Compact G_2-orbifolds via Twisted Connected Sums and Associative 3-folds

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This thesis is submitted for the degree of
Doctor of Philosophy

October 2025

Declaration

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Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor, Dr. Alexei Kovalev, whose guidance, support, expertise, and patience have been invaluable in shaping this thesis and, more importantly, myself. Our weekly meetings have been a cornerstone of the past few years, and I will deeply miss them.

I would also like to extend my gratitude to my parents, Ileana and Mircea Barbuta, whose constant support has been essential throughout the completion of my PhD. I am grateful for having my partner, Louise Erena, by my side. Her encouragement, optimistic outlook and belief in me positively impacted the process of writing my thesis.

Last but not least, I would like to thank my dog, Maple, from the bottom of my heart. You came into my life about two years ago and it is already hard to imagine how my life looked before meeting you. Because of this, I would like to dedicate this thesis to you. Your influence, though lacking an understanding of differential geometry and its subtleties, resonates throughout this entire text.

Abstract

In this thesis, we study the possibility of extending the well-established construction of compact 7-manifolds carrying an irreducible torsion-free G_2-structure, known as the *Twisted Connected Sum*, to the setting of 7-orbifolds, spaces locally modeled on $\mathbb{R}^7/\Gamma$, quotients of $\mathbb{R}^7$ by finite subgroups Γ of the group G_2. Our work extends previous results by Alexei Kovalev ([44]) and Dominic Joyce ([34]) on the existence of torsion-free G_2-structures and establishes a topological criterion for the irreducibility of such structures in the case of orbifolds. The strategy for the existence part is to lift the problem locally to a Γ-invariant problem on a manifold. For irreducibility, the strategy is to adapt a criterion due to Joyce by considering a topological invariant for orbifolds called the *orbifold fundamental group*. We also investigate the irreducibility of a number of examples found in the literature, prove that the irreducibility of a global quotient of a G_2-manifolds is equivalent to the irreducibility of the manifold, and construct a few dozen examples by using weighted projective spaces as the building blocks of the twisted connected sum.

Another result in the thesis is a classification of associative 3-folds in product G_2-manifolds of the form $X \times T^3$, and related G_2 orbifolds of the form $(X \times T^3)/\mathbb{Z}_2^2$ where X is a hyper-Kähler $K3$ surface. The defining condition for this class is that the derivative of the torus projection has constant rank. We prove that under these assumptions, up to isometry of the ambient G_2 7-fold, this class consists of associative 3-folds which are given by the quotients of either products of the form $\Sigma \times \gamma$, where Σ is a complex curve in X and γ is an appropriately chosen embedded circle in T^3, or by 3-tori, $\{x_0\} \times T^3$, where $x_0 \in X$.

Contents

Introduction

The field of G_2 geometry can be approached from at least 2 different directions: the study of the normed division algebras ([28]) and the study of reduced holonomy groups of 7-manifolds ([10]). The classification of normed division algebras (due to Hurwitz) tells us that there are, up to isomorphism, only 4 possibilities: the real numbers, the complex numbers, the quaternions, and the octonions. If $\mathbb{A} \cong \mathbb{R}^n$ is a normed division algebra, we can define its real part, $Re(\mathbb{A}) = \{a1 | a \in \mathbb{R}\}$, i.e., the $\mathbb{R}$-linear span of the multiplicative identity and its imaginary part, $Im(\mathbb{A}) = Re(\mathbb{A})^{\perp}$, i.e., the orthogonal complement of the real part with respect to the standard Euclidean inner product. It turns out that the imaginary part of each normed division algebra can be equipped with a "cross product", a bilinear map $\times : Im(\mathbb{A})^2 \to Im(\mathbb{A})$ given by $a \times b = Im(ab)$ and that normed division algebras are in one-to-one correspondence with spaces admitting a "vector cross-product". Moreover, we can define a 3-form, φ, on $Im(\mathbb{A})$ by:

$$\varphi(a, b, c) = \langle a \times b, c \rangle \tag{1}$$

In the case of the octonions, this 3-form is called the associative 3-form, or the standard G_2 3-form. For our purposes, its most important property is given by a result due to Bryant ([14]) which states that the subgroup of $GL(7; \mathbb{R})$ which stabilises φ, is precisely the group G_2, and hence, any stabiliser of φ also preserves the Euclidean metric and the standard volume form on $\mathbb{R}^7$. One can then equip manifolds with 3-forms that are "given pointwise" by the standard G_2 form. Such a 3-form will induce the metric and the orientation on the manifold and we aim to study the properties of manifolds equipped with these structures.

Berger's Theorem ([10]) on reduced holonomy states that the holonomy group of a simply connected, irreducible and non-symmetric Riemannian manifold must belong to one of 5 infinite series or to one of 2 exceptional cases. G_2 and $Spin(7)$ are the exceptional cases in dimensions 7 and 8, respectively. Due to the nature of the proof of Berger's Theorem it is impossible to determine if there actually exist manifolds that carry Riemannian metrics whose holonomy is one of the exceptional ones. Nonetheless, examples of both 8-manifolds with holonomy $Spin(7)$ and 7-manifolds with holonomy G_2 have since been found. All these examples have to carry a G_2 3-form as discussed above and the induced structures arising from it. In the G_2 case, there are two well-established constructions of compact G_2-manifolds due to Dominic Joyce ([33]) and Alexei Kovalev ([44]).

In recent years, there has been an increasing interest in singular spaces carrying full G_2 holonomy, in particular those carrying isolated conical singularities. There has been some progress in this direction due to Joyce and Karigiannis ([37]), however, there still remain many unanswered questions. Our aim is to construct 7-orbifolds, spaces locally modeled on quotients of $\mathbb{R}^7$ by finite groups, which have holonomy G_2 and carry so-called "torsion-free" G_2-structures. Reidegeld has successfully constructed orbifolds carrying a torsion-free G_2-structure in [55] by a few methods: Firstly, he considers quotients of the form $(X \times T^3)/\Gamma$, where X is a $K3$ surface, T^3 is a flat 3-torus and Γ is a finite group of isometries. To obtain orbifolds he either chooses X to be an orbifold $K3$ surface or he chooses Γ to not act freely. By this method Reidegeld demonstrates how one might construct G_2 orbifolds with varied orbifold singularity types. His second construction was achieved by adapting Joyce's Kummer-type-construction. These examples are more restricted in the singularity types that they can support, but nonetheless Reidegeld manages to construct examples for all the possible singularity types. Finally, he also briefly mentioned how Kovalev's construction could be adapted to the setting of orbifolds; we will discuss more about his contribution here when we get to examples.

We adapt Kovalev's construction in [44], the so-called Twisted Connected Sum, to orbifolds. Reidegeld's work does not include any computation of the holonomy group of the found orbifolds, his main focus being the torsion-free G_2-structure, so it could be the case that the holonomy group of all his examples is a proper subgroup of G_2. We find a general topological criterion for when a compact orbifold carrying a torsion-free G_2 has holonomy exactly equal to G_2 inspired by a similar result for manifolds proved by Joyce.

The thesis is organised into 5 chapters. The first chapter provides the reader with all the necessary background results on G_2 Geometry, orbifolds, smooth $K3$ surfaces, orbifold $K3$ surfaces, and the Twisted Connected Sum.

The next chapter is concerned with generalising the backbone of the construction; we prove an existence result for asymptotically cylindrical manifolds (ACyl manifolds), which is closely related to an ACyl version of the Calabi-Yau Theorem, by adapting the proof of Haskins, Hein, and Nordström ([27]). The next result in this section concerns the existence of torsion-free G_2-structures on glued pairs of ACyl manifolds times a circle; the gluing here is done with respect to what is known as the "Donaldson matching condition" (or, simply, the matching condition) on which we easily get a closed G_2-structure. We get a torsion-free G_2-structure by solving a certain PDE via an iterative approach as was done in [33] (or in [34] chapter 11). We investigate the matching problem for orbifolds and examine basic topological invariants of our orbifold Twisted Connected Sums; we compute their Betti numbers, their fundamental group, and their orbifold fundamental group by using the Seifert-van Kampen Theorem and the Mayer-Vietoris sequence as was done in [45]. Lastly, we look in detail at Reidegeld's examples ([55]) of so-called "non-symplectic type" orbifold Twisted Connected Sums. The results in this chapter are mainly routine generalisations of their smooth counterparts. Nonetheless, we go through the arguments found in the literature for the smooth case so that we accurately state the orbifold analogues and identify any differences; a particularly delicate part to extend being the matching problem, as we shall see.

The third chapter is dedicated to proving a result about when a manifold carrying a torsion-free G_2-structure has holonomy exactly G_2, rather than a proper subgroup of G_2; in the smooth case Joyce ([34, Prop 10.2.2]) gives a simple topological condition, namely the finiteness of the fundamental group. This result is known to fail in the case of orbifolds: in Joyce's original construction ([33]), he considers the 7-torus, T^7, equipped with the flat G_2-structure induced by the standard one on $\mathbb{R}^7$. He then chooses a finite group of affine transformations of $\mathbb{R}^7$, Γ, and aims to resolve the singularities on the quotient orbifold T^7/Γ to get G_2-manifolds. For certain choices of Γ, we have that T^7/Γ is simply connected (see [33]), but the G_2-structure is the flat one so the holonomy is trivial. Our result uses a more suitable invariant for orbifolds, namely the orbifold fundamental group, π_1^{orb}, to detect if the holonomy is a proper subgroup of G_2 or not; the finiteness of the orbifold fundamental group is a necessary and sufficient condition for the orbifold to have full G_2 holonomy. Returning to Joyce's examples, the orbifold fundamental group fits into a short exact sequence:

$$1 \to \pi_1(T^7) \to \pi_1^{orb}(T^7/\Gamma) \to \Gamma \to 1 \tag{2}$$

so it is easy to see that in this case $\pi_1^{orb}(T^7/\Gamma)$ is in fact infinite. We then apply our theorem to various G_2 orbifolds found in the literature and investigate their holonomy. We also apply our theorem to quotients, M/G, of G_2-manifolds, M, by finite groups of isometries, G, to prove that the holonomy of the quotient is G_2 if and only if the holonomy of the manifold M is G_2.

The next chapter is concerned with examples. We first introduce background results on weighted projective spaces, which will constitute a basis for new examples of orbifold G_2-manifolds and then construct G_2-manifolds from pairs of weighted projective spaces as our ACyl parts and hypersurfaces in them as the required $K3$ divisors. We use our matching theorem to construct a few dozen examples. We also exploit a particular weighted projective space for which the $K3$ divisor is smooth to construct examples by gluing it to other known smooth families in the literature (particularly those considered by Kovalev in [44]).

