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1/N versus Mean-Variance:  
What if we can forecast?

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## 1/N versus Mean-Variance: What if we can forecast?

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### ABSTRACT

Mean-variance optimisation has been roundly criticised by financial economists and practitioners alike, leading many to advocate a simple 1/N weighting heuristic. We investigate the performance of the Markowitz technique conditional on investor forecasting ability. Using a novel analytical approach, we demonstrate that investors with a modicum of forecasting ability can employ mean-variance to significantly increase their ex ante utility, outperforming the 1/N rule.

The goal of our research is to investigate the performance of mean-variance when investors have forecasting ability. In the seminal *Portfolio Selection*, Markowitz (1952) proposed mean-variance optimisation as the normative method for allocating capital to risky assets by maximising the expected return for a given level of variance. The Markowitz approach was the basis for several important advances in financial economics including the Capital Asset Pricing Model (Sharpe, 1963 and Lintner, 1965) and the understanding of the dichotomy between systematic and diversifiable risk. A considerable controversy has raged over the efficacy of the technique ever since. Two key criticisms have been levelled at the mean-variance approach. Firstly, the mean-variance approach assumes that either returns are normally distributed or that investors have quadratic utility, neither of which assumptions hold in practice<sup>1,2</sup>. The second criticism is the phenomenon known as error-maximisation where return and covariance forecast errors are magnified in the estimated portfolio weights leading to poor out of sample performance. In this paper we explore the second criticism, error maximisation and the role of forecasting ability. Estimation error and forecasting ability are of course two sides of the same coin. If we have perfect foresight, then we have no need for estimation, and hence no estimation error.

By construction, the mean-variance approach will overweight those assets that have large estimated returns, low correlations, and small variances, i.e. precisely the assets that are most likely to have large estimation errors. In the case of forecast returns, a top-quartile portfolio manager has an information coefficient<sup>3</sup> of only 0.05<sup>4</sup>. In the case of forecast covariance, Chan, Karceski and Lakonishok (1999) using sophisticated risk models, could only achieve a 0.18 correlation with realised covariance<sup>5</sup>. This lack of forecast accuracy is potentially problematic for the mean-variance

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<sup>1</sup> It has been known since Mandelbrot (1963) that the return distributions of financial assets are fat tailed relative to the normal distribution (leptokurtic) and have more extreme negative values than positive values (negatively skewed)

<sup>2</sup> The quadratic utility function has an increasing level of absolute risk aversion. Hence in a two asset framework an increase in wealth will lead to a decrease in the allocation to the risky asset. In some circumstances it is also possible for the mean-variance investor to strictly prefer less to more.

<sup>3</sup> Correlation between forecast returns and actual returns

<sup>4</sup> Merton (1980) demonstrated that we need a very long data history to estimate expected returns with any precision

<sup>5</sup> For 12-month forecasting horizons

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approach because the optimisation algorithm takes the inputs as if they are known with certainty when in fact they are noisy estimates.

There is an extensive body of work investigating mean-variance and error-maximisation. Bloomfield, Leftwich and Long (1977) demonstrate that mean-variance optimisation does not outperform equally weighted portfolios. Jobson and Korkie (1981) quantify the effect of error-maximisation using Monte Carlo simulation. Damningly, they demonstrate that an equally weighted portfolio that ignores the return, variance and correlation estimates altogether, thereby eliminating estimation error, outperforms classical mean-variance. Jorion (1991) shows that the equally-weighted and value weighted portfolios have similar out of sample performance to the minimum variance portfolio and outperform the mean-variance portfolio. In equity space, using an S&P 500 investment universe, Jagannathan and Ma (2003) show that the mean variance approach is inferior to both minimum variance and the so called “naive” 1/N heuristic. DeMiguel, Garlappi and Uppal (2009) utilising fourteen portfolio construction models and seven datasets, conclude that “none is consistently better than the 1/N rule in terms of Sharpe ratio, certainty equivalent, or turnover.” DeMiguel et al (2009) conclude that there are “many miles” to go before the gains promised by optimal portfolio-choice can actually be realised out of sample.

Kritzman, Page and Turkington (2010) take issue with these findings. The authors demonstrate that with simple estimates for expected returns and covariance, such as long-term historical averages, mean-variance optimisation generates Sharpe ratios that are higher out of sample than the 1/N heuristic. Indeed, several studies have looked at the performance of mean-variance based on predictive regression models. Solnik (1993), finds that mean-variance conditioned on fundamental variables outperform passive market benchmarks. Klemkosky and Bharati (1995), Fletcher (1997), Marquering and Verbeek (2004), and Herold and Maurer (2006) all arrive at the same conclusion: empirically, mean-variance allowing for forecasting outperforms passive strategies.

The strong performance of the 1/N rule provided the impetus for Tu and Zhou (2011) to investigate combining the weights derived through Bayesian approaches with the 1/N weights. Tu and Zhou (2011) find that combining portfolio rules outperforms both mean-variance and 1/N in the majority of scenarios. In a related piece of work, Paye (2010) argues that combining estimators of portfolio weights is itself equivalent to an asset allocation problem. Paye (2010) finds this form of shrinkage applied to an ultra-conservative strategy and the non-parametric estimator of Brandt (1999), leads to an improvement over the Brandt (1999) approach used in isolation.

Considering the prominent role the mean-variance model plays in the standard MBA curriculum, the empirical literature hardly provides a ringing endorsement of the approach. It is noteworthy that a unifying feature of the negative analyses<sup>6</sup> is the exclusive reliance on sample estimates for forecasting. Further, short histories are almost always employed. Bloomfield, Leftwich and Long (1977) use a 30 month window to estimate expected returns; Jorion (1991), and Jagannathan and Ma (2003) use 60 month windows; Duchin and Levy (2009) use 60 months of data while DeMiguel et al (2009) use 60 and 120 month windows. The arbitrary reliance on such short time periods is surprising given that the predictive power of sample moments in particular is known to be poor. Jobson and Korkie (1980) established the asymptotic properties of the sample mean and covariance matrix,

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<sup>6</sup> Excluding Kritzman, Page and Turkington (2010)

concluding that the estimators of  $\mu$  and  $\sigma$  were inappropriate for making inferences out of sample<sup>7</sup>. In the same vein, Jorion (1985) evaluates the predictive validity of sample return estimates and concludes that:

“estimation risk due to uncertain mean returns has a considerable impact on optimal portfolio selection and that alternative estimators for expected returns should be explored”

The extrapolation of sample moments is particularly puzzling given Markowitz (1952) never advocated the use of sample estimates as inputs to the optimisation algorithm. Given the volume of work that almost universally employs noisy sample estimates it is easy to forget that this practice is a departure from the original theory. Indeed, Markowitz proposed the “E-V” rule where the “E” signifies expected return and not the “M-V” rule where “M” signifies the mean. Markowitz (1952) is explicit on this point from the opening paragraph to the last. He states:

“ to use the M-V rule in the selection of securities, we must have procedures for finding reasonable  $\mu$  and  $\sigma$ . These procedures, I believe should combine statistical techniques and the judgement of practical men.”<sup>8</sup>

There are two strands of research that evaluate the performance of mean-variance in a predictive context using an analytical approach as we do here. On one hand there is Kan and Zhou (2008) and DeMiguel et al (2009) where no allowance is made for forecasting ability and the focus is the role of estimation error. On the other hand Grinold (1989), Grinold and Kahn (2001) and Campbell and Thompson (2008) ignore the role of estimation error and allow for forecasting. We draw together these strands of research in a coherent utility maximisation framework. The aim is to derive analytically the expected utility of mean-variance conditional on forecasting ability while simultaneously accounting for estimation error. This is the key contribution. Intuitively, as forecasting ability increases and estimation error decreases the expected utility of mean-variance increases. The performance of sample-based mean-variance explored in the empirical literature can be seen as a sub-case of this framework, where forecasting ability equals zero. A further contribution of our paper is to show how much skill is required for a given asset universe and estimation window for mean-variance to outperform the much vaunted 1/N strategy. This is achieved by extending the existing case to include a budget constraint. We also provide the probability of mean-variance outperforming 1/N for given skill levels and estimation windows.

Allowing for forecasting is rooted in the voluminous “anomalies” literature. Indeed there is a growing acceptance, even amongst devout proponents of efficient markets<sup>9</sup> that asset returns are not purely random and may contain a predictable component. DeBondt and Thaler (1985) find that the long-term (three to five years) “loser” portfolio outperforms the long-term “winner” portfolio by 8% per year. Banz (1981) shows that the smallest 50 stocks outperformed the largest 50 stocks by 12% per year. The predictive power of scaled price-ratios dates back to Graham (1949). Lakonishok, Schleifer and Vishny (1993) demonstrate that low price-earnings stocks outperform high price-earnings stocks by 10% per year. Jegadeesh and Titman (1993) find that medium term (6-12 months) “winners” outperform medium term “losers” by 10% per year. Asness, Moskowitz and Pedersen (2009) provide compelling evidence of predictability for commodities, government bonds, and stock indices. The

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<sup>7</sup> Merton (1980) also demonstrated that we need a very long data history to estimate expected returns with any precision.

<sup>8</sup> And of course women

<sup>9</sup>Fama and French (1992)

debate has largely shifted from whether or not predictability exists to the explanation behind the predictability. On the one hand behavioural theorists argue that cognitive and behavioural biases together with limits to arbitrage are behind the persistency of the so-called anomalies. On the other hand, efficient market theorists argue that the return premia are the compensation for bearing systematic risk. We do not attempt to resolve this debate here; rather, in line with the current state of thinking, we assume that predictability may exist. The basis for this predictability in our models may be public information sources, such as those mentioned above, or private information.

We proceed as follows. In section II we discuss the approach for estimating expected utility conditional on forecasting ability. In section III, we derive expected utilities under different assumptions assuming no budget constraint; in section IV we incorporate the budget constraint in our calculations. In section V we evaluate our results using empirical return distributions and Monte Carlo calculations. Section VI concludes.

## II. Analytical Framework

The goal of our analytical approach is to provide a closed form evaluation for expected utility conditional on forecasting ability and estimation window length. Armed with this expression we can calculate the loss function for using mean-variance relative to 1/N. We can then calculate the required level of forecasting ability for mean-variance to outperform 1/N on average, and we can also estimate the probability that mean-variance outperforms 1/N for a given forecasting ability. We hope these insights prove valuable to practitioners in determining how to allocate capital and manage risk.

The 1/N benchmark is relevant for three reasons. Firstly, the 1/N rule is not dependent on expected returns or covariances and is therefore devoid of estimation risk. Secondly, humans have an innate behavioural tendency to equal weight the options presented to them. Bernatzi and Thaler (2001) find that many investors equally weight the investment choices they are presented with<sup>10</sup>. It appears few are immune from this bias. Markowitz himself, when probed about how he allocated his retirement investments in his TIAA-CREF account, confessed: “I should have computed the historic covariances of the asset classes and drawn an efficient frontier. Instead...I split my contributions fifty-fifty between bonds and equities”<sup>11</sup>. Finally, the 1/N rule is easy for investors to apply, and thus a viable alternative in practice.

We begin by formalising a metric for forecasting ability,  $d$ , as the correlation between forecast returns  $f_{t,t+1}$ , and realised returns,  $r_{t+1}$ . This of course is the information coefficient. We use  $d$  to denote the information coefficient instead of  $IC$ , as the latter can create confusion with the identity matrix  $I$ , and the covariance vector  $C$  in our derivations. In the context of a regression of future returns on forecasts:

$$d = \frac{Cov(f_{t,t+1}, r_{t+1})}{Std(f_{t,t+1})Std(r_{t+1})} = \frac{\sqrt{SS_{exp}}}{\sqrt{SS_{tot}}} = \sqrt{1 - \frac{SS_{err}}{SS_{tot}}}$$

It can be seen that, as forecasting ability increases, forecasting error,  $SS_{err}$ , decreases. When forecasting error is zero,  $d=1$ , the perfect foresight case.

Empirically, we can measure forecasting ability as the sample value of  $d$ :

$$\hat{d} = \frac{\sqrt{\sum_{i=1}^n (\hat{f}_{t,t+1} - \bar{r})^2}}{\sqrt{\sum_{i=1}^n (r_{t+1} - \bar{r})^2}} = \frac{\sqrt{\overline{SS}_{exp}}}{\sqrt{\overline{SS}_{tot}}} = \sqrt{1 - \frac{\overline{SS}_{err}}{\overline{SS}_{tot}}}$$

In a ground-breaking piece of work, Grinold (1989) related the level of forecasting ability to the expected performance of the mean-variance investor, in what is known as the fundamental law of active management:

$$IR = d\sqrt{N}$$

where  $IR$  is the information ratio, and  $N$  is the number of independent “bets” per year. The fundamental law pertains to active returns and makes no allowance for estimation error.

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<sup>10</sup> See Huberman and Jiang (2006) for a more recent example.

<sup>11</sup> Zweig (1998)

Kan and Zhou (2007) on the other hand derive the expected utility loss of using sample data to estimate expected returns and covariance assuming that investors have no forecasting ability. Whereas Kan and Zhou (2007) take expectations over the true distribution of returns, we will take expectations with respect to the conditional distribution of returns. For the case where  $\mu$  is estimated and  $\Sigma$  is known, Kan and Zhou (2007) show that the expected loss function  $L(x^*, \hat{x})$  of using  $\hat{x}$  instead of the actual optimal weight vector  $x^*$  where  $U(\cdot)$  is the relevant utility function and  $E[\cdot]$  is the expectations operator taken over sample data, is given by:

$$L(x^*, \hat{x}) = U(x^*) - E[U(\hat{x})] = \frac{N}{2\lambda T}$$

Here  $T$  is the number of data points used to estimate the sample mean,  $N$  is the number of assets, and  $\lambda$  is the constant absolute risk aversion (CARA) parameter, assuming an exponential utility function and iid normally distributed data. We use the CARA utility function for several reasons. Numerous studies have found that expected utility obtained through direct optimisation on an ex-post basis is very similar to the expected utility of the CARA approach (Levy and Markowitz, 1979, Pulley, 1981, Kroll, Levy, and Markowitz, 1984, Simaan, 1993, Cremers, Kritman and Page, 2005). Further, the mean-variance approach is widely applied by practitioners (Amenc, Goltz and Lioui, 2011) and is arguably the standard approach for asset allocation (Black and Litterman, 1991). In addition, the CARA function, and its transform, the mean-variance utility function, are highly tractable with a well-established analytical apparatus that has proved useful for the development of equilibrium arguments and statements regarding expected utility. Finally, it is the exponential function that is used by Grinold (1989), Kan and Zhou (2008), and DeMiguel, Garlappi and Uppal (2009) the clearest antecedents of the current work. We therefore employ the CARA utility function as the cornerstone of our work.

The loss function is interpreted in the literature as a function of ex-post expected utility and corresponds to the idea that if we used the particular portfolio weights, we would get a random variable ex-post,  $U(\hat{x})$  and its mean, which would correspond to the mean of some sample, based on an iid distribution and fixed weights would necessarily be less than the optimal non-stochastic quantity  $U(x^*)$ . However, in such an iid world, both values could be interpreted as ex-ante expected utilities as long as the assumed distribution of the investor corresponds to nature's measure. This distinction is important because when we come to compare equal weights with forecasts, we can think of the problem as the ex-ante expected utility of two separate prospects exactly as one would in formulating preferences over lotteries.

DeMiguel, Garlappi and Uppal (2009) is the closest antecedent of the current work; the authors provide the condition for mean-variance to outperform  $1/N$ . When  $\mu$  is unknown and  $\Sigma$  is known DeMiguel et al (2009) show that mean-variance will outperform  $1/N$  when the following inequality, expressed in terms of generalised Sharpe ratios, holds:

$$S^{*2} - S_{1/N}^2 - \frac{N}{T} > 0$$

Several points are noteworthy here: DeMiguel et al (2009) do not allow for forecasting and do not use a budget constraint<sup>12</sup>. Ignoring the role of forecasting departs from the practice of investors and the

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<sup>12</sup>While keeping the maths simple, the results are only relevant for unconstrained agents that do not have a set level of gross exposure, such as a hedge fund manager. In contrast, a pension fund manager typically has a gross exposure of 100%.

original Markowitz (1952) theory. DeMiguel et al (2009) focus on the critical value  $T_{mv}^*$  defined as the sample size that is necessary for mean-variance to outperform 1/N on average:

$$T_{mv}^* \equiv \inf\{T: L_{MV}(w^*, \hat{w}) < L_{1/N}(w^*, w^{ew})\}$$

The use of  $T_{mv}^*$  to facilitate comparison between 1/N and mean-variance is consistent with the use of the sample mean as the input to the optimisation problem. As discussed in section I, the use of sample estimators as optimisation inputs is a particular application of mean-variance analysis, and differs both from the original theory and the behaviour of practitioners. Thus, instead of a critical  $T_{mv}^*$ , which frames the problem in terms of sample estimates, we set up a general framework where forecasting accuracy depends on the information coefficient,  $d$ : the correlation between forecast returns and realised returns<sup>13</sup>. Forecasting ability reflects either the ability to manipulate public information in a superior way, access to privileged information, or both<sup>14</sup>. Thus our analytical results are not contingent on assumptions regarding market efficiency.

