Boundaries, States and Cohomology in Three-Dimensional $\mathcal{N} = 4$ Theories

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared below in the statement of contributions and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared below and specified in the text.

Statement of Contributions

This thesis is based on work published in the following papers:

- Chapters 3 and 4 of this thesis, as well as section 2.5 of chapter 2, are based on material from the published paper [1], co-authored with Mathew Bullimore.

- Chapter 4 of this thesis is based mainly on material from the published paper [2], co-authored with Mathew Bullimore and Samuel Crew.

Not included in this thesis are the works [3–5], also completed during the author’s doctoral studies.

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Abstract

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This thesis studies geometric and algebraic aspects of 3d $\mathcal{N} = 4$ theories. We first focus on 3d $\mathcal{N} = 4$ gauge theories compactified on an elliptic curve, and provide physical realisations of the equivariant elliptic cohomology of symplectic resolutions, and recent constructions therein. The Berry connection for supersymmetric ground states in the presence of mass parameters and flat connections for flavour symmetries is analysed, resulting in a natural construction of the equivariant elliptic cohomology variety of the Higgs branch. Supersymmetric boundary conditions are investigated in this set-up. From an analysis of boundary ’t Hooft anomalies, their boundary amplitudes are demonstrated to represent equivariant elliptic cohomology classes.

We then investigate two distinguished classes of $\mathcal{N} = (2,2)$ boundary conditions, each in 1-1 correspondence with the set of isolated massive vacua, known as exceptional Dirichlet and enriched Neumann. The former mimic isolated vacua at infinity in the presence of real mass and FI parameters. The two classes are further shown to be exchanged under mirror symmetry, via collision with a mirror symmetry interface. By computing boundary amplitudes, the enriched Neumann boundary conditions reproduce the elliptic stable envelopes of Aganagic-Okounkov, and the mirror symmetry interface the mother function in equivariant elliptic cohomology. Finally, correlation functions of Janus interfaces for varying mass parameters are considered, recovering the chamber R-matrices of elliptic integrable systems.

We then study the factorisation of partition functions of $\mathcal{N} = 4$ theories on closed 3-manifolds (such as the superconformal index, twisted index and $S^3$ partition function) in terms of $S^1 \times HS^2$ partition functions. We demonstrate the latter, equipped with exceptional Dirichlet boundary conditions, realised this factorisation exactly, and can be unambiguously defined and computed using supersymmetric localisation. We show certain limits of these hemisphere partition functions yield characters of lowest weight Verma modules over the quantised Higgs and Coulomb branch chiral rings. This leads to
expressions for the closed 3-manifold partition functions in terms of such characters. On the way we uncover new connections between boundary 't Hooft anomalies, hemisphere partition functions and lowest weights of Verma modules.
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Chapter 1

Introduction

Quantum field theory is a powerful theoretical framework used to describe the world around us. It explains an incredibly broad range of physical phenomena, and has enjoyed unprecedented experimental success. Famously, the experimental discovery of the existence of the Higgs boson \([6, 7]\), predicted almost fifty years prior \([8–10]\), completed the verification of the standard model as an accurate descriptor of three of four known fundamental forces, establishing it as one of the great scientific achievements of our time.

Although extraordinarily successful in the description of weakly interacting particles,\(^1\) many aspects of quantum field theories of strongly interacting particles are poorly understood, and evade rigorous mathematical description. For example, the strong coupling behaviour of the quantum chromodynamics sector of the standard model remains mysterious; a proper understanding of quark confinement is still beyond our reach. A rigorous formulation of non-abelian Yang-Mills theory on \(\mathbb{R}^4\) and a demonstration of the existence of a mass gap is one of the Millennium Prize problems of the Clay Mathematics Institute.

The difficulty in describing such phenomena largely rests on the failure of perturbation series and Feynman diagrams to provide an accurate description of strongly coupled processes. In fact, quantum field theories with no Lagrangian description exist, for example the 6d \((2, 0)\) theory. It is also increasingly evident that even if a field theory admits a Lagrangian description, a full description involves not only the algebra of local operators, but also that of higher-dimensional defect operators \([12]\).

As is usual in physics and mathematics, to make progress it is useful to study simpler examples which are more tractable, but still exhibit phenomena of interest.

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\(^1\)In the quantum theory of the electromagnetic field, the theoretical prediction of the magnetic dipole moment agrees with experiment up to one part in a trillion \([11]\).
A powerful simplification is to assume the existence of symmetries. Coleman and Mandula \[13\] showed in 1967 that, under reasonable assumptions, the maximal bosonic symmetry group of a quantum field theory is the direct sum of the Poincaré group with a group of global symmetries. However, there are ways to side-step this. For example, supersymmetry evades the no-go theorem by enlarging the Poincaré group to the super Poincaré group, which contains fermionic generators. The analog to the Coleman-Mandula theorem for supersymmetric field theories was proven in 1974 by Haag, Lopuszanski and Sohnius \[14\].

Despite the fact that supersymmetry has not been observed experimentally, supersymmetric field theories serve as a large class of toy models which allow us to compute observables at strong coupling exactly, without turning to perturbation theory. In this way, they can provide tremendous insight into the dynamics of strongly interacting systems. Further, supersymmetric quantum field theories have been found to have incredibly deep connections to pure mathematics. This has not been simply a case of physics borrowing from mathematics; there has been a considerable flow of ideas in the opposite direction. This has in recent decades led to the field of ‘physical mathematics’, as coined by Moore in 2014 \[15\]. Understanding the reason for the existence of this rich cross-fertilisation between fields is undeniably an important first step towards a full understanding of quantum field theory.

A Correspondence

Let us describe an example of the connection between supersymmetric quantum field theories with geometry and topology, which plays a fundamental role throughout this thesis. We will be rather heuristic in this introduction; focusing on the main ideas and motivation behind these constructions. Broadly speaking, we rely on the relationship between two foundational objects in the formulation of quantum field theory: the Hilbert space and the path integral.

We first consider the Hilbert space of the theory. By definition, in a supersymmetric field theory there are conserved supercharges $Q, Q^\dagger$ which generate supersymmetry transformations. One may consider states annihilated by the supercharges:

$$Q |\psi\rangle = Q^\dagger |\psi\rangle = 0,$$

which are the same as those annihilated by a Hamiltonian $H = \{Q, Q^\dagger\}$. By a standard argument, these states are in one-to-one correspondence with the cohomology $H^*(Q)$ of the supercharge $Q$ on the Hilbert space of theory. For theories with multiple
supercharges (higher supersymmetry), where the additional supercharges generate
different supersymmetry transformations, we may consider BPS states which are
annihilated by (some fraction of) the supercharges. These assemble into shortened
multiplets of the supersymmetry algebra.

One can construct a supersymmetric index [16] which counts supersymmetric states
up to their particle statistics. This is given by:

$$\mathcal{I} = \text{Tr}_\mathcal{H} (-)^F e^{-\beta \{Q,Q^\dagger\}}.$$  \hfill (1.2)

Here the trace is over the full Hilbert space of the theory. However, Witten showed
that the index recieves contributions solely from the states of the theory annihilated
by $Q$ and $Q^\dagger$, or equivalently from the cohomology classes $H^*(Q)$. In particular, it
computes the Euler characteristic of the $Q$-complex:

$$\mathcal{I} = \sum_{p \in \mathbb{Z}} \dim H^p(Q).$$  \hfill (1.3)

Part of the beauty of this object is that it is invariant under continuous deformations
of the theory, due to the insertion of $(-)^F$ and the pairing of excited bosonic and
fermionic states due to the supersymmetry.

Now let us turn to the path integral description of the field theory. The index (1.2)
can be additionally described by the path integral:

$$\mathcal{I} = \int_{\mathcal{C}} D\phi \ e^{-S[\phi]},$$  \hfill (1.4)

assuming that $\{Q, Q^\dagger\} = H$, the Hamiltonian of the theory. The path integral is over
the configuration space $\mathcal{C}$ of fields living on the space-time $S^1_{\beta} \times M^{d-1}$, where $S^1_{\beta}$ is a
circle of radius $\beta$ and $M^{d-1}$ is a $d-1$ dimensional spatial slice on which the Hilbert
space $\mathcal{H}$ is defined. For example, for $d$-dimensional conformal field theories we are
often interested in the Hilbert space on $M^{d-1} = S^{d-1}$, as states in this Hilbert space
are in one-to-one correspondence with local operators. Periodic boundary conditions
on the fermions are imposed due to the insertion $(-)^F$ in the trace (1.2).

By a standard argument involving the Ward-Takahashi identity for $Q$, this index is
invariant under a deformation of the action:

$$S[\phi] \to S[\phi] + t S_{\text{loc}}[\phi], \quad S_{\text{loc}}[\phi] = Q \mathcal{P}_{\text{loc}}[\phi],$$  \hfill (1.5)
i.e. the addition of a $Q$-exact term, where $t$ is arbitrary. One can take the limit $t \to \infty$, whereupon the index localises to the saddle point locus $C_{\text{loc}}$ of $S_{\text{loc}}$:

$$I = \int_{C_{\text{loc}}} D\phi_0 \ e^{-S[\phi_0]} \ \text{SDet} \left[ \frac{\delta^2 S_{\text{loc}}[\phi_0]}{\delta \phi_0^2} \right]^{-1}.$$  (1.6)

The super-determinant is the ratio of determinants of operators appearing at quadratic orders of the expansion of the action about a point $\phi_0$ in $C_{\text{loc}}$ in bosonic and fermionic fluctuations. For appropriate choice of localising action, the saddle point locus is in fact finite-dimensional, reducing the problematic infinite-dimensional integral to a regular integral over this locus. See e.g. [17] for a very readable introduction to supersymmetric localisation.

Supersymmetric quantum field theories generically have flat directions in their scalar potentials which are preserved under quantum corrections, leading to extended moduli spaces of vacua. Often, the localising action $S_{\text{loc}}$ can be chosen such that its saddle points correspond to BPS (supersymmetric) configurations which are related to the aforementioned vacuum moduli spaces in some way. This of course depends heavily on the theory, choice of space-time, and choice of localising action. For example, in three dimensions, one can localise to moduli spaces of vortices, which are geometrically quasi-maps to the Higgs branch of the theory [18].

This circle of ideas connects the supersymmetric index of the theory to properties of the vacuum moduli space: the index counting supersymmetric ground states of the theory is related to an integral over a BPS locus over certain field configurations, which in many cases is related to the vacuum geometry of the theory. In fact, often this statement can be categorified; the supersymmetric states of the theory may themselves have some interpretation in terms of the vacuum geometry. We shall see many examples of thereof in this thesis.

The ur-example of these ideas occurs in one dimension. Witten [19] demonstrated that for the supersymmetric sigma model to a Riemannian manifold $M$, the Hilbert space is described by the space of differential forms $\Omega^*(M)$, and the supercharge is canonically quantised to the de Rham differential (exterior derivative). Thus the supersymmetric ground states are described by de Rham cohomology:

$$Q = d, \quad H |\psi\rangle = 0 \Rightarrow |\psi\rangle \in H^*_{\text{de Rham}}(M),$$  (1.7)

and the supersymmetric index coincides with the Euler characteristic. Beautifully, the spectrum of supersymmetric ground states encodes topological properties of the target
manifold for which the model is defined. In an early example of physical mathematics, Witten provided an analytic proof of the Morse inequalities by introducing a potential in the sigma model, and studying the cohomology of the deformed supercharge. Additionally, Alvarez-Gaumé demonstrated that localisation of the path integral form of the Witten index yields a ‘physical proof’ of the Atiyah-Singer index theorem [20]:

$$\chi(M) = \int_M e(TM)$$  \hspace{1cm} (1.8)

The left hand side (the Euler characteristic) arises from the interpretation of the ground states as de Rham cohomology classes. The right hand side (an integral over $M$ of the Euler class $e(TM)$ of the tangent bundle of $M$) results from a careful localisation of the quantum mechanical path integral to the saddle point locus of constant modes. The fluctuations normal to this locus cancel out between bosons and fermions, and the Euler class arises from fermionic integration of the quartic fermion interaction in the sigma model action.

### Equivariance and the Berry Connection

Supersymmetric theories often admit deformations which preserve supersymmetry. A particularly interesting class of such deformations are those which correspond naturally to a global flavour symmetry $G_f$. For example, for the supersymmetric sigma model above, if the target manifold $M$ is Kähler, the theory has an enhanced $\mathcal{N} = (2,2)$ supersymmetry. Suppose there is a group of Hamiltonian isometries $G_f$ acting on $M$. One may turn on a triplet of masses $(m_1, m_2, m_3)$ which are the scalar components of a background vector multiplet weakly gauging this symmetry. Denote $m_\mathbb{R} = m_1$ and $m = m_2 + im_3$. The supercharge is then deformed to:

$$Q = e^{-h_{m_\mathbb{R}}} (d + \iota_{V_m}) e^{h_{m_\mathbb{R}}},$$  \hspace{1cm} (1.9)

where $h_{m_\mathbb{R}}$ is the moment map for the Hamiltonian isometry generated by $m_\mathbb{R}$, and $V_m$ the vector field generating the $G_f$ action with complexified parameter $m$. By using standard arguments introduced by Witten [19], one may scale potential $h_{m_\mathbb{R}}$ by a constant prefactor arbitrarily, since the $Q$-complexes are isomorphic. This localises the supersymmetric ground states to the critical locus $\text{Crit } h_{m_\mathbb{R}}$. The ground states are then obtained as the cohomology of the operator $d + \iota_{V_m}$ on $\Omega^*(\text{Crit } h_{m_\mathbb{R}})$. Thus one has that the ground states are described by what is known as the equivariant
cohomology of $\text{Crit } h_{m_R}$, \textit{localised} at the equivariant parameter $m$

$$\mathcal{H}^{(0)} \cong H_{d+i_{\text{vm}}} (\text{Crit } h_{m_R}).$$

(1.10)

Of course, for $m_R = 0$ this yields the equivariant cohomology of $\mathcal{M}$. Localisation now recovers the Atiyah-Singer equivariant index formula for the equivariant Euler characteristic, a $G_f$-graded character of the equivariant cohomology groups, see \textit{e.g.} [21, 22]. The (exponentiated) masses form the fugacities in the character, see \textit{e.g.} [23].

Note that the ordinary definition of equivariant cohomology $H_{G_f}(X)$ of some variety $X$ in mathematics (see \textit{e.g.} [24]), is as an algebra over a ring $H_{G_f}(pt) = \mathbb{C}[m]$, where $m$ is allowed to vary. Since $G_f$ is a global symmetry, we fix $m$ to a constant value in $H_{G_f}(\text{Crit } h_{m_R})$, resulting in the localised equivariant cohomology. Extremely heuristically, the (delocalised) equivariant cohomology $H_{G_f}(\text{Crit } h_{m_R})$ can be regarded as the union of the spaces of supersymmetric ground states for a fixed value of $m_R$ and varying values of $m$. Thinking about how the space of supersymmetric ground states changes as we vary over the space of parameters $(m_R, m)$ of the theory naturally leads us to a quintessential object in quantum mechanics.

The Berry phase [25] was introduced as the non-dynamical contribution to the phase a non-degenerate energy eigenstate acquires as it is undergoes an adiabatic evolution in such a way that a loop is performed in the parameter space of the theory. There is a non-Abelian generalisation [26]; if the energy eigenstates are $N$-fold degenerate, the state may now be rotated by an element of $U(n)$. For each parameter $\vec{\lambda}$, denoting an arbitrary choice of ground state basis $|\psi_i(\vec{\lambda})\rangle$, the rotation of a state following the adiabatic evolution is given by the holonomy

$$U = P \exp \left( - \oint \vec{A} \cdot d\vec{\lambda} \right), \quad \vec{A}_{ij} = \langle \psi_j | \nabla_{\vec{\lambda}} | \psi_i \rangle.$$ 

(1.11)

The Berry connection $\vec{A}$ is genuinely a connection for the principal $U(N)$ bundle over the parameter space defined by the Hilbert space for each value of $\vec{\lambda}$, as realised by Simon in [27]. The canonical example is that of a spin-$\frac{1}{2}$ particle in a magnetic field $\vec{B}$ with Hamiltonian $H = -\vec{B} \cdot \vec{\sigma}$. Beautifully, the Berry connection over $\vec{B}$ turns out to be the Dirac monopole.

A natural conjecture to make is that there should be a relationship:

$$H_{G_f}(\text{Crit } h_{m_R}) \text{ as } (m_R, m) \text{ vary} \quad (\text{thus incl. } H_{G_f}(\mathcal{M})) \quad \Leftrightarrow \quad \text{Berry connection over } (m_R, m) \text{ for the } \mathcal{N} = (2, 2) \text{ } \sigma\text{-model to } \mathcal{M}$$ 

(1.12)
between the $G_f$ equivariant cohomology of a Kähler manifold $\mathcal{M}$, as well as its $m_R$-fixed submanifolds and the Berry connection (taking the form of some generalised monopole) over the space of parameters $(m_R, m)$ of the supersymmetric sigma model with target space $\mathcal{M}$. To the best of our knowledge, this correspondence has not yet been drawn in the literature. In fact, a large part of the work presented here involves making this precise in the three dimensional context, where the corresponding mathematical object is the equivariant *elliptic* cohomology of $\mathcal{M}$. The above correspondence for supersymmetric quantum mechanics is then recoverable by dimensional reduction. We return to this momentarily.

**A Brief Four Dimensional Detour**

Before moving on to discussing supersymmetric field theories in three dimensions, the primary focus of this thesis, it would be remiss of us to not mention (albeit extremely briefly) the powerful applications the ideas discussed above to supersymmetric quantum field theories in four dimensions, in order to compare and contrast. These have led to numerous developments in mathematics and physics alike.

In [28, 29] Seiberg and Witten utilised the constraining power of holomorphy to exactly solve for the strongly coupled IR dynamics of 4d $\mathcal{N} = 2$ gauge theories. Holomorphy here [30] refers to the fact that the low energy action of the theory is completely specified by a prepotential $\mathcal{F}$:

$$\mathcal{L} = \text{Im} \int d^2\theta d^2\bar{\theta} \mathcal{F}(\Phi),$$

(1.13)

where $\mathcal{F}(\Phi)$ is a holomorphic function of the $\mathcal{N} = 2$ vector superfield for the low energy abelian gauge symmetry. Their solution utilised and illustrated the power of duality; the idea that superficially different UV theories may describe the same IR dynamics. In their case, in certain regions of the vacuum moduli space, the electrically charged UV degrees of freedom were an insufficient description, and the electromagnetic duality of Montonen and Olive [31] exchanging electric and magnetic excitations is required to describe the effective low energy superpotential on the entire moduli space.

In later work, Nekrasov [32] recovered the results of Seiberg and Witten by performing supersymmetric localisation on the path integral directly. To do so he first introduced an omega background; deforming the theory to live on a particular 4d $\mathcal{N} = 2$ supergravity background, parametrised by $\epsilon_{1,2}$. These parameters act as regulators of the partition function, and can be seen as introducing a potential for space-time rotations. A topological twist is required to ensure there are unbroken supersymmetries
in this background. The path integral is localised to saddle points corresponding to solutions of $F = *F$, which are precisely the instanton equations. Thus the path integral is reduced to an equivariant integral over the moduli space of instantons, which itself splits as a union of moduli spaces of instantons with a fixed instanton number. As finite dimensional integrals, these integrals may be computed using known techniques in equivariant integration (such as the finite-dimensional version of localisation). See [33] for a review of these constructions.

The original form of the AGT correspondence [34] relates these partition functions to the conformal blocks of a 2d conformal field theory. This arises from a general construction which compactifies the aforementioned 6d $\mathcal{N} = (2, 0)$ theory on a Riemann surface $C$, leading to a relation between the 4d $\mathcal{N} = 2$ theory obtained via compactification, and a Liouville-Toda CFT on $C$. Since the symmetry group of the CFT is mathematically a $W$-algebra (a generalisation of the usual Virasoro algebra), this hints at an action on the moduli space of instantons; an early incarnation of the connections between supersymmetric field theories and geometric representation theory. See [35, 36] for introductory reviews on these topics.

It was also in four dimensions that the program of localisation on curved backgrounds was initiated. Pestun [37] first localised four dimensional $\mathcal{N} = 2$ theories on $S^4$. The methods therein have since been expanded to cover a wide range of theories and observables, in different dimensions, systematised by the rigid supersymmetry formalism of Festuccia and Seiberg [38]. The essential idea is to first take the flat space supersymmetric field theory and couple it to supergravity. The supergravity multiplet typically contains the metric $g_{\mu\nu}$, its fermionic partner the gravitino $\psi_{\mu\alpha}$, as well as some auxiliary fields. We then take the ‘rigid’ limit where the metric is set to some fixed curved metric on the background $M$ of choice, and the fermions, as well as their supersymmetry variations, vanish:

$$\psi_{\mu\alpha} = \delta \psi_{\mu\alpha} = 0.$$  \hspace{1cm} (1.14)

The solutions to the latter yield equations for Killing spinors, which generate supersymmetry for the theory, now placed on the curved space $M$ with a fixed metric. This powerful technique has allowed the formulation of supersymmetric observables and their computation via localisation on a variety of curved backgrounds. See e.g. [39] for a review for the three dimensional case, containing many of the objects we meet in this thesis.
Three Dimensions & Eight Supercharges

In this thesis, we study three dimensional $\mathcal{N} = 4$ gauge theories with eight supercharges, which have been an incredibly rich resource for physicists and mathematicians alike. They exhibit many interesting phenomena which are not present in their, perhaps physically better motivated, four dimensional cousins. We defer a more in-depth discussion of their properties to the introductory chapter 2, although even there we will not be able to cover many interesting aspects of these theories, due to their sheer extent. For now we just mention briefly a few of the reasons why such theories are interesting, followed by a description of the contributions in this thesis to their study.

Dimensional analysis tells us that these theories are free in the UV, and often flow to strongly coupled supersymmetric conformal field theories in the IR. Their deformation parameters (which we introduce momentarily) are not necessarily complex, so one does not always have access to the same holomorphy properties as in four dimensions. Thus, there is the possibility of phase transitions in these parameters.

Their vacuum moduli space is parameterised by non-zero vacuum expectation values (vevs) for the scalar fields in the $\mathcal{N} = 4$ multiplets. These are either hypermultiplets or vector multiplets. Interestingly, the scalars corresponding to the vector multiplet do not consist solely of those appearing naively in the supersymmetry multiplet, but also include a periodic scalar for each abelian factor in the maximal torus of the gauge group $G$. These may be regarded as corresponding to monopole operators, whose insertion at a point is defined in the path integral formalism by requiring the integration is over field configurations with Dirac monopole insertion at that point.

The flavour symmetry of a 3d $\mathcal{N} = 4$ theory is of the form $G_H \times G_C$, where $G_H$ acts on the hypermultiplets, and $G_C$ is the topological symmetry rotating the aforementioned dual photons (or equivalently the corresponding monopole operator). As mentioned previously, it is natural to turn on deformation parameters for these symmetries [40]. Turning on scalars in a background vector multiplet for the $G_H$ symmetry gives rise to masses $m$ for the hypermultiplets. One can similarly turn on scalars $\zeta$, the so-called Fayet-Iliopoulos parameters, for a background vector multiplet for the topological $G_C$ symmetry. In the Lagrangian, this coupling arises as a mixed Chern-Simons term coupling the background vector multiplet for $G_C$ to the dynamical vector multiplet for the gauge symmetry $G$.

For an $\mathcal{N} = 4$ theory, supersymmetry constrains the vacuum moduli space to be hyper-Kähler [41]. The precise form depends heavily on the values of the parameters $m$ and $\zeta$. If both are zero there is generically a Higgs branch $\mathcal{M}_H$ and a Coulomb branch.

\footnote{Strictly speaking, this should read twisted vector multiplet. See chapter 2 for more details.}
\( \mathcal{M}_C \), where only scalars in the hypermultiplets or the vector multiplets attain vevs, respectively. They are hyper-Kähler cones, singular (at least) at the origin. The full moduli space consists of these branches along with singular submanifolds corresponding to mixed branches, where both types of scalars obtain vevs, stretching between the two. A cartoon of this is displayed in figure 1.1.

![Cartoon of vacua](image)

**Fig. 1.1** A cartoon of the vacua of a 3d \( \mathcal{N} = 4 \) theory with vanishing mass and FI parameters. The Higgs and Coulomb branches are hyper-Kähler cones intersecting at the point where all scalars have zero vevs.

A generic value of \( \zeta \) will typically lift the Coulomb (and mixed) branch, and resolve the Higgs branch. Likewise, a generic value of \( m \) will lift the Higgs (and mixed) branch and resolve the Coulomb branch. Turning on both types of parameters, the resulting vacua may be regarded as fixed loci of the \( G_H \)-action (generated by \( m \)) on the resolved Higgs branch, or fixed loci of the \( G_C \)-action (generated by \( \zeta \)) on the resolved Coulomb branch. In this thesis we will be concerned only with theories where, once generic FI and mass parameters are turned on, there remain only \( N \) isolated, massive vacua (fixed points which can be considered to lie on either branch), which we always index by \( \{ \alpha \} \). Importantly, non-renormalisation theorems [42, 43] dictate that the classical description of the Higgs branch is not corrected in the IR. However, quantum corrections can modify the geometry and topology of the Coulomb branch, as described in [44].

Much of the above is highly suggestive; the Higgs and Coulomb branches are both hyper-Kähler, with the roles of parameters in the theory as resolution or equivariant parameters exchanged. In fact, there is a powerful duality known as 3d mirror symmetry [45] which exchanges the Higgs and Coulomb branches for mirror dual pairs of theories, together with their corresponding symmetries.\(^3\) In fact, 3d mirror symmetry is expected to be a true IR duality, in that all observables should also be mapped across the duality.

\(^3\)This is different from what is referred to by mirror symmetry in string theory, an early example of physical mathematics relating the (enumerative) geometry of pairs of Calabi-Yau threefolds. Its discovery stemmed from the observation that string theory compactifications on such pairs were found to give rise to the same physics in four dimensions [46]. They were subsequently studied using dual \( A \) and \( B \) pairs of 2d \( \mathcal{N} = (2, 2) \) topological string theories [47], prompting the mathematical development
Evidence for this came from type IIB string theory where mirror symmetry is realised as the action of $S$-duality on D3-D5-NS5 brane intersections whose world-volume theories are the mirror dual pairs [50].

As noted in [42], $3d \mathcal{N} = 4$ theories admit BPS vortex solutions, leading to vortex states charged under the topological symmetry. It is thus natural to expect that such vortex states are created and annihilated by monopole operators, whose vevs parameterise the Coulomb branch.\(^4\) Since mirror symmetry exchanges Higgs and Coulomb branches, the fundamental excitations (particles) of the dual theory are also those charged under the topological symmetry of the original theory. This leads to an interpretation of mirror symmetry as a particle-vortex duality.

One way mirror symmetry enters mathematics is through the study of supersymmetric boundary conditions. We will primarily be concerned with boundary conditions preserving $2d \mathcal{N} = (2,2)$ supersymmetry. Building on a description for Rozansky-Witten theory (3d topologically-twisted sigma models) in [54, 55], they were first studied for supersymmetric gauge theories in [56]. The ring of holomorphic functions on the Higgs branch may be identified with the gauge-invariant local operators formed from the hypermultiplet scalars, and those on the Coulomb branch by monopole operators dressed by vector multiplet scalars [57, 58]. In [56] there are then quantised by introducing an omega background, which may be regarded as introducing a potential for rotations, forcing the operators to live on an axis. For a given boundary condition $B$, bringing in the quantised bulk Higgs (or Coulomb) operators to act on the boundary Higgs (or Coulomb) operators surviving at the boundary, one recovers modules of the Higgs or Coulomb branch algebra respectively. This is illustrated in figure 1.8.

The above construction is precisely the physical construction underpinning symplectic duality [59, 60], an area of active mathematical interest, which is an equivalence between certain categories $\mathcal{O}_H$ and $\mathcal{O}_C$ attached to pairs $(X, X^!)$ of symplectic resolutions. In a particular simplification, these categories correspond precisely to lowest-weight modules for deformation quantisations of the coordinate rings of $X$ and $X^!$. Thus far, all known examples of symplectic dual pairs arise as the Higgs and Coulomb branch of a given $3d \mathcal{N} = 4$ theory. Throughout this thesis we will e.g. interchangeably use $\mathcal{M}_H$ and $X$ to refer to the Higgs branch. Different elements of $\mathcal{O}_H$ and $\mathcal{O}_C$ may recovered by taking three classes of boundary conditions preserving varying degrees of gauge and homological mirror symmetry [48]. A physical proof of mirror symmetry based on T-duality in $\mathcal{N}' = (2,2)$ supersymmetric field theory was subsequently found by Hori and Vafa [49].

\(^4\)In fact, in [51] the algebra of monopole operators is re-constructed from its action on the cohomology of the vortex moduli space of the theory, which is closely related to the rigorous mathematical formulation of the Coulomb branch by Nakajima et. al. [52, 53].
flavour symmetry at the boundary: Neumann, Dirichlet and exceptional Dirichlet. We
give more details for these in chapter 2.

In [56], a systematic action of 3d mirror symmetry was found on boundary conditions:
a given boundary condition $\mathcal{B}$ is mapped to one for the dual theory, with corresponding
Higgs and Coulomb branch modules exchanged. This was derived by the introduction
of a mirror duality interface (at least for abelian theories); an interface between two
mirror dual theories defined in the UV, which flows to the trivial interface in the
IR. It was constructed explicitly, using ideas from Hori-Vafa mirror symmetry [49].
The mirrors of boundary conditions can then be constructed explicitly in the UV by
colliding a boundary condition with the interface, as in figure 1.2.

\[ \tilde{T} \xrightarrow{\mathcal{B}} T \xrightarrow{\text{Mirror Interface}} \tilde{T} \]

**Fig. 1.2** Colliding a boundary condition $\mathcal{B}$ in theory $T$ with the mirror interface to derive
the dual boundary condition $\tilde{\mathcal{B}}$ in $\tilde{T}$.

In addition to the above, there are numerous algebro-geometric constructions arise
naturally from quantising these theories on different spacetime backgrounds. A repre-
sentative example is given in the works [56, 61], where a beautiful interpretation
of topologically twisted 3d $\mathcal{N} = 4$ theories on $\mathbb{R} \times \Sigma$ (where $\Sigma$ is a closed Riemann
surface) was derived. The topological twist may be considered as a particular rigid
supersymmetry background, as described previously. It was found that supersymmetric
ground states of this twisted theory on $\Sigma$ correspond to (approximately) the equivariant
cohomology of the moduli space of solutions to generalised vortex equations, which are
geometrically quasi-maps $\Sigma \to \mathcal{M}_H$ to the Higgs branch of the theory. The twisted
index, *i.e.* the path integral on $S^1 \times \Sigma$ is shown to localise directly to moduli spaces of
these quasi-maps, and as such computes their Euler characteristic, in the same spirit
as the quantum mechanical example introduced earlier. Since 3d mirror symmetry
identifies the twisted indices of mirror pairs of theories, a beautiful relation between
enumerative invariants for symplectic dual pairs is implied.

Having given some background, we now describe the work in this thesis, and how
it fits into the tradition of physical mathematics. We will be rather heuristic for now,
directing the reader to the next subsection for a more technical outline.
Elliptic Cohomology, Ground States & Berry Connections

A large part of our main results may be regarded as continuing in the spirit of Witten’s aforementioned analysis of supersymmetric quantum mechanics. We study 3d \( \mathcal{N} = 4 \) supersymmetric gauge theories on \( E_\tau \times \mathbb{R} \), where \( E_\tau \) is an elliptic curve (a torus) with complex structure parameter \( \tau \), and draw for the first time a precise connection between the Hilbert space of the theory on \( E_\tau \) and the equivariant elliptic cohomology of the Higgs branch.\(^5\) Although (equivariant) elliptic cohomology has been a topic of considerable study in pure mathematics for some years \([62–66]\), it has attracted considerable recent interest due to interplay with ideas from theoretical physics \([67–70]\). Our work aims to give a precise physical underpinning to many of these results.

We cannot hope to give even close to a complete view of equivariant elliptic cohomology here (an overview is provided in appendix A), however we mention a few of its salient features in order to motivate why one might expect there to be a relationship with supersymmetric theories in 3d. Throughout, we will restrict to an abelian group action \( T \). Heuristically, one may think of equivariant elliptic cohomology as the ‘top’ in a hierarchy of cohomology theories:

\[
H_T^*(X) \rightarrow K_T(X) \rightarrow \text{Ell}_T(X)
\]  

(1.15)

We have already met \( H_T^*(X) \): it describes the ground states of quantum mechanical supersymmetric sigma models with target space \( X \) and flavour symmetry \( T \) (and, in fact, the gauged linear sigma models which flow to them \([23]\)). In the \( K \)-theory case, the relationship between sheaves and D-branes (boundary conditions, which generate states when a 2d theory on a cylinder is viewed as a quantum mechanics along the length) has a storied history, beginning in \([71]\). See also \textit{e.g.} \([72]\) for a review, and \([73–75]\) for statements in gauge theory. Our analysis fills out the last part of this correspondence.

Let us add a few more motivating factors. We have introduced equivariant cohomology as an algebra over the ring of polynomials in the \( \mathbb{C} \)-valued equivariant parameters of \( \mathcal{N} = (2, 2) \) quantum mechanics. The parameters can be regarded as coordinates on the complexified Lie algebra \( t_{\mathbb{C}} \cong \mathbb{C}^n \) of \( T \), and thus formally on \( \text{Spec} \, H^*_T(\text{pt}) \cong \mathbb{C}^n \). Thus, \( \text{Spec} \, H^*_T(X) \) may be regarded as an affine scheme over \( \mathbb{C}^n \), which is often called the equivariant cohomology scheme. Equivariant cohomology classes can be regarded as elements of the coordinate ring on this scheme. Physically, we may think of them as (roughly) specifying a supersymmetric ground state at each value of the equivariant pa-

\(^5\)Note that the theory is not twisted, so we expect something different to the results of \([56, 61]\).
rameters. In equivariant K-theory (whose elements, for our purposes, are $T$-equivariant vector bundles over $X$), similar statements can be made by replacing $\mathbb{C} \to \mathbb{C}^\ast$.

The equivariant elliptic cohomology $\text{Ell}_T(X)$ is constructed as a scheme over $E_T := (E_\tau)^n$, in a way which is locally modelled on the equivariant cohomology scheme (again, see appendix A). If $N$ is the number of $T$-fixed points on $X$, $\text{Ell}_T(X)$ turns out to be an $N$-sheeted cover of $E_T$, each sheet corresponding to a fixed point, where the sheets are identified on loci in $E_T$ where the fixed points are no longer isolated in $X$ but linked by an extended fixed submanifold. Whilst there are no holomorphic functions on it (it is compact), there are still line bundles, and elliptic cohomology classes are defined as holomorphic line bundles over $\text{Ell}_T(X)$.

We note that the base algebraic groups $\mathbb{C}, \mathbb{C}^\ast, E_\tau$ precisely match the equivariant parameters appearing in the analogue of the supercharge (1.9), for 1d, 2d, and 3d theories respectively. For a 2d theory quantised on a cylinder, it is the supersymmetric combination $m + 2\pi i \oint_{S^1} A$ of a mass and flat connection for a background vector field for the $T$-symmetry. In 3d, it is the complex flat connection:

$$z := \oint_t A_t dt - \tau \oint_s A_s ds,$$

where $t$ and $s$ are coordinates on the two cycles of $E_\tau$.

Additional motivation comes from the mathematical literature [62, 76–78] on the relationship between the equivariant K-theory of $X$ and the equivariant cohomology of the loop space $LX = \text{Map}(S^1 \to X)$, and the equivariant elliptic cohomology with the equivariant K-theory of $LX$. It is reasonable to conjecture that the elliptic cohomology is related to the equivariant cohomology of the double loop space $\mathcal{X} = \text{Map}(E_\tau \to X)$. Indeed, such constructions have appeared in [79]. Physically such relations are natural to expect, particularly in the case of the 2d and 3d generalisations of the supersymmetric sigma model, since we may heuristically consider these as infinite-dimensional quantum mechanics to $LX$ or $\mathcal{X}$ respectively. See for example figure 3.2.

Now that we have the motivation, let us describe in more detail how the correspondence occurs. For a 3d theory on $E_\tau \times \mathbb{R}$ with Higgs branch $X$, we can supersymmetrically turn on a real mass $m_\mathbb{R}$ and complex holonomy $z$ (as above), corresponding to a real scalar and holonomy for a background vector multiplet for the Higgs branch flavor symmetry $T$. We subsequently drop the subscript on the real mass $m$. Turning on a real FI parameter, the theory flows to a sigma model onto the the Higgs branch $X$. As above, this may be considered as a $\mathcal{N} = (2, 2)$ quantum mechanics onto $\mathcal{X}$. In this

---

6It is $\mathcal{N} = (2, 2)$ as we introduce a parameter in the main text which softly breaks the 3d $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 2$. 
picture there is a supercharge $Q$ of the form (1.9), but now with the vector field $V_z$ on $X$. This is a vector field generating the $S^1 \times S^1$ action on $X$ induced from translations around $E_\tau$, as well as the $T$-action with parameter $z$. The moment map $h_m$ is the one induced on $X$ by that on $X$.

One can then perform an analysis of the supersymmetric ground states by using the same arguments in quantum mechanics on this infinite-dimensional model. In fact, for generic values of the mass parameters and holonomies, we recover a set of supersymmetric ground states $|\alpha\rangle$ on $E_\tau$ which are 1-1 with the massive vacua of the theory. This is identical to the analysis for quantum mechanics. This is related to the fact that the equivariant elliptic cohomology is, in a sense reviewed in appendix A, locally modelled on usual equivariant cohomology. The real meat in this problem, and how equivariant elliptic cohomology arises, comes from analysing how the supersymmetric ground states vary as one moves globally in the space of background parameters $(m, z) \in t \times E_T$. To do so, we turn to the Berry connection.

It has been shown in [80–83] that the $U(N)$ Berry connection over $(m, z)$ (recall that $N$ is the number of vacua) is in fact enhanced to a solution of the generalised Bogomolny equations. This implies the existence of a holomorphic vector bundle $\mathcal{E}$ of ground states on the $z$-space, for each fixed value of $m$, which depends piece-wise on $m$. Let $T_m = S^1 \subset T$ be the one parameter subgroup generated by the real mass. Through a localisation argument, we demonstrate that the $\mathcal{E}$ is encoded in a single holomorphic line bundle $\mathcal{L}$ over $\text{Ell}_T(X^{T_m})$, the elliptic cohomology variety of the fixed locus $X^{T_m}$. The variety is, as mentioned previously, an $N$ sheeted cover over $E_T$. The line bundle restricted to each sheet $\mathcal{L}|_{\alpha}$ corresponds to a supersymmetric ground state associated to $\alpha$, and the factors of automorphy specifying it are determined by the effective Chern-Simons levels $K_\alpha$ in the vacuum.\footnote{In three dimensions, effective Chern-Simons levels for a symmetry group can be generated in the IR by integrating out massive fermions under which it is charged [84–86].} Thus by varying $m$, our constructions recover the equivariant elliptic cohomology $\text{Ell}_T(X)$ of $X$, as well as the submanifolds of $X$ fixed by one parameter subgroups of $T$.

Continuing, we turn to supersymmetric boundary conditions, considered as states in the quantum mechanics along $\mathbb{R}$. We demonstrate that they correspond to elliptic cohomology classes. In particular, we compute the collection of boundary amplitudes $\langle \alpha | \mathcal{B} \rangle$, for a given boundary condition $\mathcal{B}$. This is given by the path integral on $E_\tau \times \mathbb{R}_+$, with the boundary condition at $x^3$ and the vacuum $\alpha$ at infinity, which we compute via localisation. The amplitudes transform in $z$ with a factor of automorphy specified by the effective Chern-Simons levels of $\alpha$ and the boundary 't Hooft anomalies of $\mathcal{B}$, as
implied by the path integral description. We find that the collection \( \{\langle \alpha | \mathcal{B} \rangle \} \) glue to a section of a single holomorphic line bundle \( \mathcal{L}_B \) over \( \text{Ell}_T(X^T) \), and hence represent an equivariant elliptic cohomology class.

See table 1.1 for a summary of the physical objects we construct in this thesis, and the mathematical structures with which we identify them.

<table>
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<td>Elliptic cohomology schemes ( \text{Ell}<em>T(X^{T</em>{\text{na}}}) )</td>
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<td>Boundary conditions ( \mathcal{B} ) &amp; amplitudes ( {\langle \alpha</td>
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<td>Enriched Neumann BCs ( N_{\alpha} )</td>
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Table 1.1 The mathematical and physical objects which we draw an equivalence between in this thesis. The exact correspondences are more complex but we omit the details for now.

**Boundaries & Interfaces: Stable Envelopes & Symplectic Duality**

To make a connection with the more recent constructions in equivariant cohomology, we turn to specific classes of boundary conditions and interfaces. As mentioned already, exceptional Dirichlet boundary conditions were first derived in [56], and are 1-1 with massive vacua \( \alpha \). Their support on the Higgs branch is the attracting set of the gradient flow for the moment map \( h_m \) of the Higgs branch flavour symmetry. For the purposes of computing supersymmetric observables, they mimic the presence of an isolated vacuum at infinity (see figure 1.6). We derive for the first time the mirror dual to these boundary conditions, which we dub enriched Neumann boundary conditions, so-called due to couplings to boundary \( \mathbb{C}^* \)-valued \( \mathcal{N} = (2, 2) \) chiral multiplets (enriched) and the preservation of gauge symmetry at the boundary (Neumann). They are found by collision with the aforementioned mirror duality interface of [56], as in figure (1.2).
Translating these results into the language of equivariant elliptic cohomology, we compute the boundary amplitudes of both the enriched Neumann boundary conditions and the mirror symmetry interface. They yield the elliptic stable envelopes of Aganagic-Okounkov [67], and the mother function [70] respectively. The elliptic stable envelopes are a particularly nice class of elliptic cohomology classes which are 1-1 with fixed points of conical symplectic resolutions. They are, amongst other things, related to the transition matrix between enumerative invariants of symplectic dual pairs (more on this in the main body of the thesis), integral solutions of qKZ equations [87, 88], and elliptic integrable systems (more on this momentarily). The mother function is an elliptic cohomology class in \(\text{Ell}_{T_H \times T_C}(X \times X')\), i.e. the elliptic cohomology of the product of the Higgs and Coulomb branch of a given theory, considered equivariantly with respect to the Higgs and Coulomb branch symmetries acting on them. It is shown in the mathematical literature that it relates the elliptic stable envelopes of symplectic dual pairs. We obtain a natural interpretation of these facts via the physical picture of a mirror symmetry interface and collisions with boundaries.

A Connection to Integrability

We conclude the story for theories quantised on \(E_\tau\) by turning to a connection with integrability. We consider Janus interfaces for the mass parameters, which are interfaces involving a position dependent profile \(m(x^3)\) along \(\mathbb{R}\). In fact, we show that the exact profile of \(m(x^3)\) is unimportant, since they realise \(Q\)-exact deformations in the action, as in equation (1.5). The dependence of the path integral on an interval \(E_\tau \times [0, l]\) on \(m(x^3)\) is solely on the chambers in the mass parameter space containing \(m(0)\) and \(m(l)\). Exploiting this, we demonstrate that the correlation functions of Janus interfaces (path integrals sandwiched between enriched Neumann boundary conditions) for masses interpolating between different chambers are precisely the so-called chamber \(R\)-matrices of elliptic integrable systems, obeying the Yang-Baxter equation. In a way made precise in the main text, by cutting the path integral computing the Janus correlation function, we may express it as a product of stable envelopes.

All of the above constructions have a precise mathematical analogue, going back to the constructions of Maulik and Okounkov [89]. There, a Yangian action (a hallmark of an integrable system) is constructed on the equivariant cohomology of quiver varieties (examples of conical symplectic resolutions). The \(R\)-matrix is built out of the cohomological stable envelopes. These constructions have been generalised to the \(K\)-theoretic and elliptic cases. These results were inspired by the Bethe/Gauge correspondence of Nekrasov and Shatashvilli [90, 91], which relates the vacua of
supersymmetric field theories (whose Higgs branches are the above quiver varieties) with solutions to the Bethe equations for corresponding spin chains (which are the above integrable systems). In this context the stable envelopes are identified as wavefunctions of off-shell Bethe states in the up-down basis, first noted in [92]. Our work makes some of these constructions precise in the elliptic case.

**Holomorphic Factorisation & Representation Theory**

Finally, we switch gears slightly and study $\mathcal{N} = 4$ theories on certain compact three-folds $\mathcal{M}_3$. In particular, we consider the phenomenon of holomorphic factorisation: where the partition function computed on $\mathcal{M}_3$ factorises as $Z_{\mathcal{M}_3} = \sum_\alpha H_\alpha \tilde{H}_\alpha$, where $H_\alpha$ is fundamental building block associated to a massive vacuum $\alpha$, and $\tilde{H}_\alpha$ is a conjugation of $H_\alpha$ depending on $\mathcal{M}_3$. There have been checks for the $S^4$ partition function in [93], the superconformal index in [94–96] and the $S^1 \times S^2$ twisted index in [5, 97]. These are examples of localisation computations for 3d $\mathcal{N} \geq 2$ theories formulated on rigid supersymmetry backgrounds.

One way to construct $\{H_\alpha\}$ is as the holomorphic blocks of [98], defined in the IR as partition functions on a twisted product $S^1 \times_q D^2$ (the $D^2$ rotates as it goes around the $S^1$ by an amount parametrised by $q$), which are appealingly manifestly solutions to certain $q$-difference equations. These equations correspond physically to identities obeyed by line operators inserted at the tip of $D^2$ and wrapping the $S^1$ factor. An artefact of this construction is that there are some normalisation ambiguities in the perturbative contributions to the partition function.

Instead, we take a direct, UV approach by defining the building blocks $H_\alpha$ as partition functions on $S^1 \times H S^2$, where $H S^2$ is the hemisphere, with the aforementioned exceptional Dirichlet boundary conditions placed on the boundary torus $E_\tau = \partial(S^1 \times H S^2)$. We compute these partition functions using supersymmetric localisation, and demonstrate holomorphic factorisation exactly up to perturbative contributions.

We also demonstrate a relationship between equivariant elliptic cohomology classes (boundary conditions on $E_\tau$) and the representation theory of quantum algebras, which proceeds as follows. The state-operator correspondence relates the hemisphere partition function to the half superconformal index which counts boundary local operators for the exceptional Dirichlet boundary condition. We find they are the same up to a ‘Casimir’ prefactor determined by the boundary ’t Hooft anomalies. Further, we show that two different limits of the partition function produce characters of the (Verma) modules of the Higgs and Coulomb branch algebras furnished by the boundary Higgs and Coulomb branch operators respectively, as described previously. Interestingly, the
prefactor (hence the boundary ’t Hooft anomalies) reproduces the highest weights of these modules in the limits.

Combined with our results on holomorphic factorisation, the above yields a description of various limits of the $S^3$ partition function, and the $S^1 \times S^2$ superconformal and twisted indices as sums of products of characters of Verma modules of the Higgs and Coulomb branch algebras. This reproduces a conjectured formula in [99] for the $S^3$ partition function.

## 1.1 Summary of results

We now give a more technical outline of the material presented in thesis.

### Preliminaries

In chapter 2 we give a broad overview of 3d $\mathcal{N} = 4$ theories, introducing background material that will be used throughout the thesis. In section 2.5 we draw some novel connections between Chern-Simons couplings, BPS domain walls, and objects in the geometry of the Higgs branch, building on the work [56]. These connections will be of vital importance throughout the remainder of the thesis.

### 3d $\mathcal{N} = 4$ Gauge Theories on Elliptic Curves

Chapters 3 and 4, study 3d $\mathcal{N} = 4$ supersymmetric gauge theories on $E_\tau \times \mathbb{R}$, where $E_\tau$ is a complex elliptic curve with complex structure parameter $\tau$ and Ramond-Ramond boundary conditions are imposed. These chapters are based on material from [1]. We give a precise physical construction of the equivariant elliptic cohomology of conical symplectic resolutions, and constructions therein, such as elliptic stable envelopes and elliptic $R$-matrices [62–65, 67–70].

![Fig. 1.3](image)

**Fig. 1.3** The set-up of chapters 3 and 4 is supersymmetric gauge theories on $E_\tau \times \mathbb{R}$.

In chapter 3, we first establish the fundamental correspondence to equivariant elliptic elliptic cohomology, proceeding as follows. A supersymmetric gauge theory on $E_\tau \times \mathbb{R}$ can be regarded as an infinite-dimensional supersymmetric quantum mechanics
on $\mathbb{R}$. An important question is to determine the space of supersymmetric ground states, which can be understood as the cohomology of a supercharge provided the system remains gapped. This has a straightforward answer in the presence of generic mass deformations. Suppose we can turn on mass deformations such that the gauge theory has only isolated massive and topologically trivial vacua, indexed by $\alpha$. Then there are corresponding supersymmetric ground states $|\alpha\rangle$ on $E_r \times \mathbb{R}$ labelled in the same manner.

The richness of this problem arises in determining supersymmetric ground states for non-generic mass deformations and more broadly how they depend on background fields on $E_r \times \mathbb{R}$ associated to flavour symmetries.

Consider a supersymmetric gauge theory with an abelian flavour symmetry $T$ acting on elementary matter supermultiplets. This flavour symmetry acts on the Higgs branch moduli space of the supersymmetric gauge theory, which we denote by $X$, and the isolated massive vacua are the fixed points $X^T = \{\alpha\}$. The computation of supersymmetric ground states on $E_r \times \mathbb{R}$ is then compatible with the following parameters associated to the flavour symmetry:

- Real mass parameters valued in $t = \text{Lie}(T)$.
- A background flat connection on $E_r$ for the flavour symmetry $T$, parameterised by the $\text{rk}T$-dimensional complex torus

$$E_T := t \otimes_{\mathbb{R}} E_r.$$  \hfill (1.17)

The total parameter space is therefore

$$t \times E_T \cong (\mathbb{R} \times E_r)^{\text{rk}(T)}$$  \hfill (1.18)

and these parameters can be regarded as expectation values of scalar fields in a background vector multiplet for the symmetry $T$ in the supersymmetric quantum mechanics.

The dependence of supersymmetric ground states on these parameters is controlled by a Berry connection. A consequence of supersymmetry is that the Berry connection is enhanced to a solution of generalised Bogomolny equations [80–83]. The asymptotic behaviour of the solution is controlled by the effective supersymmetric Chern-Simons couplings in the vacua and there are ’t Hooft monopole singularities at loci where the supersymmetric quantum mechanics fails to be gapped.

---

8Topological symmetries transforming monopole operators are incorporated in the main text.
In fact, we do not utilise the full structure of the supersymmetric Berry connection. Instead, we use a consequence of the generalised Bogomolny equations that there is a complex flat Berry connection in the real directions $t$ and a holomorphic Berry connection in the complex directions $E_T$, which commute with each other. This induces the structure of a holomorphic vector bundle $\mathcal{E}$ on each complex torus $\{m\} \times E_T$, which has a piece-wise constant dependence on the mass parameters $m \in t$.

![Fig. 1.4 An example of a hyperplane arrangement in $t \cong \mathbb{R}^2$ arising in supersymmetric QED with Higgs branch $X = T^*\mathbb{CP}^2$ and flavour symmetry $T \cong U(1)^2$ generated by mass three parameters $(m_1, m_2, m_3)$ obeying the constraint $m_1 + m_2 + m_3 = 0$. There are six faces of maximal dimension where the fixed locus $X^T_m$ consists of three isolated points $\alpha = 1, 2, 3$, six faces of dimension one where it is a union of a point $\alpha = 1$ and a moduli space $T^*\mathbb{CP}^1$ connecting $\alpha = 2, 3$ or permutations thereof, and the origin where it is the whole $T^*\mathbb{CP}^2$.](image)

The piece-wise constant dependence is controlled by a hyperplane arrangement in the space of mass parameters $t$ constructed from hyperplanes where the gauge theory is no longer completely massive. To describe the hyperplanes geometrically, let us denote by $T_m \subset T$ the 1-parameter subgroup of the flavour symmetry generated by a mass parameter $m \in t$. We then have

- For a generic mass parameter, $X^T_m = \{\alpha\}$.
- For a non-generic mass parameter, $X^T_m \neq \{\alpha\}$ and an extended moduli space opens up. This happens along hyperplanes through the origin.

In general, the fixed locus $X^T_m$ and the structure of the holomorphic vector bundle $\mathcal{E}$ on $\{m\} \times E_T$ depends on a face of the hyperplane arrangement. As example of a hyperplane arrangement for supersymmetric QED is illustrated in figure 1.4.

For a generic mass parameter, lying in a face of maximal dimension or chamber of the hyperplane arrangement, $X^T_m = \{\alpha\}$ and the holomorphic vector bundle admits a holomorphic filtration

$$0 \subset \mathcal{E}_{\alpha_1} \subset \mathcal{E}_{\alpha_2} \subset \cdots \subset \mathcal{E}_{\alpha_N} = \mathcal{E},$$

(1.19)
giving a complete flag on each fibre. The factors of automorphy of the holomorphic line bundles $\mathcal{L}_{\alpha_i} \cong \mathcal{E}_{\alpha_i}/\mathcal{E}_{\alpha_{i-1}}$ are fixed by the effective supersymmetric Chern-Simons
couplings in the massive vacuum $\alpha$. The collection of line bundles $\{L_\alpha\}$, or equivalently the associated graded $G(\mathcal{E}) = \bigoplus_\alpha L_\alpha$, can also be regarded as a section of a holomorphic line bundle on the union of identical copies $E_T^{(\alpha)} \cong E_T$ associated to each of the vacua $\alpha$,

$$\text{Ell}_T(\{\alpha\}) = \bigcup_\alpha E_T^{(\alpha)}.$$  

(1.20)

This is the $T$-equivariant elliptic cohomology variety of the fixed point set $X^T = \{\alpha\}$.

As the mass parameters are specialised to lie on faces of the hyperplane arrangement of lower dimension this structure becomes more intricate. We argue that on a general face of the hyperplane arrangement, $\mathcal{E}$ is encoded in a holomorphic line bundle on the equivariant elliptic cohomology variety of the fixed locus $T_m \subset T$,

$$\text{Ell}_T(X^{T_m}) \longrightarrow E_T.$$  

(1.21)

This is an $N$-sheeted cover of the space of $T$-flat connections $E_T$, which is obtained by making certain identifications on the sheets of $\text{Ell}_T(\{\alpha\})$. Here $N$ is the number of massive vacua $\alpha$. At the origin of the hyperplane arrangement, $m = 0$, this is the equivariant elliptic cohomology variety $\text{Ell}_T(X)$ of the entire Higgs branch $X$. This is why equivariant elliptic cohomology arises in this problem.

With the structure of the supersymmetric Berry connection in hand, we study boundary conditions compatible with the flavour symmetry $T$ and the supercharge whose cohomology computes supersymmetric ground states. The fundamental objects of study are boundary amplitudes $\langle B|\alpha\rangle$, defined as the path integral on $E_\tau \times \mathbb{R}_+$ with the boundary condition $B$ at $x^3 = 0$ and a supersymmetric ground state $|\alpha\rangle$ at $x^3 \to \infty$, as illustrated in figure 1.5.

**Fig. 1.5** A boundary amplitude is computed from a partition function on $E_\tau \times \mathbb{R}_+$ with the boundary condition $B$ at $x^3 = 0$ and a massive vacuum $\alpha$ at $x^3 \to \infty$.

The boundary amplitudes can be regarded as elliptic genera of effective two-dimensional theories obtained by reduction on $\mathbb{R}_+$. From an analysis of boundary mixed ‘t Hooft anomalies and effective supersymmetric Chern-Simons couplings, we can determine how boundary amplitudes transform under global background gauge
transformations on $E_\tau$ for the flavour symmetry $T$. This identifies boundary amplitudes as sections of holomorphic line bundles on $E_T$.

Furthermore, suppose a boundary condition $B$ is compatible with a mass parameter $m$ on some face of the hyperplane arrangement. Then we show that the collection of boundary amplitudes defined using supersymmetric ground states on that face of the hyperplane arrangement transform in such a way that they glue to a section of a holomorphic line bundle on $Ell_T(X^{T_m})$. This provides a recipe to construct equivariant elliptic cohomology classes from suitable boundary conditions. We illustrate this by computing the boundary amplitudes of Neumann boundary conditions representing the elliptic cohomology classes of holomorphic Lagrangian sub-manifolds in $X$.

In chapter 4, we move on to consider two distinguished collections of UV boundary conditions whose elements are in 1-1 correspondence with vacua $\alpha$. They depend on a choice of chambers $\mathcal{C}_H, \mathcal{C}_C$ in the spaces of real mass and FI parameters. In abelian gauge theories, they have an explicit construction as follows:

1. **Exceptional Dirichlet**

Exceptional Dirichlet boundary conditions $D_\alpha$ mimic the presence of a vacuum $\alpha$ at infinity in the presence of a mass parameter $m$ and are supported on the attracting set of the vacuum $\alpha$ under gradient flow for the moment map of the $T_m$-action on $X$. This is illustrated schematically in figure 1.6. This allows boundary amplitudes to be computed as interval partition functions on $E_\tau \times [0,\ell]$ with the boundary condition $B$ at $x^3 = 0$ and the exceptional Dirichlet boundary condition $D_\alpha$ at $x^3 = \ell$, opening up the possibility of using results from supersymmetric localisation.

![Fig. 1.6](image_url) The distinguished set of boundary conditions $D_\alpha$ mimic the presence of an isolated vacuum $\alpha$ at infinite distance, at least for computations amenable localisation.

Boundary conditions of this type were first studied in two dimensions for Landau-Ginzburg models and massive sigma models in [100] and play an important part in 2d mirror symmetry. A systematic description in massive 2d theories was developed in [101, 102]. The importance of such boundary conditions in 3d $\mathcal{N} = 4$
theories was first discussed in [56], which also gave an explicit UV construction in abelian gauge theories.

2. Enriched Neumann

Enriched Neumann boundary conditions $N_\alpha$ involve Neumann boundary conditions for the gauge fields and couplings to $\mathbb{C}^*$-valued chiral multiplets via boundary superpotentials and twisted superpotentials. We introduce an explicit construction of these boundary conditions for abelian gauge theories. They are supported on unions of attracting sets corresponding to the stable envelopes introduced in [89]. We demonstrate that their boundary wavefunctions and amplitudes reproduce the construction of elliptic stable envelopes [67].

We make the observation that these two distinguished classes of boundary conditions are exchanged under 3d mirror symmetry [45]. In fact, we derive the form of enriched Neumann boundary conditions $N_\alpha$ by colliding exceptional Dirichlet boundary conditions $D_\alpha$ with the mirror symmetry interface introduced in [56]. In the process, we identify correlation functions of this mirror symmetry interface with the mother function in equivariant elliptic cohomology [70].

To make the connection with elliptic stable envelopes more precise, we consider supersymmetric Janus interfaces for the real mass parameters $m$. These are interfaces representing a position dependent profile $m(x^3)$ connecting different faces of the hyperplane arrangement. We mainly focus on interfaces interpolating between chambers of the hyperplane arrangement or between the origin and a chamber, as illustrated in figure 1.7. A crucial observation is that the computation of boundary amplitudes and overlaps are independent of the profile in the intermediate region. This allows flexibility in computing the overlaps of boundary conditions compatible with mass parameters on different faces on the hyperplane arrangement.

An important example is the correlation function of the Janus interface interpolating between a supersymmetric ground state $\langle \alpha \rangle$ at $m = 0$ and an enriched Neumann boundary $|N_\beta \rangle$ condition for a generic mass parameter $m$ in some chamber of the hyperplane arrangement. The matrix of such boundary amplitudes represents a collection of $T$-equivariant elliptic cohomology classes on $X$ labelled by $\beta$. We show that they reproduce precisely the elliptic stable envelopes [67].

Finally, we consider the correlation functions of Janus interfaces between enriched Neumann boundary conditions defined for mass generic mass parameters in two different chambers of the hyperplane arrangement. We argue that independence of the profile function implies that they obey the same algebraic relations as chamber R-matrices of
1.1 Summary of results

**Fig. 1.7** The left figure illustrates schematically a Janus interpolation function $m(x^3)$ between chambers $\mathcal{C}, \mathcal{C}'$. The right shows the projection onto the mass parameter plane in supersymmetric QED implementing between chambers $\mathcal{C} = \{m_1 < m_2 < m_3\}$ and $\mathcal{C}' = \{m_3 < m_1 < m_2\}$.

elliptic quantum integrable systems and check this correspondence explicitly in examples. We further exploit the independence of the profile to reproduce the decomposition of such R-matrices into elliptic stable envelopes.

Appendix A provides a brief review of some of the constructions in equivariant elliptic cohomology encountered in this work. Appendices B and C contain details of calculations required for results in the main body.

**N.B.** As the work in this part of the thesis was completed, we became aware of related ongoing work of Mykola Dedushenko and Nikita Nekrasov [103], which provides a powerful complementary IR point of view to our UV constructions. They graciously agreed to coordinate the publication of these results.

**Boundaries, Vermas & Factorisation**

In chapter 5, based on the author’s collaborative work [2], we turn our attention to the subject of holomorphic factorisation. This is the phenomenon, for supersymmetric gauge theories in three dimensions with at least $\mathcal{N} = 2$ supersymmetry, in which partition functions on supersymmetric backgrounds on a compact manifold $\mathcal{M}_3$ admit a factorisation of the form

$$Z_{\mathcal{M}_3} = \sum_\alpha H_\alpha \tilde{H}_\alpha. \quad (1.22)$$

Here the sum is over a finite set of vacua $\{\alpha\}$ and $H_\alpha$ is a partition function associated to the geometry $S^1 \times \{\text{hemisphere}\}$ or a twisted product $S^1 \times_q D^2$ with a boundary condition determined by the vacuum $\alpha$. 
This factorisation of supersymmetric partition functions originated in computations of the $S^3$ partition function \cite{93} and has also been checked in many examples for the superconformal index \cite{94–96} and $S^1 \times S^2$ twisted index \cite{5, 97}. Factorisation also plays an important role in the 3d-3d correspondence \cite{96, 104, 105}. It can be derived using Higgs branch localisation \cite{106, 107} and from the gluing construction of \cite{108, 109}.

The individual building blocks $H_\alpha$ of factorisation have a number of different interpretations in both physics and mathematics. A systematic approach is holomorphic blocks \cite{98}, which are defined in the IR as partition functions of massive theories on a twisted product $S^1 \times_q D^2$. This provides an elegant prescription to construct the building blocks $H_\alpha$ as solutions to certain difference equations but suffers from some ambiguities in the determination of classical and 1-loop contributions.

In this chapter, we revisit the factorisation of supersymmetric partition functions from a UV perspective for gauge theories with $N = 4$ supersymmetry. We will define the building blocks $H_\alpha$ as the hemisphere partition functions of the $\mathcal{N} = (2, 2)$ exceptional Dirichlet boundary conditions $\{D_\alpha\}$ above, which are labelled by isolated massive vacua $\alpha$

$$H_\alpha = Z_{D_\alpha}. \quad (1.23)$$

We show how these hemisphere partition functions can be computed exactly using supersymmetric localisation.

Recall that the set of boundary conditions $\{D_\alpha\}$ depend on a choice of chambers $\mathcal{C}_H, \mathcal{C}_C$ in the spaces of real mass and FI parameters. The walls separating chambers correspond to mass and FI parameters where the theory no longer has isolated vacua. As a consequence, the factorisation jumps across these walls in such a way that the partition function $Z_{M_3}$ is unchanged.

The hemisphere partition functions depend on four parameters,

$$H_\alpha = H_\alpha(q, t, x, \xi), \quad (1.24)$$

where $q$, $t$ are fugacities dual to to combinations of isometries and R-symmetries while $x$, $\xi$ are fugacities dual to Higgs and Coulomb branch global symmetries. The hemisphere partition functions of the boundary conditions $\{D_\alpha\}$ in a given chamber $\mathcal{C}_H, \mathcal{C}_C$ are characterised by their common analytic properties in the fugacities $x$, $\xi$. They differ from the holomorphic blocks of 3d $\mathcal{N} = 4$ gauge theories presented in \cite{110–112} in the classical and 1-loop contributions.

The hemisphere partition function can be related via the state-operator correspondence to a half superconformal index counting local operators at the origin of $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. 


We find that the relation between these objects is

\[ H_\alpha = e^{\phi_\alpha} I_\alpha, \]  

(1.25)

where \( I_\alpha \) is the half superconformal index of the boundary condition \( D_\alpha \) and the pre-factor \( e^{\phi_\alpha} \) is determined by boundary 't Hooft anomalies for global and R-symmetries.

We focus on two limits of the hemisphere partition function with enhanced supersymmetry. They correspond to limits of the half superconformal index that count boundary operators transforming as the scalar components of boundary chiral and twisted chiral multiplets respectively. They are defined respectively by

\[
\begin{align*}
X^H_\alpha (x) := \lim_{t^{\frac{1}{2}} \to q^{-1/4}} H_\alpha(q, t, x, \xi), \\
X^C_\alpha (\xi) := \lim_{t^{\frac{1}{2}} \to q^{1/4}} H_\alpha(q, t, x, \xi).
\end{align*}
\] 

(1.26)

Although our notation indicates that these limits depend only on a single parameter, they retain a small additional dependence on the remaining parameters due to boundary mixed 't Hooft anomalies contributing to the pre-factor \( e^{\phi_\alpha} \).

These boundary operators counted by this limit of the half superconformal index transform as modules for the quantised algebras \( \mathcal{A}_H, \mathcal{A}_C \) of functions on the Higgs and Coulomb branch respectively [56], as illustrated in figure 1.8. The quantisations are manifested by the \( \Omega_A \) and \( \Omega_B \) deformations respectively, studied in [44, 51, 113–116]. Boundary conditions compatible with real mass and FI parameters in chambers \( \mathcal{E}_H, \mathcal{E}_C \) generate modules that are lowest weight with respect to these chambers. In particular, boundary operators on the boundary conditions \( D_\alpha \) generate lowest weight Verma modules \( \mathcal{H}^{(B)}_{D_\alpha}, \mathcal{H}^{(A)}_{D_\alpha} \) for the algebras \( \mathcal{A}_H, \mathcal{A}_C \) respectively.

Fig. 1.8 Bulk operators in either omega background acting on boundary operators, defining a module for \( \mathcal{A}_H \) or \( \mathcal{A}_C \). The above represents \( \mathcal{O}^{bulk}(\mathcal{O}^{bdy}) \).
These limits of the hemisphere partition function are then expected to reproduce the characters of the modules formed by boundary chiral or twisted chiral operators. Indeed, we show that these limits reproduce traces over Verma modules

\[ \chi^H_\alpha(x) = \text{Tr}_{\mathcal{H}^{(B)}_{\alpha}} x^{J_H}, \]
\[ \chi^C_\alpha(\xi) = \text{Tr}_{\mathcal{H}^{(C)}_{\alpha}} \xi^{J_C}, \]

where \( J_H, J_C \) denote complex moment map operators generating the Higgs and Coulomb branch symmetries. It is important here to work with the hemisphere partition function rather than half superconformal index: boundary ’t Hooft anomalies encoded in \( e^{\phi_\alpha} \) are crucial to reproduce the correct lowest weights of the Verma modules. We check this proposal explicitly for abelian gauge theories.

Returning to factorisation, we explore the implications of this result for partition functions \( Z_{\mathcal{M}_3} \) on compact spaces. Following from the general structure of factorisation, we show that certain limits of the superconformal index, \( S^1 \times S^2 \) twisted index and \( S^3 \) partition function preserving additional supercharges can be expressed in terms of the characters of lowest weight Verma modules. In particular,

\[ Z^B_{\text{SC}} = \sum_\alpha \chi^H_\alpha(x) \chi^H_\alpha(x^{-1}), \quad Z^A_{\text{SC}} = \sum_\alpha \chi^C_\alpha(\xi) \chi^C_\alpha(\xi^{-1}), \]
\[ Z^H_{\text{tw}} = \sum_\alpha \chi^H_\alpha(x) \chi^H_\alpha(x), \quad Z^A_{\text{tw}} = \sum_\alpha \chi^C_\alpha(\xi) \chi^C_\alpha(\xi), \]
\[ Z_{S^3} = \sum_\alpha \hat{\chi}^H_\alpha(x) \hat{\chi}^C_\alpha(\xi), \]

where \( A \) and \( B \) denote two different limits of the superconformal and twisted index preserving additional supercharges. In the factorisation of the \( S^3 \) partition function, the hatted characters involve an additional \( \mathbb{Z}_2 \) twist by the centre of the R-symmetry.\(^9\) This reproduces the conjectured form of the \( S^3 \) partition function in [99] from the perspective of factorisation and extends it to the superconformal and twisted index. We illustrate these factorisations explicitly for supersymmetric QED with \( N \) hypermultiplets.

Appendices D and E cover boundary conditions and localisation on \( S^1 \times HS^2 \), the relation to the work [117], and the proof of some of our claims for general abelian theories.

\(^9\)There are some additional phases that we omit in the introduction.
Chapter 2

Preliminaries on 3d $\mathcal{N} = 4$ Theories

The aim of this chapter is to introduce some background, notation and assumptions on 3d $\mathcal{N} = 4$ gauge theories that will be used throughout this thesis. Importantly, it will introduce some constructions which are not commonly covered in the literature, but will be essential to establishing connections to the mathematical literature on equivariant elliptic cohomology. It is not in any way comprehensive, the reader is encouraged to consult the many works referenced throughout for further detail.

We begin by reviewing the basic data of these theories; their supersymmetry algebras, UV Lagrangians, and global symmetries. We then move onto their vacuum moduli spaces and, closely related, their chiral rings. We then examine the effective Chern-Simons couplings for flavour symmetries in massive vacua, making some novel connections with constructions in Higgs branch geometry. We introduce BPS boundary conditions. Finally, we discuss 3d mirror symmetry, a powerful duality between pairs of 3d $\mathcal{N} = 4$ theories.

Contributions The results in section 2.5 are based on material from:


2.1 Supersymmetry Algebra & Lagrangians

2.1.1 Supersymmetry Algebra

The 3d $\mathcal{N} = 4$ supersymmetry algebra in Euclidean space takes the form (see e.g. [118, 119]):

$$\{Q^{A\dot{A}}, Q^{B\dot{B}}\} = \epsilon^{AB} \epsilon^{\dot{A}\dot{B}} p_{\alpha\beta} - \epsilon^{\dot{A}\dot{B}} \epsilon_{\alpha\beta} Z^{AB} - \epsilon^{\alpha\beta} \epsilon_{\dot{A}\dot{B}} \bar{Z}^{\dot{A}\dot{B}} + C^{AB;\dot{A}\dot{B}}_{\alpha\beta}$$

(2.1)
where $\alpha, \beta$ are $SU(2)_E$ Lorentz index, $A, B$ are indices for a doublet of the $SU(2)_H$ R-symmetry, and $\dot{A}, \dot{B}$ for a doublet of the $SU(2)_C$ R-symmetry. Here, $\epsilon^{+-} = \epsilon_{--} = 1$ and $P_{\alpha\beta} = P_\mu \sigma^{\mu}_{\alpha\beta}$ where $(\sigma^{\mu})_{\alpha\beta}$ are the standard Pauli matrices. The supercharges are complex, and obey relations under conjugation depending on a choice of quantisation i.e. Euclidean time direction. For instance, choosing $x^3$ to be the time direction imposes $(Q^A_{\dot{A}} \pm Q^{\dot{B}}_B) = \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} Q^B_{\dot{B}} \mp .$

The (Lorentz) scalar central charges $Z$ and $\tilde{Z}$ transform in vector representations of $SU(2)_H$ and $SU(2)_C$ respectively, and are associated to Coulomb and Higgs branch flavour symmetries respectively. We return to these in section 2.2. $C$ is a vector central charge carried by half-BPS $\mathcal{N} = (2,2)$ domain walls. We will give more detail on these central charges in section 2.5.

It will often prove convenient to choose an $\mathcal{N} = 2$ subalgebra, corresponding to a choice of maximal torus $U(1)_H \times U(1)_C \subset SU(2)_H \times SU(2)_C$. Throughout, we will choose $Q^+_\pm = Q^\pm_{\pm}$ and $Q^-_\pm = Q^-_{\mp}$, obeying:

$$\{Q^a_{\alpha}, Q^b_{\beta}\} = \delta^{ab} P_{\alpha\beta} - \epsilon_{\alpha\beta} Z + C^{ab}_{\alpha\beta} (2.2)$$

where $Z = Z^{+-} + \tilde{Z}^{+-}$, placing the Higgs and Coulomb flavour symmetries on the same footing, and $C^{ab}_{\alpha\beta} = C^{ab,\alpha\beta}$. From the perspective of a fixed $\mathcal{N} = 2$ subalgebra, the diagonal combination of $U(1)_H \times U(1)_C$ is an R-symmetry, and the anti-diagonal combination $U(1)_H - U(1)_C$ is an $\mathcal{N} = 2$ flavour symmetry, which we denote by $T_t$. We will often introduce a mass deformation associated to this flavour symmetry, which will deform the central charges in a way to be made precise later.

### 2.1.2 Lagrangians

Let us now discuss renormalisable Lagrangian gauge theories which realise the above supersymmetry, mostly following the conventions of [56, 44]. Such a theory is specified by the following data:

- A compact gauge group $G$ (and associated dimensionful couplings $g^2$).
- A linear quaternionic representation $Q \simeq \mathbb{H}^N$ of $G$.

This means that $G$ acts as a subgroup of the hyper-Kähler isometry group $USp(N) = U(2N) \cap Sp(2N, \mathbb{C})$ preserving the canonical hyper-Kähler structure on the quaternionic space $\mathbb{H}^N \simeq \mathbb{C}^{2N} \simeq \mathbb{R}^{4N}$. We will restrict to quaternionic representations of the form $Q = R \oplus \bar{R} \simeq T^* R$, with $R$ a unitary representation of $G$. Throughout this thesis, we will also only work with gauge groups which are a product of unitary groups.
Associated to the gauge group $G$, we have an $\mathcal{N} = 4$ vector multiplet, whose (on-shell) bosonic components consist of the gauge connection $A_\mu$, together with a triplet of $g$-valued vector multiplet scalars $\varphi^{AB}$ transforming in the adjoint of $SU(2)_C$. The remaining fields consist of $N$ hypermultiplets, whose (on-shell) bosonic components consist of $4N$ real scalars. These may be written as $N$ doublets $(X^A)_{i=1}^{N}$ of the $SU(2)$ R-symmetry.

We will also, on occasion, consider twisted vector and hypermultiplets [120]. These are the same as their untwisted counterparts, only with the roles of $SU(2)_H$ and $SU(2)_C$ interchanged.

After choosing an $\mathcal{N} = 2$ subalgebra (2.2), we may decompose into $\mathcal{N} = 2$ vector and chiral multiplets. See e.g. [42] for more on $\mathcal{N} = 2$ theories.

- The $\mathcal{N} = 4$ vector multiplet scalars decompose as $(\sigma, \varphi) = (\varphi^{++}, \varphi^{++}) \in g \oplus g_C$. The real scalar $\sigma$ transforms, together with $A_\mu$, in a vector multiplet $\mathcal{V}$, and the complex scalar $\varphi$ in an adjoint chiral multiplet.

- The $\mathcal{N} = 4$ hypermultiplet scalars decompose into pairs $(X, Y)_{i=1}^{N} = (X^1, \bar{X}^2)_{i=1}^{N} \in R \oplus \bar{R}$, transforming in chiral multiplets.

See table 2.1 for the charges of these fields (we have chosen the R-symmetry so that the charges are integer quantised).

<table>
<thead>
<tr>
<th></th>
<th>$G$</th>
<th>$U(1)_H$</th>
<th>$U(1)_C$</th>
<th>$T_1$</th>
</tr>
</thead>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\varphi$</td>
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<td>2</td>
<td>$-2$</td>
</tr>
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<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$Y$</td>
<td>$\bar{R}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.1 Gauge and R-symmetry representations of bosonic fields

We can now write down the basic Lagrangian for an $\mathcal{N} = 4$ super Yang-Mills theory with matter. First note that the $\mathcal{N} = 2$ vector multiplet may be rewritten in terms of a linear field-strength multiplet $\Sigma$ dependent explicitly on the field strength $F$ of $A$.

In terms of $\mathcal{N} = 2$ superfields, the $\mathcal{N} = 4$ vector multiplet Lagrangian is

$$\mathcal{L}_{\text{gauge}}^{\mathcal{N}=4} = \mathcal{L}_\mathcal{V} + \mathcal{L}_\varphi = \frac{1}{g^2} \text{Tr} \int d^4\theta \Sigma^2 + \varphi^+ e^{-2\nu} \varphi,$$  \hspace{1cm} (2.3)
and the $\mathcal{N} = 4$ hypermultiplet Lagrangian is

$$\mathcal{L}_{\text{matter}}^{\mathcal{N}=4} = \mathcal{L}_X + \mathcal{L}_Y + [\mathcal{L}_W + h.c.]$$

$$= \int d^4 \theta X^\dagger e^{-2\mathcal{V}} X + Y^\dagger e^{-2\mathcal{V}} Y + \sqrt{2} \int d^2 \theta \varphi X + h.c. \tag{2.4}$$

In the above, $\mathcal{V}$ and $\varphi$ are understood to act in the appropriate representation, and we have abused notation by representing $\mathcal{N} = 2$ chiral superfields with their lowest scalar components. From the perspective of $\mathcal{N} = 2$ supersymmetry, $\mathcal{L}_W$ is a complex superpotential.

Note that there is no way of coupling $\mathcal{N} = 4$ vector multiplets together in (mixed or pure) Chern-Simons interactions in a way which preserves $\mathcal{N} = 4$ supersymmetry. However, it is possible to couple twisted vector multiplets to ordinary vector multiplets in an $\mathcal{N} = 4$ mixed Chern-Simons interaction. We will see an example of this in the next section.

### 2.2 Symmetries and Deformations

The theories constructed above have a flavour symmetry commuting with the supersymmetry algebra, of the form $G_H \times G_C$, where:

- The Higgs branch flavour symmetry:
  $$G_H = N_{USp(N)}(G)/G \tag{2.5}$$
  is the normaliser of the unitary representation $G$ in $USp(N)$, modulo $G$. It is the residual symmetry on the hypermultiplets after gauging. We will denote its maximal torus by $T_H$.

- The Coulomb branch flavour symmetry $G_C$ has an abelian subgroup given by the topological symmetry:
  $$T_C = \text{Hom}(\pi_1(G), U(1)), \tag{2.6}$$
  which arises in the following way. For each $U(1)$ factor of the gauge group, there is a conserved current $J = *dA_{U(1)}$, due to the Bianchi identity. We can introduce a dual photon $\gamma$ with $J = d\gamma$, where $\gamma$ is a real scalar field which is periodic: $\gamma \sim \gamma + 2\pi g^2$, due to Dirac quantisation. For each $U(1)$ factor of $G$ we therefore have a corresponding $U(1)$ topological symmetry which rotates the
2.2 Symmetries and Deformations

dual photon, or equivalently under which a corresponding monopole (disorder) operator \( \sim e^{i\gamma/g^2} \) is charged. For \( G = \Pi_{i \in I} U(V_i) \) a product of unitary groups, \( \pi_1(G) = \mathbb{Z}^{|I|} \) and so \( T_C = U(1)^{|I|} \). It is possible that the topological symmetry undergoes an enhancement in the IR to a non-abelian group \( G_C \) such that \( T_C \) is its maximal torus. We will meet an example of this later.

The origin of the names of these symmetries will be elucidated in section 2.3.

### 2.2.1 Mass and Fayet–Iliopoulos Deformations

We can deform the theories under consideration in a supersymmetry-preserving way by coupling the fields to non-dynamical background fields \([40]\) for the flavour symmetry \( G_H \times G_C \), setting the background values of the fermions to zero, and demanding that the supersymmetry variations of the fermions vanish.

- **Mass parameters** \( m^{AB} \) are expectation values for the \( SU(2)_C \) triplet of scalars in a background vector multiplet for \( G_H \). In a choice of \( \mathcal{N} = 2 \) subalgebra, they enter as real and complex mass parameters \( (m^{\bar{R}}, m^{\bar{C}}) = (m^+, m^{++}) \in t_H \oplus (t_H)_C \).

- **Fayet–Iliopoulos (FI) parameters** \( \zeta^{AB} \) are expectation values for the \( SU(2)_H \) triplet of scalars in a background twisted vector multiplet for \( G_C \). In a choice of \( \mathcal{N} = 2 \) subalgebra, they enter as real and complex FI parameters \( (\zeta^{\bar{R}}, \zeta^{\bar{C}}) = (\zeta^+, \zeta^{++}) \in t_C \oplus (t_C)_C \).

In the Lagrangian, the mass parameters enter by replacing \( \varphi^{AB} \rightarrow \varphi^{AB} + m^{AB} \) where it is understood that the fields act in the appropriate representations. The FI parameters enter in a mixed Chern-Simons level at level 1, coupling a twisted vector multiplet for \( G_C \) to the dynamical vector multiplet for \( G \). It is the supersymmetric completion of the 3d FI term

\[
\mathcal{L}_{\text{FI}} = \frac{i}{2\pi} \sum_i \zeta_{\bar{R},i} D_i, \tag{2.7}
\]

where \( i \) runs over each abelian factor of \( G \), and \( D_i \) is the associated auxiliary field, and the associated associated superpotential coupling \( W_{\text{FI}} = \zeta_{\bar{C},i} \varphi_i \).