In chapter 5 we look at a particular class of G_2 orbifolds, namely those of the form $(X \times T^3)/\mathbb{Z}_2^2$, where X is a hyper-Kähler $K3$ surface and $\mathbb{Z}_2^2$ is generated by two commuting involutions of the non-symplectic type. Our main concern here is to give a systematic description of a class of associative 3-folds in the aforementioned G_2 orbifolds, namely those for which the derivative of the projection to the torus factor has constant rank. We initially concern ourselves with the constant rank 1 case and prove that associative 3-folds satisfying this condition are always given by products $\Sigma \times \gamma$ where $\Sigma \subset X$ is a complex curve with respect to some complex structure in the S^2 family of complex structures on X induced by the hyper-Kähler structure and $\gamma \subset T^3$ is an embedded circle. We then investigate the other constant rank possibilities and provide a full classification of the class of constant rank associative 3-folds, ending with a couple of remarks on the statement and proof of the main theorem.

We end the introduction by talking a bit about the applications of G_2-manifolds and orbifolds in pure mathematics and beyond. G_2 geometry is interesting to study in its own right, due to the interplay between the G_2-structure and the Riemannian metric. 7-manifolds equipped with a torsion-free G_2-structure and holonomy exactly G_2 are both examples proving that one of the special cases in Berger's Theorem is not just an artefact of the proof, but does in fact arise "in the wild", and examples of Ricci-flat real manifolds in odd dimensions, of which we do not have many; most of our examples of Ricci-flat manifolds come from Yau's solution to the Calabi conjecture and from manifolds carrying a hyper-Kähler structure, both of which exist in the realm of complex geometry.

Outside of pure mathematics, G_2-manifolds and -orbifolds play a role in high-energy physics, specifically in the so-called compactification of M-theory [38]. This is a theory aiming to explain phenomena of quantum gravity on an 11-dimensional manifold of the form:

$$M^{11} = \mathbb{R}^{1,3} \times M^7 \tag{3}$$

where $\mathbb{R}^{1,3}$ is Minkowski space and M^7 is a compact Ricci-flat Riemannian manifold with “small” enough total volume. We call a manifold of this type a *compactification of spacetime* and the quantum physics we observe on $\mathbb{R}^{1,3}$ is called the *low energy limit of M-theory compactified on* M^7. It is commonly believed that M^7 is in fact a G_2-manifold, however, if M^7 is a manifold, then this theory is not compatible with physical observations. Nonetheless, if one allows M^7 to have suitable singularities then M-theory compactified on M^7 does match observations. Such suitable singularities are, for example, conical singularities; as of today, no explicit examples of compact G_2-manifolds with isolated conical singularities are known. For a good overview of the role of G_2 orbifolds in high-energy physics one can consult [2].

Chapter 1

Background Results

The aim of this chapter is to introduce the background results in the areas of G_2 geometry, orbifold theory, the Twisted Connected Sum, and $K3$ surfaces. We also introduce notation and key ideas we will be using in the following chapters. The material for this chapter is standard and the main references will be [34], [31], [3], [17], [20] and [9]. We also mention [55] for the exposition on lattice theory.

1.1 G_2 Geometry

G_2 Geometry is a branch of Riemannian Geometry where we study the properties of a special class of 7-folds carrying what is known as a G_2-structure. These have some remarkable properties and are connected to other areas of mathematics such as normed division algebras, in particular with the algebra of the octonions, $\mathbb{O}$. In this section we introduce some of the important properties of G_2-structures on 7-manifolds.

Consider $\mathbb{R}^7$ equipped with the standard Euclidean metric g_0. Denote the standard orthonormal basis by $\{e_1, \ldots, e_7\}$ and write $e^{ijk} = e^i \wedge e^j \wedge e^k$, then we can define the *associative 3-form* or the *standard G_2-structure on $\mathbb{R}^7$*, ϕ_0, to be (note that there are two sign conventions and we adopt the one due to Bryant, [14, p. 539]; for more details one can check [40, p. $15-20$]):

$$\phi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \tag{1.1}$$

Now we can define the group G_2 to be the group that stabilises the associative 3-form:

$$G_2 = \{A \in GL(7, \mathbb{R}) | A^* \phi_0 = \phi_0\} \tag{1.2}$$

One reason why G_2-geometry is interesting is the next result which essentially says that G_2 is a subgroup of $SO(7)$ (by viewing $SO(7)$ as the subgroup of $GL(7,\mathbb{R})$ which preserves both the standard inner product, g_0, and the standard orientation, $dVol_0$, associated to g_0).

Theorem 1.1 ([14, Theorem 2])**.** *If $A \in G_2$, then $A^*g_0 = g_0$ and $A^*dVol_0 = dVol_0$, where $dVol_0$ is the standard volume form on $\mathbb{R}^7$.*

Moving to the setting of smooth manifolds, let M be a smooth 7-manifold. We wish to equip each tangent space with an associative 3-form as in **Equation (1.1)** in a smooth way.

Definition 1.2. Let M be a smooth 7-manifold. A *G_2-structure* on M is a smooth 3-form, ϕ, on M such that for all $p \in M$, there exists a linear isomorphism $A_p : T_pM \to \mathbb{R}^7$ which identifies $\phi_p \in \Lambda^3(T_p^*M)$ with $\phi_0 \in \Lambda^3(\mathbb{R}^7)^*$ via pullback, $A_p^*(\phi_p) = \phi_0$.

Since $G_2 \subset SO(7)$, then a G_2-structure on M will induce a metric, g_ϕ, and a corresponding volume form $dVol_\phi$, hence also a Hodge star, $\star_\phi$. For tangent vectors u, v, these satisfy:

$$6\langle u, v\rangle_{g_\phi} dVol_\phi = (u \lrcorner \phi) \wedge (v \lrcorner \phi) \wedge \phi \tag{1.3}$$

Remark 1.3. A connected 7-manifold, M, admits a G_2-structure if and only if it is orientable and spinnable ([15]); this is equivalent to the first two Stiefel-Whitney classes of M vanishing.

Proposition 1.4 (Fernández and Gray, [23])**.** *Consider a G_2-structure $\phi \in \Omega^3(M)$. Then $Hol(g_\phi) \subseteq G_2$ if and only if $\nabla_\phi \phi = 0$ if and only if $d\phi = 0$ and $d \star_\phi \phi = 0$, where ∇_ϕ denotes the Levi-Civita connection of g_ϕ. In this case we say that ϕ is a* ***torsion-free G_2-structure*** *and we call (M, ϕ) a* ***torsion-free G_2-manifold****.*

Remark 1.5. If we denote the formal L^2-adjoint of the exterior derivative by d^*, then $d \star_\phi \phi = -\star_\phi d^*\phi$ so if ϕ is torsion-free, then $d\phi = 0$ and $d^*\phi = 0$ which says that ϕ is a harmonic 3-form.

Note that the second condition,

$$d \star_\phi \phi = 0 \tag{1.4}$$

is **not** linear, since the Hodge star operator depends on the G_2-structure, ϕ, and is the main difficulty to finding torsion-free G_2-structures. Another important property of metrics induced from a torsion-free G_2-structure is that they are Ricci-flat:

Theorem 1.6 ([58, Lemma 11.8])**.** *Let (M, g) be a Riemannian 7-manifold. If $Hol(g) \subseteq G_2$, then g is Ricci-flat.*

Our best-understood examples of Ricci-flat metrics come from Yau's solution to the Calabi Conjecture (e.g. $K3$ surfaces carrying a *hyper-Kähler structure*), however, irreducible G_2-structures provide examples of Ricci-flat metrics on manifolds which are real manifolds of **odd** dimension. If (M, ϕ) is a torsion-free G_2-manifold, then $Hol(g_\phi) \subseteq G_2$. We say (M, ϕ) is an *irreducible torsion-free G_2-manifold* if $Hol(g_\phi) = G_2$.

From now on we shall refer to compact manifolds carrying a torsion-free G_2-structure simply as G_2-manifolds. There are two well-known constructions of irreducible G_2-manifolds, one due to Joyce and one due to Kovalev. Joyce's construction involves starting with a flat 7-torus and taking a quotient by a discrete group of isometries preserving the G_2-structure ϕ_0. Then he resolves the singularities of this quotient space to obtain a 7-manifold with a closed G_2-structure ($d\phi = 0$) and "small" torsion. The idea is to find a small 2-form, η, such that $\star_\phi d(\phi + d\eta) = 0$ by using a carefully designed iteration argument in perturbative analysis. Finally, for irreducibility, he shows that the fundamental group must be finite.

We shall focus on the construction by Kovalev, commonly known as the Twisted Connected Sum. Informally, the idea is to obtain non-compact, asymptotically cylindrical Riemannian manifolds with holonomy $SU(3)$ of real dimension 6, such that a Riemannian product with a circle yields a 7-manifold with the same holonomy and with an end asymptotic to a half-cylinder. The cross section of the cylindrical neck is a product of a $K3$ surface and 2 circles. Using a pair of these we can truncate the cylindrical end, cut off the cylindrical metric and identify the boundaries via an orientation-reversing isometry to get a compact Riemannian 7-manifold. The gluing map interchanges the S^1 factors and the complex structures on the $K3$ surfaces such that the fundamental group remains finite and that we get a well-defined G_2-structure, which we view as "approximating" some torsion-free G_2-structure. Initial examples for the Twisted Connected Sum were given in [44], with Corti and co-authors providing many more examples in [18].

More recently, there have been new methods of obtaining new examples of G_2-manifolds due to Joyce and Karigiannis ([37]), and due to Nordström ([53]). We will briefly review the construction in [37] in **Section** 3.2 below, where we will also compute the holonomy groups of the G_2-orbifolds needed in the construction. We also briefly outline the construction in [53] below.

Nordström generalises the Twisted Connected Sum, generalisation which has become known as the *extra-Twisted Connected Sum*. The idea is that in the Twisted Connected Sum, the cross-sections we get are products of a torus and a $K3$ surface, where the torus is given by a product of two circles of the same radius (this corresponds to viewing the torus as the quotient of $\mathbb{R}^2$ by the lattice generated by the standard basis vectors), however, there is no reason for these two circles to have the same radius. In fact, he shows that by considering circles of different radii, i.e., by considering different lattices defining the T^2 factor, one can obtain new G_2-manifolds. The main difference in this case lies in the matching condition which becomes a rotation by some angle θ of the hyper-Kähler structure on the $K3$ factors (the regular Twisted Connected Sum corresponds to the case $\theta = \frac{\pi}{2}$).