The critical level of  $d^*$  is the level of forecasting ability where the expected utility of mean-variance is equal to the expected utility of 1/N. If  $d < d^*$ , then the investor would be better off employing 1/N.

$$d^* \equiv \inf\{d: L_{MV/d}(w, \hat{w}) < L_{1/N}(w^*, w^{ew})\}$$

The 1/N approach that is widely used to benchmark the performance of mean-variance always satisfies the budget constraint. If we compare the classical unconstrained mean-variance portfolio to 1/N we are creating an inconsistency tantamount to comparing the performance characteristics of a hedge fund to a long-only pension fund.

This incongruity has been ignored in the literature to date. DeMiguel et al (2009) for example compare the expected utility of 1/N with the classical unconstrained expected utility. This portfolio rule tends to generate heavily leveraged portfolios with non-unit net exposure rendering any comparison with the 1/N rule questionable. We incorporate a budget constraint directly to solve this problem.

Our analytical approach consists of three stages. Firstly we obtain solutions for expected utility conditional on forecasting and estimation error. We then solve for  $d_{mv}^*$  by setting the expected utility of the mean-variance investor to the expected utility of the 1/N investor, or equivalently the loss function to zero. Finally, we provide the probability of mean-variance outperforming 1/N conditional on forecasting ability and estimation error. This is achieved analytically or through simulation where appropriate.

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<sup>13</sup>Our approach is in the spirit of Campbell and Thompson (2008) who derive the expected increase in return,  $k$ , for the mean-variance investor for a given forecasting ability.

$$k = \left( \frac{d^2}{1 - d^2} \right) \left( \frac{1 + S^2}{S^2} \right)$$

where  $d$  is the correlation between forecasted and realised returns and  $S$  is the unconditional Sharpe ratio of the risky asset.

<sup>14</sup> We do not exclude the case where the investor has savant capabilities or mystical powers of prediction!

In this paper, we make several contributions to the literature. Firstly we unify the effect of forecasting and estimation error in a coherent utility maximisation framework. Secondly, we provide the critical amount of skill and estimation data required by a mean-variance investor to outperform 1/N ex-ante by incorporating a budget constraint. As we show, the Kan and Zhou (2007) conclusion that increasing the number of assets leads to a deterioration in performance does not typically hold when we allow for forecasting. In this context we derive how much skill is required for the performance of mean-variance to be increasing in the number of assets,  $N$ . To achieve this we develop a novel mechanism for evaluating expected utility conditional on forecasting. We now discuss the derivation of our propositions.

### III. Derivation of Propositions: The Unconstrained Case

#### A. Introducing Forecasting Ability

We assume we have forecasts  $a$  ( $M \times 1$ ), of returns  $r_{t+1}$  ( $N \times 1$ ), a mean vector,  $\mu$  ( $N \times 1$ ), a covariance matrix,  $\Sigma$  ( $N \times N$ ) and a covariance vector,  $C$  ( $N \times 1$ ) between forecast and realised returns. The joint density function of  $a$  and  $r_{t+1}$  is:

$$\begin{pmatrix} r_{t+1} \\ a \end{pmatrix} \sim N \left[ \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \begin{bmatrix} \Sigma, C \\ C', \Omega \end{bmatrix} \right]$$

In DeMiguel, Garlappi, and Uppal (2009),  $\mu = \bar{r}$ ,  $C = 0$  and  $\Omega = \frac{1}{T}\Sigma$ , under the assumption of i.i.d. returns. We instead allow for forecasts  $a$  with  $C \neq 0$ . In common with other work in this area (Grinold, 1989), we treat forecasts  $a$  as i.i.d. normally distributed variables with a mean of zero and a standard-deviation of one.

$$\begin{pmatrix} r_{t+1} \\ a \end{pmatrix} \sim N \left[ \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \begin{bmatrix} \Sigma, C \\ C', I_N \end{bmatrix} \right]$$

The conditional pdf of  $r_{t+1}|a$  under these assumptions is:

$$N(\mu + Ca, \Sigma - CC') \tag{1}$$

As in Markowitz (1952) we assume the practitioner's utility function is not quadratic but exponential with constant relative risk aversion parameter  $\lambda$ . As we show in appendix E, if portfolio returns are normal, then after several manipulations, transformed expected utility (the certainty equivalent) can be written as a simple function quadratic in asset weights,  $w$ <sup>15</sup>:

$$E[U] = \omega' \mu - \frac{\lambda}{2} \omega' \Sigma \omega \tag{2}$$

where  $\lambda$  is the coefficient of absolute risk aversion.

The usual calculation of expected loss is to replace unknown parameters by estimators of the portfolio weights and compute expected utility, taking expectations over the sample distribution. If we ignore the information in the sample data, and equally weight the assets we have, where  $i$  is a vector of ones:

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<sup>15</sup> For the interested reader, we produce the derivation in appendix F

$$\omega = \frac{1}{N} \iota$$

Expected utility for this case is:

$$V_{1/N} = E[U_{1/N}] = \frac{\iota' \mu}{N} - \frac{\lambda}{2N^2} \iota' \Sigma \iota \quad (3)$$

which is non-stochastic.

We now derive the expected utility of the mean-variance investors by treating  $a$  as a forecast of  $r_{t+1}$ . An alternative approach, akin to Kan and Zhou (2007), could be developed which treats  $a$  as an estimate of  $\mu$ <sup>16</sup>.

### B. *Expected Utility Conditional on Forecasting Ability*

We consider six cases. We build these cases up in order of complexity, beginning with a single forecasting variable, no estimation error, and no budget constraint, and progressing to multiple forecasting variables, with estimation error and a budget constraint. Estimation error arises because agents do not know the true inputs of the utility function. We exclusively focus on the estimation error in the means, ignoring the estimation error in the covariance structure because errors in the former tend to be much more influential than errors in the latter. Chopra and Ziemba (1993) show that errors in expected returns are over ten times more important than errors in variances and over twenty times more important than errors in covariance. The six cases we consider are as follows:

Single forecasting variable

1. No budget constraint, no estimation error

Multiple forecasting variables

2. Variable forecasting ability, no estimation error, no budget constraint
3. Constant forecasting ability, no estimation error, no budget constraint
4. Constant forecasting ability, estimation error, no budget constraint
5. Constant forecasting ability, no estimation error, budget constraint
6. Constant forecasting ability, estimation error, budget constraint

The base case is well-known and we report the results; Returning to our utility function given by (2), the first order conditions can be solved to give:

$$w = \frac{\Sigma^{-1} \mu}{\lambda} \quad (4)$$

Substituting (4) into (2) we see that expected utility is:

$$V = E[U] = \frac{\alpha}{2\lambda} \quad (5)$$

where  $\alpha = \mu' \Sigma^{-1} \mu$

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<sup>16</sup> See appendix E

### C. *Expected Utility with a Single Forecasting Variable*

We begin with the case of a single forecasting variable. Of course this has many applications, for example the single-factor model of Sharpe (1963). In line with the CAPM, each asset has a sensitivity to the single factor,  $a$ , which equates to our normalised forecast over the next period. We define conditional moments as:

$$\mu^* = \mu + Ca \quad (6)$$

$$\Sigma^* = (\Sigma - CC') \quad (7)$$

$$C = \Sigma^{\frac{1}{2}}D \quad (8)$$

where  $\mu$  ( $N \times 1$ ) is the vector of population means,  $\Sigma$  ( $N \times N$ ) is the population covariance matrix,  $C$  ( $N \times 1$ ) is the covariance between forecast returns and realised returns,  $D$  ( $N \times M$ ) is the matrix of information coefficients, and  $a$  ( $M \times 1$ ) is a vector of  $N(0,1)$  variables. Whilst in this first case we only have one forecast variable per period ( $M=1$ ), the vector  $D$  leads to a different forecast for each asset.

By substituting the conditional expectations (6) and (8) into (5) we solve for the expected utility:

**Proposition 1:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of a Single Forecasting Variable, a ( $M=1$ ), with forecasting ability given by the vector  $D$ , no Estimation Error and in the absence of a Budget Constraint is given by:*

$$E[U] = \frac{1}{2\lambda} \left( \alpha + \frac{D'D + (\mu'\Sigma^{-\frac{1}{2}}D)^2}{1 - D'D} \right) \quad (9)$$

Proof: See Appendix A

The conditional expected utility with a single forecasting variable  $a(M=1)$  ignoring estimation error and without a budget constraint is given by:

$$E[U|a] = \frac{1}{2\lambda} \left( a^2 \left( D'D + \frac{(D'D)^2}{1 - D'D} \right) + a \left( 2\mu'\Sigma^{-\frac{1}{2}}D + \frac{2D'DD'\Sigma^{-\frac{1}{2}}\mu}{1 - D'D} \right) + \left( \alpha + \frac{(\mu'\Sigma^{-\frac{1}{2}}D)^2}{1 - D'D} \right) \right) \quad (10)$$

Proof: See Appendix A

We can now derive the unconditional probability of the utility, given by (10), of mean-variance exceeding the utility of the 1/N rule. From Proposition 1, we see that expected utility is quadratic in  $a$ .

Hence we have:

$$E[U|a] = \phi_2 a^2 + \phi_1 a + \phi_0$$

where

$$\phi_2 = \frac{D'D}{1 - D'D}$$

This is strictly positive as long as  $D'D$  is bounded by 1.

$$\phi_1 = \frac{2\mu'\Sigma^{-\frac{1}{2}}d}{1 - D'D}$$

$$\phi_0 = \alpha + \frac{\left(\mu'\Sigma^{-\frac{1}{2}}D\right)^2}{1 - D'D}$$

We are interested in the unconditional expected utility of the optimised strategy exceeding the 1/N case. Thus, the following analysis is valid, where the probability is taken over the  $a$  distribution.

$$\begin{aligned} \text{Prob}(E[U|a] < E[V_{1/N}]) &= \text{Prob}(\phi_2 a^2 + \phi_1 a + \phi_0 < V_{1/N}) \\ &= \text{Prob}\left(a^2 + \frac{\phi_1}{\phi_2} a + \frac{\phi_0}{\phi_2} - \frac{V_{1/N}}{\phi_2} < 0\right) \end{aligned}$$

we see that,

**Corollary 1:** *The Probability of Mean-Variance Outperforming the 1/N rule under the assumptions of a Single Forecasting Variable, a ( $M=1$ ), no Estimation Error and in the absence of a Budget Constraint is given by:*

$$\text{Prob}(E[U|a] < E[V_{1/N}]) = \Phi(\lambda_1) - \Phi(\lambda_2)$$

supposing  $\lambda_1 > \lambda_2$  without loss of generality, where

$$\lambda_i = \left(-\phi_1 \pm \sqrt{\phi_1^2 - 4(\phi_0 - V_{1/N})\phi_2}\right) \frac{1}{2\phi_2}$$

where  $\Phi(\cdot)$  is the standardised normal distribution function. Solving for  $\lambda_i$  yields between zero and two real-roots enabling us to calculate the probability of 1/N outperforming mean-variance. If  $\phi_1^2 > 4(\phi_0 - V_{1/N})\phi_2$  then we have two real roots  $\lambda_1$  and  $\lambda_2$  and there is a non-zero probability of 1/N outperforming mean-variance. If  $\phi_1^2 \leq 4(\phi_0 - V_{1/N})\phi_2$  then there is a zero probability of mean-variance outperforming mean-variance.

### C. Expected Utility of with a Variable Forecasting Ability

We now consider multiple forecasting variables, mimicking the behaviour of investors. We can draw a parallel here with the arbitrage pricing theory of Ross (1976) where there are multiple sources of systematic risk and return premia.

In all of the cases that follow:

$$\mu^* = \mu + Ca$$

$$C = \Sigma^{\frac{1}{2}}D$$

where  $\mu$  is  $N \times 1$ ,  $C$  is  $N \times N$ ,  $a$  is  $N \times 1$ , and  $D$  is  $N \times N$  and is diagonal. The significance of  $D$  being diagonal is that forecast  $a_{i,t}$  is correlated only with asset return  $r_{i,t+1}$ , and not with asset return  $r_{j,t+1}$  where  $j \neq i$ .

**Proposition 2:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a diagonal forecasting ability matrix,  $D$ , no Estimation Error and in the absence of a Budget Constraint is approximated by:*<sup>17</sup>

$$E[U] \cong \frac{\alpha + \text{tr}(D^2) + \text{tr}(D^4)}{2\lambda} + O(D^6) \quad (11)$$

Proof: See Appendix A

Again, we see that as the Sharpe ratio of the investment universe increases, expected utility increases. Expected utility is also positively related to forecasting ability as anticipated. The expected utility always exceeds the expected utility in the absence of forecasting.

When we have more than one information variable per period, the probability of mean-variance outperforming  $1/N$  cannot be expressed in terms of simple functions. In section V, we instead employ a simulation approach.

#### D. Expected Utility with Constant Forecasting Ability

We now turn to the special case where we have a constant expected information coefficient,  $d$ , for all assets. Setting the forecasting ability to a constant across all assets is consistent with the literature (Grinold, 1989 and Williams and Satchell, 2011), and practitioners' beliefs.<sup>18</sup>

We assume that the information sources are orthogonal as follows:

$$D = \begin{bmatrix} d_1 & 0 & \dots \\ 0 & \ddots & \dots \\ \vdots & \dots & d_n \end{bmatrix}$$

where we set  $d_1 = d_2 \dots = d_n = d$ .

This can be summarised as  $D = dI_N$

**Proposition 3:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , no Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance Matrix is given by:*

$$E[U] = \frac{(\alpha + d^2N)}{2\lambda(1 - d^2)} \quad (12)$$

Proof: See Appendix A

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<sup>17</sup>To an order of approximation  $O(d^4)$

<sup>18</sup>Quantitative investors for example do not tend to have a priori heterogeneous views on the predictive validity of a factor for different securities within the same asset class.

The numerator is quadratic in forecasting ability indicating that as forecasting ability increases, expected utility increases at an increasing rate. Interestingly, expected utility is positively related to the number of assets  $N$ . This makes sense in that as the investor's opportunity set expands she is able to capture incremental uncorrelated excess return streams. This is reminiscent of the fundamental law of active management (Grinold, 1989).

Grinold's (1989) fundamental law refers to the information ratio - we focus on utility. Information ratios can give misleading results under certain circumstances. For example an investment may have a very high information ratio, comprised of a low return and a very low risk level. If we cannot apply leverage, the benefit to the investor will be limited, despite the ostensibly attractive information ratio. The investment adage "you can't eat information ratios" comes to mind. Utility functions do not suffer from this drawback. A further distinction between Grinold (1989) and the current work is that we assess the welfare of the investor in total risk space, taking into account "beta" and "alpha" returns whereas Grinold (1989) focuses on active space, relative to a benchmark. In the following sections we further extend the relation to include a budget constraint and estimation error.

We can now solve for the skill level  $d$ , where the expected utility of the mean-variance investor equals the expected utility of the  $1/N$  investor:

$$E[U_{1/N}] = \frac{\mu}{N} - \frac{\lambda}{2N^2} \mu' \Sigma \mu$$

The loss function for using mean-variance in preference to  $1/N$  is therefore, as before, denoting  $V_{1/N} = E(U_{1/N})$

$$E[L(w_{1/N}, w_{mv} | \mu, \Sigma)] = V_{1/N} - \frac{(\alpha + d^2 N)}{2\lambda(1 - d^2)}$$

Setting the expected loss function to zero and solving for  $d$ ,

$$V_{1/N} = \frac{(\alpha + d^2 N)}{2\lambda(1 - d^2)}$$

$$2\lambda V_{1/N} - \alpha = d^2(N + 2\lambda V_{1/N})$$

yields,

**Corollary 2:** *Critical Forecasting Ability,  $d^*$ , required for the Mean-Variance and  $1/N$  investor to have the same Expected Utility under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, no Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance is given by:*

$$d^* = \sqrt{\frac{2\lambda V_{1/N} - \alpha}{2\lambda V_{1/N} + N}} \quad (13)$$

If  $V_{1/N} < \frac{\alpha}{2\lambda}$ , then mean-variance with zero forecasting ability outperforms  $1/N$  on average

If  $V_{1/N} > \frac{\alpha}{2\lambda}$ , then a positive level of skill is required for mean-variance to outperform  $1/N$  on average

We also note that as  $N$  increases, the required level of forecasting ability  $d^*$  to outperform  $1/N$  decreases while the converse is true for the squared Sharpe ratio,  $\alpha$ . In addition, because the numerator is strictly greater than the denominator,  $d^*$  is strictly increasing in  $V_{1/N}$ .

