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DIFFERENTIAL SYSTEMS, MOVING FRAMES, STRUCTURE-PRESERVING SUBMERSIONS AND GEOMETRICAL PROBLEMS IN PHYSICS

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Summary

The present work applies the theories of exterior differential systems, method of equivalence and moving frames to the study of geometrical problems arising in physics, especially the class of problems that can be described as "structure-preserving submersions". A novel feature of our approach is the formulation of an algorithm which we have named "the method of involutive seeds". By using this method, we can rapidly determine the number of free functions that we must specify in order to completely specify the problem, which we will call the "degree of arbitrariness" of the problem, and which for many physical systems is linked to the physical degree of freedom. This algorithm is especially helpful in dealing with systems with many constraints such as structure-preserving submersions. We also give other examples of calculations using this algorithm: in particular, we used it to investigate the degree of arbitrariness of the theory of general very special relativity, based on Riemannian geometry with a holonomy constraint, and thus argue that such a theory is not a suitable candidate for a physical theory.

As for structure-preserving submersions, which we propose as a generalisation for Riemannian submersions to other geometrical structures, after investigating the properties and degrees of arbitrariness of the general construction we use it to study the problem of flows, especially rigid flows in relativity. We generalise the classical Herglotz–Noether theorem, which states that rotational rigid flow in Minkowski spacetime must be isometric, to all dimensions and to all conformally flat spacetimes in all dimensions, and also to shear-free flows in conformally flat spacetimes; we generalize a partial result of the Ellis conjecture that a self-gravitating shear-free perfect fluid in geodesic motion must be either expansion-free or vorticity-free to all dimensions, and we will see clearly the origin of this result from the group structure of spacetime; we discuss an approach for the general Ellis conjecture, and show the relation between the Herglotz–Noether theorem and the Ellis conjecture; we show that for a free point particle lagrangian to have a Galilean boost symmetry, it is necessary and sufficient that we have a totally flat direction decoupled from the rest; finally, we give a rough, heuristic reasoning for why some of the Pauli reductions in which we attempt to get a larger gauge group than usually allowed from dimensional reduction are consistent.

Declaration of originality

This dissertation is the result of my own work and contains no part that is done in collaboration with others.

Contents

Prologue

i. Motivation of this work

The underlying aim of this work is an attempt to investigate the problem of rigid flows in relativity, especially the classical Herglotz–Noether theorem. Along the way, methods and frameworks are developed, which in turn helped to solve a number of other problems.

The classical Herglotz–Noether theorem states that *all rotational Born-rigid flows in* 4*-dimensional Minkowski spacetime are isometric.* Suppose a relativistic flow is along the normalised timelike vector field $\mathbf{I}$, this means that

$$\mathcal{L}_{\mathbf{I}} h_{\mu\nu} = 0,$$

where $h_{\mu\nu}$ is the projection of the Minkowski metric $\eta_{\mu\nu}$ onto the orthogonal complement of the vector field $\mathbf{I}$: if we write $\mathbf{I} = v^\mu \partial_\mu$, this means that $h^\mu{}_\nu = \delta^\mu{}_\nu - v^\mu v_\nu$. The statement that this flow is *rotational* amounts to a non-singular condition, and the conclusion is that, up to a factor e^λ, we have

$$\mathcal{L}_{e^\lambda \mathbf{I}} \eta_{\mu\nu} = 0$$

as well, i.e., the vector field $e^\lambda \mathbf{I}$ is a Killing vector.

While the proof of this classical theorem has been known for a very long time, it involves rather messy calculations and *ad hoc* reasoning, which makes it difficult to generalise. A first step towards a better way of tackling this problem is to observe that the condition that $h_{\mu\nu}$ is preserved along the flow implies that a *Riemannian structure* is defined on the quotient space of Minkowski spacetime by the vector field. By distancing ourselves a little bit from detailed calculations and focusing instead on the underlying geometrical structures, some order can be brought to the study of this problem.

While we are studying the structures of geometry, it often helps to consider more general settings than what is immediately needed in order to obtain results and insights that are applicable to a wider range of problems. Here the study a vector field preserving the orthogonal component of the metric can be immediately generalised to the study of *Riemannian submersions*, which in turn can be further generalised to *structure-preserving submersions*.

The main difficulty in the study of structure-preserving submersions is that such systems are obtained by giving constraints to existing problems, hence equations and quantities are almost always interdependent and hence difficult to analyse. For simpler problems such as the classical Herglotz–Noether theorem we can still navigate in this mess, but for more complicated settings, for example, when one goes to higher dimensions, the complexity of the interdependences rapidly explodes. Hence what is needed here is an *algorithmic* approach to obtain all the relations that are present in a specific setting. The way we will achieve this is by employing the method of moving frames. Fundamentally, the method of moving frames is based on the study of the underlying group structure of the problem, and consequently a complete system of all relations are obtainable from the *structural equations* of the co-frame, which can be viewed as an inhomogeneous version of the Maurer–Cartan equations for Lie groups. These equations give rise to *differential invariants* of the problem, which completely characterise the problem, and the algebraic inderdependence of these invariants are simply consequences of exterior differentiation.

In fact we can already prove the classical Herglotz–Noether theorem with minimal efforts using the outline given above, as well as making non-trivial generalisations and gaining considerable insight of the geometrical structure, for example we can immediately generalise the theorem to all dimensions ≥ 3, and also to all cases where the constraint is that the space is conformally flat instead of being homogeneous. But there is still one short-coming: even though we know the complete set of algebraic relations of the invariants that comes from the moving frame, this does not answer the question of the *functional dependence* among them. What we want here is a way to study the *Cauchy problem* of such problems: namely, the question that, in order to completely specify the problem, how many *free functions* are needed, and what they can be.

Since the structural equations for the co-frame are written in terms of exterior differential systems, this naturally leads us to consider the framework designed for study the Cauchy problem of exterior differential systems, namely the Cartan–Kähler theory. Again we will proceed from a more general setting: first we show that we can derive from the structural equations a system of Pfaffian equations which completely determine the set of differential equations, and then we show that by mere manipulation of the indices on the invariants, we can use Cartan's involutivity theorem to obtain the degree of arbitrariness of the general solution of the problem, and hence solving the Cauchy problem in this case. We formulate this procedure into an algorithm, which we have named *the method of involutive seeds*.

With the Cauchy problem solved, we have a much clearer way of interpreting the functional independence among the invariants. Returning to the problem of rigid flow, we can now easily show that in the generic case, a Born-rigid flow in a generic pseudo-Riemannian spacetime of any dimension ≥ 3 either does not exist, or there exists only solutions parametrised by constants. There are exceptions, as we know: for example, in Minkowski spacetime, a set of singular solutions, namely those with vanishing vorticity, can be constructed which depends on one free function.

In summary, in order to solve the problem of rigid flow, we constructed the framework for structure-preserving submersions, and we used three geometrical methods to investigate this framework: the method of moving frames which we used to write down the equations of the problem, the method of equivalence which we used to extract the invariants from the moving frames, and the method of exterior differential systems which we used to extract the data concerning functional independence from the problem, for which we have also formulated an algorithm in order to achieve this aim rapidly and easily.

II. Structure of this work and new results

The above discussion outlines the order in which the present work was carried out, but the presentation of this work does not actually proceed in that order, and there are more problems discussed and solved in the presentation. Here we give a summary of the contents of this work by chapter.

Chapter 1 is mainly review material, and is concerned with summarising the results of Cartan's theory of exterior differential systems and the theory of equivalence, together with the study of Riemannian geometry in this language.

In chapter 2 we formulate the method of involutive seeds, which generalises the method we used in the last chapter for calculating the degree of arbitrariness of the Riemannian metric to a much wider range of problems. We state the algorithm, and then give the proof that the algorithm really does what it claims to do. As a check, we apply this method to various problems where the degree of arbitrariness is known: we calculate the degree of arbitrariness of Riemannian spaces with torsion, of gauge theories, and of scalar field theories. These calculations should show how much easier it is to use our method than the conventional approach.

At the end of the second chapter, we apply the method of involutive seeds to a new problem, the problem of general very special relativity with holonomy constraint, and we obtain a degree of arbitrariness that we use to argue that this theory is untenable as a physically viable candidate for describing gravity.

In chapter 3 we first show how we can apply our methods to the problem of Riemannian submersions in a systematic way. The structure equations are derived, and the degree of arbitrariness of the involutive seeds is calculated. The relevant Cauchy problem is also studied. The case where the total space is of dimension two is special and is treated separately here, and it is proved that a Riemannian submersion always exists in this case. Then we step back a little bit again and formulate the general framework for dealing with all structure-preserving submersions. The end of the chapter deals with the problems of isometries in Riemannian spaces, which are always Riemannian submersions. The principal method of calculation employed in this case is different: we need to use the Lie derivative in addition to the exterior derivative. The use of Lie derivatives this way is also essential later for dealing with the physical problems of flows. Finally we briefly outlines how we can reformulate the geometrical problems of Riemannian spaces with isometries as submersion problems with very stringent constraints such that most or all degrees of arbitrariness are restricted to the reduced to the reduced space. This is, in a way of speaking, the process of dimensional reduction, to which we will come back later.

In chapter 4 we are concerned with the problems of rigid flows of various kinds. First we deal with rigid flow in Newtonian spacetimes, which mainly serves to illustrate how our methods of structure-preserving submersions can be applied to non-Riemannian geometries and to give physical interpretations of various quantities we define in order to see how they generalize to relativity. Next we study the problem of time-like Born rigid flows in relativity, which is just Riemannian submersions of codimension one. After a brief study of the degree of arbitrariness of such a system formed by perfect fluid, we generalize the classical Herglotz Noether theorem first to all homogeneous spaces of dimensions greater than two, then to all conformally flat spaces of dimensions greater than three.

Next, we embark on a detour and see how the results and methods of Born-rigid flow we just developed can be applied in an unexpected way: we show that for a free point particle lagrangian in classical mechanics to have a Galilean boost symmetry, it is necessary and sufficient that one or several directions in the space is flat and decouples from the rest directions.

Returning to flows, we specialise our general framework to the framework dealing with Weyl rigid flows, which are structure-preserving flows in Weyl's geometry, the affine geometry with a metric and a scaling factor. We then show that how the problems of shear-free flows with possibly non-zero expansions can be reformulated as a Weyl rigid flow, with the additional constraint that the total space is derivable from a Riemannian space. With this framework, we generalize a partial result of the Ellis conjecture and prove the following: a self-gravitating shear-free perfect fluid under geodesic motion must be either expansion-free or vorticity-free or both, in all dimensions greater than three. A method that can in principle be used to check the validity of the conjecture in the general case is formulated, and we also investigate the relation of the generalised Herglotz–Noether theorem and the Ellis conjecture for the case of Petrov type O.

The final part of this chapter is more of a sketchy nature, and shows how the methods of structure-preserving submersions can be applied to the problems of dimensional reduction. In particular, it is noted that if we want a scaling factor in the dimensional reduction, the correct model is that of Weyl submersion, not Riemannian submersion. At the end we propose an explanation for why some of the Pauli reductions are "consistent reductions" whereas most of them are not.

This work contains results in the papers [26, 28, 27, 25] by the present author. The papers [28, 27] have been recently submitted to *General Relativity and Gravitation*. The following is a summary of the new results appearing in this work, together with page numbers:

1° The method of involutive seeds: calculating the degree of arbitrariness (or the full set of Cartan characters) easily from a formulation in terms of moving frames, without explicitly invoking the Cartan–Kähler theorem (§**50**, p. 76–§**62**, p. 88).

2° The general framework of structure-preserving submersions, which are applicable to a wide range of physical situations and which furthermore stipulates what one should do to obtain basic properties of the system algorithmically (§**89**, p. 123–§**91**, p. 126).

   a) Calculations for the degree of arbitrariness of general Riemannian submersions (§**85**, p. 116).

   b) Calculations for the degree of arbitrariness of Weyl submersions of codimensional one, useful for studying fluid flows and dimensional reductions (§**118**, p. 168).

3° "Gauged" general very special relativity has degree of arbitrariness exactly one, and the usual approach for writing down the dynamics (Einstein–Hilbert action) yields unphysical results (§**67**, p. 92–§**71**, p. 97).

4° Various generalisations to the Herglotz–Noether theorem ("all rotating Born-rigid flow in special relativity must be isometric"):

   a) Generalisation to flat or homogeneous spacetime for all large dimensions ($\dim M \geq 3$) (§**107**, p. 148);

   b) The vanishing degree of arbitrariness for Born-rigid flow for general spacetimes implies that in a spacetime of arbitrary dimension ≥ 3 and given an arbitrary metric, the non-singular solutions are either isolated (i.e., there is no continuous families of solutions), or there are no solutions at all (§**109**, p. 151);

c) Generalisation for conformally flat spacetime of large dimensions $\dim M \geq 4$ ("all rotating Born-rigid flow in conformally flat spacetime must be conformally isometric") (§**111**, p. 154);

d) Generalisation for shear-free flow ("all shear-free flow in conformally flat spacetime must be conformally isometric") (§**129**, p. 184).

5° Some partial understanding of the Ellis conjecture ("in four-dimensional general relativity all shear-free flow due to self-gravitating, barotropic perfect fluid must be either expansion-free or vorticity-free"):

a) For the case of vanishing acceleration (geodesic flow), the conjecture holds for all large dimensions ($\dim M \geq 4$), without reference to the equation of state, and is a consequence of the local symmetry properties of the space alone (§**127**, p. 182).

b) The Herglotz–Noether theorem for shear-free flow implies the Ellis conjecture for Petrov type O (§**129**, p. 184).

c) A test that could be in principle checked by using computer algebra system for the validity of the conjecture in the generic case (§**128**, p. 183).

6° For a free particle lagrangian to enjoy Galilean symmetries, the relevant spatial direction must be flat and "completely decoupled": this is valid for all dimensions, and the case where the lagrangian is not free is also discussed (§**114**, p. 161–§**117**, p. 165).

7° The Old Kaluza–Klein theory has good degree of arbitrariness despite being physically "inconsistent". The attempt to cure this inconsistency by adding a scaling field makes the situation far more difficult. The geometrical consistency of a Pauli reduction can be understood as the solvability of a non-Pfaffian exterior differential system (§**131**, p. 185–§**137**, p. 194).

III. The methods of Cartan

As mentioned above, this work centres around the three methods of exterior differential systems, equivalence methods, and moving frames. All three methods developed by the French mathematician Élie Cartan. As they play fundamental roles in the present work, here we give a very brief overview of them and their present applications to physics.

The first method, that of exterior differential systems, is concerned with differential equations written in the form of exterior systems. The major theorem of this method, the Cartan–Kähler theorem, is at the root just the Cauchy–Kowalewski theorem applied in a particular setting. But this particular setting, in itself containing systems even more general than those covered by the direct application of the Cauchy–Kowalewski theorem, gives the theory of exterior differential equations a particular simplicity: instead of studying the problems of *differential calculus*, we are led to the study of *linear and exterior algebra*. Cartan's test, which forms an integral part of Cartan–Kähler theory, furthers this simplification: the ultimate information concerning the solutions of such systems can be deduced from an *arithmetic* structure defined in the infinitesimal linear systems.

So far, the application of this theory to problems of physics have been scarce, mainly due to the reason that physicists are not used to writing the equations of physics in terms of exterior differential systems, as the usual framework of jet spaces widely used in the study of partial differential equations seems too clumsy for physical theories.

The second method, the method of equivalence, is concerned with the problem of when two differential systems are locally "the same". The method consists in, again, reformulating the problem in terms of exterior differential forms, and then it can be solved easily and algorithmically with the method of exterior differential systems. The real usefulness of this method does not lie in really comparing two given systems, but rather deriving the *differential invariants* associated with any one system: since the equivalence problem is solved by comparing these invariants, these invariants are independent of any inessential variables that we introduced when attempting to solve the problem, for example, the choice of coordinates. This method also links together with the study of Lie groups through the use of Maurer–Cartan forms, and as a consequences, in this framework any symmetry or potential symmetry of the problem is usually explicit. As with the first method, the second method also barely appears in the physics literature.

The third method is the method of moving frames. Strictly speaking this method is just the second method applied to geometrical settings, by choosing a *general* frame on the manifold and use the associated coframe as the basis of the exterior differential system for computations. The real power of this method lies in its simplicity of application and invariance properties: the gist of the method is choosing the most general frame possible, that means, whenever we encounter some arbitrariness of choice (and such a choice is invariably parametrised by a Lie group), we formalise this choice by *prolonging* the space we are working with by going to the principle bundle formed by the product of the original manifold and the Lie group. As such, all results obtained have geometrical interpretations independent of all particular choices of coordinates, frames, etc.

Compared with the first two methods, the method of moving frames is much more familiar to the physicists, though in a watered-down form: physicists usually think of tetrad, vierbein and moving frames as synonyms. The theory of tetrad, introduced to the physicists by Hermann Weyl, consists of simply of choosing an orthonormal frame for the tangent bundle. First of all, moving frames are not restricted to orthonormal frames or even frames (some authors even consider conceptualising moving frame in terms of frame bundles harmful to its understanding). Second, even in the more restricted setting, a tetrad and a moving frame are not the same: a frame is "moving" not because it varies from point to point, but because it varies *at the same point* as well, i.e., there are more variables involved than in the tetrad formulation, and the theory of tetrad is really the theory of *un*moving frame.

The intriguing fact about these three methods is that there is a certain unity among them, and it is one of the aims of the present work to make this unity manifest and applicable to physical and geometrical problem. In physics, the tetrad method is widely used when we want to compute quantities that are too messy by using coordinates. From here, after some concepts of the equivalence method, we can transform this method into the real structural equations of the moving frame. It is well know that such structural equations contains all invariants of the problem, and it is also well known that these invariants are linked by algebraic and differential relations among them, but the method of studying them is often clumsy. What we do is that we form an exterior differential system of these invariants, obtainable directly from the structural equations to the geometrical problem, and then the problem of their algebraic and differential relations can be studied with the method of exterior differential systems. It is much more useful to have a method that works well in particular settings than a method that works for all settings, but only poorly, and after observing certain remarkable shortcuts we can take when applying the Cartan–Kähler theorem to some defining examples of such problems, we formalise such shortcuts by formulating the method of involutive seeds for geometrical problems with a covariant derivative defined. In this way, the three methods allow us to first write down the equations of the physical problem, and then obtain the invariants of the problem, which are, roughly speaking, the *physical variables* of the theory, and finally we solve the Cauchy problems for these invariants, which tell us, very roughly speaking, which of these variables can be taken as dynamical. As we will see, these methods will also allow us to calculate the *degree of freedom* of physical systems.

iv. Notations and conventions

In terms of notation, most will be clear from the context or defined as we use them. To avoid writing an unreasonable amount of summation signs, we will use the Einstein summation convention in that repeated indices are presumed to be summed for all indices, though sometimes for emphasis we still write out the sign explicitly. As we are not using the tensor calculus of Ricci, there are a few important differences. First, the indices are what are called the "tangent space indices" in the literature, even though they may look the same as indices in the usual tensor calculus. Second in a moving frame, *often the individual components of a tensor has an invariant meaning* (this depends on if our frame is suited to the problem), and such a component may have two of its indices taking the same value. To distinguish this case from the case where the indices are summed over all possible values, we will either explicitly state that there is no sum, or we will put bars on one of the indices, so i and $\bar{\imath}$ denote the same indices but for $T^i{}_{\bar{\imath}}$ this index is not summed over all values. Third, when dealing with an orthonormal frame, whether the indices are up or down is immaterial so we will usually be careless in such cases, and hence *indices are summed even though both of repeated ones appear upstairs or downstairs.*

Chapter 1

Preliminaries

i. Integral varieties, Cartan–Kähler theorem

1 Integral varieties. Ultimately, all physical problems must be formulated as equations, usually differential equations. In this work we will be exclusively concerned with differential equations in the form of *exterior differential systems*, i.e., systems whose solutions are defined by the vanishing of a number of differential forms. Such solutions are also called *integral varieties.*

In our case, the differential systems that we will study will most often be given in terms of problems that are formulated as *coframe problems.* Thus suppose on a manifold M we have a *coframe* ω_i, $i = 1, \ldots, n$ which forms a basis of the cotangent space. This naturally restricts our differential systems to contain no forms of degree 3 or more, of which all 2-forms are closed. However, in reviewing the general theory, we shall not impose this restriction.

2 Frobenius integrability. The basic problem in the study of integral varieties is the following: given a coframe with basis θ_i, ω_α, we want to find the integral varieties of

$$\theta_i = 0, \qquad i = 1, 2, \ldots, r.$$

Frobenius theorem states that, if the exterior derivative of these forms vanish *when we use these relations themselves*,

$$d\theta_i = 0 \pmod{\theta_1, \theta_2, \ldots, \theta_r},$$

then solution is guaranteed. The cotangent space of the integral varieties is spanned by ω_α where we understand that they have been pulled back onto the submanifolds, and they remain independent. In other words, ω_α can be taken as the independent conditions when the problem is Frobenius integrable. For such systems, the manifold is foliated by the solution submanifolds of dimension $n - r$. The converse is trivially true.

At any given point P in M, we call a subspace $\mathcal{V}$ of the tangent space $T_P M$ an *integral element* of the differential system if and only if $\theta_i(\mathcal{V}) = 0$. It is easy to see how to obtain the maximal integral element at any point: it is the kernel of the forms θ_i at that point[†]. Through such an integral element, the integral variety having it as the tangent space at the point is unique. Hence Frobenius integrable systems are also called *completely integrable systems*, and we say that *the solutions of completely integrable systems depend on constants at a point.*

Frobenius theorem is applicable when the system include zero-forms: we can first restrict ourselves to the *algebraic varieties* defined by the vanishing of these functions (zero-forms). It is not applicable when higher order forms are included.

3 Integrability for ideals of differential forms. When Frobenius theorem does not apply, we need to consider the generators obtained by differentiating the one-forms, modulo the generators themselves. Hence we consider the *ideal* $\mathcal{I}$ generated by

functions:	$f_1, \ldots, f_{r_0}$,
one-forms:	$\theta_1, \ldots, \theta_{r_1}$,
two-forms:	$\Theta_1^{(2)}, \ldots, \Theta_{r_2}^{(2)}$,
three-forms:	$\Theta_1^{(3)}, \ldots, \Theta_{r_2}^{(3)}$,
$\cdots$	$\cdots$

Again we can restrict attention to ideals containing no functions, since at generic points we can first form the varieties determined by the vanishing of the functions and restrict our attention only to the variety. We also assume that that the given system has already been transformed into a *closed system*:

$$d\theta_i = 0 \pmod{\Theta_1^{(2)}, \ldots, \Theta_{r_2}^{(2)}}, \qquad \text{etc.}$$

so all relations that can be obtained by differentiation have already been incorporated.

[†]This specification of the integral element by the kernel of a linear form means that in our applications, to specify a subspace of a vector space we will usually give the set of linear maps for which the subspace is the kernel. For example, instead of saying that the subspace is spanned by $\partial/\partial x_1$ and $\partial/\partial x_2$, we will say that the subspace is the kernel of $dx_3, dx_4, \ldots, dx_n$, in other words the solution of the system $dx_2 = dx_3 = \cdots = dx_n = 0$ when this equation is restricted to the tangent space of the point under consideration.

4 Characteristics. A concept that is often useful for reasoning is the one of *characteristic directions* $\mathcal{V}$. A direction given by the vector $\mathbf{v}$ is characteristic if

$$\mathbf{v} \lrcorner \Theta = 0 \qquad \text{for all } \Theta \in \mathcal{I},$$

i.e., characteristics directions are the ones having the property that if an element contains this direction, then it is automatically an integral element. Not all systems contain characteristic directions.

The concept of characteristics is useful when we are solving the Cauchy problem. Suppose we are integrating a differential system and we want to go from dimension k to dimension $k+1$. We need to specify the initial data on the integral variety of dimension k which we have already found. But if this variety contains a characteristic variety, then we know two things: first, the data along these varieties are well-defined when we know them *at any point* on them, hence there are consistency issues when specifying the initial data; second, we need specify extra functions in order to effect the integration. Hence for the Cauchy problem, specifying initial data on varieties containing characteristic varieties is problematic and is best avoided.

5 Recursive construction of integral elements. Let us now find the integral varieties of a differential system generated by a differential ideal. We will construct the integral variety by a recursive procedure: at a given point, construct a 1-dimensional integral variety through this point, and through this 1-dimensional integral variety, construct a 2 dimensional integral variety containing it, etc. Hence first consider *integral elements at a single point.* Every one-dimensional integral element $\mathbf{v}$ must satisfy

$$\theta_i(\mathbf{v}) = 0.$$

There are no more conditions: the vanishing of higher order forms is automatic due to the antisymmetric properties of differential forms.

Using a coframe $\pi_1, \pi_2, \ldots, \pi_n$ and its dual frame $\mathbf{I}_1, \ldots, \mathbf{I}_n$ on our manifold, we have $\theta_i = A_{ij}\pi_j$, and $\mathbf{v} = v_i\mathbf{I}_i$ is contained in the integral element if and only if

$$A_{ij}v_j = 0.$$

Hence we see that the space of integral element of dimension 1 is given by a *linear equation*: A_{ij} are constants at the points we consider and any solution v_j forms just a vector.

Now suppose we already have a determined 1 dimensional linear element $\mathbf{v}_1$ and we would like to extend it into a 2 dimensional linear element: this amounts to finding another direction $\mathbf{v}_2 = \bar{v}_i\mathbf{I}_i$. We need to ensure $A_{ij}\bar{v}_j = 0$: this direction must itself be a solution to the one-dimensional problem. But there is more: suppose the 2-forms in the system are

$$\Theta_i^{(2)} = A_{ijk}\pi_j \wedge \pi_k, \qquad A_{ijk} = -A_{ikj},$$

then we must have

$$A_{ijk}v_j\bar{v}_k = 0.$$

v_j is data already given to us: this, together with $A_{ij}\bar{v}_j = 0$, forms a *linear system* whose solution space gives all possible directions extending $\mathbf{v}_1$.

The general case should now be clear: given a set $\mathbf{v}_1, \ldots, \mathbf{v}_p$ forming an integral element of dimension p, the extension to dimension $p+1$ is obtained by a suitable linear system on the free tangential directions. This linear system is deduced from all the exterior forms in the system of dimension up to $p+1$. As we go up in the dimension of the integral element, the dimension of the solution space will decrease: for every step it will decrease by at least one: for example, at the second step, $\mathbf{v}_1$ and $\mathbf{v}_2$ must be independent for them to constitute an extension of $\mathbf{v}_1$. Thus we will eventually come to a dimension where the integral element can no longer be extended.

6 Cartan characters. Obviously the integral varieties we are looking for have something to do with the ranks of the various linear systems we have just described. But this immediately poses a question: how do we know that, for all choices of the point P in M, the rank of the system $A_{ij}v_j$ remains the same, and how do we know that for all choices of $\mathbf{v}_1$, the rank of the system $A_{ij}\bar{v}_j$ and $A_{ijk}v_j\bar{v}_k$ are the same, etc.? We cannot know or ensure this *a priori*, so we formulate our definition in order that such questions do not arise.

Thus let us call a point P an *integral point* if it is on the algebraic variety defined by the vanishing of functions $f_i = 0$ in the differential system. An integral point P is *generic* if, at the point, $df_i = f_{i,j}dx^j$ has maximal rank in a neighbourhood.

On a generic point, let us find the one-dimensional integral elements $\mathbf{v}$. If the rank of A_{ij} defining the integral element is maximal *in a neighbourhood*, then the integral element is said to be *ordinary* and this generic point is said to be a *regular* point. The rank of the matrix A_{ij} is called the *zeroth Cartan character* (or simply zeroth character) of the system at the point P and is denoted s_0.

Given an ordinary one-dimensional integral element $\mathbf{v}_1$ at a point, let us try to extend it by one dimension. This element is said to be *regular* if the rank of the system $A_{ij}\bar{v}_j$, $A_{ijk}v_j\bar{v}_k$, where v_j is now given, is maximal in a neighbourhood. The solutions for such a maximal-rank system $\mathbf{v}_1$, $\mathbf{v}_2$, are said to be *ordinary* integral elements. Since the rank of the system $A_{ij}\bar{v}_j$, $A_{ijk}v_j\bar{v}_k$ cannot be less than the rank of a part of itself, $A_{ij}\bar{v}_j$, we denote the rank as $s_0 + s_1$. The integer s_1 is called the *first Cartan character*.

The general pattern should be clear: for the linear system defined at a particular dimension p, the maximality of its rank defines the *ordinary* elements at this dimension, and the *regular* elements at one less dimension. Then, as the dimension of the integral elements cannot increase indefinitely, we will come to a dimension p where we have ordinary elements that cannot be further extended, and hence there are no regular elements at this dimension. The rank of this final system is

$$s_0 + s_1 + \cdots + s_p,$$

and for such a system we have a set of $p + 1$ Cartan characters.

7 Cartan–Kähler theorem: necessity. On a manifold of total dimension n, for a given ordinary integral element of dimension k, the rank of the linear system is

$$s_0 + s_1 + \cdots + s_k,$$

so a necessary condition for the existence of integral element of one dimension higher is that

$$n - (s_0 + s_1 + \cdots + s_k) \geq k + 1$$

or

$$s_0 + s_1 + \cdots + s_k < n - k,$$

otherwise there is not enough dimension to "squeeze in" our desired integral element.

On the other hand, by dimension counting again, if k is the largest possible dimension for such integral manifold, we necessarily have

$$s_0 + s_1 + \cdots + s_k = n - k.$$

There are no other cases: we cannot have greater sign in the above equations.

The condition

$$s_0 + s_1 + \cdots + s_k \leq n - k. \tag{1.1}$$

is the *necessary condition* for the existence of the class of integral varieties with *ordinary* tangent elements of dimension k: this is the necessary part of the Cartan–Kähler theorem. This says *nothing* about integral varieties with non-ordinary tangent elements: we call this latter class of integral varieties the *singular* integral varieties.

8 Cartan–Kähler theorem: sufficiency. The Cartan–Kähler theorem states that the condition (1.1) is also *sufficient* for the existence of analytic integral manifolds of dimension $k+1$ when all data given are *analytic* functions. It is proved by invoking the Cauchy–Kowalewski existence theorem for solutions of systems of partial differential equations in the so-called Cauchy–Kowalewski form.

The Cartan–Kähler theorem, through the application of Cauchy–Kowalewski theorem, in addition specifies the number of free functions we need to specify: when considering integral manifolds of dimension k starting with integral manifolds with dimension $k-1$, let us define the Cartan *pseudo-character* σ_k by the formula

$$s_0 + s_1 + \cdots + s_{k-1} + \sigma_k = n - k.$$

It is obviously always non-negative. Then *the integration process depends on σ_k arbitrary functions of k variables.*

9 Degree of arbitrariness. The Cartan–Kähler theorem states that the pseudo-character gives the number of free functions on k variables we need to specify when we integrate from dimension $k-1$ to dimension k. On the other hand, if p is the largest dimension of ordinary integral elements, we have

$$s_0 + s_1 + \cdots + s_{p-1} + s_p = n - p.$$

Compare this with the definition of the pseudo-character, we see that at this largest dimension, $\sigma_p = s_p$.

If, instead, we are only interested in integrating up to dimension $k-1$, then by comparing formulae, $\sigma_{p-1} = s_n + s_{n-1} + 1$. Indeed, by carrying this calculation further, we have a whole set of relations

$$\begin{cases} \sigma_p = s_p, \\ \sigma_{p-1} = s_p + s_{p-1} + 1, \\ \sigma_{p-2} = s_p + s_{p-1} + s_{p-2} + 2, \\ \quad \cdots \\ \sigma_1 = s_p + s_{p-1} + \cdots + s_1 + (p-1), \\ n = s_p + s_{p-1} + \cdots + s_1 + s_0 + p. \end{cases}$$

The pseudo-characters $\sigma_1, \ldots, \sigma_p$ gives us the *degrees of arbitrariness* we have when we integrate up to a certain dimension [†]. One or more of them can be zero, in which

[†] We do not call them *degree of freedom*, since this word is used somewhat differently when we deal with physical systems. The name is also justified when we consider the equivalence problem of coframes: see §**29**.

case the integration depends only on functions of less than the maximal number of variables. The last equation is an identity.

Let us investigate in more details the case where we integrate to obtain an integral variety of dimension k from a point, such that this integral variety has the integral element spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_k$. We further specify, on this integral variety, coordinates such that $\mathbf{v}_1 = \partial/\partial x_1$ such that the one dimensional integral variety obtained by the first iteration of integration is the submanifold $x_2 = x_3 = \cdots = x_k = 0$ on the k dimensional integral variety, $\mathbf{v}_2 = \partial/\partial x_2$ such that the two dimensional integral variety obtained by the second iteration of integration is the submanifold of $x_3 = \cdots = x_k = 0$, etc. Granted this, let us turn the reasoning around and find the number of arbitrary functions we can specify when integrating to obtain such a system.

The one-dimensional integral variety specified can be thought of the section of the section of the k dimensional integral variety by the flat variety $x_2 = x_3 = \cdots = x_k = 0$. For this, the number of constraint equations is s_0. But we have *already* specified the values of x_2, x_3, $\ldots$, so we can no longer choose them as "arbitrary functions" of x_1: the space in which we have to apply the Cartan–Kähler theorem has been reduced by $k-1$ dimensions, so the number of arbitrary functions is the reduced character in this lower dimensional space: $\sigma_1' = (n-(k-1)) - 1 - s_0 = n - k - s_0$. Using the definition of the pseudo character σ_k, this number is $s_1 + s_2 + \cdots + s_{k-1} + \sigma_k$.

For the two dimensional section in the k-dimensional integral variety given by the constraints $x_3 = x_4 = \cdots = x_k = 0$, the number of constraints is $s_0 + s_1$. The second pseudo character in the reduced space is $\sigma_2' = (n-(k-2)) - 2 - s_0 - s_1 = n - k - s_0 - s_1$, so effecting the section depends on $s_2 + \cdots + s_{k-1} + \sigma_k$ functions of x_1 and x_2, again by using the definition of σ_k.

Carrying this analysis step by step, we see that in integrating from a point up to dimension k, we can specify

$$\begin{aligned}
s_1 + s_2 + \cdots + s_{k-1} + \sigma_k &\text{ arbitrary functions of } x^1, \\
s_2 + \cdots + s_{k-1} + \sigma_k &\text{ arbitrary functions of } x^1, x^2, \\
\cdots \qquad & \qquad\qquad\qquad\qquad \cdots \\
s_{k-1} + \sigma_k &\text{ arbitrary functions of } x^1, x^2, \ldots, x^{k-1}, \\
\sigma_k &\text{ arbitrary functions of } x^1, x^2, \ldots, x^{k-1}, x^k.
\end{aligned}$$

The numbers on the left are *not* the pseudo-characters of the various dimensions: from bottom to top, they are σ_k, $\sigma_{k-1} - 1$, $\sigma_{k-2} - 2$, etc. This ensures that x_1, x_2, $\ldots$ remain independent variables at the last stage.

Before we apply on this result, note that that these integers are obtained only by a particular choice of successive integrations with the Cauchy–Kowalewski theorem, and we have not yet shown their invariance under all changes of variables and all choices of acceptable order of integrations. Indeed, for integration up to dimension k, *only the last non-zero integers in the series* $s_1, s_2, \ldots, s_{k-1}, \sigma_k$*, when taken separately, has an intrinsic meaning.*

10 Complete integrability. Physical predictability. Time. First, an interesting property of the series of Cartan characters. *If the system we consider consists only of functions, one forms and two forms* (which is usually the case we will be concerned with), then

$$s_0 \geq s_1 \geq s_2 \geq \cdots \geq s_{p-1} \geq s_p.$$

Thus, if one of them is zero, then all subsequent ones are zero.

The relation $s_{p-1} \geq \sigma_p$ where we replace s_p by the pseudo character holds only when p is the greatest dimension where ordinary integral elements exist. In the following we will only consider systems where we carry the integration as far as possible.

Now suppose

$$s_0 \geq s_1 \geq \cdots \geq s_k > s_{k+1} = s_{k+2} = \cdots = s_p = 0.$$

The previous section tells us that the integration depends on s_k functions of k variables. But, by our procedure, after specifying s_k functions of k variables, we only have a k dimensional integral variety for which the relation on the coordinates $x_{k+1} = x_{k+2} = \cdots = x_p = 0$ holds. Since at the point which we started our integration, the vectors $\partial/\partial x_{k+1}, \ldots, \partial/\partial x_p$ are pre-determined, we see that *given any non-singular integral variety of* k *dimensions, there is a unique* p *dimensional integral variety that is its extension.* We say that such systems are *completely integrable* from dimension k onward. If we do not integrate to the highest dimension possible, the extension of k dimensional manifold still depends on free functions of more than k functions—we can freely choose which directions to extend.

If all of $s_0, s_1, \ldots$ are zero, we have no equations and the system is trivial. The next simplest case occurs where the only non-zero integer of them is s_0. Then given any *integral element* of dimension p, there is a unique extension to a p dimensional manifold. It is easy to see that $s_1 = 0$ simply means that our system consists only of one-forms, or, if the one forms are denoted by θ_α,

$$d\theta_\alpha = 0 \pmod{\theta_\alpha}.$$

We have recovered the Frobenius theorem as a special case of the Cartan–Kähler theorem.

The significance of completely integrable systems (including non-Frobenius systems) for physics is the following. Suppose we have a *dynamical* physical theory formulated as a differential system on a total space M of dimension n, which includes both the coordinates and the fields. Let p be the dimension of the integral manifolds of the system. Then we definitely want $s_p = \sigma_p = 0$, since otherwise the system does not admit a well-defined Cauchy problem: there is no "time evolution". If $s_p = \sigma_p = 0$, then the system is completely defined when we specify data on a lower dimensional submanifold: the system is *deterministic*, and the "equations of motion" has predictive power. The dimensions from 1 to k, where we have non-vanishing Cartan characters, represent the *physical coordinates* (usually just ordinary space plus some "internal" space), whereas the dimensions we have vanishing Cartan characters represent the dependent variables (which, in particular, include the gauge variables when we consider the theory of fields) and *time*. Note that in this picture, there is no requirement that space and time cannot be mixed together.

On the other hand, for *kinematical* physical theories, i.e., theories before imposing equations of motion, there is no requirement that the pseudo character σ_p is zero.

11 Addition of algebraic equations to a differential system. Given a kinematical physical theory, we obtain a dynamical theory by adding *equations of motion*, which are differential equations. For reasons that will become clear later, these differential equations are, in our framework, *algebraic equations* [(†)], meaning that involving no explicit differentiation, and the derivatives are independent variables. Hence the following problem: when adding *algebraic* equations to an existing differential system, what are the effects on the Cartan characters, integral varieties, etc.?

Assume that we add to our differential system h algebraic equations, which are, roughly speaking, *generic*, meaning that their rank of various orders are maximal. These equations together define an algebraic variety, and we must first restrict our search for integral varieties onto this variety. In other words, up to rank and regularity condition, the total number of variables, n, is reduced by h. We write $n' = n - h$.

For our h algebraic equations, we must also add the h one form equations that are obtained by exterior differentiation. But these equations simply express the fact that the new system we consider, i.e., the restriction of the old system to the algebraic variety, must not contain tangent directions that are pointing out of this variety. Hence the effect on the Cartan characters are subtle: in particular, the effect is not to simply add to the Cartan characters by h.

Let us investigate the problem by considering the pseudo characters of all orders.

[(†)] For the lack of a better terminology, in this work by *algebraic equation* we mean equations that do not involve any kind of differentials. In particular, our "algebraic equations" may contain transcendental functions of the variables.

At dimension k, the definition of the pseudo-character is

$$\sigma_k = (n - k) - (s_0 + s_1 + \cdots + s_{k-1}).$$

But $s_0 + s_1 + \cdots + s_{k-1}$ is the number of constraints we are subject to when we extend from dimension $k - 1$ to dimension k, and $n - k$ is the co-dimension after the extension. Hence, *the pseudo-character σ_k signifies, at the infinitesimal level, the degree of arbitrariness we have when we extend an integral element of dimension $k-1$ to an integral element of dimension k.* If $\sigma_k = 0$, then the extension is unique, and at dimension $k-1$ there is exactly one direction, $\mathbf{v}_0$, in which we can extend our integral variety. If $\sigma_k = 1$, then the extension depends on one constant parameter, in other words at dimension $k-1$ there are two directions, $\mathbf{v}_0$ and $\mathbf{v}_1$ in which we can effect this extension, etc. This also shows that each pseudo-character is less than the previous one by at least one.

Now we can see the effect of adding h *generic* algebraic equations. Suppose that, before adding them, extension from dimension $k-1$ to k are possible in σ_k+1 directions, namely $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{\sigma_k}$. But after adding these equations, h of these directions become unavailable. If $\sigma_k \geq h$, then, for example by changing this basis by an equivalent one using an invertible linear transformation, we are left only with $\mathbf{v}_0, \ldots, \mathbf{v}_{\sigma_k - h}$. If $\sigma_k < h$, then extension is no longer possible (the existence of integral element of dimension $k-1$ is also in question: we need to investigate the next lower order). In short, the effect is *subtracting h from all pseudo characters* (if a pseudo character becomes negative, then we take it to no longer exist).

Using the formula of pseudo-characters in terms of characters and dimension, i.e.,

$$\begin{cases} s_0 = n - \sigma_1 - 1, \\ s_1 = \sigma_1 - \sigma_2 - 1, \\ \quad \cdots \\ s_{p-1} = \sigma_{p-1} - \sigma_p - 1, \\ s_p = \sigma_p, \end{cases}$$

we see that, if before adding the h equations, we have the series

$$s_0, \qquad s_1, \qquad \ldots, \qquad s_{k-1}, \qquad \sigma_k,$$

(σ_k can be the last Cartan character instead of the pseudo character provided we are at maximal dimension), then if $\sigma_k \geq h$, the series becomes

$$s_0, \qquad s_1, \qquad \ldots, \qquad s_{k-1}, \qquad \sigma_k - h.$$

On the other hand, if $\sigma_k < h$, then integral elements at dimension k no longer exists, at best we have

$$s_0, \qquad s_1, \qquad \ldots, \qquad s_{k-2}, \qquad s_{k-1} + \sigma_k + 1 - h,$$

and if $s_{k-1} + \sigma_k + 1 - h < 0$, the effect propagates to the next order.

Let us come to the non-generic case. Suppose when extending from dimension $k-1$ to k, the existing integral element is spanned by $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_{k-1}$, the allowable directions are spanned by $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{\sigma_k}$, and the forbidden directions are spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, …. If the addition of the algebraic equations forbids directions that are spanned by $\bar{\mathbf{w}}_1$, $\bar{\mathbf{w}}_2$, …, and say a linear combination of $\bar{\mathbf{w}}_i$ can be expressed as linear combinations of $\mathbf{w}_i$, then σ_k is reduced by a number *less* than h. (The case where a linear combination of $\bar{\mathbf{w}}_i$ is expressible in therms of linear combinations of $\mathbf{u}_i$ alone is really a generic case at a lower dimension). Hence,

The effect of adding h algebraic equations to a system with Cartan characters

$$s_1, s_2, \ldots, s_p,$$

p being the largest dimension, is to change the characters to be

$$s_1, s_2, \ldots, \sigma_q - h',$$

where $h' \le h$ and depends on the equations being added, and σ_q is the last pseudo-character $\ge h'$.

Significance for physical theories. Suppose we have constructed a *kinematical* physical theory with pseudo-character σ_k at dimension k. For this theory to be *dynamical*, we want to have a theory that still admits integral varieties at dimension k but with $\sigma_k = 0$. From the above discussion we see that, we need to add *at least* σ_k equations of motion to this system.

12 Singular solutions. Up until now we have swept all our difficulties stated at the beginning of §**6**, i.e., the rank of the various systems we consider, under the big carpet of requiring all things considered to be *generic*, *ordinary* or *regular*. The solutions for which these assumptions fail are not by any means less interesting. For completeness let us mention how we can go about searching for them, and discuss a little bit of their properties.

Suppose that at a certain dimension k, the rank of the constraints (which we write as a matrix A) on the integral elements can become less than the usual value $s_0 + s_1 +$

$\cdots + s_k$ for specific values of the variables. To study the *singular* integral varieties, we simply adjoin to our system the conditions

$$\operatorname{rank} A < s_0 + s_1 + \cdots + s_k,$$

which, despite the less sign, are actually *equality* conditions on the minors of the matrix A, i.e., they are algebraic equations. The general solutions of this new system are exactly the singular solutions of the old system.

13 Comments on analyticity. A prerequisite of the application of the Cartan–Kähler theorem is that all functions involved are *analytic*. Hence it is necessary here to discuss a few things related to the issue of analyticity and what it implies for our work.

The assumption of analyticity is exceedingly convenient, in that whenever it is assumed, statements such as "a solution depends on a certain number of variables" have precise meaning. This is not so when, for example, the functions involved are only assumed to be C^∞, since one could be suffering from the so-called identification problem: for example it is possible to glue two C^∞ functions one the real line, both of which having only bounded support, into a single function, and after such a gluing, do we still consider them to be two functions, or a single function?

Luckily, in our treatment such problems will not arise: we will be almost exclusively concerned with differential systems arising from the moving frames of equivalence problems where the variables, both dependent and independent, are the various differential invariants. Now there is no ambiguity in identifying the invariants, since in the class of problems we are considering, the equivalence problem will only have a finite-dimensional symmetry group. What this means is that the differential invariants are obtained by applying only the Frobenius theorem, and this theorem, though can be considered a special case of the Cartan–Kähler theorem, does not require analyticity.

Our (implicit) use of the Cartan–Kähler theorem starts when we formally constructs an exterior differential system involving the invariants, *where the dependent and independent variables are already clear*. We will then make statements about the Cartan characters of such systems, or what we will call the *degree of arbitrariness* such systems, which is none other than the largest non-vanishing Cartan character. Such statements only make sense when all functions are analytic, and it should *always* be understood thus in the present work. But then a question arises: if in physics the usual assumption on the functions is much weaker than analyticity, and not without good reason: while it is true that the space of analytic functions is dense, such functions have the physically undesirable property that knowledge of the function at a single point implies complete knowledge of the function everywhere, and this will seem to violate the concept of *causality* very badly.

So are statements related to the degree of arbitrariness of physical systems vacuous? Are such concepts useful at all?

The answer to this question is that the degree of arbitrariness in the analytic case acts as an excellent guide, and constitutes a very important first step, towards investigating the Cauchy problem of the physical problem. The Cauchy problem is considered solved when we have proved that, by specifying certain functions (of various differentiability) on a certain submanifold of a manifold, the equations of the physical system is completely specified, and different specification of the functions yields different states of the system. Since analytic functions are always "sufficiently differentiable", when our result states that in the analytic case the general solution depends on x functions of y variables, every solution of the Cauchy problem must be within the scope of the analytic solution and cannot lie without it. Thus, after we have determined that in the analytic case specifying certain functions on a certain submanifold is sufficient, the general Cauchy problem is now well-formulated in that it has already become a problem in search of proof (or disproof) that such conditions are also sufficient for the general case, and we are no longer required to guess the necessary initial data. This is the meaning and use of our claims of "degrees of arbitrariness" of physical systems.

It should also be noted that in most cases, the solution in the analytic case and the solution in the general case under certain assumptions are exactly the same. The counter-examples are specifically constructed, and does not have much physical significance. And since all data we collect are valid only to a finite precision, physically the specification of "sufficiently differentiable" initial date is most usually indistinguishable from the specification of analytic initial data.

ii. Independence conditions. Involution

14 Independence condition. Essential torsion. The solutions guaranteed by the Cartan–Kähler theorem does not respect our requirement of *independent variables.* To make progress, let us define a differential system with independence condition as a differential system for which we require that solutions must keep certain one-forms ω_1, ω_2, …, ω_m independent. This amounts to

$$\omega_i \wedge \omega_j \wedge \cdots \wedge \omega_k \neq 0, \qquad i, j, \ldots, k \text{ all distinct}$$

on solutions for any choice of any numbers of ω_i, ω_j, …. If we require the *coordinates* x_1, x_2, …, x_m to be independently, we simply take the independent forms to be dx_1, dx_2, …, dx_m, and the vectors spanning the integral elements we look for must be of the form

$$\frac{\partial}{\partial x_a} + \sum_{i=m+1}^{n} B_a^i \frac{\partial}{\partial z_i}, \qquad a = 1, \ldots, m. \tag{1.2}$$

Immediately, we see that we *must not* end up with equations of the form

$$0 = \Omega^{(k)} = A_{ij\ldots k}\, \omega_i \wedge \omega_j \wedge \cdots \wedge \omega_k. \tag{1.3}$$

Unless $A_{ij\ldots k} = 0$, no solution will satisfy the independence conditions. Such terms are called *essential torsion*. To proceed in such cases, we need to add to our differential system the algebraic equations

$$A_{ij\ldots k} = 0.$$

Such equations might have no solution: for example when $A_{ij\ldots k}$ are non-zero constants. We say that such a differential system is *incompatible*.

15 Involution. Reduced characters. From now on, without loss of generality, we consider only systems with no essential torsion. Our aim is to obtain solutions satisfying the independence conditions as *general solutions* of the problem without independence conditions. Hence, we will call differential systems for which the constraints on ordinary m dimensional integral elements *do not require any linear relations* among the independence conditions $\omega_1, \ldots, \omega_m$ *involutive systems*. This does not mean that all ordinary m dimensional integral elements satisfy the independence conditions: it means that *almost all* satisfy, and *almost all* integral elements spanned by the vectors of the form (1.2) are ordinary, since the conditions for otherwise are both equality conditions. For involutive systems, we can obtain the integral varieties we want by applying the Cartan–Kähler theorem to ordinary integral elements of the form (1.2).

The requirement that none of the constraints on integral elements can involve any relations for ω_i means that, in calculating the Cartan characters, we can ignore all directions that correspond to ω_i. For example, if we have the two forms

$$\Omega^\alpha = A^\alpha_{ij}\omega_i \wedge \omega_j + B^\alpha_{ia}\omega_i \wedge \pi_a + C^\alpha_{ab}\pi_a \wedge \pi_b$$

and in calculating the Cartan character, the first direction is chosen as the vector $\mathbf{I}_1$ dual to ω_1. With this direction, we have

$$\Omega^\alpha(\mathbf{I}_1) = A^\alpha_{1j}\omega_j + B^\alpha_{1a}\pi_a.$$

The requirement that we must not have any constraints for the independent directions (i.e., no relations of the form $c_i\omega_i = 0$) means that the rank of the system $(A^\alpha_{1j}, B^\alpha_{ia})$

is the same as the rank of the system (B^α_{ia}). Notice we do this only *after* using the tangent directions already found.

Let us call the numbers

$$s'_0, \qquad s'_1, \qquad s'_2, \qquad \dots \qquad s'_{m-1}$$

which are calculated by omitting all terms corresponding to ω_i in the calculation for Cartan characters the *reduced Cartan characters*: this definition holds for both involutive and non-involutive system. The *reduced pseudo-character* σ'_m is defined as

$$s'_0 + s'_1 + \cdots + s'_{m-1} + \sigma'_m = n - m.$$

Involutive system has the normal characters and reduced characters equal. Conversely, if all reduced characters are equal to the non-reduced counterparts, this implies there is no relation among the independent variables, and hence the system is involutive. We therefore have a necessary and sufficient condition for involutive systems: the equality of the reduced and normal characters.

16 Cartan's test. Involutivity theorem. It is troublesome to calculate two sets of characters, especially the non-reduced ones. We need to find a simpler criterion.

In §**9**, we calculated that the number of free *functions* we can specify in order to determine an integral variety. But we can also calculate the number of free *parameters* we can specify in order to extend a point to a k-dimensional integral *element*. When choosing the first vector, there are s_0 constraints; when choosing the second, there are $s_0 + s_1$ constraints, etc., so in total we have $ks_0 + (k-1)s_1 + \cdots + s_{k-1}$ constraints. On the other hand, on our k-dimensional integral varieties, the coordinates $x_1, \dots, x_k$ are independent, and all the rest coordinates, which we denote by $z_{k+1}, z_{k+2}, \dots, z_n$, are dependent on them. Hence we can write $dz_i = B^i_a dx_a$ for $(n-k)k$ parameters B^i_a. Therefore, the number N of free parameters of B^i_a is

$$(n-k)k - ks_0 + (k-1)s_1 + \cdots + s_{k-1} = s_1 + 2s_2 + \cdots + (k-1)s_{k-1} + k\sigma_k.$$

On the other hand, *there is another way that we can obtain the number of free parameters, without calculating the Cartan characters.* In the involutive case, we can write

$$\varpi_\alpha = A_{\alpha i}\omega_i \tag{1.4}$$

and substitute this into our system. The number of free parameters N in $A_{\alpha i}$ is the kernel of the linear system after this substitution. This number is obviously

$$N = s'_1 + 2s'_2 + \cdots + (k-1)s'_{k-1} + k\sigma'_k. \tag{1.5}$$

Such a calculation for the free parameters using equation (1.4) can be done regardless of whether the system is involutive or not. It can be shown that, if the system is not involutive, then

$$N < s'_1 + 2s'_2 + \cdots + (k-1)s'_{k-1} + k\sigma'_k.$$

Hence the arithmetic test (1.5), called *Cartan's test*, can be used to check for involutivity (Cartan's involutivity theorem).

Remark. In the equations of the differential system, any terms that are formed with the wedge products of only the independent forms ω_i are called *torsion*. The *apparent torsion* are those that can be cancelled using a suitable decomposition (1.4), the rest are the *essential torsion*. The solvability of the differential system requires that all torsion must be apparent. In calculating the reduced characters, the torsion terms do not contribute since ultimately they will become linear in ω_i, but for non-involutive systems such a calculation will eventually lead to singular solutions or incompatibility.

17 The weaker Cartan's test and involutivity theorem. Calculating reduced characters is easier than calculating the normal characters, but still a pain: we need to ensure that the integral element (1.2) we specify is generic, meaning that the rank of the various systems are maximal. Now assume that we calculate the reduced characters by ignoring the step of checking the rank condition, and denote these "reduced characters" by double primes. By definition we have

$$\begin{aligned} s''_0 &\le s'_0, \\ s''_1 + s''_0 &\le s'_0 + s'_1, \\ &\dots, \\ s''_{m-1} + \cdots + s''_1 + s''_0 &\le s'_0 + s'_1 + \cdots + s'_{m-1}. \end{aligned}$$

Adding these all together yields

$$ms''_0 + (m-1)s''_1 + \cdots + s''_{m-1} \le ms'_0 + (m-1)s'_1 + \cdots + s'_{m-1}.$$

Now assume that the "reduced characters" s_i actually satisfy Cartan's test (1.5), i.e., the left hand side of the above equation gives the number of constraints on the free parameters. We know that the right hand side gives a lower bound for this number, and hence we also have

$$ms''_0 + (m-1)s''_1 + \cdots + s''_{m-1} \ge ms'_0 + (m-1)s'_1 + \cdots + s'_{m-1}.$$

The two sides are hence equal, and $s''_i = s'_i = s_i$. If we introduce the "reduced pseudo character" σ''_m as well, we also have $\sigma''_m = \sigma'_m = \sigma_m$.

Cartan's test in which we do not care about the rank conditions is called *weaker Cartan's test*, and the result that if weaker Cartan's test passes then the system is involutive is called the *weaker involutivity theorem*, weaker since it is only a sufficient condition.

Note about notation. In the following (except the next section about prolongation) we will be using exclusively the weaker Cartan's test. It is then rather cumbersome to carry the two primes on the characters around, and from now we will omit them. We will also call the (pseudo) characters with double primes simply Cartan (pseudo) characters.

18 Prolongation. That a system is not involutive means that the solutions with independence conditions are not to be found in the general solutions of the original problem. It does not necesarrily mean that there is no such solutions. In particular, we know where to look for them next: the singular solutions.

We already have a set of conditions any integral elements should satisfy. Bur for singular solutions, there are extra conditions. As usual, we should add these extra conditions as *algebraic* equations for our system and start anew. But there is a problem: the extra conditions we want to ensure are conditions on the *free parameters*, not on any variables.

The way we proceed is that we add the equation (1.4) to our set of equation

$$\varpi_\alpha = A_{\alpha i}\omega_i,$$

in which the $A_{\alpha i}$ are regarded as *new variables*, the defining equations of which are exactly those that under substitution into the original equations, the original equations being satisfied identically. This can make some of the original equations redundant: in particular, if the original equations consists only of one forms and two forms which are derivatives of the one forms, then after the addition of this new equation, we can discard all the original two form equations. But then we have a new two form equation to add to our system, namely

$$d\varpi_\alpha = dA_{\alpha i} \wedge \omega_i,$$

and for the system which originally consists of only one forms and their derivatives, *this is now the only equations involving forms of order higher than* 1, and hence the only source of the Cartan characters s_1, s_2, ..., s_{m-1} (s_0 is just the number of independent one forms equations). We can now apply Cartan's test to this new system: if this system is involutive, then we only have to look for the general solutions of this new system.

