mathrm{Rem}(U_1^{(i)}).$$

As usual, we denoted by $U_0^{(i)}$ (and $U_1^{(i)}$) the left (and right) runner of the 2-abacus $U^{(i)}$.

Remark 5.11 Let $\lambda \vdash_{p'} ap$ and let $B \in \mathcal{Z}(a)$ correspond to λ. For $i \in \{1, \dots, p - 1\}$, let $U^{(i)} = (B_{i-1}, B_i)$, and let $U^{(0)} = (B_{p-1}, B_0)$. Then $\underline{U} := (U^{(0)}, U^{(1)}, \dots, U^{(p-1)}) \in \mathcal{D}(2a)$ and $\rho(\underline{U}) = |\lambda^-_{p'}|$, by Lemma 5.8. With this in mind we define $\mathcal{D}(\mathcal{Z}(a))$ to be the subset of $\mathcal{D}(2a)$ of sequences $\underline{U} := (U^{(0)}, U^{(1)}, \dots, U^{(p-1)})$ such that $U_0^{(i)} = U_1^{(i-1)}$ for all $i \in \{0, 1, \dots, p-1\}$ (here two runners are equal if they coincide as 1-abaci; that is, they have beads in exactly the same rows). Clearly the set $\mathcal{D}(\mathcal{Z}(a))$ is naturally in bijection with $\mathcal{Z}(a)$ via the construction described above.

Lemma 5.12 *Let a and x be as in Notation 5.6. Then* $br(ap) \le M(2a) = p - 1 + \lfloor \frac{2x}{3} \rfloor$.

Proof From Remark 5.11 it is immediate that $br(ap) \le M(2a)$, so it remains to prove $M(2a) = p - 1 + \lfloor \frac{2x}{3} \rfloor$.

Let $\underline{U} = (U^{(0)}, U^{(1)}, \dots, U^{(p-1)}) \in \mathcal{D}(2a)$ be such that $\rho(\underline{U}) = M(2a)$. Let $w_i = w(U^{(i)})$. Clearly $w_1 + w_2 + \cdots w_{p-1} \le 2a$. Moreover, arguing as in the proof of Lemma 3.10 we can assume that $w(U_1^{(i)}) = w_i$ and $w(U_0^{(i)}) = 0$ for all $i \in \{1, \dots, p-1\}$. From the maximality of $\rho(\underline{U})$ we deduce using Lemma 3.10 (in the case $\lambda = \emptyset$) that

$$\mathrm{Rem}(U_1^{(i)}) = \lfloor \sqrt{w_i} \rfloor$$

and hence

$$M(2a) = \max \left\{ \sum_{i=1}^{p-1} \lfloor \sqrt{b_i} \rfloor \;\middle|\; b_1 + \cdots + b_{p-1} \le 2a \text{ and } b_i \in \mathbb{N}_0 \ \ \forall\, 1 \le i \le p-1 \right\}.$$

Let $b = (b_1, \dots, b_{p-1})$ be such that $b_i \in \mathbb{N}_0$ for all i, $\sum_i b_i \le 2a$ and $\sum_i \lfloor \sqrt{b_i} \rfloor = M(2a)$; we will call any $(p-1)$-tuple satisfying these conditions *maximal*. If there exists i such that $b_i \ge 9$, then there exists j such that $b_j \le 1$. This follows since $\sum b_i \le 2a < 2p$. Replacing b_i by $b_i' = b_i - 4$ and b_j by $b_j' = b_j + 4$ in b we obtain a new maximal sequence b'. Hence we may assume without loss of generality that our maximal sequence b has $b_i \le 8$ for all $i \in \{1, \dots, p-1\}$.

Now if there exists i such that $b_i = 0$ then there exists j such that $b_j \ge 2$, because $2a > p$. In this case, replacing b_i by $b_i' = 1$ and b_j by $b_j' = b_j - 1$ in b we obtain a new maximal sequence b'. Hence we may further assume that b has $b_i \ge 1$ for all $i \in \{1, \dots, p-1\}$. The observations above show that without loss of generality we may assume

$$\lfloor \sqrt{b_1} \rfloor = \cdots = \lfloor \sqrt{b_t} \rfloor = 2, \ \ \lfloor \sqrt{b_{t+1}} \rfloor = \cdots = \lfloor \sqrt{b_{p-1}} \rfloor = 1,$$

for some $t \in \{0, \dots, p-1\}$.

In particular, $b_i \in \{4, \dots, 8\}$ for $i \in \{1, \dots, t\}$ and $b_j \in \{1, 2, 3\}$ for $j \in \{t+1, \dots, p-1\}$. Thus $4t + (p-1-t) \le \sum b_i \le 2a$, which gives $t \le \lfloor \frac{2x}{3} \rfloor$ since t is an integer. This in turn implies that $M(2a) = 2t + (p-1-t) \le p - 1 + \lfloor \frac{2x}{3} \rfloor$.

