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Reasoning Continuously: A Formal Construction of Continuous Proofs

Abstract. We begin with the idea that lines of reasoning are continuous mental processes and develop a notion of continuity in proof. This requires abstracting the notion of a proof as a set of sentences ordered by provability. We can then distinguish between discrete *steps* of a proof and possibly continuous *stages*, defining indexing functions to pick these out. Proof stages can be associated with the application of continuously variable rules, connecting continuity in lines of reasoning with continuously variable reasons. Some examples of continuous proofs are provided. We conclude by presenting some fundamental facts about continuous proofs, analogous to continuous structural rules and composition. We take this to be a development on its own, as well as lending support to non-finitistic constructionism.

Keywords: Proof theory, Intuitionism, Constructivism, Continuity, Continuous proof.

1. Introduction

There are two strong reasons to see reasoning by proof as discrete, discontinuous. The first arises in connection with the *a priori* foundations of mathematics in Brouwer: reasoning is discrete because the fundamental mental *acts* involved in mathematical intuition are. The second emerges *a posteriori* from computation: that the fundamental *events* involved in computation are discrete. Indeed, these two strong justifications are mutually supporting, since computation is heavily reliant on intuitionist principles. We begin with two counter-arguments that reasoning can be continuous, one *a priori* and the other *a posteriori*.

L. E. J. Brouwer, the founder of the philosophy of constructivism and intuitionism, considered himself a "neo-intuitionist". According to Brouwer (1912), intuitionism originated in Kant, for whom both time and space (Euclidean geometry) were *a priori* categories of experience. The development of non-Euclidean geometry was the first blow to this original intuitionism, but not a fatal one. According to Brouwer, intuitionism "recovered by abandoning Kant's apriority of space and adhering the more resolutely to the

Presented by **Jacek Malinowski**; *Received* August 7, 2019

Studia Logica (2020) 108: 1145–1160
https://doi.org/10.1007/s11225-019-09892-z

apriority of time" (Brouwer 1912, 85). Given time as an *a priori*, the fundamental acts of mathematical intuition turn out to be (discrete) temporal divisions of "life moments", from "oneness" into "twoness" [ibid]. Significantly, Brouwer and countless later neo-intuitionists take this to imply a restrictive theory of mathematics and symbolic logic, recognizing only the existence of denumerable sets—perhaps with indefinitely refined approximations of continuity. Reasoning and proof are likewise restricted to the denumerable.

Contrary to the intuitionist perspective on the countability of constructions [2], we can begin with the intuition that mental activity is a continuous process, reflexively examinable at any time. That, although mental processes have finite duration, this does not prevent them from being a continuum. We submit that Brouwer's variety of finitistic "intuitionism" is neither "new" nor particularly "Kantian".

This form of non-finitistic intuitionism already pre-exists in Kant, in connection with "psychological grounds of explanation for what goes on in our minds" (Kant 1790, X). For the later Kant, all of the laws for explaining what goes on in our minds are empirical but one: "namely the law of the *continuity* of all changes (since time... is the formal condition of inner intuition)" ([ibid], see [7]). Even confining ourselves to the exposition of Kant, there is a deep tension in holding resolutely to both the apriority of time and the denumerability of mental constructions, since *reasoning* and *proving* are changes. If indeed we hold the aprioricity of time as the basis of mental constructions, then the (mature) Kantian perspective would require their continuity.

Outside neo-Kantianism, the idea that nature as a whole exhibits continuity in change is a common foundational principle. This takes form, in both Leibniz and (more contentiously) Darwin, in the maxim: *natura non facit saltum*, "Nature does not make a leap" (see [1], 6). Applying this naturalistic maxim to reasoning would then imply *a fortiori* that minds, as parts of nature, do not make leaps—*animi non facit saltum*. Even admitting a weaker principle, that perhaps only some aspects of Nature involve continuous processes, this opens the possibility of an *a posteriori* justification for continuity of the specific processes involved in reasoning and computation.