#### *E Expected Utility with Constant Forecasting Ability and Estimation Error*

Thus far, we have assumed that we know  $\mu$ , and can forecast with skill  $d$ . We now introduce estimation error in  $\mu$ . In practice investors do not know the true mean of the distribution and it must be estimated. It is simplest to return to equation (4) replacing the population mean,  $\mu$ , with the sample mean  $\bar{x}$ <sup>19</sup> where  $\bar{x}$  is  $N(\mu, \Sigma/T)$ .

Assuming constant forecasting ability as before, the multivariate pdf in this case is:

$$\begin{pmatrix} r_{t+1} \\ a \\ \bar{x} \end{pmatrix} \sim N \left[ \begin{pmatrix} \mu \\ 0 \\ \mu \end{pmatrix}, \begin{bmatrix} \Sigma & C & 0 \\ C & I_N & 0 \\ 0 & 0 & \Sigma/T \end{bmatrix} \right]$$

The conditional moments of  $r_{t+1}$  given  $a$  and  $\bar{x}$  are now:

$$\mu^* = \bar{x} + \Sigma^{\frac{1}{2}} da \quad (14)$$

$$\Sigma^* = \Sigma(1 - d^2) \quad (15)$$

However, the optimal weights are stochastic functions of  $a$  and  $\bar{x}$  since the investor does not know the true parameter :

$$\hat{w} = \frac{\Sigma^{-1}(\bar{x} + d\Sigma^{\frac{1}{2}}a)}{\lambda(1 - d^2)} \quad (16)$$

Substituting (16) into (2) gives:

**Proposition 4:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables,  $a$  ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance Matrix is given by:*

$$E[U] = \frac{\alpha - \frac{N}{T} + Nd^2}{2\lambda(1 - d^2)} \quad (17)$$

Proof: See Appendix A

Again this is logical. The expected utility is positively related to the underlying squared Sharpe ratio,  $\alpha$ , and the skill level,  $d$ . The expected utility is also positively related to the length of the estimation window,  $T$ : expected utility increases as estimation error decreases. The relationship between  $N$  and expected utility however is more complex.

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<sup>19</sup> Alternative methods are possible here.

**Corollary 3:** *The relationship between expected utility and the number of assets,  $N$ , under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , no Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance Matrix is given by:*

If

$$|d| > \sqrt{\frac{1}{T}} \text{ then } \frac{\partial E(U)}{\partial N} > 0$$

If

$$|d| < \sqrt{\frac{1}{T}} \text{ then } \frac{\partial E(U)}{\partial N} < 0$$

Note that this differs from the fundamental law of active management of Grinold (1989),  $IR = d\sqrt{N}$  where the ex ante information ratio ( $IR$ ), a transformation of expected utility, is a strictly positive function of the number of assets,  $N$  and the work of DeMiguel et al (2009) where expected utility is a strictly negative function of the number of assets.

The critical level of skill,  $d^*$  for mean-variance to outperform 1/N on average is given below:

**Corollary 4:** *Critical Forecasting Ability,  $d^*$ , required for the Mean-Variance and 1/N investor to have the same Expected Utility under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, no Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance Matrix is given by:*

$$d^* = \sqrt{\frac{2\lambda V_{1/N} - \alpha + \frac{N}{T}}{2\lambda V_{1/N} + N}} \quad (18)$$

If  $V_{1/N} < \frac{\alpha - \frac{N}{T}}{2\lambda}$ , then mean-variance with zero forecasting ability outperforms 1/N on average

If  $V_{1/N} > \frac{\alpha - \frac{N}{T}}{2\lambda}$ , then a positive level of skill is required for mean-variance to outperform 1/N on average

Again this rings true. If the unconstrained squared Sharpe ratio,  $\alpha$  is larger, we require less skill to outperform 1/N. As  $T$ , the estimation window increases, critical  $d^*$  decreases due to a reduction in estimation error, and converges to the result given by (13) from above and is therefore unambiguously larger than (13). Conversely, if the expected utility of the 1/N portfolio is large, ceteris paribus, a higher skill level is required for mean-variance to outperform 1/N.

#### IV. Further Propositions: The Budget Constrained Case

In section III, we did not consider the impact of the budget constraint in any of our propositions. In practice the vast majority of investors are subject to such a constraint. Moreover, the 1/N approach that we are using as our benchmark always satisfies the budget constraint. We therefore incorporate a budget constraint directly into the optimisation problem.

Consider the constant absolute risk aversion utility function, subject to a budget constraint:

$$U = \omega' \mu - \frac{\lambda}{2} \omega' \Sigma \omega - \theta(\omega' \mathbf{1} - 1) \quad (19)$$

where  $\theta$  is the Lagrange multiplier and the other notation is as before.

As we derive in appendix B, the optimal weight in the presence of a budget constraint is:

**Proposition 5:** *The Optimal Mean-Variance Weights in the Presence of a Budget Constraint with known parameters is given by:*

$$\omega = \frac{1}{\lambda} \Sigma^{-1} \mu - \frac{(\beta - \lambda)}{\lambda \gamma} \Sigma^{-1} \mathbf{1}$$

where  $\alpha = \mu' \Sigma^{-1} \mu$ ,  $\beta = \mu' \Sigma^{-1} \mathbf{1}$ ,  $\gamma = \mathbf{1}' \Sigma^{-1} \mathbf{1}$ .

The expected utility associated in this case is given by

$$V_c = E(U) = \frac{\alpha \gamma - (\beta - \lambda)^2}{2 \lambda \gamma}$$

The loss in expected utility is given by

$$\begin{aligned} V - V_c &= \frac{\alpha \gamma - (\alpha \gamma - \beta^2) - \lambda(2\beta - \lambda)}{2 \lambda \gamma} \\ &= \frac{\beta^2 - 2\beta \lambda + \lambda^2}{2 \lambda \gamma} \\ &= \frac{(\beta - \lambda)^2}{2 \lambda \gamma} = \frac{\gamma \theta^2}{2 \lambda} \end{aligned}$$

So that if  $\theta = 0$ , ( $\beta = \lambda$ ), there is no loss in constraining the portfolio, otherwise the loss is unambiguously positive. The condition ( $\beta = \lambda$ ), can be re-expressed as  $\frac{\mathbf{1}' \Sigma^{-1} \mu}{\lambda} = 1$  which is simply the requirement that the unconstrained weights given by equation (4) add up to 1.

#### A Constant Forecasting Ability with a Budget Constraint

We could derive the expected utility with a budget constraint and a constant skill level across all assets ignoring estimation error. This is analogous with proposition 3, section III. We substitute the conditional moments (6) and (7) into Proposition 5 to get the optimal portfolio conditional upon a:

$$w = \frac{1}{\lambda(1 - d^2)} \left( \Sigma^{-1} \left( \mu + \Sigma^{-\frac{1}{2}} d a \right) - \frac{\left( \beta + d a' \Sigma^{-\frac{1}{2}} \mathbf{1} - \lambda(1 - d^2) \right)}{\gamma} \Sigma^{-1} \mathbf{1} \right) \quad (20)$$

We then substitute this relation into the transformed constant absolute risk aversion utility function (2) to yield the expected utility of the budget constrained mean-variance investor conditional on forecasting ability. We use the conditional covariance for the budget constrained derivations because we can calculate it but other authors use the unconditional covariance matrix because it provides

succinct, results. Grinold and Kahn (2001) ignore the effect of the reduction in variance due to the knowledge embedded in  $a$ . Specifically, the authors ignore terms higher than order  $d^2$  leading to an approximation which is similar to using the unconditional covariance. We note in passing, that Ding (2010) points out that the unconditional covariance assumption used in the literature (Grinold, 1989, Grinold and Kahn, 2001) has negligible practical impact. For completeness, we also show give the expression for expected utility using the unconditional covariance<sup>20</sup>.

*B. Expected Utility with Constant Forecasting Ability, Estimation Error, and a Budget Constraint*

To determine the joint impact of estimation error and forecasting on expected utility, we substitute optimal conditional weight relation (20) into the utility function (2). This is analogous to proposition 4, section III.

**Unified Fundamental Law of Asset Management**

We refer to the following proposition as the Unified Fundamental Law of Asset Management. The proposition is unifying in two ways. Firstly, it unifies the impact of estimation error and forecasting ability, two distinct areas of the literature. Secondly it incorporates “alpha” and “beta” returns, whereas the prior analytical literature focussed on one or the other, for example Grinold (1989) that focussed on active management or “alpha”. The theorem also incorporates the effect of a budget constraint facilitating like-for-like comparisons with the 1/N rule.

**Proposition 6:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , Estimation Error and in the presence of a Budget Constraint using the Conditional Covariance Matrix is given by*

$$E[U] = \frac{\alpha + (N - 1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta - \lambda(1 - d^2))^2}{\gamma}}{2\lambda(1 - d^2)}$$

If we approximate this to  $O(d^2)$ , we get

$$E[U] = \frac{(\alpha - \frac{N-1}{T})\gamma - (\beta - \lambda)^2 + d^2(\gamma(N - 1) + (\alpha\gamma - \beta^2) + \lambda^2)}{2\lambda\gamma}$$

**Corollary 6:** *The Critical Forecasting Ability,  $d^*$ , required for the Mean-Variance and 1/N investor to have the same Expected Utility under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the presence of a Budget Constraint using the Conditional Covariance Matrix and approximated to  $O(d^2)$ ; is given by:*

$$d^* = \sqrt{\frac{2\lambda V_{\frac{1}{N}} - \alpha + \frac{N-1}{T} + \frac{(\beta-\lambda)^2}{\gamma}}{N - 1 + \frac{(\alpha\gamma - \beta^2) + \lambda^2}{\gamma}}} \quad (21)$$

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<sup>20</sup> See proposition 7

If  $V_{1/N} < \frac{\alpha + \frac{N-1}{T} + \frac{(\beta-\lambda)^2}{\gamma}}{2\lambda}$ , then mean-variance with zero forecasting ability outperforms 1/N on average

If  $V_{1/N} > \frac{\alpha + \frac{N-1}{T} + \frac{(\beta-\lambda)^2}{\gamma}}{2\lambda}$ , then a positive level of skill is required for mean-variance to outperform 1/N on average

We provide a proof in appendix C of the budget constrained case with forecasting, and without estimation error; however, it is clear now that it is more elegant to take the more general case that includes estimation error given by proposition 6, and let  $T$  become infinitely large. In doing so, we recover the case without estimation error.

**Corollary 7:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , no Estimation Error and in the presence of a Budget Constraint using the Conditional Covariance Matrix is given by:*

$$E[U] = \frac{\alpha + (N-1)d^2 - \frac{(\beta-\lambda(1-d^2))^2}{\gamma}}{2\lambda(1-d^2)}$$

If we approximate this to  $O(d^2)$ , we get

$$E[U] = \frac{\alpha + d^2 \left( (N-1) + \frac{(\alpha\gamma - \beta^2) + \lambda^2}{\gamma} \right) - \frac{(\beta-\lambda)^2}{\gamma}}{2\lambda}$$

As discussed in the previous section, it is typical in the literature to employ an approximation that is akin to the unconditional covariance. Employing the unconditional covariance simplifies the relevant expressions without any meaningful loss of accuracy. For example, at the limit, a highly skilled investor with an information coefficient of 0.1 only needs to scale down the unconditional covariance matrix by 0.5%<sup>21</sup>. Further, an argument can be made that using the unconditional covariance is consistent with the behaviour of practitioners; in reality, investors do not adjust their risk forecasts based on their ex ante forecasting ability.

**Proposition 7:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , Estimation Error and in the presence of a Budget Constraint using the Unconditional Covariance Matrix is given by*

$$E[U] = \frac{\alpha + (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta-\lambda)^2}{\gamma}}{2\lambda}$$

The critical forecasting ability,  $d^*$ , is then more straightforward.

**Corollary 8:** *The Critical Forecasting Ability,  $d^*$ , required for the Mean-Variance and 1/N investor to have the same Expected Utility under the assumptions of Multiple*

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<sup>21</sup> 0.5% =  $1 - \sqrt{1 - d^2}$

Forecasting Variables,  $a$  ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the presence of a Budget Constraint using the Unconditional Covariance Matrix and approximated to  $O(d^2)$ , is given by:

$$d^* = \sqrt{\frac{2\lambda V_{\frac{1}{N}} - \alpha + \frac{N-1}{T} + \frac{(\beta-\lambda)^2}{\gamma}}{N-1}} \quad (22)$$

It is useful to compare the following four expected utility functions<sup>22</sup>.

Eq:	Budget Constraint	Estimation Error	Forecasting Ability	Expected Utility Function: $E[U]$	Critical Forecasting Ability $D^*$
(1)	No	No	No	$\frac{\alpha}{2\lambda}$	N/A
(2)	No	No	Yes	$\frac{\alpha + Nd^2}{2\lambda(1-d^2)}$	$\sqrt{\frac{2\lambda V_{1/N} - \alpha}{2\lambda V_{1/N} + N}}$
(3)	Yes	No	Yes	$\frac{\alpha + (N-1)d^2 - \frac{(\beta-\lambda(1-d^2))^2}{\gamma}}{2\lambda(1-d^2)}$	$\sqrt{\frac{2\lambda V_{\frac{1}{N}} - \alpha + \frac{(\beta-\lambda)^2}{\gamma}}{N-1 + \frac{(\alpha\gamma-\beta^2)+\lambda^2}{\gamma}}}$
(4)	Yes	Yes	Yes	$\frac{\alpha + (N-1)\left(d^2 - \frac{1}{T}\right) - \frac{(\beta-\lambda(1-d^2))^2}{\gamma}}{2\lambda(1-d^2)}$	$\sqrt{\frac{2\lambda V_{\frac{1}{N}} - \alpha + \frac{N-1}{T} + \frac{(\beta-\lambda)^2}{\gamma}}{N-1 + \frac{(\alpha\gamma-\beta^2)+\lambda^2}{\gamma}}}$

From this table, we can see some clear conclusions. In the absence of a budget constraint and no estimation error, the addition of forecasting ability increases expected utility by

$$\frac{(N + \alpha)d^2}{2\lambda(1 - d^2)}$$

In the presence of a budget constraint, there is no tidy expression for the increase in utility due to forecasting ability. The addition of the budget constraint for an investor with forecasting ability leads to a loss in expected utility of

$$\frac{\gamma d^2 + (\beta - \lambda(1 - d^2))^2}{2\lambda\gamma(1 - d^2)}$$

Finally, for the budget constrained case with forecasting ability, the impact of estimation error in the historical mean leads to a reduction in expected utility of

$$\frac{N - 1}{2T\lambda(1 - d^2)}$$

<sup>22</sup> In the table we show the unconditional covariance versions of expected utility and critical forecasting ability for the budget constrained cases (3) and (4).

C. *Expected Utility with Constant Forecasting Ability, Estimation Error, and a Budget Constraint assuming Constant Correlation, and Constant Volatility across Assets*

We now provide the expression for expected utility for a simplified case where the volatility and pair-wise correlation is the same across all assets. The goal is to identify the key drivers of expected utility. As we show in appendix D, the expected utility in this case is given by

**Corollary 9:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , Estimation Error and in the presence of a Budget Constraint with a Constant Pair-wise Correlation,  $\rho$ , and a Constant Stock Volatility,  $\sigma$ , across all Assets using the Unconditional Covariance Matrix is given by*

$$E[U] = \frac{(N - 1) \left( \frac{\sigma_u^2}{\sigma^2} (1 - \rho) - \frac{1}{T} + IC^2 \right) + O(1)}{2\lambda}$$

Thus, as asset volatility,  $\sigma$ , or asset correlation,  $\rho$ , increase, expected utility increases. Conversely, as cross-sectional dispersion,  $\sigma_u$ , increases, expected utility also increases. It is comforting to see that the theory describes what rings true in investment practice. Furthermore, this result reveals why active investors using optimisers like cross-sectional dispersion; it increases their expected utility.