Is it guaranteed that the system, after prolongation, will be involutive? No, but we can prolong again, unless the system obtained is *incompatible*, in which case there are no solutions. Could it happen that a system prolongs indefinitely without being either incompatible or involutive? This is the subject of Cartan–Kuranishi theorem, and the answer is a qualified "no": in all "well-behaving" cases, after finitely many prolongations, the systems will become either involutive or incompatible [5].

More generally, we can promote only some of the free parameters as new variables: this sometimes arrives at involutive systems quicker.

19 Complete prolongations of involutive systems. Even though it seems a bit odd at this moment, we will see later that it will be useful for us to prolong systems that are *already involutive* in order to obtain more variables to work with. We consider only a specific case: *complete* prolongation of a *Pfaffian system.* We will also assume that the integral varieties with independence conditions are the ordinary integral varieties of the largest dimension, so the pseudo-character is just the last character. Then it is possible to prove the following theorem [7]:

The complete prolongation of an involutive system is also involutive, and the new Cartan characters (denoted by stars) are related to the old ones by

$$\begin{aligned} s_1^\star &= s_1 + s_2 + \cdots + s_{m-1} + s_m, \\ s_2^\star &= \phantom{s_1 + {}} s_2 + \cdots + s_{m-1} + s_m, \\ &\cdots \\ s_{m-1}^\star &= s_{m-1} + s_m, \\ s_m^\star &= s_m. \end{aligned}$$

In particular, the degree of arbitrariness of the prolonged system is equal to that of the old system.

20 An example of calculation using Cartan's test. After these rather abstract theories let us try a concrete example, which we will also revisit using another method later on.

Consider the system of partial differential equations of a single variable $u = u(x, y, z)$ defined on $\mathbb{R}^3$

$$u_{xx} = u_{yy} = u_{zz}.$$

Following the general procedure for transforming partial differential equations into exterior differential equations, the space we are working with is hence formed by the following variables:

- x, y, z (3 variables);

- u (1 variable);
- u_x, u_y, u_z (3 variables);
- $u_{xx} = u_{yy} = u_{zz}, u_{xy} = u_{yx}, u_{xz} = u_{zx}, u_{yy}, u_{yz} = u_{zy}$ (4 variables).

So the differential equation is defined on a 11-dimensional space formed by the variables above. The solution we seek for is a three-dimensional integral manifold for which $dx \wedge dy \wedge dz \neq 0$, i.e., there are 8 variables that need to become dependent on x, y, z. The contact forms are

$$\begin{cases} \omega_u = du - u_x dx - u_y dy - u_z dz \\ \omega_x = du_x - u_{xx} dx - u_{xy} dy - u_{xz} dz \\ \omega_y = du_y - u_{xy} dx - u_{yy} dy - u_{yz} dz \\ \omega_z = du_z - u_{xz} dx - u_{yz} dy - u_{zz} dz \end{cases}$$

which are set to zero. Recall that the zeroth Cartan character s_0 is the number of equations for which any *linear* integral elements of the system must be satisfied while ignoring the dx, dy, dz, i.e., it is the rank of the matrix

$$\begin{pmatrix} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

derived from the contact one-forms, where the top row is the label and is not part of the matrix. Obviously, this number is always equal to the number of independent one-form equations above (since we are not allowed to have linear dependence among the independent variables), and here we have $s_0 = 4$, even without forming the matrix explicitly.

Under exterior differentiation we have

$$\begin{cases} d\omega_u = -du_x \wedge dx - du_y \wedge dy - du_z \wedge dz \\ d\omega_x = -du_{xx} \wedge dx - du_{xy} \wedge dy - du_{xz} \wedge dz \\ d\omega_y = -du_{xy} \wedge dx - du_{yy} \wedge dy - du_{yz} \wedge dz \\ d\omega_z = -du_{xz} \wedge dx - du_{yz} \wedge dy - du_{zz} \wedge dz \end{cases}$$

We need to use the equations $\omega_u = \omega_x = \omega_y = \omega_z = 0$, which give expressions for

du, du_x, du_y, du_z to simplify these equations. After this is done, we obtain

$$\begin{cases} d\omega_u = 0 \\ d\omega_x = -du_{xx} \wedge dx - du_{xy} \wedge dy - du_{xz} \wedge dz \\ d\omega_y = -du_{xy} \wedge dx - du_{xx} \wedge dy - du_{yz} \wedge dz \\ d\omega_z = -du_{xz} \wedge dx - du_{yz} \wedge dy - du_{xx} \wedge dz \end{cases}$$

If we want to do things really carefully, then for s_1, we give dx, dy, dz the values $\delta_1 x$, $\delta_1 y$ and $\delta_1 z$ respectively, and $s_0 + s_1$ is the rank of the matrix

$$\begin{pmatrix} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 x & \delta_1 y & 0 & \delta_1 z \\ 0 & 0 & 0 & 0 & \delta_1 y & \delta_1 x & \delta_1 z & 0 \\ 0 & 0 & 0 & 0 & \delta_1 z & 0 & \delta_1 y & \delta_1 x \end{pmatrix}$$

and we have $s_0 + s_1 = 7$, hence $s_1 = 3$. For s_2, we form the matrix

$$\begin{pmatrix} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 x & \delta_1 y & 0 & \delta_1 z \\ 0 & 0 & 0 & 0 & \delta_1 y & \delta_1 x & \delta_1 z & 0 \\ 0 & 0 & 0 & 0 & \delta_1 z & 0 & \delta_1 y & \delta_1 x \\ 0 & 0 & 0 & 0 & \delta_2 x & \delta_2 y & 0 & \delta_2 z \\ 0 & 0 & 0 & 0 & \delta_2 y & \delta_2 x & \delta_2 z & 0 \\ 0 & 0 & 0 & 0 & \delta_2 z & 0 & \delta_2 y & \delta_2 x \end{pmatrix}$$

The rank is 8, so $s_2 = 1$. Note that the matrix has already attained its maximal rank, so $s_3 = 0$. Obviously if we remove the first four columns, which can be non-zero only for the first four rows since we must enforce the one-form equations, then we can calculate more easily the numbers s_1, $s_1 + s_2$, etc., which correspond to the ranks of the series of matrices stacked together.

Of course, we do not really need to do it so laboriously, since instead of applying Cartan's test we can apply the weaker Cartan's test. This corresponds to setting

$\delta_1 y = \delta_1 z = \delta_2 x = \delta_2 z = 0$ above, and then the characters can be read of directly from the two-form equations as

$$s_1 = 3 \quad (du_{xx}, du_{xy}, du_{xz}), \qquad s_2 = 1 \quad (du_{yz}), \qquad s_3 = 0.$$

s_1 corresponding to the independent forms which multiply dx, shown in parentheses, etc. Note that this shortcut works also when we have higher order forms in our equations *as long as* the differentials of dependent variables enter only linearly, i.e., we *do not* have terms such as

$$du_{xx} \wedge du, \qquad dx \wedge dy \wedge du_x \wedge du_y.$$

If such forms are present, we cannot use this shortcut and the calculation of even the reduced characters become very difficult, since first we need to find the general 3-dimensional linear elements, which is already more difficult since now there would be quadratic or higher order relations among the parameters, and then calculate using the elements, the steps of calculation required being roughly quadratic in the total number of variables. Actually in such a case, unless the number of variables is exceedingly small, a better way to proceed is to immediately effect a prolongation so as to get rid of all of the original higher order form equations, and the new systems is guaranteed to include only linear forms in the dependent variables.

Let us see what is the number N of free parameters in an integral element. We write

$$\begin{cases} du_{xx} = u_{xxx}dx + u_{xxy}dy + u_{xxz}dz \\ du_{xy} = u_{xyx}dx + u_{xyy}dy + u_{xyz}dz \\ du_{xz} = u_{xzx}dx + u_{xzy}dy + u_{xzz}dz \\ du_{yz} = u_{yzx}dx + u_{yzy}dy + u_{yzz}dz \end{cases} \tag{1.6}$$

There are 12 parameters in the above expression. Substituting this back to the two form equations, we see that the free parameters are

$$\begin{cases} u_{xxx} = u_{xyy} = u_{xzz} \\ u_{xxy} = u_{xyx} = u_{yzz} \\ u_{xxz} = u_{xzx} = u_{yzy} \\ u_{xyz} = u_{xzy} = u_{yzx} \end{cases}$$

so here $N = 4$. Actually, even this substitution is unnecessary, since it is obvious that the free parameters are just the independent third order partial derivatives of u. We have

$$N = 4 < s_1 + 2s_2 + 3s_3 = 5,$$

so Cartan's test fails, the system is *not* involutive and prolongation is necessary. We can also see where things could go wrong as we laboriously wrote down the matrices: the real characters would correspond to matrices whose top-row labels include also dx, dy, dz. Now already at s_2, the rank of the reduced polar matrix is already constrained by the number of columns, and if we have more columns the rank could grow further, and consequently imply linear dependence among dx, dy, dz, which at the same time will imply the existence of further constraints on the free parameters in the integral element than implied by the counting of reduced Cartan characters. If this happens, which is the present case, it shows that the reduced character and real characters are not equal, and we are not in the involutive case. Prolongation corresponds, on the other hand, adding to the labels du_{xx}, du_{xy}, du_{xz}, du_{yz}, so we will not be constrained by the number of columns so soon.

Now, for prolongation we take $u_{xxx}, u_{xxy}, u_{xxz}, u_{xyz}$ to be the new variables, adjoining (1.6) to the list of one-form equations (hence for the prolonged system, $s_0 = 4+4 = 8$), and we need to differentiate (1.6) to get some new two-form equations (the original one are now all identities). We now have $11 + 4 = 15$ variables, and the number of variables that we want to get rid of is 12. We have

$$\begin{cases} d^2u_{xx} = du_{xxx} \wedge dx + du_{xxy} \wedge dy + du_{xxz} \wedge dz \\ d^2u_{xy} = du_{xxy} \wedge dx + du_{xxx} \wedge dy + du_{xyz} \wedge dz \\ d^2u_{xz} = du_{xxz} \wedge dx + du_{xyz} \wedge dy + du_{xxx} \wedge dz \\ d^2u_{yz} = du_{xyz} \wedge dx + du_{xxz} \wedge dy + du_{xxy} \wedge dz \end{cases}$$

For this new system, the reduced characters are

$$s_1 = 4 \quad (du_{xxx}, du_{xxy}, du_{xxz}, du_{xyz}), \qquad s_2 = 0, \qquad s_3 = 0.$$

Again here $s_0 + s_1 + s_2 + s_3 = 12$, the number of dependent variables, as it should be.

For completeness, we give the polar matrix for calculating s_2 of which we have removed the columns corresponding to du, du_x, du_y, du_z, du_{xx}, du_{xy}, du_{xz}, du_{yz}:

$$\begin{pmatrix} du_{xxx} & du_{xxy} & du_{xxz} & du_{xyz} \\ \hline \delta_1 x & \delta_1 y & \delta_1 z & 0 \\ \delta_1 y & \delta_1 x & 0 & \delta_1 z \\ \delta_1 z & 0 & \delta_1 x & \delta_1 y \\ 0 & \delta_1 z & \delta_1 y & \delta_1 x \\ \delta_2 x & \delta_2 y & \delta_3 z & 0 \\ \delta_2 y & \delta_2 x & 0 & \delta_3 z \\ \delta_2 z & 0 & \delta_2 x & \delta_3 y \\ 0 & \delta_2 z & \delta_2 y & \delta_3 x \end{pmatrix}$$

s_0 is the number of one-form equations, which is 7; s_1 the rank of the first 4 rows, and $s_1 + s_2$ the rank of the whole matrix.

For the parameters,

$$\begin{cases} du_{xxx} = u_{xxxx}dx + u_{xxxy}dy + u_{xxxz}dz \\ du_{xxy} = u_{xxyx}dx + u_{xxyy}dy + u_{xxyz}dz \\ du_{xxz} = u_{xxzx}dx + u_{xxzy}dy + u_{xxzz}dz \\ du_{xyz} = u_{xyzx}dx + u_{xyzy}dy + u_{xyzz}dz \end{cases}$$

again there are 12 of them. The free ones can be algorithmically obtained by substituting these expressions into the two form equations, and we have

$$\begin{cases} u_{xxxx} = u_{xxyy} = u_{xxzz} \\ u_{xxxy} = u_{xxyx} = u_{xyzz} \\ u_{xxxz} = u_{xxzx} = u_{xyzy} \\ u_{xxyz} = u_{xxzy} = u_{xyzx} \end{cases}$$

so the number of *free* parameters is $N = 4$. Again, this substitution can be avoided by noting that the free parameters are just the independent fourth order partial derivatives of u. Now

$$N = 4 = s_1 + 2s_2 + 3s_3,$$

so Cartan's test is satisfied, the system is involutive, and the general solution of the differential equation depends on 4 functions of 1 variables, by the Cartan–Kähler theorem.

III. The theory of equivalence

21 Introduction. We are now ready to discuss equivalence problems. Roughly speaking, the theory of involutive systems are suitable for discussing the Cauchy problems and degrees of arbitrariness of physical theories. The theory of equivalence, on the other hand, enables us to write down the kinematics of physical theories. The method of moving frame can be thought of a particular application of the theory of equivalence. The theory of equivalence does not depend on the choice of coordinates. In calculations, if coordinates are needed, it can produce its own.

22 Equivalence of coframe. Method of graph. Our first task is understanding, given two co-frames $\theta_1, \ldots, \theta_n$ and $\bar{\theta}_1, \ldots, \bar{\theta}_n$ defined on two differential manifolds

M and $\bar{M}$ of dimension n, when they are "locally the same", meaning there is a map $f : M \to \bar{M}$ such that

$$f^*\bar{\theta}_i = \theta_i.$$

Since it is usually obvious, we will omit pullback signs from now on and just write $\theta_i = \bar{\theta}_i$.

We can treat this equivalence problem as the integrability of the form

$$\vartheta_i = \bar{\theta}_i - \theta_i$$

in the space $\bar{M} \times M$. We are interested in the m-dimensional integral manifolds which are *transverse*, meaning that either θ_i or $\bar{\theta}_i$ can be taken as the independent conditions. This is the *method of graph*, i.e., the solution of the problem is visualised as a bijective graph in the product space.

As we are dealing with coframes, we calculate their exterior derivatives

$$d\theta_i = c_{ijk}\theta_j \wedge \theta_k, \qquad d\bar{\theta}_i = \bar{c}_{ijk}\bar{\theta}_j \wedge \bar{\theta}_k,$$

so

$$d\vartheta_i = \bar{c}_{ijk}\bar{\theta}_j \wedge \vartheta_k + \bar{c}_{ijk}\vartheta_j \wedge \theta_k + (\bar{c}_{ijk} - c_{ijk})\theta_j \wedge \theta_k$$

The existence of integral manifolds then requires

$$\bar{c}_{ijk} = c_{ijk}, \tag{1.7}$$

i.e., the system must be Frobenius integrable.

The simplest case is where $\bar{c}_{ijk}$ and c_{ijk} are both constants. If they are equal, then the two systems are equivalent, otherwise (including the case where one set are constants while the other are not) they are not equivalent. *They cannot be arbitrary constants*: in this case θ_i and their bared versions must be locally equivalent to Maurer–Cartan forms on a Lie group and the Jacobi identity for the structure constants must hold. We even get the symmetry group of the problem, namely the Lie group whose structural constants are c_{ijk}.

If $\bar{c}_{ijk}$ and c_{ijk} are not constants, then we require $f^*\bar{c}_{ijk} = c_{ijk}$, but for functions under diffeomorphism we cannot directly compare them. Worse, even though we can implicitly define the submanifold of $\bar{M} \times M$ by the relation $\bar{c}_{ijk} = c_{ijk}$, it is far from certain that restricted to this submanifold what will happen to θ_i and $\bar{\theta}_i$, one set of which we require to be independent.

To make progress, we treat (1.7) as a new condition and adjoin it to our conditions for equivalence (prolongation). The closure of our system now includes the condition

$$d\bar{c}_{ijk} = dc_{ijk}, \qquad \text{or} \qquad \bar{c}_{ijk;l} = c_{ijk;l},$$

where the *coframe derivative* for a function h is given by the expansion

$$dh = \sum h_{;i}\theta_i.$$

Since θ_i forms a coframe. In general this derivative does not commute: $h_{;ij} \neq h_{;ji}$.

Let us first investigate the case where c_{ijk} and $\bar{c}_{ijk}$ both contain m independent functions among them each. Let us denote these by

$$I_1, \quad I_2, \quad \ldots \quad I_m; \qquad \bar{I}_1, \quad \bar{I}_2, \quad \ldots \quad \bar{I}_m,$$

and the rest of the quantities are expressible as functions of them. Being functional independent, their differentials are linearly independent, or the matrix $I_{i;j}$ in the following expression is invertible:

$$dI_i = I_{i;j}\theta^j.$$

In this case, $c_{ijk} = \bar{c}_{ijk}$ implies $I_i = \bar{I}_i$, which in turn implies $dI_i = d\bar{I}_i$ and hence $\theta_i = \bar{\theta}_i$. Observe that we do not even need to check the equality for the terms other than I_i. Indeed, let us differentiate the above equation. We get:

$$0 = I_{i;jk}\theta^k \wedge \theta^j + I_{i;j}c_{jkl}\theta^k \wedge \theta^l$$

so $I_i = \bar{I}_i$ implies $\theta_i = \bar{\theta}_i$ and in turn these two equations together implies the equality of everything else. Hence, in this case, we only need to check

$$I_i = \bar{I}_i, \qquad I_{i;j} = \bar{I}_{i;j}.$$

In the general case, what we do is we use the coframe derivative to differentiate the *fundamental invariants* C_{ijk} until we get no more functional independent quantities. For example, in this way we may obtain a system

$$\bar{\theta}_i = \theta_i, \qquad \bar{c}_{ijk} = c_{ijk}, \qquad \bar{c}_{ijk;l} = c_{ijk;l}, \qquad \bar{c}_{ijk;lm} = c_{ijk;lm}, \qquad \bar{c}_{ijk;lmn} = c_{ijk;lmn},$$

where $c_{ijk;lmn}$ introduce no new functionally independent quantities. These conditions are obviously necessary. They are also sufficient for equivalent, as it can be easily checked that they imply the Frobenius integrability of the system

$$\bar{\theta}_i = \theta_i, \qquad \bar{c}_{ijk} = c_{ijk}, \qquad \bar{c}_{ijk;l} = c_{ijk;l}, \qquad \bar{c}_{ijk;lm} = c_{ijk;lm},$$

and the integral manifold really is transverse.

23 Shadowing. In the above discussion we have *two* similar systems. In applications we usually only have one system, with *unspecified* variables as parameters. We then *shadow* the system by postulating second system with the same variables and study the equivalence problem for these two systems. The invariants we obtained are defined for each of the two systems separately: these invariants classify the problem.

24 Equivalence under prescribed symmetry group. The preceding consideration allows us to study the equivalence problem of coframes defined on a manifold. However, in many cases this is not the way the problem is presented to us. Often we are not interested in the one-one correspondence of coframes but more general ones, and sometimes we do not have a complete coframe. We will now investigate how we can solve the first problem. After that, the second problem is easy.

Let us state the problem of equivalence of coframes *under a prescribed symmetry group*. We still have θ and $\bar{\theta}$, but now we consider them to be equivalent as long as

$$f^*\bar{\theta}_i = r_{ij}\theta_j, \qquad r_{ij} \in G$$

where r_{ij} belongs to a linear representation of some group G.

It is more convenient if we symmetrize the problem so that there is no difference between the barred and unbarred quantities:

$$\bar{r}_{ij}\bar{\theta}_j = r_{ij}\theta_j, \qquad r_{ij}, \bar{r}_{ij} \in G. \tag{1.8}$$

We then consider the equivalence problem on the product space, locally $M \times G$, by setting

$$\omega_i = r_{ij}\theta_j$$

to be the extended forms and similarly for the barred quantities. The equivalence condition becomes simply

$$\bar{\omega}_i = \omega_i$$

but we are short of a coframe since the number of forms ω_i is equal to the dimension of M only. Nonetheless let us differentiate the above to see what we get. For the unextended forms

$$d\theta_i = c_{ijk}\theta_j \wedge \theta_k$$

and similarly for the barred version. For the extended version,

$$\begin{aligned} d\omega_i &= dr_{ij}(r^{-1})_{jk} \wedge (r_{kl}\theta_l) + r_{ij}c_{jkl}(r^{-1})_{km}(r^{-1})_{ln}(r_{mp}\theta_p) \wedge (r_{nq}\theta_q) \\ &= \alpha_{ij} \wedge \omega_j + T_{ijk}\omega_j \wedge \omega_k \end{aligned}$$

where α_{ij} is the Maurer–Cartan form on G, and

$$T_{ijk} = r_{il}c_{lmn}(r^{-1})_{jm}(r^{-1})_{kn}$$

is just the structural functions extended to the product space.

The group parameters r_{ij} and $\bar{r}_{ij}$ are undetermined parameters. This means that as long as we can find a choice for them for which (1.8) holds, we can claim equivalence. On the other hand, their introduction gives us $\dim G$ more dimensions which must be taken care of when we finally solve the problem. Now it is perfectly possible we can use some or all of theses dimensions to set the extended structural functions T_{ijk} to convenient values. If this procedure at the same time uses up all the extra degrees of arbitrariness, then ω_i with determined values of r_{ij} still form a coframe on M and we can use our procedure in the last section. Hence, our aim is to set some or all components of T_{ijk} and $\bar{T}_{ijk}$ to the same chosen constant value by choosing r_{ij} and $\bar{r}_{ij}$.

However, this is a problem with this, namely *we do not have complete control of some of the components of the T_{ijk} on the algebraic level*, even if these components depend explicitly on r_{ij}. To see what this means, suppose by choosing $r_{ij} = r_{ij}^{(0)}$ we set T_{123}, which was a variable quantity, to the constant value 1. However, since we have chosen $r_{ij} = r_{ij}^{(0)}(x)$ where x are the coordinates on M, the Maurer–Cartan form α_{ij} now descends to the base:

$$\alpha_{ij} = a_{ijk}\omega_k$$

and now

$$\alpha_{ij} \wedge \omega_j = a_{ijk}\omega^k \wedge \omega^j$$

then even though we have set T_{123} to 1, we get another term which practically has the same effect: a_{123}, and there is no reason at all $a_{123} = \bar{a}_{123}$ on the two manifolds. The quantities a_{ijk} depend on how we choose the group parameters r_{ij} in a neighbourhood. Hence we need to exclude all such forms from consideration for now and concentrate only on the terms that we have full *algebraic* control (compare this with the procedure of *absorption of torsion* in the theory of exterior differential systems: the coefficients of $\omega^k \wedge \omega^j$ are also called *torsion*).

Hence, let the indices ij of a_{ijk} has the same symmetry as the representation of the Lie algebra we are considering. We are interested in solving the following equation

$$a_{ijk} - a_{ikj} = 2\tilde{T}_{ijk} \tag{1.9}$$

where $\tilde{T}_{ijk}$ is the largest possible linear subspace of T_{ijk} for which this equation can be solved and these are the components of T_{ijk} are can be influenced by the differentials of r_{ij}. The remaining

$$U_{ijk} = T_{ijk} - \tilde{T}_{ijk}$$

we have total control and can be safely set to constants we like. We then get

$$d\omega_i = \varpi_{ij} \wedge \omega_j + U_{ijk}\omega_j \wedge \omega_k$$

where ϖ_{ij} are the Maurer–Cartan forms plus some linear combinations of $T_{ijk}\omega^k$. In equation (1.9), the dimension of the kernel of the homogeneous equation

$$a_{ijk} - a_{ikj} = 0$$

is called the *degree of indeterminacy* $r^{(1)}$ of the problem (similar to the number of free parameters in the theory of exterior differential systems: see §**26**). If we succeed in specifying all components of r_{ij} and $r^{(1)} = 0$, we have a well-defined coframe equivalence problem without any symmetry group. The problem can then be solved by the previous equivalence method for systems without any symmetry.

25 Cartan–Kähler theorem in equivalence problems. On the other hand, if we have a partially determined co-frame ω_i on M, this teaches us how we should choose our initial frame θ_i on the manifold M so that the symmetry group G is smaller than our first attempt. We should then with this better choice of θ_i carry out the whole procedure again. The group G may then be reduced even further. Eventually, we will arrive at a system where we do not have any more U_{ijk} to play with. But we may still have a non-trivial group G.

To recap, we are working in the space $M \times G$ and have the system

$$d\omega_i = \varpi_{ij} \wedge \omega_j + U_{ijk}\omega_j \wedge \omega_k$$

where now U_{ijk} are *constants*. The degree of indeterminacy is $r^{(1)}$. For the equivalence problem, we stack two copies of these together and work in $M \times G \times \bar{M} \times \bar{G}$. The equivalence condition is the vanishing of the forms

$$\vartheta_i = \omega_i - \bar{\omega}_i,$$

but now we are looking for a m-dimensional submanifold in $M \times G \times \bar{M} \times \bar{G}$ with the independence conditions $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \neq 0$ or $\bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \cdots \wedge \bar{\omega}_n \neq 0$. The constants U_{ijk} and $\bar{U}_{ijk}$ are now manifestly the same. Let us calculate $d\vartheta_i$:

$$\begin{aligned} d\vartheta_i &= \varpi_{ij} \wedge \omega_j + U_{ijk}\omega_j \wedge \omega_k - \bar{\varpi}_{ij} \wedge \bar{\omega}_j - U_{ijk}\bar{\omega}_j \wedge \bar{\omega}_k \\ &\equiv (\varpi_{ij} - \bar{\varpi}_{ij}) \wedge \omega_j \pmod{\vartheta_i}. \end{aligned}$$

So in addition to the equations $\vartheta_i = 0$ we also need the condition of two-forms

$$(\varpi_{ij} - \bar{\varpi}_{ij}) \wedge \omega_j = 0.$$

The dimension of the integral varieties we are looking for now is smaller than the codimension of the forms ϑ_i: here it is not necessary that we have a Frobenius-integrable system and we need to resort to the Cartan–Kähler theorem.

26 Cartan characters without explicit shadowing. The Cartan characters we need are with respect to the "shadowed" version of the space $M \times G$, but due to the special form of the problem, it is not necessary to explicitly carry out the process of shadowing in order to obtain them.

We have a m-dimensional space M and a r-dimensional Lie group G. By shadowing, our total space is $M \times \bar{M} \times G \times \bar{G}$. In our case the coframe is

ω^i	independent conditions,
$\vartheta^i = \bar{\omega}^i - \omega^i$	with equations $\vartheta^i = 0$,
$\varpi^\alpha = \pi^{*\alpha} - \pi^\alpha$	with equations $d\vartheta^i = A^i{}_{j\alpha}\pi^\alpha \wedge \omega^j + T^i{}_{jk}\omega^j \wedge \omega^k = 0$,
$\varpi^{*\alpha} = \pi^{*\alpha} + \pi^\alpha$	with no equations.

We can calculate the series of Cartan characters

$$s_0 = m, \qquad s_1, \qquad s_2, \qquad \dots, \qquad s_{m-1}.$$

Using the criteria without introducing the reduced character, the involutive condition is that the number of constraints on the free variables are

$$ms_0 + (m-1)s_1 + \dots + s_{m-1}.$$

On the integral manifold, ϑ^i, ϖ^α and $\varpi^{*\alpha}$ are all expressed linearly in terms of ω^i. This gives a total of $m^2 + 2rm$ variables. But we must have $\vartheta^i = 0$, so this gives immediately m^2 constraints. As for ϖ^α, the number of constraints is by assumption $mr - r^{(1)}$, i.e., the number of free variables. Finally the parameters for $\varpi^{*\alpha}$ is not subject to any constraint. Hence the total number of constraints is $m^2 + rm - r^{(1)}$, which gives the involutive condition

$$ms_0 + (m-1)s_1 + \dots + s_{m-1} = m^2 + rm - r^{(1)}.$$

Using $s_0 = m$, we get

$$(m-1)s_1 + \dots + s_{m-1} = rm - r^{(1)},$$

and comparing with the general theory, we see that a more convenient choice of the "reduced character" is

$$s_1 + s_2 + \dots + \tilde{\sigma}_m = r.$$

which gives the involutive condition

$$s_1 + 2s_2 + \dots + n\tilde{\sigma}_m = r^{(1)}.$$

On the other hand, since the only two-form in our theory is

$$d\vartheta^i = A^i{}_{j\alpha}(\varpi^\alpha - \bar{\varpi}^\alpha) \wedge \omega^j + T^i{}_{jk}\omega^j \wedge \omega^k$$

and both ϖ^α and $\bar{\varpi}^\alpha$ contains the same degree of arbitrariness, the reduced characters can be calculated by ignoring $\bar{\varpi}^\alpha$ altogether. Hence the involutive condition can be checked using *only* the structural equations of one copy of the system.

27 Involutivity and prolongation. The calculation of the reduced Cartan characters can lead to two outcomes. The first is that the system is involutive. This means that a solution to our problem exists in the current setting, and the last non-vanishing (pseudo-)character tells us how many free functions we can specify in the solution of the problem. In particular, since any particular solution provides an equivalence of our original problem, the free functions can be interpreted as parametrizing the self-equivalence, or symmetry, of our system. The symmetry in this case is an infinite-dimensional Lie pseudo-group parametrised by the arbitrary functions obtained.

The other outcome is that the system is not involutive. This simply means that the space we are working with is too small: we need to prolong the system we are considering. The prolonged space is the most natural setting for discussing any physical problems: here there is no "hidden constraints".

28 Symmetries of solutions, special and general solutions. Above we have see that the total number of variables minus the number of functionally independent invariants gives us the dimension of the symmetry group. It might now seem that for this to be practically applicable, we must be given an explicit example of a theory, where all invariants can be calculated as functions of the variables. However, in theory-building in physics, we are usually given only the *form* a system should have, with the greatest number of unspecified variables. It then seems that we must divide our problem into cases according to the number of independent invariants in each order, and study each case separately.

Luckily, not all cases are created equal, and in practice only cases at the two extremes are of interest to us. At one extreme, *we have as few invariant as possible*. Such is the case of the solutions with the greatest number of symmetries and the interest arises precisely due to its symmetrical properties.

The other extreme, often overlooked, is actually of more fundamental importance: cases where *we have as many invariants as possible*, and *these invariants appear as early as possible*. These cases are important since *they are the general cases*. Consider a system of unspecified variables with first order structural functions. We can calculate, in this case, the rank of the structural functions *in the general case* by taking the largest possible rank of the structural functions, which is usually the rank of a matrix with unspecified variables in it. Now, to say that a system does not satisfy our assumption of maximum generality, is to say that the rank of this matrix is less than its maximum value, which can be formulated as the vanishing of suitable minors of this matrix. Topologically speaking such solutions form a closed set, and the dimension of the space of such solutions is strictly less than that of the general solutions (if we really want to study these solutions: we simply adjoin the rank conditions to our initial conditions and study the *general solutions* of this new system).

Physically speaking, this means that once a physical theory is set up, then any *real world* initial conditions that we will get are conditions of maximal generality, since *almost all possible initial conditions are general.* Hence in theory building this is the case that we should focus on. If the other cases have different properties contradicting the properties of the general case, *these should be considered as anomalies and studied separately.*

29 Physical degrees of arbitrariness, integration of invariants. We already know how to compare two systems to see whether they are locally "the same". We know that the system is uniquely determined by its system of invariants (which is included in the system of structural functions), but the invariants cannot be chosen arbitrarily: for the case where they are constants, we have already remarked that they must satisfy the Jacobi identity. We will now answer the following question which arises naturally:

What is the most general way that we can choose these structural functions, on which we may impose some a priori *relations?*

The answer to this question then tells us the *physical degrees of arbitrariness* of a system, which is the number of arbitrary functions that can be freely specified [(†)]. These functions themselves can be taken to be physical fields (in this raw form as scalar fields defined only formally they are not useful for practical computations: it is much better to find suitable equivalent tensor fields). We will solve this problem by applying the Cartan–Kähler theorem.

Let us consider the case with minimal symmetry: we assume that the space is already suitably extended and the equivalence problem is defined by n one-forms ω^i, $i = 1, \ldots, n$. We have the structural functions

$$d\omega^i = c^i{}_{jk}\omega^j \wedge \omega^k,$$

where there are some algebraic constraints on the structural constants:

$$f_\alpha(c^i{}_{jk}) = 0,$$

which defines the algebraically independent invariants. In the following calculations, whenever we define new derived invariants, we need to ensure the derived forms of

[(†)] Note that this is different from physical *degrees of freedom.* There are at least two senses in which the word degree of freedom is used for physical systems: for the kinematical sense, this is roughly the same as the degree of arbitrariness, but when we come to dynamics, what is usually understood by the degree of freedom is the number of free functions *together with their time derivatives* (also assumed to be free) that we need to specify on a Cauchy surface: thus the dynamical degree of freedom is, very roughly speaking, half of the degree of arbitrariness.

this relation is satisfied: this partially constrains the independent invariants of higher orders. From now on we will do this implicitly.

Assume that in $c^i{}_{jk}$, we can already find n independent invariants, which we call I_i, $i = 1, \dots, n$. Then

$$dI_i = I_{i;j}\omega^j \tag{1.10}$$

as we defined previously. Since I_i are assumed to be independent, *the matrix of $I_{i;j}$ is invertible*, and we can solve for ω^i as a linear homogeneous functions in dI_i.

Now take the system of $c^i{}_{jk}$, $c^i{}_{jk;l}$ where only algebraic constraints have been applied. We have

$$dc^i{}_{jk} = c^i{}_{jk;l}\omega^l$$

which this system must satisfy. The identity $d^2c^i{}_{jk;l} = 0$ may contain additional algebraic constraints on $c^i{}_{jk;l}$, which we assume from now on to be taken into account. For reasons that will become apparent in a moment, we prolong this system once, by including also the equations of the next order

$$dc^i{}_{jk;l} = c^i{}_{jk;lm}\omega^m,$$

where $c^i{}_{jk;lm}$ are for the moment only subject to the necessary algebraic constraints, obtained by calculating $d^2c_{ij;lm} = 0$ and maybe also $d^2c_{ij;l} = 0$. If we use the solution of (1.10) to substitute for ω^i, then we obtain a system of differential equations where the variables are $c^i{}_{jk}$ and $c^i{}_{jk;l}$, and whose independent conditions is simply that ω^i, now considered as linear combinations of dI_i, do not vanish. For this system we can apply the usual Cartan–Kähler treatment, and as usual the last non-vanishing Cartan (pseudo) character is the degree of arbitrariness. Once the integration of this system is done, (1.10) can be used to solve for ω^i as functions of the invariants, where n of them are now considered as local coordinates.

We are not quite done yet—for our solution, all differential relations among the invariants of various orders are satisfied. But we still have another relation to satisfy, namely the structural relation that we began with:

$$d\omega^i = c^i{}_{jk}\omega^j \wedge \omega^k.$$

For this, let us differentiate (1.10). We obtain

$$0 = dI_{i;j} \wedge \omega^j + I_{i;j}d\omega^j = I_{i;jk}\omega^j \wedge \omega^k + I_{i;j}d\omega^j. \tag{1.11}$$

The $I_{i;jk}$, which are found among $c^i{}_{jk;lm}$, satisfy certain algebraic defining relations: namely, when on the right hand side of

$$d^2c^i{}_{jk} = c^i{}_{jk;lm}\omega^l \wedge \omega^m + c^i{}_{jk;l}d\omega^l$$

we replace $d\omega^l$ by $c^l{}_{jk}\omega^j \wedge \omega^k$, the expression vanishes identically as an algebraic consequence:

$$0 = c^i{}_{jk;lm}\omega^l \wedge \omega^m + c^i{}_{jk;l}c^l{}_{mn}\omega^m \wedge \omega^n.$$

Specialising to I_i, this means that

$$0 = I_{i;jk}\omega^j \wedge \omega^k + I_{i;j}c^j{}_{mn}\omega^m \wedge \omega^n.$$

This shows that, by using the defining relations of $I_{i;jk}$ alone, we have, by continuing calculating (1.11),

$$0 = I_{i;j}(d\omega^j - c^j{}_{kl}\omega^k \wedge \omega^l) \equiv I_{i;j}\Pi^j.$$

Recall that, since we have the maximum number of independent invariants, $I_{i;j}$ is invertible. Hence we have $\Pi^i = 0$, which is the same as the structural relations that we start with. Our claim is hence proved.

This reasoning obviously goes through even if not all independent invariants appear in the first order, as long as the total number of them is equal to the dimension of the space. Hence *in the most general systems, we can use the independent invariants themselves as local coordinates.*

30 Avoiding unnecessary prolongation. From the above consideration, whenever we have a system that is guaranteed to contain all *functionally independent* invariants in its parameters (which is perhaps obtained by one or more prolongations), we need to prolong the system twice further to obtain a differential systems for the invariants alone, and the solution to this system tells us the degree of arbitrariness of the theory. The problem is that, except in the simplest examples, prolongation is very tedious. Suppose that we have p invariants on a n dimensional space. Two prolongations potentially introduce a maximum of $np(n+3)/2$ new quantities.

Luckily for us, for practical applications it is not always necessary to effect this prolongation: the theoretical tools we need is in §**19**. We have the structural equations

$$d\omega^i = c^i{}_{jk}\omega^j \wedge \omega^k,$$

the constraints

$$f_\alpha(c^i{}_{jk}) = 0,$$

and the defining relations for the derived invariants

$$dc^i{}_{jk} = c^i{}_{jk;l}\omega^l,$$

and similar ones for $dc^i{}_{jk;l}$, etc., depending on the order of prolongation we are considering. What we did was to omit the structural equations from the system. The structural equations, when derived again, $d^2\omega^i = 0$ will in general give new constraints for ${c^i}_{jk;l}$, which are not derivable from the other two relations, so we always need to include ${c^i}_{jk;l}$ as independent variables.

Once these are included, note that the system

$$f_\alpha({c^i}_{jk}) = 0, \qquad d{c^i}_{jk} = {c^i}_{jk;l}\omega^l$$

implicitly contains all algebraic relations for all higher order invariants: these relations are obtained by the usual process of *complete prolongation*. Thus, if this system is involutive, by §**19**, all higher order systems are involutive as well. Hence in practice we usually only need to prolong once, instead of twice. We can also deduce what the Cartan characters would be under any prolongation by the results of §**19**.

What if we are only able to obtain a set of invariants that are guaranteed to contain all functionally independent invariants, in number equal to the number of dimensions of the original problem, after k prolongations? The preceding section tells us that in this case, the system that allows us to rederive the structural equations is of order $k+2$. It may be tempting to apply the following reasoning in this case: we know that the structural equations, which are omitted in our differential system, contributes to at most order 1, and hence we still only needs to prolong once: if the once prolonged system is involutive, we obtain all the results we want. However, this reasoning has the fatal problem that, *we need to know what the independent variables are in order to apply the theory for involutive systems*. In this case it is still necessary to prolong to at least order $k+1$.

IV. Riemannian geometry

31 The equivalence problem. Consider a Riemannian metric, which may be positive definite or indefinite

(1.12) $$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

We first need to find a suitable co-frame. We can decompose into squares:

(1.13) $$ds^2 = \sum_\mu \epsilon_\mu(\theta^\mu)^2.$$

The θ^μ is hence a suitable co-frame for our problem. The scalar quantity $\epsilon_\mu = \pm 1$, and is used to keep track of the minus signs in our theory: if the metric is positive definite, then $\epsilon_\mu = 1$ for all indices. It should be easy to see that the value of ϵ_μ, for what we do below, does not change the form of the theory we develop and only adds minus signs in various places, so for convenience we will omit writing it out explicitly.

The equivalence problem is then the equivalence of the co-frame θ^μ up to a symmetry that preserves (1.12), i.e.,

$$\bar\theta^\mu = r^\mu{}_\nu\theta^\nu, \qquad r^\mu{}_\nu \in SO(p,q)$$

where we have implicitly restricted the orientation of the coframe. Symmetrising the problem, the lifted basic co-frame is

$$\omega^\mu = r^\mu{}_\nu\theta^\nu, \qquad r^\mu{}_\nu \in SO(p,q). \tag{1.14}$$

For the frame θ^μ, we have

$$d\theta^\mu = a^\mu_{\nu\rho}\theta^\nu \wedge \theta^\rho$$

thus

$$d\omega^\mu = g^\mu{}_\nu d\theta^\nu + dg^\mu{}_\nu \wedge \theta^\nu = \alpha^\mu{}_\nu \wedge \omega^\nu + \tilde a^\mu{}_{\rho\nu}\omega^\rho \wedge \omega^\nu,$$

$\omega^\mu{}_\nu$ being the Maurer–Cartan form of $SO(n)$. The torsion $\tilde a^\mu{}_{\rho\nu}$ can be completely absorbed: indeed, write $\alpha^\mu{}_\nu = z^\mu{}_{\nu\lambda}\omega^\lambda$, then,

$$d\omega^\mu = (z^\mu{}_{\nu\lambda} + \tilde a^\mu{}_{\lambda\nu})\omega^\lambda \wedge \omega^\nu, \qquad z^\mu{}_{\nu\lambda} = -z^\nu{}_{\mu\lambda}, \quad \tilde a^\mu{}_{\rho\lambda} = -\tilde a^\mu{}_{\lambda\rho}, \quad \tilde a^\mu{}_{\rho\lambda} = -\tilde a^\rho{}_{\mu\lambda}.$$

We want the following to vanish:

$$(z^\mu{}_{\nu\lambda} - z^\mu{}_{\lambda\nu}) + \tilde a^\mu{}_{\lambda\nu} = 0,$$

and the unique solution for $z^\mu{}_{\nu\lambda}$ is

$$z^\mu{}_{\nu\lambda} = \tfrac{1}{2}(-\tilde a^\mu{}_{\lambda\nu} + \tilde a^\nu{}_{\lambda\mu} + \tilde a^\lambda{}_{\nu\mu}).$$

Uniqueness implies the degree of indeterminacy $r^{(1)} = 0$.

The structural equations after absorption are

$$d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu, \qquad \omega^\mu{}_\nu = -\alpha^\mu{}_\nu + z^\mu{}_{\nu\lambda}\omega^\lambda. \tag{1.15}$$

We see that the Cartan characters are (c.f. §**26**):

$$s_1 = n-1, \qquad s_2 = n-2, \qquad \dots, \qquad s_{n-1} = 1.$$

For σ_n, the dimension of the group $SO(n)$ is $\frac{n(n-1)}{2}$. So

$$\sigma_n = \frac{n(n-1)}{2} - \sum_{k=1}^{n-1} k = 0 \ (= s_n).$$

Therefore,

$$s_1 + 2s_2 + \cdots + ns_n = \frac{n(n^2-1)}{6},$$

and unless $n = 0$ or $n = 1$, which are trivial cases we exclude from the discussions, we need to prolong the system.

32 Prolongation. Principal bundle. Prolongation amounts to, in this case, adding the parameters of the group $SO(n)$ as variables. The coframe is completed by adjoining the forms $\omega^\mu{}_\nu$: they are, up to linear terms in ω^μ, the Maurer–Cartan forms of $SO(n)$. The equivalence problem now can be treated as a basic equivalence problem. We call the extended space, which is locally $M \times SO(p,q)$, the principal bundle of the problem.

The equivalence condition

$$\omega^\mu = \bar{\omega}^\mu, \qquad d\omega^\mu = d\bar{\omega}^\mu$$

implies

$$(\omega^\mu{}_\nu - \bar{\omega}^\mu{}_\nu) \wedge \omega^\nu = 0$$

or

$$\omega^\mu{}_\nu - \bar{\omega}^\mu{}_\nu = a^\mu{}_{\nu\lambda}\omega^\lambda, \qquad a^\mu{}_{\nu\lambda} = -a^\nu{}_{\mu\lambda} = a^\mu{}_{\lambda\mu}$$

but using the symmetries we get $a^\mu{}_{\nu\lambda} = 0$. Hence the equivalence problem requires

$$\omega^\mu = \bar{\omega}^\mu, \qquad \omega^\mu{}_\nu = \bar{\omega}^\mu{}_\nu.$$

For the structural equations, the first equation

$$d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu$$

is given. For the differential of the second set of forms, in principle we could have

$$d\omega^\mu{}_\nu = -\omega^\mu{}_\lambda \wedge \omega^\lambda{}_\mu + \tfrac{1}{2}R^\mu{}_{\nu\rho\lambda}\omega^\lambda \wedge \omega^\lambda + R^\mu{}_{\nu\rho\lambda\gamma}\omega^\rho{}_\lambda \wedge \omega^\gamma + \tfrac{1}{2}R^\mu{}_{\nu\rho\lambda\gamma\delta}\omega^\rho{}_\lambda \wedge \omega^\gamma{}_\delta,$$

but differentiating the first equation again, we get

$$d\omega^\mu{}_\nu \wedge \omega^\nu - \omega^\mu{}_\nu \wedge \omega^\nu{}_\lambda \wedge \omega^\lambda = 0$$

using the last two equations together, we deduce that

$$R^\mu{}_{[\nu\rho\lambda]} = 0, \qquad R^\mu{}_{[\nu|\rho\lambda|\gamma]} = 0, \qquad R^\mu{}_{\nu\rho\lambda\gamma\delta} = 0.$$

again by using the combination of symmetry and antisymmetry of indices we have $R^\mu{}_{\nu\rho\lambda\gamma} = 0$ identically. Hence our second structural equation is

$$d\omega^\mu{}_\nu = -\omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu + \tfrac{1}{2}R^\mu{}_{\nu\rho\lambda}\omega^\rho \wedge \omega^\lambda, \tag{1.16}$$

The quantities $R^\mu{}_{\nu\rho\lambda}$ are the *fundamental invariants* of the problem, which are nothing more than the usual Riemann curvature tensor in the bundle. It has symmetries

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} = -R_{\mu\rho\lambda\rho} = R_{\rho\lambda\mu\nu}, \qquad R_{\mu[\nu\rho\lambda]} = 0,$$

the first set comes from the antisymmetry of terms $\omega^\rho \wedge \omega^\lambda$ and the antisymmetry of the indices in $\omega^\mu{}_\nu$, whereas the second set comes from $d^2\omega^\mu = 0$ above (the Bianchi identity).

33 Derived invariants, coframe derivatives, covariant derivatives We now investigate some special properties of $R_{\mu\nu\rho\lambda}$. First, ω^μ and $\omega^\mu{}_\nu$ are both defined on the bundle $M \times SO(p,q)$. To go from one point on the bundle to another point on the bundle corresponding to the same point on the base, the frames ω^μ are simply rotated: $\omega'^\mu = r^\mu{}_\nu \omega^\nu$. Using the first structural equation, it is easily seen that the forms $\omega^\mu{}_\nu$ undergoes similar rotation: $\omega'^\mu{}_\nu = (r^{-1})^\mu{}_\lambda \omega^\lambda_\gamma r^\gamma{}_\mu$. Then by using the structural equation, we see that the transformation law for $R_{\mu\nu\rho\lambda}$ is

$$R'_{\mu\nu\rho\lambda} = r^\alpha{}_\mu r^\beta{}_\nu r^\gamma{}_\rho r^\delta{}_\lambda R_{\alpha\beta\rho\lambda}$$

which shows that $R_{\mu\nu\rho\lambda}$ *transforms under the* $(0,4)$ *tensor representation of* $SO(p,q)$. Indeed, all indices we have been using are tensor indices in the sense that they furnish a tensor representation of the symmetry group $SO(p,q)$ [†].

Let us for the moment regress to our unextended frame θ^μ on the manifold M. Suppose that f is a scalar function defined on M, meaning that when lifted into $M \times SO(p,q)$ it is constant on each fibre. In other words, it is a $(0,0)$ tensor. Let $\mathbf{v}_\mu$ be the frame dual to θ^μ on M. We have

$$\begin{aligned} df &= \mathbf{v}_\mu(f)\theta^\mu \\ &= (r^{-1})^\mu{}_\nu \mathbf{v}_\mu(f)\omega^\nu. \end{aligned}$$

So in this simple case the coframe derivatives with respect to ω^μ are simply the rotated version of the frame derivatives with respect to $\mathbf{v}_\mu$ on the base (i.e., tensor of rank $(0,1)$), and there are no coframe derivatives with respect to $\omega^\mu{}_\nu$. Next we consider a tensor of type $(1,1)$. Let it have the value of $t^\mu{}_\nu$ on the base with respect to the frame θ^μ. As it is a tensor, its value in the bundle is already determined:

$$T^\mu{}_\nu = r^\mu{}_\rho t^\rho{}_\lambda (r^{-1})^\lambda{}_\nu = r^\mu{}_\rho r^\nu{}_\lambda t^\rho{}_\lambda, \tag{1.17}$$

the last equality follows because we are dealing with the orthogonal group. Now

$$\begin{aligned} dT^\mu{}_\nu &= \mathbf{v}_\lambda(T^\mu{}_\nu)\theta^\lambda + \frac{\partial T^\mu{}_\nu}{\partial r^\alpha{}_\beta} dr^\alpha{}_\beta \\ &= \left((r^{-1})^\lambda{}_\delta \mathbf{v}_\lambda(T^\mu{}_\nu) + r^\gamma{}_\beta \frac{\partial T^\mu{}_\nu}{\partial r^\alpha{}_\beta} z^\alpha{}_{\gamma\delta} \right) \omega^\delta + \left(r^\gamma{}_\beta \frac{\partial T^\mu{}_\nu}{\partial r^\alpha{}_\beta} \right) \omega^\alpha{}_\gamma. \end{aligned}$$

[†] Another, more algorithmic, way to show this is to derive the structural equations and show that the coframe derivatives of $R_{\mu\nu\rho\lambda}$ do not have independent terms in the group direction. Indeed, all the results of this section can be obtained by exterior differentiating the structural equations alone, though more calculations would be involved.

Let us first investigate the derivatives with respect to $\omega^\mu{}_\nu$. The form of the coframe derivative

$$r^\gamma{}_\beta \frac{\partial T^\mu{}_\nu}{\partial r^\alpha{}_\beta}$$

together with the formula (1.17), allow us to deduce an importance consequence of the fact that $T^\mu{}_\nu$ is a *tensor*: *its coframe derivatives in the vertical direction is linear in components of itself.* In particular, this means that vertical coframe derivatives will never introduce any new functionally independent quantities. What we get from this is that even though they are still invariant derivatives, for equivalence problems in this and similar geometrical settings we only need to be concerned with the horizontal derivatives. What are the horizontal derivatives? The first clue is that $z^\alpha{}_{\beta\gamma}$ is nothing more than the rotated components of the Christoffel symbols on the base, and hence the form of the formula

$$T_{\mu\nu;\delta} \equiv (r^{-1})^\lambda{}_\delta \mathbf{v}_\lambda(T^\mu{}_\nu) + r^\gamma{}_\beta \frac{\partial T^\mu{}_\nu}{\partial r^\alpha{}_\beta} z^\alpha{}_{\gamma\delta} \tag{1.18}$$

is the bundle version of the covariant derivative using the Levi–Civita connection. Notice that the part

$$r^\gamma{}_\beta \frac{\partial T^\mu{}_\nu}{\partial r^\alpha{}_\beta}$$

together with (1.17) succinctly summarises how we should multiply the Christoffel symbols with the various indices of $T^\mu{}_\nu$.

Let us think a bit more about how we can practically calculate covariant derivatives. We already know that the vertical coframe derivatives are not very useful. But they are certainly easy enough to calculate! So schematically we have the following

$$dT - (\text{vertical coframe derivatives})\omega^\mu{}_\nu = (\text{covariant derivatives})\omega^\mu \equiv \nabla T.$$

Using (1.17) and (1.18), it is easy to see that this implies, in particular, for a $(1,0)$ tensor,

$$\nabla T^\mu \equiv T^\mu{}_{;\lambda}\omega^\lambda = dT^\mu + \omega^\mu{}_\nu T^\nu,$$

whereas for a $(0,1)$ tensor, it is

$$\nabla T_\mu \equiv T_{\mu;\lambda}\omega^\lambda = dT_\mu - \omega^\nu{}_\mu T_\nu.$$

The general rule is that to form the covariant derivative of T, we take dT and add to it terms of product of itself with $\omega^\mu{}_\nu$, each product corresponding to an index of T with the sign analogous to the above two simple cases.

34 Algebraic independence. Our next task is to use the various symmetries to choose among the fundamental and derived invariants $R_{\mu\nu\rho\lambda}$, $R_{\mu\nu\rho\lambda;\alpha}$, ... a set that is *algebraically independent.*

Let us first start with the fundamental invariants $R_{\mu\nu\rho\lambda}$. The relations are

$$R_{\mu\nu\rho\lambda} = R_{\nu\mu\rho\lambda}, \qquad R_{\mu\nu\rho\lambda} = R_{\mu\nu\lambda\rho}, \qquad R_{\mu[\nu\rho\lambda]} = 0.$$

If we expand the identities

$$R_{\mu[\nu\rho\lambda]} - R_{\nu[\mu\rho\lambda]} - R_{\rho[\mu\nu\lambda]} + R_{\lambda[\mu\nu\rho]} = 0$$

and use the first two antisymmetries, we obtain an identity which is dependent on the above but is nonetheless very useful:

$$R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu}.$$

We will arrange the four indices in a way such that every combination of indices in this class is guaranteed to be independent, whereas combinations not in this class are dependent. Take $R_{\mu\rho\rho\lambda}$. Using the first two identities, we can have

$$\mu > \nu, \qquad \rho > \lambda,$$

and using the last identity, we can have further

$$\mu \geq \rho.$$

After this, the only identity that may contain relations between quantities that are still left is $R_{\mu[\nu\rho\lambda]} = 0$. For the quantities that are still left, this relation is

$$R_{\mu\nu\rho\lambda} - R_{\mu\rho\nu\lambda} + R_{\mu\lambda\nu\rho} = 0$$

Observe that this relation is an identity unless all four indices are different. If all indices are different, this relation contains exactly one term for which $\nu < \lambda$ and two terms for which $\nu > \lambda$. Hence, using these relations, we can arrange that

$$\mu > \nu, \qquad \rho > \lambda, \qquad \mu \geq \rho, \qquad \nu \geq \lambda. \tag{1.19}$$

Quantities whose indices satisfy the above relation are called *normal.* We take normal quantities to be independent and non-normal ones to be dependent. Enumerating all possibilities for the normal quantities, we have

$$\begin{cases} \mu > \nu > \rho > \lambda : & C_4^n \text{ terms,} \\ \mu > \rho > \nu > \lambda : & C_4^n \text{ terms,} \\ \mu > \nu = \rho > \lambda : & C_3^n \text{ terms,} \\ \mu = \rho > \nu > \lambda : & C_3^n \text{ terms,} \\ \mu > \rho > \nu = \lambda : & C_3^n \text{ terms,} \\ \mu = \rho > \nu = \lambda : & C_2^n \text{ terms.} \end{cases}$$

We have deliberately arranged such that the last index λ is the smallest.

Adding these together, we get

$$2C_4^n + 3C_3^n + C_2^n = \frac{n^2(n^2-1)}{12}$$

normal terms for the fundamental invariants.

Next let us deal with the first derived invariants $R_{\mu\nu\rho\lambda;\delta}$. In addition to the symmetries which are the same as in the case of fundamental invariants, there is an additional symmetry relating the last index: exterior differentiating the relation

$$d\omega^\mu{}_\nu = -\omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu + \tfrac{1}{2}R_{\mu\nu\rho\lambda}\omega^\rho \wedge \omega^\lambda$$

and equating independent terms, we obtain the second Bianchi identity

$$R_{\mu\nu[\rho\lambda;\delta]} = 0.$$

Using this identity, noting that only derivatives of normal quantities can be normal and applying a reasoning similar to before, we can arrange, in addition to (1.19), the relation

$$\rho \geq \delta.$$

The counting is then straightforward: the number of algebraically independent first order invariants is

$$2C_2^n + 9C_3^n + 12C_4^n + 5C_5^n = \frac{n^2(n^2-1)(n+2)}{24}.$$

We can pursue this programme further to investigate higher order invariants. Note that there are no longer any more new identities (all identities can be obtained by covariantly differentiating those that we already have), except the identities for the derivatives of any tensors:

$$T^{\mu\nu\cdots}{}_{\rho\lambda\ldots;\gamma\delta} = T^{\mu\nu\cdots}{}_{\rho\lambda\ldots;\delta\gamma} + \cdots$$

where the final dots represent terms linear in $T^{\mu\nu\cdots}{}_{\rho\lambda\ldots;\delta}$, i.e., one order lower, whose coefficients are $R_{\mu\nu\rho\lambda}$. Hence *for the purpose of counting normal invariants*, the order of derivative is immaterial. Therefore we can arrange it such that the normal invariants are those:

1° whose non-derived indices satisfy (1.19),

2° whose derived indices are non-increasing,

3° whose largest derived index is no greater than ρ, the third index.

35 Functional independence. The number of linearly independent functions on a manifold can never exceed the dimension of the manifold. Here we will show that, *in the general case*, all functionally independent invariants occur within the underived Riemann tensors, c.f. §**29**.

