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Decentralized Cooperative Network Games in Economics and Finance

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Abstract

This paper introduces a new class of cooperative graphical games, where multiple coalitions engage in decentralized bargaining processes to optimize their collective goals. Unlike traditional cooperative game theory, which often overlooks the dynamics of internal bargaining within coalitions, our approach incorporates a graphical communication structure that includes multigraphs and hypergraphs. This framework introduces a novel concept, the “importance value,” which quantifies the influence of agents and coalitions in the bargaining process. By leveraging distributed optimization techniques, we employ a dual decomposition algorithm that guarantees convergence in a single iteration of bargaining. Additionally, combinatorial methods are used to determine the optimal allocation of agents to bargaining relationships, ensuring an efficient solution to the distributed optimization problem.

Keywords: Cooperative Network Games, Hypergraph Distributed Optimization,
Economic & Financial Networks

JEL Classification: C61 , C62 , C71 , D85

1 Introduction

The study of coalition formation and behavior is central to social sciences. In competitive environments, cooperative game theory provides a formal framework for analyzing coalitions, focusing on coalition structure, strategies, and payoffs. However, this approach often remains at a high level and does not delve deeply into the internal bargaining dynamics within coalitions. In contrast, non-cooperative game theory examines how bargaining affects the distribution of payoffs within coalitions. This paper aims to address this gap by modeling coalitions as collections of agents who can engage in non-cooperative bargaining with agents from other coalitions. This bargaining process influences the payoffs of all coalitions. Our approach generalizes the “position value” introduced by [Meessen \(1988\)](#), applying it to both agents and coalitions.

Coalitions are common in the real world, with agents interacting with those from other coalitions. Agents within coalitions often have varying levels of influence, so we introduce the “importance value” of each agent. This is defined as the product of their coalition’s importance and their individual influence within it. We define the “power value” of a coalition as the inner product of the vector of agents’ importance values and a vector representing the coalition’s collective importance. Both importance and power values are essential for modeling bargaining relationships between agents from different coalitions, using network theory. Previous studies, such as [Bramoullé et al. \(2014\)](#) and [Jackson \(2003\)](#), have explored network theory within cooperative game theory, including the distribution of power in networked environments as seen in [Bienenstock and Bonacich \(1993\)](#) and [Myerson \(1991\)](#). In our model, transferable utility is represented by a weighted multigraph, and we compute the Shapley values [Shapley \(1951\)](#) of each subcoalition using the method proposed by [Mercik et al. \(2021\)](#). Alternative approaches for Shapley value calculation in cooperative games are presented in [Cesari et al. \(2018\)](#) and [Singal et al. \(2021\)](#).

The communication among coalitions is captured by a hypergraph [Berge \(1973\)](#), where agents within a coalition may engage in more complex interactions beyond simple pairwise exchanges. Hypergraphs in cooperative game theory have been explored in [Bulò and Pelillo \(2009\)](#) and [Le Breton et al. \(1992\)](#), and their advantages for communication are discussed in [Heintz et al. \(2019\)](#) and [Wolf et al. \(2016\)](#). Centrality measures, which quantify a coalition’s importance in a network, are applied to hypergraphs and have been widely studied [Bavelas \(1950\)](#) and [Bonacich \(1987\)](#). In our setting, the centrality of a coalition is determined by the degree of the hyperedges attached to it. We define the importance value of each agent as the product of their Shapley value and their coalition’s centrality, combining Shapley values with centrality measures to capture both individual and collective influence within coalitions.

Our analysis focuses on quadratic objective functions, which reflect agents’ importance values and simplify the solution process. While the framework could be extended to non-linear functions, stability analysis would follow a similar approach to [Papastaikoudis et al. \(2024\)](#). The goal is to optimize the welfare of all coalitions through decentralized bargaining. This non-cooperative setting allows agents to use their importance values as reference points in the bargaining process. Depending on the

coalitions' objectives, the optimization may be framed as either a minimization or maximization problem. The algorithm is decentralized, meaning outcomes are determined solely by the agents within each hyperedge, without any centralized intervention. For more on decentralized bargaining, see [Elliott and Nava \(2019\)](#) and [Blouin and Serrano \(2001\)](#). We solve the optimization problem using distributed techniques, as discussed in [Tsitsiklis \(1984\)](#) and [Rao \(1987\)](#). Dual decomposition and first-order methods, common in distributed optimization, are explained in [Kelly et al. \(1998\)](#) and [Palomar and Chiang \(2006\)](#).

We propose a dual decomposition algorithm for quadratic objective functions, which converges quickly. Additionally, we introduce a modification incorporating Hessian matrices, allowing agents to reach agreements after just one iteration by using local Shapley values and centrality information. If Hessians are unavailable, the algorithm remains decentralized, though convergence requires more iterations. The communication matrix of the hypergraph is represented by Bolla's Laplacian for hypergraphs [Bolla \(1993\)](#), which serves as a projection matrix in our information structure. These properties, together with quadratic objective functions, enable fast convergence. The optimal allocation of agents within a coalition is determined using combinatorial techniques.

This framework has real-world applications, such as in the European Union policymaking process, where countries form coalitions based on shared interests. The importance values reflect each country's bargaining power, helping to analyze fair and efficient negotiation outcomes. Similarly, in the Eurozone, where countries' banks form coalitions, the framework can fairly allocate bailout costs by considering each country's financial system importance. The distributed optimization approach models decentralized bargaining among countries and the European Central Bank, ensuring fair negotiations and stable outcomes.

The paper is structured as follows: Section 2 introduces the mathematical tools used, including cooperative game theory [Osborne and Rubinstein \(1994\)](#), graph and hypergraph theory [Vitaly \(2009\)](#), and matrix theory [Horn and Johnson \(2012\)](#). Section 3 discusses the information structure, calculating the importance values of agents and coalitions using Shapley values and centrality measures. Section 4 formulates the bargaining game as a distributed optimization problem, solved using dual decomposition. Section 5 presents the combinatorial distributed optimization case and provides numerical examples.

2 Preliminaries

2.1 Notation

The set of real numbers is $\mathbb{R}$. For $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) means that all components of x are nonnegative (positive). We use $\|\cdot\|_2$ to denote the Euclidean norm in $\mathbb{R}^n$. We use $|A|$ to denote the cardinality of set A . The factorial of a number $n \in \mathbb{Z}$ is denoted by $n!$ and is given by $n! = 1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n$.

2.2 Cooperative Game Theory

Definition 1. A coalitional game with transferable utility is a pair $G : \{N, v\}$, where N is a finite set of agents and is called the grand coalition of G . The payoff function $v : 2^{|N|} \rightarrow \mathbb{R}$ associates with each coalition $S \subset N$ a real valued payoff $v(S)$ that the coalition's members can distribute among themselves. We assume that $0 \leq v(S) \leq v(D) \forall S \subseteq D \subseteq N$ and $v(\emptyset) = 0$.

Definition 2. The function $\phi : \mathbb{N} \times \mathbb{R}^{2^{|N|}} \rightarrow \mathbb{R}^{|N|}$ defines a payoff division game over $G : \{N, v\}$. As a result, $\phi(G)$ is a vector of payoffs to each agent, explaining how they divide the payoff of the grand coalition where with $\phi_i(N, v)$ we define i 's agent payoff.

Theorem 1. Given a coalitional game (N, v) , there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and satisfies the axioms of efficiency, symmetry, dummy player and additivity [Osborne and Rubinstein \(1994\)](#). This payoff division $\phi(N, v)$ is the Shapley value and for i th agent is given by the following formula

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N/\{i\}} |S|!(|N| - |S| - 1)! [v(S \cup \{i\}) - v(S)]. \quad (1)$$

The Shapley value is defined as a fair way of dividing the grand coalition's payoff among its members.

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2.3 Dynamical Systems

We will study a discrete time, autonomous linear system,

$$x(k+1) = Ax(k) + Bu(k) \quad (2)$$

where $x(k), u(k) \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$.