## V. Empirical Application

We now investigate the expected utility of the mean-variance investor for the budget constrained case, with forecasting ability and estimation error using the Unified Fundamental Law of Asset Management. This case aligns well with the problem of the institutional investment manager. We employ two data sets to replicate the asset allocation and the security selection problem. These problems are the two most common applications of mean-variance analysis in Asset Management.

### A *Expected Utility and Asset Allocation*

We use the six key assets commonly employed by sophisticated institutional investors: domestic equities, foreign equities, government bonds, corporate bonds, real-estate and commodities for the period 1975-2012. The summary statistics and correlation matrix are shown in tables 1 and 2.

**Table 1**  
**Asset Class Summary Statistics**

Table 1 provides the summary statistics for the S&P 500, MSCI EAFE, Barclays US Aggregate Bond, FTSE-NAREIT, Barclays US Corporate Bond and Goldman Sachs Commodities Index for the period 12/1975-12/2011 (N=6, T=433). The average annual return is calculated arithmetically. The annual standard deviation is calculated by multiplying the monthly standard deviation by  $\sqrt{12}$ . The Sharpe ratio is calculated as the annual excess return over three-month treasury bills divided by the standard deviation. The certainty equivalent is calculated using the quadratic utility function with a risk aversion of 0.025 as calculated in section II. The maximum drawdown is calculated as the maximum peak-to-trough return. Autocorrelation is defined as the correlation of returns in period  $t$  to  $t-1$ .

	S&P 500	MSCI EAFE	US Aggregate Government Bonds	FTSE/ NAREIT	US Corporate Bonds	GSCI
Annual Return	10.64	9.26	9.79	10.15	9.09	8.82
Standard Deviation	15.82	18.26	9.93	17.76	8.50	20.68
Sharpe Ratio	0.28	0.17	0.36	0.22	0.34	0.13
Certainty Equivalent	0.34	0.05	0.60	0.16	0.60	0.19
Positive Months	200	188	200	193	202	179
Negative Months	113	125	113	120	110	134
% Positive Months	64%	60%	64%	62%	65%	57%
Maximum	13.47	15.61	11.77	28.07	14.12	22.94
Minimum	- 21.54	- 20.17	- 9.17	- 30.54	- 11.21	- 28.20
Maximum Drawdown	- 50.92	- 56.40	- 13.82	- 68.18	- 22.54	- 67.65
Skewness	- 0.78	- 0.35	0.20	- 0.91	- 0.13	0.19
Kurtosis	5.22	3.78	4.44	11.39	7.93	5.16
Autocorrelation	0.06	0.12	0.06	0.10	0.11	0.17

The S&P 500 has the highest return over the period followed by REITs. Government bonds however have the highest Sharpe ratio and lowest maximum drawdown. The S&P 500, MSCI EAFE, FTSE/NAREIT indices are all highly correlated. Government bonds have low correlations with the other asset classes except for corporate bonds. Commodities have a low correlation with the major asset classes.

**Table 2**  
**Asset Class Correlations**

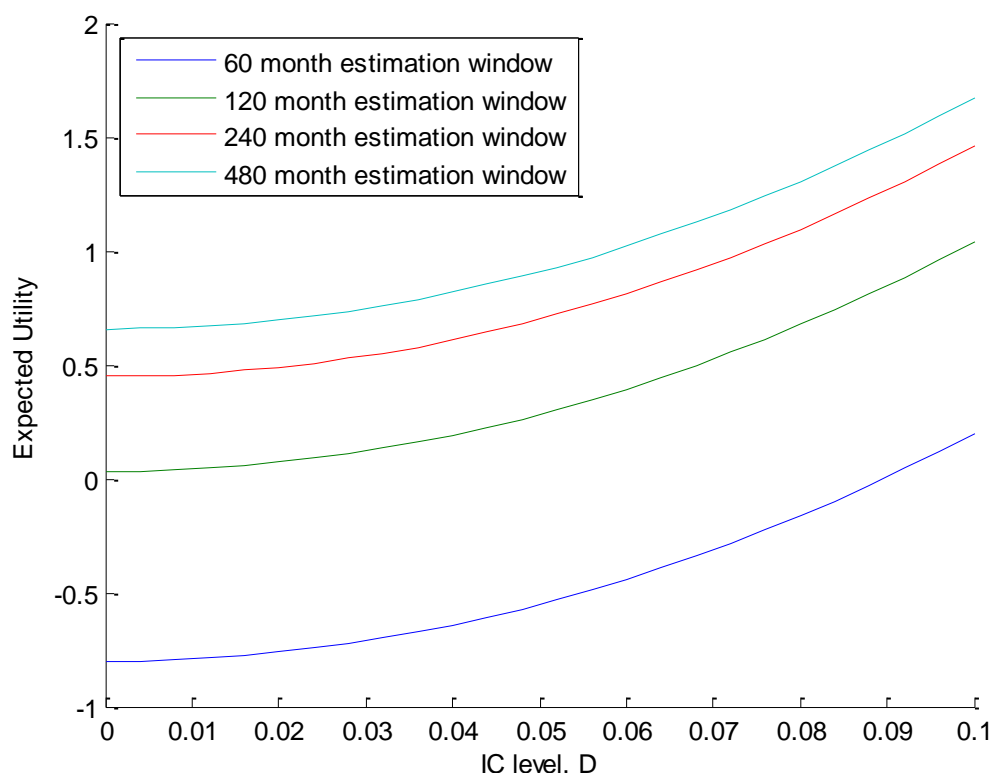
Table 2 provides the correlation matrix for the S&P 500, MSCI EAFE, Barclays US Aggregate Bond, FTSE/NAREIT, Barclays US Corporate Bond and Goldman Sachs Commodities Index for the period 12/1975-12/2011. (N=6, T=433).

	S&P 500	MSCI EAFE	Govt. Bonds	FTSE/ NAREIT	Corp. Bonds	Commodities
S&P 500	1					
MSCI EAFE	0.63	1				
Govt Bonds	0.12	0.04	1			
FTSE/ NAREIT	0.60	0.46	0.11	1		
Corp. Bonds	0.32	0.25	0.84	0.36	1	
Commodities	0.16	0.25	-	0.15	0.02	1

Figure 1 shows the expected utility of mean-variance using the unified fundamental law reproduced below and derived in proposition six for five estimation windows ranging from  $T=60$  to 300 months with  $N=6$ . We employ the estimated means and covariance structure shown in tables 1 and 2. We use a risk aversion level of 0.025 as derived in appendix E.

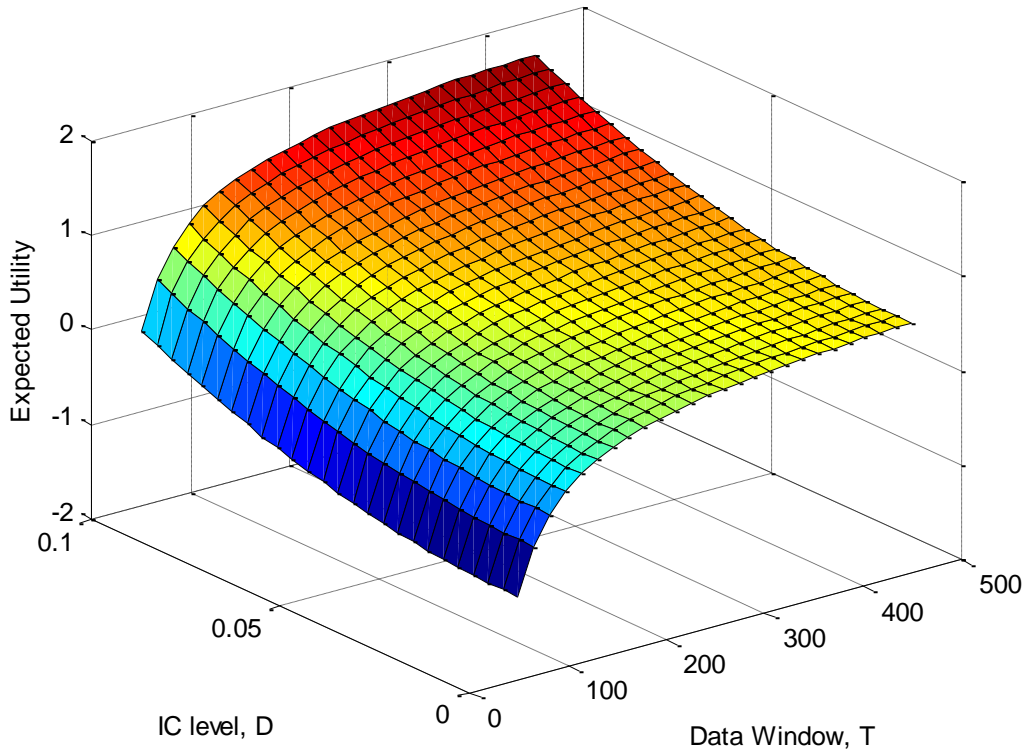
$$E[U] = \frac{\alpha + (N - 1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta - \lambda(1 - d^2))^2}{\gamma}}{2\lambda(1 - d^2)}$$

In the empirical literature a 60 month estimation window is very common. It is therefore of no surprise that these studies were not complimentary of the mean-variance technique, especially given they ignore the role of forecasting ability altogether.



**Figure 1 – Expected Utility vs. Forecasting ability: Asset Allocation.** Figure 1 provides the expected utility of the budget constrained investor using the Unified Fundamental Law of Asset Management for estimation windows of 60, 120, 240, and 480 months. The horizontal line shows the expected utility of the  $1/N$  investor. To calibrate the model we use the estimated mean and covariance for the S&P 500, MSCI EAFE, Barclays US Govt. Bond Index, Barclays US Corp. Bond Index, Commodities (GSCI), FTSE/NAREIT Index for the period 1975-2011.

We now investigate the interplay between forecasting ability and the estimation window length. In figure 2 we show the expected utility conditional on forecasting of the budget constrained investor in the presence of estimation error utilising the unifying investment theorem for the asset allocation problem described above. Note the hyperbolic relationship between the length of the estimation window and expected utility and the quadratic relationship between forecasting ability and expected utility. It is evident that a large gain in welfare accompanies both an increase in forecasting ability and an increase in the estimation window.

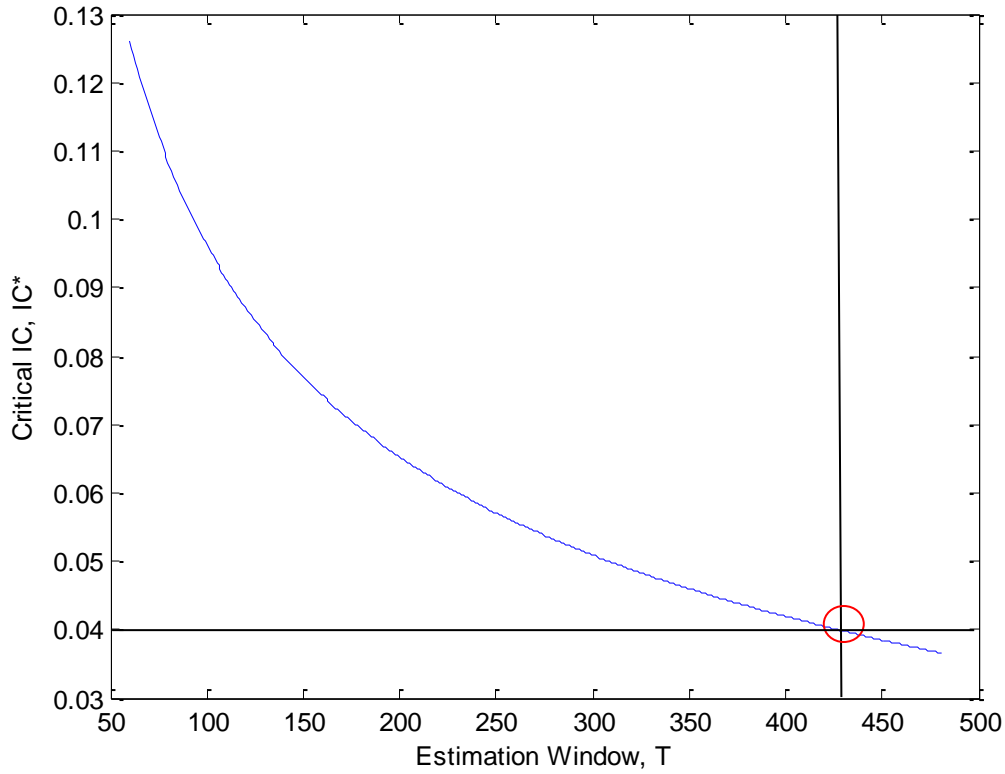


**Figure 2 – Expected Utility vs. Forecasting ability and Estimation Window: Asset Allocation.** Figure 2 shows the expected utility of the budget constrained investor derived using the Unified Fundamental Law of Asset Management for the asset allocation problem. To calibrate the model we use the estimated mean and covariance for the S&P 500, MSCI EAFE, Barclays US Govt. Bond Index, Barclays US Corp. Bond Index, Commodities (GSCI), FTSE/NAREIT Index for the period 1975-2011.

In figure 3 we show the level of forecasting ability required for the budget constrained, mean-variance investor subject to estimation error to outperform 1/N using corollary 6:

$$d^* = \sqrt{\frac{2\lambda V_1 - \alpha + \frac{N-1}{T} + \frac{(\beta-\lambda)^2}{\gamma}}{N-1 + \frac{(\alpha\gamma - \beta^2) + \lambda^2}{\gamma}}}$$

For the asset allocation data used thus far we have 433 estimation months,  $T$ . The expected utility of the 1/N rule using equation (3) is 0.79. This leads to a critical skill level,  $d^*$ , of 0.0396. Note, that corollary 8, which uses the unconditional covariance forecast also gives us a critical skill level of 0.0396, with a slightly more straightforward calculation. This level of forecasting ability is consistent with the level of the information coefficients of widely studied variables such as price-multiples and momentum.



**Figure 3 – Critical Forecasting Ability Level, IC\*: Asset Allocation.** Figure 3 shows the required skill level IC\* for mean-variance to outperform 1/N for the asset allocation problem using the Unified Fundamental Law of Asset Management. To calibrate the model we use the estimated mean and covariance for the S&P 500, MSCI EAFE, Barclays US Govt. Bond Index, Barclays US Corp. Bond Index, Commodities (GSCI), FTSE/NAREIT Index for the period 1975-2011.

Up to this point we have developed a mechanism for determining the expected utility of a mean-variance investor that is subject to estimation error and a budget constraint. This has enabled us to derive expected utility and the amount of skill we require to outperform 1/N on average. To gain a richer understanding of the relative performance of mean-variance and 1/N, we now explore the probability of mean-variance outperforming 1/N in a given period.

In section II, we were able to derive the solution for the probability of mean-variance outperforming 1/N for the case of a single forecasting variable,  $M=1$ <sup>23</sup>. In the case of multiple forecasting variables,  $M \neq 1$ , we derive the probability through simulation as it is not possible as far as we are aware to express the probability in terms of simple functions as in corollary (1). Specifically, we investigate the probability of mean-variance outperforming 1/N conditional on forecasting ability, in the presence of estimation error and subject to a budget constraint, consistent with the unified fundamental law. Our simulations are constructed as follows. We generate 10,000 return histories with 10,000 monthly periods for each forecasting level ( $d$ ), estimation window length ( $T$ ) combination using the following ten steps.