We will now introduce orbifolds and some of the foundational results that will be useful for us, and then we review $K3$ surfaces and orbifold $K3$ surfaces, which are crucial ingredients for the Twisted Connected Sum and its orbifold generalisation. We end this chapter with a review of the Twisted Connected Sum.

1.2 Orbifolds

This section will introduce the simplest possible singular spaces on which we can hope to generalise the Twisted Connected Sum, namely orbifolds. These are spaces locally modeled on quotients of $\mathbb{R}^n$ by finite groups, and a great deal of results from the theory of smooth manifolds apply (almost) verbatim to orbifolds. Most of the material in the following subsections is contained in [17], [3] and [20].

We begin with the definition of an orbifold which we will be using:

Definition 1.7. Let $\mathcal{M}$ be a Hausdorff, second countable topological space. We say that (U, Γ, ϕ) is an *orbifold chart* for $\mathcal{M}$ if:

- U is a connected, open neighbourhood of $0 \in \mathbb{R}^n$,
- $\Gamma \subset GL(n, \mathbb{R})$ is a finite group such that $\forall \gamma \in \Gamma$, $\gamma(U) \subset U$ and the codimension of the fixed point set of γ is greater than or equal to 2,
- the map $\phi : U \to \mathcal{U}$ is continuous, $\forall \gamma \in \Gamma$ and $x \in U$, $\phi(x) = \phi(\gamma.x)$ and it induces a homeomorphism $\phi_\Gamma : U/\Gamma \to \mathcal{U}$, where $\mathcal{U} \subset \mathcal{M}$ is an open subset.

Given two orbifold charts (U, Γ, ϕ) and (V, Δ, ψ), we say that a map $\lambda : U \to V$ is an *injection of charts* (sometimes we will also say a *chart injection*) if it is smooth, linear, injective and it satisfies $\psi \circ \lambda = \phi$. We say $\mathcal{M}$ is an *orbifold* if there exists a family of orbifold charts $\{(U_i, \Gamma_i, \phi_i)\}$ such that $\mathcal{M} = \cup_i \phi_i(U_i)$ and $\forall (U_i, \Gamma_i, \phi_i)$ and (U_j, Γ_j, ϕ_j) with $p \in \phi_i(U_i) \cap \phi_j(U_j)$, there exists an orbifold chart (U_k, Γ_k, ϕ_k) with $p \in \phi_k(U_k)$ and injections of charts $\lambda_{ki} : U_k \to U_i$ and $\lambda_{kj} : U_k \to U_j$.

Note that given an orbifold chart, (U, Γ, ϕ), $\forall \gamma \in \Gamma$, the map induced by the $GL(n, \mathbb{R})$ action, $\gamma : U \to U$, is an injection of charts. For a chart, (U, Γ, ϕ), the *isotropy subgroup* of Γ at $x \in U$ is defined to be

$$\Gamma_x = \{\gamma \in \Gamma | \gamma . x = x\} \tag{1.5}$$

and we say that x is a *singular point* if Γ_x is non-trivial. We say a point $p \in \mathcal{U} \subset \mathcal{M}$ is a *singularity* if $p = \phi(x)$ for a singular point $x \in U$. The set of all singularities will be denoted by $\mathcal{M}_{sing}$ and points in $\mathcal{M}_{reg} = \mathcal{M} \setminus \mathcal{M}_{sing}$ will be called *regular*. Sometimes, when there is no chance of confusion, we will abuse the terminology and call p a singular point. We also note that each connected component of the singular locus is a manifold without boundary ([17]). Given this, it will be useful for us to define the dimension of $\mathcal{M}_{sing}$, n_{sing}, to be

$$n_{sing} = max\{dimC | C \text{ is a connected component of } \mathcal{M}_{sing}\} \tag{1.6}$$

We will return to this point and expand a bit on it in **Chapter** 3. We denote the set of all orbifold charts on an orbifold $\mathcal{M}$ by $\mathcal{F}_{\mathcal{M}}$ and the set of all injections of charts by $\mathcal{L}_{\mathcal{M}}$.

Definition 1.8. Let $\mathcal{M}$ and $\mathcal{P}$ be orbifolds. A collection $\{f_U\}_{(U,\Gamma,\phi) \in \mathcal{F}_{\mathcal{M}}}$ such that for each $(U, \Gamma, \phi) \in \mathcal{F}_{\mathcal{M}}$ there exists an $(V, \Delta, \psi) \in \mathcal{F}_{\mathcal{P}}$ and a map $f_U : U \to V$ satisfying that for each injection of charts $\lambda : U \to U' \in \mathcal{L}_{\mathcal{M}}$ there exists an injection of charts $\tau : V \to V' \in \mathcal{L}_{\mathcal{P}}$ such that the following diagram commutes:

$$\begin{tikzcd}
U \arrow[r, "f_U"] \arrow[d, "\lambda"] & V \arrow[d, "\tau"] \\
U' \arrow[r, "f_{U'}"] & V'
\end{tikzcd}$$

will be called an *orbifold map* and we call the f_U's local orbifold maps. Note that this implies that we have a map of topological spaces $f : \mathcal{M} \to \mathcal{P}$, such that, for each chart $(U, \Gamma, \phi) \in \mathcal{F}_{\mathcal{M}}$, $f \circ \phi = \psi \circ f_U$, which we also refer to as an orbifold map. We say that an orbifold map, f, is of class C^k, C^∞, etc. if each f_U is.

The tangent space over a point $p \in \mathcal{M}$ in an orbifold chart (U, Γ, φ) is defined to be:

$$T_p\mathcal{M} = T_{\varphi^{-1}(p)}U \tag{1.7}$$

and it comes equipped with an action of Γ, induced from the action of Γ on U. We also define *the tangent cone over* p to be:

$$C_p\mathcal{M} = T_p\mathcal{M}/\Gamma_p \tag{1.8}$$

where Γ_p is the isotropy group at p. If p is a smooth point, then the tangent cone is, in fact, the regular tangent space. We can define the derivative of a smooth map of orbifolds $f : \mathcal{M} \to \mathcal{P}$ at a point $p \in \mathcal{M}$ to be the unique linear map $D_pf : T_p\mathcal{M} \to T_{f(p)}\mathcal{P}$ defined in terms of an orbifold chart, (U, Γ, φ), around p, as $D_pf = D_{\varphi^{-1}(p)}f_U$. Note that this definition is independent of the chosen orbifold chart. We also define the *rank of* D_pf to be the rank of $D_{\varphi^{-1}(p)}f_U$. We also have a notion of vector bundles over an orbifold, called *orbibundles*:

Definition 1.9. Let $\mathcal{M}$ be an orbifold and $\mathcal{F}_{\mathcal{M}}$ the set of all orbifold charts on $\mathcal{M}$. An orbibundle of rank r over $\mathcal{M}$ consists of a rank r vector bundle, B_{U_i} over U_i, for every orbifold chart $(U_i, \Gamma_i, \phi_i) \in \mathcal{F}_{\mathcal{M}}$, with fibre a r-dimensional vector space, V, independent of the orbifold chart, together with a homomorphism $h_{U_i} : \Gamma_i \to GL(r; V)$ satisfying:

- if b lies in the fibre over $x_i \in U_i$, then for each $\gamma \in \Gamma_i$, $bh_{U_i}(\gamma)$ lies in the fibre over $\gamma^{-1}x_i$
- if the map $\lambda_{ji} : U_i \to U_j$ is a chart injection, then there is a bundle map $\lambda^*_{ji} : B_{U_j}|_{\lambda_{ji}(U_i)} \to B_{U_i}$ satisfying the condition that if $\gamma \in \Gamma_i$, and $\gamma' \in \Gamma_j$ is the unique element of Γ_j such that $\lambda_{ji} \circ \gamma = \gamma' \circ \lambda_{ji}$, then $h_{U_i}(\gamma) \circ \lambda^*_{ji} = \lambda^*_{ji} \circ h_{U_i}(\gamma')$. Moreover, if $\lambda_{kj} : U_j \to U_k$ is another injection of charts, then $(\lambda_{kj} \circ \lambda_{ji})^* = \lambda^*_{ji} \circ \lambda^*_{kj}$

Remark 1.10. We can construct a tangent orbibundle, a cotangent orbibundle and (r, s)-tensor orbibundles, which we shall denote in the usual way (for example, $T^*\mathcal{M}$ will denote the cotangent orbibundle), using the tangent cones. We can also define other (vector) orbibundles where the general fibres are quotients V/Γ_p of a vector space V, or principal G-orbibundles, for G a Lie group, by slightly tweaking the definition above.

Remark 1.11. If $\mathcal{E} \to \mathcal{M}$ is an orbibundle we shall denote its space of smooth sections by $C^\infty(\mathcal{E})^{orb}$ and similarly for other spaces of sections of different regularities. All such sections will be given as collections similar to the case of orbifold maps. The '*orb*' superscript is superfluous here, in fact, we could have easily denoted the space of smooth sections as $C^\infty(\mathcal{E})$. We add the '*orb*' superscript as a means to emphasise that we are considering sections of orbibundles over orbifolds; however, when it is clear from the context that we are working with orbifolds we shall omit the '*orb*' superscript. It is also important to note that this distinction sometimes is very important, for example, $\pi_1(\mathcal{M})$ is the fundamental group of the orbifold which we obtain by ignoring the orbifold structure on the underlying topological space, but $\pi_1^{orb}(\mathcal{M})$ denotes the orbifold fundamental group which takes into account the orbifold structure. Whenever the distinction will be important we will explicitly state this to avoid any confusion that may arise.