The binary switching operations characteristic of digital computers are thought to support the idea that continuity is impossible in computation—except as indefinite approximation. Nonetheless, these are not the only sorts of computers and, it must be admitted, evidentially underdetermine the impossibility claim. This sort of justification is forgetful of the long history of analogue computing and neglectful of contemporary sophisticated attempts to embed computation in continuous processes. The field of evolutionary electronics, for instance, holds promise of exploiting analogue relationships of charge distribution across computer hardware, even when those machines do not differ in constitution from ordinary circuits (see [10]). Various aspects of computation have also been reconstructed within biological systems (see [5,8]), where both the comparative parallelism and continuity of processes is exploited to solve logical problems, e.g., the satisfiability of Boolean formulae. Moreover, returning to the case of mental processes, Leng and Ludwig [6] discuss the neurophysiological role of "analogue signals (i.e. signals that vary continuously in time and which elicit concentration-dependent responses)" as cases of analogue computation.

There is, at least, some conceptual and empirical merit to the idea that precise thought can involve a continuous move along, not a chain, but a genuine line of reasoning. We here develop a notion of continuity in proofs, lines of reasoning with finite *length* but uncountably many *stages*. We do not intend to further defend the idea that mental processes *are* continuous; one of us believes the greatest barriers to this are conceptual, the other empirical. Our claim is conditional. *If* mental processes are continuous, the correct account of proof will be one admitting continuity.

Our presentation of this idea proceeds in three main parts. Section 2 describes the notion of a continuous proof, first as a matter of being able to define an indexing function to pick out stages of reasoning from a continuum, and second via the definition of a continuum of rules for inferring intermediate stages. Section 3 presents examples of continuous proofs, and Section 4 presents some continuous versions of structural facts about proofs.

2. The Notion of a Continuous Proof

2.1. Abstraction of the Notion of Proof

A proof P that α follows from Γ, can be considered as an ordered set of sets of sentences.[1]

$$P = \{\Gamma \ldots \{\alpha\}\} \tag{1}$$

[1] The following conventions are used throughout the text. Uppercase Roman letters from the end of the alphabet (X, Y) denote sets; H is reserved for a homotopy and $[R]$ for an arbitrary rule; Lowercase Roman from the beginning of the alphabet $(a, b, \ldots, i, j, k)$ denote elements of sets, unless otherwise specified or reserved for function symbols (f, g, h, s); Uppercase Greek $(\Gamma, \Delta, \Sigma \ldots)$ are used for arbitrary sets of sentences, while lowercase $(\alpha, \beta, \gamma, \phi)$ denote arbitrary sentences.

With the following properties,[2]

1. $\min(P) = \Gamma$
2. $\max(P) = \{\alpha\}$
3. $p_i \leq p_j \iff p_i \vdash p_j$

Equivalently, if $\mathcal{L}$ is the language of P, we can *index* a proof using an ordered[3] set I and indexing function $f : I \to 2^{\mathcal{L}}$ such that

1. $f(\inf(I)) = \Gamma$
2. $f(\sup(I)) = \{\alpha\}$
3. $i \leq j \iff f(i) \vdash f(j)$

Ordinarily the set of indices for a finite length proof would also be finite—i.e., an infinite set of indices would map to an infinitely long proof. Nonetheless, we need not equate the cardinality of the indices for a proof with its length—the unit interval has an infinite cardinality but finite length. We use this abstraction below to present the notion of continuous yet finitely long proofs.

2.2. Distinguishing Proof Steps from Stages

Reasoning is ordinarily thought to consist of a series of *steps*, numbered naturally. The aim here is to give meaning to the notion of *stages* of reasoning indexed by the reals.

Ordinarily, if a proof has length n, $\ell(P) = n$, then it has n steps, and each $m < n$ corresponds to some such step—the assumptions having index 0. We could likewise set the length of a proof to 1, $\bar{\ell}(P) = 1$, then extract stages as fractions of the former step index over proof length: then for each $m < n$, m/n is a proof stage corresponding to a discrete step. But what corresponds to stages that fall between steps, or irrational stages that cannot be a ratio of any finite step index to proof length? We can give meaning to these stages, and thus to continuity of proof, using the above abstraction of the notion of proof.

A proof *has steps* if, for some *finite* n, there is an indexing function $f : [0 \ldots n] \to 2^{\mathcal{L}}$, such that,

$$\forall_{i,j} \quad i \leq j \iff f(i) \vdash f(j) \tag{2}$$

[2]Note that $f(i) \vdash f(j)$ carries the usual meaning that $(\forall_{\alpha \in f(j)})(\exists_{\Delta \subset f(i)})\Delta \vdash \alpha$.

[3]A linear order is presumed throughout the following. Though more complex orders (such as partially ordered or pre-ordered) may be used to index proofs, we set these aside for ease of presentation.