On our manifold (principal bundle) of $n(n+1)$ dimensions, we already have a coframe formed with ω^μ and $\omega^\mu{}_\nu$. Therefore, if we can find a set of equations that enables us to solve ω^μ and $\omega^\mu{}_\nu$ in terms of $dR_{\mu\nu\rho\lambda;\gamma\ldots}$, then we only need to consider up to the order of the invariants contained in this equation, since their differentials already form a coframe. The simplest relation where all terms in the coframe appear is just the formula for the first order covariant derivative of the fundamental invariants:

$$dR_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\alpha}\omega^\alpha{}_\lambda + R_{\mu\nu\alpha\lambda}\omega^\alpha{}_\rho + R_{\mu\alpha\rho\lambda}\omega^\alpha{}_\nu + R_{\alpha\nu\rho\lambda}\omega^\alpha{}_\mu + R_{\mu\nu\rho\lambda;\gamma}\omega^\gamma.$$

In general, this already suffices to solve for all ω^μ and $\omega^\mu{}_\nu$. First, let us single out the equations in the above formulae in which $\mu > \nu > \rho = \lambda$. We have

$$dR_{\mu\nu\rho\nu} = (R_{\mu\nu\mu\nu} - R_{\rho\nu\rho\nu})\omega^\mu{}_\rho + \cdots \qquad \text{(no summation)}.$$

The ω^μ are easier: it suffices to note

$$dR_{n\mu n\mu} = R_{n\mu n\mu;\mu}\omega^\mu + \cdots \qquad \text{(no summation)},$$

and

$$dR_{n,n-1,n,n-2} = R_{n,n-1,n,n-2;n}\omega^n + \cdots \qquad \text{(no summation)}.$$

which are to be shown.

Hence, by the discussion of §**29** and §**30**, the Riemann tensors and their derivatives up to second order are sufficient to investigate the degree of arbitrariness of the space in the general case.

36 The involutive system of Riemann tensors. By applying our results in §**30**, we take the variables to be

$$R_{\mu\nu\rho\lambda}, \qquad R_{\mu\nu\rho\lambda;\gamma},$$

where the independent forms are ω^μ, $\omega^\mu{}_\nu$, now considered as forms in these variables. The differential system for them is

$$dR_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda;\gamma}\omega^\gamma - R_{\gamma\nu\rho\lambda}\omega^\mu{}_\gamma - R_{\mu\gamma\rho\lambda}\omega^\nu{}_\gamma - R_{\mu\nu\gamma\lambda}\omega^\rho{}_\gamma - R_{\mu\nu\rho\gamma}\omega^\lambda{}_\gamma.$$

This contains the variables ω^μ and $\omega^\mu{}_\nu$ which are not part of the differential system: we need to first choose $n(n+1)/2$ of the equations to solve for ω^μ and $\omega^\mu{}_\nu$ and substitute the result back into the rest of the equations: this has been shown to be possible in the general case above.

According to the general procedure, important information is contained in the derived equation:

$$d^2 R_{\mu\nu\rho\lambda} = (dR_{\mu\nu\rho\lambda;\gamma} + \cdots) \wedge \omega^\gamma$$

where the dots are linear in ω^μ, ${\omega^\mu}_\nu$. Since we are calculating the reduced characters, these terms do not matter. On the other hand, since all functionally independent invariants occur at first order, none of $dR_{\mu\nu\rho\lambda;\gamma}$ appear in the independence conditions. The number of Cartan characters are

s_1	# independent terms of	$R_{\mu\nu\rho\lambda;1}$,
$s_1 + s_2$	# independent terms of	$R_{\mu\nu\rho\lambda;1}$, $R_{\mu\nu\rho\lambda;2}$,
...	...	...
$s_1 + s_2 + \cdots + s_n$	# independent terms of	$R_{\mu\nu\rho\lambda;1}$, $R_{\mu\nu\rho\lambda;2}$, ..., $R_{\mu\nu\rho\lambda;n}$,

with $s_{n+1}, \ldots, s_{n(n+1)/2}$ all zero: $s_1 + s_2 + \cdots + s_n$ gives the total number of normal invariants of $R_{\mu\nu\rho\lambda;\gamma}$.

It remains to check that the pseudo-character is also zero. The total number of normal $R_{\mu\nu\rho\lambda}$ is $s_0 + n(n+1)/2$, since the equation for d^2R contains linear combinations which can be solved for the $n(n+1)/2$ independent forms ω^μ, ${\omega^\mu}_\nu$, which must be excluded from our differential system, and the rest gives s_0. The defining relation for the pseudo-character $\sigma_{n(n+1)/2}$

$$s_0 + s_1 + \cdots + s_{n(n+1)/2-1} + \sigma_{n(n+1)/2} = N - \frac{n(n+1)}{2}$$

then shows $\sigma_{n(n+1)/2} = 0$, where N is the total number of variables, including the normal $R_{\mu\nu\rho\lambda}$ and $R_{\mu\nu\rho\lambda;\delta}$ *except* those that have been used to solve for ω^μ and ${\omega^\mu}_\nu$. This result holds as long as all of the normal first order quantities appear independently on the right hand side of the derived equation.

Due to the manner in which we labelled the normal terms, we see easily that

s_1	is the number of normal terms of	$R_{\mu\nu\rho\lambda;1}$,
s_2	is the number of normal terms of	$R_{\mu\nu\rho\lambda;2}$,
...	...	...
s_n	is the number of normal terms of	$R_{\mu\nu\rho\lambda;n}$.

On the other hand, we know that the general solution for the integral element is given by the covariant derivative formula for $R_{\mu\nu\rho\lambda;\gamma\delta}$:

$$dR_{\mu\nu\rho\lambda;\gamma} = R_{\mu\nu\rho\lambda;\gamma\delta}\omega^\delta + \cdots$$

where the dots indicate terms in quantities already defined. Here $R_{\mu\nu\rho\lambda;\gamma\delta}$ are considered as *free parameters* for this differential system. We have, due to the way we have arranged the indices,

$$\begin{array}{llll}
\#\text{ normal} & R_{\mu\nu\rho\lambda;\gamma 1} & \text{is } \#\text{ normal} & R_{\mu\nu\rho\lambda;1},\, R_{\mu\nu\rho\lambda;2},\, \ldots,\, R_{\mu\nu\rho\lambda;n}, \\
\#\text{ normal} & R_{\mu\nu\rho\lambda;\gamma 2} & \text{is } \#\text{ normal} & R_{\mu\nu\rho\lambda;2},\, \ldots,\, R_{\mu\nu\rho\lambda;n}, \\
\ldots & \ldots & \ldots & \ldots \\
\#\text{ normal} & R_{\mu\nu\rho\lambda;\gamma n} & \text{is } \#\text{ normal} & R_{\mu\nu\rho\lambda;n}.
\end{array}$$

Hence the number of free parameters is exactly

$$s_1 + 2s_2 + \cdots + ns_n,$$

and the system is involutive.

37 Degree of arbitrariness of a Riemannian geometry. The last non-vanishing Cartan character is s_n: this shows that, in this case, given a n dimensional integral variety where the independent forms are ω^μ, the integral variety of $n(n+1)/2$ dimensions whose independent forms are ω^μ, ${\omega^\mu}_\nu$ is uniquely determined. This is none other than the property of a section of the principal bundle. In §**10**, we called such systems completely integrable from dimension n.

The character s_n is the number of normal quantities of $R_{\mu\nu\rho\lambda;n}$. Due to the way normal quantities are defined, this must have the form $R_{n\nu n\lambda;n}$, where μ, λ can take values from 1 to $n-1$, and $\nu \geq \lambda$. Hence this number is

$$s_n = \frac{n(n-1)}{2}.$$

This is the number of independent components of the metric tensor (we need to subtract the freedom of choosing coordinates: the number of independent components of the metric is $n(n+1)/2 - n$). The merit of this procedure is that it can be applied to cases where the independent fields are not obvious at all: one has in mind especially those theories that are defined by specifying a *connection*. This also solves the following problem: what is the maximum number of components of Riemannian tensor that can be specified in general? (Note that for the way we carried our analysis, the result is only guaranteed for $n \geq 4$, i.e. when we are guaranteed that we can solve for ω_μ, $\omega_{\mu\nu}$ using the differentials of the fundamental invariants.) The number of components that can be freely specified *in the general case* is then $n(n-1)/2 + n(n+1)/2 = n^2$. This result does not tell us *which components can be specified freely* (not all can), and gives only the *number* of components (or linear combinations thereof) that can be freely specified *in the general case*, and some of them might well be first order instead of zeroth order. We have already said that the cases where $n = 1, 2, 3$ needs to be considered as separate cases, but actually even in these low dimensional cases the formula holds.

Another question to be asked is if at a *single point*, we specify the components of $R_{\mu\nu\rho\lambda}$ and their covariant derivatives to arbitrary order, is the system consistent? The general theory of Cartan–Kähler integrability tells us that such systems always have solutions in the analytic case.

38 Determinism. Gravity. In §**10**, we discussed the procedure of adding algebraic equations to the differential systems in order to obtain a deterministic theory. Here, since the degree of arbitrariness at dimension n is $n(n-1)/2$, the minimal number of equation required is also $n(n-1)/2$.

In general the equations we add can be of any functions of the invariants (if we want to add invariants of higher orders, we must prolong the system further). But we can restrict the highest order of the invariants that occur: let us now require that only the fundamental (underived) invariants appear in the equations we add. In addition, we can also require the form of these equations: let us require that the equations are linear in the fundamental invariants (these two requirements are equivalent to the assumptions made by Einstein in writing down the theory of relativity: no derivatives of the metric of order greater than 2 appears in the equations of motion, and the equations are linear in the second order derivatives).

Further, we know that $SO(p,q)$ is the local symmetry of the theory. Hence it is desirable that the equations we add are invariant under this symmetry. Invariance and linearity together implies that in the equations, the invariants must appear as "blocks" of invariant subspaces of the representation. In the present case, there are three such subspaces for the Riemann tensor:

- The Weyl tensor;
- The traceless part of the Ricci tensor;
- The Ricci scalar.

We will restrict our attention to the unique equation satisfying the additional assumption of energy-momentum conservation, namely the Einstein equation:

$$R_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}R + \Lambda\delta_{\mu\nu} = T_{\mu\nu}$$

where $R_{\mu\nu}$ denotes the Ricci tensor, R the Ricci scalar and $T_{\mu\nu}$ denotes a *given* symmetric rank 2 tensor (the energy momentum tensor) and Λ is the cosmological constant.

First note that if $\Lambda = 0$ and $T_{\mu\nu} = 0$, then the equation is equivalent to $R_{\mu\nu} = 0$. Indeed, it is not hard to see that *for the purpose of counting degree of arbitrariness*, we only need to study the equation

$$R_{\mu\nu} = 0, \tag{1.20}$$

which are now thought as additional *algebraic equations* added to our system.

For $\mu, \nu < n$, the equations we have added are

$$R_{1\mu 1\nu} + R_{2\mu 2\nu} + \cdots + R_{n\mu n\nu} = 0,$$

so now $R_{n\mu n\nu}$ for $\mu, \nu < n$ are no longer considered independent terms. For $\mu = n$, $\nu < n-1$,

$$R_{1n1\nu} + R_{2n2\nu} + \cdots + R_{n-1,n,n-1,\nu} = 0,$$

so $R_{n,n-1,n-1,\nu}$ for $\nu < n-1$ are no longer considered independent. For $\mu = n$, $\nu = n-1$,

$$R_{1,n,1,n-1} + R_{2,n,2,n-1} + \cdots + R_{n-2,n,n-2,n-1} = 0,$$

so $R_{n,n-2,n-1,n-2}$ is no longer considered independent. Finally, for $\mu = n$, $\nu = n$,

$$R_{n1n1} + R_{n2n2} + \cdots + R_{n,n-1,n,n-1} = 0.$$

However, all of these terms have already been declared dependent in the case of $\mu, \nu < n$. Hence we need to substitute them with independent terms. After this substitution, $R_{n-1,n-2,n-1,n-2}$ is no longer independent.

In summary, the independent (normal) components of the Riemann tensor are now $R_{\mu\nu\rho\lambda}$ for which

$$\mu > \nu, \qquad \rho > \lambda, \qquad \mu \geq \rho, \qquad \nu \geq \lambda,$$

from which we exclude the following

$$R_{n\mu n\lambda}, \qquad R_{n,n-1,n-1,\lambda}, \qquad R_{n,n-2,n-1,n-2}, \qquad R_{n-1,n-2,n-1,n-2}.$$

Next, we also need to take the equations $dR_{\mu\nu} = 0$ into account. But

$$dR_{\mu\nu} = R_{\mu\nu;\lambda}\omega^{\lambda} + R_{\mu\lambda}\omega^{\nu}{}_{\lambda} + R_{\lambda\nu}\omega^{\mu}{}_{\lambda},$$

the terms linear in $\omega^{\mu}{}_{\nu}$ cancel each other, and hence we require $R_{\mu\nu;\lambda} = 0$. Hence, the derived equations only tell us that *independent derived invariants comes from derivation of independent fundamental invariants.*

The equation that allows us to calculate the Cartan reduced characters is still

$$d^2 R_{\mu\nu\rho\lambda} = (dR_{\mu\nu\rho\lambda;\gamma} + \cdots) \wedge \omega^{\gamma},$$

and we see that s_i is still the number of independent terms $R_{\mu\nu\rho\lambda;i}$. On the other hand, the independent terms of $R_{\mu\nu\rho\lambda;\gamma}$ and $R_{\mu\nu\rho\lambda;\gamma\delta}$ are still those that satisfy the same relations as the fundamental normal expressions with the addition of $\rho \geq \gamma \geq \delta$, and hence the previous analysis does not change its form. The system is hence involutive, and the last non-vanishing character is s_{n-1}: the number of independent terms of $R_{\mu\nu\rho\lambda;n-1}$. We have the following possibilities:

$$R_{n-1,\mu,n-1,\nu;n-1}, \qquad R_{n,\mu,n-1,\nu;n-1}.$$

The first possibility gives

$$\frac{(n-1)(n-2)}{2} - 1$$

terms (we need to exclude $R_{n-1,n-2,n-1,n-2;n-1}$), and the second possibility also gives

$$\frac{(n-1)(n-2)}{2} - 1$$

terms (we must have $\mu < n-1$, and exclude $R_{n,n-2,n-1,n-2}$). Hence

$$s_{n-1} = n(n-3).$$

The thing to note is that, for dimensions that we know the analysis is correct ($n \geq 4$), this number is greater than the number of degree of arbitrariness for a $n-1$ dimensional Riemannian space, but less than the degree of arbitrariness of a $n-1$ dimensional Riemannian space plus a symmetric bilinear form. (Actually this formula, which reads $s_2 = 0$ for $n = 3$, is also correct for $n = 3$: we can easily check that in three dimensions $R_{\mu\nu} = 0$ implies $R_{\mu\nu\rho\lambda} = 0$, so actually $s_3 = s_2 = s_1 = 0$, showing that it really does not make much dynamical sense to discuss Einstein gravity in dimensions less than 4). Roughly speaking, the s_{n-1} degree of arbitrariness, which can be interpreted as the degree of arbitrariness of Cauchy data we need to specify, decomposes into the metric on the $n-1$ dimensional "section" and the second fundamental form on this section, subject to certain constraints.

Note that, when $n = 4$, $s_{n-1} = 4$, and this shows that the *degree of freedom* of a graviton is $\frac{4}{2} = 2$, as when we calculate the degree of freedom of a dynamical theory, a field, together with its time derivative which are both freely specifiable on a Cauchy surface counts as a single degree of freedom.

Remark. The Einstein equations consist of $n(n+1)/2$ independent equations. Using the reasoning in §**11**, we can get an estimate of the effects of adding these equations by assuming the $n(n+1)/2$ equations to be added are *generic*. Since $n(n+1)/2$ is greater than $n(n-1)/2$, in the generic case $s_n = 0$ after adding the equations. Before adding the equations, s_{n-1} comes from the following terms which are independent:

$$R_{n,\nu,n,\lambda;n-1}, \qquad R_{n,\nu,n-1,\lambda;n-1}, \qquad R_{n-1,\nu,n-1,\lambda;n-1}$$

which, by an easy calculation, is

$$\frac{n(3n-5)}{2}$$

in number. Thus after adding the equations, the *lower bound* of s_{n-1} is

$$\frac{n(3n-5)}{2} - \frac{n(n+1)}{2} = n(n-3),$$

exactly the number we obtained before, showing that the Einstein equations are actually *generic* in this sense.

V. Generalised spaces

39 Homogeneous space. To discuss generalised spaces in the sense of Cartan, we need to first discuss homogeneous spaces. The construction of homogeneous space should be familiar: we have a Lie group G and a subgroup $H \subset G$. The quotient G/H is called a *homogeneous space*, or a Klein geometry. It is usually assumed that H contains no non-trivial normal subgroups of G, since otherwise by applying the quotient by this normal subgroup simultaneously to G and H gives a homogeneous space that is essentially the same.

To connect to the theory that we have already developed for equivalence problems, let the Lie algebra of $\mathfrak{g}$ split to $\mathfrak{h} \oplus \mathfrak{p}$. Then the Maurer–Cartan forms of G can be arranged such that they are divided to a part ω_i corresponding to $\mathfrak{p}$, and a part ω_α corresponding to $\mathfrak{h}$. The structural equations for the forms ω_i and ω_α are in this case just the Maurer–Cartan equations for the Lie group G [(†)]:

$$d\omega + [\omega \wedge \omega] = 0,$$

where ω is the *Lie algebra-valued Maurer–Cartan form.* In our view, the construction of a homogeneous space is the extension of the completely isotropic base manifold M by the Lie group H, with the property that after this extension, the principal bundle, locally $M \times H$, is itself a Lie group.

40 Generalised spaces. Geometric torsion. Mutation. In a homogeneous space a point cannot be distinguished from any other point by its geometrical property alone. We can deform a homogeneous space such that the symmetry G still acts on an infinitesimal level to obtain *generalised spaces* where the symmetry is broken globally. This means that the Maurer–Cartan relation is replaced with

$$d\omega + [\omega \wedge \omega] = \Omega$$

where Ω is a $\mathfrak{g}$-valued 2-form. The division of ω into $\mathfrak{h}$ part and $\mathfrak{p}$ part still makes sense, and $\omega_{\mathfrak{p}}$ are forms which remain independent on the base manifold, and $\omega_{\mathfrak{h}}$ are Maurer–Cartan forms of H, up to linear terms in $\omega_{\mathfrak{p}}$.

A special class of generalised spaces are those that have Ω elements of the Lie subalgebra $\mathfrak{h}$ instead of the full Lie algebra $\mathfrak{g}$. Such geometries are said to be *torsion-*

[(†)]Notation: $[\alpha\wedge\beta]$ for Lie algebra-valued forms α and β means that we have to apply simultaneously the exterior product on α and β considered as one-forms, *and* the Lie bracket on them considered as elements of the Lie algebra.

free [(†)]. In general, the notion of absorption of torsion still applies to this kind of torsion, and in generalised geometries based on Euclidean model we can absorb all torsion, c.f. §**31**.

Geometries whose curvature vanishes completely are even more special. Here is a trick that we can do: if the curvature is itself homogeneous, we can change our model $\mathfrak{g}$ so as to obtain a curvature-free geometry, i.e., a homogeneous space. For example, if in Riemannian geometry we have

$$d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + \kappa\omega_i \wedge \omega_j,$$

where κ is a constant, then we can use the coframe written in the Lie algebra matrix form

$$\begin{pmatrix} 0 & \kappa\omega_i \\ \omega_i & \omega_{ij} \end{pmatrix}.$$

By a suitable scaling, we can make $\kappa = 1$ or -1 (where we have introduced a length scale), so in this case we have passed from the model of the Euclidean space to the model of the sphere or hyperbolic space. Such a change of model is called *mutation.* Note that whatever model we use, the geometrical data contained therein are equivalent.

41 Reduction of the principal bundle. A very useful technique in the study of geometrical problems based on generalised geometries is the so-called *reduction of principal bundle.* Roughly speaking, we find that for a *particular problem*, the complete set of additional symmetries given by the group of the principal bundle inconvenient, and we would like to kill off some of the degrees of arbitrariness.

Let us first recall how principal bundles arise: in the problem of equivalence problems we find that the base variables x^i on our manifold M are not enough to parametrize our theory and are forced to adjoin the variables y^α of the group G, which parameterises the arbitrariness in choosing our coframes, as new independent variables, and we obtain the principal bundle which locally is $M \times G$. Suppose that we find that for a particular problem we actually introduced too many new parameters, for example since the particular problem, perhaps by introducing more geometrical structure, has only a symmetry group H that is a subgroup of the general theory. Going through the trouble of starting with the base and the group H and construct the theory again is often difficult since imposition of the condition that this theory must be a *sub-theory* of the general theory could be hard.

[(†)] We now have three different notions of torsion: torsion in the sense of exterior differential systems, torsion in the sense of equivalence problem and torsion in the sense of generalised geometry. These three notions are obviously related by the way we obtain them. A fourth notion, that in the theory of monoids, is apparently unrelated.

Since H is a subgroup of G, the Maurer–Cartan forms of G can be parametrized in a special way: we can have a basis that splits into two parts,

$$\alpha^\alpha \quad (\alpha = 1, 2, \dots, h), \qquad \text{and} \qquad \alpha^\mu \quad (\mu = h+1, h+2, \dots, g)$$

where $h = \dim H$ and $g = \dim G$. Recalling that the Lie algebra of a Lie group is a linear space, we can choose coordinates y^α, y^μ on G such that

$$\alpha^\alpha = 0 \pmod{dy^\alpha}, \qquad \alpha^\mu = 0 \pmod{dy^\mu}.$$

The gist of prolongation is that we come from the relation

$$y = y(x)$$

which is *unspecified*, to a situation that we treat y as independent variable in is own right. Hence, under a partial prolongation, in which only y^α are taken as independent variables, we still have $y^\mu = y^\mu(x)$. *In particular, y^μ are not functions of y^α*:

$$dy^\mu = 0 \pmod{dx^i}$$

or

$$\alpha^\mu = 0 \pmod{\omega^i}, \qquad \alpha^\mu = c_{\mu i}\omega^i,$$

where ω^i are a basis on the base M and $c_{\mu i}(x^i, y^\alpha)$ are functions that may be subject to further constraints according to the particular problem at hand. This property will remain true when to α^α and α^μ we add linear combinations of ω^i, as in our absorption process in the equivalence problem. We hence have

In a reduction of the principal bundle, the vertical basis forms which are no longer taken to be independent can be expressed linearly in terms of the basis forms on the base manifold alone.

This opens up a new problem of knowing if a quantity $\bar{Q}$ (which may be function or forms or other objects) on $M \times H$ is "the same" as a quantity Q of the same kind on $M \times G$. Since we have a natural inclusion map $\iota : M \times H \to M \times G$, obviously the criteria should be

$$\iota^*(Q) = \bar{Q}.$$

In this language, our lemma is of the following form:

$$\iota^*(\omega^\mu) = 0 \pmod{\omega^i}.$$

If $H = \{e\}$ the trivial group, then we are reduced back to the base manifold.

Remark. There is in principle nothing that prevents us to take y^μ to depend on y^α. If we allow this, we have a well-defined section $M \times H \to M \times G$. What we are saying here is that *this does not arise in processes which we have called "prolongations"*. In other words, "reduction of the principal bundle" is a special subclass of sections $M \times H \to M \times G$, and the lemma is really a definition in disguise, its value being that it tells us the circumstances that such special cases are applicable.

42 Tensors. It is elementary that on a manifold M with coframe ω^i, for every function f, we can write

$$df = f_{,i}\omega^i.$$

Thus in principle, in a bundle $M \times G$ where the coframe is ω^i, ω^α, we have, for an arbitrary function f,

$$df = f_{,i}\omega^i + f_{,\alpha}\omega^\alpha.$$

However the usefulness of bundles arise from just the fact that almost all functions that we will consider on it are of a very special class. First, if f is a function on M, i.e., if x^i are the coordinates for M and u^α the coordinates for G, $f = f(x)$, so we have in this case

$$df = f_{,i}\omega^i.$$

More specifically, suppose our group G acts by some linear representation on M. In this case we can replace the forms ω^α by the same forms written in accordance with the representation of the group: namely, we can write ${\omega^i}_j$. Depending on the group, there may be relations among ${\omega^i}_j$, for example for $SO(n)$ with the defining representation, ${\omega^i}_j = -{\omega^j}_i$. In this case we can define *tensors* on the principal bundle. They are constructed in the following way: first, we decide on the type of the tensor (p, q), meaning that we specify its multilinear representation under G. Then, for any section $M \to M \times G$, we choose the correct number of functions corresponding to the components of such a multilinear representation, optionally with required algebraic symmetries (symmetric or anti-symmetric under some indices, etc.). Labelling these functions as ${T^{ijk\dots}}_{lmn\dots}(x, y_0(x))$ where $y_0(x)$ determines the section, the tensor in the bundle is defined as

$$T^{ijk\dots}_{lmn\dots}(x, y_0) = {g^i}_{i'}{g^j}_{j'}{g^k}_{k'}\dots {(g^{-1})^{l'}}_l {(g^{-1})^{m'}}_m {(g^{-1})^{n'}}_n \dots T^{i'j'k'\dots}_{l'm'n'\dots}(x, y) \tag{1.21}$$

where ${g^i}_j = {g^i}_j(x)$ are the parameters of the group element g such that $y = g \cdot y_0$. Now let us write using multi-index notation, T^I_J. We still must have [†]

$$dT^I_J = T^I_{J;i}\omega^i + T^I_{J;\alpha}\omega^\alpha,$$

[†]Here, in accordance with usual usage, we have changed the comma into semicolon, but note that we do not have any concept of "ordinary differentiation", i.e., differentiation with respect to coordinates, which are denoted by commas.

but as a consequence of (1.21), the terms $T^I_{J;\alpha}$ really contains no derivatives at all. Indeed, we can easily verify that, as a consequence of (1.21), for example

$$dT^i{}_j = T^i{}_{j;k}\omega^k - \omega^i{}_{i'}T^{i'}{}_j + \omega^{j'}{}_j T^i{}_{j'}, \tag{1.22}$$

where it matters if an index on a tensor is upstairs or downstairs [(†)].

43 Geometry of connections and covariant derivatives. For our coframe ω^i, ω^α, let us form the dual frame $\mathbf{I}_i$, $\mathbf{I}_\alpha$. Note that due to the way duality works, we must have a whose set of ω^i and ω^α before we can completely determine the frame. In particular, if we change ω^α by adding linear combinations of ω^i, then $\mathbf{I}_\alpha$ will remain the same: it is the $\mathbf{I}_i$ that will change accordingly. This shows that $\mathbf{I}_\alpha$ are just the invariant Maurer–Cartan vector field on the Lie group G without involvement with the coordinates of the base x^i. On the other hand, using (1.22), it is easy to see that

$$T^I_{J;k} = \mathbf{I}_k(T^I_J).$$

(We also have a similar relation for $T^I_{J;\alpha}$, but as we have remarked, it contains no new information.) This, together with the fact that $\mathbf{I}_i$ form a horizontal subspace of the tangent space (since their complements, $\mathbf{I}_\alpha$ are vertical), motivates us to define the distribution of vector fields spanned by $\mathbf{I}_i$ as a geometrical *connection*. The covariant derivative operator ∇ operating on any tensor is thus defined in terms of (1.22):

$$\nabla T^I_J = dT^I_J \pmod{\omega^\alpha}, \qquad \nabla T^I_J = 0 \pmod{\omega^i}.$$

We can be more specific since the above two equations, together with the coframe ω^i, ω^α, completely determines the expression of the covariant derivative: it is just (1.22) rearranged.

44 Existence of covariant derivatives. Reductive geometry. To be sure that these covariant derivatives we just defined make sense we need to check that they are really invariant under choosing our co-frames. Indeed, let T_I be a tensor and ω_i be the horizontal coframes, whereas ω_α are the vertical coframes, which together forms the Lie algebra-valued coframe matrix ω. We can change the way we choose the coframes by

$$\omega' = h^{-1}(x)\omega h(x)$$

where $h(x)$ is an element of the subgroup H that can be dependent on the coordinates x on M. Now if the definition of covariant derivative

$$dT_I = T_{I;i}\omega_i + T_{I;\alpha}\omega_\alpha$$

[(†)] Again, if the group is the orthogonal group, it can easily be checked that due to the relation $\omega^i{}_j = -\omega^j{}_i$ the position is again immaterial.

makes invariant sense, then $T_{I;i}$ must change under a linear representation of H under such a change of co-frame. In other words, the horizontal derivatives must not mix with the vertical ones under any change of co-frame. This requires that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be invariant under adjoint action of $\mathfrak{h}$, so whether covariant derivatives are well-defined depends only on the model homogeneous space. We can such geometries *reductive geometries*. Furthermore, any mutation of a reductive geometry remains reductive. When dealing with a new geometry we need to first check if it is reductive before we can utilise the notion of covariant derivative.

Chapter 2

The method of involutive seeds

i. Simplification of Cartan's test for certain Pfaffian systems

45 In the previous chapter, we used a combination of techniques from the method of moving frames and Cartan–Kähler theorem to obtain the degree of arbitrariness of the metric and of solutions to Einstein's equation. The value of this derivation lies more in the method itself than in the results obtained: after all, the results are well-known. In this chapter our aim is to formalise this method, in order that it can be applied easily and fruitfully to a wide range of problems, in particular, to our problems dealing with submersions later.

In all of the applications, we will see the importance of the process that we called prolongation: although this process is very cumbersome when we try to effect it and do the subsequent Cartan–Kähler calculations by hand, in many cases prolongations give us exterior differential systems of a very particular form, and for such exterior differential systems the direct applications of Cartan's test is unnecessary and there is often a much simpler way to calculate the characters.

46 Finding patterns in the application of Cartan's test. As we have already seen, the application of Cartan's test is a rather time-consuming activity, and when the dimension of the system becomes high, or is left generic, doing the relevant calculations quickly becomes impossible, at least when resorting to pen and paper. As the reader might have already noticed, however, that in a large class of problems the exterior differential system takes a very particular form, and whenever the system is in such a form it is not necessary to carry out the calculations as what we did above: the form of the differential system implies that, when certain further conditions are satisfied, we can obtain the set of Cartan characters of the system from combinatorical manipulations on suitably set up indices of certain dependent variables.

Let Σ be a Pfaffian system defined on an m-dimensional manifold M with the independent condition given by $\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n$ where the θ_i are one-forms. We make the following assumptions:

1° We can enumerate a finite set S of dependent variables of the system whose differential cannot be expressed linearly in terms of the θ_i when using the equations of the Pfaffian system, i.e., $f \in S$ are such that

$$df \neq 0 \pmod{\theta_1, \theta_2, \ldots, \theta_n},$$

and whose differentials are all linearly independent.

2° We can label the variables in the set S using multi-indices from the set $\{1, 2, \ldots, n\}$, i.e., the same indices as those labeling the θ_i, such that if for example $u_I \in S$ has multi-index I ending in k (e.g., it is of the form $abc \ldots k$), then it occurs in the following form in the two form equations in the system:

$$a\, du_I \wedge \theta_k, \tag{2.1}$$

where a is a *non-zero* function or constant, and it does not occur in the form

$$b\, du_I \wedge \theta_l$$

for $b \neq 0$ and $l < k$. (Note: the cases where the *functions* a vanish must be studied separately, i.e., $a = 0$ must be adjoined to the list of equations and we must start again.)

3° For the linear integral elements of system, which must be of the following form,

$$du_{Ik} = u_{Ik1}\theta_1 + u_{Ik2}\theta_2 + \cdots + u_{Ikk}\theta_k + u_{Ik,k+1}\theta_{k+1} + \cdots + u_{Ikn}\theta_n,$$

The *parameters* u_{Ik1} to u_{Ikk} are free parameters for all $u_{Ik} \in S$, and the rest of the parameters are dependent on the free parameters (which include the case where some of them are constants).

4° When we arrange the two form equations by factoring out the factors θ_k and discarding all two forms linear in $\theta_i \wedge \theta_j$, we obtain by our assumptions above

$$0 = \sum_k (a_1^k du_{I_1}^k + a_2^k du_{I_2}^k + \cdots + a_p^k du_{I_p}^k) \wedge \theta_k$$

where the a_i^k are non-zero constant coefficients and $u_{I_i}^k$ are the variables labeled by their multi-indices. We assume that, for each equation and each k, there is only one $u_{I_p}^k$ whose multi-index has its last index equal to k.

47 A lemma for a very simple verification of Cartan's test. Now we will state and prove the following

LEMMA. *Under the above assumptions, the original Pfaffian system is involutive, and its set of Cartan characters can be calculated in the following way: the k-th character s_k is equal to the number of variables in the set S with last index k.*

Proof. First of all, the fact that the set S is formed by variables satisfying the assumption 1 means that only the variables in the set S contributes to the reduced Cartan characters when we apply the weaker Cartan's test: recall that in using the weaker test, we ignore all terms linear in $\theta_i \wedge \theta_j$ where θ_i, θ_j are the independent forms for the differential system.

As usual, the character s_0 is just the number of one-form equations. But now we know that this character has another signification: Let T be the set of dependent variables. The set $T - S$ is the set of dependent variables whose differential can be expressed linearly in terms of the θ_i. Now these linear relations must come from the one-form equations. However, by assumption 1 there are no linear relations among the differentials of the variables in S, hence the number of variables in the set $T - S$ is exactly the number of one-form equations, i.e., s_0.

The condition 2 together with the condition 4 imply that each variable in the set S contributes exactly one to one of the reduced character $s_1, s_2, \ldots, s_n$, and if the variable u_I has its last index equal to k, it contributes to s_k. This can be seen if we construct the integral element, step by step, by going through the directions $\theta_1, \theta_2, \ldots, \theta_n$. For s_k, it is equal to the number of independent coefficients of θ_k which are also independent of all coefficients of θ_l for $l < k$ as written out in condition 4, and condition 2 implies that such independent coefficients are contributed exactly by the variables u_I with the last index equal to k. (Observe also that the condition 2 implies $s_1 \geq s_2 \geq \ldots$, and since any two form equations are obtained by differentiating a one-form equation, we also have $s_0 \geq s_1$, which is by the way guaranteed to hold for any Pfaffian system.)

Now, since the number of variables in $T - S$ is equal to s_0, the number of variables in S is equal to $s_1 + s_2 + \cdots + s_n$, the total number of variables is hence equal to $s_0 + s_1 + \cdots + s_n$. What this shows is that the *pseudo-character*, σ_n, is just equal to the last character s_n, so in applying Cartan's test we can use the last character in place of the pseudo-character.

Now we apply Cartan's test by counting the number of free parameters of the system. The condition 3 in effect lists all the free parameters, and we will count them also by their last index. Observe that every variable u_I in S whose last index is k generates exactly one free parameter for all indices $l \leq k$ and non-others. Hence, the free parameters with last index 1 are generated by all the variables in S whose last index is ≥ 1, the free parameters with last index 2 are generated by the variables in S whose last index is ≥ 2, etc. Thus, the variables in S whose last index is k is counted exactly k times in the enumeration of free parameters. This means that Cartan's test

$$N = s_1 + 2s_2 + \cdots + ns_n$$

is automatically satisfied, and hence the system is involutive and the reduced Cartan characters that we calculated by counting indices above is equal to the real characters.

Q.E.D.

Of the four conditions required by the lemma, the condition 3 is most difficult to check since it requires us to construct the linear integral element of the system. There are, however, systems for which we know before hand that the linear integral elements must be of a certain form, for which it is more convenient to replace the condition 3 by some other conditions.

COROLLARY. *Assume that we have a Pfaffian system satisfying the conditions 1, 2, 4 stated above and the following conditions:*

- *We can find a partition of the set S into $\{S_\lambda\}$ satisfying the following: the set of variables in a partition S_λ have the same symbol with the same indices except the last one, and the last index of them goes from* 1 *to a certain integer l, where l is the number of variables in S_λ.*
- *For each S_λ in which the last index goes from* 1 *to l, the same symbol with last indices going from $l+1$ to n are also variables of the system, but they are not in the set S.*
- *The parameters for a linear integral element are obtained by adjoining one index to the variables, and the parameters satisfy* only *the following relations:*
  - *Any parameter is equal to the parameter obtained by permuting its last two indices, up to known functions of the variables: $u_{Ipq} = u_{Iqp} + \ldots$.*
  - *Any parameter obtainable by adjoining an index to a variable* not *in the set S is not free.*

Then the system is involutive, and its set of Cartan characters can be calculated in the following way: the k-th character s_k is equal to the number of variables in the set S with last index k.

Proof. It suffices to show that the conditions listed above implies the condition 3. To do this, we use the relations $u_{Ipq} = u_{Iqp} + \ldots$ to remove all parameters that can be obtained from variables not in the set S. But since in each partition $\{S_\lambda\}$ the indices runs from 1 to l, what remains are exactly the parameters satisfying the condition 3. Q.E.D.

48 The Frobenius integrable case and the inconsistent case. Then there remains a special case, not covered by our consideration above, which needs special consideration, namely the case where the set S is empty. There are two possibilities now: either the system is Frobenius integrable (corresponding to the case where we have s_0 equal to the number of dependent variables and all other Cartan characters vanish), or the system is in need of prolongation or inconsistent. How do we distinguish between these cases?

Since now $df = 0 \pmod{\theta_i}$ for any dependent variable f, we can express the two form equations entirely in terms of $\theta_i \wedge \theta_j$. The question is: after this substitution, are the two-form equations identities? If they are, then the Frobenius theorem applies: the forms defining the differential system include the one forms $df - f_{ij}\theta_j$, and to say that all the two form equations become identities when such one form equations are taken into account implies precisely the assumptions of the Frobenius theorem. If, on the other hand, we get non-vanishing terms in the two-forms, then there are two further cases to consider. If the coefficients of $\theta_i \wedge \theta_j$ can be made to vanish by some special choices of the dependent variables, then these equations must be adjoined to the differential system and we need to start from the beginning again. If this is not possible, for example because the coefficients of $\theta_i \wedge \theta_j$ are explicit functions of the independent variables whose vanishing would imply relations among the independent variables, or constants, then the system is inconsistent.

Note that there is nothing that prevents us from prolonging an inconsistent system, but when such systems are prolonged, the new system will already contain inconsistent *linear* equations among them.

49 Examples of application to differential equations. The lemma and its corollary only applies to very specifically constructed Pfaffian systems. Their real power lies in the fact that for many systems, we can derive an equivalent system that satisfies the assumptions of this lemma. Both of them deal with variables with indices attached, and a principal way that such variables arise is by differentiating old variables and adjoining the derivatives as new variables to the differential system. We can also see how the last condition specified in the corollary usually arises: if the indices are differentiation indices, then $u_{Ipq} = u_{Iqp} + \dots$ is simply the relation for the exchange of indices: for ordinary derivatives we simply have $u_{Ipq} = u_{Iqp}$, and the dots arise when we have a connection and the derivative is covariant, and the connection coefficients are dependent variables not in the set S. In such cases care must be taken so that the third condition of the corollary are really the *only* relations satisfied by the parameters. When we find that there are additional relations, this could be due to the fact that the there are hidden relations among our variables and the best way to proceed is to immediately prolong the system.

Let us try the example that we have already dealt with by applying the Cartan's test: the equation

$$u_{xx} = u_{yy} = u_{zz}.$$

The independent variables are x, y, z, the dependent ones are $u_x, u_y, u_z, u_{xx} = u_{yy} = u_{zz}, u_{xy} = u_{yx}, u_{xz} = u_{zx}, u_{yz} = u_{zy}$. By condition 1, The set S is formed by the variables with no accompanying contact forms, namely $u_{xx}, u_{yx}, u_{zx}, u_{zy}$. We number the indices as

$$x = 1, \qquad y = 2, \qquad z = 3$$

When this choice is made, we are forced to take the following as the set S

$$u_{32}, \quad u_{31}, \quad u_{21}, \quad u_{11},$$

and we *cannot*, say, exchange u_{23} for u_{32}, since in the differential of the contact form both $du_{32} \wedge dx_2$ and $du_{23} \wedge dx_3$ occur, and condition 2 forces us to take the one with the lower last index as the "canonical" variable.

Then we can see that the conditions 2 and 4 are satisfied due to the form of the contact forms and their differentials, which are respectively the one and two form equations in the system. In order to apply the corollary, we divide the set S into three subsets:

$$\{u_{21}\}, \qquad \{u_{11}\}, \qquad \{u_{32}, u_{31}\}.$$

Now all except the third condition of the corollary are satisfied. If the third condition is also satisfied, then the following should be independent parameters:

$$u_{211}, \quad u_{111}, \quad u_{322}, \quad u_{321}, \quad u_{311},$$

but this is not true since $u_{322} = u_{zyy} = u_{yyz} = u_{xxz} = u_{zxx} = u_{311}$. Hence we immediately prolong the system by including

$$u_{321}, \quad u_{311}, \quad u_{211}, \quad u_{111}$$

as new variables. The new set S is then formed by these four variables. We divide them into four sets, each containing a single variable. If the conditions of the corollary are satisfied, the independent parameters should now be

$$u_{3211}, \quad u_{3111}, \quad u_{2111}, \quad u_{1111},$$

and this is now true. To see this rigorously we can write down all of the parameters, and then use all the relations that we have on them to remove the dependent ones, and verify that the remaining ones are really the set written above. In our case, the relations are of two kinds: first, we can exchange any two indices, and second, we can exchange the indices 33, 22 and 11 among themselves, and the assumptions of the lemma are satisfied whenever these relations are vacuous for the chosen parameters above. For the way we choose the parameters, the first kind of relations are always vacuous, whereas the second is vacuous for the prolonged system since the last index of the invariants in the set S is always 1, whereas for the unprolonged system since there are invariants in the set S whose last index is larger than 1 there are non-trivial relations among the chosen parameters. Hence by the corollary, we have $s_1 = 4$, since all four variables have last index equal to 1, and $s_2 = s_3 = 0$.

By applying the lemma, we can immediately read off the Cartan characters of many systems. For example, we can generalize the above to a single dependent variable u depending on n variables $x_1, x_2, \dots, x_n$, for which the differential equation is

$$u_{11} = u_{22} = u_{33} = \cdots = u_{nn}.$$

The conditions of the corollary will only be verified when we prolong the system such that variables now have n indices, otherwise a situation similar to $u_{322} = u_{311}$ occurring in the parameters that we have seen before will arise. Then we see that the set S is formed by all variables $u_{k_1 k_2 \dots k_n}$ for which

$$k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_{n-1} \geq k_n = 1,$$

and the only case where equality holds between any adjacent k_i and k_{i+1} is when $k_i = k_{i+1} = 1$. Since the last index is always 1, $s_2 = s_3 = \cdots = s_n = 0$. A little arithmetic shows that

$$s_1 = 2^{n-1}.$$

Here we can see the advantage of utilizing the lemma instead of applying the Cartan test directly, especially when n becomes larges.

Next, let us try the wave equation $u_{xx} + u_{yy} - u_{tt} = 0$. The variables in the set S are

$$u_{ty}, \quad u_{tx}, \quad u_{yy}, \quad u_{yx}, \quad u_{xx},$$

and setting $x = 1, y = 2, t = 3$, we divide them into

$$\{u_{32}, u_{31}\}, \qquad \{u_{22}, u_{21}\}, \qquad \{u_{11}\},$$

the parameters

$$u_{322}, \quad u_{321} \quad u_{311}, \quad u_{222}, \quad u_{221}, \quad u_{211}, \quad u_{111}$$

are really independent. Hence immediately we see that $s_1 = 3, s_2 = 2$, the general solution depends on two functions of two variables.

II. Application to equivalence problems

50 The equivalence problem. Besides differential equations, another way by which differential systems will take the form satisfying the lemma and corollary that will allow us to read off the Cartan characters directly from suitably arranged indices is when dealing with the degree of arbitrariness of equivalence problems in the analytical case. Here since we are only interested in the degree of arbitrariness, which has a precise meaning in the analytical case and which acts as a guide to the dimension of the Cauchy data in other cases, it is not even necessary to formulate the exterior differential system explicitly. In the following, we will formulate an algorithm that allows us to extract this information directly from the moving frame of the equivalence problem.

We will start by structuring a generic equivalence problem a bit.

51 The moving frame. First of all we place the system under consideration, which is an equivalence problem that we want to study, into the language of moving frames. Assume that this has been done: thus let ω_μ, $\mu = 1, 2, \ldots, N$ be a co-frame for a suitable manifold P constructed from the geometrical problem. We can then proceed to write down the structural equations

$$d\omega_\mu = I_{\mu\nu\lambda}\omega_\nu \wedge \omega_\lambda, \tag{2.2}$$

The quantities $I_{\mu\nu\lambda}$, which are scalar functions on P, which we will call the *fundamental invariants* of the system. In addition, we may have a set of functions J_α subject to certain constraints. We further assume that we can find a subset of the 1-forms, denoted by ω_i, $i = 1, 2, \ldots, n$, such that all ω_μ, $I_{\mu\nu\lambda}$ and F_α are labelled using indices $i, j, k \ldots$ instead of $\mu, \nu, \rho \ldots$, and those indices labelling the additional functions.

From now on we will treat the additional functions J_α (which may or may not involve explicit functional dependence among themselves and the $I_{\mu\nu\lambda}$) as additional fundamental differential invariants, and unless explicitly mentioned, "fundamental invariants" means both $I_{\mu\nu\lambda}$ and J_α.

Here note that we can take the indices $i, j, k \ldots$ to run over all values of $\mu, \nu, \lambda \ldots$, and then the assumption of the existence of this subset of indices will be valid for all systems. The utility of our assumption, on the other hand, will be clear in a moment.

Remark. In cases where we have additional functions J_α, these are considered further constraints: two systems are considered equivalent if and only if we can set $\omega_\mu = \omega'_\mu$ and $J_\alpha = J'_\alpha$. Allowing for the possibility of additional functions gives us great flexibility in application of our theory. For example, if for example, we have a system with one-forms ω_μ spanning the cotangent space but are not independent, we can choose a subset of these one forms as the co-frame and the rest can be written in terms of the co-frame with the help of additional functions, and thus this system can be treated by our algorithm. On the other hand, if a system has certain one-forms ω_μ which do not span the cotangent space, we can choose the other one-forms arbitrarily in order to form a coframe, and then prolong the problem to obtain a coframe on a suitable principal bundle. Thus in this case our algorithm is also applicable.

52 The derived invariants. We can differentiate the fundamental invariants:

$$dI_{\mu\nu\lambda} = I_{\mu\nu\lambda;\delta}\omega_\delta,$$

where we will call the functions $I_{\mu\nu\lambda;\delta}$ *first order derived invariants*. As ω_μ form a co-frame, formally they are uniquely determined in terms of the co-frame. Derivations can be carried out further:

$$dI_{\mu\nu\lambda;\delta} = I_{\mu\nu\lambda;\delta\gamma}\omega_\gamma,$$

where we will call the functions $I_{\mu\nu\lambda;\delta\gamma}$ *second order derived invariants*. Derived invariants of all order are defined recursively.

By our assumption, the one-forms ω_μ other than ω_i are labelled using indices $i, j, k \ldots$, and we denote such forms by $\omega_I, \omega_J, \ldots$ where $I, J \ldots$ are multi-indices in $i, j, k \ldots$. Thus, for any fundamental or derived invariant $I_{I;J}$, we have

$$dI_{I;J} = I_{I;Jk}\omega_k + I_{I;J;K}\omega_K,$$

which is nothing more than the above formulae rewritten using different indices.

We assume that the derived invariants $I_{I;J;K}$ can be expressed explicitly in terms of $I_{I;Jk}$ and lower order invariants, i.e., the capital indices, when adjoined to invariants, do not give rise to any new independent invariants.

Remark. As before, if we take $i, j, k \ldots$ to run over all possible indices, then the above assumption is trivially verified since there is then no $I_{I;J;K}$. In the applications that we have in mind, however, the ω_i will be the *horizontal* one-forms in a principal bundle and the above assumption means that only horizontal derivatives can be independent, i.e., we have a *connection* at work here. The derived quantities $I_{I;Jk}$ are then the *covariant derivatives* of $I_{I;J}$ in the bundle if the bundle is reductive.

53 Algebraic relations. Next we need to take into account algebraic relations of the invariants. When we write down the structural equations, the fundamental invariants appearing in the structural equations are subject to certain relations (symmetries). For example, for the most general structural equation (2.2), the symmetry is $I_{\mu\nu\lambda} =$

$-I_{\mu\lambda\nu}$. When we stipulate the additional functions, they may also be subject to certain relations. We call such algebraic relations the *fundamental algebraic relations.*

We can derive the structural equations:

$$0 = d^2\omega_\mu = dI_{\mu\nu\lambda} \wedge \omega_\nu \wedge \omega_\lambda + I_{\mu\nu\lambda} d\omega_\nu \wedge d\omega_\lambda - I_{\mu\nu\lambda}\omega_\nu \wedge d\omega_\lambda,$$

and, after using the structural equations themselves and the resolution of $dI_{\mu\nu\lambda}$ in terms of derived invariants, we obtain

$$0 = F_{\mu\nu\lambda}\omega_\mu \wedge \omega_\nu \wedge \omega_\lambda,$$

where $F_{\mu\nu\lambda}$ are functions in the differential invariants. We call the algebraic relations

$$F_{[\mu\nu\lambda]} = 0$$

the *Bianchi algebraic relations.*

If $I_{I;J}$ is a differential invariant, deriving it we get

$$dI_{I;J} = I_{I;Jk}\omega_k + I_{I;J;K}\omega_K,$$

and deriving again we get

$$0 = I_{I;Jkl}\omega_k \wedge \omega_l + C_{IJkl}\omega_k \wedge \omega_l + A_{IJAk}\omega_A \wedge \omega_k + B_{IJAB}\omega_A \wedge \omega_B,$$

where the functions C_{IJkl} contains only invariants of lower order.

We further assume that $A_{IJAk} = 0$ *and* $B_{IJAB} = 0$ *identically.*

Remark. Again, if $i, j, k\ldots$ runs over all values the assumption is trivially verified. If, as we have mentioned, the choice of indices $i, j, k\ldots$ comes from the existence of a connection, this assumption is also verified easily.

The relations

$$I_{I;Jkl} = I_{I;Jlk} + C_{IJkl}$$

are called the *generic algebraic relations.*

Obviously, if an invariant I_I is actually one of the additional functions in the system, there will not be any Bianchi algebraic relation for it.

If $R = 0$ is an algebraic relation, we can derive it to obtain

$$dR = R_i\omega_i + R_A\omega_A = 0,$$

and we have the new relations

$$R_i = 0, \qquad R_A = 0$$

which are called the *derived algebraic relations.*

Remark. As before, the relation $R_A = 0$ is usually either vacuous or an identity, which is the case whenever the coframe comes from a connection.

The defining relations, the Bianchi relations, the generic relations and their derived relations are all the algebraic relations of the invariants.

54 The involutive seeds and the degree of arbitrariness. Now our equivalence problem is formulated in the proper moving frame. We will now hand-pick some differential invariants of the moving frame, which we will call the involutive seeds of the system, that will allow us to calculate the Cartan characters of an exterior differential system that we will set up.

First we define the concept of *pre-seeds.* The choice of the set of pre-seeds is either the set of fundamental invariants (i.e., what we called $I_{\mu\nu\lambda}$, without any differentiation indices) together with the set of additional functions (i.e., what we called J_α before), or a set of invariants derived from another choice of the set of pre-seeds by replacing one or more invariants by all of its derivatives using the co-frame. In other words, a set of pre-seeds is obtained by a recursive procedure that starts from the set $S_0 = \{I_{\mu\nu\lambda}, J_\alpha\}$ and at each step replaces some of the invariants in S_i by all of its derivatives to obtain a new set S_{i+1}. We say that a set of pre-seeds S_k *covers* all of the invariants of the system that are in the set

$$C_k = \bigcup_{i=0}^{k} S_i.$$

The set $\tilde{C}_k = C_k - S_k$ is the set of invariants *strictly covered* by S_k.

For example, the set $\{I_{\mu\nu\lambda}, J_\alpha\}$ is a choice of pre-seeds, from this we can derive the sets $\{I_{\mu\nu\lambda;\delta}, J_\alpha\}$, $\{I_{\mu\nu\rho;\lambda\delta}, J_{\alpha;\mu}\}$, etc. It is also allowed to replace a single invariant by all of its derivatives. All invariants in this set as well as any invariant obtainable from invariants from this set by removing one or more indices from the end are covered.

A set of seeds is a subset S_k^* of a given choice of pre-seeds S_k such that the following conditions are satisfied:

I1. (Covering.) All invariants occurring explicitly in the structural equations $\omega_\mu = \cdots$ and the derived structural equations $d\omega_\mu = \cdots$, as well as in any algebraic relations to be enforced, together with all additional functions, are covered by the set of pre-seeds from which it is derived;

I2. (Independence.) In S_k^* there are no invariants that are algebraically expressible using only the invariants in the strictly covered set $\tilde{C}_k$. Furthermore, the invariants in S_k^* are functionally independent, and all of the invariants S_k can be expressed using S_k^* and $\tilde{C}_k$ subject to all of the algebraic relations being imposed on the invariants, i.e., S_k^* is a subset of S_k whose differentials form a base in the linear space formed by all of the differentials in S_k which are independent of the differentials of $\tilde{C}_k$.

I3. (Derivation index.) Every invariant in the set contains at least one derivation index. (This condition can be non-vacuous only for invariants coming from additional functions.)

We note that in specifying a choice of seeds, it is also necessary to specify how the set of pre-seeds from which it is derived is obtained from the set $\{I_{\mu\nu\lambda}, J_\alpha\}$ (although in practice this is usually obvious). This is particularly important since the choice of the set of seeds may be *empty* for a system with a non-zero number of fundamental invariants, in which case the derived invariants from a certain order are all dependent on invariants of lower order, and only by specifying how we obtain the set of pre-seeds do we know which invariants are covered and which are strictly covered.

Suppose now we have chosen a set of seeds. Now we give an ordering to the indices $i, j, \ldots, n$ labelled by the first few natural numbers, i.e., a function s such that $s(i) \in \mathbb{N}$, which amounts to a permutation, or relabelling, of the numerical indices if the indices themselves run from 1 to some number without jumps. Such an ordering makes a choice of seeds a set of *involutive seeds* if the following conditions are satisfied:

O1. (Minimality of lower indices.) If two invariants I_{Ii} and J_{Jj} occur together in a relation for some value of i and j and $i > j$, then I_{Ii} must *not* be an involutive seed.

O2. (Counting condition.) The set obtained from the set of seeds by replacing every seed $I_{ijk\ldots l}$ by all of its derivatives $I_{ijk\ldots lm}$ with $m \leq l$ consists of a set of independent invariants, independent among themselves and of the set C_k.

Remark. That an ordering we write down is an involutive ordering with respect to a system of involutive seeds is the *test* of our algorithm, and we will obtain information about the degree of arbitrariness only if this test passes. In the next section we will describe, by examples, of how to proceed in order to have a good chance of arriving at an involutive ordering.

We are now nearly done: armed with an involutive ordering for a system of involutive seeds, in many cases we can read off the degree of arbitrariness of the system directly. Let us describe one further test:

R. (Rank condition.) We can find a number of strictly covered algebraically independent invariants I_α, $\alpha = 1, 2, \dots$, which are *not* additional functions introduced and which when derived give

$$dI_\alpha = C_{\alpha\mu}\omega_\mu,$$

and the rank of the matrix $C_{\alpha\mu}$ is equal to the number of one-forms ω_μ.

Assume that this test is satisfied. Let S' be the subset of the set of involutive seeds S^*_k such that all elements are such that their last index is maximal with respect to the involutive ordering among elements of the set. Let this index be d and let k be the number of elements of S'. Then we say that the system has degree of arbitrariness k, occurring at dimension $s(d)$. If the set of involutive seeds is the empty set, then the system does not have any degree of arbitrariness. The system may or may not be inconsistent (later we will see how to distinguish between the two cases).

Granted, there are systems for which the test **R** fails. For such systems, we can still apply our algorithms, but the number we obtained is only meaningful if the system does not involve additional functions that we put in by hand, and still the number is only guaranteed to be an *upper bound* of the degree of arbitrariness of the system.

Remark. As we will see, if the test **R** fails, then the question we are posing are not formulated on the best space possible: we can reduce the problem into a lower dimensional one, for which the test **R** holds and we can obtain the degree of arbitrariness, not merely an upper bound. If **R** fails and we have additional functions defined on the space, we may be able to use the additional functions to specialise the moving frames so as to reduce the dimension. Note that such a reduction of dimension need not be a reduction of the dimension of the *base manifold*: for example, in the principal bundle over a manifold Riemannian, we may be able to reduce the bundle from an $SO(n)$ bundle to a suitable G bundle where G is a subgroup of $SO(n)$, and in extreme cases G may even be a discrete group.

As any well-formulated problem will satisfy the test **R**, most often the verification of this condition is easy.