Example 1.12. Let (U, Γ, ϕ) be a chart. Let $\Omega^k(U)_\Gamma \subset \Omega^k(U)$ denote the subset of differential k-forms which are invariant under the action of Γ. Explicitly, $\Omega^k(U)_\Gamma$ consists of those differential k-forms satisfying:

$$\omega_x(X_1, \ldots, X_k) = \omega_{\gamma.x}(\mathcal{J}(\gamma)(x)X_1, \ldots, \mathcal{J}(\gamma)(x)X_k) \tag{1.9}$$

for all $x \in U$, $\gamma \in \Gamma$ and $X_1, \ldots, X_k \in T_xU$, where $\mathcal{J}(\gamma)(p) = \left(\frac{\partial x^i \circ \gamma}{\partial x^j}\right)$ is the Jacobian of γ at p. We say that $\omega = \{\omega_U\}_{\mathcal{F}_\mathcal{M}}$ is an orbifold differential k-form on $\mathcal{M}$ if for each chart (U, Γ, ϕ) the form ω_U is in $\Omega^k(U)_\Gamma$ and for each injection of charts $\lambda : U \to U'$ the form ω_U satisfies:

$$(\omega_U)_x(X_1, \ldots, X_k) = (\omega_{U'})_{\lambda(x)}(\mathcal{J}(\lambda)(p)X_1, \ldots, \mathcal{J}(\lambda)(p)X_k) \tag{1.10}$$

where, as before, $\mathcal{J}(\lambda)(p) = \left(\frac{\partial x'^i \circ \lambda}{\partial x^j}\right)$ is the Jacobian of λ at p and x'^i are local coordinates on U'. We denote the space of orbifold differential k-forms on $\mathcal{M}$ by $\Omega^k(\mathcal{M})^{orb}$. Note in particular that any operation on differential forms which preserves Γ-invariance gives an analogous well-defined operation on orbifold differential forms; of interest being the wedge product, symmetrisation and anti-symmetrisation. Moreover the usual exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$ preserves Γ-invariance and is compatible with chart injections so it gives a well-defined exterior derivative for orbifold differential forms which we shall also denote by d. Hence we get a cohomology theory for $\Omega^*(\mathcal{M})^{orb}$, which we shall denote by $H^*_{dR}(\mathcal{M})$. It has been proven in [59] that $\forall k = 0, \ldots, dim(\mathcal{M})$ we have $H^k_{dR}(\mathcal{M}) \cong H^k(\mathcal{M}; \mathbb{R})$, that is, the deRham-type cohomology for $\Omega^*(\mathcal{M})^{orb}$ is isomorphic to the usual cohomology with real coefficients.

Moreover, as previously noted, a lot of results concerning manifolds generalise to orbifolds. We mention just a few, which will prove to be important in this thesis: the existence and uniqueness of the Levi-Civita connection, Stokes' Theorem, the Hodge Decomposition Theorem ([8]), the Calabi-Yau Theorem; we will discuss the latter in more detail below.

It is rather easy to see, but important to note that the spaces of sections of oribundles are vector spaces. For example, given an oriented Riemannian orbifold $\mathcal{M}$, $C^k(\mathcal{M})^{orb}$ will be the space of bounded continuous functions $\mathcal{M} \to \mathbb{R}$ which have k-times continuous bounded derivative. As for manifolds, it will be a vector space and we can equip it with the standard norm $||f||_{C^k} = \sum_{i=0}^{k} sup_{\mathcal{M}}|\nabla^i f|$, where ∇ denotes the Levi-Civita connection on $\mathcal{M}$. Similarly, we can define Hölder spaces, $C^{k,\alpha}(\mathcal{M})^{orb}$, or Sobolev spaces $L^p_k(\mathcal{M})^{orb}$ which will generalise to the spaces of sections of an orbifold vector bundle in the same way they do for manifolds. If $\mathcal{E} \to \mathcal{M}$ is an orbibundle we shall denote, for example, the corresponding Sobolev space of sections by $L^p_k(\mathcal{E})^{orb}$.

Since the corresponding spaces of sections are vector spaces plenty of results related to the analysis of sections on orbifolds carry over from the manifold case:

Example 1.13. As an example we shall show that if $(\mathcal{M}, g)$ is a compact Riemannian orbifold (i.e., g is a Riemannian metric on $\mathcal{M}$) and $k \geq 0$ then $C^k(\mathcal{M})^{orb}$ is a Banach space. First note that it is a vector space under the usual addition

$$f + g = \{f_U + g_U\}_{\mathcal{F}_{\mathcal{M}}} \tag{1.11}$$

and let $f_i = \{(f_i)_U\}_{\mathcal{F}_{\mathcal{M}}}$ be a Cauchy sequence in $C^k(\mathcal{M})^{orb}$. For each orbifold chart (U, Γ, ϕ), $C^k(U)$ is a Banach space so the sequence $(f_i)_U$ converges to a function $f_U \in C^k(U)$. For $\gamma \in \Gamma$ the function $||(f_i)_U(x) - (f_i)_U(\gamma.x)||_{C^k}$ is continuous in x so

$$||f_U(x) - f_U(\gamma.x)||_{C^k} = lim_{i\to\infty}||(f_i)_U(x) - (f_i)_U(\gamma.x)||_{C^k} = 0 \tag{1.12}$$

so indeed $f_U \in C^k(U)_\Gamma$ for each orbifold chart and each $\gamma \in \Gamma$ so indeed $f = \{f_U\}_{\mathcal{F}_{\mathcal{M}}} \in C^k(\mathcal{M})^{orb}$.

We define a *complex orbifold*, $(\mathcal{M}, J)$, as a real $2n$-dimensional orbifold with an *orbifold complex structure*, $J = \{J_U\}_{\mathcal{F}_{\mathcal{M}}}$, for which each local chart (U, Γ, ϕ) satisfies that $U \subset \mathbb{C}^n$, $\Gamma \subset GL(n, \mathbb{C})$, the complex structure on U, J_U, is Γ-invariant and compatible with injections of charts, meaning that $\forall x \in U$, $X \in T_xU$ and all injections of charts $\lambda : U \to U'$ we have:

$$(J_U)_x(X) = (J_{U'})_{\lambda(x)}(\mathcal{J}(\lambda)(x)X) \tag{1.13}$$

A *Kähler orbifold* is a complex orbifold, $(\mathcal{M}, J)$, equipped with an orbifold Kähler form, $\omega = \{\omega_U\}_{\mathcal{F}_{\mathcal{M}}}$. Note that the existence of a Kähler form on $(\mathcal{M}, J)$ implies that for every orbifold chart (U, Γ, ϕ), we have $\Gamma \subset U(n) \subset GL(n, \mathbb{C})$. The Hodge Decomposition Theorem has been proven for orbifolds in [8], from which it follows that the $\partial\overline{\partial}$-Lemma extends to Kähler orbifolds. We state and prove the orbifold $\partial\overline{\partial}$-Lemma below to showcase one of the major tools we will use, namely "averaging over the local isotropy groups".

Lemma 1.14. *Let* $(\mathcal{M}, J, \omega)$ *be a compact Kähler orbifold of complex dimension* n *and let* ω' *be a smooth closed real* $(1,1)$*-form on* $\mathcal{M}$*. Then for every point* $p \in \mathcal{M}$ *there exists a neighbourhood,* $\mathcal{U}$ *and* $f \in C^\infty(\mathcal{U})$ *such that* $\omega'|_{\mathcal{U}} = \frac{i}{2\pi}\partial\overline{\partial} f$

Proof. If $p \in \mathcal{M}$ is a smooth point, then there exists a neighbourhood $\mathcal{V} \subset \mathcal{M}$ such that $\mathcal{V}$ is biholomorphic to $\mathbb{C}^n$. Using the regular $\partial\overline{\partial}$-Lemma yields a neighbourhood $\mathcal{W}$ such that $\omega'|_{\mathcal{W}} = \frac{i}{2\pi}\partial\overline{\partial} f$. Set $\mathcal{U} = \mathcal{V} \cap \mathcal{W}$, then $\omega'|_{\mathcal{U}} = \frac{i}{2\pi}\partial\overline{\partial} f$. If p is singular, then there exists an orbifold chart (V, Γ, ϕ) and $\omega'|_{\mathcal{V}} \in \Omega^{1,1}(V)_\Gamma$. Using the $\partial\overline{\partial}$-Lemma on V, we have that there exists a $U \subset V$ and $f \in C^\infty(U)$ with $\omega'|_U = \frac{i}{2\pi}\partial\overline{\partial} f$. Now $\omega'|_{\mathcal{U}}$ is Γ-invariant so it suffices to construct f^Γ from f, such that f^Γ is Γ invariant (so it gives a well-defined element of $C^\infty(U)_\Gamma$) and $\partial\overline{\partial} f = \partial\overline{\partial} f^\Gamma$. This follows immediately since $\omega'|_U = \frac{i}{2\pi}\partial\overline{\partial} f$ and $\omega'|_U$ is Γ invariant so that $\partial\overline{\partial} f$ is Γ invariant. Now consider "averaging f over Γ":

$$f^\Gamma = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma^* f \tag{1.14}$$

Note that f^Γ is Γ invariant and we have that:

$$\begin{aligned}\partial\overline{\partial} f^\Gamma = \partial\overline{\partial}\left(\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma^* f\right) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\partial\overline{\partial}(\gamma^* f) = \\ = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma^*(\partial\overline{\partial} f) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\partial\overline{\partial} f = \partial\overline{\partial} f\end{aligned} \tag{1.15}$$

where we have used Γ invariance of $\partial\overline{\partial} f$ in the second line. Hence we have $\frac{i}{2\pi}\partial\overline{\partial} f^\Gamma = \omega'|_{\mathcal{U}}$ □

Moreover, note that we can define the *canonical orbibundle* just as we would for complex manifolds.

Definition 1.15. Let $\mathcal{M}$ be a complex orbifold of dimension n. Then we define the *canonical orbibundle* of $\mathcal{M}$ to be:

$$K_{\mathcal{M}} = \Lambda^n T^*\mathcal{M}^{1,0} \tag{1.16}$$

where $T^*\mathcal{M}^{1,0}$ is the holomorphic cotangent orbibundle. In general, this will be an invertible sheaf ([34, Chapter 6.3]); however, if we have that $\forall p \in \mathcal{M}$, the isotropy group Γ_p is a subgroup of $SL(n;\mathbb{C})$, then this is indeed a line bundle in the sense of complex geometry ([3, Chapter 1.3]).

Remark 1.16. Invertible sheaves are the natural generalization of line bundles over singular algebraic varieties; to see why note that if M is a complex manifold, L is a line bundle over M, and $\mathcal{O}(L)$ is the sheaf of holomorphic sections of L, then $\mathcal{O}(L)$ is an invertible sheaf. Conversely, if $\mathcal{F}$ is an invertible sheaf, then it can be identified with the sheaf of holomorphic sections of a line bundle $L_{\mathcal{F}}$.

However, in this thesis we will not need invertible sheaves as can be seen from the remark below.

Remark 1.17. What we mean by line bundle in the sense of complex geometry is the following:

Note that $K_{\mathcal{M}}$ is a complex orbibundle so the fibre over each point, p, is of the form $\mathbb{C}/\Gamma_p$. Now note that the fibres of the cotangent bundle can be identified with $\mathbb{C}^n/\Gamma_p$ using the complex structure of $\mathcal{M}$ ([3, Chapter 1.3]). The action induced on the fibre $\mathbb{C}$ is the one obtained by taking the determinant of the above representation so indeed if $\Gamma_p \subset SL(n;\mathbb{C})$ for all $p \in \mathcal{M}$, then the fibre over every point will be a copy of $\mathbb{C}$. This gives the structure of a topological line bundle. Moreover, the transition functions for this topological line bundle will be given by the determinants of transition functions for $T^*\mathcal{M}^{1,0}$, which are indeed holomorphic maps so indeed $K_{\mathcal{M}}$ is a complex line bundle.

