A Nonequilibrium Equation of State for a Turbulent 2D Bose Gas

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Nonequilibrium equations of state can provide an effective thermodynamic-like description of far-from-equilibrium systems. We experimentally construct such an equation for a direct energy cascade in a turbulent two-dimensional Bose gas. Our homogeneous gas is continuously driven on a large length scale and, with matching dissipation on a small length scale, exhibits a nonthermal but stationary power-law momentum distribution. Our equation of state links the cascade amplitude with the underlying scale-invariant energy flux, and can, for different drive strengths, gas densities, and interaction strengths, be recast into a universal power-law form using scalings consistent with the Gross–Pitaevskii model.

The framework of thermodynamics provides an effective description of many-particle systems at or near equilibrium, encapsulating complex microscopic behavior with a small number of macroscopic state variables that are related by a system-dependent equation of state (EOS) [1–5]. There is a lot of interest, and also progress, in transposing these ideas to far-from-equilibrium systems that may not attain even local thermodynamic equilibrium, including active matter [6–8], glasses [9], plasmas [10], and turbulence [11–17].

Driven turbulent cascades exhibit stationary far-from-equilibrium states, which are natural candidates for a thermodynamic-like characterization. They feature scale-invariant transport of conserved quantities, such as energy [18–20] or enstrophy [21], between an injection scale, k_I (where k is a wavenumber), and a dissipation scale, k_D. In the inertial range, between k_I and k_D, spectra of system-dependent quantities such as energy or wave action generically exhibit power laws with universal exponents [19–25], that are insensitive to microscopic details of the system. The state of the system (in this k-range) is then specified by the amplitude of the (statistically averaged) power-law spectrum. However, it is in general an open question what this spectrum amplitude depends on.

Famously, for a Kolmogorov direct energy cascade (from large to small length scales, *i.e.*, with $k_\mathrm{I} < k_\mathrm{D}$) in incompressible fluids [19, 20], one can predict the amplitude of the energy spectrum (up to the Kolmogorov constant [26]) using dimensional analysis; this analysis dictates that the spectrum amplitude depends only on the underlying energy flux, and gives the amplitude-flux relation. However, dimensional analysis is insufficient for a compressible fluid (with finite chemical potential), such as a weakly interacting Bose gas. Nevertheless, for a direct energy cascade in a 3D Bose gas, it was experimentally established that the amplitude of the power-law momentum distribution (wave-action spectrum) can be uniquely related to the underlying energy flux, in a way that does not depend on the details of the energy injection or dissipation, or the system's history [16]. The emergence of such a reduced macroscopic description makes the amplitude-flux relation analogous to an equilibrium EOS [16, 27–29], but with intrinsically nonequilibrium state variables; see Fig. 1. Understanding and predicting the form of such amplitude-flux relations is an area of active theoretical research [27–32].

In this Letter, we experimentally establish a nonequilibrium EOS for a direct energy cascade in a homogeneous two-dimensional (2D) Bose gas [33], which in a turbulent steady state exhibits a power-law momentum distribution n_k with a universal exponent γ and an amplitude n_0. Using scalings consistent with the Gross–Pitaevskii model, we show that our measurements for different drive strengths, gas densities, and strengths of the interparticle interactions, can be recast into a dimensionless power-law EOS that universally links n_0 to the cascade energy flux ϵ. Our measured ϵ satisfies the zeroth law of turbulence [34–36] and, by quenching the drive strength, we verify that the steady-state n_0 satisfies the same EOS independently of the system's history.

We start with a quasipure homogeneous 2D condensate of ^{39}K atoms in the $|F = 1, m_F = -1\rangle$ hyperfine state, confined to the x-y plane by a harmonic trap with frequency $\omega_z = 2\pi \times 1.2\,\mathrm{kHz}$, and within the plane by a square optical box trap of side length $L = 30\,\mu\mathrm{m}$ and depth $U_\mathrm{D} = k_\mathrm{B} \times 175\,\mathrm{nK}$ [37]. For a 2D gas, the interaction strength is given by the dimensionless $\tilde{g} = a\sqrt{8\pi m\omega_z/\hbar}$ [38, 39], where m is the atom mass and a is the s-wave scattering length, which we tune using a Feshbach resonance at 33.6 G [40]. For our variable atom number, $N = (20-36)\times 10^3$, corresponding to density $n = N/L^2 = (22-40)\,\mu\mathrm{m}^{-2}$, and $\tilde{g} = (0.02-0.08)$, the characteristic interaction energy per particle is $\zeta = \hbar^2\tilde{g}n/m = k_\mathrm{B} \times (6.6-42)\,\mathrm{nK}$.