Remark. The method that we have outlined above relies on the weaker Cartan test. Recall that even if the weaker Cartan test fails, the system may still be involutive. A very real consequence this has for the method of involutive seeds is that there are systems for which we cannot find any seeds which are involutive, but at the same time the system is not inconsistent, the simplest example is given by the system of two fundamental invariants I and J, two independent forms θ_1 and θ_2, and for which the algebraic relations are $I_{;1} = 0$, $J_{;2} = 0$. In these cases, changing the set of independent forms to another equivalent set formed by their linear combinations will always solve the problem, which can be shown by showing that whenever Cartan's weaker test fails but the system is involutive we can find another base for which the test passes. For example, we can replace the independent forms θ_1 and θ_2 by $\vartheta_1 = \frac{1}{2}(\theta_1 + \theta_2)$ and $\vartheta_2 = \frac{1}{2}(\theta_1 - \theta_2)$ above (we can see that such problems are aggravated by the fact that in such situations the forms θ_1 and θ_2 would seem to be the "natural" ones to use since the equations then become particularly simple). There are other more *ad hoc* ways of dealing with such systems, for example we can try to impose the algebraic conditions not all at the same time, but "step by step" instead. We will meet such examples in our study of structure-preserving submersions.

55 The directions of evolution. There is one additional bit of information that we can obtain from our manipulation of indices. Assume that for a system, our algorithm applies with the condition **R** satisfied. Then for a system of involutive seeds together with its involutive ordering, let S be the set of last indices of the involutive seeds, and let T be the set of all possible indices. Then the set of indices $T-S$ gives the *directions of evolution* of the problem. The interpretation is as follows: if we consider the Cauchy problem of the system, then the system is specified completely by specifying d functions on a k dimensional submanifold of the manifold in which the system is defined, where d is the degree of arbitrariness and k is the dimension at which the degree of arbitrariness occurs. But in general, this submanifold cannot be chosen at will: it must be transverse to the system of vectors having indices taken from the set $T-S$ for a certain choice of involutive seeds and ordering (this choice is in general not unique). As a consequence of our assumptions that the one-forms splits into two sets ω_i and ω_α and all independent invariants take the indices of ω_i, we see that all directions corresponding to ω_α are automatically directions of evolution.

Suppose now that we add some equations of motion to our system (see §**11**). Then we will have two sets of directions of evolution: one set for before adding the equations of motion, and one set for after. We can say hat, roughly, the common directions in these two sets are the "gauge directions", whereas the rest are the "physical directions of evolution".

If the space for which the system is defined is a principal bundle and ω_i are the horizontal forms whereas ω_α are the vertical forms, then the directions of evolution will automatically include the directions of ω_α, and this just affirms the fact that the data in the bundle is completely determined once we specify the data on a section of the bundle. For real, physical equations of motion, the single physical direction of evolution when compared with the system without the equations added must be the time direction.

There is a further constraint on the choice of the submanifold: it must not contain the so-called characteristic directions. For this constraint, see the proof of our algorithm.

III. Proof of the algorithm

56 Now we prove our algorithm. The proof consists of three parts:

1° Construct an exterior differential system for all the invariants of the system that must be satisfied by any solution of the system;

2° Calculate the degree of arbitrariness (the number of free functions) of the general solution to this exterior differential system;

3° Prove that the solution that we obtained is compatible with the *original* exterior differential system, i.e., the structural equations of the moving frame, hence the degree of arbitrariness we obtained really is the degree of arbitrariness of the system we are interested in.

57 The exterior differential system for the invariants. First we shall assume that the condition **R** is satisfied, and there are no additional functions added to the system. We now consider all the algebraically independent invariants to be independent variables. A subspace formed by a subset of these invariants will be the space on which we work.

Since two functions f, g, are *functionally independent* if and only if their differentials df, dg are *linearly independent*, for systems satisfying the condition **R**, we can find a full set of independent invariants such that the co-frame ω_μ is solvable in terms of functions and differentials of these invariants, e.g., see §**29** and §**35**. Now for any invariant $I_{I;J}$, we have the expansion

$$dI_{I;J} = I_{I;Jk}\omega_k + C_{I;J\alpha}\omega_\alpha$$

which by our assumption, only $I_{I;Jk}$ may contain new algebraically independent invariants. We will take such equations for a certain set of invariants to be the differential system for the invariants, taking care to constrain them by all the Bianchi relations of the systems. The forms ω_k, ω_α are now considered to be nothing more than shorthands for some linear combinations of the differentials of the invariants. This is an exterior differential system with independence condition: the independent one-forms are exactly the one-forms ω_i, ω_α, which gives an implicit independence condition for the invariants themselves. For the moment we will ignore the original structural equations.

Our differential system is closed by adding the equations obtained by exterior differentiation:

$$0 = d^2 I_{I;J} = d(I_{I;Jk}\omega_k + C_{I;J\alpha}\omega_\alpha). \tag{2.3}$$

Observe that, if the invariants $I_{I;Jkl}$ for all l are included in our sets of invariants, then such equations are identically satisfied: parts of the algebraic relations satisfied by $I_{I;Jkl}$ are exactly those that ensures this equation holds. Thus, for the two-form equations in our differential system, we only need to consider those coming from the differentiation of one-form equations consisting of the most number of derivation indices.

Expanding (2.3) and using the equations of the differential systems themselves, we get

$$0 = dI_{I;Jk} \wedge \omega_k + \cdots \tag{2.4}$$

where the dots denote two-forms formed with the independent one-forms. By the previous remark, such equations are non-vacuous only in the cases where derived invariants of $I_{I;Jk}$ are not included in our system, in which case $dI_{I;Jk}$ are now independent one-forms, and hence they are the only two-form equations in our differential system.

The crucial observation now is that, for such an exterior differential system, the set of independent variables that appear in the two-form equations $d^2 I_{I;J}$ which are non-zero $\pmod{\omega_k}$ is formed exactly by a choice of seeds, and for this differential system the total space of variables in the set of invariants covered by the seeds: the procedure which we used to choose the seeds is implicitly just the procedure in which we prolong the differential system involving fewer invariants to one that involves more invariants, where once all derivatives of an invariant is included as variables the differential invariant itself will no-longer occur as an independent one-form in the two-form equations of the system, and the condition **I1** ensures that the system includes all the invariants that must be included, **I2** ensures independence of the seeds.

58 The degree of arbitrariness of the system of invariants. We have now at our disposal an exterior differential system whose variables are the set of invariants covered by the seeds and whose independent conditions are the ω_i, ω_α. We now check that this exterior differential system satisfies the condition of the lemma.

Condition 1 is satisfied if we take the set S to be the set of seeds.

Condition 2 is satisfied when we find an *involutive* ordering for the indices: the requirement that the last index of the invariant k matches the ω_k for which a term $dI_{I;J} \wedge \omega_k$ occurs in the two form equations follows from the definition of the derived invariants, and the fact that it does not occur together with any ω_l for $l < k$ follows from our requirement of minimality of last indices (**O1**).

Condition 3 is ensured by our counting condition **O2** on the set of involutive seeds.

Lastly, condition 4 is always automatically satisfied, since for our differential systems are only one term $dI \wedge \omega_k$ for each k for each equation, as our equations are actually all equations defining the derivatives of invariants.

Thus, we can apply our lemma, and the set of Cartan characters are obtained by counting the indices on the seeds, as detailed by the conclusion of the lemma, which corresponds to our statement of the algorithm.

Note that since the coordinates we use are provided by the invariants themselves, we do not have to worry about any degree of arbitrariness coming from the "arbitrariness" in choosing the coordinates. This is what is meant by saying that our system is "coordinate-free": the price to pay being the condition **R**.

The claims of the "directions of evolution" follows directly from the interpretation of the Cartan–Kähler theorem.

Obviously, from the involutive seeds we can also read off the other Cartan characters besides the last one, but as we have discussed they do not have independent meaning.

59 The original equations. We have shown that the exterior differential system we constructed above is involutive. It remains to show that the original equations are also satisfied, and the degree of arbitrariness of the system we constructed can be interpreted as the degree of arbitrariness of the original system. This concerns the equivalence problem of the original system, and the conclusions we need have already been shown in §**29**: due to the way our exterior differential system is set up and that the condition **R** is satisfied, the original equations are satisfied by all equations of the solution of the constructed system and *vice versa*, and the degree of arbitrariness of the constructed system parametrises the freedom of the equivalence problem of the original system.

60 The Frobenius case and the inconsistent case. As in §**48**, it can also happen here that the set of involutive seeds is empty. Is the system inconsistent or Frobenius integrable? We know that the fact that the set of involutive seeds is empty simply shows that from a certain order all of the pre-seeds are expressible as the other invariants. Consequently, when all linear equations are taken into account, the two-form equations in the system contains only two-forms constructed using only the independent forms. Again, only when all of these terms vanish do we have Frobenius integrability. However, in the present case, since one of the requirement that needs to be satisfied by a choice of seeds is that, in **I2**, all of the invariants in the set S_k of pre-seeds which are *not* seeds need to be expressible using lower level invariants and the seeds, but if such incompatibility occurs this is impossible: there would not be such expressions subject to all of the algebraic relations being imposed.

The problem is that, we often realise that we cannot have any non-empty set of seeds before we have analysed the system of invariants and relations completely. This is the case where from a certain order all of the invariants are already expressible using lower order invariants by the relations that we need to impose. What we need to do here is thus to determine whether the empty set is a system of seeds for the given order at all. By the requirement **I2**, this now boils down to checking if all of the algebraic relations that we need to enforce are compatible, and the simplest way to proceed is again try to determine the *parameters* $I_{I;k}$ occurring in $dI_I = I_{I;k}\theta_k$ for *every* invariant I_I up to the order considered. As in §**48**, incompatible systems manifest themselves in that choosing such parameters are impossible due to the constraints of the differential system: for example, one equation may require a certain $I_{I;k} = 0$, whereas another one may require $I_{I;k} = 1$, and there is no way to reconcile these contradicting requirements.

61 The case with non-maximal number of invariants. The problem with the case where the condition **R** fails means that, first, it is impossible to solve all of ω_μ in terms of the differentials of the invariants, and thus it is not possible to form a differential system for the invariants in the way that we did for the non-singular case, and second, even if the first difficulty is somehow overcome, the matrix $I_{\mu,\nu}$ which we uses to solve for the forms is singular, and thus there is no way to ensure that the original structural equations are really satisfied.

For systems with a non-maximal number ρ of invariants, we can find $m - \rho$ vector fields $\mathbf{v}_\alpha$, corresponding to the symmetry group, for which

$$\mathcal{L}_{\mathbf{v}}\omega_\mu = 0, \qquad \mathbf{v}(c_{ijk,lm\ldots}) = 0.$$

(For why this is the case, see §**92**.) Now suppose we take a submanifold N of dimension ρ transverse to all of the the vectors $\mathbf{v}_\alpha$, and we find a solution of our system (i.e., a functional dependence of the forms ω_μ and the invariants $c_{\mu\nu\rho}$ on m coordinates x_μ satisfying the structural equations) valid in an infinitesimal neighbourhood of N, then using the system of vector fields $\mathbf{v}_\alpha$, this solution can be extended to the whole space. Also, any solution valid for the whole space M, when restricted to such a transverse submanifold N, will also satisfy the structural equations: the equations that are to be satisfied now are just the pullbacks of the equations on the total space, and exterior differentiation commutes with pullbacks. On the other hand, the converse is in general not true: for simplicity, let $x_1, x_2, \ldots, x_\rho$ be the coordinates on the submanifold N, and $x_{\rho+1}, \ldots, x_m$ be the transverse coordinates. Then our structural equations are written in the differentials of these coordinates. That we have a solution on N means that, for the structural equation,

$$d\omega_\mu(\partial_{x_\alpha}, \partial_{x_\beta}) = c_{\mu\nu\rho}\omega_\nu \wedge \omega_\rho(\partial_{x_\alpha}, \partial_{x_\beta}), \qquad (\alpha, \beta \le \rho)$$

is satisfied on at points on N. We can even show that for certain coordinates and for $\alpha, \beta > \rho$, this equation also holds, due to the action of the Lie group. However, in general, when $\alpha \leq \rho$ and $\beta > \rho$, there is no reason that this equation will hold. Hence, that the structural equations are satisfied when pulled back onto any submanifold N thus chosen is a necessary, but in general not sufficient condition for a solution of the original equations.

Now let us return to our differential invariants and involutive seeds. First observe that, if $I_{I;Jk}$ is an involutive seed, then ω_k depends on the differentials of the invariants: indeed,

$$dI_{I;J} = I_{I;Jk}\omega_{\bar{k}} + \cdots$$

and due to the requirements of the involutive seeds, when we prolong the problem,

$$dI_{I;Jk} = I_{I;Jk\bar{k}}\omega_{\tilde{k}} + \cdots$$

where $I_{I;Jk\bar{k}}$ is an *independent* invariant. This shows that for all solutions, we can find a submanifold N such that all the forms ω_k occurring with a differential of the involutive seed in equation (2.4) remain independent. On other other hand, clearly if for the system (2.4), if we take any $m - \rho$ forms ω_μ other than those ω_k occurring explicitly with the involutive seeds in (2.4) to be forms written in terms of the invariants, with *arbitrary* functional dependence, we can now solve the remaining ρ one-forms ω_μ, which contains the ω_k, in terms of the invariants. By a reasoning exactly the same as in the maximal rank case we see that this system is involutive, with degree of arbitrariness given by the last non-zero Cartan character. Observe also that if for a system of involutive seeds and ordering, if the maximal last index is k, then $\rho \geq k$: this can be seen easily from (2.4). This means that the degree of arbitrariness always occurs at a dimension $\leq \rho$.

Using the Lie group action, such a system of values of the invariants can be extended to the whole space. But as remarked earlier, there is no guarantee that after extension all of the structural equations will be satisfied, hence in this case we have obtained only an upper bound.

Remark. For the following special case, the upper bound is realised: the structural equations reads

$$\begin{cases} d\omega_\alpha = C_{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma, \\ d\omega_i = c_{ijk}\omega_j \wedge \omega_k, \end{cases}$$

where $C_{\alpha\beta\gamma}$ are constants, and all of ω_i can be solved in terms of c_{ijk} and their invariants: it suffices to first ignore the first set of equations and obtain the degree of arbitrariness for this system. This degree of arbitrariness is realised since the first set of equations is consistent (otherwise the Bianchi relations will have something that reduces to $1 = 0$), and from the theory of Lie groups we know that there exists a Lie group satisfying the first set of equations. The solution space is then the product space formed by the Lie group and the solution whose degree of arbitrariness we know.

62 Additional functions in the equivalence problem. We only consider the case satisfying condition **R**, as our algorithm states. In this case, we simply adjoin the additional functions and their derivatives to the system (2.4), and the result is obvious. The condition **I3** is necessary since only those functions satisfying **I3** will appear in the written out part in the right hand side of (2.4), for which the Cartan characters are calculated.

IV. Examples of calculations in physical problems

63 Degree of arbitrariness of Riemannian geometry with torsion. Einstein–Cartan gravity. For a Riemannian manifold, if, instead of writing the usual structural relations, we write

$$\begin{cases} d\omega_i = -\omega_{ij} \wedge \omega_j + \frac{1}{2} T_{ijk} \omega_j \wedge \omega_k, \\ d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l, \end{cases} \tag{2.5}$$

we have a Riemannian geometry with torsion $T_{ijk} = -T_{ikj}$. We can carry out the above procedure: the algebraic relations for R_{ijkl} are, up to terms in T_{ijk} and $T_{ijk;l}$, which has no effect in our theory, exactly the same as before, and there are no separate Bianchi relations for T_{ijk}. Thus, for the involutive seeds, we use $R_{ijkl;m}$ with symmetry as before, and $T_{ijk;l}$. As $T_{ijk;l}$ satisfies no Bianchi relation of its own, there is no restriction on l, and for $l = n$ the number of seeds is just the number of independent T_{ijk}. As j and k are antisymmetric by the defining relation, the additional contribution is $n^2(n-1)/2$.

Actually, since we already know the degree of arbitrariness of Riemannian geometry without torsion, we can reason as follows: the structural equation (2.5) defines the two one-forms ω_i and ω_{ij}, from which we can define uniquely the two one-forms ω_i and ω'_{ij}, such that these two forms satisfy the structural equations without torsion, by setting

$$\omega'_{ij} = \omega_{ij} + (-T_{ikj} + T_{jki} + T_{kji})\omega_k.$$

Hence the system (2.5) is completely equivalent to the structural equations without torsion together with the additional function T_{ijk} with the relations

$$T_{ijk} = -T_{ikj},$$

from which we can easily deduce the additional contribution to the degree of arbitrariness: it is $n^2(n-1)/2$.

Now for the Einstein–Cartan theory of gravity, in addition to the usual Einstein equations, we couple the torsion to spin density directly:

$$T_{ijk} = S_{ijk}$$

so that all degree of arbitrariness generated by T_{ijk} are killed. Thus, the theory has exactly the same degree of arbitrariness as the usual Einstein theory. Physically, we say that "spin density does not propagate": there are no "spin waves".

64 Gauge theories. Here we want to find the degree of arbitrariness of a classical gauge theory over a Riemannian geometry. A gauge theory on a Riemannian manifold is usually specified by writing down some gauge potentials A_{ai}, for which a is a group index (omitted if the group is one-dimensional) and i is a spacetime index, and A_{ai} depends on the spacetime coordinates only. This requires pn functions to define, where p is the dimension of the group and n is the dimension of spacetime. However, there is also the "gauge invariance" of A_{ai} that must be taken into account: thus, the true degree of arbitrariness is, by this intuitive argument, $p(n-1)$.

Let us calculate this number by our algorithm. In the formulation of moving frames, a gauge theory is constructed as follows: we have the structural equations for the Riemannian space, and we couple a Lie group G to it. Let us assume that the Lie group has Maurer-Cartan structural equations

$$d\alpha_a = -C_{abc}\alpha_b \wedge \alpha_c,$$

where α_a are the Maurer-Cartan forms and C_{abc} are the structure *constants* for the Lie group. We form the product space of the Riemannian principal bundle and the Lie group, and change the structural equation to

$$d\gamma_a = -C_{abc}\gamma_b \wedge \gamma_c + \tfrac{1}{2}F_{aij}\omega_i \wedge \omega_j.$$

The defining relations for F_{aij} are as follows: for the index a, the symmetry is the same as that of the form α_a. For the index i, j, we have $F_{aij} = -F_{aji}$. There is also an additional relation occurring at first order, namely

$$dF_{aij} = F_{aij;k}\omega_k + G_{bij}\gamma_b + (\text{terms in } \omega_{ij}),$$

where G_{bij} are functions of F_{aij}, the exact form depending on the group. For example, for the group $SO(p)$, the additional one-forms are γ_{ab} with a, b antisymmetric,

$$dF_{abij} = F_{abij;k}\omega_k - F_{cbij}\gamma_{ac} - F_{acij}\gamma_{bc} - F_{abkj}\omega_{ik} - F_{abik}\omega_{jk}.$$

Basically, the form of these relations just means that the group directions are not dynamical, as differentiations in these directions do not generate new independent invariants.

The Bianchi relations for F_{aij} is

$$F_{a[ij;k]} = 0,$$

and the rest of the relations are obvious. Thus, we can take the independent terms of F_{aij} to be those that have $i > j$, and using the Bianchi relation, take the independent terms of $F_{aij;k}$ to be those that have

$$i > j, \qquad i \geq k.$$

Then we take the set of involutive seeds be the independent $R_{ijkl;m}$ as before, and also the independent $F_{aij;k}$. The involutive order is the order that all indices $a, b, c\dots$ are considered greater than $i, j, k\dots$. The number of terms $F_{ani;n}$ is thus

$$p(n-1),$$

where p is the dimension of the Lie group, since the index a in F_{aij} is a group index. The condition **R** can be easily verified. Hence this is the additional contribution to the degree of arbitrariness, occurring at dimension n, as the not-so-precise argument at the beginning of this section shows. For example, for $SO(2)$ gauge theory (electromagnetism) in dimension $3+1$, this number is simply 3.

65 Yang–Mills equations. We now study the degree of arbitrariness of adding to the above system the classical Yang–Mills equation, which in our notation, reads

$$\sum_i F_{aij;i} = \text{source terms},$$

which has a total of $pn(n-1)/2$ equations: more than the degree of arbitrariness of the original system. For the equations of motion, when $j < n$, the constraints are just that

$$F_{anj;n}$$

are no longer independent by condition **O1**. When $j = n$, we have (omitting the group indices)

$$F_{1n;1} + F_{2n;2} + \cdots + F_{n-1,n;n-1} = \cdots,$$

so $F_{n,n-1;n-1}$ is no longer independent. Under Yang–Mills equation for the gauge fields *and* Einstein's equations for the gravitational field, the additional contribution to the degree of arbitrariness from the gauge fields is therefore the number of remaining normal terms of $F_{aij;n-1}$, which are

$$F_{a,n,j;n-1} \quad (j = 1, \dots, n-2), \qquad F_{a,n-1,j;n-1} \quad (j = 1, \dots, n-2),$$

giving the degree of arbitrariness

$$2(n-2)r.$$

occurring at dimension $n - 1$. This gives the number of fields we must specify on a hypersurface to have a well-defined Cauchy problem for Yang–Mills equations coupled to Einstein gravity, or in flat spacetime. Note that it is essential to impose the Einstein equations or some other conditions that completely specifies the spacetime geometry using Cauchy data on submanifolds of dimensions less than n, otherwise the $2(n-2)r$ degree of arbitrariness we get here at dimension $n-1$ will be "eclipsed" by the degree of arbitrariness of the Riemannian geometry at dimension n. Again, an essential feature is that when going up one dimension, we require *two* additional copies of the Lie algebra: this corresponds to *one* degree of freedom for the boson. When $r = 1$ and $n = 4$, we see that the photon has 2 degrees of freedom.

Remark. For specific dimensions and specific groups, Estabrooks [17] has set up explicit exterior differential systems for the Yang–Mills equations using coordinates, and, with the help of computer algebra programs, calculates the Cartan characters for the systems (which, as we saw in the proof of our algorithm, is intimately related to the degree of freedom). For example, for $SU(2)$ Yang–Mills equations of dimensions $3, 4, 5, 6$, the last non-vanishing Cartan characters occur at dimensions $2, 3, 4, 5$, and are $9, 15, 21, 27$. However, since Estabrooks used coordinates, the gauge degree of arbitrariness (3 for $SU(2)$ Yang–Mills theory) is still present. If we subtract it from his answers, we get $6, 12, 18, 24$, which is just our answer (c.f. §**11**). The Cartan characters obtained for Maxwell theory for dimensions $3, 4, 5, 6$ by Estabrooks can also be shown to be in complete agreement with our result by analogous reasoning. Of course, our algorithm is so simple so that it is unnecessary to resort to computers for the calculations, we not need to reason with the gauge degrees of arbitrariness since no coordinate is used, and our result does not depend on the dimension nor on the geometry of the underlying space (Estabrooks considered only theories set up in flat spaces).

66 Scalar field theory on Riemannian manifold. We have checked that our algorithm gives the correct answers for gravitons and gauge bosons. Let us now very briefly check the case of scalar fields. In our approach, adding a scalar field corresponds to having a free function to a Riemannian manifold. From our discussion about additional functions, it is obvious that at the kinematical level, the extra degree of arbitrariness is 1: denoting the field be the scalar function f, this degree of arbitrariness comes from $f_{;n}$. Now we study what happens when we specify the equations of motion for the scalar field, i.e., its dynamics.

We add the Klein–Gordon equation, which is, in moving frames, the single equation

$$\sum_{i=1}^{n} f_{;ii} = m^2 f.$$

Assuming that we have already killed all degrees of arbitrariness coming from the Riemannian metric. For any system of involutive seeds and ordering, $f_{;nn}$ is no longer considered a seed. Hence, by the generic relations, the degree of arbitrariness on a Cauchy surface comes from the two terms

$$f_{;n,n-1}, \qquad f_{;n-1,n-1}$$

and thus the degree of arbitrariness of a scalar field under Klein–Gordon equation is exactly 2, independent of the dimension of the manifold, occurring at dimension $n-1$. Of course, this translates to a degree of freedom of scalar particles $\frac{2}{2} = 1$.

V. The degree of arbitrariness of general very special relativity

67 Holonomy. Two classes of general very special relativity. So far we have been applying our method to rather old problems. Let us now focus on a recent problem, which concerns the fundamental symmetry of nature: the programme of very special relativity (VSR) and its generalisation to curved spacetime: general very special relativity (GVSR) [13, 20].

In short, the programme of VSR assumes that the true physical symmetry of nature is not the full Lorentz group, but only a maximal subgroup, taken to be the four-parameter similitude group $SIM(2)$. If we write the Lorentz group as

$$\begin{pmatrix} 0 & K_x & K_y & K_z \\ K_x & 0 & J_z & -J_y \\ K_y & -J_z & 0 & J_x \\ K_z & J_y & -J_x & 0 \end{pmatrix},$$

then $SIM(2)$ is

$$\begin{pmatrix} 0 & T_1 & T_2 & K_z \\ T_1 & 0 & J_z & -T_1 \\ T_2 & -J_z & 0 & -T_2 \\ K_z & T_1 & T_2 & 0 \end{pmatrix}.$$

At the observational level, it is argued that no large discrepancy with the usual theory is expected and only rather delicate experiments can decide whether $SIM(2)$ or the Lorentz group is the true symmetry group of nature. The theory is often characterised by "a metric plus a null vector field", since there is a null direction that is preserved by such a group:

$$\begin{pmatrix} 0 & T_1 & T_2 & K_z \\ T_1 & 0 & J_z & -T_1 \\ T_2 & -J_z & 0 & -T_2 \\ K_z & T_1 & T_2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} K_z \\ 0 \\ 0 \\ K_z \end{pmatrix}.$$

Instead of working in orthonormal frames, since we now have a null direction that is preserved, we can instead work in a light-cone frame. The full Lorentz group in the light-cone frame is

$$\begin{pmatrix} a & 0 & -T_1 & -T_2 \\ 0 & -a & -P_1 & -P_2 \\ P_1 & T_1 & 0 & J_z \\ P_2 & T_2 & -J_z & 0 \end{pmatrix}.$$

We require the null direction $(1,0,0,0)^T$ be preserved, and it is easy to see that this implies that $SIM(2)$ is obtained from Lorentz group by setting $P_1 = P_2 = 0$, and a is just the scaling factor in this direction.

Going from very special relativity to general very special relativity amounts to requiring that the $SIM(2)$ is only local, analogous to the case where the Lorentz group is only local in general relativity. Now we can immediately distinguish two kinds of theories, which form the starting points of generalising VSR: that is, when viewed as a Riemannian space, do we require the holonomy group of the spacetime to be restricted to $SIM(2)$?

First consider the case where no such restriction is placed. Then actually we do have a scenario which can be accurately described as "metric plus null vector field": we need the full metric to derive the differential invariants, which includes all components of the Riemann curvature, and as we know the degree of arbitrariness from this is $\frac{4(4-1)}{2} = 6$. On top of this we have the freedom of choosing the null direction at each point, and we can calculate this by an intuitive argument: the light-cone is a 3-dimensional hypersurface, and hence and null vector depends on three parameters, and hence any null direction depends on 2 parameters. Therefore the total degree of arbitrariness for this kind of general very special relativity is $6 + 2 = 8$. Since we now have even more degree of arbitrariness than in conventional relativity, we also need more equations of motions, and thus additional mechanisms are required to determine these degrees of arbitrariness in a dynamical theory.

The other scenario, where a holonomy restriction is applied, is what we will be concerned with here. Roughly speaking, here there is no way to obtain anything that transforms under the "forbidden" symmetries of the full Lorentz group, whereas in the scenario without holonomy restriction the "forbidden symmetries" can be seen by transporting a vector around a closed loop. Hence here the metric does not retain its full degree of arbitrariness, and instead of a "metric plus null vector field", the more accurate description is "metric minus null vector field". Investigating what is the degree of arbitrariness of the theory at the kinematical level is what we shall do now.

68 GVSR as Cartan's generalised space. GVSR can be construed as a generalised space by prolonging a four dimensional manifold M with $SIM(2)$. We can actually consider a more general setting: we do not restrict to four dimensions and take as our starting point the Cartan connection matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^+ & \alpha & 0 & \theta^i \\ \omega^- & 0 & -\alpha & 0 \\ \omega^i & 0 & \theta^i & {\omega^i}_j \end{pmatrix}$$

where ω^i, θ^i are both $n-2$ one-forms, and ${\omega^i}_j$ is in $SO(n-2)$. The first structural equations now read

$$\begin{cases} d\omega^+ = -\alpha \wedge \omega^+ - \theta^i \wedge \omega^i, \\ d\omega^- = \alpha \wedge \omega^-, \\ d\omega^i = -\theta^i \wedge \omega^- - {\omega^i}_j \wedge \omega^j. \end{cases}$$

Immediately there is something very peculiar: the distribution $\omega^- = 0$ is completely integrable. Remember that ω^+, instead of ω^-, is the dual form of the null direction.

The second structural equations include the curvatures and read

$$\begin{cases} d\alpha = & -\frac{1}{2}R_{-+ij}\omega^i \wedge \omega^j - R_{-++-}\omega^+ \wedge \omega^- \\ & - R_{-++i}\omega^+ \wedge \omega^i - R_{-+-i}\omega^- \wedge \omega^i, \\ d\theta^i = & -\alpha \wedge \theta^i - {\omega^i}_j \wedge \theta^j + \frac{1}{2}R_{i-jk}\omega^j \wedge \omega^k \\ & + R_{i-+-}\omega^+ \wedge \omega^- + R_{i-+j}\omega^+ \wedge \omega^j + R_{i--j}\omega^- \wedge \omega^j, \\ d{\omega^i}_j = & -{\omega^i}_k \wedge {\omega^k}_j + \frac{1}{2}R_{ijkl}\omega^k \wedge \omega^l \\ & + R_{ij+-}\omega^+ \wedge \omega^- + R_{ij+k}\omega^+ \wedge \omega^k + R_{ij-k}\omega^- \wedge \omega^k. \end{cases}$$

69 Bianchi identities. The first Bianchi identities are obtained by differentiating the horizontal forms twice. The equation $d^2\omega^- = 0$ gives

$$R_{-+ij} = 0, \qquad R_{-++i} = 0,$$

the equation $d^2\omega^+ = 0$ gives

$$R_{[i|-|jk]} = 0, \qquad R_{-+-i} = R_{i-+-}, \qquad R_{i-+j} = R_{j-+i}, \qquad R_{i--j} = R_{j--i},$$

and the equation $d^2\omega^i = 0$ gives

$$R_{i[jkl]} = 0, \qquad R_{ij+k} = R_{ik+j}, \qquad R_{i-+j} = 0, \qquad R_{i-jk} + R_{ijk-} + R_{ik-j} = 0.$$

Notice that R_{ij+k} is antisymmetric in the indices i and j but symmetric in j and k, hence it vanishes identically. Thus we can simplify our second structural equations to be

$$\begin{cases} d\alpha = & -R_{-++-}\omega^+ \wedge \omega^- - R_{i-+-}\omega^- \wedge \omega^i, \\ d\theta^i = & -\alpha \wedge \theta^i - \omega^i{}_j \wedge \theta^j + \frac{1}{2}R_{i-jk}\omega^j \wedge \omega^k + R_{i-+-}\omega^+ \wedge \omega^- + R_{i--j}\omega^- \wedge \omega^j, \\ d\omega^i{}_j = & -\omega^i{}_k \wedge \omega^k{}_j + \frac{1}{2}R_{ijkl}\omega^k \wedge \omega^l + R_{ij-k}\omega^- \wedge \omega^k. \end{cases}$$

For the second Bianchi identities, $d^2\alpha = 0$ gives

$$R_{i-+-;j} = R_{j-+-;i}, \qquad R_{-++-;i} = -R_{i-+-;+},$$

$d^2\theta^i = 0$ gives

$$R_{i-[jk;l]} = 0, \quad R_{i-+-;j} = -R_{i--j;+}, \quad R_{i-jk;+} = 0, \quad R_{i-jk;-} + R_{i--j;k} - R_{i--k;j} = 0,$$

and $d^2\omega^i{}_j = 0$ gives

$$R_{ij[kl;m]} = 0, \qquad R_{ijkl;+} = 0, \qquad R_{ijkl;-} + R_{ij-k;l} - R_{ij-l;k} = 0, \qquad R_{ij-k;+} = 0.$$

70 Differential invariants. Normal invariants. As we have done in the Riemannian case, we can try to find a system of algebraically independent invariants from the total set of invariants which we will call normal. By using the first Bianchi identities, we can stipulate that the following to be the normal zeroth order invariants:

Invariant	Normal terms
R_{-++-}	all
R_{-+-i}	all
R_{jk-i}	$j > k, k \geq i$
R_{i--j}	$i \geq j$
R_{ijkl}	$i > j,\ k > l,\ i \geq k,\ j \geq l$

For four dimensions, there are 1 normal R_{-++-}, 2 normal R_{-+-i} (R_{-+-2} and R_{-+-1}), 1 normal R_{jk-i} (R_{21-1}), 3 normal R_{i--j} (R_{2--2}, R_{2--1} and R_{1--1}), and 1 normal

R_{ijkl} (R_{2121}), giving a total 8 independent zeroth order invariants. We know that for a four dimensional Riemannian geometry, the number of algebraically independent components of the Riemann tensor is 20: this gives a first hint that the degree of arbitrariness of GVSR is much less than its Riemannian counterpart.

To be more precise on this point, let us go one order higher and consider the first order invariants. By using both the first and second Bianchi identities, we can stipulate the following normal invariants *in four dimensions*:

Invariant	Normal terms
$R_{-++-;-}$	all
$R_{-++-;i}$	all
$R_{-++-;+}$	all
$R_{-+-i;-}$	all
$R_{-+-i;j}$	$i \geq j$
$R_{21-1;-}$	all
$R_{21-1;1}$	all
$R_{i--j;-}$	$i \geq j$
$R_{i--j;k}$	$i \geq j \geq k$
$R_{2121;-}$	all
$R_{2121;i}$	all

This table also tells us in what order we must integrate our system: first the $-$ direction, then the i directions, and finally the $+$ direction, which is the direction of the null vector field. We see that this system satisfies all the requirements for involutive seeds, and the degree of arbitrariness is the number of independent derivatives in $+$, of which we have only one, namely $R_{-++-;+}$. *Hence GVSR has degree of arbitrariness exactly one*, down from 6 in the Riemannian case, and this is the degree of arbitrariness we have mentioned intuitively as "the degree of arbitrariness of the metric *minus* that of the vector field". We also see that this number is less than any intuitive and simplistic calculations: in particular, this degree of arbitrariness at the kinematical level is the same as the theory of Newtonian gravity (see later, §**104**, p. 144, concerning the degree of arbitrariness of the Newtonian setting).

Let us also mention that this calculation does not carry over directly to higher dimensions. To see what the problem is, note that we the following two relations:

$$R_{i-[jk;l]} = 0, \qquad R_{ijkl;-} + R_{ij-k;l} - R_{ij-l;k}.$$

Using only the first Bianchi identities, we can already make $R_{ij-k;l}$ satisfy $i > j \geq k$. By also using the second relation above, we can make $R_{ij-k;l}$ satisfy $i > j \geq k \geq l$, at the expense of introducing terms in $R_{ijkl;-}$. But then the first equation above gives relations among $R_{ijkl;-}$, and since when we lower the last index of $R_{ijkl;m}$ we get $R_{ijkl;-}$, this shows that the requirements for involutive seeds do not hold for our choice.

71 The initial data for the Cauchy problem. The first observation is that setting R_{-++-} to functions of coordinates gives a well-defined Cauchy problem ($R_{-++-;+}$ is the only independent invariant with a derivation index +), and in this case the degree of arbitrariness comes from the terms with derivative index 2, and among the normal first order invariants we have

$$R_{-+-2;2}, \qquad R_{2--2;2}, \qquad R_{2121;2} \tag{2.6}$$

so we need to specify three functions on the hypersurface. However, such a Cauchy data represents too big a departure from the usual theories: in particular, the degree of arbitrariness would then be odd, and hence this does not correspond to any physical degree of freedom of the "graviton", since in that case the degree of arbitrariness would always be even. Furthermore, it is hard to see whether such an "equation of motion" R_{-++-} comes from any action principle.

We can attempt to do more conventional things. First let us remark that we *can* set up orthonormal frames and hence we do have the metric in our theory, only that it does not have the full degree of arbitrariness. In orthonormal frames we can set up the Einstein–Hilbert action,

$$S_{EH} = \int \epsilon_{\mu\nu\rho\lambda}\Omega^{\mu\nu} \wedge \omega^{\rho} \wedge \omega^{\lambda},$$

where $\Omega^{\mu\nu}$ is the curvature 2-form,

$$\Omega^{\mu\nu} = R^{\mu\nu}{}_{\rho\lambda}\omega^{\rho} \wedge \omega^{\lambda},$$

and $\epsilon_{\mu\nu\rho\lambda}$ is totally antisymmetric in all indices (the Levi–Civita tensor). The usual variation gives the usual Einstein equations, there is no problem with that. But when we try to investigate the Cauchy problem for this equation, we encounter problems. Let us investigate only the vacuum case for which the equation is equivalent to the vanishing of the Ricci tensor $R_{\mu\nu} = 0$. In our light-cone frame, we have the following expressions for the Ricci tensor

$$\begin{cases} R_{++} = 0, \\ R_{+-} = R_{-++-}, \\ R_{--} = R_{2-2-} + R_{1-1-}, \\ R_{22} = R_{1212}, \\ R_{11} = R_{1212}, \\ R_{12} = 0. \end{cases}$$

If we are working in GVSR, the identities $R_{++} = R_{12} = 0$, $R_{11} = R_{22}$ are automatic. We see that besides setting R_{-++-}, this also sets R_{2121} and R_{2-2-}. Since we have in this case implicitly set R_{-++-}, the previous discussion shows that the degree of arbitrariness is obtained by adding to the system (2.6) two additional equations. By our discussion in §**11**, the new degree of arbitrariness is at least one at dimension $n-1$. But from the form of (2.6), it is also at most one: since by setting R_{2121} and R_{2-2-} we have set the last two terms in (2.6). This allows us to immediately reject the theory as a viable candidate for physical theories: the degree of arbitrariness 1 is unacceptable, since as an odd number it cannot be interpreted as coming from the degree of freedom of a physical particle, and if we interpret it as something else then there is no propagation for gravitation.

There is another problem with trying to use the Einstein–Hilbert action in GVSR: if instead of doing a simple variation, we use Palatini's procedure and try to derive the Einstein–Cartan theory for coupling to the spin density of matter, we get the variation of the rotational part,

$$\delta S_{EH} = 2\int \epsilon_{\mu\nu\rho\lambda}\xi^{\mu\nu}\wedge T^{\rho}{}_{\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta}\wedge\omega^{\lambda} + \text{boundary terms},$$

and in the usual Riemannian case, this action (not coupled to matter) ensures that the torsion tensor $T^{\rho}{}_{\gamma\delta}=0$. But here the variation of the connections, $\xi^{\mu\nu}=\delta\omega^{\mu\nu}$, is constrained by the Lie algebra of $SIM(2)$, and as a consequence we cannot choose them as freely as in the Riemannian case and even in vacuum we cannot ensure that the torsion tensor vanishes.

In view of all these difficulties associated with GVSR with holonomy constraint, we conclude that such a theory is not a viable alternative to general relativity. Breaking Lorentz invariance in this way also breaks the delicate balance in the dynamical theory of Einstein equations. Our method here is also applicable for any proposed gravity theory with a different local symmetry group, and provides a first test any potential candidate for the physical theory needs to pass: i.e., very roughly speaking, gravity must propagate, and there must be enough degree of arbitrariness allowing us to have an interpretation of the free functions of the theory as coming from the degree of freedom of particles.

For GVSR without holonomy constraint, the problems we have encountered here do not arise and the Einstein equations are well-defined. At the dynamical level, challenges arise from the opposite direction: there the degree of arbitrariness of null vector field is completely free and we also need to find an equation of motion for the vector field.

Chapter 3

Riemannian submersions and structure-preserving submersions

I. The structural equations

72 The dual of immersion. In this chapter we will study the problem of structure-preserving submersions, and first the restricted problem of Riemannian submersions.

In a sense, a structure-preserving submersion is the "dual" of a structure-preserving immersion, the most well-known example of which is isometric embedding in Riemannian spaces. For isometric embedding, we have a target space M, a object space B, and a map

$$\iota : B \to M$$

which is, first of all, an immersion: the rank of this map is maximal and equal to the dimension of B, and in addition it preserves the Riemannian metric under pull-back, namely, let ds^2 be the metric on M and $d\bar{s}^2$ be the metric on B, we have

$$\iota^* ds^2 = d\bar{s}^2. \tag{3.1}$$

For many structures, given any well-behaving immersion, such an embedding preserving structure can be defined uniquely.

The gist of an isometric embedding is the map (3.1). Hence, for our present purpose, the gist of a structure-preserving submersion is the dual of the map (3.1), which concerns the additional "structure" defined on the total space.

Let M be a Riemannian space (the total space), with a Riemannian metric ds^2 defined on it. Let B be another Riemannian space, of less dimension than M, with its own metric $d\bar{s}^2$, and let $\pi : M \to B$ be a submersion map: it has maximal rank, equal to the dimension of B. We say that π is a Riemannian submersion, or a submersion preserving Riemannian structure, if the dual of (3.1) is satisfied:

$$\pi_*(ds^2) = d\bar{s}^2 \tag{3.2}$$

where $\pi_*(ds^2)$ denotes the *push-forward* of ds^2 under π. It is well known that pullback has much nicer properties than push-forward: most of the time the push-forward of a tensor is not even well-defined. Indeed, the requirement that (3.2) be well-defined already places severe constraints on the problem.

73 The moving frames of a Riemannian submersion. Since M and B are both Riemannian spaces, we set up moving frames, as we did before, on them. Let

$$ds^2 = \sum_{\mu=1}^{m}(\omega_\mu)^2, \qquad d\bar{s}^2 = \sum_{i=1}^{p}(\pi_i)^2$$

be the decomposition of the respective metrics into squares, where $m = \dim M$, $p = \dim B$ and $p < m$ (we do not consider the trivial case of $p = m$, which corresponds to local isomorphisms of metrics, not necessarily continuous). Using the $SO(n)$ freedom on M, we can further specialised the form of decomposition of ds^2 so that we have

$$ds^2 = \sum_{i=1}^{p}(\omega_i)^2 + \sum_{a=p+1}^{m}(\omega_a)^2, \tag{3.3}$$

such that ω_a span the kernel of the map (3.2). Then a further use of the $SO(n)$ symmetry on M (more precisely, the symmetry of the subgroup $SO(p)$), we can arrange that we always have

$$\pi_*(\omega_a) = 0, \qquad \pi_*(\omega_i) = \pi_i.$$

This now establishes an algebraic bijective correspondence between ω_i and π_i. Since this is a bijection, we can also write it as

$$\pi^*(\pi_i) = \omega_i, \qquad \pi^*(d\bar{s}^2) = \sum_{i=1}^{p}(\omega_i)^2. \tag{3.4}$$

These equations contain the same information as (3.2), but are now written in pull-backs. In writing (3.4):

- On B, there is no change to the symmetry group $SO(p)$;
- On M, we first did a reduction of principal bundle and reduced the symmetry group $SO(m)$ to $SO(p) \times SO(m-p)$ (this allows us to define (3.3)), and then we used up the $SO(p)$ degree of arbitrariness completely (this allows us to write (3.4)).

So our residual symmetries are $SO(p)$ and $SO(m-p)$, *which act on different spaces.*

74 Preliminary determination of the structural equations. From now on we will omit the pullback signs.

The forms π_i satisfy the structural equations

$$\begin{cases} d\pi_i = -\pi_{ij} \wedge \pi_j, \\ d\pi_{ij} = -\pi_{ik} \wedge \pi_{kj} + \frac{1}{2} S_{ijkl}\, \pi^k \wedge \pi^l. \end{cases} \tag{3.5}$$

The structural equations for ω_μ are

$$\begin{cases} d\omega_\mu = -\omega_{\mu\nu} \wedge \omega_\nu, \\ d\omega_{\mu\nu} = -\omega_{\mu\lambda} \wedge \omega_{\lambda\nu} + \frac{1}{2} R_{\mu\nu\rho\lambda}\, \omega^\rho \wedge \omega^\lambda, \end{cases}$$

which, under (3.3), becomes

$$\begin{cases} d\omega_i = & -\,\omega_{ij} \wedge \omega_j - \omega_{ia} \wedge \omega_a, \\ d\omega_a = & -\,\omega_{ab} \wedge \omega_b - \omega_{ai} \wedge \omega_i, \\ d\omega_{ij} = & -\,\omega_{ik} \wedge \omega_{kj} - \omega_{ia} \wedge \omega_{aj} \\ & + \frac{1}{2} R_{ijkl} \omega^k \wedge \omega^l + R_{ijka}\, \omega^k \wedge \omega^a + \frac{1}{2} R_{ijab}\, \omega^a \wedge \omega^b, \\ d\omega_{ab} = & -\,\omega_{ac} \wedge \omega_{cb} - \omega_{ai} \wedge \omega_{ib} \\ & + \frac{1}{2} R_{abcd}\, \omega_c \wedge \omega_d + R_{abci}\, \omega_c \wedge \omega_i + \frac{1}{2} R_{abij}\, \omega_i \wedge \omega_j, \\ d\omega_{ia} = & -\,\omega_{ij} \wedge \omega_{ja} - \omega_{ib} \wedge \omega_{ba} \\ & + \frac{1}{2} R_{iajk}\, \omega_j \wedge \omega_k + R_{iajb}\, \omega_j \wedge \omega_b + \frac{1}{2} R_{iabc}\, \omega_b \wedge \omega_c. \end{cases} \tag{3.6}$$

In principle, given a Riemannian submersion, the quantities $R_{\mu\nu\rho\lambda}$ and S_{ijkl} appearing in the equations (3.5) and (3.6) and their covariant derivatives in each space form a complete system of differential invariants for the Riemannian submersion (how the submersion is situated in M is contained in the equation (3.3)). But this has two shortcomings: first, these equations are *redundant*; second, these equations are useful if we already know a system is a Riemannian submersion and want to study its equivalence with another, but is of not too much help when we want to study the general properties of such systems, since we do not yet know if the given system of equations really determine a Riemannian submersion.

75 Submersion as geometrical structure on M alone. The reasoning above, though it gives us a picture of what is going on, does not help us very much when we want to calculate quantities in the problems. To proceed, let us start anew by forming the product space $M \times B$ and its suitable principal bundle.

First recall our discussion in §**41** (p. 64) about reductions of the principal bundle: by aligning the moving frame we have reduced the $SO(m)$ symmetry into $SO(p) \times$

$SO(n-p)$. This implies we can write

$$\omega_{ai} = -\omega_{ia} = K_{iab}\,\omega_b - M_{ija}\,\omega_j. \tag{3.7}$$

The symmetry for the system is now $SO(p) \times SO(p) \times SO(m-p)$, hence our bundle is $M \times B \times SO(p) \times SO(p) \times SO(m-p)$.

In this bundle, we are interested in finding the integral variety of the differential system

$$\vartheta_i = \pi_i - \omega_i = 0. \tag{3.8}$$

On these integral varieties, the submersion is a Riemannian submersion since (3.4) is satisfied. As usual, we also need to take into account that its exterior derivative also vanishes on the integral manifold, for which

$$d\vartheta_i = -(\pi_{ij} - \omega_{ij} + M_{ija}\,\omega_a) \wedge \omega_j + K_{iab}\,\omega_a \wedge \omega_b = 0.$$

We are only interested in integral varieties of the system (3.8) such that ω_i, ω_a remains independent. Hence, we must have [(†)]

$$\begin{cases} K_{iab} = K_{iba}, \\ M_{ija} = -M_{jia}, \\ \omega_{ij} = \pi_{ij} + M_{ija}\omega_a. \end{cases} \tag{3.9}$$

Under these conditions, (3.8) is completely integrable by Frobenius theorem. If these conditions are not satisfied, then there do not exist any integral manifolds under the independence requirements.

The equation (3.8) gives π_i linearly in terms of ω_i (they are equal), whereas the last equation of (3.9) gives ω_{ij} linearly in terms of π_{ij} up to linear terms in ω_i. Hence the integral manifolds we have found are of

$$m + \frac{p(p-1)}{2} + \frac{(m-p)(m-p-1)}{2}$$

dimensions and our final bundle is $M \times SO(p) \times SO(m-p)$ in which the $SO(p)$ components acts in a very special way. The coframe is taken to be formed by

$$\omega_i, \qquad \omega_a, \qquad \pi_{ij}, \qquad \omega_{ab}.$$

[(†)] The third relation require some care: indeed, the most general solution is of the form

$$\omega_{ij} = \pi_{ij} + M_{ija}\omega_a + c_{ijk}\omega_k, \qquad (c_{ijk} = c_{ikj})$$

but c_{ijk} must also satisfy $c_{ijk} = -c_{jik}$ for it to be absorbable into ω_{ij}. Using its two symmetries for a total of six times, we obtain $c_{ijk} = 0$.

This also accords with our previous reasoning where we have preliminarily determined the correct symmetry group of the theory. The reason that we have chosen π_{ij} instead of ω_{ij} as our variable will become obvious as we go on. Thus we have arrived at a coframe on M alone which completely characterises the Riemannian submersion. We have yet to derive the structural equation for the co-frame itself but at least we know how to proceed: these structural equations are consequences of (3.5), (3.6), (3.7), (3.8) and (3.9). The complete system of differential invariants are M_{ija}, K_{iab}, $R_{\mu\nu\rho\lambda}$ and S_{ijkl} and all of their covariant derivatives, but among these invariants there are many algebraic relations.

Before we go and derive these invariants, let us do some further clean up works to make our life easier in subsequent calculations.

76 The modified connection on M. According to §**43**, we can treat ω_i, ω_a, ω_{ij}, ω_{ab} together with

$$\omega'_{ai} = -\omega'_{ia} = 0$$

as a connection on M. But using (3.9), ω_i, ω_a, π_{ij}, ω_{ab} also form a connection on M. These two connections are related: the structural equations for ω_i can be written

$$d\omega_i = -\omega_{ij} \wedge \omega_j + M_{ija}\, \omega_a \wedge \omega_j = -\pi_{ij} \wedge \omega_j. \tag{3.10}$$

We see from this that even if we are given only the ω_{ij}, it is still helpful for us to define π_{ij} in the way that we have done, since *it is the torsion-free version of the part of the connection* ω_{ij}. In our case, since π_{ij} has its own independent geometrical origin (it was the connection on B), it is even more useful: if we define the covariant derivative ∇ using the formula of the connection with π_{ij}, for example,

$$\nabla a_{ai} \equiv a_{ai;j}\, \omega_j + a_{ai;b}\, \omega_b = da_{ai} + \omega_{ab}\, a_{bi} + \pi_{ij}\, a_{aj},$$

then to say a quantity Q_I (which may have any indices) on M is independent of the vertical coordinates and hence actually well-defined on B (i.e., on M it is the pullback of some set of quantity on B for which the $a, b, \ldots$ indices acts simply as labels for functions on B and not as tensor indices), is just to say that

$$Q_{I;a} = 0 \qquad \text{for all } a.$$

In particular, the invariants S_{ijkl} satisfy such relations [†].

An important property of this connection is that, with respect to it, all of the invariants that occur in our structural equations are *tensors*. This means that in the

[†] This is not an additional assumption—we will derive this below.

coframe derivatives of the invariants, only those coframe derivatives with respect to the horizontal forms are algebraically independent. We can prove this easily: expanding the relation for $d^2\pi_i = 0$, we have

$$\begin{aligned} 0 = {} & \tfrac{1}{2} S_{ijkl;m} \pi_m \wedge \pi_k \wedge \pi_l + \tfrac{1}{2} S_{ijkl;\underline{mn}} \pi_{mn} \wedge \pi_k \wedge \pi_l \\ & - \tfrac{1}{2} S_{iklm} \pi_l \wedge \pi_m \wedge \pi_{kj} + \tfrac{1}{2} S_{kjlm} \pi_{ik} \wedge \pi_l \wedge \pi_m \\ & - \tfrac{1}{2} S_{ijkl} \pi_{km} \wedge \pi_m \wedge \pi_l + \tfrac{1}{2} S_{ijkl} \pi_k \wedge \pi_{lm} \wedge \pi_m, \end{aligned}$$

the underlined indices $\underline{mn}$ meaning that it should be considered a *single* index: it is the derivation index with respect to π_{mn}. If we focus on the terms involving forms like $\pi_{mn} \wedge \pi_k \wedge \pi_l$, we see that $S_{ijkl;\underline{mn}}$ is expressed linearly in terms of S_{ijkl} itself, which is what we want to show. But this expression is just the transformation of a tensor quantity with the indices $ijkl$ in the principal bundle!

Using entirely similar argument, by considering the relation $d^2\omega_{ia} = 0$ for example, it is straightforward, though a bit tedious, to show directly that the functions M_{ija} and K_{iab} transforms as tensors in the bundle A, with all their indices tensor indices. If we denote the connection on A by ∇, we see that for a tensor a_{ai}, we have

$$\nabla a_{ai} \equiv a_{ai;j}\, \omega_j + a_{ai;b}\, \omega_b = d a_{ai} + \omega_{ab}\, a_{bi} + \pi_{ij}\, a_{aj}.$$

The transformation laws for all other indices can be deduced by extending this formula linearly.

77 The geometry of the fibres. The structural equation for ω_a is now

$$d\omega_a = -\omega_{ab} \wedge \omega_b - K_{iab}\, \omega_b \wedge \omega_i - M_{ija}\, \omega_i \wedge \omega_j. \tag{3.11}$$

Notice that unlike in the ω_i case, this structural equation is already free of any absorbable torsion: the M_{ija} term is definitely not absorbable, and though we can write

$$d\omega_a = -(\omega_{ab} - K_{iab}\, \omega_i) \wedge \omega_b - M_{ija}\, \omega_i \wedge \omega_j.$$

this is no absorption either, since K_{iab} are symmetric in a, b whereas ω_{ab} are antisymmetric in a, b, so they do not appear at the same time in any term in the expansion. This also shows that we need to take both M_{ija} and K_{iab} as structural functions in the structural equation: M_{ija} not because that it appears in $d\omega_i$ (it is absorbable there), but because it also appears in $d\omega_a$.

Nonetheless we can still simplify things in the vertical direction a bit. In (3.6) there is the structural function R_{abcd}, and as we shall see, up to the usual symmetry of Riemannian tensor *with indices a, b, c, d only*, all of its normal components are independent. Nonetheless it is hard to say what this quantity represents, except that it is some linear combination of sectional curvatures for the total space M. We will find another set of quantities which play the same role as this R_{abcd}.

The structural equation (3.10) shows that in the space M, the distribution defined by $\omega_i = 0$ is completely integrable. We know what they are: they are the fibres of the submersion, and every fibre is projected into a single point of B under the submersion. Note that a Riemannian metric is induced on the fibre, and if we work in the bundle, the fibre retains the full $SO(p) \times SO(m-p)$ symmetry in its tangent space at a single point. The equation

$$d\omega_{ab} = -\omega_{ac} \wedge \omega_{cb} - \omega_{ai} \wedge \omega_{ib} + \tfrac{1}{2} R_{abcd}\, \omega_c \wedge \omega_d + R_{abci}\, \omega_c \wedge \omega_i + \tfrac{1}{2} R_{abij}\, \omega_i \wedge \omega_j,$$

gives

$$d\omega_{ab} = -\omega_{ac} \wedge \omega_{cb} + K_{iac} K_{ibd}\, \omega_c \wedge \omega_d + \tfrac{1}{2} R_{abcd}\, \omega_c \wedge \omega_d \pmod{\omega_i}.$$

Hence if we define

$$S_{abcd} = R_{abcd} + K_{iac} K_{ibd} - K_{iad} K_{ibc} \tag{3.12}$$

then S_{abcd} is the Riemannian curvature on the fibres. Unlike S_{ijkl}, it can vary in both the ω_i and ω_a directions. In our calculations, we will prefer to use S_{abcd} over R_{abcd}.

Remark. It is *wrong* to write (c.f. the case with π_{ij})

$$d\omega_{ab} = -\omega_{ac} \wedge \omega_{cb} + \tfrac{1}{2} S_{abcd}\, \omega_c \wedge \omega_d,$$

where there is no modulus by ω_i at the end. If such an equation holds, it means that ω_{ab} depends on neither the coordinates whose differentials are ω_i, nor the group coordinates whose differentials are π_{ij}. The difference with the case of π_{ij} is that π_{ij} (and $\omega_i = \pi_i$) can be viewed as forms on the space B, so we are guaranteed that they depend only on the coordinates on B and on the group coordinates of $SO(p)$. On the other hand, ω_{ab} (and ω_a) can in general depend on all coordinates of M and on all the group variables of $SO(p) \times SO(m-p)$.

On the other hand, the relation (3.12) is consistent. The equation

$$d\omega_a = -\omega_{ab} \wedge \omega_b - K_{iab}\, \omega_b \wedge \omega_i - M_{ija}\, \omega_i \wedge \omega_j.$$

gives

$$d\bar{\omega}_a = -\bar{\omega}_{ab} \wedge \bar{\omega}_b,$$

where the bar indicates restriction onto the fibre. Hence $\bar{\omega}_{ab}$ is the unique torsion free connection on the fibre. Granted this, we have

$$d\bar{\omega}_{ab} = -\bar{\omega}_{ac} \wedge \bar{\omega}_{cb} + \tfrac{1}{2} \bar{S}_{abcd}\, \bar{\omega}_c \wedge \bar{\omega}_d. \tag{3.13}$$

Compare with

$$d\bar{\omega}_{ab} = -\bar{\omega}_{ac} \wedge \bar{\omega}_{cb} + \bar{K}_{iac}\bar{K}_{ibd}\,\bar{\omega}_c \wedge \bar{\omega}_d + \tfrac{1}{2}\bar{R}_{abcd}\,\bar{\omega}_c \wedge \bar{\omega}_d,$$

we obtain the barred version of equation (3.12). But as (3.12) is an algebraic equation, it also holds when we remove the bars.