Data Generation Algorithm:

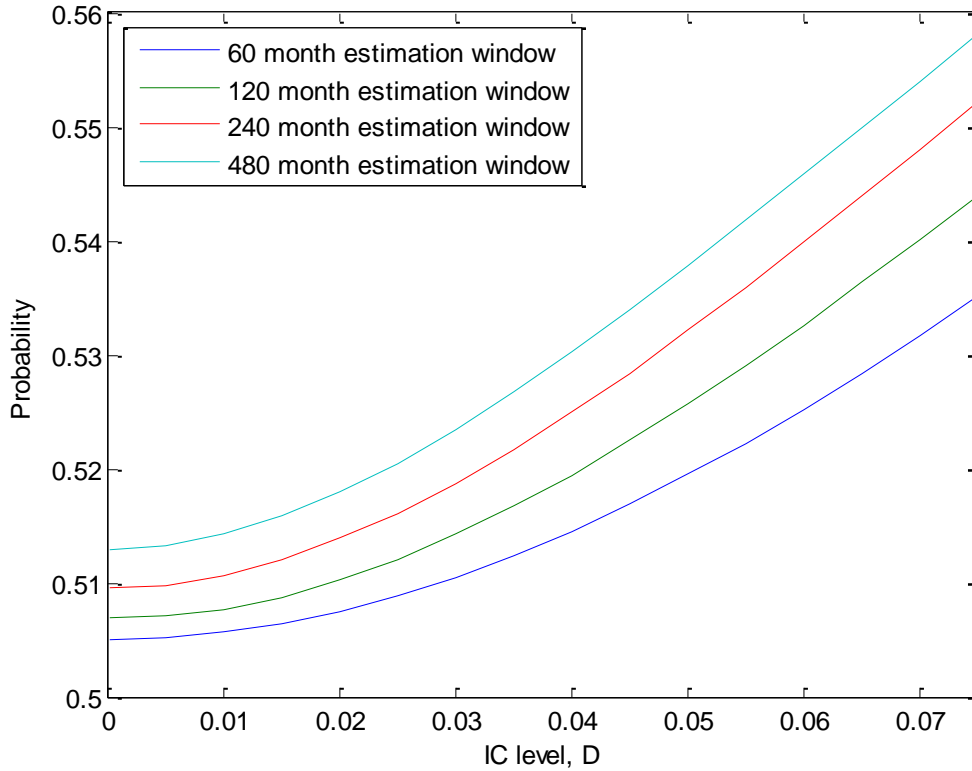
1. Generate information matrix:  $N(0,1)$  variables,  $a$  (N x L)
2. Generate random matrix:  $N(0,1)$  variables,  $z$  (N x L)

<sup>23</sup> See corollary (1)

3. Generate estimation window matrix:  $N(0,1)$  variables,  $e$  (N x K)
4. Calculate historic sample moments,  $\hat{\mu}$  and  $\hat{\Sigma}$
5. Generate estimation window:  $r_e = \hat{\mu} + \hat{\Sigma}^{\frac{1}{2}}e$  (N x K)
6. Calculate sample mean of estimation window returns:  $\bar{\mu}$
7. Combine random variables:  $x_t = da_t + \sqrt{1-d^2}z_t$  (N x L)
8. Simulate measurement window:  $r_t = \hat{\mu} + \hat{\Sigma}^{\frac{1}{2}}x_t$  (N x L)
9. Calculate covariance of estimation window returns:  $\bar{\Sigma}$
10. Simulate expected returns:  $\varphi_t = \bar{\mu} + \bar{\Sigma}^{\frac{1}{2}}a_t$  (N x L)

Having simulated the expected and realised return histories, we then calculate the realised utility of the mean-variance and the 1/N investor each period using the derived weights of the budget constrained investor, as in proposition 5, and the quadratic utility function.

Figure 4 shows the probability of mean-variance outperforming 1/N for four estimation window lengths and twenty forecasting levels. We see here that the probability of mean-variance outperforming 1/N is greater than 50% for all estimation window lengths and skill levels.



**Figure 4 – Probability of Mean-Variance Outperforming 1/N: Asset Allocation.** Figure 4 shows the probability of mean-variance generating a higher utility than the 1/N rule in a monthly period. We attained the probabilities through simulating 10,000 sets of asset allocation data of 10,000 months, for each level of skill, D, and estimation window length, T. To calibrate the model we use the estimated mean and covariance for the S&P 500, MSCI EAFE, Barclays US Govt. Bond Index, Barclays US Corp. Bond Index, Commodities (GSCI), FTSE/NAREIT Index for the period 1975-2011.

## B Expected Utility and Stock Selection

We now turn to the stock selection problem. We use the largest 50 stocks from the S&P 500 index<sup>24</sup>. The summary statistics are shown in table 3 for the largest 50 stocks. The median Sharpe ratio is 0.29. Note that because we have taken the largest stocks at a given point in time, the return figures may be biased upwards. Nevertheless, the 9.12% median return is in line with long-term averages. The median pairwise correlation is 0.32. The 5<sup>th</sup> and 95th percentile correlations are 0.12 and 0.56 respectively.

**Table 3**  
**Asset Class Summary Statistics**

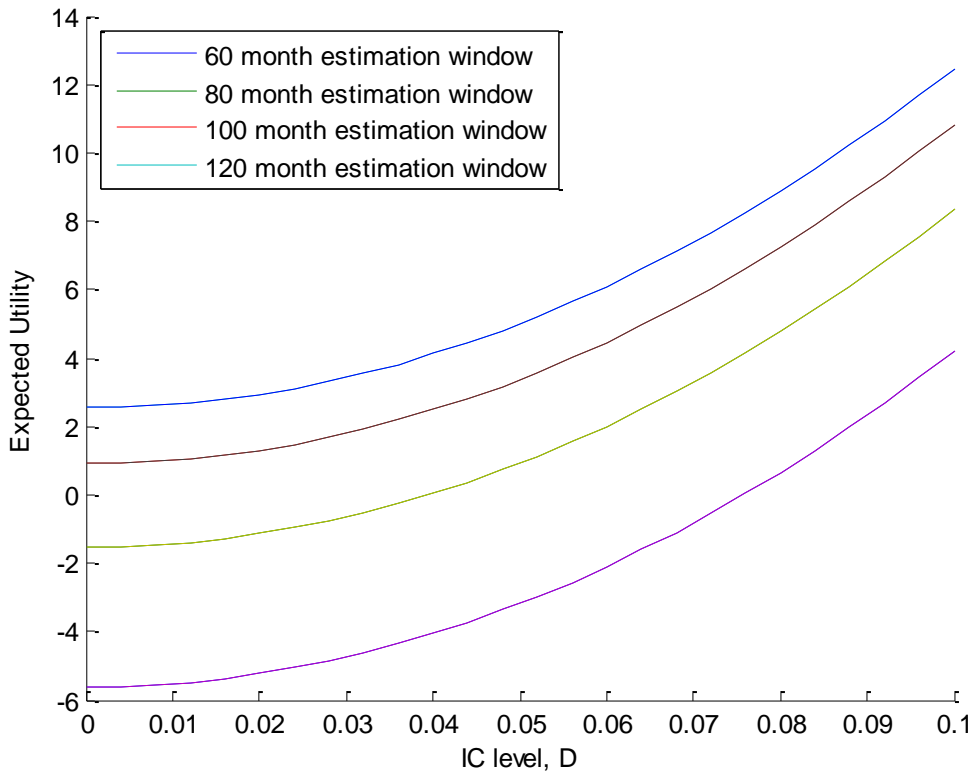
Table 3 provides the summary statistics for the 50 largest stocks in the S&P 500 for the period 05/2002-05/2012. The average annual return is calculated arithmetically. The annual standard deviation is calculated by multiplying the monthly standard deviation by  $\sqrt{12}$ . The Sharpe ratio is calculated as the annual excess return over three-month treasury bills divided by the standard deviation. The certainty equivalent is calculated using the quadratic utility function with a risk aversion of 0.025 as calculated in appendix E. The maximum drawdown is calculated as the maximum peak-to-trough return. Autocorrelation is defined as the correlation of returns in period  $t$  to  $t-1$ .

	Median		Minimum		Maximum	
Annual Return	9.12	-	10.25		47.00	
Standard Deviation	25.84		14.49		105.13	
Sharpe Ratio	0.29	-	0.24		1.17	
Certainty Equivalent	-	0.49	-	23.44	0.75	
Positive Months	68.00		57.00		79.00	
Negative Months	52.00		41.00		63.00	
% Positive Months	0.57		0.48		0.66	
Maximum	24.63		10.14		244.98	
Minimum	-	21.90	-	84.35	-	11.61
Maximum Drawdown	-	52.02	-	99.40	-	25.37
Skew	-	0.16	-	1.14		4.89
Kurtosis		4.20		2.60		40.16
Autocorrelation	-	0.01	-	0.21		0.20

In figure 5, we show the expected utility conditional on forecasting ability and the length of estimation window employing the unified fundamental law for the 50 stock universe. If the estimation window is short (long), the expected utility is much lower (higher) than in the asset allocation case. This is driven by the increase in estimation error due to the larger number of assets. The expected utility is much lower (higher) for low (high) levels of skill than for the asset allocation problem. In general, the expected utility of the stock selector, where  $N$  is large, is more sensitive to both estimation error and the level of forecasting ability than the expected utility of the asset allocator.

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<sup>24</sup> As of 5/2012



**Figure 5– Expected Utility vs. Forecasting ability: Stock Selection.** Figure 4 provides the expected utility of the budget constrained investor using the Unified Fundamental Law of Asset Management for estimation windows of 60, 80, 180, 100, and 120 months. The horizontal line shows the expected utility of the 1/N investor. To calibrate the model we use the estimated mean and covariance for the largest 50 stocks (as at 5/2012) in the S&P 500 for the period 5/2002-5/2012.

In figure 6 we show the expected utility of the budget constrained mean-variance investor as the number of assets increases for four estimation window lengths. In line with the analysis in section II, corollary 3, the effect of increasing the number of assets,  $N$  depends on the relationship between forecasting ability and estimation window.

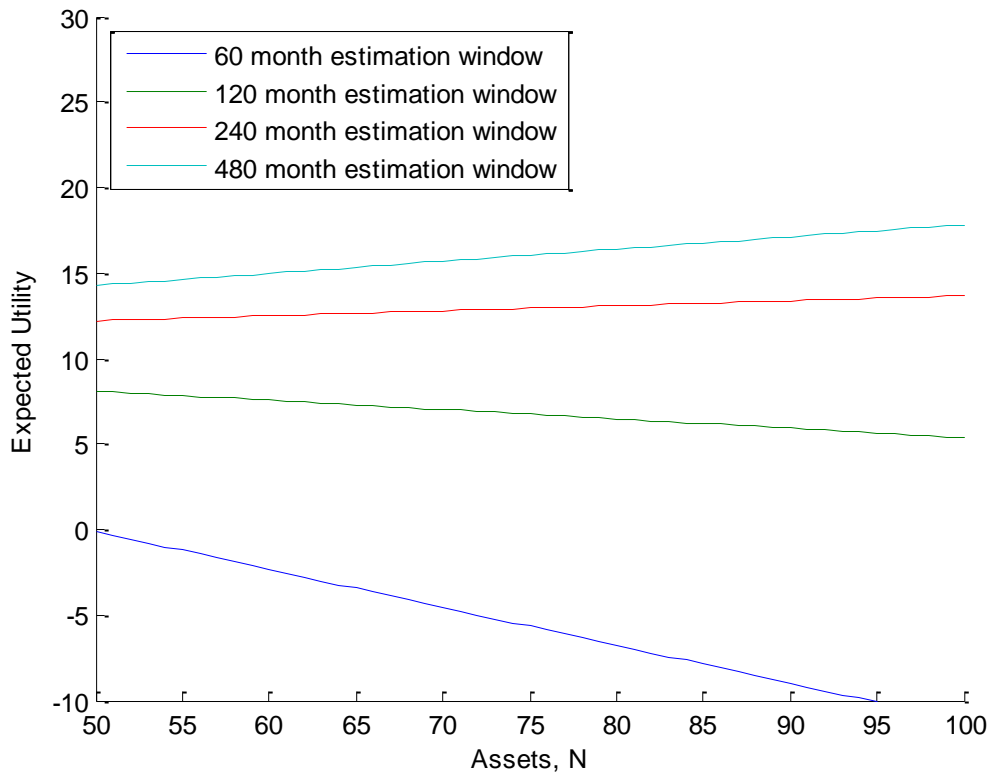
If

$$d > \sqrt{\frac{1}{T}} \text{ then } \uparrow N \Rightarrow \uparrow E[U]$$

If

$$d < \sqrt{\frac{1}{T}} \text{ then } \uparrow N \Rightarrow \downarrow E[U]$$

This runs contrary to the Kan and Zhou (2008) and DeMiguel et al (2009) conclusion that as the number of assets increases, expected utility decreases. This has implications for the appropriate portfolio construction technique for practitioners.

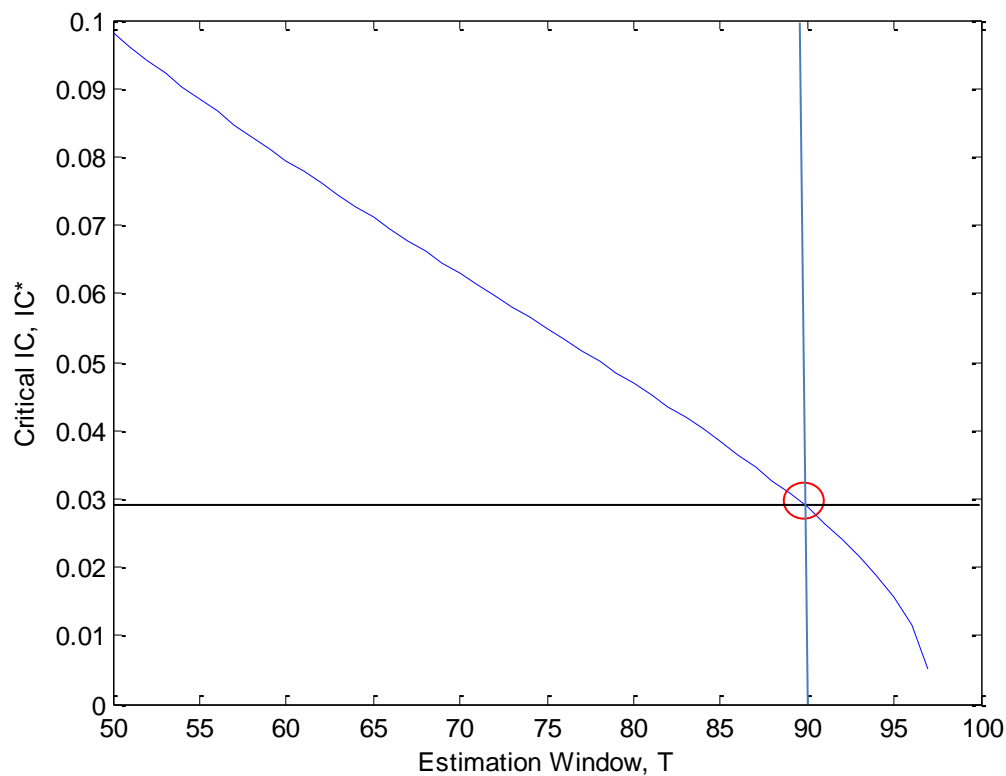


**Figure 6 – Expected Utility vs. Assets, N given Estimation Window: 50 largest stocks in S&P 500.** Figure 5 shows the expected utility of the budget constrained investor derived using the Unified Fundamental Law of Asset Management for a forecasting ability,  $D=0.075$ . To calibrate the model we use the estimated mean and covariance for the largest 50 stocks (as at 5/2012) in the S&P 500 for the period 5/2002-5/2012.

In figure 7 we use corollary 6 to show the level of forecasting ability required for the mean-variance stock selector to outperform the  $1/N$  investor. The expected utility of the  $1/N$  rule using equation (3) is 0.64, i.e. lower than the corresponding asset allocation figure of 0.79. This lower value makes sense given the natural diversification inherent in the asset allocation case. For the purpose of illustration we assume below that we have 90 estimation months available to us<sup>25</sup>. This leads to a critical skill level  $d^*$  of approximately 0.029. This implies that for subject to the assumptions we have made above, the asset allocator with  $N_1$  assets requires a larger amount of skill than the stock selector with  $N_2$  stocks, where  $N_1 < N_2$ .

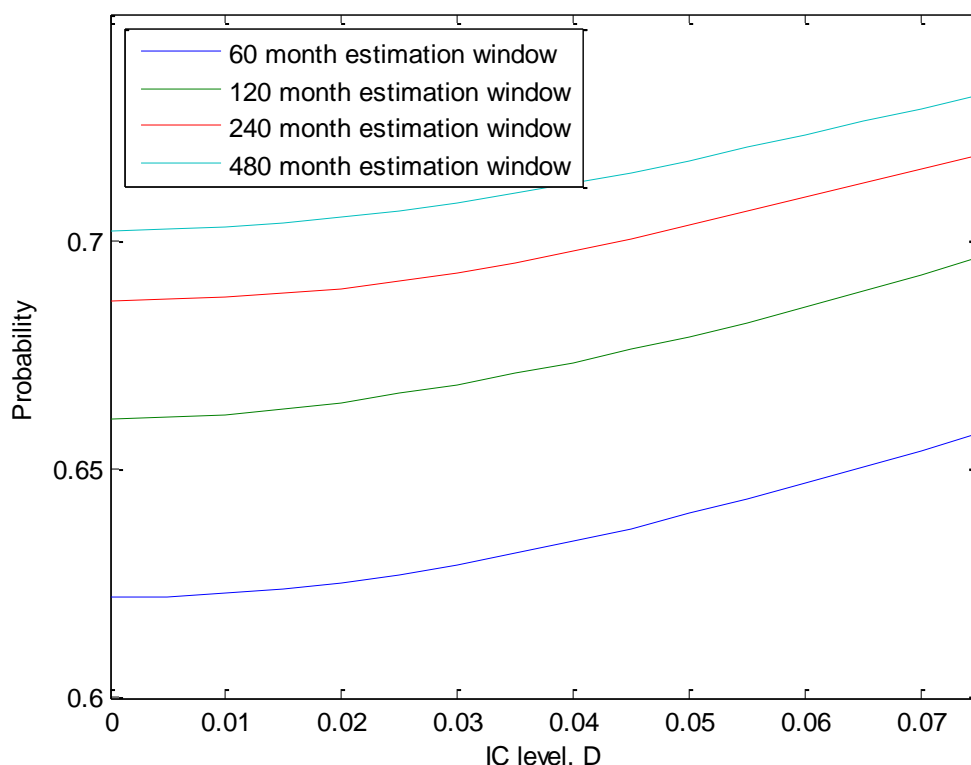
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<sup>25</sup> Note that in practice, this is a conservative figure for the majority of stocks.