For us, it will be the case that the canonical orbibundle will be a complex line bundle. This is because we shall only consider complex orbifolds which carry a metric with holonomy $SU(n)$. For such a metric to exist we need, for every point p of our complex orbifolds, the isotropy group at p to be a finite subgroup of $SU(n)$. As such, the canonical orbibundle is a complex line bundle.

Definition 1.18 (see [13], section 4)**.** Let $(\mathcal{M}, J)$ be a complex orbifold, $\mathcal{E} \to \mathcal{M}$ be a complex orbibundle and ∇ a covariant derivative on $\mathcal{E}$. We define the *orbifold first Chern class* of ∇ on $\mathcal{E}$ to be:

$$c_1(\mathcal{E}, \nabla)^{orb} = \left[\frac{i}{2\pi} tr_{\mathbb{C}}(F_{\nabla})\right] \in H^2(\mathcal{M};\mathbb{R}) \tag{1.17}$$

where F_∇ is the curvature corresponding to ∇. We define the *orbifold first Chern class* of $(\mathcal{M}, J)$ to be $c_1(T^{1,0}\mathcal{M})^{orb}$. It only depends on the complex structure, J, so we shall denote it as $c_1(J)^{orb}$ or $c_1(\mathcal{M})^{orb}$.

Remark 1.19 (cf. [34], p. 135). If $\mathcal{M}$ is a complex orbifold such that all orbifold groups lie in $SL(n, \mathbb{C})$, then $K_\mathcal{M}$ is a genuine line bundle over $\mathcal{M}$. This is the case, for example, when our complex orbifold is equipped with a metric with holonomy $SU(n)$. In such an instance, the first Chern class is a well-defined element of $H^2(\mathcal{M}; \mathbb{Z})$ in the usual way. If not all the isotropy groups are subgroups of $SL(n, \mathbb{C})$, then in fact $K_\mathcal{M}$ is a genuine orbibundle, with fibres $\mathbb{C}$ over smooth points and $\mathbb{C}/\Gamma$ over singular points. In this case, we have that $c_1(\mathcal{M})^{orb} \in H^2(\mathcal{M}; \mathbb{Q})$. Either way, we have that $c_1(\mathcal{M})^{orb} \in H^2(\mathcal{M}; \mathbb{R})$. Moreover, in either case, $c_1(\mathcal{M})^{orb} = 0$ implies that $K_\mathcal{M}$ is trivial (either as a genuine line bundle or as an orbibundle).

We also mention that the Calabi-Yau Theorem holds for orbifolds. This was known for some time (see for example [34, Section 6.5.1]), however, the first detailed proof seems to be given in [48, Section 3.5]:

Theorem 1.20 (Calabi-Yau, [48, Theorem 3.5.1])**.** *Let $\mathcal{M}$ be a compact Kähler orbifold with Kähler form ω. Then there exists a unique Ricci-flat Kähler metric $\overline{\omega}$ in the same cohomology class as ω if and only if the orbifold first Chern class vanishes, i.e., $c_1(\mathcal{M})^{orb} = 0$*

We end this section by discussing the *orbifold fundamental group.* We will first discuss why we need a separate notion of fundamental group for orbifolds and then we will formally introduce it as is done in [17], sections 2.1 and 2.2. Let's consider an example first:

Example 1.21. Consider the quotient of $\mathbb{R}^2$ by the action of $\mathbb{Z}_3$ whose generator acts as a rotation by $\frac{2\pi}{3}$ about the origin. We denote by $\mathcal{M} = \mathbb{R}^2/\mathbb{Z}_3$ the resulting quotient space. The orbifold has singular locus given by the origin $\mathcal{M}_{sing} = \{(0, 0)\}$. Consider a loop γ_1 based at some $y \neq [(0, 0)] \in \mathcal{M}$ that circles the origin once. This loop lifts to a path $\widetilde{\gamma}_1$ in $\mathbb{R}^2$ from, say, $\widetilde{y}_1$ to $\widetilde{y}_2$, where $\{\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3\} \subset \mathbb{R}^2$ are the pre-images of y through the quotient map. Note however, that keeping the end-points of $\widetilde{\gamma}_1$ fixed, it can not be homotoped to a constant loop in $\mathbb{R}^2$. The same is true for the loop, γ_2 in $\mathcal{M}$, which loops twice about the origin. The loop, γ_3 which loops 3 times about the origin however, lifts to a concatenation of 3 paths in $\mathbb{R}^2$, one from $\widetilde{y}_1$ to $\widetilde{y}_2$, another from $\widetilde{y}_2$ to $\widetilde{y}_3$ and finally a last piece from $\widetilde{y}_3$ to $\widetilde{y}_1$; hence $\widetilde{\gamma}_3$ forms a loop in $\mathbb{R}^2$, which we can homotope to a constant loop. Note however, that $\mathcal{M}$ is in fact homeomorphic to $\mathbb{R}^2$ so $\pi_1(\mathcal{M})$ is trivial. Nonetheless, this doesn't quite capture the notion that the loops γ_i for $i = 1, 2, 3$ should be seen as distinct.

The problem in the example is fixed by considering the orbifold fundamental group, $\pi_1^{orb}(\mathcal{M})$. Our above example suggests $\pi_1^{orb}(\mathcal{M}) = \mathbb{Z}_3$. In fact, by similar arguments to those made in the example, we can find that for any orbifold chart (U, Γ, ϕ), where U is simply connected, we have $\pi_1^{orb}(\mathcal{U}) = \Gamma$ where $\mathcal{U} = \phi(U)$. Heuristically we can understand this by arguing that loops in $\mathcal{U}$ lift to paths between pre-images of a non-singular base-point. For such a base-point, we get $|\Gamma|$ pre-images. Each path connecting a different number of these pre-images will contribute a distinct loop in $\pi_1^{orb}(\mathcal{U})$. So we would expect $\pi_1^{orb}(\mathcal{U}) = \Gamma$. We make this precise as follows using what is known as the *pseudogroup of orbifolds*:

Definition 1.22 (Pseudogroup of Orbifolds)**.** Let $\mathcal{M}$ be an orbifold with $\mathcal{A} = \{(U_i, \Gamma_i, \phi_i)\}$ its atlas of charts. We define

$$U_{\mathcal{A}} = \bigsqcup_i U_i \tag{1.18}$$

$$\phi = \bigsqcup \phi_i : U_{\mathcal{A}} \to \mathcal{M} \tag{1.19}$$

so that if $x \in U_i$, then $\phi(x) = \phi_i(x)$. A *change of charts of* $\mathcal{A}$ is a diffeomorphism $h : V \to W$, where $V, W \subset U_{\mathcal{A}}$ are open sets, such that $\phi \circ h = \phi|_V$. In particular, notice how all chart injections in $\mathcal{L}_{\mathcal{M}}$ and all elements $\gamma \in \Gamma_i$ are changes of charts. The collection of all changes of charts generates what is known as a pseudogroup $\mathcal{H}_{\mathcal{A}}$ of local diffeomorphisms of $U_{\mathcal{A}}$ ([17]) and we have that ϕ induces a homeomorphism $U_{\mathcal{A}}/\mathcal{H}_{\mathcal{A}} \to \mathcal{M}$ We can use changes of charts as an equivalent definition for the compatibility of charts in an orbifold atlas. Taking the atlas $\mathcal{A}$ to be the maximal atlas of the orbifold $\mathcal{M}$, we obtain the *pseudogroup of the orbifold* $\mathcal{M}$, denoted by $\mathcal{H}_{\mathcal{M}}$, and we let $U_{\mathcal{M}}$ be the corresponding manifold on which this pseudogroup is defined i.e., $U_{\mathcal{M}}$ is the disjoint union of all the open sets in the charts contained by the maximal atlas.

The pseudogroup of an orbifold consists of a manifold, $U_{\mathcal{M}}$, and a collection of local diffeomorphisms of $U_{\mathcal{M}}$. $U_{\mathcal{M}}$ will play the role of $\mathcal{U}$ above, it is the space where we lift loops in our orbifold so that we can determine their homotopy type, ϕ plays the role of the quotient map, and the change of charts, roughly speaking, plays the role of the group elements in the previous examples.

Now we can use the pseudogroup of the orbifold, $\mathcal{H}_{\mathcal{M}}$, to define $\mathcal{H}_{\mathcal{M}}$-*loops* and a notion of homotopy for them that will allow us to define the orbifold fundamental group:

Definition 1.23. Let $\mathcal{M}$ be an orbifold and $\mathcal{H}_{\mathcal{M}}$ be its pseudogroup. A *$\mathcal{H}_{\mathcal{M}}$-loop based at $p \in \mathcal{M}$* consists of:

(i) A partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1\}$ of $[0;1]$.

(ii) for each $i = 1, \ldots, n$ a continuous path $c_i : [t_{i-1}; t_i] \to U_{\mathcal{M}}$

(iii) for each $i = 1, \ldots, n$ an element $h_i \in \mathcal{H}_{\mathcal{M}}$ defined around $c_i(t_i)$, such that for $i = 1, \ldots, n-1$ we have $h_i c_i(t_i) = c_{i+1}(t_i)$ and $c_1(0) = h_n c_n(1) = p$

A *subdivision* of an $\mathcal{H}_{\mathcal{M}}$-loop is a new $\mathcal{H}_{\mathcal{M}}$-loop obtained by adding points to our partition, taking restrictions of the c_i on the new intervals formed and taking $h_i = id$ at the new points. We say two $\mathcal{H}_{\mathcal{M}}$-loops are *equivalent* if they admit subdivisions (h_i, c_i) and (h'_i, c'_i) such that for each $i = 1, \ldots, n$ there exists $g_i \in \mathcal{H}_{\mathcal{M}}$ defined on a neighbourhood of c_i such that:

(a) $g_1 = id$ and $g_i c_i = c'_i$

(b) for $i = 1, \ldots, n-1$, $h'_i g_i$ and $g_{i+1} h_i$ have the same germ at $c_i(t_i)$, and $h'_n g_n$ and h_n have the same germ at $c_n(1)$

A *deformation* of a $\mathcal{H}_{\mathcal{M}}$-loop, (h_i, c_i) consists of deformations c_i^s of the paths c_i such that for all $s \in [0;1]$, (h_i, c_i^s) is a $\mathcal{H}_{\mathcal{M}}$-loop.