78 Reduction of the structural functions. At this stage we have the bundle $M \times SO(p) \times SO(m-p)$, the coframe formed by the one-forms

$$\omega_i, \qquad \omega_a, \qquad \pi_{ij}, \qquad \omega_{ab},$$

and the structural equations

$$(3.14) \qquad \begin{cases} d\omega_i = & -\,\pi_{ij} \wedge \omega_j, \\ d\omega_a = & -\,\omega_{ab} \wedge \omega_b - K_{iab}\,\omega_b \wedge \omega_i - M_{ija}\,\omega_i \wedge \omega_j, \\ d\pi_{ij} = & -\,\pi_{ik} \wedge \pi_{kj} + \tfrac{1}{2}S_{ijkl}\,\omega_k \wedge \omega_l, \\ d\omega_{ab} = & -\,\omega_{ac} \wedge \omega_{cb} + \tfrac{1}{2}S_{abcd}\,\omega_c \wedge \omega_d + \tfrac{1}{2}(R_{abij} + 2M_{kia}M_{kjb})\omega_i \wedge \omega_j, \\ & +\,(R_{abci} + K_{jbc}M_{jia} - K_{jac}M_{jib})\omega_c \wedge \omega_i. \end{cases}$$

The problem is that these structural equations are not the only ones. We have other equations that we must satisfy, namely (3.6), (3.7), (3.8) and (3.9) and their exterior derivatives, and all components of $R_{\mu\nu\rho\lambda}$ are differential invariants, which do not all appear in (3.14). We would like to replace (3.14) by a system for which all the relations are taken care of automatically.

First consider the consequences of the equations themselves.

Equations for ω_{ia}:

$$d\omega_{ia} = \; -\,\omega_{ij} \wedge \omega_{ja} - \omega_{ib} \wedge \omega_{ba} + \tfrac{1}{2}R_{iajk}\,\omega_j \wedge \omega_k + R_{iajb}\,\omega_j \wedge \omega_b + \tfrac{1}{2}R_{iabc}\,\omega_b \wedge \omega_c.$$

The left hand side gives

$$\begin{aligned} d\omega_{ia} &= d(M_{ija}\,\omega_j - K_{iab}\,\omega_b) \\ &= M_{ija;b}\,\omega^b \wedge \omega^j + M_{ija;k}\,\omega^k \wedge \omega_j - K_{iab;c}\,\omega_c \wedge \omega_b - K_{iab;j}\,\omega_j \wedge \omega_b \\ &\quad - M_{kja}\,\pi_{ik} \wedge \omega_j - M_{ika}\,\pi_{jk} \wedge \omega_j - M_{ijc}\,\omega_{ac} \wedge \omega_j \\ &\quad + K_{kab}\,\pi_{ik} \wedge \omega_b + K_{icb}\,\omega_{ac} \wedge \omega_b + K_{iac}\,\omega_{bc} \wedge \omega_b \\ &\quad + M_{ija}(-\pi_{jk} \wedge \omega_k) - K_{iab}(-\omega_{bc} \wedge \omega_c - K_{jbc}\,\omega_c \wedge \omega_j - M_{ijb}\,\omega_i \wedge \omega_j). \end{aligned}$$

while the right hand side gives

$$
\begin{aligned}
d\omega_{ia} = {} & K_{ibc}\,\omega_c \wedge \omega_{ba} - M_{ikb}\,\omega_k \wedge \omega_{ba} \\
& + K_{jac}\pi_{ij} \wedge \omega_c - M_{jka}\,\pi_{ij} \wedge \omega_k + M_{ijb}K_{jac}\,\omega_b \wedge \omega_c - M_{ijb}M_{jka}\,\omega_b \wedge \omega_k \\
& + \tfrac{1}{2}R_{iabc}\,\omega_b \wedge \omega_c + R_{iajb}\,\omega_j \wedge \omega_b + \tfrac{1}{2}R_{iajk}\,\omega_j \wedge \omega_k.
\end{aligned}
$$

Equating the two sides, all terms containing ω_{ab} or π_{ij} cancel [(†)]. The rest gives three relations

$$
\begin{cases}
R_{aibc} = -K_{iab;c} + K_{iac;b} - M_{kib}K_{ack} + M_{kic}K_{kab}, \\
R_{aibj} = M_{ikb}M_{jka} - M_{ija;b} - K_{iab;j} - K_{iac}K_{jbc}, \\
R_{aijk} = M_{ija;k} - M_{ika;j} - 2M_{jkb}K_{iab}.
\end{cases}
$$

Equations for ω_{ij} (not π_{ij}):

$$
d\omega_{ij} = -\,\omega_{ik} \wedge \omega_{kj} - \omega_{ia} \wedge \omega_{aj}.
$$

The left hand side gives:

$$
\begin{aligned}
d\omega_{ij} &= d(\pi_{ij} + M_{ija}\,\omega_a) \\
&= -\,\pi_{ik} \wedge \pi_{kj} + \tfrac{1}{2}S_{ijkl}\,\omega_k \wedge \omega_l + M_{ija;b}\,\omega_b \wedge \omega_a + M_{ija;k}\omega_k \wedge \omega_a \\
&\quad - M_{kja}\,\pi_{ik} \wedge \omega_a - M_{ika}\,\pi_{jk} \wedge \omega_a - M_{ijc}\,\omega_{ac} \wedge \omega_a \\
&\quad + M_{ija}(-\omega_{ab} \wedge \omega_b - K_{kab}\,\omega_b \wedge \omega_k - M_{kla}\omega_k \wedge \omega_l).
\end{aligned}
$$

The right hand side gives:

$$
\begin{aligned}
d\omega_{ij} = {} & -\,\pi_{ik} \wedge \pi_{kj} - M_{kjb}\,\pi_{ik} \wedge \omega_b - M_{ika}\,\omega_a \wedge \pi_{kj} - M_{ika}M_{kjb}\,\omega_a \wedge \omega_b \\
& + K_{iab}K_{jac}\,\omega_b \wedge \omega_c - (K_{iab}M_{jma} - K_{jab}M_{ima})\omega_b \wedge \omega_m + M_{ika}M_{jma}\omega_k \wedge \omega_m \\
& + \tfrac{1}{2}R_{ijkl}\,\omega_k \wedge \omega_l + \tfrac{1}{2}R_{ijab}\,\omega_a \wedge \omega_b + R_{ijka}\,\omega_k \wedge \omega_a.
\end{aligned}
$$

Equating the two sides, again terms that contain π_{ij} or ω_{ab} cancel. The rest gives

$$
\begin{aligned}
R_{ijkl} &= S_{ijkl} + M_{ila}M_{jka} - M_{ika}M_{jla} - 2M_{ija}M_{kla}, \\
R_{ijab} &= M_{ikb}M_{jka} - M_{ika}M_{jkb} - M_{ija;b} + M_{ijb;a} + K_{jac}K_{ibc} - K_{iac}K_{jbc}, \\
R_{ijkb} &= M_{ijb;k} - M_{jka}K_{iab} + M_{ika}K_{jab} + M_{ija}K_{kab}.
\end{aligned}
$$

Equations for ω_{ab}:

$$
d\omega_{ab} = -\,\omega_{ac} \wedge \omega_{cb} - \omega_{ai} \wedge \omega_{ib}.
$$

[(†)]This is because we know that the invariants transform as tensors: see **§76**. Actually, it is obvious that if we do not make this assumption, then we will derive the fact that all the invariants are tensors here.

Unlike the previous two cases, here there is no second way that we can calculate the left hand side independently, see the remark in §**77**. Hence the only equation that we can draw from it is the definition of S_{abcd}:

$$R_{abcd} = S_{abcd} - K_{iac}K_{ibd} + K_{iad}K_{ibc}.$$

The other equations are already taken care of by the form of the equation (3.14).

For convenience, we collect what we obtain from above here:

$$\text{(3.15)}\qquad \begin{cases} R_{abcd} = S_{abcd} - K_{iac}K_{ibd} + K_{iad}K_{ibc}, \\ R_{ijkl} = S_{ijkl} + M_{ila}M_{jka} - M_{ika}M_{jla} - 2M_{ija}M_{kla}, \\ R_{ijab} = M_{ikb}M_{jka} - M_{ika}M_{jkb} - M_{ija;b} + M_{ijb;a} + K_{jac}K_{ibc} - K_{iac}K_{jbc}, \\ R_{ijkb} = M_{ijb;k} - M_{jka}K_{iab} + M_{ika}K_{jab} + M_{ija}K_{kab}, \\ R_{aibc} = -K_{iab;c} + K_{iac;b} - M_{kib}K_{ack} + M_{kic}K_{kab}, \\ R_{aibj} = M_{ikb}M_{jka} - M_{ija;b} - K_{iab;j} - K_{iac}K_{jbc}, \\ R_{aijk} = M_{ija;k} - M_{ika;j} - 2M_{jkb}K_{iab}. \end{cases}$$

It looks as if the left hand sides contain all components of the Riemannian tensor for the space M, but actually at this stage we can only be sure of the symmetries $R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} = -R_{\mu\nu\lambda\rho}$: in these equations, when the indices to which the symmetries apply, for example, μ and ν, are of the same kind, i.e., being i and j or a and b, then such symmetries can be explicitly checked. When they are of different kind we have left them implicit by, for example, writing the equation for only a, i, with the understanding that the equation for i, a is obtained by, e.g., $R_{ia\mu\nu} = -R_{ai\mu\nu}$. We need additional symmetries, i.e., the Bianchi identity $R_{\mu[\nu\rho\lambda]} = 0$, to really obtain all components of the Riemannian tensor. This will be taken care of in §**79**. By jumping ahead a little bit and assume we have derived the Bianchi identities and hence can express all components of the Riemann tensor, *we can regard all components of the Riemann tensor of the total space as expressible algebraically in terms of the rest, and express everything in terms of the quantities appearing on the right hand side.* On the other hand, equations (3.15) are useful when we want to study submersions where the geometry of M is *given*.

Eliminating the quantities $R_{\mu\nu\rho\lambda}$, our structural equations are now

$$\text{(3.16)}\qquad \begin{cases} d\omega_i = -\pi_{ij} \wedge \omega_j, \\ d\omega_a = -\omega_{ab} \wedge \omega_b - K_{iab}\,\omega_b \wedge \omega_i - M_{ija}\,\omega_i \wedge \omega_j, \\ d\pi_{ij} = -\pi_{ik} \wedge \pi_{kj} + \frac{1}{2}S_{ijkl}\,\omega_k \wedge \omega_l, \\ d\omega_{ab} = -\omega_{ac} \wedge \omega_{cb} + \frac{1}{2}S_{abcd}\,\omega_c \wedge \omega_d \\ \qquad\quad - 2K_{ic[a;b]}\,\omega_c \wedge \omega_i + \frac{1}{2}(-2M_{ij[a;b]} - K_{iac}K_{jbc} + K_{ibc}K_{jac})\omega_i \wedge \omega_j. \end{cases}$$

Looking at these equations, we can now deduce a set of quantities that, together with their covariant derivatives, are *guaranteed* to contain all algebraically or functionally independent differential invariants for our problem: they are

$$M_{ija}, \qquad K_{iab}, \qquad S_{ijkl}, \qquad S_{abcd}.$$

And all of these quantities appear as independent terms in the structural equation (3.16). If we want all the invariants that appear in the equations (not counting their derivatives, unless explicitly included), we need to add the following two quantities to the list [(†)]

$$K_{ia[b;c]}, \qquad M_{ij[a;b]}$$

which are contained in the first covariant derivatives of the first set.

II. ALGEBRAIC RELATIONS AMONG INVARIANTS

79 First Bianchi identities. We know that for any problem involving differential forms we need to exterior differentiate the given equations once to obtain additional information. The equations (3.6), (3.5) and (3.13), when differentiated, will give the so-called Bianchi identities which places additional algebraic constraints on the differential invariants. Conforming to the usual practice, we will call the relations obtained from differentiating the horizontal forms ω_i and ω_a the *first Bianchi identities*, and that of differentiating ω_{ab}, ω_{ij} and ω_{ai} *second Bianchi identities*.

Note that, instead of calculating the Bianchi identities explicitly, we could use the observation that all these equations are nothing more than the usual first Bianchi identities $R_{\mu[\nu\rho\lambda]}$ expressed in other variables, see §**34**. Hence we can just require that all relevant Riemannian tensors have this symmetry to obtain our relations. But there are too many relations to take care of and it is very messy.

Differentiating the first equations in (3.5) and (3.13) simply tells us that S_{ijkl} and S_{abcd} satisfy the relevant Bianchi identity in the subspaces. It only remains to calculate

[(†)]Note that these have the symmetries of R_{iabc} and R_{ijab} respectively. In particular, $K_{ia[b;c]}$ is no longer symmetric under exchange of the indices a and b.

$d^2\omega_a = 0$. We have

$$
\begin{aligned}
0 = d^2\omega_a &= d(-\omega_{ab}\wedge\omega_b - K_{iab}\,\omega_b\wedge\omega_i - M_{ija}\,\omega_i\wedge\omega_j) \\
&= -\tfrac{1}{2}R_{abcd}\,\omega_c\wedge\omega_d\wedge\omega_b - \tfrac{1}{2}R_{abij}\,\omega_i\wedge\omega_j\wedge\omega_b - R_{abci}\,\omega_c\wedge\omega_i\wedge\omega_b \\
&\quad - K_{iac}K_{ibd}\,\omega_c\wedge\omega_d\wedge\omega_b + K_{iac}M_{imb}\,\omega_c\wedge\omega_m\wedge\omega_b \\
&\quad - K_{ibc}M_{ima}\,\omega_c\wedge\omega_m\wedge\omega_b - M_{ika}M_{imb}\,\omega_k\wedge\omega_m\wedge\omega_b \\
&\quad + K_{iab}K_{jbc}\,\omega_c\wedge\omega_j\wedge\omega_i + K_{iab}M_{jkb}\,\omega_j\wedge\omega_k\wedge\omega_i \\
&\quad - M_{ija;b}\,\omega_b\wedge\omega_i\wedge\omega_j - M_{ija;k}\,\omega_k\wedge\omega_i\wedge\omega_j
\end{aligned}
$$

By equating independent three-forms, we obtain (the one from $\omega_a\wedge\omega_b\wedge\omega_c$ just gives us the usual Bianchi identity for R_{abcd}):

$$
\begin{cases}
0 = M_{ija;k}\,\omega_k\wedge\omega_i\wedge\omega_j - K_{iab}M_{jkb}\,\omega_i\wedge\omega_j\wedge\omega_k, \\
0 = -R_{abci}\,\omega_b\wedge\omega_c\wedge\omega_i + K_{jac}M_{jib}\,\omega_b\wedge\omega_c\wedge\omega_i + K_{iab;c}\,\omega_b\wedge\omega_c\wedge\omega_i \\
0 = -\tfrac{1}{2}R_{abij}\,\omega_i\wedge\omega_j\wedge\omega_b - M_{ika}M_{imb}\,\omega_k\wedge\omega_m\wedge\omega_b + K_{iab}K_{jbc}\,\omega_c\wedge\omega_j\wedge\omega_i \\
\qquad - K_{iab;j}\,\omega_j\wedge\omega_b\wedge\omega_i - M_{ija;b}\,\omega_b\wedge\omega_i\wedge\omega_j.
\end{cases}
$$

Equations for $\omega_i\wedge\omega_j\wedge\omega_k$. This gives

$$M_{ija;k} + M_{jka;i} + M_{kia;j} = M_{ijb}K_{kab} + M_{jkb}K_{iab} + M_{kib}K_{jab}$$

which shows that the quantities $M_{[ij|a;|k]}$ totally antisymmetric in the horizontal indices are not independent (they are expressible as quadratic functions of zeroth order invariants).

Equations for $\omega_i\wedge\omega_a\wedge\omega_b$. This gives

$$R_{abci} - R_{acbi} = K_{jac}M_{jib} - K_{jab}M_{jic} + K_{iab;c} - K_{iac;b}.$$

Let us do some index manipulation. We can write

$$
\begin{cases}
R_{abci} - R_{acbi} = +R_{abci} + R_{cabi} = K_{jac}M_{jib} - K_{jab}M_{jic} + K_{iab;c} - K_{iac;b}, \\
R_{baci} - R_{bcai} = -R_{abci} + R_{cbai} = K_{jbc}M_{jia} - K_{jba}M_{jic} + K_{iba;c} - K_{ibc;a}, \\
R_{cabi} - R_{cbai} = +R_{cabi} - R_{cbai} = K_{jcb}M_{jia} - K_{jca}M_{jib} + K_{ica;b} - K_{icb;a}.
\end{cases}
$$

Subtract the second and third equation from the first and divide the result by 2, we have

$$R_{abci} = K_{ibc;a} - K_{iac;b} + K_{jac}M_{jib} - K_{jbc}M_{jia}.$$

This is an independent equation, but comparing with the equation for R_{aibc} in (3.15), we see that this one simply expresses that

$$R_{abci} = R_{ciab}.$$

Equations for $\omega_i \wedge \omega_j \wedge \omega_b$. This gives

$$R_{abij} = -M_{kia}M_{kjb} + M_{kja}M_{kib} - K_{iab;j} + K_{jab;i} - K_{iac}K_{jcb} + K_{jac}K_{icb} - 2M_{ija;b}.$$

There is a hidden condition in this equation: from the way we obtain R_{abij}, we have $R_{abij} = -R_{baij}$. This means that in the right hand side of this equation, when we symmetrize a and b, we get a vanishing quantity, which gives

$$-K_{iab;j} + K_{jab;i} = M_{ija;b} + M_{ijb;a},$$

or simply $M_{ij(a;b)} = -K_{[i|ab;|j]}$. So we can take either of them as independent, but not both. For reasons that will become clear later, we will take $K_{[i|ab;|j]}$ as independent.

On the other hand, if we use this relation, by comparing with (3.15), we simply obtain

$$R_{abij} = R_{ijab}.$$

Note that the equation in (3.15)

$$R_{aibj} = M_{ikb}M_{jka} - M_{ija;b} - K_{iab;j} - K_{iac}K_{jbc}$$

is not symmetric under exchange of the two pairs of indices ai and bj. Using the relations we have just obtained, we can rewrite it as

$$R_{aibj} = M_{ikb}M_{jka} - K_{iac}K_{jbc} - \tfrac{1}{2}(M_{ija;b} + M_{jib;a} + K_{iab;j} + K_{jba;i}),$$

which manifestly has the required symmetry.

Remark. If we differentiate (3.16) to obtain the first Bianchi identities, we get the same two relations

$$\begin{cases} M_{ija;k} + M_{jka;i} + M_{kia;j} = M_{ijb}K_{kab} + M_{jkb}K_{iab} + M_{kib}K_{jab}, \\ -K_{iab;j} + K_{jab;i} = M_{ija;b} + M_{ijb;a}, \end{cases}$$

and the usual relations for the Riemannian tensors S_{ijkl} and S_{abcd}. The calculation is easier, but we need to do more work to relate them to the original manifold.

80 Second Bianchi identities. Algebraic relations. For convenience in calculations, it is preferable to define additional symbols in our calculations. We define

$$A_{abci} = -2K_{ic[a;b]}, \qquad A_{abij} = -2M_{ij[a;b]} - K_{aic}K_{jbc} + K_{ibc}K_{jac}.$$

The structural equations are now

$$\begin{cases} d\omega_i = & -\,\pi_{ij} \wedge \omega_j, \\ d\omega_a = & -\,\omega_{ab} \wedge \omega_b - K_{iab}\,\omega_b \wedge \omega_i - M_{ija}\,\omega_i \wedge \omega_j, \\ d\pi_{ij} = & -\,\pi_{ik} \wedge \pi_{kj} + \frac{1}{2}S_{ijkl}\,\omega_k \wedge \omega_l, \\ d\omega_{ab} = & -\,\omega_{ac} \wedge \omega_{cb} + \frac{1}{2}S_{abcd}\,\omega_c \wedge \omega_d + A_{abci}\,\omega_c \wedge \omega_i + \frac{1}{2}A_{abij}\,\omega_i \wedge \omega_j. \end{cases} \tag{3.17}$$

Note the simpler form of the last equation. We now want to calculate $d^2\pi_{ij} = 0$ and $d^2\omega_{ab} = 0$. The first gives simply that $S_{ij[kl;m]} = 0$, $S_{ijkl;a} = 0$, i.e., the usual second Bianchi identity for the base. The second gives us

$$\begin{aligned} d^2\omega_{ab} = & -d\omega_{ac} \wedge \omega_{cb} + \omega_{ac} \wedge d\omega_{cb} + \tfrac{1}{2}dS_{abcd} \wedge \omega_c \wedge \omega_d + \tfrac{1}{2}S_{abcd}\,d\omega_c \wedge \omega_d \\ & - \tfrac{1}{2}S_{abcd}\,\omega_c \wedge d\omega_d + dA_{abci}\,\omega_c \wedge \omega_i + A_{abci}\,d\omega_c \wedge \omega_i \\ & - A_{abci}\,\omega_c \wedge d\omega_i + \tfrac{1}{2}dA_{abij} \wedge \omega_i \wedge \omega_j + \tfrac{1}{2}A_{abij}\,d\omega_i \wedge \omega_j - \tfrac{1}{2}\,A_{abij}\,\omega_i \wedge d\omega_j. \end{aligned}$$

Equating independent three-forms, this gives us

$$\begin{cases} S_{ab[cd;e]} = 0, \\ S_{abcd;i} = 2A_{abci;d} - S_{abe[d}K_{|i|c]e} - S_{abde}K_{iec}, \\ A_{ab[ij;k]} = 2A_{abc[i}M_{jk]c}, \\ A_{abc[i;j]} = -\frac{1}{2}A_{abij;c} - A_{abd[i}K_{j]dc} - S_{abcd}M_{ijd}. \end{cases}$$

81 Order of covariant derivation. Ordinary derivatives commute, covariant derivatives not necessarily so. Let us try an examples. Let I be a scalar quantity. For its second order covariant derivatives, any algebraic relations are obtained by calculating d^2I. We have

$$d^2I = I_{;ab}\,\omega_a \wedge \omega_b + (I_{;ai} - I_{;ia} + K_{iab}I_{;b})\omega_i \wedge \omega_a + (I_{;jk} - I_{;a}M_{jka})\omega_k \wedge \omega_j.$$

So we have

$$\begin{cases} I_{;ab} - I_{;ba} = 0, \\ I_{;ai} - I_{;ia} = -K_{iab}I_{;b}, \\ I_{;kl} - I_{;kj} = I_{;a}M_{jka}. \end{cases}$$

The non-zero right hand sides show non-commutativity. However, note that the right hand side contains only derivatives of order 1 or less. This means that *for the purpose* *of counting independent components of covariant derivatives, it is only necessary to consider them as if the order of derivation is immaterial.*

The same holds for tensor quantities. The general rule for exchanging orders of derivations is complicated, but it can be seen from the following example:

$$\begin{aligned} d^2 T_{ia} = {} & (T_{ia;kj} - T_{ia;b} M_{jkb} - \tfrac{1}{2} T_{la} S_{iljk} - \tfrac{1}{2} T_{ib} A_{abjk}) \omega_j \wedge \omega_k \\ & + (T_{ia;dc} - \tfrac{1}{2} T_{ib} S_{abcd}) \omega_c \wedge \omega_d \\ & + (T_{ia;bj} - T_{ia;jb} + T_{ia;c} K_{jc;b} + T_{ic} A_{acbj}) \omega_j \wedge \omega_b . \end{aligned}$$

We see that for counting purposes the order of derivation still does not matter. Note also that in the above expression, all of the fundamental invariants M_{ija}, K_{iab}, S_{ijkl}, S_{abcd}, A_{abij} and A_{abci} appear.

This result also has the following importance for us. The first and second Bianchi identities are all identities that can be obtained from the original equations. But if we include derivatives of higher orders of the invariants, there are additional constraints obtained by differentiating the defining equations of the derivatives. As we have already seen, these equations merely gives us the commutativity properties of covariant derivatives.

For example, for M_{ija},

$$dM_{ija} = M_{ija;b}\,\omega_b + M_{ija;k}\,\omega_k - M_{kja}\,\pi_{ik} - M_{ika}\,\pi_{jk} - M_{ijb}\,\omega_{ab},$$

is the formula for covariant derivative. Differentiating this relation again we obtain the formula for exchanging two indices of derivation, which is exactly the *algebraic relation of the free parameters of the theory*, from the point of view of integral elements. If we include these free parameters as dependent variables by prolonging again, then these relations become the defining relations for these quantities.

82 Other identities. Every equation that we have should be differentiated to obtain new relations. We still have (3.15): do its derivatives give us new relations? When differentiated, the left hand sides will contain the derived Riemannian tensors, for example, $R_{abcd;e}\omega_e + R_{abcd;i}\omega_i$, whereas the right hand sides will contain derivatives of M_{ija}, K_{iab}, S_{abcd}, S_{ijkl}, linear in ω_i and ω_a. Hence if we are only interested in relations among M_{ija}, K_{iab}, S_{abcd}, S_{ijkl}, these relations do not give us any new information.

III. Involutivity and degree of arbitrariness

83 Functionally independent invariants. With the above results we are now finally ready to study the general algebraic relations among all invariants of the theory. For this purpose, it is best to forget the geometrical origin of the theory and treat the system (3.17)

$$\begin{cases} d\omega_i = & -\pi_{ij} \wedge \omega_j, \\ d\omega_a = & -\omega_{ab} \wedge \omega_b - K_{iab}\,\omega_b \wedge \omega_i - M_{ija}\,\omega_i \wedge \omega_j, \\ d\pi_{ij} = & -\pi_{ik} \wedge \pi_{kj} + \frac{1}{2}S_{ijkl}\,\omega_k \wedge \omega_l, \\ d\omega_{ab} = & -\omega_{ac} \wedge \omega_{cb} + \frac{1}{2}S_{abcd}\,\omega_c \wedge \omega_d + A_{abci}\,\omega_c \wedge \omega_i + \frac{1}{2}A_{abij}\,\omega_i \wedge \omega_j. \end{cases}$$

as *a priori* given. We know that this contains implicitly all required symmetries. The relations among the covariant derivatives of the quantities can all be deduced by differentiating again, and those relations between this system and the geometry on the total space (3.15) can be required by hand, if needed. The differential invariants that directly appear in these equations are

$$(3.18) \qquad M_{ija}, \qquad K_{iab}, \qquad S_{ijkl}, \qquad S_{abcd}, \qquad A_{abci}, \qquad A_{abij}.$$

As usual, a first question that should be asked is that, *in the general case*, assuming that the independence condition is given by the one-forms ω_i, ω_{ij}, ω_a, ω_{ab}, does this system already contain all the functionally independent differential invariants? The answer is yes. Let p denote the number of indices of $i, j, k, \dots$ and q the number of indices $a, b, c, \dots$. We know that, when p and q are sufficiently large, S_{ijkl} contains all invariants that can be solved for ω_i, ω_{ij}, and S_{abcd} contains all invariants that can be solved for ω_a, ω_{ab} (see §**30**). In our present case we can do even better and eliminate the "sufficiently large dimension" restriction from most cases: R_{ijkl} can always be solved for ω_{ij} and R_{abcd} for ω_{ab}, regardless of dimensions (in the smallest dimension we allow, $p = q = 1$, all of them vanish). K_{iab} can be used to solve for ω_i and M_{ija} can be used to solve for ω_a if $p > 1$. If $p = 1$ but $q > 1$, then K_{iab} contains additional terms that can be solved for ω_a. Thus it only remains the case $p = q = 1$, which will be treated separately later: §**87**. Observe that we have chosen not to use the two quantities A_{abci} and A_{abij}.

84 Setting up the differential system. From §**29**, we know that, since all functionally independent invariants already appear at the zeroth order, the differential system are the system of formulae expressing how to calculate the covariant derivatives of the quantities (3.18).

Unfortunately, the system

$$\begin{cases} dM_{ija} = M_{ija;k}\omega_k + M_{ija;b}\omega_b + \cdots \,, \\ dK_{iab} = K_{iab;j}\omega_j + K_{iab;c}\omega_c + \cdots \,, \\ dS_{ijkl} = S_{ijkl;m}\omega_m + \cdots \,, \\ dS_{abcd} = S_{abcd;i}\omega_i + S_{abcd;e}\omega_e + \cdots \,, \\ dA_{abci} = A_{abci;j}\omega_j + A_{abci;d}\omega_d + \cdots \,, \\ dA_{abij} = A_{abij;k}\omega_k + A_{abij;c}\omega_c + \cdots \,, \end{cases}$$

as it stands is no good for applying the method of involutive seeds: we want the non-vanishing functions that are written explicitly on the right hand sides to be *seeds*, so that the exterior derivatives of the *independent terms* among them satisfy no *a priori* given relations. This is not true here: the relations [†]

$$A_{abci} = 2K_{ic[a;b]}, \qquad A_{abij} = -2M_{ij[a;b]} - \cdots$$

allow us to resolve some of the dM_{ijab} in terms of $A_{abij;k}$ and $A_{abij;c}$, and some of $dK_{ica;b}$ in terms of $A_{abci;j}$ and $A_{abci;d}$.

To circumvent this difficulty, let us immediately effect a *partial* prolongation of our differential system: we include $M_{ija;k}$, $M_{ija;b}$, $K_{iab;j}$, $K_{iab;c}$ as new variables (note that this means that A_{abij} and A_{abci} and their covariant derivatives are no longer considered variables: they are expressible in terms of the ones we have just defined). Our system is now

$$\begin{cases} dM_{ija} = M_{ija;k}\omega_k + M_{ija;b}\omega_b + \cdots \,, \\ dK_{iab} = K_{iab;j}\omega_j + K_{iab;c}\omega_c + \cdots \,, \end{cases} \tag{3.19}$$

together with

$$\begin{cases} dM_{ija;k} = M_{ija;kl}\omega_l + M_{ija;bk}\omega_b + \cdots \,, \\ dM_{ija;b} = M_{ija;bk}\omega_k + M_{ija;bc}\omega_c + \cdots \,, \\ dK_{iab;j} = K_{iab;jk}\omega_k + K_{iab;cj}\omega_c + \cdots \,, \\ dK_{iab;c} = K_{iab;cj}\omega_j + K_{iab;cd}\omega_d + \cdots \,, \\ dS_{ijkl} = S_{ijkl;m}\omega_m + \cdots \,, \\ dS_{abcd} = S_{abcd;i}\omega_i + S_{abcd;e}\omega_e + \cdots \,. \end{cases} \tag{3.20}$$

The exterior derivation of (3.19) vanishes identically if we use (3.20). The *algebraically independent* quantities that are written explicitly on the right hand side of (3.20) are the *seeds*.

[†] These relations (and also similar ones) are not new assumptions: they are consequences of (3.17), obtained by exterior differentiating and equating independent forms.

85 The algebraically independent invariants and the seeds. The consideration of the previous section needs to be made precise by knowing what exactly the seeds are, using the algebraic relations among the invariants, which we already know (§**79**, §**80**, §**81**, §**82**).

To begin, the recognition of the following *normal* invariants should present no problem:

Invariant	Normal terms
M_{ija}	$i > j$
K_{iab}	$a \geq b$
S_{ijkl}	$i > j,\ k > l,\ i \geq k,\ j \geq l$
S_{abcd}	$a > b,\ c > d,\ a \geq c,\ b \geq d$
$S_{ijkl;m}$	$i > j,\ k > l,\ i \geq k,\ j \geq l,\ k \geq m$
$S_{abcd;e}$	$a > b,\ c > d,\ a \geq c,\ b \geq d,\ c \geq e$

At this moment let us recall another requirement for seeds: for a seed I_α, if $I_{\alpha;i}$ is an independent *parameter*, then $I_{\alpha;j}$ is independent as well for all $j < i$. Here we have two sets of indices, $i, j, k, \dots$ and $a, b, c, \dots$ and we need to determine whether we take $i > a$ or $i < a$. Here the choice is already made for us: look at (3.20), we see that $S_{ijkl;a} = 0$. Hence to satisfy the requirements for the involutive seeds and ordering, here it is necessary to take $i < a$. This also dictates our preferences for indices $i, j, k, \dots$ over $a, b, c, \dots$ when counting independent quantities. *For our following discussion, the "order" of constructing the integral variety is therefore first constructing the base manifold extended by ω_i, and then adding the leaves extended by ω_a.* The reverse order does not work.

The first Bianchi identity gives us two relations of the form

$$M_{[ij|a;|k]} = \cdots, \qquad M_{ij(a;b)} = -K_{[i|ab;|j]},$$

and these are the only additional relations for the first derivatives of M_{ija} and K_{iab}. So we can add another four entries to our table of normal invariants:

Invariant	Normal terms
$M_{ij[a;b]}$	$i > j,\ a > b$
$M_{ija;k}$	$i > j,\ i \geq k$
$K_{iab;c}$	$a \geq b$
$K_{iab;j}$	$a \geq b$

The only one that requires explanation is the second one. Indeed, for $M_{ija;k}$, consider the indices i, j, k to be all distinct. To be concrete, we can write them as 1, 2, 3. Then we can list all quantities with these indices:

$$M_{12a;3}, \quad M_{23a;1}, \quad M_{31a;2}, \quad M_{32a;1}, \quad M_{21a;3}, \quad M_{13a;2}.$$

Using $M_{(ij)a;b} = 0$, all those with $i < j$ can be expressed in terms of those with $i > j$. Hence we are left with

$$M_{31a;2}, \quad M_{32a;1}, \quad M_{21a;3}.$$

There is exactly one relation among these three quantities:

$$M_{31a;2} - M_{21a;3} - M_{32a;1} = \text{ functions of zeroth order invariants,}$$

so we can express $M_{21a;3}$ in terms of the other two and zeroth order invariants. Hence in this case we can take all normal expressions to satisfy $i > j$ and $i > k$.

Consider the case where there are only two distinct indices, and to be concrete let us assume that they are 1 and 2. Then we have the terms

$$M_{12a;1}, \quad M_{12a;2}, \quad M_{21a;1}, \quad M_{21a;2},$$

Using the antisymmetry in the first two indices, we can reduce this set to

$$M_{21a;1}, \qquad M_{21a;2}.$$

In this case the relation $M_{[ij|a;|k]}$ is satisfied identically. Since $M_{(ij)a} = 0$, we cannot have all indices i, j, k identical. Hence, we see easily that in all these cases, the quantities are normal if and only if

$$i > j, \qquad i \geq k.$$

The second Bianchi identities, together with the symmetry of derivation indices, gives us the symmetries for all the remaining invariants. For the second Bianchi identities, the "interesting" ones (i.e., the ones that are not of the form of a symmetry of a Riemannian tensor) are

$$\begin{aligned} S_{abcd;i} &= 2A_{abci;d} + \cdots , \\ A_{ab[ij;k]} &= \cdots , \\ A_{abc[i;j]} &= -\tfrac{1}{2}A_{abij;c} + \cdots . \end{aligned}$$

where dots denote terms of lower order. Expressing A_{abij} and A_{abci} in terms of the derivatives of M_{ija} and K_{iab}, these become

$$\begin{aligned} S_{abcd;i} &= -2K_{ic[a;b]d} + \cdots , \\ M_{[ij|[ab];|k]} &= \cdots , \\ K_{ic[a;b]j} &= M_{ij[a;b]c} + \cdots . \end{aligned}$$

The ghastly notation $M_{[ij|[ab];|k]}$ simply means

$$\tfrac{1}{3}(M_{ij[ab];k} + M_{jk[ab];i} + M_{ki[ab];c}).$$

Hence for the remaining invariants: $S_{ijkl;a}$ all vanish, $S_{abcd;i}$ we take to be independent. There remains

$$M_{ij[a;b]c}, \qquad M_{ij[a;b]k}, \qquad M_{ija;kl}, \qquad K_{iab;cd}, \qquad K_{iab;cj}, \qquad K_{iab;jk}.$$

$M_{ija;kl}$ will have normal terms satisfying

$$i > j, \qquad i \geq k, \qquad k \geq l.$$

Using the relation for $M_{[ij|[a;b]|k]}$, $M_{ij[a;b]k}$ will have normal terms satisfying

$$a > b, \qquad i > j, \qquad i \geq k.$$

For $M_{ij[a;b]c}$, it contains no normal terms since by our index preference they are expressed in terms of $K_{ic[a;b]j}$.

$K_{iab;jk}$ will have normal terms

$$a \geq b, \qquad j \geq k,$$

where as $K_{iab;cj}$ simply has

$$a \geq b.$$

The most important term is $K_{ica;bd}$. First of all, $K_{ic[a;b]d}$ is not independent since it is expressible in terms of $S_{abcd;i}$. Hence we should only consider $K_{ic(a;b)d}$. For $K_{ica;bd}$, we can swap the first two or last two indices. For the middle two, we have

$$K_{ica;bd} = K_{icb;ad} - S_{abcd;i} + \cdots$$

hence for counting purposes, *these four indices are totally symmetric.* We can arrange $K_{iab;cd}$ such that

$$a \geq b \geq c \geq d.$$

So finally, we are able to write the complete table of our invariants

Invariant	Normal terms
M_{ija}	$i > j$
K_{iab}	$a \geq b$
S_{ijkl}	$i > j,\ k > l,\ i \geq k,\ j \geq l$
S_{abcd}	$a > b,\ c > d,\ a \geq c,\ b \geq d$
$M_{ija;b}$	$i > j,\ a > b$
$M_{ija;k}$	$i > j,\ i \geq k$
$K_{iab;c}$	$a \geq b$
$K_{iab;j}$	$a \geq b$
$S_{abcd;i}$	$a > b,\ c > d,\ a \geq c,\ b \geq d$
$S_{ijkl;m}$	$i > j,\ k > l,\ i \geq k,\ j \geq l,\ k \geq m$
$S_{abcd;e}$	$a > b,\ c > d,\ a \geq c,\ b \geq d,\ c \geq e$
$M_{ija;kl}$	$i > j,\ i \geq k,\ k \geq l$
$M_{ija;bk}$	$a > b,\ i > j,\ i \geq k$
$K_{iab;jk}$	$a \geq b,\ j \geq k$
$K_{iab;cj}$	$a \geq b$
$K_{iab;cd}$	$a \geq b \geq c \geq d$

The second block contains the involutive seeds. It can be verified that the conditions for involutive seeds and ordering are satisfied: the seeds has the last index satisfying the condition that when this index is reduced, the quantity obtained is still a seed (for $S_{abcd;e}$ and $K_{iab;cd}$—the only two with last index of the set $a, b, c, \ldots$, it is necessary to verify that this condition holds when we change the last index into an index from the set $i, j, k, \ldots$). Hence this system is involutive. The degree of arbitrariness is given by the number of seeds whose last index is maximal. These are obtained only from

$$S_{abcd;e} \qquad \text{for} \qquad a = c = e = q,\ d \leq b < q$$

and

$$K_{iab;cd} \qquad \text{for} \qquad a = b = c = d = q.$$

These give a total of

$$s_{p+q} = \frac{q(q-1)}{2} + p$$

functions of $(p+q)$ variables, which constitutes the degree of arbitrariness in the general case.

There are two other characters that may be of interest:

$$\begin{cases} s_{p+1} = \dfrac{q^2(q^2-1)}{2} + \dfrac{pq^2(q+1)}{2}, \\ s_p = \dfrac{p(p-1)}{2} + \dfrac{q(q+1)[q^2-q-1+p(q+2)]}{2}. \end{cases}$$

If $p = 0$, s_{p+q} gives the degree of arbitrariness for q dimensional Riemannian space. If $q = 0$, s_p gives the degree of arbitrariness for p dimensional Riemannian space (in this case the formula for s_{p+q} does not make sense). The character s_{p+1} gives the minimal number of equations we need so as to kill all degree of arbitrariness on the leaves. Caveat: this counting includes all derived equations up to the order we are considering, for example, if we specify $M_{ija} = 0$, we automatically have also $M_{ija;k} = 0$, $M_{ija;b} = 0$, etc. Needless to say, these two characters, being non-maximal characters in the general case, depend on the prolongation we have used.

Remark. It is easy to see that the degree of arbitrariness calculation is independent of the connection we use to take the covariant derivatives (see §**76**).

86 Existence of structural preserving submersions. We now ask the following question: given a Riemannian geometry, does there exist a structural preserving submersion on it? A first attempt would be to use (3.15) and carry out the involutive procedure, where the right hand sides are now taken to be given functions. This would immediately lead to difficult calculations.

Actually, we can resolve the question in another way. Recall that in §**37** we have shown that the degree of arbitrariness of a general $p + q$ dimensional Riemannian geometry is

$$s_{p+q} = \frac{(p+q)(p+q-1)}{2}.$$

The difference of this degree of arbitrariness and the one we have found for a general structure preserving Riemannian submersion is

$$\frac{p(p+2q-3)}{2}.$$

This number is greater than zero except for the case of $p = q = 1$ (if p or q is zero, then the submersion is trivial; the formula for the degree of arbitrariness of Riemannian submersion does not hold for the case $q = 0$). Thus, we see that *except for the case of $p = q = 1$, Riemannian spaces that admit structure-preserving submersions are exceptional* (we can say roughly that they have measure zero in the space of all Riemannian geometries).

87 Existence of structural preserving submersions: case of $p = q = 1$. For $p = q = 1$, the two sets of degree of arbitrariness match, so it is possible that all 2 dimensional Riemannian spaces admit structural preserving submersions: this reasoning shows only that it is possible, since the equality of degree of arbitrariness shows only that the dimension of their solution space is the same. We will now prove that in the analytic case, this possibility is locally realised (globally, there might be topological obstructions).

Indeed, in two dimensions, the structural equation for a Riemannian submersion is exceptionally simple:

$$\begin{cases} d\omega_0 = K\omega_1 \wedge \omega_0, \\ d\omega_1 = 0, \end{cases}$$

where ω_0 lives on the leaf, ω_1 lives on the base, and there is no principal bundle: the reduction of the principal bundle of $SO(2)$ is complete. In other words, as long as we can choose a section of the bundle of 2 dimensional Riemannian geometry such that the structural equation takes the above form, this section, with its distinguished directions ω_0 and ω_1, furnishes a Riemannian submersion.

A general section gives

$$\begin{cases} d\theta_0 = a\theta_1 \wedge \theta_0, \\ d\theta_1 = b\theta_1 \wedge \theta_0. \end{cases}$$

We want to find a function t of two variables such that

$$d(\cos t\,\theta_0 + \sin t\,\theta_1) = 0,$$

then we can set $\omega_0 = \cos t\,\theta_0 + \sin t\,\theta_1$, and we are done. Expanding the above, we get

$$(-t_{,1}\sin t - t_{,0}\cos t + a + b)\theta_1 \wedge \theta_0 = 0.$$

Since now both $d\theta_0 = 0 \pmod{\theta_0}$ and $d\theta_1 = 0 \pmod{\theta_1}$, we can set $\theta_0 = dx$, $\theta_1 = dy$ for a certain system of coordinates (x, y). Then the equation in question becomes

$$\sin t\frac{\partial t}{\partial y} + \cos t\frac{\partial t}{\partial x} = a(x, y) + b(x, y),$$

and this system is of Cauchy–Kowalewski form, hence provided $a(x, y)$ and $b(x, y)$ are analytic functions, solution always exists. Q.E.D.

88 The Cauchy data for Riemannian submersions. The degree of arbitrariness of the submersion we have calculated gives us the number of functions we need to specify to have a well defined Cauchy problem. However, taken at face value, it requires us to specify $K_{iab;cd}$ for $a = b = c = d = q$ and $S_{abcd;e}$ for $a = c = e = q$, $d \leq b < q$, and these data are neither convenient nor very invariant. We will now propose some better ways of specifying the Cauchy data, which yields a well-defined Cauchy problem.

Our aim is to kill the above two terms in the list of involutive seeds. For $S_{abcd;e}$, we know what to do: by our discussion of Riemannian geometry, it suffices to specify the Ricci tensor. We need to check that when we lower the index $S_{abcd;e}$ when $e = 1$ to $S_{abcd;i}$, we maintain independence: this does hold. For $K_{iab;cd}$, we need q equations, and we will simply specify the contraction of K_{iab}:

$$K_i \equiv \sum_a K_{iaa}.$$

Hence for K_{iab}, we now take the invariant terms to be those where not both a and b take the maximal value. At first order, $K_{iab;c}$ and $K_{iab;j}$ also cannot have both a and b taking maximal value. At second order, we have an equation of the form

$$S_{abcd;i} = K_{icb;ad} - K_{ica;bd} + \cdots = 2K_{ic[b;a]d} + \cdots .$$

If the indices a, b both take maximal value, this is an identity. If a and c take maximal value, then $K_{icb;ad}$ is no longer considered independent. This means for $K_{iab;cd}$ to be independent, when a is maximal, we require $q = a > b \geq c \geq d$. There is also no problem when we lower $K_{iab;cd}$ to $K_{iab;cj}$.

Granted these, the character s_{p+q} is zero now. The contribution to s_{p+q-1} now comes from two parts. The first part, having its origin in $S_{abcd;e}$ on which the Ricci tensor condition has been imposed, is

$$q(q-3)$$

when $q \geq 3$. When $q = 2$ it is zero, and for $q = 1$ the case needs to be treated separately, since we pass directly to the reduced space (this case will be treated in a later chapter). There are only two contributions from $K_{iab;cd}$ now, namely

$$K_{iab;cd}, \qquad a = q \text{ or } q-1, \quad b = c = d = q-1.$$

So the degree of arbitrariness is now

$$s_{p+q-1} = 2p + (s_{q-1} \text{ for the Einstein theory of dimension } q).$$

Instead of specifying the Ricci tensor, we can also directly specify the metric of the fibre at each point of the space. Then S_{abcd} and all its derivatives are no longer independent, and it is easy to see that for this case, the degree of arbitrariness is simply $(q > 1)$

$$s_{p+q-1} = 2p,$$

the system is still involutive, showing that *it is always consistent to specify any geometry of the fibres independently at each point on the reduced manifold.* On the other hand, attempting to reduce the order where the first non-vanishing character occur down to the reduced manifold by directly specifying values for K_{iab} would lead to compatibility problems, which may be shown by the fact that under such constraints the system of invariants can no longer be considered a system of involutive seeds.

iv. A framework for general structure-preserving submersions

89 The definition. We have studied the problem of Riemannian submersions essentially as a problem in exterior differential systems. Now let us reflect on what we have done using more geometrical considerations, and at the same time reformulate our procedure so that it applies to more general structure-preserving submersions.

First: what do we mean by a "structure-preserving submersion"? We will first give the definition, then explain:

Definition. Let $\mathrm{pr}_M : P \to M$ and $\mathrm{pr}_N : Q \to N$ be two Cartan's generalised spaces, namely, M and N are the base manifolds, and P and Q are the principal bundles over M and N respectively. Let π_i, $i = 1, \dots, \dim N$ and π_μ be the Cartan connection on Q, in which the π_i are the horizontal forms and π_μ are the vertical forms. Let ω_i, $i = 1, \dots, \dim N$, ω_a, $a = \dim N + 1, \dots, \dim M$ and ω_α be the Cartan connection on P, in which the ω_i and ω_a are the horizontal forms and the ω_α are the vertical forms. A structure-preserving submersion is a solution (i.e., an *integral variety*) of the exterior differential system

$$\pi_i = \omega_i \tag{3.21}$$

with the independence conditions given by the forms

$$\pi_i, \qquad \pi_\mu, \qquad \omega_a$$

together with those of the forms

$$\omega_\alpha$$

corresponding to the Lie algebra having trivial actions on the forms π_i, and having the space of forms ω_a as an invariant subspace.

This definition is formulated such that it is as concise as possible and if we accept it, we can rapidly do calculations on a structure-preserving submersion without discussing many subtle points about structure-preserving submersions (note in particular that this definition makes no mention of any reduction of principal bundles: the integral variety of the proposed exterior differential system implicitly contains information of all such reductions). As a price to pay, the definition is not very intuitive.

90 Consequences of the definition. Let us first check that it is, first and foremost, a submersion. First, generically, assume that $f : A \to B$ is a submersion, then we can form the graph of this map, which is a submanifold of $S \subset A \times B$. It is clear that

$\dim S = \dim A$. If we have a coframe ω_A on A and ω_I on B, then ω_A, ω_I, or rather their pullbacks under the projection maps $\mathrm{pr}_A : A \times B \to A$ and $\mathrm{pr}_B\, A \times B \to B$, together form a coframe on $A \times B$. As the submanifold S arises from the function f, the forms $\iota^*\omega_A$ are independent one-forms on S where $\iota : S \to A \times B$ is the canonical inclusion map. On the other hand, the rank condition of the submersion means that $\iota^*\omega_A$ are also independent one-forms on S. Conversely, if these conditions on the forms are satisfied, then the submanifold S arises locally as the graph of a submersion map.

For our problem, it is thus clear that the solution of the differential system (3.21) arises from a submersion for which the map is $f : A \to B$, and $B = Q$. The manifold A is a little bit more complicated: it has first of all the coframe π_i, π_μ, ω_a and ω^*_α, where the star over ω_α meaning that only those corresponding to trivial actions on π_i and preserving the subspace of ω_a are included. It is also a sub-bundle of the principal bundle P: notice that the forms π_i and ω_a can be taken as a set of horizontal forms.

If, instead of dealing with the principal bundles P and Q, we deal with sections on them, and assume that for a certain section the definition still holds when we substitute the forms with the pullbacks of forms onto the section, with the independent conditions now only given by π_i and ω_a, since the vertical forms for any section are expressible linearly in terms of the pullbacks of the horizontal forms. Then we see that under this section, the integral variety we have found corresponds to a submersion $M \to N$.

To summarise, the integral variety we have found corresponds to the following situation:

$$\begin{array}{ccccccc} M \times H & \xrightarrow{\approx} & A & \xrightarrow{f} & Q & \xleftarrow{\approx} & N \times H_N \\ & \searrow^{\mathrm{pr}_1} & \downarrow{\scriptstyle \mathrm{pr}_M} & & \downarrow{\scriptstyle \mathrm{pr}_N} & \swarrow_{\mathrm{pr}_1} & \\ & & M & \xrightarrow{\sigma} & N & & \end{array}$$

in the diagram above, $\sigma : M \to N$ is a submersion on the base manifolds M and N, and we have found a submersion f covering σ from a certain principal bundle A over M to the principal Q over N.

What is the principal bundle A and what is the group H? Since the map f is a submersion, it is clear that the group H_N is a subgroup of the group H. This is also clearly seen from the fact that ω_μ is the Maurer–Cartan forms when restricted to a vertical subspace, which is isomorphic to H_N, and ω_μ, ω^*_α together can be taken as the Maurer–Cartan forms on H. The forms ω^*_α are practically found as follows: for any Cartan connection ω_A of a principal bundle with Lie algebra $\mathfrak{g}$, the transformation under the right action of the principal group itself is

$$h : \omega_A \to \mathrm{Ad}_h(\omega_A), \qquad h \in \mathfrak{g}.$$

This is the equation that we can use to determine what are the forms ω^*_α. For example, if the principal bundle P and Q both correspond to the principal bundle for geometries with a projective connection, then we can write the Cartan connection on P as a matrix

$$\begin{pmatrix} \omega_{00} & \omega_{0i} & \omega_{0a} \\ \omega_i & \omega_{ij} & \omega_{ia} \\ \omega_a & \omega_{ai} & \omega_{ab} \end{pmatrix}$$

where not all forms are independent: in particular, since we are dealing with projective geometry, $\omega_{00} + \omega_{11} + \cdots = 0$. Under the principal right action, the above connection matrix is transformed to

$$\begin{pmatrix} h_{00} & h_{0i} & h_{0a} \\ 0 & h_{ij} & h_{ia} \\ 0 & h_{ai} & h_{ab} \end{pmatrix} \begin{pmatrix} \omega_{00} & \omega_{0i} & \omega_{0a} \\ \omega_i & \omega_{ij} & \omega_{ia} \\ \omega_a & \omega_{ai} & \omega_{ab} \end{pmatrix} - \begin{pmatrix} \omega_{00} & \omega_{0i} & \omega_{0a} \\ \omega_i & \omega_{ij} & \omega_{ia} \\ \omega_a & \omega_{ai} & \omega_{ab} \end{pmatrix} \begin{pmatrix} h_{00} & h_{0i} & h_{0a} \\ 0 & h_{ij} & h_{ia} \\ 0 & h_{ai} & h_{ab} \end{pmatrix}.$$

This means that we have one equation of the form

$$\begin{pmatrix} h_{00} & h_{0i} & h_{0a} \\ 0 & h_{ij} & h_{ia} \\ 0 & h_{ai} & h_{ab} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \omega_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \omega_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_{00} & h_{0i} & h_{0a} \\ 0 & h_{ij} & h_{ia} \\ 0 & h_{ai} & h_{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.22}$$

and

$$\begin{pmatrix} h_{00} & h_{0i} & h_{0a} \\ 0 & h_{ij} & h_{ia} \\ 0 & h_{ai} & h_{ab} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \omega_a & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \omega_a & 0 & 0 \end{pmatrix} \begin{pmatrix} h_{00} & h_{0i} & h_{0a} \\ 0 & h_{ij} & h_{ia} \\ 0 & h_{ai} & h_{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \star \end{pmatrix} \tag{3.23}$$

for each ω_i, $i = 1, 2, \ldots, \dim N$. These will tell us which linear combinations of h_{00}, h_{0i}, h_{0a}, h_{ij}, h_{ia}, h_{ai} and h_{ab} need to be set to determined constants (here zero). The complement of those that are set to constants gives the linear combinations of the forms that are retained in the bundle A.

Thus, we see that the group H satisfies $H_N \subset H \subset H_M$, where the subset symbol means subgroup, and is uniquely determined by the procedure above. Another way of saying the same thing is that A is obtained from P by a *reduction of the principal bundle* from the structural group H_M to H. Intuitively, the significance of this reduction is as follows: for any structure-preserving submersions, the horizontal one-forms ω_i are replaced with the one-forms π_i arising from the submersion. The π_i satisfies its own structural equations (and hence its structure is "preserved" in M), and consequently any right action in P that "moves" π_i in any non-trivial way is forbidden. Or, in the language of moving frames (instead of coframes), a part of the frame is already fixed, so any transformation of the frames not preserving completely this part of the frame is no longer allowed. Yet another way of saying the same thing is: we have the frame on M, but also the frame deduced from the one on N, which can be interpreted as a partial frame on M. We need to use some of the degree of arbitrariness of the group H_M in order to *align* the frame on M with the partial frame on N, and hence after this alignment, some of the degree of arbitrariness of H_M is lost and we obtain the subgroup H (this is how we argued in §**73**). The reason that we also disallow mixing of ω_a and π_i is that, we know that for any principal bundle, the right action of the group just changes the choice of identity of the group in the bundle and should not have any real effect. However, if ω_a can be changed into π_i by such an action, then the definition of structure-preserving submersion will depend on such a choice, and hence our definition would make no sense.

Thus, in summary, we have the following commutative diagram:

$$\begin{array}{ccccc} M\times H_M & \xleftarrow{\tilde\rho} & M\times H & \xrightarrow{\tilde f} & N\times H_N \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow\simeq \\ P & \xleftarrow{\rho} & A & \xrightarrow{f} & Q \\ \mathrm{pr}_M\downarrow & & \mathrm{pr}_M\downarrow & & \downarrow\mathrm{pr}_N \\ M & \xleftarrow{=} & M & \xrightarrow{\sigma} & N \end{array}$$

where ρ is the inclusion map arising from the reduction of the principal bundle (note the direction of arrow). From this diagram, we also have the following interpretation of the integral variety in our definition: using the maps f and ρ, we pull back the coframes on P and Q to A, and the independent forms form a co-frame on A. The map f, which covers the submersion σ, is the *structure-preserving* submersion.

Note that our definition does not explicitly state that the group H_N must be a subgroup of H_M, but if this is not the case, it is impossible to find any solution of the required differential system: there must be non-trivial relations among the π_μ. If this case arises in applications, we need to start again and try to find structure-preserving submersions preserving a subgroup of H_N.

Note also it is in general impossible to define a covering submersion map directly from P to Q: this requires us finding a submersion from the group H_M to H_N, which also *preserves the group structure*, i.e., the map must be a surjective homomorphism. Such maps do not in general exist, even when H_N is a subgroup of H_M. Thus we see that the reduction of the principal bundle ρ is essential.

On the other hand, our definition, which only explicitly talks about the exterior differential system (3.21), makes all these discussions about reductions of bundles, etc., redundant, even though they are certainly helpful for an intuitive understanding.