Lemma 2. The system (2) is asymptotically stable if for all eigenvalues λ_i of A we have $|\lambda_i| < 1$.

2.4 Graphs and Hypergraphs

A directed weighted multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is a graph that contains weighted oriented edges and self loops among the nodes. By $\mathcal{V}$ we denote the node set, by $\mathcal{E}$ the edge set where each edge is of the form $e_{i,j} \in \mathcal{E}$ with $i, j \in \mathcal{V}$ and the element i is called the tail of the edge while the element j is called the head of the edge. We will use the symbol $\sim$ in order to denote the adjacency among nodes and edges. The total number of nodes in the network is called the order of the network and is denoted by $|\mathcal{V}|$ while the total number of edges in the network is called the size of the network and is denoted by $|\mathcal{E}|$. By $\mathcal{W}$ we denote the respective set of edge weights where a weight $w_{ij} \in \mathcal{W}, \forall 1 \leq i, j \leq |\mathcal{E}|$ is assigned to each edge of the multigraph. The adjacency

$|\mathcal{V}| \times |\mathcal{V}|$ matrix of the multigraph is given by

$$A = a_{ij} = \begin{cases} w_{i,j}, & \forall e_{i,j} \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

A hypergraph is a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the node set and $\mathcal{E} = \{\mathcal{E}_1, \dots, \mathcal{E}_m\}$ is the hyperedge set. A hyperedge can join any number of nodes and not just two as it is in the case of a graph. The degree of a node v_i denoted by $|v_i|$ in a hypergraph is the total number of hyperedges that are adjacent to this node and the degree of a hyperedge $\mathcal{E}_j$ denoted by $|\mathcal{E}_j|$ is the total number of nodes that are adjacent to this hyperedge. We define by D_V the diagonal $|\mathcal{V}| \times |\mathcal{V}|$ matrix whose diagonal entries are the degrees of each node and by D_E the diagonal $|\mathcal{E}| \times |\mathcal{E}|$ matrix whose diagonal entries are the degrees of each hyperedge. For a hypergraph $\mathcal{H}$, the incidence matrix denoted by E , is a $|\mathcal{V}| \times |\mathcal{E}|$ matrix whose (i, j) -th entry is given by

$$E_{ij} = \begin{cases} 1, & v_i \in \mathcal{E}_j \\ 0, & \text{otherwise.} \end{cases}$$

The hypergraph Laplacian (Bolla's Laplacian) Q is a $|\mathcal{V}| \times |\mathcal{V}|$ matrix given by the formula:

$$Q = D_V - ED_E^{-1}E^T.$$

2.5 Convex Analysis

A differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be convex if $f(x) - f(y) \leq (\nabla f(x))^T(x - y) \forall x, y \in \mathbb{R}^n$. If strict inequality holds then f is a strictly convex function. A function $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz if for each $x, x_0 \in A$, there exist constant $M > 0$ and $\delta_0 > 0$ such that $\|x - x_0\| < \delta_0 \Rightarrow \|f(x) - f(x_0)\| \leq M\|x - x_0\|$.

2.6 Matrix Theory

For a matrix $M \in \mathbb{R}^{n \times n}$ by M^T and $S(M)$ we denote its transpose and its spectrum respectively. Matrix $P \in \mathbb{R}^{n \times n}$ is an orthogonal projection matrix when $P^2 = P = P^T$ while matrix $P \in \mathbb{R}^{n \times n}$ is called a projection matrix if $P^2 = P$. Both types of projection matrices have spectrum $S(P) = \{0, 1\}$. A symmetric matrix M is called positive (semi)definite $M(\succeq) \succ 0$ if and only if $x^T M x (\geq) > 0$ for every nonzero $x \in \mathbb{R}^n$.

3 Importance Values

In this section we will define the importance value of the agents of all coalitions as the product of their importance within the coalition that they belong and the importance of their coalition in the global bargaining game. For the importance of the agents within their coalition we will make use of the Shapley values which we will calculate from the adjacency matrix of the multigraph that describes the transferable

utility among the agents of the coalition. For the importance of each coalition we will make use of the node degree centrality measure for the hypergraph that connects the different coalitions. We will also define the power value of the coalitions as the inner product of the vector with the importance values of its respective agents and the vector of common components equal with the node centrality measure that describes the importance of the coalition.

3.1 Multigraph Shapley Values

We assume that we have K coalitions $C_1, \dots, C_K$ where each coalition C_i has p_i agents. The agents in each coalition communicate via a multigraph $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i, \mathcal{W}_i)$. A cooperative game is being played in each coalition $G_i\{p_i, v_i\}$ where v_i is the payoff function of i th coalition. We assume that the transferable utility among agents of each coalition can be described with the use of the adjacency matrix of the multigraph $A_i = [a_{kl}^i] \in \mathbb{R}^{p_i \times p_i}$ which satisfies the following properties,

$$a_{kl}^i \geq 0 \text{ and } a_{kk}^i > \max\left\{\sum_{k \neq l} a_{kl}^i, \sum_{k \neq l} a_{lk}^i\right\} \forall k, l.$$

These assumptions practically mean that all the transferred utilities among the agents are positive and the total utility that an agent transfers or is transferred to him is always less than his own utility. Following the methodology proposed in [Mercik et al. \(2021\)](#), we will now present a quantity that will help us define the payoff function of the cooperative game and consequently the Shapley values of the coalitions. We denote by $w_i \in \mathbb{R}^{p_i \times 1}$ the vector of coalition C_i that describes the multigraph interdependencies for each agent of the coalition. The vector $w_i = (w_i(1), \dots, w_i(p_i))$ is given by the formula

$$w_i = A_i \cdot 1_{p_i} - A_i^T \cdot 1_{p_i} \quad (3)$$

where 1_{p_i} is a vector of ones of dimension p_i . It is important to note that $w_i \cdot 1_{p_i}^T = 0$, e.g., the sum of multigraph interdependencies among all agents of coalition C_i sums to zero. The payoffs of each agent of the coalition C_i are equal with the respective diagonal elements of matrix A_i , e.g., $(v_i(1), \dots, v_i(p_i)) = (a_{11}^i, \dots, a_{p_i p_i}^i)$ respectively. For the payoffs of each subcoalition within the coalition C_i we have

$$v_i(\{1, \dots, k\}) = \sum_{j=1}^k v_i(\{j\}) + \sum_{j=2}^k w_i(j), \forall k \leq p_i. \quad (4)$$

Using (1) and (4) we will define the Shapley values for each coalition C_i as $p_i \times p_i$ diagonal matrices $A_i^* = (a_i^{1*}, \dots, a_i^{p_i*})$ and use them to represent the importance of the agents within their coalitions. The Shapley values of all the agents for the global bargaining game can be described by the diagonal matrix $A^* = \text{diag}\{A_1^*, \dots, A_K^*\}$.

3.2 Hypergraph Node Centrality of Coalitions

The interactions among the coalitions $C_1, \dots, C_K$ is conducted with the use of a hypergraph $\mathcal{H} = \{\mathcal{V}, \mathcal{E}\}$ where the node set $\mathcal{V} = \{C_1, \dots, C_K\}$ corresponds to the K different

coalitions while the hyperedge set $\mathcal{E}$ corresponds to their interactions. The hypergraph node centrality measure for each coalition will be denoted by b_i , $\forall 1 \leq i \leq K$ and will correspond to the respective element of the diagonal node degree matrix $D_K = \text{diag}\{|C_1|, \dots, |C_K|\}$. Each agent within each coalition has as reference the node centrality measure of his coalition and as a result, $b_i^1 = \dots = b_i^{p_i} = b_i \forall 1 \leq i \leq K$.