That is, there is a function from the integer length of a proof to the power of its language that is monotonic.[4] Likewise, a proof *has stages* if there is an indexing function $f : [0, 1] \rightarrow 2^{\mathcal{L}}$ that is monotonic.

For example, assuming that P is a single-step proof that α follows from Γ, i.e. $\ell(P) = 1$, there will be no step corresponding to stage 1/2—an indexing function with a single step will not be defined for 1/2. But, if P has stages, then there will be a stage $\{\alpha^{1/2}\}$ such that,

1. $f(1/2) = \{\alpha^{1/2}\}$
2. $\Gamma \vdash \alpha^{1/2} \quad \& \quad \alpha^{1/2} \vdash \alpha$

Moreover, if we consider the stage given by $f(\pi/6)$, since $1/2 < \pi/6 < 1$, we conclude that $f(1/2) \vdash f(\pi/6)$, and that $f(\pi/6) \vdash \alpha$.

Abstractly at least, there is nothing wrong with considering stages in proofs that do not correspond to steps of a known proof, nor with considering stages that could not correspond to steps in any finite proof. Since stages are particular sets of sentences, it is natural to wonder about the stage corresponding to a real number, beyond the facts about its order in a proof. We should be able to say which sentences are in $f(r)$, for any $r \in [0, 1]$. Particular examples will be given below (Section 3) after considering the use of continuous and discrete rules.

2.3. Continuizing Discrete Rules

We can now consider introducing explicit reference to the rules by which a proof proceeds. One way of interpreting a proof is as a single step, where the intermittent steps show that a rule is valid. So let us consider only single step proofs that deploy a rule.

We know well enough what it means to iteratively apply a rule, the aim here will be to develop what it means to partially or continuously apply a rule. Let us say that $[R]^n$ indicates the n-fold application of the rule $[R]$, then the aim is to define the fractional- and real-fold application of a rule: e.g. $[R]^{m/n}$ and $[R]^r$. Just as there are rules that may be applied iteratively[5] there are rules that can be applied fractionally or a real number of times. In general, we can think of a rule that can be written as a sequent, where

[4]'Monotonic' here means "monotonically increasing with respect to provability" and should not be confused with the structural notion of a "monotonic" logic, i.e. one with "weakening on the left" (see Section 4.1).

[5]Such as the Rule of Necessitation: $\vdash \alpha \implies \vdash \Box\alpha$ [RN], which may be applied n-fold to yeild $\vdash \alpha \implies \vdash \Box_n \ldots \Box_1\alpha$ $[RN]^n$.

f picks out some set of sentences at i and proceeds to pick out the result of applying a rule $[R]$ at $i+1$, i.e., at the next *step* of a proof,

$$\frac{f(i)}{f(i+1)}[R]^1 \tag{3}$$

The aim in this section is just to characterize what it would mean to further generalize this form of presentation for a rule to the case where proofs have stages instead of steps, and rules may proceed by any increment r, i.e. rules of the following form (see examples in § 3 structural rules in § 4.1).

$$\frac{f(i)}{f(i+n)}[R]^n \tag{4}$$

Consider that P is a proof that α follows from Γ by the rule $[R]^n$ (to be written "$\Gamma \vdash \alpha\ [R]^n$";). If $[R]^n$ is the n-fold application of a rule $[R]$, then we can state some facts relating different steps of this application, and generalize on these.

If $0 < m < n$, then P can be separated into two proofs, one of length m, the other of length $n-m$,

$$\Gamma \vdash \alpha\ [R]^n \iff (\exists \Delta)\ \Gamma \vdash \Delta\ [R]^m \ \&\ \Delta \vdash \alpha\ [R]^{n-m} \tag{5}$$

Likewise, if $n = m * k$, then one can consider[6] the rule as the k-fold product application of the rule $[R]^m$,

$$\Gamma \vdash \alpha\ [R]^n \iff \Gamma \vdash \alpha\ [[R]^m]^k \tag{6}$$

As with the length of a proof, we can set the n-fold application of a rule to n=1, and extract stages as fractional applications of the rule.[7] For an n-fold rule $[R']^n$, we can consider the rule $[R]^1$, and re-state the above facts about separation and products in fractional terms, where $m \in \mathbb{R}$ and $m < n$,

$$\Gamma \vdash \alpha\ [R]^1 \iff (\exists \Delta)\ \Gamma \vdash \Delta\ [R]^{\frac{m}{n}} \ \&\ \Delta \vdash \alpha\ [R]^{1-\frac{m}{n}} \tag{7}$$

$$\Gamma \vdash \alpha\ [R]^1 \iff \Gamma \vdash \alpha\ [[R]^{\frac{m}{n}}]^{\frac{n}{m}} \tag{8}$$