91 The structural equations. The use of coframes, which is more general than connections, leads us naturally to the equivalence problem. We also know that the *struc-*

tural equations of the coframe contains all the differential invariants of the problem. For any structure-preserving submersions, our definition already gives us a co-frame, namely the set

$$\pi_i, \qquad \pi_\mu, \qquad \omega_a, \qquad \omega^*_\alpha. \tag{3.24}$$

The problem *may* also give us additional functions that must be included in the determination of the equivalence problem. Let us now study, without specialising to specific groups, their structural equations.

The structural equations of structure-preserving submersions are deduced from four sets of equations: the first set is the structural equations on N:

$$d\pi_i = \cdots, \qquad d\pi_\mu = \cdots, \tag{3.25}$$

the second set is the structural equations on M:

$$d\omega_i = \cdots, \qquad d\omega_a = \cdots, \qquad d\omega_\alpha = \cdots, \tag{3.26}$$

the third set is the exterior differential system itself:

$$\omega_i = \pi_i, \qquad d\omega_i = d\pi_i, \tag{3.27}$$

and finally, the fourth set is the decomposition of the ω_α that are not independent, which we write as $\omega^\dagger_\alpha$:

$$\omega^\dagger_\alpha = A_{\alpha i}\pi_i + B_{\alpha a}\omega_a + C_{\alpha\mu}\pi_\mu + D_{\alpha\beta}\omega^*_\beta, \qquad d\omega^\dagger_\alpha = \cdots. \tag{3.28}$$

These equations are not all independent: due to the dimension of the integral variety, we know that from them we should deduce a set of independent structural equations, on the left hand sides of which are the exterior derivatives of the independent one-forms (3.24). On the other hand, all functions appearing on the right hand sides of (3.25), (3.26), (3.27), (3.28) must be taken as differential invariants of the system. The redundant equations among them then gives the *algebraic relations* among the differential invariants.

The first equations of (3.27) and (3.28) are the only redundant one-form equations in our system ("redundant" in terms of forming a coframe). We should immediately use them to substitute all occurrences of ω_i and $\omega^\dagger_\alpha$ with the independent forms. Once this is done, we see that both (3.25) and (3.26) contain expressions for $d\pi_i$, both (3.27) and (3.28) contain expressions for $d\omega^\dagger_\alpha$. Using these equalities, we obtain all the constraints of the system at this level. These constraints may make all the differential invariants that do not occur explicitly in the new coframe structural equations whose left hand sides are the exterior derivatives of (3.24) completely expressible in terms of the invariants that occur explicitly in the new coframe structural equations. If this is not the case, then any invariants that do not explicitly occur at this level must be included as additional scalar functions in the equivalence problem, which would then mean that the co-frame does not uniquely determine the structure-preserving submersion. This is analogous to the cases in immersion where the induced structure on the submanifold is not uniquely defined.

The relations among the differential invariants obtained above will be called the *defining relations* for them. There are other kinds of relations for them.

One kind of such relations is called the *generic relations* and they involve the *coframe derivatives* of the differential invariants. For example, if I is any invariant, possibly with indices, then its coframe derivative is defined as

$$dI = I_{;i}\pi_i + I_{;a}\omega_a + I_{;\mu}\omega_\mu + I_{;\alpha}\omega^*_\alpha, \tag{3.29}$$

but this defining equation can be immediately differentiated again, which may generate higher order coframe derivatives. The generic relations are just the relations $d^2I = 0$. Note that the coframe derivatives $I_{;i}, I_{;a}, I_\mu, I_\alpha$ are considered algebraically *independent* quantities unless there is an explicit relation for them, which.

The next kind is called the *Bianchi relations*: these are obtained by exterior differentiating (3.25), (3.26), (3.27), (3.28) and use the identity $d^2 = 0$. Deriving these relations is usually a rather tedious process, so it is important to note the following in order to reduce unnecessary work: for example, we can calculate $d^2\pi_i$ by either exterior differentiating the equation in (3.26), or the equation in (3.27), but *we only need to differentiate one of them* since they imply each other. Indeed, we have already obtained relations for the invariants which makes the relation $d\pi_i = d\omega_i$ and identity, so $d^2\pi_i = d^2\omega_i = 0$ is an identity as well. The same reasoning applies to the equations involving $\omega^\dagger_\alpha$.

The last kind of these relations is called the *derived relations*. For example, let $f(I) = 0$ be an algebraic relation (zero form equation) of the previous kinds. Then a relation can be obtained by exterior differentiating: $df(I) = 0$. Of course, derived relations can be further derived to obtain an infinite tower of relations, but if we truncate the tower of differential invariants by only considering invariants whose number of indices is less than a given number, then the total number of relations at this stage is finite.

If at any stage a relation we obtain is *incompatible*, for example of the form $1 = 0$, this simply means that no required integral variety exists. In particular, it is easy to show by considering the structural constants of the groups that, if H_N is not a subgroup of H_M, then incompatibility will occur.

It should be noted that many of the procedures treated here are completely analogous to the corresponding procedures in dealing with the methods of involutive seeds. In particular, the classes of relations in the two theories correspond to each other exactly.

V. Riemannian submersion and isometries

92 Systems with less than maximal number of independent invariants. Isometries. By using the moving frame and applying the method of involutive seeds, we can already study all Riemannian geometries where the maximal set of invariants eventually occur. It remains other cases, namely where we have a set of invariants that is fewer in number than the dimension of the bundle.

Instead of restricting to Riemannian geometry, let us first consider the more general situation and assume that we have a general equivalence problem. The total number of functionally independent invariants c_{ijk}, $c_{ijk;l}$, $c_{ijk;lm}$, ... is a well-defined number ρ. It satisfies $0 \leq \rho \leq m$, where m is the dimension of the coframe. We will now show that this number gives information about the dimension of the symmetry group of the equivalence problem.

To show this, we need to be a bit more abstract in our approach. For an equivalence problem formulated on a base manifold M, let us consider the (Euclidean) space $\mathcal{C}$ of the invariants $c^i{}_{jk}$ up to sufficiently high order so that all functionally independent invariants are guaranteed to be included. The exact order is immaterial, that we have considered a lot of redundant variables does not matter either. Then, for every concrete system, we have a *classifying map* $T : M \to \mathcal{C}$. Up to the usual regularity considerations, this map defines a *classifying manifold* in the space of invariants. Then if two points P and $\bar{P}$ of two systems defined on M and $\bar{M}$ map to the same point in the classifying manifold (the classifying manifolds of the two problems are identified in the obvious manner), the two systems are equivalent at the points P and $\bar{P}$ by the general procedure of the equivalence problem; if this condition holds for all points in the open sets $S \subset M$ and $\bar{S} \subset \bar{M}$, then the identification of S and $\bar{S}$ that makes the condition holds provides an equivalence of the two systems.

Now consider the image of the classifying map $T : M \to \mathcal{C}$. At regular points, this image is a submanifold of $\mathcal{C}$: its dimension is an integer ρ. If $\rho < m$, then the pre-image of a point $Q \in \mathcal{C}$ is non-trivial: $T^{-1}(Q)$ is locally a $m - \rho$ dimensional submanifold in M. But if we set up the equivalence problem in trying to deduce the equivalence of M with itself, but identifying a point P in M with a nearby point P' in a neighbourhood, we see that as long as P and P' are in the same pre-image, i.e., as long as $T(P) = T(P')$, the equivalence problem has a solution. As P and P' can be connected by a path not going out of the pre-image, we see that this self-equivalence is actually a *infinitesimal symmetry* under a finite dimensional Lie group, and $m - \rho$ gives the dimension of the symmetry group of the problem.

Thus we see that the relation $\rho < m$ implies the existence of a certain vector field on the space on which the co-frame is set up, and this vector field constitutes a symmetry of the coframe. On the other hand, whenever $\rho = m$, no such symmetry group can exist.

Now let us return to the Riemannian case. Whenever we have a non-maximal number of invariants, this means that we can find a vector field which constitutes a symmetry of our co-frame. But recall that our co-frame is set up on the prolonged space, in other words there are also variables describing the freedom in choosing the frames. Consequently, only a subset of such co-frame symmetries "descends" onto the base manifolds. On the other hand, whenever such a symmetry descends onto the base, we know what they are: the vector field then constitutes a simple coordinate transformation, and under this transformation the old and new geometrical structure defined on the base manifold, i.e., the metric, remains the same, hence such vector fields are what we call Killing vector fields, and the symmetries of coframes descending onto the base are none other than the isometries.

In this section our main interest lies in the study of isometries. As we will see, for any particular isometry defined on the base manifold, there is a unique extension of the Killing vector field to the principal bundle, and the special form of such a vector field living in the bundle gives us intricate connections between the study of Killing vector fields and the study of Riemannian submersions.

93 Riemannian spaces admitting Killing vectors. Let us come to the study of Killing vectors now. The metric can be written as

$$ds^2 = \sum_\mu \omega_\mu \otimes \omega_\mu.$$

Let $\mathbf{v}$ be a vector field on the manifold. If $\mathbf{v}$ is a Killing vector field, then

$$\mathcal{L}_{\mathbf{v}} \left(\sum \omega_\mu \otimes \omega_\mu \right) = 0, \tag{3.30}$$

where $\mathcal{L}$ denotes the Lie derivative.

Let us reconsider our approach. The definition of the Killing vector is (3.30), which involves the tensor product of the horizontal forms. Why not simply $\mathcal{L}_{\mathbf{v}}\omega_\mu = 0$? Requiring that each horizontal form is separately invariant under the Killing vector is too strong: we only requires the invariance of the bilinear form. But it is obvious that for every Killing vector field that satisfies (3.30), we can find a section of the principal bundle such that $\mathcal{L}_{\mathbf{v}}\omega_\mu = 0$ holds: it suffices to use this relation as the definition of the section that we want.

But the relation $\mathcal{L}_{\mathbf{v}}\omega_\mu = 0$, which is now required to hold in a particular section, has the following significance in the bundle: it is not only necessary to specify in which horizontal direction to move in order to obtain an isometry, but also to specify how we rotate the frames in this direction. In other words, let $\mathbf{V}$ be a vector field *on the bundle* (in particular, it is *not* assumed to be a tensor in any way *a priori*, and it has components $\mathbf{V} = V_\mu \mathbf{I}_\mu + V_{\mu\nu}\mathbf{I}_{\mu\nu}$, where $\mathbf{I}_\mu$ and $\mathbf{I}_{\mu\nu}$ are the dual basis for the coframe ω_μ, $\omega_{\mu\nu}$). The existence of an isometry then requires

$$\mathcal{L}_{\mathbf{V}}\omega_\mu = 0$$

on the bundle. Let us see what this implies by calculating the Lie derivative. We have

$$\begin{aligned}\mathcal{L}_{\mathbf{V}}\omega_\mu &= -(V_\rho \mathbf{I}_\rho + V_{\rho\lambda}\mathbf{I}_{\rho\lambda}) \lrcorner (\omega_{\mu\nu} \wedge \omega_\nu) + d[(V_\rho \mathbf{I}_\rho + V_{\rho\lambda}\mathbf{I}_{\rho\lambda}) \lrcorner \omega_\mu] \\ &= -V_{\mu\nu}\omega_\nu + V_\nu \omega_{\mu\nu} + V_{\mu,\nu}\omega_\nu + V_{\mu,\rho\lambda}\omega_{\rho\lambda}\end{aligned}$$

where in the last line we have expanded dV_μ in terms of the coframe. We see that

$$\begin{cases} V_{\mu\nu}\omega_\nu = V_{\mu,\nu}\omega_\nu, \\ V_\nu \omega_{\mu\nu} = -V_{\mu,\rho\lambda}\omega_{\rho\lambda}. \end{cases} \tag{3.31}$$

The second equation is rather curious. Indeed, let v_μ be a vector on the base manifold, i.e., a tensor on the bundle. We know that

$$dv_\mu \equiv v_{\mu;\nu}\omega_\nu + v_{\mu,\nu\rho}\omega_{\nu\rho} = v_{\mu;\nu}\omega_\nu - v_\nu \omega_{\mu\nu},$$

giving us

$$v_{\mu,\nu\rho}\omega_{\nu\rho} = -v_\nu \omega_{\mu\nu}.$$

Hence the second equation of (3.31) just tells us that V_μ are the components of a tensor, i.e., a vector on the base. Then the first equation gives us

$$\begin{cases} V_{[\mu;\nu]} = V_{\mu\nu}, \\ V_{(\mu;\nu)} = 0. \end{cases}$$

The second equation gives us the usual Killing's equation in the bundle, whereas the first tells us how we need to *lift* the Killing vector field on the base into a vector field on the bundle.

94 Derived magic formula. Constant curvature along the Killing vector field. In our study of exterior differential systems we know it is essential that we include the derived equations of everything we do. Even though when we use vector fields, strictly speaking we are not doing calculations with exterior differential systems, let us do the same. Deriving the magic formula $\mathcal{L}_{\mathbf{v}}\omega = \mathbf{v} \lrcorner (d\omega) + d(\mathbf{v} \lrcorner \omega)$ gives

$$d\mathcal{L}_{\mathbf{v}}\omega = \mathcal{L}_{\mathbf{v}}(d\omega),$$

which is the well-known fact that exterior derivative and Lie derivatives commute on differential forms (deriving again yields an identity). Hence, *when we have an equation concerning Lie derivatives, we should always include the derived equations as well.* Do this for $\mathcal{L}_{\mathbf{V}}\omega_\mu = 0$, we have

$$\mathcal{L}_{\mathbf{V}}(d\omega_\mu) = -(\mathcal{L}_{\mathbf{V}}\omega_{\mu\nu}) \wedge \omega_\nu,$$

so

$$\mathcal{L}_{\mathbf{V}}\omega_{\mu\nu} = c_{\mu\nu\lambda}\omega_\lambda, \qquad c_{\mu\nu\lambda} = c_{\mu\lambda\nu},$$

but as $c_{\mu\nu\lambda} = -c_{\nu\mu\lambda}$, it vanishes identically. Hence even though we have only required $\mathcal{L}_{\mathbf{V}}\omega_\mu = 0$, the vector field we have found on the bundle satisfies $\mathcal{L}_{\mathbf{V}}\omega_{\mu\nu} = 0$ as well.

Now we derive the equation we have just obtained:

$$\mathcal{L}_{\mathbf{V}}(d\omega_{\mu\nu}) = \tfrac{1}{2}(\mathcal{L}_{\mathbf{V}}R_{\mu\nu\rho\lambda})\omega_\rho \wedge \omega_\lambda,$$

giving

$$\mathcal{L}_{\mathbf{V}}R_{\mu\nu\rho\lambda} = 0,$$

since for the equation to be satisfied $R_{\mu\nu\rho\lambda}$ has to be both symmetric and antisymmetric in the ρ and λ indices.

Now we have obtained that the curvature *tensor* $R_{\mu\nu\rho\lambda}$ is constant under the Killing vector field. We can derive these conditions further. For example, if T_μ is a tensor and $\mathcal{L}_{\mathbf{V}}T_\mu = 0$, then

$$\mathcal{L}_{\mathbf{V}}dT_\mu = \mathcal{L}_V(T_{\mu;\nu}\omega_\nu + \cdots) = (\mathcal{L}_{\mathbf{V}}T_{\mu;\nu})\omega_\nu = 0,$$

where dots indicate terms that are zero under Lie derivatives. Hence, by carrying out more derivations, we see that *all covariant derivatives of the tensor are constant under the Lie derivative.* This result is easily seen to hold for tensors of all valences.

95 Homogeneous space. Suppose now $R_{\mu\nu\rho\lambda}$ is actually constant. Then the equations $d\omega_\mu$ and $d\omega_{\mu\nu}$ actually form the Maurer–Cartan structural equations of a Lie group, and hence the space is a homogeneous space obtained by the quotient of a Lie group by a subgroup. In this case, we have the maximal number of Killing vector fields, which together form a Lie algebra that is exactly the Lie algebra of the Lie group. We can also deduce that if a tensor has vanishing Lie derivatives under all these Killing vector fields, then its components with respect to the basis formed by the tensor products of ω_μ, $\omega_{\mu\nu}$, $\mathbf{I}_\mu$, $\mathbf{I}_{\mu\nu}$ are constants: indeed, the tensor is a left invariant tensor on the Lie group, and all left invariant tensors on a Lie group can be obtained by pushing out a tensor defined at the identity by the left invariant vector fields, which are exactly $\mathbf{I}_\mu$, $\mathbf{I}_{\mu\nu}$ and their linear combinations with constant coefficients. All such tensors obtained have constant coefficients with respect to the left-invariant bases.

96 Defining structural preserving Riemannian submersion by vector fields. We will use the techniques we have developed for Riemannian submersion to study Riemannian spaces admitting Killing vector fields. First, we will give another defining property of Riemannian submersions that makes it evident that isometries can be interpreted as Riemannian submersions.

We have seen that the condition $\mathcal{L}_\mathbf{V}\omega_\mu = 0$ (which implies $\mathcal{L}_\mathbf{V}\omega_{\mu\nu} = 0$) can be taken as the condition for isometry. Suppose that we have a Riemannian submersion. Then roughly speaking, the forms ω_i, which are pullbacks of forms from the reduced manifold, are aligned along each leaf of the foliation (see §**73**). This suggests that we could try the condition $\mathcal{L}_\mathbf{U}\omega_i = 0$, where $\mathbf{U}$ points only along the leaves on the manifold, as the condition for structure preserving submersion.

Since we have distinguished two subsets of horizontal forms ω_i and ω_a, this amounts to a reduction of the principal bundle. Hence we have

$$\omega_{ai} = -\omega_{ia} = K_{iab}\omega_b - M_{ija}\omega_j$$

as before, but for the moment there is no constraints on K_{iab} and M_{ija}. As for $\mathbf{U}$, we now have

$$\mathbf{U} = U_a\mathbf{I}_a + U_{ij}\mathbf{I}_{ij} + U_{ab}\mathbf{I}_{ab},$$

with $U_a \neq 0$ and no term in $\mathbf{I}_i$. We can now calculate

$$\begin{aligned}\mathcal{L}_\mathbf{U}\omega_i &= d(\mathbf{U} \lrcorner\, \omega_i) + \mathbf{U} \lrcorner\, d\omega_i \\ &= (M_{ija}U_a - U_{ij})\omega_j + (K_{iab} - K_{iba})U_b\omega_a,\end{aligned}$$

which requires

$$M_{ija}U_a = U_{ij}, \qquad K_{i[ab]}U_b = 0.$$

Now if there is only a single index for $a, b, \ldots$, we obviously have

$$M_{(ij)a} = 0, \qquad K_{i[ab]} = 0, \tag{3.32}$$

since we require $U_a \neq 0$. If there are several indices for $a, b, \ldots$, recall how these indices arise: these indices arise because we can do a submersion along $\mathbf{U}$. Hence there are as many $\mathbf{U}$ satisfying $\mathcal{L}_{\mathbf{U}}\omega_i = 0$ as there are indices for $a, b, \ldots$. Then we see that (3.32) is also satisfied in this case. On the other hand, if (3.32) is satisfied, obviously we can choose U_a arbitrarily, and after this choice is made, U_{ij} is uniquely determined (U_{ab} does not enter anywhere in these equations). But (3.32) is just the condition we found for structural preserving submersions using the method of integral varieties within product spaces. The two definitions are hence equivalent.

Note that now in general $\mathcal{L}_{\mathbf{U}}$ acting on ω_a, ω_{ab} and ω_{ij} are not zero. But if we define

$$\pi_{ij} = \omega_{ij} - M_{ija}\omega_a,$$

then

$$\mathbf{U} \lrcorner \pi_{ij} = 0$$

and the derived equation of $\mathcal{L}_{\mathbf{U}}\omega_i$ gives

$$\mathcal{L}_{\mathbf{U}}\pi_{ij} = 0,$$

we have recovered the modified connection we have defined in §**76**. Carrying out the derivation further simply gives

$$\mathcal{L}_{\mathbf{U}}S_{ijkl} = 0,$$

which should not surprise us now. We should not carry out further derivations: it is easier to carry out the analysis using the method of exterior differential systems as we have done before.

Remark. We can also do the analysis on the base if we like. On the base, the structural preserving submersion condition is that there exists some vector field

$$\mathbf{u} = u_a\mathbf{I}_a$$

such that

$$\mathcal{L}_{\mathbf{u}}\left(\sum \omega_i \otimes \omega_i\right) = 0.$$

Obviously, the calculation is more involved.

97 Killing vector field along the submersion direction. To make calculation easier, first we change from the coframe with ω_{ij} to the coframe with ϖ_{ij}. The corresponding change for the frame is

$$\mathbf{I}'_i = \mathbf{I}_i, \qquad \mathbf{I}'_a = \mathbf{I}_a + M_{ija}\mathbf{I}_{ij}, \qquad \mathbf{I}'_{ij} = \mathbf{I}_{ij}, \qquad \mathbf{I}'_{ab} = \mathbf{I}_{ab},$$

hence for a vector field $\mathbf{V}$ defining a direction of structural preserving submersion,

$$\mathbf{V} = V_a \mathbf{I}'_a + V_{ab} \mathbf{I}_{ab}.$$

Now suppose that $\mathbf{V}$ is also a Killing vector field. First recall a property of Lie derivatives: if $\mathcal{L}_{\mathbf{X}}\omega = 0$ and $\mathcal{L}_{\mathbf{Y}}\omega = 0$, then $\mathcal{L}_{a\mathbf{X}+b\mathbf{Y}}\omega = 0$ where a and b are *constants*. This means that if we have several Killing vector fields, we can form their linear combinations with constant coefficients which are still Killing vector fields. Next, the condition $\mathcal{L}_{\mathbf{V}}\omega_\mu = 0$ is independent of whatever coframe we choose on the bundle. The easiest way to check this is to recall the properties enjoyed by the components of $\mathbf{V}$. Since $V_{\mu\nu}$ is obtained by lifting V_μ, and this lifting needs to be done again once we change coframe, we only need to check the conditions on V_μ. The condition that V_μ is a tensor is obviously invariant under change of frame, as well as $V_{(\mu;\nu)} = 0$. Thus, for a vector field that already is a vector field generating a structure preserving submersion, we only need to specify that it satisfies in addition $\mathcal{L}_{\mathbf{V}}\omega_a = 0$, since $\mathcal{L}_{\mathbf{V}}\omega_i = 0$ are already satisfied. Calculating,

$$\mathcal{L}_{\mathbf{V}}\omega_a = (V_{a,b} - V_{ab})\omega_b + (V_{a,i} - K_{iab}V_b)\omega_i + V_{a,ij}\varpi_{ij} + (V_{a,bc}\omega_{bc} + V_b\omega_{ab}) = 0,$$

giving the conditions

$$\begin{cases} V_{[a,b]} = V_{ab}, \\ V_{(a,b)} = 0, \\ V_{a,i} = K_{iab}V_b, \\ V_{a,ij} = 0, \\ V_{a,bc} = V_b\delta_{ac}. \end{cases}$$

As usual, some of these conditions just mean that V_a is the components of a tensor. Henceforth we will write

$$dV_a = V_{a;b}\omega_a + K_{iab}V_b\omega_i - V_b\omega_{ab}.$$

For the derived relations, in addition to the relations for general structure preserving submersions, we have

$$\mathcal{L}_{\mathbf{V}}(d\omega_a) = -(\mathcal{L}_{\mathbf{V}}\omega_{ab}) \wedge \omega_b - (\mathcal{L}_{\mathbf{V}}K_{iab})\omega_b \wedge \omega_i - (\mathcal{L}_{\mathbf{V}}M_{ija})\omega_i \wedge \omega_j$$

which immediately gives

$$\mathcal{L}_{\mathbf{V}}K_{iab} = 0, \qquad \mathcal{L}_{\mathbf{V}}M_{ija} = 0,$$

(the terms $\mathcal{L}_{\mathbf{V}}\omega_{ab}$ and $\mathcal{L}_{\mathbf{V}}K_{iab}\omega_i$ cannot mix, since the symmetries on the indices are opposite.)

As for ω_{ab},

$$\mathcal{L}_{\mathbf{V}}\omega_{ab} = c_{abc}\omega_c,$$

but c_{abc} has to be symmetric in b, c but antisymmetric in a, b, so it vanishes identically. We have the additional condition

$$\mathcal{L}_{\mathbf{V}}\omega_{ab} = 0.$$

Now for any tensor $T_{ab\ldots ij\ldots}$, if $\mathcal{L}_{\mathbf{V}}T_{ab\ldots ij\ldots} = 0$, we have

$$\begin{aligned}\mathcal{L}_{\mathbf{V}}(dT_{ab\ldots ij\ldots}) &= \mathcal{L}_{\mathbf{V}}(T_{ab\ldots ij\ldots;c}\omega_c + T_{ab\ldots ij\ldots;k}\omega_k + \cdots) \\ &= (\mathcal{L}_{\mathbf{V}}T_{ab\ldots ij\ldots;c})\omega_c + (\mathcal{L}_{\mathbf{V}}T_{ab\ldots ij\ldots;k})\omega_k = 0,\end{aligned}$$

so all covariant derivatives of these tensors are also invariant under the action of $\mathbf{V}$.

Now differentiate the relation $\mathcal{L}_{\mathbf{V}}\omega_{ab} = 0$:

$$\mathcal{L}_{\mathbf{V}}(d\omega_{ab}) = \tfrac{1}{2}(\mathcal{L}_{\mathbf{V}}S_{abcd})\omega_c \wedge \omega_d = 0,$$

so *for an isometry interpreted within the framework of structure preserving submersion, all of the invariants S_{ijkl}, S_{abcd}, M_{ija}, K_{iab} and all of their covariant derivatives are invariant under any of the Killing vector fields.*

98 Necessary and sufficient condition for a Riemannian submersion to contain a Killing submersion. We have learned that the condition that all differential invariants and their covariant derivatives are invariant under a vector field is the necessary condition for the vector field, which is along a submersion direction, to be an isometry. This condition is also sufficient: it suffices to note that the Riemann tensor and their derivatives of the whole space can be reconstructed by using all these invariants and their derivatives, with (3.15). Then according to the theory of equivalence, since the differential invariants of the whole space match up to all orders, the group action generated by the vector field is a symmetry of the theory, and symmetry in this theory is exactly isometry.

However, checking equality of differential invariants to all orders is impractical and unnecessary. Since we already have a submersion, we only need to ensure the equality of ω_i, ω_a, π_{ij}, ω_{ab} to its copy under the vector field. Then according to the general theory of equivalence, since in the bundle we do not have any excessive symmetry group at our disposal and the Frobenius theorem is sufficient, we only need to check the vanishing under Lie derivative of the invariants that appear directly in equation (3.16). Hence, in addition to M_{ija}, K_{iab}, S_{abcd}, S_{ijkl}, we need to check the vanishing of the Lie derivative of $K_{ia[b;c]}$ and $M_{ij[a;b]}$. The vanishing of the Lie derivatives of these quantities hence constitute the necessary and sufficient condition for a vector which already generates parts of a Riemannian submersion to be an isometry.

A special case occurs when we have a single vector field, then since we only have a one co-dimensional foliation, $K_{ia[b;c]}$ and $M_{ij[a;b]}$ vanish identically, as well as S_{abcd}. As S_{ijkl} is invariant under the vector field automatically (condition for Riemannian submersion), we only need to check the vanishing of Lie derivatives of M_{ija} and K_{iab}. But now we have $\mathbf{V} = \lambda \mathbf{I}_0$ ($\mathbf{I}_0$ is the single tangent vector along the leaves and λ is a positive scalar function), and

$$\mathcal{L}_{\mathbf{V}} M_{ij} = \lambda \dot{M}_{ij}, \qquad \mathcal{L}_{\mathbf{V}} K_i = \lambda \dot{K}_i,$$

where we have suppressed all 0 indices and used a dot to denote covariant derivation in the fibre direction. Hence, we require $\dot{M}_{ij} = 0$ and $\dot{K}_i = 0$, *and in this co-dimension* 1 *case, in general when we want to check a quantity is invariant under the submersion vector field, we only need to check that its covariant derivative in this direction vanishes.*

We can also integrate to obtain the parameter λ as a function of the coordinates. *Working with a section of the principal bundle*, we know that our vector field $\mathbf{v} = \lambda \mathbf{I}_0$ already satisfies

$$\mathcal{L}_{\lambda \mathbf{I}_0} \sum \omega_i \otimes \omega_i = 0.$$

With our condition $M_{ija} = -M_{jia}$, $K_{iab} = K_{iba}$, this is an identity. Hence we only need to require

$$\begin{aligned} 0 &= \mathcal{L}_{\lambda \mathbf{I}_0}(\omega_0 \otimes \omega_0) \\ &= 2\dot{\lambda}\, \omega_0 \otimes \omega_0 + 2(\lambda_{;i} - \lambda K_i)(\omega_i \otimes_S \omega_0), \end{aligned}$$

so the positive function λ must also be constant along the leaves. As for $\lambda_{;i} = \lambda K_i$, using any coordinates where x^i are the coordinates on the reduced manifold, it suffices to integrate the equation

$$\frac{\partial \log \lambda}{\partial x^i} = K_i.$$

If we do not yet know the condition for isometry, we can see it from this equation: K_i is the derivative on the reduced manifold of something independent of the fibre coordinates, so $\dot{K} = 0$. Since K_i is obtained from a differential, $K_{[i;j]} = 0$, but we have from the general equations of submersion, $K_{[i|ab;|j]} = -M_{ij(a;b)}$, so in this case $\dot{M}_{ij} = -K_{[i;j]} = 0$.

On the other hand, if the co-dimension is greater than one, attempting to carry out the same explicit integration yields complicated partial differential equations containing higher derivatives.

99 Isometries interpreted as structure preserving submersions. Let us suppose that there are r independent Killing vector fields on the manifold. These generate a local action of a r dimensional Lie group G on the manifold, whose Lie algebra is exactly the Lie algebra of the Killing vector fields. Also, as with all group action on manifolds, away from singular points the action generates a foliation of the space. Restricted to each leaf, the group acts transitively. Now work on the base: let ω_a be parts of the orthonormal coframe aligned on the leaves and ω_i be the complement in the orthonormal coframe. The group must map $\sum \omega_a \otimes \omega_a$ to itself since this quantity remains the same when restricted to each leaf. But as the group action also maps $\sum \omega_a \otimes \omega_a + \sum \omega_i \otimes \omega_i$ to itself, $\sum \omega_i \otimes \omega_i$ is mapped to itself also. We see that this coframe realises a *structural preserving submersion* corresponding to the whole set of isometry.

Now, restricted to each leaf, the structural equations for ω_a, ω_{ab}, *where the space is considered as a homogeneous space obtained by the quotient of G by the isotropy group at each point on the leaf*, contain only constant coefficients: otherwise the points would not be indistinguishable under the group action. Note that the specification of the group is essential: the group $\mathbb{R} \times SO(3)$ is perfectly acceptable, and on the leaf we can have the structural equation

$$\begin{cases} d\omega_1 = 0, \\ d\omega_2 = -\omega_{23} \wedge \omega_3, \\ d\omega_3 = \omega_{23} \wedge \omega_2, \\ d\omega_{23} = \frac{1}{2}\rho\, \omega_2 \wedge \omega_3. \end{cases}$$

which is a homogeneous space $\mathbb{R} \times SO(3)/SO(2)$, and the bundle has group $SO(2)$. In other words, if the symmetry group is not the full subgroup $SO(q)$ for the forms ω_a, in making R_{abcd} constant along each leaf we have already effected a further reduction of the principal bundle.

Note that for the existence of a q co-dimensional foliation by such Killing vector fields, the number of Killing vector fields must be equal or greater than q, in which exactly q of them are pointwise linearly independent at each point. *In the further reduced bundle*, all of the quantities are invariant under the Lie group formed by the Killing vector fields. As they are also tensors, we have a practical method for constructing them: it suffices to enumerate the invariant tensors with the correct indices of a, b, … (up to a constant factor), and then the various quantities splits into a product: for example, if the group is the maximal rotational group, then K_{iab} must be $k_i\delta_{ab}$, where k_i is now a rank 1 tensor *defined on the reduced manifold*, i.e., restricted to each leaf it is becomes q constants. If $q > 1$, then M_{ija} must vanish, since there is no invariant vector on the sphere. Of course if as in the above example we have a non-maximal group, the situation is more complicated. In general, if the tensor under consideration has indices of the leaves represented as a whole by I and indices of the reduced manifold represented as J, then it must be of the form

$$\sum_{\alpha} B^{\alpha}_{J} L^{\alpha}_{I},$$

where each L^{α}_{I} is a invariant tensor on the leaf of the correct index, and each B^{α}_{J} is a tensor on the reduced manifold of the correct index. Note that since invariant tensors contains no degree of arbitrariness, *all degrees of arbitrariness of such tensors are restricted to the reduced manifold.*

In addition to the quantities M_{ija}, K_{iab}, S_{ijkl} and S_{abcd}, if the group generated by the Killing vectors are not the maximal possible symmetry group, *there will be additional differential invariants for the isometry.* Let $\mathfrak{h}$ be the Lie algebra of the Killing vector fields. Then the Lie algebra of the symmetry group of the leaves, $\mathfrak{so}(q)$, decomposes into $\mathfrak{so}(q) = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a vector space. Then we can divide linear combinations of the forms ω_{ab} into those that are in the Lie subalgebra $\mathfrak{h}$ and those that are not. Denote the latter by ω_α, $\alpha = 1, 2, \ldots, \dim \mathfrak{so}(p) - \dim \mathfrak{h}$. As for any reduction of the principal bundle, we have

$$\omega_\alpha = P_{\alpha a}\omega_a + Q_{\alpha i}\omega_i.$$

But each of the form ω_{ab} (and hence their linear combinations with constant coefficients) satisfies $\mathcal{L}_{\mathbf{V}}\omega_{ab} = 0$, so we immediately have

$$\mathcal{L}_{\mathbf{V}} P_{\alpha a} = 0, \qquad \mathcal{L}_{\mathbf{V}} Q_{\alpha i} = 0,$$

so again, these invariants that are introduced do not depend on the coordinates on the fibres. *In all cases, we have reduced the geometry of a higher dimensional problem to the geometry of a lower dimensional problem together with a set of fields defined on the lower dimensional space.*

We can also construct all spaces with a given isometry: it suffices to integrate the appropriate system of differential invariants. This will be illustrated now.

100 Degree of arbitrariness of a Riemannian submersion due to rotational isometry. We could calculate the degree of arbitrariness of a Riemannian submersion due to rotational isometry by first writing down the structural equations specific for this case, which are now much simplified compared to the general case, and then do the analysis. Here, however, we will simply consider the problem as adding constraints to our system of invariants of the general theory.

Note that covariant derivative in the vertical direction of any quantity are no longer considered independent. The quantity S_{abcd} is just the metric to the sphere, so $S_{abcd;i}$

can be written as a vector on the base. We can take this to mean that S_{abcd} is independent only for $a = b = c = d = 1$.

Next, since there exists no invariant vector field on the sphere, $M_{ija} = 0$. On the other hand, $K_{iab} = k_i \delta_{ab}$.

Next, we had the relations

$$M_{ij(a;b)} = -K_{[i|ab;|j]}, \qquad S_{abcd;i} = -2K_{ic[a;b]d} + \cdots , \qquad K_{ic[a;b]j} = M_{ij[a;b]c} + \cdots$$

and as before, we have preferred derivations with respect to the i, j indices. But now these derivations do not generate independent terms. So the terms on both sides are now no longer considered independent.

Hence now, the contributions to s_p comes from

$$S_{ijkl;m}, \qquad K_{iab;jk},$$

and $K_{iab;jk}$ has for counting purposes its indices i, j, k totally symmetric. Therefore

$$s_p = \frac{p(p-1)}{2} + 1,$$

there is a single degree of arbitrariness in addition to the Riemannian tensor on the reduced manifold.

For the special case $p = 1$, $q = 3$, we have, roughly speaking, a series of spheres packed along a line. If we add the constraint that the Ricci tensor of the total space vanishes, i.e., the space needs to be a vacuum solution of Einstein's equation, then we obtain the Schwarzschild solution.

Chapter 4

The problems of rigid flow

101 Overview. In this chapter we shall apply the method of structure preserving submersion to the study of rigid flow. Intuitively, a rigid flow is a vector field on some space that preserves some structure, usually a Riemannian (or even Euclidean) metric. We shall first begin with the historical roots of all rigid flow, namely the motion of a rigid body in Galilean space.

I. Newtonian rigid body

102 Galilean rigid motion within the framework of structure preserving submersion. The usual way of thinking about a rigid body is that we have some body B with a Euclidean metric defined on it, a space M also with a Euclidean metric, together with a one-parameter family of embeddings [†]

$$\iota(t) : B \to M$$

such that for any t, ι pulls back the metric on M to the metric on B. The parameter t is obviously interpreted as the physical time. This picture is not incorrect, but it is incomplete: spacetime is now the product space $M \times \mathbb{R}$ where $\mathbb{R}$ is the range of values of t. There is a structure on M: the Euclidean metric, but this picture does not specify any additional structure when we take spacetime as a whole.

A better way is to define the *Galilean space*, characterised by its inertial coordinates. If we have a coordinate system $(t, \mathbf{x})$ that is *inertial* (meaning that the equations of

[†] Note that we are dealing with rigid *body* here, not rigid *flow*, the difference being that the flow has an additional piece of data when compared with the body: the velocity of the flow at each time. As we shall see later, the motion of rigid body in Newtonian spacetime is more analogous to rigid flow in relativistic spacetime than that of Newtonian rigid flow, since in relativistic rigid flow the velocity can be normalised.

motion takes some simple form: the coordinates x_i form a Cartesian coordinate system and t is properly normalised), we can obtain any other inertial coordinate system by the Galilean transformation

$$\begin{pmatrix} 1 \\ t' \\ x'_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Delta t & 1 & 0 \\ \Delta x_i & v_i & r_{ij} \end{pmatrix} \begin{pmatrix} 1 \\ t \\ x_i \end{pmatrix}.$$

This is a group, where Δt is the time translation, Δx_i is the space translation, v_i is the boost and r_{ij} is the space rotation. As we saw in §**39**, we can realise Galilean spacetime as a homogeneous space, by forming the quotient of the Galilean group by the subgroup that keeps the origin invariant: namely the subgroup with $\Delta t = 0$, $\Delta x_i = 0$.

The Maurer–Cartan form of the Galilean group, which is also the Cartan connection of a Galilean space, is then in matrix form

$$\begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ \theta_i & \omega_i & \omega_{ij} \end{pmatrix}$$

by expanding the matrix Maurer–Cartan equation $d\omega + [\omega \wedge \omega] = 0$, we see that the structural equations for Galilean space are

$$\begin{cases} d\tau = 0, \\ d\theta_i = -\omega_i \wedge \tau - \omega_{ij} \wedge \theta_j, \\ d\omega_i = -\omega_{ij} \wedge \omega_j, \\ d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}. \end{cases}$$

These equations determine the geometry on the space $M \times \mathbb{R}$ where the body moves. The body itself is also represented locally as a homogeneous space (an Euclidean space). On the body, we have the Cartan connection

$$\begin{pmatrix} 0 & 0 \\ \pi_i & \pi_{ij} \end{pmatrix},$$

with the structural equations of the Euclidean group

$$\begin{cases} d\pi_i = -\pi_{ij} \wedge \pi_j, \\ d\pi_{ij} = -\pi_{ik} \wedge \pi_{kj}. \end{cases}$$

By our general method, for the rigid motion, we form the product space of $M \times \mathbb{R} \times B$, require $\theta_i = \pi_i$, and reduce the bundle to the appropriate subgroup. After the reduction of the bundle, mixing of space (reduced space coordinates) and time (fibre coordinate) is no longer allowed. So we write

$$\omega_i = K_i \tau + M_{ij} \theta_j.$$

On the other hand, differentiating $\theta_i = \pi_i$ gives

$$(\pi_{ij} - \omega_{ij} + M_{ij}\tau) \wedge \theta_j = 0,$$

so [†]

$$M_{(ij)} = 0, \qquad \omega_{ij} = \pi_{ij} + M_{ij}\tau.$$

Writing the structural equation with the remaining one forms,

$$\begin{cases} d\tau = 0, \\ d\theta_i = -\pi_{ij} \wedge \theta_j, \\ d\pi_{ij} = -\pi_{ik} \wedge \pi_{kj}. \end{cases}$$

This is just the structural equation of the space B together with $d\tau = 0$. However, recall that the submersion is defined by differential invariants, not only of the reduced bundle, but of the total space and reduced space as well. Hence K_i and M_{ij} are invariants in the theory. They satisfy certain relations among them. First, we expand the equation $d\omega_i = -\omega_{ij} \wedge \omega_j$, which tells us

$$K_{i;j} = M_{ij;0} + M_{ik} M_{kj}, \qquad M_{[ij;k]} = 0,$$

next, expansion of $d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}$ gives

$$M_{ij;k} = 0.$$

The system of involutive seeds occur at first order. The zeroth order normal invariants are

$$M_{ij} \quad (i > j), \qquad K_i,$$

and the first order normal invariants are

$$M_{ij;0} \quad (i > j), \qquad K_{i;0}.$$

So, if the space is n dimensional (hence the Galilean spacetime is $n+1$ dimensional), then

$$s_1 = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}, \qquad s_2 = s_3 = \cdots = 0.$$

[†] Here we need to use again Cartan's lemma and the trick that a tensor with three indices, one pair of which being symmetric and another pair antisymmetric, vanishes identically.

103 Physical significance of the invariants. We can interpret the differential invariants physically. Let a moving frame be represented by $\mathbf{M}$, $\mathbf{I}_0$, $\mathbf{I}_i$, where $\mathbf{M}$ is its position in the spacetime, $\mathbf{I}_0$ is a unit vector along the time direction based on $\mathbf{M}$ and $\mathbf{I}_i$ are a set of orthonormal vectors. The structural equations for the moving frame is

$$d(\mathbf{M}, \mathbf{I}_0, \mathbf{I}_i) = (\mathbf{M}, \mathbf{I}_0, \mathbf{I}_i) \begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ \theta_i & \omega_i & \omega_{ij} \end{pmatrix}.$$

If the frame is chosen such that $\mathbf{M}$ represents a point in the rigid body, then $\mathbf{I}_0$ just points along the flow lines of the rigid body. For such a frame, using the formula above

$$d\mathbf{I}_0 = \omega_i \mathbf{I}_i = K_i \mathbf{I}_i \tau + M_{ij} \mathbf{I}_i \theta_j.$$

This formula shows that, as we move forward in time infinitesimally, the unit vector along the flow line undergoes the change $K_i \mathbf{I}_i$, i.e., $K_i \mathbf{I}_i$ measures the *acceleration* undergone by the point $\mathbf{M}$ under the motion. If on the other hand we move in the space direction $\mathbf{I}_j$, the change undergone by $\mathbf{I}_0$ is $M_{ij} \mathbf{I}_i$. Hence $M_{ij} \mathbf{I}_i$ represents the expansion, vorticity and shear of the motion. We can decompose it into

$$M_{ij} = M_{[ij]} + (M_{(ij)} - \delta_{ij} M) + \delta_{ij} M, \qquad M = \frac{\operatorname{tr} M_{ij}}{n},$$

where the three terms in the expansion are respectively the vorticity, the shear, and the expansion. But we know that $M_{(ij)} = 0$, so a rigid motion must be expansion-free and shear-free, which we should already know from fluid mechanics.

We also know what we need to specify in order to obtain a well determined rigid motion: we specify $M_{ij;0}$ and $K_{i;0}$ along a one dimensional line, which amounts to specify the rotation M_{ij} and acceleration K_i *at a single point* at each instant. This corresponds to our intuition.

104 Rigid motion in Newtonian spacetime. Instead of studying rigid motion in the completely homogeneous Galilean spacetime, we can go a small step further and study rigid motion in Newtonian gravity. First, we need to formulate Newtonian gravity in an invariant coframe. First note that *any* Newtonian spacetime can be written in a frame such that

$$\begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ \theta_i & \omega_i & \omega_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ dt & 0 & 0 \\ dx_i & G_i dt & 0 \end{pmatrix}.$$

Physically, we require G_i to be the components of the acceleration due to gravity. This can be written in equations as

$$dG_i \wedge dx_i \wedge dt = 0,$$

for derivation, see [12]. Now, prolong this particular frame into the most general frame allowed, we see that a Newtonian spacetime is characterised by the structural equations

$$\begin{cases} d\tau = 0, \\ d\theta_i = -\omega_i \wedge \tau - \omega_{ij} \wedge \theta_j, \\ d\omega_i = -\omega_{ij} \wedge \omega_j + \Gamma_{ij}\theta_j \wedge \tau + \frac{1}{2}\Gamma_{ijk}\theta_j \wedge \theta_k, \\ d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}, \end{cases}$$

and the relation

$$(d\omega_i + \omega_{ij} \wedge \omega_j) \wedge \theta_i = 0,$$

which implies

$$\Gamma_{[ijk]} = 0, \qquad \Gamma_{ij} = \Gamma_{ji}.$$

The equation $d^2\theta_i = 0$ gives the "first Bianchi identity", which is simply that

$$\Gamma_{ijk} = 0,$$

and the equation $d^2\omega_i = 0$ gives the "second Bianchi identity", which is

$$\Gamma_{i[j;k]} = 0,$$

but as $\Gamma_{[ij]} = 0$, $\Gamma_{ij;k}$ is totally symmetric in all indices.

For the calculation of the Cartan characters, we see that again we need to take the time index to be smaller than the space index in order to apply our procedure of involutive seeds. Then s_{n+1} is the number of normal quantities of $\Gamma_{ij;k}$ where $k = n$, i.e., $\Gamma_{nn;n}$. This degree of arbitrariness is hence $s_{n+1} = 1$, the degree of arbitrariness of a single Newtonian gravitational potential (i.e., a scalar field).

Now for the rigid flow, again we have

$$\omega_i = K_i\tau + M_{ij}\theta_j,$$

and $d\theta_i = d\pi_i$ straight away gives $M_{(ij)} = 0$, as before. The equation $d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}$ again gives $M_{ij;k} = 0$. The equation for $d\omega_i$ gives

$$d\omega_i + \omega_{ij} = (K_{i;j} - M_{ij;0} - M_{ik}M_{kj})\theta_j \wedge \tau + M_{ij;k}\theta_k \wedge \theta_j = \Gamma_{ij}\theta_j \wedge \tau,$$

so

$$\begin{cases} K_{[i;j]} = M_{ij;0}, \\ K_{(i;j)} = M_{ik}M_{kj} + \Gamma_{ij}. \end{cases}$$

If we now specify the gravitational potential, this amounts to specifying Γ_{ij} as known functions of the coordinates, and in this case we see that the involutive seeds are exactly the same as in the case of Galilean spacetime. If, on the other hand, we consider an unspecified Newtonian potential, then there is no further constraints on Γ_{ij}, whereas $K_{i;j}$ is completely determined in terms of the rest. Hence there is also exactly one degree of arbitrariness, the same degree of arbitrariness as in the case with no rigid motion.

This shows that *Newtonian gravity places no restriction, and at the same time grants no new degree of arbitrariness to rigid motion, compared to rigid motion in Galilean spacetime.*

With our framework we can easily discuss rigid motion in more general spacetime with a Galilean connection: for example, we can waive the requirement of the existence of a single gravitational potential $(d\omega_i+\omega_{ij}\wedge\omega_j)\wedge\theta_i = 0$, or the requirement that space is flat $d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} = 0$. Since the physical significance of such spacetime is rather obscure, we shall simply note that, when the space is not homogeneous, then any non-trivial rigid motion (i.e., except those in which the body just sits stationary in time) requires a symmetry in the direction the body moves (for a related problem, see §**114**), and that the gravitational potentials Γ_{ij}, which are now not subject to constraints, neither adds or subtracts from the degree of arbitrariness of rigid motion.

ii. Born rigid flow

105 The notion of Born rigid flow. The notion of rigidity in Newtonian spacetime is intuitive and straightforward. It is natural to extend this notion to the theory of relativity discovered by Einstein. Such a definition is given by Born [4], which reads:

A body is called rigid if the distance between neighbouring pair of particles, measured orthogonal to the worldlines of either of them, remains constant along the worldline.

Let us immediately note a few things. First, the same wording can be used to define rigidity in Newtonian spacetime, if the notion of worldline and orthogonality are defined in the obvious manner. Second, in relativity, *this condition should be taken to be infinitesimal*, since the distance "orthogonal" to a worldline only has a precise meaning in such a limit (in Newtonian spacetime, however, this condition makes sense even with respect to finite distance). If we denote the vector field along the worldlines to be $\lambda\mathbf{I}_0$ and an orthonormal co-frame to be ω_0, ω_i, we see that this condition for rigidity is simply

$$\mathcal{L}_{\lambda\mathbf{I}_0}(\omega_i \otimes \omega_i) = 0,$$

and comparing with our discussion in §**96**, we see that this condition is just the condition for a Riemannian submersion of codimension one. Hence we can use our results about Riemannian submersions to study rigid flow in relativity (of course, since we have been doing Riemannian submersions, we need the so-called "Wick rotation" trick, and pretend that we are working with a positive definite metric, which we will do implicitly).

106 The structural equations and the general degree of arbitrariness. The structural equations of rigid flow in relativity is a much simplified version of the structural equations for Riemannian submersion. Let ω_0, ω_i be the basic coframe, parts of the vertical forms decomposes:

$$\omega_{0i} = -\omega_{i0} = K_{i00}\omega_0 - M_{ij0}\omega_j \equiv K_i\omega_0 - M_{ij}\omega_j,$$

and rigidity requires $M_{(ij)} = 0$. Now we can immediately give physical interpretations to M_{ij} and K_i: since we have

$$d\mathbf{I}_0 = K_i\omega_0\mathbf{I}_i + M_{ij}\omega_j\mathbf{I}_i,$$

we see that K_i and M_{ij} have exactly the same interpretations as in Newtonian spacetime: they are the acceleration and vorticity of the flow respectively, and the flow is shear-free and expansion-free. The modified connection is defined by

$$\pi_{ij} = \omega_{ij} - M_{ij}\omega_0,$$

and the structural equations now read

$$\begin{cases} d\omega_i = -\pi_{ij} \wedge \omega_j, \\ d\omega_0 = -K_i\omega_0 \wedge \omega_i - M_{ij}\omega_i \wedge \omega_j, \\ d\pi_{ij} = -\pi_{ik} \wedge \pi_{kj} + \frac{1}{2}S_{ijkl}\omega_k \wedge \omega_l. \end{cases}$$

We can also write the Riemann tensors in terms of the invariants:

$$\begin{cases} R_{ijkl} = S_{ijkl} + M_{il}M_{jk} - M_{ik}M_{jl} - 2M_{ij}M_{kl}, \\ R_{ijk0} = M_{ij;k} - M_{jk}K_i + M_{ik}K_j + M_{ij}K_k, \\ R_{0i0j} = M_{ik}M_{jk} - K_{(i;j)} - K_iK_j, \end{cases} \tag{4.1}$$

and for the algebraic relations, in addition to those that involve exchanges of derivation indices, the usual ones for Riemann tensors, and those that are obtained by covariant differentiation, we have

$$\begin{cases} M_{[ij;k]} = 3M_{[ij}K_{k]}, \\ M_{ij;0} = -K_{[i;j]}. \end{cases}$$

There are no Bianchi relations for second order derivations—this is in contrast with the higher codimensional case. We can write the table of the involutive seeds

Invariant	Normal terms
M_{ij}	$i > j$
K_i	all
S_{ijkl}	$i > j,\ k > l,\ i \geq k,\ j \geq l$
$M_{ij;k}$	$i > j,\ i \geq k$
$K_{i;0}$	all
$K_{i;j}$	all
$S_{ijkl;m}$	$i > j,\ k > l,\ i \geq k,\ j \geq l,\ k \geq m$

If the space is n dimensional, $s_n = n - 1$. Except for $n \leq 2$, this is less than the degree of arbitrariness of a Riemannian space, $n(n-1)/2$, showing that not all spaces admit rigid flow (the case where $n = 2$ has already been done in §**87**). This contrasts with the case of spaces with a Galilean connection, in which there always exist the trivial flow.

107 Rigid flow in homogeneous spacetime. By analogy with the case in Galilean spacetime, we can now study the problem of the existence of rigid flow in Minkowski spacetime. Since there is no essential difference, we should at the same time include the study of rigid flow in all homogeneous time, i.e., de Sitter and anti-de Sitter spacetime. Our result would generalise the following classical theorem to all dimensions and to all homogeneous spacetimes (and later to all conformally flat spacetimes of dimension ≥ 4):

The Herglotz–Noether Theorem. *In the spacetime of* $(3+1)$*-dimensional special relativity, every rotational Born-rigid flow must be isometric.*

See [24, 30, 30, 22] for more details about the proof of the classical theorem.

Specifying the geometry of the total spacetime amounts to specifying the quantities R_{ijkl}, R_{ijk0} and R_{0i0j} in (4.2). Since the total spacetime is now assumed to be homogeneous, these quantities are constant. In particular, they do not depend on the fibre coordinates. Then we can immediately see that M_{ij} also does not depend on the fibre coordinates: it suffices to take the first equation of (4.2)

$$R_{ij\bar{i}\bar{j}} = S_{ij\bar{i}\bar{j}} - 3M_{ij}M_{\bar{i}\bar{j}}$$

where $\bar{i}$ and i represent the same index, with no summation over them.

Now there are two cases that have to be discussed separately.

First case. Assume that M_{ij} does not vanish identically. Then the second equation contains the equations

$$R_{ij\bar{i}0} = M_{ij;\bar{i}} + 2M_{ij}K_{\bar{i}},$$

since M_{ij} does not depend on the fibre coordinates, $M_{ij;k}$ does not neither. Then the above equation shows that if $M_{ij} \neq 0$, then K_k does not depend on the fibre coordinates where k can take any index that appears in non-vanishing M_{ij}. Now suppose l is an index that does not appear in the index of any non-vanishing M_{ij}, then for a certain non-vanishing M_{ij}, for example M_{12}, we have

$$R_{12l0} = M_{12;l} + M_{12}K_l,$$

showing that K_l does not depend on the fibre coordinates either.

Now we see that both M_{ij} and K_k does not depend on the fibre coordinates. By our discussion of §**98**, we see that the submersion is actually generated by a Killing vector field. It is easy to see that in order to ensure $M_{ij} \neq 0$, this Killing vector field must contain some rotational part.

Since specifying a Killing vector field in a homogeneous space it suffices to specify a few constants at a point, in this case

$$s_1 = s_2 = \cdots = 0,$$

and the exterior differential system that we have set up for the invariants is Frobenius integrable. (We will see later in §**109** that such Frobenius integrability is actually quite exceptional.)

Second case. We need to consider only the case where $M_{ij} = 0$ for all i, j, since otherwise by using the right action of the principal bundle we can transform the system locally into an equivalent one for which $M_{ij} \neq 0$ for all i, j. Then, the equations (4.2) become

$$\begin{cases} R_{ijkl} = S_{ijkl}, \\ R_{ijk0} = 0, \\ R_{0i0j} = -K_{(i;j)} - K_iK_j. \end{cases}$$

Immediately we see from the second equation above that, for homogeneous space, unless the space has vanishing curvature, i.e., Minkowski space, there is no solution. Hence assume that we are in Minkowski space. Then

$$S_{ijkl} = 0, \qquad K_{(i;j)} = -K_iK_j,$$

which together with $K_{[i;j]} = 0$, forces us to take the involutive seeds as

$$K_{i;0}.$$

Let us now come to the verification of condition **R** (§**54**). In the general case, we use K_i to solve for ω_i, $K_{i;0}$ to solve for ω_0. As for ω_{ij}, we use the series of invariants

$K_{i;00}, K_{i;000}, K_{i;0000}, \ldots$: this is possible, since all of these transform under $SO(n-1)$ and are independent.

The degree of arbitrariness is thus $n-1$, occurring at dimension 1. Intuitively, this is the motion of hyperplanes in Minkowski spacetime, the hyperplane always being orthogonal to the worldline of any one of its point. The $n-1$ degree of arbitrariness is the acceleration of any one of its point as a function of time.

Remark. This reasoning cannot be extended to the higher codimensional case: if we attempt to do this, we only get $\sum_a M_{ija} M_{\bar{i}\bar{j}a}$ independent of the fibre coordinates. Roughly speaking, the proof goes through because on $\mathbb{R}$ there is no non-trivial connected isotropy group.

108 Geometrical interpretations. First case. Here we simply have a rotational Killing vector on spacetime. Since $M_{ij} \neq 0$, the distribution $\omega_0 = 0$ gives

$$d\omega_0 = -M_{ij}\omega_i \wedge \omega_j,$$

i.e., it is not completely integrable. This shows that it is impossible to find a coordinate system $(\mathbf{x}, t)$ on the spacetime such that t is the parameter along each fibre, $\mathbf{x}$ is the parameter on the reduced space and *for every constant t section we have a section isometric to the reduced space* (such a picture would correspond to our intuition in the Newtonian case: the constant time sections are just the "moving rigid body").

The equation

$$S_{ij\bar{i}\bar{j}} = R_{ij\bar{i}\bar{j}} + 3M_{ij}M_{\bar{i}\bar{j}}$$

shows that the reduced space is more positively curved than the spacetime, since M_{ij}^2 is always positive (this is in the Riemannian sense: in the pseudo-Riemannian case the statement still has some content under a Wick rotation). However, the reduced space is not homogeneous. For example, for dimension $3+1$ Minkowski spacetime, we can use a coframe such that $M_{12} \neq 0$ but $M_{13} = M_{23} = 0$. Then it is easy to see that

$$S_{ijkl} = 0 \qquad \text{except for} \qquad S_{1212} > 0,$$

and ω_3 is actually a flat direction.