3.3 Importance value of the agents

We define the importance of the agents of the global bargaining game as the product of their importance within the coalition that they belong (a_i^{j*}) $\forall 1 \leq i \leq K$, $1 \leq j \leq p_i$ and the importance of their coalition (b_i) $\forall 1 \leq i \leq K$. We will denote the importance of the agents of the global game by I_i^j and the following relationship holds:

$$I_i^j = a_i^{j*} \cdot b_i \forall 1 \leq i \leq K, 1 \leq j \leq p_i. \quad (5)$$

An immediate property is that for the agents of the same coalition we have that

- $I_i^l \geq I_i^m$ iff $a_i^{l*} \geq a_i^{m*} \forall 1 \leq i \leq K, 1 \leq l, m \leq p_i$.

We will denote the global importance of all agents by the $p \times 1$ vector I where $p = p_1 + \dots + p_K$.

3.4 Power value of the coalitions

We define the power value of the coalitions as the inner product of their agents' Shapley values vector $a_i^* = (a_i^{1*}, \dots, a_i^{p_i*})$ and the vector of the respective coalition importance which has all its components equal with the node centrality measure of the respective coalition $\mathbf{b}_i = (b_i^1, \dots, b_i^{p_i}) \forall 1 \leq i \leq K$. We denote the power value of the i th coalition by I_i and the following relationship holds:

$$I_i = \langle a_i^*, \mathbf{b}_i \rangle \forall 1 \leq i \leq K. \quad (6)$$

Some properties regarding the power values of the coalitions are given below

- In the case that there is only one coalition we have that $I_1 = 0$ since $b_1 = 0$.
- In the case that there are only two coalitions we have that $I_1 \geq I_2$ iff $a_1^* \geq a_2^*$.
- If $I_l \geq I_m$ and $I_m \geq I_n \Rightarrow I_l \geq I_n$.

Example 1. We assume that we have 5 coalitions, $C_1 = \{x_1^1\}$, $C_2 = \{x_2^1, x_2^2, x_2^3\}$, $C_3 = \{x_3^1, x_3^2\}$, $C_4 = \{x_4^1, x_4^2\}$ and $C_5 = \{x_5^1\}$ with their multigraph representation given in Figure 7 and their respective adjacency matrices to be:

$$A_1 = (12), A_2 = \begin{pmatrix} 11 & 3 & 1 \\ 2 & 12 & 4 \\ 3 & 1 & 10 \end{pmatrix}, A_3 = \begin{pmatrix} 6 & 1 \\ 2 & 4 \end{pmatrix}, A_4 = \begin{pmatrix} 7 & 3 \\ 1 & 5 \end{pmatrix}, A_5 = (9).$$

The first and fifth coalitions are trivial since they are formed by a single agent. The Shapley values of these coalitions coincide with their actual value as they are given

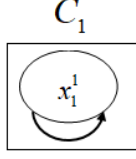


Fig. 1: 1st coalition

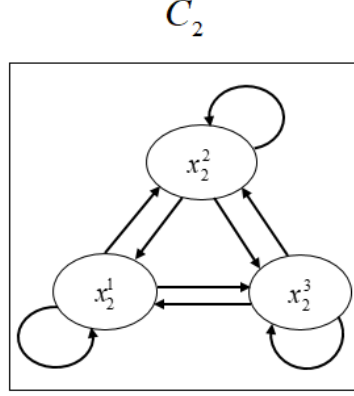


Fig. 2: 2nd Coalition

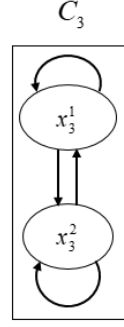


Fig. 3: 3rd Coalition

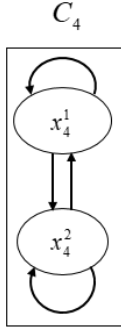


Fig. 4: 4th Coalition

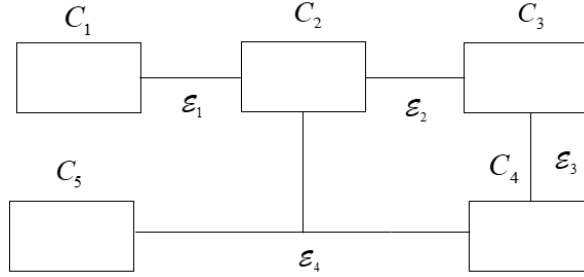


Fig. 5: Hypergraph Coalitions communication.

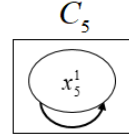


Fig. 6: 5th coalition

Fig. 7: Multigraph Representations of Coalitions

by the scalar matrices A_1 and A_5 . As a result, $a_1^{1*} = 12$ and $a_5^{1*} = 9$. The third and fourth coalitions are two agent coalitions. The Shapley values of these coalitions will be calculated by the formulas (3) and (4) for the matrices A_3 and A_4 . For the third coalition we have from matrix A_3 that $v_3(\{1\}) = 6$, $v_3(\{2\}) = 4$, $w_3(1) = 1 - 2 = -1$ and $w_3(2) = 2 - 1 = 1$. We calculate the payoff for the subcoalitions as $v_3(\{1, 2\}) = v_3(\{1\}) + v_3(\{2\}) + w_3(2) = 6 + 4 + 1 = 11$ and $v_3(\{2, 1\}) = v_3(\{2\}) + v_3(\{1\}) + w_3(1) = 4 + 6 - 1 = 9$. The Shapley values for the third coalition can be calculated by table 1 and as a result,

$$a_3^{1*} = \frac{1}{2}[6 + 5] = 5.5, \quad a_3^{2*} = \frac{1}{2}[5 + 4] = 4.5$$

Similarly, for the fourth coalition from matrix A_4 we have that $v_4(\{1\}) = 7$, $v_4(\{2\}) = 5$, $w_4(1) = 3 - 1 = 2$ and $w_4(2) = 1 - 3 = -2$. We calculate the payoff for the subcoalitions as $v_4(\{1, 2\}) = v_4(\{1\}) + v_4(\{2\}) + w_4(2) = 7 + 5 - 2 = 10$ and

Probability	Order of Arrival	1st Marginal Contribution	2nd Marginal Contribution
1/2	1-2	$v_3(\{1\}) = 6$	$v_3(\{1, 2\}) - v_3(\{1\}) = 5$
1/2	2-1	$v_3(\{2, 1\}) - v_3(\{2\}) = 5$	$v_3(\{2\}) = 4$

Table 1: Table of 3rd Coalition

$v_4(\{2, 1\}) = v_4(\{2\}) + v_4(\{1\}) + w_4(1) = 7 + 5 + 2 = 14$. The Shapley values for the fourth coalition can be calculated by table 2 and as a result,

$$a_4^{1*} = \frac{1}{2}[7 + 9] = 8 \quad a_4^{2*} = \frac{1}{2}[5 + 3] = 4.$$

Probability	Order of Arrival	1st Marginal Contribution	2nd Marginal Contribution
1/2	1-2	$v_4(\{1\}) = 7$	$v_4(\{1, 2\}) - v_4(\{1\}) = 3$
1/2	2-1	$v_4(\{2, 1\}) - v_4(\{2\}) = 9$	$v_4(\{2\}) = 5$