Finally, equality holds because we can construct $\underline{U} \in \mathcal{D}(2a)$ such that $w(U^{(1)}) = \cdots = w(U^{(t)}) = 4$, $w(U^{(t+1)}) = \cdots = w(U^{(p-1)}) = 1$ and $w(U^{(0)}) = 2a - 3t - (p-1)$, where $t = \lfloor \frac{2x}{3} \rfloor$, with $\mathrm{Rem}(U_1^{(j)}) = 2$ for $j \in \{1, \dots, t\}$ and $\mathrm{Rem}(U_1^{(j)}) = 1$ for $j \in \{t+1, \dots, p-1\}$. □

Lemmas 5.9 and 5.12 show that $p - 1 + 2\lfloor \frac{x}{3} \rfloor \le br(ap) \le p - 1 + \lfloor \frac{2x}{3} \rfloor$. In particular if $\delta \ne 2$ then we have that $\lfloor \frac{2x}{3} \rfloor = 2q + \lfloor \frac{2\delta}{3} \rfloor = 2q = 2\lfloor \frac{x}{3} \rfloor$. In this case we have $br(ap) = M(2a) = p - 1 + 2q$. To deal with the remaining case of $\delta = 2$ where $p - 1 + 2q \le br(ap) \le M(2a) = p - 1 + 2q + 1$, we have the following lemma.

Lemma 5.13 *Let* $a \in \mathbb{N}$ *be as in Notation 5.6 and suppose that* $\delta = 2$. *Then* $br(ap) \le M(2a) - 1$.

Proof From Remark 5.11 it is enough to show that if $\underline{U} \in \mathcal{D}(2a)$ and $\rho(\underline{U}) = M(2a)$, then $\underline{U} \notin \mathcal{D}(\mathcal{Z}(a))$. To do this we will show that if $\rho(\underline{U}) = M(2a)$ then there exists $i \in \{0, 1, \ldots, p-1\}$ such that $U_0^{(i)} \neq U_1^{(i-1)}$.

For $i \in \{0, 1, \ldots, p-1\}$, let $b_i = w(U^{(i)})$. Arguing as in the proof of Lemma 5.12 we see that $\rho(\underline{U}) = \sum_{i=1}^{p-1} \lfloor \sqrt{b_i} \rfloor$. Moreover, given any composition $\underline{w} = (w_1, \ldots, w_{p-1})$ such that $w_1 + \cdots + w_{p-1} \leq 2a$ there exists $\underline{V} \in \mathcal{D}(2a)$ such that $w(V^i) = w_i$ for all $i \in \{1, \ldots, p-1\}$, $w(V^0) = 2a - (w_1 + \cdots + w_{p-1})$ and $\rho(\underline{V}) = \sum_{i=1}^{p-1} \lfloor \sqrt{w_i} \rfloor$.

Let $\underline{b} = (b_1, \ldots, b_{p-1})$ and suppose that $b_i \geq 9$, for some $i \in \{1, \ldots p-1\}$.

- If there exists j such that $b_j = 0$, then replacing (b_i, b_j) by $(b_i', b_j') := (b_i - 4, 4)$ in $\underline{b}$ we obtain a new composition $\underline{b'}$ such that $\sum_{i=1}^{p-1} \lfloor \sqrt{w_i} \rfloor > \rho(\underline{U})$, contradicting the maximality of $\rho(\underline{U})$.
- If $b_i \geq 10$, then there exists $j \neq l$ such that $b_j = b_l = 1$ since $a < p$. But then we may replace (b_i, b_j, b_l) by $(b_i - 6, 4, 4)$ in $\underline{b}$ to obtain a similar contradiction as before.
- If there exists $i' \neq i$ such that $b_{i'} \geq 9$, then as above we deduce that $b_{i'} = 9$. In particular, $2a \geq 18$ so $p > 3$. Since $a < p$, there exist distinct j, j', j'' such that $b_j = b_{j'} = b_{j''} = 1$. But then we may replace $(9, 9, 1, 1, 1)$ by $(5, 4, 4, 4, 4)$ in $\underline{b}$ to obtain a contradiction.