We are now in a position to describe the scalar central charges in the \( \mathcal{N} = 4 \) supersymmetry algebra (2.1). They are given by:

\[
Z^{AB} = \zeta^{AB} \cdot J_C, \quad \bar{Z}^{AB} \cdot J_H, \tag{2.8}
\]

where \( J_C \) and \( J_H \) are generators of \( T_C \) and \( T_H \) respectively.
In the remainder of this thesis, with a few notable exceptions, we turn on only real parameters $m_R$ and $\zeta_R$ which leave unbroken the maximal tori $U(1)_H \times U(1)_C$ and $T_H \times T_C$ of the R-symmetry and flavour symmetry respectively. We will therefore drop the $\mathbb{R}$ subscript and simply denote these parameters by $m$ and $\zeta$.

We can break to $\mathcal{N} = 2$ supersymmetry by introducing a real mass parameter $\epsilon$ for $T_t$, which we noted above is a distinguished flavour symmetry from the perspective of $3d \, \mathcal{N} = 2$ supersymmetry. $\mathcal{N} = 2$ theories obtained this way are often referred to as $\mathcal{N} = 2^*$ theories in the literature. In the $\mathcal{N} = 2$ algebra (2.2), the central charge $Z$ is given by:

$$Z = m \cdot J_H + \zeta \cdot J_C + \epsilon \cdot J_t \tag{2.9}$$

where $J_t$ is the generator for $T_t$.

**Comment on Notation**

It will be useful to make the following definitions:

- We define $T := T_H \times T_t$ as the maximal torus of the $3d \, \mathcal{N} = 2$ flavour symmetry acting on elementary chiral multiplets. We denote the corresponding Lie algebra by $t$ and use a shorthand notation $x = (m, \epsilon) \in t$ to denote collectively the associated real mass parameters.

- We define $T_f := T_C \times T_H \times T_t$ as the total $3d \, \mathcal{N} = 2$ flavour symmetry acting on elementary chiral multiplets and monopole operators. We denote the corresponding Lie algebra by $t_f$ and use a shorthand notation $x_f = (\zeta, m, \epsilon) \in t_f$ to denote collectively the associated real parameters.

**2.2.2 Example**

It is convenient time to introduce a running example we will use throughout this thesis; supersymmetric QED. This is a sufficiently complex theory to illustrate all of the main ideas on connections to the mathematical literature throughout this work.

The theory is defined as in section 2.1.2 by $G = U(1)$ and $Q = T^* \mathbb{C}^N$ where $\mathbb{C}^N$ denotes $N$ copies of the charge $+1$ representation. The flavour symmetries are $G_C = U(1)$ and $G_H = PSU(N)$. Correspondingly, we introduce an FI parameter $\zeta \in \mathbb{R}$, mass parameters $(m_1, \ldots, m_N) \in \mathbb{R}^{N-1}$ with $\sum \alpha m_\alpha = 0$.

The hypermultiplets decompose into chiral multiplets with complex scalar components $(X_\alpha, Y_\alpha)$, which transform with charge $(+1, -1)$ under the gauge symmetry, in the anti-fundamental and fundamental representation of $G_H = PSU(N)$ and with
charge \((+1,+1)\) under \(T_t\). The chiral multiplets \(\varphi, X_\alpha, Y_\alpha\) have total real mass \(-2\epsilon, \sigma - m_\alpha + \epsilon, -\sigma + m_\alpha + \epsilon\) respectively. The theory is represented as a quiver in figure 2.1.

2.3 Vacuum Moduli Spaces

We now discuss the vacuum moduli spaces of 3d \(\mathcal{N} = 4\) theories, which \(\mathcal{N} = 4\) supersymmetry constrains to be hyper-Kähler [41]. There are generically flat directions in the classical potentials for these theories, which descend in the IR to quantum moduli spaces of vacua. As before, we will consider only the case where real mass and FI parameters are turned on. See e.g. [44, 56] for the relevant statements involving complex parameters.

The solutions to the classical vacuum equations are given by the simultaneous critical loci of the following two superpotentials.

- The complex 3d \(\mathcal{N} = 2\) superpotential \(W\) given by:
  \[
  W = \varphi \cdot \mu_C
  \]  
  where \(\mu_C : T^*R \rightarrow g^* \otimes_{\mathbb{R}} \mathbb{C}\) is the complex moment map for the \(G\) action on the hypermultiplet representation \(T^*R\). We recognise this as the F-term contribution in the matter Lagrangian for the hypermultiplets (2.4).

- The real superpotential \(h\) given by:
  \[
  h = \sigma \cdot (\mu_R + [\varphi, \varphi^\dagger] - \zeta) + m \cdot \mu_{H,R} + \epsilon \cdot \mu_{t,R}
  = \sigma \cdot (\mu_R + [\varphi, \varphi^\dagger] - \zeta) + x \cdot \mu_{T,R},
  \]
  where \(\mu_R, \mu_{H,R}, \mu_{t,R}\) are the real moment maps for the \(G, T_H, T_t\) action on \(T^*R\). In the second line, we have used the shorthand notation \(x = (m, \epsilon)\) and \(\mu_{T,R} = (\mu_{H,R}, \mu_{t,R})\) for the mass parameters and moment maps for \(T = T_H \times T_t\).

In general, both the hypermultiplet and the vector multiplet scalars may obtain vacuum expectation values (vevs), and the vacuum moduli space consists of ‘mixed branches’. However, for specific ranges of the parameters, there are distinct branches.
known as the Higgs branch $\mathcal{M}_H$ and Coulomb branch $\mathcal{M}_C$ where the hypermultiplet scalars and the vector multiplet scalars separately obtain vevs. Throughout this thesis, we will also make the key assumption that:

**Assumption:** The gauge theory flows to an interacting superconformal fixed point (without a decoupled free sector) and, upon introducing generic real FI and mass parameters, has isolated, massive, topologically trivial vacua. We label the isolated vacua by indices, $\alpha, \beta, \ldots$ and denote the number of them by $N$.

This translates to the assumption that $\mathcal{M}_H$ and $\mathcal{M}_C$ are conical symplectic resolutions, with resolution parameters $\zeta$ and $m$ respectively, and have isolated fixed points under infinitesimal $T_H$ and $T_C$ actions respectively. We will unpack this further in the remainder of this section, including what we mean by genericity.

Let us also state that an additional motivation for the assumption comes from the fact that such theories transform nicely under 3d mirror symmetry, and thus play an important role in connections to the mathematical literature on symplectic duality [59, 60]. We will return to this idea at various points in this thesis.

**The Higgs Branch**

Let us first set the real masses $m$ and $\epsilon$ to zero. The vacuum equations are then:

\begin{align}
\mu_R &= \zeta, \quad \mu_C = 0, \\
\sigma \cdot X &= 0, \quad \sigma \cdot Y = 0, \\
\varphi \cdot X &= 0, \quad \varphi^\dagger \cdot X = 0, \quad \varphi \cdot Y = 0, \quad \varphi^\dagger \cdot Y = 0, \\
[\sigma, \varphi] &= 0, \quad [\varphi, \varphi^\dagger] = 0.
\end{align}

(2.12)

Then under our key assumption, the FI parameter $\zeta$ can be chosen such that $G$ acts freely on the solutions to vacuum equations. This generally requires that $\zeta$ lies in the complement of an arrangement of linear hyperplanes through the origin in the space $t_C$ of FI parameters. This means that it lies in some chamber $\mathcal{C}_C$: a connected component of the complement of the hyperplanes, or alternatively a face of the hyperplane arrangement of maximal dimension. We return to give further detail on this chamber structure later in the chapter.

This implies that $(\sigma, \varphi) = 0$ on the solutions of the vacuum equations, i.e. that the gauge group is completely broken. This is what gives the Higgs branch its name. Then
the remaining vacuum equations describe the Higgs branch as a hyper-Kähler quotient:

\[ \mathcal{M}_H = \mu_C^{-1}(0) \cap \mu_R^{-1}(\zeta)/G, \]  

(2.13)

Under our assumptions, for a generic \( \zeta \) this is a smooth hyper-Kähler manifold. The assumption requires that:

\[ \pi : \mathcal{M}_H \to \mathcal{M}_H|_{\zeta=0} \]  

(2.14)

is, for a choice of generic \( \zeta \), a conical symplectic resolution of the hyper-Kähler cone \( \mathcal{M}_H|_{\zeta=0} \) given by the quotient (2.13) with \( \zeta = 0 \). The inverse image \( \pi^{-1}(0) \), where 0 is the conical singularity (and a fixed point of \( T_T \)), is a compact holomorphic Lagrangian.

Importantly, the non-renormalisation theorems of [42, 43] tell us that this classical description of the Higgs branch is not corrected in the IR, i.e. that the quantum Higgs branch is simply the classical one. Thus, in a phase with zero mass and generic FI, the theory flows (if placed on a compact space) to a non-linear sigma model to the Higgs branch.

**The Coulomb Branch**

When the FI parameters \( \zeta \) vanish the vacuum equations admit a Coulomb branch \( \mathcal{M}_C \), where \( (\sigma, \varphi) \) obtain vevs in the Cartan subalgebra \( \mathfrak{h} \) of \( g \). Let \( r = \text{rk}G \). For generic values of \( (\sigma, \varphi) \in \mathbb{R}^{3r} \), the gauge group is broken to a maximal torus of \( G \), giving the Coulomb branch its name. The massless abelian gauge fields in the \( r \, U(1) \) factors of the maximal torus can be dualised to periodic scalars \( \gamma \), and one obtains a classical description of the Coulomb branch:

\[ \mathcal{M}^{\text{class}}_C \approx \left[ (\mathbb{R}^{3r} - \Delta) \times (S^1)^r \right]/\text{Weyl}(G) \]  

(2.15)

where \( \Delta \) is the discriminant locus in \( \mathbb{R}^{3r} \) of values of \( (\sigma, \varphi) \) which do not fully break \( G \) to its maximal torus.

The Coulomb branch is also hyper-Kähler, of complex dimension \( \text{dim}_C \mathcal{M}^{\text{class}}_C = 2\text{rk}G \). However, unlike the Higgs branch, quantum corrections (both perturbative and non-perturbative) can modify the geometry and topology of the Coulomb branch. This was first described precisely in [44]. It turns out that the real masses \( m \) act as resolution parameters for the Coulomb branch. We return to discuss the Coulomb branch in more detail in section 2.7, although it will not be the focus of this thesis.
Symmetries

The R-symmetries and flavour symmetries descend to actions on the hyper-Kähler varieties $\mathcal{M}_H$ and $\mathcal{M}_C$. The R-symmetries $SU(2)_H$ and $SU(2)_C$ rotate the triplet of complex structures on $\mathcal{M}_H$ and $\mathcal{M}_C$ respectively. The flavour symmetries $G_H$ and $G_C$ act via tri-Hamiltonian isometries of $\mathcal{M}_H$ and $\mathcal{M}_C$ respectively.

From the $\mathcal{N} = 2$ perspective, a choice of maximal torus $U(1)_H \times U(1)_C$ singles out a preferred complex structure on $\mathcal{M}_H$ and $\mathcal{M}_C$. The Higgs and Coulomb branch are then viewed as Kähler manifolds equipped with a holomorphic symplectic form, transforming with weight $+2$ under $U(1)_H$ and $U(1)_C$ respectively (and therefore $+2$ and $-2$ under $T_t$). The flavour symmetries $G_H$ and $G_C$ act via Hamiltonian isometries of $\mathcal{M}_H$ and $\mathcal{M}_C$ which leave the respective holomorphic symplectic forms invariant.

2.3.1 Isolated Massive Vacua

Let us now consider turning on both real masses $(m, \epsilon)$ and FI parameters $\zeta$. The vacuum equations 2.12 are modified by:

$$\sigma \to \sigma + m + \epsilon,$$

(2.16)

where $(m, \epsilon)$ are understood to act in the appropriate representation of $T = T_H \times T_t$. The vacua now correspond to configurations of $X, Y$ and $\varphi$ solving the updated vacuum equations, such that there exists a $\sigma$ such that the combined infinitesimal $G \times T_H \times T_t$ transformations generated by $\sigma + m + \epsilon$ leaves the configuration invariant. They are simultaneous critical points of $(h, W)$ given in equations (2.10) and (2.11).

Such vacua can be viewed as the fixed locus of $T_t = T_H \times T_t$ on $\mathcal{M}_H$ generated by $m + \epsilon$, or alternatively the fixed locus of $T_C \times T_t$ on $\mathcal{M}_C$ generated by $\zeta + \epsilon$. Since the Higgs branch is not quantum-corrected, it will be convenient for us to take the former point of view. Due to our key assumption, for generic mass parameters $m$ (and any $\epsilon$), the fixed locus

$$\mathcal{M}_H^{T_H \times T_t} = \{\alpha\}$$

(2.17)

is a set of isolated fixed points, which we index by $\alpha \in \{1, ..., N\}$.

It will be convenient to give the following combinatorial description of the supersymmetric vacua. Each vacuum $\alpha$ is specified by a set of $r := \text{rank}(G)$ distinct weights

$$\alpha : \{\varrho_1, \ldots, \varrho_r\}$$

(2.18)
appearing in the $G \times T_H \times T_t$ weight decomposition of the hypermultiplet representation $T^* R$. These weights are the charges of the hypermultiplet fields that do not vanish in the vacuum. If we decompose into components

$$\varrho_i = (\rho_i, \rho_{H,i}, \rho_{t,i}), \quad (2.19)$$

the vector multiplet scalar $\sigma$ is fixed by the equations

$$\rho_i \cdot \sigma + \rho_{H,i} \cdot m + \rho_{t,i} \cdot \epsilon = 0 \quad (2.20)$$

for all $i = 1, \ldots, r$. This is simply the statement that the action of $\sigma + m + \epsilon$ on the hypermultiplet fields which attain vevs must be trivial.

To define a supersymmetric vacuum, the gauge components must satisfy

1. $\{\rho_1, \ldots, \rho_r\}$ span $\mathfrak{h}^*$,
2. $\zeta \in \text{Cone}^+(\rho_1, \ldots, \rho_r) \subset \mathfrak{h}^*$,

where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra of $G$. Here we regard the FI parameter $\zeta$ as an element of $\mathfrak{h}^*$ through the inclusion $t_C = Z(\mathfrak{h}^*) \subset \mathfrak{h}^*$.

The first condition can be understood as requiring that the gauge group is completely broken, and the second that the FI parameter allows the appropriate hypermultiplet scalar vev to be turned on in solving $\mu_R = \zeta$. These conditions coincide with JK residue prescriptions appearing in the computation of supersymmetric observables. This is consistent with the realisation of supersymmetric vacua as fixed points on the Higgs branch.

Aside: Due to the hypermultiplet field expectation values, the Higgs branch flavour and R-symmetries preserved in a massive vacuum $\alpha$ are shifted compared to the UV gauge theory. These shifts are explicitly defined in section 2.5.1. When needed, we distinguish the unbroken symmetries in a massive vacuum $\alpha$ by a superscript $U(1)^{(\alpha)}_H$, $T^{(\alpha)}_H$, or from the perspective of 3d $\mathcal{N} = 2$ flavour symmetries $T^{(\alpha)} = T^{(\alpha)}_H \times T^{(\alpha)}_t$. However, we mostly drop the superscript to lighten the notation with the understanding that these symmetries are shifted as appropriate for the supersymmetric massive vacuum.
Example

In supersymmetric QED, the complex and real superpotentials are

\[
W = \phi \sum_{\alpha=1}^{N} X_\alpha Y_\alpha, \\
R = \sum_{\alpha=1}^{N} (\sigma - m_\alpha + \epsilon)|X_\alpha|^2 + \sum_{\alpha=1}^{N} (-\sigma + m_\alpha + \epsilon)|Y_\alpha|^2 - \zeta \sigma. 
\] (2.21)

There are \( N \) isolated critical points for generic mass and FI parameters. The critical points \( \alpha = 1, \ldots, N \) corresponds to non-vanishing expectation values \( |X_\alpha|^2 = \zeta \) and \( \sigma = m_\alpha - \epsilon \) when \( \zeta > 0 \), and \( |Y_\alpha|^2 = -\zeta \) and \( \sigma = m_\alpha + \epsilon \) when \( \zeta < 0 \).

2.4 Higgs Branch Geometry

For the purposes of connecting with equivariant elliptic cohomology, we view a supersymmetric gauge theory through the lens of its Higgs branch. This means we first set \( m = 0 \), keeping \( \zeta \) generic, such that the theory flows to a smooth sigma model onto the Higgs branch. We can then turn the mass parameter back on as a deformation of the sigma model.

Comment on notation: From here on out, associated to a given 3d \( N = 4 \) theory \( T \), we will mainly denote the Higgs branch by \( X \) and the Coulomb branch by \( X' \). This is consistent with the mathematical literature on equivariant elliptic cohomology and symplectic duality.

Recall that the Higgs Branch is a hyper-Kähler quotient:

\[
X = \mu_C^{-1}(0) \cap \mu_\mathbb{R}^{-1}(\zeta)/G, 
\] (2.22)

which may be viewed as a Kähler manifold with a holomorphic symplectic form, descending from that on \( Q \): \( dX \wedge dY \). The FI parameter \( \zeta \) is a Kähler parameter and \( X \) depends on the chamber \( C \). The holomorphic symplectic structure is independent of \( \zeta \) within each chamber. We typically fix a chamber and omit this from the notation.

Let us review the role of flavour symmetries from this perspective.
• The topological symmetry is $T_C = \text{Pic}(X) \otimes \mathbb{Z} U(1)$ and has co-character lattice $\Gamma_C = \text{Pic}(X)$. The chamber $\mathcal{C}_C \subset \text{Pic}(X) \otimes \mathbb{Z} \mathbb{R}$ containing $\zeta$ is the ample cone of $X$.

• The flavour symmetry $T_H$ is a maximally commuting set of Hamiltonian isometries of $X$ leaving the holomorphic symplectic form invariant.

• The flavour symmetry $T_t$ is a Hamiltonian isometry of $X$ transforming the holomorphic symplectic form with weight $+2$.

Recall that we have defined $T = T_H \times T_t$ and $T_f = T_C \times T_t$.

Let us clarify the above description of the topological symmetry. As $X$ is a quotient by $G$ (which we assume to be a product of unitary groups $\prod_{i \in I} U(V_i)$), there is a natural principal $G$-bundle over $X$, and associated $T$-equivariant tautological bundles $\mathcal{V}_i$. A result known as Kirwan surjectivity, proven for Nakajima quiver varieties [121] and hyper-toric varieties [122, 123] (which are the main types of theories we consider in this work, defined below), states that the equivariant $K$-theory $K_T(X)$ is generated by the Schur functors of the tautological bundles $\mathcal{V}_i$. Consequently the Picard group of $X$ is generated by the tautological line bundles $L_i = \det V_i$ for $i \in I$ and so $\text{Pic}(X) \cong \mathbb{Z}^{|I|}$. We may therefore regard $T_C \cong U(1)^I \cong \text{Pic}(X) \otimes \mathbb{Z} U(1)$ and $t_C \cong \text{Pic}(X) \otimes \mathbb{Z} \mathbb{R}$, with $\zeta$ an element of the latter.

### Massive Vacua

As before, the Hamiltonian isometries $T_H$ have isolated fixed points that are identified with the massive supersymmetric vacua $\{\alpha\}$. However, we can now offer up an alternative, equivalent, description of the vacua in the sigma model description, by re-introducing the mass parameters. A mass parameter $x = (m, \epsilon)$ is a deformation of the sigma model by the real moment maps $h_m, h_\epsilon : X \to \mathbb{R}$ for the corresponding Hamiltonian isometries of $X$,

$$
\begin{align*}
    h_m &= m \cdot \mu_{H,\mathbb{R}}, \\
    h_\epsilon &= \epsilon \cdot \mu_{t,\mathbb{R}},
\end{align*}
\tag{2.23}
$$

where $\mu_{H,\mathbb{R}}, \mu_{t,\mathbb{R}}$ now denote the moment maps for the $T = T_H \times T_t$ action on $X$, which descend from those on $Q$ appearing in the real superpotential (2.11). Provided $m, \epsilon$ are generic, the critical points are the massive vacua,

$$
\text{Crit}(h_m) = \text{Crit}(h_m + h_\epsilon) = \{\alpha\}. \tag{2.24}
$$
2.4.1 Algebraic Description

For many computations, it is convenient to introduce an algebraic description of the Higgs branch as a holomorphic symplectic quotient

\[ X = \mu_c^{-1}(0)^*/G_c, \tag{2.25} \]

where \( G_c \) is the complexification of the gauge group and the superscript denotes a stability condition depending on the chamber \( \mathcal{C}_C \) containing \( \zeta \). This is the content of the Kempf-Ness theorem [124]. We do not describe the stability condition for a general theory, instead giving it on a case-by-case basis for the examples we consider.

From this perspective, \( T_H, T_t \) combine with gradient flow for the associated moment maps to give an action of the complexification of \( T_H, T_t \) on \( X \) by complex isometries transforming the holomorphic symplectic form with weight 0, +2 respectively. The vacua \( \alpha \) are again the fixed points of these actions.

The algebraic description provides a convenient way to enumerate the collections of weights \( \Phi_\alpha \) of the tangent spaces \( T_\alpha X \), which will play an important part in this paper. First, considering fluctuations \( (\delta X, \delta Y) \) of the hypermultiplets, we may consider the complex

\[ 0 \to g_c \overset{\alpha}{\to} T^*R \overset{\beta}{\to} g^*_c \to 0, \tag{2.26} \]

where the maps:

\[ \alpha : (\delta g) \mapsto (\delta g \cdot X, \delta g \cdot Y), \quad \beta : (\delta X, \delta Y) \mapsto (X \cdot \delta Y + \delta X \cdot Y) \tag{2.27} \]

at a point \((X, Y) \in T^*R\) are infinitesimal complex gauge transformations and the differential of the complex moment map. Restricting to the stable locus \( \mu_c^{-1}(0)^* \), the complex becomes exact and descends to a complex of vector bundles on \( X \), whose cohomology gives the tangent bundle:

\[ TX = \text{Ker}(\beta)/\text{Im}(\alpha). \tag{2.28} \]

The terms in this complex transform as representations of \( G \times T_H \times T_t \). We introduce formal grading parameters \( w = (s, v, t) \) such that a weight \( \varrho = (\rho, \rho_H, \rho_t) \) is represented by a Laurent monomial \( w^\varrho = s^\rho v^{\rho_H} t^{\rho_t} \). Recall from section 2.2 that fixed points \( \alpha \) are in 1-1 correspondence with collections of weights \( (\varrho_1, \ldots, \varrho_r) \) appearing in the weight decomposition of \( T^*R \). Since the gauge components span \( \mathfrak{h}^* \), we may solve the
2.4 Higgs Branch Geometry

\[ w^s = s^R_{\rho H, i} t^{\rho H, i} = 1 \quad (2.29) \]

uniquely for the \( r \) components of \( s \). We denote the solution associated to a supersymmetric vacuum \( \alpha \) by \( s_\alpha \). The character of \( T_\alpha X \) is obtained from that of the tangent complex by the substitution \( s = s_\alpha \),

\[ \begin{align*}
    \text{Ch} T_\alpha X &= \text{Ch} Q - \text{Ch} g_C - t^{-2} \text{Ch} g_C^* \bigg|_{s=s_\alpha} \\
    &= \sum_{\lambda \in \Phi_\alpha} v^{\lambda_H, \lambda_t} \lambda_t^H, \\
    \end{align*} \quad (2.30) \]

from which the weights \( \lambda = (\lambda_H, \lambda_t) \in \Phi_\alpha \) can be determined.

Let us note here that there is an important pairing of weights in \( \Phi_\alpha \) given by the action of the holomorphic symplectic form. As the holomorphic symplectic form transforms with weight +2 under \( T_t \), if \( \lambda \in \Phi_\alpha \), then also \( \lambda^* \in \Phi_\alpha \), where we define \( \lambda^* = -2e_t - \lambda \) and \( e_t \) is the fundamental weight of \( T_t \).

2.4.2 Examples

Let us now give two examples of classes of 3d \( \mathcal{N} = 4 \) theories and their associated Higgs branch geometries. Theories in these classes often behave nicely under mirror symmetry. All the examples we meet throughout this thesis will fall into either category.

- **Abelian gauge theories and hyper-toric varieties**

  We will consider abelian gauge theories with gauge group \( G = U(1)^r \) and \( N \) hypermultiplets \( (X_i, Y_i) \). We will assume that no nontrivial subgroup of \( G \) acts trivially on the hypermultiplets, and hence \( N \geq r \) and \( \text{rk} G_H = r' := N - r \).

  In general we will only work with the maximal tori of the flavour symmetries, and set \( G_H = T_H = U(1)^{r'} \), \( G_C = T_C = U(1)^r \). The data of this theory is then given simply by a charge matrix \( (Q)_{1 \leq i \leq N}^{1 \leq a \leq r} \) which specifies the linear quaternionic representation \( Q = T^* R \).

  There are many gauge-equivalent choices of matrix \( (q)_{1 \leq i \leq N}^{1 \leq a \leq r} \) encoding the \( T_H \) charges of the hypermultiplets. They are given by those matrices which satisfy

\[ \left( \begin{array}{c}
    \star \\
    \bar{Q}
  \end{array} \right) = \left( \begin{array}{c}
    Q \\
    q
  \end{array} \right)^{-1, T} \quad (2.31) \]
such that

\[ 0 \rightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^N \xrightarrow{\tilde{Q}} \mathbb{Z}^{r'} \rightarrow 0 \]  

(2.32)

is a short exact sequence of lattices.

The resulting Higgs branch is, for generic values of the FI parameter \( \zeta \), a smooth hyper-toric variety. See [56], in particular chapter 6, for further details on these theories. The geometry of hypertoric varieties have been studied intensely in the mathematics literature, see *e.g.* [122, 125–129], and can be organised into the combinatoric data of so-called hyperplane arrangements, or hypertoric diagrams. We will introduce these ideas on a need-to-know basis throughout this thesis, referring to the above sources when required.

- **Quiver gauge theories and Nakajima quiver varieties**

We will also encounter unitary quiver gauge theories with gauge group \( G = \prod_I U(V_I) \) and \( Q = T^*R \) with:

\[ R = \bigoplus_I \text{Hom}(W_I, V_I) \oplus \bigoplus_{I \leq J} \text{Hom}(V_I, V_J) \otimes Q_{IJ} \]  

(2.33)

where \( V_I \) and \( W_I \) are complex vector spaces whilst \( Q_{IJ} \) is a matrix of multiplicities. They are ubiquitous throughout the study of supersymmetric field theory, and can be geometrically engineered as the gauge theory living on a system of branes [130].

The Higgs branches of these theories are quiver varieties, introduced by Nakajima [131]. The stability conditions were formulated by King [132], see also *e.g.* [133].

**Supersymmetric QED**

We now give some explicit details for our main example of supersymmetric QED, which conveniently falls into both of the categories above. Its quiver diagram was given in figure 2.1. We will, unless otherwise stated, work with quiver gauge theory conventions for supersymmetric QED throughout this thesis.

The Higgs branch is the holomorphic symplectic quotient obtained by imposing the real and complex moment map equations,

\[ \sum_{\alpha=1}^N (|X_\alpha|^2 - |Y_\alpha|^2) = \zeta, \quad \sum_{\alpha=1}^N X_\alpha Y_\alpha = 0, \]  

(2.34)
and quotienting by $G = U(1)$ gauge transformations. The holomorphic symplectic form on $X$ is induced from $\Omega = \sum \alpha \ dX_{\alpha} \wedge dY_{\alpha}$ under the quotient. In both chambers $\mathcal{C}_C = \{ \pm \zeta > 0 \}$, we have
\[
X = T^* \mathbb{CP}^{N-1}
\]
which is a symplectic resolution of the minimal nilpotent orbit in $\mathfrak{sl}(N, \mathbb{C})$. The two chambers are distinguished by the identification of the ample cone $\mathcal{C}_C = \text{Pic}(X) \otimes \mathbb{Z} \mathbb{R}_\pm$.

The stable locus for $\zeta > 0$ and $\zeta < 0$ consists of solutions where $X \neq 0$ and $Y \neq 0$ respectively, i.e. when\(^1\) $X$ or $Y$ define a complex line in $W = \mathbb{C}^N$. For example, for $\zeta > 0$ we have the holomorphic symplectic quotient
\[
\{X_{\alpha}, Y_{\alpha} \mid \sum X_{\alpha} Y_{\alpha} = 0, X \neq 0 \} / \mathbb{C}^*
\]
recovering the algebraic description (2.25) of $X$. The $\{X_{\alpha}\}$ may be regarded as homogeneous coordinates on the base $\mathbb{CP}^{N-1}$.

The flavour symmetry acts as:
\[
T_H : (X_{\alpha}, Y_{\alpha}) \mapsto (e^{-i\theta_{\alpha}} X_{\alpha}, e^{+i\theta_{\alpha}} Y_{\alpha}), \quad T_t : (X_{\alpha}, Y_{\alpha}) \mapsto (e^{i\theta} X_{\alpha}, e^{i\theta} Y_{\alpha})
\]
There are $N$ fixed points $\{\alpha\}$ corresponding to the image of $X_{\alpha} = \sqrt{\zeta}$ when $\zeta > 0$ and $Y_{\alpha} = \sqrt{-\zeta}$ when $\zeta < 0$ under the quotient. They are the coordinate lines in the base $\mathbb{CP}^{N-1}$.

The mass parameters $(m_1, \ldots, m_N)$, $\epsilon$ act by Hamiltonian isometries induced by the transformations of the coordinates $X_{\alpha}$, $Y_{\alpha}$, with moment maps
\[
h_m = - \sum_{\alpha=1}^{N} m_{\alpha} (|X_{\alpha}|^2 - |Y_{\alpha}|^2),
\]
\[
h_{\epsilon} = \epsilon \sum_{\alpha=1}^{N} (|X_{\alpha}|^2 + |Y_{\alpha}|^2).
\]
These equal $h_m(\alpha) = -m_{\alpha} \zeta$, $h_{\epsilon}(\alpha) = \epsilon |\zeta|$ at the critical loci.

\section{2.5 Chern-Simons Couplings & Higgs Branches}

We now turn our attention to two objects associated to the massive vacua of 3d $\mathcal{N} = 4$ theories, Chern-Simons couplings and BPS domain walls. We will see that they are

\(^1\)Note we have overloaded the notation $X$ for both an $\mathcal{N} = 2$ chiral as well as the Higgs branch, we hope that the intended meaning is clear from context.
Preliminaries on 3d $\mathcal{N} = 4$ Theories

intimately connected, and demonstrate some novel connections to associated objects in the geometry of the Higgs branch. These constructions will be of central importance in our connections to equivariant elliptic cohomology in chapters 3 and 4.

2.5.1 Chern-Simons Levels

Recall that in the UV, the only bare Chern-Simons coupling is between the vector multiplet for the gauge symmetry $G$ and a background twisted vector multiplet for the topological symmetry $T_C$. However, effective Chern-Simons levels can be generated in the IR from the breaking of $G$, and integrating out massive degrees of freedom [84–86]. In a massive supersymmetric vacuum $\alpha$, we may have effective supersymmetric Chern-Simons terms for flavour and R-symmetries.

We keep track of supersymmetric Chern-Simons terms for $\mathcal{N} = 2$ flavour symmetries $T_f$, which are encapsulated in pairings

$$K_\alpha : \Gamma_f \times \Gamma_f \rightarrow \mathbb{Z}$$

(2.39)

where $\Gamma_f \subset t_f$ denotes the co-character lattice of $T_f$. As mentioned above, here $T_f = T_C \times T$ denotes the symmetries preserved in the massive vacuum $\alpha$, which may be shifted compared to the UV gauge theory definition. We will choose slightly unusual conventions such that the induced Chern-Simons term in a vacuum $\alpha$ is:

$$\mathcal{L}_\alpha^{\text{eff}} = -\frac{i}{2\pi}K_\alpha(A^f, \wedge dA^f)$$

(2.40)

where $A^f = (A^C, A^H, A^t)$ are the background connections for $T_f$.

As we will see, the Chern-Simons levels are piece-wise constant in the parameters $x_f$ and may jump across loci where an extended moduli space of supersymmetric vacua opens up.

We will mainly consider 3d $\mathcal{N} = 4$ gauge theories broken to $\mathcal{N} = 2$ broken to $\mathcal{N} = 2$ by a small (in a way to be defined) mass parameter $\epsilon$ for the $\mathcal{N} = 2$ flavour symmetry $T_t$. In this case, the potential supersymmetric Chern-Simons terms for flavour symmetries are restricted to the following:

- A mixed $T_H$-$T_C$ supersymmetric Chern-Simons term $\kappa_\alpha : \Gamma_H \times \Gamma_C \rightarrow \mathbb{Z}$.

- A mixed $U(1)_H$-$T_C$ coupling, which becomes a mixed flavour $T_t$-$T_C$ supersymmetric Chern-Simons term $\kappa_\alpha^C : \Gamma_t \times \Gamma_C \rightarrow \mathbb{Z}$.

$^2$Note in this convention a normal Chern-Simons level must be $\frac{1}{2}\mathbb{Z}$-quantised for gauge-invariance.
• A mixed $T_H$-$U(1)_C$ coupling, which becomes a mixed flavour $T_H$-$T_t$ supersymmetric Chern-Simons term $\kappa^H_{\alpha} : \Gamma_H \times \Gamma_t \to \mathbb{Z}$.

• A mixed $U(1)_H$-$U(1)_C$ coupling, which becomes a flavour $T_t$-$T_t$ supersymmetric Chern-Simons term $\tilde{\kappa}_\alpha : \Gamma_t \times \Gamma_t \to \mathbb{Z}$.

where $\Gamma_C$, $\Gamma_H$, $\Gamma_t$ denote the co-character lattices of $T_C$, $T_H$, $T_t$ respectively. We write collectively

$$K_\alpha = (\kappa_\alpha, \kappa^C_\alpha, \kappa^H_\alpha, \tilde{\kappa}_\alpha). \quad (2.41)$$

To proceed, let us first describe the re-definition of flavour symmetries in a vacuum. Recall the combinatoric characterisation of supersymmetric vacua $\alpha$ in section 2.2.

There are $r = \text{rk}G$ chiral multiplet scalars (each belonging to different hypermultiplets) which attain vevs, with corresponding $G \times T \times T_H$ weights $g_i = (\rho_i, \rho_{H,i}, \rho_{t,i})$. Note that we will work in the convention that these weights are opposite to the charges of the corresponding chiral.\(^3\) This fixes the vector multiplet scalars:

$$\rho_i^a \sigma_a + (\rho_{H})^i s_m + (\rho_t)i \epsilon = 0. \quad (2.42)$$

Thus a combination of $G$, $T_H$ and $T_t$ is broken. This can be seen as a breaking of $G$ and a re-definition of $T_H$ and $T_t$ from their UV definitions:

$$J^{(\alpha)}_H = J_H - J \cdot \rho^{-1} \cdot \rho_H, \quad J^{(\alpha)}_t = J_t - J \cdot \rho^{-1} \cdot \rho_t, \quad (2.43)$$

having suppressed indices. Here, $J, J_H, J_t$ are the generators of $G \times T_H \times T_t$. Note that we may invert $\rho^a_i$ due to the first condition in section 2.2, corresponding to the fact that the vacuum breaks the gauge symmetry completely. In the remainder of this section, we will drop the the $(\alpha)$ superscript, with this re-definition understood.

**Generation**

We are now in a position to describe how the effective Chern-Simons levels are generated, which we do so first from a gauge theory point of view. It will be convenient to introduce fundamental weights $\{e_1, \ldots\}$ for $T_H$, $e_t$ for $T_t$ and $\{e^C_1, \ldots\}$ for $T_C$.

The $\kappa_\alpha$ and $\kappa^C_\alpha$ terms come simply from the mixed Chern-Simons term (2.7) in the UV coupling $G$ and $T_C$, and the re-definition of symmetries (2.43). The UV FI coupling corresponds to $-e_C \otimes e_G$ where $e_G$ are fundamental weights for the gauge

\(^3\)This is geometrically motivated by the mathematics literature: the corresponding weight to a chiral $X$ in $T^*R$ is $dX/dX$.\)
symmetry. Therefore, in terms of the symmetries in the vacuum $\alpha$:

$$\kappa_\alpha + \kappa^C_\alpha = (\rho^{-1})^i_a(\rho_H)^s_i e^a_C \otimes e_s + (\rho^{-1})^i_a(\rho_i)^s_i e^a_C \otimes e_t. \quad (2.44)$$

The remaining Chern-Simons levels in $K_\alpha$, which we denote $\kappa_{\alpha}^{\text{matter}}$, are obtained by integrating out massive degrees of freedom [84–86]. They are given by [42, 134]:

$$\kappa_{\alpha}^{\text{matter}} = \frac{1}{4} \sum_{N=2}^{\mathcal{N}=2} (\lambda_H \cdot e + \lambda_t e_t) \otimes (\lambda_H \cdot e + \lambda_t e_t) \text{sign}(m^{eff}) \quad (2.45)$$

where the sum is over 3d $\mathcal{N} = 2$ chirals, $\lambda_H$ and $\lambda_t$ are the charges of the chiral under the flavour symmetries $T_H$ and $T_t$ in the vacuum, and $m^{eff}$ is the effective mass of the chiral in the vacuum $\alpha$. This is given by substituting $\sigma$ in the effective mass of the chiral, for the values (2.42) fixed by the vacuum. Note here we must remember to include $\mathcal{N} = 2$ adjoint chiral multiplets coming from the $\mathcal{N} = 4$ vector multiplet. This expression clearly exhibits a piece-wise dependence on the mass parameters $x = (m, \epsilon)$.

If $\epsilon$ is sufficiently small, it is not hard to see (see section 2.5.2), that $\kappa_{\alpha}^{\text{matter}}$ is of the form $\kappa^{H}_\alpha + \tilde{\kappa}_\alpha$.

**Example**

We again return to our example of supersymmetric QED. We have fundamental weights $e_1, \ldots, e_N$ for $T_H$ and $e_t$ for $T_t$. We choose a default chamber $\mathfrak{C}_C = \{\zeta > 0\}$, so that in vacuum $\alpha$, $X_\alpha$ gets a vev. One therefore has:

$$\kappa_\alpha = -e_\alpha \otimes e_C, \quad \kappa^{C}_\alpha = e_t \otimes e_C. \quad (2.46)$$

Notice that $\kappa_\alpha$ is independent of the chamber $\mathfrak{C}_C$ (in the opposite chamber $Y_\alpha$ gets a vev), but $\kappa^{C}_\alpha$ is dependent on $\mathfrak{C}_C$.

If $\epsilon = 0$, then we find that the mass parameters are split into $N!$ chambers labelled by a permutation of $m_1 > m_2 > \cdots > m_N$, separated by codimension one hyperplanes when $m_\alpha = m_\beta$. Let us choose a default chamber

$$\mathfrak{C}_H = \{m_1 > m_2 > \cdots > m_N\}. \quad (2.47)$$

This hyperplane arrangement is illustrated for $N = 3$ in figure 2.2.
The hyperplane arrangement in $t_H \cong \mathbb{R}^2$ for supersymmetric QED with $N = 3$ with horizontal coordinate $m_2 - m_1$ and vertical coordinate $m_3 - m_2$. The chambers are labelled by permutations of $\{1, 2, 3\}$. The default chamber $m_1 > m_2 > m_3$ is shaded.

The remaining supersymmetric Chern-Simons couplings in the default chambers are

$$
\kappa^H_\alpha = \left( \sum_{\beta < \alpha} (e_\alpha - e_\beta) + \sum_{\beta > \alpha} (e_\beta - e_\alpha) \right) \otimes e_t \quad (2.48)
$$

$$
\tilde{\kappa}_\alpha = (N - 2\alpha + 1)e_t \otimes e_t.
$$

These couplings are unchanged when $\epsilon \neq 0$, provided $|\epsilon| \ll |m_\alpha - m_\beta|$.

### 2.5.2 Higgs Branch Interpretation

The mixed Chern-Simons levels can be given a concrete realisation from the geometry of the Higgs branch, which will elucidate the meaning of the chamber structure in the $x = (m, \epsilon)$ parameters. Before we continue, we introduce an additional assumption about the Higgs branch:

**Assumption:** We will assume that $X$ is a Goresky-Kottwitz-MacPherson (GKM) manifold [135] for the Hamiltonian $T$-action. This means the following.

- Let $\Phi_\alpha$ denote the collection of weights in the $T$ weight decomposition of $T_\alpha X$. Then the elements of $\Phi_\alpha$ are pairwise linearly independent for all fixed points $\alpha$.

- Equivalently, for every two fixed points $\alpha, \beta$ in $X^T$, there is no more than one $T$-equivariant curve connecting them.
To see the equivalence,\(^{4}\) take \(\lambda \in \mathfrak{t}^*\) in the \(T\)-weight decomposition of \(T_\alpha X\) and consider the codimension 1 subtorus \(T^\lambda\) of \(T\) generated by \(\text{Ker}(\lambda)\). Let \(\Sigma_\lambda\) be the connected component of \(X^{T^\lambda}\) containing \(\alpha\). Since the weights are pairwise linearly independent, \(\Sigma_\lambda\) must have complex dimension 1. Further, since it is oriented, \(T\)-equivariant, and have an isolated fixed point, we must have \(\Sigma_\lambda \cong \mathbb{CP}^1\) or \(\mathbb{C}\). If it is \(\mathbb{CP}^1\) then it must join \(\alpha\) to another fixed point \(\beta\), and by pairwise linearity it is the unique \(T\)-equivariant curve which does so. The corresponding weight in \(T_\beta X\) is \(-\lambda\).

This assumption is satisfied in the supersymmetric QED example and in more general abelian theories where \(X\) is hyper-toric \([122]\). It is also satisfied for supersymmetric quiver gauge theories \(T_\rho[SU(N)]\) where \(X\) is the cotangent bundle of a partial flag variety, which includes the case where \(X\) is \(T^k Gr(k, N)\) where \(Gr(k, N)\) is the Grassmannian. It will hopefully be clear that arguments can be generalised to non-GKM varieties, although the chamber structures we introduce will become more complicated.

Let us now proceed to recover the effective Chern-Simons couplings in a vacuum from the geometry of the Higgs branch, \(i.e.\) from the perspective of a sigma model to \(X\). We will fix an FI parameter \(\zeta\), or equivalently a chamber \(\mathcal{C}_C\). The supersymmetric Chern-Simons terms \(\kappa_\alpha\), \(\kappa^C_\alpha\) arise from classical contributions to the vector central charge given by the values of the moment maps at critical points,

\[
\begin{align*}
\kappa_\alpha(m, \zeta) &= h_m(\alpha) \\
\kappa^C_\alpha(\epsilon, \zeta) &= h_\epsilon(\alpha).
\end{align*}
\]

(2.49)

This can be seen in the following way. In the vacuum \(\alpha\), let \(\{Z_i\}\) denote the \(i = 1, \ldots, r\) \(\mathcal{N} = 2\) chiral multiplet scalars which obtain vevs. Note that \(Z_i = X_i\) or \(Y_i\) in the decomposition \((X_i, Y_i) \in T^*R\) in a way determined by the stability parameter, or equivalently \(\mathcal{C}_C\). In the vacuum \(\alpha\) we have (summing over indices):

\[
\begin{align*}
\mu_R = \rho_i^0 |Z_i|^2 = \zeta^a, \\
m_m = m_s(\rho_H)_i^a |Z_i|^2, \\
h_\epsilon = \epsilon(\rho_t)_i |Z_i|^2,
\end{align*}
\]

(2.50)

from which the (2.49) may be recovered straightforwardly, matching (2.44). Note that we could also write:

\[
\kappa_\alpha(m, \zeta) + \kappa^C_\alpha(\epsilon, \zeta) = h|_\alpha
\]

(2.51)

where \(h\) is the real superpotential (2.11).

---

\(^{4}\)See \textit{e.g.} \([136]\)
The remaining Chern-Simons levels $\kappa_{\text{matter}}^\alpha$ arise from a 1-loop contribution from integrating out massive fluctuations around a critical point. To describe these contributions cleanly, we introduce some notation for tangent weights.

**Aside on Tangent Spaces and Chambers:** As above, let $\Phi_\alpha$ denote the collection of weights in the $T$ weight decomposition of $T_\alpha X$, and we make the GKM assumption that elements of $\Phi_\alpha$ are pair-wise linearly independent. Introducing mass parameters $x = (m, \epsilon)$, there is a decomposition

$$\Phi_\alpha = \Phi_\alpha^+ \cup \Phi_\alpha^-.$$  

(2.52)

where

$$\Phi_\alpha^+ = \{ \lambda \in \Phi_\alpha \mid \lambda \cdot x > 0 \}$$

(2.53)

$$\Phi_\alpha^- = \{ \lambda \in \Phi_\alpha \mid \lambda \cdot x < 0 \}.$$ 

denote the collections of positive and negative weights. We denote the corresponding decomposition of the tangent space by

$$T_\alpha X = N_\alpha^+ \oplus N_\alpha^-.$$  

(2.54)

Notice that this decomposition changes precisely when one crosses a hyperplane

$$\mathbb{W}_\lambda = \{ \lambda \cdot x = 0 \},$$  

(2.55)

for any $\lambda \in T_\alpha X$. From our above description of GKM manifolds, we see that this is precisely when some extended moduli space of supersymmetric vacua opens up (i.e. $X^{T_\lambda}$ is no longer isolated but consists of the fixed points and $\Sigma_\lambda$). The hyperplane arrangement divides the space of $x = (m, \epsilon)$ parameters $t$ into chambers $\mathcal{C}$, faces of maximal dimension, in which the decomposition (2.52) is constant.

Setting $\epsilon = 0$ (the $\mathcal{N} = 4$ limit), and denoting the $T_H$ component of a weight $\lambda$ by $\lambda_H$, then one similarly constructs the set of hyperplanes

$$\mathbb{W}_\lambda = \mathbb{W}_{\lambda^*} = \{ \lambda_H \cdot x = 0 \}.$$  

(2.56)

Note that as indicated above, in this limit the hyperplane for a weight $\lambda$ and its dual $\lambda^* = -2\epsilon t - \lambda$ coincide. The hyperplane arrangement now divides $t_H$ into chambers $\mathcal{C}_H$, in which the decomposition (2.52) is constant. If the mass parameters $m$ lie on a
hyperplane, an extended moduli space of vacua, or fixed locus, opens up. This can have complex dimension $\geq 1$.

These hyperplane arrangements are a fundamental structure throughout this paper. Various quantities depend in a piece-wise constant way on the mass and FI parameters, such that they depend only on a choice of face of the hyperplane arrangement.

We are now in a position to provide an expression for the remaining effective Chern-Simons levels $\kappa_{\text{matter}}$ in terms of the Higgs branch geometry. They are given by:

$$
\kappa_{\text{matter}} = \frac{1}{4} \left( \sum_{\lambda \in \Phi_+^\alpha} \lambda \otimes \lambda - \sum_{\lambda \in \Phi_-^\alpha} \lambda \otimes \lambda \right). \quad (2.57)
$$

It is not hard to see that this expression matches the gauge-theoretic expression (2.45). Note that (2.45) also contains contributions from the chirals in the same hypermultiplets as those that attain vevs in the vacuum $\alpha$, as well as the adjoint chiral multiplets coming from the $\mathcal{N} = 4$ vector multiplet. However these cancel out, to give (2.57), much as in the computation of the tangent space character (2.30). We may interpret this from the sigma model perspective as integrating out the massive fluctuations corresponding to tangent directions at the critical point $\alpha$. The sign of the correction is correlated with the sign of the mass of a fluctuation, with the result above. The result is clearly dependent on the choice of chamber $C$, and is constant within it.

If $\epsilon = 0$ or if $|\epsilon| \ll |\lambda \cdot x|$ for all $\lambda$, then $\lambda \in \Phi_+^\alpha$ implies that $\lambda^* \in \Phi_-^\alpha$. The decomposition (2.52) depends on $\mathcal{C}_H$ and is constant within it. Then one may rewrite:

$$
\kappa_{\text{matter}} = \kappa^H_{\alpha} + \tilde{\kappa}_{\alpha} = \frac{1}{2} \left( \sum_{\lambda \in \Phi_+^\alpha} \lambda \otimes e_t - \sum_{\lambda \in \Phi_+^\alpha} e_t \otimes \lambda \right) \quad (2.58)
$$

where

$$
\lambda_{\alpha}^\pm = \sum_{\alpha \in \Phi_+^\alpha} \lambda, \quad (2.59)
$$

and we regard the sum $\kappa^H_{\alpha} + \tilde{\kappa}_{\alpha}$ as a pairing $\Gamma \times \Gamma \to \mathbb{Z}$. In the first line, we re-arranged the sum in (2.57) using the transformation properties of the holomorphic symplectic form. Despite the factor of half, this defines an integer pairing due to contributions of pairs of weights $\lambda, \lambda^*$ with opposite sign.
Chamber Dependence

It is interesting to note the dependence of the Chern-Simons couplings $K_\alpha$ on the chambers $\mathcal{C}_H$ and $\mathcal{C}_C$, having turned off $\epsilon$. Let us label a supersymmetric vacuum $\alpha$ by the hypermultiplets which attain vevs, indexed by $i = 1, \ldots, r$. Depending on the FI parameter $\zeta$, or more precisely the chamber $\mathcal{C}_C$, one or the other of the chirals $X_i$ or $Y_i$ in those hypermultiplets attains a vev. From (2.44) it is clear that $\kappa_\alpha$ is independent of $\mathcal{C}_C$, but $\kappa^C_\alpha$ is dependent. Clearly neither are dependent on $\mathcal{C}_H$. From (2.29), (2.30) and the symplectic pairing of weights in $T_{\alpha, X}$, it is not hard to see that the collection of weights $\Phi_\alpha$ is dependent on the choice of $\mathcal{C}_C$, but not once we restrict to considering just the $T_H$ weights $\lambda_H$. Therefore, $\kappa^H_\alpha$ depends on $\mathcal{C}_H$ but not $\mathcal{C}_C$, and $\tilde{\kappa}_\alpha$ depends on both. This is summarised in Table 2.2.

<table>
<thead>
<tr>
<th>$\mathcal{C}_H$</th>
<th>$\kappa_\alpha$</th>
<th>$\kappa^C_\alpha$</th>
<th>$\kappa^H_\alpha$</th>
<th>$\tilde{\kappa}_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{C}_C$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
</tbody>
</table>

Table 2.2 The dependence of Chern-Simons levels on $\mathcal{C}_H$ and $\mathcal{C}_C$, for zero or small $\epsilon$.

Example

We again return to supersymmetric QED. One can immediately recover the Chern-Simons levels $\kappa_\alpha$ and $\kappa^C_\alpha$ (2.46) from the values of the moment maps $h_m$ and $h_\epsilon$ at the critical loci (2.38):

$$\kappa_\alpha(m, \zeta) = h_m(\alpha) = -m_\alpha \zeta,$$

$$\kappa^C_\alpha(\epsilon, \zeta) = h_\epsilon(\alpha) = \epsilon |\zeta|.$$  

(2.60)

To describe the remaining supersymmetric Chern-Simons levels, let us fix the default chamber $\mathcal{C}_C = \{ \zeta > 0 \}$ and determine the weight spaces $\Phi_\alpha$. We have

$$\text{Ch} Q - \text{Ch} g_C - t^{-2} \text{Ch} g^*_C = t^{-1} s^{-1} \sum_{\beta=1}^N v_\beta + t^{-1} s \sum_{\beta=1}^N v^{-1}_\beta - t^{-2} - 1.$$  

(2.61)

The expectation value of $X_\alpha$ determines $s^{-1}v_\alpha t^{-1} = 1$, and thus

$$\text{Ch} T_\alpha X = \prod_{\beta \neq \alpha} \frac{v_\beta}{v_\alpha} + t^{-2} \prod_{\beta \neq \alpha} \frac{v_\alpha}{v_\beta}.$$  

(2.62)

The tangent weights at a vacuum $\alpha$ are therefore

$$\Phi_\alpha = \{ e_\beta - e_\alpha, \beta \neq \alpha \} \cup \{-2 e_\alpha - e_\beta + e_\alpha, \beta \neq \alpha\},$$  

(2.63)
coinciding with the tangent weights of $X = \mathbb{CP}^{N-1}$ at coordinate hyperplanes in the base.

Suppose $\epsilon = 0$ with the mass parameters in the default chamber $\mathcal{C}_H = \{m_1 > m_2 > \cdots > m_N\}$, or turning on a small mass parameter $\epsilon$, i.e. the corresponding chamber $\mathcal{C}$ where $|\epsilon| < |m_\alpha - m_\beta|$. Then the decomposition into positive and negative weights is
\begin{align}
\Phi_+^\alpha &= \{e_\beta - e_\alpha, \beta < \alpha\} \cup \{-2\epsilon_t - e_\beta + e_\alpha, \beta > \alpha\}, \\
\Phi_-^\alpha &= \{e_\beta - e_\alpha, \beta > \alpha\} \cup \{-2\epsilon_t - e_\beta + e_\alpha, \beta < \alpha\}.
\end{align}
\[(2.64)\]

We have
\begin{align}
\lambda_+^\alpha &= \sum_{\beta<\alpha} (e_\beta - e_\alpha) + \sum_{\beta>\alpha} (-2\epsilon_t - e_\beta + e_\alpha), \\
\lambda_-^\alpha &= \sum_{\beta>\alpha} (e_\beta - e_\alpha) + \sum_{\beta<\alpha} (-2\epsilon_t - e_\beta + e_\alpha),
\end{align}
\[(2.65)\]
and
\[\frac{1}{2}(\lambda_-^\alpha - \lambda_+^\alpha) = \sum_{\beta<\alpha} (e_\alpha - e_\beta) + \sum_{\beta>\alpha} (e_\beta - e_\alpha) + (N - 2\alpha + 1)\epsilon_t.\]
\[(2.66)\]

We therefore recover the remaining supersymmetric Chern-Simons couplings,
\begin{align}
\kappa_+^H &= \left(\sum_{\beta<\alpha} (e_\alpha - e_\beta) + \sum_{\beta>\alpha} (e_\beta - e_\alpha)\right) \otimes \epsilon_t, \\
\tilde{\kappa}_\alpha &= (N - 2\alpha + 1)\epsilon_t \otimes \epsilon_t.
\end{align}
\[(2.67)\]

The supersymmetric Chern-Simons couplings in other chambers for the mass parameters $(m_1, \ldots, m_N)$, $\epsilon$ can be computed similarly.

### 2.5.3 Domain Walls, Central Charges & Hyperplanes

We now turn to elucidate a relationship between Chern-Simons levels and domain walls on the Higgs branch, first predicted in [56]. We will discuss the relationship between the so-called GKM graph of $X$, families of domain walls and the aforementioned hyperplane arrangements. It will be convenient to take the perspective of a sigma model to the Higgs branch, fixing a chamber $\mathcal{C}_C$.

We introduce the vector central charge function:
\[C_\alpha := K_\alpha(x_f, x_f).\]
\[(2.68)\]
For zero or small $\epsilon$ (characterised above), we may decompose as:

$$C_\alpha = \kappa_\alpha(m, \zeta) + \kappa_\alpha^C(\epsilon, \zeta) + \kappa_\alpha^H(m, \epsilon) + \tilde{\kappa}(\epsilon).$$  \hspace{1cm} (2.69)

This function is thus called because of its appearance in the supersymmetry algebra, which we explain now. First, a domain wall preserving $\mathcal{N} = (0, 2)$ supersymmetry in the deformed sigma model is a solution of the gradient flow equations on $X$ for the moment map $h_x = h_m + h_\epsilon$ connecting critical points $\alpha, \beta$. The presence of a domain wall interpolating between massive vacua contributes to the central charge $C_{\alpha\beta}^{ab}$ in the $\mathcal{N} = 2$ supersymmetry algebra (2.2).\footnote{See appendix C of [56].} See there also for an interpretation of the $C_{\pm\pm}$ central charges. Note here $\alpha$ and $\beta$ are Lorentz indices, not vacuum labels. For example, a domain wall in the $x^{1,2}$ plane interpolating between a massive vacuum $\alpha$ at $x^3 \to -\infty$ and $\beta$ and $x^3 \to \infty$ contributes a central charge in the $\mathcal{N} = 2$ supersymmetry algebra:

$$C_+^- = C_3^+ = C_\alpha - C_\beta.$$  \hspace{1cm} (2.70)

We can see that these domain walls have tension proportional to $|C_\alpha - C_\beta|$. This can be seen from the following two facts:

- Take $\lambda \in T_\alpha \Sigma_\lambda \subset T_\alpha X$, where $\Sigma_\lambda \cong \mathbb{CP}^1$ is the $T_\lambda$-fixed curve containing $\alpha$ and $\beta$, as defined in section 2.5.2. If the mass parameter $x = (m, \epsilon)$ lies on the hyperplane $\mathcal{W}_\lambda$, i.e. $\lambda \cdot x = 0$, then $x$ generates an element of $T_\lambda$. The moment map $h_x$ is constant on the connected components of the fixed locus action. In particular we must have $h_x|_\alpha = h_x|_\beta$.

- By construction the $T^\lambda$-weight of $T_\alpha \Sigma_\lambda \subset T_\alpha X$ (and thus $T_\beta \Sigma_\lambda$) is 0. Since $T^\lambda$ fixes point-wise the curve $\Sigma_\lambda$, the (quantised) weights of $T^\lambda$ are constant over $\Sigma_\lambda$ by continuity, and in particular coincide at $\alpha$ and $\beta$. Thus the $T_\lambda$ characters of $T_\alpha X$ and $T_\beta X$ agree.

From the geometric constructions of the Chern-Simons levels (2.49) and (2.57) this implies that $C_\alpha|_{\mathcal{W}_\lambda} = C_\beta|_{\mathcal{W}_\lambda}$.

Solutions of the gradient flow equation on $X$ are not isolated. Acting with $T$ generates an $S^1$ family of gradient flows between vacua $\alpha, \beta$ that sweep out a compact curve $\mathbb{CP}^1 \subset X$. Furthermore, there are non-compact families of gradient flows extending out from a supersymmetric vacuum $\alpha$ to infinity. These possibilities are illustrated in figure 2.3. These are precisely the the surfaces $\Sigma_\lambda$ associated to each weight $\lambda$ in the tangent spaces fixed by $T^\lambda$, in the notation of section 2.5.2.
These families of gradient flows generate the 1-skeleton of $X$, which can be represented by the GKM graph \cite{135, 136}. This consists of the following elements:

- Vertices labelled by supersymmetric vacua $\{α\}$.
- Internal edges $α \rightarrow β$ representing curves $Σ_λ \cong \mathbb{C}P^1$ labelled by a tangent weight $λ \in Φ_α^+ \cap (-Φ_β^-)$.
- External edges $α \leftarrow \infty$ and $α \rightarrow \infty$ representing curves $Σ_λ \cong \mathbb{C}$ labelled by a tangent weight $λ \in (-Φ_α^-)$ and $Φ_α^+$ respectively.

Physically, this is a representation of the supersymmetric vacua and families of domain walls connecting them. The arrows represent directions of positive gradient flow for fixed parameters $x = (m, ϵ)$, and depend on the choice of chamber $C \subset t$. The GKM assumption ensures there is at most one internal edge connecting any pair of vertices. Supersymmetric QED with $N = 3$ is illustrated in figure 2.4.

Now consider the hyperplane arrangement in the space $t \cong t_H \oplus \mathbb{R}$ of mass parameters $x = (m, ϵ)$. The hyperplanes where the critical locus of the moment map $h_x$ (i.e. the
fixed point set of the 1-parameter subgroup of $T$) is larger than the set of isolated critical points \{\alpha\} take the form

$$\mathcal{W}_\lambda = \{\lambda \cdot x = 0\}, \quad (2.71)$$

where $\lambda$ is a tangent weight in the GKM graph. When such a hyperplane is crossed, the arrow on the corresponding edge flips orientation. As before, we may distinguish two types of hyperplane:

- If $\lambda$ labels an internal edge $\alpha \to \beta$, the hyperplane $\mathcal{W}_\lambda$ corresponds to a locus where a compact Higgs branch $\Sigma_\lambda \cong \mathbb{C}P^1$ opens up, and $C_\alpha = C_\beta$.

- If $\lambda$ labels an external edge connecting $\alpha$, the hyperplane $\mathcal{W}_\lambda$ corresponds to a locus where a non-compact Higgs branch $\Sigma_\lambda \cong \mathbb{C}$ opens up.

We are primarily interested in the limit $\epsilon \to 0$. Then gradient flows for the moment map $h_m$ correspond to domain walls preserving 2d $\mathcal{N} = (2, 2)$ supersymmetry whose tension receives a classical contribution only from $C_\alpha = h_m(\alpha)$. Furthermore, the hyperplane arrangement degenerates as follows. Recall that an incoming edge with weight $\lambda \in -\Phi^-_\alpha$ is always paired with an outgoing edge with weight $\lambda^* \in \Phi^+_\alpha$. In the limit $\epsilon \to 0$,

$$\mathcal{W}_\lambda = \mathcal{W}_{\lambda^*} = \{\lambda_H \cdot m = 0\} \quad (2.72)$$

and there is no distinction between internal and external hyperplanes. The hyperplanes are all of the form $\mathcal{W}_\lambda = \{C_\alpha = C_\beta\}$ and reproduce the hyperplane arrangement in the space $t_H$ of mass parameters discussed previously.

Let us now briefly mention some aspects of studying chamber structure in the full space of parameters, i.e. including the FI $\zeta$, although we will not need it much in the remainder of this thesis.

**Coulomb Branch Interpretation**

The supersymmetric Chern-Simons levels $K_\alpha = (\kappa_\alpha, \kappa^C_\alpha, \kappa^H_\alpha, \tilde{\kappa}_\alpha)$ must also have an interpretation in terms of Coulomb branch geometry, obtained by first fixing a generic mass $m$ to obtain a sigma model to $X^!$, and then turning on the equivariant parameters on the Coulomb branch $(\zeta, \epsilon)$. The roles of $\kappa^C_\alpha$ and $\kappa^H_\alpha$ are switched, whilst those of $\kappa_\alpha$ and $\tilde{\kappa}_\alpha$ remain the same. That is, $\kappa_\alpha + \kappa^H_\alpha$ may be obtained by evaluating the real moment map for the Coulomb branch flavour symmetry $h_\zeta \sim \zeta^a \sigma_a$ at the vacuum, and $\kappa^C_\alpha + \tilde{\kappa}_\alpha$ can be derived from the characters of tangent spaces $T_\alpha X^!$ analogously to
This is consistent with the chamber dependencies of these couplings given in table 2.2.

This is readily checked for an abelian theory with a hyper-toric Higgs branch $X$, since the $X'$ is also hyper-toric, and its data can be given explicitly (see e.g. [56]). In general, this is a non-trivial statement on the geometry of symplectic dual pairs $(X, X')$.

We may also study the loci in the full range of parameters for $\mathcal{N} = 2$ flavour symmetries $t_f$, where an extended moduli space of supersymmetric vacua opens up, which can now lie on mixed branches. This again consists of loci \{ $C_\alpha - C_\beta = 0$ \} $\subset t_f$ where the tension of a domain wall connecting $\alpha$ and $\beta$ vanishes and a compact moduli space opens up, and the additional loci where a non-compact moduli space attached to a single vacuum $\alpha$ opens up.

$\mathcal{N} = 4$ Limit

In the limit $\epsilon \to 0$ where $\mathcal{N} = 4$ supersymmetry is restored, the vector central charge function simplifies dramatically to

$$ C_\alpha = \kappa_\alpha(m, \zeta) $$

and depends only on the $\mathcal{N} = 4$ supersymmetric Chern-Simons coupling $\kappa_\alpha : \Gamma_H \times \Gamma_C \to \mathbb{Z}$ between a vector multiplet and a twisted vector multiplet. In this limit, loci in the space of mass and FI parameters where supersymmetric vacua fail to be isolated are all of the form

$$ \{ C_\alpha = C_\beta \} \subset t_H \times t_C. $$

Their projections onto the two factors are linear hyperplanes through the origin, forming hyperplane arrangements in the spaces of mass and FI parameters $t_H, t_C$. The connected components of the complement of the hyperplanes, are the chambers $\mathcal{C}_H, \mathcal{C}_C$. Again, this may be checked easily in the case of hyper-torics.

### 2.6 Boundary Conditions

The study of supersymmetric boundary conditions lies at the heart of many connections between quantum field theory and mathematics. In the spirit of this thesis, we will focus our attention to three dimensions, where we consider boundary conditions preserving at least $\mathcal{N} = (0,2)$ supersymmetry. Building on the description of boundary conditions in Rozansky-Witten theory [54, 55], the study of $\mathcal{N} = (2,2)$ boundary conditions in
supersymmetric gauge theories was initiated in [56], where they were found to underpin the physical realisation of symplectic duality [59, 60]. Various other aspects of such boundary conditions have also been discussed in [2, 51, 109, 137]. As we will consider breaking the bulk $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 2$, we also consider $\mathcal{N} = (0, 2)$ boundary conditions, studied in [117, 138–140]. We may think of $\mathcal{N} = (2, 2)$ boundary conditions as a special class of these boundary conditions.

We will see in chapters 3 and 4 that such boundary conditions form the basis of the construction of classes in equivariant elliptic cohomology. We state some of the basic aspects of $(0, 2)$ and $(2, 2)$ boundary conditions here, introducing more on a need-to-know basis in the main body of the thesis, and referring the reader in particular to [56] and [117] for further details.

### 2.6.1 $\mathcal{N} = (0, 2)$ Boundary Conditions

We start first by describing $(0, 2)$ boundary conditions for 3d $\mathcal{N} = 2$ theories and their associated anomalies, systematically analysed in [117]. Such boundary conditions preserve the supercharges $Q_++$ and $Q_-$, shown in table 2.3. Such boundary conditions may be found by first decomposing the bulk 3d multiplets into multiplets under the $(0, 2)$ subalgebra. Varying the action, requiring that the boundary terms vanish will demand that a choice of fields are set to zero at the boundary. Preserving $(0, 2)$ supersymmetry then demands that the entire $(0, 2)$ multiplet containing those fields is set to zero. We choose $x^{1,2}$ to be coordinates parallel to the plane, and $x^3$ perpendicular to it.

Free Chiral

We start with the simplest example of a free 3d $\mathcal{N} = 2$ chiral multiplet, which we denote:

$$\Phi_{3d} = \phi + \theta^+ \psi_+ + \theta^- \psi_- + \theta^+ \theta^- F + \ldots,$$ (2.75)
with remaining components fixed by chirality $D_{\pm} \Phi_{3d} = 0$. The equations of motion set $F = 0$. The 3d chiral decomposes into:

- A $(0, 2)$ chiral multiplet $\Phi = \phi + \theta^+ \psi_+ - i \theta^+ \bar{\theta}^+ \partial_+ \phi$.
- A $(0, 2)$ Fermi multiplet $\Psi = \bar{\psi}_- + \theta^+ f - i \theta^+ \bar{\theta}^+ \partial_+ \psi_-$.

These obey $D_+ \Phi = 0$ and $D_+ \Psi = 0$. On-shell, the auxiliary field $f$ equals $\partial_3 \bar{\phi}$.

Requiring that the boundary terms to vanish in the variation of the bulk action $\int d^3 x d^4 \theta \Phi^\dagger_{3d} \Phi_{3d}$ demands that either $\psi_+$ or $\psi_-$ is zero. Supersymmetrising either choice, we obtain either:

\begin{align}
\text{Neumann (N)} : & \quad |\Psi| = 0 \implies \partial_3 \phi = |\psi_-| = 0 \\
\text{Dirichlet (D)} : & \quad |\Phi| = c \implies |\phi| = |\psi_+| = 0.
\end{align}

In the Dirichlet boundary condition, $c$ is a constant. Turning on a non-zero $c$ often explicitly breaks some symmetries, this will be important in what follows. If we have multiple chirals, we have a choice of $N$ or $D$ for each of them.

**Boundary Interactions**

The above boundary conditions may be enriched by adding $(0, 2)$ chiral or Fermi multiplets at the boundary, coupled to each other and bulk fields with a fermionic superpotential, which corresponds to a choice of ‘$E$’ and ‘$J$’ terms [141]. We will not require the full use of this technology, instead giving here the two main examples we will use in this thesis.

Starting with a $N$ boundary condition, we introduce a boundary Fermi multiplet $\Gamma$ with a $J$-term $\Phi$, i.e. we modify the action by adding a boundary term

\[ i \int d^2 x d\theta^+ \Phi \Gamma + \text{h.c.} + \Gamma \text{ kinetic terms}. \]  

The boundary variation, at low energy, sets:

\[ |\Psi| = \Gamma, \quad |\Phi| = 0. \]  

(2.78)

Thus, adding in the boundary interaction is equivalent to ‘flipping’ the boundary condition for the bulk chiral from Neumann to Dirichlet. Similarly, we may start with a $D$ boundary condition and introduce a boundary chiral multiplet $C$ with a $J$-term $C$ for $\Psi$, i.e. modifying the action by adding a
boundary term
\[ i \int d^2xd\theta^+ \Psi |C + h.c. + C \text{ kinetic terms.} \] (2.79)
which sets:
\[ \Phi| = C, \quad \Psi| = 0 \] (2.80)
which is equivalent to ‘flipping’ the boundary condition for the bulk chiral from Dirichlet to Neumann. Analogous boundary conditions for 4d \( \mathcal{N} = 2 \) theories were introduced in [104].

We note briefly that a bulk superpotential \( \int d^4x d^2\theta W(\{\Phi_{3d}\}) \) is generically problematic, as preserving \( (0, 2) \) supersymmetry demands that \( W(\{\Phi_{3d}\})| = 0 \) on the boundary [117]. If not, one needs to introduce additional boundary Fermi multiplets with \( E \) and \( J \)-terms which ‘factorise’ the bulk superpotential at the boundary. However, we only consider \( \mathcal{N} = 4 \) theories, whose superpotential is constrained to be of the form (2.10). In all of the boundary conditions we consider in this thesis, either \( \varphi = 0 \) or \( \mu_C = 0 \) and thus this modification is not required, and supersymmetry is automatically preserved.

Gauge Fields

The 3d \( \mathcal{N} = 2 \) vector multiplet given as a superfield is:
\[ V = \theta \sigma^a \bar{\theta} A_\mu + i \theta \bar{\theta} \sigma - i \theta^2 \bar{\theta} \lambda - i \bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \bar{\theta}^2 D. \] (2.81)
On shell, the auxiliary field \( D \) satisfies: \( \frac{1}{g^2} D = \mu_R - \zeta - k \sigma \), where \( \mu_R \) is the moment map of the \( G \) action on chiral multiplets, \( \zeta \) is the FI (both the real ones, from the perspective of \( \mathcal{N} = 4 \)) and \( k \) is Chern-Simons term. Since we work with \( \mathcal{N} = 4 \) theories softly broken to \( \mathcal{N} = 2 \), we set \( k = 0 \).

We may decompose the \( \mathcal{N} = 2 \) vector multiplet as:
- A \((0, 2)\) chiral superfield \( S = \sigma + i A_3 - 2 \theta^+ \bar{\lambda}_+ \).
- A \((0, 2)\) field-strength superfield \( \Upsilon = \lambda_- + \theta^+(F_{12} + i D_{2d}) \).

The latter is constructed from a \((0, 2)\) gauge multiplet. On-shell, the 2d auxiliary field is given by \( D_{2d} = D - \partial_3 \sigma \). Both \((0, 2)\) constituents are covariantly chiral \( \mathcal{D}_+ S = \mathcal{D}_+ \Upsilon = 0 \).

There are two possibilities for boundary conditions for the gauge fields which preserve the Yang-Mills equations of motion. A Neumann boundary condition sets \( F_{3z}| = 0 \) (using left and right-moving coordinates \( \pm \) in the \( x^{1,2} \) plane) and preserves the gauge symmetry on the boundary. A Dirichlet boundary condition set \( F_{+-}| = 0 \), breaking
the gauge symmetry $G$ to a global symmetry $G_\partial$ at the boundary. Supersymmetrising, we obtain the following $(0,2)$ boundary conditions, obtained by gauge covariantising setting $S$ or $\Upsilon$ to zero at the boundary:

\[
\begin{align*}
\text{Neumann } \mathcal{N} : & \quad F_{3\pm}| = \lambda_+| = \sigma| = 0 \\
\text{Dirichlet } \mathcal{D} : & \quad F_{+-}| = \lambda_-| = D_{2d}| = 0.
\end{align*}
\] (2.82)

On-shell, the $D_{2d}$ constraint becomes $\frac{1}{\sigma} \partial_3 \sigma| = \mu_{R}| - \zeta$.

In general, one may also choose to preserve a subgroup $H \subset G$ of the gauge group at the boundary, however we will not consider such boundary conditions in this work.

**Anomalies**

The boundary conditions described above preserve some fermionic degrees of freedom at the boundary, and will suffer from anomalies. We will initially treat ’t Hooft anomalies for global and R-symmetries and gauge anomalies on the same footing. We may keep track of the boundary anomalies using an anomaly polynomial which is bilinear in the curvatures:

\[
P = k(f, f)
\] (2.83)

where $f$ is a vector of field strengths and $K = \Gamma \times \Gamma \rightarrow \mathbb{Z}$ is a pairing on the co-character lattice of the symmetry group $G$. This matches our notation for Chern-Simons couplings in 2.5.1. Therefore, in these conventions:

- A purely 2d left/right-handed chiral fermion $\chi_{\pm}$ charged $+1$ under a $U(1)$ symmetry contributes $\pm \frac{1}{2} f^2$ to $P$, or equivalently $\pm \frac{1}{2} e \otimes e$ to $K$, where $e$ is the fundamental weight of $G$.

- A matrix of Chern-Simons terms $K$, as in (2.39) induces, via anomaly inflow, a boundary anomaly $\pm K(f, f)$. Here, the + sign corresponds to a left boundary condition on a bulk $x^3 \geq 0$. The − sign corresponds to a right boundary condition on a bulk $x^3 \leq 0$. Thus in particular, a $U(1)$ Chern-Simons level $K$ (which in our conventions must be $\frac{1}{2} \mathbb{Z}$ quantised for gauge-invariance) induces a boundary anomaly $\pm K f^2$.

Note that this differs from the conventions of [117] by a factor of $\frac{1}{2}$, and while annoying now, will simplify the expressions for boundary anomalies for $\mathcal{N} = (2,2)$ boundary conditions later. In what follows, for simplicity, we will stick to abelian symmetries, see loc. cit. for details on the non-abelian case. We will also work with anomaly polynomials instead of bilinear forms, to save notation.
A 3d fermion $\psi = (\psi_+, \psi_-)$ has left and right handed components, one of which is set to zero for a boundary condition with $(0, 2)$ supersymmetry. By introducing a real mass, flowing to the IR and matching UV and IR anomalies, the authors of [117] derive that:

- A 3d fermion, charged $+1$ under a $U(1)$ symmetry, with boundary condition $\psi_+ = 0$, and $\psi_-$ surviving at the boundary, contributes $\pm \frac{1}{4} f^2$ to the anomaly polynomial.

**Chiral Multiplet:** We can now write down the anomalies associated to the boundary conditions we introduced above. Consider a 3d chiral $\Phi_{3d}$ with charge $+1$ under a $U(1)_F$ flavour symmetry, and $\Delta$ under a $U(1)_R$ R-symmetry. Then the contributions from either type of boundary condition are:

\[
(N) : -\frac{1}{4} (f_F + (\Delta - 1)f_R)^2, \quad (D) : \frac{1}{4} (f_F + (\Delta - 1)f_R)^2.
\]

(2.84)

This is because, in a Neumann boundary condition, $\psi_+$ survives, and has $U(1)_F$ charge $+1$ and $U(1)_R$ charge $\Delta - 1$. Similarly, for a Dirichlet boundary condition, $\bar{\psi}_-$ survives, with $U(1)_F$ charge $-1$ and $U(1)_R$ charge $1 - \Delta$.

Note that this is consistent with the flips modifying $D \leftrightarrow N$. Flipping $D$ to $N$ requires a boundary chiral $C$ with charges $(1, \Delta)$ (fixed by the boundary coupling), which contributes a purely 2d right-handed fermion and anomaly $-\frac{1}{2} (f_F + (\Delta - 1)f_R)^2$, which exactly modifies the $D$ anomaly above the $N$ anomaly. Similarly flipping $N$ to $D$ requires a Fermi multiplet of charge $(-1, -\Delta - 1)$, containing a left-handed fermion $\gamma_-$ of the same charges, contributing $\frac{1}{2} (f_F + (\Delta - 1)f_R)^2$. This exactly modifies the $N$ anomaly to the $D$ anomaly.

**Vector Multiplet:** We also note the boundary anomalies for Neumann $N$ and Dirichlet $D$ boundary conditions for the vector multiplet. Restricting to rank $r$ abelian gauge groups for simplicity, we have:

\[
N : \frac{1}{4} f^2_R, \quad D : -\frac{1}{4} f^2_R.
\]

(2.85)

arising from $\lambda_-$ and $\lambda_+$ surviving at the boundary respectively. The R-charges of $\lambda_{\pm}$ are both fixed to be 1 as the R-charges of $A$ and $\sigma$ vanish.

Let us now distinguish between ’t Hooft anomalies and gauge anomalies. For Neumann boundary conditions for the gauge multiplet, the total boundary anomaly for $G$ must be cancelled, since the gauge symmetry is preserved at the boundary. This
cancellation may come from both bulk and boundary matter. For Dirichlet boundary conditions, the gauge symmetry is broken to a boundary flavour symmetry $G_\partial$. Its anomalies are normal ’t Hooft anomalies, and it need not be cancelled.

Note that there is a boundary anomaly of $-f_C \cdot f$ arising from the FI coupling, where $f_C$ and $f_G$ are curvatures for the topological symmetry $T_C$ and gauge symmetry $G$ respectively. Thus, for a Neumann boundary condition for the vector multiplet, unless we are able to redefine the topological symmetry (see the discussion on $\mathcal{N} = (2, 2)$ boundary conditions below), or add boundary matter charged under $T_C$ (see chapter 4), the $T_C$ symmetry is broken at the boundary.\footnote{Another way of seeing that $T_C$ is broken at the boundary is by noticing that the Neumann boundary condition for $V$ can be rephrased as a Dirichlet boundary condition for the 3d dual-photon multiplet \cite{117}, imposing $\sigma + i\gamma = 0$. $T_C$ shifts $\gamma$ and is therefore broken by this boundary condition.}

Finally, let us discuss what happens when a linear combination of symmetries are broken when we turn on a boundary value $c \neq 0$ for a 3d chiral multiplet $\Phi_{3d}$ charged under those symmetries with Dirichlet boundary conditions. Let us consider the case where we have Dirichlet boundary conditions $D$ for the vector multiplet, and $\Phi_{3d}$ has charge $+1$ under $G_\partial$, and charge $-1$ under some flavour symmetry $U(1)_F$. We encounter this case frequently in the rest of the thesis, and hope that the generalisation is obvious. Turning on a boundary value $c \neq 0$ for $\Phi_{3d}$ breaks a combination of $G$ and $U(1)_F$, which may be seen as a breaking of $G$ and a redefinition of $U(1)_F$ at the boundary:

$$J_{bdy}^F = J_{bdy}^G + J_G.$$  \hfill (2.86)

In the anomaly polynomial, this may be implemented by setting $f_G - f_F = 0$, then taking $f_F$ to be the field strength for $J_{bdy}^F$.

Note that many aspects we have discussed of boundary anomalies closely mirror those of supersymmetric Chern-Simons levels. Both will have a key role in the construction of key objects in equivariant elliptic cohomology from 3d $\mathcal{N} = 4$ gauge theory. In chapter 4 we will discuss boundary conditions engineered to mimic a supersymmetric vacuum $\alpha$ at infinity, whose boundary anomalies coincided with the effective Chern-Simons levels $K_\alpha$ introduced in section 2.5.

### 2.6.2 $\mathcal{N} = (2, 2)$ Boundary Conditions

We now consider $\mathcal{N} = (2, 2)$ boundary conditions for $\mathcal{N} = 4$ theories \cite{56}. The $\mathcal{N} = (2, 2)$ algebra is generated by $Q_+^{++}, Q_+^{+-}, Q_+^{-+}, Q_-^{++}, Q_-^{+-}, Q_-^{-+}$, as shown in table 2.3. The boundary conditions are specified by:
A Neumann or Dirichlet boundary condition for the gauge fields - supersymmetry then determines the rest of the boundary conditions for the $\mathcal{N} = 4$ vector multiplet.\footnote{Again, one may also consider only preserving a subgroup of the gauge group at the boundary.}

A Neumann-Dirichlet boundary condition for the hypermultiplets, specified by a holomorphic Lagrangian splitting or polarisation $\varepsilon$ of the representation $Q = T^* R$.\footnote{This is a decomposition of the representation $T^* R \cong L \oplus L^*$, where $L$ is a Lagrangian. We will use the two terminologies interchangeably.} Let us write $R = \mathbb{C}^N$ with polarisation denoted by a sign vector $\varepsilon \in \{\pm\}^N$ specifying

\[
(X_{\varepsilon \beta}, Y_{\varepsilon \beta}) = \begin{cases} 
(X_\beta, Y_\beta) & \text{if } \varepsilon_\beta = + \\
(Y_\beta, -X_\beta) & \text{if } \varepsilon_\beta = -
\end{cases} \text{ for } \beta = 1, \ldots, N. \tag{2.87}
\]

The boundary condition specifies that

\[
D_{\perp} X_{\varepsilon \beta} |_{\varepsilon \beta} = 0, \quad Y_{\varepsilon \beta} |_{\varepsilon \beta} = c_{\varepsilon \beta}, \tag{2.88}
\]

and in particular $X_{\varepsilon \beta}$ transform in $\mathcal{N} = (2, 2)$ chiral multiplets at the boundary.