As in the case of a proof possessing stages, a rule $[R]$ is continuous iff, there is a triplet $\langle [R]^i, [R]^{1-i}, \Delta_i \rangle$ for all $i \in [0,1]$, such that,

$$\Gamma \vdash \alpha[R] \iff \Gamma \vdash \Delta_i\ [R]^i \ \&\ \Delta_i \vdash \alpha\ [R]^{1-i} \tag{9}$$

For familiarity, we have presented continuous rules by, in effect, interpolating the consequences of applying an ordinary rule up to the point of continuity.

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[6]For example, $\Gamma \vdash \alpha\ [R]^6 \iff \Gamma \vdash \Delta\ [R]^3 \quad \& \quad \Delta \vdash \alpha\ [R]^3$

[7]Note that $[R]^0$ should always be mere reiteration, regardless of what $[R]^1$ is.

On the other hand, we can think of providing a rule that is defined infinitesimally and extending this up to applications over an interval. In Section 3 we provide concrete examples of such rules.

2.4. Connecting Stages to Continuous Rules

We can consider proofs as having real-valued stages and we can consider rules with real-fold applications. To connect these two parallel notions, we can consider the rth stage of a proof $P \subset 2^{\mathcal{L}}$, that α follows from Γ by the rule $[R]$, as the set containing the sentence proved by the rth-fold application of $[R]$, i.e. if P has stages given by $f : [0,1] \to 2^{\mathcal{L}}$, then

$$f(r) = \Delta_r \iff \Gamma \vdash \Delta_r \; [R]^r \tag{10}$$

We can thus connect the abstract notion of a continuous line of reasoning, proof, with the variable application of a rule. If we are able to reflect on a line of reasoning at any point, then there is an extent to which we can follow a rule to conclude at exactly that stage; if we can follow a rule to any extent, then there is a stage in a line of reasoning corresponding to the result of that rule.

Like intuitionism and construction, we can consider the provision of a continuous proof as a standard indicating that the methods, rules, of proof more closely match the mental activities involved. If mental processes are continuous processes, proofs *ought* to be given with continuously variable rules or functions for extracting real-valued stages.

2.5. Topological Requirements for Continuous Indexing Functions

We here define *proofs* as continuous when the set used to index them is as large as the continuum [0,1]. To refer to individual proofs it is often sufficient to refer to their *indexing function*, a function $f : [0,1] \to 2^{\mathcal{L}}$ that picks out a subset of some particular language $\mathcal{L}$ for each index $i \in [0,1]$. This should not, however, be taken to license identifying a proof with its indexing function. *A proof* is a mental event, a series of propositions related in a certain way, not a manner of picking out such sequences.

At this point, it is natural to wonder about the continuity of the indexing function itself, since an indexing function may have [0,1] as domain without itself being a continuous function. Note that, on the other hand, the continuity of indexing functions does not by itself imply the continuity *of a proof*; an indexing function may be continuous while the proof is not. Indeed, any indexing function with steps, i.e. from a space of discrete indices, is automatically continuous but no function with steps indexes a continuous proof. That said, we can extend an indexing function for a continuous proof up to a continuous indexing function in an intuitive way. We briefly digress on this point before returning to the construction of continuous proofs.

For a function $f : X \to Y$ itself to be continuous requires that f^{-1} maps open sets in Y to opens in X, which makes sense only given some topology on each. While the natural topology on the interval $[0, 1]$ is clear, defining "continuous indexing functions" for proofs also requires the imposition of a particular topology on $2^{\mathcal{L}}$. There may of course be a variety of more or less interesting ways to impose, define, such a topology that depend on features internal to the language $\mathcal{L}$, but it is sufficient for our purposes to consider a topology that can be imposed given only a choice of provability relation $\vdash$. This is because we can impose a topology sufficient to define open sets by treating $(2^{\mathcal{L}}, \vdash)$ as an order, supposing only that '$\vdash$' and '$\neq$' are defined on $\mathcal{L}$.