We say that two $\mathcal{H}_{\mathcal{M}}$-loops are in *the same homotopy class* if they can be obtained from each other by a finite sequence of subdivisions, equivalences and deformations.

We define the *orbifold fundamental group based at* p, $\pi_1^{orb}(\mathcal{M}, p)$, to be the set of homotopy classes of $\mathcal{H}_{\mathcal{M}}$-loops based at p, together with concatenation. If $\mathcal{M}$ is path-connected, then the orbifold fundamental groups based at different points are isomorphic and we talk only about the orbifold fundamental group and denote it by $\pi_1^{orb}(\mathcal{M})$.

We also have a surjective group homomorphism $\pi_1^{orb}(\mathcal{M}) \to \pi_1(\mathcal{M})$, which "forgets the orbifold structure" ([17]). Moreover, using $\mathcal{H}_{\mathcal{M}}$-loops we can easily see how to formalise our previous informal argument about "lifting loops to U" which we used to argue that, for an orbifold chart (U, Γ, ϕ), with simply connected U, we should have $\pi_1^{orb}(\mathcal{U}) = \Gamma$, where $\mathcal{U} = \phi(U)$. For the orbifold fundamental group, we have the following version of the Seifert-van Kampen Theorem:

Theorem 1.24. *(Seifert-van Kampen Theorem for orbifold fundamental groups, [17, Thm. 2.2.3]) Let $\mathcal{M}$ be a connected orbifold, $\mathcal{U}$ and $\mathcal{V}$ be open connected subsets such that $\mathcal{M} = \mathcal{U} \cup \mathcal{V}$ as topological spaces and $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ is connected. Let $\iota_{\mathcal{U}\mathcal{M}} : \mathcal{U} \to \mathcal{M}$ and $\iota_{\mathcal{V}\mathcal{M}} : \mathcal{V} \to \mathcal{M}$ denote the inclusions of $\mathcal{U}$ and $\mathcal{V}$ into $\mathcal{M}$, and, $\iota^*_{\mathcal{U}\mathcal{M}}$ and $\iota^*_{\mathcal{V}\mathcal{M}}$ denote the induced homomorphisms on the respective orbifold fundamental groups (for example, $\iota^*_{\mathcal{U}\mathcal{M}} : \pi_1^{orb}(\mathcal{U}) \to \pi_1^{orb}(\mathcal{M})$). Then we have:*

$$\pi_1^{orb}(\mathcal{M}) \cong \pi_1^{orb}(\mathcal{U}) *_{\pi_1^{orb}(\mathcal{W})} \pi_1^{orb}(\mathcal{V}) \tag{1.20}$$

where the right-hand side is the amalgamated product, i.e., the quotient of the free product $\pi_1^{orb}(\mathcal{U}) * \pi_1^{orb}(\mathcal{V})$ by the normal subgroup generated by $\{\iota^*_{\mathcal{W}\mathcal{U}}(\gamma)\iota^*_{\mathcal{W}\mathcal{V}}(\gamma) | \gamma \in \pi_1^{orb}(\mathcal{W})\}$, where $\iota^*_{\mathcal{W}\mathcal{U}}$ and $\iota^*_{\mathcal{W}\mathcal{V}}$ denote, as above, the homomorphisms induced by the inclusions of the respective sets.

Similar to the fundamental group, one can also consider the orbifold fundamental group as the group of *deck transformations* of the *orbifold universal cover* ([17]).

Definition 1.25. Let $\mathcal{M}$ be a connected orbifold. An *orbifold covering* of $\mathcal{M}$ is a pair $(\widetilde{\mathcal{M}}, \rho)$ where $\widetilde{\mathcal{M}}$ is another orbifold and $\rho : \widetilde{\mathcal{M}} \to \mathcal{M}$ is a surjective smooth map such that:

(i) For every $p \in \mathcal{M}$, there is an orbifold chart (U, Γ, ϕ) such that $\rho^{-1}(\mathcal{U})$ is a disjoint union of open subsets $\mathcal{V}_i \subset \widetilde{\mathcal{M}}$,

(ii) Each $\mathcal{V}_i$ admits an orbifold chart of the form (U, Γ_i, ϕ_i) with Γ_i a subgroup of Γ and ρ_U is the identity.

The *orbifold universal covering* of $\mathcal{M}$ is an orbifold covering $(\widehat{\mathcal{M}}, \widehat{\rho})$ such that given any other orbifold covering, $(\widetilde{\mathcal{M}}, \rho)$, and points $\widehat{x} \in \widehat{\mathcal{M}}$, $\widetilde{x} \in \widetilde{\mathcal{M}}$ mapping to the same point in $\mathcal{M}$ under the covering maps, $\widehat{\rho}(\widehat{x}) = \rho(\widetilde{x}) = x \in \mathcal{M}$, then there exists a unique covering $\pi : \widehat{\mathcal{M}} \to \widetilde{\mathcal{M}}$ such that $\rho \circ \pi = \widehat{\rho}$ and $\pi(\widehat{x}) = \widetilde{x}$.

A *deck transformation* is a diffeomorphism $f : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ such that $\widehat{\rho} \circ f = \widehat{\rho}$. We denote the group of deck transformations by $Aut(\widehat{\rho})$.

We also mention that the orbifold universal cover always exists ([17, Theorem 2.3.4]) and that it is unique up to covering isomorphism. Moreover, we collect some of the important properties of the orbifold universal cover in the proposition below:

Proposition 1.26 ([17, Proposition 2.3.5])**.** *Let* $\mathcal{M}$ *be a connected orbifold and let* $\rho : \widehat{\mathcal{M}} \to \mathcal{M}$ *denote its orbifold universal cover. Then*

(i) $Aut(\rho)$ *is isomorphic to* $\pi_1^{orb}(\mathcal{M})$*.*

(ii) ρ *induces a diffeomorphism* $\widehat{\mathcal{M}}/Aut(\rho) \cong \mathcal{M}$

(iii) $\pi_1^{orb}(\widehat{\mathcal{M}}) = 1$ *i.e.* $\widehat{\mathcal{M}}$ *is orbifold simply connected*

We note that item (iii) above implies that the orbifold universal cover is also simply connected as a topological space, given the surjective group homomorphism $\pi_1^{orb}(\widehat{\mathcal{M}}) \to \pi_1(\widehat{\mathcal{M}})$.

1.3 $K3$ Surfaces and Orbifold $K3$ Surfaces

In view of our goal of looking at the Twisted Connected Sum construction in-depth, in this section, we shall review some important facts about $K3$ surfaces and their moduli spaces. Most of the results on smooth $K3$ surfaces are taken from [9, Chapter VIII] and [55]. After that, we will introduce orbifold $K3$ surfaces and present some of the differences and similarities from their smooth cousins. Firstly, we note that a lot of results concerning $K3$ surfaces are derived from considering the second cohomology, $H^2(K3;\mathbb{Z})$, together with the intersection form. For complex surfaces, the second cohomology together with the intersection form is a lattice, and as such we will begin with a reminder of the important lattice theory facts we will need before defining what a $K3$ surface is:

A *lattice* is a free abelian group L of finite rank together with a symmetric bilinear form $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}$. The *rank* of a lattice, $rk(L)$, is the same as the rank of the underlying group. We say L is *even* if $x^2 = \langle x, x\rangle \in 2\mathbb{Z}$ $\forall x \in L$. Let $\{x_1, \dots, x_n\}$ be a basis for L, then the *Gram matrix* of L is $G_L = (\langle x_i, x_j\rangle)_{1\leq i,j\leq n}$. We say L is *unimodular* if $|det(G_L)| = 1$ and *non-degenerate* if $det(G_L) \neq 0$. Note that $det(G_L)$ is independent of the chosen basis. A *sublattice* of L is a subgroup $N \subset L$ together with the restriction of the bilinear form of L to N.

Given 2 lattices, $(L, \langle\cdot,\cdot\rangle_L)$ and $(M, \langle\cdot,\cdot\rangle_M)$, a *lattice morphism* (or *lattice isometry*) is a $\mathbb{Z}$-linear map $\phi : L \to M$ such that $\langle x, y\rangle_L = \langle\phi(x), \phi(y)\rangle_M$ for all $x, y \in L$. We say a lattice morphism is a *lattice embedding* if it is injective; if it is bijective we say it is a *lattice isomorphism.* The *direct sum* of 2 lattices is the direct sum of the underlying groups, $L\oplus M$, together with the bilinear form, $\langle(x, y), (x', y')\rangle_\oplus = \langle x, x'\rangle_L + \langle x', y'\rangle_M$ for all $x, x' \in L$ and $y, y' \in M$.

Definition 1.27. Let $(L, \langle \cdot, \cdot \rangle_L)$ be a lattice. A sublattice $N \subset L$ is *primitive* if the quotient group, L/N, has no torsion i.e., if there are no elements (besides the zero element) that are of finite order. A lattice N is *primitively embedded* in L, if there exists an embedding $\phi : N \to L$ such that $\phi(N)$ is a primitive sublattice. An element, $x \in L$, is *primitive* if the sublattice generated by x, $\{kx | k \in \mathbb{Z}\}$, is primitive. Note that this is equivalent with the requirement that $\forall n > 1$ and $y \in L$ we have $x \neq ny$

The *dual* of a lattice L is given by $L^* = Hom_{\mathbb{Z}}(L, \mathbb{Z}) = \{\psi : L \to \mathbb{Z} | \psi$ is $\mathbb{Z}$-linear$\}$. Suppose L is non-degenerate and let G_L be the Gram matrix of L with respect to a basis $\{x_1, \ldots, x_n\}$. With respect to the dual basis we have that $G_{L^*} = G_L^{-1}$. Note that in general the dual bilinear form, which has matrix representation G_{L^*} takes values in $\mathbb{Q}$ rather than $\mathbb{Z}$; however, if L is unimodular, then all the entries of G_{L^*} are integers so we get a well-defined lattice. We also have the *correlation map*, $\phi : L \to L^*$ given by $x \mapsto \langle \cdot, x \rangle_L$. The *discriminant group* of L is the quotient $L^*/\phi(L)$.

Lemma 1.28 ([51])**.** *Let L be a lattice and $\{x_1, \ldots, x_n\}$ a basis of L. The discriminant group of L is a finite group of order $|det(G_L)|$. Moreover, the minimal number of generators of the discriminant group $l(L)$ satisfies $l(L) \leq rk(L)$. In particular, if L is unimodular then the discriminant group of L is trivial.*

If L is a lattice, then the tensor product $L_{\mathbb{R}} = L \otimes \mathbb{R}$ is a real vector space and we extend the bilinear form on L to a $\mathbb{R}$-bilinear form on $L_{\mathbb{R}}$, $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. The *signature* of the lattice is the signature of $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. Now we can return to $K3$ surfaces.