For a Neumann boundary condition for the gauge fields, the splitting must be compatible with the preserved boundary gauge symmetry, and the constants $c_{\varepsilon}$ necessarily vanish.

For Dirichlet boundary conditions for the vector multiplet, we will generically choose $c_{\varepsilon}$ such that the boundary $G_{\beta}$ gauge symmetry is completely broken. This is because one cannot quotient by $G$ at the boundary (as it becomes a flavour symmetry at the boundary), or impose $D$-term constraints (as they are absorbed the boundary condition for $\sigma$ as below equation (2.82)). One may therefore only impose $F$-term constraints. Therefore unless the choice of $c$ completely breaks $G$, there will be a complexified gauge orbit (an unbounded moduli space) of 2d vacua fibred above each bulk vacuum.

**As $\mathcal{N} = (0, 2)$ Boundary Conditions**

It will be convenient to describe $(2, 2)$ boundary conditions as a subclass of $(0, 2)$ boundary conditions. To do this we note the following decompositions. A 3d $\mathcal{N} = 4$ vector multiplet decomposes into a 3d $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$ and an adjoint chiral multiplet with scalar component $\varphi$. A 3d $\mathcal{N} = 4$ hypermultiplet decomposes into...
a pair of 3d $\mathcal{N} = 2$ chiral multiplets $X$ and $Y$. Recall that under 2d $\mathcal{N} = (0,2)$ supersymmetry:

- $\mathcal{V}$ decomposes into a $\mathcal{N} = (0,2)$ chiral superfield $S$ containing $A_3 - i\sigma$, and a $\mathcal{N} = (0,2)$ Fermi field strength multiplet $\Upsilon$, containing $F_{12}$.

- The 3d $\mathcal{N} = 2$ chiral multiplets $\varphi, X, Y$ decompose into $\mathcal{N} = (0,2)$ chiral multiplets $\Phi_{\varphi}, \Phi_X, \Phi_Y$, and $\mathcal{N} = (0,2)$ Fermi multiplets $\Psi_{\varphi}, \Psi_X, \Psi_Y$.

Alternatively, we could have first decomposed under $\mathcal{N} = (2,2)$ supersymmetry [56], before further decomposing under $\mathcal{N} = (0,2)$ supersymmetry.

- The 3d $\mathcal{N} = 4$ vector multiplet decomposes into a $\mathcal{N} = (2,2)$ chiral multiplet $(S, \bar{\Psi}_{\varphi})$, and a twisted chiral field strength multiplet $(\Phi_{\varphi}, \Upsilon)$.

- The 3d $\mathcal{N} = 4$ hypermultiplets decompose into $\mathcal{N} = (2,2)$ chiral multiplets $(\Phi_X, \Psi_Y)$ and $(\Phi_Y, -\Psi_X)$.

The (2,2) boundary conditions can then be obtained by gauge-covariantly setting one or the other of the (2,2) multiplets appearing in the decompositions of the $\mathcal{N} = 4$ bulk multiplets to zero (or a constant) at the boundary. We may then read off the boundary conditions on the component fields using the description of (0,2) boundary conditions in section 2.6.1. In particular:

- A (2,2) Neumann boundary condition for the vector multiplet imposes Neumann $\mathcal{N}$ for the 3d $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$, and Neumann $(\mathcal{N})$ for the adjoint $\mathcal{N} = 2$ chiral multiplet $\varphi$.

- A (2,2) Dirichlet boundary condition imposes Dirichlet $\mathcal{D}$ for for the 3d $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$, and Dirichlet $(\mathcal{D})$ for the adjoint $\mathcal{N} = 2$ chiral multiplet $\varphi$.

- The boundary conditions on the hypermultiplets must be of the Neumann-Dirichlet type described in equations (2.87) and (2.88). In particular $X_{\varepsilon\beta}$ are assigned Neumann $(\mathcal{N})$, and $Y_{\varepsilon\beta}$ Dirichlet $(\mathcal{D})$ boundary conditions.

**Symmetries and Anomalies**

We discuss some aspects of symmetries preserved at a boundary, which will be of use in the rest of this thesis. The boundary vector and axial R-symmetry may be the same as the bulk R-symmetry $U(1)_H \times U(1)_C$. However, the bulk R-symmetry may be spontaneously broken at the boundary, but a linear combination of the bulk
2.6 Boundary Conditions

R-symmetries and flavour symmetries is preserved and becomes $U(1)_V \times U(1)_A$. Let us give two examples, the generalisations should be clear.

- For a Neumann boundary condition for a $G = U(1)$ vector multiplet for the gauge symmetry, there is again an anomaly inflow from the FI coupling $-f_C f$. Further, it is not hard to check that the hypermultiplets $(X_\beta, Y_\beta)$ of charge $Q_\beta$ under $G$, with boundary conditions (2.88), contribute in total $(\varepsilon \cdot Q)f f_A$. Here $f_A$ is initially the field-strength for the $U(1)_C$ R-symmetry. Therefore, it looks like both $T_C$ and $U(1)_C$ are broken at the boundary by an anomaly. However, if one redefines:

$$J_A = J_{U(1)_C} - (\varepsilon \cdot Q)J_C$$

(2.89)

then $U(1)_A$ may be preserved at the boundary, and we may define it as the boundary axial R-symmetry. In the anomaly polynomial, this sets $f_C - (\varepsilon \cdot Q)f_A = 0$, and $f_A$ is now considered as the field-strength for the boundary $U(1)_A$ symmetry.

- Recall that for a Dirichlet boundary condition for the vector multiplet, we must turn on boundary vevs $c_\varepsilon$ which break the boundary gauge symmetry $G_\partial$ completely. However, in some cases, we may turn on just enough vevs such that although $G_\partial$ is broken, $G_H$ is conserved. As an example, we may take supersymmetric QED and turn on only a boundary vev $X_\alpha = c$. This breaks a linear combination of $G$, $T_H$ and $U(1)_H$, which may be seen as a breaking of $G_\partial$ and a redefinition of boundary symmetries $T_H$ and $U(1)_V$:

$$J^\beta_H = J^\beta_{H,\text{bulk}} + \delta^{\alpha\beta}J_G,$$

$$J_V = J_{U(1)_H} - J_G.$$  

(2.90)

In the anomaly polynomial, this sets $f_\partial = f_H^\alpha - f_V$.

Images

Let us state the images of these boundary conditions on the Higgs branch, obtained by the authors of [56].

For a Neumann boundary condition for the vector multiplet, the Higgs branch image is given by:

$$\frac{(L \cap \mu_C^{-1}(0) \cap \mu^R_{\varepsilon}(\zeta))/G,}{(L \cap \mu_C^{-1}(0) \cap \mu^R_{\varepsilon}(\zeta))/G},$$

(2.91)

where $L$ is the Lagrangian in the space of hypermultiplet scalars $T^* R$ defined by the polarisation $\varepsilon$, i.e. the space spanned by the scalars $X_{\varepsilon,\beta}$ which have Neumann
boundary conditions. This is the image under the hyper-Kähler quotient defining the Higgs branch, and is a holomorphic Lagrangian submanifold.

Let us now consider a Dirichlet boundary condition for the vector multiplet, with constants $c_\varepsilon$ specifying the Neumann-Dirichlet boundary condition for the hypermultiplets. We will as above assume $c_\varepsilon$ break the boundary gauge symmetry (which is really a boundary flavour symmetry) completely. The Higgs branch image is the image of $Y_\varepsilon = c_\varepsilon$ under the complex symplectic quotient:

$$ (Y_\varepsilon = c_\varepsilon) \cap \mu_{c_\varepsilon}^{-1}(0). \quad (2.92) $$

There is no quotient in the above because of our assumption on $c_\varepsilon$. This is again a holomorphic Lagrangian submanifold.

The Coulomb branch images of these boundary conditions are much more difficult to determine, requiring the incorporation of quantum corrections for Neumann boundary conditions, and a semi-classical analysis of bulk $(2,2)$ BPS equations in the case of Dirichlet boundary conditions. We do not consider them here, returning to give some details for supersymmetric QED in chapter 4.

**Boundary Superpotentials**

Finally, we discuss the effect of adding boundary interactions in the form of a $(2,2)$ superpotential. Suppose we start with a Neumann-Dirichlet boundary condition for the hypermultiplets of the form (2.88). The $X_\varepsilon$ transform as $(2,2)$ chiral multiplets at the boundary, and we may thus introduce boundary chiral multiplets $\phi$, and a boundary superpotential $W(X_\varepsilon, \phi)$. Then analysis in [56] shows that requiring the boundary variations in the combined bulk-boundary action requires that the boundary conditions are modified to:

Right boundary: $Y_{\varepsilon,\beta} = \frac{\partial W}{\partial X_{\varepsilon,\beta}}$, $\frac{\partial W}{\partial \phi} = 0$,

Left boundary: $\left| Y_{\varepsilon,\beta} = -\frac{\partial W}{\partial X_{\varepsilon,\beta}} \right|$, $\frac{\partial W}{\partial \phi} = 0$, \quad (2.93)

where the boundary condition for $X_{\varepsilon,\beta}$ is also modified in order to satisfy the second equation.

The Higgs branch image of the boundary condition is then modified to the image under the holomorphic symplectic quotient of the holomorphic Lagrangian of $T^*R$ defined by (2.93).
As an example, we consider the $(2, 2)$ ‘flip’. We introduce a boundary superpotential $W = X_{\varepsilon\beta}\phi$. Then (2.93) sets $\frac{\partial W}{\partial \phi} = X_{\varepsilon\beta} = 0$, and $Y_{\varepsilon\beta} = \frac{\partial W}{\partial X_{\varepsilon\beta}} = \phi$, allowing $Y_{\varepsilon\beta}$ to fluctuate. This essentially flips $\varepsilon_{\beta}$, modifying the polarisation.

One can also study the effect of adding in a boundary twisted superpotential, coupled to the the twisted chiral field strength multiplet $(\Phi_\varphi, \upsilon)$ (in $(0, 2)$ language), and the twisted chiral $(S^\vee, \Psi_\varphi)$. Here $S^\vee$ is the $(0, 2)$ chiral multiplet obtained by T-dualising $S$, and contains $\sigma + i\gamma$, where $\gamma$ is the dual photon. We will not require such boundary conditions in this work, see [56] for more details.

2.7 Chiral Rings

So far we have not yet discussed the spectrum of local operators appearing in 3d $\mathcal{N} = 4$ gauge theories. We will consider two particular classes of protected half-BPS operators, which form commutative operator algebras known as chiral rings [142]. In chapter 5, we will discuss a quantisation of these operator algebras, and discuss their modules realised by local operators on a boundary of space-time.

We define the Higgs and Coulomb branch chiral rings (whose etymology will become clear shortly), to be the operators annihilated by the following supercharges:\footnote{Note that actually we have made a choice of $\mathcal{N} = 2$ subalgebra here, corresponding to a choice of $U(1)_H \times U(1)_C \subset SU(2)_H \times SU(2)_C$. The operators annihilated these supercharges turn out to be holomorphic functions on the Higgs and Coulomb branches corresponding to the complex structure induced by this choice. We have a $\mathbb{P}^1 \times \mathbb{P}^1$ of choices leading to holomorphic functions in the corresponding complex structures. See e.g. appendix C of [56].}

$$\begin{array}{c|cccccccc} & Q_{++} & Q_{+-} & Q_{-+} & Q_{--} & Q_{+'} & Q_{-'} & Q_{-'} & Q_{-'} \\
M_H \text{ chiral ring} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
M_C \text{ chiral ring} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}$$

Table 2.4 Supercharges preserved by chiral ring operators

Recall that we have assumed that the theories we study flow in the IR to a 3d $\mathcal{N} = 4$ superconformal field theory. We will further, throughout this thesis, make the assumption that the theory is good or ugly in the Gaiotto-Witten classification [143].

This means the theory flows to an IR theory whose superconformal R-symmetry is the one manifest in the UV, plus, in the case of ugly theories a decoupled free sector. The superconformal theory has an enlarged $\mathcal{N} = 4$ superconformal symmetry, the super Lie algebra $\mathfrak{osp}(4|4, \mathbb{R})$ [119]. In addition to the supercharges $Q_{\alpha\dot{A}}$, there are 8
additional supercharges $S_{AA}^\alpha$. As is usual in the study of conformal field theory, we may radially quantise and obtain a state-operator correspondence between local operators inserted at the origin and states on an $S^2$ containing the origin. In this quantisation, the Hermitian conjugation acts as $(Q_{\alpha}^{AA})^\dagger = S_{AA}^\alpha$.

**Operators in Cohomology**

Let us now define:

\[ Q_H = Q_+^+ + Q_-^- , \quad Q_C = Q_+^+ + Q_-^-. \]  

(2.94)

These are scalars under the (anti-)diagonal of $U(1)_E \times U(1)_C$ and $U(1)_E \times U(1)_H$ respectively, and are the ones preserved in a topological twist of the theory. The corresponding sigma models are the usual [144], and twisted [145] Rozansky-Witten theories.

Under the above assumption, the cohomologies of these supercharges coincide with the $\mathcal{M}_H$ and $\mathcal{M}_C$ chiral rings. We describe the argument for $Q_H$, the argument for $Q_C$ is the same after exchanging $H$ and $C$ labels.

Any unitary representation of the superconformal group must satisfy the following unitary bounds:

\begin{align}
\{ Q_+^+, S_+^+ \} &= D - J_3 - J_{U(1)_H}/2 - J_{U(1)_C}/2 \geq 0, \\
\{ Q_-^-, S_+^- \} &= D + J_3 - J_{U(1)_H}/2 + J_{U(1)_C}/2 \geq 0,
\end{align}

(2.95)

and so also:

\[ \{ Q_H, Q_H^\dagger \} = 2D - J_{U(1)_C} \geq 0. \]

(2.96)

In the above, $D$ is the dilatation, $J_3$ is the generator of rotations, and $J_{U(1)_H}$ and $J_{U(1)_C}$ of $U(1)_H$ and $U(1)_C$.

By the usual argument, an operator or state in $Q_H$ cohomology has a representative obeying $D = J_{U(1)_H}/2$. From (2.95) it must also have $J_3 = -J_{U(1)_C}/2$. From Dolan’s classification of $osp(4|4, \mathbb{R})$ multiplets [119], such operators are precisely those annihilated by the supercharges in table 2.4, and thus coincide with the Higgs branch chiral ring.

Further, the $Q_H$ cohomology is generated by gauge-invariant local operators formed from the hypermultiplet scalars in the UV Lagrangian, modulo the complex moment map constraint.
Similarly, the $Q_C$ cohomology is (at least in the UV) generated by monopole ‘dressed’ by vector multiplet scalar fields [57, 58]. We will review some further details shortly.

**Holomorphic Functions**

Let us now show that the operators in the chiral rings/the cohomology of the above supercharges are position-independent. This follows directly from that fact that the momentum $P_\mu$ is exact with respect to either of the supercharges (2.94):

\[
\{Q_{H,C}, Q_\mu^{H,C}\} = P_\mu \quad \text{where}
\]

\[
Q_\mu^H = \frac{1}{2} (\sigma_\mu)^\alpha_A Q^{-A}_\alpha, \quad Q_\mu^C = \frac{1}{2} (\sigma_\mu)^\alpha_A Q^A_\alpha.
\]

Therefore, an operator $\mathcal{O}$ in $Q = Q_{H,C}$ cohomology obeys

\[
[P_\mu, \mathcal{O}] = \{[Q, \mathcal{O}], Q_\mu\} + \{Q, [Q_\mu, \mathcal{O}]\}.
\]

The first term vanishes as $\mathcal{O}$ is in cohomology, and the second term is $Q$-exact and so vanishes inside correlation functions as $Q$ is a symmetry.

Correlation functions of $Q$-closed operators are therefore independent of position. In flat space, we may then arbitrarily separate such operators without altering correlators, and then apply cluster decomposition [146]. This defines a product structure:

\[
\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle
\]

which is also commutative, as in 3d there is no notion of operator ordering.

The Higgs and Coulomb branch chiral rings then define holomorphic functions on the Higgs and Coulomb branches in the following way. In flat 3d space-time, defining the theory requires a definition of a vacuum state, or path integral asymptotics. This is a superselection sector. We may choose a Higgs or Coulomb branch vacuum state $|\Omega_{H,C}\rangle$, parameterised by points on the (quantum) Higgs and Coulomb branches. Then an element $\mathcal{O}$ of either the Higgs or Coulomb branch chiral ring defines a holomorphic function $f_{\mathcal{O}_{H,C}} : \mathcal{M}_{H,C} \to \mathbb{C}$ by:

\[
f_{\mathcal{O}_{H,C}} : |\Omega_{H,C}\rangle \mapsto \langle \mathcal{O} |\Omega_{H,C}\rangle,
\]

i.e. the expectation value computed in the vacuum $|\Omega_{H,C}\rangle$. 

Higgs Branch

For the Higgs branch, from the arguments above, the map $O \mapsto f_O$ from the chiral ring to holomorphic functions on $\mathcal{M}_H$ is clearly surjective. We may therefore identify the Higgs branch chiral ring with the coordinate ring $\mathbb{C}[\mathcal{M}_H]$. It is given by the complex symplectic reduction

$$\mathbb{C}[\mathcal{M}_H] = \mathbb{C}[X_i, Y_i]^G/(\mu_C - \zeta_C = 0),$$

where we have allowed for a complex FI parameter. This is the ring of gauge-invariant polynomials in the free hypermultiplet ring $\mathbb{C}[X_i, Y_i]$, modulo the ideal generated by the complex moment map. The Higgs branch is not quantum corrected, so neither is the Higgs branch chiral ring.

Coulomb Branch

To the best of the author’s knowledge, there does not yet exist a proof that the chiral ring of dressed monopole operators gives a complete set of holomorphic functions on the quantum-corrected Coulomb branch, *i.e.* that it coincides with $\mathbb{C}[\mathcal{M}_H]$. However, this has been the case in all known examples, see *e.g.* [17, 147], and we will assume it to be true.

Let us briefly describe the aforementioned dressed monopole operators. These admit a path integral description, which for non-abelian theories is also valid in strongly coupled regions of the moduli space of vacua with unbroken non-abelian gauge symmetry. A half-BPS monopole operator (annihilated by the second row of supercharges in (2.4)) inserted at a point $x$ is defined by requiring that the path integral is restricted to integrate over gauge field configurations with a Dirac monopole singularity at $x$. The profile of $\sigma$ near $x$ is then determined by supersymmetry. More precisely, using spherical coordinates $(r, \theta, \phi)$ centered at $x$, the monopole operator imposes the following singular boundary conditions in the path integral as $r \to 0$ [148]

$$A_\pm \sim \frac{m}{2}(\pm 1 - \cos \theta)d\phi, \quad \sigma \sim \frac{m}{2r}.$$

Here, $A_\pm$ is the gauge connection in the north and south patch patch of an $S^2$ surrounding $x$. The monopole operator is labelled by its magnetic charge $m \in \mathfrak{h}/\text{Weyl}G$, specifying an embedding $U(1) \hookrightarrow G$. For the gauge bundle to be well-defined, standard Dirac quantisation implies that $e^{2\pi i m} = \mathbb{I}_G$. Thus, $m \in \Gamma_L G/\text{Weyl}(G)$, where $\Gamma_L G$ is...
the weight lattice of the GNO dual group $L G$, and so equivalently a cocharacter in $\text{Hom}(U(1), G)$.

The monopole operator breaks the gauge group $G$ to a subgroup $G_m$. One can then dress the monopole operator by a polynomial $p$ of the complex scalars $\varphi$ lying in $\mathfrak{g}_m = \mathcal{L}(G_m)$. We can therefore label the dressed monopole operators by $M_{m,p}$.

Our above assumption states that the dressed monopole operators $M_{m,p}$ generate the coordinate ring of the Coulomb branch. The Coulomb branch receives both perturbative and non-perturbative corrections (the latter occurring in non-abelian gauge theories), and so the chiral ring is also deformed. The quantum-corrected ring relations between monopole operators operators and the Poisson algebra structure was first determined systematically in [44]. This then determines the Coulomb branch as a complex symplectic manifold

$$\mathcal{M}_H := \text{Spec} \mathbb{C}[\mathcal{M}_H].$$

For abelian gauge theories, this was done by taking into account the 1-loop quantum corrections [45, 149, 150]. For non-abelian gauge theories, it was accomplished by an abelianisation map, relating the vevs of monopole operators to linear combinations of vevs of monopoles of the low-energy abelian gauge theory. Mathematically this embeds $\mathbb{C}[\mathcal{M}_C]$ into a larger algebra of holomorphic functions $\mathbb{C}[\mathcal{M}_C^{\text{abelian}}]$ on an abelian patch $\mathcal{M}_C^{\text{abelian}}$ of the full Coulomb branch where non-abelian gauge symmetry is restored.

The Coulomb branch algebra has also been formulated via its action on the cohomology of the moduli space of vortices in the theory [51], which is the physical counterpart of the rigorous mathematical formulation of the Coulomb branch due to Braverman-Finkelberg-Nakajima [52, 53].

Example

We now spell out the details of the above constructions in our example of supersymmetric QED. The Higgs branch chiral ring (2.102) is generated by the gauge-invariant bilinears $X_\alpha Y_\beta$ subject to the vanishing of $\sum_\alpha X_\alpha Y_\alpha = \zeta_C$. For $\zeta_C = 0$ this coincides with the coordinate ring of $T^* \mathbb{CP}^{N-1}$.

Classically, the Coulomb branch is parameterised by $(\sigma, \varphi)$ and the dual photon $\gamma$, obeying $d\gamma = *F$. Topologically then we have $\mathcal{M}_C^{\text{class}} = \mathbb{R}^3 \times S^1$. In our choice of complex structure, the holomorphic functions on $\mathcal{M}_C^{\text{class}}$ are given by vevs of $\varphi$ and the

\[11\] Note that classically, provided the complex scalars $(\sigma, \varphi)$ break $G$ to its maximal torus, one can dualise the resulting massless abelian gauge fields to a dual photon $\gamma$, and one can express the monopole operator (2.103) as $\sim \exp \frac{2\pi i}{g} (\sigma + i\gamma)$.
monopole operators $v^\pm = e^{\pm \frac{1}{2} (\sigma + i\gamma)}$, obeying $v^+v^- = 1$. The monopole operators are $\mathbb{C}^*$-valued. We may identify $\mathcal{M}_C^{\text{class}} = \mathbb{C} \times \mathbb{C}^*$ as a complex symplectic manifold with holomorphic symplectic form

$$\Omega = d\varphi \wedge d\log v^+.$$  \hspace{1cm} (2.105)$$

To study the quantum-corrected Coulomb branch, for clarity, we consider the general case where the triplet $(m_{R,\alpha}, m_{C,\alpha})$ of masses for $T_H$ have been turned on (but not a mass $\epsilon$ for $T_i$). The Coulomb branch recieves only 1-loop corrections [45, 149, 150], which shrinks the dual photon circle $S^1$ at all values of $(\sigma, \varphi)$ where the effective mass of a hypermultiplet vanishes. These are the points where $(\sigma, \varphi) = (m_{R,\alpha}, m_{C,\alpha})$ for any $\alpha$. In the IR, the vevs of the monopole operators become $\mathbb{C}$-valued functions and obey the relation [57]:

$$v^+v^- = N \prod_{\alpha=1}^N (\varphi + m_{C,\alpha})$$  \hspace{1cm} (2.106)$$

with the same holomorphic symplectic form as above, which accounts for the shrinking of the $S^1$. This specifies the quantum-corrected Coulomb branch $\mathcal{M}_C$ as a complex symplectic manifold. When the masses are turned off, this is the singularity $\mathbb{C}^2/\mathbb{Z}_N$. The complex masses $\{m_{C,\alpha}\}$ provide a complex deformation thereof, and the real masses $\{m_{R,\alpha}\}$ resolve the space, although this is not captured by the coordinate ring. For details on the hyper-Kähler metric on the Coulomb branch, see [45, 149, 150].

2.8 Mirror Symmetry

3d $\mathcal{N} = 4$ theories may experience a powerful duality known as 3d mirror symmetry, discovered in [45]. For two theories $\mathcal{T}$ and $\tilde{\mathcal{T}}$ related by mirror symmetry, the objects in table 2.5 are identified (the tilded objects are the associated ones in $\tilde{\mathcal{T}}$). The sign on the the identification of mirror dual masses $\tilde{m}$ with the FI parameters $\zeta$ is just a convention. In words, at the zeroth order, 3d mirror symmetry exchanges the Higgs and Coulomb branches of two theories, together with their corresponding symmetries.

3d mirror symmetry is expected to be an infrared duality of theories $(\mathcal{T}, \tilde{\mathcal{T}})$, and as such all observables should also be mapped across the duality. This is in part motivated by the realisation of 3d $\mathcal{N} = 4$ theories as world-volume theories of D3-D5-NS5 brane intersections in type IIB string theory and 3d mirror symmetry as the application of $S$-duality [50]. See also [151, 152]. Since then, many non-trivial checks of 3d mirror
2.8 Mirror Symmetry

<table>
<thead>
<tr>
<th>$\mathcal{T}$</th>
<th>$\tilde{\mathcal{T}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{M}_H$</td>
<td>$\tilde{\mathcal{M}}_C$</td>
</tr>
<tr>
<td>$\mathcal{M}_C$</td>
<td>$\tilde{\mathcal{M}}_H$</td>
</tr>
<tr>
<td>$SU(2)_H \times SU(2)_C$</td>
<td>$SU(2)_C \times SU(2)_H$</td>
</tr>
<tr>
<td>$G_H$</td>
<td>$\tilde{G}_C$</td>
</tr>
<tr>
<td>$G_C$</td>
<td>$\tilde{G}_H$</td>
</tr>
<tr>
<td>$(m_R,m_C)$</td>
<td>$(\tilde{\zeta}_R,\tilde{\zeta}_C)$</td>
</tr>
<tr>
<td>$(\zeta_R,\zeta_C)$</td>
<td>$-(\tilde{m}_R,\tilde{m}_C)$</td>
</tr>
</tbody>
</table>

Table 2.5 The 3d $\mathcal{N}=4$ mirror symmetry dictionary

duality have been made, by matching correlation functions, partition functions and more. As a small sample see [57, 58, 153, 154].

Although the action of mirror symmetry on abelian theories is systematic [45, 152], the mirror dual of a general gauge theory is difficult to calculate in general, even for quiver theories, and may not even be Lagrangian. Sometimes we are lucky; for example the class of $T^*_\rho$ theories [155] is closed under mirror symmetry, and the 3d ADHM quiver gauge theory is also mirror dual. Where the mirror theory is known, this enables us to express the Coulomb branch of the original theory as a hyper-Kähler quotient. We saw in equation (2.106) that the Coulomb branch of supersymmetric QED is a resolution and/or deformation of the $\mathbb{C}^2/\mathbb{Z}_N$ singularity. In chapter 4, we will meet the mirror dual of supersymmetric QED, whose Higgs branch is this space. It is an abelian $A_{N-1}$ quiver gauge theory, whose Coulomb branch flavour symmetry enhances in the IR to the $PSU(N)$ Higgs branch flavour symmetry of QED.

We now turn to discuss briefly two topics of active mathematical interest related to 3d mirror symmetry, which we return to in the remainder of this thesis.

Symplectic Duality

Symplectic duality [59, 60] is an equivalence between certain categories $\mathcal{O}_H$ and $\mathcal{O}_C$ attached to pairs $(X, X')$ of symplectic resolutions. All known examples can be realised physically as the Higgs and Coulomb branches $(\mathcal{M}_H, \mathcal{M}_C)$ of a 3d $\mathcal{N}=4$ theory $\mathcal{T}$.

After a simplification, $\mathcal{O}_H$ and $\mathcal{O}_C$ correspond to lowest-weight modules for the quantised algebras $\mathcal{A}_H$ and $\mathcal{A}_C$, which are deformation quantisations of the coordinate (chiral) rings $\mathbb{C}[\mathcal{M}_H]$ and $\mathbb{C}[\mathcal{M}_C]$. Symplectic duality then gives collections of pairs of modules $(\mathcal{H}_B^{(B)}, \mathcal{H}_B^{(A)})$ mapped to each other under duality.
In [56], the above mathematical constructions are realised beautifully in the context of 3d $\mathcal{N} = 4$ theory, involving both types of object discussed in sections 2.6 and 2.7. The chiral rings are quantised, by a standard ($\Omega_A$) or twisted ($\Omega_B$) omega background, giving $\mathcal{A}_H$ and $\mathcal{A}_C$ respectively. Physically, this reduces the 3d theory to a quantum mechanics, such that the chiral ring operators are forced to lie on a line and thus become non-commutative. A $\mathcal{N} = (2, 2)$ boundary condition $\mathcal{B}$ supports boundary Higgs and Coulomb branch operators, which then furnishes $\mathcal{A}_{H,C}$ modules ($\mathcal{H}_B^{(B)}, \mathcal{H}_B^{(A)}$) respectively. The $\mathcal{A}_{H,C}$ action is realised physically by bringing in bulk operators to act on boundary operators. Under 3d mirror symmetry, the boundary condition $\mathcal{B}$ is mapped to one in the dual theory, and the modules ($\mathcal{H}_B^{(B)}, \mathcal{H}_B^{(A)}$) are exchanged. We return to these constructions in chapter 5.

In chapter 4, introduce a new class of boundary conditions, which we called enriched Neumann $N_\alpha$, which are in one-to-one correspondence with supersymmetric vacua $\alpha$. We find they are mirror dual to a class of boundary conditions $D_\alpha$, the so-called exceptional Dirichlet boundary conditions constructed in [56], whose Higgs branch image coincide with attracting Lagrangian submanifolds of fixed points $\alpha$ under the gradient flow we discussed in sections 2.4 and 2.5. The $N_\alpha$ boundary conditions turn out to be supported on the Higgs branch on the stable envelopes of Maulik-Okounkov [89]. Compactifying the theory on a torus, they yield classes in equivariant elliptic cohomology known as the elliptic stable envelopes of Aganagic-Okounkov [67].

**Enumerative Geometry**

There has been intense recent study in relating the enumerative geometry of symplectic dual pairs $(X, X^!)$ . The central object in this program of study is the vertex function, which is, very roughly:

$$V_\alpha(X) = \sum_d \zeta^d \chi_T(\text{QM}^d_\alpha(X)).$$

This is a generating function of equivariant Euler characteristics of $\text{QM}^d_\alpha(X)$: the moduli space of quasi-maps [156, 157] from $\mathbb{P}^1$ to $X$, based at a fixed point $\alpha$ at $\infty \in \mathbb{P}^1$. Physically, quasi-maps are vortices, which tend to a fixed vacuum $\alpha$ at $\infty$, and so the vertex function $V_\alpha$ has an interpretation physically as a vortex partition function. We will meet another physical avatar of these objects in chapter 5. They are given by the hemisphere partition functions on $S^1 \times HS^2$ of a 3d theory with Higgs branch $X$, equipped with the aforementioned $D_\alpha$ boundary conditions at $\partial(S^1 \times HS^2) = T^2$. This will be more rigorously established in upcoming work by the author and collaborators [158]. See also [159] for further details on progress thus far.
In the context of enumerative geometry, mirror symmetry relates the vertex functions $V_\alpha(X), \tilde{V}_\alpha(X^!)$ of symplectic dual pairs $(X, X^!)$, made precise in the works [70, 123, 160–163]. The vertex functions of the dual pair are related by a transition matrix, closely related to the elliptic stable envelopes mentioned above. Physically, this arises from the equality of hemisphere partition function for mirror dual theories equipped with dual pairs of boundary conditions, of $D_\alpha$ and $N_\alpha$ type. This will also appear in the work [158].
Chapter 3

Theories on Elliptic Curves I: Elliptic Cohomology

In this chapter, we study elementary aspects of 3d $\mathcal{N} = 4$ supersymmetric gauge theories compactified on $\mathbb{R} \times E_\tau$, where $E_\tau$ is a complex torus with complex structure modulus $\tau$. In particular, we focus on the computation of supersymmetric ground states and their supersymmetric Berry connection over the space of mass parameters and flat connections on $E_\tau$ for flavour symmetries. We show that this leads naturally to algebraic constructions appearing in equivariant elliptic cohomology. For example, we demonstrate that the elliptic cohomology scheme of the Higgs branch $X$ can be recovered from a spectral curve associated to the Berry connection, and that the supersymmetric ground states furnish a holomorphic line bundle over this scheme. We demonstrate how supersymmetric boundary conditions realise equivariant elliptic cohomology classes.

In appendix A we include a brief review of elliptic cohomology, introducing the main structures whose physical avatars in 3d $\mathcal{N} = 4$ gauge theory we meet in the remainder of this chapter. It provides some motivation for the constructions we make in the main body of this thesis, and we will refer to it throughout. We note that for comparison to much of the mathematics literature, e.g. [67, 70, 123, 160–163], the complexification of $T_H$ is often called $A$, and that of $T_i$ by $\hbar$.

In this chapter we assume knowledge of the material from chapter 2. The geometric description of effective Chern-Simons levels plays a particularly important role.

Contributions The results in this chapter are based on material from:

3.1 Supersymmetry

It will be helpful to first decompose the 3d $\mathcal{N} = 4$ supersymmetry algebra (2.1) on $\mathbb{R} \times \mathbb{R}^2$ as a 1d $\mathcal{N} = (4, 4)$ supersymmetric quantum mechanics on $\mathbb{R}$. The supersymmetry algebra can be written

$$\{Q_+^{AA}, Q_-^{BB}\} = \epsilon^{AB} \epsilon^{\dot{A}\dot{B}} P$$
$$\{Q_+^{AA}, Q_-^{BB}\} = \epsilon^{AB} \epsilon^{\dot{A}\dot{B}} H + \epsilon^{AB} \tilde{Z}^{\dot{A}\dot{B}} + \epsilon^{\dot{A}\dot{B}} Z^{AB} + C^{AB,\dot{A}\dot{B}}$$

(3.1)

where $H := P_{++}$ is the Hamiltonian and $P := P_{++}$ and $\bar{P} := P_{--}$ become central charges from the perspective of supersymmetric quantum mechanics. Recall that the remaining scalar central charges are associated to global symmetries, $Z^{AB} = \zeta^{AB} \cdot J_C$ and $\tilde{Z}^{\dot{A}\dot{B}} = m^{\dot{A}\dot{B}} \cdot J_H$, while $C^{AB,\dot{A}\dot{B}}$ arises from a vector central charge associated to domain walls and is bi-linear in the mass and FI parameters. We have assumed the existence of domain walls only in the $x^3$ direction.

As in chapter 2, we set the complex parameters to zero, $m^{++} = 0$ and $t^{++} = 0$, and define $m := m^{+-}$ and $\zeta = \zeta^{+-}$. We then consider a 3d $\mathcal{N} = 2$ subalgebra commuting with $T_t := U(1)_H - U(1)_C$, which becomes a 1d $\mathcal{N} = (2, 2)$ subalgebra in supersymmetric quantum mechanics. The supercharges are $Q_+^{++}, Q_+^{+-}, Q_-^{--}$ and the non-vanishing commutators are

$$\{Q_+^{++}, Q_-^{--}\} = P,$$
$$\{Q_+^{++}, Q_-^{--}\} = H + Z + C,$$
$$\{Q_+^{+-}, Q_-^{--}\} = H - Z + C,$$

(3.2)

where we define $Z := Z^{+-} + \tilde{Z}^{+-}$ and $C = C^{+-,+-}$. The Higgs and Coulomb flavour symmetries are now on the same footing with $Z = m \cdot J_H + \zeta \cdot J_C$. Recall from section 2.5.3 that in the presence of a domain wall in the $x^{1,2}$-plane interpolating between massive vacua $\alpha, \beta$, $C = C_\alpha - C_\beta$ where $C_\alpha = \kappa_\alpha(m, \zeta)$ is the central charge function (2.73).

As the combination $T_t$ is now a flavour symmetry, it is possible to turn on a real mass parameter $\epsilon$. This deforms the central charges further to $Z = m \cdot J_H + \zeta \cdot J_C + \epsilon \cdot J_t$ and $C_\alpha$ has a more general form (2.69). In shorthand notation, $Z = x_f \cdot J_f$ and $C_\alpha = K_\alpha(x_f, x_f)$. 
3.2 Reduction on Elliptic Curve

Let us now place such a theory on $E_\tau \times \mathbb{R}$ where $E_\tau$ is the elliptic curve with complex structure modulus $\tau = \tau_1 + i\tau_2$ and area $\tau_2 > 0$. This is implemented by forming the complex combination $x^1 + ix^2$ and making the identifications $(x^1, x^2) \sim (x^1 + 1, x^2)$ and $(x^1, x^2) \sim (x^1 + \tau^1, x^2 + \tau^2)$.

It is also convenient to introduce real coordinates $(s, t)$ and identify $s \sim s + 1$ and $t \sim t + 1$, such that $x^1 = s + \tau_1 t$ and $x^2 = \tau_2 t$. This gives a continuous isomorphism of groups $E_\tau \to S^1 \times S^1$ induced by the transformation $x^1 + ix^2 \mapsto (e^{2\pi is}, e^{2\pi it})$. These coordinates are illustrated in figure 3.1.

We impose R-R boundary conditions, which preserves the full supersymmetry. We can also now introduce background flat connections $A_f = (A_C, A_H, A_t)$ on $E_\tau$ for all 3d $\mathcal{N} = 2$ flavour symmetries. This background preserves the same supersymmetry algebra (3.2), but now

$$
P = \frac{i}{\tau_2} (\partial_t - \tau \partial_s) + \frac{i}{\tau_2} (z_f \cdot J_f)$$

$$
\bar{P} = -\frac{i}{\tau_2} (\partial_t - \bar{\tau} \partial_s) - \frac{i}{\tau_2} (\bar{z}_f \cdot J_f)
$$

where

$$
z_f := \oint A_{f,t} \, dt - \tau \oint A_{f,s} \, ds
$$

and we again combine $z_f = (z_C, z_H, z_t)$ in shorthand notation. From the perspective of $\mathcal{N} = (2, 2)$ supersymmetric quantum mechanics, the real parameters $x_f$ and complex flat connections $z_f$ combine as expectation values for the triplet of scalar fields in a background vector multiplet for the 3d $\mathcal{N} = 2$ flavour symmetry $T_f$.

The total space of background parameters is $t_f \times E_{T_f}$. Here $t_f$ parametrises the real parameters $x_f$ and

$$
E_{T_f} := \Gamma_f \otimes_{\mathbb{Z}} E_\tau
$$
is the complex torus parametrising background flat connections $z_f$ modulo gauge transformations $z_f \to z_f + (\nu_f + \tau \mu_f)$. It will also be convenient to introduce the notation $x = (m, \epsilon)$ and $z = (z_H, z_t)$, parameterised by $t$ and:

$$E_T := (\Gamma_H \oplus \Gamma_t) \otimes_{\mathbb{Z}} E_{\tau}. \quad (3.6)$$

The above is consistent with the notation we have used in reviewing equivariant elliptic cohomology of $X$ in appendix A.

It is often convenient to restrict attention to a sector with fixed KK momentum and flavour charge, whereupon $Z = x_f \cdot \gamma_f$ and

$$P = + \frac{i}{\tau_2} (n - \tau m) + \frac{i}{\tau_2} (z_f \cdot \gamma_f)$$
$$\bar{P} = - \frac{i}{\tau_2} (n - \bar{\tau} m) - \frac{i}{\tau_2} (\bar{z}_f \cdot \gamma_f) \quad (3.7)$$

where $(m, n) \in \mathbb{Z}^2$ are KK momenta conjugate to the coordinates $(s, t)$ and $\gamma_f \in \Gamma_f$ is a weight of the flavour symmetry $T_f$.

A global background gauge transformation $z_f \mapsto z_f + (\nu_f - \tau \mu_f)$ is specified by a pair of co-characters $\mu_f, \nu_f \in \Gamma_f$ associated to the cycles with coordinates $s, t$. In the sector with flavour weight $\gamma_f$, this can be absorbed by shifting the KK momenta $(m, n) \to (m - \gamma_f \cdot \mu_f, n - \gamma_f \cdot \nu_f)$.

### 3.2.1 Infinite-Dimensional Model

It is often useful to invoke an infinite-dimensional model for the effective $\mathcal{N} = (2,2)$ supersymmetric quantum mechanics.

Let us fix a generic FI parameter $\zeta$, such that in the absence of mass parameters the theory flows to a smooth sigma model onto the Higgs branch $X$ in flat space. Passing to $E_{\tau} \times \mathbb{R}$ and setting the background connection $z_C = 0$, the system is described by an $\mathcal{N} = (2,2)$ quantum mechanics whose target is the space of smooth maps

$$\mathcal{X} = \text{Map}(E_{\tau} \to X), \quad (3.8)$$

which is an infinite-dimensional Kähler manifold. This is illustrated in figure 3.2.

The kinetic terms involving derivatives along $E_{\tau}$ are obtained by coupling to a background vector multiplet for the $S^1 \times S^1$ symmetry of $\mathcal{X}$ induced by translations of the coordinates $(s, t)$.
The mass parameters \( x = (m, \epsilon) \) and background connections \( z = (z_H, z_t) \) are introduced by coupling to a background vector multiplet for the induced action of \( T = T_H \times T_t \) on \( \mathcal{X} \) and turning on expectation values for the scalar fields. The background flat connection \( z_C \) is expected to induce a flat connection on the target \( \mathcal{X} \) that deforms the action of the supercharges.

### 3.3 Supersymmetric Ground States

Let us now consider states of the supersymmetric quantum mechanics annihilated by all four of the supercharges generating the 1d \( \mathcal{N} = (2, 2) \) supersymmetry algebra (3.2). We refer to such states as supersymmetric ground states. Supersymmetric ground states are necessarily annihilated by \( H + C, Z, P, \bar{P} \).

As usual in supersymmetric quantum mechanics, if the spectrum is gapped, it is convenient to introduce a cohomological description of supersymmetric ground states. For this purpose, we consider the supercharge

\[ Q := Q_+^+ + Q_{-}^- , \tag{3.9} \]

which satisfies \( Q^2 = 2P \) and commutes with the central charges \( Z, P \). We then restrict attention to states in the supersymmetric quantum mechanics annihilated by \( Z, P \), which requires

\[
\begin{align*}
  x_f \cdot \gamma_f &= 0 , \\
  (n - m\tau) + z_f \cdot \gamma_f &= 0 ,
\end{align*}
\]

where \( (m, n) \) are the KK momenta conjugate to the coordinates \( (s, t) \) and \( \gamma_f \in \Gamma_f^\gamma \) is a weight of \( T_f \). The supercharge \( Q \) becomes a differential and its cohomology is an alternative description of supersymmetric ground states.
We can provide a heuristic picture of supersymmetric ground states using the infinite-dimensional quantum mechanics with target space $\mathcal{X} = \text{Map}(E_\tau, X)$. For simplicity we set the flat connection for the topological symmetry to zero, $z_C = 0$. The supercharge $Q$ is then a twisted equivariant deformation of the de Rham differential on $\mathcal{X}$,

$$Q = e^{-h_x}(d + \iota_{V_z})e^{h_x},$$  \hspace{1cm} (3.11)

where

- $h_x$ is the moment map for the Hamiltonian isometry generated by the mass parameters $m, \epsilon$. Here we abuse notation and write $h_x = h_m + h_\epsilon$ for the moment map on both $X$ and $\mathcal{X} = \text{Map}(E_\tau, X)$.

- $V_z$ is a combination of the vector field $\partial_t - \tau \partial_s$ generating the $S^1 \times S^1$ group action on $\mathcal{X}$ induced by translations of the coordinates $(s, t)$ and the vector field generating the $S^1 \subset T$ action on $\mathcal{X}$ with parameters $z = (z_H, z_t)$.

This type of supercharge was already encountered in [19] and has recently been further studied in quiver supersymmetric quantum mechanics [164, 165]. It arises whenever an $\mathcal{N} = (2, 2)$ supersymmetric quantum mechanics is coupled to background vector multiplets.

The supersymmetric ground states can be analysed by applying standard arguments in supersymmetric quantum mechanics to this infinite-dimensional model. First, we can scale the superpotential $h_x$ in order to localise supersymmetric ground states around $\text{Crit}(h_x) \subset \mathcal{X}$. The supersymmetric ground states can then be obtained from the cohomology of the equivariant differential $d + \iota_{V_z}$ on $\text{Crit}(h_x)$, which is the equivariant cohomology of $\text{Crit}(h_x)$ localised at the equivariant parameter $z$.

Introducing a background connection $z_C$ would further deform the supercharge $Q$ by the addition of a background flat connection on $\mathcal{X}$. This does not materially change the outcome for supersymmetric ground states provided the FI parameter $\zeta$ is generic and therefore we set it to zero for simplicity.

This computation of supersymmetric ground states is clearly sensitive to the mass parameters $x \in t$ and will jump along the loci $W_\lambda \subset t$ introduced in section 2.5. We consider various cases in turn before presenting the general construction.

### 3.3.1 Generic Mass Parameters

First suppose the mass parameters lie in the complement of all of the loci $W_\lambda = \{ \lambda \cdot x = 0 \}$. Then the condition that supersymmetric ground states are annihilated by the
central charge $Z = x_f \cdot J_f$ implies that they are uncharged under flavour symmetries, or $\gamma_f = 0$. In turn, $P = 0$ implies they have zero KK momentum, $n, m = 0$.

From the perspective of supersymmetric quantum mechanics to $\mathcal{X}$, the critical points of $h_x$ are constant maps $E_\tau \rightarrow \alpha$. In the limit that the coefficient in front of the superpotential is sent to infinity, there are normalisable perturbative ground states given by Gaussian wavefunctions localised at constant maps $E_\tau \rightarrow \alpha$, which may be chosen to be orthonormal. Since $h_x$ is the moment map for a Hamiltonian isometry of a Kähler manifold $\mathcal{X}$ and the spectrum is gapped, there are no instanton corrections and this is an exact description of supersymmetric ground states.

We can introduce another description of the supersymmetric ground states as follows. Let us fix a generic mass parameters in some chamber $\mathcal{C} \subset t$. We then define

- $| \alpha \rangle_\mathcal{C}$ is the supersymmetric ground state obtained from the path integral on $E_\tau \times \mathbb{R}_+$ with the supersymmetric vacuum $\alpha$ at infinity $x^3 \rightarrow +\infty$,
- $\mathcal{C} \langle \alpha | \beta \rangle$ is the supersymmetric ground state obtained from the path integral on $E_\tau \times \mathbb{R}_-$ with the supersymmetric vacuum $\alpha$ at infinity $x^3 \rightarrow -\infty$.

These states are orthonormal,

$$\mathcal{C} \langle \alpha | \beta \rangle = \delta_{\alpha,\beta}, \quad (3.12)$$

which is interpreted as the partition function on $E_\tau \times \mathbb{R}$ with vacuum $\alpha$ at $x^3 \rightarrow -\infty$ and vacuum $\beta \rightarrow +\infty$. Note that the normalisation is independent of the potential background connection $z$. This basis of supersymmetric ground states depends on the chamber $\mathcal{C}$ for the real mass parameters $x = (m, \epsilon)$.

### 3.3.2 Mass Parameters on Walls

Now consider supersymmetric ground states when the real mass parameters lie on a wall $\mathcal{W}_\lambda = \{ \lambda \cdot x = 0 \}$ for some weight $\lambda \in \Phi_\alpha$ of the tangent space at the vacuum $\alpha$.

Then, we claim that provided $\lambda \cdot z \notin \mathbb{Z} + \tau\mathbb{Z}$, there are again $N$ supersymmetric ground states of zero KK momentum and flavour charge, but whose properties now depend on whether $\lambda$ corresponds to an internal or external edge in the GKM diagram, as discussed in section 2.5. Thus there is a doubly-periodic array of distinguished points in the space of background flat connections $z$

$$\lambda \cdot z \in \mathbb{Z} + \tau\mathbb{Z}, \quad (3.13)$$

where supersymmetric ground states may carry KK momenta and flavour weight, and the nature of these points again depends on whether $\lambda$ is an internal or external edge.
We now prove this using our infinite-dimensional description of the effective $\mathcal{N} = (2,2)$ quantum mechanics, considering external and internal edges in turn.

**External Edge**

If $\lambda \in \Phi_\alpha$ is an external edge, the critical locus of the superpotential is

$$\text{Crit}(h_x) = \{ \gamma \neq \alpha \} \cup \text{Map}(E_\tau, \Sigma_\lambda), \quad (3.14)$$

where $\Sigma_\lambda \cong \mathbb{C}$. There are $N-1$ ground states localised around constant maps $E_\tau \to \gamma$ with $\gamma \neq \alpha$, as in the discussion of generic mass parameters. However, the ground state associated to $\alpha$ is different. We must now consider the cohomology of the remaining differential $Q = d + \iota_{V_z}$, which is the equivariant cohomology of the critical locus $\text{Map}(E_\tau, \Sigma_\lambda)$, localised at the background flat connection $z$.

Provided the background flat connection is not a distinguished point, $\lambda \cdot z \notin \mathbb{Z} + \tau \mathbb{Z}$, the vector field $V_z$ has a single fixed point corresponding to the constant map $E_\tau \to \alpha$, with associated supersymmetric ground state $|\alpha\rangle$. There is now some ambiguity in the normalisation as unitarity is lost in describing supersymmetric ground states cohomologically [51]. A natural choice is to define $|\alpha\rangle$ as the Poincaré dual of the equivariant fundamental class of the constant map $\{ E_\tau \to \alpha \}$ inside the critical locus $\text{Map}(E_\tau, \Sigma_\lambda)$. This is normalised such that

$$\langle \alpha|\alpha\rangle = \prod_{n,m \in \mathbb{Z}} (n + m\tau + \lambda \cdot z)$$

$$= i \frac{\vartheta_1(\lambda \cdot z, \tau)}{\eta(\tau)}, \quad (3.15)$$

where the first line is the equivariant Euler class of the normal bundle to the constant map $E_\tau \to \alpha$ inside $\text{Map}(E_\tau, \Sigma_\lambda)$. In the second line, we use zeta-function regularisation to define this in terms of the Jacobi theta function $\vartheta_1(z, \tau)$ and Dedekind eta function $\eta(\tau)$.

When the background flat connection satisfies $\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}$, the fixed locus of $V$ is non-compact, the supersymmetric quantum mechanics is not gapped and the cohomological description of supersymmetric ground states breaks down.
Internal Edge

If $\lambda \in \Phi_\alpha \cap (-\Phi_\beta)$ labels an internal edge connecting $\alpha$ and $\beta$, the critical locus of the real superpotential is

$$\text{Crit}(h) = \{ \gamma \neq \alpha, \beta \} \cup \text{Map}(E_\tau, \Sigma_\lambda),$$

(3.16)

where now $\Sigma_\lambda \cong \mathbb{CP}^1$. There are now $N-2$ supersymmetric ground states corresponding to constant maps $E_\tau \to \gamma$ with $\gamma \neq \alpha, \beta$. However, for the supersymmetric ground states associated to $\alpha, \beta$, we must again consider the cohomology of the remaining differential $d + \iota_V$, which is the equivariant cohomology of the component $\text{Map}(E_\tau, \Sigma_\lambda)$ localised at the background flat connection $z$.

This component of the critical locus is compact, so the supersymmetric quantum mechanics is gapped and the cohomological construction of supersymmetric ground states is valid for any background flat connection $z$. Nevertheless, there are interesting phenomena at the loci where the fixed locus of $V_z$ does not consist of isolated points.

Provided $\lambda \cdot z \notin \mathbb{Z} + \tau \mathbb{Z}$, the vector field $V_z$ has isolated fixed points on $\text{Map}(E_\tau, \Sigma_\lambda)$ corresponding to the constant map $E_\tau \to \alpha$ and $E_\tau \to \beta$. There are therefore two supersymmetric ground states, which are normalised such that

$$\langle \alpha | \alpha \rangle = \prod_{n,m \in \mathbb{Z}} (n + m \tau + \lambda \cdot z) = i \frac{\vartheta_1(\lambda \cdot z, \tau)}{\eta(\tau)},$$

$$\langle \beta | \beta \rangle = \prod_{n,m \in \mathbb{Z}} (n + m \tau - \lambda \cdot z) = i \frac{\vartheta_1(-\lambda \cdot z, \tau)}{\eta(\tau)},$$

$$\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle = 0.$$

(3.17)

When $\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}$ the fixed locus of $V_z$ is non-isolated and the above supersymmetric ground states are not linearly independent. To find a linearly independent basis of supersymmetric ground states that extends across this locus one can, for example, pass to the linear combinations

$$|1\rangle = -i \frac{\eta(\tau)}{\vartheta_1(\lambda \cdot z, \tau)} (|\alpha\rangle - |\beta\rangle),$$

$$|2\rangle = \frac{1}{2} (|\alpha\rangle + |\beta\rangle),$$

(3.18)

which mix the contributions from the supersymmetric vacua $\alpha, \beta$. 
3.3.3 Vanishing Mass Parameters

We may continue this process to construct supersymmetric ground states on the intersection series of loci \( W_\lambda, W_{\lambda'}, \ldots \). Instead, we skip to the endpoint and consider the case of vanishing mass parameters \( x = 0 \), or equivalently the intersection of all hyperplanes \( W_\lambda \) with \( \lambda \) running over all edges of the GKM diagram.

The real superpotential now vanishes and we must consider the equivariant cohomology of the whole \( \mathcal{X} = \text{Map}(E_\tau, X) \), localised at the background flat connection \( z \). Provided the background flat connection is generic, now meaning \( \lambda \cdot z \notin \mathbb{Z} + \tau \mathbb{Z} \) for all tangent weights, the vector field \( V_z \) has only isolated fixed points corresponding to constant maps \( E_\tau \to \alpha \). Following the discussion above, there are then \( N \) supersymmetric ground states \( |\alpha\rangle \), normalised such that

\[
\langle \alpha | \beta \rangle = \prod_{\lambda \in \Phi_\alpha^-} i \frac{\vartheta_1(\lambda \cdot z, \tau)}{\eta(\tau)} \delta_{\alpha\beta}.
\]  

They are the equivariant fundamental classes of the constant maps \( \{E_\tau \to \alpha\} \) inside \( \mathcal{X} \). We will see below that these supersymmetric ground states play the role of the fixed point basis in \( T \)-equivariant elliptic cohomology of \( X \).

If \( \lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z} \), this construction breaks down. If \( \lambda \) is an external edge of the GKM diagram, the supersymmetric quantum mechanics is not gapped. If \( \lambda \in \Phi_\alpha^+ \cap (-\Phi_\beta^-) \) is an internal edge, we must take linear combinations corresponding to de Rham cohomology classes on \( \text{Map}(E_\tau, \Sigma_\lambda) \). These loci can of course further overlap leading to more intricate structures.

Finally, let us consider the relationship between the supersymmetric ground states \( |\alpha\rangle_{\mathcal{C}} \) for generic mass parameters in some chamber \( \mathcal{C} \) and the supersymmetric ground states \( |\alpha\rangle \) at the origin. We have seen that in the limit \( x \to 0 \), the supersymmetric ground states \( |\alpha\rangle_{\mathcal{C}} \) are no longer appropriate. However, we claim that

\[
|\alpha\rangle_{\mathcal{C}} \prod_{\lambda \in \Phi_\alpha^+} i \frac{\vartheta_1(\lambda \cdot z)}{\eta(\tau)} \rightarrow |\alpha\rangle,
\]

\[
\prod_{\lambda \in \Phi_\alpha^+} i \frac{\vartheta_1(\lambda \cdot z)}{\eta(\tau)} |\alpha\rangle \rightarrow \langle \alpha |,
\]

with the understanding that this holds for computations preserving the supercharge \( Q \) used in the cohomological construction of supersymmetric ground states. This is compatible with the normalisations set out above.
We will discuss this relation further using boundary conditions and supersymmetric localisation in section 4.1. In that context, computations involving supersymmetric ground states are independent of the real mass parameters, so it is convenient to write this as an equality

\[ |\alpha\rangle_C \prod_{\lambda \in \Phi^-} i \frac{\vartheta_1(\lambda \cdot z)}{\eta(\tau)} = |\alpha\rangle, \]

\[ \prod_{\lambda \in \Phi^+} i \frac{\vartheta_a(\lambda \cdot z)}{\eta(\tau)} \varepsilon(\langle \alpha |) = \langle \alpha |. \]

(3.21)

However, as we discuss further in section 4.5 it is more accurate to say that the sets of supersymmetric ground states are related by the action of a Janus interface interpolating between vanishing mass parameters and mass parameters in a chamber \( c \).

### 3.3.4 Example

In supersymmetric QED, there are supersymmetric vacua \( \alpha = 1, \ldots, N \), mass parameters \( m_1, \ldots, m_N, \epsilon \) and background flat connections \( z_1, \ldots, z_N, z_t \). Let us choose the default chambers \( C_C = \{ \zeta > 0 \} \) and \( C = \{ m_1 > \cdots > m_N, |\epsilon| < |m_\alpha - m_\beta| \} \). Then

\[ |\alpha\rangle_C \prod_{\beta > \alpha} i \frac{\vartheta_1(z_\beta - z_\alpha)}{\eta(\tau)} \prod_{\beta < \alpha} i \frac{\vartheta_1(-2z_t - z_\beta + z_\alpha)}{\eta(\tau)} = |\alpha\rangle, \]

\[ \varepsilon(\langle \alpha |) \prod_{\beta > \alpha} i \frac{\vartheta_1(z_\beta - z_\alpha)}{\eta(\tau)} \prod_{\beta < \alpha} i \frac{\vartheta_1(-2z_t - z_\beta + z_\alpha)}{\eta(\tau)} = \langle \alpha |, \]

(3.22)

and

\[ \langle \alpha | \beta \rangle = \delta_{\alpha \beta} \prod_{\beta \neq \alpha} i \frac{\vartheta_1(z_\beta - z_\alpha)}{\eta(\tau)} i \frac{\vartheta_1(-2z_t - z_\beta + z_\alpha)}{\eta(\tau)}. \]

(3.23)

### 3.4 Spectral Data

A more systematic approach to supersymmetric ground states and their dependence on the background parameters is via the supersymmetric Berry connection. The form of the Berry connection is dictated by the fact that the mass parameters \( x_f = (\zeta, m, \epsilon) \) and background connections \( z_f = (z_C, z_H, z_t) \) transform as the real and complex scalar components of 1d \( \mathcal{N} = (2, 2) \) vector multiplets [80–82].

Let us denote the number of supersymmetric ground states by \( N \). Then there is a Berry connection on the rank-\( N \) vector bundle of supersymmetric ground states over
$t_f \times E_{T_f}$. Recall that $t_f$ parametrises the real parameters $x_f$ and

$$E_{T_f} := \Gamma_f \otimes_{\mathbb{Z}} \mathbb{E}_\tau$$  \hspace{1cm} (3.24)

is the complex torus parametrising background flat connections $z_f$ modulo gauge transformations $z_f \rightarrow z_f + (\nu_f + \tau \mu_f)$.

The Berry connection is enhanced to a solution of the generalised $U(N)$ Bogomolny equations on $t_f \times E_{T_f}$, which is perhaps best described as a rank-$N$ hyper-holomorphic connection on $t_f \times t_f \times E_{T_f}$ that is invariant under translations in the additional $t_f$ direction.

Concretely, this involves a pair $(A, \Phi)$ consisting of

- a connection $A$ on a principal $U(N)$ bundle $P$ on $t_f \times E_{T_f}$,
- a $t_f^\vee$-valued section $\Phi$ of $\text{Ad}(P)$, which arises from the components of the hyper-holomorphic connection in the additional directions.

The asymptotic behaviour in the non-compact $t_f$-directions is that of a generalised doubly-periodic abelian monopole whose charges are controlled by the effective supersymmetric Chern-Simons couplings [82]. In an asymptotic region $|x_f| \rightarrow \infty$ in some chamber $\mathcal{C}_f$, $P$ splits as a direct sum of principal $U(1)$ bundles $P_\alpha$ and the solution is abelian $(A_\alpha, \Phi_\alpha)$ with leading growth

$$\Phi_\alpha \rightarrow \frac{2\pi}{\tau_2} K_\alpha(x_f) + \cdots,$$  \hspace{1cm} (3.25)

where we regard the effective $\mathcal{N} = 2$ supersymmetric Chern-Simons couplings in the chamber $\mathcal{C}_f$ as a linear map $K_\alpha : \Gamma_f \rightarrow \Gamma_f^\vee$. In particular, contracting with the real parameters shows that

$$x_f \cdot \Phi_\alpha \rightarrow C_\alpha + \cdots,$$  \hspace{1cm} (3.26)

where $C_\alpha = K_\alpha(x_f, x_f)$ is the vector central charge function in section 2.5. This type of boundary condition when $\text{rk} T_f = 1$ was introduced in the construction of doubly-periodic monopole solutions in [166–168].

Let us fix parameters $(\zeta, z_C)$ and focus on the supersymmetric Berry connection for the remaining parameters $(x, z) \in t \times E_T$. The analysis of supersymmetric ground states in the previous section indicates the supersymmetric Berry connection will have important features at real co-dimension three loci $S_\lambda \subset t \times E_T$ labelled by tangent
weights $\lambda \in \Phi_\alpha$ and defined by
\[
\begin{align*}
\lambda \cdot m &= 0, \\
\lambda \cdot z &\in \mathbb{Z} + \tau \mathbb{Z}.
\end{align*}
\] (3.27)

Based on the considerations of the previous section and the explicit form of the supersymmetric Berry connections for supersymmetric quantum mechanics with targets $\Sigma_\lambda \cong \mathbb{C}\mathbb{P}^1$ and $\Sigma_\lambda \cong \mathbb{C}$, we expect the following behaviour:

- If $\lambda \in \Phi_\alpha$ corresponds to an external edge of the GKM diagram, there is a singular 't Hooft monopole configuration centred on $S_\lambda$, where the spectrum of the supersymmetric quantum mechanics fails to be gapped.

- If $\lambda \in \Phi_\alpha \cap (-\Phi_\beta)$ corresponds to an internal edge of the GKM diagram, there is a smooth $SU(2)$ monopole configuration centred on $S_\lambda$, which mixes the supersymmetric ground states associated to $\alpha, \beta$.

These loci can intersect in higher co-dimension leading to more intricate configurations, for example smooth $SU(k)$ monopole configurations with $1 < k \leq N$.

We will not attempt a full analysis of the Berry connection and its connection to doubly-periodic monopoles here. Instead, we will focus on a particular algebraic construction that makes direct contact with equivariant elliptic cohomology.

### 3.4.1 Spectral Data

Let us now consider the supersymmetric Berry connection using the picture of supersymmetric ground states as elements of the cohomology of the supercharge $Q$. We consider the real and complex parameters in turn:

- The supercharge depends on the real parameters $x_f$ such that

\[
\partial_{x_f} Q = -[\Phi, Q]
\] (3.28)

where $\Phi \in \mathfrak{t}_f^\vee$ are hermitian operators independent of $x_f$. They play the role analogous to a moment map for the symmetry $T_f$ in the supersymmetric quantum mechanics. This descends to a complexified flat connection $\mathcal{D}_{x_f} = D_{x_f} + \Phi$ for supersymmetric ground states along $t_f$.

- The supercharge $Q$ depends holomorphically on the background connection $z_f$, so the anti-holomorphic derivative commutes with $Q$ and descends to a holomorphic Berry connection $\mathcal{D}_{\bar{z}_f}$ on supersymmetric ground states along $E_{T_f}$.
In the language of [169], this is consistent with the effective quantum mechanics being of BAA-type. The Berry connections commute,

\[ [\mathcal{D}_{z_f}, \mathcal{D}_{x_f}] = 0, \tag{3.29} \]

which form part of the generalised Bogomolny equations. Thus the Berry connection \( \mathcal{D}_{z_f} \) determines the structure of a rank \( N \) holomorphic vector bundle \( \mathcal{E} \) on each slice \( \{x_f\} \times E_{T_f} \), that varies in a covariantly constant way with the mass parameters \( x_f \).

The asymptotic boundary conditions imply that for parameters \( x_f \) in a given chamber \( \mathcal{C}_f \subset t_f \) in the space of mass and FI parameters the holomorphic vector bundle admits a holomorphic filtration

\[ 0 \subset \mathcal{E}_{\alpha_1} \subset \mathcal{E}_{\alpha_2} \subset \cdots \subset \mathcal{E}_{\alpha_N} = \mathcal{E}, \tag{3.30} \]

where \( \mathcal{E}_{\alpha_i} \) is a rank \( i \) holomorphic subbundle labelled by a vacuum \( \alpha_i \), generated by holomorphic sections of \( \mathcal{E} \) with a decay rate fixed by \( C_{\alpha_i} \). This follows from [170] for the doubly periodic monopoles we consider, following classic analogous results for monopoles in \( \mathbb{R}^3 \) [171–173].

One can take the associated graded bundle:

\[ \mathcal{G}(\mathcal{E}) = \bigoplus_{\alpha} \mathcal{L}_\alpha, \tag{3.31} \]

which splits by construction as a sum of holomorphic line bundles \( \mathcal{L}_{\alpha_i} \cong \mathcal{E}_{\alpha_i}/\mathcal{E}_{\alpha_{i-1}} \). A section of \( \mathcal{L}_{\alpha} \) transforms with factor of automorphy

\[ s_\alpha(z_f + \nu_f + \tau \mu_f) = e^{-i\theta_\alpha(z_f, \mu_f)} s_\alpha(z_f) \tag{3.32} \]

where

\[ \theta_\alpha(z_f, \mu_f) = 2\pi (K_\alpha(z_f, \mu_f) + K_\alpha(\mu_f, z_f) + \tau K_\alpha(\mu_f, \mu_f)). \tag{3.33} \]

The supersymmetric ground states \( |\alpha\rangle_{\mathcal{E}} \) introduced above will transform as sections of the holomorphic line bundles \( \mathcal{L}_{\alpha} \). Note that the factor of automorphy depends on the chamber \( \mathcal{C}_f \subset t_f \) through the Chern-Simons couplings \( K_\alpha \).

There are a number of algebraic approaches to the generalised Bogomolny equations obeyed by the supersymmetric Berry connection. For example, the scattering method would study the scattering problem for \( \mathcal{D}_{x_f} \) and the associated spectral data. This would generalise the classical scattering methods [171, 174, 175] and correspond to the \( z \)-spectral data for doubly periodic monopoles [167].
3.4.2 Elliptic Cohomology Variety

Here we present an alternative spectral construction that makes direct contact with equivariant elliptic cohomology. Let us again fix parameters \((\zeta, z_C)\) and focus on the supersymmetric Berry connection for the remaining parameters \((x, z) \in t \times E_T\). We denote the chamber containing the mass parameters \(x\) by \(\mathfrak{c} \subset t\).

In place of the scattering problem for \(D_x\) across the origin of the mass parameter space, recall that our analysis of supersymmetric ground states using the infinite-dimensional supersymmetric quantum mechanics model showed that

\[
|\alpha\rangle_{\mathfrak{c}} \prod_{\lambda \in \Phi_{\alpha}} i \frac{\vartheta_1(\lambda \cdot z)}{\eta(\tau)} \rightarrow |\alpha\rangle
\]

as \(x \to 0\) in each chamber \(\mathfrak{c}\), where \(|\alpha\rangle\) denotes the common set of supersymmetric ground states at the origin of \(0 \in t\) of the space of mass parameters.

Let us first check that this is consistent with the supersymmetric Berry connection. Recall that the supersymmetric ground states \(|\alpha\rangle_{\mathfrak{c}}\) transform as sections of the holomorphic line bundles \(L_{\alpha}\) whose factors of automorphy are fixed by the Chern-Simons levels \(K_{\alpha}\). The supersymmetric ground states \(|\alpha\rangle\) will then transform as sections of holomorphic line bundles \(L'_{\alpha}\) whose factors of automorphy are shifted by the additional Jacobi theta functions

\[
\left(\prod_{\lambda \in \Phi_{\alpha}} i \frac{\vartheta_1(\lambda \cdot z)}{\eta(\tau)}\right) \cdot \prod_{\lambda \in \Phi_{\alpha}} i \frac{\vartheta_1(\lambda \cdot z)}{\eta(\tau)}.
\]

This is equivalent to shifting the supersymmetric Chern-Simons couplings as follows,

\[
K'_{\alpha} = K_{\alpha} + \frac{1}{2} \sum_{\lambda \in \Phi_{\alpha}} \lambda \otimes \lambda,
\]

\[
= \kappa_{\alpha} + \kappa_{\alpha}^C + \frac{1}{4} \sum_{\lambda \in \Phi_{\alpha}} \lambda \otimes \lambda.
\]

where in the second line we have used (2.57) for the supersymmetric Chern-Simons levels \(\kappa_{\alpha}^H, \tilde{\kappa}_{\alpha}\). Since \(\kappa_{\alpha}, \kappa_{\alpha}^C\) are independent of the chamber \(\mathfrak{c}\) for the mass parameters, the factors of automorphy of \(|\alpha\rangle\) have the same property and therefore (3.34) is consistent with the supersymmetric Berry connection.

With this observation in hand, the supersymmetric ground states at the origin \(0 \in t\) of the mass parameter space have a remarkable property:
The holomorphic line bundles $L'_\alpha, L'_\beta$ are isomorphic on restriction to the locus $\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}$ where the weight $\lambda \in \Phi_\alpha \cap (-\Phi_\beta)$ labels an internal edge of the GKM diagram of $X$.

We provide a detailed argument for this result in appendix B. Concretely, the factors of automorphy defined by the shifted Chern-Simons couplings $K'_\alpha, K'_\beta$ are equivalent (in a way made precise in the appendix) on restriction to the locus $\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}$ in the space of background flat connections $E_{T_f}$.

This means the collection of holomorphic line bundles $L'_\alpha$ on the space of background flat connections $E_{T_f}$ is equivalent to a single holomorphic line bundle on an $N$-sheeted cover

$$E_T(X) := \left( \bigsqcup_{\alpha} E^{(\alpha)}_{T_f} \right) / \Delta,$$

where:

- $E^{(\alpha)}_{T_f} \cong \Gamma_f \otimes \mathbb{Z} \ E_\alpha$ are $N$ copies of the torus of background flat connections for the full flavour symmetry $T_f = T_C \times T$ associated to the supersymmetric vacua $\alpha$.

- $\Delta$ identifies the copies $E^{(\alpha)}_{T_f}$ and $E^{(\beta)}_{T_f}$ at points $\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}$ where $\lambda \in \Phi_\alpha \cap (-\Phi_\beta)$ labels an internal edge of the GKM diagram.

This is the extended $T$-equivariant elliptic cohomology variety$^1$ of $X$ [62–65], whose construction is reviewed in appendix A. Note that copies of the space of flat connections are only identified along the components parametrising flat connections for the non-topological flavour symmetry $T \subset T_f$. It is therefore sometimes convenient to consider the non-extended equivariant elliptic cohomology variety by removing the factors of $E_{T_C}$, which is denoted by $\text{Ell}_T(X)$.

More generally, on a generic face of the hyperplane arrangement in the space of mass parameters, the holomorphic line bundles associated to supersymmetric ground states combine to a section of a line bundle on the equivariant elliptic cohomology variety

$$E_T(X^{T_m})$$

where the $X^{T_m}$ is the fixed locus of the symmetry $T_m \subset T$ generated by the mass parameters $m$. In particular, if $m$ lies in a chamber of the hyperplane arrangement and $X^{T_m} = \{\alpha\}$, then $E_T(\{\alpha\})$ consists of $N$ independent copies of $E_T$ without identifications.

$^1$Note that in general, the elliptic cohomology of a variety $X$ is a scheme. However, we assume the GKM property which implies that $E_T(X)$ is in fact a variety [70].
We could regard the collection of varieties $E_T(X^{T^m})$ as $m \in \mathfrak{t}$ varies over the space of mass parameters together with the holomorphic line bundles generated by supersymmetric ground states as as a kind of spectral data for the supersymmetric Berry connection on $\mathfrak{t} \times E_T$. It would be interesting to pin down its relation to the usual spectral data associated to the generalised Bogomolny equations, for example using the scattering method.

Let us compare to the mathematical construction of the equivariant elliptic cohomology variety, reviewed in appendix A. The fact that $E_T(X)$ is locally modelled, i.e. being identical on a generic fibre (A.13), by the normal equivariant cohomology variety is very natural from the physical point of view. The equivariant cohomology describes the ground states of a purely 1d sigma model to the $X$, where the equivariant parameters are complex masses, which are $\mathfrak{t}_C$-valued, instead of our holonomies $z$. At a generic value of $z$, our cohomological arguments show that we simply obtain $N$ ground states in 1-1 with the massive vacua, identically to what we would have obtained by analysing the analogous 1d sigma model at a generic value of the complex masses. The key difference in 3d lies in the global behaviour of these ground states as we perform large background gauge transformations in $E_T$, which captures the full parameter dependence of the states, i.e. the Berry connection.

3.4.3 Gauge Theory Picture

There is an another description of the elliptic cohomology variety from the perspective of supersymmetric gauge theory, without passing to a sigma model on $X$.

We first imagine the un-gauged theory with target space $T^*R$ and regard $G$ as an additional flavour symmetry with real mass parameter $\sigma$ and background flat connection specified by $u$. In this case, the parameter space of background flat connections is

$$E_{T_f} \times E_G,$$

where

$$E_G = (E_{\tau} \otimes_{\mathbb{R}} \mathfrak{h})/W$$

parametrises background flat connections for $G$. The coordinates on the latter are Weyl-invariant functions of the coordinates $u$. For generic mass parameters and flat connections there is a single supersymmetric ground state corresponding to the fixed point at the origin of $T^*R$. The elliptic cohomology variety as constructed above is $E_{T_f} \times E_G$. 
If we now fix a generic FI parameter $\zeta$ and gauge the symmetry $G$, recall that there are $N$ supersymmetric vacua $\alpha$ in flat space labelled by sets of weights $\{\varrho_1, \ldots, \varrho_r\}$ of $G \times T$ satisfying conditions in section 2.3.1. This fixes the components of the real vector multiplet scalar in a supersymmetric vacuum $\alpha$ via the equations

$$\rho_a \cdot \sigma + \rho_{H,a} \cdot x = 0 \quad (3.41)$$

for $a = 1, \ldots, r$. Similarly, in the effective supersymmetric quantum mechanics on $\mathbb{R} \times E\tau$ this fixes the gauge holonomy in a supersymmetric ground state, up to gauge transformations, via

$$\rho_a \cdot u + \rho_{H,a} \cdot z_H + \rho_{t,a} \cdot z_t \in \mathbb{Z} + \tau\mathbb{Z}. \quad (3.42)$$

The set of $N$ solutions modulo gauge transformations, $u_\alpha(z)$, generate an $N$-sheeted cover of $E_T$ as the background flat connections $z$ is varied. Trivially including the flat connection for the topological symmetry, this becomes an $N$-sheeted cover of $E_{T_f}$. This gives a construction of the extended equivariant elliptic cohomology variety

$$c : E_T(X) \hookrightarrow E_{T_f} \times E_G. \quad (3.43)$$

In this construction, the coordinates $u$ are identified with the (logarithm of the) elliptic Chern roots. This matches the statement of Kirwan surjectivity for equivariant elliptic cohomology (A.21). This perspective on the elliptic cohomology variety will be useful in our discussion of Dirichlet boundary conditions in section 3.9.

### 3.4.4 Example

Let us consider supersymmetric QED with $N$ flavours in the default chamber $\mathcal{C}_C = \{\zeta > 0\}$. There are background flat connections $z_f = (z_C, z_1, \ldots, z_N, z_t)$ for the flavour symmetries $T_f = T_C \times T_H \times T_t$. They are subject to $\sum_\alpha z_\alpha \in \mathbb{Z} + \tau\mathbb{Z}$.

The supersymmetric ground states $|\alpha\rangle$ transform under background gauge transformations with factors of automorphy determined by the shifted levels

$$K'_\alpha = -e_\alpha \otimes e_C + e_t \otimes e_C$$

$$+ \frac{1}{4} \sum_{\gamma \neq \alpha} (e_\gamma - e_\alpha) \otimes (e_\gamma - e_\alpha)$$

$$+ \frac{1}{4} \sum_{\gamma \neq \alpha} (-2e_t - e_\gamma + e_\alpha)(-2e_t - e_\gamma + e_\alpha) \quad (3.44)$$
3.5 Boundary Conditions

If we restrict to background flat connections with \( z_\alpha - z_\beta \in \mathbb{Z} + \tau \mathbb{Z} \), it is straightforward to check that \( \theta_\alpha - \theta_\beta \in \mathbb{Z} \) and therefore the factors of automorphy of the supersymmetric ground states \(|\alpha\rangle\) and \(|\beta\rangle\) coincide.

The spectral curve \( E_T(X) \) is therefore constructed from \( N \) identical copies \( E^{(\alpha)}_{T_f} \) of the space of background flat connections parameterised by \( z_f = (z_C, z_1, \ldots, z_N, z_t) \). The copies \( E^{(\alpha)}_{T_f}, E^{(\beta)}_{T_f} \) are identified along the loci \( z_\alpha - z_\beta \in \mathbb{Z} + \tau \mathbb{Z} \) for all distinct pairs, say \( \alpha < \beta \).

3.5 Boundary Conditions

We now consider boundary conditions preserving at least \( \mathcal{N} = (0, 2) \) supersymmetry. Some aspects of such boundary conditions in section 2.6, see therein for further details and references.

To a boundary condition \( B \) preserving the flavour symmetry \( T_f \), we will associate a state \(|B\rangle\) in the supersymmetric quantum mechanics on \( \mathbb{R} \times E_\tau \) studied earlier in this chapter. We will study the boundary amplitudes \( \langle B|\alpha\rangle \) formed from the overlap with supersymmetric ground states and how they transform under large background gauge transformations on \( E_\tau \) according to the boundary 't Hooft anomalies of \( B \). We show that if a boundary condition is compatible with real mass parameters \( m \), the collection of boundary amplitudes assemble into a section of a holomorphic line bundle on the elliptic cohomology variety \( \text{Ell}_{T_f}(X^{T_f}) \), focusing on the cases where the mass parameters are zero or generic. In this way, we associate equivariant elliptic cohomology classes to boundary conditions.

3.5.1 Assumptions

We consider boundary conditions preserving at least 2d \( \mathcal{N} = (0, 2) \) supersymmetry in the \( x^{1,2} \)-plane, generated by \( Q^+_+ , Q^-_- \). In many cases, they will preserve a larger \( \mathcal{N} = (2, 2) \) supersymmetry generated by \( Q^+_+ , Q^-_+, Q^+_-, Q^-_- \). All such boundary conditions preserve the combination \( Q = Q^+_+ + Q^-_- \) and are compatible with cohomological construction of supersymmetric ground states introduced in section 3.3.

For \( \mathcal{N} = (2, 2) \) boundary conditions, we require that they preserve a boundary vector and axial R-symmetry \( U(1)_V \times U(1)_A \) and at least a boundary flavour symmetry \( T_H \times T_C \). For \( \mathcal{N} = (0, 2) \) boundary conditions, we require a boundary R-symmetry and at least a boundary flavour symmetry \( T_f = T_C \times T_H \times T_L \). In the case of an \( \mathcal{N} = (2, 2) \)
boundary condition,

\[ T_L := U(1)_V - U(1)_A, \]  

which is twice the left-moving R-symmetry.

The boundary vector and axial R-symmetry may be the same as the bulk R-symmetry \( U(1)_H \times U(1)_C \) and hence \( T_L = T_t \). However, as described in section 2.6.2, the bulk R-symmetry may be spontaneously broken at the boundary, but a linear combination of the bulk R-symmetries and flavour symmetries is preserved and becomes \( U(1)_V \times U(1)_A \). When this happens, we will draw attention to this distinction, but will abuse notation and still denote \( T_f = T_C \times T_H \times T_L \).

We will also encounter boundary conditions where a mixed gauge-flavour anomaly breaks some subgroup of the above symmetries, and deal with this subtlety as it arises in the article. We will see this problem is immaterial when passing to the boundary amplitudes associated to the boundary condition.

A boundary condition preserving \( T_H \times T_C \) may or may not be compatible with turning on the associated real mass and FI parameters. In this section, we exclusively set \( \epsilon = 0 \) and denote the chambers of the hyperplane arrangements for the remaining FI and mass parameters \( \zeta, m \) by \( \mathcal{C}_C, \mathcal{C}_H \). If a boundary condition is compatible with FI and mass parameters on a face of the hyperplane arrangement, it will be compatible with all such parameters on that face. We then say the boundary condition is compatible with that face. We mostly consider boundary conditions compatible with mass and FI parameters in given chambers of the hyperplane arrangements.

Higgs Branch Image

An important characteristic of a boundary condition is the Higgs branch image, which is a rough description of the boundary condition in the regime where the bulk gauge theory flows to a sigma model on \( X \). A \( \mathcal{N} = (0, 2) \) boundary condition satisfying the conditions above has support on a Kähler sub-manifold in \( X \) invariant under \( T \). For a \( \mathcal{N} = (2, 2) \) boundary condition, the additional supersymmetry ensures the support is a holomorphic Lagrangian in \( X \) [54, 55].