It is clear how an indexing function $f : [0, 1] \to 2^{\mathcal{L}}$ should be defined on an interval $(i, j) \subset [0, 1]$,

$$f((i, j)) = \{f(k)|i < k < j\} \tag{11}$$

And likewise how its inverse should be defined on open intervals (a, b) for $a, b \in 2^{\mathcal{L}}$,

$$f^{-1}((a, b)) = (f^{-1}(a), f^{-1}(b)) \tag{12}$$

The definition of an open interval around some "point" $f(k)$ of $2^{\mathcal{L}}$ is less obvious, though also fairly intuitive once we come to treat $\vdash$ as an order on subsets of $\mathcal{L}$. One way to produce a topology sufficient to define continuous indexing functions is to consider $(2^{\mathcal{L}}, \vdash)$ as a preordered set and impose the Alexandrov topology on it. This is the topology wherein the open sets are just the upper sets with respect to $(2^{\mathcal{L}}, \vdash)$, where u is upper in $(2^{\mathcal{L}}, \vdash)$ iff for $x \in u$, $x \vdash y \implies y \in u$. Intuitively, upper sets are those that contain everything provable from any of their members. Constraining ourselves to the image of f, we can define the open sets as all sets g such that $g = u \cap Im(f)$ and u is open in the Alexandrov topology. This constraint to the image of f is necessary to ensure that the potentially unbounded upper sets in the Alexandrov topology on $(2^{\mathcal{L}}, \vdash)$ are bounded in the resulting topology, e.g., if f is an indexing function for a proof that $\Gamma \vdash \alpha$, $f(1) = \{\alpha\}$, so constraining to the image of f will force opens to terminate in $\{\alpha\}$. Call this the *provability topology* on $2^{\mathcal{L}}$.

What remains is just to defined an indexing function as continuous iff $f^{-1}((a, b))$ is open in the natural topology on $[0, 1]$ whenever (a, b) is open in the provability topology on the domain of f. Definition of the conditions under which the subsets of some particular class of languages can be the codomain of continuous indexing functions are beyond the scope of this article; we simply note here that this richer notion of continuous proof can arise whenever the preorder $(2^{\mathcal{L}}, \vdash)$ is equipped with any provability topology sufficient to define open intervals.

3. Examples of Continuous Proofs

3.1. Continuous Proof of an Inequality

Consider the simple fact that $x \geq 1 \vdash x \geq 0$. A continuous mental construction of this fact might proceed by taking a real line with x at 1, then *sliding* leftwards until 0, knowing that x is greater than anything encountered. Consider the following rule $[R]^m$

$$x = n \implies x \geq n - n * m : m \geq 0,$$

For which the stage-function is,

$$f(m) = \{x \geq n - n * m\}$$

Since, taking $n = 1$,

$$f(0) = \{x \geq 1 - 1 * 0\} = \{x \geq 1\} \tag{13}$$

$$f(1) = \{x \geq 1 - 1 * 1\} = \{x \geq 0\} \tag{14}$$

Finally, notice that $\lim_{m \to 0}[R]^m$ remains valid, defining $[R]$ infinitesimally.

3.2. Continuous Proof of a Conjunct

Consider the rule of conjunction elimination $[\wedge E]$ which says that from any conjunction one can infer either conjunct, i.e., that $\phi \wedge \psi \vdash \phi$ $[\wedge E]$. In a fuzzy-logic context, where truth-values are allowed to vary continuously, the truth of either conjunct can likewise be specified continuously. A continuous mental construction of a conjunction elimination might thus proceed by asserting the truth-value (v) of the conjunction of ϕ and ψ, then *shifting* gradually to specify only the value of ϕ. Consider the rule $[\wedge E]^m$,

$$v(\phi) = 1 \text{ and } v(\psi) = 1 \implies v(\phi) = 1 \quad \text{and} \quad v(\psi) \geq 1 - m$$

And the stage-function, as above,

$$f(m) = \{v(\phi) = 1 \quad \text{and} \quad v(\psi) \geq 1 - m\}$$

3.3. Continuous Proof of Homotopy

In our view, the proof of that two morphisms are homotopic is a prime example of continuity in proof. Consider a proof that two functions f and g, with the same domain X and codomain Y, are homotopic, i.e., that $f \cong g$. This proof must begin by asserting one function as an object, then showing that it can be continuously deformed into the other, e.g. that f can be continuously deformed into g. This is done by defining a homotopy, which is a continuous function $H : X \times [0,1] \longrightarrow Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for $x \in X$.