Definition 1.29. A compact complex surface, S, is called a *(smooth) $K3$ surface* if it is simply connected and has topologically trivial canonical bundle, K_S.

One could alternatively define a smooth $K3$ surface, S, as a smooth compact complex surface with trivial canonical bundle and with vanishing Hodge number $h^{1,0}(S) = 0$. The underlying real manifold of any $K3$ surface is of a fixed diffeomorphism type so in particular, the intersection form on the second cohomology and the Betti numbers are the same for all $K3$ surfaces. A great deal of information about a $K3$ surface can be obtained purely algebraically by just considering the lattice $H^2(S; \mathbb{Z})$ (see, for example, the results below concerning various moduli spaces of $K3$ surfaces). The isomorphism type of this lattice is fixed and we call it the $K3$ *lattice*; it is given by:

$$L = 3H \oplus 2(-E_8) \tag{1.21}$$

where H is the hyperbolic plane lattice with bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.22}$$

and $-E_8$ is the root lattice of E_8 together with the negative of the usual bilinear form.

Theorem 1.30. *If S is a $K3$ surface, then the Hodge numbers of S are given by:*

$$\begin{aligned} h^{0,0}(S) &= h^{2,0}(S) = 1 \\ h^{1,0}(S) &= 0 \\ h^{1,1}(S) &= 20 \end{aligned} \tag{1.23}$$

Moreover, the second integral cohomology, $H^2(S, \mathbb{Z})$ together with the intersection form is an even unimodular lattice of signature $(3, 19)$. Up to isometry, the only lattice having these properties is the $K3$ lattice, L.

Remark 1.31. Using the result concerning the Hodge numbers of a $K3$ surface from the previous theorem, together with complex conjugation and Serre duality, we can fully determine the Hodge diamond of all $K3$ surfaces and hence their Betti numbers:

$$\begin{aligned} b^0(S) &= 1 \\ b^1(S) &= 0 \\ b^2(S) &= 22 \end{aligned} \tag{1.24}$$

with the remaining Betti numbers being given by Poincaré duality.

A $K3$ surface, S, together with a lattice isometry $\phi : H^2(S, \mathbb{Z}) \to L$ is called a *marked $K3$ surface*. We say two marked $K3$ surfaces, (S, ϕ) and (S', ϕ'), are *isomorphic* if there exists a biholomorphic map $f : S \to S'$ such that $\phi \circ f^* = \phi'$ where $f^* : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ denotes the pullback.

Example 1.32. Every smooth quartic, S, in $\mathbb{CP}^3$ is a $K3$ surface. To see this note that the canonical bundle of S is trivial by the adjunction formula: if $S \subset \mathbb{CP}^n$ is a smooth hypersurface, then the canonical bundle of S is given by $K_S = (K_{\mathbb{CP}^n} \otimes L_S)|_S$ where L_S is the line bundle associated to the divisor S. Moreover, Lefschetz's Hyperplane Theorem tells us that any hypersurface with a positive associated line bundle in $\mathbb{CP}^3$ is simply connected. As a particular example, consider the Fermat quartic:

$$\{[x:y:z:w] \in \mathbb{CP}^3 | x^4 + y^4 + z^4 + w^4 = 0\} \tag{1.25}$$

It can also be shown that, for example, the smooth intersection of a quadric and a cubic in $\mathbb{CP}^4$ is also a $K3$ surface, where we note that the adjunction formula can be generalised to: if $S \subset \mathbb{CP}^n$ is a transverse intersection of divisors $D_1, \ldots, D_r$, the canonical bundle is given by $K_S = (K_{\mathbb{CP}^n} \otimes L_{D_1} \otimes \cdots \otimes L_{D_r})|_S$.

Another important concept for us will be that of a *hyper-Kähler* manifold, which we define below:

Definition 1.33. Let (M, g) be a $4m$ dimensional Riemannian manifold and (I, J, K) a triple of integrable complex structures on M. We say that $(M; I, J, K)$ is a *hyper-Kähler* manifold if $IJ = K$ and g is Kähler with respect to each of the I, J and K. Such a manifold has holonomy contained in $Sp(m)$. Moreover, for any $a = (a_1, a_2, a_3) \in S^2$, g is a Kähler metric with respect to any linear combination complex structure $a_1 I + a_2 J + a_3 K$ so that (M, g) admits an S^2 family of complex structures.

We will also sometimes denote a hyper-Kähler structure by the triple of Kähler forms associated to (I, J, K), $(\omega_I, \omega_J, \omega_K)$, since as proved in [35, Section 10.1.2] we know that the triple $(\omega_I, \omega_J, \omega_K)$ determines a hyper-Kähler structure.

Remark *1.34.* All (smooth) $K3$ surfaces admit a Kähler metric ([60]; moreover, the same proof shows that this is also true for orbifold $K3$ surfaces) and by Yau's Theorem there exists a unique Ricci-Flat Kähler metric in each Kähler class. Given such a Ricci-Flat metric, one can define a hyper-Kähler structure on every $K3$ surface ([9, Chapter VIII, Section 13]). We will later prove that the same is true for every $K3$ orbifold.

We are interested in the various moduli spaces whose points represent $K3$ surfaces with some extra structure. The first one is the moduli space of marked $K3$ surfaces $\mathcal{M}_{K3}^m$ which is the set of all marked $K3$ surfaces modulo isomorphism.

Now on any marked $K3$ surfaces, (S, ϕ), we have a global holomorphic $(2,0)$-form, $\Omega = \kappa_J + i\kappa_K$, where κ_J and κ_K are real 2-forms; they are also Kähler forms with respect to the additional complex structures, J and K, which make S hyper-Kähler. The form is unique up to multiplication by a complex constant. Let L be the $K3$ lattice, $L_{\mathbb{C}} = L \otimes \mathbb{C}$ and $\phi_{\mathbb{C}} : H^2(S, \mathbb{C}) \to$

$L_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of ϕ. Moreover, let $\mathbb{P}(L_{\mathbb{C}})$ denote the projective space of complex lines in $L_{\mathbb{C}}$. Consider the complex line spanned by $\phi_{\mathbb{C}}([\Omega])$; this defines a point $p(S,\phi) \in \mathbb{P}_{\mathbb{C}}$. We call $p(S,\phi)$ the *period point* of (S,ϕ). This assignment defines a map $p : \mathcal{M}^m_{K3} \to \mathbb{P}(L_{\mathbb{C}})$ called *the period map.*

Since κ_J and κ_K are Kähler forms, then they are positive and we have that $\langle [\Omega], [\Omega] \rangle = 0$ and $\langle [\Omega], [\bar{\Omega}] \rangle > 0$ where $\langle \cdot, \cdot \rangle$ denotes the extension of the intersection form to $H^2(S,\mathbb{C})$. Now we look for points in the complexified $K3$ lattice, $x \in L_{\mathbb{C}}$, such that the complex lines they span, $l_x \in \mathbb{P}(L_{\mathbb{C}})$, satisfy these properties:

$$D = \{ l_x \in \mathbb{P}(L_{\mathbb{C}}) | \langle x, x \rangle = 0, \langle x, \bar{x} \rangle > 0 \} \tag{1.26}$$

we call D, the *period domain* and reducing the target domain of the period map so that it becomes a map $p : \mathcal{M}^m_{K3} \to D$ we get an important result on the structure of $\mathcal{M}^m_{K3}$:

Theorem 1.35. *The period map $p : \mathcal{M}^m_{K3} \to D$ is surjective.*

Another important ingredient when it comes to understanding $\mathcal{M}^m_{K3}$ are the so-called *Torelli Theorems*, which are answers to the following question: What does the period point of a marked $K3$ surface tell us about the complex structure of S? One of these theorems is the following:

Theorem 1.36 (Weak Torelli theorem)**.** *If (S,ϕ) and (S',ϕ') are two marked $K3$ surfaces with the same period point, then there exists a biholomorphic map $f : S \to S'$*

Note that, this is not enough for us to conclude that (S,ϕ) and (S',ϕ') represent the same point in $\mathcal{M}^m_{K3}$, since they need not be isomorphic; in particular, the pullback induced by the biholomorphism need not send one marking to the other. To get more information on this moduli space, we need a few more definitions.

Definition 1.37. Let S and S' be $K3$ surfaces. A lattice isometry $\psi : H^2(S;\mathbb{Z}) \to H^2(S';\mathbb{Z})$ is a *Hodge-isometry* if its $\mathbb{C}$-linear extension preserves the Hodge decomposition $H^2(S;\mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)$. We say $x \in H^2(S;\mathbb{Z})$ is *effective* if there is an effective divisor D of S with $c_1(\mathcal{O}_S(D)) = x$. An effective class x is *nodal* if $x^2 = -2$. The *positive cone* of S is the connected component of $\{x \in H^{1,1}(S) | x^2 > 0\}$ which contains a Kähler class. We say a Hodge-isometry is *effective* if it maps the positive cone of S to the positive cone of S' and if it maps effective classes in $H^2(S;\mathbb{Z})$ to effective classes in $H^2(S';\mathbb{Z})$.

Then we can refine the weak Torelli theorem to:

Theorem 1.38. *Let S and S' be two unmarked $K3$ surfaces. If there exists a Hodge isometry $\psi : H^2(S,\mathbb{Z}) \to H^2(S';\mathbb{Z})$, then S and S' are isomorphic.*

Moreover, we get the global Torelli theorem:

Theorem 1.39 (Global Torelli Theorem, see [55])**.** *Let S and S' be two unmarked $K3$ surfaces. If there exists an effective Hodge isometry, $\psi : H^2(S;\mathbb{Z}) \to H^2(S';\mathbb{Z})$, then ψ is the pull-back of a unique biholomorphism $f : S' \to S$. The converse also holds so that there is a bijection between effective Hodge isometries and biholomorphisms.*

Remark 1.40*.* The period map as defined above is a local isomorphism of complex manifolds. In particular, this shows that $\mathcal{M}^m_{K3}$ is locally a complex manifold of dimension 20; however, it is not Hausdorff ([9], pp. 334 – 338).

Lastly, we think of two more closely related moduli spaces, which will be useful later on when we talk about orbifold $K3$ surfaces.