**Figure 7 – Expected Utility vs. Forecasting ability: Stock Selection.** Figure 6 shows the required skill level  $IC^*$  for mean-variance to outperform  $1/N$  for the stock selection problem. To calibrate the model we use the estimated mean and covariance for the largest 50 stocks (as at 5/2012) in the S&P 500 for the period 5/2002-5/2012.

Finally, in figure 8 we show the probability of mean-variance outperforming  $1/N$  in a given month for the stock selector. Consistent with the finding that a lower level of skill is required to outperform  $1/N$  for the stock selection problem, the probability of outperforming  $1/N$  in a given month is universally higher than for the asset allocator. Even for low levels of skill and estimation data, the probability of mean-variance outperforming  $1/N$  is above 60%.



**Figure 8 – Probability of Mean-Variance Outperforming 1/N: Stock Selection.** Figure 7 shows the probability of mean-variance generating a higher utility than the 1/N rule in a monthly period. We attained the probabilities through simulating 10,000 sets of stock selection data of 10,000 months, for each level of skill,  $D$ , and estimation window length,  $T$ . To calibrate the model we use the estimated mean and covariance for the largest 50 stocks (as at 5/2012) in the S&P 500 for the period 5/2002-5/2012.

## IV. Conclusions

The academic literature would appear to be virtually unanimous in promoting equal weighted investment as preferable to mean-variance optimisation. However, this literature tends to make two key assumptions, namely, that the investor has no forecasting ability and that there is no budget-constraint operating for the investor.

We depart from previous work by allowing for forecasting skill and we also impose a budget constraint on the problem; as a consequence we get a number of new results for both ex-ante expected utility, conditional on the forecasts used and also for ex-post expected utility which is now averaged over the forecast distribution. Our results present a much more compelling case for optimised portfolios. In particular, we now find that, for plausible skill levels, increasing the number of assets increases expected utility; a result first shown by Grinold (1989), but neglected by the academic literature.

The analytical results of Kan and Zhou (2008) and DeMiguel et al (2009) on the performance of mean-variance suggest that vast amounts of data are required for mean-variance to outperform 1/N. DeMiguel et al (2009) conclude that there are “many miles to go” before the promised benefits of optimal portfolio choice can be realised out of sample. Our results are in stark contrast to this. We show that even for modest levels of forecasting ability and realistic estimation windows we can expect mean-variance to outperform 1/N.

## Appendix A – Derivation of Propositions: The Unconstrained Case

A Proof of Proposition 1:

The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of a Single Forecasting Variable, a ( $M=1$ ), no Estimation Error and in the absence of a Budget Constraint:

Given equation (1), the conditional density of  $r_{t+1}|a$ :

$$N(\mu + Ca, \Sigma - CC')$$

We let the conditional squared Sharpe ratio equal:

$$\alpha^* = \mu^{*'} \Sigma^{*-1} \mu^{*'} \quad \text{A1}$$

where

$$\mu^* = \mu + Ca \quad \text{A2}$$

$$C = \Sigma^{\frac{1}{2}} D \quad \text{A3}$$

where D is ( $N \times 1$ )

Using the relation:

$$(\Sigma - CC')^{-1} = \Sigma^{-1} + \frac{\Sigma^{-1} CC' \Sigma^{-1}}{1 - C' \Sigma^{-1} C}$$

which holds since C is  $N \times 1$ . Substituting equation A3 into the above, we have:

$$(\Sigma - CC')^{-1} = \Sigma^{-1} + \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D}$$

So

$$\begin{aligned} \alpha^* &= (\mu + Ca)' \left( \Sigma^{-1} + \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D} \right) (\mu + Ca) \\ &= \left( \mu + \Sigma^{\frac{1}{2}} D a \right)' \left( \Sigma^{-1} + \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D} \right) \left( \mu + \Sigma^{\frac{1}{2}} D a \right) \\ &= \alpha + \mu' \Sigma^{-\frac{1}{2}} D a + \left( \Sigma^{\frac{1}{2}} D a \right)' \Sigma^{-1} \mu + \left( \Sigma^{\frac{1}{2}} D a \right)' \Sigma^{-1} \left( \Sigma^{\frac{1}{2}} D a \right) + \\ &\quad \left( \Sigma^{\frac{1}{2}} D a \right)' \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D} \mu + \left( \Sigma^{\frac{1}{2}} D a \right)' \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D} \mu + \left( \Sigma^{\frac{1}{2}} D a \right)' \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D} \left( \Sigma^{\frac{1}{2}} D a \right) + \mu' \frac{\Sigma^{-\frac{1}{2}} D D' \Sigma^{-\frac{1}{2}}}{1 - D' D} \mu \end{aligned}$$

So

$$\begin{aligned}
E[U|a] &= \frac{1}{2\lambda} \left( \alpha + 2\mu' \Sigma^{-\frac{1}{2}} D a + D' D a^2 + \frac{2a D' D D' \Sigma^{-\frac{1}{2}} \mu}{1 - D' D} + \frac{a^2 (D' D)^2}{1 - D' D} + \frac{(\mu' \Sigma^{-\frac{1}{2}} D)^2}{1 - D' D} \right) \\
&= \frac{1}{2\lambda} \left( \alpha^2 \left( D' D + \frac{(D' D)^2}{1 - D' D} \right) + a \left( 2\mu' \Sigma^{-\frac{1}{2}} D + \frac{2D' D D' \Sigma^{-\frac{1}{2}} \mu}{1 - D' D} \right) + \left( \alpha + \frac{(\mu' \Sigma^{-\frac{1}{2}} D)^2}{1 - D' D} \right) \right) \quad A4
\end{aligned}$$

which is quadratic in  $a$ .

Taking expectations over  $a$ , using  $E[a] = 0$ ,  $E[a^2] = 1$ , we arrive at the formulae below:

**Proposition 1:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of a Single Forecasting Variable,  $a$  ( $M=1$ ), no Estimation Error and in the absence of a Budget Constraint is given by:*

$$E[U] = \frac{1}{2\lambda} \left( \alpha + \frac{D' D + (\mu' \Sigma^{-\frac{1}{2}} D)^2}{1 - D' D} \right)$$

*B Proof of Proposition 2:*

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables,  $a$  ( $M=N$ ), no Estimation Error and in the absence of a Budget Constraint*

Because  $C$  is a matrix in this case and  $D$  is symmetric we have:

$$(\Sigma - C C')^{-1} \cong \Sigma^{-\frac{1}{2}} (I - D^2)^{-1} \Sigma^{-\frac{1}{2}}$$

and

$$(I - D^2)^{-1} = \sum_{j=0}^{\infty} (D^2)^j$$

If  $D$  is a matrix with eigenvalues less than 1. Since  $D$  is assumed to be a diagonal matrix with correlations on the diagonal, this is satisfied.

Using  $(\Sigma - C C')^{-1} \cong \Sigma^{-\frac{1}{2}} (I + D^2) \Sigma^{-\frac{1}{2}}$  which is an approximation to  $O(D^4)$ .

So

$$\begin{aligned}
\alpha^* &= \left( \mu + \Sigma^{\frac{1}{2}} D a \right)' \Sigma^{-\frac{1}{2}} (I + D^2) \Sigma^{-\frac{1}{2}} \left( \mu + \Sigma^{\frac{1}{2}} D a \right) \\
&= \alpha + 2\mu' \Sigma^{\frac{1}{2}} \Sigma^{-1} D a + \left( \Sigma^{\frac{1}{2}} D a \right)' \Sigma^{-1} \Sigma^{\frac{1}{2}} D a + \\
&\quad \mu' \Sigma^{-\frac{1}{2}} D^2 \Sigma^{-\frac{1}{2}} \mu + 2\mu' \Sigma^{-\frac{1}{2}} D^2 \Sigma^{-\frac{1}{2}} \Sigma^{\frac{1}{2}} D a + \left( \Sigma^{\frac{1}{2}} D a \right)' \Sigma^{-\frac{1}{2}} D^2 \Sigma^{-\frac{1}{2}} \left( \Sigma^{\frac{1}{2}} D a \right) \\
&= \alpha + 2\mu' \Sigma^{-\frac{1}{2}} D a + a' D^2 a + 2\mu' \Sigma^{-\frac{1}{2}} D^3 a + a' D^4 a
\end{aligned}$$

$$\alpha^* = \alpha + a \left( 2\mu' \Sigma^{-\frac{1}{2}} D + 2\mu' \Sigma^{-\frac{1}{2}} D^3 \right) + a' (D^2 + D^4) a \quad A5$$

Taking expectations over  $a$ , we have  $[a] = 0$ ,  $E[a' X a] = \text{tr}(X)$

**Proposition 2:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables,  $a$  ( $M=N$ ), no Estimation Error and in the absence of a Budget Constraint is approximated by:*

$$E[U] \cong \frac{\alpha + \text{tr}(D^2) + \text{tr}(D^4)}{2\lambda} + O(D^6)$$

C *Proof of Proposition 3:*

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables,  $a$  ( $M=N$ ), with a Constant Forecasting Ability level  $d$ , no Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance*

In this case:

$$C = \Sigma^{\frac{1}{2}} D = \Sigma^{\frac{1}{2}} d$$

as

$$D = dI_N$$

$$\Sigma^{*-1} = (\Sigma - CC')^{-1}$$

$$\left( \Sigma - d^2 \Sigma^{\frac{1}{2}} I \Sigma^{\frac{1}{2}} \right)^{-1} = \frac{\Sigma^{-1}}{1 - d^2}$$

Thus we have:

$$\alpha^* = (\mu + Ca)' \Sigma^{*-1} (\mu + Ca)$$

$$= \frac{1}{1 - d^2} (\mu + \Sigma^{\frac{1}{2}} da)' \Sigma^{-1} (\mu + \Sigma^{\frac{1}{2}} da)$$

$$= \frac{1}{1 - d^2} \left( \alpha + 2\mu' \Sigma^{-\frac{1}{2}} da + (\Sigma^{\frac{1}{2}} da)' \Sigma^{-1} (\Sigma^{\frac{1}{2}} da) \right)$$

$$\alpha^* = \frac{1}{1 - d^2} \left( \alpha + 2d\mu' \Sigma^{-\frac{1}{2}} a + d^2 a' a \right) \quad A6$$

Taking expectations we have,  $[a] = 0$ ,  $E[a' X a] = \text{tr}(X)$ :

**Proposition 3:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables,  $a$  ( $M=N$ ), with a Constant Forecasting Ability level, no Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance is given by:*

$$E[U] = \frac{(\alpha + d^2N)}{2\lambda(1 - d^2)}$$

*D Proof of Proposition 4*

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a (M=N), with a Constant Forecasting Ability level, Estimation Error and in the absence of a Budget Constraint using the Unconditional Covariance*

The estimated weights are:

$$w = \frac{\Sigma^{-1}\mu}{\lambda}$$

The conditional moments are:

$$\mu^* = \bar{x} + \Sigma^{\frac{1}{2}}da$$

$$\Sigma^* = \frac{\Sigma}{(1 - d^2)}$$

$$\hat{w} = \frac{\Sigma^{-1}(\bar{x} + \Sigma^{\frac{1}{2}}da)}{\lambda(1 - d^2)}$$

The expected return of the mean-variance investor, conditional on  $a$  is thus:

$$\begin{aligned} E[r_p|a] &= \frac{(\bar{x} + \Sigma^{\frac{1}{2}}da)' \Sigma^{-1} (\mu + \Sigma^{\frac{1}{2}}da)}{\lambda(1 - d^2)} \\ &= \frac{\bar{x}' \Sigma^{-1} \mu + d\bar{x}' \Sigma^{-\frac{1}{2}} a + d\mu' \Sigma^{-\frac{1}{2}} a + a' ad^2}{\lambda(1 - d^2)} \end{aligned}$$

Averaging over  $a$  we have:

$$E[r_p] = \frac{\alpha + Nd^2}{\lambda(1 - d^2)} \tag{A7}$$

Expected risk in this case is:

$$\begin{aligned} E[\sigma_p^2|a, \bar{x}] &= E(\omega' \Sigma^* \omega | a) = \frac{(\bar{x} + \Sigma^{\frac{1}{2}}da)' \Sigma^{-1} (\bar{x} + \Sigma^{\frac{1}{2}}da)}{\lambda^2(1 - d^2)} \\ &= \frac{\bar{x}' \Sigma^{-1} \bar{x} + 2\bar{x}' \Sigma^{-\frac{1}{2}} ad + a' ad^2}{\lambda^2(1 - d^2)} \end{aligned}$$

$$E[\sigma_p^2] = \frac{\alpha - \frac{N}{T} + Nd^2}{\lambda^2(1 - d^2)} \tag{A8}$$

**Proposition 4:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance is given by:*

$$E[U] = \frac{\alpha - \frac{N}{T} + Nd^2}{2\lambda(1 - d^2)}$$

*E Expected Utility using the Unconditional Covariance*

We have already shown the expected utility results for the unconstrained mean-variance investor using the conditional covariance. For completeness, we now show the expected utility results using the unconditional covariance.

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, no Estimation Error and in the absence of a Budget Constraint using the Unconditional Covariance Matrix is given by*

$$E[U] = \frac{\alpha + Nd^2}{2\lambda}$$

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the absence of a Budget Constraint using the Conditional Covariance Matrix is given by*

$$E[U] = \frac{\alpha - \frac{N}{T} + Nd^2}{2\lambda}$$

## Appendix B – Utility, Optimal Weights and the Budget Constraint

A *Proof of Proposition 5:*

*Optimal Mean-Variance Weights in the Presence of a Budget Constraint*

The Lagrangian of the budget constrained constant absolute risk aversion investor is as follows:

$$U = \omega' \mu - \frac{\lambda}{2} \omega' \Sigma \omega - \theta (\omega' \mathbf{1} - 1) \quad \text{A9}$$

where  $\theta$  is the Lagrange multiplier.

Our first order conditions are:

$$\mu - \lambda \Sigma \omega - \theta \mathbf{1} = 0$$

So

$$\omega = \frac{1}{\lambda} \Sigma^{-1} (\mu - \theta \mathbf{1})$$

$$\mathbf{1}' \omega = \frac{1}{\lambda} (\mathbf{1}' \Sigma^{-1} \mu - \theta \mathbf{1}' \Sigma^{-1} \mathbf{1})$$

Let

$$\alpha = \mu' \Sigma^{-1} \mu \quad \text{A10}$$

$$\beta = \mu' \Sigma^{-1} \mathbf{1} \quad \text{A11}$$

$$\gamma = \mathbf{1}' \Sigma^{-1} \mathbf{1} \quad \text{A12}$$

In the literature,  $\alpha$  is often referred to as the squared Sharpe ratio.

$\alpha > 0$ ,  $\beta > 0$  and  $\alpha \gamma - \beta^2 > 0$  for  $\mu$  not proportional to  $\mathbf{1}$ .

So

$$1 = \frac{1}{\lambda} (\beta - \theta \gamma)$$

$$\beta = \lambda + \theta \gamma$$

Thus

$$\theta = \frac{(\beta - \lambda)}{\gamma}$$

Finally, we have the optimal weights in the presence of a budget constraint

**Proposition 5:** *The Optimal Mean-Variance Weights in the Presence of a Budget Constraint with known parameters is given by*

$$\omega = \frac{1}{\lambda} \Sigma^{-1} \mu - \frac{(\beta - \lambda)}{\lambda \gamma} \Sigma^{-1} \mathbf{1} \quad \text{A13}$$

It has been brought to our attention that this result was also given by Jorion (1985).