It should also be noted that the solutions we obtained are local, and actually in general *global solutions cannot exist* for Minkowski spacetime. Indeed, suppose that we choose an inertial frame in a Minkowski spacetime such that the Killing vector consists of pure rotation. Then in this frame the velocity of the vector field is proportional to the distance from the centre of rotation, and as we go further and further this velocity will exceed the speed of light.

Second case. Now $M_{ij} = 0$, the distribution $\omega_0 = 0$ is completely integrable: the picture of some rigid body moving in spacetime is valid. Furthermore, since

$$S_{ijkl} = R_{ijkl} = 0,$$

we know what these moving bodies are: they are just hyperplanes. The non-zero Cartan character comes from the time derivative $K_{i;0}$. This shows that to specify completely the motion, we can take an arbitrary point on the moving body, specify its acceleration K_i at a certain instant and the change of its acceleration $K_{i;0}$ at all time. This can be visualised as an ordinary plane moving arbitrarily in three dimensional space, but we need to remember that the "time" in this picture is intrinsic and depends only on how the planes at different time are stacked together and independent of the parameter time in our model.

109 Degree of arbitrariness of Born-rigid flow in general spacetime. The generalised Herglotz–Noether theorem tells us that any generic Born-rigid flow must be isometric in homogeneous spacetime. We can understand this problem even better by investigating the degree of arbitrariness of the system where we specify the invariants $R_{\mu\nu\rho\lambda}$ completely, but without assuming that the spacetime is in anyway special. The invariants $R_{\mu\nu\rho\lambda}$ of the total space are, as we know,

$$\begin{cases} R_{ijkl} = S_{ijkl} + M_{il}M_{jk} - M_{ik}M_{jl} - 2M_{ij}M_{kl}, \\ R_{ijk0} = M_{ij;k} - M_{jk}K_i + M_{ik}K_j + M_{ij}K_k, \\ R_{0i0j} = M_{ik}M_{jk} - K_{(i;j)} - K_iK_j. \end{cases} \tag{4.2}$$

To completely specify a spacetime means simply to take $R_{\mu\nu\rho\lambda}$, as well as all of their derivatives, as known functions. The first relation can be used to express S_{ijkl} completely in terms of M_{ij} and R_{ijkl}, the second can be used to express $M_{ij;k}$ completely in terms of R_{ijk0}, M_{ij} and K_i (the reader can check that this is really the case, by verifying that the symmetries of the equation does not make us to miss any terms), and the third equation expresses $K_{(i;j)}$ completely in terms of R_{0i0j}, K_i and K_j. By considering $R_{ijkl;0}$, we see that *in the general case* $M_{ij;0}$ is completely specified in terms of $R_{ijkl;0}$: in more details, the first equation implies

$$R_{ij\bar{i}\bar{j}} = S_{ij\bar{i}\bar{j}} - 3M_{ij}M_{\bar{i}\bar{j}}$$

where a bar over i means that there is no summation between i and $\bar{i}$, but otherwise $i = \bar{i}$. This implies

$$M_{ij;0}M_{\bar{i}\bar{j}} = -\tfrac{1}{6}R_{ij\bar{i}\bar{j};0}$$

and if $M_{ij} \neq 0$ for all i, j, which is what we mean by the general case, $M_{ij;0}$ is completely determined by $R_{ij\bar{i}\bar{j};0}$, which by our assumption is a known function. Now we see that

S_{ijkl}, $M_{ij;k}$, $M_{ij;0}$, $K_{(i;j)}$ are all completely determined. By $K_{[i;j]} = -M_{ij;0}$, $K_{[i;j]}$ is as well. Differentiating the second equation of (4.2) and taking those linear in ω_0 shows that *in the general case*, $K_{i;0}$ is completely determined. By our definition in §**54**, we cannot have any non-empty set as the set of seeds, and hence this system has no degree of arbitrariness at all. This means that the system either has solutions depending on constants, or has no solutions at all. This is in accordance with our remark earlier that, since the degree of arbitrariness of the system without specifying the geometry is less than the degree of arbitrariness of a general Riemannian space, not all configurations admit solutions.

But we also need to know, given a particular spacetime and a particular geometry on it, whether the system is inconsistent or Frobenius integrable. The general procedure, as discussed in §**60**, involves trying to determine if all the algebraic relations being imposed are compatible. For the present case, the equations (4.2) together with all of their covariant derivatives are added to the old system of relations, so now for example the quantity $M_{ij;k0}$ can be calculated by either considering the expression $dM_{ij;k}$ or $dM_{ij;0}$, the expression of the first involves the quantity $R_{ijk0;l}$, while the second involves $R_{ijkl;0}$, and these two expressions need to be checked to be compatible. Of course in practice this is a difficult verification, but in general, when the specification of $R_{\mu\nu\rho\lambda}$ is *generic*, they will not be compatible since the number of equations greater than the number of variables, and compatibility requires further constraints on the specification of $R_{\mu\nu\rho\lambda}$. Hence the cases where the exterior differential system for the invariants is Frobenius integrable, as in the Minkowski case, are actually *exceptional*: in general we have inconsistent systems.

In this way, the Herglotz–Noether theorem is really a “well-anticipated” theorem: as we know that moving frames having no degree of arbitrariness must have structural equations reducible to Maurer–Cartan equations of Lie groups, we expect only completely symmetrical solutions. The Herglotz–Noether theorem simply tells us that the symmetries are given by just the rotating Killing vectors.

110 The normal form of the metric for rigid flow. For some applications it might be advantageous to have at our disposal the normal form of the metric for a certain rigid flow. To find this normal form, it is best to start from an orthonormal frame defined on the manifold. In our case, let the frame be formed by

$$\mathbf{e}_0, \qquad \mathbf{e}_i,$$

and the corresponding coframe be formed by

$$\omega_0, \qquad \omega_i.$$

We further assume that the normalised vector field formed by $\mathbf{e}_0$ is along the flow. Now it is possible to choose coordinates such that $\mathbf{e}_0 = \partial_{x_0} \equiv \partial_t$: this just corresponds to a normalised local coordinates on the flowlines. We choose the other coordinates such that they locally label the flowlines uniquely, so the subset of an open ball in this manifold defined by $t =$ constant label these flowlines.

Now we investigate what form the coframe would take. In general,

$$\omega_0 = a\,dt + b_i dx_i, \qquad \omega_i = c_i dt + f_{ij} dx_j.$$

On the other hand, we must have $\omega_0(\mathbf{e}_0) = 1$, $\omega_i(\mathbf{e}_0) = 0$, the first of which gives $a = 1$, and the second of which gives $c_i = 0$. Thus, for any flow, we can find an adapted coordinate to the flow such that the metric takes the form

$$g = -\omega_0 \otimes \omega_0 + \sum_i \omega_i \otimes \omega_i = -(dt + b_i dx_i)^2 + f_{ki} f_{kj} dx_i dx_j.$$

Since f_{ij} cannot be determined by the specification of our coordinates and do not have independent meaning, we will write the above formula instead as

$$g = -(dt + \varphi_i dx_i)^2 + h_{ij} dx_i dx_j, \qquad h_{ij} = h_{ji}.$$

Now we will try to investigate what restriction on this metric applies when the vector field ∂_t defines a Riemannian submersion. The vector field is $\mathbf{e}_0$, i.e., has components $(1, 0, 0, 0)$, and in the othornomal frame the component is the same for the covariant and contravariant compoments. Thus, we immediately see that the projection of the metric into the horizontal subspace is

$$h \equiv g + \omega_0 \otimes \omega_0 = h_{ij} dx_i dx_j.$$

Since the components of the vector field $\mathbf{e}_0$ is constant in the coordinate system we have set up, we have

$$\mathcal{L}_{\mathbf{e}_0} h = \frac{\partial h_{ij}}{\partial t} dx_i dx_j = 0.$$

Thus the condition for rigid flow simply means that $h_{ij} = h_{ij}(x_k)$ is independent of the time coordinates, i.e., *it defines a Riemannian metric on the quotient space with coordinates* x_i. Thus the normal form of the metric in the case of a Riemannian submersion is

$$g = -(dt + \varphi_i(t, x_k) dx_i)^2 + h_{ij}(x_k) dx_i dx_j. \tag{4.3}$$

In particular, there is no restriction on what φ_i should be. It is also obvious that whenever we write down a metric in the form of (4.3), then ∂_t is a normalised vector field and defines a Riemannian submersion.

Note that, for any positive scalar function $\lambda = \lambda(t, x^i)$, we also have

$$\mathcal{L}_{\lambda \mathbf{e}_0} h = \lambda \frac{\partial h_{ij}}{\partial t} dx_i dx_j = 0.$$

If instead of Riemannian submersion we consider Weyl submersion, which will be defined later in §**122**, then the condition for Weyl-rigid flow

$$\mathcal{L}_{\mathbf{e}_0} h = \frac{\partial h_{ij}}{\partial t} dx_i dx_j = k h_{ij} dx_i dx_j, \qquad k > 0$$

and in this case the normal form of the metric is clearly

$$g = -(dt + \varphi_i(t, x_k) dx_i)^2 + e^{\lambda(t, x_k)} h_{ij}(x_k) dx_i dx_j.$$

111 Rigid flow in conformally flat spacetime. Note that in the calculations for proving the Herglotz–Noether theorem we only used a few of the equations (4.2): more precisely, equations whose left hand sides are the following:

$$R_{ij\bar{i}\bar{j}}, \qquad R_{ijk0},$$

and the rest of the equations are easily seen to be satisfied identically. On the other hand, since the degree of arbitrariness for a general spacetime is zero, as we have seen above, in general we cannot set any rigid relativistic disk into rotating. These two observations suggest that the property that a Born-rigid flow is isometric could be valid for situations where we specify a weaker condition on the total spacetime. Note that in a general spacetime the Herglotz–Noether theorem is false: there is a counter-example given by Pirani in [38].

Here we investigate the validity of the theorem when the total space is conformally flat. For this, we first need the expressions for the Ricci tensor and scalar in terms of invariants of submersion, which are easily calculated to be

$$\begin{cases} R_{ij} = S_{ij} + 2M_{ik}M_{kj} - K_{(i;j)} - K_i K_j, \\ R_{00} = -K_{i;i} - K_i K_i + M_{ij} M_{ij}, \\ R_{0i} = M_{ji;j} - 2M_{ij} K_j, \\ \quad R = S - 2K_{i;i} - 2K_i K_i - M_{ij} M_{ij}. \end{cases}$$

Now we can form the Weyl tensor, by subtracting various traces from the Riemann tensor. The general formula is

$$\begin{aligned} W_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} &- \tfrac{1}{n-2}(\delta_{\mu\rho} R_{\lambda\nu} - \delta_{\mu\lambda} R_{\rho\nu} - \delta_{\nu\rho} R_{\lambda\mu} + \delta_{\nu\lambda} R_{\rho\mu}) \\ &+ \tfrac{1}{(n-1)(n-2)} R(\delta_{\mu\rho}\delta_{\lambda\nu} - \delta_{\mu\lambda}\delta_{\rho\nu}). \end{aligned}$$

Specialising to our present case, we have

$$\begin{cases} W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(\delta_{ik}R_{lj} - \delta_{il}R_{kj} - \delta_{jk}R_{li} + \delta_{jl}R_{ki}) \\ \qquad\qquad + \frac{1}{(n-1)(n-2)}R(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj}), \\ W_{i0j0} = R_{i0j0} - \frac{1}{n-2}(R_{ij} + R_{00}\delta_{ij}) + \frac{1}{(n-1)(n-2)}R\delta_{ij}, \\ W_{ijk0} = R_{ijk0} - \frac{1}{n-2}(\delta_{ik}R_{0j} - \delta_{jk}R_{0i}). \end{cases}$$

Now what we do with this mess? First observe that we can divide the Riemann tensor, Ricci tensor and Ricci scalars into two classes. The first class comprises of

$$R_{ijkl}, \qquad R_{0i0j}, \qquad R_{ij}, \qquad R_{00}, \qquad R$$

which contain only terms in

$$M_{ij}M_{kl}, \qquad K_{(i;j)} + K_iK_j \equiv Q_{ij},$$

and those involving S_{ijkl}. We shall also write $Q = \operatorname{tr} Q_{ij} = K_{i;i} + K_iK_i$. The second class comprises of

$$R_{ijk0}, \qquad R_{0i},$$

which contain only terms in

$$M_{ij;k}, \qquad M_{ij}K_k.$$

Then observe that W_{ijkl} and W_{i0j0} also belongs to the first class, whereas W_{ijk0} belongs to the second class. We can hence mimic our procedure in the homogeneous case, by first using equations of the first class to solve for M_{ij} in terms of the quantities which are independent of the fibre coordinates, and then using equations of the second class to solve for K_i.

In a sense, this result is also well-anticipated, and the problem is actually easier when the dimension is large: when all four indices of W_{ijkl} are different, it is just the Riemann tensor R_{ijkl}. Even though we cannot get nice quadratic terms such as $M_{ij}M_{\overline{ij}}$ now, the number of independent equations coming from W_{ijkl} is $O(n^4)$, whereas the number of variables of M_{ij} is only $O(n^2)$, and they can be solved completely. Then use the equations W_{ijk0} for which all indices are different, which are $O(n^3)$ in number, to solve for the $O(n)$ terms K_i. Again this can be solved completely, except for the case where $M_{ij} = 0$, as before.

In low dimensions, however, we do not have this luxury, and we need to be more precise of what we do. We will divide our procedure into two steps.

Step 1. First let us calculate the quantity W_{i0j0}:

$$W_{i0j0} = \tfrac{n}{n-2}(M_{ik}M_{jk} - \tfrac{1}{n-1}\delta_{ij}M_{kl}M_{kl}) - \tfrac{1}{n-2}(S_{ij} - \tfrac{1}{n-1}\delta_{ij}S) - \tfrac{n-3}{n-2}(Q_{ij} - \tfrac{1}{n-1}\delta_{ij}Q),$$

which we can write as (since the space is conformally flat, the Weyl tensor vanishes, $W_{\mu\nu\rho\lambda} = 0$)

$$Q_{ij} - \tfrac{1}{n-1}\delta_{ij}Q = \tfrac{n}{n-3}(M_{ik}M_{jk} - \tfrac{1}{n-1}\delta_{ij}M_{kl}M_{kl}) - \tfrac{1}{n-3}(S_{ij} - \tfrac{1}{n-1}\delta_{ij}S). \tag{4.4}$$

On the other hand, W_{ijkl} can be written

$$W_{ijkl} = R_{ijkl} - \tfrac{1}{n-2}(\delta_{ik}F_{lj} - \delta_{il}F_{kj} - \delta_{jk}F_{li} + \delta_{jl}F_{ki})$$

where we have defined

$$F_{ij} = R_{ij} - \tfrac{1}{2(n-1)}\delta_{ij}R.$$

We can use (4.4) to derive the expression of F_{ij} involving only the curvatures and M_{ij}:

$$F_{ij} = \tfrac{1}{n-3}((n-2)S_{ij} - \tfrac{1}{2}\delta_{ij}S) - \tfrac{3}{n-3}((n-2)M_{ik}M_{jk} - \tfrac{1}{2}M_{kl}M_{kl}\delta_{ij}).$$

Now investigate the expression of W_{ijkl} with only two distinct indices:

$$\begin{aligned} W_{ij\bar{i}\bar{j}} = S_{ij\bar{i}\bar{j}} &- \tfrac{1}{n-3}(S_{\bar{i}\bar{i}} + S_{\bar{j}\bar{j}}) + \tfrac{1}{(n-2)(n-3)}S \\ &- 3M_{ij}M_{\bar{i}\bar{j}} + \tfrac{3}{n-3}(M_{ik}M_{\bar{i}k} + M_{jk}M_{\bar{j}k}) - \tfrac{3}{(n-2)(n-3)}M_{kl}M_{kl}. \end{aligned}$$

Let the indices i and j go through all permutations and sum, we get

$$\tfrac{n^2-4n+5}{(n-2)(n-3)}M_{kl}M_{kl} = (\cdots),$$

where $(\cdots)$ represents quantities that are independent of the fibre coordinates. Substitute back, we get

$$3M_{ij}M_{\bar{i}\bar{j}} - \tfrac{3}{n-3}(M_{ik}M_{\bar{i}k} + M_{jk}M_{\bar{j}k}) = (\cdots).$$

Now let j go through all possible values and sum, we get

$$\tfrac{6}{n-3}M_{ik}M_{\bar{i}k} = (\cdots),$$

and back substitute again gives

$$-3M_{ij}M_{\bar{i}\bar{j}} = (\cdots),$$

which says that *the quantities M_{ij} are independent of the fibre coordinates.*

Step 2. The expression for W_{ijk0} is

$$\begin{aligned} W_{ijk0} = M_{ij;k} &- \tfrac{1}{n-2}(\delta_{ik}M_{lj;l} - \delta_{jk}M_{li;l}) \\ &- M_{jk}K_i + M_{ik}K_j + M_{ij}K_k + \tfrac{2}{n-2}\delta_{ik}M_{jl}K_l - \tfrac{2}{n-2}\delta_{jk}M_{il}K_l, \end{aligned}$$

and we now know that the first line contain only quantities independent of the fibre coordinates. Consideration of the term $W_{ij\bar{i}0}$ gives

$$2M_{ij}K_{\bar{j}} - \tfrac{2}{n-2}M_{il}K_l = (\cdots),$$

and summing over all j gives

$$-\tfrac{2}{n-2}M_{il}K_l = (\cdots),$$

back substituting gives

$$2M_{ij}K_{\bar{j}} = (\cdots),$$

so if $M_{ij} \neq 0$ for a certain pair i, j, then for these two values K_i and K_j are both independent of the fibre coordinates. If i, j and k are all distinct, then W_{ijk0} is just R_{ijk0}. Let k be an index such that $M_{lk} = 0$ for all choices of l, and choose a pair of indices i, j such that $M_{ij} \neq 0$. Then consideration of W_{ijk0} gives

$$M_{ij}K_k = (\cdots),$$

hence *as long as $M_{ij} \neq 0$ for any component, K_i is independent of the fibre coordinates.* We have successfully proved the following:

Any rotational rigid flow in a conformally flat spacetime is a Killing flow.

Remark. Unlike the homogeneous case, here the result does not assert the existence of any rigid flow, due to the fact that a conformally flat spacetime does not necessarily admit any rotational Killing vector field.

Cases of $n = 2, 3$. The above proof holds only for the dimension of the spacetime $n \geq 4$: the factors $(n-1)$, $(n-2)$ and $(n-3)$ appear in the numerator at various places. At a deeper level, for dimension less than 4, the Weyl tensor is trivial. For $n = 2$, we know that all spaces are conformally flat, so this case has already been studied in §**87**. For $n = 3$, the condition for a spacetime to be conformally flat is that the Cotton tensor

$$C_{\mu\nu\rho} = R_{\mu\nu;\rho} - R_{\mu\rho;\nu} + \tfrac{1}{4}(\delta_{\mu\rho}R_{;\nu} - \delta_{\mu\nu}R_{;\rho})$$

vanishes. This condition involves higher derivatives of M_{ij} and K_i, and in general does not imply that we must have Killing vector fields under rigid flow.

112 The relation of the generalised Herglotz–Noether theorem to the AdS–CFT correspondences. In recent works there are attempts to predict universal features of large rotating black holes in AdS backgrounds through the AdS–CFT correspondence[3].

The strategy is first to consider the dual description of black holes in terms of a quantum field theory on a conformal background, which is usually taken to be the Einstein static universe $S^{D-2}\times$ time[29, 41], and then pass to the thermodynamic limit of the quantum field theory, which was then argued to yield a *classical* and *dissipationless* relativistic fluid flowing on the conformal boundary of AdS. Physically, the most important cases are $\mathrm{AdS}_5 \times S^5$, $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_4 \times S^7$ which arise from type IIB and M theory, but the general case is also interesting in its own right. Thus, assuming the use of duality and the passing to the limit are legitimate in the realm one is considering, understanding of the classical fluid immediately yields an understanding of the original black holes.

To model a relativistic flow, we specify the time-like unit vector u^μ, which represents the set of the worldlines of the fluid particles. A flow described by such a vector field is completely general, and the key point in the analysis is that a flow on the CFT space given by the dual of a black hole must, to some first approximation, be *dissipationless*, where as usual in the fluid mechanics case, being dissipationless means that the flow has no shear nor expansion,

$$0 = \sigma^{\mu\nu} = \frac{1}{2}(h^{\mu\sigma}u^{\nu}{}_{;\sigma} + h^{\nu\sigma}u^{\mu}{}_{;\sigma} - \frac{1}{n-1}\theta h^{\mu\nu})$$

and

$$0 = \theta = u^{\mu}{}_{;\mu},$$

where the induced metric with respect to the fluid is defined as

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{4.5}$$

As can be easily seen, in such a case the assumption that a fluid is dissipationless is equivalent to that the flow is Born-rigid.

Physically, the no-shear condition is argued in a straightforward manner by recognising that in the perfect fluid approximation, we cannot have entropy production and a non-zero $\sigma^{\mu\nu}$ produces entropy. The expansion-free condition $\theta = 0$, on the other hand, is geometrical in nature. It should be recognised that in [3] we are actually calculating conformal invariant quantities using the language of Riemannian geometry, and in the proper setting, the trace of the stress tensor is not physical at all. This is perhaps most clearly seen by recognising, as in [3], that the trace is related to the (Riemannian) scalar curvature of the background space, but the scalar and Ricci part of the curvature can be set to arbitrary value by applying a scaling to the space. Hence, assuming that such a transformation has already been applied, we can safely assume that $\theta = 0$. There is a caveat, however: even if we started with a particular Riemannian model of a conformal space, say $S^3\times\mathbb{R}$, after setting $\theta = 0$ we can no longer safely working using the usual Riemannian metric on $S^3 \times \mathbb{R}$, otherwise inconsistencies might occur. Therefore, any geometrical results that we need to apply in such a situation has to be results valid for *all* conformally flat spacetime; validity for $S^3 \times \mathbb{R}$ alone is insufficient.

Now, if we are satisfied that in the fluid approximation of the dual description of black holes, the relevant fluid is a dissipationless (shear-free and expansion-free) fluid flowing in a conformally flat spacetime, then the following assumption, implicit in the paper of Bhattacharyya et al., further simplifies the matter so that concrete calculations can now be carried out:

Assumption. *In a conformally flat spacetime, the only dissipationless and rotating ($\omega^{\mu\nu} \neq 0$) fluid flow are those flow that coincide with Killing flows (isometric flows).*

Of course, this assumption is none other than the generalisation of Herglotz–Noether theorem to conformally flat spacetime, which we have proved moments ago. Thus we see the relevance of the extension of the Herglotz–Noether theorem to the conformally flat case to the study of black holes by AdS–CFT correspondences: it is the key ingredient that we need so that the fluid description of black holes on the boundary of AdS is tenable.

113 The existence of rigid flow in the rotation-free case. We have already dealt with the extensions of the Herglotz–Noether theorem in the case where the flow is rotational. It remains to discuss the other case which is classically believed to be easier, even trivial, namely when our $M_{ij} = 0$. In the classical setting we have seen that in such cases solutions occur as a family depending on one parameter, and we construct this family by choosing any time-like curve in the Minkowski spacetime, and construct its family of orthogonal hyperplanes. We have also seen that such solution is *impossible* in curved homogeneous spacetimes.

A question can now be asked: though in a general spacetime the concept of hyperplanes is not applicable, certainly we can construct geodesics going out from the curve that we have chosen and orthogonal to it, and the geodesics going out from the same point on the curve forms a submanifold. Why does this *not* define a rigid-motion in general? We know that a necessary condition for such a construction to define a rigid motion is that the submanifolds are all isometric to each other. In Minkowski spacetime this is ensured by the fact that all such constructions give planes and all planes are isometric, but in a general spacetime this condition will fail.

To get a deeper understanding of the irrotational case, we start from the normal form of the Riemannian metric which we have found:

$$g = -(dt + \varphi_i(t, x_k)dx_i)^2 + h_{ij}(x_k)dx_i dx_j.$$

We have already found that we can interpret M_{ij} as the integrability tensor of the distribution $\omega_0 = 0$, i.e., when this happens, the reduced space can be embedded in spacetime so that this family of embedded spaces all have the metric h_{ij} defined on it. Due to the existence of this family of embedded spaces, we can find a function τ labelling these embedded spaces (the first integral of the integral submanifolds due to $\omega_0 = 0$), and for this function the distribution $\omega_0 = 0$ is equivalent to $d\tau = 0$. This means that $\omega_0 \propto d\tau$, and we can write the metric in the form

$$g = f(\tau, x_k)d\tau^2 + h_{ij}(x_k)dx_i dx_j.$$

Note that not all specifications of the functions $f(\tau, x_k)$ give different geometries for the spacetime (though they give different flows). For example, take h_{ij} to be Euclidean, we can calculate the Riemann tensor to be

$$R^0{}_{i0j} = -\tfrac{1}{2} f^{-1} f_{,ij},$$

with all other components vanishing, and all affine functions f of the x^i give rise to the flat metric.

To generalise the above statements and to make them more precise, we consider the invariants of the submersion are K_i and S_{ijkl}, with the relation

$$S_{ijkl;0} = 0, \qquad K_{[i;j]} = 0,$$

and the equations relating the spacetime geometry to the submersion reads

$$\begin{cases} R_{ijkl} = S_{ijkl}, \\ R_{ijk0} = 0, \\ R_{0i0j} = -K_{i;j} - K_i K_j. \end{cases}$$

Note that as it stands, using this frame, it is not possible to find a system of involutive seeds for this system, since $\dot{S}_{ijkl} = 0$ requires us to take 0 to be the largest index, but since $\dot{K}_i$ is free we cannot take 0 to be the largest index. We could proceed by using a frame where none of the vectors forming the frame is along the flow direction. However, it is at the same time clear that if we first specify S_{ijkl}, then $K_{(i;j)}$ and $K_{i;0}$ are both unconstrained. The method of involutive seeds now dictates that we take both of them to be the involutive seeds, for the ordering 0 is now taken to be the *smallest* index, and the degree of arbitrariness is 1 occurring at dimension n, contributed by

$$K_{n-1;n-1}.$$

This corresponds to choosing $h_{ij}(x_k)$ first, *then* choose $f(\tau, x_k)$. We also see that the necessary condition for a space to admit a rotation-free rigid motion is that we can find frames in which R_{ijkl} forms a Riemannian tensor of a lower dimensional space whose value is independent of where on the flowline we calculate it, and $R_{ijk0} = 0$. Of course, this condition is not very useful for actually checking if a space admits a rotation-free rigid motion, but the remarkable thing follows from the fact that the condition is also sufficient: since if it is satisfied, we are now dealing with the system

$$K_{i;j} = -R_{0i0j} - K_i K_j$$

where R_{0i0j} is now considered to be given. We can then calculate the degree of arbitrariness of this system using the involutive seeds, the calculation is exactly the same as in the Minkowski case, and hence the general solution depends on $n-1$ functions of one variable. Thus, *if a space admits a rotation-free Born-rigid flow on it, it admits also a whole family of solutions parametrised by $n-1$ functions of one variable.*

III. The problem of Galilean invariance for particle lagrangian

114 Galilean invariance. Besides the usual applications, the methods of a rigid flow in Riemannian spacetime apply in unexpected circumstances. Let us now study the problem of Galilean invariance for a point particle lagrangian in classical mechanics. The lagrangian is

$$L = \tfrac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j.$$

A Galilean invariance is just a local symmetry for the equation of motion under boosts, i.e., the invariance of the lagrangian up to a divergence. Under such a symmetry, infinitesimally we can set

$$\delta x^i = f^i(x,t), \qquad \delta \dot{x}^i = \partial_t f^i + f^i{}_{,k} \dot{x}^k,$$

so we have

$$\delta L = \tfrac{1}{2}(g_{ij,k} f^k + g_{kj} f^k{}_{,i} + g_{ik} f^k{}_{,j}) \dot{x}^i \dot{x}^j + g_{ij} \partial_t f^i \dot{x}^j = \tfrac{1}{2}(\mathcal{L}_f g_{ij}) \dot{x}^i \dot{x}^j + g_{ij} \partial_t f^i \dot{x}^j.$$

This change is a symmetry of the system if it is a total time derivative of a function $F(x,t)$

$$\delta L = \frac{dF}{dt} = \partial_t F + F_{,k} \dot{x}^k.$$

Now, the velocity is an independent variable. Hence the coefficients of the powers of $\dot{x}$ must be equal. Therefore

$$\begin{cases} \partial_t F = 0, \\ F_{,i} = g_{ij} \partial_t f^j, \\ 0 = \mathcal{L}_f g_{ij} \end{cases}$$

The first equation means that F is a function of space only: $F = F(x)$, and the third means that at any instant of time, the vector field f is a Killing vector field. So the most general form of f is

$$f^i = \varphi_a(t)v_a^i(x)$$

where the v_a^i form a basis of the Killing vector fields, $a = 1, 2, \dots k$ where k is the number of Killing vector fields on the space (might as well be zero), and they depend on space only. Then the second equation gives us

$$\varphi'_a(t)v_{ai}(x) = F_{,i}(x)$$

and we immediately have:

$$\varphi_a(t) = a_a t + b_a,$$

which are affine functions of t (†). But then $a_a v_{ai}$ is a Killing vector, since now a_a are constants. Let us write $v_i = a_a v_{ai}$, then we also have

$$v_{[i,j]} = 0.$$

Hence the system admits a certain number of Galilean boost invariance if and only if the system has the same number of *gradient Killing vectors*. To find such a symmetry is to find a vector field v^i such that, first, it is Killing, and second, its covariant form is given by a total derivative. This problem can now be studied completely within the geometry of the space with metric g_{ij} and without consideration of the effects of time.

115 Galilean symmetry from rigid flow. It is easy to check that in flat space, only the translation Killing vector fields are gradient Killing vector fields, and the only homogeneous space with positive definite metric that admits gradient Killing vector fields at all is Euclidean space. We would like to investigate the problem in the case of a space with a general positive definite Riemannian metric.

Let us now set up orthonormal moving frames on the Riemannian space with metric g_{ij}. If the space is n dimensional, let us align $\omega^\star$ along v_i:

$$e^{\lambda(x)}\omega^\star = v_i dx^i$$

where $\exp\lambda$ is a scalar function depending on position and is fixed up to a global constant by the orthonormal condition of the moving frame and the norm of the Killing vector field at different points. The quantity $\exp\lambda$ is required to be non-negative

(†)The case $a_a = 0$ is the trivial case: the symmetry does not involve time and the Lagrangian only has the spatial symmetries, not the boost symmetries.

everywhere, and positive almost everywhere, and hence we write it in this exponential form. The rest of the moving frame is ω^i where now $i = 1, 2, \ldots, n-1$ and

$$ds^2 = \omega^\star \otimes \omega^\star + \sum_{i=1}^{n-1} \omega^i \otimes \omega^i,$$

which leads us to the formalism of a rigid flow. We have the first Maurer–Cartan relations

$$\begin{cases} d\omega^\star = -K_i \omega^\star \wedge \omega^i - M_{ij} \omega^i \wedge \omega^j, \\ d\omega^i = -(\omega_{ij} - M_{ij}\omega^\star) \wedge \omega^j \end{cases}$$

and as usual, $M_{ij} = -M_{ji}$.

As we are in the case where the submersion comes from an isometry, we must have that

$$K_{[i;j]} = 0, \qquad K_{i;\star} = 0,$$

together with the relations

$$\lambda_{;\star} = 0, \qquad \lambda_{;i} = K_i,$$

as we have seen in §**98**. The condition that the Killing vector is locally a gradient vector is, by Poincaré's lemma,

$$0 = d(e^\lambda \omega^\star) = e^\lambda (2K_i \omega^i \wedge \omega^\star - M_{ij} \omega^i \wedge \omega^j)$$

and hence we are required to have

$$K_i = 0, \qquad M_{ij} = 0.$$

in other words, there is now a complete decoupling between the Killing vector field direction and the "horizontal" direction, and every space that admits a gradient Killing vector is uniquely determined by giving its reduced space after the reduction, and the total space is simply the product space of the reduced Riemannian space and $\mathbb{E}^1$, the Euclidean line, with the product metric on it.

116 The case of several boost symmetries We now know that if we have a boost symmetry, then we have the frame $\omega^\star, \omega^i$ on the space, and

$$\begin{cases} d\omega^\star = 0, \\ d\omega^i = -\omega^i{}_j \wedge \omega^j. \end{cases}$$

Now we prove by induction that *the existence of additional gradient Killing vectors implies the flat directions can be extended to totally flat, uncoupled fibrations of* *higher dimensions.* Assume we already have ω^a, $a = 1, \dots, k$ the flat directions. The Maurer–Cartan equations are

$$\begin{cases} d\omega^a = 0, \\ d\omega^i = -\omega^i{}_j \wedge \omega^j. \end{cases}$$

where now $i = 1, 2, \dots, n-k$. Now suppose we have another gradient Killing vector field, which we write as

$$\mathbf{u} = a^a \mathbf{I}_a + b^i \mathbf{I}_i$$

and with the indeterminacy of an overall constant. We can actually fix this constant: since this Killing vector field must also be a pure translation as the first one, the vector field has constant norm, and we can set

$$\sum_a (a^a)^2 + \sum_i (b^i)^2 = 1.$$

We now have two conditions that we must impose. First, $\mathbf{u}$ is a Killing vector field, and second, the form

$$\mu = a^a \omega^a + b^i \omega^i$$

satisfies $d\mu = 0$. For the first condition, we have (working with a section)

$$\begin{cases} \mathcal{L}_{\mathbf{u}} \omega^\star = da^a = a^a{}_{,b} \omega^b + a^a{}_{,i} \omega^i \\ \mathcal{L}_{\mathbf{u}} \omega^i = -\mathbf{u} \lrcorner (\omega^i{}_j \wedge \omega^j) + db^i = b^i{}_{,a} \omega^a + c_i \omega^i \end{cases}$$

by the magic formula, where c_i are functions we do not really care about. Then

$$\mathcal{L}_{\mathbf{u}} ds^2 = (a^a{}_{,i} + b^i{}_{,a})(\omega^i \otimes \omega^a + \omega^a \otimes \omega^i) + \cdots$$

where the dots are terms not containing the cross terms in ω^a and ω^i. Hence

$$a^a{}_{,i} + b^i{}_{,a} = 0.$$

For the second condition,

$$0 = d(a^a \omega^a + b^i \omega^i) = (a^a{}_{,i} - b^i{}_{,a}) \omega^i \wedge \omega^\star + \cdots$$

where again we have omitted terms not containing the cross terms. Hence, we need

$$a^a{}_{,i} - b^i{}_{,a} = 0,$$

which together with the earlier condition implies $b^i{}_{,a} = 0$, b^i are functions actually defined on the reduced space.

We also have $a_{a,i} = 0$. If we consider now the terms in $\omega^a \otimes_S \omega^b$ and $\omega^a \wedge \omega^b$ in the above calculations, the conditions we get are

$$a_{a,b} + a_{b,a} = 0, \quad a_{a,b} - a_{b,a} = 0 \quad \Rightarrow \quad a_{a,b} = 0$$

which together with the previous condition implies that that a_a are constants. Then, using the residual $SO(n-1)$ symmetry for the frames with respect to ω^i and $\mathbf{I}_i$, we can write

$$\mathbf{u} = a_a \mathbf{I}_a + b\mathbf{I}_{n-k}$$

where now a_a and b are all constants. Since sums of Killing vector fields with constant coefficients are Killing vector fields themselves, this shows that $\mathbf{I}_{n-k}$ is a Killing vector field, and in this case it is a gradient vector field as well. The Cartan connection matrix in this frame now becomes

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^a & 0 & 0 & 0 \\ \omega^{n-k} & 0 & 0 & 0 \\ \omega^i & 0 & 0 & {\omega^i}_j \end{pmatrix}$$

where now $i, j = 1, 2, \ldots, n-k-1$. This shows that any n dimensional space with k gradient Killing vector fields can be constructed by a $n-k$ dimensional Riemannian space multiplied with $\mathbb{E}^k$, with the direct product metric. In particular, *the only n dimensional space with n independent gradient Killing vectors is Euclidean space itself.*

117 Generalisation to time-dependent systems There are several ways that we can generalise the Lagrangian that we started with: first, by allowing a potential term, second, by allowing space geometry to be time dependent, and third, by allowing magnetic force terms. The most general Lagrangian for these cases combined can be written

$$L = \tfrac{1}{2} g_{ij}(x,t)\dot{x}^i \dot{x}^j + A_i(x,t)\dot{x}^i - V(x,t)$$

where now $i = 1, 2, \ldots, nk$ where n is the dimension of the manifold the dynamics takes place, and k is the number of particles. As usual,

$$\delta x^i = f^i(x,t), \qquad \delta \dot{x}^i = \partial_t f^i + {f^i}_{,k}\dot{x}^k.$$

Now we have

$$\delta L = \tfrac{1}{2}(\mathcal{L}_f g_{ij})\dot{x}^i \dot{x}^j + g_{ij}\partial_t f^i \dot{x}^j + (\mathcal{L}_f A_i)\dot{x}^i + A_i \partial_t f^i - V_{,i} f^i,$$

as in mechanics, the variation does not involve the term δt.

The change is a symmetry of the system if it is a total time derivative of a function $F(x,t)$

$$\delta L = \frac{dF}{dt} = \partial_t F + F_{,k}\dot{x}^k.$$

Now, the velocity is an independent variable. Hence the coefficients of the powers of $\dot{x}$ must be equal. Therefore

$$\begin{cases} \partial_t F = A_i \partial_t f^i - V_{,i} f^i, \\ F_{,i} = g_{ij}\partial_t f^j + \mathcal{L}_f A_i, \\ 0 = \mathcal{L}_f g_{ij} \end{cases}$$

Again, assume that v_a^i is a basis of Killing vectors on the space, which now in principle can be time-dependent. For example, locally we can identify a small neighbourhood of Euclidean space with a small neighbourhood of the sphere or the hyperbolic space. Let the deformation of the space be only due to the change of the curvature of the space. Then we see the family of Killing vectors also change smoothly. Actually, in this case, we see that for the canonical coordinates on the sphere, the Euclidean space and the hyperbolic space, the only expression for the Killing vector fields remain the same in a small neighbourhood.

It is in principle also possible that as the space deforms as time goes by there may come up with new Killing vectors, and some of the old Killing vectors may disappear. It is obvious that these "non-persistent" Killing vectors will not play a role in what follows and our basis is a basis only for the persistent ones. Then we can write

$$f^i = \varphi_a(t) v_a^i(x,t).$$

Now let us turn on the three assumptions one by one, to investigate their effects on the problem.

If we only have $V \neq 0$, then our equations become

$$\begin{cases} f^i = \varphi_a(t) v_a^i(x) \\ \partial_t F = -V_{,i}\varphi_a v_a^i, \\ F_{,i} = g_{ij}\varphi_a' v_a^i, \\ 0 = \mathcal{L}_f g_{ij}. \end{cases}$$

Again, at any instant of time, the vector field $\varphi_a' v_a^i$ is a gradient Killing vector field. If the potential is time-dependent, the φ_a will no longer be an affine function of t, but with more complicated time-dependence. If on the other hand V is time-independent as well, $V = V(x)$, then once again φ_a must be affine functions of t. In this case we write $\varphi_a = a_a t + b_a$, and

$$\begin{cases} \partial_t F = -V_{,i}(a_a t + b_a) v_a^i, \\ F_{,i} = g_{ij} a_a v_a^i, \end{cases}$$

and consistency requires

$$-\partial_i[(a_a t + b_a)(V_{,j} v_a^j)] = 0.$$

If $a_a = 0$, then we are in the trivial case and no boost symmetry arises. In the non-trivial case, we already have v_a^j gradient Killing vectors, and by our above arguments in the canonical Cartesian coordinates $\partial_i v_a^j = 0$. Therefore we must require, in a canonical Cartesian coordinates,

$$V_{,ij} = 0,$$

i.e., the potential can be at most linear in the positions. This is the case for the linear gravitational potential on the surface of the earth, but not for the inverse gravitational potential for large bodies. However *this does not really exclude the possibility of boost invariance in these cases*, since under boosts, potentials that are originally time-independent will become time-dependent.

Now consider the case that the metric g_{ij} is time-dependent, without any potential terms. We have

$$F_{,k} = g_{ij}\partial_t(\varphi_a v_a^i) = g_{ij}\varphi_a' v_a^i + g_{ij}\varphi_a \partial_t v_a^i.$$

If the local expression of the Killing vector fields do not change (it is not true that this can always be upheld by a change of coordinates: in particular, we need to have $\mathcal{L}_{v_a}(g_{ij,k} f^k) = 0$, though such coordinates obviously exists for Robertson-Walker style scaling and the above "mutation" of homogeneous spaces mentioned above), that is to say, v_a^i does not depend on t, then we return to our previous case

$$F_{,k} = g_{ij}\varphi_a' v_a^i,$$

and

$$\begin{cases} f^i = \varphi_a(t) v_a^i(x) \\ \partial_t F = 0, \\ F_{,i} = g_{ij}\varphi_a' v_a^i, \\ 0 = \mathcal{L}_f g_{ij}. \end{cases}$$

then for surfaces where $t =$ constant, $\varphi_a' v_a^i$ must still be a gradient Killing vector field. Hence in this case we are back to the business of finding gradient Killing vector fields on a space. The t dependence in $\varphi_a(t)$ is in this case no longer affine: it must be chosen to cancel the t dependence in g_{ij} so as to ensure that the partial derivatives on F commutes.

Note that, if we are only interested in finding some Galilean boosts, we can set some of the φ_a to vanish by hand, which means the vector field $\varphi_a' v_a^i$ depends only on some of the vector fields. Then, as long as these vector fields remain form-invariant under change in time, the above analysis is still valid. For example, the following metric

$$ds^2 = \sum_i a_i(t) dx^i dx^i$$

where the different directions are scaled differently, reproduces all the translation Killing vectors if we change variables as time goes by.

Finally, if now we allow A_i to be non-zero, then as

$$F_{,i} = g_{ij}\partial_t f^j + \mathcal{L}_f A_i,$$

we are required to have gradient Killing vector fields only if $\mathcal{L}_f A_i = 0$, i.e., A_i is constant in the boost directions. If the system is boost-invariant in all directions, then $A_i = 0$.

IV. Weyl rigid flow

118 The structure of Weyl geometry as a generalised space. Weyl geometry is constructed by adding a scaling freedom to Riemannian geometry (†) and, unlike the more general conformal geometry, remains an affine geometry. On the base manifold, we set up an orthonormal frame ω^μ (the "normal" part now only makes limited sense: it is no longer possible to compare the length of two covectors not situated in the same cotangent space). As the local symmetry group is now larger than the rotational group, from now on we need to pay attention if an index is upstairs or downstairs. On the bundle, which is now $M \times SO(n) \times \mathbb{R}^+$, the coframe is formed by ω^μ, $\omega^\mu{}_\nu$ and $\omega^\lambda{}_{\bar\lambda} \equiv \tau$, with the structural equation

$$\begin{cases} d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu - \tau \wedge \omega^\mu, \\ d\omega^\mu{}_\nu = -\omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu + \frac{1}{2} R^\mu{}_{\nu\rho\lambda}\omega^\rho \wedge \omega^\lambda, \\ d\tau = \frac{1}{2} F_{\mu\nu}\omega^\mu \wedge \omega^\nu, \end{cases}$$

where $F_{\mu\nu}$ is the *scaling curvature*.

As we know, the *covariant derivative* for any affine theory, acting on tensor and form components, are defined by

$$dv^i = v^i{}_{;j}\omega^j - v^k\omega^i{}_k, \qquad dw_i = w_{i;j}\omega^j + w_k\omega^k{}_i$$

(†) Due to the history of the discovery of gauge theories, this is often compared with the $U(1)$ principal bundle, i.e., electromagnetism. It is important to note that there are a few differences here and there. In particular, the electromagnetic connection is separate from the spacetime connection.

so for our present case

$$dv^\mu = v^\mu{}_{;\nu}\omega^\nu - v^\lambda\omega^\mu{}_\lambda - v^\mu\tau, \qquad dw_\mu = w_{\mu;\nu}\omega^\nu + w_\lambda\omega^\lambda{}_\mu + w_\mu\tau,$$

the placement of indices dictates whether we get a plus or a minus term linear in τ, the scaling connection. In particular,

$$\begin{aligned} dF_{\mu\nu} &= F_{\mu\nu;\lambda}\omega^\lambda + F_{\lambda\nu}\omega^\lambda{}_\mu + F_{\mu\lambda}\omega^\lambda{}_\nu + 2F_{\mu\nu}\tau, \\ dR^\mu{}_{\nu\rho\lambda} &= R^\mu{}_{\nu\rho\lambda;\gamma}\omega^\gamma - R^\gamma{}_{\nu\rho\lambda}\omega^\mu{}_\gamma + R^\mu{}_{\gamma\rho\lambda}\omega^\gamma{}_\nu + R^\mu{}_{\nu\gamma\lambda}\omega^\gamma{}_\rho + R^\mu{}_{\nu\rho\gamma}\omega^\gamma{}_\lambda + 2R^\mu{}_{\nu\rho\lambda}\tau. \end{aligned}$$

Besides the "defining" symmetries,

$$F_{\mu\nu} = -F_{\nu\mu}, \qquad R^\mu{}_{\nu\rho\lambda} = -R^\nu{}_{\mu\rho\lambda} = -R^\mu{}_{\nu\lambda\rho},$$

we have the Bianchi identities:

$$\begin{cases} d^2\omega^\mu: & R^\mu{}_{[\nu\rho\lambda]} = -\delta^\mu{}_{[\rho}F_{\nu\lambda]}, \\ d^2\tau: & F_{[\mu\nu;\rho]} = 0, \\ d^2\omega^\mu{}_\nu: & R^\mu{}_{\nu[\rho\lambda;\gamma]} = 0. \end{cases}$$

Hence we have the table of invariants

Invariant	Normal terms
$F_{\mu\nu}$	$\mu > \nu$
$R^\mu{}_{\nu\rho\lambda}$	Riemann tensor symmetry
$F_{\mu\nu;\lambda}$	$\mu > \nu,\ \mu \geq \lambda$
$R^\mu{}_{\nu\rho\lambda;\gamma}$	Riemann tensor symmetry

so the degree of arbitrariness is

$$s_n = \frac{n(n-1)}{2} + (n-1) = \frac{(n+2)(n-1)}{2}.$$

We also know that the maximal number of symmetries of the space is equal to the dimension of the symmetry group of the homogeneous version of the space, which is in the present case $n + \dim(SO(n)) + 1$, corresponding to translation, rotation and scaling. For example, in Cartesian coordinates with the Euclidean metric, the "Killing" vector fields are

$$\frac{\partial}{\partial x^i}, \qquad x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}, \qquad \sum_i x^i\frac{\partial}{\partial x^i}.$$

119 The structural equations of Weyl rigid flow. Now we can construct the theory of structure preserving submersion in Weyl geometry. For simplicity we shall restrict to the codimension one case. As in the Riemannian case, let the horizontal forms on the total space to be divided into two classes, ω^0 and ω^i, and let the horizontal forms on the reduced space be π^i. On the product space we require

$$\omega^i = \pi^i.$$

The reduction of the principal bundle entails, as usual

$$\omega^0{}_i = -\omega^i{}_0 = K_i{}^0{}_0\omega^0 - M_{ij}{}^0\omega^j - B_{ij}{}^0\omega^j + V^0{}_{ijk}\pi^{jk},$$

where

$$M_{ij}{}^0 = -M_{ij}{}^0, \qquad B_{ij}{}^0 = B_{ji}{}^0, \qquad V^0{}_{ijk} = -V^0_{ikj}.$$

and the first structural equations of the total space are, after reduction

$$\begin{cases} d\omega^0 = -K_i{}^0{}_0\omega^0 \wedge \omega^i + M_{ij}{}^0\omega^j \wedge \omega^i - \tau \wedge \omega^0 - V^0{}_{ijk}\pi^{jk} \wedge \omega^i, \\ d\omega^i = -\omega^i{}_j \wedge \omega^j - M^i{}_{j0}\omega^j \wedge \omega^0 - B^i{}_{j0}\omega^j \wedge \omega^0 - \tau \wedge \omega^i + V_0{}^{ijk}\pi_{jk} \wedge \omega^0, \end{cases}$$

we explicitly indicate all indices, including 0, since now in general $M_{ij0} \neq M_{ij}{}^0$.

For the reduced space, the first structural equation is

$$d\pi^i = -\pi^i{}_j \wedge \pi^j - \varpi \wedge \pi^i.$$

Requiring $d\omega^i = d\pi^i$ now, we get

$$V_{0ijk} = 0$$

and

$$(\pi^i{}_j - \omega^i{}_j + M^i{}_{j0}\omega^0) + (B^i{}_{j0}\omega^0 + \delta^i{}_j\varpi - \delta^i{}_j\tau) = C^i{}_{jk}\omega^k,$$

where $C^i{}_{jk} = C^i{}_{kj}$. First let us antisymmetrise the indices i, j in this equation. This gives us

$$\omega^i{}_j = \pi^i{}_j + M^i{}_{j0}\omega^0$$

as usual. For the symmetric part, if i and j are distinct, we have

$$B^i{}_{j0}\omega^0 = C^i{}_{jk}\omega^k,$$

and since ω^0 and ω^i are independent, both sides vanish. In particular, this shows that the only components of $C^i{}_{jk}$ that may be non-zero are those that have all three indices the same. If i and j are the same, then

$$B^i{}_{\bar{i}0}\omega^0 + \varpi - \tau = C^i{}_{\bar{i}\bar{i}}\omega^{\bar{i}}.$$

This must hold for all choice of indices i, hence

$$C^i{}_{jk} = 0, \qquad B^i{}_{\bar{i}0} \equiv E_0, \qquad B^i{}_{j0} = \delta^i{}_j E_0$$

and

$$\tau = \varpi + E_0 \omega^0.$$

Using the quantities $M_{ij}{}^0$ and E_0, we can exchange the forms $\pi^i{}_j$ and ϖ for $\omega^i{}_j$ and τ. Henceforth we take ω^i, ω^0, $\pi^i{}_j$, τ to be the coframe (connection) on the total space, so that the coframe derivative (covariant derivative) in the i direction is independent of the fibre coordinates.

Now the complete structural equations for the coframe are

$$\begin{cases} d\omega^0 = -K_i{}^0{}_0 \omega^0 \wedge \omega^i - M_{ij}{}^0 \omega^i \wedge \omega^j - \varpi \wedge \omega^0, \\ d\omega^i = -\pi^i{}_j \wedge \omega^j - \varpi \wedge \omega^i, \\ d\pi^i{}_j = -\pi^i{}_k \wedge \pi^k{}_j + \frac{1}{2} S^i{}_{jkl} \omega^j \wedge \omega^l, \\ d\varpi = \frac{1}{2} G_{ij} \omega^i \wedge \omega^j. \end{cases}$$

The last three equations are equations on the reduced space.

120 Bianchi identities. It is now customary to derive the Bianchi identities for the submersion. First,

$$d^2\omega^0 = -(\tfrac{1}{2} G_{ij} - K_i{}^0{}_{0;j} - M_{ij}{}^0{}_{;0}) \omega^i \wedge \omega^j \wedge \omega^0 - (M_{ij}{}^0{}_{;k} - K_i{}^0{}_0 M_{jk}{}^0) \omega^i \wedge \omega^j \wedge \omega^k,$$

so

$$G_{ij} = -2K_{[i|}{}^0{}_{0|;j]} - 2M_{ij}{}^0{}_{;0}, \qquad M_{[ij|}{}^0{}_{;|k]} = K_{[i|}{}^0{}_0 M_{|jk]}{}^0. \tag{4.6}$$

Next,

$$d^2\varpi = \tfrac{1}{2} G_{ij;k} \omega^i \wedge \omega^j \wedge \omega^k + \tfrac{1}{2} G_{ij;0} \omega^i \wedge \omega^j \wedge \omega^0,$$

giving us

$$G_{[ij;k]} = 0, \qquad G_{ij;0} = 0.$$

In particular, this shows that G_{ij} is independent of the fibre coordinates, which should be expected.

Next,

$$d^2\omega^i = -\tfrac{1}{2} (S^i{}_{jkl} - \delta^i{}_l G_{jk}) \omega^j \wedge \omega^k \wedge \omega^l,$$

so

$$S^i{}_{[jkl]} = \delta^i{}_{[l} G_{jk]},$$

which is the extension of the usual first Bianchi identity. We can get a clearer picture of how these two quantities are related by forming the contraction of the Ricci tensor:

$$S_{[jl]} = \tfrac{p-2}{2} G_{jl} \tag{4.7}$$

where p is the dimension of the reduced space, or $n-1$ in our case. Hence the scaling curvature contributes to the antisymmetric part of the Ricci tensor. Finally,

$$d^2\pi^i{}_j = \tfrac{1}{2} S^i{}_{jkl;m} \omega^m \wedge \omega^k \wedge \omega^l + \tfrac{1}{2} S^i{}_{jkl;0} \omega^0 \wedge \omega^k \wedge \omega^l,$$

which gives

$$S^i{}_{j[kl;m]} = 0, \qquad S^i{}_{jkl;0} = 0,$$

which are the usual second Bianchi identity and the condition that $S^i{}_{jkl}$ is independent of the fibre coordinates.

Armed with these identities, the calculation of the Cartan characters present no problem

Invariant	Normal terms
E_0	all
M_{ij0}	$i > j$
K_{i00}	all
S_{ijkl}	$i > j,\ k > l,\ i \geq k,\ j \geq l$
G_{ij}	$i > j$
$E_{0;i}$	all
$E_{0;0}$	all
$M_{ij0;k}$	$i > j,\ i \geq k$
$K_{i00;0}$	all
$K_{i00;j}$	all
$S_{ijkl;m}$	$i > j,\ k > l,\ i \geq k,\ j \geq l,\ k \geq m$
$G_{ij;k}$	$i > j,\ i \geq k$
$M_{ij0;kl}$	$i > j,\ i \geq k,\ k \geq l$
$K_{i00;jk}$	$j \geq k$
$K_{i00;0j}$	all
$K_{i00;00}$	all
$E_{0;ij}$	$i \geq j$
$E_{0;0i}$	all
$E_{0;00}$	all

Now in addition to $K_{i00;00}$, $E_{0;00}$ also contributes to the Cartan character. We see that, in the generic case, the problem of structure preserving submersion in Weyl geometry has

$$s_n = n$$

degree of arbitrariness, one more than in the Riemannian case.

121 Invariants of the total space in terms of invariants of the flow. We also want to get expressions of the curvatures $F_{\mu\nu}$ and $R^{\mu}{}_{\nu\rho\lambda}$ in terms of the curvatures G_{ij}, $S^i{}_{jkl}$ and the invariants $K_i{}^0{}_0$, $M_{ij}{}^0$ and E_0. We just have to calculate.