We continuously inject energy into the gas by resonantly driving a lowest-energy phonon mode, with wavenumber

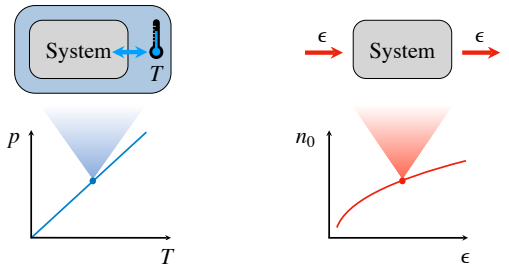

FIG. 1. The equation of state (EOS) for an equilibrium system (left), and an analogous description for a flux-carrying turbulent system in a steady state, with matching injection and dissipation rate ϵ (right). For an equilibrium system, coupled to a reservoir, state variables such as pressure, p, are functions of the reservoir temperature T. For a flux-carrying turbulent system, the amplitude, n_0, of the emergent power-law spectrum is a function of the underlying flux ϵ.

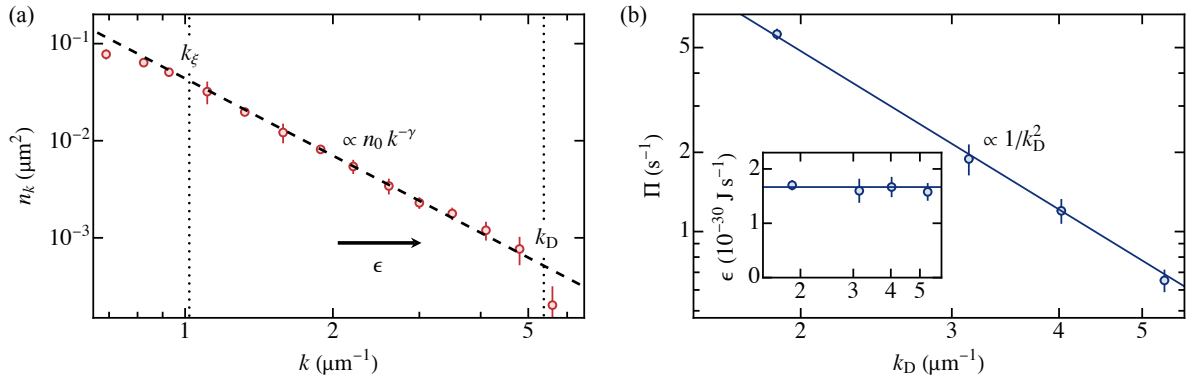

FIG. 2. Turbulent steady state and the underlying energy flux. (a) An example of a steady-state momentum distribution $n_k(k)$ for our turbulent 2D gas, held in a square box trap of side length $L = 30\,\mu$m and driven on a large length scale ($\sim L$) by a time-periodic force. Between the inverse healing length k_ξ and the dissipation scale k_D, set by our trap depth, n_k exhibits a power-law spectrum, $\propto n_0\, k^{-\gamma}$, with $\gamma \approx 2.6$ (dashed line). Here, the gas has density $n = 25\,\mu\text{m}^{-2}$ and dimensionless interaction strength $\tilde{g} = 0.04$, and the force amplitude is $F_0 = 1.6\zeta/L$, where $\zeta = \hbar^2\tilde{g}n/m$. (b) The corresponding energy flux ϵ. The steady-state particle flux Π is measured through atom loss for various k_D. In agreement with the zeroth law of turbulence, we observe $\Pi \propto 1/k_D^2$ (solid line), and thus deduce the k_D-independent $\epsilon = \Pi\hbar^2 k_D^2/(2m)$ (inset). Note that n_k, Π, and ϵ are normalized per particle, and the spectrum in (a) was measured with the largest k_D value in (b).