Table 2: Table of 4th Coalition

For the second coalition, following the same methodology for matrix A_2 , we have $v_2(\{1\}) = 11, v_2(\{2\}) = 12, v_2(\{3\}) = 10, w_2(1) = 4 - 5 = -1, w_2(2) = 6 - 4 = 2$ and $w_2(3) = 4 - 5 = -1$. For the pairwise subcoalitions we have $v_2(\{1, 2\}) = v_2(\{1\}) + v_2(\{2\}) + w_2(2) = 11 + 12 + 2 = 25$ and similarly for the other pairwise combinations we have $v_2(\{2, 1\}) = 22, v_2(\{1, 3\}) = 20, v_2(\{3, 1\}) = 20, v_2(\{2, 3\}) = 21$ and $v_2(\{3, 2\}) = 21$. For the combinations of the grand coalition we have $v_2(\{1, 2, 3\}) = v_2(\{1\}) + v_2(\{2\}) + v_2(\{3\}) + w_2(2) + w_2(3) = 11 + 12 + 10 + 2 - 1 = 34$ and for the rest combinations we have $v_2(\{1, 3, 2\}) = 34, v_2(\{2, 1, 3\}) = 31, v_2(\{2, 3, 1\}) = 31, v_2(\{3, 1, 2\}) = 34, v_2(\{3, 2, 1\}) = 34$. The Shapley values for the third coalition can be calculated by table 3 and as a result,

$$a_2^{1*} = \frac{1}{6}[11 + 11 + 10 + 10 + 10 + 13] = \frac{71}{6} = 10.83$$

$$a_2^{2*} = \frac{1}{6}[14 + 14 + 12 + 12 + 14 + 11] = \frac{77}{6} = 12.83$$

$$a_2^{3*} = \frac{1}{6}[9 + 9 + 9 + 9 + 10 + 10] = \frac{56}{6} = 9.34$$

The Shapley values of all the agents for the global bargaining game can be described as

$$A^* = \text{diag}\{A_1^*, \dots, A_5^*\} = \text{diag}\{12, 10.83, 12.83, 9.34, 5.5, 4.5, 8, 4, 9\}.$$

Prob.	Arrival Order	1 st Marginal Contribution	2 nd Marginal Contribution	3 rd Marginal Contribution
1/6	1-2-3	$v_2(\{1\}) = 11$	$v_2(\{1, 2\}) - v_2(\{1\}) = 14$	$v_2(\{1, 2, 3\}) - v_2(\{1, 2\}) = 9$
1/6	1-3-2	$v_2(\{1\}) = 11$	$v_2(\{1, 3, 2\}) - v_2(\{1, 3\}) = 14$	$v_2(\{1, 3\}) - v_2(\{1\}) = 9$
1/6	2-1-3	$v_2(\{2, 1\}) - v_2(\{2\}) = 10$	$v_2(\{2\}) = 12$	$v_2(\{2, 1, 3\}) - v_2(\{1, 2\}) = 9$
1/6	2-3-1	$v_2(\{2, 3, 1\}) - v_2(\{2, 3\}) = 10$	$v_2(\{2\}) = 12$	$v_2(\{2, 3\}) - v_2(\{2\}) = 9$
1/6	3-1-2	$v_2(\{3, 1\}) - v_2(\{3\}) = 10$	$v_2(\{3, 1, 2\}) - v_2(\{3, 1\}) = 14$	$v_2(\{3\}) = 10$
1/6	3-2-1	$v_2(\{3, 2, 1\}) - v_2(\{3\}) = 13$	$v_2(\{3, 2\}) - v_2(\{3\}) = 11$	$v_2(\{3\}) = 10$

Table 3: Table of 2nd Coalition

For the calculation of the importance of the coalitions we have to calculate the hypergraph node degree matrix of the 5th subfigure in Figure 7 which is given below,

$$D_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have that the node centrality measures of the coalitions and their respective agents are:

$$\begin{aligned} b_1^1 &= b_1 = 1 \\ b_2^1 &= b_2^2 = b_2^3 = b_2 = 3 \\ b_3^1 &= b_3^2 = b_3 = 2 \\ b_4^1 &= b_4^2 = b_4 = 2 \\ b_5^1 &= b_5 = 1. \end{aligned}$$

The coalition centrality measure can be written more compactly in vector form $b^T = (1, 3, 3, 3, 2, 2, 2, 2, 1)$. As a result, the importance values of the agents are:

$$\begin{aligned} I_1^1 &= 12 \cdot 1 = 12 \\ I_2^1 &= 10.83 \cdot 3 = 32.49, I_2^2 = 12.83 \cdot 3 = 38.49, I_2^3 = 9.34 \cdot 3 = 28.02 \\ I_3^1 &= 5.5 \cdot 2 = 11, I_3^2 = 4.5 \cdot 2 = 9 \\ I_4^1 &= 8 \cdot 2 = 16, I_4^2 = 4 \cdot 2 = 8 \\ I_5^1 &= 9 \cdot 1 = 9 \end{aligned}$$

Now regarding the power values of the coalitions we have:

$$I_1 = 12, I_2 = 99, I_3 = 20, I_4 = 24, I_5 = 9.$$

4 Hypergrpah Distributed Optimization

The global bargaining game can also be described with the use of a new hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E})$ that results by incorporating the agents of each coalition in the information structure of the hypergraph $\mathcal{H}$ that describes the coalitions bargaining relationships. In this new hypergraph the node set does not represent just the coalitions but the agents and the coalitions at the same time via appropriate partitions. We have that $\mathcal{V}' = \{\mathcal{C}_1 = \{x_1^1, \dots, x_1^{p_1}\}, \dots, \mathcal{C}_K = \{x_K^1, \dots, x_K^{p_K}\}\}$, i.e., the nodes represent

the agents partitioned according to the coalitions that they belong. The hyperedges represent the interactions among the coalitions via their respective agents and each hyperedge from $\mathcal{E} = \{\mathcal{E}_1, \dots, \mathcal{E}_N\}$ corresponds to a bargaining relationship. For the optimization of the objective function of each coalition given the importance parameters, the bargaining relationships among the coalitions and the augmented hypergraph $\mathcal{H}'$ we have the following distributed optimization problem for K coalitions,

$$\begin{aligned} \min_{x=[x_1, \dots, x_K]} \quad & \sum_{i=1}^K f_i(x_i, A_i^*, b_i) \\ \text{s.t.} \quad & x = Ez \quad i = 1, \dots, K \end{aligned} \tag{7}$$

where

- $f_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}$ is the objective function of i th coalition and is considered to be strictly convex, continuously differentiable with its gradient ∇f_i being locally Lipschitz.
- Vectors $x_i, \forall 1 \leq i \leq K$ denote the importance values of the agents of the K coalitions in the global bargaining game and the component $x_i^l, \forall 1 \leq l \leq p_i$ of vector x_i denotes the importance value of the l th agent of i th coalition. The variables of all coalitions can be expressed in compact form as $x = (x_1, \dots, x_K) \in \mathbb{R}^p$, where $p = p_1 + \dots + p_K$.
- Matrix A_i^* corresponds to the $p_i \times p_i$ diagonal matrix with the importance values (Shapley values) of the agents of i th coalition. A compact expression of the importance values of all the coalitions is given by the $p \times p$ diagonal matrix $A^* = \text{diag}\{A_1^*, \dots, A_K^*\}$.
- The $p_i \times 1$ vector b_i is the vector of the coalition importance (hypergraph node centrality) of i th coalition. A compact expression of the importance values of all the coalitions is given by the $p \times 1$ vector $b = \text{vec}(b_i)_{i=1}^K$.
- Vector z is a $N \times 1$ vector where each component $z_j, \forall 1 \leq j \leq N$ corresponds to the agreed value among the agents for the j th bargaining relationship. We have assumed that there are N bargaining relationships in total among the coalitions and also that each agent from each coalition is responsible for exactly one bargaining relationship.
- Matrix E has dimensions $p \times N$ and is the incidence matrix of $\mathcal{H}'$ that allocates the agents of the coalitions to their respective bargaining relationships.