The compatibility with FI and mass parameters can be neatly understood from this perspective. First, compatibility with FI parameters in a fixed chamber \( \mathcal{C}_C \) is necessary for the boundary condition to preserve supersymmetry and define a reasonable boundary condition in a regime where the bulk gauge theory flows to a sigma model to \( X \). Second, compatibility with mass parameters in a chamber \( \mathcal{C}_H \) requires that

- a right boundary condition on \( x^3 \leq 0 \) has support \( S \subset \bigcup_{\alpha} X^-_\alpha \),
• a left boundary condition on \( x^3 \geq 0 \) has support \( S \subset \bigcup_\alpha X^+_\alpha \).

where \( X^+_\alpha \) denotes the attracting and repelling sets of the critical point \( \alpha \) generated by a positive gradient flow for the moment map \( h_m : X \to \mathbb{R} \) for all mass parameters \( m \in \mathcal{C}_H \).

The origin of the latter characterisation is that the BPS equations for the supercharges \( Q_+^+, Q_-^- \) generating the \( \mathcal{N} = (0, 2) \) supersymmetry algebra are inverse gradient flow for the moment map \( h_m \) in the \( x^3 \)-direction [56]. With our conventions, the moment map \( h_m \) decreases as \( x^3 \to \infty \), and increases as \( x^3 \to -\infty \).

### Anomalies

Boundary conditions are subject to mixed ’t Hooft anomalies for the R-symmetries \( U(1)_V, U(1)_A \) and flavour symmetries \( T_C, T_H \), which are of paramount important in the presence of background connections on \( E_\tau \). They may also suffer from gauge anomalies.

We keep track of boundary ’t Hooft anomalies using an anomaly polynomial [117], using conventions outlined in section 2.6. The anomaly polynomial of an \( \mathcal{N} = (0, 2) \) boundary condition is bilinear in the curvatures \( f_V, f_A, f_C, f_H \), associated to \( U(1)_V, U(1)_A, T_C, T_H \). If the boundary condition preserves a boundary gauge symmetry, it may also depend on an associated curvature \( f \).

The computation of boundary amplitudes on \( E_\tau \) will yield elliptic genera that involve a background for the left-moving \( R \) symmetry \( T_L \). They will therefore only detect boundary anomalies of \( T_L \), rather than those of the vector and axial R-symmetries separately. With this in mind, we only turn on a field strength \( f_L \), which may be implemented in the anomaly polynomial by substituting \( f_V \sim f_L \) and \( f_A \sim -f_L \).

We therefore consider boundary anomalies for \( T_C \times T_H \times T_L \). We will later encounter Neumann boundary conditions where the gauge anomaly does not vanish, and also Dirichlet boundary conditions where the gauge symmetry becomes a boundary flavour symmetry, which we treat as they arise. Putting aside these cases for now, the boundary anomaly polynomial of an \( \mathcal{N} = (0, 2) \) boundary condition takes the form

\[
\mathcal{P} = K(f_f, f_f) \tag{3.46}
\]

where we have introduced a shorthand notation \( f_f = (f_C, f_H, f_L) \) and \( K : \Gamma_f \times \Gamma_f \to \mathbb{Z} \) is a pairing on the co-character lattice of the boundary flavour symmetry \( T_f = T_C \times T_H \times T_L \). For an \( \mathcal{N} = (2, 2) \) boundary condition, the anomaly polynomial specialises to

\[
\mathcal{P} = k(f_H, f_C) + k^C(f_L, f_C) + k^H(f_H, f_L) + \tilde{k}(f_L, f_L), \tag{3.47}
\]
where the coefficients $k, k^C, k^H, \tilde{k}$ are pairings with the same structure as the supersymmetric Chern-Simons terms $\kappa_\alpha, \kappa^C_\alpha, \kappa^H_\alpha, \tilde{\kappa}_\alpha$ in section 2.5.

For convenience, we define $\mathcal{P}_\alpha$ to be the polynomial above with pairings set to the corresponding supersymmetric Chern-Simons terms, encoding the contribution to the boundary anomaly from anomaly inflow from a massive vacuum $\alpha$ at $x^3 \to +\infty$. Thus $-\mathcal{P}_\alpha$ encodes the anomaly inflow from the vacuum at $x^3 \to -\infty$.

### 3.6 Boundary Amplitudes & Cohomology Classes

A boundary condition preserving at least 2d $\mathcal{N} = (0, 2)$ supersymmetry in the $x^{1,2}$-plane shares a common pair of supercharges $Q^{++}_+, Q^{-+}_+$ with the 1d $\mathcal{N} = (2, 2)$ subalgebra along $x^3$ annihilating supersymmetric ground states on $E_\tau$. In particular, the boundary condition preserves the combination $Q = Q^{++}_+ + Q^{+-}_+$, whose cohomology we use to compute supersymmetric ground states.

To a right or left $\mathcal{N} = (0, 2)$ boundary condition $B$, we can therefore associate boundary state $|B\rangle$ or $\langle B|$ respectively in the effective supersymmetric quantum mechanics on $\mathbb{R} \times E_\tau$. The overlaps of boundary states with supersymmetric ground states associated to vacua $\alpha$ are known as boundary amplitudes. Boundary amplitudes can be represented as a path integral on $E_\tau \times \mathbb{R}_{\leq 0}$ or $E_\tau \times \mathbb{R}_{\geq 0}$ with the boundary condition at $x^3 = 0$ and and the vacuum $\alpha$ at $x^3 \to -\infty$ or $x^3 \to +\infty$. This is illustrated in figure 3.3.

![Fig. 3.3 Boundary amplitudes for left and right boundary conditions](image)

The presence of the vacuum $\alpha$ at infinity breaks the gauge symmetry of the theory on $E_\tau \times \mathbb{R}_{\geq 0}$ or $E_\tau \times \mathbb{R}_{\leq 0}$. One is led to consider boundary ’t Hooft anomalies and anomaly inflow from the remaining $T_f = T_C \times T_H \times T_L$ flavour symmetry.\(^2\) This may be computed by making the analogous substitutions to (2.20) in the boundary anomaly polynomial for the boundary condition (if the boundary condition supports a boundary

\(^2\)Note that strictly speaking there is the usual subtlety in that the $T_H$ and $U(1)_V$ appearing here are actually the unbroken symmetries $T_H^{(\alpha)}$ and $U(1)_H^{(\alpha)}$ discussed in section 2.5.1.
3.6 Boundary Amplitudes & Cohomology Classes

gauge symmetry), replacing \((\sigma, m, \epsilon)\) with \((f, f_H, f_t)\), and then adding the anomaly polynomial \(\mathcal{P}_\alpha\) encoding anomaly inflow from \(\alpha\).

The boundary amplitudes can be regarded as the elliptic genera on \(E_\tau\) of effective 2d \(\mathcal{N} = (0, 2)\) or \(\mathcal{N} = (2, 2)\) theories obtained by reduction on a half-line. The mixed anomalies of the effective theory are simply the sum of the boundary mixed \(\text{’t} \text{Hooft}\) anomalies and the anomaly inflow from the supersymmetric Chern-Simons terms associated to the isolated massive vacua. As they are elliptic genera, the amplitudes therefore transform as sections of holomorphic line bundles on the torus \(E_{\Gamma_f}\) of background flat connections \(z_f = (z_C, z_H, z_t)\), whose quasi-periodicities are fixed by this sum \([176]\).

In section 4.1, we will compute boundary amplitudes using supersymmetric localisation on \(E_\tau \times I\) where \(I\) is a finite interval, replacing the vacuum at infinity by a distinguished class of boundary conditions at finite distance that generate states in the same \(Q\)-cohomology class. The anomaly inflow from supersymmetric Chern-Simons terms is reproduced by the boundary \(\text{’t} \text{Hooft}\) anomalies of these boundary conditions.

The boundary amplitudes transform under large gauge transformations as follows,

\[
f_\alpha(z_f + \nu_f + \tau \mu_f) = (-1)^{\ell_\alpha(\mu_f + \nu_f)} e^{-i\theta_\alpha(\mu_f, z_f)} f_\alpha(z_f), \tag{3.48}\]

where

\[
\theta_\alpha(\mu_f, z_f) = 2\pi \left( k_\alpha(\mu_f, z_f) + k_\alpha(z_f, \mu_f) + \tau k_\alpha(\mu_f, \mu_f) \right) \tag{3.49}
\]

and \(k_\alpha\) is the total boundary mixed \(\text{’t} \text{Hooft}\) anomaly from the boundary condition and anomaly inflow from the vacuum \(\alpha\).

The contribution \(\ell_\alpha : \Gamma_f \to \mathbb{Z}\) is known as the linear anomaly \([176]\). We expect that with a careful identification of the \(\mathbb{Z}_2\) fermion number in the elliptic genus with an R-symmetry whose background flat connection implements R-R boundary conditions, the linear anomaly may be considered as a mixed anomaly between flavour and R-symmetry, and placed on the same footing as \(\theta_\alpha\). In this work, we follow the conventions of \([177, 178]\) for \(\mathcal{N} = (0, 2)\) boundary conditions and give a concrete geometric description of the linear anomaly for the amplitudes we consider.

To write down expressions for boundary amplitudes, it is convenient to view the elliptic curve as \(E_\tau \cong \mathbb{C}^\times / q^\mathbb{Z}\) with \(q = e^{2\pi i \tau}\) and write the background flat connections as\(^3\)

\[
\xi = e^{2\pi i z_C}, \quad \nu = e^{2\pi i z_H}, \quad t = e^{2\pi i z_t}, \tag{3.50}
\]

\(^3\)In the remainder of this article, if the bulk T\(_L\) symmetry is re-defined to a boundary T\(_L\), we will still use \(z_t\) and \(t\) to denote the (exponentiated) background holonomies for T\(_L\), with this understanding implicit.
or collectively \( a_f = e^{2\pi i z_f} \) and \( a = e^{2\pi i z} \), where recall \( z_f = (z_C, z_H, z_t) \) and \( z = (z_H, z_t) \).

In this notation, the quasi-periodicities of boundary amplitudes becomes

\[
\begin{align*}
\hat{f}_{\alpha}(e^{2\pi i \nu_f} a_f) &= (-1)^{f_{\alpha}(\nu_f)} f_{\alpha}(a_f), \\
\hat{f}_{\alpha}(q^{\mu_f} a_f) &= (-1)^{f_{\alpha}(\mu_f)} q^{-k_{\alpha}(\mu_f)} a_f^{-2k_{\alpha}(\nu_f)} f_{\alpha}(a_f).
\end{align*}
\]

(3.51)

It is useful to define the normalised theta function,

\[
\vartheta(a) := i \frac{\vartheta_1(z; \tau)}{\eta(\tau)} = -q^{1/2} a^{-1/2} (a;q)_{\infty} (qa^{-1};q)_{\infty},
\]

(3.52)

where \( \vartheta_1(z; \tau) \) is the Jacobi theta function, \( \eta(\tau) \) is the Dedekind eta function and

\[
(a; q) = \prod_{k=0}^{\infty} (1 - aq^k)
\]

(3.53)

is the \( q \)-Pochhammer symbol. It transforms under large background gauge transformations as \( \vartheta(qx) = -x^{-1}q^{-1/2} \vartheta(x) \). This combination appears naturally in the computation of the elliptic genera of 2d \( \mathcal{N} = (0, 2) \) supersymmetric gauge theories [177, 178] and, up to a factor of \( q^{1/2} \) to ensure modularity, it is the same combination used in reference [67]. See [179] for more on how line bundles on elliptic curves can be described by factors of automorphy, and their sections by theta functions.

As in the discussion of supersymmetric ground states in section 3.3, our construction of boundary amplitudes will depend on the real mass parameters \( m \). We consider the cases of generic mass parameters and zero mass parameters in turn.

### 3.6.1 Generic Masses

First consider generic mass parameters in some chamber, \( m \in \mathcal{C}_H \). Recall from section 3.3 that the supersymmetric ground states \( |\alpha\rangle_{\mathcal{C}} \) and \( \langle \alpha| \) are defined by placing a massive supersymmetric vacuum at \( x^3 \to +\infty \) and \( x^3 \to -\infty \).

For a boundary condition \( B \) that is compatible with mass parameters in the chamber \( \mathcal{C}_H \), the boundary amplitudes are then defined as follows.

- The boundary amplitude \( \epsilon \langle \alpha|B \rangle \) is defined by the path integral on \( E_{\tau} \times \mathbb{R}_{\leq 0} \) with a right boundary condition \( B \) at \( \tau = 0 \) and the massive vacuum \( \alpha \) at \( \tau \to -\infty \).
- The boundary amplitude \( \langle B|\alpha\rangle_{\epsilon} \) is defined by the path integral on \( E_{\tau} \times \mathbb{R}_{\geq 0} \) with a left boundary condition \( B \) at \( \tau = 0 \) and the massive vacuum \( \alpha \) at \( \tau \to +\infty \).
The boundary amplitudes transform as in (3.51) with
\[ k^\pm_\alpha := k_B \pm K_\alpha, \]  
(3.54)
where \( k_B \) is the mixed 't Hooft anomaly of the boundary condition \( B \). If \( B \) suffers from a gauge anomaly, by \( k_B \) we mean the anomalies in the unbroken flavour symmetries in the vacuum \( \alpha \) as discussed at the beginning of section 3.6. The \( \pm \) sign is for the vacuum at \( \pm \infty \).

The contribution to the linear anomaly is more tricky to pin down. In section 4.1, we will formulate boundary amplitudes as elliptic genera of effective 2d \( \mathcal{N} = (0,2) \) theories obtained by reduction on an interval. If these only involve standard \( \mathcal{N} = (2,2) \) multiplets, the linear anomaly is determined by the difference between the sum of the \( T \) weights of chiral multiplets and Fermi multiplets that contribute to the boundary amplitude. Since the linear anomaly is only defined mod 2, we have the relation
\[ \ell^\pm_\alpha \otimes e_t = -\tilde{k}^\pm_\alpha. \]  
(3.55)
We will also see that equation (3.55) is true for the more exotic periodic boundary matter considered in sections 4.2 and 4.3.

### 3.6.2 Vanishing Mass Parameters

We now consider boundary amplitudes obtained from the overlaps with supersymmetric ground states \( |\alpha\rangle \) and \( \langle \alpha| \) appropriate for vanishing mass parameters.

Such boundary amplitudes may be computed even if the boundary condition \( B \) is incompatible with turning on mass parameters. However, if the boundary condition \( B \) is compatible with mass parameters in the chamber \( \mathcal{C}_H \), using the relationship between supersymmetric ground states (3.21) we have
\[ \langle \alpha|B \rangle = \varepsilon \langle \alpha|B \rangle \times \prod_{\lambda \in \Phi^+} \vartheta (a^\lambda), \]  
(3.56)
\[ \langle B|\alpha \rangle = \langle B|\alpha \rangle \varepsilon \times \prod_{\lambda \in \Phi^-} \vartheta (a^\lambda). \]

They transform as in equation (3.51) with
\[ k^\pm_\alpha := k_B \pm (\kappa_\alpha + \kappa^C_\alpha) + \frac{1}{4} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda, \]  
(3.57)
using the form of $\kappa^H_\alpha + \tilde{\kappa}$ in equation (2.57) and the additional contribution from the normalisation in (3.56). As in our discussion of supersymmetric ground states, an important compatibility condition is that this is independent of the chamber $C_H$. The normalisation also modifies the linear anomaly of the boundary amplitudes to

$$
\ell^\pm_\alpha = -\tilde{k}_B + \sum_{\lambda \in \Phi^+_\alpha} \lambda_H \pm \frac{1}{2} \sum_{\lambda \in \Phi_\alpha} \lambda, \quad (3.58)
$$

which is again independent of the chamber $C_H$ due to the symplectic pairing of weights. Additionally, both $\ell^\pm_\alpha$ give equivalent factors of automorphy due to the symplectic pairing and the fact that the linear anomaly is sensitive only to parity.

The factors of automorphy of the boundary amplitudes are now independent of the chamber for the mass parameters. Additionally, on the loci

$$
\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z} \quad (3.59)
$$

where $\lambda \in \Phi_\alpha \cap (-\Phi_\beta)$ is any internal edge of the GKM diagram of $X$, $k_\alpha$ and $k_\beta$ define isomorphic line bundles (see appendix B). Following the discussion in section 3.4.2, this implies that the boundary amplitudes transform as a section of a holomorphic line bundle on the spectral curve $E_T(X)$.

The upshot is that:

**Proposition 1** The collection of boundary amplitudes $\{\langle \alpha | B \rangle\}$ of a given boundary condition $B$ therefore represent a class in the $T$-equivariant elliptic cohomology of $X$.

### 3.6.3 Lagrangian Branes

Suppose we have a left or right $\mathcal{N} = (2,2)$ boundary condition $B$ that flows to a Lagrangian boundary condition $L \subset X$ in the sigma model to $X$.

First, suppose that the boundary condition is compatible with the mass parameters in some chamber $C_H$. This means concretely that $L \subset \bigcup_\alpha X^\pm_\alpha$ for a left / right boundary condition. Then we propose that the boundary amplitudes with mass parameters

---

\(^4\)Note that in section 3.4 we ignored the contribution of the linear anomaly for the sake of brevity. However it is easy to check that the factors of automorphy arising from the linear anomaly coincide on the loci $\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}$ in the sense of appendix B.
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turned on are given by

\[
\begin{align*}
&\langle \alpha | B \rangle = \prod_{\lambda \in \Phi_+^\alpha (L)} \frac{\theta (a^{\lambda^*})}{\theta (a^{\lambda})} = \prod_{\lambda \in \Phi_+^\alpha (L)} \frac{\theta (t^{2-\lambda} u^{-\lambda} v^{-\lambda})}{\theta (t^{\lambda} v^{\lambda})}, \\
&\langle B | \alpha \rangle = \prod_{\lambda \in \Phi_+^\alpha (L)} \frac{\theta (a^{\lambda})}{\theta (a^{\lambda^*})} = \prod_{\lambda \in \Phi_+^\alpha (L)} \frac{\theta (t^{2-\lambda} u^{-\lambda} v^{-\lambda})}{\theta (t^{\lambda} v^{\lambda})},
\end{align*}
\]

(3.60)

where \( \Phi_+^\alpha (L) \subset \Phi_+^\alpha \) denotes the weights of the tangent space \( T_\alpha L \subset T_\alpha X^\pm \). This is the elliptic genus of the \( \mathcal{N} = (2, 2) \) chiral multiplets parametrising the fluctuations in \( T_\alpha L \).

This will be derived using supersymmetric localisation in section 4.1, by introducing boundary conditions that generate boundary states in the same \( Q \) cohomology classes as the supersymmetric ground states \( |\alpha\rangle_C \).

When the mass parameters are set to zero, we instead consider the overlaps with the supersymmetric ground states \( |\alpha\rangle \). The right boundary amplitude may be computed from the above result as follows,

\[
\begin{align*}
\langle B | \alpha \rangle &= \langle B | \alpha \rangle_C \prod_{\lambda \in \Phi_+^\alpha (L)} \theta (a^{\lambda}) \\
&= \prod_{\lambda \in \Phi_+^\alpha (L)} \theta (a^{\lambda}) \prod_{\lambda \in \Phi_+^\alpha (L)^\perp} \theta (a^{\lambda}) \\
&= \prod_{\lambda \in \Phi_+^\alpha (L)^\perp} \theta (a^{\lambda}) \prod_{\lambda \in \Phi_+^\alpha (L)^\perp} \theta (a^{\lambda}) \\
&= \prod_{\lambda \in \Phi_+^\alpha (L)^\perp} \theta (a^{\lambda})
\end{align*}
\]

(3.61)

where \( \Phi_+^\alpha (L)^\perp \) denotes the complement of \( \Phi_+^\alpha (L) \subset \Phi_+^\alpha \). A similar computation yields the same result for \( \langle B | \alpha \rangle \). Note again the consistency check that there is no dependence on a chamber for the mass parameters.

This boundary amplitude corresponds to the elliptic genus of \( \mathcal{N} = (0, 2) \) Fermi multiplets parametrising the normal directions to \( T_\alpha L \subset T_\alpha X \) with weights \( \Phi_+^\alpha (L)^\perp \).

The set of boundary amplitudes \( \langle \alpha | B \rangle \) represent a section of a holomorphic line bundle on \( E_T(X) \), which is the elliptic cohomology class of \( L \subset X \).

3.6.4 Example

Let us consider an example from supersymmetric QED. We fix the default chambers and consider the boundary condition \( N \), defined by a \( \mathcal{N} = (2, 2) \) Neumann boundary
condition for the vector multiplet:

\[ F_{3 \mu} = 0, \quad \sigma = 0, \quad D_3 \sigma = 0, \quad (3.62) \]

together with the hypermultiplet boundary condition

\[ D_3 X_\beta = 0, \quad Y_\beta = 0, \quad \beta = 1, \ldots, N. \quad (3.63) \]

This flows to a compact Lagrangian brane supported on \( L \cong \mathbb{C}P^{N-1} \subset X \) and is therefore compatible with any chamber for the mass parameters.

The Neumann boundary condition has a mixed ’t Hooft anomaly between the boundary gauge symmetry and the bulk \( U(1)_C \) R-symmetry and \( T_C \) flavour symmetry, encoded in a contribution to the anomaly polynomial,

\[
\begin{align*}
    f(-f_C + N f_{U(1)_C}) & \text{ for a right boundary condition,} \\
    f(+f_C + N f_{U(1)_C}) & \text{ for a left boundary condition.}
\end{align*}
\quad (3.64)
\]

Here, \( f_{U(1)_C} \) denotes the field strength for the \( U(1)_C \) R-symmetry at the boundary.

If we were to consider the boundary condition in isolation, we could define an unbroken boundary axial R-symmetry \( U(1)_A \) generated by the current \( J_A = J_{U(1)_C} \pm N T_C \), which does not suffer a mixed gauge anomaly. This would be implemented in the anomaly polynomial by setting \( f_{U(1)_C} = f_A \) and \( f_C = \pm N f_A \).

However, since we ultimately consider boundary amplitudes where the gauge symmetry is broken anyway, we instead consider the full boundary anomaly with \( U(1)_V = U(1)_H \) and \( U(1)_A = U(1)_C \). For example, for the left boundary condition we have

\[
\begin{align*}
\mathcal{P}[N] &= f \cdot f_C + f_V \cdot f_A + \sum_{\beta=1}^{N} (f - f_H^\beta) \cdot f_A \\
&= f \cdot f_C - f_L \cdot f_L - \sum_{\beta=1}^{N} (f - f_H^\beta) \cdot f_L + \ldots
\end{align*}
\quad (3.65)
\]

where in the second line we only keep track of \( \mathcal{N} = 2 \) flavour symmetries. The first term arises from anomaly inflow, the second from gauginos surviving the Neumann boundary condition for the vector multiplet, and the remaining terms from fermions in the hypermultiplets.
Let us now consider the boundary amplitudes. The spaces of positive and negative weights in the default chamber are

\[
\Phi^{-}_\alpha(L) = \{ e_\beta - e_\alpha, \beta > \alpha \},
\]

\[
\Phi^{+}_\alpha(L) = \{ e_\beta - e_\alpha, \beta < \alpha \},
\]

and therefore

\[
\varepsilon \langle N | \alpha \rangle = \prod_{\beta < \alpha} \frac{\vartheta(t^{-2}v_\alpha/v_\beta)}{\vartheta(v_\beta/v_\alpha)}, \quad \langle N | \alpha \rangle \varepsilon = \prod_{\beta > \alpha} \frac{\vartheta(t^{-2}v_\alpha/v_\beta)}{\vartheta(v_\beta/v_\alpha)}. \tag{3.67}
\]

These are the elliptic genera of the \( N = (2, 2) \) chiral multiplets corresponding to the positive and negative weight fluctuations in \( T_\alpha \mathbb{CP}^{N-1} \) respectively.

It is straightforward to check that these boundary amplitudes transform according to (3.54), where \( k_B \) is obtained via substituting \( f = f_H - f_L \) in the anomaly for \( N \). For example, for the boundary amplitude \( \langle N | \alpha \rangle \varepsilon \), the total boundary mixed \('t Hooft\) anomaly is

\[
P^+_\alpha = \mathcal{P}[N]|_{f = f_H - f_L} + \mathcal{P}_\alpha
= 2(N - \alpha)f_L^2 + 2 \sum_{\beta > \alpha} (f_H^\beta - f_H^\alpha)f_L, \tag{3.68}
\]

which reproduces the quasi-periodicity of the boundary amplitude. Note now \( f_L \) and \( f_H \) are field strengths for the symmetries \( T^{(\alpha)}_L \) and \( T^{(\alpha)}_H \) in the vacuum \( \alpha \), as defined in section 2.5.1.

The boundary amplitudes at the origin of the mass parameter space are

\[
\langle \alpha | N \rangle = \langle N | \alpha \rangle = \prod_{\beta \neq \alpha} \vartheta(t^{-2}v_\alpha v_\beta^{-1}), \tag{3.69}
\]

which is the elliptic genus of Fermi multiplets parametrising the cotangent directions at each fixed point \( \alpha \subset \mathbb{CP}^{N-1} \). They represent the elliptic cohomology class of the compact Lagrangian submanifold \( \mathbb{CP}^{N-1} \subset X \). By construction they transform according to quasi-periodicities (3.57).

### 3.7 Boundary Overlaps

The overlap \( \langle B^l | B^r \rangle \) of boundary states can be defined as the partition function on \( E_\tau \times [0, \ell] \) with R-R boundary conditions on \( E_\tau \), with a left boundary condition \( B^l \) at
\( x^3 = 0 \) and right boundary condition \( B^r \) at \( x^3 = \ell \). The computation of such partition functions has been addressed using supersymmetric localisation in [180].

The overlap of boundary conditions is independent of the length \( \ell \). This gives two ways to interpret the boundary overlap:

1. Sending \( \ell \to 0 \), it is the elliptic genus of the effective 2d \( \mathcal{N} = (0, 2) \) or \( \mathcal{N} = (2, 2) \) theory obtained by colliding the boundary conditions \( B^l, B^r \).

2. Sending \( \ell \to \infty \) and expanding in isolated massive vacua \( \alpha \) in the intermediate region, it can be expressed in terms of the boundary amplitudes

\[
\langle B^l | B^r \rangle = \sum_\alpha \langle B^l | \alpha \rangle \langle \alpha | B^r \rangle
\]

\[
= \sum_\alpha \langle B^l | \alpha \rangle \langle \alpha | B^r \rangle \prod_{\lambda \in \Phi_\alpha} \frac{1}{\vartheta(a \lambda)}.
\]

In the first line, we assume we can turn on mass parameters in a chamber \( \mathcal{C} \) compatible with both boundary conditions.

These interpretations are both compatible with the transformation properties under large background gauge transformations, namely that the boundary overlap transforms with a factor of automorphy fixed by the sum \( k_l + k_r \) of boundary anomalies from \( B^l \) and \( B^r \).

This is because the only possible 't Hooft anomalies arise from boundary chiral fermions and anomaly inflow. In colliding the boundary conditions, the contributions from anomaly inflow to the left and the right cancel out. In the decomposition into boundary amplitudes, the factor of automorphy of each term in the first line are, using (3.54)

\[
k_l + k_r = (k_l - k_\alpha) + (k_r + k_\alpha)
\]

where \( k_{l,r} \) are the boundary 't Hooft anomalies of \( B^{l,r} \). The second decomposition also has the same factors of automorphy,

\[
k_l + k_r = (k_l - \kappa_\alpha - \kappa_\alpha^C + \frac{1}{4} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda) + (k_r + \kappa_\alpha + \kappa_\alpha^C + \frac{1}{4} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda)
\]

\[
- \frac{1}{2} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda,
\]

and so all of these interpretations are compatible.

Let us finally mention an important subtlety. We have already encountered the fact that Neumann boundary conditions for the vector multiplet generically have mixed 't
Hooft anomalies for the unbroken boundary gauge symmetry. This is not problematic for boundary amplitudes as the gauge symmetry is completely broken in a massive vacuum $\alpha$ anyway. However, for overlaps $\langle B^l | B^r \rangle$ between pairs of Neumann boundary conditions, a mixed 't Hooft anomaly between a boundary gauge symmetry and flavour symmetry $T_f$ will require a specialisation of the background flat connections $z_f$ for consistency. An example is presented below.

### 3.7.1 Example

Let us continue with the example of supersymmetric QED with $N$ flavours, and compute the overlap $\langle N | N \rangle$ of the Neumann boundary condition $N$ supported on $\mathbb{C}P^{N-1} \subset X$.

In the limit $\ell \to 0$, we recover a 2d $\mathcal{N} = (2, 2)$ gauge theory with $G = U(1)$ and $N$ chiral multiplets of charge $+1$. The computation of the elliptic genus is subtle due to the mixed $G - T_L$ anomaly with coefficient $2N$. This is an example presented in [177]. The result is

$$\langle N | N \rangle = - \oint_{\Gamma} \frac{ds}{2\pi i s} \frac{\eta(\tau)^2}{\vartheta(t^{-2})} \prod_{\alpha=1}^{N} \frac{\vartheta(t s^{-1} v_\beta)}{\vartheta(t s v_\beta^{-1})},$$

where the JK contour $\Gamma$ selects poles at $s = v_\alpha t^{-1}$, and single-valuedness of the integrand requires $t^{2N} = 1$ due to the mixed anomaly.

The latter is a consequence of considering the overlap between left and right Neumann boundary conditions $N$; one cannot simultaneously make both of the redefinitions below equation (3.64) to define boundary axial $R$-symmetries with no mixed gauge anomaly. The $G - U(1)_C$ anomaly of the $N$ chiral multiplets in the limit $\ell \to 0$ is equal, as expected, to the sum of the anomalies of the left and right $N$ boundary conditions given in (3.64).

The same result is reproduced by the decomposition into boundary amplitudes given in equations (3.67) and (3.69) obtained in the opposite limit $\ell \to \infty$, but global consistency requires $t^{2N} = 1$.

### 3.8 Boundary Wavefunctions

We will introduce another decomposition by cutting the path integral using an auxiliary set of Dirichlet supersymmetric boundary conditions. This type of construction has
been used extensively in the literature on supersymmetric localisation [96, 154, 181–185] and analysed systematically in [108, 109]. We will make contact with the ‘off-shell’ form of equivariant elliptic cohomology classes, reviewed in appendix A.

There is significant freedom in the choice of auxiliary Dirichlet boundary conditions. Different choices have advantages and disadvantages, especially in how the auxiliary boundary conditions interact with mass parameters. We describe two choices of auxiliary boundary conditions that preserve $N = (2, 2)$ and $N = (0, 2)$ supersymmetry.

### 3.8.1 $\mathcal{N} = (2, 2)$ Cutting

The first method uses boundary conditions that preserve 2d $\mathcal{N} = (2, 2)$ supersymmetry and involves Dirichlet boundary conditions for the vector multiplet [56]. Specifically, we consider the following Dirichlet boundary conditions $D_\varepsilon$ (see section 2.6.2):

- An $\mathcal{N} = (2, 2)$ Dirichlet boundary condition for the 3d $\mathcal{N} = 4$ vector multiplet, where the complex scalar vanishes at the boundary $\varphi = 0$, as does the parallel component of the field strength $F_{12} = 0$.

- A $\mathcal{N} = (2, 2)$ Neumann-Dirichlet boundary condition for the hypermultiplet, specified by the polarisation or Lagrangian splitting $\varepsilon$ of the representation $Q = T^* R$. Recall that if we write $R = \mathbb{C}^N$ with polarisation denoted by a sign vector $\varepsilon \in \{\pm\}^N$ specifying

$$
(X_{\varepsilon\beta}, Y_{\varepsilon\beta}) = \begin{cases} 
(X_\beta, Y_\beta) & \text{if } \varepsilon_\beta = + \\
(Y_\beta, -X_\beta) & \text{if } \varepsilon_\beta = -
\end{cases} \quad \text{for } \beta = 1, \ldots, N. \quad (3.74)
$$

then the boundary condition specifies that

$$
D_\perp X_{\varepsilon\beta} | = 0 \quad Y_{\varepsilon\beta} | = 0. \quad (3.75)
$$

The scalars $X_{\varepsilon\beta}$ transform as bottom components of $\mathcal{N} = (2, 2)$ chiral multiplets at the boundary.

In addition to the bulk flavour symmetry $T_f$, the boundary conditions support a boundary $G_\partial$ flavour symmetry, generated by global gauge transformations at the boundary. Placing the boundary condition on $E_\tau$, in addition to the background flavour connection with holonomy $a_f = e^{2\pi i z_f}$, we may also introduce a background flat connection for the boundary $G_\partial$ symmetry with holonomy $s = e^{2\pi i u}$. We therefore denote the boundary conditions by $D_\varepsilon(s)$. 
We now cut the path integral as follows. We first impose the above Dirichlet boundary conditions on the left and right of the cut. This introduces a pair of boundary flavour symmetries $G_\partial$, $G'_\partial$. We then introduce $N$ boundary $\mathcal{N}=(2,2)$ chiral multiplets $\Phi_\beta$ coupled to the bulk hypermultiplet fields to the boundary via a (2,2) superpotential

$$W = \sum_{\beta=1}^{N} X_{\epsilon^{\beta}} |\phi_\beta - \phi_\beta'| X'_{\epsilon^{\beta}}, \quad (3.76)$$

which involves the hypermultiplet fields with Neumann boundary conditions. The boundary superpotential identifies the boundary flavour symmetries $G_\partial$, $G'_\partial$ and imposes (omitting the subscript on the polarisation)

$$Y_{\epsilon^\beta} = \frac{\partial W}{\partial X_{\epsilon^\beta}} = \phi, \quad |Y'_{\epsilon^\beta} = - \frac{\partial W}{\partial |X'_{\epsilon^\beta}|} = \phi, \quad 0 = \frac{\partial W}{\partial \phi} = X_{\epsilon^\beta} - |X'_{\epsilon^\beta}|, \quad (3.77)$$

thus identifying the hypermultiplet fields on each side. We then gauge the remaining diagonal $G_\partial$ boundary symmetry to $G$, by introducing a dynamical 2d $\mathcal{N}=(2,2)$ vector multiplet.

This leads to a decomposition of boundary overlaps

$$\langle B^{|B} \rangle = (-)^r \oint_{\text{JK}} du Z_\epsilon(s) \langle B^{|D_\epsilon(s)} \rangle \langle D_\epsilon(s)|B^r \rangle \quad (3.78)$$

where $Z_\epsilon(s)$ is the elliptic genus of the boundary vector multiplet and chiral multiplets. We refer to $\langle D_\epsilon(s)|B \rangle$ as the wavefunction of a right boundary condition $B$ and $\langle B|D_\epsilon(s) \rangle$ as the wavefunction of a left boundary condition $B$. Note that the integrand is independent of the choice of polarisation, with the dependence on $\epsilon$ in $Z_\epsilon(s)$ cancelled by the dependence of the wavefunctions.

The integral is over a real contour in the parameter space $E_G$ of flat connections and performed according to the JK residue prescription for elliptic genera [177, 178]. Provided there is no anomaly for the total boundary $G$ symmetry obtained by summing the contributions from $B^l$, $B^r$, the integrand is invariant under $s \to qs$ and there are a finite set of poles.

**Example**

For supersymmetric QED with $N$ hypermultiplets,

$$Z_\epsilon(s) = \frac{\eta(q)^2}{\vartheta(t-2)} \prod_{\beta=1}^{N} \frac{\vartheta(t \epsilon^\beta \bar{v}_\beta \epsilon^\beta \beta)}{\vartheta(t s^{-\epsilon^\beta} \bar{v}_\beta \epsilon^\beta \beta)}.$$

$$\quad (3.79)$$
Let us reconsider the normalisation of the Neumann boundary condition $N$ supported on the base $\mathbb{CP}^{N-1} \subset X$ from this perspective. For this boundary condition, it is convenient to choose the polarisation $\varepsilon = (+, \ldots, +)$. Then we also have

$$\langle N|D_\varepsilon(s)\rangle = \langle D_\varepsilon(s)|N\rangle = \prod_{\beta=1}^{N} \frac{\vartheta(ts^{-1}v_\beta)}{\vartheta(tsv_\beta^{-1})}, \quad (3.80)$$

which reproduces the elliptic genus of the $N = (2,2)$ chiral multiplets arising from the hypermultiplet fields $X_\beta$ that have Neumann boundary conditions at both boundaries. Putting these components together we find

$$\langle N|N\rangle = -\oint_{JK} du \left[ \eta(q)^2 \prod_{\beta=1}^{N} \frac{\vartheta(tsv_\beta^{-1})}{\vartheta(ts^{-1}v_\beta)} \frac{\vartheta(ts^{-1}v_\beta)}{\vartheta(tsv_\beta^{-1})} \right] = -\oint_{JK} du \left[ \eta(q)^2 \prod_{\beta=1}^{N} \frac{\vartheta(ts^{-1}v_\beta)}{\vartheta(tsv_\beta^{-1})} \right], \quad (3.81)$$

which agrees with the previous computation (3.73).

### 3.8.2 $\mathcal{N} = (0,2)$ Cutting

The $\mathcal{N} = (2,2)$ cutting has some inconvenient features. First, it depends on a choice of polarisation $\varepsilon$. Second, there may not exist a polarisation that is compatible with all the supersymmetric massive vacua $\{\alpha\}$, meaning that some of the overlaps $\langle D_\varepsilon(s)|\alpha\rangle$ break supersymmetry and vanish. Finally, the auxiliary boundary conditions $D_\varepsilon(s)$ may not be compatible with introducing mass parameters in the same chamber as a given boundary condition $B$.

To circumvent these difficulties, we consider an alternative set of auxiliary boundary conditions preserving only $\mathcal{N} = (0,2)$ supersymmetry. The additional flexibility will allow us to make more canonical choices that are compatible with all supersymmetric massive vacua and mass parameters in any chamber.

To proceed, it is instructive to decompose the 3d $\mathcal{N} = 4$ multiplets under $(0,2)$ supersymmetry in the $x^{1,2}$ plane as in section 2.6.2, which we reproduce here for convenience. Let us first note that a 3d $\mathcal{N} = 4$ vector multiplet decomposes into a 3d $\mathcal{N} = 2$ vector multiplet and an adjoint chiral multiplet with scalar component $\varphi$. A 3d $\mathcal{N} = 4$ hypermultiplet decomposes into a pair of 3d $\mathcal{N} = 2$ chiral multiplets $X$ and $Y$. They decompose further under 2d $\mathcal{N} = (0,2)$ supersymmetry as follows:
• The 3d $\mathcal{N} = 2$ vector multiplet decomposes into a $\mathcal{N} = (0, 2)$ chiral superfield $S$ containing $A_3 - i\sigma$ as its scalar component, and a $\mathcal{N} = (0, 2)$ Fermi field strength multiplet $T$, containing $F_{12}$.

• The 3d $\mathcal{N} = 2$ chiral multiplets $\varphi, X, Y$ decompose into $\mathcal{N} = (0, 2)$ chiral multiplets $\Phi_{\varphi}, \Phi_X, \Phi_Y$, and $\mathcal{N} = (0, 2)$ Fermi multiplets $\Psi_{\varphi}, \Psi_X, \Psi_Y$.

Alternatively, we could have first decomposed under $\mathcal{N} = (2, 2)$ supersymmetry, before further decomposing under $\mathcal{N} = (0, 2)$ supersymmetry. From this perspective, the above supermultiplets arise from a chiral multiplet $(S, \bar{\Psi}_{\varphi})$, a twisted chiral field strength multiplet $(\Phi_{\varphi}, T)$, and chiral multiplets $(\Phi_X, \Psi_Y)$ and $(\Phi_Y, -\Psi_X)$.

Let us now describe the auxiliary $\mathcal{N} = (0, 2)$ boundary conditions. First, we always assign a Dirichlet boundary condition for the 3d $\mathcal{N} = 2$ vector multiplet. This supports a boundary $G_3$ flavour symmetry and allows us to introduce a background flat connection with holonomy $s = e^{2\pi i u}$. We then assign a Neumann boundary condition for the $\mathcal{N} = 2$ chiral multiplet containing $\varphi$,

$$\Psi_{\varphi}| = 0 \quad D_3 \Phi_{\varphi} = 0. \quad (3.82)$$

This is in contrast with the $\mathcal{N} = (2, 2)$ Dirichlet boundary condition for a 3d $\mathcal{N} = 4$ vector multiplet, which would assign a Dirichlet boundary condition to $\varphi$.

We then introduce two sets of auxiliary boundary conditions, with Neumann and Dirichlet boundary conditions for all the hypermultiplet scalar fields. In terms of $\mathcal{N} = (0, 2)$ supermultiplets, they are:

$$D_C(s) : \quad \Psi_X| = \Psi_Y| = 0 \quad D_3 \Phi_X| = D_3 \Phi_Y| = 0,$$

$$D_F(s) : \quad \Phi_X| = \Phi_Y| = 0 \quad D_3 \Psi_X| = D_3 \Psi_Y| = 0. \quad (3.83)$$

The subscripts therefore signify whether $\mathcal{N} = (0, 2)$ chiral or Fermi multiplets obey Neumann boundary conditions. We note that the $D_C$ boundary condition is compatible with all supersymmetric vacua. We can then associate wavefunctions $\langle D_C(s)|B\rangle$ and $\langle D_F(s)|B\rangle$ to a right boundary condition $B$ and wavefunctions $\langle B|D_C(s)\rangle$ and $\langle B|D_F(s)\rangle$ to a left boundary condition.

There are now four ways to cut the path integral by introducing the boundary conditions $D_C, D_F$ on each side with appropriate superpotential couplings. Different choices will reflect different mathematical interpretations of the overlaps. We describe two of the four explicitly.
First, let us assign a left $D_F$ boundary condition on the right of the cut and a right $D_C$ on the left of the cut. They are then coupled by a boundary superpotential given in terms of boundary superfields as

$$
\int d^2 x d\theta^+ \Phi_X \cdot |\Psi_{X'} + \Phi_Y| \cdot |\Psi_{Y'} + \Phi_\varphi| \cdot \Gamma_\varphi - \Gamma_\varphi \cdot |\Phi_\varphi'|, \quad (3.84)
$$

where $\Gamma_\varphi$ is an auxiliary boundary Fermi multiplet in the adjoint representation of $G$, whose $\mathcal{N} = 2$ flavour charges are fixed by invariance of the superpotential. As described in 2.6.1, this identifies the boundary $G$ symmetries and imposes

$$
\Psi_X = |\Psi_{X'}|, \quad \Phi_X = |\Phi_{X'}|, \quad \Psi_Y = |\Psi_{Y'}|, \quad \Phi_Y = |\Phi_{Y'}|, \\
\Phi_\varphi - |\Phi_\varphi'| = 0, \quad \Psi_\varphi = \Gamma_\varphi = |\Psi_\varphi'|, \quad (3.85)
$$

which identifies $X| = |X'|$, $Y| = |Y'$ and $\varphi| = |\varphi'$ and their super-partners across the interface. We then gauge the remaining diagonal $G$ boundary symmetry by introducing a dynamical 2d $\mathcal{N} = (0, 2)$ vector multiplet.

This interface leads to the decomposition of overlaps into boundary amplitudes,

$$
\langle B' | B' \rangle = \oint d\theta^+, Z_{V,\Gamma_\varphi}(s) \langle B' | D_C(s) \rangle \langle D_F(s) | B' \rangle, \quad (3.86)
$$

where $Z_{V,\Gamma_\varphi}(s)$ is the contribution of the dynamical $\mathcal{N} = (0, 2)$ vector multiplet and Fermi multiplet $\Gamma_\varphi$ at the boundary together with a minus sign $(-)^r$ from the gauge integral,

$$
Z_{V,\Gamma_\varphi}(s) = (\eta(q)^2 \vartheta(s^2))^r \prod_{\alpha \in G} \vartheta(s^\alpha) \vartheta(t^2 s^\alpha). \quad (3.87)
$$

The product is over roots $\alpha$ of $G$.

The second type of interface assigns the boundary conditions $D_C$ to both sides of the cut and introduces the boundary superpotential

$$
\int d^2 x d\theta^+ \Phi_X \cdot \Gamma_X - \Gamma_X \cdot |\Phi_{X'} + \Phi_Y| \cdot \Gamma_Y - \Gamma_Y \cdot |\Phi_{Y'} + \Phi_\varphi| \cdot \Gamma_\varphi - \Gamma_\varphi \cdot |\Phi_\varphi'|. \quad (3.88)
$$

where $\Gamma_X$, $\Gamma_Y$, and $\Gamma_\varphi$ are boundary Fermi multiplets in the appropriate representations. The superpotential couplings identify

$$
\Phi_X - |\Phi_{X'}| = 0, \quad \Phi_Y - |\Phi_{Y'}| = 0, \quad \Psi_X = |\Psi_{X'}|, \quad \Psi_Y = |\Psi_{Y'}|, \\
\Phi_\varphi - |\Phi_{\varphi'}| = 0, \quad \Psi_\varphi = |\Psi_{\varphi'}|, \quad (3.89)
$$
3.8 Boundary Wavefunctions

which again identifies the fields across the interface. We then gauge the remaining diagonal $G$ symmetry by introducing an $\mathcal{N} = (0, 2)$ vector multiplet. This interface allows overlaps to be constructed from wavefunctions,

$$\langle B^l | B^r \rangle = \oint_{\mathcal{J}K} du \, Z_{V, \Gamma_x} \, Z_{\Gamma} (s) \, \langle B^l | D_C (s) \rangle \, \langle D_C (s) | B^r \rangle,$$

(3.90)

where $Z_{\Gamma} (s)$ is the elliptic genus of the boundary Fermi multiplets $\Gamma_X$ and $\Gamma_Y$.

It is straightforward to check that this decomposition is equivalent to the first. The Fermi multiplets $\Gamma_X$, $\Gamma_Y$ implement a flip of the left boundary condition for the 3d $\mathcal{N} = 2$ chiral multiplets $X$, $Y$ from Dirichlet to Neumann. For the decompositions of overlaps, the ratio of the wavefunctions $\langle D_C (s) | B \rangle$ and $\langle D_F (s) | B \rangle$ is precisely the contribution $Z_{\Gamma} (s)$ of the boundary Fermi multiplets.

The remaining two decompositions are constructed in a similar manner. In summary, the four possible decompositions of an overlap into wavefunctions are

$$\langle B^l | B^r \rangle = \oint_{\mathcal{J}K} du \, Z_{V, \Gamma_x} \, Z_{\Gamma} (s) \, \langle B^l | D_C (s) \rangle \, \langle D_F (s) | B^r \rangle \, \langle D_C (s) | B^r \rangle \, \langle D_C (s) | B^r \rangle \, \langle D_F (s) | B^r \rangle \, \langle D_F (s) | B^r \rangle .$$

(3.91)

Here $Z_C (s)$ is the elliptic genus of auxiliary $\mathcal{N} = (0, 2)$ chirals $C_X$ and $C_Y$ coupled to $\Psi_X$ and $\Psi_Y$ at the analogous interface to (3.88) with the roles of chirals and Fermis interchanged. It is easy to check from the charge assignments that $Z_C = Z_{\Gamma}^{-1}$.

If both $B^l$, $B^r$ prescribe Neumann boundary conditions for the vector multiplet, the integral is a JK residue prescription [177, 178, 180]. As before, if the effective 2d $\mathcal{N} = (0, 2)$ theory has mixed ’t Hooft anomalies involving the gauge symmetry, it is necessary to restrict the background flat connections to ensure the integrand is periodic and the contour integral is well-defined. In section 4.1, we will consider boundary conditions involving Dirichlet for the vector multiplet, which enforce a different pole prescription.

For the wavefunctions for the auxiliary boundary conditions themselves, by setting $B^l = D_F (s')$ in the top line of (3.91), consistency requires that

$$\langle D_F (s') | D_C (s) \rangle = Z_{V, \Gamma_x} (s)^{-1} \delta^{(s)} (u - u').$$

(3.92)
Here $\delta^{(r)}(u - u')$ should be considered as a pole prescription around a pole of rank $r$ at $u = u'$ of residue 1. The wavefunctions involving other combinations of $D_C, D_F$ are related by a normalisation by $Z_C$ or $Z_F$.

The wavefunction (3.92) is consistent with its path integral representation on $E_r \times [0, \ell]$. If $s \neq s'$ the system breaks supersymmetry and the path integral vanishes. If $s = s'$, sending $\ell \to 0$, the remaining fluctuating 2d $\mathcal{N} = (0, 2)$ supermultiplets are the adjoint chiral $\Phi_{\varphi}$ charged under $T_t$, and an adjoint chiral $S$ neutral under $T_t$. The Cartan components of the latter naively gives a factor

$$(-\vartheta(1)^{-1})^r, \quad (3.93)$$

which is singular. However, noting that

$$2\pi i \text{Res}_{u=0} \vartheta(q; e^{2\pi i u})^{-1} = \eta(q)^{-2}, \quad (3.94)$$

we replace the contribution of the $S$ by

$$(-)^r \frac{\delta^{(r)}(u - u')}{-\eta(q)^{2r}}, \quad (3.95)$$

where the delta function is regarded as a pole prescription as above. If we combine this with the off-diagonal contribution of $S$ and the adjoint chiral $\varphi$, we reproduce (3.92).

**Example**

Let us take supersymmetric QED with $N$ hypermultiplets and again consider the overlap of the Neumann boundary condition supported on $\mathbb{C}P^{N-1} \subset X$. By taking the limit $\ell \to 0$, one has the following wavefunctions

$$\langle D_F(s)|N\rangle = \langle N|D_F(s)\rangle = \frac{-1}{\vartheta(t^2)} \prod_{\beta=1}^{N} \vartheta(t^{-1}sv_{\beta}^{-1}),$$

$$\langle D_C(s)|N\rangle = \langle N|D_C(s)\rangle = \frac{-1}{\vartheta(t^2)} \prod_{\beta=1}^{N} \frac{-1}{\vartheta(tsv_{\beta}^{-1})}. \quad (3.96)$$

In the first line, the remaining degrees of freedom after collapsing the interval are the chiral multiplet $\Phi_{\varphi}$ and the $N$ Fermi multiplets $\Psi_{Y_{\beta}}$. Similarly, in the second line, the remaining degrees of freedom are $\Phi_{\varphi}$ and the $N$ chiral multiplets $\Phi_{X_{\beta}}$. 
The various contributions to the cutting formula are

\[ Z_{V,\Gamma, \varphi} = (\eta(q)^2 \vartheta(t^2))^r, \]

\[ Z_\Gamma = \prod_{\beta=1}^{N} \vartheta(t^{-1}sv_\beta) \vartheta(t^{-1}s_v), \quad Z_C = \prod_{\beta=1}^{N} \frac{1}{\vartheta(tsv_\beta) \vartheta(ts_v)}, \quad \text{(3.97)} \]

and using any of the four decompositions in equation (3.91), the normalisation of the Neumann boundary condition agrees with (3.73).

### 3.9 Wavefunctions to Amplitudes

We now explain how to pass from wavefunctions to boundary amplitudes. We present the results for boundary amplitudes constructed from the supersymmetric ground states \( |\alpha\rangle \) and wavefunctions constructed from the \( \mathcal{N} = (0, 2) \) boundary conditions \( D_F(s) \) and \( D_C(s) \). These combinations are canonical in the sense that they do not depend on a choice of chamber or Lagrangian splitting. Other choices are found from the relations presented in previous sections.

We will derive the results using consistency between the decompositions into boundary amplitudes and wavefunctions considered thus far. We will introduce boundary condition representatives of the supersymmetric ground states and a derivation of the same results utilising supersymmetric localisation in section 4.1.

Let us then compare the decomposition of an overlap into wavefunctions and boundary amplitudes. For this purpose, it is most convenient to start from the decomposition into wavefunctions using the auxiliary Dirichlet boundary condition \( D_F(s) \),

\[ \langle B^l | B^r \rangle = \oint_{JK} du \ Z_{V,\Gamma, x} \ Z_C(s) \ \langle B^l | D_F(s) \rangle \langle D_F(s) | B^r \rangle. \quad \text{(3.98)} \]

Let us assume \( B^l \) and \( B^r \) prescribe a Neumann boundary condition for the vector multiplet. Then the poles contributing to the JK residue prescription arise entirely from the contribution \( Z_C(s) \) of the auxiliary chiral multiplets and are in 1-1 correspondence with supersymmetric vacua \( \alpha \).

Let us describe this concretely, returning to the description of supersymmetric vacua in section 2.3.1. The contribution of the auxiliary chiral multiplets may be expressed in terms of the weight decomposition of the matter representation \( T^*R \) as

\[ Z_C(s) = \prod_{\varphi \in T^*R} \frac{1}{\vartheta(w\varphi)}, \quad \text{(3.99)} \]
where we have denoted the $G \times T_H \times T_t$ fugacities collectively as $w = (s, v, t)$, and weights as $\varrho = (\rho, \rho_H, \rho_t)$. The JK residue prescription for the elliptic genus [177, 178] selects the rank $r$ poles given by

$$w^{\varrho_i} = 1, \quad i = 1, \ldots, r, \quad (3.100)$$

where the collection of $G$ weights $\{\rho_1, \ldots, \rho_r\}$ obey the conditions outlined in section 2.3.1. Such collections are in 1-1 correspondence with supersymmetric vacuum $\alpha$. Recall we may invert the weights to obtain a unique value of the boundary gauge flat connection $u = u_\alpha$, and denote $s_\alpha = e^{2\pi i u_\alpha}$.

This implies the following crucial property of the contribution of the integrand:

$$(2\pi i)^r \text{Res}_{u = u_\alpha} Z_{V,\Gamma_\varphi}(s) Z_C(s) = \prod_{\lambda \in \Phi_\alpha} \frac{1}{\vartheta(a^\lambda)}. \quad (3.101)$$

In the above, the contribution of the $\mathcal{N} = (0, 2)$ vector multiplet and adjoint Fermi $\Gamma_\varphi$ play the role of the complex moment map and quotient in (2.30).

This is consistent with the decomposition into boundary amplitudes (3.70) provided the wavefunction $\langle D_F(s) | B \rangle$ evaluates to the boundary amplitude $\langle \alpha | B \rangle$ at $s = s_\alpha$. In summary, consistency demands that

$$\langle D_F(s_\alpha) | B \rangle = \langle \alpha | B \rangle, \quad (3.102)$$

$$(2\pi i)^r \text{Res}_{u = u_\alpha} Z_{V,\Gamma_\varphi}(s) \langle D_C(s) | B \rangle = \prod_{\lambda \in \Phi_\alpha} \frac{1}{\vartheta(a^\lambda)} \langle \alpha | B \rangle. \quad (3.103)$$

Note that this implies the supersymmetric ground states $|\alpha\rangle$ lie in the same $Q$-cohomology classes as the boundary states generated by a boundary condition of the form $D_F(s_\alpha)$. We return to this observation in the next section.

It is also useful to consider the wavefunctions of the supersymmetric ground states themselves. Compatibility with the above results and the normalisation of supersymmetric ground states requires that

$$\langle D_F(s_\alpha) | \alpha \rangle = \prod_{\lambda \in \Phi_\alpha} \vartheta(a^\lambda), \quad \quad (3.104)$$

$$(2\pi i)^r \text{Res}_{u = u_\alpha} Z_{V,\Gamma_\varphi}(s) \langle D_C(s) | \alpha \rangle = 1. \quad (3.105)$$
3.9.1 Mathematical Interpretation

Let us now discuss the interpretation of the boundary wavefunctions in terms of equivariant elliptic cohomology. We have already seen that the boundary amplitudes $\langle \alpha | B \rangle$ transform as sections of holomorphic line bundles on $E_T^{(\alpha)}$ that glue to a section of a holomorphic line bundle on $E_T(X)$. The wavefunction repackages this information using the gauge theory description of $E_T(X)$ described in section 3.4.3.

First note that the auxiliary Dirichlet boundary conditions break the gauge symmetry, leaving a boundary $G$ flavour symmetry, which we have denoted $G_{\partial}$. The corresponding wavefunction $\langle D_F(s) | B \rangle$ therefore transforms as a section of a line bundle on the spectral curve $E_T \times E_G$, where the flat connection $s$ parametrises $E_G$. The associated boundary amplitudes obtained by setting $s = s_{\alpha}$,

$$\langle \alpha | B \rangle = \langle D_F(s_{\alpha}) | B \rangle,$$  \hspace{1cm} (3.106)

represent the equivariant elliptic cohomology class obtained by pull back via the inclusion $c : E_T(X) \hookrightarrow E_F \times E_G$. This is precisely the ‘off-shell’ form of elliptic cohomology classes, reviewed in A. In the mathematics literature we reference, e.g. [67, 70, 123, 186] equivariant elliptic cohomology classes are often given in this form, with the pull back implicit.

Example

Let us return to supersymmetric QED and check the relation between the boundary amplitudes and wavefunctions of the Neumann boundary condition $N$ supported on $\mathbb{CP}^{N-1} \subset X$. The wavefunctions were given in (3.96). We then have $s_{\alpha} = v_{\alpha}t^{-1}$ and

$$\langle D_F(s_{\alpha}) | N \rangle = \prod_{\beta \neq \alpha} \vartheta(t^{-2}v_{\alpha}v_{\beta}^{-1})$$  \hspace{1cm} (3.107)

which agrees with the boundary amplitude $\langle \alpha | N \rangle$ in (3.69). Similarly, we find

$$\vartheta(t^2)\eta(q)^2 \text{Res}_{s=s_{\alpha}} \langle D_C(s) | N \rangle = \prod_{\beta \neq \alpha} \frac{1}{\vartheta(v_{\beta}v_{\alpha}^{-1})}.$$  \hspace{1cm} (3.108)
Chapter 4

Theories on Elliptic Curves II: More Boundaries & Interfaces

In this chapter, we consider two distinguished collections of UV boundary conditions whose elements are in 1-1 correspondence with vacua $\alpha$. The first are a class of Dirichlet boundary conditions, which mimic the presence of a vacuum $\alpha$ at infinity. They give a convenient prescription for the computation of boundary amplitudes we discussed in the previous chapter. The second are a class of Neumann boundary conditions and involve couplings to boundary $\mathbb{C}^*$-valued chiral multiplets. They are supported on unions of attracting sets corresponding to the stable envelopes introduced in [89]. We observe that these classes are exchanged under 3d mirror symmetry. In fact, we derive the form of enriched Neumann boundary conditions $N_\alpha$ by colliding exceptional Dirichlet boundary conditions $D_\alpha$ with the mirror symmetry interface introduced in [56], which we review.

We demonstrate that that, upon compactification on $E_\tau$, the correlation functions of this mirror symmetry interface coincide with the mother function in equivariant elliptic cohomology [70], and the boundary wavefunctions and amplitudes of enriched Neumann boundary conditions with the elliptic stable envelopes of [67]. The latter implies that the enriched Neumann boundary conditions generate states in the same $Q$-cohomology class as the elliptic stable envelopes.

Finally, we consider supersymmetric Janus interfaces for the real mass parameters. We discuss the relationship between their correlation functions between different pairs of boundary conditions, with the elliptic stable envelopes, and the chamber $R$-matrices of elliptic quantum integrable systems.

The importance and construction of elliptic stable envelopes is reviewed in appendix A.2. They were introduced in [67], studied further in [68, 70, 123, 186–188], and play
an important role at the heart of modern geometric representation theory, symplectic duality and enumerative geometry.

**Contributions** The results in this chapter are based on material from:


### 4.1 Exceptional Dirichlet

In this section, we consider a distinguished class of boundary conditions that are equivalent to a vacuum at $x^3 \to \pm \infty$, at least for the purpose of computations preserving the supercharge $Q$. This introduces another perspective on the supersymmetric ground states. This is illustrated in figure 4.1. This can lead to a convenient method to compute boundary amplitudes using supersymmetric localisation. Such boundary conditions preserving $\mathcal{N} = (2,2)$ supersymmetry were first considered in [56], and have been studied further in [2, 3].

![Fig. 4.1 Vacuum boundary conditions.](image)

Analogous boundary conditions for 2d $\mathcal{N} = (2,2)$ Landau-Ginzburg models and massive sigma models have been studied in [100] and play an important part in 2d mirror symmetry. A systematic description in massive 2d $\mathcal{N} = (2,2)$ theories has also been developed in [101, 102].

#### 4.1.1 $\mathcal{N} = (2,2)$ Exceptional Dirichlet

First consider generic mass parameters in some chamber $\mathcal{C}$. We consider $\mathcal{N} = (2,2)$ boundary conditions $D^\alpha_{\alpha}, D^\alpha_{\beta}$ which mimic a vacuum $\alpha$ at $x^3 \to \pm \infty$ for computations preserving the supercharge $Q$. Wrapping such boundary conditions on $E_\tau$ will produce boundary states in the same $Q$-cohomology class as the supersymmetric ground states $|\alpha\rangle_{\mathcal{C}}, \epsilon(\alpha)$.

In particular:
• The boundary amplitude $\varepsilon\langle \alpha | B \rangle = \langle D^r_\alpha | B \rangle$ is the path integral on $E_\tau \times [-\ell, 0]$ with R-R boundary conditions with the right boundary condition $B$ at $x^3 = 0$ and the distinguished boundary condition $D^l_\alpha$ at $x^3 = -\ell$.

• The boundary amplitude $\langle B | \alpha \rangle_\varepsilon = \langle B | D^r_\alpha \rangle$ is the path integral on $E_\tau \times [0, \ell]$ with R-R boundary conditions with the left boundary condition $B$ at $x^3 = 0$ and the distinguished boundary condition $D^r_\alpha$ at $x^3 = \ell$.

If we can find UV gauge theory constructions of the distinguished boundary conditions $D^l_\alpha, D^r_\alpha$ this will provide a convenient method to compute boundary amplitudes using supersymmetric localisation on $E_\tau$ times an interval.

There are two important consistency checks on any proposal for such distinguished boundary conditions:

• From the perspective of a massive sigma model to $X$, the BPS equations for 2d $\mathcal{N} = (2, 2)$ supersymmetry are gradient flow for the real moment map $h_m : X \to \mathbb{R}$. Therefore, to mimic a massive vacuum $\alpha$ at $x^3 \to \pm \infty$, the boundary conditions $D^{cl}_\alpha$ must flow to Lagrangian branes supported on the repelling/attracting manifolds $X^\pm_\alpha \subset X$.

• By anomaly inflow, the boundary anomalies of $D^r_\alpha, D^l_\alpha$ must match the effective supersymmetric Chern-Simons couplings in the vacua $\alpha$. In our conventions, if we denote the boundary anomalies of $D^r_\alpha$ by $k_\alpha, k^C_\alpha, k^H_\alpha, \tilde{k}_\alpha$, they should match the effective supersymmetric Chern-Simons couplings $\kappa_\alpha, \kappa^C_\alpha, \kappa^H_\alpha, \tilde{\kappa}_\alpha$ introduced in section 2.5 in the chamber $\mathcal{C}$. The boundary anomalies of $D^l_\alpha$ should be minus these.

If the latter condition is satisfied, it is guaranteed that overlaps with other boundary conditions $B^l, B^r$ will transform in the same way as the boundary amplitudes $\langle B^l | \alpha \rangle_\varepsilon, \varepsilon\langle \alpha | B^r \rangle$ under large background gauge transformations.

**Construction in Abelian Theories**

A UV gauge theory construction of the distinguished boundary conditions $D^l_\alpha, D^r_\alpha$ in abelian 3d $\mathcal{N} = 4$ gauge theories was found in [56] and are known as exceptional Dirichlet. This proposal passes the first consistency check by construction. That they pass the second consistency check was proven in [2].
To construct exceptional Dirichlet boundary conditions, recall that in the vacuum $\alpha$ the real vector multiplet scalar $\sigma$ is uniquely determined by (having turned off $\epsilon$)

$$\rho_i \cdot \sigma + \rho_{H,i} \cdot m = 0,$$

where $\varrho_i = (\rho_i, \rho_{H,i}, \rho_{t,i})$ with $i = 1, \ldots, r$ are the set of weights associated to the vacuum $\alpha$. This in turn determines the effective real masses in the vacuum $\alpha$ of all remaining hypermultiplets. Fixing a chamber for the real mass parameters, we may split the hypermultiplet fields into those with zero, positive and negative mass.

For the right exceptional Dirichlet boundary condition, we define a splitting $\varepsilon^r_\alpha$ as in equation (2.87) such that $Y_{\varepsilon^r_\alpha}$ consist of hypermultiplet fields with negative real mass in the vacuum $\alpha$, or those which both have zero real mass and attain an expectation value in $\alpha$. Then the right exceptional Dirichlet boundary condition $D^r_\alpha$ is defined as follows.

- A $\mathcal{N} = (2,2)$ Dirichlet boundary condition for the vector multiplet. The boundary value of $\varphi$ is fixed by requiring the effective complex mass of all hypermultiplets with expectation values in $\alpha$ vanish. For $m_\mathcal{C} = 0$, $\varphi = 0$.

- A Neumann-Dirichlet boundary condition for hypermultiplets

$$D^\perp X_{\varepsilon^r_\alpha} \vert_0 = 0, \quad Y_{\varepsilon^r_\alpha} \vert_0 = \begin{cases} 0 & \text{if } Y_{\varepsilon^r_\alpha} \text{ has negative real mass in } \alpha \\ c & \text{if } Y_{\varepsilon^r_\alpha} \text{ has zero real mass in } \alpha \text{ and attains a vev} \end{cases}$$

where $c$ is the expectation value in the vacuum $\alpha$.

The Higgs branch image of $D^r_\alpha$ is precisely the repelling set $X^-_\alpha \subset X$, since the hypermultiplet fields $X_{\varepsilon^r_\alpha}$ are exponentially suppressed under the inverse gradient flow.

For the left exceptional Dirichlet boundary condition $D^l_\alpha$, we instead take $Y_{\varepsilon^l_\alpha}$ to be chirals of positive real mass in the vacuum $\alpha$, or those which have zero real mass and attain a vev. Similarly, the image of the boundary condition $D^l_\alpha$ is $X^+_\alpha \subset X$.

Wrapping the theory on $E_\tau$, a feature common of both left and right boundary conditions is that the boundary gauge flat connection $u$ is fixed in terms of $z_H, z_t$ to the value determined by the vacuum, i.e. the unique solution of

$$\rho_i \cdot u + \rho_{H,i} \cdot z_H + \rho_{t,i} z_t = 0.$$  

This condition is not required to be invariant under shifts of $\mathbb{Z} + \tau \mathbb{Z}$, as the gauge symmetry is broken to a flavour symmetry at the boundary due to the Dirichlet
4.1 Exceptional Dirichlet

boundary condition for the vector multiplet. We denote the distinguished value of the boundary holonomy \( u_\alpha \), and also \( s_\alpha = e^{2\pi i u_\alpha} \).

The above construction can be elegantly rephrased in terms of weights. The splitting corresponds to introducing the decomposition

\[
Q = Q^+_\alpha \sqcup Q^-_\alpha \sqcup Q^0_\alpha
\]

(4.4)
of the matter representation \( Q = T^*R \) into weight spaces, which after the evaluation at the fixed point (2.30), correspond to positive, negative and zero weights respectively. Note that if a chiral \( X \) has positive real mass, its corresponding weight in \( T^*R \) is in fact \( d/dX \) and thus corresponds to an element of \( Q^-_\alpha \).

We note that

\[
Q^0_\alpha = \{ \varrho_i, \varrho^*_i = -2e_t - \varrho_i \quad \text{for } i = 1, \ldots, r \}
\]

(4.5)
where \( \varrho_i \) are the weights which label the vacuum \( \alpha \), and \( \varrho^*_i \) the weights corresponding to their partners from the same hypermultiplet. After evaluation at the vacuum, \( w^{\varrho_i} = 1 \) for \( i = 1, \ldots, r \) (or equivalently \( s = s_\alpha \)), the character of \( Q^0_\alpha \) is precisely cancelled by \( \mu_C \) and the gauge group quotient in (2.30). One also has

\[
\varrho \in Q^+_\alpha \iff \varrho^* \in Q^-_\alpha
\]

(4.6)
again corresponding to pairs of chirals in the same hypermultiplet.

The polarisation is then rephrased in terms of weight spaces as

\[
\begin{align*}
\varepsilon^r_\alpha : & \\
\quad d/dX \varepsilon^r_\alpha & \in Q^-_\alpha \cup \{ \varrho_i, i = 1, \ldots, r \} \\
\quad d/dY \varepsilon^r_\alpha & \in Q^+_\alpha \cup \{ \varrho^*_i, i = 1, \ldots, r \}, \\
\varepsilon^l_\alpha : & \\
\quad d/dX \varepsilon^l_\alpha & \in Q^+_\alpha \cup \{ \varrho_i, i = 1, \ldots, r \} \\
\quad d/dY \varepsilon^l_\alpha & \in Q^-_\alpha \cup \{ \varrho^*_i, i = 1, \ldots, r \}.
\end{align*}
\]

(4.7)

Anomalies

Let us derive the boundary anomalies of \( D^r_\alpha \), and check they match the supersymmetric Chern-Simons levels in the vacuum \( \alpha \).

First, the boundary anomalies involving the topological symmetry come purely from anomaly inflow and therefore trivially match the Chern-Simons couplings \( \kappa_\alpha, \kappa^C_\alpha \). As in section 2.5, they coincide with the bilinear couplings appearing in the moment
maps $h_m$ and $h_e$ at the fixed point $\alpha$. This matching was shown in detail for abelian theories in [2], which we review in appendix E.

Let us therefore focus on anomalies arising from bulk fermions, computed using the results in [117]. We focus on $\mathcal{N} = 2$ flavour symmetries, and first compute the anomaly polynomial for zero boundary expectation values $c = 0$. The boundary condition initially supports an additional flavour symmetry $G_\partial$ with field strength $f_\partial$, generated by global gauge transformations at the boundary. Using the description (4.7), the undeformed anomaly is

$$r f_L^2 + \frac{1}{4} \prod_{\varrho \in Q_+^\alpha} (\varrho \cdot F)^2 + \frac{1}{4} \prod_{i=1}^r (\varrho_i \cdot F)^2 - \frac{1}{4} \prod_{\varrho \in Q_-^\alpha} (\varrho \cdot F)^2 - \frac{1}{4} \prod_{i=1}^r (\varrho_i \cdot F)^2,$$

where we have denoted $F = (f_\partial, f_H, f_L)$. Turning on the expectation value $c$, $G_\partial$ is broken and $f_\partial$ is set to the value determined by solving $\varrho_i \cdot F = 0$ for $i = 1, \ldots, r$. This is analogous to the substitution in the character (2.30).

Evaluating the above and re-introducing the terms from anomaly inflow, we recover

$$\mathcal{P}[D_\alpha] = \kappa_\alpha(f_H, f_C) + \kappa_\alpha^C(f_L, f_C) + \frac{1}{4} \sum_{\lambda \in \Phi^+} (\lambda \cdot f)^2 - \frac{1}{4} \sum_{\lambda \in \Phi^-} (\lambda \cdot f)^2,$$

which matches the Chern-Simons levels (2.57), i.e. $\mathcal{P}[D_\alpha] = \mathcal{P}_\alpha$.

One similarly recovers $\mathcal{P}[D_\alpha] = -\mathcal{P}_\alpha$, due to the opposite Lagrangian splitting and contribution from anomaly inflow due to the orientation of the boundary. This agrees with anomaly inflow from placing a massive supersymmetric vacuum $\alpha$ on the left.

**Orthonormality**

Another consistency check is to show that the interval partition functions of left and right exceptional Dirichlet boundary are orthonormal:

$$\langle D_\alpha^l | D_\beta^r \rangle = \delta_{\alpha, \beta}.$$  

The configurations contributing to the localised path integral on $E_\tau \times I$ have constant profiles for the hypermultiplet scalars [189]. This implies that if $\alpha \neq \beta$, the boundary expectation values and holonomies are incompatible and break supersymmetry. If $\alpha = \beta$, taking $\ell \to 0$, the remaining fluctuating degrees of freedom consist of a neutral $\mathcal{N} = (2, 2)$ chiral multiplet $(S, \bar{\Psi}_\phi)$ of $R$-charge 0 and $r$ chiral multiplets $(\Phi_{X_{s_\alpha}}, \Psi_{Y_{s_\alpha}})$.
for each $X_{e^\alpha}$ of vanishing mass. The former is naively singular, with a contribution
\[
\left( \frac{\vartheta(t^{-2})}{\vartheta(1)} \right)^r,
\]
however this is cancelled by the contribution from the $r$ chiral multiplets
\[
\prod_{i=1}^r \left. \frac{\vartheta(w^{\phi_i})}{\vartheta(w^{\phi_i})} \right|_{w^{\phi_i}=1}^{w^{\phi_i}=1},..,r
\]
when evaluated at the value of the boundary holonomy of $G_\beta$ determined by both $D^l_\alpha$ and $D^r_\alpha$. This recovers the expected normalisation (4.10).

Example

Let us consider supersymmetric QED in the default chambers. The exceptional Dirichlet boundary conditions are given by Dirichlet for the vector multiplet and the following boundary condition for the hypermultiplets,
\[
D^r_\alpha : \begin{align*}
  D_3 Y_\beta &= 0, \quad X_\beta = c \delta_{\alpha \beta}, \quad \beta \leq \alpha, \\
  D_3 X_\beta &= 0, \quad Y_\beta = 0, \quad \beta > \alpha,
\end{align*}
\]
\[
D^l_\alpha : \begin{align*}
  D_3 X_\beta &= 0, \quad Y_\beta = 0, \quad \beta < \alpha, \\
  D_3 Y_\beta &= 0, \quad X_\beta = c \delta_{\alpha \beta}, \quad \beta \geq \alpha,
\end{align*}
\]
where $c \neq 0$. In both cases, as $X_\alpha = c$, the effective real and complex mass parameters of this hypermultiplet field must vanish. This requires requires $\sigma = m_\alpha$ and $\varphi = m_{\alpha,C}$, where $m_{\alpha,C}$ are the complex mass parameters for $T_H$.

Wrapping on $E_r$, the choice of vacuum $\alpha$ uniquely determines the value of the holonomy $u$ of the gauge field at the boundary
\[
u = z_{H,\alpha} - z_t =: u_\alpha \quad \Rightarrow \quad s v^{-1}_\alpha t = 1
\]
according to the hypermultiplet scalar $X_\alpha$ with a non-vanishing expectation value.

Next, we check the support of the right boundary conditions in $X = T^* \mathbb{CP}^{N-1}$. Let $\langle e_{\alpha_1} e_{\alpha_2} \cdots \rangle \subset \mathbb{CP}^{N-1} \subset X$ denote the projective subspace generated by the fundamental weights $e_{\alpha_1}, e_{\alpha_2}, \cdots$ of $T_H$. The supersymmetric vacua or fixed points are $\alpha = \langle e_\alpha \rangle$.

\footnote{Although we ultimately set the complex mass and FI parameters parameters to zero, which determines a fixed maximal torus $U(1)_H \times U(1)_C$, it is sometime convenient to include them when discussing boundary conditions and interfaces in flat space.}
Now consider the subspaces $U_{\alpha} = \langle e_\alpha, \ldots, e_N \rangle$ and inclusions $\iota_\alpha : U_{\alpha+1} \rightarrow U_{\alpha}$. Then the right exceptional Dirichlet boundary conditions have support

$$N^\perp U_\alpha - \iota_\alpha^* N^\perp U_\alpha. \quad (4.16)$$

where $N^\perp$ denotes the co-normal bundle. This has the following interpretation:

- The coordinates $X_\beta$ for $\beta \geq \alpha$ parameterise the base $U_\alpha$.
- The coordinates $Y_\beta$ for $\beta < \alpha$ parameterise the co-normal directions to $U_\alpha$.
- The pull back $\iota_\alpha^* N^\perp U_\alpha$ is excluded due the constraint $X_\alpha = c \neq 0$.

This support is the attracting set $X^-_\alpha$ in the default chamber. Note that the closure of the attracting set is the whole co-normal bundle $X^-_\alpha = N^\perp U_\alpha$. This is illustrated in terms of the hyper-toric diagram in figure 4.2. A similar argument shows that the left boundary conditions are supported on $X^+_\alpha$.

Finally, we check the boundary anomalies reproduce the supersymmetric Chern-Simons coupling in the vacuum $\alpha$ in equations (2.46) and (2.48). We initially introduce separate field strengths $f_V$ and $f_A$ for the vector and axial R-symmetries. For the right boundary condition, starting with $c = 0$, the anomaly polynomial is

$$-f_\partial f_C - f_V f_A + \left( \sum_{\beta \leq \alpha} (f_\beta^\beta - f_\partial) + \sum_{\beta > \alpha} (f_\partial - f_\beta^\beta) \right) f_A. \quad (4.17)$$

The first term comes from anomaly inflow from the mixed Chern-Simons term coupling the gauge symmetry to the topological symmetry, or equivalently the FI parameter. The second comes from gauginos, and the third from hypermultiplet fermions.

Now turning on $c \neq 0$, a combination of $T_H, U(1)_H$ and $G_\partial$ is broken. This can be seen as breaking $G_\partial$ and a re-definition of the boundary symmetries $T_H$ and $U(1)_V$,

$$J_H^\beta = J_H^{\beta, \text{bulk}} + \delta^{\alpha\beta} J_G;$$
$$J_V = J_{U(1)_H} - J_G. \quad (4.18)$$

\(^2\)As outlined in section 2.6.2, these Higgs branch images are found by taking the intersections of the Lagrangian submanifold of $T^* \mathbb{R} \cong \mathbb{C}^{2N}$ specified by the splittings (4.13) and (4.14) with the moment map constraint $\mu_C = 0$. One does not quotient by the gauge group because it is broken at the boundary by the Dirichlet boundary condition for the vector multiplet. Neither does one impose the real moment map $\mu_R = \zeta_R$, which arises as a D-term constraint, because it is absorbed into the boundary condition for $\sigma$. 

Fig. 4.2 The hyper-toric diagram for the Higgs branch of supersymmetric QED with $N = 3$, see e.g. [56]. The slice $X_\beta Y_\beta = 0$ for all $\beta = 1, 2, 3$ is a fibration over $\mathbb{R}^2$ with typical fibre $(S^1)^2$, where one combination of the circles degenerates along each of the hyperplanes $\mathcal{H}_\beta$ defined by the vanishing of the hypermultiplet $(X_\beta, Y_\beta)$. The diagram is an illustration of the base. The fibre fully degenerates to a point at each vacuum $\{1, 2, 3\}$. The support of each exceptional Dirichlet boundary condition $D_\alpha^r$ in the default chamber is shown. For $D_1^r$ and $D_2^r$ we subtract the intersection with hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. For $D_3^r$ there is no such intersection. The direction of inverse gradient flow in the default chamber is indicated by the dotted arrow.

In the anomaly polynomial, this sets $f_\beta = f_H^\alpha - f_V$. Thus the boundary 't Hooft anomaly polynomial for the right exceptional Dirichlet boundary condition is

$$\mathcal{P}[D_\alpha^r] = -f_H^\alpha f_C + f_V f_C + \left( \sum_{\beta < \alpha} (f_H^\beta - f_H^\alpha) + \sum_{\beta > \alpha} (f_H^\alpha - f_H^\beta) \right) f_A + f_V (2\alpha - N - 1) f_A. \quad (4.19)$$

Note now $f_H$ and $f_V$ are field strengths for the boundary symmetries (4.18). This matches the supersymmetric Chern-Simons levels in the default chamber, after replacing $f_A = -f_L$ and $f_V = f_L$. Similarly $\mathcal{P}[D_\alpha^l] = -\mathcal{P}[D_\alpha^r]$.

4.1.2 $\mathcal{N} = (0, 2)$ Exceptional Dirichlet

We now consider the case of vanishing mass parameters and construct boundary conditions $D_\alpha$, whose boundary states on $E_\tau$ lie in the same $Q$-cohomology class as the supersymmetric ground states $|\alpha\rangle$. Such boundary conditions will have the property that:
The boundary amplitude \( \langle \alpha | B \rangle = \langle D_\alpha | B \rangle \) is given by the path integral on \( E_\tau \times [-\ell, 0] \) with R-R boundary conditions with right boundary condition \( B \) at \( x^3 = 0 \) and the distinguished boundary condition \( D_\alpha \) at \( x^3 = -\ell \).

The boundary amplitude \( \langle B | \alpha \rangle = \langle B | D_\alpha \rangle \) is given by the path integral on \( E_\tau \times [0, \ell] \) with R-R boundary conditions with left boundary condition \( B \) at \( x^3 = 0 \) and the distinguished boundary condition \( D_\alpha \) at \( x^3 = \ell \).

Note that we have not introduced separate notation for left and right boundary conditions, as we will see momentarily that they take the same form. With explicit UV gauge theory constructions of such boundary conditions, this provides a convenient way to compute the boundary amplitudes via supersymmetric localisation. Recall these amplitudes glue to a section of a holomorphic line bundle on \( E_T(X) \).

Three consistency checks on a proposal such a class of boundary conditions are:

- The boundary condition \( D_\alpha \) should flow to Dirichlet boundary conditions in a massive sigma model to \( X \) supported at the fixed points \( \alpha \in X \).

- By anomaly inflow, the boundary anomalies should match the shifted supersymmetric Chern-Simons couplings given in (3.36),

\[
\begin{align*}
|D_\alpha\rangle & : \quad \kappa_\alpha + \kappa^C_\alpha + \frac{1}{4} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda, \\
\langle D_\alpha| & : \quad -\kappa_\alpha - \kappa^C_\alpha + \frac{1}{4} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda.
\end{align*}
\]

which are independent of the chamber for the mass parameters.

- The corresponding boundary states are normalised with respect to \( \mathcal{N} = (2, 2) \) exceptional Dirichlet boundary conditions such that

\[
\begin{align*}
\langle D_\alpha | B \rangle &= \langle D^l_\alpha | B \rangle \times \prod_{\lambda \in \Phi^+_\alpha} \vartheta(a^\lambda), \\
\langle B | D_\alpha \rangle &= \langle B | D^r_\alpha \rangle \times \prod_{\lambda \in \Phi^-_\alpha} \vartheta(a^\lambda),
\end{align*}
\]

in agreement with (3.56).