We can compute our expected utility  $V_c$ , as follows:

$$\begin{aligned}
V_c &= \omega' \mu - \frac{\lambda}{2} \omega' \Sigma \omega \\
&= \frac{\alpha}{\lambda} - \frac{\beta(\beta - \lambda)}{\lambda\gamma} - \frac{\lambda}{2\lambda^2} (\mu' - \frac{(\beta - \lambda)}{\gamma} i') \Sigma^{-1} (\mu - \frac{(\beta - \lambda)}{\gamma} i) \\
&= \frac{\beta}{\gamma} - \frac{\beta(\beta - \lambda)}{\lambda\gamma} - \frac{1}{2\lambda} (\alpha - \frac{2\beta(\beta - \lambda)}{\gamma} + \frac{(\beta - \lambda)^2}{\gamma}) \\
&= \frac{\alpha\gamma - \beta^2}{\lambda\gamma} + \frac{\beta}{\gamma} - \frac{1}{2\lambda\gamma} (\alpha\gamma - (\beta - \lambda)(\beta + \lambda)) \\
&= \frac{\alpha\gamma - \beta^2}{\lambda\gamma} + \frac{\beta}{\gamma} - \frac{\alpha\gamma - \beta^2}{2\lambda\gamma} - \frac{\lambda}{2\lambda} \\
V_c &= \frac{\alpha\gamma - (\beta - \lambda)^2}{2\lambda\gamma} \tag{A14}
\end{aligned}$$

## Appendix C – Derivation of Propositions: The Budget Constrained Case

A Proof of Corollary 7:

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, no Estimation Error and in the presence of a Budget Constraint using the Conditional Covariance Matrix.*

We begin by restating the conditional moments:

$$\mu^* = \mu + Ca$$

$$\Sigma^* = (\Sigma - CC')$$

where

$$C = d\Sigma^{\frac{1}{2}}$$

and the optimal weight relation in the presence of a budget constraint:

$$\omega = \frac{1}{\lambda} \Sigma^{-1} \mu - \frac{(\beta - \lambda)}{\lambda\gamma} \Sigma^{-1} i$$

where

$$\beta^* = \mu'^* \Sigma^{*-1} i$$

$$\gamma = i' \Sigma^{*-1} i$$

Substituting the conditional moments into the optimal weight relation yields:

$$\omega = \frac{1}{\lambda(1-d^2)} \left( \Sigma^{-1} \left( \mu + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \beta + da' \Sigma^{-\frac{1}{2}} i - \lambda(1-d^2) \right)}{\gamma} \Sigma^{-1} i \right)$$

The expected return of the mean-variance portfolio conditional on  $a$  is:

$$\begin{aligned} E[r_p|a] &= \frac{1}{\lambda(1-d^2)} \left( \Sigma^{-1} \left( \mu + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \beta + da' \Sigma^{-\frac{1}{2}} i - \lambda(1-d^2) \right)}{\gamma} \Sigma^{-1} i \right)' \left( \mu + \Sigma^{\frac{1}{2}} da \right) \\ &= \frac{1}{\lambda(1-d^2)} \left( \left( \mu + \Sigma^{\frac{1}{2}} da \right)' \Sigma^{-1} \left( \mu + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \beta + da' \Sigma^{-\frac{1}{2}} i - \lambda(1-d^2) \right)}{\gamma} \left( \mu + \Sigma^{\frac{1}{2}} da \right)' \Sigma^{-1} i \right) \\ &= \frac{1}{\lambda(1-d^2)} \left( \left( \mu' \Sigma^{-1} \mu + d \mu' \Sigma^{-\frac{1}{2}} a + d \mu' \Sigma^{-\frac{1}{2}} a + d^2 a' a \right) \right. \\ &\quad \left. - \frac{\left( \beta + da' \Sigma^{-\frac{1}{2}} i - \lambda(1-d^2) \right)}{\gamma} \left( \beta + da' \Sigma^{-\frac{1}{2}} i \right) \right) \\ E[r_p|a] &= \frac{1}{\lambda(1-d^2)} \left( \left( \mu' \Sigma^{-1} \mu + d \mu' \Sigma^{-\frac{1}{2}} a + d \mu' \Sigma^{-\frac{1}{2}} a + d^2 a' a \right) \right. \\ &\quad \left. - \frac{\beta^2 + 2\beta da' \Sigma^{-\frac{1}{2}} i + d^2 a' \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} a - \lambda(1-d^2)\beta - \lambda(1-d^2) da' \Sigma^{-\frac{1}{2}} i}{\gamma} \right) \end{aligned}$$

Taking expectations over  $a$ , we have:

$$E[r_p] = \frac{1}{\lambda(1-d^2)} \left( (\alpha + Nd^2) - \frac{\beta(\beta - \lambda(1-d^2)) + d^2 \text{tr}(\Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}})}{\gamma} \right)$$

$$\text{Now } \text{tr}(\Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}}) = \text{tr}(i \Sigma^{-1} i)$$

$$\text{tr}(\gamma) = \gamma, \text{ so}$$

$$E[r_p] = \frac{1}{\lambda(1-d^2)} \left( (\alpha + (N-1)d^2) - \frac{\beta(\beta - \lambda(1-d^2))}{\gamma} \right)$$

The expected risk of the portfolio conditional on  $a$  is:

$$\begin{aligned}
E[\sigma_p^2|a] &= \frac{1}{\lambda^2(1-d^2)} \left( \left( \mu + \Sigma^{\frac{1}{2}}da \right) - \frac{\left( \beta + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2) \right)}{\gamma} i \right)' \Sigma^{-1} \left( \left( \mu + \Sigma^{\frac{1}{2}}da \right) \right. \\
&\quad \left. - \frac{\left( \beta + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2) \right)}{\gamma} i \right) \\
&= \frac{1}{\lambda^2(1-d^2)} \left( \left( \mu + \Sigma^{\frac{1}{2}}da \right)' \Sigma^{-1} \left( \mu + \Sigma^{\frac{1}{2}}da \right) - 2 \frac{\left( \beta + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2) \right)}{\gamma} \left( \mu + \Sigma^{\frac{1}{2}}da \right)' \Sigma^{-1} i \right. \\
&\quad \left. + \frac{\left( \beta + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2) \right)^2}{\gamma^2} i' \Sigma^{-1} i \right) \\
&= \frac{1}{\lambda^2(1-d^2)} \left( \mu' \Sigma^{-1} \mu + 2d\mu' \Sigma^{-\frac{1}{2}}a + d^2a'a \right. \\
&\quad \left. - 2 \frac{\left( \beta^2 + 2\beta da'\Sigma^{-\frac{1}{2}}i + d^2a'\Sigma^{-\frac{1}{2}}i' \Sigma^{-\frac{1}{2}}a - \lambda(1-d^2)\beta - \lambda(1-d^2)da'\Sigma^{-\frac{1}{2}}i \right)}{\gamma} \right. \\
&\quad \left. + \frac{\beta^2 + d^2a'\Sigma^{-\frac{1}{2}}i' \Sigma^{-\frac{1}{2}}a + \lambda^2(1-d^2)^2 + 2\beta da'\Sigma^{-\frac{1}{2}}i - 2\beta\lambda(1-d^2) - 2\lambda(1-d^2)da'\Sigma^{-\frac{1}{2}}i}{\gamma} \right)
\end{aligned}$$

Taking expectations over  $a$ , we have

$$\begin{aligned}
&= \frac{1}{\lambda^2(1-d^2)} \left( \alpha + Nd^2 - 2 \frac{\beta(\beta - \lambda(1-d^2)) + d^2 \text{tr} \left( a' \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} a \right)}{\gamma} \right. \\
&\quad \left. + \frac{(\beta - \lambda(1-d^2))^2 - d^2 \text{tr} \left( \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} \right)}{\gamma} \right)
\end{aligned}$$

$$E[\sigma_p^2] = \frac{1}{\lambda^2(1-d^2)} \left( \alpha + (N-1)d^2 + \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right)$$

Thus,

$$\begin{aligned}
E[U] &= \frac{1}{\lambda(1-d^2)} \left( (\alpha + (N-1)d^2) - \frac{\beta(\beta - \lambda(1-d^2))}{\gamma} \right) \\
&\quad - \frac{\lambda}{2} \left( \frac{1}{\lambda^2(1-d^2)} \left( \alpha + (N-1)d^2 + \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right) \right) \\
&= \frac{1}{\lambda(1-d^2)} \left( \left( (\alpha + (N-1)d^2) - \frac{\beta(\beta - \lambda(1-d^2))d^2}{\gamma} \right) \right. \\
&\quad \left. - \frac{1}{2} \left( \alpha + (N-1)d^2 + \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right) \right) \\
&= \frac{1}{2\lambda(1-d^2)} \left( \alpha + (N-1)d^2 - \frac{2\beta(\beta - \lambda(1-d^2))}{\gamma} - \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right) \\
E[U] &= \frac{\alpha\gamma + (N-1)d^2\gamma - (\beta - \lambda(1-d^2))^2}{2\lambda(1-d^2)\gamma}
\end{aligned}$$

**Corollary 7:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , Estimation Error and in the presence of a Budget Constraint using the Unconditional Covariance Matrix is given by:*

$$E[U] = \frac{\alpha + (N-1)d^2 - \frac{(\beta - \lambda(1-d^2))^2}{\gamma}}{2\lambda(1-d^2)}$$

*B Proof of Proposition 6:*

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the presence of a Budget Constraint using the Conditional Covariance Matrix.*

To determine the joint impact of estimation error and forecasting on expected utility in the presence of a budget constraint, we substitute the conditional moments (14) and (15) into the optimal weight relation:

$$\omega = \frac{1}{\lambda} \Sigma^{-1} \mu - \frac{(\beta - \lambda)}{\lambda\gamma} \Sigma^{-1} \mathbf{i}$$

So this becomes

$$w = \frac{1}{\lambda(1-d^2)} \left( \Sigma^{-1} \left( \bar{x} + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \bar{x}' \Sigma^{-1} \mathbf{i} + da' \Sigma^{-\frac{1}{2}} \mathbf{i} - \lambda(1-d^2) \right)}{\gamma} \Sigma^{-1} \mathbf{i} \right)$$

The expected return of the mean-variance portfolio conditional on  $a$  and  $\bar{x}$  is:

$$\begin{aligned}
E[r_p|a, \bar{x}] &= \frac{1}{\lambda(1-d^2)} \left( \Sigma^{-1} (\bar{x} + \Sigma^{\frac{1}{2}} da) - \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2))}{\gamma} \Sigma^{-1}i \right)' (\mu + \Sigma^{\frac{1}{2}}da) \\
&= \frac{1}{\lambda(1-d^2)} \left( (\bar{x} + \Sigma^{\frac{1}{2}}da)' \Sigma^{-1} (\mu + \Sigma^{\frac{1}{2}}da) - \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2))}{\gamma} (\mu + \Sigma^{\frac{1}{2}}da)' \Sigma^{-1}i \right) \\
&= \frac{1}{\lambda(1-d^2)} \left( \bar{x}'\Sigma^{-1}\mu + d\bar{x}'\Sigma^{-\frac{1}{2}}a + d\mu'\Sigma^{-\frac{1}{2}}a + d^2a'a \right) \\
&\quad - \frac{\bar{x}'\Sigma^{-1}i\beta + (\beta + \bar{x}'\Sigma^{-1}\mu)da'\Sigma^{-\frac{1}{2}}i + d^2a'\Sigma^{-\frac{1}{2}}ii'\Sigma^{-\frac{1}{2}}a - \lambda(1-d^2)\beta - \lambda(1-d^2)da'\Sigma^{-\frac{1}{2}}i}{\gamma}
\end{aligned}$$

Taking expectations over both conditioning variables, we have:

$$E[r_p] = \frac{1}{\lambda(1-d^2)} \left( (\alpha + Nd^2) - \frac{\beta(\beta - \lambda(1-d^2)) + d^2 \text{tr}(\Sigma^{-\frac{1}{2}}ii'\Sigma^{-\frac{1}{2}})}{\gamma} \right)$$

$$\text{Now } \text{tr}(\Sigma^{-\frac{1}{2}}ii'\Sigma^{-\frac{1}{2}}) = \text{tr}(i'\Sigma^{-1}i)$$

$$= \text{tr}(\gamma) = \gamma, \text{ so}$$

$$E[r_p] = \frac{1}{\lambda(1-d^2)} \left( (\alpha + (N-1)d^2) - \frac{\beta(\beta - \lambda(1-d^2))}{\gamma} \right)$$

The expected risk of the portfolio conditional on  $a$  is:

$$\begin{aligned}
E[\sigma_p^2|a, \bar{x}] &= \frac{1}{\lambda^2(1-d^2)} \left( \left( \bar{x} + \Sigma^{\frac{1}{2}}da \right) - \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2))}{\gamma} i \right)' \Sigma^{-1} \left( \bar{x} + \Sigma^{\frac{1}{2}}da \right) \\
&\quad - \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda(1-d^2))}{\gamma} i
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\lambda^2(1-d^2)} \left( (\bar{x} + \Sigma^{\frac{1}{2}} da)' \Sigma^{-1} (\bar{x} + \Sigma^{\frac{1}{2}} da) \right. \\
&\quad - 2 \frac{\left( \bar{x}' \Sigma^{-1} i + da' \Sigma^{-\frac{1}{2}} i - \lambda(1-d^2) \right) \left( \bar{x}' \Sigma^{-1} i + da' \Sigma^{-\frac{1}{2}} i \right)}{\gamma} \\
&\quad \left. + \frac{\left( \bar{x}' \Sigma^{-1} i + da' \Sigma^{-\frac{1}{2}} i - \lambda(1-d^2) \right)^2}{\gamma^2} i' \Sigma^{-1} i \right) \\
&= \frac{1}{\lambda^2(1-d^2)} \left( \bar{x}' \Sigma^{-1} \bar{x} + 2d \bar{x}' \Sigma^{-\frac{1}{2}} a + d^2 a' a \right. \\
&\quad - 2 \frac{\left( \bar{x}' \Sigma^{-1} i i' \Sigma^{-1} \bar{x} + d \bar{x}' \Sigma^{-1} i a' \Sigma^{-\frac{1}{2}} i + d \bar{x}' \Sigma^{-1} i a' \Sigma^{-\frac{1}{2}} i + d^2 a' \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} a - \lambda(1-d^2) \bar{x}' \Sigma^{-1} i - \lambda(1-d^2) da' \Sigma^{-\frac{1}{2}} i \right)}{\gamma} \\
&\quad \left. + \frac{\bar{x}' \Sigma^{-1} i i' \Sigma^{-1} \bar{x} + d^2 a' \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} a + \lambda^2(1-d^2)^2 + 2 \bar{x}' \Sigma^{-1} i da' \Sigma^{-\frac{1}{2}} i - 2 \bar{x}' \Sigma^{-1} i \lambda(1-d^2) - 2 \lambda(1-d^2) da' \Sigma^{-\frac{1}{2}} i}{\gamma} \right)
\end{aligned}$$

Taking expectations over both conditioning variables, we have

$$\begin{aligned}
&= \frac{1}{\lambda^2(1-d^2)} \left( \alpha + \frac{N}{T} + Nd^2 - 2 \frac{\left( \beta^2 + \frac{\text{tr}(i \Sigma^{-1} i)}{T} - \beta \lambda(1-d^2) + d^2 \text{tr} \left( \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} \right) \right)}{\gamma} \right. \\
&\quad \left. + \frac{(\beta - \lambda(1-d^2))^2 + \frac{\text{tr}(i \Sigma^{-1} i)}{T} - d^2 \text{tr} \left( \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} \right)}{\gamma} \right)
\end{aligned}$$

Now  $\text{tr}(i' \Sigma^{-1} i)$  and  $\text{tr} \left( \Sigma^{-\frac{1}{2}} i i' \Sigma^{-\frac{1}{2}} \right)$  both equal  $\gamma$ , thus:

$$E[\sigma_p^2] = \frac{1}{\lambda^2(1-d^2)} \left( \alpha + (N-1) \left( d^2 - \frac{1}{T} \right) + \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right)$$

and,

$$\begin{aligned}
E[U] &= \frac{1}{\lambda(1-d^2)} \left( \left( \alpha + (N-1)d^2 \right) - \frac{\beta(\beta - \lambda(1-d^2))}{\gamma} \right) \\
&\quad - \frac{\lambda}{2} \left( \frac{1}{\lambda^2(1-d^2)} \left( \alpha + (N-1) \left( d^2 - \frac{1}{T} \right) + \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\lambda(1-d^2)} \left( \left( (\alpha + (N-1)d^2) - \frac{\beta(\beta - \lambda(1-d^2))d^2}{\gamma} \right) \right. \\
&\quad \left. - \frac{1}{2} \left( \alpha + (N-1) \left( d^2 - \frac{1}{T} \right) + \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right) \right) \\
&= \frac{1}{2\lambda(1-d^2)} \left( \alpha - (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{2\beta(\beta - \lambda(1-d^2))}{\gamma} - \frac{\lambda^2(1-d^2)^2 - \beta^2}{\gamma} \right) \\
E[U] &= \frac{\alpha + (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta - \lambda(1-d^2))^2}{\gamma}}{2\lambda(1-d^2)}
\end{aligned}$$

**Proposition 6:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the presence of a Budget Constraint using the Conditional Covariance Matrix is given by:*

$$E[U] = \frac{\alpha + (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta - \lambda(1-d^2))^2}{\gamma}}{2\lambda(1-d^2)}$$

*B Proof of Proposition 7:*

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level, Estimation Error and in the presence of a Budget Constraint using the Unconditional Covariance Matrix.*

Instead of substituting the conditional covariance matrix into the optimal weight expression (A13), we now use the unconditional covariance matrix.