The solution of (7) shows how the importance values of the agents affect the bargaining processes of the coalitions and vice versa. The Lagrangian of (7) using general objective function f_i for each coalition is

$$\mathcal{L}(x, z, v) = \sum_{i=1}^K f_i(x_i) - v^T(x - Ez) \tag{8}$$

where $v \in \mathbb{R}^p$ is the Lagrange multiplier associated with $x = Ez$. The relationship (8) can also be written in the form

$$\mathcal{L}(x, z, v) = \sum_{i=1}^K (f_i(x_i) - v_i^T x_i) + v^T Ez$$

where v_i is the subvector of v associated with the i th coalition. The optimality conditions are:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \nabla f_i = v_i \text{ (coalitions interconnections)} \quad (9a)$$

$$\frac{\partial L}{\partial v} = 0 \Rightarrow x = Ez \text{ (primal feasibility)} \quad (9b)$$

$$\frac{\partial L}{\partial z} = 0 \Rightarrow E^T v = 0 \text{ (dual feasibility)}. \quad (9c)$$

We will study the case where the objective function of each coalition is of the quadratic form $f_i(x_i) = \frac{1}{2}x_i^T(A_i^*)^{-1}x_i - b_i^T x_i + c_i$, $1 \leq i \leq K$ where c_i is a real scalar assumed to be a constant. The quadratic hypergraph distributed optimization problem of the coalitions in (7) can also be written in the following form,

$$\begin{aligned} \min_{x=[x_1, \dots, x_K]} \quad & \frac{1}{2}x^T(A^*)^{-1}x - b^T x + c \\ \text{s.t.} \quad & x = Ez \end{aligned} \quad (10)$$

where $c = \sum_{i=1}^K c_i \in \mathbb{R}$.

Remark 1. We included in the quadratic version of the objective function the inverse of A_i^* so that an unconstrained optimization of (10) to result to the importance values of the agents, i.e., $x^* = A^*b = I$.

4.1 Dual Decomposition Algorithm

Now we will use a discrete-time dual decomposition algorithm that was proposed in Samar et al. (2007) and solves problem (7). To find the dual function we first minimize (8) over z , which results in the condition (9c). Then we solve the following subproblems independently for each i given v_i

$$\min_{x_i} f_i(x_i) - v_i^T x_i \quad (11)$$

Since the objective functions $f_i(x_i)$ in their general form are assumed strictly convex, the solution to each coalition is unique. The dual of the original problem (7) is

$$\max_{v=[v_1, \dots, v_K]} \sum_{i=1}^K q_i(v_i)$$

$$\text{s.t. } E^T v = 0 \quad (12)$$

where $q_i(v_i) = \min_{x_i} \{f_i(x_i) - v_i^T x_i\}$. The dynamical system of the dual decomposition algorithm that we will use is presented below:

Dynamical System

$$E^T v(0) = 0 \quad (13a)$$

$$\nabla f(x(k)) = v(k) \quad (13b)$$

$$z(k) = (E^T E)^{-1} E^T x(k) \quad (13c)$$

$$v(k+1) - v(k) = -\rho(x(k) - Ez(k)). \quad (13d)$$

where ρ is the stepsize of the algorithm. Substituting (13c) in (13d) we have

$$v(k+1) - v(k) = -\rho(x(k) - E(E^T E)^{-1} E^T x(k)) \Rightarrow \quad (14a)$$

$$v(k+1) - v(k) = -\rho Q x(k) \quad (14b)$$

where

$$Q = I - E(E^T E)^{-1} E^T. \quad (15)$$

Lemma 3. *Matrix Q in (20) is an orthogonal projection matrix.*

Proof. We have that

$$\begin{aligned} Q^2 &= [I - E(E^T E)^{-1} E^T]^2 \\ &= [I - E(E^T E)^{-1} E^T][I - E(E^T E)^{-1} E^T] \\ &= I - 2E(E^T E)^{-1} E^T \\ &\quad + E(E^T E)^{-1} E^T E(E^T E)^{-1} E^T \\ &= I - 2E(E^T E)^{-1} E^T + E(E^T E)^{-1} E^T \\ &= I - E(E^T E)^{-1} E^T \\ &= Q. \end{aligned}$$

As a result, Q is an orthogonal projection matrix. The spectrum of Q is $S(Q) = \{0, 1\}$. $\square$

Using the fact that the gradient of the objective quadratic function in (10) is equal to $\nabla f(x) = (A^*)^{-1} x - b = v \Rightarrow x = A^*(v + b)$ the algorithm (18a)-(18d) can also be written compactly as follows:

$$E^T v(0) = 0 \quad (17a)$$

$$v(k+1) = (I - \rho Q A^*) v(k) - \rho Q A^* b \quad (17b)$$

For reasons of completeness we will provide proofs regarding the convergence of the proposed algorithm and the optimality of its equilibrium point.

Theorem 4. *The algorithm in (13a)-(13d) converges asymptotically.*

Proof. We consider the compact form in (17a)-(17b). Since matrix Q is a projection matrix with its maximum eigenvalue $\lambda_{\max}(Q) = 1$ we have that $\lambda_{\max}(QA^*) \leq \lambda_{\max}(A^*)$. By choosing the stepsize to be $\rho = \frac{1}{\lambda_{\max}(A^*)}$, all the eigenvalues of matrix $I - \rho QA^*$ satisfy the following relationship: $0 \leq \lambda_i(I - \rho QA^*) < 1$, $\forall 1 \leq i \leq p$. As a result, the algorithm converges asymptotically. $\square$

Theorem 5. *The algorithm in (13a)-(13d) satisfies the optimality conditions (9a)-(9c) at equilibrium.*

Proof. Conditions, (9a), (9b) are trivially satisfied at equilibrium from (13b), (13d), respectively. It remains to show that $E^T v = 0$ at equilibrium. Since we have the initial conditions $E^T v(0) = 0$ it is sufficient to show $E^T v(k+1) = E^T v(k)$ for all $k \geq 0$. Premultiplying (13d) by E^T we need to show that

$$-\rho E^T(x(k) - Ez(k)) = 0.$$

This holds since

$$\begin{aligned} E^T(x(k) - Ez(k)) &= E^T(I - E(E^T E)^{-1} E^T)x(k) = \\ &= (E^T - E^T E(E^T E)^{-1} E^T)x(k) = \\ &= 0. \end{aligned}$$

$\square$

4.2 Convergence after one Bargaining Iteration

We will now present a modification of the dual decomposition algorithm which finds optimal solution in one iteration of bargaining for quadratic objective functions by using the Hessian matrix $H = \nabla^2 f(x) = (A^*)^{-1}$. The dynamical system of the algorithm that we will use is presented below:

Dynamical System

$$E^T v(0) = 0 \tag{18a}$$

$$\nabla f(x(k)) = v(k) \tag{18b}$$

$$z(k) = (E^T H E)^{-1} E^T H x(k) \tag{18c}$$

$$v(k+1) - v(k) = -H(x(k) - Ez(k)). \tag{18d}$$

Substituting (18c) in (18d) we have

$$v(k+1) - v(k) = -H(x(k) - E(E^T H E)^{-1} E^T H x(k)) \Rightarrow \tag{19a}$$

$$v(k+1) - v(k) = -H Q x(k) \tag{19b}$$

where

$$Q' = I - E(E^T H E)^{-1} E^T H. \quad (20)$$

Lemma 6. *Matrix Q' in (20) is a projection matrix.*

Proof. We have that

$$\begin{aligned} (Q')^2 &= [I - E(E^T H E)^{-1} E^T H]^2 \\ &= [I - E(E^T H E)^{-1} E^T H][I - E(E^T H E)^{-1} E^T H] \\ &= I - 2E(E^T H E)^{-1} E^T H \\ &\quad + E(E^T H E)^{-1} E^T H E (E^T H E)^{-1} E^T H \\ &= I - 2E(E^T H E)^{-1} E^T H + E(E^T H E)^{-1} E^T H \\ &= I - E(E^T H E)^{-1} E^T H \\ &= Q'. \end{aligned}$$

As a result, Q' is a projection matrix. The spectrum of Q' is $S(Q') = \{0, 1\}$. $\square$

Using the fact that the gradient of the quadratic function f is equal to $\nabla f(x) = Hx - b = v \Rightarrow x = H^{-1}(v + b)$ the algorithm can also be written compactly as follows:

$$E^T v(0) = 0 \quad (22a)$$

$$v(k+1) = (I - HQ'H^{-1})v(k) - HQ'H^{-1}b. \quad (22b)$$

It is important to note that algorithm (22a)-(22b) is fully decentralized and the same holds for algorithm (17a)-(17b), i.e., no central intervention from the coalitions and as a result, the agents attached to the same hyperedge can be viewed as representatives of their coalitions in a non cooperative bargaining game whose outcome affects the final payoff of the coalitions. The following theorems present results which are associated with the convergence of the proposed algorithm in one iteration and the optimality of its equilibrium point.