The last two compatibility checks are of course intimately related. The second ensures that the overlaps with boundary conditions \( B^l \) and \( B^r \) transform in the correct way under large background gauge transformations.
4.1 Exceptional Dirichlet

Construction

The $\mathcal{N} = (0, 2)$ exceptional Dirichlet boundary condition $D_\alpha$ has a simple construction that is valid for any supersymmetric gauge theory, and is the same for both left and right. Decomposing into 3d $\mathcal{N} = 2$ supermultiplets, the boundary conditions are specified as follows:

- The vector multiplet has a Dirichlet boundary condition.
- The adjoint chiral multiplet $\varphi$ has a Neumann boundary condition (3.82).
- The chiral multiplets $X, Y$ are all assigned Dirichlet boundary conditions with boundary expectation values as in the vacuum $\alpha$, completely breaking the boundary $G_3$ symmetry. The remaining fluctuating degrees of freedom at the boundary are the $\mathcal{N} = (0, 2)$ Fermi multiplets $\Psi_X$ and $\Psi_Y$.

The support of this boundary condition is the vacuum $\alpha \in X$.

We note that this construction is compatible with the formula (3.102), reproduced below
\[
\langle D_F(s_\alpha)|B\rangle = \langle \alpha|B\rangle, \tag{4.22}
\]
relating boundary wavefunctions and amplitudes. The $\mathcal{N} = (0, 2)$ exceptional Dirichlet boundary condition $D_\alpha$ is obtained from the $\mathcal{N} = (0, 2)$ auxiliary Dirichlet boundary condition $D_F(s)$ by turning on expectations values for hypermultiplet scalars as in the vacuum $\alpha$. This fixes the boundary holonomy to $s = s_\alpha$.

Anomalies

Now consider the boundary anomalies of $D_\alpha$. Those involving the topological symmetry $T_C$ are the same as for the $\mathcal{N} = (2, 2)$ exceptional Dirichlet boundary conditions and arise from anomaly inflow. The remaining anomalies arise from fermions in chiral multiplets surviving at the boundary. In terms of the matter representation $T^*R$, their contribution to the anomaly polynomial for $c = 0$ are
\[
-rf_L^2 + \frac{1}{4} \prod_{\varrho \in T^*R} (\varrho \cdot F)^2 \tag{4.23}
\]
where we have again denoted $F = (f_\vartheta, f_H, f_L)$. Turning on $c \neq 0$, we must again eliminate $f_\vartheta$ by solving $\varrho_i \cdot F = 0$. The above contribution becomes
\[
\frac{1}{4} \prod_{\lambda \in \Phi_\alpha} (\lambda \cdot f)^2. \tag{4.24}
\]
Adding in the contributions from anomaly inflow, one obtains

\[ |D_\alpha\rangle : \quad \mathcal{P} = +\kappa_\alpha(f_H, f_C) + \kappa^C_\alpha(f_L, f_C) + \frac{1}{4} \prod_{\lambda \in \Phi_\alpha} (\lambda \cdot f)^2, \]

\[ \langle D_\alpha | : \quad \mathcal{P} = -\kappa_\alpha(f_H, f_C) - \kappa^C_\alpha(f_L, f_C) + \frac{1}{4} \prod_{\lambda \in \Phi_\alpha} (\lambda \cdot f)^2 \]

(4.25)

where the anomaly polynomials of the right and left boundary conditions are related by flipping the contributions \( \kappa_\alpha, \kappa^C_\alpha \). This reproduces the expectation (4.20)

**Orthogonality**

We now check that the path integral on \( E_\tau \times [0, \ell] \) with boundary conditions \( D_\alpha \) at both boundaries reproduces the normalisation of supersymmetric ground states \( |\alpha\rangle \) in (3.19). If \( \alpha \neq \beta \), the partition function will vanish as before. Assuming the contrary, taking \( \ell \to 0 \), there is a contribution from Fermi multiplets \( \Psi_X, \Psi_Y \),

\[ \prod_{\varrho \in T^* R} \vartheta(w^\varrho) \bigg|_{s = s_\alpha} = (\vartheta(1)\vartheta(t^2))^r \prod_{\lambda \in \Phi_\alpha} \vartheta(a^\lambda). \]

(4.26)

The first factor has a zero of order \( r \), but is cancelled by the remaining contribution of the adjoint chiral multiplets \( S \) and \( \Phi_\varphi \). In summary,

\[ \langle D_\alpha | D_\beta \rangle = \delta_{\alpha\beta} \prod_{\lambda \in \Phi_\alpha} \vartheta(a^\lambda). \]

(4.27)

**Example**

Again let us consider supersymmetric QED in the default chambers. The \( \mathcal{N} = (0, 2) \) exceptional Dirichlet boundary conditions are found by imposing a Dirichlet boundary condition for the 3d \( \mathcal{N} = 2 \) vector multiplet, a Neumann boundary condition for the chiral multiplet \( \varphi = (\Phi_\varphi, \Gamma_\varphi) \), together with

\[ D_3 \Psi_{X_\beta} = 0, \quad X_\beta = c\delta_{\alpha\beta}, \quad \beta = 1, \ldots, N \]

\[ D_3 \Psi_{Y_\beta} = 0, \quad Y_\beta = 0, \]

(4.28)

for the hypermultiplets. The Higgs branch image is the fixed point \( \alpha \), in which \( X_\alpha \neq 0 \). We do not impose \( \varphi - m_{\alpha, C} = 0 \) as for \( \mathcal{N} = (2, 2) \) exceptional Dirichlet boundary conditions, as here \( \varphi \) is assigned a Neumann boundary condition.
4.1 Exceptional Dirichlet

Setting the expectation value to zero, \( c = 0 \), the boundary anomaly polynomial is, initially keeping track of separate \( U(1)_V \) and \( U(1)_A \) anomalies:

\[
\mp f_\theta \cdot f_C - \frac{1}{2} \left( f_V^2 + f_A^2 \right) + \frac{1}{4} \sum_{\beta=1}^{N} \left[ (f_\theta - f_H^\beta - f_A)^2 + (-f_\theta + f_H^\beta - f_A)^2 \right],
\]

(4.29)

where the \( + \) sign is for \( \langle \alpha \rangle \) and the \( - \) is for \( |\alpha\rangle \). Turning on the expectation value, \( c \neq 0 \), we again make the re-definitions of boundary symmetries (4.18) and set \( f_\theta = f_H^\alpha_\theta \). Let us also pass to considering only 3d \( \mathcal{N} = 2 \) flavour symmetries, and set \( f_V = -f_A = f_L \).

We obtain

\[
\mp (f_H^\alpha - f_L) \cdot f_C + \frac{1}{4} \sum_{\beta \neq \alpha} \left[ (f_H^\beta - f_H^\alpha)^2 + (-2f_L - f_H^\beta + f_H^\alpha)^2 \right]
\]

(4.30)

which agrees with (4.20) with \( \kappa_\alpha = -\epsilon_\alpha \otimes e_C \) and \( \kappa_C^\alpha = e_t \otimes e_C \) as expected.

4.1.3 Boundary Amplitudes

The exceptional Dirichlet boundary conditions \( D_{l,r}^\alpha \) and \( D_\alpha \) provide an independent way to compute boundary amplitudes using supersymmetric localisation for the interval partition function [180]. In this section, we show such computations agree with the formulae derived via general consistency constraints in section 3.9.

We begin with the boundary amplitudes with vanishing mass parameters. From the explicit form of exceptional Dirichlet \( D_\alpha \) boundary condition as \( D_F(s) \) with boundary expectation values and \( s = s_\alpha \), we immediately find that

\[
\langle \alpha | B \rangle = \langle D_F(s_\alpha)|B \rangle, \quad \langle B | \alpha \rangle = \langle B|D_F(s_\alpha) \rangle,
\]

(4.31)

as required by consistency in (3.102).

Now let us consider the boundary amplitudes with mass parameters in some chamber. Specialising to an abelian gauge theory, the exceptional Dirichlet boundary condition \( D_\alpha \) is obtained from \( D_\alpha \) by a coupling to the boundary superpotential

\[
\int d^2x d\theta^+ \left( \Phi_\varphi | \cdot \Gamma_\varphi + \Psi_{X_{\alpha}} \cdot C_{\varepsilon_\alpha} \right),
\]

(4.32)

where \( C_{\varepsilon_\alpha} \) and \( \Gamma_\varphi \) are boundary chiral multiplets and an adjoint Fermi multiplet respectively. There is an implicit sum over each component of the polarisation \( \varepsilon_\alpha \) that specifies the boundary condition \( D_\alpha^r \). The boundary coupling implements a flip [117] of the boundary conditions for the chiral multiplets \( \varphi, X, Y \) to those specified in \( D_\alpha^r \).
The elliptic genus of the additional boundary contributions \( C_{\epsilon, \Gamma_{\phi}} \) is
\[
\vartheta(7-2)^r \prod_{\rho \in Q_{L}} \frac{1}{\vartheta(w^{\rho})} \prod_{i=1}^{r} \frac{1}{\vartheta(w^{\rho_i})} \bigg|_{s=s_{\alpha}} \prod_{\lambda \in \Phi_{\alpha}} \frac{1}{\vartheta(a^{\lambda})}. \quad (4.33)
\]
We therefore have
\[
\langle B|D_{\alpha} \rangle = \prod_{\lambda \in \Phi_{\alpha}} \frac{1}{\vartheta(a^{\lambda})} \langle B|D_{\alpha} \rangle, \quad (4.34)
\]
which reproduces (4.21). A similar argument applies to boundary amplitudes involving the left exceptional Dirichlet boundary condition \( D_{\alpha}^{l} \).

**Lagrangian Branes**

To illustrate the utility of this approach, we derive the formulae proposed in section 3.6.3 for the boundary amplitudes of boundary conditions flowing to smooth Lagrangian branes \( L \subset X \) in the sigma model to \( X \).

An \( \mathcal{N} = (2,2) \) boundary condition \( N_{L} \) flowing to a smooth Lagrangian brane \( L \subset X \) can be constructed by imposing Neumann boundary conditions for the vector multiplet, together with a standard boundary condition for the hypermultiplet specified by a polarisation \( \varepsilon_{L} \). The Lagrangian \( L \) is the image under the hyperKähler quotient of the Lagrangian \( Q_{L} \subset Q = T^{*}R \) specified by the polarisation.

First note that for the boundary amplitude with \( D_{\alpha} \) to be non-vanishing, the polarisation \( \varepsilon_{L} \) must be compatible with the vacuum \( \alpha \). This means any hypermultiplet scalar which has a non-zero expectation value in the vacuum \( \alpha \) is one of the \( \{X_{\varepsilon_{L}}\} \), and not the \( \{Y_{\varepsilon_{L}}\} \). Equivalently, this means that \( Q_{L} \) contains the weights \( \varrho_{i}, i = 1, \ldots r \) which label the vacuum \( \alpha \).

Let us then consider the boundary amplitude \( \langle N_{L}|D_{\alpha} \rangle \). Sending the length of the interval to zero, the remaining degrees of freedom on \( E_{\tau} \) consist of the \( \mathcal{N} = (0,2) \) adjoint chiral multiplet \( \Phi_{\varphi} \) and the Fermi multiplets \( \Psi_{Y_{\varepsilon_{L}}} \). The holonomy of the gauge connection is fixed to \( s_{\alpha} \). Thus,
\[
\langle N_{L}|D_{\alpha} \rangle = \frac{1}{\vartheta(t-2)^r} \prod_{e \in Q_{L}} \vartheta(w^{e}) \bigg|_{s=s_{\alpha}} \prod_{\lambda \in \Phi_{\alpha}(L)} \vartheta(a^{\lambda}), \quad (4.35)
\]

Note that the contribution from $\Phi_\varphi$ is cancelled by the Fermi multiplets paired with hypermultiplet scalars which get expectation values in the vacuum $\alpha$. This reproduces the formula proposed in equation (3.61), which is the elliptic genus of Fermi multiplets parametrising $(T_\alpha L)^\perp \subset T_\alpha$. These boundary amplitudes represent the equivariant elliptic cohomology class of $L \subset X$. Note the result vanishes unless $\{\varrho_1, \ldots, \varrho_r\} \subset Q_L$. The boundary amplitude $\langle D_\alpha|N_L \rangle$ gives the same answer.

Let us now assume the left boundary condition $N_L$ is compatible with mass parameters in some chamber, and consider the boundary amplitude $\langle N_L|D_\alpha^* \rangle$. Sending the length of the interval to zero, the remaining degrees of freedom on $E_\tau$ consist of the $N=2,2$ chiral multiplets compatible with both the splitting $\varepsilon_\alpha^r$ of $D_\alpha^*$ and $\varepsilon_L$ of $N_L$. Assuming $\{\varrho_1, \ldots, \varrho_r\} \subset Q_L$, they are precisely the $N=2,2$ chirals containing scalars dual to the weights in $Q_L \cap Q_\alpha$. Therefore $\langle N_L|D_\alpha^* \rangle$ is given by the $(2,2)$ elliptic genus

$$\langle N_L|D_\alpha^* \rangle = \prod_{\varrho \in Q_L \cap Q_\alpha} \left. \frac{\partial (w^\varrho^*)}{\partial (w^\varrho)} \right|_{s=s_\alpha} = \prod_{\lambda \in \Phi_\alpha(L)} \left. \frac{\partial (a^\lambda^*)}{\partial (a^\lambda)} \right|_{s=s_\alpha}.$$  (4.36)

At the fixed point, these become the weights in $\Phi_\alpha^r(L) \subset \Phi_\alpha$ of the tangent space $T_\alpha L \subset T_\alpha X^\perp$. Therefore $\langle N_L|D_\alpha^* \rangle$ is given by the $(2,2)$ elliptic genus

$$\langle N_L|D_\alpha^* \rangle = \prod_{\varrho \in Q_L \cap Q_\alpha} \left. \frac{\partial (w^\varrho^*)}{\partial (w^\varrho)} \right|_{s=s_\alpha} = \prod_{\lambda \in \Phi_\alpha^r(L)} \left. \frac{\partial (a^\lambda^*)}{\partial (a^\lambda)} \right|_{s=s_\alpha},$$  (4.37)

and similarly

$$\langle D_\alpha|N_L \rangle = \prod_{\varrho \in Q_L \cap Q_\alpha^*} \left. \frac{\partial (w^\varrho^*)}{\partial (w^\varrho)} \right|_{s=s_\alpha} = \prod_{\lambda \in \Phi_\alpha^*(L)} \left. \frac{\partial (a^\lambda^*)}{\partial (a^\lambda)} \right|_{s=s_\alpha}.$$  (4.38)

This reproduces the boundary amplitudes (3.60).

**Example**

Let us return to the Neumann boundary condition $N$ for supersymmetric QED that flows to the compact Lagrangian brane $L = \mathbb{C}P^{N-1} \subset X$. In the default chamber, this corresponds to the polarisation $\varepsilon_L = \{+, \cdots, +\}$.

In computing the boundary amplitudes with $D_\alpha$, the remaining degrees of freedom on $E_\tau$ are the $N$ Fermi multiplets $\Psi_{Y,\beta}$, $\beta = 1, \ldots, N$, and a neutral chiral multiplet $\Phi_\varphi$. Thus

$$\langle D_\alpha|N_L \rangle = \frac{1}{\partial (t^{-2})} \prod_{\beta=1}^N \left. \frac{\partial (t^{-1}sv_\beta^{-1})}{\partial (t^{-1}sv_\beta^{-1})} \right|_{s=v_\alpha t^{-1}} = \prod_{\beta \neq \alpha} \left. \frac{\partial (t^{-2}v_\alpha v_\beta^{-1})}{\partial (t^{-2}v_\alpha v_\beta^{-1})} \right|_{s=v_\alpha t^{-1}}.$$  (4.39)
with an identical result for $\langle N_L | D_\alpha \rangle$. This reproduces the previous formula (3.69). Note that the parameter $t$ corresponds to the left-moving boundary R-symmetry $T_L = U(1)_V - U(1)_A$, where $U(1)_A$ is the boundary axial R-symmetry defined in section 3.6.4 and $U(1)_V$ is the boundary vector R-symmetry defined in (4.18).

The boundary condition $N$ is compatible with mass parameters in any chamber. In the overlap $\langle N | D^r_\alpha \rangle$, there are no fluctuating degrees of freedom from the vector multiplet. The remaining contribution comes from the $\mathcal{N} = (2, 2)$ chiral multiplets containing the scalars $X_\beta$ for $\beta > \alpha$ in the default chamber, evaluated at $u = u_\alpha$. Thus:

$$\langle N | D^r_\alpha \rangle = \prod_{\beta > \alpha} \frac{\vartheta (t^{2 \frac{v_{\alpha}}{v_{\beta}}})}{\vartheta (\frac{v_{\beta}}{v_{\alpha}})},$$

which reproduces the previous formula (3.67).

### 4.1.4 Wavefunctions of Exceptional Dirichlet

We now consider the wavefunctions of exceptional Dirichlet boundary conditions, either on the left or right, and at the origin or in a chamber of the mass parameter space,

$$\langle D^l_\alpha | D_C(s) \rangle, \quad \langle D_C(s) | D^r_\alpha \rangle,$$

$$\langle D_\alpha | D_C(s) \rangle, \quad \langle D_C(s) | D_\alpha \rangle.$$  

(4.41)  

(4.42)

Here we use the $\mathcal{N} = (0, 2)$ boundary condition $D_C$ to write down wavefunctions, since it is compatible with the non-vanishing expectation value for $X_\alpha$.

A common feature is that, similarly to the auxiliary Dirichlet boundary conditions (3.92), the wavefunction vanishes unless $s = s_\alpha$. Provided $s = s_\alpha$, collapsing the interval there is fluctuating adjoint $\mathcal{N} = (0, 2)$ chiral multiplet $S$, whose elliptic genus becomes singular. Using identical reasoning to the discussion surrounding equations (3.93) to (3.95) we replace this contribution by

$$(-)^r \frac{\delta^{(r)}(u - u_\alpha)}{\eta(q)^{2r}},$$

(4.43)

where the delta function is understood as a contour prescription around an order $r$ pole at $u = u_\alpha$ with unit residue.

Let us first consider the wavefunction of the $\mathcal{N} = (0, 2)$ exceptional Dirichlet boundary conditions $D_\alpha$. In addition to the contribution above, for both left and right boundary conditions, the only other contribution comes from the adjoint chiral
4.1 Exceptional Dirichlet

multiplet $\Phi$. In summary,

$$\langle D_\alpha|D_C(s)\rangle = \langle D_C(s)|D_\alpha\rangle = \frac{\delta^{(r)}(u - u_\alpha)}{(\vartheta(t^2)\eta(q)^2)^r}.$$  \hfill (4.44)

This wavefunction obeys the expected property (3.105).

For the $\mathcal{N} = (2,2)$ exceptional Dirichlet boundary conditions, there are additional contributions of the $\mathcal{N} = (0,2)$ chiral multiplets arising from the hypermultiplet fields with Neumann boundary conditions. For the right boundary condition $D_r^\alpha$, their contribution may be written in terms of $T^*R$ weights as

$$\prod_{i=1}^r \frac{1}{\vartheta(w^i_\alpha)} \prod_{\varrho \in \mathbb{Q}_{\mathbb{C}}} \frac{1}{\vartheta(w^\varrho)}|_{s=s_\alpha} = \frac{1}{\vartheta(t^{-2})^r} \prod_{\lambda \in \Phi_\alpha} \frac{1}{\vartheta(a^\lambda)}. \hfill (4.45)$$

Thus

$$\langle D_C(s)|D_r^\alpha\rangle = \frac{\delta^{(r)}(u - u_\alpha)}{(\eta(q)^2\vartheta(t^2))^r} \prod_{\lambda \in \Phi_\alpha} \vartheta(a^\lambda), \hfill (4.46)$$

and similarly

$$\langle D_l^\alpha|D_C(s)\rangle = \frac{\delta^{(r)}(u - u_\alpha)}{(\eta(q)^2\vartheta(t^2))^r} \prod_{\lambda \in \Phi_\alpha^+} \vartheta(a^\lambda). \hfill (4.47)$$

These wavefunctions satisfy the relative normalisations of the boundary states created by the $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ boundary conditions (4.21).

It is easy to check that these wavefunctions are consistent with the formulae for boundary amplitudes (3.102). For example, let us consider the right $\mathcal{N} = (0,2)$ exceptional Dirichlet $D_\alpha$. Then

$$\langle D_\alpha|B\rangle = \int du (\eta(q)^2\vartheta(t^2))^r \langle D_\alpha|D_C(s)\rangle \langle D_F(s)|B\rangle$$

$$= \langle D_F(s_\alpha)|B\rangle,$$  \hfill (4.48)

where $s_\alpha = e^{2\pi i u_\alpha}$ as before. This again is simply the evaluation of the ‘Fermi’ wavefunction of $B$ evaluated at values of the boundary holonomy $s$ fixed by the vacuum. The collection $\{\langle D_\alpha|B\rangle\}_{\alpha \in \text{f.p.}}$ is guaranteed to glue to a single holomorphic line bundle on $E_T(X)$. Analogous statements hold for left and $(2,2)$ exceptional Dirichlet boundary conditions.

Let us also briefly check that the wavefunctions are consistent with orthogonality. Note that if $\alpha \neq \beta$, the contour prescriptions are not compatible. On the other hand,
we have

$$\langle D^l_\alpha | D^r_\alpha \rangle = \oint du \left( \eta(q)^2 \vartheta(t^2) \right)^r Z_T \langle D^l_\alpha | D_C(s) \rangle \langle D_C(s) | D^r_\alpha \rangle = 1. \quad (4.49)$$

This is because in $Z_T$, at $u = u_\alpha$ there is a zero of order $r$ arising from $(0, 2)$ Fermis which become neutral in the vacuum $\alpha$, multiplied by a factor $\left( \eta(q)^2 \vartheta(t^2) \right)^r \prod_{\lambda \in F_\alpha} \vartheta(a^\lambda)$. The $\eta(q)^2$ comes from the non-zero factor in $\vartheta(1)$, and $\vartheta(t^2)$ from the Fermi multiplets in the same hypermultiplets as the Fermis which become neutral. Recalling the description of $\delta^{(r)}(u - u_\alpha)$ in the exceptional Dirichlet wavefunctions as a contour prescription around a pole at $u = u_\alpha$ of order $r$ and unit residue, we recover the correct normalisation above.

**Example**

We return to the example of supersymmetric QED. The wavefunctions of $D_\alpha$ are obtained from (4.44) by setting $r = 1$.

For the wavefunction of $D^r_\alpha$, from equation (4.13) the remaining fluctuating degrees of freedom are Fermi multiplets $\Phi_Y \beta$ for $\beta \leq \alpha$ and $\Phi_X \beta$ for $\beta > \alpha$. These contribute

$$\prod_{\beta \leq \alpha} \vartheta(t^{-1} v_\beta) \prod_{\beta > \alpha} \vartheta(t v_\beta^{-1}) \Bigg|_{sv_\alpha^{-1} t=1}$$

and combining with the contribution of the chiral $S$,

$$\langle D_C(s) | D^r_\alpha \rangle = \frac{\delta(u - u_\alpha)}{\eta(q)^2 \vartheta(t^2) \prod_{\beta < \alpha} \vartheta(t^{-2} \frac{v_\alpha}{v_\beta}) \prod_{\beta > \alpha} \vartheta(v_\alpha \frac{v_\beta}{v_\alpha})}, \quad (4.51)$$

which agrees with the general formula with repelling weights $\Phi^-_{\alpha}$ in (2.64).

Similarly for $D^l_\alpha$, from (4.14), contributing degrees of freedom from the hypermultiplets are the $\mathcal{N} = (2, 2)$ chirals $\Phi_X \beta$ for $\beta < \alpha$ and $\Psi_Y \beta$ for $\beta \geq \alpha$. So similarly

$$\langle D^l_\alpha | D_C(s) \rangle = \frac{\delta(u - u_\alpha)}{\eta(q)^2 \vartheta(t^2) \prod_{\beta < \alpha} \vartheta(\frac{v_\alpha}{v_\beta}) \prod_{\beta > \alpha} \vartheta(t^{-2} \frac{v_\alpha}{v_\beta})}, \quad (4.52)$$

where the denominator is constructed from the attracting weights $\Phi^+_{\alpha}$. 
4.1 Exceptional Dirichlet

4.1.5 Mirror Image

An interesting problem is to understand the mirror dual of the $\mathcal{N} = (2, 2)$ exceptional Dirichlet boundary conditions $\{D_\alpha\}$ of a theory $\mathcal{T}$. A hint is provided by the Coulomb branch image of these boundary conditions. Let us denote the Coulomb branch of $\mathcal{T}$ by $X^l$, known in the mathematics literature as the symplectic dual. The dual theory $\tilde{\mathcal{T}}$ has a Higgs branch $X^h$, and Coulomb branch $X$. There is a canonical isomorphism of fixed points, and we denote them by $\{\alpha\}$ in both $X$ and $X^l$. These are the images of the same massive vacua of $\mathcal{T}$ on its Higgs and Coulomb branches.

The mirror dual boundary condition for $\tilde{\mathcal{T}}$ must have a Higgs branch image in $X^h$ coinciding with the Coulomb branch image in $\mathcal{T}$ of $D_\alpha$. We consider the latter first, in the case of $\mathcal{T} = \text{SQED}[N]$. The generalisation to arbitrary abelian theories follows similarly.

For generic real masses, and zero complex masses, the Coulomb branch $X^l$ of $\mathcal{T}$ is the $A_{N-1}$ surface (a resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_N$). The Coulomb branch image of $D^r_\alpha$ is supported on the fibre $\varphi = 0$, which we denote $S_0$. This is a fibration, with base $\mathbb{R}$ parameterised by $\sigma$, and typical fibre $S^1$ by the dual photon $\gamma$. The photon circle shrinks where hypermultiplets become massless, i.e. when $\{\sigma - m_\alpha = 0\}$. These are the images of the vacua $\{\alpha\}$ on $X^l$. Thus $S_0$ is a chain of $N - 1$ copies of $\mathbb{P}^1$ capped on both ends by a copy of $\mathbb{C}$, see figure 4.3.

![Fig. 4.3](image)

The real moment map for $\mathcal{T}_C$, $h_\zeta = -\zeta \cdot \sigma$ decreases from left to right, in our choice of $\mathcal{E}_C$. The semi-classical analysis of section 3.4 of [56] implies the Coulomb branch support of $D^r_\alpha$ is necessarily contained in the locus of $S_0$ where the effective real mass of $X_\alpha$ is negative, i.e. $\sigma - m_\alpha < 0$. This consists of the union of all the divisors $\mathbb{P}^1$ and copy of $\mathbb{C}$ to the left of $\alpha$ in $S_0$ in figure 4.3. These are not just the repelling Lagrangian $(X^l)_{\alpha}^c$ for the Morse flow generated by $h_\zeta = -\zeta \cdot \mu_C = -\zeta \cdot \sigma$ on $X^l$, which is just the single $\mathbb{P}^1$ (or $\mathbb{C}$ for $\alpha = N$) to the left of $\alpha$. Analogous statements hold for $D^l_\alpha$.

Realising $X^l$ as the Higgs branch of $\tilde{\mathcal{T}}$, this locus is precisely the support of the stable envelope [67, 89] of the $A_{N-1}$ surface, in the chambers $\tilde{\mathcal{E}}_H$ and $\tilde{\mathcal{E}}_C$ of mirror mass and FI parameters under the identifications $(m_\alpha, \zeta) = (\tilde{\zeta}_\alpha, -\tilde{m})$. The Morse flow
on $X^l$ is with respect to the function $\tilde{h}_m = \tilde{m} \cdot \tilde{\mu}_H$, which is equal to $h_C$ after these identifications.

The above suggests that the mirror boundary conditions in $\tilde{T}$ should be supported on the stable envelope associated to the fixed point $\alpha$. We have not proved this yet; the semi-classical analysis only implies a necessary condition for the Coulomb branch support. However, in the next section we produce the mirror dual boundary conditions explicitly by studying a mirror symmetry interface, and show that indeed the Higgs branch image of the mirror dual boundary condition coincides with stable envelopes.

For abelian theories, where $X$ and $X^l$ are hyper-toric varieties, the stable envelope contains the attracting Lagrangian $(X^l)_\alpha$, but is generically larger [190]. Thus the dual of exceptional Dirichlet is generically not another exceptional Dirichlet.

### 4.2 Mirror Symmetry Interface

We now consider the $\mathcal{N} = (2,2)$ mirror symmetry interface between theories $T, \tilde{T}$, which flows to a trivial interface in the IR whilst exchanging Higgs and Coulomb branch data [56]. The mirrors of boundary conditions can be constructed by collision with the interface. Our aim is to construct boundary conditions mirror to the exceptional Dirichlet $D^r_\alpha$ and $\tilde{D}^r_\alpha$. This provides an alternative basis of supersymmetric ground states and, and we show in section 4.4 that this reproduces the construction of elliptic stable envelopes from supersymmetric gauge theory, and that the interface itself defines the mother function [70] in the equivariant elliptic cohomology of symplectic dual pairs. From here on out, we give explicit constructions for abelian theories, leaving non-abelian examples to future work.

#### 4.2.1 Definition

We consider the mirror symmetry interface between a pair of mirror abelian gauge theories $T$ on the left and $\tilde{T}$ on the right.

- The theory $T$ has $G = U(1)^r$ and $R = \mathbb{C}^N$ with $r \times N$ charge matrix $Q$. It has a Higgs branch flavour symmetry $T_H = U(1)^{N-r}$ with $(N-r) \times N$ charge matrix $q$.

- The theory $\tilde{T}$ has $\tilde{G} = U(1)^{N-r}$ and $\tilde{R} = \mathbb{C}^N$ with $(N-r) \times N$ charge matrix $\tilde{Q}$. It has a Higgs branch flavour symmetry $\tilde{T}_H = U(1)^r$ with $r \times N$ charge matrix $\tilde{q}$.
The Coulomb branch flavour symmetries are $T_C = \tilde{T}_H$ and $\tilde{T}_C = T_H$ and the mass and FI parameters are related by $(\tilde{\zeta}, \tilde{m}) = (m, -\zeta)$. The charge matrices are related by

$$
\begin{pmatrix} \tilde{q} \\ Q \end{pmatrix}^T = \begin{pmatrix} Q \\ q \end{pmatrix}^{-1}
$$

(4.53)

and further details can be found in [56]. Note that the multiplets in $\tilde{T}$ are twisted vector multiplets and twisted hypermultiplets, with the roles of $SU(2)_H$ and $SU(2)_C$ interchanged.

As is ubiquitous with $\mathcal{N} = (2,2)$ boundary conditions, the mirror symmetry interface depends on choice of polarisation $\varepsilon \in \{\pm\}^N$, see equation 2.87. We begin by imposing right Neumann boundary conditions $N_\varepsilon$ on $x^3 \leq 0$ and left Neumann boundary conditions $\tilde{N}_{-\varepsilon}$ on $x^3 \geq 0$,

$$
N_\varepsilon : \ D_3 X_\varepsilon = 0, \quad Y_\varepsilon = 0, \\
\tilde{N}_{-\varepsilon} : \ D_3 \tilde{X}_{-\varepsilon} = 0, \quad \tilde{Y}_{-\varepsilon} = 0.
$$

(4.54)

We then introduce $N$ 2d $\mathcal{N} = (2,2)$ chiral multiplets $\Phi_i$ and their T-duals $\tilde{\Phi}_i$ at $x^3 = 0$, whose scalar components $\phi_i, \tilde{\phi}_i$ are valued in $\mathbb{C}^* \cong \mathbb{R} \times S^1$. Finally, we introduce boundary superpotentials

$$
W = \sum_{i=1}^N X_{\varepsilon_i} e^{-\varepsilon_i \phi_i} - \phi_i (\tilde{Q}_i \cdot |\tilde{\varphi} + \tilde{q}_i \cdot \tilde{m}_C), \\
\tilde{W} = \sum_{i=1}^N e^{\varepsilon_i \tilde{\phi}_i} |\tilde{X}_{-\varepsilon_i} - (Q_i \cdot \varphi) + q_i \cdot m_C| \tilde{\phi}_i.
$$

(4.55)

where $\varphi, \tilde{\varphi}$ are the complex vector multiplet scalars. We have also introduced complex mass and FI parameters with $(\tilde{\zeta}_C, \tilde{m}_C) = (m_C, -\zeta_C)$ as this is useful to discuss anomalies, with the understanding that we will ultimately set them to zero.

The superpotentials identify

- $G, T_H$ and $U(1)_V$ as translation symmetries of $\phi$, or winding symmetries of $\tilde{\phi}$,
- $\tilde{G}, T_C$ and $U(1)_A$ as winding symmetries of $\phi$, or translation symmetries of $\tilde{\phi}$.

The charges of $\phi_i, \tilde{\phi}_i$ are fixed by the first terms in the superpotentials (4.55). The second terms break these symmetries explicitly. They may be interpreted as 2d $\theta$-angles

---

3We note that such multiplets and similar superpotentials appear ubiquitously in Hori-Vafa mirror symmetry [49]. In the superpotentials we have represented $\mathcal{N} = (2,2)$ multiplets by their scalar components, and will continue to make the same abuse of notation in the remainder of this section.
encoding the contribution to mixed anomalies. This interface reproduces the mirror map between Higgs and Coulomb branch chiral rings [56].

Another consistency check is that the mirror symmetry interface is anomaly-free for any polarisation $\varepsilon$. This requires anomalies of the Neumann boundary conditions $N_\varepsilon$ and $\tilde{N}_{-\varepsilon}$ are cancelled by those of the boundary chiral multiplets $\Phi_i, \tilde{\Phi}_i$. The latter can be seen from the superpotentials. Recall that the combination

$$\tilde{Q}^i \cdot \tilde{\phi} + \tilde{q}^i \cdot \tilde{m}_C$$

appearing in the superpotential $W$ is the effective complex mass of $\tilde{\phi}_i$ and encodes its charges under its translation symmetries $\tilde{G}, T_C$. The superpotential $W$ displays a mixed anomaly under translation symmetries $G, T_H$ or $U(1)_V$ of $\phi_i$. Similarly,

$$Q^i \cdot \phi + q^i \cdot m_C$$

is the effective complex mass for $\phi_i$ and encodes its charges under its translation symmetries $G, T_H$ and one obtains the same mixed anomaly by shifting $\tilde{\phi}_i$ under $\tilde{G}, T_C$ or $U(1)_A$.

Overall, the contribution to the mixed anomaly from $\phi_i, \tilde{\phi}_i$ is

$$\sum_{i=1}^{N} \left( f_V - \varepsilon_i (Q^i \cdot f + q^i \cdot f_H) \right) \left( f_A + \varepsilon_i (\tilde{Q}^i \cdot \tilde{f} + \tilde{q}^i \cdot \tilde{f}_H) \right) - 2N f_A f_V.$$  

We have denoted by $\tilde{f}$ the field strength for $\tilde{G}$ and identify $\tilde{f}_H = -f_C$ under mirror symmetry. In the summation, the first and second factors are the charges of the operators $e^{-\varepsilon_i \phi_i}$ and $e^{\varepsilon_i \tilde{\phi}_i}$, which create translation and winding modes respectively for $\phi_i$. The final term comes from the fermions $\psi_{\pm,\phi_i}$ in the $N = (2, 2)$ chiral multiplet $\Phi_i$, which are only charged under $R$ symmetries. This precisely cancels the mixed anomalies coming from the boundary conditions $N_\varepsilon$ and $\tilde{N}_{-\varepsilon}$ at the interface.

### 4.2.2 Example

We will consider the mirror symmetry interface for $\mathcal{T}$ supersymmetric QED with $N$ flavours. The mirror $\tilde{\mathcal{T}}$ is an abelian $A_{N-1}$-type quiver gauge theory with $G = U(1)^{N-1}$, and $X^i$ is the resolution of the $A_{N-1}$ singularity $\mathbb{C}^2/\mathbb{Z}_N$. We choose charge matrices corresponding to quiver conventions, as illustrated in figure 4.4.\footnote{Note that the action of T-duality on $\Phi_i$ dualises the scalar $\phi_i$, and leaves the fermions $\psi^{\phi_i}_{\pm}$ alone. The dual scalar $\tilde{\phi}_i$ and the same fermions comprise the dual $\mathcal{N} = (2, 2)$ twisted chiral multiplet to $\Phi_i$.}
The mass and FI parameters are identified according to \((m_\alpha, \zeta) = (\tilde{\zeta}_\alpha, -\tilde{m})\) and similarly for the chambers. In the default chambers \(C_C = \{\zeta > 0\}\) and \(C_H = \{m_1 > m_2 > \ldots > m_N\}\) for supersymmetric QED, the vacua \(\alpha\) in the mirror are

\[
\tilde{Y}_\beta = \sqrt{m_\beta - m_\alpha} \quad \text{for} \quad \beta < \alpha, \\
\tilde{X}_\beta = \sqrt{m_\alpha - m_\beta} \quad \text{for} \quad \beta > \alpha,
\]

with all other hypermultiplet fields vanishing.

Let us derive the weight space decomposition of the tangent space \(T_\alpha X^!\). We denote the matter representation of \(\tilde{G} \times \tilde{T}_H \times T_t\) of \(\tilde{T}\) by \(T^* \tilde{R}\), and work with the fundamental weights of \(\mathcal{T}\). Note \(T_t\) transforms the holomorphic symplectic form of \(X^!\) with weight \(-2\).

\[
\text{Ch} T^* \tilde{R} = t \sum_{\beta=1}^{N} \left( \frac{\tilde{s}_\beta}{\tilde{s}_{\beta-1}} + \frac{\tilde{s}_{\beta-1}}{\tilde{s}_\beta} \right),
\]

\[
\text{Ch} \tilde{\mathcal{G}}_C + t^2 \text{Ch} \tilde{\mathcal{G}}^*_C = (N - 1)(1 + t^2).
\]

where we have identified \(\tilde{s}_0 = \xi^{\frac{1}{2}}\) and \(\tilde{s}_N = \xi^{-\frac{1}{2}}\). Note that \(\xi\) is the formal parameter we have associated to \(T_C\), and will be identified as \(e^{2\pi i z_C}\) in our computations on \(E_\tau \times I\).

The choice of vacuum (4.59) determines

\[
t^{-\frac{\tilde{s}_{\beta-1}}{\tilde{s}_\beta}} = 1, \quad \beta < \alpha, \quad \Rightarrow \quad \tilde{s}_\beta = t^{-\beta} \xi^{\frac{1}{2}}, \quad \beta < \alpha, \\
t^{-\frac{\tilde{s}_{\beta}}{\tilde{s}_{\beta-1}}} = 1, \quad \beta > \alpha, \quad \Rightarrow \quad \tilde{s}_\beta = t^{-(N-\beta)} \xi^{-\frac{1}{2}}, \quad \beta \geq \alpha.
\]

Thus one has

\[
\text{Ch} T_\alpha X^! = \text{Ch} T^* \tilde{R} - \text{Ch} \tilde{\mathcal{G}}_C + t^2 \text{Ch} \tilde{\mathcal{G}}^*_C|_\alpha \\
= \xi^{-1} t^{-N+2\alpha} + \xi t^{N-2\alpha+2}.
\]
Therefore in our choice of $\mathfrak{C}_C$:

$$\tilde{\Phi}_\alpha^+ = \{e_C + (N - 2\alpha + 2)e_t\}, \quad \tilde{\Phi}_\alpha^- = \{-e_C + (-N + 2\alpha)e_t\}. \quad (4.64)$$

### 4.3 Enriched Neumann

We now use the mirror symmetry interface to derive the mirror of the collection of right exceptional Dirichlet boundary conditions $\tilde{D}_\alpha^r$. The mirror is a collection of right Neumann boundary conditions $N^r_\alpha$, defined by a polarisation and enriched by $\mathbb{C}^*$-valued chiral multiplets coupled via boundary superpotentials and twisted superpotentials. We refer to these boundary conditions as enriched Neumann. In section 4.4, we show that they generate states in the same $Q$-cohomology class as the elliptic stable envelopes of [67].

It was conjectured in [56] that the mirror of exceptional Dirichlet boundary conditions are again exceptional Dirichlet. This proposal reproduces the same boundary chiral rings, anomalies and Higgs branch support for generic complex FI parameters. However, it fails to capture the correct Higgs branch support with vanishing complex FI parameter and quarter-BPS boundary operators contributing to the general half superconformal index. Enriched Neumann may flow to exceptional Dirichlet in special cases, but this is not generically the case.

In summary, as was in fact partially anticipated in appendix B of [56], the mirror of exceptional Dirichlet boundary conditions are enriched Neumann boundary conditions.

#### 4.3.1 Definition and Derivation

We focus here on the case where $\mathcal{T}$ is supersymmetric QED, the extension to general abelian theories considered in [56] follows straightforwardly. We use the default chambers $\mathfrak{C}_C = \{\zeta > 0\}$ and $\mathfrak{C}_H = \{m_1 > m_2 > \ldots > m_N\}$ with corresponding chambers $\tilde{\mathfrak{C}}_C, \tilde{\mathfrak{C}}_H$ in the mirror obtained under the identifications $(m_\alpha, \zeta) = (\tilde{\zeta}_\alpha, -\tilde{m})$.

We start from the right exceptional Dirichlet boundary conditions for $\tilde{T}$ and collide with the mirror interface to derive right exceptional Neumann boundary conditions for $T$. This is shown in figure 4.5. The right exceptional Dirichlet boundary conditions $\tilde{D}_\alpha^r$ in the default chamber are

$$D_3\tilde{X}_\beta = 0, \quad \tilde{Y}_\beta = \tilde{c}_\beta, \quad \beta = 1, \ldots, \alpha - 1,$$

$$D_3\tilde{Y}_\alpha = 0, \quad \tilde{X}_\alpha = 0,$$

$$D_3\tilde{Y}_\beta = 0, \quad \tilde{X}_\beta = \tilde{c}_\beta, \quad \beta = \alpha + 1, \ldots N. \quad (4.65)$$
where $\tilde{c}_\beta$ are non-vanishing constants. The boundary condition on the complex scalars in the twisted vector multiplet requires

$$\tilde{\phi}_\beta = \begin{cases} 
+ \frac{1}{2} \zeta_C & \beta < \alpha \\
- \frac{1}{2} \zeta_C & \beta \geq \alpha 
\end{cases} \quad (4.66)$$

where $\zeta_C = -\tilde{m}_C$ is the complex FI parameter.

It is convenient to tailor the choice of polarisation in the mirror symmetry interface with the exceptional Dirichlet boundary condition. To collide with the right exceptional Dirichlet $\tilde{D}_r^\alpha$ associated to the vacuum $\alpha$ it is convenient to use the polarisation $\varepsilon = (+ \ldots + - \ldots -)$, where the first $-$ is in position $\alpha$. The interface then has superpotentials

$$W_{\text{int}} = \sum_{\beta < \alpha} (X_\beta | e^{-\phi_\beta} - \phi_\beta | \tilde{M}_C^\beta) + \sum_{\beta \geq \alpha} (Y_\beta | e^{\phi_\beta} - \phi_\beta | \tilde{M}_C^\beta),$$

$$\tilde{W}_{\text{int}} = \sum_{\beta < \alpha} (\tilde{e}^{-\tilde{\phi}_\beta} | \tilde{Y}_\beta - \tilde{M}_C^\beta | \tilde{\phi}_\beta) + \sum_{\beta \geq \alpha} (\tilde{e}^{-\tilde{\phi}_\beta} | \tilde{X}_\beta - \tilde{M}_C^\beta | \tilde{\phi}_\beta), \quad (4.67)$$

where $M_C^\beta = \varphi - m_{\alpha,c}, \tilde{M}_C^\beta = \tilde{\varphi}_\beta - \tilde{\varphi}_{\beta-1}$ denote the total complex masses of $X^\beta, \tilde{X}^\beta$. Here we have abused notation and denoted $\tilde{\varphi}_0 = -\tilde{\varphi}_N = \frac{1}{2} \zeta_C$.

On collision with the right exceptional Dirichlet boundary condition $\tilde{D}_r^\alpha$, we obtain a right Neumann boundary condition for $\mathcal{T}$, coupled to the boundary (twisted) superpotential

$$W = \sum_{\beta < \alpha} (X_\beta | e^{-\phi_\beta}) + (Y_\alpha | e^{\phi_\alpha} - (-\zeta_C) \phi_\alpha) + \sum_{\beta \geq \alpha} (Y_\beta | e^{\phi_\beta}),$$

$$\tilde{W} = \sum_{\beta < \alpha} (\tilde{c}_\beta e^{\tilde{\phi}_\beta} - M_C^\beta | \tilde{\phi}_\beta) + (-M_C^\alpha | \tilde{\phi}_\alpha) + \sum_{\beta \geq \alpha} (\tilde{c}_\beta e^{-\tilde{\phi}_\beta} - M_C^\beta | \tilde{\phi}_\beta), \quad (4.68)$$
Colliding with the exceptional Dirichlet boundary breaks the gauge symmetry $\tilde{G} = U(1)^{N-1}$ at the interface, shifting the $U(1)_A$ and $T_C$ weights of boundary operators charged under $\tilde{G}$. From the perspective of $\mathcal{T}$, this can be seen as redefinition of the boundary $U(1)_A$ and $T_C$ symmetries by the addition of a generator of $\partial|\tilde{G}$. This redefinition only alters the charges of the boundary operators constructed from $\Phi_\alpha, \tilde{\Phi}_\alpha$, which are are modified to those in table 4.1. The fermions $\psi^{\phi_\alpha}_{\pm}$ are not charged under the gauge symmetry $\tilde{G}$ and are therefore unaffected by this shift.

<table>
<thead>
<tr>
<th>Operator</th>
<th>$G$</th>
<th>$T_{H,\alpha}$</th>
<th>$T_C$</th>
<th>$U(1)_V$</th>
<th>$U(1)_A$</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{\phi_\alpha}$</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$s v_\alpha^{-1} t$</td>
</tr>
<tr>
<td>$e^{-\tilde{\phi}_\alpha}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-(N-2\alpha)$</td>
<td>$\xi t^{N-2\alpha}$</td>
</tr>
<tr>
<td>$\psi^{\phi_\alpha}_{\pm}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>$\mp 1$</td>
<td>$1, t^{-2}$</td>
</tr>
</tbody>
</table>

Table 4.1 Charges of operators in the boundary chiral multiplet $\Phi_\alpha$ in the construction of right enriched Neumann boundary condition $N^r_\alpha$ in supersymmetric QED.

We can now integrate out the boundary chiral multiplets $\Phi_\beta$ with $\beta \neq \alpha$. Let us do this for $\beta > \alpha$; the $\beta < \alpha$ case is treated similarly. The term $\tilde{c}_\beta e^{-\tilde{\phi}_\beta}$ in the twisted superpotential removes $e^{-\phi_\beta}$, promoting $\eta_\beta \equiv e^{\phi_\beta}$ to a $\mathbb{C}$-valued chiral multiplet. We can also integrate out $\tilde{\phi}_\beta$ using $\partial W / \partial \tilde{\phi}_\beta = 0$. The remaining contributions to the superpotentials are

$$
W_\beta = Y_\beta |\eta_\beta|,
\tilde{W}_\beta = M_\beta^\beta \left( \log M_\beta^\beta - 1 \right) - M_\beta^\beta \log(-\tilde{c}_\beta).
$$

The boundary superpotential imposes $X_\beta = 0$. The first term in the twisted superpotential is the 1-loop correction from integrating out the boundary chiral multiplet $\eta_\beta$. In combination they implement a flip of the boundary condition for the hypermultiplet $(X_\alpha, Y_\beta)$ - see section 5.3 of [56]. The remaining second term in the twisted superpotential contributes to the complexified 2d FI parameter.

This remaining boundary is a right Neumann boundary condition (3.62) for the vector multiplet, together with the Lagrangian splitting for hypermultiplets

$$
D_\perp Y_\beta = 0, \quad X_\beta = 0, \quad \beta \leq \alpha,
D_\perp X_\beta = 0, \quad Y_\beta = 0, \quad \beta > \alpha,
$$

(4.70)
4.3 Enriched Neumann

coupled to boundary $\mathbb{C}^*$-valued chirals multiplets $\Phi_\alpha, \tilde{\Phi}_\alpha$ with charges summarised in table 4.1 via the boundary superpotential and twisted superpotential

$$ W = Y_\alpha |e^{\phi_\alpha} - (-\zeta_C)\phi_\alpha|, \quad \tilde{W} = -(\varphi - m_{\alpha,\mathbb{C}})\tilde{\phi}_\alpha - t_{2d}\varphi. \quad (4.71) $$

The boundary condition supports a complexified 2d FI parameter $t_{2d}$, such that the boundary conditions on the real scalar $\sigma$ and dual photon $\gamma$ are

$$ \sigma + i\gamma = t_{2d}. \quad (4.72) $$

For later, we note that due to the first term in the boundary superpotential only $e^{-m\tilde{\phi}_\alpha}$ with $m \geq 0$ are genuine boundary chiral operators. We refer to this collection of boundary conditions as enriched Neumann and denote them by $N^r_\alpha$.

We can similarly construct the left enriched Neumann boundary conditions $N^l_\alpha$, mirror to the exceptional Dirichlet boundary conditions $\tilde{D}^l_\alpha$. We find that $N^l_\alpha$ is defined by a left $\mathcal{N} = (2,2)$ Neumann boundary condition for the vector multiplet (3.62), together with the Lagrangian splitting for the hypermultiplets

$$ D_\perp X_\beta = 0, \quad Y_\beta = 0, \quad \beta < \alpha, $$
$$ D_\perp Y_\beta = 0, \quad X_\beta = 0, \quad \beta \geq \alpha, \quad (4.73) $$

coupled to a boundary $\mathbb{C}^*$-valued chiral multiplet $\Phi_\alpha, \tilde{\Phi}_\alpha$ with charges summarised in table 4.2 via boundary superpotentials

$$ W = |Y_\alpha e^{\phi_\alpha} - \zeta_C\phi_\alpha|, \quad \tilde{W} = -\tilde{\phi}_\alpha(|\varphi - m_{\alpha,\mathbb{C}}). \quad (4.74) $$

Note the opposite sign of the contribution $\zeta_C\phi_\alpha$ to the boundary superpotential compared to (4.71), which reflects a twist operator with opposite topological charge.

<table>
<thead>
<tr>
<th>Operator</th>
<th>$G$</th>
<th>$T_{H,\alpha}$</th>
<th>$T_C$</th>
<th>$U(1)_V$</th>
<th>$U(1)_A$</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{\phi_\alpha}$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$sv^{-1}_{\alpha}t$</td>
</tr>
<tr>
<td>$e^{-\tilde{\phi}_\alpha}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$N + 2 - 2\alpha$</td>
<td>$\xi^{-1}t^{-N-2+2\alpha}$</td>
</tr>
<tr>
<td>$\psi^{\phi_\pm}_\alpha$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$\mp 1$</td>
<td>$1, t^{-2}$</td>
</tr>
</tbody>
</table>

Table 4.2 Charges of operators in the boundary chiral multiplet $\Phi_\alpha$ in the construction of right enriched Neumann boundary condition $N^l_\alpha$ in supersymmetric QED.

---

In the above derivation, $t_{2d} = -\sum_{\beta < \alpha} \log(-\tilde{c}_\beta) + \sum_{\beta > \alpha} \log(-\tilde{c}_\beta)$. 
We now provide two consistency checks by computing the boundary \('t Hooft\) anomalies and the Higgs branch support of the enriched Neumann boundary conditions $N^r_\alpha$.

### 4.3.2 Anomalies

We now compute the boundary mixed \('t Hooft\) anomalies for the enriched Neumann boundary conditions $N^r_\alpha$. As the mirror interface is trivial in the IR and anomaly free, this will agree by construction with the anomalies of exceptional Dirichlet boundary conditions $\tilde{D}^r_\alpha$ in the mirror. Moreover, as the anomalies of exceptional Dirichlet boundary conditions reproduce the effective Chern-Simons terms in the vacua $\alpha$, which match under mirror symmetry, this must reproduce the anomalies of the exceptional Dirichlet $D^r_\alpha$ in the same theory. We will check this for supersymmetric QED.

A key observation is that the $\mathbb{C}^\ast$-valued chiral multiplets have mixed anomalies between translation and winding symmetries. This can be seen, for example, from the twisted superpotential, which contains a coupling between $\tilde{\Phi}_\alpha$ and the twisted chiral field strength multiplet containing $\varphi$, as well as the background multiplet for $U(1)_\alpha \subset T_H$ containing $m_{\alpha,C}$. Thus $\tilde{\Phi}_\alpha$ can be interpreted as a dynamical complexified theta angle. The mixed anomalies can be seen by shifting $\tilde{\phi}_\alpha$ or performing a gauge or flavour transformation.

With this in mind, we compute the anomaly polynomial of $N^r_\alpha$. First, the contributions from the boundary conditions on the bulk vector and hypermultiplets are computed following [117], with the result

$$P(N^r_\alpha)_{\text{bulk}} = -ff_C + f_V f_A + \left( -\sum_{\beta \leq \alpha} (f - f^\beta_H) + \sum_{\beta > \alpha} (f - f^\beta_H) \right) f_A. \quad (4.75)$$

The first term comes from anomaly inflow from the bulk mixed Chern-Simons term between $G \cong U(1)$ to $T_C \cong U(1)$, or equivalently the bulk FI coupling. The second term comes from the gauginos surviving on Neumann boundary condition for the vector multiplet. The remaining terms arise from fermions in the hypermultiplets. Second, the contribution of the boundary $\mathbb{C}^\ast$-valued matter can be computed using table 4.1,

$$P(N^r_\alpha)_{\text{bdy}} = (f - f^\beta_H + f_V)(f_C - (N - 2\alpha)f_A) - 2f_A f_V, \quad (4.76)$$
where the first term comes from mixed translation-winding anomalies of $\phi_\alpha$ and $\tilde{\phi}_\alpha$, and the second from boundary fermions $\psi_{\phi_\alpha}$. Summing the two contributions,

$$\mathcal{P}[\mathcal{N}_\alpha^r] = -f_\alpha^c f_c + \left( \sum_{\beta<\alpha} (f_\beta^h - f_\alpha^h) + \sum_{\beta>\alpha} (f_\alpha^h - f_\beta^h) \right) f_A \tag{4.77}$$

Note that, beautifully, the gauge anomalies from bulk supermultiplets have been completely cancelled by the boundary chiral multiplets. This coincides with the boundary mixed ‘t Hooft anomalies for the exceptional Dirichlet boundary conditions $D^r_\alpha$, or equivalently the mixed supersymmetric Chern-Simons terms in a supersymmetric massive vacuum $\alpha$.

### Higgs Branch Support

We now determine the Higgs branch support of the enriched Neumann boundary conditions $\mathcal{N}_\alpha^r$. To contrast with exceptional Dirichlet, it is convenient to first consider $\zeta_C \neq 0$, where the Higgs branch is a complex deformation of $T\ast \mathbb{C}P^{N-1}$. We then set $\zeta_C = 0$ to recover $X = T\ast \mathbb{C}P^{N-1}$.

First, the boundary superpotential requires

$$X_\beta = 0 \ (\beta < \alpha), \quad X_\alpha Y_\alpha = \zeta_C, \quad Y_\beta = 0 \ (\beta > \alpha), \tag{4.79}$$

by $G = U(1)$. For non-zero complex FI parameter $\zeta_C \neq 0$, this is the attracting submanifold of the vacuum $\alpha$ in the complex deformation of the Higgs branch and coincides with the support of exceptional Dirichlet $D^r_\alpha$. However, sending $\zeta_C$ to zero, for enriched Neumann we now have $X_\alpha = 0$ and/or $Y_\alpha = 0$ and the support becomes

$$\mathcal{N}_\alpha^r \cap \mathcal{N}_\alpha^r+1 = X^{-}_\alpha \cup \overline{X^{-}_{\alpha+1}}, \tag{4.80}$$

where we have recycled the notation from section 4.1. This differs from the support of exceptional Dirichlet, which remains the attracting submanifold $X^{-}_\alpha \subset X$. The support
of enriched Neumann and exceptional Dirichlet boundary conditions for supersymmetric QED with 3 hypermultiplets are contrasted in figure 4.6.

![Diagram](image_url)

**Fig. 4.6** The support of right enriched Neumann boundary conditions (right) and exceptional Dirichlet (left) for supersymmetric QED with $N = 3$ hypermultiplets. See figure 4.2 for the interpretation of the hyper-toric diagram.

The support of enriched Neumann boundary conditions $N^r_\alpha$ coincides with the cohomological stable envelopes of $X = T^*\mathbb{C}P^{N-1}$ introduced in [89] in the default chamber. We will see shortly that the boundary amplitudes of enriched Neumann on $E_\tau$ reproduce the corresponding elliptic stable envelopes. A similar computation shows that the support of left enriched Neumann boundary conditions is the cohomological stable envelope in the opposite chamber for the mass parameters.

Let us summarise this picture more broadly for exceptional Dirichlet $D^r_\alpha$ and enriched Neumann $N^r_\alpha$ boundary conditions. Although the names of these boundary conditions refer to explicit UV constructions in abelian theories, we expect the same considerations to apply to analogous distinguished sets of boundary conditions labelled by vacua $\alpha$ in theories satisfying our assumptions in section 2.

- **Exceptional Dirichlet $D^r_\alpha$**
  
  - $\zeta_c$ generic: supported on the attracting submanifold $X_{\alpha,\zeta_c}^- \subset X_{\zeta_c}$.
  
  - $\zeta_c = 0$: supported on the attracting submanifold $X_\alpha^- \subset X$.
  
  - While $X_{\alpha,\zeta_c}^- \subset X_{\zeta_c}$ is closed for generic $\zeta_c$, $X_\alpha^- \subset X$ is generally not closed at $\zeta_c = 0$. The closure $\overline{X_\alpha^-}$ is not generally stable under perturbations away from $\zeta_c = 0$. In other words, the Higgs branch support jumps upon turning on a complex FI parameter.

- **Enriched Neumann $N^r_\alpha$**
4.3 Enriched Neumann

- $\zeta_C$ generic: supported on the attracting set $X_{\alpha,\zeta_C} \subset X_{\zeta_C}$.
- $\zeta_C = 0$: supported on the stable envelope $\text{Stab}(\alpha) \subset X$, which contains $X_\alpha^-$ and generically a union of some $X_\beta^-$ where $h_m|_\beta > h_m|_\alpha$.
- The support is given by closing $X_{\alpha,\zeta_C}^- \subset X_{\zeta_C}$ in the whole family of complex deformations labelled by $\zeta_C \in H^2(X, \mathbb{C})$, including $\zeta_C = 0$. In other words, the Higgs branch is smoothly deformed upon turning on a complex FI parameter.

It is straightforward to extend our derivation to general abelian theories, whose Higgs branches are hyper-toric varieties. For a vacuum $\alpha$, denote by $S \subset \{1, \ldots, N\}$ the subset of size $r$ corresponding to hypermultiplets $(X_i, Y_i)$ of zero real mass in $\alpha$. Then we may define the polarisation

$$
(\varepsilon_{\alpha}^r)_j = \begin{cases} 
- (\varepsilon_{\alpha}^r)_j & j \in S \\
(\varepsilon_{\alpha}^r)_j & j \notin S.
\end{cases}
$$

In [56] it was shown that for $\zeta_C = 0$, the Higgs branch image of $D_{\alpha}^r$ is the toric variety associated to the chamber $\Delta_{S,\varepsilon_{\alpha}^r}$ of the hyper-toric diagram of $X$, using notation therein. Repeating our analysis for a general abelian theory, we find that the image of enriched Neumann boundary conditions $N_{\alpha}^r$, for $\zeta_C = 0$, are toric varieties associated to the orthants $V_{S,\varepsilon_{\alpha}^r}$, obeying

$$
V_{S,\varepsilon_{\alpha}^r} = \bigcup_{\substack{\varepsilon_j = (\varepsilon_{\alpha}^r)_j \\ \forall j \notin S}} \Delta_{\varepsilon} \supset \Delta_{S,\varepsilon_{\alpha}^r}
$$

These are precisely the cohomological stable envelopes of hyper-toric varieties [190].

We note that the cohomological stable envelopes appeared as an orthonormal basis of boundary conditions of $\mathcal{N} = 4$ quantum mechanics in a work by one of the authors [92], containing 1d analogues of some of the results we discuss in the remainder of this paper.

Finally, we mention that the difference between exceptional Dirichlet and enriched Neumann boundary conditions is also detected by the general half index [117, 138, 139] counting quarter-BPS boundary local operators. We will explore this topic, connecting with the mathematical literature on vertex functions [157] and exploring their mirror symmetry properties [5, 161, 162] in a future work [158].
4.3.3 Enriched Neumann for the Dual

We can also employ these techniques to construct enriched Neumann boundary conditions for $\tilde{T}$, which will realise elliptic stable envelopes of the resolution of the $A_{N-1}$-type singularity. We will construct the left enriched Neumann boundary conditions $\tilde{N}_l^{\alpha}$ by acting with the same mirror symmetry interface on the left exceptional Dirichlet boundary conditions $D_{l}^{\alpha}$ given in (4.14).

To do so, we collide with a duality interface specified by the (twisted) superpotential

$$W = \sum_{\beta < \alpha} (Y_{\beta}|e^{-\phi_{\beta}} - \phi_{\beta}|\tilde{M}_{C}^{\beta}) + \sum_{\beta \geq \alpha} (X_{\beta}|e^{-\phi_{\beta}} - \phi_{\beta}|\tilde{M}_{C}^{\beta}),$$

$$\tilde{W} = \sum_{\beta < \alpha} (e^{-\tilde{\phi}_{\beta}}|\tilde{X}_{\beta} - M_{C}^{\beta}|\tilde{\phi}_{\beta}) + \sum_{\beta \geq \alpha} (e^{\tilde{\phi}_{\beta}}|\tilde{Y}_{\beta} - M_{C}^{\beta}|\tilde{\phi}_{\beta}).$$

Colliding with $D_{l}^{\alpha}$, similarly to before, the $\alpha$th terms flip the boundary condition for the twisted hyper $(\tilde{X}_{\alpha}, \tilde{Y}_{\alpha})$ in $\tilde{T}$. In summary, the dual is found to be the left enriched Neumann boundary condition, which we denote by $\tilde{N}_l^{\alpha}$, specified by the splitting:

$$D_{\perp}^{\alpha} \tilde{X}_{\beta} = 0, \quad \tilde{Y}_{\beta} = 0, \quad \beta \leq \alpha,$$

$$D_{\perp}^{\alpha} \tilde{Y}_{\beta} = 0, \quad \tilde{X}_{\beta} = 0, \quad \beta > \alpha,$$

(4.84)

coupled to $N - 1$ boundary $\mathbb{C}^*$-valued $N = (2,2)$ twisted chirals $\tilde{\Phi}_{\beta}$ for $\beta \neq \alpha$, and their $T$-duals, via (twisted) boundary superpotentials

$$W = -\sum_{\beta < \alpha} \phi_{\beta}|(\tilde{\varphi}_{\beta} - \tilde{\varphi}_{\beta-1}) - \sum_{\beta > \alpha} \phi_{\beta}|(\tilde{\varphi}_{\beta} - \tilde{\varphi}_{\beta-1}),$$

$$\tilde{W} = \sum_{\beta < \alpha} \left( e^{-\tilde{\phi}_{\beta}}|\tilde{X}_{\beta} - (m_{\alpha,C} - m_{\beta,C})\tilde{\phi}_{\beta} \right) + \sum_{\beta > \alpha} \left( e^{\tilde{\phi}_{\beta}}|\tilde{Y}_{\beta} - (m_{\alpha,C} - m_{\beta,C})\tilde{\phi}_{\beta} \right),$$

(4.85)

where we have abused notation and let $\tilde{\varphi}_{0} = -\tilde{\varphi}_{N} = \frac{1}{2}\zeta_{C}$. As before, collision with $D_{l}^{\alpha}$ shifts the charges of operators at the boundary charged under the gauge symmetry of $\tilde{T}$. These are just the scalars $\phi_{\beta}$ in the $\mathbb{C}^*$ valued chirals $\Phi_{\beta}$. Note from the perspective of $\tilde{T}$ the valid twist operators are $e^{n\phi_{\beta}}$ for $\beta < \alpha$ and $e^{-n\phi_{\beta}}$ for $\beta > \alpha$, where $n \in \mathbb{Z}_{\geq 0}$. The shift in charges can be encoded in the substitution $sv_{\alpha}^{-1}t = 1$ in the characters of these operators:

$$e^{\phi_{\beta}} : \quad sv_{\beta}^{-1}t \rightarrow v_{\alpha}v_{\beta}^{-1} \quad \text{for} \quad \beta < \alpha,$$

$$e^{-\phi_{\beta}} : \quad s^{-1}v_{\beta}t \rightarrow v_{\alpha}^{-1}v_{\beta}t^{2} \quad \text{for} \quad \beta > \alpha.$$

(4.86)

One can also determine the Higgs branch support of the above left enriched Neumann boundary conditions for $\tilde{T}$, following section 5 of [56]. The calculation is similar to
that of $\mathcal{T}$. The result is that the Higgs branch image is given by the $\tilde{G} = U(1)^{N-1}$ quotient of the following Lagrangian of the matter representation $\tilde{Q}$

$$\tilde{Y}_\alpha = 0, \quad D_\perp \tilde{X}_\alpha = 0,$$

$$\tilde{X}_\beta \tilde{Y}_\beta = m_{\alpha,\mathcal{C}} - m_{\beta,\mathcal{C}} \quad \beta \neq \alpha.$$  \hspace{1cm} (4.87)

In our case, we set the complex masses to zero, $m_C = 0$.

It is then not hard to check that the Higgs branch support of $\tilde{N}_\alpha$ in the $A_{N-1}$ surface matches the putative Coulomb branch images of $D^i_\alpha$ described in section 4.1.5. In particular, the slice $\mathcal{S}_0$ of the Coulomb branch of $\mathcal{T}$ is mapped by mirror duality to a slice $S^0 = \{ \tilde{X}_\gamma \tilde{Y}_\gamma = 0 \quad \forall \gamma = 1, \ldots, N \}$, and the axis labelled by the Coulomb branch moment map $\sigma$ is mapped to the dual Higgs branch moment map $\tilde{\mu}_H = \frac{1}{2}(|\tilde{X}_N|^2 - |\tilde{Y}_N|^2 + |\tilde{X}_1|^2 - |\tilde{Y}_1|^2)$. This is illustrated in the figure 4.7 for the case with 3 hypermultiplets.

![Figure 4.7](image)

**Fig. 4.7** The slice $S^0$ of the Higgs branch of $\tilde{T}$. The support of $\tilde{N}_\alpha$ is the blue region, and $\tilde{N}^i_\alpha$ the red, for $\alpha = 1$. The real moment map $h_{\tilde{\omega}} = \tilde{m} \cdot \tilde{\mu}_H = -\zeta \cdot \tilde{\mu}_H$ on the Higgs branch of $\tilde{T}$ decreases from left to right in our choice of $\mathcal{C}_H = \mathcal{C}_C$.

### 4.4 Amplitudes and Wavefunctions

We now place the theory on an elliptic curve $E_\tau$, and compute the boundary amplitudes and wavefunctions of both the mirror symmetry interface and enriched Neumann boundary conditions. We will show that they coincide with the mother function (reviewed in the main text), and elliptic stable envelopes (reviewed in appendix A.2), respectively. We will expand on the connection to elliptic stable envelopes in section 4.5.
4.4.1 Mirror Symmetry Interface & Mother Function

We first consider the wavefunction of the mirror symmetry interface, obtained by forming a sandwich with auxiliary Dirichlet boundary conditions. This provides an integration kernel that may be used to compute the action of mirror symmetry on the wavefunctions of other boundary conditions. We will identify this kernel with the mother function in equivariant elliptic cohomology.

Let us then consider the partition function on $E_\tau \times [-\ell, \ell]$, with the mirror interface at $x^3 = 0$, a left reference Dirichlet boundary condition for $\mathcal{T}$ at $x^3 = -\ell$, and a right reference Dirichlet boundary condition for $\tilde{T}$ at $x^3 = \ell$. This is illustrated in figure 4.8.

$$
\mathcal{I}_F (s, \tilde{s}) = \langle D_F(s) | W_{\text{int}}, \bar{W}_{\text{int}} | I | \tilde{D}_F(\tilde{s}) \rangle
$$

![Fig. 4.8](image-url)

The path integral on $E_\tau \times [-\ell, \ell]$ which yields the mother function.

There is a choice of $N = (2, 2)$ or $N = (0, 2)$ reference Dirichlet boundary conditions, which will lead to different normalisations of the mirror interface kernel function. To ensure the set-up is compatible with real masses in any chamber and also to match results in the mathematical literature [70, 123], we will choose the $N = (0, 2)$ boundary conditions:

$$
D_F(s) : |\Phi_X| = |\Phi_Y| = 0,
\tilde{D}_F(\tilde{s}) : |\tilde{\Phi}_X| = |\tilde{\Phi}_Y| = 0.
$$

Here $s_a$, $a = 1, \ldots, r$ are holonomies around $E_\tau$ for $\mathcal{T}$, and $\tilde{s}_{a'}$, $a' = 1, \ldots, N - r$ for $\tilde{T}$. Recall that these boundary conditions also impose a Dirichlet boundary condition for the 3d $\mathcal{N} = 2$ vector multiplet, where the gauge connections at the boundary are set equal to $s_a$ and $\tilde{s}_{a'}$ respectively, and a Neumann boundary condition for the adjoint chiral $\varphi$, $\bar{\varphi}$.

We denote the kernel function with this choice of auxiliary Dirichlet boundary conditions as

$$
\mathcal{I}_F (s, \tilde{s}) = \langle D_F(s) | \mathcal{I} | \tilde{D}_F(\tilde{s}) \rangle.
$$

The kernel functions for other choices of reference Dirichlet boundary conditions are related by overall normalisations as in section 3.8.
This kernel function may be computed explicitly for abelian gauge theories and coincides with the *mother function* \[70, 123\] in equivariant elliptic cohomology for hyper-toric varieties, reviewed in section 4.4.3. There we will also see the reason for its mathematical name; by using the mother function as an integration kernel to compute the action of mirror symmetry on exceptional Dirichlet boundary conditions; it relates the elliptic stable envelopes of a pair of symplectic dual varieties \((X, X^\dagger)\) in a particular way.

**Computation**

Here we focus on the wavefunction when \(\mathcal{T}\) is supersymmetric QED with \(N\) flavours and \(\widetilde{\mathcal{T}}\) is mirror the abelian \(A_{N-1}\)-type quiver gauge theory, the extension to the abelian case is straightforward. This will receive contributions from the following sources:

- Colliding the left Dirichlet boundary condition \(D_F(s)\) with the right Neumann boundary condition \(N_\varepsilon\), there remains Fermi multiplets \(\Psi_{Y_\varepsilon}\). Similarly, colliding the right Dirichlet boundary condition \(\widetilde{D}_F(\tilde{s})\) with the left Neumann boundary condition \(\widetilde{N}_{-\varepsilon}\) leaves behind Fermi multiplets \(\Psi_{\tilde{Y}_{-\varepsilon}}\).

- The remaining fluctuating degrees of freedom from colliding the boundary conditions for the vector multiplets are adjoint chiral multiplets \(\Phi_\varphi, \Phi_{\tilde{\varphi}}\), with scalar components \(\varphi\) and \(\tilde{\varphi}\).

- The \(\mathbb{C}^*\)-valued \(\mathcal{N} = (2, 2)\) boundary chiral multiplets \(\Phi_\beta\).

Combining the elliptic genera of these contributions will yield the kernel function of the mirror symmetry interface.

We enumerate the elliptic genera of these contributions in turn. First, the contributions from the \(\mathcal{N} = (0, 2)\) Fermi multiplets arising from bulk hypermultiplets are

\[
\Psi_{Y_\varepsilon} : \prod_{\beta=1}^{N} \vartheta(t^{-1}(sv^{-1}_\beta)_{\varepsilon\beta}), \quad \Psi_{\tilde{Y}_{-\varepsilon}} : \prod_{\beta=1}^{N} \vartheta(t(\tilde{s}_\beta/\tilde{s}_{\beta-1})^{-\varepsilon\beta}).
\] (4.90)

The contributions from the adjoint \((0, 2)\) chiralas are

\[
\Phi_\varphi : \vartheta(t^{-2})^{-1}, \quad \Phi_{\tilde{\varphi}} : \vartheta(t^2)^{-(N-1)}.
\] (4.91)

The contribution from \(\mathbb{C}^*\)-valued chiral multiplets \(\{\Phi_\beta\}_{\beta=1,\ldots,N}\) is more difficult to compute. The description of the duality interface is slightly non-Lagrangian as it involves boundary superpotentials and twisted superpotentials coupling to \(\Phi_\beta\) and its
T-dual $\tilde{\Phi}_\beta$ simultaneously. However, since these couplings are exact for the elliptic genus, it is reasonable to treat the contributions as isolated 2d $\mathbb{C}^*$-valued chiral multiplets.

It is unclear how to compute the R-R sector elliptic genus of these chiral multiplets using supersymmetric localisation. Instead, we first compute their elliptic genus in the NS-NS sector by employing an operator counting argument, before performing a spectral flow back to the R-R sector. This computation is performed in appendix C. The result is

$$\Phi, \tilde{\Phi} : \prod_{\beta=1}^N \vartheta(t^2) \vartheta((sv_\beta^{-1})^{-\varepsilon_\beta} (\tilde{s}_\beta / \tilde{s}_{\beta-1})^{\varepsilon_\beta})$$

(4.92)

We note that the arguments of the theta-functions in the denominators correspond to the weights of the boundary operators $e^{-\varepsilon_\beta \phi_\beta}$ and $e^{\varepsilon_\beta \tilde{\phi}_\beta}$.

Combining the three contributions (4.90), (4.91) and (4.92), yields the result

$$\mathcal{I}_F(s, \tilde{s}) = - \prod_{\beta=1}^N \vartheta \left( (sv_\beta^{-1})^{-\varepsilon_\beta} (\tilde{s}_\beta / \tilde{s}_{\beta-1})^{\varepsilon_\beta} \right).$$

(4.93)

We emphasise that different choices of polarisation $\varepsilon$ in the definition of the mirror interface yield the same mirror interface kernel up to a sign, when sandwiched between the $\mathcal{N} = (0, 2)$ reference Dirichlet boundary conditions. Up to an overall sign and a re-definition of gauge and R-symmetries, this is the mother function for $T^*\mathbb{P}^{N-1}$ [70, 123].

Had we sandwiched with reference $\mathcal{N} = (2, 2)$ Dirichlet boundary conditions $D_{-\varepsilon}(s)$ and $\tilde{D}_{\varepsilon}(\tilde{s})$, upon shrinking the interval the remaining fluctuating degrees of freedom are solely the $\mathbb{C}^*$-valued chirals, and the interface kernel would just yield (4.92).

**Equivariant Elliptic Cohomology**

We now explain why, from our physical perspective, it is natural to identify the duality interface as generating an element of the $T_f$ equivariant elliptic cohomology of $X \times X^!$, i.e. a holomorphic line bundle over the variety $\text{Ell}_{T_f}(X \times X^!)$.

One may view the partition function of the interface as that of a doubled theory $\mathcal{T} \times \tilde{\mathcal{T}}$ on $E_r \times [0, \ell]$, by folding together the two theories $\mathcal{T}$ and $\tilde{\mathcal{T}}$. The doubled theory consists of two sectors which are decoupled in the bulk, but are coupled at the duality interface which becomes a boundary condition.

The Higgs branch of $\mathcal{T} \times \tilde{\mathcal{T}}$ is $X \times X^!$, with an equivariant action of $T_f = T_C \times T_H \times T_L$. For the purposes of this discussion, we denote the image of the massive vacua of $\mathcal{T}$ on its Higgs branch $X$ by $\alpha$, and its Coulomb branch $X^!$ by $\tilde{\alpha}$. Then $\mathcal{T} \times \tilde{\mathcal{T}}$ has $N^2$ vacua.
4.4 Amplitudes and Wavefunctions

given by a choice of \((\alpha, \tilde{\alpha})\). Following section 3.4 (or appendix A), the equivariant elliptic cohomology variety of \(X \times X^!\), or spectral curve for the space of supersymmetric ground states of \(T \times \tilde{T}\), is:

\[
\text{Ell}_{T_f}(X \times X^!) := \left( \bigcup_{(\alpha, \tilde{\alpha})} E^{(\alpha, \tilde{\alpha})}_{T_f} \right) / (\Delta \times \tilde{\Delta}).
\]  (4.94)

In the above

- \(E^{(\alpha, \tilde{\alpha})}_{T_f} \cong \Gamma_f \otimes_{\mathbb{Z}} E_*\) are \(N^2\) copies of the torus of background flat connections for \(T_f\), associated to the supersymmetric vacua \((\alpha, \tilde{\alpha})\).
- \(\Delta \times \tilde{\Delta}\) identifies:
  - The copies \(E^{(\gamma, \tilde{\alpha})}_{T_f}\) and \(E^{(\gamma, \tilde{\beta})}_{T_f}\) for all \(\gamma\), at points \(\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}\), where \(\lambda \in \Phi_\alpha \cap (-\Phi_\beta)\) labels an internal edge of the GKM diagram of \(X\).
  - The copies \(E^{(\gamma, \alpha)}_{\tilde{T}_f}\) and \(E^{(\gamma, \beta)}_{\tilde{T}_f}\) for all \(\gamma\), at points \(\tilde{\lambda} \cdot \tilde{z} \in \mathbb{Z} + \tau \mathbb{Z}\) where \(\tilde{\lambda} \in \tilde{\Phi}_\alpha \cap (-\tilde{\Phi}_\beta)\) labels an internal edge of the GKM diagram of \(X^!\). Here \(\tilde{z} = (z_C, z_t)\).

As a boundary condition for \(T \times \tilde{T}\), the interface naturally defines a \(Q\)-cohomology class, or supersymmetric ground state. The mother function \(I_F(s, \tilde{s})\) is now interpreted as its wavefunction, with reference boundary condition \(D_F(s) \times \tilde{D}_F(\tilde{s})\). See figure 4.9.

\[I_F(s, \tilde{s}) = W_{\text{int}}, \tilde{W}_{\text{int}}\]

\(\mathcal{T} \times \tilde{\mathcal{T}}\)

\(D_F(s) \times \tilde{D}_F(\tilde{s})\)

\(N_e \times N_e\)

\textbf{Fig. 4.9} The wavefunction of the duality interface.

By evaluating \(I_F(s, \tilde{s})\) at the values of \(s\) and \(\tilde{s}\) specified by \((\alpha, \tilde{\alpha})\), one recovers an \(N \times N\) matrix of boundary amplitudes, glued along its rows according to the GKM description of \(\mathcal{T}\), and its columns according to that of \(\tilde{\mathcal{T}}\). These form a section of a holomorphic line bundle over \(\text{Ell}_{T_f}(X \times X^!)\). We do not reproduce the amplitudes explicitly here in the interest of brevity, since we shall see shortly that we effectively compute them as the boundary amplitudes of enriched Neumann boundary conditions.
We note that the mother function expressed as in (4.89) is, in the equivariant elliptic cohomology terminology, in off-shell form. To the best of the author’s knowledge, off-shell mother functions have only been derived explicitly for hyper-toric varieties [123]. For more complicated spaces, obtained by a non-abelian gauge quotient, the mother function is often given in terms of the matrix of amplitudes described above [70]. This is presumably related to the difficulty of deriving the mirror symmetry interface in terms of explicit boundary matter for mirror pairs of non-abelian gauge theories.

4.4.2 Enriched Neumann & Elliptic Stable Envelopes

We will first compute the wavefunctions of enriched Neumann boundary conditions from first principles. We then show that colliding the mirror symmetry interface with exceptional Dirichlet boundary conditions produces the same wavefunctions using the kernel function. We again stick to supersymmetric QED, the generalisation to general abelian theories is easy.

We first consider the wavefunction of right boundary enriched Neumann,

\[
\langle D_F(s) | N_\alpha^r \rangle .
\]  

(4.95)

After taking the interval length \( \ell \to 0 \), the remaining fluctuating degrees of freedom are the chiral multiplet \( \Phi_\phi \), Fermi multiplets \( \Psi_X, \Psi_Y \) for \( \gamma \leq \alpha \), and the boundary \( \mathbb{C}^* \) chiral multiplet \( \Phi_\alpha \). The contribution of the \( \mathbb{C}^* \) chiral multiplet\(^6\) is computed using the same method as for the mother function and gives the contribution (4.92) for \( \gamma = \alpha \), except with \( \tilde{s} = \tilde{s}|_\alpha \). We return to this observation in the next section.