This is intuitive enough when we see that $t \in [0,1]$ can be interpreted as the "time elapsed during a deformation of f to g". If no time has elapsed f should retain its form and if the deformation is complete we should have arrived at g. Likewise, we can interpret H as a stage-function for the mental construction that deforms f into g to prove they are homotopic:[8] for each $i \in [0,1]$, H continuously defines the object arrived at by that stage in a continuous proof that $f \cong g$.

This is clear if we look at an alternative definition of homotopy. For f, g as above, we can consider an indexed family of functions $h_{t\in[0,1]}$ such that $h_0 = f$, $h_1 = g$ and the map from $t \rightarrow h_t$ is continuous. Now consider a language $\mathcal{L}$ containing at least function symbols for the functions Y^X from X to Y and existential quantification. A continuous proof that $f \cong g$ is indexed by a function $s : [0,1] \rightarrow 2^{\mathcal{L}}$, such that,

$$s(i) = \{(\exists_{h_t})h_t : X \rightarrow Y : 0 \leq t \leq i\} \tag{15}$$

Equivalently, assuming H as defined above, $s(i) = \{(\exists_{h_i})\ h_i(x) = H(x,i)\}$ indexes a proof that $f \cong g$.

4. Fundamental Facts About Continuous Proofs

To conclude, we show that familiar notions of structural rules (*Axiom, Weakening* and *Cut*) can be defined in continuous terms, and offer some useful notions for the construction of continuous proofs (*Ties, Binds* and *Slurs*).

[8] Moreover, this is a particularly interesting reinterpretation when considering proofs as morphisms in syntactic categories, since then the homotopy of two continuous proofs would itself be a continuous proof of their equivalence.

4.1. Continuous Structure

The structural rule called *Axiom* is essentially just the assertion that a constant function for any Γ is a proof indexing function.

$$\frac{}{\Gamma \vdash \Gamma} \quad \text{[Axiom]} \tag{16}$$

$$(\exists f)\ f(i) = \Gamma \ :\ i \in [0,1] \quad \text{[c-Axiom]} \tag{17}$$

Any constant function is continuous, and it is immediate that the above f satisfies the conditions for being a stage function.

Further, *Weakening* on the left and right are essentially just claims about the existence of indexing functions satisfying certain criteria about the union and intersection of their stages. The definitions below are simplified assuming that any stage-function can be scaled according to any continuous real-valued function, i.e. so that if f and g agree on any k, m, they can be scaled to agree precisely at i. For any $i < j \in [0,1]$,

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Delta} \quad \text{[Left-Weakening]} \tag{18}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Sigma} \quad \text{[Right-Weakening]} \tag{19}$$

$$\frac{f(i) = \Gamma \qquad f(j) = \Delta}{(\exists g)\ g(i) = \Gamma \cup \Sigma \qquad g(j) = \Delta} \quad \text{[c-Left-Weakening]} \tag{20}$$

$$\frac{f(i) = \Gamma \qquad f(j) = \Delta}{(\exists g)\ g(i) = \Gamma \qquad g(j) = \Delta \cap \Sigma} \quad \text{[c-Right-Weakening]} \tag{21}$$

Lastly, *Cut* is essentially a claim about the constructibility of an indexing function out of a pair of others, given some information about subsets of the codomains of the latter. This is a special case of what we will term a *Tie*, characterized in the next section. In the following definitions Δ is the formula that is "Cut" out of the proof.

$$\frac{\Gamma \vdash \Delta \qquad \Sigma, \Delta \vdash \Theta}{\Gamma, \Sigma \vdash \Theta} \quad \text{[Cut]} \tag{22}$$

$$\frac{f(i) = \Gamma \qquad f(j) = \Delta \qquad g(i) = \Sigma \cup \Delta \qquad g(j) = \Theta}{(\exists h)\ h(i) = \Gamma \cup \Sigma \qquad h(j) = \Theta} \quad \text{[c-Cut]} \tag{23}$$

4.2. Constructing and Combining Proofs: Ties, Binds and Slurs

4.2.1. Ties Restall [9] refers to the rule *Cut* as the "sequent calculus analogue of composing proofs". Intuitively, that is right, but it is clear why composition as usually understood cannot be applied to the indexing functions for proofs. Considering two such functions f and g, their composition

$g \circ f$ will never be defined, since $dom(g) = [0, 1]$ while $cod(f) = 2^{\mathcal{L}}$. One can nonetheless *Tie* proofs together wherever they share stages, i.e., when, for proofs indexed by f and g, there is some i, j such that $f(i) = g(j)$.