Theorem 7. *The algorithm in (18a)-(18d) converges in one iteration.*

Proof. We consider the compact form in (22a)-(22b). By considering an iteration to find $v(k+2)$ we have

$$\begin{aligned} v(k+2) &= (I - HQ'H^{-1})v(k+1) - HQ'H^{-1}b \\ &= (I - HQ'H^{-1})((I - HQ'H^{-1})v(k) - HQ'H^{-1}b) \\ &\quad - HQ'H^{-1}b \\ &= (I - HQ'H^{-1})^2 v(k) - (I - HQ'H^{-1})HQ'H^{-1}b \\ &\quad - HQ'H^{-1}b \\ &= (I - HQ'H^{-1})^2 v(k) - 2HQ'H^{-1}b \\ &\quad + (HQ'H^{-1})^2 b. \end{aligned}$$

We notice that

$$\begin{aligned}
(I - HQ'H^{-1})^2 &= (I - HQ'H^{-1})(I - HQ'H^{-1}) \\
&= I - 2HQ'H^{-1} + HQ'H^{-1}HQ'H^{-1} \\
&= I - 2HQ'H^{-1} + H(Q')^2H^{-1} \\
&= I - HQ'H^{-1}
\end{aligned}$$

since Q' is a projection matrix which implies $(Q')^2 = Q'$. Similarly for $(HQ'H^{-1})^2$ we have that

$$(HQ'H^{-1})^2 = HQ'H^{-1}HQ'H^{-1} = H(Q')^2H^{-1} = HQ'H^{-1}.$$

As a result,

$$v(k+2) = (I - HQ'H^{-1})v(k) - HQ'H^{-1}b = v(k+1).$$

Therefore by induction $v(k) = v(1)$ for $k \geq 1$, hence the algorithm converges in one iteration. $\square$

Theorem 8. *The algorithm in (18a)-(18d) satisfies the optimality conditions (9a)-(9c) at equilibrium.*

Proof. Conditions, (9a), (9b) are trivially satisfied at equilibrium from (18b), (18d), respectively. It remains to show that $E^T v = 0$ at equilibrium. Since we have the initial conditions $E^T v(0) = 0$ it is sufficient to show $E^T v(k+1) = E^T v(k)$ for all $k \geq 0$. Premultiplying (18d) by E^T we need to show that

$$E^T H(x(k) - Ez(k)) = 0.$$

This holds since

$$\begin{aligned}
E^T H(x(k) - Ez(k)) &= E^T H(I - E(E^T H E)^{-1} E^T H)x(k) \\
&= 0.
\end{aligned}$$

$\square$

We would like to mention two important remarks regarding the algorithm that we used.

Remark 2. *If we did not use the diagonal matrices $(A_i^*)^{-1}$ with the respective Shapley values of the coalitions in the objective functions of the distributed optimization formulation and we used instead the transferable utility matrices A_i then the modified dual decomposition algorithm would not be decentralized since the communication matrix Q' in (20) would be a full block matrix, i.e., the algorithm would require global information. This practically means that all the agents would get involved in all the bargaining interactions and the problem would be centralized.*

Remark 3. The fact that the original dual decomposition algorithm converges to the optimum after multiple iterations in contrast of the one step convergence of the modified dual decomposition algorithm can be explained as the lack of local information regarding the importance of the agents and their coalitions in the bargaining relationships that provides the use of the Hessian matrix in the modified dual decomposition algorithm.

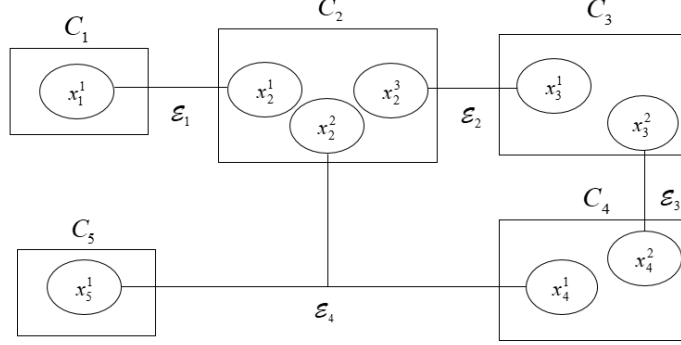


Fig. 8: Interconnections of coalitions

Example 2. Continuing the previous example and combining the subfigures of Figure 7 our new hypergraph is given in Figure 8 with the coalitions and their agents to be represented by the node set $\mathcal{V} = \{C_1, C_2, C_3, C_4, C_5\}$ where $C_1 = \{x_1^1\}$, $C_2 = \{x_2^1, x_2^2, x_2^3\}$, $C_3 = \{x_3^1, x_3^2\}$, $C_4 = \{x_4^1, x_4^2\}$ and $C_5 = \{x_5^1\}$. The bargaining relationships are represented by the hyperedge set $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ where each hyperedge represents a bargaining relationship among the agents of the five coalitions. Agents $\{x_1^1, x_2^1\}$ are attached to bargaining relationship $\mathcal{E}_1$, agents $\{x_2^3, x_3^1\}$ are attached to bargaining relationship $\mathcal{E}_2$, agents $\{x_3^2, x_4^2\}$ are attached to bargaining relationship $\mathcal{E}_3$ and agents $\{x_2^2, x_4^1, x_5^1\}$ are attached to bargaining relationship $\mathcal{E}_4$. The node degree matrix is $D_V = I_{9 \times 9}$ and the hyperedge degree matrix is $D_E = \text{diag}\{2, 2, 2, 3\}$. The hypergraph incidence matrix is

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{bmatrix}.$$

where each E_i matrix corresponds to the respective coalition C_i . We assume the following distributed optimization problem for the given information structure:

$$\min_{x_1, x_2, x_3, x_4, x_5} \sum_{i=1}^5 f_i(x_i)$$

$$s.t. \quad x_i = E_i z, \quad 1 \leq i \leq 5$$

where

$$\begin{aligned} f_1(x_1) &= \frac{12}{2}x_1^2 + x_1 + c_1 \\ f_2(x_2) &= \frac{1}{2}x_2^T \begin{pmatrix} 10.83 & 0 & 0 \\ 0 & 12.83 & 0 \\ 0 & 0 & 9.34 \end{pmatrix} x_2 + (3 \ 3 \ 3)^T x_3 + c_2 \\ f_3(x_3) &= \frac{1}{2}x_3^T \begin{pmatrix} 5.5 & 0 \\ 0 & 4.5 \end{pmatrix} x_3 + (2 \ 2)^T x_3 + c_3 \\ f_4(x_4) &= \frac{1}{2}x_4^T \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix} x_4 + (2 \ 2)^T x_3 + c_4 \\ f_5(x_5) &= \frac{9}{2}x_4^2 + x_4 + c_5 \end{aligned}$$

with $x_1, x_5 \in \mathbb{R}$, $x_3, x_4 \in \mathbb{R}^2$ and $x_2 \in \mathbb{R}^3$. According to Figure 8 we can describe the bargaining agreement values with the vector $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \in \mathbb{R}^4$. The problem can also be written in the following form

$$\min_x \frac{1}{2}x^T A^* x - b^T x + c$$

$$s.t. \quad x = Ez$$

where

$$A^* = \text{diag}\{12, 10.83, 12.83, 9.34, 5.5, 4.5, 8, 4, 9\}, b^T = (1, 3, 3, 3, 2, 2, 2, 2, 1), E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and constant $c \in \mathbb{R}$ to be randomly chosen. For the standard dual decomposition algorithm we have that matrix Q is given by the formula $Q = I - E(E^T E)^{-1}E^T$