Combining these contributions, we find

\[
\langle D_F(s) | N_\alpha^r \rangle = \frac{1}{\vartheta(t^{-2})} \prod_{\gamma \leq \alpha} \frac{\vartheta(t^{-1}s^{-1}v_\gamma)}{\vartheta(s^{-1}v_\alpha)} \frac{\vartheta(t^2)\vartheta(s^{-1}t\xi t^{N-2}\alpha)}{\vartheta(s^{-1}t)\vartheta(\xi t^{N-2}\alpha)} \prod_{\gamma > \alpha} \vartheta(t^{-1}s^{-1}v_\gamma),
\]

\[
= \prod_{\gamma < \alpha} \frac{\vartheta(t^{-1}s^{-1}v_\gamma)}{\vartheta(s^{-1}v_\alpha)} \frac{\vartheta(s^{-1}t\xi t^{N-2}\alpha)}{\vartheta(\xi t^{N-2}\alpha)} \prod_{\gamma > \alpha} \vartheta(t^{-1}s^{-1}v_\gamma),
\]

(4.96)

\(^6\)Note that similar ratios of theta functions appeared in [98] in the IR treatment of holomorphic blocks: partition functions on \( S^1 \times HS^2 \). There they arise as replacements of contributions of mixed Chern-Simons levels, motivated by quasi-periodicity and meromorphicity arguments. They occur naturally in our UV perspective as contributions of \( \mathbb{C}^* \)-valued matter on the boundary torus \( E_\tau = \partial(S^1 \times HS^2) \). We return to this observation in a future work [158].
This expression matches the elliptic stable envelope for $X = T^*\mathbb{P}^{N-1}$ \cite{67}, in ‘off-shell’ form:

$$\langle D_F(s)|N^r_\alpha \rangle = \text{Stab}(\alpha)\xi_{H,\xi}. \quad (4.97)$$

Recall that the boundary amplitudes at the origin of mass parameter space are constructed using $(0, 2)$ exceptional Dirichlet, and are, using any of the equivalent results of chapter 3 or section 4.1:

$$\langle \beta|N^r_\alpha \rangle = \langle D_F(v_\beta t^{-1})|N^r_\alpha \rangle = \prod_{\gamma < \alpha} \vartheta(v_\gamma v_{\beta}^{-1}) \vartheta(v_\alpha v_\gamma^{-1}) \prod_{\gamma > \alpha} \vartheta(t^{-2}v_\beta v_\gamma^{-1}). \quad (4.98)$$

They are the restrictions $\text{Stab}(\alpha)\xi_{H,\xi}|_\beta := \text{Stab}(\alpha)\xi_{H,\xi}|_{E_{Tf}}$ to the sheets $E_{Tf}$ of $E_T(X)$.

**Consistency**

We now provide some physical reasoning for why the overlaps $\langle \beta|N^r_\alpha \rangle$ coincide with the elliptic stable envelopes.

The collection $\{\langle \beta|N^r_\alpha \rangle\}_{\beta = 1, \ldots, N}$ glues to give a section of a line bundle over $E_T(X) = \text{Ell}_T(X) \times E_{Tc}$. This is consistent with our description of supersymmetric ground states in section 3.4. In particular, by construction, it transforms with the correct factors of automorphy as the elliptic stable envelopes, as outlined in appendix A.2.

The matrix of boundary amplitudes is lower-triangular: it vanishes for $\beta < \alpha$. This reflects that the Higgs branch support of the enriched Neumann boundary condition $N^r_\alpha$ contains only the fixed points $\beta \geq \alpha$. This is consistent with the support condition of elliptic stable envelopes in appendix A.2.

The boundary amplitudes constructed using $N = (2, 2)$ exceptional Dirichlet boundary conditions $D^i_\alpha$ are simply related to the $(0, 2)$ boundary amplitudes by a normalisation. For example, for QED:

$$\langle D^j_\beta|N^r_\alpha \rangle = \langle D_F(s_\beta)|N^r_\alpha \rangle \prod_{\lambda \in \Phi^+_{H,\xi}} \frac{1}{\vartheta(a^\lambda)}$$

$$= \begin{cases} \vartheta(t^2)\vartheta(\xi_{Tc}) \prod_{\alpha < \gamma < \beta} \vartheta(t^{-2}v_\gamma v_\beta^{-1}) & \text{if } \beta > \alpha \\ 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta < \alpha. \end{cases} \quad (4.99)$$

The quasi-periodicities of the non-vanishing overlaps must be consistent with the anomalies of the boundary conditions $D^j_\beta$, $N^r_\alpha$, and are determined by the difference of
Chern-Simons levels $K_\alpha - K_\beta$. In particular, the diagonal $D_\alpha^l$, $N_\alpha^r$ should be non-zero (as the supports overlap), and transform with trivial factor of automorphy, and thus must be a constant. Since $\langle D_\beta^l \rangle$ and $\langle D_\beta^r \rangle = \langle \beta \rangle$ are related via the normalisation (4.21), this determines the normalisation of $\langle \alpha | N_\alpha^r \rangle = \prod_{\lambda \in \Phi^+} \vartheta (a^\lambda)$, in agreement with the normalisation condition of elliptic stable envelopes in appendix A.2.

In summary, the boundary amplitudes obey all the defining characteristics of elliptic stable envelopes. Since elliptic stable envelopes are unique [67], the boundary amplitudes should in fact exactly generate the elliptic stable envelopes as a cohomology class.

Note that the matrix of boundary amplitudes (4.99) is nothing but the pole-subtraction matrix introduced in [67]. The reason for this identification is explained in [158].

**Mirror Theory**

One can similarly compute the wavefunction of enriched Neumann for the mirror $\tilde{T}$. We consider the wavefunction of left enriched Neumann boundary condition,

$$\langle \tilde{N}_\alpha^l | \tilde{D}_F (\tilde{s}) \rangle.$$  (4.100)

After taking interval length $\ell \to 0$, there are the following contributions from Fermi and chiral multiplets arising from the boundary conditions for bulk supermultiplets,

$$\Psi_{\tilde{Y}_\gamma} : \vartheta \left( t \tilde{s}_\gamma / \tilde{s}_{\gamma - 1} \right) \quad \gamma \leq \alpha,$n

$$\Psi_{\tilde{X}_\gamma} : \vartheta \left( t \tilde{s}_{\gamma - 1} / \tilde{s}_\gamma \right) \quad \gamma > \alpha,$n

$$\Phi_{\tilde{\phi}} : \vartheta \left( t^2 \right)^{- (N-1)}.$$  (4.101)

The contribution of the $N - 1 \mathbb{C}^*$-valued twisted chiral multiplets is

$$\prod_{\gamma < \alpha} \vartheta (t^2) \vartheta \left( t^{-1} \frac{\tilde{s}_{\gamma - 1} \tilde{v}_\alpha}{\tilde{s}_\gamma} \right) \prod_{\gamma > \alpha} \vartheta (t^2) \vartheta \left( t^{-1} \frac{\tilde{s}_{\gamma - 1} \tilde{v}_\alpha}{\tilde{s}_\gamma} \right) \prod_{\gamma < \alpha} \vartheta \left( t^{-1} \frac{\tilde{s}_{\gamma - 1} \tilde{v}_\alpha}{\tilde{s}_\gamma} \right) \vartheta \left( t^2 \frac{\tilde{v}_\alpha}{\tilde{v}_\gamma} \right).$$  (4.102)

Putting these contributions together

$$\langle \tilde{N}_\alpha^l | \tilde{D}_F (\tilde{s}) \rangle = \prod_{\gamma < \alpha} \vartheta \left( t^{-1} \frac{\tilde{s}_{\gamma - 1} \tilde{v}_\alpha}{\tilde{s}_\gamma} \right) \vartheta \left( t \frac{\tilde{s}_\alpha}{\tilde{s}_{\alpha - 1}} \right) \prod_{\gamma > \alpha} \vartheta \left( t^{-1} \frac{\tilde{s}_{\gamma - 1} \tilde{v}_\alpha}{\tilde{s}_\gamma} \right) \vartheta \left( t^2 \frac{\tilde{v}_\alpha}{\tilde{v}_\gamma} \right).$$  (4.103)
4.4 Amplitudes and Wavefunctions

which, recalling the identifications \((m, \zeta) = (\tilde{\zeta}, -\tilde{m})\), may be identified as the elliptic stable envelope for the \(A_{N-1}\) surface [67, 70]

\[
\langle \tilde{N}_\alpha^l | \tilde{D}_F(\tilde{s}) \rangle = \hat{\text{Stab}}(\alpha)_{\tilde{m}^{opp}} \tilde{\zeta}^{-1}
\]

in the chambers

\[
\tilde{\mathcal{C}}_H = \{ \tilde{m} < 0 \}, \quad \tilde{\mathcal{C}}_C = \{ \tilde{\zeta}_1 > \ldots > \tilde{\zeta}_N \}.
\]

The inverted \(\tilde{\xi}\) and flipped mass chamber in (4.104) are consistent with the fact we are considering wavefunctions of left boundary conditions.

The boundary amplitudes of \(\tilde{N}_\alpha^l\) are given by evaluating (4.103) at (4.62), the value of the boundary \(\tilde{G}\) fugacities \(\tilde{s}\) in the vacuum \(\beta\) of \(\tilde{T}\) fixed by the expectation values (4.65):

\[
\langle \tilde{N}_\alpha^l | \tilde{D}_\beta \rangle = \begin{cases} 
0 & \text{if } \beta > \alpha \\
(-1)^{N-1} \partial(g^{-1} t^{-N+2\alpha}) & \text{if } \beta = \alpha \\
(-1)^{N-\alpha+\beta-1} \partial(t^2) \frac{\theta(\frac{\tilde{\zeta}^{-1} t^{-N+2\beta}}{\tilde{m}})}{\theta(\frac{\tilde{\zeta}^{-1} t^{-N+2\alpha}}{\tilde{m}})} \prod_{\beta < \gamma < \alpha} \frac{\theta(t^{-2\tilde{m}_{\gamma}})}{\theta(t^{-2\tilde{m}_{\gamma}})} & \text{if } \beta < \alpha.
\end{cases}
\]

This is upper-triangular, reflecting the fact that \(\tilde{N}_\alpha^l\) contains only the fixed points \(\beta \leq \alpha\), see figure 4.7. These amplitudes glue to a section of line bundle over \(\text{Ell}_{\tilde{T}}(X^1) \times E_{T_H} \equiv E_{\tilde{T}}(X^1)\), where we have defined \(\tilde{T} = T_C \times T_i\). This is the extended elliptic cohomology of \(X^1\), or equivalently the spectral curve for \(\tilde{T}\).

4.4.3 Elliptic Stable Envelopes from Duality Interface

We now derive the same wavefunctions and boundary amplitudes for enriched Neumann boundary conditions by applying the mirror symmetry interface to exceptional Dirichlet. We shall see this has a well-defined mathematical origin in elliptic cohomology.

Let us first recover the wavefunction of the enriched Neumann boundary condition \(N_\alpha^r\) by colliding the interface on the right with \(\tilde{D}_\alpha^r\). The relevant set-up is illustrated in figure 4.10.

To evaluate the wavefunction, we cut the path integral using auxiliary Dirichlet boundary conditions in the mirror theory. We choose the polarisation \(\varepsilon = \{+ \ldots + - \ldots -\}\) for the mirror interface, where the first minus sign is at position \(\alpha\), as in equation (4.67). This gives a decomposition of the wavefunction of enriched Neumann
as follows,
\[ \langle D_F(s)|N^\alpha \rangle = \oint \frac{d\tilde{s}}{2\pi i} \left( \eta(q)^2 \vartheta(t^2) \right)^{N-1} \mathcal{I}_F(s, \tilde{s}) \langle \bar{D}_F(\tilde{s})|\bar{D}^r_\alpha \rangle, \]  
(4.107)

where
\[ \mathcal{I}_F(s, \tilde{s}) = (-)^{\alpha} \prod_{\beta=1}^{N} \vartheta \left( sv_\beta^{-1} \frac{\tilde{s}_\beta^{-1}}{\tilde{s}_\beta} \right), \]  
(4.108)

and the minus sign comes from the choice of polarisation. The wavefunction of exceptional Dirichlet in the mirror is
\[ \langle \bar{D}_F(\tilde{s})|\bar{D}^r_\alpha \rangle = \delta^{(N-1)}(\tilde{u} - \tilde{u}|_\alpha) \left( \eta(q)^2 \vartheta(t^2) \right)^{N-1} \prod_{\lambda\in\tilde{\Phi}_\alpha} \vartheta(\tilde{\alpha}/^\lambda), \]  
(4.109)

where we have used the tangent weights (4.64). The delta function is regarded as a contour prescription around a rank \( N-1 \) pole at (4.62) which is the value in the vacuum \( \alpha \) in the mirror theory. Evaluating the residue reproduces the wavefunction (4.96). A similar computation yields the wavefunction of left enriched Neumann boundary conditions in the mirror (4.103).

The Mother Function

Let us now briefly review the constructions in the mathematical literature, see in particular section 6 of [70], of which this is the precise physical analog. For this discussion we deviate slightly in notation, and denote the image of the massive vacua of \( \mathcal{T} \) on its Higgs branch \( X \) by \( \alpha \) and its Coulomb branch \( X^! \) by \( \tilde{\alpha} \). Regarding \( X \times X^! \) as a \( T_f = T_C \times T_H \times T_t \) variety, there are equivariant embeddings
\[ X = X \times \{ \tilde{\alpha} \} \overset{i_\alpha}{\longrightarrow} X \times X^! \overset{j_{\alpha^!}}{\longleftarrow} \{ \alpha \} \times X^! = X^!, \]  
(4.110)
Taking the $T_f$ equivariant elliptic cohomology

$$\text{Ell}_{T_f}(X \times \{\tilde{\alpha}\}) = \text{Ell}_T(X) \times \text{Ell}_{T_C}(\{\tilde{\alpha}\}) = \text{Ell}_T(X) \times E_{T_C} = E_T(X). \quad (4.111)$$

The latter is precisely what we identified as the extended elliptic cohomology. Similarly

$$\text{Ell}_{T_f}(\{\alpha\} \times X^! \times \{\tilde{\alpha}\}) = \text{Ell}_{\tilde{T}}(X^!) \times E_{T_H} = E_{\tilde{T}}(X^!), \quad (4.112)$$

where we have defined $\tilde{T} = T_C \times T_t$, such that $\text{Ell}_{\tilde{T}}(X^!)$ is the extended elliptic cohomology of $X^!$, and the associated spectral curve for the space of supersymmetric ground states of $\tilde{T}$.

The functoriality of elliptic cohomology then gives maps

$$E_T(X) \xrightarrow{i^*_\alpha} \text{Ell}_{T_f}(X \times X^! \times \{\tilde{\alpha}\}) \xleftarrow{i^*_{\tilde{\alpha}}} E_{\tilde{T}}(X^!). \quad (4.113)$$

The main result of [70] is that there exists a holomorphic section $m$ of a particular line bundle on $\text{Ell}_{T_f}(X \times X^!)$ which can be pulled back by the maps above to give

$$\frac{(i^*_\alpha)^*(m)}{\prod_{\lambda \in \Phi^+} \vartheta(e^{2\pi i \lambda \cdot \tilde{\alpha}})} = \text{Stab}(\alpha)_{\xi, \zeta}, \quad \frac{(i^*_{\tilde{\alpha}})^*(m)}{\prod_{\lambda \in \Phi^+} \vartheta(e^{2\pi i \lambda \cdot \tilde{z}})} = \widehat{\text{Stab}}(\alpha)_{\xi, \zeta}^{-1}, \quad (4.114)$$

which are sections of line bundles on $E_T(X)$ and $E_{\tilde{T}}(X^!)$ respectively.

**Physical Correspondence**

The physical correspondence is now as follows. The existence of the mother function $m$ is equivalent to the existence of a duality interface between theories $T$ and $\tilde{T}$. The mother function $m$ is itself the wavefunction $I_F(s, \tilde{s})$, or the mirror duality interface kernel, described previously. The pullbacks, or fixed point evaluations (4.114) are simply the collision of the duality interface with exceptional Dirichlet boundary conditions.

### 4.4.4 Orthogonality and Duality of Stable Envelopes

We demonstrated in section 4.1 that left and right $\mathcal{N} = (2, 2)$ exceptional Dirichlet boundary conditions are orthonormal, as expected as they represent normalisable
supersymmetric ground states at $x \to \pm \infty$. Since enriched Neumann boundary conditions are mirror to exceptional Dirichlet, we also expect that they are orthonormal,

$$\langle N^l_\alpha | N^r_\beta \rangle = \delta_{\alpha\beta}. \quad (4.115)$$

Let us demonstrate this in supersymmetric QED.

First we write down the wavefunction of left enriched Neumann, using the same reasoning as before. The overlap with auxiliary Dirichlet boundary conditions receives contributions from the Fermi multiplets $\Psi_{Y_\beta}$ for $\beta < \alpha$ and $\Psi_{X_\beta}$ for $\beta \geq \alpha$, the chiral $\Phi_\phi$, as well as the $\mathbb{C}^*$ boundary chiral $\Phi_\alpha$. The result is

$$\langle N^l_\alpha | D_F(s) \rangle = \prod_{\beta < \alpha} \vartheta(t^{-1}sv^{-1}_\beta) \vartheta(sv^{-1}_\alpha t^{-1}t^{N-2+2\alpha} - 1) \prod_{\beta > \alpha} \vartheta(t^{-1}s v^{-1}_\beta). \quad (4.116)$$

We note the relation

$$\langle N^l_\alpha | D_F(s) \rangle = \langle D_F(s) | N^r_{t\alpha} \rangle |_{v_\beta \mapsto v_\beta \cdot \alpha, \xi \mapsto \xi^{-1}} = \text{Stab}(t \cdot \alpha) \epsilon_{H, \xi^{-1}} |_{v_\beta \mapsto v_\beta \cdot \alpha} \quad (4.117)$$

where $t : \{1, \ldots, N\} \mapsto \{N, \ldots, 1\}$ is the longest permutation in $S_N$. This is as expected, changing orientation from right to left in the Morse flow is given by inverting the chamber to $\mathcal{C}^{opp}_H$ for mass parameters. The inversion of $\xi$ arises from anomaly inflow from an oppositely oriented boundary.

With this wavefunction in hand, we can compute the overlap of left and right enriched Neumann boundary conditions by cutting the path integral,

$$\langle N^l_\alpha | N^r_\beta \rangle = \oint_{s=v_t^{-1}} \frac{d\eta(q)}{2\pi i s} \eta(q)^2 \vartheta(t^2) Z_C(s) \langle N^l_\alpha | D_F(s) \rangle \langle D_F(s) | N^r_\beta \rangle. \quad (4.118)$$

The integrand is periodic in $s$, reflecting the absence of gauge anomalies at either boundary. The JK contour selects poles in $Z_C(s)$ at $s = v_\gamma t^{-1}$ with $\gamma = 1, \ldots, N$. This generates the decomposition into boundary amplitudes,

$$\langle N^l_\alpha | N^r_\beta \rangle = \sum_{\gamma=1}^N \langle N^l_\alpha | \gamma \rangle \langle \gamma | N^r_\beta \rangle \prod_{\lambda \in \Phi_\gamma} \frac{1}{\vartheta(a^\lambda)}. \quad (4.119)$$
Note that $\langle N_a^\dagger | \gamma \rangle$ vanishes for $\gamma > \alpha$, and $\langle \gamma | N_a^\dagger \rangle$ for $\gamma < \beta$, as in our construction of the pole subtraction matrix. Thus if $\alpha < \beta$, the overlap vanishes trivially. It is straightforward to check the diagonal components $\alpha = \beta$ evaluate to 1. For $\alpha > \beta$, the vanishing reduces to a $(\alpha - \beta + 1)$-term theta function identity. These identities are straightforward to check for small $N$ by hand, and can be checked using a computer for larger values of $N$. One thus reproduces orthonormality (4.115).

Mathematically, this is precisely the duality result of Aganagic and Okounkov [67]

$$
\sum_{\gamma \in \text{fp.}} \frac{\text{Stab}(\alpha)_{\xi^\nu, \xi^{-1}}}{\prod_{\lambda \in \Phi_\gamma} \vartheta(a^\lambda)} \text{Stab}(\beta)_{\xi, \xi^{-1}} = \delta_{\alpha\beta}
$$

of orthonormal elliptic cohomology classes, realised as the orthonormality of left and right enriched Neumann boundary conditions in fixed chambers $\mathcal{C}_H$ and $\mathcal{C}_C$.

### 4.5 Janus Interfaces

Thus far we have tacitly assumed that the real FI and mass parameters are constant. However, the set-up is consistent with varying profiles $\zeta(x^3)$, $m(x^3)$ for the mass parameters in a way that preserves the supercharge $Q$. The computation of boundary amplitudes and overlaps are independent of the profiles of the FI and mass parameters in the $x^3$-direction, except through constraints that their terminal or asymptotic values place on compatible boundary conditions.

Indeed, such non-trivial profiles for the mass parameters are required to correctly interpret some of the computations we have performed. For example, suppose we have left and right boundary conditions $B^l$, $B^r$ compatible with chambers $\mathcal{C}$, $\mathcal{C}'$.\(^8\) Then computing the overlap $\langle B^l|B^r \rangle$ as a partition function on $E_\tau \times [-\ell, \ell]$, we must assume a non-trivial profile $m(x^3)$ for the mass parameters interpolating between the chambers $\mathcal{C}$ at $x^3 = 0$ and $\mathcal{C}'$ at $x^3 = \ell$.

Since the computations are independent of the profile $m(x^3)$, we can imagine that the mass parameters vary from $\mathcal{C}$ to another $\mathcal{C}'$ in a vanishingly small region $\Delta \subset [-\ell, \ell]$. This is known as a supersymmetric Janus interface, which we denote by $J_{\mathcal{C}, \mathcal{C}'}$. The overlap of boundary conditions is more correctly expressed as a correlation function of the Janus interface $\langle B^l|J_{\mathcal{C}, \mathcal{C}'}|B^r \rangle$. This picture also applies to boundary amplitudes.

In this section, we review some aspects of Janus interfaces and re-visit some of the computations done thus far in a new light, especially in the study of exceptional

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\(^8\)We drop the $H$ subscript on the chamber $\mathcal{C}_H$ of mass parameters in the rest of this section.
Dirichlet and enriched Neumann boundary conditions. We explain how the computation of correlation functions
\[ \langle N_\alpha^c J_{E,F} N_\beta^c \rangle \quad (4.121) \]
reproduces chamber R-matrices. We exploit independence of the profile \( m(x^3) \) to express the R-matrices in terms of elliptic stable envelopes and explain why they satisfy Yang-Baxter.

**4.5.1 Explicit Constructions**

Supersymmetric Janus interfaces have been studied extensively in the literature, beginning with the case of 4d \( \mathcal{N} = 4 \) Super Yang-Mills [143, 191–193].

Let us consider Janus interfaces from a 3d \( \mathcal{N} = 2 \) perspective (we follow the notation of [180]) and consider a background vector multiplet \( (A^F, \sigma^F, \lambda^F, \bar{\lambda}^F, D^F) \) for a flavour symmetry \( F \). The configurations preserving half of the supercharges \( Q^+_+ \), \( Q^+_\) are given by
\[
A^F = a^F dx^3, \quad \sigma^F = \sigma^F(x^3), \quad D^F = i\partial_3 \sigma^F, \\
\lambda^F = 0, \quad \bar{\lambda}^F = 0. 
\]

Crucially, the profile \( \sigma^F(x^3) \) for the real vector multiplet scalar is allowed to vary in the \( x^3 \)-direction provided a compensating profile for the auxiliary field \( D^F(x^3) \) is turned on.

In the following, we argue that computations preserving the supercharge \( Q = Q^+_+ + Q^+_\) are independent of the profile in the bulk \( x^3 \in (-\ell, \ell) \), and depends only on the terminal values \( \sigma^F(\pm \ell) \). We do this by showing that perturbing the action by \( \delta \sigma^F(x^3) \) such that \( \delta \sigma^F(\pm \ell) = 0 \) results in a \( Q \)-exact deformation of the action. The usual localisation argument then finishes the argument.

We apply this to the case where \( F = T_H, T_C \) and the vector multiplet scalar \( \sigma^F = m, \zeta \). However, since in this paper we fix a chamber for the FI parameter and thus a Higgs branch \( X \), we are primarily interested in varying profiles for the mass parameters.

**Mass Janus**

Let us first consider the case of \( F = T_H \) with varying mass parameters \( \sigma^H = m(x^3) \) and background auxiliary field \( D^H = im'(x^3) \). We consider the action of a 3d \( \mathcal{N} = 2 \) chiral multiplet \( (\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F}) \) charged under the flavour symmetry \( T_H \). The part of
the Lagrangian that depends on \( m(x^3) \) is

\[
\mathcal{L}_m = \bar{\phi} m^2(x^3) \phi + 2 \bar{\phi} \sigma m(x^3) \phi - \bar{\phi} m'(x^3) \phi + i \bar{\psi} m(x^3) \psi
\]

(4.123)

where \( m \) and \( \sigma \) act in the appropriate \( G \) and \( G_H \) representations. A variation in the profile \( m(x^3) \rightarrow m(x^3) + \delta m(x^3) \) results in a change in the Lagrangian

\[
\delta \mathcal{L}_m = \bar{\phi} \delta m^2 \phi + 2 \bar{\phi} \delta m(\sigma + m) \phi + \bar{\phi} \delta m \partial_3 \phi + \partial_3 (\bar{\phi} \delta m \phi) - \partial_3 (\bar{\phi} \partial_3 m \phi) + i \bar{\psi} \delta m \psi.
\]

(4.124)

It is not hard to show that

\[
Q \cdot V_m = \delta \mathcal{L}_m + \partial_3 (\bar{\phi} \delta m \phi) - i (\bar{\phi} \delta m F + \bar{F} \delta m \phi) - \bar{\phi} \delta m^2 \phi,
\]

(4.125)

where

\[
V_m = \frac{\delta m}{2t} \left( \bar{\phi} \delta m \psi + \bar{\psi} \delta m \phi \right).
\]

(4.126)

Although it appears as given that \( \delta \mathcal{L}_m \) is not \( Q \)-exact, we now show that the last two terms in \( Q \cdot V_m \) can be absorbed into a re-definition of the auxiliary fields.

Let \( \mathcal{Z}_m \) denote the partition function with the initial mass profile \( m(x^3) \), and \( S[\Phi]_m \) the corresponding action. Then by the usual localisation argument, we have

\[
\mathcal{Z}_m = \int D\Phi e^{-S[\Phi]_m} = \int D\Phi e^{-S[\Phi]_m - Q \cdot V_m},
\]

(4.127)

where \( \Phi \) denotes collectively all the dynamical fields.

The auxiliary fields \( F \) and \( \bar{F} \), included to ensure closure of the supersymmetry algebra on fermions, appear in the action only through the quadratic term \( \int_{E_\tau \times I} \bar{F} F \). Denoting \( \Phi' \) to be the collection of fields without \( F, \bar{F} \), and \( S[\Phi']_m \) the action minus this term, then

\[
S[\Phi]_m + Q \cdot V_m = S[\Phi']_m + \int_{E_\tau \times I} \delta \mathcal{L}_m + \partial_3 (\bar{\phi} \delta m \phi) + (\bar{F} - i \bar{\phi} \delta m)(F - i \delta m \phi),
\]

(4.128)

using (4.125). Substituting the above into (4.127) and redefining

\[
\bar{F} - i \bar{\phi} \delta m \rightarrow \bar{\Phi}, \quad F - i \delta m \phi \rightarrow F,
\]

(4.129)
then we have\(^9\)

\[ Z_m = \int D\Phi' DFD\bar{F}e^{-S[\Phi']_m - \int (\bar{F}F + \delta L_m) - \int \partial_3 (\bar{\delta} m_\phi)} \]

\[ = \int D\Phi e^{-S[\Phi]_{m + \delta m} - \int \partial_3 (\bar{\delta} m_\phi)}. \quad (4.130) \]

So provided that \(\delta m(\pm \ell) = 0\), it is true that \(Z_{m + \delta m} = Z_m\). We have shown that \(Z_m\) is independent of the interior profile of \(m(x^3)\) and only on the values \(m(\pm \ell)\) at the end points of the interval.

**FI Janus**

The FI parameter consists of the real scalar of a background \(\mathcal{N} = 2\) vector multiplet \((A^C, \sigma^C, \lambda^C, \bar{\lambda}^C, D^C)\) coupled to the gauge symmetry via a mixed Chern-Simons term. In an \(\mathcal{N} = 4\) theory, it is the real scalar \(\zeta = \zeta^{++}\) in an \(\mathcal{N} = 4\) twisted vector multiplet, coupled to a \(\mathcal{N} = 4\) vector multiplet for the gauge symmetry via an \(\mathcal{N} = 4\) CS term.

In either case, we will be interested in giving the background scalar \(\sigma^C\) a non-trivial profile in \(x^3\), i.e \(\zeta(x^3)\). For this to be BPS we require \(D^C = i\partial_3 \sigma^C\). Therefore the (bulk) FI term is

\[ S_{FI} = \int_{E_7 \times I} i\zeta(x^3)D - \zeta'(x^3)\sigma. \quad (4.131) \]

For the coupling of the gauge field to a background for \(A^C\), see e.g. [2, 189].\(^{10}\) Note that

\[ Q \cdot S_{FI} = \int_{E_7 \times I} \left( \zeta \bar{\epsilon} \gamma^\mu D_\mu \lambda + \zeta D_\mu \bar{\lambda} \gamma^\mu \epsilon - \zeta' \bar{\epsilon} \lambda + \zeta' \bar{\lambda} \epsilon \right) \]

\[ = -\frac{1}{2} \int_{E_7 \times I} \partial_3 \left( \zeta (\lambda_+ + \bar{\lambda}_+) \right), \quad (4.132) \]

so that \(S_{FI}\) is supersymmetric under \(Q\), provided Neumann boundary conditions for the vector multiplet (implying \(\lambda_+ = \bar{\lambda}_+ = 0\)) are prescribed at both boundaries. In passing to the second line we have dropped the total derivatives in the \(x^1, x^2\) directions, as \(\lambda\) and \(\bar{\lambda}\) obey R-R boundary conditions. For Dirichlet boundary conditions one

\(^9\)We have assumed that since the redefinition (4.129) is linear, the measure is invariant. There is an additional assumption in the following. In Euclidean signature all fields are complexified and barred and unbarred fields are independent. In the path integral there is a choice of middle dimensional contour in field space for the bosonic fields, the canonical choice relating bar and unbarred fields by complex conjugation: \(\bar{F} = F^\dagger\). Then the term \(\int \bar{F}F\) is positive definite in the action. The redefinition (4.129) deforms away from this contour. To retain a positive definite action we assume the contour can be deformed back to the canonical choice without changing the answer.

\(^{10}\)These references include couplings for NS-NS boundary conditions on \(E_7\); the analogous results for the R-R sector are obtained from dropping the dependence on \(E_7\) coordinates in the boundary conditions and spinors, and taking the curvature of \(HS^2\) to infinity.
must add boundary Chern-Simons terms to preserve supersymmetry, as in \cite{2}. We will not consider this case.

Under a variation $\zeta(x^3) \rightarrow \zeta(x^3) + \delta \zeta(x^3)$, the variation is $Q$-exact up to boundary terms:
\[
\delta S_{\text{FI}} = \int_{E_r \times I} i \delta \zeta (D - i \partial_3 \sigma) - \partial_3 (\delta \zeta \sigma)
\]
\[
= Q \cdot \left[ \int_{E_r \times I} \frac{i \delta \zeta}{4} \left( \lambda_+ - \lambda_- \right) \right] - \int_{E_r \times I} \partial_3 (\delta \zeta \sigma).
\]

Thus the path integral is independent of the interior profile of the FI parameter $\zeta(x^3)$. If $\sigma$ is fixed to a non-zero boundary 2d FI parameter $\sigma|_{t_{2d}}$, as is the case for the enriched Neumann boundary conditions encountered in this work, then the above shows that the partition function does depend on the terminal value(s) of the FI Janus $\zeta(\pm \ell)$.

### 4.5.2 Mass Parameter Janus

Let us first reconsider the boundary amplitudes of exceptional Dirichlet and enriched Neumann boundary conditions. Both sets of boundary conditions are defined in the presence of mass parameters in some chamber $C$. As we now keep track of chamber structure let us write the left boundary conditions compatible with mass parameters this chamber as $\langle D^{|C}_\alpha \rangle$, $\langle N^{|C}_\alpha \rangle$, and right boundary conditions as $\langle D^{|C}_{\alpha} \rangle$, $\langle N^{|C}_{\alpha} \rangle$.

We considered the boundary amplitudes representing equivariant cohomology classes on $X$ by taking overlaps with the supersymmetric ground states $\langle \alpha \rangle$ defined at vanishing mass parameters. We would previously have denoted such boundary amplitudes as $\langle \beta|D^{|C}_{\alpha} \rangle$, $\langle \beta|N^{|C}_{\alpha} \rangle$. However, in light of the discussion above, it is more appropriate to regard them as correlation functions
\[
\langle \alpha|J_{0,|C_\alpha} |D^{|C}_\beta \rangle = \Theta(\Phi^+_\beta \alpha) \delta_{\alpha \beta}, \quad \langle D^{|C}_{\alpha} |J_{|C,0} |\beta \rangle = \Theta(\Phi^-_{\beta} \alpha) \delta_{\alpha \beta},
\]
\[
\langle \alpha|J_{0,|C_\alpha} |N^{|C}_\beta \rangle = \text{Stab}(\beta)_{\epsilon_{\alpha}} \xi_{\alpha} \rangle, \quad \langle N^{|C}_{\alpha} |J_{|C,0} |\beta \rangle = \text{Stab}(\alpha)_{\epsilon_{\alpha} \epsilon_{\beta}^{-1}} \xi_{\alpha} \rangle,
\]

where $J_{0,|C_\alpha}$ and $J_{|C,0}$ are Janus interfaces interpolating between vanishing mass parameters and mass parameters in the chamber $C$, from left to right and vice versa. In the above we have emphasised the chamber dependence in the sets of positive and negative weights $\Phi^\pm_{\alpha}$. Additionally we have used the shorthand where if $W$ is a set of weights, then
\[
\Theta(\pm W) \equiv \prod_{\lambda \in W} \vartheta(a^\lambda)^{\pm 1}.
\]
It will also be convenient to define the matrix of boundary amplitudes

\[ S_{\alpha\beta}^\xi = \text{Stab}(\beta)\xi|_{\alpha}. \] (4.136)

We may then use the orthogonality of exceptional Dirichlet and enriched Neumann boundary conditions to write

\[
\langle \alpha | J_{0, \xi} \rangle = \langle D_{\alpha}^\xi | \Theta(\Phi_{\alpha}^{x,}) \rangle, \quad J_{\xi, 0} | \beta \rangle = \Theta(<\beta^{-1} D_{\beta}^\xi >) \]

\[
= \sum_{\gamma \leq \alpha} \langle N_{\gamma}^\xi | S_{\alpha\gamma}^\xi , \xi \rangle \]

\[
= \sum_{\gamma \geq \beta} S_{\beta\gamma}^{\xi, opp, -1} | N_{\gamma}^\xi \rangle \], \] (4.137)

where in the second line we recall for example that \( S_{\beta\gamma}^{\xi} \) is lower triangular with respect to the partial ordering induced by the Morse flow with respect to mass parameters in the chamber \( C \).

The first line for exceptional Dirichlet is the correct interpretation of the relationship between supersymmetric ground states with \( m = 0 \) and \( m \in C \) derived using the infinite-dimensional quantum mechanics model in section 3.3. Namely, in transporting the supersymmetric ground states from \( m = 0 \) to \( m \in C \), the supersymmetric ground states are related by a factor (3.20).

The second equation shows that the elliptic stable envelope provides the matrix elements that express how to decompose the supersymmetric ground states at \( m = 0 \) in terms of states generated by enriched Neumann boundary conditions when transported to \( m \in C \).

Let us go one step further and apply these equations to a general boundary condition compatible with mass parameters in the chamber \( C \). We first consider a right boundary condition, which we denote at \( m = 0 \) by \( B \) and at \( m \in C \) by \( B_{C}^\xi \). We find immediately:

\[
\langle \alpha | B \rangle = \Theta(\Phi_{\alpha}^{x,}) \langle D_{\alpha}^\xi | B_{C}^\xi \rangle, \quad \langle \alpha | B \rangle = \sum_{\beta} S_{\alpha\beta}^{\xi, opp, -1} | N_{\alpha}^\xi \rangle \]. \] (4.138)

Note that both of these equations relate quantities on the right, \{\langle D_{\alpha}^\xi | B_{C}^\xi \rangle\} and \{\langle N_{\alpha}^\xi | B_{C}^\xi \rangle\}, which transform as sections of holomorphic line bundles on \( E_{T}(\{\alpha\}) = \bigsqcup_{\alpha} E_{T_{j}}^{(\alpha)} \) to boundary amplitudes on the left that glue to a section of a line bundle on \( E_{T}(X) \).

This is the physical realisation of the construction in [67] which realises the stable envelope as map of sections of holomorphic line bundles on \( \bigsqcup_{\alpha} E_{T_{j}}^{(\alpha)} \) to those on \( E_{T}(X) \), as reviewed in appendix A.2. In that language, we have a map of sections of the line.
bundles:

\[ \mathcal{N}' \otimes \mathcal{N}_B \to \mathcal{N} \otimes \mathcal{N}_B, \]  

(4.139)

where \( \mathcal{N}_B \) is a line bundle encoding the boundary anomalies of the boundary condition (which are independent of \( \mathcal{E} \)). Since \( \mathcal{N}_B \) is a common factor, we may remove it to recover the description in equation (A.25).

### 4.5.3 \( R \)-matrices

Correlation functions of particular interest are the overlap of enriched Neumann \( \mathcal{N}'^\alpha \), \( \mathcal{N}_\beta^\xi \) in a pair of distinct chambers. Following the discussion above, we regard this as a correlation function

\[ R_{\alpha \beta}^{\mathcal{E}', \mathcal{E}} := \langle \mathcal{N}'^\alpha | J_{\mathcal{E}'}, \mathcal{E}_\beta \rangle. \]  

(4.140)

Note that if we choose \( \mathcal{E} = \mathcal{E}' \) then \( R_{\alpha \beta}^{\mathcal{E}, \mathcal{E}} = \delta_{\alpha \beta} \), as the enriched Neumann boundary conditions are orthonormal.

These correlation functions obey an important property as a consequence of the independence of the profile \( m(x^3) \) connecting the chambers \( \mathcal{E}, \mathcal{E}' \). Suppose that \( \mathcal{E}, \mathcal{E}' \) are not neighbouring chambers, meaning they are not separated by a single hyperplane \( \mathcal{W}_{\alpha \beta} \subset t_H \). Then by deforming the profile \( m(x^3) \), we can decompose

\[ J_{\mathcal{E}'}, \mathcal{E} = J_{\mathcal{E}', \mathcal{E}_1} J_{\mathcal{E}_1, \mathcal{E}_2} \cdots J_{\mathcal{E}_n, \mathcal{E}}, \]  

(4.141)

as a composition of elementary Janus interfaces connecting neighbouring chambers. By expanding in enriched Neumann boundary conditions in each of the intermediate chambers \( \mathcal{E}_1, \cdots, \mathcal{E}_n \), we find

\[ R_{\alpha \beta}^{\mathcal{E}', \mathcal{E}} = \sum_{\gamma_1, \gamma_2, \cdots, \gamma_n} R_{\alpha \gamma_1}^{\mathcal{E}', \mathcal{E}_1} R_{\gamma_1 \gamma_2}^{\mathcal{E}_1, \mathcal{E}_2} \cdots R_{\gamma_n \beta}^{\mathcal{E}_n, \mathcal{E}}. \]  

(4.142)

Moreover, different decompositions must yield the same result due to invariance under deformations of the profile \( m(x^3) \). Special cases of this relation include the Yang-Baxter equation and unitarity condition obeyed by \( R \)-matrices of quantum integrable models. Therefore, following [92], we refer to these correlation functions as chamber \( R \)-matrices.

The chamber \( R \)-matrices can be computed directly using supersymmetric localisation on \( E_\tau \times [0, \ell] \) with the left and right enriched Neumann boundary conditions. However, it is also convenient to decompose the result in terms of boundary amplitudes. For
this purpose, we decompose the Janus interface as

\[ \mathcal{J}_{\varepsilon,\varepsilon} = \mathcal{J}_{\varepsilon,0}\mathcal{J}_{0,\varepsilon}, \]

(4.143)

where we think about smooth deforming the profile \( m(x^3) \) as illustrated in figure 4.11. This allows us to decompose the chamber R-matrices as follows

\[ R^e_{\alpha\beta} = \langle N^e_\alpha | \mathcal{J}^e_{\varepsilon,0} | N^e_\beta \rangle \]

\[ = \langle N^e_\alpha | \mathcal{J}^e_{\varepsilon,0} | N^e_\beta \rangle \]

\[ = \sum_\gamma \Theta(-\Phi_\gamma) \langle N^e_\alpha | \mathcal{J}^e_{\varepsilon,0} | \gamma \rangle \langle \gamma | \mathcal{J}_{0,\varepsilon} | N^e_\beta \rangle \]

\[ = (S^e)_{\alpha\gamma}^{-1} S^e_{\gamma\beta}, \]

which reproduces the construction of chamber R-matrices in terms of elliptic stable envelopes introduced in [67]. We have used the orthogonality of enriched Neumann boundary conditions:

\[ \Theta(-\Phi_\beta) \langle N^e_\alpha | \mathcal{J}^e_{\varepsilon,0} | \beta \rangle = \Theta(-\Phi_\beta) S^{e,\xi}\xi^{-1} = (S^e,\xi)_{\alpha\beta}^{-1}, \]

(4.145)

as discussed in section 4.4.4.

Note that one could have instead proceeded here by decomposing into wavefunctions to get an integral formula:

\[ R^e_{\alpha\beta} = \oint \frac{ds}{2\pi i s} \mathcal{Z}_\mathcal{V} \mathcal{Z}_C(s) \langle N^e_\alpha | D_F(s) \rangle \langle D_F(s) | N^e_\beta \rangle. \]

(4.146)

One could also arrive at the same integral formula by collapsing the interval directly and evaluating the elliptic genus of the effective 2d theory consisting of boundary \( \mathbb{C}^* \) chiral multiplets and surviving bulk matter, as in section 3.7. Of course, evaluating the residues decomposes this into boundary amplitudes.
Example: Supersymmetric QED

Let us specialise to our example of supersymmetric QED. Let $\mathcal{C}_0 = \{m_1 > m_2 > \ldots > m_N\}$ be our initial choice of chamber. A generic chamber $\mathcal{C}$ can therefore be expressed as $\mathcal{C} = \pi \cdot \mathcal{C}_0$ where $\pi \in S_N$. In a way analogous to the arguments in section 4.4.4, it is not hard to see that the wavefunction of a right enriched Neumann boundary condition in the chamber $\mathcal{C}$ is:

$$
\langle D_F(s) | N^\mathcal{C}_\alpha \rangle = \text{Stab}(\pi^{-1} \cdot \alpha)\xi_0, \xi \big|_{v_{\beta} \rightarrow v_{\alpha, \beta}}
$$

(4.147)

It is of course also possible to derive this directly by using the mirror symmetry interface.

Let us now compute the chamber $R$-matrix explicitly for supersymmetric QED with $N = 2$. The default chamber is $\mathcal{C} = \{m_1 > m_2\}$. The only non-trivial chamber $R$-matrix $R^\mathcal{C}_{\alpha, \beta}$ comes from choosing the opposite chamber $\mathcal{C}' = \{m_1 < m_2\}$.

Using the boundary amplitudes, obtained via (4.117) and (4.147),

$$
R_{\mathcal{C}, \mathcal{C}'} = S_{\mathcal{C}'}^{-1} S_{\mathcal{C}} = \left( \begin{array}{ccc} 1 & \frac{\vartheta(t^{-2})\vartheta(\xi^{-1}\frac{v_2}{v_1})}{\vartheta(\xi^{-1})\vartheta(\xi^{-1}\frac{v_2}{v_1})} \vartheta(\xi^{-1}\frac{v_2}{v_1}) & 0 \\ 0 & \frac{\vartheta(t^{-2})\vartheta(\xi^{-1}\frac{v_2}{v_1})}{\vartheta(\xi^{-1})\vartheta(\xi^{-1}\frac{v_2}{v_1})} \vartheta(\xi^{-1}\frac{v_2}{v_1}) & \vartheta(\xi^{-1}\frac{v_2}{v_1}) \end{array} \right)
$$

(4.148)

which may be simplified by the 3-term theta function identity $\vartheta(ab)\vartheta(a/b)\vartheta(c)^2 + \text{cyclic} = 0$ to recover the $\mathfrak{sl}_2$ $R$-matrix

$$
R_{\mathcal{C}, \mathcal{C}'} = \left( \begin{array}{cc} \vartheta(\xi^{-1}\frac{v_2}{v_1}) & \vartheta(t^{-2})\vartheta(\xi^{-1}\frac{v_2}{v_1}) \\ \vartheta(\xi^{-1}\frac{v_2}{v_1}) & \vartheta(t^{-2})\vartheta(\xi^{-1}\frac{v_2}{v_1}) \end{array} \right)
$$

(4.149)

as in [67]. Similarly, by computing $\text{Stab}(\alpha)\xi_{\mathcal{C}}|_{\beta}$, the $R$-matrix in the opposite direction is given by

$$
R_{\mathcal{C}, \mathcal{C}'} = \left( \begin{array}{cc} \vartheta(t^{-2})\vartheta(\xi^{-1}\frac{v_2}{v_1}) & \vartheta(\xi^{-1}\frac{v_2}{v_1}) \\ \vartheta(t^{-2})\vartheta(\xi^{-1}\frac{v_2}{v_1}) & \vartheta(\xi^{-1}\frac{v_2}{v_1}) \end{array} \right)
$$

(4.150)

Then using the same cyclic identity as before, we recover the unitarity condition

$$
R_{\mathcal{C}, \mathcal{C}'} R_{\mathcal{C}', \mathcal{C}} = I_{2 \times 2}
$$

(4.151)

obeyed by $R$-matrices, as expected.
Chapter 5

Hemisphere Blocks & Verma Modules

In this chapter, we return to discuss another aspect of the exceptional Dirichlet boundary conditions discussed in chapters 2 and 4. In particular, we revisit the factorisation of supersymmetric partition functions of 3d $\mathcal{N} = 4$ gauge theories, finding that the building blocks are hemisphere partition functions on $S^1 \times HS^2$ with exceptional Dirichlet boundary conditions. These can be unambiguously defined, and computed using supersymmetric localisation.

We show that certain limits of these partition functions coincide with characters of lowest weight Verma modules over the quantised Higgs and Coulomb branch chiral rings. This leads to expressions for the superconformal index, twisted index and $S^3$ partition functions in terms of such characters. On the way we will uncover new connections between boundary ’t Hooft anomalies, hemisphere partition functions, and lowest weights of Verma modules.

Contributions The results in this chapter are based on material from:


5.1 Assumptions

As before, we work on $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ with coordinates $\{x^1, x^2, x^3\}$ where $x^3 \geq 0$. We will again consider boundary conditions $B$ preserving a 2d $\mathcal{N} = (2, 2)$ supersymmetry, which in the IR flow to superconformal boundary conditions preserving $(2, 2)$ superconformal
supersymmetry. The latter is the subalgebra of the full 3d $\mathcal{N} = 4$ superconformal algebra $\mathfrak{osp}(4|4, \mathbb{R})$ generated by the four supercharges $Q^+_+, Q^-_-, Q^+_-, Q^-_+$ and their conjugates in radial quantisation $S^+_{++}, S^+_{+-}, S^-_{-+}, S^-_{+-}$, as in section 2.7.

The boundary conditions support a global symmetry containing a subgroup of the bulk global symmetry $G_H \times G_C$ and any additional symmetries arising from boundary degrees of freedom. In this chapter, we focus on boundary conditions preserving at least a maximal torus $T_H \times T_C$ of the bulk theory.

As discussed through this thesis, e.g. in section 2.6.2, the boundary R-symmetry $U(1)_V \times U(1)_A$ is sometimes, but not always, identified with a maximal torus $U(1)_H \times U(1)_C$ of the bulk R-symmetry. It can happen that $U(1)_H \times U(1)_C$ is spontaneously broken at the boundary but a linear combination involving boundary flavour symmetries is preserved, which we again denote by $U(1)_V \times U(1)_A$. The boundary conditions introduced in section 5.5 will be of the latter type.

As before, the boundary global and R-symmetries are subject to boundary mixed ’t Hooft anomalies, which will play an important role throughout. See sections 2.5 and 3.5 for our conventions.

**Example**

Let us introduce a simple example we will use throughout this chapter: the free hypermultiplet. A hypermultiplet contains two complex scalar fields $X, Y$ such that $(X, Y^\dagger)$ transforms as a doublet of $SU(2)_H$ R-symmetry while $(X, Y)$ transform as a doublet of $G_H = SU(2)$. Recall that the basic Neumann-Dirichlet boundary conditions are

$$
B_X : \quad \partial_\perp X|_\partial = 0 \quad Y|_\partial = 0, \\
B_Y : \quad \partial_\perp Y|_\partial = 0 \quad X|_\partial = 0, 
$$

(5.1)

together with appropriate boundary conditions for the fermions. They break the global symmetry to $T_H = U(1)$ with a boundary mixed ’t Hooft anomaly $k^H = -1$, $k^H = +1$ for $B_X$, $B_Y$. As before, this is normalised such that the contribution from a boundary $\mathcal{N} = (2, 2)$ chiral multiplet of $U(1)$ charge +1 to the mixed anomaly is 2.

### 5.2 Half Superconformal Index

An important observable associated to the boundary conditions we have met throughout this thesis is the half superconformal index [117, 138, 139], which is a character of...
the vector space of local operators at the boundary. Although it is defined for (0, 2) boundary conditions for 3d $\mathcal{N} = 2$ theories, in this chapter we will only be concerned with indices and partition functions (2, 2) boundary conditions.

We define the half superconformal index by

$$I_B = \text{Tr}_{\mathcal{H}_B}(-1)^F q^{J_3 + \frac{J_V + J_A}{2}} t^{\frac{J_V - J_A}{2}} x^{F_H F_C},$$

(5.2)

where $J_3$ is the generator of rotations in the $x^{1,2}$-plane, $J_V, J_A$ are the generators of the boundary R-symmetry $U(1)_V \times U(1)_A$ and $F_H, F_C$ denotes the Cartan generator of the boundary flavour symmetry $T_H \times T_C$. The fermion number is chosen to be $(-1)^F = (-1)^{2J_3}$. Finally, $\mathcal{H}_B$ denotes the space of states in radial quantisation annihilated by the pair of conjugate supercharges $Q_{++}^+$ and $S_{++}^+$ or equivalently their anti-commutator

$$\{Q_{++}^+, S_{++}^+\} = D - J_3 - J_V/2 - J_A/2 \geq 0.$$  

(5.3)

Unitarity bounds of the four supercharges preserved by the boundary condition imply that operators contributing to the index satisfy the inequality

$$J_3 + \frac{1}{4}(J_V + J_A) \geq 0,$$

(5.4)

which is saturated only by the unit operator. The half superconformal index is therefore a formal Taylor series in $q^{1/4}$ starting with 1, whose convergence requires $|q| < 1$. These half indices can be computed as in [117] and can be interpreted as a character of the boundary chiral algebra [194].

Here we have assumed that $U(1)_V \times U(1)_A$ is identified with a maximal torus $U(1)_H \times U(1)_C$ of the bulk R-symmetry. If there is mixing with boundary global symmetries then unitarity bounds are modified. In such cases, the half superconformal index may not start with 1 and convergence may require additional constraints on the flavour fugacities. Examples of this phenomenon are discussed in section 5.5.

Comment on Notation

The notation used in this chapter differs from that used in previous chapters in the following ways:

- In the index (5.2), we have used the fugacity $t$ to grade by the symmetry $\frac{1}{2}(J_V - J_A)$. In chapter 3 and 4, $t$ was the exponentiated holonomy for a background gauge field for the symmetry $J_V - J_A$, i.e. without the factor of $\frac{1}{2}$. This content is too long to fit into a single line.
Our choice in this chapter is mainly due to the fact it will simplify many of the formula we meet.

- We use $x$ now to grade the Higgs branch flavour symmetries, as opposed to $v$ in 3 and 4. This is again primarily for aesthetic reasons.

Example

For the basic hypermultiplet boundary conditions (5.1), the half superconformal index is given by

$$I_{B_X} = \frac{(q^{3/4} t^{-1/2} x; q)_{\infty}}{(q^{1/2} t^{1/2} x; q)_{\infty}} = 1 + q^{1/2} t^{1/2} x + \cdots, \quad (5.5)$$

$$I_{B_Y} = \frac{(q^{3/4} t^{-1/2} x^{-1}; q)_{\infty}}{(q^{1/2} t^{1/2} x^{-1}; q)_{\infty}} = 1 + q^{1/2} t^{1/2} x^{-1} + \cdots.$$  

Note that the leading contributions to the index beyond the unit operator are the boundary operators $X|_{\partial}, Y|_{\partial}$ supported on $B_X, B_Y$.

We are primarily interested in two limits $t^{\frac{1}{2}} \to q^{-\frac{1}{4}}$, where the remaining combinations of generators commute with additional supercharges. These limits require additional constraints on the flavour fugacities to maintain convergence, which is related to the response of boundary conditions to turning on bulk real mass and FI parameters.

5.2.1 B-Limit

The $B$-index is defined by

$$I_B^{(B)} := \lim_{t^{\frac{1}{2}} \to q^{-\frac{1}{4}}} I_B = \text{Tr}_H x^F_H. \quad (5.6)$$

In the limit $t^{\frac{1}{2}} \to q^{-\frac{1}{4}}$, the generator $J_3 + \frac{J_4}{2}$ conjugate to $q$ commutes with an additional supercharge $Q^+_-$. The index therefore receives contributions only from operators in the subspace $H_B^{(B)} \subset H_B$ annihilated by both supercharges $Q^+_+, Q^+_-$ and their conjugates in radial quantisation, or equivalently by the anti-commutators

$$\{Q^+_+, S^+_{++}\} = D - J_3 - \frac{J_V}{2} - \frac{J_A}{2},$$

$$\{Q^-_+, S^-_{+-}\} = D + J_3 - \frac{J_V}{2} + \frac{J_A}{2}. \quad (5.7)$$
Such operators transform as the scalar components of $\mathcal{N} = (2, 2)$ chiral multiplets and include the images of bulk Higgs branch operators under the bulk to boundary map, see section 2.7. Their quantum numbers obey

$$D = \frac{J_V}{2}, \quad J_3 + \frac{J_A}{2} = 0$$

and therefore the index is independent of $q$. They are uncharged under $T_C$ so it is also independent of $\xi$. Finally, we can remove the $(-1)^F$ as such operators are bosons.

To maintain convergence, there must clearly be a constraint on $x$. We can regard this parameter as an element of the complexified maximal torus $T_H \otimes \mathbb{R} \mathbb{C}$. If a boundary condition preserves $\mathcal{N} = (2, 2)$ supersymmetry in the presence of a real mass $m$, boundary operators contributing to the $B$-limit of the half superconformal index obey:

$$\langle m, F_H \rangle \geq 0.$$

To see this, note from section 3.5.1 (and from [56]), that if $B$ is compatible with the mass $m$, i.e. it preserves $(2, 2)$ supersymmetry when $m$ is turned on, the Higgs branch image of $B$ must be contained in the repelling sets of the critical loci of the action generated by the mass parameter. Therefore, the holomorphic functions on the Higgs branch image, i.e. the boundary local operators counted by this limit, are therefore charged as in (5.9).

The index will therefore converge if $-\log |x|$ lies in the same chamber as $m$. In summary:

- If a boundary condition is compatible with a real mass $m \in \mathfrak{C}_H$, the $B$-limit of the half superconformal index converges for $-\log |x| \in \mathfrak{C}_H$.

Example

We illustrate this statement for a hypermultiplet. The $B$-limit of the half superconformal indices of the basic boundary conditions are

$$I^{(B)}_{B_X} = 1 + x + x^2 + \cdots = \frac{1}{1-x},$$

$$I^{(B)}_{B_Y} = 1 + x^{-1} + x^{-2} + \cdots = \frac{1}{1-x^{-1}}.$$  \hfill (5.10)

These expansions arise from monomials in the boundary Higgs branch operators $X|_0$ and $Y|_0$ respectively. The index of $B_X$ converges for $|x| < 1$, while that of $B_Y$ converges...
for $|x| > 1$. This is consistent with the fact that the $B_X$ is compatible with real mass parameter $m > 0$, while $B_Y$ is compatible with $m < 0$ [56].

In section 5.6.4, we will also encounter the closely related limit $t^\frac{1}{2} \rightarrow e^{-\pi i} q^{-\frac{1}{2}}$. Almost identical arguments hold except the differing sign leads to an additional factor of $(-1)^{J_V}$ in equation (5.6), such that the bottom components of chiral multiplets are counted with an additional sign depending on their vector R-charge.

## 5.2.2 A-Limit

The $A$-index is similarly defined by

$$I^{(A)}_B := \lim_{t^\frac{1}{2} \rightarrow q^\frac{1}{2}} I = \text{Tr} \mathcal{H}^{(A)}_B \xi^{F_C}. \tag{5.11}$$

In the limit $t^\frac{1}{2} \rightarrow q^\frac{1}{2}$, the generator $J_3 + J_V$ conjugate to $q$ now commutes with an additional supercharge $Q^{+\pm}$. The index therefore receives contributions only from operators in the subspace $\mathcal{H}^{(A)}_B \subset \mathcal{H}_B$ annihilated by both supercharges $Q^{+\pm}, Q^{-\pm}$ and their conjugates in radial quantisation, or equivalently by the anti-commutators

$$\{Q^{+\pm}, S^{+\pm}\} = D - J_3 - \frac{J_V}{2} - \frac{J_A}{2},$$

$$\{Q^{-\pm}, S^{-\pm}\} = D + J_3 + \frac{J_V}{2} - \frac{J_A}{2}. \tag{5.12}$$

Such operators transform as the scalar component of $\mathcal{N} = (2, 2)$ twisted chiral multiplets and include the images of bulk Coulomb branch operators under the bulk to boundary map. The quantum numbers of such operators obey

$$D = \frac{J_A}{2}, \quad J_3 + \frac{J_V}{2} = 0 \tag{5.13}$$

and therefore the index is independent of $q$. They are not charged under $T_H$ so it is also independent of $x$. Finally, we can again remove the $(-1)^F$ as such operators are bosons.

To maintain convergence, we now need a constraint on $\xi$. We can regard this parameter as an element of the complexified maximal torus $T_C \otimes_R \mathbb{C}$. If a boundary condition preserves $\mathcal{N} = (2, 2)$ supersymmetry in the presence of a real FI parameter $\zeta$, boundary operators contributing to the $A$-limit of the half superconformal index obey

$$\langle \zeta, F_C \rangle \geq 0. \tag{5.14}$$
This follows from analogous arguments as the B-limit, replacing Higgs branch with Coulomb branch.

The index will therefore converge if $-\log |\xi|$ lies in the same chamber as $\zeta$. In summary:

- If a boundary condition is compatible with a real FI parameter $\zeta \in \mathbb{C}_C$, the $A$-limit of the half superconformal index converges for $-\log |\xi| \in \mathbb{C}_C$.

For hypermultiplet boundary conditions,

$$I_{Bx}^{(A)} = I_{By}^{(A)} = 1,$$  \hspace{1cm} (5.15)

which simply reflects the absence of bulk Coulomb branch operators that could supply twisted chiral operators at the boundary. This index is independent of $\xi$ so there is no issue with convergence in this case.

In section 5.6.4, we will also encounter the closely related limit $t^2 \to e^{\pi i} q^{1/4}$. Almost identical arguments hold except the differing sign leads to an additional factor of $(-1)^J$ in equation (5.11), such that the bottom components of twisted chiral multiplets are counted with an additional sign depending on their axial R-charge.

### 5.3 Hemisphere Partition Function

The half superconformal index can be computed from a UV description by invoking the state-operator correspondence to relate it to a hemisphere partition function on $S^1 \times HS^2$ and applying supersymmetric localisation. This essentially builds on similar computations for the bulk superconformal index, using either Coulomb branch or Higgs branch localisation. We give the details of this computation, and the form of boundary conditions on this geometry in appendix D.

From one perspective, the $S^1 \times HS^2$ background is a product

$$ds^2 = d\tau^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$ \hspace{1cm} (5.16)

where $\tau \sim \tau + \beta r$ and $0 \leq \theta \leq \pi/2$ and the boundary condition $B$ supported at $\theta = \pi/2$. The boundary conditions around $S^1$ are then twisted according to the fugacities in the superconformal index (5.2). Another perspective is to replace the metric by an $S^1$-fibration over $HS^2$ together with an appropriate background connection for the boundary global and R-symmetries around $S^1$. The fugacity $q$ is set to $e^{-2\beta}$, see appendix D.
The result of supersymmetric localisation leads to the computation of 1-loop determinants that require regularisation in a way compatible with the supersymmetry preserved. A consequence is that the hemisphere partition function \( Z_B \) of an \( \mathcal{N} = (2, 2) \) boundary condition is related to the superconformal index by a multiplicative factor,

\[
Z_B = e^{\phi_B I_B},
\]

where \( \phi_B \) is determined by the boundary mixed 't Hooft anomalies. In fact, this is true for \( \mathcal{N} = 2 \) theories with \((0, 2)\) boundary conditions, as we show in appendix D. An analysis of the 1-loop determinants then shows that

\[
\phi_B = \frac{1}{\log q} \log y \cdot K \cdot \log y,
\]

where \( y \) denotes the fugacities for all boundary global and \( R \)-symmetries, and \( K \) the bilinear form encoding mixed 't Hooft anomaly coefficients. Specialising to the \( \mathcal{N} = 4 \) case, and using our notation for the possible anomaly coefficients from section 3.5.1 (which are the same as the allowed effective Chern-Simons levels in 2.5), this becomes

\[
\phi_B = \frac{1}{\log q} \left[ \log \xi \cdot k \cdot \log x \right] \\
+ \frac{1}{\log q} \left[ \log \xi \cdot k^C \cdot \log \left( q^{\frac{1}{2}} t^{\frac{3}{2}} \right) \right] \\
- \frac{1}{\log q} \left[ \log \left( q^{\frac{1}{2}} t^{\frac{1}{2}} \right) \cdot k^H \cdot \log x \right] \\
- \frac{1}{\log q} \left[ \log \left( q^{\frac{1}{2}} t^{\frac{1}{2}} \right) \cdot \tilde{k} \cdot \log \left( q^{\frac{1}{2}} t^{-\frac{1}{2}} \right) \right],
\]

where from our definition (5.2) of the half superconformal index the fugacities associated to \( U(1)_V \) and \( U(1)_A \) are \( q^{\frac{1}{2}} t^{\frac{3}{2}} \) and \( q^{\frac{1}{2}} t^{-\frac{1}{2}} \) respectively. The signs are due simply to our conventions for the anomaly coefficients as descending to mixed anomalies with \( U(1)_V - U(1)_A \). We will frequently abuse notation and think of the coefficients as maps: \( k^C : \Gamma_C \to \mathbb{Z} \), \( k^H : \Gamma_H \to \mathbb{Z} \), and \( \tilde{k} \in \mathbb{Z} \).

Let us illustrate this result for the basic boundary conditions (5.1) for a hypermultiplet. Combining the results for Neumann and Dirichlet boundary conditions for 3d \( \mathcal{N} = 2 \) chiral multiplets found in [117] we find

\[
\phi_{B_X} = + \frac{1}{\log q} \log x \log (q^{\frac{1}{4}} t^{-\frac{1}{2}}), \\
\phi_{B_Y} = - \frac{1}{\log q} \log x \log (q^{\frac{1}{4}} t^{-\frac{1}{2}}),
\]

(5.20)
which reproduces the boundary mixed 't Hooft anomaly between the $U(1)$ global symmetry and $U(1)_A$ axial R-symmetry with coefficients $k^H = -1, +1$ for $B_X, B_Y$.

### 5.3.1 B-Limit

In the limit $t^1 \rightarrow q^{-\frac{1}{4}}$, the fugacity conjugate to $U(1)_A$ becomes $q^\frac{1}{2}$ while the fugacity conjugate to $U(1)_V$ becomes $1$. The overall factor relating the hemisphere partition function and the half superconformal index therefore no longer detects boundary 't Hooft anomalies involving $U(1)_V$. Explicitly, it becomes

\[
\phi_B^{(B)} = -\frac{1}{2} k^H \cdot \log x + \frac{\log x \cdot k \cdot \log \xi}{\log q},
\]

and exponentiating

\[
e^{\phi_B^{(B)}} = x^{-\frac{k^H}{2} + \frac{k \cdot \log \xi}{\log q}}.
\]

Note that although the $B$-limit of the half superconformal index is independent of $\xi$, the hemisphere partition function may retain some dependence on $\log \xi$ through the boundary mixed anomaly between $T_C$ and $T_H$. We denote

\[Z_B := X_B^H. \]

For the hypermultiplet, this limit is

\[X_B^{H_x} = \frac{x^{1/2}}{1 - x}, \quad X_B^{H_y} = \frac{x^{-1/2}}{1 - x^{-1}}, \]

which encodes the anomaly coefficients $k^H = -1$ for $B_X$ and $k^H = +1$ for $B_Y$.

### 5.3.2 A-Limit

In the limit $t^2 \rightarrow q^{\frac{1}{4}}$, the fugacity conjugate to $U(1)_V$ becomes $q^\frac{1}{2}$ while the fugacity conjugate to $U(1)_A$ becomes $1$. The overall factor relating the hemisphere partition function and the half superconformal index therefore no longer detects boundary 't Hooft anomalies involving $U(1)_A$. Explicitly, it becomes

\[
\phi_B^{(A)} = \frac{1}{2} k^C \cdot \log \xi + \frac{\log \xi \cdot k \cdot \log x}{\log q},
\]

and exponentiating

\[e^{\phi_B^{(A)}} = \xi^{-\frac{k^C}{2} + \frac{k \cdot \log \xi}{\log q}}.\]
Note that although the $A$-limit of the half superconformal index is independent of $x$, the hemisphere partition function may retain some dependence on $\log x$ through the boundary mixed anomaly between $T_C$ and $T_H$. We denote
\[
\lim_{t_\frac{3}{2} - q^2} \mathcal{Z}_B := \mathcal{X}_B^C.
\]
(5.27)
For the hypermultiplet,
\[
\mathcal{X}_B^C_{S} = 1, \quad \mathcal{X}_B^C_{Y} = 1,
\]
(5.28)
as the only boundary mixed 't Hooft anomalies involve $U(1)_A$.

### 5.4 Characters of Modules

Let us return temporarily to the half superconformal index on $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. We have considered two limits of the half superconformal index,
\[
I_B^{(A)} = \text{Tr}_{\mathcal{H}_B^{(A)}} \xi^F_C,
\]
\[
I_B^{(B)} = \text{Tr}_{\mathcal{H}_B^{(B)}} x^F_H,
\]
(5.29)
where $\mathcal{H}_B^{(B)}$ and $\mathcal{H}_B^{(A)}$ denote respectively boundary operators that are the scalar components of $\mathcal{N} = (2, 2)$ chiral and twisted chiral multiplets.

These setups admit deformations that can be described either as an omega background [44, 51, 113, 114] or passing to a 'Q + S' type construction as in [115, 116, 154, 195, 196]. For concreteness, we will focus here on the omega background perspective.

There are two possible omega backgrounds $\Omega_A, \Omega_B$ in the $x^{1,2}$-plane. These deformations break superconformal symmetry but the boundary operators at the origin of the $x^{1,2}$-plane remain the same. However, bulk local operators are now constrained to the $x^3$-axis and generate non-commutative algebras $\mathcal{A}_H, \mathcal{A}_C$ that act on boundary operators. In this way, $\mathcal{H}_B^{(A)}, \mathcal{H}_B^{(B)}$ become modules for $\mathcal{A}_H, \mathcal{A}_C$, as described in [56]. This is illustrated in figure 5.1.

The algebras $\mathcal{A}_H, \mathcal{A}_C$ are equivariant deformation quantisations of the Poisson algebras of functions $\mathbb{C}[\mathcal{M}_H]$ and $\mathbb{C}[\mathcal{M}_C]$ on the Higgs and Coulomb branch respectively. In the following, we will describe this for the examples of the free hypermultiplet, and supersymmetric QED. They are determined by periods $\zeta_C \in \mathfrak{t}_C \otimes \mathbb{C}$ and $m_C \in \mathfrak{t}_H \otimes \mathbb{C}$, which are complex mass and FI parameters.\(^1\) The algebras include operators $J_H, J_C$.

\(^1\)In the previous chapter, we turned off complex mass and FI parameters. In this chapter, it will be helpful to turn them on in our discussion of modules and characters.
whose commutators measure $T_H, T_C$ charge. For example, in $A_H$ we have

$$[J_H, \mathcal{O}_\gamma] = \gamma \mathcal{O}_\gamma$$  \hspace{1cm} (5.30)

where $\mathcal{O}_\gamma$ is a Higgs branch operator of charge $\gamma \in \Gamma_H^\vee$. This provides a grading of the non-commutative algebras $A_H, A_C$ by the character lattices $\Gamma_H^\vee, \Gamma_C^\vee$. Similarly, there is a weight decomposition of any module generated by a boundary condition preserving global symmetry $T_H, T_C$.

Now consider the operators

$$J_m = m \cdot J_H, \quad J_\zeta = \zeta \cdot J_C,$$

(5.31)

where $m$ and $t$ are the real mass and FI parameters. An observation of [56] is that boundary conditions compatible with real parameters $m, t$ determine modules that are lowest weight for the operators $J_m, J_\zeta$, meaning their weights are bounded below.

This property only depends on the chamber: if a module is lowest weight for $m \in \mathcal{C}_H$, it is lowest weight for any other $m' \in \mathcal{C}_H$ in the same chamber. Therefore, having fixed $\mathcal{C}_H, \mathcal{C}_C$, we simply refer to modules associated to compatible boundary conditions as lowest weight.

The modules $\mathcal{H}^{(A)}_B, \mathcal{H}^{(B)}_B$ will therefore have lowest weight states that we denote by $|B\rangle^{(A)}, |B\rangle^{(B)}$. If we were to add constants to the operators $J_H, J_C$ such that the lowest weight states have charge 0, this would correspond to the charge measured by the generators $F_H, F_C$ appearing in the definition of the half superconformal index. The condition of lowest weight is then equivalent to the inequalities (5.9) and (5.14) and the characters of these modules coincide with the half superconformal indices in (5.29).
However, as we show for a general abelian theory in appendix E, the charges of the lowest weight states measured by the operators $J_H, J_C$ are determined by boundary mixed 't Hooft anomalies:

$$J_H |B\rangle^{(B)} = \left( -\frac{1}{2} k^H + \frac{1}{\epsilon} \zeta_C \cdot k \right) |B\rangle^{(B)},$$

$$J_C |B\rangle^{(A)} = \left( \frac{1}{2} k^C + \frac{1}{\epsilon} m_C \cdot k \right) |B\rangle^{(A)}.$$  \hspace{1cm} (5.32)

Here $\epsilon$ is used to denote the omega deformation parameter in agreement with the literature, in contrast to the previous chapters where it was used to denote a real mass for $T_t$, which we do not consider in this chapter.

Let us now define the equivariant characters of these modules by

$$X_B^H = \text{Tr}_{H_B} x^{J_H},$$

$$X_B^C = \text{Tr}_{H_B} \xi^{J_C}.$$  \hspace{1cm} (5.33)

Then the lowest weight states contribute the following multiplicative factors

$$x^{-\frac{1}{2} k^H + \frac{1}{\epsilon} \zeta_C \cdot k}, \quad \xi^{\frac{1}{2} k^C + \frac{1}{\epsilon} m_C \cdot k}$$  \hspace{1cm} (5.34)

to these equivariant characters.

If we now compare to the multiplicative factor relating the hemisphere partition function to the half superconformal index in (5.22) and (5.26), we can identify the hemisphere partition function with the character

$$\lim_{t^\frac{1}{2} \to q^{-\frac{1}{4}}} Z_B = X_B^H,$$

$$\lim_{t^\frac{1}{2} \to q^{-\frac{1}{4}}} Z_B = X_B^C.$$  \hspace{1cm} (5.35)

under the following identification of variables

$$\epsilon \leftrightarrow -\log q, \quad m_C \leftrightarrow -\log x, \quad \zeta_C \leftrightarrow -\log m_C.$$  \hspace{1cm} (5.36)

It would be desirable to give a more direct derivation of this correspondence by carefully understanding the map from the operator counting picture to the $S^1 \times HS^2$ background used for supersymmetric localisation. Nevertheless, this relation will play an important role in the remainder of this paper.
Example

We briefly consider the $\Omega_B$ deformation of the free hypermultiplet. The quantised algebra $\mathcal{A}_H$ is generated by the complex scalar fields $\hat{X}, \hat{Y}$ subject to $[\hat{Y}, \hat{X}] = \epsilon$, i.e. the Heisenberg algebra. The basic boundary conditions correspond to the modules

$$\mathcal{H}^{(B)}_{B_X} : |n\rangle = \hat{X}^n|0\rangle \quad n \geq 0,$$
$$\mathcal{H}^{(B)}_{B_Y} : |n\rangle = \hat{Y}^n|0\rangle \quad n \geq 0,$$

where for convenience we write $|0\rangle := |B_X\rangle^{(B)}$ or $|B_Y\rangle^{(B)}$, which obeys $\hat{Y}|0\rangle = 0$ and $\hat{X}|0\rangle = 0$ respectively.

The global symmetry $T_H = U(1)$ preserved by both boundary conditions is generated by the complex moment map

$$J_H := \frac{1}{\epsilon} \hat{X}\hat{Y} + \frac{1}{2} = \frac{1}{\epsilon} \hat{Y}\hat{X} - \frac{1}{2}.$$  (5.38)

such that

$$\mathcal{H}^{(B)}_{B_X} : J_H |n\rangle = + \left(n + \frac{1}{2}\right) |n\rangle,$$
$$\mathcal{H}^{(B)}_{B_Y} : J_H |n\rangle = - \left(n + \frac{1}{2}\right) |n\rangle.$$  (5.39)

Note that the normal ordering of the moment map reproduces the expected shifts due to the boundary mixed anomaly $k^H = -1, +1$. We also see explicitly that $B_X$ is compatible with $m > 0$ and lowest weight in the chamber $\mathcal{C}_H = \{m > 0\}$, while $B_Y$ is compatible with $m < 0$ and lowest weight in opposite chamber $\mathcal{C}_H = \{m < 0\}$.

The characters of these modules are

$$\chi^{H}_{B_X} = x^{1/2} \sum_{n \geq 0} x^n = \frac{x^{1/2}}{1-x},$$
$$\chi^{H}_{B_Y} = x^{-1/2} \sum_{n \geq 0} x^{-n} = \frac{x^{-1/2}}{1-x^{-1}},$$

which converge to the function on the right when $|x| < 1$ for $B_X$ and $|x| > 1$ for $B_Y$. This is in perfect agreement with the hemisphere partition functions (5.24).
5.5 Exceptional Dirichlet Boundary Conditions

We now focus on the exceptional Dirichlet boundary conditions $D^{l,r}_\alpha$ introduced in section 4.1. For clarity, in this chapter we consider right boundary conditions and thus denote $B_\alpha = D^r_\alpha$.

Recall that the collection of UV boundary conditions $\{B_\alpha\}$ are simultaneously compatible with mass and FI parameters in the chambers $\mathcal{C}_H$, $\mathcal{C}_C$ and mimic the presence of an isolated massive vacuum $\alpha$ at infinity, at least for the purpose of computations preserving supersymmetry. The collection $\{B_\alpha\}$ depends on the chambers and may jump across walls in the space of mass and FI parameters. Also recall that a generic feature of such boundary conditions is that the boundary ’t Hooft anomalies coincide with the effective supersymmetric Chern-Simons levels in the vacuum $\alpha$.

5.5.1 Abelian Theories

Recall that for abelian gauge theories, there is a proposal for constructing the collections $\{B_\alpha\}$ using ‘exceptional Dirichlet’ boundary conditions [56], reviewed in section 4.1. This involves a Dirichlet boundary condition for the $\mathcal{N} = 4$ vector multiplet and a standard boundary condition for the hypermultiplets, deformed by non-vanishing expectation values such that a maximal torus $T_H \times T_C$ of the bulk global symmetry is preserved. We focus here on supersymmetric QED, leaving general abelian theories to appendix E. In [3], these ideas were applied to a non-abelian theory with adjoint matter.

For ease of reference in this chapter, let us briefly review supersymmetric QED with gauge group $G = U(1)$ and $N$ fundamental hypermultiplets $(X_\beta, Y_\beta)$. The bulk global symmetries are $G_H = PSU(N)$ and $G_C = U(1)$ (enhanced to $SU(2)$ when $N = 2$). Correspondingly, we can introduce real mass parameters $(m_1, \ldots, m_N)$ obeying $\sum_\beta m_\beta = 0$ and a real FI parameter $\zeta$.

The classical vacua are solutions of

$$\sum_{\alpha=1}^N |X_\alpha|^2 - |Y_\alpha|^2 = \zeta,$$
$$\sum_\alpha X_\alpha Y_\alpha = 0,$$
$$(\sigma - m_\alpha) X_\alpha = 0,$$
$$(\sigma - m_\alpha) Y_\alpha = 0,$$

where $\sigma$ and $\varphi$ are the real and complex scalar fields in the vector multiplet respectively.
5.5 Exceptional Dirichlet Boundary Conditions

Assuming generic real mass and FI parameters, there are \( N \) isolated massive vacua:

\[
\alpha : \quad |X_\beta|^2 - |Y_\beta|^2 = \begin{cases} 
\zeta & \text{if } \beta = \alpha \\
0 & \text{if } \beta \neq \alpha
\end{cases}, \quad X_\beta Y_\beta = 0, \quad \sigma = m_\alpha, \quad \varphi = 0, \quad (5.42)
\]

which we index by \( \alpha = 1, \ldots, N \). The massive vacua have central charges

\[
C_\alpha = - \sum_{\beta=1}^{N} m_\beta \left( |X_\beta|^2 - |Y_\beta|^2 \right) \bigg|_\alpha = -m_\alpha \zeta \quad (5.43)
\]

or equivalently mixed supersymmetric Chern-Simons term with components \( \kappa = -e_\alpha \otimes e_C \), where \( e_1, \ldots, e_N \) are fundamental weights for \( T_H \) and \( e_C \) for \( T_C \).

In this case, generic parameters means concretely that \( m_\beta \neq m_\gamma \) for \( \beta \neq \gamma \) and \( \zeta \neq 0 \). There are therefore \( N! \) chambers \( \mathcal{C}_H \subset t_H \) specified by an ordering of the real masses and two chambers \( \mathcal{C}_C \subset t_H \) specified by the sign of \( \zeta \). Henceforth, as in the previous chapters, we fix

\[
\mathcal{C}_H = \{ m_1 > m_2 > \ldots > m_N \}, \quad \mathcal{C}_C = \{ \zeta > 0 \}. \quad (5.44)
\]

In this choice of chamber, the exceptional Dirichlet boundary condition \( B_\alpha \) imposes Dirichlet boundary conditions for the vector multiplet together with Neumann-Dirichlet boundary conditions for the hypermultiplets:

\[
B_\alpha : \quad D_3 Y_\beta = 0, \quad X_\beta = c \delta_{\alpha \beta}, \quad \beta \leq \alpha, \\
D_3 X_\beta = 0, \quad Y_\beta = 0, \quad \beta > \alpha, \quad (5.45)
\]

where \( c \neq 0 \). The boundary conditions \( \{ B_\alpha \} \) associated to the opposite chamber \( \mathcal{C}_H = \{ \zeta < 0 \} \) for the FI parameter are obtained by interchanging the boundary conditions for \( X_\beta \) and \( Y_\beta \) for all \( \beta = 1, \ldots, N \). Similarly, the boundary conditions associated to other chambers \( \mathcal{C}_H \) for the mass parameters are related by permutations of the hypermultiplets. For the boundary conditions (5.45), the boundary mixed 't Hooft anomalies are:

\[
\kappa_\alpha = -e_\alpha \otimes e_C, \\
\kappa^C_\alpha = e_V \otimes e_C, \\
\kappa^H_\alpha = - \left( \sum_{\beta < \alpha} (e_\alpha - e_\beta) + \sum_{\beta > \alpha} (e_\beta - e_\alpha) \right) \otimes e_A, \quad (5.46) \\
\bar{\kappa}_\alpha = -(N - 2\alpha + 1) e_V \otimes e_A.
\]
where we have denoted the fundamental weights of $U(1)_V$ and $U(1)_A$ by $e_V$ and $e_A$. Replacing $e_V \rightarrow e_t$ and $e_A \rightarrow -e_t$ we recover the effective supersymmetric Chern-Simons levels for $\mathcal{N} = 2$ flavour symmetries in the vacuum $\alpha$ given in (2.46) and (2.48). The general abelian case is derived in appendix E.

### 5.5.2 Half Superconformal Index

We compute the half superconformal index of the boundary conditions $B_\alpha$ in two steps. We first compute the half superconformal index of a Dirichlet boundary condition with $c = 0$ and then deform to $c \neq 0$. The second step involves a redefinition of the boundary symmetries, as discussed in section 2.6.2, which we also review here. This is similar in spirit to the construction of surface defects in [197].

Suppose we have a Dirichlet boundary condition $B$ in a $U(1)$ gauge theory preserving a maximal torus $U(1)_V \times U(1)_A$ of the bulk R-symmetry and a distinguished boundary symmetry $U(1)_\partial$ arising from the bulk gauge symmetry. The half superconformal index of this boundary condition has the form

$$ I_B = \text{Tr}_{\mathcal{H}_B} (-1)^F q^{J_V + \frac{J_V + J_A}{4}} t^{\frac{J_V - J_A}{2}} x^{F_H} \xi^{F_C} z^{F_g}, \quad (5.47) $$

where $z$ and $F_g$ denote respectively the fugacity and generator of $U(1)_\partial$. Suppose we now initiate a boundary RG flow to a new superconformal boundary condition $B_c$ by turning on an expectation value $c$ for a hypermultiplet scalar of charge $+1$ under $U(1)_\partial$ and weight $Q_H$ under $T_H$. A hypermultiplet scalar also has charge $1$ under $J_V$ and therefore a linear combination of $U(1)_V$, $U(1)_\partial$, $T_H$ is spontaneously broken. However, the linear combinations

$$ J'_V := J_V - F_g $$
$$ F'_H := F_H - Q_H F_g \quad (5.48) $$

are preserved along the RG flow and become the boundary vector R-symmetry and Higgs branch flavour symmetry of boundary condition $B_c$.