The above observations show that if $b_i \geq 9$ for some $i \in \{1, \ldots, p-1\}$ then in fact $b_i = 9$ and $1 \leq b_i \leq 8$ for all $j \neq i$. In particular, there exists $t \in \{0, 1, \ldots, p-2\}$ such that $\underline{b}$ has t parts satisfying $\lfloor \sqrt{b_j} \rfloor = 2$ and $p - 2 - t$ parts satisfying $\lfloor \sqrt{b_j} \rfloor = 1$. Hence $M(2a) = 3 + 2t + (p - 2 - t) = p - 1 + \lfloor \frac{2x}{3} \rfloor = p - 1 + 2q + 1$, so $t = 2q - 1$. But this implies that

$$2a \geq \sum_{m=1}^{p-1} b_m \geq 9 + 4t + (p - 2 - t) = p - 1 + 6q + 5.$$

Therefore we have $6q + 5 \leq 2a - (p-1) = 2x = 6q + 4$, a contradiction. Thus $b_i \leq 8$ for all $i \in \{1, \ldots, p-1\}$.

So suppose there are t values of i for which $\lfloor \sqrt{b_i} \rfloor = 2$, s values for which it is 1, and $p - 1 - s - t$ values for which it is 0. Then

$$p + 2q = M(2a) = 2t + s \leq p - 1 + t,$$

so $t \geq 2q + 1$. In particular $t \geq 1$, so there exists i with $\lfloor \sqrt{b_i} \rfloor = 2$. If there exists $j \neq l$ such that $b_j = b_l = 0$, then we may replace (b_i, b_j, b_l) by $(b_i - 2, 1, 1)$ in $\underline{b}$ to obtain a contradiction to the maximality of $\rho(\underline{U})$. So there is at most one $b_j = 0$ and thus $s + t \in \{p-2, p-1\}$.

If $s + t = p - 2$, then $p + 2q = M(2a) = 2t + s$ implies $t = 2q + 2$, and so

$$6q + 4 - b_0 = 2x - b_0 = \sum_{m=1}^{p-1} b_m - (p-1) \geq 4t + s - (p-1) = 6q + 5,$$

which is a contradiction. Thus $s + t = p - 1$ and $t = 2q + 1$. Since

$$6q + 4 - b_0 = \sum_{m=1}^{p-1} b_m - (p-1) \geq 4t + s - (p-1) = 6q + 3,$$

one of the following must hold:

(i) $|\{i : b_i = 4\}| = t$, $|\{i : b_i = 1\}| = s$ and $b_0 = 1$; or

(ii) $|\{i : b_i = 4\}| = t - 1$, $|\{i : b_i = 5\}| = 1$, $|\{i : b_i = 1\}| = s$ and $b_0 = 0$; or
(iii) $|\{i : b_i = 4\}| = t$, $|\{i : b_i = 2\}| = 1$, $|\{i : b_i = 1\}| = s - 1$ and $b_0 = 0$.

Now, suppose for a contradiction that $\underline{U} \in \mathcal{D}(\mathcal{Z}(a))$. Then we have that the bead configurations on $U_1^{(i-1)}$ and $U_0^{(i)}$ are equal for all i; call this property $(\star)$. The key in the following will be that $t = |\{i : b_i \geq 4\}| = 2q + 1$ is odd.

In case (i), let $i \in \{1, \ldots, p-1\}$ be such that $b_i = 4$. Then $(w(U_0^{(i)}), w(U_1^{(i)})) = (j, 4-j)$ for some $0 \leq j \leq 4$. If $j = 2$ then $(\star)$ would imply $b_{i+1} \geq 2$, and hence $b_{i+1} = 4$, since $b_l \in \{1, 4\}$ for all l. This then gives $w(U_0^{(i+1)}) = w(U_1^{(i+1)}) = 2$. We can iterate this argument to deduce that $w(U_0^{(y)}) = w(U_1^{(y)}) = 2$ for all $y \in \{0, 1, \ldots, p-1\}$, which is a contradiction. Thus $j \in \{0, 1, 3, 4\}$.

If $j = 0$, then $w(U_1^{(i)}) = 4$, so $(\star)$ implies that $w(U_0^{(i+1)}) = 4$ and hence $b_{i+1} = 4$ also. Similarly if $j = 1$, then $w(U_0^{(i+1)}) = 3$ and hence $b_{i+1} = 4$. On the other hand, if $j = 3$ or $j = 4$ then similarly we deduce that $b_{i-1} = 4$. These observations imply that t is an even natural number (because if $j \in \{0, 1\}$ then we may pair off i and $i+1$ where $b_i = b_{i+1} = 4$, and if $j \in \{3, 4\}$ then we may pair off i and $i-1$ where $b_i = b_{i-1} = 4$). This gives a contradiction, and so $\underline{U} \notin \mathcal{D}(\mathcal{Z}(a))$, as desired. The analyses of cases (ii) and (iii) are similar. □

Thus when $\delta = 2$ we also have that $br(ap) = p - 1 + 2\lfloor \frac{x}{3} \rfloor$, by Lemmas 5.9, 5.12 and 5.13. This proves Proposition 5.3.