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{-1}{3} & \frac{-1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & 0 & \frac{-1}{3} & 0 & 0 & 0 & 0 & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}$$

By implementing the original dual decomposition algorithm (13a)-(13d) with initial conditions $v(0)^T = 0_{9 \times 1}$ and stepsize $\rho = \frac{1}{12}$ we see in Figure 9 the convergence of the bargaining relationships and their respective dual variables in Figure 10

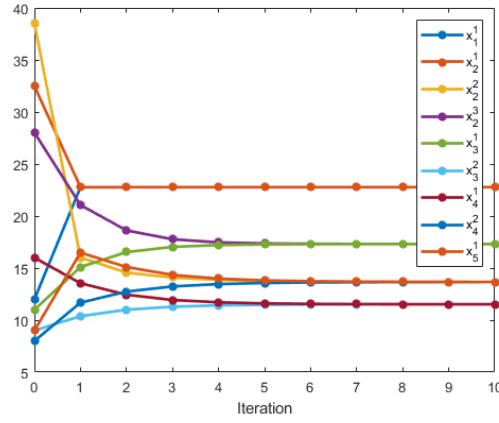


Fig. 9: Convergence of Bargaining Processes

For the modified dual decomposition algorithm we have that matrix Q' is given by the formula

$$Q = I - E(E^T A E)^{-1} E^T A = \begin{pmatrix} 0,47 & -0,47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0,53 & 0,53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0,50 & 0 & 0 & 0 & 0 & -0,15 & -0,35 & 0 \\ 0 & 0 & 0 & 0,37 & -0,37 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0,63 & 0,63 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0,64 & -0,64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0,36 & 0,36 & 0 & 0 & 0 \\ 0 & 0 & -0,50 & 0 & 0 & 0 & 0 & 0,85 & -0,35 & 0 \\ 0 & 0 & -0,50 & 0 & 0 & 0 & 0 & -0,15 & 0,65 & 0 \end{pmatrix}$$

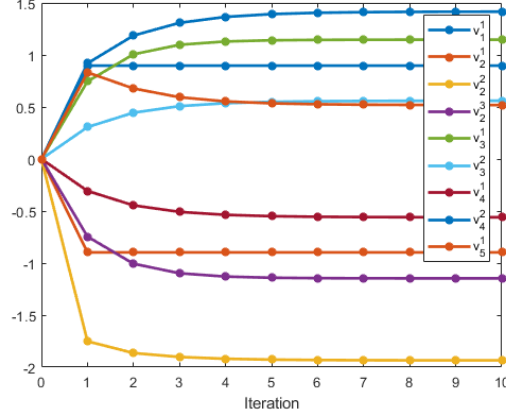


Fig. 10: Convergence of Dual variables

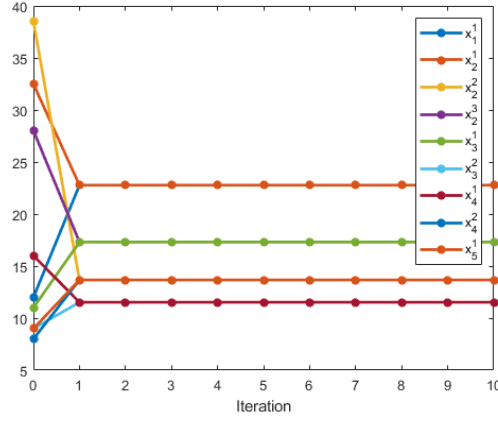


Fig. 11: Convergence of Bargaining Processes

By implementing the modified dual decomposition algorithm (18a)-(18d) with initial conditions $v(0)^T = 0_{9 \times 1}$ we can observe from Figure 11 that all the agents from all coalitions reach an agreement in one bargaining iteration and the same holds for their respective dual variables as we can see in Figure 12.

Remark 4. It is important to note that the dual variables can be viewed as “shadow prices” for the primal variables. Since we have not impose constraints on the primal variables it is possible the dual variables to become negative. This can be viewed as a form of “sacrifice” from the respective agents, i.e., for the greater good of the coalition some of its agents may have losses. Also the dual variables have to sum to zero at each iteration, i.e., $E^T v(k) = 0, k \geq 0$. This property can be interpreted as a simultaneous zero sum game for the shadow prices of the agents.

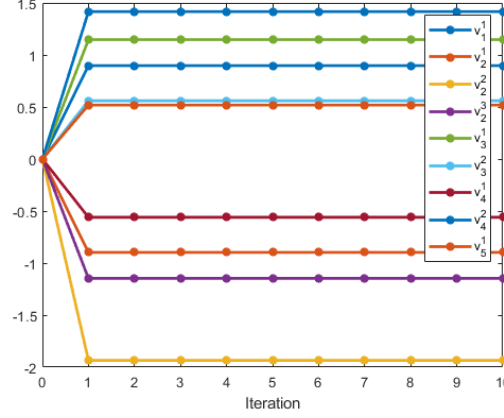


Fig. 12: Convergence of Dual Variables

5 Combinatorial Decentralized Cooperative Graphical Games

In this section with the use of combinatorial techniques we will find the optimum allocation of agents to the corresponding agreements/hyperedges in terms of the optimum value for the distributed optimization problem (10). Since we have that only one agent from each coalition is responsible for exactly one bargaining relationship we have the following result:

Lemma 9. *All possible allocation combinations of agents to the respective bargaining relationships of their coalitions that are described by a hypergraph $\mathcal{H} = \{\mathcal{V}, \mathcal{E}\}$ with $|\mathcal{V}| = p_1 + \dots + p_K = p$ and $\mathcal{E} = N$ are given by the following formula:*

$$\mathcal{C}\{\mathcal{H}\} = \prod_{i=1}^K p_i! = p_1! \cdots p_K!. \quad (25)$$

Example 3. *If we assumed the previous example we would have $\mathcal{C}\{\mathcal{H}\} = p_1! \cdots p_5! = 1!3!2!2!1! = 24$ possible combinations which for reasons of space we will not present but instead we introduce a new example that has less possible combinations. Let's assume that we have the coalitions that are depicted in the first subfigure of Figure 13 where the Shapley values of each coalition are given in the following matrices*

$$A_1^* = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}, A_2^* = (3), A_3^* = \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}, A_4^* = (2), A_5^* = (9)$$

and the node centrality measures of the coalitions are given below:

$$b_1^1 = b_1^1 = b_1 = 2, b_2^1 = b_2 = 1, b_3^1 = b_3^2 = b_3 = 2, b_4^1 = b_4 = 1, b_5^1 = b_5 = 1.$$

The variables of the coalitions are $x_1, x_4, x_5 \in \mathbb{R}$ and $x_1, x_3 \in \mathbb{R}^2$. We notice that we have three agreement common values represented by the vector $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{R}^3$. The possible combinations of this example are $\mathcal{C}\{\mathcal{H}\} = 2!1!2!1!1! = 4$ and all these combinations are depicted in Figure 13.