At the level of the half superconformal index, this is implemented by setting the weight of this field to unity, $q^{\frac{1}{2}} t^{\frac{1}{2}} x^{Q_H} z = 1$ and eliminating $z$. Indeed, we find

$$ I_B(z \rightarrow q^{-1/4} t^{-1/2} x^{-Q_H}) = \text{Tr}_{\mathcal{H}_B} (-1)^F q^{J'_V + \frac{J'_V + J_A}{4}} t^{\frac{J'_V - J_A}{2}} x^{F'_H} \xi^{F_C} = \text{Tr}_{\mathcal{H}_{B_c}} (-1)^F q^{J'_V + \frac{J'_V + J_A}{4}} t^{\frac{J'_V - J_A}{2}} x^{F'_H} \xi^{F_C}. \quad (5.49) $$
In making this argument, we assume that any difference between $\mathcal{H}_c$ and $\mathcal{H}_B$ (with the gradings shifted by setting $z = q^{-\frac{1}{4}} t^{-\frac{1}{2}} x^{-Q}$) cancels out in the trace. This follows from the fact that $c$ is an exact deformation of the boundary action.

Let us now implement this procedure for exceptional Dirichlet boundary conditions. The first step is to evaluate the half superconformal index of the Dirichlet boundary condition with $c = 0$ in equation (5.45) and preserves an additional boundary symmetry $U(1)_d$ with fugacity $z$. This is given by

$$\sum_{m \in \mathbb{Z}} \left( \xi \left( q^{\frac{1}{4}} t^{-\frac{1}{2}} \right)^{2\alpha-N} \right)^m \left( \frac{t q^{\frac{1}{4}}; q}{q^{\frac{1}{4}}}; q \right) \prod_{\beta \leq \alpha} \left( \frac{q^{\frac{1}{4}} t^{-\frac{1}{2}} z^\beta; q}{q^{\frac{1}{4}} t^{-\frac{1}{2}} z^\alpha; q} \right) \prod_{\beta > \alpha} \left( \frac{q^{\frac{1}{4}} t^{-\frac{1}{2}} z^\beta; q}{q^{\frac{1}{4}} t^{-\frac{1}{2}} z^\alpha; q} \right).$$

where we have fugacities $\xi$ and $x_1, \ldots, x_N$ for $T_C$ and $T_H$ respectively and the $q$-Pochhammer symbols $(a, q)_{\infty}$ should be understood as expansions in $q$. The summation over $m \in \mathbb{Z}$ arises from boundary monopole operators. The power of $q^{\frac{1}{4}} t^{-\frac{1}{2}}$ multiplying $\xi$ is due a boundary mixed ’t Hooft anomaly between $U(1)_A$ and $U(1)_B$.

The second step is introduce an expectation value $c \neq 0$ for $X_\alpha$ and flow to the exceptional Dirichlet boundary condition $B_\alpha$. As described above, this is implemented by setting $z = x_\alpha^{-1} t^{-\frac{1}{2}} q^{-\frac{1}{2}}$. Performing this substitution in equation (5.50), the half superconformal index is

$$I_{B_\alpha} = \prod_{\beta=1}^{\alpha-1} \frac{(q^\frac{1}{2} t^{-\frac{1}{2}} x_\beta; q)_{\infty}}{(q^\frac{1}{2} t^{-\frac{1}{2}} x_\alpha; q)_{\infty}} \prod_{\beta=\alpha+1}^{N} \frac{(q^\frac{1}{2} t^{-\frac{1}{2}} x_\alpha; q)_{\infty}}{(q^\frac{1}{2} t^{-\frac{1}{2}} x_\beta; q)_{\infty}} \times \sum_{m \geq 0} \left( \left( q^{\frac{1}{4}} t^{-\frac{1}{2}} \right)^N \xi \right)^m \prod_{\beta=1}^{N} \frac{(q^\frac{1}{2} t^{-\frac{1}{2}} x_\alpha; q)_{\infty}}{(q^\frac{1}{2} t^{-\frac{1}{2}} x_\beta; q)_{\infty}},$$

where the summation now only extends over $m \in \mathbb{Z}_{\geq 0}$.

**Remark**

In the above, the second line coincides with the vortex partition function for $\mathcal{N} = 4$ supersymmetric QED [110–112, 198] and can be interpreted geometrically as a K-theoretic vertex function [67, 68, 123]. We gave a heuristic description of these objects around equation (2.107). In fact, the computation of vertex functions usually proceeds via equivariant localisation to a Jackson $q$-integral [199], which is precisely mirrored by the procedure above to compute half-indices. We will return to give a more rigorous derivation of the equality between vertex functions and half-indices (and the hemisphere partition functions below) for exceptional Dirichlet boundary conditions in [158].
Limits

Let us now consider limits of the half superconformal index preserving additional supercharges. First, in the $A$-limit $t^{\frac{1}{2}} \rightarrow q^{\frac{1}{4}}$, the contributions from ratios of $q$-Pochhammer symbols cancel out completely leaving

$$I_{B_a}^{(A)} = \sum_{m \geq 0} \xi^m = \frac{1}{1 - \xi},$$

(5.52)

for all $i = 1, \ldots, N$. This converges when $|\xi| < 0$, corresponding to the fact that the collection of exceptional Dirichlet boundary conditions $\{B_\alpha\}$ are compatible with a real FI parameter in the chamber $\mathfrak{C}_C = \{\zeta > 0\}$.

Second, in the $B$-limit $t^{\frac{1}{2}} \rightarrow q^{-\frac{1}{4}}$, the contributions from $m > 0$ vanish and the remaining contribution from $m = 0$ converges to

$$I_{B_{\text{al}}}^{(B)} = \prod_{\beta=1}^{\alpha-1} \frac{1}{1 - x_\beta/x_\alpha} \prod_{\beta=\alpha+1}^{N} \frac{1}{1 - x_\alpha/x_\beta},$$

(5.53)

provided that $|x_\beta| < |x_\gamma|$ for $\beta < \gamma$. This corresponds to the fact that the collection of exceptional Dirichlet boundary conditions $\{B_\alpha\}$ are compatible with real mass parameters in the chamber $\mathfrak{C}_H = \{m_1 > m_2 > \cdots > m_N\}$.

5.5.3 Hemisphere Partition Function

We can now upgrade these computations to the hemisphere partition function. The ratios of $q$-Pochhammer symbols are replaced by regularised 1-loop determinants. The details are included in appendix D. The result is an additional prefactor $e^{\phi_\alpha}$ encoding the boundary mixed ’t Hooft anomalies obtained by substituting (5.46) into (5.18). Explicitly

$$Z_{B_\alpha} = Z^{\text{Cl}}_{\alpha} Z^{1\text{-loop}}_{\alpha} Z^{\text{Vortex}}_{\alpha},$$

(5.54)

where:

$$Z^{\text{Cl}}_{\alpha} = e^{\phi_\alpha},$$

(5.55)

$$Z^{1\text{-loop}}_{\alpha} = \prod_{\beta=1}^{\alpha-1} \left( \frac{q^{\frac{1}{2}} t^{\frac{1}{2}} x_\beta}{x_\alpha} ; q \right)_\infty \prod_{\beta=\alpha+1}^{N} \left( \frac{q^{\frac{1}{2}} t^{-\frac{1}{2}} x_\alpha}{x_\beta} ; q \right)_\infty,$$

(5.56)

$$Z^{\text{Vortex}}_{\alpha} = \sum_{m \geq 0} \left( \left( q^\frac{1}{2} t^{\frac{1}{2}} \right)^N \xi \right)^m \prod_{\beta=1}^{N} \left( \frac{q^{\frac{1}{2}} t^{\frac{1}{2}} x_\beta}{x_\alpha} ; q \right)_m.$$
The prefactor is given by

$$\phi_\alpha = (2\alpha - N - 1) \frac{\log \left( q^{\frac{1}{2} t^{-\frac{1}{2}}} \right) \log \left( q^{\frac{1}{2} t^{\frac{1}{2}}} \right)}{\log q} - \frac{\log \xi \log(x_\alpha)}{\log q} + \frac{\log \xi \log \left( q^{\frac{1}{2} t^{\frac{1}{2}}} \right)}{\log q}$$

$$+ \sum_{\beta < \alpha} \frac{\log \left( q^{\frac{1}{2} t^{-\frac{1}{2}}} \right) \log(x_\beta / x_\alpha)}{\log q} + \sum_{\beta > \alpha} \frac{\log \left( q^{\frac{1}{2} t^{-\frac{1}{2}}} \right) \log(x_\alpha / x_\beta)}{\log q}. \quad (5.58)$$

We have limits:

$$X_A^H = \lim_{t^{\frac{1}{2}} \to q^{-\frac{1}{2}}} Z_{B_\alpha} = e^{-\frac{\log \xi \log(x_\alpha)}{\log q}} \prod_{\beta < \alpha} \frac{(x_\beta / x_\alpha)^{1/2}}{1 - x_\beta / x_\alpha} \prod_{\beta > \alpha} \frac{(x_\alpha / x_\beta)^{1/2}}{1 - x_\alpha / x_\beta}, \quad (5.59)$$

$$X_A^C = \lim_{t^{\frac{1}{2}} \to q^{-\frac{1}{2}}} Z_{B_\alpha} = e^{-\frac{\log \xi \log(x_\alpha)}{\log q}} \frac{\xi^{\frac{1}{2}}}{1 - \xi}. \quad (5.60)$$

### 5.5.4 Characters of Verma Modules

The exceptional Dirichlet boundary conditions $B_\alpha$ define lowest weight Verma modules for the quantised algebra of functions on the Coulomb branch and Higgs branch in the $\Omega_A$ and $\Omega_B$ deformations respectively. We now show that the $A$ and $B$-limits of the hemisphere partition function reproduces the characters of these representations.

#### Higgs Branch

The quantised Higgs branch chiral ring in supersymmetric QED can be constructed via quantum symplectic reduction. It is generated from $N$ commuting copies of the Heisenberg algebra

$$[\hat{Y}_\beta, \hat{X}_\alpha] = \epsilon \delta_{\alpha\beta}, \quad \alpha, \beta \in \{1, \ldots, N\}, \quad (5.61)$$

restricting to gauge invariant combinations, and imposing the constraint

$$\sum_{\beta=1}^N :\hat{X}_\beta \hat{Y}_\beta: = \zeta_C. \quad (5.62)$$

where the normal ordering is $:\hat{X}_\beta \hat{Y}_\beta: = \hat{X}_\beta \hat{Y}_\beta + \frac{\xi}{2} = \hat{Y}_\beta \hat{X}_\beta - \frac{\xi}{2}$. These are the quantisations of the complex moment maps for the $U(1)$ subgroup of $T_H$ rotating the $\beta^{th}$ hypermultiplet. The complex FI parameter $\zeta_C$ determines the period of the deformation quantisation.
It is convenient to introduce gauge-invariant generators

\[
\begin{align*}
    h_\beta &= \hat{X}_\beta \hat{Y}_\beta - \hat{X}_{\beta+1} \hat{Y}_{\beta+1}, \\
    e_\beta &= \hat{X}_\beta \hat{Y}_{\beta+1} \quad \beta = 1, \ldots, N - 1, \\
    f_\beta &= \hat{X}_{\beta+1} \hat{Y}_\beta \quad \beta = 1, \ldots, N - 1,
\end{align*}
\]

such that

\[
\begin{align*}
    [e_\beta, f_\gamma] &= \epsilon \delta_\beta \gamma h_\gamma, \\
    [h_\beta, e_\gamma] &= +\epsilon A_{\beta\gamma} e_\gamma, \\
    [h_\beta, f_\gamma] &= -\epsilon A_{\beta\gamma} f_\gamma,
\end{align*}
\]

and

\[
\begin{align*}
    \text{ad}(e_\beta)^{1-A_{\beta\gamma}} e_\gamma &= 0, \\
    \text{ad}(f_\beta)^{1-A_{\beta\gamma}} f_\gamma &= 0.
\end{align*}
\]

where \( A_{\beta\gamma} \) is the Cartan matrix of \( \mathfrak{sl}_N \). The complex moment map equation then determines all of the Casimir elements of the enveloping algebra of \( \mathfrak{sl}_N \) in terms of the period \( \zeta_C \). We therefore find a central quotient of \( U(\mathfrak{sl}_N) \).

More generally, it is convenient to introduce generators

\[
\begin{align*}
    e_{\beta,\gamma} &= \hat{X}_\beta \hat{Y}_\gamma \quad \text{for} \quad \beta < \gamma, \\
    f_{\beta,\gamma} &= \hat{X}_\beta \hat{Y}_\gamma \quad \text{for} \quad \beta > \gamma,
\end{align*}
\]

such that for example, \( e_{\gamma,\gamma+1} = e_\gamma \) and \( e_{\gamma,\gamma+2} = \frac{1}{\epsilon}[e_\gamma, e_{\gamma+1}] \). We also note that the generator of the global symmetry \( U(1)_m \subset T_H \) generated by real mass parameters \( m_1, \ldots, m_N \) is

\[
J_m := -\sum_{\beta=1}^{N} m_\beta \hat{X}_\beta \hat{Y}_\beta
\]

\[
= \sum_{\beta,\gamma=1}^{N-1} (m_{\beta+1} - m_\beta) A_{\beta\gamma}^{-1} h_\beta
\]

such that

\[
\begin{align*}
    [J_m, e_{\beta,\gamma}] &= \epsilon (m_\gamma - m_\beta) e_{\beta,\gamma} \quad \text{for} \quad \beta < \gamma, \\
    [J_m, f_{\beta,\gamma}] &= \epsilon (m_\gamma - m_\beta) f_{\beta,\gamma} \quad \text{for} \quad \beta > \gamma.
\end{align*}
\]

This means that inside our chosen chamber \( \mathfrak{c}_H = \{ m_1 > m_2 > \cdots > m_N \} \) for the real mass parameters, \( e_{\beta,\gamma} \) and \( f_{\beta,\gamma} \) are lowering and raising operators respectively for the weight associated to \( J_m \).
Let us now consider the modules associated to the exceptional Dirichlet boundary conditions $B_\alpha$ defined in equation (5.45). These modules are generated by acting on a vacuum state $|B_\alpha\rangle$ satisfying

$$\hat{X}_\beta |B_\alpha\rangle = \delta_{\alpha\beta} c |B_\alpha\rangle \quad \text{for} \quad \beta = 1, \ldots, \alpha,$$

$$\hat{Y}_\beta |B_\alpha\rangle = 0 \quad \text{for} \quad \beta = \alpha + 1, \ldots, N,$$

where $c$ is a non-zero constant. In the action of gauge-invariant generators on the vacuum state, the constant $c$ can always be absorbed using the fact that the complex moment map equation annihilates this vacuum state. First, we find

$$J_m |B_\alpha\rangle = \left[ \frac{\epsilon}{2} \left( \sum_{\beta<\alpha} m_\beta - \sum_{\beta>\alpha} m_\beta \right) + \frac{(N - 2\alpha + 1 - \epsilon - \zeta c)}{2} m_\alpha \right] |B_\alpha\rangle$$

which encodes the boundary mixed 't Hooft anomalies for the global symmetry $U(1)_m$ in (5.46) as claimed in section 5.4, after identifying $J_{H,\beta} = -\hat{X}_\beta \hat{Y}_\beta$; and fugacities $x_\beta = e^{-m_\beta}$. In addition, the boundary state is annihilated by $e_{\beta,\gamma}$ for all $\beta < \gamma$. Finally, the operators not annihilating the boundary state are

$$f_{\alpha,\beta} \quad \text{for} \quad \beta < \alpha$$

$$f_{\gamma,\alpha} \quad \text{for} \quad \gamma > \alpha$$

and therefore their action on the boundary state generates a lowest weight Verma module in our chamber for the mass parameters.

We can now compute the character of this module using equation (5.70) and the commutators (5.68) to find

$$\text{Tr} \ e^{-2m_\alpha} = x_\alpha^{-\zeta c} \prod_{\beta<\alpha} \left( \frac{x_\beta}{x_\alpha} \right)^{\frac{1}{2}} \prod_{\gamma>\alpha} \left( \frac{x_\alpha}{x_\gamma} \right)^{\frac{1}{2}} \prod_{\beta<\alpha} \left( 1 + \frac{x_\beta}{x_\alpha} + \frac{x_\beta^2}{x_\alpha^2} + \ldots \right) \prod_{\gamma>\alpha} \left( 1 + \frac{x_\alpha}{x_\gamma} + \frac{x_\alpha^2}{x_\gamma^2} + \ldots \right)$$

$$= x_\alpha^{-\zeta c} \prod_{\beta<\alpha} \frac{(x_\beta/x_\alpha)^{\frac{1}{2}}}{1 - x_\beta/x_\alpha} \prod_{\gamma>\alpha} \frac{(x_\alpha/x_\gamma)^{\frac{1}{2}}}{1 - x_\alpha/x_\gamma}.$$
Coulomb Branch

The quantised Coulomb branch chiral ring of supersymmetric QED is generated by the complex scalar $\varphi$ and the monopole operators $v^{\pm}$ subject to

$$[\hat{\varphi}, \hat{v}_\pm] = \pm \epsilon \hat{v}_\pm,$$

$$\hat{v}_+ \hat{v}_- = \prod_{\beta=1}^N \left( \hat{\varphi} - m_{\beta,C} + \frac{\epsilon}{2} \right),$$

$$\hat{v}_- \hat{v}_+ = \prod_{\beta=1}^N \left( \hat{\varphi} - m_{\beta,C} - \frac{\epsilon}{2} \right),$$

which is a spherical rational Cherednik algebra [56].

The topological global symmetry generated by a real FI parameter $\zeta \in \mathbb{R}$ is generated by the operator $J_{\zeta} = -\zeta \hat{\varphi}$ such that

$$[J_{\zeta}, \hat{v}_\pm] = \mp \epsilon \zeta \hat{v}_\pm.$$

This means that in our chamber $\mathcal{C}_H = \{ \zeta > 0 \}$, the monopole operator $\hat{v}_+$ is a lowering operator and $\hat{v}_-$ is a raising operator with respect to $J_{\zeta}$. The minus sign in $J_{\zeta}$ compared to $J_m$ comes from our convention for the FI parameter.

Let us now consider the modules for the quantised Coulomb branch algebra associated to the exceptional Dirichlet boundary conditions $B_\alpha$. The modules are generated by boundary states $|B_\alpha\rangle$ that obey

$$\left( \hat{\varphi} - m_{\alpha,C} + \frac{\xi}{2} \right) |B_\alpha\rangle = 0, \quad \hat{v}_+ |B_\alpha\rangle = 0.$$  

Note that the expression in the brackets is the effective complex mass in the $\Omega_A$-deformation of the complex scalar $X_\alpha$, which arises because $X_\alpha$ receives a non-vanishing expectation value at the boundary. The second arises from an analysis of boundary monopole operators. The boundary condition therefore generates a lowest weight Verma module by acting with $\hat{v}_-$. The character of this module is

$$\text{Tr} e^{-\frac{J_{\zeta}}{\epsilon}} = \xi^{-\frac{m_{\alpha,C}}{\epsilon} + \frac{1}{2}} (1 + \xi + \xi^2 + \ldots)$$

$$= \xi^{-\frac{m_{\alpha,C}}{\epsilon} + \frac{\xi^{1/2}}{1 - \xi}},$$

which converges to the second line for $|\xi| < 1$. This agrees with the result for the $A$-limit (5.60) of the hemisphere partition function.
5.6 Holomorphic Factorisation

We now consider the factorisation of 3d $\mathcal{N} = 4$ partition functions on closed 3-manifolds in terms of hemisphere partition functions associated to vacua. As a corollary to our analysis for hemisphere partition functions, we show that 3-manifold partition functions can be factorised in terms of Verma module characters of $\mathcal{A}_H$ and $\mathcal{A}_C$.

5.6.1 Preliminaries

For theories with $\mathcal{N} \geq 2$ supersymmetry, partition functions on many 3-manifolds $\mathcal{M}_3$ admit a factorisation schematically of the form

$$Z_{\mathcal{M}_3} = \sum_{\alpha} H_{\alpha} \bar{H}_{\alpha}$$  \hspace{1cm} (5.77)

where $\alpha$ correspond to isolated vacua. The ‘$\sim$’ operation implements a transformation of fugacities corresponding to the orientation reversal and element $g$ of $SL(2, \mathbb{Z})$ gluing the boundary tori $\partial(S^1 \times HS^2) = T^2$ in the Heegaard decomposition of $\mathcal{M}_3$:

$$\mathcal{M}_3 = \left(S^1 \times HS^2\right) \cup_g \left(S^1 \times HS^2\right).$$  \hspace{1cm} (5.78)

In this work we focus on factorisations of the $S^1 \times S^2$ superconformal and twisted indices, and the partition function on the squashed sphere or ellipsoid.

Our proposal for theories with $\mathcal{N} = 4$ supersymmetry is to identify the components $H_{\alpha}$ with hemisphere partition function on $S^1 \times HS^2$ computed with the particular boundary condition $B_{\alpha}$:

$$H_{\alpha} = Z_{B_{\alpha}}.$$  \hspace{1cm} (5.79)

This identification depends on a choice of chambers $\mathcal{C}_H$, $\mathcal{C}_C$ in the spaces of real mass and FI parameters and the blocks $H_{\alpha}$ may differ from the traditional holomorphic blocks in classical and 1-loop contributions. This gives a clean geometric interpretation of factorisation where each block is associated to a vacuum in a systematic way.

It is then natural to examine factorisation in limits that preserve additional supercharges. This yields various formulae for such partition functions as sums over vacua of pairs of characters of Verma modules for $\mathcal{A}_H$, $\mathcal{A}_C$. Such a formula was proposed for the $S^3$ partition function in [99]. The present work shows that this arises naturally from the more general factorisation in equation (5.77). We check this explicitly for a free hypermultiplet and supersymmetric QED. In [3], this was investigated for the (non-abelian) ADHM quiver gauge theory.
Partial factorisations have been demonstrated explicitly using Coulomb branch localisation in a number of examples [5, 93–97]. Higgs branch localisation offers a more direct approach where the path integral is localised to a sum over vortex contributions [106, 107]. We note in contrast the factorisation we propose is exact, in the sense that the perturbative pieces of $Z_{M_3}$ are fully factorised into those of $Z_{B_\alpha}$.

Relationship to Holomorphic Blocks

A systematic approach to realising the factorisation (5.77) for 3d $\mathcal{N} = 2$ theories was developed in [98], where $H_\alpha$ are realised as holomorphic blocks. These are defined in the IR as partition functions of massive theories on a twisted product $S^1 \times D^2$, where $D^2$ is a cigar, and the theory is partially topologically twisted with respect to a choice of $U(1)_R$ symmetry. The blocks are elegantly constructed as solutions to difference equations arising from line operator identities, but suffer from ambiguities in the classical and 1-loop contributions.

Nevertheless it is interesting to consider why, up to the perturbative contributions, the holomorphic blocks and the hemisphere partition functions agree in many cases. The confusion arises due to the fact that they are seemingly computed on different supersymmetry backgrounds - the hemisphere partition functions on $S^1 \times HS^2$ are formulated using rigid supersymmetry [38] and are not topologically twisted. We sketch a resolution to this problem here, leaving the full details to [158].

One can consider a 1-parameter squashing deformation $HS^2_b$ of the hemisphere, preserving the same supersymmetry [200]. Just as for the the squashed index on $S^1 \times S^2_b$, one expects that the resulting partition function does not depend on the squashing parameter. It is likely one could show this using the techniques of [201, 202]. For $b \neq 1$, the R-symmetry background $A^R$ has a non-trivial profile. In the infinite squashing limit $b \to 0$, the hemisphere is deformed to an infinitely long cigar. The R-symmetry background gauge field $A^R$ tends to $\frac{1}{2}w$ near the pole, and $\frac{1}{2}d\phi$ in the infinitely long region around the equator. This coincides with the holomorphic block geometry. Similar arguments for 2d theories have appeared in [203].

5.6.2 Superconformal Index

The superconformal index on $S^1 \times S^2$ is defined analogously to the half superconformal index introduced in section 5.2 and so our discussion here is brief. The superconformal
The superconformal index is defined by

\[ Z_{SC} = \text{Tr}_{\mathcal{H}_{SC}} (-1)^F q^{J_3 + \frac{J_3 + f_A}{4}} t^{\frac{J_3 - f_A}{2}} x^{F_H} \xi^{F_C}, \tag{5.80} \]

where \( \mathcal{H}_{SC} \) is the space of local operators annihilated by the pair of conjugate supercharges \( Q^+_1 \) and \( S^+_1 \), or equivalently states in radial quantisation. The index can be computed as a path integral on \( S^1 \times S^2 \) [106, 107, 204–206].

We propose an exact factorisation of the superconformal index into hemisphere partition functions for the distinguished boundary conditions \( B_\alpha \) associated to vacua,

\[ Z_{SC}(q, t, x, \xi) = \sum_\alpha Z_{B_\alpha}(q, t, x, \xi) Z_{\bar{B}_\alpha}(\bar{q}, \bar{t}, \bar{x}, \bar{\xi}), \tag{5.81} \]

where

\[ \bar{q} = q^{-1}, \quad \bar{t} = t^{-1}, \quad \bar{x} = x^{-1}, \quad \bar{\xi} = \xi^{-1} \tag{5.82} \]

is the transformation of variables implementing the splitting of \( S^1 \times S^2 \).

We are interested in limits of the superconformal index as \( t^{\frac{1}{2}} \to q^{\pm \frac{1}{4}} \), where the remaining generators commute with additional supercharges. These limits were also studied in [207], where it was noted that the superconformal index reproduces the Hilbert series of the Higgs and Coulomb branch, and thus depend only on fugacities \( x \) and \( \xi \) respectively. We make a connection here to characters of Verma modules for \( \mathcal{A}_H, \mathcal{A}_C \).

The arguments are the same as in section 5.2, and using the exact factorisation (5.81), in the limit we recover the equivariant Coulomb and Higgs branch Hilbert series

\[ \lim_{t^{\frac{1}{2}} \to q^{\frac{1}{4}}} Z_{SC}(\xi) = \sum_\alpha \chi^C_\alpha(q, x, \xi) \chi^C_\alpha(q^{-1}, x^{-1}, \xi^{-1}), \]
\[ \lim_{t^{\frac{1}{2}} \to q^{-\frac{1}{4}}} Z_{SC}(x) = \sum_\alpha \chi^H_\alpha(q, x, \xi) \chi^H_\alpha(q^{-1}, x^{-1}, \xi^{-1}), \tag{5.83} \]

expressed as a sum of products of Verma module characters for \( \mathcal{A}_H, \mathcal{A}_C \) respectively. Note that although \( \chi^C_\alpha \) retain a residual \( q \) and \( x \) dependence due to the mixed \( T_H \times T_C \) boundary ’t Hooft anomaly, these contributions cancel in the gluing such that the limit \( t^{\frac{1}{2}} \to q^{\frac{1}{2}} \) of the superconformal index depends only on \( \xi \). Analogous statements hold in the other limit.
Example: Hypermultiplet

We briefly consider factorisation of the superconformal index of a free hypermultiplet. In the chamber \( \mathcal{C}_H = \{ m > 0 \} \), the factorisation is in terms of the boundary condition \( B_X \). In the absence of background flux, the superconformal index is

\[
Z_{SC} = \left( \frac{q^{\frac{3}{2}} x t^{-\frac{1}{2}}; q}{q^{\frac{1}{2}} x^{-1} t^{-\frac{1}{2}}; q} \right) \left( \frac{q^{\frac{1}{2}} x^{-1} t^{-\frac{1}{2}}; q}{q^{\frac{3}{2}} x t^{-\frac{1}{2}}; q} \right) = \| Z_{B_X} \|^2_{SC}, \tag{5.84}
\]

where

\[
Z_{B_X} = e^{\frac{1}{\log q} \log x \log (q^{\frac{1}{2}} x^{-\frac{1}{2}}) \left( \frac{q^{\frac{1}{2}} t^{-\frac{1}{2}} x; q}{q^{\frac{1}{2}} t x; q} \right)} \tag{5.85}
\]

is the full hemisphere partition function of \( B_X \) and we have used the analytic continuation \( (a; q)_\infty = (aq^{-1}; q^{-1})_\infty^{-1} \). Note that the contribution of boundary anomalies to the hemisphere partition function (5.85) cancels out in the superconformal index.

The superconformal index in the \( A \)-limit \( t^{\frac{1}{2}} \to q^{\frac{1}{2}} \) is 1, reflecting the absence of a Coulomb branch. The superconformal index in the \( B \)-limit \( t^{\frac{1}{2}} \to q^{-\frac{1}{2}} \) is

\[
\lim_{t^{\frac{1}{2}} \to q^{-\frac{1}{2}}} Z_{SC} = \lambda^H_{B_X} (x) \lambda^H_{B_X} (x^{-1}) = -\frac{x}{(1 - x)^2}, \tag{5.86}
\]

which coincides with the equivariant Hilbert series of the Higgs branch \( T^* \mathbb{C} \).

Example: SQED

The superconformal index of supersymmetric QED with \( N \) hypermultiplets can be computed by localisation and was factorised into holomorphic blocks in [95]. After an appropriate redefinition of parameters, shifting the fugacity \( t \) to grade by the \( N = 4 \) superconformal R-charge and including the contribution of the \( N = 2 \) adjoint chiral

\footnote{We use \( \| \|^2_{SC} \) throughout this section to denote the gluing (5.82), and similar notation for the twisted index and ellipsoid partition function.}
multiplet we have:

\[
Z_{SC} = \sum_{m \in \mathbb{Z}} \xi^m \left( \frac{q^{\frac{1}{2}}}{t} \right)^{\frac{N|m|}{2}} \left( \frac{tq^{\frac{1}{2}}; q}{(t^{-1}q^{\frac{1}{2}}; q)_{\infty}} \right) m \in \mathbb{Z} \sum_{m \geq 0} \left( \frac{(q^{\frac{1}{2}} t^{-\frac{1}{2}})^N \xi}{m} \right) \prod_{\beta = 1}^{N} \left( \frac{\theta(tq^{\frac{1}{2}} x_{\beta}; q)}{(q^{\frac{1}{2}} x_{\beta}; q)_{\infty}} \right) \right]_{SC} (5.87)
\]

where the contour encloses the poles

\[
s = x_{\alpha} t^{-\frac{1}{2}} q^{-\frac{1}{4} - \frac{|m|}{2} - l} \quad \alpha = 1, \ldots, N; \quad l \in \mathbb{Z}_{\geq 0}. \quad (5.88)
\]

The holomorphic block decomposition is not automatically written in terms of hemisphere partition functions of the boundary conditions \( B_{\alpha} \). In order to do so, we can rewrite the 1-loop contribution to \( Z^{1\text{loop}}_{\alpha} \) of \( Z_{B_\alpha} \) given in equation (5.56) as

\[
Z^{1\text{loop}}_{\alpha} = \prod_{\beta \neq \alpha} \left( \frac{q x_{\beta}}{tq^{\frac{1}{2}} x_{\beta}; q} \right)_{\infty} \prod_{m \geq 0} \left( \frac{(q^{\frac{1}{2}} t^{-\frac{1}{2}})^N \xi}{m} \right) \prod_{\beta = 1}^{N} \theta(tq^{\frac{1}{2}} x_{\beta}; q) \theta(\frac{q x_{\beta}}{x_{\alpha}}; q)_{\infty} \prod_{\beta = \alpha + 1}^{N} \theta(\frac{q x_{\beta}}{q x_{\alpha}}; q), \quad (5.89)
\]

where we define \( \theta(x; q) := (x; q)(qx^{-1}; q) \) (note the difference in convention to chapter 3). Then we note that the theta functions in (5.89) fuse trivially using the identity

\[
\theta(aq^{\frac{m}{2}}; q) \theta(a^{-1}q^{-\frac{m}{2}}; q^{-1}) = 1, \quad (5.90)
\]

and also that the anomaly contribution to the hemisphere partition function in equation (5.55) satisfies \( \|Z^{Cl}_{\alpha}\|_{SC}^2 = 1 \). Combining these results we find

\[
Z_{SC} = \sum_{\alpha = 1}^{N} \|Z_{B_\alpha}\|_{SC}^2, \quad (5.91)
\]

as required. This computation for the superconformal index had the simple feature that the classical or anomaly contribution glues to 1 and we could have worked with the half-superconformal index \( I_{B_\alpha} \). However, this will not be the case for the twisted index, where it plays a crucial role in recovering an exact factorisation.
In the two limits with enhanced supersymmetry (5.83) we find

\[
\lim_{t^2 \to q^\frac{1}{2}} Z_{SC} = \sum_{\alpha = 1}^{N} \mathcal{X}_\alpha^C(q, x, \xi) \mathcal{X}_\alpha^C(q^{-1}, x^{-1}, \xi^{-1}) = -\frac{N \xi}{(1 - \xi)^2},
\]

\[
\lim_{t^2 \to q^{-\frac{1}{2}}} Z_{SC} = \sum_{\alpha = 1}^{N} \mathcal{X}_\alpha^H(q, x, \xi) \mathcal{X}_\alpha^H(q^{-1}, x^{-1}, \xi^{-1}) = (-1)^{N-1} \sum_{\alpha = 1}^{N} \prod_{\beta \neq \alpha} \frac{x_\alpha / x_\beta}{(1 - x_\alpha / x_\beta)^2},
\]

which coincide with the equivariant Coulomb and Higgs branch Hilbert series for supersymmetric QED respectively, up to an overall sign. We note that as expected these depend only on $\xi$ and $x$ respectively.

### 5.6.3 $S^1 \times S^2$ Twisted Index

We next consider the twisted index of 3d $\mathcal{N} = 4$ supersymmetric gauge theories on $S^1 \times S^2$ [18, 208, 209]. There are two versions of the twisted index depending on which R-symmetry is used to twist along $S^2$:

- The $A$-twisted index $Z_{tw}^A$ twists using $U(1)_H$.
- The $B$-twisted index $Z_{tw}^B$ twists using $U(1)_C$.

These two indices preserve a common pair of supercharges $Q^1_3$, $Q^2_3$ that commute with the combinations $J_3 + \frac{\mathcal{J}_H}{2}$ and $J_3 + \frac{\mathcal{J}_A}{2}$, and the anti-diagonal combination $J_V - J_A$. The twisted indices are then defined by

\[
Z_{tw}^A = \text{Tr}_{\mathcal{H}^A_{S^2}} (-1)^F q^{J_3 + \frac{\mathcal{J}_H}{2}} x^{\mathcal{J}_A} \xi^{\mathcal{J}_C},
\]

\[
Z_{tw}^B = \text{Tr}_{\mathcal{H}^B_{S^2}} (-1)^F q^{J_3 + \frac{\mathcal{J}_A}{2}} x^{\mathcal{J}_H} \xi^{\mathcal{J}_C},
\]

where $\mathcal{H}^A_{S^2}$ denote respectively states in the $A$, $B$ twisted theory on $S^2$ that are annihilated by the supercharges $Q^1_3$ and $Q^2_3$.

It was shown in [18] that the twisted indices are generating functions for a certain virtual Euler character of moduli spaces of twisted quasi-maps $\mathcal{Q}$ from $S^2$ to $\mathcal{M}_H$. The twisted index can be computed by Coulomb branch localisation and factorised into
holomorphic blocks \cite{5,97}. Geometrically, this can be understood as a factorisation

\[ Z_{\text{tw}} \simeq \sum_{\alpha} \left\| \chi (\mathcal{Q}^{(\alpha)}) \right\|_{\text{tw}}^2 \]  

where \( \chi (\mathcal{Q}^{(\alpha)}) \) denotes schematically a generating function for a virtual Euler character of the moduli space \( \mathcal{Q}^{(\alpha)} \) of based quasi-maps tending to the vacuum \( \alpha \) \cite{5}.

In this section we propose an exact factorisation of the twisted indices in terms of hemisphere partition functions of the distinguished boundary conditions \( B_\alpha \). In this work we do not consider turning on background fluxes for the flavour symmetries. In order to express this factorisation, it is first convenient to introduce \( A \)- and \( B \)-shifted hemisphere partition functions defined by

\[ Z_{A,B_\alpha} (q, t, x, \xi) = Z_{A,B_\alpha} (q, t^{1/2}, x, \xi) = Z_{A,B_\alpha} (\bar{q}, \bar{t}, \bar{x}, \bar{\xi}), \]  

\[ Z_{B,tw} (q, t, x, \xi) = Z_{B,tw} (q, t, x, \xi), \]  

\[ Z_{B,\bar{t}w} (q, t, x, \xi) = Z_{B,\bar{t}w} (q, t, x, \xi), \]  

\[ \sum_{\alpha} Z_{A,B_\alpha} (q, t, x, \xi) Z_{A,B_\alpha} (\bar{q}, \bar{t}, \bar{x}, \bar{\xi}), \]  

\[ \sum_{\alpha} Z_{B,B_\alpha} (q, t, x, \xi) Z_{B,B_\alpha} (\bar{q}, \bar{t}, \bar{x}, \bar{\xi}), \]  

where the gluing is

\[ \bar{q} = q^{-1}, \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{\xi} = \xi. \]  

We are again interested in the limit \( t^{1/2} \rightarrow 1 \) of the \( A, B \) twisted indices, which preserves the four supercharges commuting with \( J_3 + \frac{J_V}{2}, J_3 + \frac{J_3}{2} \). Supersymmetry implies \( Z_{A,tw} \) and \( Z_{B,tw} \) are independent of the fugacities \( x \) and \( \xi \) respectively and (in the absence of background flux) both are independent of \( q \). Therefore

\[ Z_{A,tw} = \text{Tr}_{\mathcal{H}_{A,S^2}} (-1)^F \xi^{F_C}, \]  

\[ Z_{B,tw} = \text{Tr}_{\mathcal{H}_{B,S^2}} (-1)^F x^{F_H}, \]  

where \( \mathcal{H}_{A,B} \) now denotes respectively states in the \( A, B \) twisted theory on \( S^2 \) annihilated by all four supercharges commuting with \( J_3 + \frac{J_V}{2}, J_3 + \frac{J_3}{2} \).

These limits compute the partition function of the fully topologically twisted theory, or equivariant Rozansky-Witten invariant, on \( S^1 \times S^2 \). In this case, the topological

\[ \text{It would be interesting to verify this with a Higgs branch localisation scheme including the angular momentum deformation } q. \]
state-operator map can be invoked to show that the index counts operators in the
cohomology of the scalar supercharges

\[ Q_A := Q_+^{11} + Q_-^{21}, \]
\[ Q_B := Q_+^{11} + Q_-^{12}. \]  

(5.99)

In ‘good’ and ‘ugly’ theories in the sense of [143], this coincides with local operators in
the Coulomb and Higgs branch chiral ring and therefore the twisted indices \( Z_{tw}^A \) and
\( Z_{tw}^B \) are expected to again reproduce the equivariant Hilbert series of the Coulomb and
Higgs branch respectively. For example, the integral representation of the \( B \)-twisted
index reproduces the Molien integral for the Hilbert series of the Higgs branch [209].

The \( t^{1\over 2} \rightarrow 1 \) limits of the \( A \) and \( B \) twisted indices therefore coincide with the \( t^{1\over 2} \rightarrow q^{1\over 4} \)
and \( t^{1\over 2} \rightarrow q^{-1\over 4} \) limits of the superconformal index respectively.

From our proposed factorisation (5.96) we recover the formulae

\[
\lim_{t^{1\over 2} \rightarrow 1} Z_{tw}^A(\xi) = \sum_\alpha \chi_C^\alpha(q, x, \xi) \chi_C^\alpha(q^{-1}, x, \xi), \\
\lim_{t^{1\over 2} \rightarrow 1} Z_{tw}^B(x) = \sum_\alpha \chi_H^\alpha(q, x, \xi) \chi_H^\alpha(q^{-1}, x, \xi).
\]  

(5.100)

This gives clean formulae for the Hilbert series of 3d \( N = 4 \) chiral rings in terms of
Verma modules constructed out of boundary operators, with a different gluing to the
corresponding expressions for the limits of the superconformal index (5.83). Again,
the gluing is such that, for example in the \( A \)-twist, the \( x \) and \( q \) dependence (which is
solely in the classical piece of the \( A_C \) Verma characters) cancels.

**Example: Hypermultiplet**

We briefly consider the twisted indices of a free hypermultiplet. In the absence of
background flux for the flavour symmetry, the \( B \)-twisted index is

\[
Z_{tw}^B = \frac{t^{1\over 2}}{(1 - xt^{1\over 2})(1 - x^{-1}t^{1\over 2})} = - \|Z_{Bx}^B\|_{tw}^2 
\]  

(5.101)

where

\[
Z_{Bx}^B = e^{\frac{1}{\log q} \log x \log(q^{1\over 2}t^{1\over 2})(qt^{1\over 4}x; q)_{\infty}} (t^{1\over 2}x; q)_{\infty}, 
\]  

(5.102)

using the same analytic continuation of the \( q \)-Pochhammer as for the superconformal
index.
The \( t^\frac{1}{2} \to 1 \) limit preserving additional supersymmetry is

\[
\lim_{t^\frac{1}{2} \to 1} Z^B_{tw} = -x^H B_x(x) A^H B_x(x) = -\frac{x}{(1-x)^2},
\]

which coincides with the equivariant Hilbert series of the free hypermultiplet. The \( A \)-twisted index is 1 and is reproduced by the factorisation \( \|Z^A_{Bx}\|^2_{tw} = 1 \). The \( t^\frac{1}{2} \to 1 \) limit is therefore trivial and compatible with the absence of a Coulomb branch.

**Example: SQED**

We now demonstrate this factorisation explicitly in supersymmetric QED in the absence of background fluxes for global symmetries.\(^4\) The twisted indices can be expressed as the following contour integrals

\[
Z^A_{tw} = \frac{1}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})} \sum_{m \in \mathbb{Z}} ((-1)^N \xi)^m \int_{\Gamma_A} \frac{ds}{2\pi i s} s^{Nm} N \prod_{\beta=1}^N \left( \frac{q^{1-m} t^\frac{1}{2} s^{-1} x_\beta, q_m}{q^{1-m} t^\frac{1}{2} s x_\beta^{-1}, q_m} \right)
\]

\[
Z^B_{tw} = -(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \sum_{m \in \mathbb{Z}} ((-1)^N \xi)^m \int_{\Gamma_B} \frac{ds}{2\pi i s} s^{Nmt} N \prod_{\beta=1}^N \left( \frac{q^{1-m} t^\frac{1}{2} s^{-1} x_\beta, q_m-1}{q^{1-m} t^\frac{1}{2} s x_\beta^{-1}, q_m+1} \right)
\]

where the contour \( \Gamma_A \) surrounds the poles at \( s = x_\alpha t^{-\frac{1}{2}} q^{1-m} x^{1-k} \) for \( k = 0, \ldots, m-1, \alpha = 1, \ldots, N \) and \( \Gamma_B \) surrounds the poles at \( s = x_\alpha t^{-\frac{1}{2}} q^{-k} x^{1-k} \) for \( k = 0, \ldots, m, \alpha = 1, \ldots, N \).

We now demonstrate the factorisation of these twisted indices according to (5.96). The \( B \)-twisted index factorises naturally when evaluated on the aforementioned poles [5] as:

\[
Z^B_{tw} = (-1)^N \sum_{\alpha=1}^N t^{\frac{1}{2}(\alpha-1)} x^{-N}_\alpha \left[ \prod_{\beta\neq\alpha} \left( \frac{x_\alpha}{x_\beta} ; q \right) \right] \left[ \prod_{\beta \leq \alpha} \left( \frac{x_\alpha}{x_\beta} ; q \right) \right] \left[ \prod_{\beta > \alpha} \left( \frac{x_\beta}{x_\alpha} ; q \right) \right] \cdot Z^B_{B_{\alpha}}(q, t, x, \xi)^2}_{tw}
\]

where for instance \( Z^1_{\alpha}(q, t, x, \xi) = Z^B_{B_{\alpha}}(q, t q^{\frac{1}{2}}, x, \xi) \). The twisted index as written in the first equality in (5.105) is partially factorised in terms of vortex partition

\(^4\)Non-trivial background fluxes for global symmetries can be incorporated easily and factorisation is in terms of the same hemisphere partition functions but with \( q \)-shifts of the fugacities by the appropriate fluxes [5].
functions or holomorphic blocks. In passing from the first to the second line, the 1-loop piece has been re-organised as in (5.89) (but with the shift of \(t\)) and we have used the following identity

\[
\theta_q(a q^{m/2}; q) \theta_q(a q^{-m/2}, q^{-1}) = (-1)^{m-1} a^{1-m}
\]

(5.106)

to fuse the theta functions. In passing to the last line, the remaining monomial is identified with the \(\|Z_{B, Cl}^\alpha\|_{tw}^2\). We thus produce a full factorisation (up to an overall sign) in terms of hemisphere partition functions \(B_\alpha\) associated to vacua in a fixed chamber for the mass parameters.

Similarly, the \(A\)-twisted index can be fully factorised:

\[
Z^A_{tw} = \sum_{\alpha=1}^N t^{(-N+1)\xi} \left[ \prod_{\beta \neq \alpha} \left( \frac{q^{x_{\beta}/x_\alpha}}{(t q^{x_{\beta}/x_\alpha}; q)_1} \right) \right] \left\| \sum_{m \geq 0} \left( t^{-N/2} \xi \right)^m \left[ \prod_{\beta = 1}^N \left( \frac{t q^{x_{\beta}/x_\alpha}}{(q^{x_{\beta}/x_\alpha}; q)_m} \right) \right] \right\|_{tw}^2
\]

(5.107)

In the \(t^2 \to 1\) limit with enhanced supersymmetry, the twisted indices become

\[
\lim_{t^2 \to 1} Z^A_{tw} = \sum_{\alpha=1}^N X^{C}_\alpha(q, x, \xi) X^{C^\ast}_\alpha(q^{-1}, x, \xi) = \frac{N \xi}{(1-\xi)^2},
\]

\[
\lim_{t^2 \to 1} Z^B_{tw} = (-1)^N \sum_{\alpha=1}^N X^{H}_\alpha(q, x, \xi) X^{H^\ast}_\alpha(q^{-1}, x, \xi) = (-1)^N \prod_{\beta \neq \alpha} \frac{x_\alpha/x_\beta}{(1-x_\alpha/x_\beta)^2},
\]

(5.108)

in agreement with the equivariant Hilbert series and results (5.92) for the \(A\)-limit and \(B\)-limit of the superconformal index.

### 5.6.4 \(S^3_b\) Partition Function

The final case we consider is the partition function on the squashed sphere or ellipsoid \(S^3_b\) [210]. In reference [93] it was shown that the supersymmetric localisation computation of such partition functions can be factorised into holomorphic blocks. We propose that the sphere partition functions of 3d \(\mathcal{N} = 4\) theories, deformed by an axial mass \(T\),
admits the following factorisation into hemisphere partition functions

\[ Z_{S^3_b} = \sum_{\alpha} Z_{B_{\alpha}}(q, t, x, \xi) Z_{B_{\alpha}}(\bar{q}, \bar{t}, \bar{x}, \bar{\xi}) \]  

(5.109)

up to an overall phase. The parameters are identified by

\[ q = e^{-2\pi bQ}, \quad t = e^{2\pi bT}, \quad x = e^{-2\pi bm}, \quad \xi = e^{-2\pi \zeta}, \]

\[ \bar{q} = e^{-2\pi iQ}, \quad \bar{t} = e^{2\pi T}, \quad \bar{x} = e^{-2\pi m}, \quad \bar{\xi} = e^{-2\pi \zeta} \]  

(5.110)

where \( Q := b + \frac{1}{b} \). Note we have used \( \zeta \) in this section for the FI parameter in order to avoid confusion. In writing expressions from \( S^3_b \) in terms of exponentiated parameters, rational powers are defined such that \( q^r := e^{-2\pi i rQ} \) where \( r \in \mathbb{Q} \).

We now consider the limit of the axial mass

\[ T \to \pm \frac{i}{2} (b - 1/b) \]  

(5.111)

preserving additional supersymmetry [207]. The resulting partition function depends on \( b \) in a trivial way: this parameter can either be absorbed into masses and FI parameters or sent to 1 to recover the matrix model for the round \( S^3 \) partition function as studied in [211, 212]. It has been proposed that the \( S^3 \) partition function in this limit can be expressed as a sum over massive vacua \( \alpha \) of products of twisted characters of Verma modules of \( \mathcal{A}_H \) and \( \mathcal{A}_C \) [99]. Explaining this proposal was one of the original motivations for the present work.

Beginning from the general factorised form of the \( S^3_b \) partition function (5.109), we note first that the limit \( T \to \frac{i}{2} (b - 1/b) \) sends

\[ t^{\frac{1}{2}} \to e^{-\pi i q^{-\frac{1}{4}}}, \quad \bar{t}^{\frac{1}{2}} \to e^{\pi i \bar{q}^{\frac{1}{4}}}. \]  

(5.112)

It is important to keep track of the minus sign in the exponentials, since this corresponds to a choice of branch in the logarithms appearing in the anomaly contribution to the hemisphere partition function. Taking the limit we find the following expression,

\[ \lim_{T \to \frac{i}{2} (b - 1/b)} Z_{S^3_b} = \sum_{\alpha} \hat{\mathcal{X}}^H_{\alpha}(q, x, \xi) \hat{\mathcal{X}}^C_{\alpha}(\bar{q}, \bar{x}, \bar{\xi}). \]  

(5.113)
Here the twisted characters are defined by
\[
\hat{X}_H^a(q, x, \xi) := \lim_{t \to e^{-\pi i q^2}} Z_{B^a}(q, t, x, \xi) = e^{\hat{\phi}_{B^a}} Tr_{H^{(B)}}(-1)^{j_V} x^{F_H},
\]
\[
\hat{X}_C^a(q, x, \xi) := \lim_{t \to e^{-\pi i q^2}} Z_{B^a}(q, t, x, \xi) = e^{\hat{\phi}_{B^a}} Tr_{H^{(A)}}(-1)^{j_A} \xi^{F_C}.
\]
These differ from the characters used previously by a \( \mathbb{Z}_2 \) twist by the centre of the Higgs or Coulomb branch R-symmetry. This is implemented in the trace by the additional factors of \((-1)^{j_V} \) and \((-1)^{j_A} \) respectively and in the classical or boundary anomaly contributions, which become
\[
e^{\hat{\phi}_{B^a}} = e^{-\frac{H}{2} + k \log \frac{\xi}{q}} e^{-\frac{\pi i k C \log \xi}{\log q}} e^{-\frac{\pi i k H \log x}{\log q}} e^{-\frac{\pi i k \log \xi}{\log q}},
\]
\[
e^{\hat{\phi}_{B^a}} = \xi \frac{e^{\frac{C}{2} + k \log \frac{\xi}{q}} e^{-\frac{\pi i k C \log \xi}{\log q}} e^{-\frac{\pi i k H \log x}{\log q}} e^{-\frac{\pi i k \log \xi}{\log q}}}{\mathbb{Z}^3_b}.
\]
The anomaly coefficients are those of the boundary condition \( B^a \), but we omit the index \( \alpha \) to avoid clutter. Note the presence of additional phases compared to \( e^{\hat{\phi}_{B^a}} \), \( e^{\hat{\phi}_{B^a}} \).

To recover the proposal of [99], we write equation (5.113) in terms of sphere parameters as follows,
\[
\sum_{\alpha} \left( e^{\pi i T/2} \xi \right)^{\hat{X}_H^a} e^{-\frac{\pi i k C \log \xi}{\log q}} e^{2\pi i m \log \xi} \left\| Z_{B^a} \right\|_2 \mathbb{Z}^3_b.
\]
(5.116)

after conjugating fugacities and gluing.

Finally, we note that the alternative limit \( T \to -\frac{i}{2} (b - 1/b) \) is obtained simply by exchanging \( b \leftrightarrow 1/b \) and thus barred and unbarred fugacities.

**Example: Hypermultiplet**

For a free hypermultiplet the partition function may be factorised using well-known double sine function identities
\[
\mathbb{Z}^3_b = \frac{s_b (-m + T/2 + iQ/4)}{s_b (-m - T/2 - iQ/4)} = \left\| Z_{B^a} \right\|_2^2.
\]
(5.117)
Then in the twisted trace limit we have
\[
\lim_{T \to \frac{i}{2}(b-1/b)} Z_{S^3_b} = \hat{X}^H_{B_X}(q, x) \hat{X}^C_{B_X}(\bar{q}, \bar{x}) = \frac{1}{2\cosh \pi bx},
\]
(5.118)
where
\[
\hat{X}^H_{B_X}(q, x) = e^{\frac{\pi \log x}{\log \bar{x}} - 1}, \quad \hat{X}^C_{B_X}(\bar{q}, \bar{x}) = e^{-\frac{\pi \log \bar{x}}{\log x}}.
\]
(5.119)
Note that the plus sign in the denominator of the twisted Higgs branch character arises from the additional weight \((-1)^{J_{V}}\) and the fact that the raising operator is the scalar field \(\hat{X}\) with \(J_{V} = 1\). The Coulomb branch Verma module is trivial and the Coulomb branch twisted character simply counts the identity operator, whose contribution is a phase due to the \(\mathbb{Z}_2\) twist by the centre of \(J_A\).

**Example: SQED**

A partially factorised form of the \(S^3_b\) partition function of supersymmetric QED is found by a computation in [93],
\[
Z_{S^3_b} = \frac{1}{s_b(T)} \int dz e^{2\pi i cz} \prod_{\beta=1}^{N} \left[ \frac{s_b(z + m_{\beta} + T/2 + iQ/4)}{s_b(z + m_{\beta} - T/2 - iQ/4)} \right]
= \sum_{\alpha=1}^{N} e^{-2\pi i \left( \zeta + \frac{N}{2}\left(T+iQ/2\right) \right)} \left( m_{\alpha} + \frac{1}{2} \left(T-iQ/2\right) \right) \left( \prod_{\beta \neq \alpha} \left( \frac{q^{\frac{x_{\beta}}{x_{\alpha}}}}{q^{\frac{x_{\alpha}}{x_{\beta}}}} \right) \right) \left( \prod_{m=0}^{\infty} \left( q^{1/2} t^{1/2} \right)^N \right) \left( \prod_{\beta=1}^{N} \left( \frac{q^{x_{\beta}}}{m_{\beta}} \right) \right)^{m} \left( \prod_{\beta=1}^{N} \left( \frac{q^{x_{\beta}}}{m_{\beta}} \right) \right)^{m} \left( \prod_{\beta=1}^{N} \left( \frac{q^{x_{\beta}}}{m_{\beta}} \right) \right)^{m}
\]
(5.120)
where the contour surrounds simple poles of the numerator at
\[
z = -m_{\alpha} - T/2 + iQ/4 + imb + in/b, \quad m, n \geq 0, \quad \alpha = 1, \ldots, N,
\]
(5.121)
and in the second line
\[
x_{\beta} = e^{-2\pi bm_{\beta}}, \quad \bar{x}_{\beta} = e^{-2\pi bm_{\beta}}.
\]
(5.122)
Notice this factorisation corresponds to hemisphere partition functions for boundary conditions compatible with mass parameters in different chambers.

In order to bring this expression into the factorised form of (5.109), it is necessary to re-arrange the classical and 1-loop contributions. For the hemisphere partition function \(Z_{B_a}\) given in section 5.5.3, we rewrite the 1-loop piece as in (5.89) and use
the identity:

$$\theta \left( e^{2\pi im_0 \hat{q} \hat{n}} q; q \right) \theta \left( e^{2\pi im_0 \hat{q} \hat{n}} \hat{q}; \hat{q} \right) = e^{-\frac{\pi i}{12} \left( b^2 + \frac{1}{b^2} \right)} e^{-i\pi \left( m + i\frac{1}{2}(1-n)Q \right)^2}$$

(5.123)

(see e.g. [213]) to glue the theta functions in (5.89). Then identifying $\sum m_0 = 0$, we fuse under $b \leftrightarrow \frac{1}{b}$ to obtain an exact factorisation

$$Z_{S^3_b} = e^{-\frac{\pi i}{12} Q^2} \sum_{\alpha=1}^{N} \|Z_{B_{\alpha}}(q, t, x, \xi)\|_{S^3_b}^2,$$

(5.124)

up to an overall phase which we now drop.

We now take the twisted character limit to find:

$$\lim_{T \to \frac{i}{2}(b-1/b)} Z_{S^3_b} = \sum_{\alpha=1}^{N} \hat{\chi}^H_{\alpha}(q, x, \xi) \hat{\chi}^C_{\alpha}(\hat{q}, \hat{x}, \hat{\xi}),$$

(5.125)

where the twisted Verma characters are

$$\hat{\chi}^H_{\alpha}(q, x, \xi) = e^{\hat{\delta}^{(H)}_{\alpha}} \prod_{\beta<\alpha} \frac{1}{1 - \frac{x_\beta}{x_\alpha}} \prod_{\beta>\alpha} \frac{1}{1 - \frac{x_\alpha}{x_\beta}},$$

$$\hat{\chi}^C_{\alpha}(q, x, \xi) = \frac{1}{1 - (-)^N \xi},$$

(5.126)

and the prefactors $e^{\hat{\delta}^{(H)}_{\alpha}}$ and $e^{\hat{\delta}^{(A)}_{\alpha}}$ are given by (5.115) and the anomaly coefficients (5.46). The $(-1)^N$ in the denominator of the twisted character of the Coulomb branch Verma module is because the raising operator is the monopole $\hat{v}_-$ with $J_A = N$. The raising operators in the Higgs branch Verma module are gauge invariant combinations (5.71) with $J_V = 2$, so there is no additional sign in the twisted character. Also note that the result for $N = 1$ is consistent with a hypermultiplet (5.118) under mirror symmetry.

The result of [99] is recovered explicitly by gluing the pre-factors in (5.126). Writing everything in terms of the sphere parameters we have

$$\lim_{T \to \frac{i}{2}(b-1/b)} Z_{S^3_b} = \sum_{\alpha=1}^{N} \left( e^{-\frac{\pi i}{2} B_{\alpha} \xi} \right)^{2N-1} \frac{e^{-\frac{\pi i}{2} B_{\alpha} \xi}}{1 - (-1)^N e^{-\frac{2\pi i}{T}}} \prod_{\beta<\alpha} \frac{e^{-\pi b(m_\beta-m_\alpha)}}{1 - e^{-2\pi b(m_\beta-m_\alpha)}} \prod_{\beta>\alpha} \frac{e^{-\pi b(m_\alpha-m_\beta)}}{1 - e^{-2\pi b(m_\alpha-m_\beta)}},$$

(5.127)
Chapter 6

Further Directions

Let us conclude with a brief description of directions for further research, building mainly on the ideas presented in this thesis.

Direct Generalisations

We first mention some rather obvious generalisations to the constructions described in this work. First, it would be natural to attempt to construct the equivariant cohomology and K-theory of symplectic resolutions via the Berry connection of 1d and 2d theories respectively, analogously to our constructions in chapters 3 and 4. The ability to recover cohomology from K-theory, and K-theory from elliptic cohomology via a limiting procedure (see e.g. [67]) should also have a precise physical analog in the dimensional reduction of 3d theories. It would also be interesting to investigate how objects such as stable envelopes (realised via boundary conditions) and the chamber structure track through the dimensional reduction.

Of course, it would also be interesting to generalise many of the constructions outlined in this work explicitly to non-abelian theories, namely the mirror duality interface and enriched Neumann boundary conditions. The states they generate on the elliptic curve should yield the mother function and elliptic stable envelopes of the appropriate varieties.

An alternative potentially fruitful extension is to consider theories with less supersymmetry, such as 3d $\mathcal{N} = 1$, or $\mathcal{N} = 2$ theories. Much of our analysis should go through for $\mathcal{N} = 2$ theories, and should lead to the equivariant elliptic cohomology of Kähler manifolds. Tentatively, $\mathcal{N} = (0,1)$ boundary conditions for $\mathcal{N} = 1$ theories may lead to the theory of topological modular forms, an exciting area of active study.
Berry Connections, Periodic Monopoles and Riemann-Hilbert

We noted in chapter 3 that the Berry connection for 3d $\mathcal{N}=4$ theories, which takes the form of a doubly periodic monopole on the parameter space, encodes not only the elliptic cohomology of the whole Higgs branch, but also that of submanifolds fixed by all 1-parameter subgroups of isometries. This gives a kind of spectral data for the monopole, consisting of holomorphic filtrations of the rank $N$ holomorphic vector bundle $\mathcal{E}$ at fixed values of the masses $m$. This should actually also the case for $\mathcal{N}=2$ theories. It would be interesting to uncover the relation with the spectral data obtained by the conventional scattering method approach to generalised Bogomolny equations. In particular, it should be related to the scattering data for doubly periodic monopoles in the $m$-directions.

An alternative set of spectral data, associated to the periodic directions in $E_T$ (the ‘$x$-spectral data’ of [167]), seems to instead realise the equivariant quantum K-theory of $X$. This suggests fascinatingly that the Berry connection provides a unification of these mathematical structures.

Mochizuki [170] has established a Hitchin-Kobayashi type correspondence between doubly periodic monopoles on $\mathbb{R} \times E_\tau$ (corresponding to theories with $\dim T = 1$) and $q$-difference modules, and further a Riemann-Hilbert type correspondence between such difference modules and pairs of filtrations on holomorphic vector bundles on $E_\tau$ induced by the monopole.1 Our discussion above should precisely be the physical analog of these constructions, generalised to doubly periodic monopoles in higher dimensions (i.e. on $t \times E_T$). The higher-dimensional case is studied in ongoing work by Kontsevich-Soibleman [214]. The connection to equivariant elliptic cohomology, and quantum equivariant K-theory is new, to the best of our knowledge. These connections are illustrated below in figure 6.1. It would be extremely interesting to establish these correspondences in detail and investigate the additional structure we have proposed.

The natural arena to study this set-up is through the brane amplitudes on $S^1 \times D^2$ of [82], originally studied in the 2d context [100, 215, 216]. The quantum equivariant K-theory is realised by line operator insertions wrapping the $S^1$ at the tip of the cigar, and equivariant elliptic cohomology by boundary conditions. A conformal limit of these amplitudes studied in [82] yields the holomorphic blocks, or equivalently the hemisphere partition functions studied in this work. Difference equations for these objects have been studied in [96, 104, 217].

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1Mochizuki’s construction depends on an extra complex parameter $\lambda$, parametrising a family of complex structures on the moduli space of background parameters for the theory. Physically this labels a family of supercharges, whose cohomology contains the objects realising the corresponding spectral data.
**Fig. 6.1** The web of connections arising from studying the Berry connection for 3d $\mathcal{N} \geq 2$ theories.

It is also interesting to study the analogous problem for 2d $\mathcal{N} = (2, 2)$ theories, which should relate equivariant K-theory and quantum equivariant cohomology. Mochizuki has studied the Hitchin-Kobayashi correspondence for singly periodic monopoles in [218].

**Generalised Global Symmetries**

It should be possible to incorporate recently studied generalised global symmetries [12], that is 1-form and 2-group symmetries (see e.g. [219]), into our 3d setup. One can then consider the dependence of the spectrum of ground states on the values of background fields for these higher symmetries. This should lead to a generalisation of the Berry connection, and a corresponding generalisation of equivariant elliptic cohomology, which is currently formulated using only 0-form symmetries. Boundary conditions (naturally forming a 2-category), which preserve the 2-group symmetry should then generate the cohomology classes. This is extremely suggestive, and tantalisingly similar to a proposal for a 2-equivariant version of elliptic cohomology due to Lurie [66]. It would be interesting to formalise these constructions for 3d $\mathcal{N} = 2$ gauge theories and explore the connections to 2-equivariant elliptic cohomology.

**Enumerative Geometry**

There is a clear application of the results in thesis to enumerative geometry, arising from considering the holomorphic block geometry, or the hemisphere partition functions computed on $S^1 \times HS^2$. Recall that we analysed two distinguished classes of boundary conditions in 3d $\mathcal{N} = 4$ theories which are exchanged under mirror symmetry, exceptional Dirichlet and enriched Neumann (the latter generating the elliptic stable envelopes of Aganagic-Okounkov [67]). We plan to establish rigorously that the hemisphere partition functions with exceptional Dirichlet boundary conditions are
Further Directions

precisely the vertex functions in equivariant quantum K-theory [157]:

$$Z_{S^1 \times H^2}^{D_r} \propto V_{\alpha}(X) = \sum_{d} \zeta^{d} \chi_{T}(Q\text{M}^{d}_{\alpha}(X)),$$

(6.1)

dwhere geometric invariants which equivariantly count quasi-maps (vortices) into the Higgs branch $X$ of the theory. This could be accomplished potentially via a Higgs branch localisation.

Further, by mirror symmetry, the hemisphere partition function of enriched Neumann should equal a vertex function for the mirror theory. It is also interesting to relate the half-indices of exceptional Dirichlet and enriched Neumann in a given theory. This may be done by inserting the identity $\sum_{\alpha} |D_{\alpha}^{r}\rangle \langle D_{\alpha}^{r}|$ into the hemisphere partition function for enriched Neumann boundary conditions. More precisely, we may consider $Z_{S^1 \times H^2}(N_{\alpha}^{r})$ as the overlap $\langle \Omega |N_{\alpha}^{r}\rangle$ between the states $\langle \Omega |$ generated by the path integral on the hemisphere, and $|N_{\alpha}^{r}\rangle$ by the boundary condition on $\partial(S^1 \times H^2) = E_{r}$. We have:

$$\langle \Omega |N_{\alpha}^{r}\rangle = \sum_{\beta} \langle \Omega |D_{\beta}^{r}\rangle \langle D_{\beta}^{r}|N_{\alpha}^{r}\rangle.$$

(6.2)

By equating the above with the hemisphere partition function with exceptional Dirichlet boundary conditions for the mirror dual theory, which is in turn the vertex function for the symplectic dual variety $\tilde{V}_{\alpha}(X)$, one obtains physically the following relation:

$$\tilde{V}_{\alpha}(X) = \sum_{\beta} V_{\beta}(X) \langle D_{\beta}^{r}|N_{\alpha}^{r}\rangle.$$

(6.3)

In chapter 4, we identified $\langle D_{\beta}^{r}|N_{\alpha}^{r}\rangle = \text{Stab}_{\alpha \mid \beta} \Theta(\beta + \frac{1}{2})$, which is precisely the pole-subtraction matrix [67, 68]. This would recover the relationship between vertex functions of symplectic dual varieties [70, 123, 160–163]. It would also be interesting to explore the consequences of this result in studying solutions to quantum difference equations.

Boundary Chiral Algebras

There is almost certainly an interesting interplay of the results in this thesis with the chiral algebras of boundary local operators [194], which are mathematically vertex operator algebras. These algebras categorify the half indices considered in chapter 5. For pure 3d $\mathcal{N} = 2$ Chern-Simons theory, the boundary chiral algebra (for a given boundary condition) and Hilbert space are respectively the vacuum module and space

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This may be accomplished by stretching the hemisphere into an infinitely long cigar, possible by the argument at the beginning of section 5.6, and then expanding in a basis of vacua in the intermediate flat region between the boundary and near the pole.
of conformal blocks of the WZW model appearing in Witten’s geometric quantisation of Chern-Simons theory. It may be that similar results hold for more general theories, and it will be interesting to see how our description of ground states as the elliptic cohomology of the Higgs branch fits into this discussion.

**Black Hole Microstates**

Finally, there is an important application of the results in thesis, in particular chapter 5, to the study of black hole entropy in four dimensions. The AdS/CFT correspondence [220] suggests that supersymmetric black hole microstates in $AdS_4$ can be accounted for by BPS states of a 3d supersymmetric conformal field theory living on the boundary. Whilst the degeneracy of these states has been verified by computing indices of 3d superconformal field theories [221, 222], as of yet there has been no significant progress in describing the Hilbert space of these states explicitly (a categorification), in terms of algebraic or geometric data. Our understanding of indices in terms of the quasi-map geometry of the Higgs branch may provide insight to rectifying this problem, with some progress made in this direction for a 3d SCFT with an AdS dual (the ADHM quiver theory) in the work [3]. Preliminary investigations suggest one may be able to recover black hole entropy as the trace of a certain operator in the quantum K-theory of the Higgs branch of the theory ($\text{Hilb}^N(\mathbb{C}^2)$, the Hilbert scheme of $N$ points in $\mathbb{C}^2$).
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Appendix A

Elliptic Cohomology Background

In this appendix, we review the construction of equivariant elliptic cohomology and some associated structures (such as elliptic stable envelopes [67] and mother functions [70]), to which we give a precise physically underpinning in chapters 3 and 4 of this thesis. It is intended for non-experts, focusing on giving intuition and motivation for these constructions. Equivariant elliptic cohomology was developed in [62–66, 223], see therein for further details. The author also found the review material on equivariant elliptic cohomology of symplectic resolutions in [67, 70, 186] helpful. For comparison with much of the literature on equivariant elliptic cohomology of symplectic resolutions, we note that what we call $T_H$ is often called $A$, and $T_t$ by $\hbar$.

A.1 Equivariant Elliptic Cohomology

A.1.1 Motivation

Suppose that $X$ is a smooth $T$-variety, i.e. it is equipped with the action of a torus $T \simeq (\mathbb{C}^*)^n$. We will further assume that $X$ is a hyper-Kähler quotient by a product of unitary groups $G = \prod_{i \in I} U(V_i)$, a symplectic resolution, and has isolated fixed points under $T$. These are properties satisfied by the Higgs branches we consider in this thesis.

The equivariant cohomology $H^*_T(X)$ is an algebra over the ring

$$H^*_T(pt) \cong \mathbb{C}[m_1, \ldots, m_n],$$

(A.1)

1In our case, we have an $T = T_H \times T_t$ acting on the Higgs branch of a theory $\mathcal{T}$. We can combine with gradient flow for the associated moment maps to obtain a natural complexification of $T$, and we do not distinguish between them in this section.
where \( \{ m_i \} \) are coordinates on \( t_C \cong \mathbb{C}^n \). Equivalently, Spec \( H^*_T(X) \) can be viewed as an affine scheme over Spec \( H^*_T(pt) \cong \mathbb{C}^n \). We call this the *equivariant cohomology scheme*, and may then think of equivariant cohomology classes as functions over this scheme. We will take this latter interpretation as our guide for motivating the construction of elliptic cohomology. Note that in general we may need to consider super-commutative schemes for varieties with odd cohomologies, however symplectic resolutions have even cohomology only. Replacing \( \mathbb{C} \) with \( \mathbb{C}^* \) there are analogous statements for the equivariant K-theory \( K_T(X) \). For example, Spec \( K_T(X) \) is an affine scheme over the algebraic torus Spec \( K_T(pt) \).

If \( X \) satisfies *Kirwan surjectivity*, proven for Nakajima quiver varieties \([121]\) and hyper-toric varieties \([122, 123]\), then we can be even more explicit. Recall that there are natural \( T \)-equivariant tautological bundles \( V_i \) arising from the hyper-Kähler quotient. Kirwan surjectivity tells us that \( K_T(X) \) is generated by tautological bundles, and \( H^*_T(X) \) by (equivariant) Chern classes of tautological bundles. Let us work with cohomology to simplify presentation, analogous statements can be made for K-theory. Let \( \{ \sigma \} \) be collectively the (equivariant) Chern roots of the tautological bundles \( V_i \), and \( \mathfrak{S} = \prod_{i \in I} S_{v_i} \), where \( S_{v_i} \) is the symmetric group of \( v_i := \dim \mathbb{C} V_i \) elements. Then:

\[
H^*_T(pt) \otimes \mathbb{C}[\{ \sigma \}]^\mathfrak{S} \to H^*_T(X) \tag{A.2}
\]

is a surjection. In other words, there is an embedding:

\[
\text{Spec } H^*_T(X) \hookrightarrow t_C \times \prod_{i \in I} \text{Sym} V_i \cong \mathbb{C}^n \times \prod_{i \in I} S^{v_i} \mathbb{C} \tag{A.3}
\]

Additionally, equivariant localisation in cohomology then tells us that:

\[
H^*_T(X) \to \bigoplus_{\alpha \in X^T} H^*_T(\alpha) \tag{A.4}
\]

is an injection, where the right hand side is the equivariant cohomology of the fixed point set. This is the pullback in equivariant cohomology of the inclusion \( \{ \alpha \} \hookrightarrow X \). The map is given by the evaluation of the equivariant Chern roots at the fixed points \( \{ \alpha \} \), *i.e.* fixing \( \sigma \) to be the \( T \)-weights of the tautological bundles at \( \{ \alpha \} \).\(^2\) By application of the first isomorphism theorem one therefore has the explicit description

\[
H^*_T(X) \cong \mathbb{C}[m_1, \ldots, m_n] \otimes \mathbb{C}[\{ \sigma \}]^\mathfrak{S} / I, \tag{A.5}
\]

\(^2\)The corresponding evaluation in K-theory is precisely the one described in equation (2.29) to evaluate the tangent space character.
where $I$ is the ideal of polynomials vanishing at the fixed points (in the way described above).

As an example, take $X = T^*\mathbb{P}^{N-1}$. There is a rank 1 tautological bundle, with Chern class $\sigma$. Let us denote the parameters for $T = T_H \times T_t$ by $(m_1, \ldots, m_N, t)$. Recall $T_t$ is the action scaling the cotangent directions in $T^*\mathbb{P}^1$, and the holomorphic symplectic form with weight 2. There are $N$ fixed points, indexed by $\alpha$, given by the $N$ coordinate hyperplanes in $\mathbb{P}^{N-1}$, where $\sigma = m_\alpha - t$. Thus:

$$H^*_T(T^*\mathbb{P}^{N-1}) \cong \mathbb{C}[m_1, \ldots, m_N, t, \sigma] / \left\{ \prod_{\alpha=1}^{N} (\sigma - m_\alpha + t) \right\}, \quad (A.6)$$
equivalently polynomial functions over the equivariant cohomology scheme:

$$\text{Spec } H^*_T(T^*\mathbb{P}^{N-1}) = \left\{ \prod_{\alpha=1}^{N} (\sigma - m_\alpha + t) = 0 \right\} \subset t_\mathbb{C} \times \mathbb{C}. \quad (A.7)$$

As mentioned above, the analogous statements can be obtained in equivariant K-theory by replacing $\mathbb{C}$-valued parameters with $\mathbb{C}^*$-valued parameters, and taking Laurent (instead of usual) polynomials over the equivariant K-theory scheme. One is led to thinking about the natural generalisation to the third 1-dimensional algebraic group $E_\tau = \mathbb{C}^*/q\mathbb{Z}$, where $q = e^{2\pi i \tau}$. If we take the above philosophy seriously, as a starting point we might try to construct a scheme over $E_\tau$ associated to $X$, and then build cohomology classes as functions over it. Notice however that $E_\tau$ is compact; there are no holomorphic functions on it. But there are sections of line bundles. We will now see that equivariant elliptic cohomology is defined in this way.

### A.1.2 Elliptic Cohomology Scheme

Let us now describe the construction of the equivariant elliptic cohomology scheme, following closely the presentation in [67]. We fix an elliptic curve $E_\tau = \mathbb{C}^*/q\mathbb{Z}$, where $q = e^{2\pi i \tau}$ with $0 < |q| < 1$. Then the equivariant elliptic cohomology defines a functor:

$$\text{Ell}_T : \{ T\text{-variety } X \} \rightarrow \{ \text{schemes} \} \quad (A.8)$$
covariant in $T$ and $X$, such that $\text{Ell}_{\mathbb{C}^*}(\text{pt}) = E_\tau$. Covariance in $T$ implies that

$$E_T := \text{Ell}_T(\text{pt}) = T/\text{qcochar}(T) \cong E^{\text{dim}(T)}. \quad (A.9)$$
Covariance in $X$ provides the canonical projection $X \to \text{pt}$ with a map $\pi : \text{Ell}_T(X) \to E_T$ which we now describe.

Consistently with the the main text, we will use coordinates $z = (z_H, z_t)$ on $E_T = E_{T_H} \times E_{T_t}$, which are identified up to shifts in $\Gamma_H \oplus \Gamma_t$. We also use the exponentiated coordinates $v = e^{2\pi i z_H}$, $t = e^{2\pi i z_t}$, and collectively $a = e^{2\pi i z}$, which are thus identified up to $q$ shifts.

Let $z \in E_T$ and $U_z \subset E_T$ a small analytic neighbourhood of it. $U_z$ is isomorphic to a small analytic neighbourhood in $t_C = \mathbb{C}^{\dim(T)}$ where $z$ is mapped to the origin. Then $\pi$ is determined by the following diagram:

$$
\begin{array}{ccc}
\text{Spec } H^*_T(X_T, \mathbb{C}) & \xleftarrow{\pi^{-1}(U_z)} & \text{Ell}_T(X) \\
\downarrow & & \downarrow \\
t_C & \xleftarrow{U_z} & E_T
\end{array}
$$

where both left and right squares are commutative. Here

$$
T_z = \bigcap_{\substack{\chi \in \text{char}(T) \\ \chi(z) = 0}} \ker \chi \subset T.
$$

Now, Nakajima quiver varieties and hyper-toric varieties obey a property known as equivariant formality, see [67, 135], which implies:

$$
H^*_T(X_T) \cong H^*(X_T) \otimes H^*_T(\text{pt}) \quad \forall z.
$$

This means the fibre of $\pi$ at $z$ may be obtained by sending the equivariant parameters in a neighbourhood of $z$ to 0, i.e.

$$
\pi^{-1}(z) = \text{Spec } H^*(X_T, \mathbb{C}).
$$

The equivariant elliptic cohomology scheme $\text{Ell}_T(X)$ can then be constructed by ‘gluing’ the fibres of $\pi$. More formally, for each $U_z$ one has an algebra $\mathcal{A}|_{U_z} := H^*_T(X_T, \mathbb{C}) \otimes H^*_T(\text{pt}) \mathcal{O}_{U_z}^{\text{analytic}}$, which glue to a sheaf $\mathcal{A}$ of algebras over $E_T$. Then $\text{Ell}_T(X) = \text{Spec}_{E_T} \mathcal{A}$. As alluded above, one then defines elliptic cohomology classes to be holomorphic (or meromorphic) sections of line bundles over $E_T(X)$. We will see a more explicit description of these shortly.
Example

Let us consider the example of $X = T^*\mathbb{P}^1$, the Higgs branch of supersymmetric QED with two hypermultiplets. Recall the algebraic description (2.36) of $X$. On the pre-quotient affine space with coordinates $(X_1, Y_1, X_2, Y_2)$, $T = T_H \times T_t = (\mathbb{C}^*)^2 \times \mathbb{C}^*$ acts as

$$(X_1, Y_1, X_2, Y_2) \rightarrow (v_1^{-1}tX_1, v_1tY_1, v_2^{-1}tX_2, v_2tY_2).$$

(A.14)

Let us use the coordinates $z = (z_1, z_2, z_t)$ on $E_T \sim E_3^t$. If $z_1$ and $z_2$ are generic, one has that $X^T_z$ are just the poles of $\mathbb{P}^1$, where either $X_1$ or $X_2$ is non-zero and all other coordinates vanish. The fibre $\pi^{-1}(z) = \text{Spec} H^*(X^T_z, \mathbb{C})$ is just the set of two points.

Now consider points on the diagonal:

$$\Delta = \{(z_1, z_2, z_t) \mid z_1 = z_2\} \subset E_T.$$

(A.15)

Technically, we take $z_1$ and $z_2$ to be coordinates on $E_T$ identified up to $z_i \sim z_i + m + n\tau$ with $m, n \in \mathbb{Z}$, so $\Delta$ is really the locus where $z_1 = z_2 \mod \mathbb{Z} + \tau\mathbb{Z}$. For these points, $X^T_z = X$, and one has from above that:

$$H^*_T(T^*\mathbb{P}^1) = \mathbb{C}[\sigma, \delta z_1, \delta z_2, \delta z_t]/(\sigma - \delta z_1 + \delta z_t)(\sigma - \delta z_2 + \delta z_t)$$

(A.16)

where $\delta z_i$ are local coordinates on $t_\mathbb{C}$. Thus $\text{Spec} H^*_T(T^*\mathbb{P}^1)$ is given by two intersecting hyperplanes $\{\sigma = \delta z_i - \delta z_t\}$ in $\mathbb{C}^4$, and models Ell$_T(X)$ in a neighbourhood of $z \in \Delta$.

We conclude that Ell$_T(X)$ is the union of two copies of $E_T^{(i)} \cong E_T$ which intersect along the diagonal $\Delta$:

$$\text{Ell}_T(X) = \left( E_T^{(1)} \cup E_T^{(2)} \right) / \Delta.$$

(A.17)

The intersection is transversal so the scheme Ell$_T(X)$ is actually a variety with a simple normal crossing singularity along the diagonal.

A.1.3 GKM Varieties

In general then, it is not hard to see that:

$$\text{Ell}_T(X) = \left( \bigcup_{\alpha \in X_T} E_T^{(\alpha)} \right) / \Delta$$

(A.18)
where \( \{\alpha\} \) is the set of fixed points, \( E_T^{(\alpha)} \cong E_T \) are \( |X^T| \) copies of \( E_T \), and \( \Delta \) denotes the gluing of these varieties along the subschemes \( \text{Spec} \ H^*(X^T_z) \) for subtori \( T_z \) such that \( X^T_z \) is larger than \( X^T \). Generically, this gluing can be very complicated.