Proof of Theorem B This follows directly from Propositions 5.1, 5.2 and 5.3. □

We devote the final part of this section to the description of the set $\mathcal{E}_{ap}$ for any $\frac{p}{2} < a < p$.

Proposition 5.14 *Let $a \in \mathbb{N}$ be such that $\frac{p}{2} < a < p$. Then $\mathcal{E}_{ap} = \{1, 2, \ldots, br(ap)\}$.*

Proof Let $\beta \vdash_{p'} ap$ with p-abacus configuration $B := B_\beta$ be as defined in Lemma 5.9. In particular we proved that $|\beta_{p'}^{-}| = br(ap) = p - 1 + 2q$, with notation as in Notation 5.6.

Denote by b the bead in position $(1+\delta, 1)$ of B. For $i \in \{1, 2, \ldots, \frac{p-1}{2}\}$ let c_i be the bead in position $(0, p-2i)$ of B and let $B(i)$ be the p-abacus configuration obtained from B by sliding b down to position $(1+\delta+i, 1)$ and by sliding c_j up to position $(-1, p-2j)$ for all $j \in \{1, \ldots, i\}$. Let $\mu(i) \vdash ap$ be the partition corresponding to the p-abacus configuration $B(i)$. From Theorem 2.7 we have that $\mu(i) \vdash_{p'} ap$ and $|\mu(i)_{p'}^{-}| = |\beta_{p'}^{-}| - 2i$. It follows that

$$\{2q, 2q+2, \cdots, br(ap) - 2, br(ap)\} \subseteq \mathcal{E}_{ap}.$$

Now let $A := B(\frac{p-1}{2})$. For $i \in \{1, 2, \ldots, q-1\}$ let $A(i)$ be the p-abacus configuration obtained from A by sliding down bead b from position $(1 + \delta + \frac{p-1}{2}, 1)$ to position $(1 + \delta + \frac{p-1}{2} + 3i, 1)$ and by replacing runner A_{2j+1} with $A_{2j+1}^{\uparrow}$ for all $j \in \{1, \ldots, i\}$. (For convenience, this step is depicted below.)

$$\begin{array}{cccc} & 2j & 2j+1 & 2j+2 \\ -2 & \times & \circ & \times \\ -1 & \times & \times & \times \\ 0 & \circ & \circ & \circ \\ 1 & \circ & \times & \circ \end{array} \quad \longrightarrow \quad \begin{array}{cccc} & 2j & 2j+1 & 2j+2 \\ -2 & \times & \times & \times \\ -1 & \times & \times & \times \\ 0 & \circ & \circ & \circ \\ 1 & \circ & \circ & \circ \end{array}$$

Let $\nu(i) \vdash ap$ be the partition corresponding to the p-abacus configuration $A(i)$. Since $w(A_{2i+1}) = 3$ for all $i \in \{1, 2, \ldots, q-1\}$, it follows from Theorem 2.7 that $\nu(i) \vdash_{p'} ap$ and $|\nu(i)^-_{p'}| = |\mu(\frac{p-1}{2})^-_{p'}| - 2i$. Thus $\{2, 4, 6, \cdots, 2q-2\} \subseteq \mathcal{E}_{ap}$, and so it remains to show $\{1, 3, \ldots, br(ap) - 1\} \subseteq \mathcal{E}_{ap}$.

First suppose $q \geq 1$. Consider the p-abacus configuration C obtained from B by sliding down the bead in position $(-1, 0)$ to position $(0, 0)$ and by sliding up the bead in position $(0, 1)$ to position $(-1, 1)$.

Let γ be the partition corresponding to C. It is easy to see that $\gamma \vdash_{p'} ap$ and that $|\gamma^-_{p'}| = br(ap) - 1$. We can now repeat the strategy used above to see that $\{3, 5, \ldots, br(ap) - 1\} \subseteq \mathcal{E}_{ap}$. Of course, $1 \in \mathcal{E}_{ap}$ by considering the trivial partition $(ap) \vdash_{p'} ap$.

If $q = 0$ we begin with the p-abacus configuration C' obtained from B by swapping runners 0 and 1, instead of C. The same argument then shows $\{1, 3, \ldots, br(ap) - 1\} \subseteq \mathcal{E}_{ap}$. □

We conclude by observing that Propositions 5.1, 5.4 and 5.14 together prove Theorem 3.13.

Acknowledgements We thank Jason Long for suggesting Definition 5.10 and for many useful conversations on this topic. We are also grateful to the anonymous reviewer for several helpful comments.

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