$k_L = \pi/L \approx 0.1\,\mu\text{m}^{-1}$ and angular frequency $\omega_B = \sqrt{\zeta/m}\,k_L$, using a spatially uniform time-periodic force along one of the box axes [33, 37, 41]; for our different measurements, we vary the force amplitude F_0 in the range $(0.7-12)\,\zeta/L$. This injection is anisotropic, but the steady-state momentum distribution n_k shows an emergent statistical isotropy for $k \gtrsim k_\xi$, where $k_\xi = \sqrt{\tilde{g}n} = (0.7-1.8)\,\mu\text{m}^{-1}$ is the inverse healing length [33]. From a theoretical point of view [27, 28, 32], k_ξ is the effective injection scale, k_I, for a statistically isotropic cascade, and the condensate acts as a particle reservoir [42]. The continuous energy dissipation occurs in the form of a small particle-loss rate due to atoms escaping the trap at $k_D = \sqrt{2mU_D}/\hbar$ [36, 43]; for our full trap depth, $k_D \approx 5.3\,\mu\text{m}^{-1}$, but note that for some measurements we also reduce k_D.

In Fig. 2(a), for one set of system parameters, we show the azimuthally averaged n_k for a steady-state turbulent cloud (see also [33, 44]); we measure n_k using matter-wave focusing [45, 46], average over experimental repetitions, and normalize it such that $\int n_k\, 2\pi k\,\mathrm{d}k = 1$. For $k_\xi \lesssim k \lesssim k_D$, we observe a power-law $n_k \propto n_0\, k^{-\gamma}$, with $\gamma = 2.6$ (dashed line). In the theory of weak-wave turbulence (WWT), the Kolmogorov–Zakharov (KZ) analytical solution for a direct energy cascade in a Bose gas predicts (for $k \gtrsim k_\xi$) a power-law n_k with $\gamma = d$, where d is the system dimensionality [22, 25]. However, this solution is not valid for $d = 2$ because $n_k \propto k^{-2}$ coincides with the equilibrium (Rayleigh–Jeans) distribution and carries no flux. In various theories [22, 47–49], this results in logarithmic corrections that make n_k steeper than k^{-2}. These corrections are scale-dependent, but over a finite experimental k range the spectrum is fitted well by a power law with an exponent larger than 2 (see also [16, 32]).

In Fig. 2(b), we show our measurement of the corresponding energy flux ϵ, deduced from the scale-invariant particle flux Π [36], which is directly seen in the rate at which atoms leave the trap at k_D [36, 44]; normalizing per particle, $\Pi = -\dot{N}/N$. We measure Π for various k_D (by reducing the trap depth) and, in agreement with the zeroth law of turbulence [34, 35], observe $\Pi \propto 1/k_D^2$ (solid line), which corresponds to a k_D-independent $\epsilon = \Pi\hbar^2 k_D^2/(2m)$ (see inset).

In Fig. 3, we summarize our measurements for various $\tilde{g}$, n, and ϵ, and construct a universal EOS. Note that here we always measure n_k for our largest k_D and that, for fixed $\tilde{g}$ and n, we vary ϵ using F_0 as the experimental control parameter. The emergent ϵ is monotonic in F_0, but their relationship is in general complicated, and to construct the universal EOS, one needs to relate n_0 to the measured ϵ rather than the externally imposed F_0 [16].

In Fig. 3(a) we show that we always observe γ consistent with 2.7(1) (see also [33]), without any systematic dependence on $\tilde{g}$, n, or ϵ. From here on, we define a dimensionless $n_0 = k_\xi^{2-\gamma}\langle k^\gamma n_k\rangle$ [16] with fixed $\gamma = 2.7$ and always take the average $\langle k^\gamma n_k\rangle$ for k in the range $(2-4)\,\mu\text{m}^{-1}$.

In Fig. 3(b), we show all our measurements of $n_0(\epsilon)$ for different $\tilde{g}$ and n, and in Fig. 3(c) we show that they all collapse onto a single universal curve when we plot n_0 versus the dimensionless energy flux $\hbar\epsilon/\zeta^2$, using a normalization of ϵ that is natural within the Gross–Pitaevskii model [16]. This universal EOS is captured well by a power-law fit, $n_0 \propto (\hbar\epsilon/\zeta^2)^b$ (solid line), with $b = 0.47(5)$.