The optimal weight expression, conditional on  $a$  and  $\bar{x}$  becomes

$$w = \frac{1}{\lambda} \left( \Sigma^{-1} \left( \bar{x} + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \bar{x}' \Sigma^{-1} i + da' \Sigma^{-\frac{1}{2}} i - \lambda \right)}{\gamma} \Sigma^{-1} i \right)$$

The expected return of the mean-variance portfolio conditional on  $a$  and  $\bar{x}$  is:

$$\begin{aligned}
E[r_p | a, \bar{x}] &= \frac{1}{\lambda} \left( \Sigma^{-1} \left( \bar{x} + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \bar{x}' \Sigma^{-1} i + da' \Sigma^{-\frac{1}{2}} i - \lambda \right)}{\gamma} \Sigma^{-1} i \right)' \left( \mu + \Sigma^{\frac{1}{2}} da \right) \\
&= \frac{1}{\lambda} \left( \left( \bar{x} + \Sigma^{\frac{1}{2}} da \right)' \Sigma^{-1} \left( \mu + \Sigma^{\frac{1}{2}} da \right) - \frac{\left( \bar{x}' \Sigma^{-1} i + da' \Sigma^{-\frac{1}{2}} i - \lambda \right)}{\gamma} \left( \mu + \Sigma^{\frac{1}{2}} da \right)' \Sigma^{-1} i \right)
\end{aligned}$$

$$= \frac{1}{\lambda} \left( \left( \bar{x}'\Sigma^{-1}\mu + d\bar{x}'\Sigma^{-\frac{1}{2}}a + d\mu'\Sigma^{-\frac{1}{2}}a + d^2a'a \right) \right. \\ \left. - \frac{\bar{x}'\Sigma^{-1}i\beta + (\beta + \bar{x}'\Sigma^{-1}\mu)da'\Sigma^{-\frac{1}{2}}i + d^2a'\Sigma^{-\frac{1}{2}}ii'\Sigma^{-\frac{1}{2}}a - \lambda\beta - \lambda da'\Sigma^{-\frac{1}{2}}i}{\gamma} \right)$$

Taking expectations over both conditioning variables, we have:

$$E[r_p] = \frac{1}{\lambda(1-d^2)} \left( (\alpha + Nd^2) - \frac{\beta(\beta - \lambda) + d^2 \text{tr}(\Sigma^{-\frac{1}{2}}ii'\Sigma^{-\frac{1}{2}})}{\gamma} \right)$$

$$\text{Now } \text{tr}(\Sigma^{-\frac{1}{2}}ii'\Sigma^{-\frac{1}{2}}) = \text{tr}(i'\Sigma^{-1}i)$$

$$= \text{tr}(\gamma) = \gamma, \text{ so}$$

$$E[r_p] = \frac{1}{\lambda} \left( (\alpha + (N-1)d^2) - \frac{\beta(\beta - \lambda)}{\gamma} \right)$$

The expected risk of the portfolio conditional on  $a$  is:

$$E[\sigma_p^2|a, \bar{x}] = \frac{1}{\lambda^2} \left( \left( \bar{x} + \Sigma^{\frac{1}{2}}da \right) - \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda)}{\gamma} i \right)' \Sigma^{-1} \left( \bar{x} + \Sigma^{\frac{1}{2}}da \right) \\ - \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda)}{\gamma} i \right) \\ = \frac{1}{\lambda^2} \left( \left( \bar{x} + \Sigma^{\frac{1}{2}}da \right)' \Sigma^{-1} \left( \bar{x} + \Sigma^{\frac{1}{2}}da \right) - 2 \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda)(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i)}{\gamma} \right. \\ \left. + \frac{(\bar{x}'\Sigma^{-1}i + da'\Sigma^{-\frac{1}{2}}i - \lambda)^2}{\gamma^2} i'\Sigma^{-1}i \right)$$

$$\begin{aligned}
&= \frac{1}{\lambda^2} \left( \bar{x}'\Sigma^{-1}\bar{x} + 2d\bar{x}'\Sigma^{-\frac{1}{2}}a + d^2a'a \right. \\
&\quad \left. - 2 \frac{\left( \bar{x}'\Sigma^{-1}i' \Sigma^{-1}\bar{x} + d\bar{x}'\Sigma^{-1}ia' \Sigma^{-\frac{1}{2}}i + d\bar{x}'\Sigma^{-1}ia' \Sigma^{-\frac{1}{2}}i + d^2a' \Sigma^{-\frac{1}{2}}ii' \Sigma^{-\frac{1}{2}}a - \lambda\bar{x}'\Sigma^{-1}i - \lambda da' \Sigma^{-\frac{1}{2}}i \right)}{\gamma} \right) \\
&\quad \left. + \frac{\bar{x}'\Sigma^{-1}ii' \Sigma^{-1}\bar{x} + d^2a' \Sigma^{-\frac{1}{2}}ii' \Sigma^{-\frac{1}{2}}a + \lambda^2 + 2\bar{x}'\Sigma^{-1}ida' \Sigma^{-\frac{1}{2}}i - 2\bar{x}'\Sigma^{-1}i\lambda - 2\lambda da' \Sigma^{-\frac{1}{2}}i}{\gamma} \right)
\end{aligned}$$

Taking expectations over both conditioning variables, we have

$$\begin{aligned}
&= \frac{1}{\lambda^2(1-d^2)} \left( \alpha + \frac{N}{T} + Nd^2 - 2 \frac{\left( \beta^2 + \frac{\text{tr}(i'\Sigma^{-1}i)}{T} - \beta\lambda + d^2 \text{tr} \left( \Sigma^{-\frac{1}{2}}ii' \Sigma^{-\frac{1}{2}} \right) \right)}{\gamma} \right) \\
&\quad \left. + \frac{\left( \beta - \lambda \right)^2 + \frac{\text{tr}(i'\Sigma^{-1}i)}{T} - d^2 \text{tr} \left( \Sigma^{-\frac{1}{2}}ii' \Sigma^{-\frac{1}{2}} \right)}{\gamma} \right)
\end{aligned}$$

Now  $\text{tr}(i'\Sigma^{-1}i)$  and  $\text{tr} \left( \Sigma^{-\frac{1}{2}}ii' \Sigma^{-\frac{1}{2}} \right)$  both equal  $\gamma$ , thus:

$$E[\sigma_p^2] = \frac{1}{\lambda^2} \left( \alpha + (N-1) \left( d^2 - \frac{1}{T} \right) + \frac{\lambda^2 - \beta^2}{\gamma} \right)$$

and,

$$\begin{aligned}
E[U] &= \frac{1}{\lambda} \left( \left( \alpha + (N-1)d^2 - \frac{\beta(\beta-\lambda)}{\gamma} \right) - \frac{\lambda}{2} \left( \frac{1}{\lambda^2} \left( \alpha + (N-1) \left( d^2 - \frac{1}{T} \right) + \frac{\lambda^2 - \beta^2}{\gamma} \right) \right) \right) \\
&= \frac{1}{\lambda} \left( \left( \left( \alpha + (N-1)d^2 - \frac{\beta(\beta-\lambda)d^2}{\gamma} \right) - \frac{1}{2} \left( \alpha + (N-1) \left( d^2 - \frac{1}{T} \right) + \frac{\lambda^2 - \beta^2}{\gamma} \right) \right) \right) \\
&= \frac{1}{2\lambda(1-d^2)} \left( \alpha - (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{2\beta(\beta-\lambda)}{\gamma} - \frac{\lambda^2 - \beta^2}{\gamma} \right) \\
E[U] &= \frac{\alpha + (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta-\lambda)^2}{\gamma}}{2\lambda}
\end{aligned}$$

**Proposition 7:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a (M=N), with a Constant Forecasting Ability level, Estimation Error and in the presence of a Budget Constraint using the Unconditional Covariance Matrix is given by:*

$$E[U] = \frac{\alpha + (N-1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta-\lambda)^2}{\gamma}}{2\lambda}$$

## Appendix D – The Unified Fundamental Law As under a Simplified Model of Return Generation

A *Proof of Corollary 9:*

*The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , Estimation Error and in the presence of a Budget Constraint with a Constant Pair-wise Correlation,  $\rho$ , and a Constant Stock Volatility,  $\sigma$ , across all Assets using the Conditional Covariance Matrix*

The Unified Fundamental Law as derived in appendix C is given by

$$E[U] = \frac{\alpha + (N - 1) \left( d^2 - \frac{1}{T} \right) - \frac{(\beta - \lambda(1 - d^2))^2}{\gamma}}{2\lambda(1 - d^2)}$$

For the  $1/N$  investor we have

$$V_{\frac{1}{N}} = \hat{\mu}_1 - \frac{\lambda}{2N^2} i' \Sigma i$$

where

$$\hat{\mu}_1 = \sum \frac{\mu_i}{N}, \hat{\mu}_2 = \sum \frac{\mu_i^2}{N}, \hat{\sigma}_\mu^2 = \hat{\mu}_2 - \hat{\mu}_1^2$$

we assume that  $\text{plim}_{N \rightarrow \infty} \hat{\mu}_1 = \mu_1$  and so forth.

To explicate the key drivers of expected utility, we assume a constant correlation,  $\rho$ , and constant variance,  $\sigma^2$ , structure such that

$$\Sigma = \sigma^2(1 - \rho)I_N + \sigma^2 p i i'$$

where  $p$  is an  $(N \times 1)$  vector of ones.

Using the standard inversion result, we can show that

$$\Sigma^{-1} = \frac{1}{\sigma^2(1 - \rho)} \left( I_N - \frac{\rho i i'}{1 + \rho(N - 1)} \right)$$

In what follows we drop the sample operator,  $\hat{\cdot}$ .

Now

$$\begin{aligned} \alpha &= \mu' \Sigma^{-1} \mu \\ &= \frac{1}{\sigma^2(1 - \rho)} \left( N \mu_2 - \frac{\rho N^2 \mu_1^2}{1 + \rho(N - 1)} \right) \\ &= \frac{1}{\sigma^2(1 - \rho)} \left( \frac{(N^2 - N) \mu_2 \rho - N^2 \mu_1^2 \rho}{1 + \rho(N - 1)} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma^2(1-\rho)} \left( \frac{N^2 \rho \sigma_\mu^2 - N \rho \mu_2}{1 + \rho(N-1)} \right) \\
&= N^2 \frac{\sigma_\mu^2}{\sigma^2} (1-\rho) + O(1)
\end{aligned}$$

By similar arguments

$$\begin{aligned}
\gamma &= i' \Sigma^{-1} i \\
&= \frac{N(1 + \rho(N-1)) - \rho N^2}{\sigma^2(1-\rho)(1 + \rho(N-1))} \\
&= \frac{N(1-\rho)}{\sigma^2(1-\rho)(1 + \rho(N-1))} \\
&= \frac{1}{\rho \sigma^2} + O\left(\frac{1}{N}\right)
\end{aligned}$$

Likewise,

$$\begin{aligned}
\beta &= \frac{1}{\sigma^2(1-\rho)} \left( \frac{N \mu_1 (1-\rho)}{1 + \rho(N-1)} \right) \\
&= \frac{\mu_1}{\sigma^2 \rho}
\end{aligned}$$

Thus,

**Corollary 9:** *The Unconditional Expected Utility of the Mean-Variance investor under the assumptions of Multiple Forecasting Variables, a ( $M=N$ ), with a Constant Forecasting Ability level,  $d$ , Estimation Error and in the presence of a Budget Constraint with a Constant Pair-wise Correlation,  $\rho$ , and a Constant Stock Volatility,  $\sigma$ , across all Assets using the Conditional Covariance Matrix*

$$E[U] = \frac{(N-1) \left( \frac{\sigma_\mu^2}{\sigma^2} (1-\rho) + d^2 - \frac{1}{T} \right) + O(1)}{2\lambda(1-d^2)}$$

Now

$$\begin{aligned}
V_{\frac{1}{N}} &= \mu_1 - \frac{\lambda}{2N^2} i' \Sigma i \\
&= \mu_1 - \frac{\lambda}{2N^2} (N\sigma^2 + (N^2 - N)\rho\sigma^2) \\
&= \mu_1 - \frac{\lambda\sigma^2}{2N} - \frac{\lambda\rho\sigma^2}{2} + \frac{\lambda\rho\sigma^2}{2N} \\
&= \mu_1 - \frac{\lambda\rho\sigma^2}{2} + O\left(\frac{1}{N}\right)
\end{aligned}$$

Thus, utility is increasing in  $N$ , for

$$T > \frac{\sigma^2}{d^2\sigma^2 + \sigma_\mu^2(1 - \rho)}$$

We can also conclude that as  $\sigma$  and  $\rho$  increase, expected utility decreases. Further, as cross-sectional dispersion,  $\sigma_u$  increases, expected utility increases.

## Appendix E – Derivation of the Coefficient of Risk Aversion

To ensure that our results are robust across the risk spectrum, we use three levels of risk aversion, representing a conservative, a balanced and an aggressive investor. Kritzman (2011) for a different purpose, shows how we can infer the level of risk aversion using actual investor allocations. For an investor that allocates to bonds and equities the constant absolute risk aversion utility function can be expressed as:

$$E(U) = x_e\mu_e + x_b\mu_b - \frac{\lambda}{2}(\sigma_e^2w_e^2 + \sigma_b^2w_b^2 + 2\rho\sigma_e\sigma_bw_e w_b)$$

The marginal utility is maximised by equating the partial derivatives:

$$\frac{dE[U]}{dx_e} = \mu_e - \lambda(\sigma_e^2w_e + \rho\sigma_e\sigma_bw_b)$$

$$\frac{dE[U]}{dx_b} = \mu_b - \lambda(\sigma_b^2w_b + \rho\sigma_e\sigma_bw_e)$$

$$\lambda = (\mu_e - \mu_b)/(\sigma_e^2w_e + \rho\sigma_e\sigma_bw_b - \sigma_b^2w_b - \rho\sigma_e\sigma_bw_e)$$

To calibrate equation x we use long-dated US government bonds and the S&P 500 since 1927. The average allocation for balanced US mutual funds is 50% equities and 50% fixed income and cash<sup>26</sup>. Morningstar, the fund rating service, define a conservative (aggressive) fund as a 20-50% (70-90%) allocation to equities, and a 50-80% (10-30%) allocation to fixed income and cash. We use the mid-points of these ranges to give the allocations of the conservative and aggressive investors. The resultant levels of  $\lambda$  are 0.015, 0.025, and 0.045 respectively.

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<sup>26</sup> Ibbotson and Kaplan (2000)

## Appendix E – Derivation of the Practitioner’s Utility Function

In this appendix we show that maximising the constant absolute risk aversion (CARA) utility function is equivalent to maximising the quadratic utility function.

The CARA utility function, is defined as:

$$U = -e^{-\lambda w}$$

where  $\lambda > 0$  equals the Arrow-Pratt coefficient of absolute risk aversion.

The utility function is positively sloped and concave indicating risk aversion.

$$U'(C) = \lambda e^{-\lambda w} > 0$$

$$U''(C) = -\lambda^2 e^{-\lambda w} < 0$$

If wealth,  $w$ , is normally distributed:

$$\tilde{w} \sim N(\mu_w, \sigma_w^2)$$

Then the certainty equivalent,  $CE$ , can be derived as follows:

$$-e^{-\lambda CE} = -E[e^{-\lambda \tilde{w}}]$$

$$= -e^{-\lambda \mu_w + \frac{\lambda}{2} \sigma_w^2}$$

Therefore:

$$-\lambda CE = -\lambda \mu_w + \frac{\lambda}{2} \sigma_w^2$$

$$CE = \mu_w - \frac{\lambda}{2} \sigma_w^2$$

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