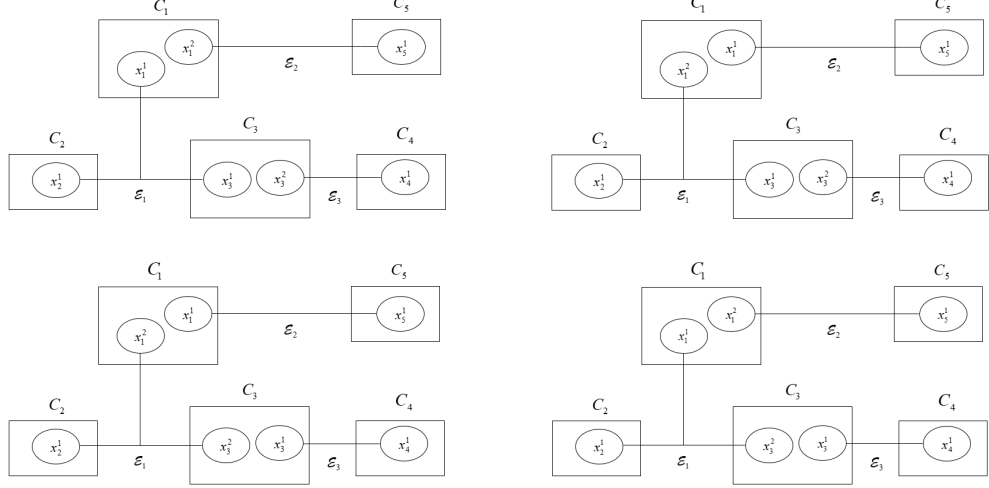


Fig. 13: Plots of the combinations of interactions among the agents of the coalitions

The problem that we study can be written in the following compact form where constant $c \in \mathbb{R}$ is fixed and randomly chosen. The distributed combinatorial optimization problem is

$$\begin{aligned} \mathcal{C}(\mathcal{H})[1 \leq i \leq 4] : \min_x & \frac{1}{2} x^T (A_i^*)^{-1} x - b^T x + c \\ \text{s.t.} \quad & x = Ez \end{aligned} \quad (26)$$

where

$$A_1^* = \text{diag}\{5, 7, 3, 8, 4, 2, 9\}, A_2^* = \text{diag}\{7, 5, 3, 8, 4, 2, 9\}, A_3^* = \text{diag}\{7, 5, 3, 4, 8, 2, 9\}, A_4^* = \text{diag}\{5, 7, 3, 4, 8, 2, 9\}$$

are the possible combinations of Shapley matrix A^* . The rest quantities b, c and E are not affected by the combinations and they are presented below.

$$b^T = (1, 3, 3, 3, 2, 2, 2, 1), E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We will solve every combination of the distributed optimization problem with the modified dual decomposition algorithm. Communication matrix Q' is affected by the combinations of matrix A and takes the following forms according to each combination:

$$Q_1 = \begin{pmatrix} \frac{11}{16} & 0 & \frac{-3}{16} & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & \frac{9}{16} & 0 & 0 & 0 & 0 & \frac{-9}{16} \\ \frac{-5}{16} & 0 & \frac{13}{16} & \frac{-1}{2} & 0 & 0 & 0 \\ \frac{-3}{16} & 0 & \frac{-3}{16} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{-2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{-7}{16} & 0 & 0 & 0 & 0 & \frac{7}{16} \end{pmatrix}, Q_2 = \begin{pmatrix} \frac{11}{18} & 0 & \frac{-1}{6} & \frac{-4}{9} & 0 & 0 & 0 \\ 0 & \frac{9}{14} & 0 & 0 & 0 & 0 & \frac{-9}{14} \\ \frac{-7}{18} & 0 & \frac{5}{6} & \frac{-4}{9} & 0 & 0 & 0 \\ \frac{-1}{18} & 0 & \frac{-1}{6} & \frac{4}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{-2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{-5}{14} & 0 & 0 & 0 & 0 & \frac{5}{14} \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{-3}{14} & \frac{-2}{7} & 0 & 0 & 0 \\ 0 & \frac{9}{14} & 0 & 0 & 0 & 0 & \frac{-9}{14} \\ \frac{-1}{2} & 0 & \frac{11}{14} & \frac{-2}{7} & 0 & 0 & 0 \\ \frac{-1}{2} & 0 & \frac{-3}{14} & \frac{2}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{-4}{5} & \frac{4}{5} & 0 \\ 0 & \frac{-5}{14} & 0 & 0 & 0 & 0 & \frac{5}{14} \end{pmatrix}, Q_4 = \begin{pmatrix} \frac{7}{12} & 0 & \frac{-1}{4} & \frac{-1}{3} & 0 & 0 & 0 \\ 0 & \frac{9}{16} & 0 & 0 & 0 & 0 & \frac{-9}{16} \\ \frac{-5}{12} & 0 & \frac{3}{4} & \frac{-1}{3} & 0 & 0 & 0 \\ \frac{-1}{12} & 0 & \frac{-1}{4} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{-4}{5} & \frac{4}{5} & 0 \\ 0 & \frac{-7}{16} & 0 & 0 & 0 & 0 & \frac{7}{16} \end{pmatrix}$$

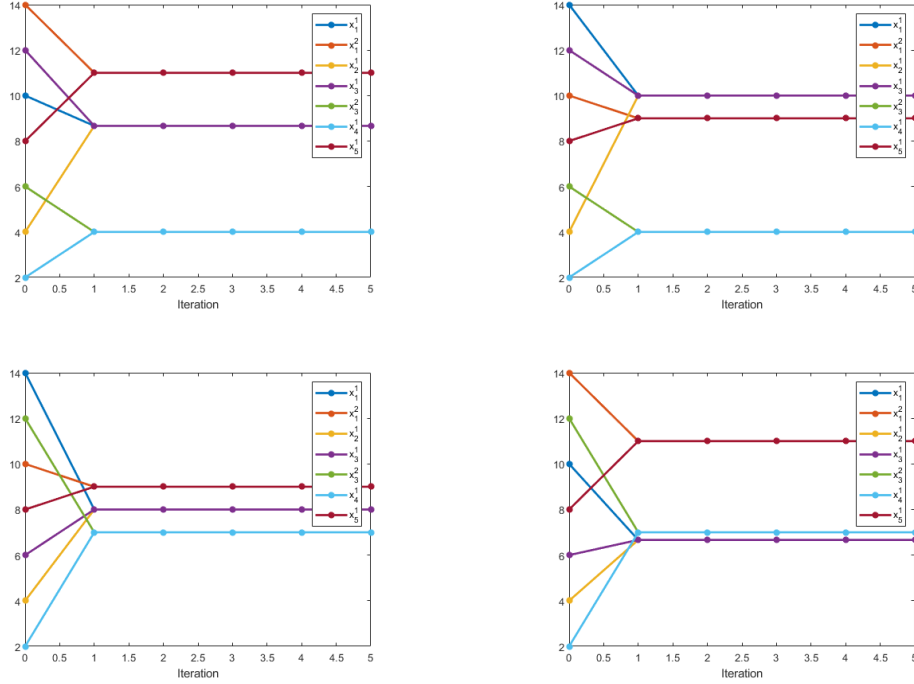


Fig. 14: Convergence plots of all four combinations of interactions among the agents of the coalitions

By implementing the modified dual decomposition algorithm (17a)-(17b) with initial conditions $v(0)^T = 0_{7 \times 1}$ for all the four combinations we can observe in Figure 14 the agreements among the agents after one bargaining iteration. We have that the optimal values of the problem (26) for all four combinations are given below

$$F_1^* = \frac{1}{2}x_1^{*T}A_1^*x_1^* - b^T x_1^* + c = 3,7992 + c$$

$$F_2^* = \frac{1}{2}x_2^{*T}A_2^*x_2^* - b^T x_2^* + c = 6,6107 + c$$

$$F_3^* = \frac{1}{2}x_3^{*T}A_3^*x_3^* - b^T x_3^* + c = 17,4679 + c$$

$$F_4^* = \frac{1}{2}x_4^{*T}A_4^*x_4^* - b^T x_4^* + c = 12,5589 + c$$

where we notice that the first combination provides the best value among the four different combinations of the distributed optimization problem.

6 Conclusion

In conclusion, this paper provides a novel framework for modeling and analyzing coalitions in competitive environments through decentralized bargaining. By combining cooperative game theory with graphical structures such as multigraphs and hypergraphs, we capture the intricate bargaining dynamics that influence payoff distributions within coalitions. The introduction of the “importance value” allows for a deeper understanding of agents’ influence, while the dual decomposition algorithm ensures efficient and fast convergence in the bargaining process. Our approach has practical applications in real-world scenarios, such as international policymaking and economic negotiations, where decentralized decision-making is essential. The combination of distributed optimization techniques and combinatorial methods paves the way for more accurate and fair allocation of resources in coalition-based settings.

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