Finally, for n_0 and ϵ to be analogous to equilibrium state variables, their relationship should be independent of the history of the system. To verify this, we performed an experiment where we prepared a steady turbulent state and then changed F_0 (from $2\zeta/L$ to $6\zeta/L$), confirming that the steady-state n_0 before and after this change satisfy the same EOS; these two measurements are included in Fig. 3, and are highlighted in the inset of Fig. 3(c).

Theoretically, the problem of the functional form of the EOS in 2D is still quite open. Some of the theories with flux-carrying n_k predict $b = 1/3$ [22, 49], based on 4-wave interactions (often assumed to dominate for $k \gtrsim k_\xi$) [22]. Others predict $b = 1/2$ [47, 48] (see also [28]), which is phys-

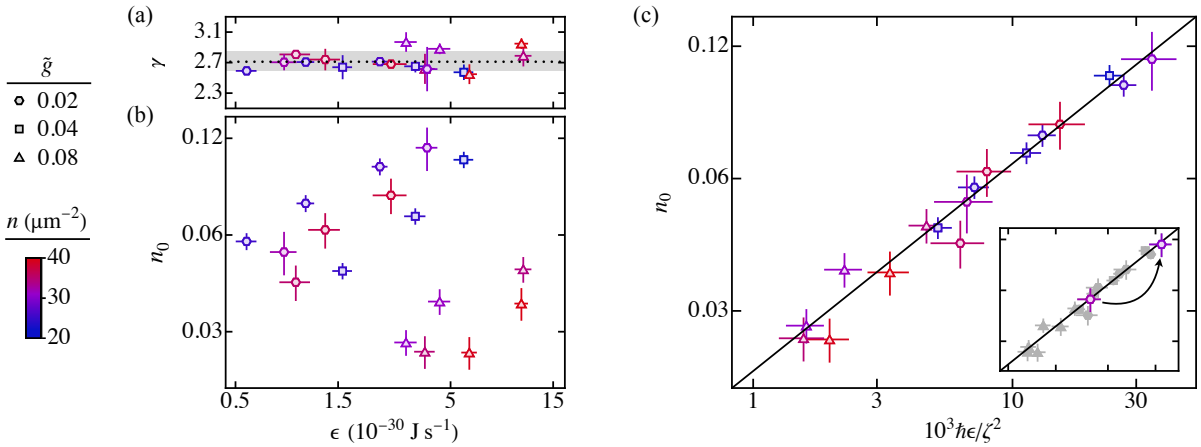

FIG. 3. The universal EOS. Here we generalize our measurements of the steady-state n_k to various $\tilde{g}$, n, and ϵ. (a) The cascade exponent γ. Our data show no systematic variation and are consistent with $\gamma = 2.7(1)$ (dotted line and shading). (b) The cascade amplitude n_0 (extracted with fixed $\gamma = 2.7$) plotted versus ϵ. (c) Plotting n_0 versus the dimensionless energy flux $\hbar\epsilon/\zeta^2$ collapses all our data onto a universal EOS. The solid line shows a power-law fit, with exponent $b = 0.47(5)$. Inset: we highlight the two data points corresponding to the steady states established before and after a change in the driving-force amplitude (see text), with the other points from the main panel shown in gray.

ically associated with a significance of (effective) 3-wave interactions in a Bose gas. The latter prediction agrees with our experiments, but this could be fortuitous. For example, beyond-WWT effects, such as vortex excitations, could also play a role, as seen in numerical studies of 3D Bose gases [27, 50].

Our results provide benchmarks for theories of quantum-gas turbulence [22, 24, 25, 27–32] and contribute to establishing a general framework of nonequilibrium thermodynamics [6–10, 16]. In the future, it would be interesting to extend studies of far-from-equilibrium EOS to inverse cascades [29, 51], and to systems with fermionic statistics or long-range interactions. Going beyond the statistically averaged n_k, which is directly related to the first-order correlation function, it would be interesting to explore higher-order correlations in the steady state, *e.g.*, using off-resonant light scattering [52] as a probe of second-order correlations.