It will help to first define the restriction and scaling of an indexing function. If there is a continuous proof indexed by f that $\Gamma \vdash \Delta$, then there is a pair of proofs for every $r \in [0, 1]$, one that $\Gamma \vdash f(r)$ and another that $f(r) \vdash \Delta$. We can think of these, respectively, as "restricting f to r from above" $f|^r$ and "restricting f to r from below" $f|_r$, defined for $i \in [0, 1]$ as follows.

$$f|^r(i) = f(r \cdot i) \tag{24}$$

$$f|_r(i) = f(r + i(1 - r)) \tag{25}$$

Now consider a case where $f(1) = g(0)$, i.e., where the conclusion of the proof that f indexes are the premises of the proof indexed by g. Clearly we can tie f and g together end-to-end, less obvious is that we can scale them so that any $r \in [0, 1]$ is their mid-stage. This can be defined for $i \in [0, 1]$ as follows.

$$\overbrace{f \quad g}^{r}(i) = \begin{cases} f(i/r) & : i < r \\ g\left(\dfrac{i-r}{1-r}\right) & : i \geq r \end{cases} \tag{26}$$

Putting these two notions together, we can see that if $f(i) = g(j)$, this determines two new proofs depending on how we perform the tie: there will be one that proceeds as f until i then carries on with g, and likewise one beggining with g and proceeding with f after j. That is, $f(i) = g(j)$ implies the existence of proofs with indexing functions $\overbrace{f|^i \quad g|_j}$ and $\overbrace{g|^j \quad f|_i}$. e.g., for $x \in [0, 1]$,

$$\overbrace{f|^i \quad g|_j}^{r}(x) = \begin{cases} f|^i(x/r) = f(r \cdot x/r) = f(x) & : x < r \\ g|_j\left(\dfrac{x-r}{1-r}\right) = g\left(r + (1-r)\dfrac{x-r}{1-r}\right) = g(x) & : x \geq r \end{cases} \tag{27}$$

Together these notions allow us to structure and combine existing continuous proofs in a manner that generalizes the discrete (limiting) case.

4.2.2. Binds It is trivial to show that for any discrete indexing-function of a two-step proof there is a function that is everywhere defined on $[0, 1]$ with an equivalent sequent form, i.e. that agrees with it on {0,1}. Consider a function f such that $f(0) = \Gamma$ and $f(1) = \alpha$, we define a "bind of f" to be $\overline{f}$,

$$\overline{f}(i) = \begin{cases} \Gamma & : \quad i \leq \frac{1}{2} \\ \Gamma \cup \{\alpha\} & : \quad \frac{1}{2} < i \leq 1 \end{cases} \tag{28}$$

E.g., the bind of f is a function $\overline{f}$ which outputs Γ up to and including $\frac{1}{2}$, then outputs $\Gamma \cup \{\alpha\}$. The term 'bind' comes from music theory, where a bind between two notes indicates that the first is played continuously up to and including when the second note begins. It is possible on any instrument, importantly, even on instruments with discrete note intervals, such as the piano. When a proof f contains more than two steps, one can always bind any two consecutive steps and tie them to produce a continuous proof $\overline{f}$. Note that with the provability topology defined in Section 2.5, binds are not only continuous proofs but have a continuous indexing function.

4.2.3. Slurs Given a discontinuous proof f, we term the (equivalence class of) monotonic continuous proofs that are sequent-equivalent to it a 'slur', and write $\tilde{f}$. This term also comes from music theory, where it indicates that a note continuously transitions into another note—such as by sliding the bow or ones fingers along a stringed instrument—and is only possible on instruments without discrete note intervals. Analogously, one can slur an indexing function only if $2^{\mathcal{L}} \cong \aleph_1$ or greater .

The method of slurring a proof has essentially been covered. First consider the equivalence classes $[2^{\mathcal{L}}]$ of elements $\Delta \in 2^{\mathcal{L}}$ where,

$$[\Delta] = \{\Sigma \mid \Delta \iff \Sigma\} \tag{29}$$

For a discrete proof indexing function f that $\Gamma \vdash \alpha$, the slurs of f are given by a function $[\tilde{f}] : [0, 1] \rightarrow [2^{\mathcal{L}}]$, if it exists, satisfying three conditions.