In this thesis, we mostly restrict to studying Higgs branches which are GKM varieties [135]. See section 2.5.2 for the definition and details. In particular, between any two fixed points \( \alpha \) and \( \beta \) there is at most one \( T \)-equivariant curve \( \Sigma_\lambda \cong \mathbb{P}^1 \) connecting them, which is labelled by a \( T \)-weight \( \lambda = T_\alpha \Sigma_\lambda \subset T_\alpha X, -\lambda = T_\beta \Sigma_\lambda \subset T_\beta X \). As can be seen from our previous example, this implies:

**Proposition 2** If \( X \) is GKM, then \( \text{Ell}_T(X) \) is of the form (A.18), where \( /\Delta \) denotes the intersections of all pairs \( E_T^{(\alpha)}, E_T^{(\beta)} \) such that there exists a \( T \)-equivariant \( \Sigma_\lambda \cong \mathbb{P}^1 \) connecting \( \alpha \) and \( \beta \), along the common hyperplane

\[
E_T^{(\alpha)} \supset \{\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}\} \subset E_T^{(\beta)}
\]  

(A.19)

where \( \lambda \) is the weight labelling the curve as above. Additionally, the intersections are transversal and therefore the scheme \( \text{Ell}_T(X) \) is a variety with simple normal crossing singularities.

For the proof of the latter point, see [70]. We meet the example of \( T^* \mathbb{P}^{N-1} \) in chapter 3.

In the case of GKM varieties then, one has a particularly simple description of line bundles (elliptic cohomology classes) on \( \text{Ell}_T(X) \).

**Proposition 3** A line bundle \( \mathcal{L} \) on a GKM variety \( X \) is a collection of line bundles \( \mathcal{L}_\alpha \) on the components \( E_T^{(\alpha)}, \alpha \in X^T \) which agree on the intersections

\[
\mathcal{L}_\alpha|_{E_T^{(\alpha)} \cap E_T^{(\beta)}} = \mathcal{L}_\beta|_{E_T^{(\alpha)} \cap E_T^{(\beta)}}.
\]  

(A.20)

A meromorphic (holomorphic) section of \( \mathcal{L} \) is a collection of meromorphic (holomorphic) section of \( \mathcal{L}_\alpha \) agreeing on the intersections in the same way as above.

One needs to be slightly careful on what ‘agreeing on the intersection’ means here. Each of the components \( E_T^{(\alpha)} \) is some power of the elliptic curve \( E_\tau \), and so isomorphism classes of line bundles over \( E_T^{(\alpha)} \) are classified by factors of automorphy up to an equivalence relation, and sections of those line bundles by functions on \( E_\tau \) which transform with said factors of automorphy [179, 224, 225]. These sections can be explicitly expressed in terms of the elliptic theta function defined in (3.52). We review these facts in more detail in appendix B.
The equality in (A.20) is then more properly an isomorphism of the line bundles on \(E_T\) obtained by restricting \(L_\alpha\) and \(L_\beta\) on to the locus \(\{\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}\}\), where \(\lambda\) is the tangent weight to the curve connecting \(\alpha\) and \(\beta\).

### A.1.4 Off-shell Formalism

One can extend Kirwan surjectivity (A.3) to equivariant elliptic cohomology, using the local modelling of the elliptic cohomology scheme on the cohomology/K-theory scheme [67]. This gives an embedding:

\[
c : \text{Ell}_T(X) \hookrightarrow E_T \times E_G := E_T \times \prod_{i \in I} S^\vee_i E_r,
\]

where the coordinates on the second factor are symmetric functions in the elliptic Chern roots, which we also denote by \(\{s = e^{2\pi i u}\}\), see [63, 64]. Each sheet \(E_T^{(\alpha)}\) of \(\text{Ell}_T(X)\) is given by fixing the elliptic Chern roots to their values at the fixed point \(\alpha\) analogously to equivariant cohomology and K-theory.

Often, the objects we encounter (such as line bundles on \(\text{Ell}_T(X)\) and associated sections) are obtained via the pullback \(c^*\) of corresponding objects on \(E_T \times E_G\). We will often call the objects in \(E_T \times E_G\) ‘off-shell’, as in [186].

We meet a natural physical interpretation of both the target of (A.21) , and sections of line bundles over it in chapter 3.

### A.1.5 Extended Elliptic Cohomology

Let us denote:

\[
E_{T_C} = \text{Pic}_T(X) \otimes \mathbb{Z} E_r,
\]

and define \(E_T = E_T \times E_{T_C} = E_{TH}\). In the mathematics literature, the parameters in \(E_T\) are called equivariant parameters, and those in \(E_{T_C}\) Kähler parameters. If \(G = \prod_{i \in I} U(V_i)\), then \(\text{Pic}(X) \cong \mathbb{Z}^{|I|}\) so \(E_{T_C} \cong E_{T_C}^{||I||} \cong E_r^{\dim(T_C)}\). We have drawn the connection to the Coulomb branch flavour symmetry here. Consistent with the main text, we will use coordinates \(z_C\) on \(E_{T_C}\), and corresponding exponentiated coordinate \(\xi = e^{2\pi i z_C}\) (which is identified up to \(q\)-shifts).

We denote by:

\[
E_T(X) := \text{Ell}_T(X) \times E_{T_C}
\]
the extended equivariant elliptic cohomology of $X$, which is trivially a scheme over $E_{T_f}$, and embeds into $E_{T_f} \times \prod_{i \in I} S^{n_i} E_r$. Further, we have:

$$E_T(X) = \left( \bigcup_{\alpha \in X_T} E_{T_f}^{(\alpha)} \right) / \Delta,$$

where now $E_{T_f}^{(\alpha)} \cong E_{T_f}$, and the gluing $\Delta$ is unchanged (it is along the same loci). The description of line bundles for GKM varieties in proposition 3 is unchanged, just with $T$ replaced by $T_f$.

The extended elliptic cohomology is clearly natural to study from the point of view of 3d $\mathcal{N} = 4$ gauge theory: the equivariant and Kähler parameters correspond to holonomies for background connections for the $T_f = T_H \times T_f \times T_C$ flavour symmetry. It is natural to turn them on and consider the behaviour of supersymmetric ground states over their moduli space, see chapter 3.

### A.2 Elliptic Stable Envelopes

#### A.2.1 Motivation

The theory of stable envelopes originated in [89] as a basis of equivariant cohomology classes for Nakajima quiver varieties. They are a generalisation of classes of attracting Lagrangian submanifolds (see the brief discussion in section 4.3.2 on this), obeying a set of desirable properties, and defined uniquely. These constructions were extended to equivariant K-theory in [157, 226], and finally to equivariant elliptic cohomology in [67]. The elliptic stable envelopes may be regarded as a nice basis of elliptic equivariant cohomology, or an assignment of an elliptic cohomology class to each fixed point. They were studied further in [68, 70, 186–188].

The elliptic stable envelope is in fact the most general structure as the K-theoretic and cohomological stable envelopes may be considered as its limits. Unlike the others, elliptic stable envelopes depend on Kähler parameters, which physically are background connections for the topological symmetry $T_C$, consistent with the fact that they are natural to study through the lens of 3d gauge theory and mirror symmetry.

Since their introduction stable envelopes have played an increasingly important role in numerous areas of mathematics, a few of which we mention here, although they will not be the focus of this thesis. Hopefully they serve to give some motivation for realising the stable envelopes physically.
• In [89], Maulik and Okounkov construct the action of a quiver Yangian on the equivariant cohomology of Nakajima quiver varieties, drawing a connection between geometry and quantum integrable systems. The \( R \)-matrix for this integrable system is demonstrated to be built out of stable envelopes. Further generalisations have been made to the K-theoretic and elliptic cases.

• The physical origin of the above is the Bethe/Gauge correspondence of Nekrasov and Shatashvili [90, 91]. One considers supersymmetric gauge theories with Nakajima quiver varieties as Higgs branches. The massive vacua are then identified with Bethe equations for spin chains, which are the same quantum integrable systems studied by Maulik and Okounkov above. Further, the (twisted) chiral rings of these gauge theories are identified with the quantum cohomology/K-theory/elliptic cohomology of the Higgs branch. In this context, the stable envelopes appear as the wavefunctions of off-shell Bethe states of the spin chain in the up-down basis, demonstrated for equivariant cohomology in [92].

• Stable envelopes have been understood as so-called weight functions for integral solutions of qKZ equations [87, 88].

• Elliptic stable envelopes have played an important role in enumerative geometry, in particular relating enumerative invariants of symplectic dual pairs \((X, X')\) [70, 123, 160–163]. The invariants are vertex functions, and are equivariant counts of quasi-maps to \((X, X')\). As the vertex functions themselves are solutions to quantum difference equations in both equivariant, and Kähler parameters [68], the elliptic stable envelopes relate bases of solutions to these equations.

A.2.2 Constructions

We now briefly review the definition of the elliptic stable envelopes of a symplectic resolution \( X \), following section 3 of [67] and section 2 of [186]. We will only consider \( X \) with isolated fixed points under \( T \); the construction in [67] is more generic.

The elliptic stable envelope, in a chamber \( \mathcal{C}_H \), defines a map of \( E_{T_f} \) modules (sections of the following line bundles):

\[
\text{Stab}_{\mathcal{C}_H} : \mathcal{N}' \to \mathcal{N}
\]  

(A.25)
where $\mathcal{N}'$ is a a particular line bundle over

$$\text{Ell}_T(X^{TH}) \times E_{TC} = \bigsqcup_{\alpha \in X^T} E^{(\alpha)}_{T_f}$$

(A.26)

and $\mathcal{N}$ is a particular line bundle over $E_T(X)$. From the above, $\mathcal{N}'$ may be described as a collection of line bundles $\mathcal{N}'_\alpha$ on each $E^{(\alpha)}_{T_f}$. The definition of $\mathcal{N}'$ is dependent on the choice of chamber $\mathcal{C}_H$. From the above, $\mathcal{N}$ is a collection of line bundles $\mathcal{N}_\alpha$ on each $E^{(\alpha)}_{T_f}$, which glue to a single line bundle on $E_T(X)$. $\mathcal{N}$ is the pull back $c^*(\mathcal{N}^{\text{os}})$ of a line bundle on $E_T \times E_G$.

The construction of $\mathcal{N}'$ and $\mathcal{N}$ is detailed in section 3 of [67] and 2 of [186]. We do not review this in full here, as this would require the introduction of a significant number of new notions and notations. The important points are:

- The factors of automorphy characterising the line bundles $\mathcal{N}'_\alpha$ are, up to adding an overall $\alpha$-independent shift, precisely equal to (minus) the effective Chern-Simons levels $-\kappa_\alpha$, in a chamber $\mathcal{C}_H$, introduced in section 2.5.

- The factors of automorphy characterising the line bundle $\mathcal{N}^{\text{os}}$ are precisely equivalent, after subtracting the same $\alpha$-independent shift, to the boundary anomalies of the auxiliary Dirichlet boundary condition $\langle DF(s)\rangle$ introduced in section 3.8.

Thus the line bundles $\mathcal{N}_\alpha$ transform with factors of automorphy determined by the boundary anomalies of $\langle DF(s_\alpha)\rangle$, where $s_\alpha$ denotes the values of $s$ fixed at the vacuum $\alpha$. Recall from section 4.1.2 that these are the same as those of the state $\langle \alpha \rangle$, generated by doing the path integral with a vacuum $\alpha$ at $x^3 \to -\infty$. They are given by $-\kappa_\alpha - \kappa_\alpha^C + \frac{1}{4} \sum_{\lambda \in \Phi_\alpha} \lambda \otimes \lambda$, see (3.57). They are $\mathcal{C}_H$ independent.

These become clear by comparing the geometric interpretation of Chern-Simons levels with the line bundles in section 3 of [67]. In particular, the universal line bundles there essentially encode the mixed anomalies with $T_C$. Technically the stable envelope also depends on a choice of polarisation, but a different choice is implemented simply by a constant translation in the $E_{TC}$ directions, i.e. a shift of $\xi = e^{2\pi izt}$ by a power of $t$. In the above we have implicitly chosen a polarisation to agree with the factors of automorphy of the specified boundary conditions.

From the above, it becomes clear that to give a map (A.25) is the same as to construct a section $\text{Stab}_{\mathcal{C}_H}(\alpha)$ of the line bundle

$$\pi^*(\mathcal{N}_\alpha)^{-1} \otimes \mathcal{N} \in \text{Pic}(E_T(X))$$

(A.27)
over $E_T(X)$, where $\pi$ is the projection $\pi: E_T(X) \to E_T^{(\alpha)}$. We refer to this as the elliptic stable envelope of a fixed point $\alpha$, and it may be regarded as an elliptic cohomology class in $E_T(X)$. These are the cohomology classes we realise through the collection of boundary amplitudes $\{\langle \beta | N_\alpha \rangle\}$ of the enriched Neumann boundary condition $N_\alpha$ in section 4.3. More accurately:

$$\text{Stab}_{\xi_\alpha}(\alpha)|_{E_T(\beta)} = \langle \beta | N_\alpha \rangle.$$  

(A.28)

The boundary anomalies appearing of $N_\alpha$ and the inflow from $\langle \beta \rangle$ are consistent with the line bundles above.

Alternatively, one may instead construct a section $\text{Stab}_{\xi_\alpha}^{os}(\alpha)$ of the line bundle

$$\pi^*(N'_\alpha)^{-1} \otimes N^{os} \in \text{Pic}(E_T \times E_G),$$  

(A.29)

and then pull it back:

$$\text{Stab}_{\xi_\alpha}(\alpha) = c^*\text{Stab}_{\xi_\alpha}^{os}(\alpha).$$  

(A.30)

The off-shell form of the stable envelope $\text{Stab}_{\xi_\alpha}^{os}(\alpha)$ is precisely the wavefunction $\langle D_F(s)|N_\alpha \rangle$ we meet in the main text.

Together with being a map of sections of these line bundles, the stable envelope must also satisfy the following conditions:

- $\text{Stab}_{\xi}(\alpha)|_{\beta} = 0$ for $\beta < \alpha$, where the ordering on fixed points coincides with the one used in the main text, that is $h_m|_{\beta} < h_m|_{\alpha}$.

- $\text{Stab}_{\xi}(\alpha)|_{\alpha} = \prod_{\lambda \in \Phi^+_+} \vartheta(a^\lambda)$.

The first condition is the analog of the support of the cohomological stable envelope being contained in the full attracting set (obtained by successively taking closures and then attracting sets of the closure) with respect to the inverse gradient flow, as in [67]. The second condition is simply a normalisation.

**Theorem A.2.1** [67] The elliptic stable envelopes, defined by the map (A.25) and the above conditions, exist for Nakajima quiver varieties and hyper-toric varieties, and are unique.
Appendix B

Line Bundles on the Elliptic Curve

In this appendix we demonstrate the gluing property of line bundles \( \{ L'_\alpha \} \) required in section 3.4.2. First, we recall some canonical results on line bundles over elliptic curves [179], using the rephrasing in [224, 225] of these results in the language of factors of automorphy.

It will be convenient to regard the background flat connections \( z_f = (z_C, z_H, z_t) \) as coordinates on \( \mathfrak{t}_f^C = \mathfrak{t}_f \otimes_\mathbb{R} \mathbb{C} \). Then global background gauge transformations \( z_f \rightarrow z_f + \nu_f + \mu_f \tau \) form a group \( \Lambda_f \) of deck transformations so that we have a universal covering:

\[
\mathfrak{t}_f^C \rightarrow \mathfrak{t}_f^C / \Lambda_f = E_{T_f}, \quad z_f \mapsto [z_f]. \tag{B.1}
\]

A factor of automorphy is a holomorphic function \( F : \mathfrak{t}_f^C \times \Lambda_f \rightarrow \mathbb{C}^* \) obeying

\[
F(z_f, \nu_f + \mu_f \tau + \nu'_f + \mu'_f \tau) = F(z_f + \nu_f + \mu_f \tau, \nu'_f + \mu'_f \tau)F(z_f, \nu_f + \mu_f \tau). \tag{B.2}
\]

There is an equivalence relation on factors of automorphy, where \( F \sim F' \) if there exists a holomorphic function \( H : \mathfrak{t}_f^C \rightarrow \mathbb{C}^* \) obeying

\[
H(z_f + \nu_f + \mu_f \tau)F(z_f, \nu_f + \mu_f \tau) = F'(z_f, \nu_f + \mu_f \tau)H(z_f). \tag{B.3}
\]

Importantly, isomorphism classes of line bundles over \( E_{T_f} \), are in 1-1 correspondence with equivalence classes \( [F] \). Sections of a line bundle associated to \( F \) are 1-1 with holomorphic functions \( s : \mathfrak{t}_f^C \rightarrow \mathbb{C} \) satisfying

\[
s(z_f + \nu_f + \mu_f \tau) = F(z_f, \nu_f + \mu_f \tau)s(z_f). \tag{B.4}
\]
In our set-up, the line bundles $\mathcal{L}'_\alpha$ have factors of automorphy

$$F_\alpha(z_f, \nu_f + \mu_f \tau) = e^{-2\pi i (K'_\alpha(z_f, \mu_f) + K'_\alpha(\mu_f, z_f) + \tau K'_\alpha(\mu_f, \mu_f))} \quad (B.5)$$

where $K'_\alpha$ are the shifted supersymmetric Chern-Simons couplings as in equation (3.36). It is not difficult to check that $F_\alpha$ obeys property (B.2).

Now consider a weight $\lambda \in \Phi_\alpha \cup (-\Phi_\beta)$ labelling an interior edge of the GKM diagram. Mass parameters $x$ lying on the hyperplane $W_\lambda = \{ \lambda \cdot x = 0 \}$ generate a codimension 1 subtorus $T^\lambda$ of $T$, which leaves point-wise fixed the irreducible curve $\Sigma_\lambda \cong \mathbb{P}^1$ connecting the fixed points $\alpha$ and $\beta$. The weight $\lambda$ then defines a doubly-periodic array of hyperplanes in $t_f^C$ where

$$\lambda \cdot z \in \mathbb{Z} + \tau \mathbb{Z}, \quad (B.6)$$

recalling that $z = (z_H, z_t)$. Let us suppose $\lambda \cdot z = a + b\tau$ for $a, b \in \mathbb{Z}$. This breaks the group of large background gauge transformations to $\Lambda^\lambda_f$, whose elements obey $\lambda \cdot \nu = \lambda \cdot \mu = 0$, where $\mu = (\mu_H, \mu_t)$ etc. Then we may define

$$E^\lambda_{T_f} = \{ z_f \in t_f^C | \lambda \cdot z = a + b\tau \} / \Lambda^\lambda_f \hookrightarrow E_{T_f}, \quad (B.7)$$

a codimension-one subtorus of $E_{T_f}$, where we have left the $a, b$ dependence implicit. Analogous definitions to the above can be made for line bundles over $E^\lambda_{T_f}$.

The result required in section 3.4.2 is that for all such $\lambda$, the restriction (more accurately the pull-back under the above inclusion) of the line bundles $\mathcal{L}'_\alpha$ and $\mathcal{L}'_\beta$ to $E^\lambda_{T_f}$ (for any $a, b \in \mathbb{Z}$) are isomorphic. We demonstrate this using their factors of automorphy.

On $E^\lambda_{T_f}$ (the pull-back of) $\mathcal{L}'_\alpha$, $\mathcal{L}'_\beta$ have factors of automorphy given by restricting $F_\alpha$, $F_\beta$ to $\lambda \cdot z = a + b\tau$ and $\Lambda_f$ to $\Lambda^\lambda_f$. In the remainder of this appendix, we use $|_{E^\lambda_{T_f}}$ to denote these restrictions. Then isomorphism of $\mathcal{L}'_\alpha$ and $\mathcal{L}'_\beta$ on the locus is equivalent to the existence of a holomorphic function $H(z_f)$ such that

$$H(z_f + \nu_f + \mu_f \tau)F_\beta(z_f, \nu_f + \mu_f \tau) = F_\alpha(z_f, \nu_f + \mu_f \tau)H(z_f) \bigg|_{E^\lambda_{T_f}}. \quad (B.8)$$

To demonstrate this, we first collect the following two results.
1. First consider the contribution from $\kappa_\alpha + \kappa_\alpha^C$ to the ratio $F_\alpha/F_\beta$ of factors of automorphy

$$e^{-2\pi i ((\kappa_\alpha + \kappa_\alpha^C)(\tau_{f, \mu_f}) + (\kappa_\alpha + \kappa_\alpha^C)(\lambda_{f}) + \tau(\kappa_\alpha + \kappa_\alpha^C)(\lambda_{f, \mu_f}))},$$  \hspace{1cm} (B.9)

where we have used the shorthand $\kappa_{\alpha\beta} = \kappa_\alpha - \kappa_\beta$. Note the relation on critical points of Morse flows

$$h|_\alpha - h|_\beta = \langle \zeta, [\Sigma_\lambda] \rangle (\lambda \cdot m),$$  \hspace{1cm} (B.10)

where $\zeta$ is identified as an element of $H^2(X, \mathbb{R})$. From equation (2.51), one has that (B.9) equals

$$e^{-2\pi i \langle \mu_C, [\Sigma_\lambda] \rangle (\lambda \cdot z)} = e^{-2\pi i b(\mu_C, [\Sigma_\lambda])\tau}$$  \hspace{1cm} (B.11)

on restriction to $E^\lambda_{T_f}$. We have used $\langle \mu_C, [\Sigma_\lambda] \rangle \in \mathbb{Z}$.

2. Now consider the contribution from the sum over $\Phi_\alpha$. By construction the $T^\lambda$-weight of $T^{\alpha}_{\Sigma_\lambda} \subset T^{\alpha}_X$ (and thus $T^{\beta}_{\Sigma_\lambda}$) is 0. Since $T^\lambda$ fixes point-wise the curve $\Sigma_\lambda$, the (quantised) weights of $T^\lambda$ are constant over $\Sigma_\lambda$ by continuity, and in particular coincide at $\alpha$ and $\beta$. Thus there is a pairing of weights $\lambda'_\alpha \in \Phi_\alpha$ and $\lambda'_\beta \in \Phi_\beta$ such that

$$(\lambda'_\alpha - \lambda'_\beta) \cdot z = C_{\alpha\beta}(\lambda \cdot z).$$  \hspace{1cm} (B.12)

The constants $\{C_{\alpha\beta}\}$ are integer, as the characters of the isotropy representations of $T^{\alpha}_X, T^{\beta}_X$ agree when $e^{2\pi i \lambda \cdot z} = 1$, in particular when $\lambda \cdot z \in \mathbb{Z}\setminus\{0\}$. Note that (B.12) also implies $\lambda'_\alpha \cdot \mu = \lambda'_\beta \cdot \mu$ for all $\nu_f + \mu_f \tau \in \Lambda_f$.

The contribution from $\Phi_\alpha, \Phi_\beta$ in the ratio $F_\alpha/F_\beta$ is:

$$-\pi i \left( \sum_{\lambda'_\alpha \in \Phi_\alpha} (\lambda'_\alpha \cdot z)(\lambda'_\alpha \cdot \mu) + \frac{\pi}{2} \sum_{\lambda'_\beta \in \Phi_\beta} (\lambda'_\beta \cdot z)(\lambda'_\beta \cdot \mu) - \frac{\pi}{2} \sum_{\lambda'_\beta \in \Phi_\beta} (\lambda'_\beta \cdot \mu)^2 \right).$$  \hspace{1cm} (B.13)

Using (B.12), on restriction to $E^\lambda_{T_f}$ this equals

$$e^{-\pi i \sum_{\lambda'_\alpha} (\lambda'_\alpha - \lambda'_\beta) \cdot z(\lambda'_\alpha \cdot \mu)} = e^{-\pi i b \sum_{\lambda'_\alpha} C_{\alpha\beta}(\lambda'_\alpha \cdot \mu)\tau},$$  \hspace{1cm} (B.14)

where the sum is over $\Phi_\alpha$ or $\Phi_\beta$, the answer is the same. The $a$-dependence drops out as the symplectic pairing of weights in $\Phi_\alpha$ (or $\Phi_\beta$) implies $\frac{1}{2} \sum_{\lambda'_\alpha} C_{\alpha\beta}(\lambda'_\alpha \cdot \mu) \in \mathbb{Z}$. 


Using these results in conjunction, we have

\[ \frac{F_\alpha(z_f, \nu_f + \mu_f \tau)}{F_\beta(z_f, \nu_f + \mu_f \tau)} \bigg|_{E^l_{f}} = e^{-2\pi ib(\langle \mu_f, [\Sigma_\lambda] \rangle + \frac{1}{2} \sum_{\lambda'} C_{\alpha \beta}(\lambda' \cdot \mu)) \tau}. \] (B.15)

Then the function

\[ H(z_f) = e^{2\pi ib(\langle z, [\Sigma_\lambda] \rangle + \frac{1}{2} \sum_{\lambda'} C_{\alpha \beta}(\lambda' \cdot z)} \] (B.16)

obeys (B.8), where the $\nu_f$-dependence drops out by using $\langle \nu_f, [\Sigma_\lambda] \rangle \in \mathbb{Z}$ for all $\nu_f$, and the symplectic pairing of the weights $\lambda'$ in $\Phi_\alpha$ or $\Phi_\beta$. We have therefore established the isomorphism of $\mathcal{L}'_\alpha$ and $\mathcal{L}'_\beta$ on $E^l_{f}$. 
Appendix C

Periodic Chiral Multiplets

In this appendix we give the computation of the contribution to the wavefunction of the mirror duality interface, i.e. the mother function in section 4.4.1, from the \( \mathbb{C}^* \)-valued (2, 2) chiral multiplets \( \Phi_\beta \) which appear in interface. Our strategy is to first compute it in the NS-NS sector. In this sector the contribution takes the form of an elliptic genus which coincides with the superconformal index, and therefore it is possible to use an operator counting argument to compute it. We then perform a spectral flow to R-R sector. We will stick to our example of supersymmetric QED, the generalisation to arbitrary abelian theories is straightforward.

In the NS-NS sector, \( q = e^{2\pi ir} \) grades the operator counting by the left-moving Hamiltonian \( H_L \), which by unitary bound arguments is equivalent to a grading by \( J_3 + \frac{1}{4}(U(1)_V + U(1)_A) \). We claim that the contribution from the \( \mathbb{C}^* \) matter is given in NS-NS sector by:

\[
\prod_{\beta=1}^{N} q^{\frac{1}{2}} t \frac{\vartheta(q^{\frac{1}{2}} t^2) \vartheta(C_\beta D_\beta)}{\vartheta(C_\beta) \vartheta(D_\beta)}
\]

where we have identified:

- \( C_\beta \equiv q^{\frac{1}{2}} t (sv_\beta^{-1})^{-\varepsilon_\beta} \) as the charge of \( e^{-\varepsilon_\beta \phi_\beta} \), which creates translation modes of \( \phi_\beta \).
- \( D_\beta \equiv q^{\frac{1}{2}} t^{-1} (\tilde{s}_\beta/\tilde{s}_{\beta-1})^{\varepsilon_\beta} \) as the charge of \( e^{\varepsilon_\beta \tilde{\phi}_\beta} \), the T-dual, which creates winding modes of \( \phi_\beta \).

To prove this, using the identity

\[
-q^{\frac{1}{12}} \frac{\vartheta(C_\beta D_\beta)}{\vartheta(C_\beta) \vartheta(D_\beta)} = \frac{1}{(q; q)_\infty^2} \sum_{n \in \mathbb{Z}} \frac{(C_\beta)^n}{1 - q^n D_\beta},
\]
we rewrite the (C.1) as

$$\prod_{\beta=1}^{N} (q^{\frac{1}{2}}t^{2}; q)_{\infty}(q^{\frac{1}{2}}t^{-2}; q)_{\infty} \sum_{n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}} q^{mn} (C_{\beta})^{n} (D_{\beta})^{m},$$

(C.3)

and justify the contribution from each component in turn.

The description for the duality interface is non-Lagrangian and involves both the $\mathcal{N} = (2, 2)$ chiral $\Phi_{\beta}$ and its T-dual $\tilde{\Phi}_{\beta}$. To perform the operator counting, we choose the duality frame of $T$. One then counts the operators in $Q$-cohomology in $\Phi_{\beta}$, as well as the twist operators $e^{i\varepsilon\tilde{T}_{\beta}}$. Due to the superpotential in (4.55), only positive powers of the twist operator $e^{me^{i\theta_{\beta}}}$ survive [49, 227], i.e. $m \geq 0$.

Note that $\phi_{\beta}$ is the bottom component of a $\mathcal{N} = (2, 2)$ chiral $\Phi_{\beta}$ which transforms as a supermultiplet under vector and axial R-symmetries as:

$$V : e^{-iaFV} \Phi_{\beta}(z, \bar{z}, \theta^{\pm}, \bar{\theta}^{\pm})e^{iaFV} = \Phi_{\beta}(z, \bar{z}, e^{-ia\theta^{\pm}}, e^{ia\bar{\theta}^{\pm}}) + i\varepsilon_{\beta}\alpha,$$

$$A : e^{-iaF_{A}} \Phi_{\beta}(z, \bar{z}, \theta^{\pm}, \bar{\theta}^{\pm})e^{iaF_{A}} = \Phi_{\beta}(z, \bar{z}, e^{+ia\theta^{\pm}}, e^{-ia\bar{\theta}^{\pm}}).$$

(C.4)

The gauge-covariant operators which contribute in $Q$-cohomology are:

<table>
<thead>
<tr>
<th>Operator</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-\varepsilon_{\beta}\phi_{\beta}}$</td>
<td>$C_{\beta}$</td>
</tr>
<tr>
<td>$D_{z}^{n}\phi_{\beta}$, $n \geq 1$</td>
<td>$q^{n}$</td>
</tr>
<tr>
<td>$D_{z}^{n}\tilde{\phi}_{\beta}$, $n \geq 1$</td>
<td>$q^{n}$</td>
</tr>
<tr>
<td>$e^{\varepsilon\phi_{\beta}}$</td>
<td>$D_{\beta}$</td>
</tr>
<tr>
<td>$D_{z}^{n}\psi_{-}^{\phi_{\beta}}$, $n \geq 0$</td>
<td>$q^{\frac{1}{2}+n}t^{-2}$</td>
</tr>
<tr>
<td>$D_{z}^{n}\bar{\psi}<em>{-}^{\phi</em>{\beta}}$, $n \geq 0$</td>
<td>$q^{\frac{1}{2}+n}t^{2}$</td>
</tr>
</tbody>
</table>

Here $D_{z}$ are gauge-covariant holomorphic derivatives on $E_{\tau}$, and $\psi_{-}^{\phi_{\beta}}, \bar{\psi}_{-}^{\phi_{\beta}}$ are fermions in $\Phi_{\beta}$. Notice that the gauge-covariant derivatives of the $\mathbb{C}^{\ast}$-valued scalars are

$$D_{z}\phi_{\beta} = \partial_{z}\phi_{\beta} + iA_{z}$$

(C.5)

where $A_{z}$ acts in the appropriate representation, and are gauge neutral since the gauge transformation acts as a shift $\phi_{\beta} \rightarrow \phi_{\beta} + i\alpha(z, \bar{z})$. Similarly $D_{z}\phi_{\beta}$ is neutral under flavour and $R$-symmetries. We now match the formulae (C.3):

- The fermions $D_{z}^{n}\psi_{-}^{\phi_{\beta}}$ and $D_{z}^{n}\bar{\psi}_{-}^{\phi_{\beta}}$ for $n \geq 0$ contribute $(q^{\frac{1}{2}}t^{2}; q)_{\infty}(q^{\frac{1}{2}}t^{-2}; q)_{\infty}$.
- The covariant derivatives $D_{z}^{n}\phi_{\beta}$ and $D_{z}^{n}\tilde{\phi}_{\beta}$ for $n \geq 1$ contribute $(q; q)_{\infty}^{2}$. 

• We interpret the sum in (C.3) as the contribution of translation modes from $e^{-\varepsilon_{\beta}\phi_{\beta}}$ and winding modes from $e^{\varepsilon_{\beta}\tilde{\phi}_{\beta}}$. Recall the BPS operators are arbitrary powers of $e^{-\varepsilon_{\beta}\phi_{\beta}}$, but only positive powers of $e^{\varepsilon_{\beta}\tilde{\phi}_{\beta}}$. Inserting $m$ powers of the twist operator $e^{\varepsilon_{\beta}\tilde{\phi}_{\beta}}$ means that any operator with charge $n$ under the translation symmetry (which is a gauge symmetry here) acquires a spin $mn$. Thus we have the contribution of composite operators

$$e^{-ns_{\beta}\phi_{\beta}}e^{m\varepsilon_{\beta}\tilde{\phi}_{\beta}} \Rightarrow q^{mn}(C_{\beta})^{n}(D_{\beta})^{m}. \quad (C.6)$$

A nice consistency check for this calculation is that the contribution (C.1) is symmetric under $t \leftrightarrow t^{-1}$, $C_{\beta} \leftrightarrow D_{\beta}$, although we chose to expand it asymmetrically in (C.3). One could alternatively expand as

$$\prod_{\beta=1}^{N} \frac{(q^{1/2}t^{-2};q)_{\infty}(q^{3/2}t^{2};q)_{\infty}}{(q;q)_{\infty}^{2}} \sum_{n,m \in \mathbb{Z}, m \geq 0} q^{mn}(D_{\beta})^{n}(C_{\beta})^{m}. \quad (C.7)$$

This exchanges translation and winding modes, and reflects the fact that physically one can choose to count operators in either the $T$ or $\tilde{T}$ duality frame: in the $\tilde{T}$ frame $\tilde{\phi}_{\beta}$ are translation modes and $\phi_{\beta}$ are winding modes.

In order to obtain the result in R-R sector, and to match onto the mother function, we perform a spectral flow. To do so, the operators must have $2\mathbb{Z}$ quantised left and right $R$-charges (concretely, in their character they must have even powers of $t$). Noting that $q^{1/2}t^{\pm 1}$ are the $U(1)_{V}$ and $U(1)_{A}$ fugacities respectively, we use gauge and flavour symmetries to redefine the $R$-charges so that

$$s \rightarrow sq^{1/2}t, \quad \tilde{s}_{\beta} \rightarrow \tilde{s}_{\beta}(q^{1/2}t^{-1})^{\beta}. \quad (C.8)$$

We can then perform the spectral flow in the genus by substituting $t \rightarrow tq^{-1/4}$, redefining back the $R$-charges\(^1\) by $s \rightarrow st^{-1}$ and $\tilde{s}_{\beta} \rightarrow \tilde{s}_{\beta}t^{\beta}$ and then normalising by a monomial to ensure the result has the correct quasi-periodicities to reflect the contribution to mixed anomalies coming from the dynamical $\theta$-angles in (4.55). The contribution in R-R sector is thus:

$$\prod_{i=1}^{N} \frac{\partial(t^{2})\partial(((sv_{\beta}^{-1})^{-\varepsilon_{\beta}}(\tilde{s}_{\beta}/\tilde{s}_{\beta-1})^{\varepsilon_{\beta}})}{\partial(t(sv_{\beta}^{-1})^{-\varepsilon_{\beta}})\partial((t^{-1}(\tilde{s}_{\beta}/\tilde{s}_{\beta-1})^{\varepsilon_{\beta}})(C.9)}.$$ 

\(^1\)In the R-R sector elliptic genus, $q$ simply grades by $H_{L}$ which is equivalent to grading by $J_{3}$ by unitary bounds.
Appendix D

Hemisphere Partition Functions & Localisation

In this appendix we discuss the formulation of 3d $\mathcal{N} = 2$ theories on $S^1 \times HS^2$, where $HS^2$ is a hemisphere with a $U(1)$ isometry, and the computation of their partition functions. We impose 2d $\mathcal{N} = (0, 2)$ boundary conditions on $S^1 \times \partial(HS^2) \simeq S^1 \times S^1 = T^2$. We show this coincides with the half superconformal index up to the Casimir energy, which is precisely the equivariant integral of the boundary ’t Hooft anomaly. The 3d $\mathcal{N} = 4$ cases of interest with $\mathcal{N} = (2, 2)$ boundary conditions can then be obtained as a specialisation.

The case where the $\mathcal{N} = 2$ vector multiplet is assigned a Neumann boundary condition on $T^2$ was analysed in [189]. We propose an extension to cover the Dirichlet boundary condition for the vector multiplet, and find the partition function is expressed as a sum over fluxes corresponding to boundary monopole configurations for the vector multiplet. We stick mostly to abelian theories for simplicity, and will return to give a fuller picture of the non-abelian case and localisation in future work.

D.1 Supersymmetry and the Index

Rigid supersymmetry on $S^1 \times S^2$ was considered in [204, 205] for the purposes of computing the superconformal index via Coulomb branch localisation. The computation of the superconformal index via Higgs branch localisation was performed in [106, 107]. As the metric on $HS^2$ is identical to the one on $S^2$, the same conformal Killing spinors can be used. In this appendix, we follow the conventions of [180, 189], as in section
4.5. The metric on the $S^1 \times HS^2$ with radius $r$ is:

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + d\tau^2,$$

where $0 \leq \theta \leq \pi/2$, $\phi \sim \phi + 2\pi$, $\tau \sim \tau + \beta r$. We also use subscripts $\mu \in \{1, 2, 3\}$ for coordinates $\{\theta, \phi, \tau\}$, and $\{\hat{1}, \hat{2}, \hat{3}\}$ for components in a frame specified by the dreibein

$$e^{\hat{1}} = r d\theta, \quad e^{\hat{2}} = r \sin \theta d\phi, \quad e^{\hat{3}} = d\tau.$$

The Killing spinor equations are

$$\nabla_\mu \epsilon = \frac{1}{2r} \gamma_\mu \gamma_3 \epsilon, \quad \nabla_\mu \bar{\epsilon} = -\frac{1}{2r} \gamma_\mu \gamma_3 \bar{\epsilon},$$

and we choose solutions

$$\epsilon_\alpha = e^{\tau/2r} \epsilon^{i\varphi/2} \begin{pmatrix} \cos \theta/2 \\ \sin \theta/2 \end{pmatrix}, \quad \bar{\epsilon}_\alpha = e^{-\tau/2r} e^{-i\varphi/2} \begin{pmatrix} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix}.$$  

The supersymmetry transformations of $N = 2$ vector and chiral multiplets are given in section 2 of [189]. The transformations generated by $\epsilon, \bar{\epsilon}$ on the boundary $T^2$ generate a $N = (0, 2)$ supersymmetry under which boundary conditions for the bulk multiplets must be compatible. We define these in the next subsection. The spinors are not periodic around $S^1$, and thus twisted boundary conditions must be imposed. This is precisely compatible with the hemisphere partition function, which is a path integral over fields on $S^1 \times HS^2$ with the twisted periodicities dictated by the fugacities:

$$\Phi(\tau + \beta r) = e^{-\beta_1 R} e^{-(\beta_1 - \beta_2) J_3} x^{-J_H} \xi^{-J_C} \Phi(\tau).$$

Note the Killing spinors (D.4) obey these conditions, e.g. $\bar{\epsilon}$ has R-charge $+1$ and $J_3 = -1/2$. From standard arguments, the path integral gives a trace over states on $HS^2$:

$$Z_{S^1 \times HS^2} = \text{Tr}_{H(HS^2)} \left[ (-)^F e^{-\beta_1 (D-R-J_3)} e^{-\beta_2 (D+J_1)} x^{-J_H} \xi^{-J_C} \right] = \text{Tr}_{H(HS^2)} \left[ (-)^F q^{J_3 + R/2} x^{J_H} \xi^{J_C} \right].$$

which is independent of $\beta_1$. Here $\beta = \beta_1 + \beta_2$, $F = 2J_3$ is the fermion number, $q = e^{-2\beta_2}$ and $J_{H,C}$ the generators of matter and topological flavour symmetry with fugacities $x, \xi$ respectively. The 1-loop determinants can be computed using these periodicities. Alternatively, noting that (D.5) contains a gauge transformation for flavour and R-symmetries, one can equivalently turn on background flat connections for
these symmetries [106]. In either case the twisted periodicity condition corresponding to
the angular momentum $J_3$ is implemented by the coordinate identification (eliminating
$\beta_1$)
\[
(\tau, \varphi) \sim (\tau + \beta r, \varphi - i(\beta - 2\beta_2)).
\] (D.7)
To evaluate the classical action properly one needs to take this identification into
account. Redefining
\[
\tilde{\tau} = \tau, \quad \tilde{\varphi} = \varphi + \frac{i(\beta - 2\beta_2)}{\beta r} \tau \quad \Rightarrow \quad (\tilde{\tau}, \tilde{\varphi}) \sim (\tilde{\tau} + \beta r, \tilde{\varphi}),
\] (D.8)
the classical actions can be evaluated by integrating separately over $\tilde{\tau}, \tilde{\varphi}$. To recover
the Casimir energy corresponding to the boundary 't Hooft anomaly, necessary for
exact holomorphic factorisation, we will see we should set $\beta_2 = \beta$ and the fugacity
$q = e^{-2\beta}$ and will do so from here on out.

D.2 Boundary Conditions

We specify a set of $\mathcal{N} = (0, 2)$ boundary conditions on $T^2$ for 3d $\mathcal{N} = 2$ multiplets.
These differ from [189] in that they involve a Dirichlet boundary condition for the
vector multiplet. We restrict to an abelian gauge group $G = U(1)^k$ with Lie algebra $\mathfrak{g}$,
for simplicity. We define the complexified covariant derivative $D = D + \sigma = \nabla + iA + \sigma$
where $A$ and $\sigma$ act in the appropriate representation.

- For the $\mathcal{N} = 2$ vector multiplet $(A_\mu, \sigma, \lambda, \bar{\lambda}, D)$, the Dirichlet boundary condition
  at $\theta = \pi / 2$ is:

\[
\begin{align*}
A_{2,3} &= a_{2,3}, & \lambda_+ + \lambda_- &= 0, \\
\partial_1 (iF_{2\bar{3}} + F_{1\bar{3}}) &= 0, &  \bar{\lambda}_+ + \bar{\lambda}_- &= 0, \\
D &= 0, & \partial_1 (\lambda_+ - \lambda_-) &= 0, \\
D_1 \sigma &= 0, & \partial_1 (\bar{\lambda}_+ - \bar{\lambda}_-) &= 0.
\end{align*}
\] (D.9)

Here $a_{2,3}$ is a constant flat connection on the boundary torus $T^2$. This breaks
the gauge symmetry to a global symmetry $G_\partial$ at the boundary.
• For $\mathcal{N} = 2$ chiral multiplets $(\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F})$ of R-charge $\Delta$, the Neumann boundary condition is:

$$\mathcal{D}_1 \phi = 0, \quad F = 0,$$
$$\mathcal{D}_1 \bar{\phi} = 0, \quad \bar{F} = 0,$$
$$\psi_+ + \psi_- = 0, \quad \mathcal{D}_1 (\psi_+ - \psi_-) + (\lambda_+ - \lambda_-) \cdot \phi = 0,$$
$$\bar{\psi}_+ + \bar{\psi}_- = 0, \quad \mathcal{D}_1 (\bar{\psi}_+ - \bar{\psi}_-) - \bar{\phi} \cdot (\bar{\lambda}_+ - \bar{\lambda}_-) = 0. \quad (D.10)$$

The basic Dirichlet boundary condition is:

$$\phi = 0, \quad D_1 (ie^{i\tau} e^{-i\varphi} D_1 \phi + F) + \frac{1}{2} (\lambda_+ - \lambda_-) \cdot (\psi_+ + \psi_-) = 0,$$
$$\bar{\phi} = 0, \quad D_1 (ie^{-i\tau} e^{i\varphi} D_1 \bar{\phi} + \bar{F}) + \frac{1}{2} (\bar{\psi}_+ + \bar{\psi}_-) \cdot (\bar{\lambda}_+ - \bar{\lambda}_-) = 0,$$
$$\psi_+ - \psi_- = 0, \quad D_1 (\psi_+ + \psi_-) = 0,$$
$$\bar{\psi}_+ - \bar{\psi}_- = 0, \quad D_1 (\bar{\psi}_+ + \bar{\psi}_-) = 0. \quad (D.11)$$

These are related to 3d lifts of the boundary conditions in [73–75]. For purposes of application to 3d $\mathcal{N} = 4$ theories with boundary conditions associated to vacua, we would like to turn on non-zero values at the boundary for the scalars $\phi$ in chiral multiplets which acquire non-zero VEVs in the vacuum, analogously to the operator picture on $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. Thus we would like to deform the basic Dirichlet boundary condition for such scalars to

$$(D_c) : \quad \phi = c, \quad \bar{\phi} = \bar{c}, \quad c \neq 0, \quad (D.12)$$

keeping the same boundary conditions for the remaining fields in the $\mathcal{N} = 2$ chiral. To preserve supersymmetry in the right column of $(D.11)$ we demand

$$\rho(A_2) = 0, \quad \rho(A_3) - \frac{i\Delta}{r} = 0, \quad (D.13)$$

at $\theta = \pi/2$, where $\rho$ is the gauge group representation of the chiral. If we choose to realise the twisted boundary conditions for flavour symmetries around $S^1$ as holonomies for background vector multiplets, then the condition becomes:

$$\rho(A_3) + \sum_l \rho_l (A_3^l) - \frac{i\Delta}{r} = 0, \quad (D.14)$$
where \( \rho_l \) is the flavour representation. In computing the path integral, a hermiticity condition is imposed on the gauge fields. Thus a constant boundary value for \( \phi \) can be turned on only for chirals of zero R-charge. To turn on a non-zero VEV, the background gauge fields must obey the constraint in (D.14), and thus the boundary condition breaks the combination of flavour symmetries (including \( G_\partial \)) dual to the charges of the chiral. The result for such chirals with arbitrary R-charges can be obtained by analytically continuing the final partition function by complexifying flavour fugacities \[106\].

D.3 Localisation

In this section we describe the localisation computation of the hemisphere partition function for general 3d \( \mathcal{N} = 2 \) theories with Dirichlet boundary conditions for the vector multiplet.

**BPS Locus**  We may use the same localising actions as in \[189, 205\] (the SYM and matter actions which are Q-exact), but restrict the saddle points to the ones compatible with the boundary condition. These saddle points coincide with the BPS locus.

- For the \( \mathcal{N} = 2 \) vector multiplets, the key feature of Dirichlet boundary conditions is that they are compatible with slicing in half a Dirac monopole on \( S^2 \). The saddle points are

\[
A = a_3 d\tau + 2m B_\alpha dx^\alpha, \quad \sigma = m/r,
\]

where \( \alpha = 1, 2, m \in \text{Hom}(U(1), G) \cong \mathbb{Z}^k \) and \( a_3 \in \mathfrak{g} \) is a constant. Note the constant value of \( A_\tau \) is fixed to its boundary value (D.9). Here \( B_i \) is the monopole of unit flux on \( S^2 \)

\[
B = \frac{1}{2} \omega,
\]

where \( \omega \) is the spin connection on \( S^2 \). The factor of two difference between (D.15) and (22) of \[205\] comes from the fact that for a \( U(1) \) monopole on the hemisphere to have a well defined flux \( \frac{1}{2\pi} \int_{H S^2} F = m \in \mathbb{Z} \), it must have the functional form of a monopole of twice the magnetic charge on \( S^2 \). Explicitly we could write

\[
A = a_3 d\tau + m(\kappa - \cos \theta) d\phi, \quad \kappa = \begin{cases} 1 & \text{for } \theta \in [0, \pi/2 - \epsilon), \\ 0 & \text{for } \theta \in (\pi/2 - 2\epsilon, \pi/2]. \end{cases}
\]
This is trivialised at the boundary and thus compatible with (D.9). Thus in the path integral we sum over monopole sectors $\mathbf{m}$, mirroring the half index computation [117].

• For an $\mathcal{N} = 2$ chiral multiplet with $\Delta \neq 0$ the BPS locus sets all components of a chiral multiplet to 0. For $\Delta = 0$, the scalar is set everywhere to the constant value it takes at the boundary (D.12), see [106] for details.

As usual in localisation, we set:

$$\Phi = \Phi^{(0)} + \frac{\Phi'}{\sqrt{\delta}}, \quad (D.18)$$

where $\Phi^{(0)}$ are BPS configurations and $\Phi'$ fluctuations around the locus. Then:

$$Z_{S^1 \times HS^2} = \lim_{\delta \to \infty} \int D\Phi e^{-S[\Phi] - \delta Q \cdot V[\Phi]} = \sum_{\mathbf{m} \in \mathbb{Z}^k} e^{-S_{\text{cl}}[\Phi^{(0)}]} Z_{1\text{-loop}}(q, s, x, \xi, \mathbf{m}), \quad (D.19)$$

where $\Phi$ denotes the set of all fields, and $Q \cdot V[\Phi]$ are the localising actions given in [189, 205], and are just the 3d $\mathcal{N} = 2$ Yang-Mills and matter actions. The path integral is over all configurations obeying boundary conditions in section D.1, and twisted periodicities defined by the trace (D.5). Here $s = e^{-i\beta \omega_3}$ is the fugacity for the gauge symmetry which is broken to a flavour symmetry by the boundary condition. $S_{\text{cl}}$ is the action evaluated on the BPS locus. We now describe each ingredient in turn.

**Classical Contribution** To implement a grading by the topological symmetry, we turn on a BPS configuration for a background vector multiplet

$$A^{(T)} = \eta d\tau, \quad \sigma^{(T)} = D^{(T)} = \lambda^{(T)} = \bar{\lambda}^{(T)} = 0, \quad (D.20)$$

in the mixed bulk-boundary Chern-Simons term for an abelian gauge group (for a non-abelian gauge group, the topological symmetry just couples to the centre)

$$S_{\text{mCS}} = \frac{i}{4\pi} \int_{S^1 \times HS^2} d^3x \left[ e^{\mu \nu \rho} \left( \partial_\mu A_\nu A^{(T)}_\rho + \partial_\mu A^{(T)}_\nu A_\rho \right) \right. \left. + \sqrt{g}(-\bar{\lambda}^{(T)} \lambda - \bar{\bar{\lambda}}^{(T)} \lambda + 2\sigma^{(T)} D + 2\sigma D^{(T)} \right) \right] \quad (D.21)$$

$$- \frac{1}{4\pi} \int_{T^2} d^2x \sqrt{g^{(2)}} \left[ A_2 A_2^{(T)} + A_3 A_3^{(T)} - 2\sigma \sigma^{(T)} \right].$$
The boundary terms involving $A$ and $\sigma$ are required for invariance under infinitesimal
gauge/flavour transformations, and supersymmetry respectively. The evaluation of the
term in the first line has the usual subtlety. Using coordinates $\tilde{\tau}, \tilde{\varphi}$ in (D.8), we write
e.g.

$$A = a_3 d\tilde{\tau} - m \cos \theta \left( d\tilde{\varphi} + \frac{i}{r} d\tilde{\tau} \right),$$

(D.22)

and extend to connections on a 4-manifold $D^2 \times HS^2$ where the $S^1$ factor is the
boundary of a flat disk $D^2$ with $\rho \in [0, 1]$:

$$\hat{A} = a_3 \rho^2 d\tilde{\tau} - m \cos \theta \left( d\tilde{\varphi} + \frac{i}{\rho^2} \eta d\tilde{\tau} \right),
\hat{A}^{(T)} = \eta \rho^2 d\tilde{\tau}.$$  

(D.23)

The action is defined to be the evaluation on the extension over $D^2 \times HS^2$

$$\frac{i}{4\pi} \int_{S^1 \times HS^2} A^{(T)} \wedge F + A \wedge F^{(T)} \equiv \frac{i}{2\pi} \int_{D^2 \times HS^2} \hat{F}^{(T)} \wedge \hat{F} = m(i\beta r \eta).$$

(D.24)

Including the boundary contribution from (D.21):

$$e^{S_{mCS}}|_{BPS} = e^{-\frac{\log(z) \log(z^m)}{\log q}},$$

(D.25)

where we defined

$$s = e^{-i\beta r a_3}, \quad \xi = e^{-i\beta r \eta},$$

(D.26)

as the fugacities for the $G_\theta$ and topological symmetries which we use throughout. We
have set $\beta_2 = \beta$, so that when this is combined with the anomalous contributions of
the vector and chiral multiplets the prefactor reproduces the anomaly polynomial, as
we shall see in section D.5. The contribution of a (diagonal) Chern-Simons term at
level $k$ can be obtained by dropping the $(T)$ superscript and multiplying by $k/2$.

1-loop Determinants. Here we give the 1-loop determinants, with the proof for the
chiral multiplet with Dirichlet boundary conditions in the next subsection. The results
are stated for a general gauge group.

- **The $\mathcal{N} = 2$ chiral in Neumann ($N$).** For an $\mathcal{N} = 2$ chiral in representation $\rho$
of the gauge group, $\rho_f$ of the flavour group and $R$-charge $\Delta$

$$Z_{1\text{-loop}}^{(N)} = e^{\xi \left[ -\log \left( q^{\frac{\Delta}{2}} + \rho(m) s^{\rho, \rho_f} \right) \right] \left( q^{\frac{\Delta}{2}} + \rho(m) s^{\rho, \rho_f}; q \right)_\infty^{-1}}$$  

(D.27)
where the function
\[ E[x] = \frac{\beta_2}{12} - \frac{x}{4} + \frac{x^2}{8\beta_2} \]  
arises from a zeta regularisation as in [189]. The factor of two difference in the way the monopole charge enters compared to the \( S^1 \times S^2 \) index is due to monopoles on the \( HS^2 \) having the same functional form as monopoles on \( S^2 \) with twice the flux.

- **The \( \mathcal{N} = 2 \) chiral in Dirichlet \((D)\).** Similarly to above we obtain
  \[ Z_{1\text{-loop}}^{(D)} = e^{-E\left[-\log\left(q^{1-\frac{\Delta}{2}-\rho(m)s-\rho x-\rho f}\right)\right]} \left(q^{1-\frac{\Delta}{2}-\rho(m)s-\rho x-\rho f}; q\right)_\infty. \] (D.29)

- **The \( \mathcal{N} = 2 \) vector multiplet in Dirichlet.** Note this is also the contribution of a Neumann chiral in the adjoint, with charge R-charge 2.
  \[ Z_{1\text{-loop}}^{\text{vector}} = e^{E\left[-\log(q)\right]} \left(q^{1+\alpha(m)s}; q\right)_\infty^{-1} \left[q^{1+\alpha(m)s}; q\right]_{\infty}^{-1}. \] (D.30)

To compute the partition function with some chirals with a deformed Dirichlet boundary condition, the procedure can be described as computing with Dirichlet boundary conditions and then setting to 1 the product of fugacities dual to the charges of the chiral, as in (D.14). This is analogous to the half-index computation for these boundary conditions [117].

### D.4 Details: Chiral Multiplet with Dirichlet B.C.

In this section we derive the 1-loop determinant of the chiral multiplet with a basic Dirichlet boundary condition about the saddle points (D.15). Contrary to [189], we do not expand in terms of monopole spherical harmonics as they do not form a complete eigenbasis on \( HS^2 \) for the differential operators in the Gaussian integrals in the presence of a monopole - we do not require regularity at the south pole. Instead the determinant is derived by matching bosonic and fermionic eigenmodes, similarly to the 2d result in [228]. We abuse notation and also denote the fluctuating parts of the scalar and fermion as \((\phi, \psi)\). The differential operators appearing at quadratic order are, after
substituting the BPS locus (D.15):

\[
D_{\text{scalar}} = \tilde{D}_{\text{scalar}} + \left[-D^3 D_3 + \frac{m^2}{r^2} + \frac{1 - 2\Delta}{r} D_3 + \frac{\Delta(1 - \Delta)}{r^2}\right],
\]

\[
D_{\text{fermion}} = \tilde{D}_{\text{fermion}} + \left[D_3 - \frac{1 - 2\Delta}{2r}\right].
\]  

We have multiplied the fermionic operator appearing in the action by \(\gamma_3\) due to the spinor product \(\epsilon \cdot \psi = \epsilon_- \psi_+ - \epsilon_+ \psi_-\), and defined

\[
\tilde{D}_{\text{scalar}} = -D^i D_i, \quad \tilde{D}_{\text{fermion}} = \gamma^3 \gamma^i D_i - \frac{m}{r} \gamma^3,
\]  

for \(i, j = 1, 2\), and \(m\) acting implicitly in the appropriate representation. All covariant derivatives are with respect to the background (D.15), for example on spinors:

\[
D_\mu = \partial_\mu + \frac{1}{2} iw_\mu \sigma_3 + i m w_\mu,
\]

where \(w_\mu = (0, -\cos \theta, 0)\) is the spin connection. The 1-loop determinant will be given by

\[
Z_{1\text{-loop}}^{(D)} = \frac{\det D_{\text{fermion}}}{\det D_{\text{scalar}}}
\]

after a suitable regularisation. The boundary condition \(\phi|_{\theta = \frac{\pi}{2}} = \psi_+ - \psi_-|_{\theta = \frac{\pi}{2}} = 0\) is imposed on the fluctuating modes. As expected, there are large cancellations between bosonic and fermionic eigenmodes.

We work in the setting where the twisted periodicities in (D.5) due to the flavour symmetries are cancelled by turning on holonomies for their background vector multiplets but retain twisted periodicities due to the R-symmetry and angular momentum. We therefore have \(D_3 = \nabla_3 + i a_3 + i \sum_l a_l^3\), where \(a_l^3\) are flat connection(s) for flavour symmetry. This operator commutes with \(\tilde{D}_{\text{scalar}}, \tilde{D}_{\text{fermion}}\) in (D.31), and so we diagonalise them simultaneously. For a field of R-charge \(R\), we expand in terms of fields:

\[
O_{n,m}(\theta, \varphi, \tau) = e^{\tau \left(2\pi i n - (R + m) \beta_1 + m \beta_2\right)} O_m(\theta, \varphi),
\]

where

\[
J_3 O_m = (-i \partial_\varphi + \kappa m) O_m = m O_m.
\]  

Then \(D_3\) acts as:

\[
\beta r D_3 O_{n,m} = \left[2\pi i n - (R + m) \beta_1 + m \beta_2 + i \beta r \rho(a_3) + i \beta r \rho_l(a_l^3)\right] O_{n,m}.
\]
**Paired Eigenmodes**  We now exhibit the pairing of fermionic and bosonic eigenmodes. If $\psi$ is a fermionic eigenmode obeying the boundary condition $\psi_+ - \psi_-|_{\theta = \pi/2} = 0$ and satisfying

$$\hat{D}_{\text{fermion}} \psi = \left( \gamma^3 \gamma^i D_i - \frac{m}{r} \gamma^3 \right) \psi = \nu \psi,$$  \hspace{1cm} (D.38)

then we can construct

$$\phi' = \bar{\epsilon} \psi$$  \hspace{1cm} (D.39)

which obeys

$$\hat{D}_{\text{scalar}} \phi' = -g^{ij}_{(2)} D_i D_j \phi' = \left( \nu(\nu + 1) - \frac{m^2}{r^2} \right) \phi'.$$  \hspace{1cm} (D.40)

Similarly, for a scalar eigenmode $\phi$ such that

$$\hat{D}_{\text{scalar}} \phi = M^2 \phi,$$  \hspace{1cm} (D.41)

one can construct two spinor eigenmodes

$$\psi^{(1,2)} = \gamma^i \epsilon D_i \phi + \frac{m}{r} \epsilon \phi - \nu \gamma^3 \epsilon \phi,$$  \hspace{1cm} (D.42)

where $\nu$ is a solution to $\nu(\nu + 1) - \frac{m^2}{r^2} = M^2$; i.e. if $\nu$ is a solution so is $-\nu - 1$. It is easy to check that eigenvalues of $\phi$ and the pair $\psi^{(1,2)}$ cancel in the determinant, noting that the Killings spinors (D.4) satisfy

$$\beta r D_3 \epsilon = \frac{\beta}{2} \epsilon, \hspace{1cm} \beta r D_3 \bar{\epsilon} = -\frac{\beta}{2} \bar{\epsilon}.$$  \hspace{1cm} (D.43)

Also $\psi^{(1,2)}$ and $\phi'$ obey the appropriate boundary condition.

**Unpaired Eigenmodes**  The non-cancelling contributions to the 1-loop determinant are the ones which do not participate in the pairing above, that is when (D.39) or (D.42) are undefined. An unpaired scalar eigenmode is a $\phi$ such that:

$$\gamma^i \epsilon D_i \phi + \frac{m}{r} \epsilon \phi - \nu \gamma^3 \epsilon \phi = 0.$$  \hspace{1cm} (D.44)

Contracting with $\bar{\epsilon}$ gives

$$\partial_\phi \phi + i(\kappa m - r \nu) \phi = 0.$$  \hspace{1cm} (D.45)
Using the ansatz $\phi = f(\theta)e^{-i(\kappa m - r\nu)\varphi}$ (suppressing $\tau$ dependence for now) and contract-ing (D.44) with $\bar{\epsilon}\gamma_3$ we obtain:

$$\sin \theta \partial_\theta f + mf - r\nu \cos \theta f = 0.$$  \hfill (D.46)

There are no non-trivial solutions obeying the boundary condition and thus no unpaired scalar eigenmodes.\(^{1}\)

We now look for unpaired spinor eigenmodes $\psi$. If $\bar{\epsilon}\psi = 0$ then we may write $\psi = \bar{\epsilon}\Phi$ where $\Phi$ is a scalar of R-charge $\Delta - 2$ (so that $\psi$ has R-charge $\Delta - 1$). Using the Killing spinor equations

$$\left(\gamma^3\gamma^i D_i - \frac{m}{r}\gamma^3\right) (\bar{\epsilon}\Phi) = \nu \psi,$$

$$\Rightarrow \gamma^3\gamma^i D_i \Phi = \left(\nu + \frac{1}{r}\right) (\bar{\epsilon}\Phi) + \frac{m}{r}\gamma^3\bar{\epsilon}\Phi.$$  \hfill (D.47)

Contracting with $\epsilon$ and $\epsilon\gamma^3$ gives

$$\sin \theta \partial_\theta \Phi = -(r\nu + 1) \cos \theta + m\Phi,$$

$$\left(\partial_\varphi + im(\kappa - \cos \theta)\right) \Phi = i(r\nu + 1)\Phi - im \cos \theta.$$  \hfill (D.48)

Using the ansatz $\Phi = f(\theta)e^{i(r\nu + 1 - \kappa m)\varphi}$, we find solutions:

$$\Phi = \sin(\theta/2)^m \cos(\theta/2)^{-m}(\sin \theta)^j e^{i(-j-\kappa m)\varphi} e^{\frac{r}{2\pi}(2\pi n - (\Delta - 2 - j)|\beta_1 - j\beta_2|)}.$$  \hfill (D.49)

where $j = -r\nu - 1$ is an integer such that $j + m \geq 0$. The last requirement is for regularity at $\theta = 0$. This is less restrictive than also requiring regularity at $\theta = \pi$ as for the $S^1 \times S^2$ index, which would require $j \geq |m|$. The $\tau$-dependent exponential ensures the twisted periodicity condition (D.5). The unpaired fermionic eigenmodes are thus

$$\psi = e^{\frac{r}{2\pi}(2\pi n - (\Delta - 1 - (j + 1/2))\beta_1 - (j + 1/2)\beta_2)} e^{i(-1 - j - \kappa m)\varphi}$$

$$\times \sin(\theta/2)^m \cos(\theta/2)^{-m}(\sin \theta)^j \left(\frac{\sin(\theta/2)}{\cos(\theta/2)}\right)^i.$$  \hfill (D.50)

\(^{1}\)Note that this means that if we have an eigenmode $\psi$ with eigenvalue $\nu$ which is paired, we may always construct the eigenmode $\tilde{\psi}$ with eigenmode $-\nu - 1$ by using first the map (D.39) to construct $\phi'$, and then (D.42) to construct $\psi_+$ proportional to $\psi$, and $\psi_- := \tilde{\psi}$.
The 1-loop Determinant  We now have all the ingredients needed to write down the
1-loop determinant for the $N = 2$ chiral multiplet with Dirichlet boundary condition.

$$ Z^{(D)}_{1\text{loop}} = \prod_{n \in \mathbb{Z}} \prod_{j \geq -\rho(m)} \left[ \beta r \left( \frac{1 - 2\Delta}{2r} + \frac{j + 1}{r} \right) - \beta r D_3 \right] $$

$$ = \prod_{n \in \mathbb{Z}} \prod_{j \geq 0} \left[ -2\pi in - i\beta r \rho(a_3) + (2j + 2 - \Delta - 2\rho(m))\beta_2 - i\beta r \rho_l(a'_3) \right] $$

$$ = e^{-[i\beta r \rho(a_3) - i\beta r \rho_l(a'_3) + (2 - \Delta - 2\rho(m))\beta_2]} \left( e^{i\beta r \rho(a_3) + i\beta r \rho_l(a'_3)} q^{1 - \Delta/2 - \rho(m)} ; q \right)_\infty. $$

The final line has been zeta function regularised as in [189].

D.5 Regularisation and Anomaly Polynomials

We now show that the results in section D.3 for the $S^1 \times HS^2$ partition function
reproduces the formula (3.31) in [117] for the half-index $I$ counting local operators
inserted at the origin of $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$, up to a prefactor encoding boundary ’t Hooft
anomalies:

$$ Z = e^\phi I. $$

We also stick to an abelian gauge group, the non-abelian generalisation
can be found by ensuring consistency with the maximal torus of the gauge group. The
$q$-Pochhammer contributions clearly match, so we need only consider the classical
contribution and the $E$ functions. Examining each in turn:

- The coupling to the topological symmetry gives

$$ e^{-\frac{2 \log \xi \log sq^m}{2 \log q}}. $$

Rewriting the term in the exponential as:

$$ \frac{1}{\log q} \begin{pmatrix} \log sq^m & \log \xi \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \log sq^m \\ \log \xi \end{pmatrix} $$

This is the same bilinear form encoding the mixed boundary ’t Hooft anomaly
between the boundary gauge symmetry and the topological symmetry in the
anomaly polynomial contribution

\[-f f_\xi, \quad (D.55)\]

where $f, f_\xi$ are field strengths for the corresponding symmetries. Isolating the $m$ dependence in (D.54) recovers $\xi^{-m}$ which appears in the half index formulae of [117]. The $m$ independent part contributes to $\phi$ with $-\log \xi \log s/\log q$.

- For a chiral with $(N)$ boundary conditions, transforming with charge $\rho$ under an abelian gauge group and $R$-charge $\Delta$, the anomalous contribution is

\[
e^{-\mathcal{E} \left[ -\log \left( q^{\frac{3}{4}+\rho(m)s}\right) \right]} = Ce^{-\frac{1}{\log q} \left[ -\frac{1}{4} \left( \rho \log sq^m + (\Delta - 1) \log q^{\frac{1}{2}} \right)^2 \right]}, \quad (D.56)
\]

where $C := e^{-\frac{\mathcal{E}}{\log q}}$. Up to this constant, this matches the bilinear form encoding the contribution of the chiral to the boundary 't Hooft anomaly polynomial

\[
-\frac{1}{4} (\rho f + (\Delta - 1)r)^2 \quad (D.57)
\]

after replacing $\log s \to f$ and $\log q^{\frac{1}{2}} \to r$. Here $r$ is the field strength of the $R$-symmetry. Again the $m$ dependence matches the half index formula, and we obtain an overall contribution to the prefactor $\phi$ of:

\[
\frac{1}{\log q} \left[ -\frac{1}{4} \left( \rho \log s + (\Delta - 1) \log q^{\frac{1}{2}} \right)^2 \right]. \quad (D.58)
\]

- Similarly for a chiral with $(D)$ boundary conditions, we have:

\[
e^{-\mathcal{E} \left[ -\log \left( q^{\frac{3}{4}+\rho(m)s}\right) \right]} = C^{-1}e^{-\frac{1}{\log q} \left[ \frac{1}{4} \left( \rho \log sq^{m} + (\Delta - 1) \log q^{\frac{1}{2}} \right)^2 \right]}, \quad (D.59)
\]

which matches the contribution to the boundary 't Hooft anomaly polynomial

\[
\frac{1}{4} (\rho f + (\Delta - 1)r)^2. \quad (D.60)
\]

- A U(1) $\mathcal{N} = 2$ vector multiplet contributes

\[
e^{\mathcal{E} \left[ -\log q \right]} = Ce^{-\frac{1}{\log q} \left[ -\frac{1}{4} \left( \log q^{\frac{1}{2}} \right)^2 \right]} \quad (D.61)
\]
matching the corresponding boundary ’t Hooft anomaly polynomial contribution
\[ -\frac{1}{4} r^2. \]  
(D.62)

In summary, up to factors of $C$, we are left with a prefactor $\phi$ given precisely by:
\[ \phi = \frac{1}{\log q} P(\log q^2, \log s, \log \xi) \]  
(D.63)

where $P(r, f, f_\xi)$ is the anomaly polynomial encoding the boundary ’t Hooft anomaly, consisting of contributions (D.55), (D.57), (D.57) and (D.62). For a non-abelian theory, it is the equivariant integral of the polynomial [229].

In an $\mathcal{N} = 4$ theory, with $(2, 2)$ boundary condition, the factors $C$ always cancel, and the cancellations of the $E$ reflect that only the mixed anomalies listed in sections 2.5 and 3.5 can occur, thus proving equation (5.18).
Appendix E

Boundary Anomalies and Vermas

With the result of section D.5 in hand, we prove the claims in section 5.4 for a general 3d $\mathcal{N} = 4$ abelian theory. That is, we show that if $\mathcal{P}_{B_\alpha}(r, t, f_H, f_C)$ is the boundary ’t Hooft anomaly polynomial for a boundary condition $B_\alpha$, the lowest weights of the corresponding Higgs and Coulomb branch Verma modules are given by:

\[
\lim_{t^{1/2} \to q^{-1/4}} \phi_{B_\alpha}^{(B)} = \lim_{t^{1/2} \to q^{-1/4}} \frac{1}{\log q} \mathcal{P}_{B_\alpha}(\log q^{1/2}, \log t, \log x, \log \xi) \\
\lim_{t^{1/2} \to q^{1/4}} \phi_{B_\alpha}^{(A)} = \lim_{t^{1/2} \to q^{1/4}} \frac{1}{\log q} \mathcal{P}_{B_\alpha}(\log q^{1/4}, \log t, \log x, \log \xi). \tag{E.1}
\]

Further, the mixed anomaly coefficient $k$ between $T_H$ and $T_C$ is equal to the central charge $\kappa_\alpha$ where $\alpha$ is the vacuum for the abelian theory associated to boundary condition $B_\alpha$. In the language of chapters 4, $B_\alpha$ is an exceptional Dirichlet boundary condition $D^*_\alpha$ for a given pair of chambers $\mathfrak{c}_H$ and $\mathfrak{c}_C$ of mass and FI parameters.

We reviewed the construction of exceptional Dirichlet boundary conditions for abelian 3d $\mathcal{N} = 4$ theories in terms of weights space decompositions in section 4.1. In this appendix we will work with charge matrix description (see section 2.4.2) as used in the original construction of exceptional Dirichlet boundary conditions in [56].

Consider a gauge group $G = U(1)^r$, with $N$ hypermultiplets $(X_i, Y_i)$. The Higgs and Coulomb branch flavour symmetries are

\[
G_H = U(1)^{N-r} := U(1)^{r'}, \quad G_C = U(1)^r. \tag{E.2}
\]

We denote by:

\[
\mathbf{Q} = \{Q^i_a\}_{1 \leq a \leq r}, \quad \mathbf{q} = \{q^i_{\beta}\}_{1 \leq \beta \leq r'} \tag{E.3}
\]
the matrices of gauge and flavour charges respectively. An exceptional Dirichlet boundary condition is labelled by a subset $S \subset (1, \ldots, N)$ (determined by the vacuum $\alpha$) such that the charge submatrix $Q^{(S)}$ is non-degenerate and a polarisation (sign vector) $\varepsilon$ so that the boundary condition sets

$$B : \begin{cases} 
Y_i | = c_i & \varepsilon_i = + \\
X_i | = c_i & \varepsilon_i = - 
\end{cases} \quad (i \in S), \quad \begin{cases} 
Y_j | = 0 & \varepsilon_j = + \\
X_j | = 0 & \varepsilon_j = - 
\end{cases} \quad (j \notin S) \quad (E.4)$$

where the $c_i$ are non-zero. The scalars fixed to non-zero values at the boundary are those with non-zero values on the vacuum $\alpha$. This boundary condition fully breaks the gauge symmetry and preserves the flavour symmetry at the boundary. The sign vector $\varepsilon$ is determined by the chamber $\mathcal{C}_H$ as described in section 4.1.

Let us define the submatrices and subvectors for later use:

$$Q^S = \{Q_a^i\}_{1 \leq a \leq r}^{i \in S}, \quad q^S = \{q_\beta^i\}_{1 \leq \beta \leq N-r}^{i \in S}, \quad \varepsilon^S = \{\varepsilon_i\}_{i \in S},$$

$$Q' = \{Q_a^i\}_{1 \leq a \leq r}^{j \in S}, \quad q' = \{q_\beta^i\}_{1 \leq \beta \leq N-r}^{j \notin S}, \quad \varepsilon' = \{\varepsilon_j\}_{j \notin S}. \quad (E.5)$$

**Anomaly Polynomial** To compute the anomaly polynomial we can first compute it for the boundary condition with zero values for $c_i$, and then deform to the anomaly polynomial for $B$ by setting to 1 the sum of field strengths dual to the charges of the $\mathcal{N} = 2$ chiral labelled by $S$, whose scalars are set to $c_i$. We define field strengths $r, t, f_\theta, f_H, f_C$ for $(R_V + R_A)/2, (R_V - R_A)/2, G_\theta, G_H$ and $G_C$ respectively. With $c = 0$, the anomaly polynomial receives contributions [117]:

- From the $\mathcal{N} = 4$ vector multiplet:

$$-f_\theta \cdot f_C + r \left( \frac{1}{4} t^2 - \frac{1}{4} r^2 \right) = -f_\theta \cdot f_C - \frac{r}{4} (r + t) (r - t). \quad (E.6)$$

- From the $j^{th} \mathcal{N} = 4$ hypermultiplet:

$$-\varepsilon_j \left[ \frac{1}{4} \left( f_\theta \cdot Q^j + f_H \cdot q^j + \frac{1}{2} t - \frac{1}{2} r \right)^2 - \frac{1}{4} \left( -f_\theta \cdot Q^j - f_H \cdot q^j + \frac{1}{2} t - \frac{1}{2} r \right)^2 \right] = \frac{\varepsilon_j}{2} (r - t) \left( f_\theta \cdot Q^j + f_H \cdot q^j \right)$$

where we sum over $a$ and $\beta$ implicitly. So from all $N$ hypers:

$$\frac{1}{2} (r - t) \left( f_\theta \cdot Q \cdot \varepsilon + f_H \cdot q \cdot \varepsilon \right). \quad (E.8)$$
So the total anomaly polynomial before deformation is
\[ P = -f_0 \cdot f_C - \frac{r}{4} (r + t) (r - t) + \frac{1}{2} (r - t) (f_0 \cdot \varepsilon + f_H \cdot q \cdot \varepsilon). \tag{E.9} \]

Now deforming to non-zero \( c \), set for each \( i \in S \)
\[ f_0 \cdot Q^i + f_H \cdot q^i - \frac{\varepsilon_i}{2} (r + t) = 0, \tag{E.10} \]
or since \( Q^{(S)} \) is invertible:
\[ f_0 = -f_H \cdot q^S \cdot Q^{S-1} + \frac{1}{2} (r + t) \varepsilon^S \cdot Q^{S-1}. \tag{E.11} \]

Substituting into the undeformed \( P \), we arrive at anomaly polynomial for \( B_\alpha \)
\[ P_{B_\alpha} = f_H \cdot q^S \cdot Q^{S-1} \cdot f_C \\
- \frac{1}{2} \varepsilon^S \cdot Q^{S-1} \cdot f_C (r + t) \\
+ \frac{1}{2} (r - t) f_H \cdot (q' - q^S \cdot Q^{S-1} \cdot Q') \cdot \varepsilon' \\
+ \frac{1}{4} (r - t) (r + t) \left( \varepsilon^S \cdot Q^{S-1} \cdot Q \cdot \varepsilon - r \right). \tag{E.12} \]

We may easily read off the coefficients defined in section 5.1 for the various mixed anomalies:
\[ k = q^S \cdot Q^{S-1}, \]
\[ k^C = -\varepsilon^S \cdot Q^{S-1}, \]
\[ k^H = (q^S \cdot Q^{S-1} \cdot Q' - q') \cdot \varepsilon', \]
\[ \tilde{k} = -\varepsilon^S \cdot Q^{S-1} \cdot Q \cdot \varepsilon + r. \tag{E.13} \]

**Central Charges** The central charge \( \kappa_\alpha \) is the bilinear pairing such that:
\[ \kappa_\alpha(m_R, \zeta_R) = \begin{cases} h_m(\alpha) = m_R \cdot \mu_{H,R}(\alpha) \\ h_\zeta(\alpha) = -\zeta_R \cdot \mu_{C,R}(\alpha). \end{cases} \tag{E.14} \]

The bilinear pairing for a general abelian theory is derived in section 7.4.2 of [56]. We briefly recap it here. Define
\[ w_j := |X_j|^2 - |Y_j|^2, \quad W_j := X_j Y_j. \tag{E.15} \]

\(^1\)The minus sign difference in these generators is due to our convention for the FI parameter.
• On $\mathcal{M}_H$ we have $h_m(\alpha) = m_R \cdot \mu_{H,R}|_\alpha = m_R \cdot q \cdot w|_\alpha$. Now $w_j = 0$ for all $j \notin S$ at the vacuum. The remaining $w_i$ for $i \in S$ are determined by the real moment map $Q \cdot w = \zeta_R$. Then one can see immediately that:

$$h_m(\alpha) = m_R \cdot q \cdot q^S \cdot Q^{S-1} \cdot \zeta_R,$$

(E.16)

so $\kappa_\alpha = q^S(Q^S)^{-1}$ coincides with the value of the anomaly coefficient $k$ in (E.13).

• Considering $\mathcal{M}_C$ yields the same answer. At the vacuum $h_\zeta(\alpha) = -\sigma \cdot \zeta_R|_\alpha$. At $\alpha$ the effective real mass of the hypermultiplets must vanish for all $i \in S$: $M^i = \sigma \cdot Q^i + m_R \cdot q^i = 0$. Thus $\sigma|_\alpha = -m_R \cdot q^S(Q^S)^{-1}$ and so $h_\zeta(\alpha) = h_m(\alpha)$.

**Lowest Weights** We show now that the anomaly coefficients (E.13) coincide with the lowest weight characters of the Verma module defined by $B_\alpha$ as described in section 5.4.

• On the Higgs branch recall that the action of $\hat{W}_j = :\hat{X}_j \hat{Y}_j:$ for $j \notin S$ is given by

$$\hat{W}_j|B_\alpha) = \frac{\epsilon}{2}\epsilon_j|B_\alpha).$$

(E.17)

For $i \in S$ it is fixed by the relation $Q \cdot \hat{W} = \zeta_C$, and on the lowest weight state

$$\hat{W}_i|B_\alpha) = -\left(Q^{S-1} \cdot Q' \cdot \hat{W}'\right)_i|B_\alpha) + \left(Q^{S-1} \cdot \zeta_C\right)_i|B_\alpha)
= \left(-\frac{1}{2} \epsilon \left(Q^{S-1} \cdot \epsilon'\right)_i + \left(Q^{S-1} \cdot \zeta_C\right)_i\right)|B_\alpha).$$

(E.18)

The Verma character for $\mathcal{A}_H$ is

$$\Tr \left[e^{-\frac{1}{2}m_{xz} q \hat{W}}\right],$$

(E.19)

where we identify fugacities for the flavour symmetry $x_i = e^{-m_{xz,i}}$. One can straightforwardly compute the character of the lowest weight state as

$$x^{-\frac{1}{2}(q^S \cdot Q^{S-1} \cdot Q' - q') \epsilon' + \frac{1}{2} q^S \cdot Q^{S-1} \cdot \zeta_C}$$

(E.20)

matching the values of $k^H$ and $k$ in (E.13).

• On the Coulomb branch the vacuum obeys $\left(\hat{M}_C^i - \frac{1}{2} \epsilon_i \epsilon\right)|B) = 0$ for all $i \in S$ where the (quantised) effective complex masses are given by $\hat{M}_C^i = \hat{\phi} \cdot Q^i + m_C \cdot q^i$. 
Thus
\[ \hat{\phi}|B\rangle = (\frac{\epsilon}{2} \epsilon_s \cdot Q^{s-1} - m_c \cdot q_s \cdot Q^{s-1}) |B\rangle. \] (E.21)

The character of the Coulomb Verma is
\[ \text{Tr} \left[ e^{\frac{1}{2} \zeta \hat{\phi}} \right], \] (E.22)

and thus the character of the lowest weight state is
\[ \xi^{\frac{1}{2} \epsilon_s s - \epsilon_s 1 + \frac{1}{2} m_c q_s Q^{s-1}}, \] (E.23)

matching the values of \( k_V \) and \( k \) in (E.13).

We conclude that the lowest weights of the Higgs and Coulomb branch algebra modules defined by \( B \) are indeed given by the limits of the prefactor/Casimir energy \( \phi_B \), which itself coincides with the anomaly polynomial describing boundary 't Hooft anomalies determined by \( B \).