From

$$d\tau = F_{i0}\omega^i \wedge \omega^0 + \tfrac{1}{2}F_{ij}\omega^i \wedge \omega^j = d(\varpi + E_0\omega^0),$$

we get

$$\begin{cases} F_{ij} = G_{ij} - 2E_0 M_{ij}{}^0, \\ F_{i0} = E_{0;i} + E_0 K_i{}^0{}_0. \end{cases} \tag{4.8}$$

From

$$d\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \tfrac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l + R^i{}_{jk0}\omega^k \wedge \omega^0 - \omega^i{}_0 \wedge \omega^0{}_j = d(\pi^i{}_j + M^i{}_{j0}\omega^0),$$

we get

$$\begin{cases} R^i{}_{jkl} = S^i{}_{jkl} - 2M^i{}_{j0}M_{kl}{}^0 + M^i{}_{l0}M_{jk}{}^0 - M^i{}_{k0}M_{jl}{}^0 \\ \qquad\quad + M_{jk}{}^0\delta^i{}_l E_0 - M_{jl}{}^0\delta^i{}_k E_0 + M^i{}_l\delta_{jk}E^0 - M^i{}_k\delta_{jl}E^0 \\ \qquad\quad + \delta^i{}_l\delta_{jk}E_0E_0 - \delta^i{}_k\delta_{jl}E_0E_0, \\ R^i{}_{jk0} = M^i{}_{j0;k} - M_{jk}{}^0K^i{}_{00} + M^i{}_{k0}K_j{}^0{}_0 + M^i{}_{j0}K_k{}^0{}_0 \\ \qquad\quad - \delta_{jk}E^0K^i{}_{00} + \delta^i{}_kE_0K_j{}^0{}_0. \end{cases}$$

From

$$\begin{aligned} d\omega^i{}_0 &= -\omega^i{}_j \wedge \omega^j{}_0 + \tfrac{1}{2}R^i{}_{0jk}\omega^j \wedge \omega^k + R^i{}_{0j0}\omega^j \wedge \omega^0 \\ &= d(-K^i{}_{00}\omega^0 + M^i{}_{j0}\omega^j + \delta^i{}_jE_0\omega^j) \end{aligned}$$

we get

$$\begin{cases} R^i{}_{0jk} = 2K^i{}_{00}M_{jk}{}^0 - M^i{}_{j0;k} + M^i{}_{k0;j} - \delta^i{}_jE_{0;k} + \delta^i{}_kE_{0;j}, \\ R^i{}_{0j0} = -M^i{}_{k0}M^k{}_{j0} - K^i{}_{00;j} - K^i{}_{00}K_j{}^0{}_0 - M^i{}_{j0;0} \\ \qquad\quad - \delta^i{}_jE_{0;0} - M^i{}_{k0}\delta^k{}_jE_0. \end{cases}$$

V. Shear-free flow as Weyl rigid flow

122 From shear-free flow to Weyl geometry. In dealing with fluids we often need to consider flows that are shear-free but may have non-zero expansion. The methods of Riemannian submersion are not directly applicable to this case, but can we conceptualise it as some other kind of structure-preserving submersion so that our general method still applies? For any flow, once we mark the flow direction as distinguished, the adapted coframe has decomposition

$$\omega_{0i} = K_i\omega_0 - M_{ij}\omega_j - B_{ij}\omega^j - E\omega^i,$$

where K_i, M_{ij}, B_{ij}, E has respective interpretation acceleration, vorticity, shear and expansion. So the condition for a flow to be shear-free is simply that $B_{ij} = 0$. Now working on the base, we can check that a shear-free flow preserves the horizontal part of the metric up to scale [(†)]:

$$\mathcal{L}_{\mathbf{I}_0}\left(\sum \omega_i \otimes \omega_i\right) = E\left(\sum \omega_i \otimes \omega_i\right).$$

For easier calculation, let us lift this unto the bundle and try to find out the lifting condition for a vector field

$$\mathbf{V} = \mathbf{I}_0 + V_{ij}\mathbf{I}_{ij}$$

to satisfy

$$\mathcal{L}_{\mathbf{V}}\omega_i \propto \omega_i.$$

We have

$$\begin{aligned}\mathcal{L}_{\mathbf{V}}\omega_i &= \mathbf{V}\,\lrcorner(-\omega_{ij}\wedge\omega_j - M_{ij}\omega_j\wedge\omega_0 - E\omega_i\wedge\omega_0)\\ &= E\omega_i - (V_{ij} + M_{ij})\wedge\omega_j,\end{aligned}$$

hence the uplifting is

$$\mathbf{V} = \mathbf{I}_0 - M_{ij}\mathbf{I}_{ij}.$$

The condition in the bundle

$$\mathcal{L}_{\mathbf{V}}\omega_i = E\omega_i$$

is still not very convenient to work with. Let us try to find some quantity that vanishes under the Lie derivative in the bundle. It is reasonable to try the scaling $\theta_i = e^{-\Lambda}\omega_i$. We have

$$\begin{aligned}\mathcal{L}_{\mathbf{V}}\theta_i &= \mathcal{L}_{\mathbf{V}}(e^{-\Lambda})\omega_i + e^{-\Lambda}E\omega_i\\ &= (E - \mathbf{I}_0(\Lambda))\theta_i,\end{aligned}$$

If we write $\mathbf{I}_0 = \partial/\partial t$, it suffices to integrate the equation

$$\frac{d\Lambda}{dt} = E$$

[(†)] We can take the vector field to be $\lambda\mathbf{I}_0$ for any function $\lambda > 0$ instead of simply $\mathbf{I}_0$ if we want: it only contributes an overall factor in all the following and does not change any of our conclusions. If the terms to be differentiated by the Lie derivation contains the form ω_0, the factor λ will have significance, as we have seen before (§**97**).

along the flow. This is always solvable on each flowline, the solution depending on one constant. On the space itself, a solution for such an equation hence depends on a function of $n-1$ parameters, in other words, on any hypersurface transverse to the flowlines we can choose these constants of integration arbitrarily (subject to the appropriate smoothness conditions, of course).

To obtain the complete set of conditions, we need, as usual, to differentiate our condition. We have

$$\begin{aligned} 0 = \mathcal{L}_{\mathbf{V}} d\theta_i &= \mathcal{L}_{\mathbf{V}}(-\omega_{ij} \wedge \theta_j - e^{\Lambda} M_{ik}\theta_k \wedge \theta_0 - e^{\Lambda} E\theta_i \wedge \theta_0 - d\Lambda \wedge \theta_i) \\ &= -\mathcal{L}_{\mathbf{V}}(-\omega_{ij} + e^{\Lambda} M_{ik}\theta_0) \wedge \theta_j - \mathcal{L}_{\mathbf{V}}(d\Lambda - e^{\Lambda} E\theta_0) \wedge \theta_i. \end{aligned}$$

Hence the forms [(†)]

$$\pi_{ij} \equiv \omega_{ij} - e^{\Lambda} M_{ik}\theta_0, \qquad \varpi \equiv d\Lambda - e^{\Lambda} E\theta_0$$

are independent of the fibre coordinates. It is also easy to see that

$$\theta_i, \qquad \pi_{ij}, \qquad \varpi$$

provide a Weyl connection on the reduced space. *Hence a shear-free flow can be interpreted as a structure-preserving submersion: a submersion that preserves the Weyl structure of the subspace.*

Remark. The most general conformal geometry is not what we want: it contains, in addition to rotations and scaling, other local symmetries: the special conformal transformations. If we include special conformal transformations, the geometry is no longer reductive, and hence there is no way we can define any covariant derivatives along directions on the base manifold—every derivative necessarily leads us upstairs, into the bundle.

123 Riemannian geometry disguised as Weyl geometry. It is rather awkward to talk about the reduction of a Riemannian geometry to a Weyl geometry since Weyl geometry is more general than Riemannian geometry. We can remedy this by putting the Riemannian geometry into the form of a Weyl geometry, and talk instead of the reduction of a Weyl geometry to another Weyl geometry. We take a coframe in the Riemannian geometry θ^μ, $\theta^\mu{}_\nu$ and its curvature $\mathcal{R}^\mu{}_{\nu\rho\lambda}$, and do the scaling

$$\omega^\mu = e^{-\Lambda}\theta^\mu$$

[(†)] We really need to assume the Lie derivatives are linear in θ_i and apply some manipulation to get the following. We omit the manipulations, which should be familiar now.

for a function Λ defined on the base. The structural equation then becomes

$$\begin{cases} d\omega^\mu = -\theta^\mu{}_\nu - d\Lambda \wedge \omega^\mu, \\ d\theta^\mu{}_\nu = -\theta^\mu{}_\lambda \wedge \theta^\lambda{}_\nu + \frac{1}{2} e^{2\Lambda} \mathcal{R}^\mu{}_{\nu\rho\lambda} \omega^\rho \wedge \omega^\lambda, \end{cases}$$

we see that

$$\omega^\mu{}_\nu = \theta^\mu{}_\nu, \qquad \tau = d\Lambda, \qquad R^\mu{}_{\nu\rho\lambda} = e^{2\Lambda}\mathcal{R}^\mu{}_{\nu\rho\lambda}.$$

In particular, this implies

$$d\tau = d^2\Lambda = 0, \qquad F_{\mu\nu} = 0.$$

This condition is also sufficient for the local reducibility of a Weyl geometry to a Riemannian geometry: it suffices to integrate these equations back to get the value of Λ. Note that for such a geometry, the invariants has *exactly the same* structure as a Riemannian geometry (they have the same value up to scaling), and hence the degree of arbitrariness is exactly the same. In Weyl geometry we have removed one degree of arbitrariness from the metric by scaling, but for the existence of the covariant derivative, we have added a scale connection. In the reducible case, this added scale connection, as we have just seen, is the differential of a function, hence we need to add back one degree of arbitrariness. In other words, a Weyl geometry reducible to Riemannian geometry amounts to taking away one degree of arbitrariness from the metric and add it to somewhere else, and the net change for the degree of arbitrariness is zero (but such a manipulation is not completely in vain: the symmetry group is now one dimension larger). This should come as a relief for us: we want to study Riemannian geometry, and if the degree of arbitrariness is not the same, we have introduced or removed degree of arbitrariness.

Then there is the question of what happens for $n = 2$, surfaces. It is a celebrated result that all two dimensional surfaces are conformally equivalent. The answer is that Weyl geometry for $n = 2$ is not the same as the problem of conformal equivalence of surfaces. To see this, let us apply the equivalence method from scratch. Let θ^1, θ^2 be an orthogonal frame on the surface. For two surfaces to be conformally equivalent, we require

$$\begin{pmatrix} \bar\theta^1 \\ \bar\theta^2 \end{pmatrix} = L \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}.$$

Let the lifted frame be

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = L \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix},$$

we can derive the structural equation

$$\begin{cases} d\omega^1 = d(\log L) \wedge \omega^1 + dt \wedge \omega^2 + a\omega^1 \wedge \omega^2, \\ d\omega^2 = -dt \wedge \omega^1 + d(\log L) \wedge \omega^2 + b\omega^1 \wedge \omega^2, \end{cases}$$

where a and b are torsion. At the linear level, the solution

$$d(\log L) = l_1\omega^1 + l_2\omega^2, \qquad dt = m^1\omega^1 + m^2\omega^2$$

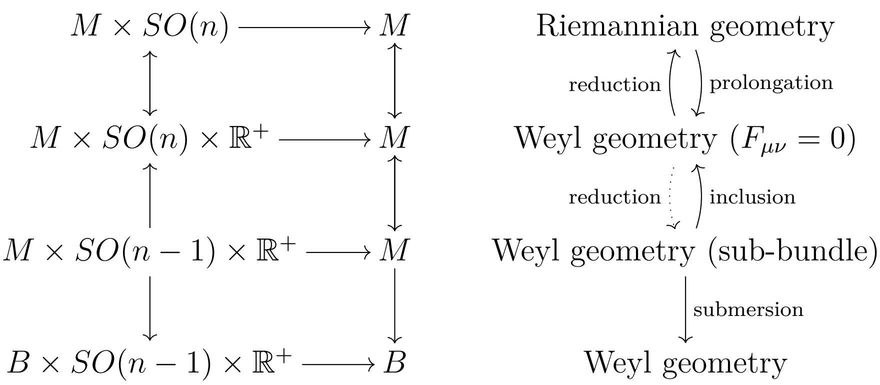
contains four variables l_1, l_2, m_1, m_2, two of which must be used to set the torsion to zero. Hence the number of free functions is 2. On the other hand, the Cartan characters have

$$s_1 = 2, \qquad s_2 = 0,$$

and since $1 \cdot 2 + 2 \cdot 0 = 2$, *this system, which is not completely integrable, is involutive*, there are no hidden conditions for the existence of integral varieties! The solution depends on 2 functions of 1 variables, and the system has an infinite dimensional symmetry group. In this case we are not justified to prolong the system. If we prolong as we did for the higher dimensional case, we get Weyl geometry for $n = 2$, but it describes a different geometry.

What we need to take away from this consideration is that in the so-called Weyl form, it is easy to deduce that the invariants $R_{\mu\nu\rho\lambda}$ and, if we do a reduction under a flow, the invariants M_{ij}, K_i and E, are just the scaled counterparts of the quantities in the Weyl frame. *In particular*, going from Riemannian geometry to the Weyl framework does not alter the vertical forms $\omega_{\mu\nu}$ in any way.

124 The structure of shear-free flow. We can draw the following diagram for what we have done so far:



the dotted line just means that we have no use of the map indicated. This diagram is curious: on the top row, we have $M \times SO(n)$, but at the bottom row we have

$B\times SO(n-1)\times\mathbb{R}^+$: by some magic, we have conjured up the $\mathbb{R}^+$ degree of arbitrariness from air. The equation (4.8) tells us that, in this case, since $F_{\mu\nu}=0$,

$$\begin{cases} G_{ij} = 2E_0 M_{ij}{}^0, \\ E_{0;i} = -E_0 K_i{}^0{}_0. \end{cases} \tag{4.9}$$

and if $G_{ij}\neq 0$, we really have magic here: we have derived inhomogeneity in the $\mathbb{R}^+$ part from a geometry where no $\mathbb{R}^+$ part exists, recalling that the group $SO(n-1)\times\mathbb{R}^+$ is *not* a subgroup of $SO(n)$. Observe that to realise this, the restrictions are huge: not only do we have $G_{ij;0}=0$, so the degree of arbitrariness of $G_{ij;0}$ lies only in the reduced space, but since $G_{[ij;k]}=0$, locally the degree of arbitrariness of G_{ij} is only that of a vector on the reduced space.

Let us note several things. First is that, from the Bianchi identity $G_{ij;0}=0$, we get

$$E_{0;0}M_{ij}{}^0 + E_0 M_{ij}{}^0{}_{0;0} = 0,$$

so if $E_0\neq 0$, we know how $M_{ij}{}^0$ scales on the fibre. From the same equation, we see that $M_{ij}{}^0$ is preserved on the fibre up to scale, so we can now write

$$M_{ij}{}^0{}_{0;0} = -\lambda_0 M_{ij}{}^0{}_0,$$

then

$$E_{0;0}M_{ij}{}^0 - E_0\lambda_0 M_{ij}{}^0{}_0 = M_{ij}{}^0(E_{0;0}-E_0\lambda_0) = 0,$$

so if in addition $M_{ij}{}^0\neq 0$, the way E_0 scales on the fibre is related to the way $M_{ij}{}^0$ scales on the fibre, given by

$$E_{0;0} = E_0\lambda_0.$$

We also have another formula for G_{ij}: (4.6), which can now be written

$$K_{[i|}{}^0{}_{0;|j]} = (\lambda_0 - E_0)M_{ij}{}^0. \tag{4.10}$$

Another nice property is that, since $G_{ij;0}=0$ and M_{ij0} is proportional to G_{ij}, if neither M_{ij0} nor E_0 vanish the principal bundle can be reduced further: basically M_{ij0} is invariant up to scale on each fibre, so it is a well-defined quantity on the reduced space up to scale. We can then use the residual $SO(n-1)$ symmetry on the reduced space to set various components of M_{ij0} to zero. For example, if the reduced space is only three dimensional, then we can always effect a reduction of the principal bundle such that $M_{130}=M_{230}=0$. Note that this is related to the (generalised) Herglotz–Noether theorem: whenever the Herglotz–Noether theorem applies, the Born-rigid flow is isometric, and for isometric flow such a reduction of bundle can be made since $M_{ij0;0}=0$.

However, as the diagram that we have drawn has shown, shear-rigid flow is rather complicated. As a consequence, calculating its degree of arbitrariness is hard. To see the problem involved, look at the relation (4.10). This relation can be taken as the definition of λ_0, but this relation holds for all indices i and j, so some of the invariants $K_{i00;j}$ and M_{ij0} are no longer independent no matter how we choose our seeds. Note also that there is no relation whatsoever involving $K_{i00;0}$ and no relation beyond those that are already present for the Born-rigid case for $S_{ijkl;m}$, consequently we want 0 to be the greatest index, but then for a system of involutive seeds we want all of $K_{i00;j}$ to remain independent. Suppose we take M_{ij0} to be dependent, and write

$$M_{ij0} = \frac{K_{[i|00;|j]}}{\lambda_0 - E_0},$$

which obviously will require that $\lambda_0 \neq E_0$. Now M_{ij0} is no longer independent, the Bianchi identity

$$M_{[ij|0;|k]} = K_{[i|00} M_{|jk]}{}^0$$

must be "propagated" to the other variables that are still independent so far. But this yields very complicated relations involving a lot of fractions and singular cases.

125 Existence of shear-free flow. Even though the degree of arbitrariness of shear-free flow is at present unclear, note that there certainly exists families of shear-free flow, and furthermore, we can construct all of them using the results we already have. To see this, assume that we have a shear-free flow on a *Riemannian* manifold and let $\mathbf{v}^0$ be the unit vector field along the flow. For the co-frame, we can write

$$\begin{cases} d\omega_0 = -\omega_{0i} \wedge \omega_i, \\ d\omega_i = -\omega_{ij} \wedge \omega_j - \omega_{i0} \wedge \omega_0. \end{cases}$$

Note that we are using the original connection on the manifold. For any section, we have

$$\omega_{0i} = -\omega_{i0} = K_i \omega_0 - M_{ij}\omega_j - H_{ij}\omega_j - E\omega_i,$$

where M_{ij} is antisymmetric and H_{ij} is symmetric and trace-free. The quantities M_{ij}, H_{ij} and E have interpretations of vorticity, shear and expansion *regardless of whether a structure-preserving submersion can be defined or not.* Now let us consider a conformal scaling applied on the Riemannian manifold. We have

$$\omega_0 \mapsto \pi_0 = e^\lambda \omega_0, \qquad \omega_i \mapsto \pi_i = e^\lambda \omega_i.$$

This will induce essential changes to the Levi–Civita connection. Let us investigate the changes to ω_{0i}. We have

$$d\pi_0 = -(\omega_{0i} + \lambda_{;i}\omega_0) \wedge \pi_i,$$

so

$$\pi_{0i} = \omega_{0i} - \lambda_{;i}\omega_0 + S_{ij}\omega_j$$

where $S_{ij} = S_{ji}$ and the derivatives are taken with respect to the *old* frame. Now consider

$$d\pi_i = -\omega_{ij} \wedge \pi_j - \omega_{i0} \wedge \pi_0 + d\lambda \wedge \pi_i,$$

and requiring $\omega_{0i} = -\omega_{i0}$, we obtain

$$\pi_{ij} = \omega_{ij} - S_{ij}\omega_0 - \delta_{ij}\lambda_{;0}\omega_0 - \lambda_{;j}\omega_i + T_{ijk}\omega_k,$$

where $T_{ijk} = T_{kji}$. Requiring $\pi_{ij} = -\pi_{ji}$, we see that

$$T_{ijk} = \lambda_{;j}\delta_{ik}, \qquad S_{ij} = \lambda_{;0}\delta_{ij}.$$

In summary, we have

$$\pi_{0i} = \omega_{0i} - \lambda_{;i}\omega_0 + \lambda_{;0}\delta_{ij},$$

in other words,

$$K_i \mapsto K_i - \lambda_{;i}, \qquad E \mapsto E - \lambda_{;0},$$

with M_{ij} and H_{ij} unchanged. Now if $H_{ij} = 0$, by solving the equation

$$\lambda_{;0} = E$$

for the variable λ, we can locally scale any shear-free flow to a Born rigid flow. Since the scaling goes both ways, we have proved that

Any local shear-free flow can be obtained by conformally scaling a Born-rigid flow.

As one Born-rigid flow can be scaled to an infinitely many number of shear-free flows, this unfortunately tells us nothing about the degree of arbitrariness of shear-free flows.

One thing is certain, though. We know that the degree of arbitrariness of a Weyl rigid flow is n for n dimensions, and the degree of arbitrariness of a Riemannian space is $n(n-1)/2$ for n dimensions. Since $n(n-1)/2 > n$ for $n \geq 4$, this shows that

For dimensions $n \geq 4$, not all Riemannian spaces admit shear-free flows.

126 Self-gravitating perfect fluid under shear-free flow. The Ellis conjecture. Now we return to the study of shear-free flow through Weyl rigid flow. The condition that the spacetime is formed by a self-gravitating perfect fluid amounts to constraints on the Einstein tensors of the total space. In our framework, these constraints amounts to

$$\begin{cases} \rho = R^0{}_0 - \frac{1}{2}R, \\ P = R^i{}_{\bar{i}} - \frac{1}{2}R, \\ 0 = R_{ij}, & (i \neq j), \\ 0 = R_{0i}. \end{cases}$$

Note that ρ and P *do not scale* under Weyl scaling, due to the above formulae: for example, $R^0{}_0 - \frac{1}{2}R = \rho$ has no uncontracted 0 index. Unless we introduce an equation of state by hand, the first equation is useless: it can be thought of the equation of state itself. The second introduces $p - 1 = n - 2$ constraints, the third $p(p-1)/2$ and the fourth p (it can be checked that $R_{ij} = R_{ji}$ and $R_{0i} = R_{i0}$ are identities under our assumption $F_{\mu\nu} = 0$).

Using the invariants we can rewrite these equations

$$(4.11)\quad \begin{cases} \rho = -\frac{1}{2}S + 2M^i{}_{j0}M_i{}^{j0} - \frac{1}{2}K^{i0}{}_{0;i} - \frac{1}{2}K^{i0}{}_0 K_{i0}{}^0 - \frac{p}{2}E^0\lambda_0 + \frac{p^2-p}{2}E^0E_0, \\ P = S^{\bar{i}}{}_i - \frac{1}{2}S - 2M^{\bar{i}j0}M_{ij0} + M_{jk0}M^{jk0} - K^{\bar{i}00}K_{i00} \\ \qquad + \frac{p-2}{2}E_0\lambda^0 + \frac{(p-1)(p-2)}{2}E^0E_0 - K^{\bar{i}0}{}_{0;i} + \frac{1}{2}K^{j0}{}_{0;j} + \frac{1}{2}K^{j00}K_{j00}, \\ 0 = \frac{1}{2}(S^i{}_j + S_j{}^i) - M^{ki0}M_{kj0} - K^{i00}K_{j00} - \frac{1}{2}(K^{i0}{}_{0;j} + K_{j0}{}^{0;i}), \qquad (i \neq j), \\ 0 = M^i{}_{j0;i} + 2M_{ij}{}^0K^i{}_{00} + pE_0K_j{}^0{}_0. \end{cases}$$

This is the subject of the celebrated Ellis conjecture (for a review and partial results, see [35]): namely these constraints together with any barotropic equation of state which satisfies $\rho + P \neq 0$ require either $E_0 = 0$ or $M_{ij}{}^0 = 0$ to hold. Of course, since $G_{ij} = M_{ij0}E^0$, this conclusion is equivalent to the statement that the reduced space also has a Riemannian connection.

There are many partial results in which additional assumptions are added, of which we will mention one here: if in four dimensions the acceleration of the fluid is zero ($K_{i00} = 0$), then the theorem holds. We shall see that, the proof of this partial result achieves a remarkable simplicity in our approach to the problem, and at the same time we get its extension to higher dimensions. In particular, the proof does not involve writing down any differential equation and is due mainly to the symmetry restrictions of the problems.

Before carrying out with our proof, let us note a trivial result that can be read off immediately from our approach, namely

For an irrotational fluid ($M_{ij0} = 0$) with vanishing energy flux and non-vanishing pressure (†)*, either the expansion or the acceleration has to vanish.*

(†)Not necessarily a perfect fluid.

Indeed, the vanishing energy flux assumption is

$$0 = R_{0j} = M^{i}{}_{j0;i} + 2M_{ij}{}^{0}K^{i}{}_{00} + pE_0K_j{}^0{}_0,$$

and the irrotational assumption reduces this to

$$0 = pE_0K_j{}^0{}_0,$$

so either $E_0 = 0$ or $K_j{}^0{}_0 = 0$. Q.E.D.

127 Shear-free geodesic perfect fluid. If $K_{i00} = 0$ then the fluid is subject to no acceleration, i.e., the flow is geodesic. Now (4.10) reads

$$(\lambda_0 - E_0)M_{ij}{}^0 = 0.$$

From now on we shall assume that neither M_{ij0} nor E_0 vanishes and try to derive a contradiction. The above equation immediately gives

$$\lambda_0 = E_0,$$

and we have

$$E_{0;0} = E_0E_0, \qquad M_{ij0;0} = -E_0M_{ij0}.$$

Now take the second equation of (4.11):

$$P = S^{\bar{i}}{}_i - \tfrac{1}{2}S - 2M^{\bar{i}j0}M_{ij0} + M_{jk0}M^{jk0} + \tfrac{p(p-2)}{2}E^0E_0,$$

and derive it in the flow direction, recalling that $S_{ijkl;0} = 0$:

$$P_{;0} = 4E_0M^{\bar{i}j0}M_{ij0} - 2E_0M_{jk0}M^{jk0} + p(p-2)E_0E_0E^0.$$

So far our consideration has been in the vertical directions. Now change our point of view and focus on the horizontal directions, we see that

$$\sum_j M^{\bar{i}j0}M_{ij0} = \text{quantity transforming trivially under } SO(n-1),$$

and the quantity is the same for all value of the index i. A simple manipulation shows that the absolute value of each component

$$|M_{ij0}|$$

has the same value, and this quantity is invariant under $SO(n-1)$. As M_{ij0} is anti-symmetric in i and j, it has to vanish for $n \geq 4$, the cases we consider, contradicting our assumption. Q.E.D.

Observe that our proof is valid for all dimensions $n \geq 4$, and we did not require any equation of state: in particular, we did not require $P + \rho \neq 0$.

It seems likely that for the other cases the equation of state and the dimension restriction are necessary. Note that the equation of state $P = P(\rho)$ gives $P_{;0} = \rho_{;0} P'$, and both sides can be calculated independently by regarding $P' = dP/d\rho$ as a new scalar variable, and these equations do not involve S_{ijkl}, so the equation of state is actually quite a lot of new constraints. We shall not pursue the calculations, since we do not yet have new things to add to the existing results in this case.

128 Comments on the Ellis conjecture. Let us first observe some general facts about the Herglotz–Noether theorem and its generalisations. The setting for the theorem is a geometrical system under heavy constraints, and the statement of the theorem is that any solution of the geometrical system must possess rather strong symmetry properties. We can understand our proof for them roughly as follows: as these systems are so overdetermined such that their degree of arbitrariness is zero, all solutions, if they exist at all, must have rather high symmetry, as we know that the degree of arbitrariness is inversely linked to symmetry.

Now for the Ellis conjecture we have a completely analogous situation: the system of a Weyl submersion in a Riemannian space is already a very heavily constrained system, and the additional assumptions of perfect fluid and equations of state places even more stringent restrictions on any solutions that may exist. The conclusion, that any solutions must be either rotation-free or expansion-free, can also be understood as saying that the solutions cannot have the most general degree of arbitrariness that we expect. In this sense, the Ellis conjecture, similar to the Herglotz–Noether theorem, is also a “well-anticipated” result. But as the geometrical system for the Ellis conjecture is considerably more complicated than that of the Herglotz–Noether theorem, we expect more work to be involved.

Let us now analyse the partial results for the conjecture a bit. These can be divided into three classes. The first class concerns the specialisation of the equation of state: for example, the case for incoherent radiation and dust is proven. The second class concerns the specialisation of space-time: for example, the case for Petrov type N is proven, and for Petrov type III there are some partial results. The third class concerns the specialisation of the flow itself, of which we have seen an example, namely the case of geodesic flows.

For the framework that we have set up, the equation of state $P = P(\rho)$ can be thought as introducing an function P, and having all covariant derivatives of P related to those of ρ in the exterior differential system. The constraint $P + \rho \neq 0$ is simply a non-singular condition, and our approach only studies non-singular solutions of the exterior differential systems.

With this set up, we see immediately that a necessary requirement for the validity of the Ellis conjecture is that, in four dimensions, the exterior differential system (4.11) has degree of arbitrariness 0. As we have mentioned in §**124**, the algebraic relations for the shear-free flows is very complicated. However, since the calculation of Cartan characters is algorithmic, it is certainly feasible that the Cartan characters, at least for given dimensions, be calculated using computer algebra packages. Thus we can in principle carry out a preliminary test for the Ellis conjecture: this would be restricted to the space of Petrov type I. As a general rule, and as we have seen an example from our investigation of Herglotz–Noether theorem, the solutions of a system with degree of arbitrariness 0, when they exist at all, are highly symmetric: this is due to the fact that the degree of arbitrariness is also related to the dimension of the symmetry group. Thus, if we have ascertained that the degree of arbitrariness 0, we can proceed to find some of the symmetrical properties of any solutions, and if such properties are untenable in the general case, then the Ellis conjecture would be proved for the general case. Of course, this would still not completely prove the Ellis conjecture, since there may be quite a number of singular cases to consider. In principle, the singular cases can be studied one by one using similar methods, but from experience we know that the calculations for singular cases are often much more difficult still.

The formulation and execution of the above plan using a computer algebra package would constitute a natural continuation of the present work.

129 The conformal Herglotz–Noether theorem again. The Ellis conjecture for spacetime of Petrov type O. Suppose that we have a rigid flow on a conformally flat spacetime. By scaling and absorption as we have done in §**125**, this is equivalent to a Weyl flow on the *totally flat version of the Weyl spacetime*. Then we can use the equations relating the invariants of the total geometry with that of the subgeometry to prove the theorem of §**111**, the complication for calculation now is that there is the torsion absorption procedure involved (and we need to derive what it means for a vector field to be a Killing vector field in the Riemannian case, interpreted in the Weyl framework). The main difficulty of the proof in §**111**, on the other hand, is that the Weyl tensor is complicated to calculate. We shall not give more details of the proof using Weyl flow, as it does not really represent a simplification.

However, there are some results that such a reasoning can give us for free. Recall (§**125**) that scaling does not change the vorticity and shear of a flow, and all conformal rigid flow in Riemannian spacetime can be obtained by scaling a Born-rigid flow. As any conformal scaling does not alter the Weyl tensor, any conformal rigid flow in a *flat* spacetime, or indeed in any *conformally flat* spacetime, can be obtained by scaling a Born-rigid flow in a certain conformally flat spacetime. As the property of being a conformal Killing vector is also unchanged by conformal scaling, we have the following theorem

In a conformally flat spacetime of dimension ≥ 4, a rotational conformal rigid flow (a shear-free flow) must be a conformal Killing flow.

With relation to the Ellis conjecture again, Coley [14] has shown that for the case in which the flow is along a conformal Killing vector, the Ellis conjecture holds. Thus the Ellis conjecture also holds for all conformally flat spacetime: this is the Ellis conjecture for spacetime of Petrov type O.

130 Generic relativistic flow. From our study of Born rigid flow and shear-free flow it would seem that we have found a pattern, and it is tempting to generalise it further: formulating arbitrary relativistic flow as a submersion preserving an affine connection, together with a condition on the total affine space requiring that it is derivable from a Riemannian space.

However, this is hardly worth the effort, due to the following reason: note that in our previous examples of Born-rigid and shear-free cases, the condition (3.22) implies (3.23). On the total space, this implies that the foliation due to the submersion itself is sufficient to split the tangent space into the direct sum of two parts, however, in the general case, only the subspace of each leaf is completely determined. Thus, at least at the linear level, for the general case *the foliation on the total space itself is insufficient to determine completely the structure-preserving submersion*: we need some additional data.

As the total space is actually derived from a Riemannian space, such a splitting is available to us, but this actually only complicates the matter, as we cannot simply require that the splitting which makes an affine connection available on reduced space is just the splitting due to the Riemannian metric: there is no reason such a splitting will give rise to a structure-preserving submersion, and there is no reason that if this splitting does not give rise to such a submersion, no other splitting will. Furthermore, such a requirement would be physically unjustified. Thus we need to introduce quite a lot of auxiliary variables to parametrise the relationship between the two splittings. Since the whole point of the framework of structure-preserving submersions is reduction of variables, this really defeats the purpose (and the auxiliary functions introduced will in general not have any nice properties).

VI. The geometry of dimensional reductions

131 Introduction. In this final section we use our methods and some of the results of rigid flow and of general structure-preserving submersions we have obtained to investigate the geometry of various schemes of *dimensional reductions*. Our aim in this chapter is not to derive new results, but rather to point out ambiguities and pitfalls in the theory and to make the geometrical nature of the various schemes more explicit.

First, let us briefly discuss why anyone bothers with dimensional reduction at all. Broadly speaking there are two kinds of reasons. The first kind is that we do not actually believe in any physical theories that is based on higher dimensional dynamics, but in trying to build theories one attempts to build a higher dimensional model of a lower dimensional theory, which may achieve remarkable simplifications in its descriptions. In the general sense, the process of prolongation, the construction of the principal bundle, etc., are this kind of "higher dimensional theories", but the word "dimensional reduction" is usually applied specifically to cases where the higher dimensional theory is a particular *metric* theory. These days the number of people who still studies metric dimensional reductions for this reason is limited, since we know that this is not the best way of going to higher dimensions in exchange for ease of calculations, etc., and the introduction of higher dimensional metric, instead of making things more elegant, usually complicates the theory.

The second kind of reason is that we have a physical theory which we like very much, but it is of the wrong dimension. We thus need to invent ways to explain why we do not observe these extra dimensions under normal circumstances. In view of the popularity of string theories and M-theories in recent decades, the number of people taking this view is considerable. Note that submitting to this view also subjects the theory into much more stringent requirement: for example, in cases when a higher dimensional theory has a certain class of solutions that make no sense, if one subscribes to the first view one can simply ignore them by saying that the model does not apply in such circumstances: in other words by adding more assumptions. However if one subscribes to the second view it is much more difficult to argue away such offending solutions: giving suitable initial data, such solutions will occur physically, and one needs to explain why such initial data do not arise.

We have already studied the problems of structure-preserving submersions by applying the method of moving frames and our analysis of the invariants using involutive seeds. The framework of structure-preserving submersions is ideal for the study of dimensional reductions, since in going from higher dimensions to lower dimensions, we would like ways to give the lower dimensional effective theory the usual structure we are accustomed to in four dimensions, and structure-preserving submersion is an efficient way to achieve this aim in a rather clean way. The other alternative is the brane-world scenario, which has the disadvantage that there is usually no way to make the lower dimensional theory a self-contained one: on a surface in a higher dimensional total space, it is very hard to make sure that there will not be things coming incessantly from the extra dimensions, which for creatures constrained to live on the surface seem to come out of nowhere.

We will study two conventional dimensional reduction schemes. The framework of structure-preserving submersions allow us to devise more general schemes than these, but we will leave this work for the future and only point only suitable extensions in a few places.

132 The old Kaluza–Klein reduction and its "inconsistency". We start with the Kaluza–Klein ansatz, which is a higher dimensional metric in a particular form:

$$ds^2 = (d\phi - \tilde{A}_i dx^i) \cdot (d\phi - \tilde{A}_j dx^j) + g_{ij} dx^i dx^j,$$

c.f. the normal form of Born-rigid flow. After dimensional reduction, g_{ij} is interpreted as the metric on the reduced space. To make contact with our theory, we first write the lower dimensional metric in an orthonormal coframe, hence

$$ds^2 = (d\phi - A_i \omega^i) \cdot (d\phi - A_i \omega^i) + \sum_i \omega^i \otimes \omega^i,$$

where the $\tilde{A}_i$ has undergone a linear transformation that transforms dx^i to θ^i. If we now define

$$\omega^0 = d\phi - A_i \omega^i = d\phi - \tilde{A}_i dx^i,$$

the horizontal coframe ω^0, ω^i is just the coframe that defines a Born-rigid flow which we discussed in §**105**. Differentiating ω^0, we obtain

$$d\omega^0 = -d\tilde{A}_i \wedge dx^i = \tilde{A}_{i;j} dx^i \wedge dx^j - \tilde{A}_{i;0} d\phi \wedge dx^i \equiv A_{i;j} \omega^i \wedge \omega^j - A_{i;0} \omega^0 \wedge \omega^i.$$

On the other hand, the structural equation for a Born-rigid flow reads

$$d\omega^0 = -K_i \omega^0 \wedge \omega^i - M_{ij} \omega^i \wedge \omega^j,$$

so we have the identification

$$A_{i;0} = K_i, \qquad A_{[i;j]} = -M_{ij}.$$

In Kaluza–Klein theory, we identify $A_{[i;j]}$ with the electromagnetic field tensor, and hence M_{ij}, with started its life as part of the connection after reduction of the bundle,

$$\omega_{0i} = K_i \omega_0 - M_{ij} \omega_j,$$

is now given the dual role of a curvature arising from an Ehresmann connection on the lower dimensional space,

$$M_{ij} \omega^i \wedge \omega^j.$$

This dual interpretation of M_{ij}, as both the connection and the curvature, is the gist of Kaluza–Klein reduction.

So far the only restriction on the Born rigid flow is that M_{ij} is derivable from a potential A_i, which locally requires $M_{[ij;k]} = 0$. Kaluza–Klein reductions as usually applied, however, stipulates additional assumptions which make the theory physically reasonable: that $\tilde{A}_i$ is independent of the coordinate θ. This implies that $A_{i;0} = 0$, so $K_i = 0$ for the Born-rigid flow. *The old Kaluza–Klein theory, when interpreted as a Born rigid flow, has no acceleration exerted on the flow.* Note also that, because of the algebraic identity arising from the first Bianchi identity of the structural equations $M_{[ij;k]} = 3M_{[ij}K_{k]}$, the local existence of the potential is now automatic.

This old Kaluza–Klein theory is usually deemed an inconsistent theory, without defining precisely what "inconsistency" really means here. The usual reasoning goes like this: say we want to stipulate the Einstein equation of the total space as the equation of motion for the system. For the vacuum equation, from our formula of the Ricci tensors, we have

$$R_{00} = -K_{i;i} - K_iK_i + M_{ij}M_{ij} = 0.$$

Now $K_i = 0$, so this requires $\sum_{i,j} M_{ij}M_{ij} = 0$, which is deemed unphysical.

Despite this "inconsistency", we can see that the old Kaluza–Klein theory really has a lot of merit by giving the correct degree of arbitrariness of gravity coupled to pure electromagnetism. In §**88**, we have given a system of constraints which gives a well-defined Cauchy problem for Riemannian submersions, namely specifying the Ricci tensor of the fibres and $\sum_a K_{iaa}$. In the present case, the Ricci tensor of the fibre is trivial, and $\sum_a K_{iaa}$ is just K_i. Since in the theory formulated thus we must have $K_i = 0$, *the system does not depend on any function of n variables*, where n is the dimension of the total space. Thus the aim of deriving a lower dimensional effective theory is really achieved. If we look at it in more details, we see that the degree of arbitrariness of the theory is

$$s_{n-1} = \text{Riemannian degree of arbitrariness of reduced space} + (n-2)$$

where $n-2$ comes from the number of normal terms of $M_{ij;k}$ where k takes the maximal value. We see that this is exactly the kinematical degree of arbitrariness of electromagnetism coupled to gravity, though here coupled not in the same way. As we have seen above, the dynamical theory (the equations of motion) is different from the usual theory.

133 Kaluza–Klein theory with warping factor. Due to the problem of "inconsistency" of the old Kaluza–Klein theory, there is an improved, more complicated version.

The ansatz is now

$$ds^2 = e^{\frac{4\sigma}{\sqrt{3}}}(d\phi - \tilde{A}_i dx^i)^2 + e^{-\frac{2\sigma}{\sqrt{3}}} g_{ij} dx^i dx^j,$$

c.f. the normal form of the Weyl rigid flow, and at this stage only the functions g_{ij} is required to be independent of the coordinate ϕ. For this theory, it seems that different people has different ways of interpreting $\tilde{A}_i$: it is interpreted as the electromagnetic potential up to different scaling factors involving σ.

The first thing to note about this theory is that if σ is dependent on ϕ, then *this theory does not amount to a Riemannian submersion*, so studies of this theory using methods of Riemannian submersion are invalid. The correct framework is, obviously, Weyl rigid flow, as we have done from §**118**.

Let us see what our theory of Weyl rigid flow can say about this type of Kaluza–Klein theory. The scaling curvature of the total space, $F_{\mu\nu}$, obviously vanishes, since the total space is a Riemannian space. Then for the equation (4.9) in §**124**, we see that

$$G_{ij} = 2E_0 M^0_{ij}$$

where G_{ij} is the scaling curvature of the reduced space. *Thus we always want non-zero scaling curvatures for the reduced space*, since otherwise either $M^0_{ij} = 0$, in which case the electromagnetism is trivial, or $E_0 = 0$, in which case we are back to the old Kaluza–Klein theory. For the original ansatz, this implies in particular that *we always want the factor σ to depend on ϕ*. It is easy to see this for our original ansatz: if σ is independent of ϕ, by scaling the coordinates x^i suitably we are back to the old Kaluza–Klein theory, and the transformed g_{ij} is still independent of ϕ.

As we have mentioned, there are different conventions of how to interpret A_i, and different interpretations gives different electromagnetic field tensor. There are often also various assumptions of whether or how A_i depends on the coordinate ϕ. However, due to the workings of the warping factor σ, from §**124** we see that now

$${M_{ij}}^0{}_{;0} = -\lambda_0 {M_{ij}}^0,$$

so M_{ij0} is now up to scale a good candidate for the electromagnetic field tensor, regardless of the dependence of A_i on ϕ. However, now M_{ij0} is no longer simply $A_{[i;j]}$: it is a fairly involved function of the derivatives of both A_i and σ. The usual approach only considers $A_{[i;j]}$ or its scaled versions to be the electromagnetic field tensor, which in our view is rather messy and not very coordinate-independent.

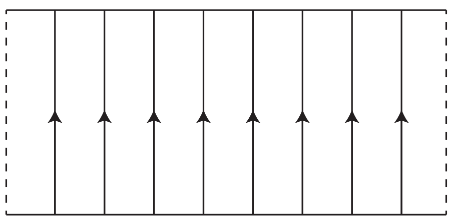
We shall not go further and study the Cauchy problem for Kaluza–Klein theory with warping factor. Different assumptions yields different results, and there is no consensus on what is the “correct” assumption on A_i. The Cauchy problem is very complicated even with our interpretation of $M_{ij}{}^0$ as the field tensor. The existence of the potential requires $M_{[ij|0|;k]} = 0$, which is the same as

$$M_{[ij|0}K_{|k]00} = 0,$$

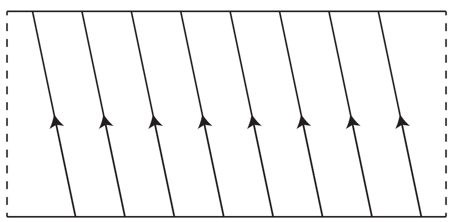
placing constraints on K_{k0} that depend on the rank of M_{ij0}. The Kaluza–Klein theory with warping factor, in its attempt to avoid the embarrassing "inconsistency", loses much of its appeal of simplicity.

134 Some potential problems for Kaluza–Klein type theories from global considerations. One is usually lured into studying Kaluza–Klein theory by being shown a picture of something which looks like a long tube, and a curve which looks just like the tube except that it is much thinner, and the mantra that "from a distance, the compact dimensions seem to disappear", or "the smaller, compact dimensions do not take part in dynamics in a first approximation". One then goes to write down the ansatz for a local Kaluza–Klein metric as we have done. If we really takes the higher dimensional theory seriously, then this approach ignores all global problems and constraints, together with all problem of identification: given such a tube-like spacetime, how to we recognise it as an effective lower dimensional spacetime?

To visualise the problem, let us cut the tube into a strip, which gives us something like the following:



the upper and lower edges are identified, and the arrows represent the vector field through which we usually think of the dimensional reduction. But locally, this is exactly the same as the following:



yet for the second picture, globally the "reduced space" may not be a space at all if the slope of the vector field is irrational.

Note that we are not saying that making a first approximation of a higher dimensional space with compact dimension as a lower dimensional space is intrinsically wrong: we are saying that the way we write down the ansatz may be problematic: as we know, structure-preserving submersions, being overdetermined systems, are heavily constraints, and if we look for the exact solutions to the ansatz we may find solutions that has special properties, which are not physically justifiable in general, and the existence and uniqueness of lower dimensional solutions *corresponding to a single higher dimensional exact solution* is also in question.

Another problem is that one usually considers the Einstein equations to be the equation of motion for the total space. Then what is really needed is a way to ensure that the reduction dimension is really very small. This must come from some additional mechanism, since the Einstein equations tends to give solutions that are of Kaluza–Klein type, even globally, save for a few singular points, but such solutions contain compact dimensions that are way too big.

For example, in such a theory it is reasonable to give the following matter content: a stationary lump of matter concentrated at a single point, and nothing elsewhere. The solution is obviously of the Schwarzschild form, as we have seen in §**100**. In addition, such solutions are all of the Kaluza–Klein form, but as we move further and further from the concentrated matter the circles representing the reduction grows unbounded. Even if we consider a ring of matter the basic property is the same: the radius of the reduction circles will diverge at infinity.

These two problems are not specific to Kaluza–Klein type reductions: all reduction schemes, if they are to be taken seriously, must address similar problems.

135 Extension of Kaluza–Klein theory to De Witt theory, viewed as a Riemannian submersion. Another way to view the old Kaluza–Klein theory is that when restricted to the fibre, the structural equation is trivial,

$$d\omega^0 = 0 \pmod{\omega^i},$$

of course this is simply a consequence that the fibre is one dimensional. De Witt reduction capitalises on this by generalising this triviality to higher dimensional fibres. Hence we require that, when restricted to a single fibre,

$$d\omega^a = c^a{}_{bc}\omega^b \wedge \omega^c \pmod{\omega^i},$$

for $c^a{}_{bc}$ constant on the fibre. Then from the theory of Lie groups we know that $c^a{}_{bc}$ cannot be chosen arbitrarily: they must satisfy the Jacobi identity and ω^a is automatically the left-invariant Maurer–Cartan form on a Lie group, and the quadratic form $\sum_a \omega^a \otimes \omega^a$ is the left-invariant metric on the Lie group.

Note that writing down the above structural equation requires that we do a complete reduction of the principle bundle in the group direction, and the existence and uniqueness of such a reduction are both questionable. If we do not do this, then the connections ${\omega^a}_b$ would come from the orthonormal frame over a Lie group regarding its left-invariant metric, which is a rather complicated thing to deal with, and further in this case ω^a no longer has the interpretation of left-invariant Maurer–Cartan forms. From the point of view of the total space, the structural equation for $d\omega^a$ is

$$d\omega^a = e^\Lambda {c^a}_{bc}\omega^b \wedge \omega^c - K_{iab}\omega^b \wedge \omega^i - {M_{ij}}^a \omega^i \wedge \omega^j,$$

where the scaling factor e^Λ means that we do not require that the Lie groups at each point of the reduced space to be of the same size, only that they are of same kind. We can relax this condition further by requiring only that ${c^a}_{bc}$ to be constant over each fibre, but allowing its structure to change. This amounts allowing the fibre to mutate from one Lie group to a related one.

Note that since now every index in the range a, b, ... are accompanied by the basis of left invariant form, they themselves have the interpretation of Lie algebra-valued quantities. Hence, the expression

$${M_{ij}}^a \omega^i \wedge \omega^j$$

has the interpretation of a Lie algebra-valued two form and is the potential candidate for the Ehresmann curvature after the reduction. The existence of the gauge potential, i.e., the Ehresmann connection, is simply ${M_{[ij|}}^a{}_{;|k]} = 0$, as in the Kaluza–Klein case, since our derivatives are covariant derivatives.

In the usual ansatz for De Witt reduction

$$ds^2 = g_{ab}(\lambda^a + 2A^a)(\lambda^b + 2A^b) + \Delta^{-1} g_{ij} dx^i dx^j,$$

where λ^a are the left-invariant one-forms on the group, A^a are Lie algebra valued one forms on the reduced space with coframe dx^i, the identification with our theory is as follows. If Δ^{-1} is constant, then dA^b is simply ${M_{ij}}^a \omega^i \wedge \omega^j$, and K_{iab} simply vanishes. In this case we again have the inconsistency as in the old Kaluza–Klein case. For non-vanishing K_{iab}, it is necessary to have a non-trivial factor Δ^{-1}, and the methods of structure-preserving submersions in Weyl geometry is needed.

Since one dimensional geometry is locally trivial, Kaluza–Klein type reductions do not place any local requirements on the fibres. De Witt reduction, on the other hand, in an attempt to obtain clean results that are not much more difficult than the Kaluza–Klein reductions in the higher dimension case, not only requires that the fibres are locally homogeneous spaces arising from Lie groups, but requires the choice of a particular left-invariant co-frame on this space, the choice furthermore affects the reduction. These assumptions are highly unnatural if the higher dimensional theory is to be taken seriously as having physical significance of its own, and as a consequence De Witt reductions suffer from even more severe *identification* problems than Kaluza–Klein reductions.

136 The history of Pauli reduction. One cannot help but have the feeling that the reduction schemes discussed above are butchering the geometry, since we are identifying parts of the Cartan *connection* with an Ehresmann *curvature*. Why not simply identify curvature with curvature? Actually this is not difficult to do, at least in principle. For example, take a Riemannian geometry under structure-preserving submersion, the curvature matrix splits into

$$d\omega + [\omega \wedge \omega] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega^a{}_b & \Omega^a{}_i \\ 0 & -(\Omega^a{}_i)^T & \Omega^i{}_j \end{pmatrix}.$$

and the part $\Omega^a{}_b$ contains forms of the following form

$$R^a{}_{bij}\omega^i \wedge \omega^j,$$

which under rather mild conditions have the interpretation of an Ehresmann connection with the Lie group $SO(q)$, where q is the dimension of the fibres. Note that for $q = 1$ the term simply vanishes identically.

This roughly looks like what Pauli set out to do [36, 32], but he actually did something different. From our view, first, the part of the curvature

$$R^a{}_{bcd}\omega^c \wedge \omega^d$$

is annoying, since it intrinsically depends on the fibre coordinates. What Pauli did is simply to make such terms trivial, by assuming every fibre is a homogeneous space (more precisely, a sphere). Then Pauli did something that seem extremely unreasonable for our modern eye: he wanted to obtain an Ehresmann connection, not for the group $SO(q)$, but for $SO(q+1)$, *since* $SO(q+1)$ *is the isometry group of the fibre,* S^q. Of course this is doomed to failure, as this goes against the group structure of the geometry: $\Omega^a{}_b$ is not linked with $SO(q+1)$.

Nonetheless let us think a little bit about why Pauli came out with such a strange idea. It must be emphasised that Pauli did this while we were still ignorant about the correct geometrical framework (principle bundles) for the particles, and his work is a search in this direction. *He would never have believed that the higher dimensional space* *has the interpretation of a higher dimensional spacetime, only as some internal space*, so in this regard we cannot blame him for introducing unnatural geometries. Second, he probably had in his mind, at the time of writing down such a reduction, the problem of a sphere rolling without slipping on a surface, which he was no doubt very familiar with. These two look quite similar intuitively: a sphere is "attached" to every point of the surface, and in such a problem we really have an Ehresmann connection taking value in the whole isometry group of the sphere: it is actually the correct framework of principle bundles. Pauli was at that time probably also influenced by the work of Kaluza–Klein and Weyl, and hence attempted to introduce a metric into the whole bundle, which ruined the theory. Finally, Pauli did not publish any of this theory as he was not satisfied with it himself.

The interesting thing is, of course, there are known systems for which, through complicated mechanisms, Pauli's aim is really achieved (see [21] and the references therein), and this is a surprise. Note that none of such systems are systems of pure geometry: we need the additional matter content to make this work. In the following we will try to provide a simple explanation of why such systems could work, and also point out the difficulties in constructing general systems which are consistent with Pauli's reduction scheme.

137 Consistent Pauli reductions. In all known consistent Pauli reductions there are implicitly or explicitly the involvement of a certain 2-form defined on the space. This two form usually comes from the B-field of string theory, but this does not need to concern us. What is important to us is that we can use this two-form to change our structural equations a little bit:

$$d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu + \tfrac{1}{2} T^\mu{}_{\nu\lambda}\omega^\nu \wedge \omega^\lambda,$$

and from §**63** we learned that there is no restriction on $T^\mu{}_{\nu\lambda}$, *any two-form can be taken as the torsion two-form* $\Omega^\mu = \frac{1}{2}T^\mu{}_{\nu\lambda}\omega^\nu \wedge \omega^\lambda$.

With torsion, the curvature matrix now becomes

$$d\omega + [\omega \wedge \omega] = \begin{pmatrix} 0 & 0 & 0 \\ \Omega^a & \Omega^a{}_b & \Omega^a{}_i \\ \Omega^i & -(\Omega^a{}_i)^T & \Omega^i{}_j \end{pmatrix}.$$

Now let us focus on the structural equation restricted to a single fibre, *without adding torsions of the form* $T^a{}_{bc}\omega^b \wedge \omega^c$:

$$\begin{cases} d\omega^a = -\omega^{ab} \wedge \omega^b, \\ d\omega^a{}_b = -\omega^a{}_c \wedge \omega^c{}_b + \kappa\omega^a \wedge \omega_b. \end{cases}$$

where κ is the constant which measures the size of the sphere. From §**40**, we know that by changing our model geometry for the total space from the Poincaré group to the rotational group, we can stipulate that the term $\kappa\omega^a \wedge \omega_b$ is part of the structural equation for the Lie group, i.e., now there is no curvature for spheres but there is curvature for planes.

This can be done for a single point over the fibre. To do it for all points at once, we need to go into the Weyl reduction formalism. For simplicity we shall not reflect this in our notations, as we will not attempt any detailed calculations, and hence the explicit use of the Weyl scheme does not add much to our understanding.

Thus, now the curvature matrix becomes

$$d\omega + [\omega \wedge \omega] = \begin{pmatrix} 0 & -\Omega^a & -\Omega^i \\ \Omega^a & \Omega^a{}_b & \Omega^a{}_i \\ \Omega^i & -(\Omega^a{}_i)^T & \Omega^i{}_j \end{pmatrix},$$

where we disregard torsion of the form $T^a{}_{bc}$. As a consequence of this and our mutation of model, $\Omega^a{}_b$ contains no terms of the form $R^a{}_{bcd}\omega^c \wedge \omega^d$. Then, the curvature matrix contains the terms

$$\Omega' = \begin{pmatrix} 0 & -\Omega^a \\ \Omega^a & \Omega^a{}_b \end{pmatrix} = \begin{pmatrix} 0 & -T^a{}_{ij}\omega^i \wedge \omega^j, \\ T^a{}_{ij}\omega^i \wedge \omega^j & R^a{}_{bij}\omega^i \wedge \omega^j, \end{pmatrix},$$

which is now a good candidate for the Ehresmann curvature of $SO(q+1)$, with enough degree of arbitrariness.

Of course, this procedure is complicated in that it is hard to see how we can construct general solutions satisfying its various constraints. The main constraint is the existence of the Ehresmann connection γ, a $\mathfrak{so}(p+1)$-valued one form in ω^i, such that

$$d\gamma + [\gamma \wedge \gamma] = \Omega'.$$

If we prolong the space we are working with by regarding γ as independent variables, we see that the solutions we are searching for is defined by the equality of a two-form. Hence we need the full power of the Cartan–Kähler theorem for general differential systems, and none of our methods for involutive seeds, etc., apply, since they are restricted to linear Pfaffian systems. It may be possible to proceed in the following way: in the Bianchi identity for the Ehresmann curvature

$$d\Omega' = [\Omega' \wedge \gamma]$$

Ω' is given if we specify a geometry, $d\Omega'$ can be calculated readily, so this equation can allow us to reduce drastically the number of free variables in γ. However, applied to a real calculation this is too complicated, and it is again quite useless if we want to find the general solution.

Another problem is that the procedure is not very “clean”. In general, Ω' will be a complicated function of M_{ija}, K_{iab} and E where the last quantity is the scaling factor in Weyl geometry. However, in general these three quantities contain more free variables than Ω' contains, and hence the question of how to interpret the “left-over” pieces such as

$$T^a{}_{bi}\omega^b \wedge \omega^i, \qquad R^a{}_{bci}\omega^c \wedge \omega^i$$

does not seem very easy to resolve.

Epilogue

We are now at the end of our work. We have developed the method of involutive seeds, which specialises Cartan's various methods to geometrical settings equipped with a covariant derivative, and we have constructed a general framework for working with structure-preserving submersions. We have also investigated various applications along the way, especially the problems of rigid flows.

Of course, many more things could have been done to expand this work further. Here we will only mention a few that the author would like to do very much but could not due to either lack of time, or lack of expertise in a certain field, or both.

From the theoretical part, the method of involutive seeds gives a way to calculate the degree of arbitrariness of a system rapidly, but this "degree of arbitrariness" is still rather vague: it is only the number of free functions we can specify, and the relation with the physical "degree of freedom" is not very clear in most circumstances. To obtain further information about the dynamics of a systems, what is needed is a unification of our present methods with Hamiltonian mechanics and symplectic methods.

Another theoretical problem that is worth investigating is the link with Noether's theorem. We have seen that the difference between the maximal possible number of functionally independent invariants and the actual number attained by a system is the dimension of the symmetry group of the problem. Here we can of course make a nodding acknowledgement to Noether's theorem, but an understanding of the precise relationship between the two, and how to *derive* these conserved quantities directly within our framework, is certainly of more value.

For more concrete applications, the author feels that much more can come out from the method of Weyl flow and Weyl submersions. In particular, other partial results and the Ellis conjecture itself should be studied in more details. More detailed studies of the various dimensional reduction schemes by our methods could also lead to new insights, etc.

The applications of the methods developed to the problems of dimensional reductions under various schemes, treated only briefly in the last part of the work, certainly deserves further work to be done.

To end this work, let us remember the following words of Ludwig Wittgenstein:

We predicate of the thing what lies in the method of representing it. Impressed by the possibility of a comparison, we think we are perceiving a state of affairs of the highest generality. [†]

THE END.

[†]From *Philosophical Investigations* (*Philosophische Untersuchungen*): *Man prädiziert von der Sache, was in der Darstellungsweise liegt. Die Möglichkeit des Vergleichs, die uns beeindruckt, nehmen wir für die Wahrnehmung einer höchst allgemeinen Sachlage.*

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