- *Continuity*: $[\tilde{f}]$ is a continuous map from $[0, 1]$ to $[2^{\mathcal{L}}]$
- *Monotonicity*: $(\forall_{i,j})\ i \leq j \iff [\tilde{f}](i) \vdash [\tilde{f}](j)$
- *Sequent Equivalence*: $[\tilde{f}](0) = [f(0)]$ and $[\tilde{f}](1) = [f(1)]$

Again, if the proof has more than two steps, one can slur each two consecutive steps then tie them together. Since we must allow that $[\tilde{f}](i) \cong \kappa$ for any cardinaily κ, we deploy the axiom of choice to provide a specific (witnessing) slur. That is, a choice function c such that $c([\Gamma]) = \Gamma$ and $c([\alpha]) = \alpha$ gives rise to a specific slur $\tilde{f}$, satisfying the above constraints.

$$c \circ [\tilde{f}] = \tilde{f} : [0, 1] \rightarrow 2^{\mathcal{L}} \tag{30}$$

Notice that it is not necessary that the slur of a proof be in the same language, i.e., we allow that $cod([\tilde{f}]) \neq cod(f)$. We here present as example the case of slurring a sequent by moving from crisp to fuzzy-sets. Consider that, if $\mathit{\Gamma} \vdash \alpha$ and $\mathit{\Delta} = \mathit{\Gamma} \cup \{\alpha\}$, then trivially $\mathit{\Gamma} \vdash \mathit{\Delta}$. Intuitively, in any case where we would say that a set proves its subsets, i.e., $\mathit{\Sigma} \subset \mathit{\Gamma} \implies \mathit{\Gamma} \vdash \mathit{\Sigma}$, we should also say that it proves its fuzzy-subsets, on grounds of generality. Now consider a set $\mathit{\Delta}^{\alpha=i}$ which is just like $\mathit{\Delta}$, except that the sentence α has a degree of membership χ of i.

$$\chi_{\mathit{\Delta}^{\alpha=i}}(\beta) = \begin{cases} 1 & : \beta \in \mathit{\Delta} \setminus \{\alpha\} \\ i & : \beta = \alpha \end{cases} \tag{31}$$

This leads to the fuzzification rule for sequents, below. We might indeed think of this as a variety of weakening,[9] since at $i = 0$ it reduces to *Axiom* and at $i \geq 0$ it makes a (potentially non-trivial) claim about the provability of α from $\mathit{\Gamma}$.

$$\frac{\mathit{\Gamma} \vdash \mathit{\Delta}}{\mathit{\Gamma} \vdash \mathit{\Delta}^{\alpha=i}} \quad : \alpha \in \mathit{\Delta},\ i \in [0, 1] \tag{32}$$

Finally, granted an indexing function for $\mathit{\Gamma} \vdash \alpha$, where $\mathit{\Delta} = \mathit{\Gamma} \cup \{\alpha\}$, a slur of f can be defined using fuzzy subsets,

$$\tilde{f}(i) = \mathit{\Delta}^{\alpha=i} \tag{33}$$

Notice that $\mathit{\Delta}^{\alpha=i}$ is one member of $[\mathit{\Delta}^{\alpha=i}]$, and that while $\mathit{\Delta}^{\alpha=1}$ and $\mathit{\Delta}^{\alpha=0}$ are crisp sets, $\mathit{\Delta}^{\alpha=i} : 0 < i < 1$ are fuzzy sets, i.e., $cod(\tilde{f}) \neq cod(f)$.

5. Conclusion

There are conceptual and empirical motivations for considering proofs with a continuity of stages. We have shown how this can be formalized using two equivalent means, stage-functions or continuized rules, to pick out real-valued proof indices. We have provided examples spanning disciplines where proofs play a major role, from arithmetic to (fuzzy) logic and homotopy theory. Finally, we have shown that a continuous approach to proofs is sufficient to formalize many familiar structural rules from proof-theory proper, and that moreover it can be used describe the conditions under which proofs can be combined in various *prima facie* interesting ways.

We take this to be a sufficiently broad coverage to motivate further research into particular uses of continuous proofs. Moreover, as a contribution

[9]Notice that it has an intuitive left and right form.

in its own right, this motivates a different sort of constructionism: continuous constructions of finite length.

Acknowledgements. We would like to thank Graham Priest for reading an earlier draft of this work.

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