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The data that support the findings of this article are openly available [53].

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[42] Note that for all our data the number of atoms at $k < k_\xi$ is still at least 50% of its initial value.
[43] The particle number in our system is slowly decreasing in time, and we always report the instantaneous values of n, n_0, and ϵ. For all our main data (taken with the largest k_D), the system is always in a quasi-static turbulent state that is well characterized by these instantaneous values because the time, t_D, that it takes for the cascade to propagate to k_D (as seen in the delay of the onset of losses after we start driving the system) is at least 4 times shorter than the characteristic time scale of the particle loss [44].
[44] See End Matter for additional details on Fig. 2, including the cascade-induced loss dynamics, measurements of the steady-state n_k at different times, and individual experimental repetitions.
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End Matter

Here, we provide additional experimental details for the data presented in Fig. 2.

In Fig. 4(a), we plot the evolution of the cascade-induced atom loss ΔN as a function of the drive time t, which we use to extract the particle flux $\Pi = -\dot{N}/N$. We fit the data with a piecewise function (solid line), which is 0 for $t < t_{\mathrm{D}}$ and an exponential for $t > t_{\mathrm{D}}$, with t_{D} and Π as the fit parameters. Here, we obtain $t_{\mathrm{D}} \approx 270\,\mathrm{ms}$ and $\Pi^{-1} \approx 1500\,\mathrm{ms} \gg t_{\mathrm{D}}$. We always measure n_k just after t_{D}, when the total loss is small. As shown in Fig. 4(b), n_k is essentially the same at $t = 330\,\mathrm{ms}$ and $450\,\mathrm{ms}$ [vertical dotted lines in Fig. 4(a)], and in Fig. 2(a) we report the average of these two measurements. In Fig. 4(c) we illustrate, for $t = 450\,\mathrm{ms}$, how n_k varies between experimental repetitions.

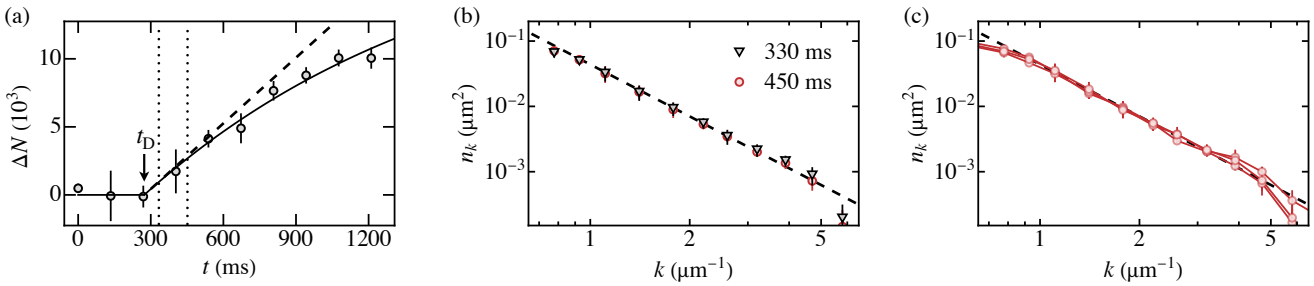

FIG. 4. Additional panels for Fig. 2. (a) Atom-loss dynamics $\Delta N(t)$ associated with the turbulent cascade shown in Fig. 2(a); the initial atom number is $2.6(2) \times 10^4$. The solid line shows our piecewise fit (see text), which gives $t_{\mathrm{D}} \approx 270\,\mathrm{ms}$ and $\Pi^{-1} \approx 1500\,\mathrm{ms}$ (for $t > t_{\mathrm{D}}$). The dashed line shows the tangent to the fit at t_{D}, and the vertical dotted lines indicate $t = 330\,\mathrm{ms}$ and $450\,\mathrm{ms}$. (b) Statistically averaged n_k at $t = 330\,\mathrm{ms}$ and $450\,\mathrm{ms}$. (c) Individual realizations of n_k for $t = 450\,\mathrm{ms}$. The dashed lines in (b) and (c) are identical to the one shown in Fig. 2(a).