A marked pair is a marked $K3$ surfaces (S, ϕ) together with a Kähler class, $y \in H^{1,1}(S;\mathbb{R})$, which we write as (S, ϕ, y). Two marked pairs, (S_1, ϕ_1, y_1) and (S_2, ϕ_2, y_2) are isomorphic if there exists a biholomorphic map $f : S_1 \to S_2$ such that $\phi_1 \circ f^* = \phi_2$ and $f^*\phi_2 = \phi_1$. The moduli space of marked pairs, denoted by $\mathcal{M}^{mp}_{K3}$ is the set of all marked pairs modulo isomorphisms. Then, if D denotes the period domain and L the $K3$ lattice (as above), we can define the *refined period map*:

$$\begin{aligned} p' : \mathcal{M}^{mp}_{K3} &\to D \times L_{\mathbb{R}} \\ (S, \phi, y) &\to (p(S, \phi), \phi_{\mathbb{R}}(y)) \end{aligned} \tag{1.27}$$

where $\phi_{\mathbb{R}}$ denotes the $\mathbb{R}$-linear extension of the marking ϕ.

Theorem 1.41. *p' as defined above is a bijection between $\mathcal{M}^{mp}_{K3}$ and the following subset of $D \times L_{\mathbb{R}}$:*

$$\begin{aligned} D' = \{(l_x, y) \in D \times L_{\mathbb{R}} | \langle x, y\rangle = 0, y^2 > 0, \\ \langle y, d\rangle \neq 0, \forall d \in L \text{ with } d^2 = -2, \langle x, d\rangle = 0\} \end{aligned} \tag{1.28}$$

Moreover, $\mathcal{M}^{mp}_{K3}$ is a real analytic Hausdorff manifold of dimension 60*.*

The last moduli space we look at is the space of hyper-Kähler metrics on $K3$ surfaces together with a marking, $\mathcal{M}^{hk}_{K3}$. The following lemma shows that $\mathcal{M}^{hk}_{K3}$ is closely related to $\mathcal{M}^{mp}_{K3}$:

Lemma 1.42. *Let M be the real 4-manifold underlying a $K3$ surface. There is a one-to-one correspondence between marked pairs and hyper-Kähler metrics on M together with a choice of parallel complex structure and a marking.*

For a proof of the lemma one can check [55, Chapter 3.3]. Lastly to describe the moduli space $\mathcal{M}_{K3}^{hk}$ in terms of lattice data, note that it is diffeomorphic to the so-called *hyper-Kähler period domain*:

$$\begin{aligned} D^{hk} = \{(x, y, z) \in L_{\mathbb{R}}^3 | x^2 = y^2 = z^2 > 0, \langle x, y\rangle = \langle x, z\rangle = \langle y, z\rangle = 0, \\ \nexists d \in L \text{ with } d^2 = -2 \text{ and } \langle x, d, \rangle = \langle y, d\rangle = \langle z, d\rangle = 0\} \end{aligned} \tag{1.29}$$

We now define what we mean by an *orbifold $K3$ surface* and a crepant(or minimal) resolution of singularities:

Definition 1.43. We say that $\mathcal{X}$ is an *orbifold $K3$ surface*, if it is a complex 2-orbifold that is simply connected and has trivial canonical orbibundle $K_{\mathcal{X}}$. A surjective map, $\pi : X \to \mathcal{X}$ is called a *resolution of singularities* if X is a smooth surface, and, away from $\pi^{-1}(\mathcal{X}_{sing})$, π is an isomorphism of complex surfaces. A resolution of singularities $\pi : X \to \mathcal{X}$ is called a *crepant (or minimal) resolution of singularities* if $K_X = \pi^* K_{\mathcal{X}}$.

Various examples of orbifold $K3$ surfaces can be found; some of the most readily available are those arising as hypersurfaces in certain weighted projective spaces (section 4.1 below). In the case of $K3$ surfaces crepant resolutions should be thought of as finite sequences of blow-ups ([55] and [34, Section 6.4.1]); this gives us a way to construct orbifold $K3$ surfaces with any prescribed singular locus, which we will explain towards the end of this section after introducing ADE singularities. We also note the following important property of resolutions:

Theorem 1.44 ([43, Thm. 7.8])**.** *Let $\mathcal{X}$ be an orbifold and let $\pi : X \to \mathcal{X}$ be a resolution of singularities. Then $\pi_1(X) \cong \pi_1(\mathcal{X})$ i.e., X and $\mathcal{X}$, as topological spaces, have isomorphic fundamental groups.*

Remark 1.45. Note first that any orbifold surface admits a unique crepant resolution ([34, Chapter 6.4.1]). If $\mathcal{X}$ is an orbifold $K3$ surface consider its crepant resolution $\pi : X \to \mathcal{X}$. Then by **Theorem 1.44** we know that X is simply connected. Moreover, $K_{\mathcal{X}}$ is trivial so K_X is trivial since the resolution is crepant. Hence, the crepant resolution of an orbifold $K3$ surface is a (smooth) $K3$ surface. This will turn out to be very important when considering generalisations of results about $K3$ surfaces to equivalent statements about orbifold $K3$ surfaces.

As we shall see, one of the crucial ingredients for the Twisted Connected Sum construction is a pair of $K3$ surfaces satisfying what is known as the *Donaldson matching condition*, or simply, the *matching condition*. We shall discuss the generalisation of the matching condition to orbifolds in section 2.3 below; nonetheless, we will include in this section the background results on orbifold $K3$ surfaces which we need to make sense of "the orbifold matching condition", in particular the existence of orbifold hyper-Kähler structures on orbifolds $K3$ surfaces and the orbifold global Torelli Theorem. We will now prove that every orbifold $K3$ surface admits a hyper-Kähler structure:

Proposition 1.46. *Let $\mathcal{X}$ be a $K3$ orbifold. Then there exists a hyper-Kähler structure, (I, J, K), on $\mathcal{X}$*

Proof. The argument is due to Hitchin and can be found in [9]; we adapt it to orbifolds below:

Recall that any $K3$ orbifold admits a Kähler metric (cf. **Remark 1.34**). Moreover, by Yau's Theorem for orbifolds, there exists a unique Ricci-flat Kähler metric in each Kähler class. Given such a Ricci-flat Kähler metric, say g, with respect to the underlying complex structure I, and associated global holomorphic $(2,0)$-form, Ω, define, for all $p \in \mathcal{X}_{reg}$, $J_1 : T_p\mathcal{X} \to T_p\mathcal{X}$ by $g(J_1 v, w) = Re\Omega(v, w)$, $\forall v, w \in T_p\mathcal{X}$. If $p \in \mathcal{X}_{sing}$, let (U, Γ, ϕ) be an orbifold chart about p and define $J_1 : T_0U \to T_0U$ by $g(J_1\phi_* v, \phi_* w) = Re\Omega(\phi_* v, \phi_* w)$, $\forall v, w \in T_0U$. Since ϕ, g, and Ω are Γ-invariant, this is well-defined and it induces a well-defined map $J_1 : T_p\mathcal{X} \to T_p\mathcal{X}$. Then we have that

$$g(IJ_1v, w) = -g(J_1v, Iw) = -Re\Omega(v, Iw) = -Re\Omega(Iv, w) = -g(J_1Iv, w) \tag{1.30}$$

so that $IJ_1 = -J_1I$. Furthermore, J_1 is skew-adjoint so J_1^2 is self-adjoint with non-positive eigenvalues and there is at least one non-zero eigenvalue, say $-\lambda^2$. Let V be the eigenspace of this eigenvalue and set $J = \lambda J_1$ so that $J^2 = -id$ on V. Now J^2 and I commute so I also preserves V. This implies that $dim_{\mathbb{R}} V \geq 4$ so $V = T_p\mathcal{X}$. Hence J is a well-defined almost complex structure so that $\{I, J, IJ\}$ is an almost hyper-Kähler structure. Furthermore, $g(Jv, Jw) = -g(v, J^2w) = g(v, w)$ and J is parallel so that it is, in particular, integrable and $\{I, J, IJ\}$ gives a well-defined hyper-Kähler structure on $\mathcal{X}$, which we denote by (I, J, K). We denote by κ_I the Kähler form and $\kappa_J + i\kappa_K$ the holomorphic volume form of this metric, where κ_I, κ_J and κ_K are real 2-forms on $\mathcal{X}$. □

We now introduce the root lattice and *the singularity number* for orbifold $K3$ surfaces and then discuss ADE singularities:

Definition 1.47. Let $\mathcal{X}$ be an orbifold $K3$ surface and $\pi : X \to \mathcal{X}$ its minimal resolution. Let $\delta_1, \dots, \delta_k \in H^{1,1}(X) \cap H^2(X;\mathbb{Z})$ be the classes of all the (-2)-curves contracted by π ($\langle \delta_i, \delta_i \rangle = -2$ for all $i = 1, \dots, k$). The *root system* of $\mathcal{X}$ is:

$$R(\mathcal{X}) = \left\{ \delta = \sum_{i=1}^{k} a_i \delta_i \in H^2(X;\mathbb{Z}) \;\middle|\; a_i \in \mathbb{Z}, \langle \delta, \delta \rangle = -2 \right\} \tag{1.31}$$

We also set $I^2(\mathcal{X})$ to be the set of all classes in $H^2(X;\mathbb{Z})$ which are orthogonal to $R(\mathcal{X})$, i.e., those classes α for which we have $\langle \alpha, \delta \rangle = 0$, $\forall \delta \in R(\mathcal{X})$. We call the sublattice generated by $\mathcal{R}(\mathcal{X})$ the *root lattice*, $\mathcal{R}(\mathcal{X})^L$. We call the rank of the root lattice the *singularity number of* $\mathcal{X}$ and we denote it by $sing(\mathcal{X})$:

$$sing(\mathcal{X}) = rk(\mathcal{R}(\mathcal{X})^L) \tag{1.32}$$

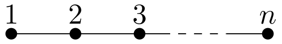
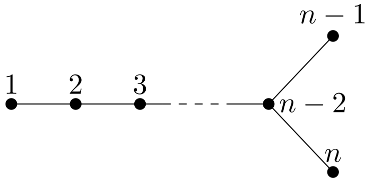
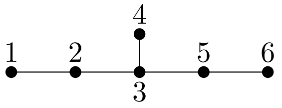
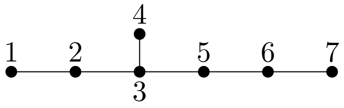
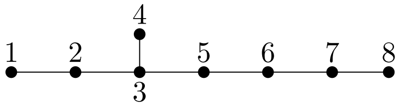
.

We now turn our attention to ADE singularities. John McKay pointed out that there is a $1-1$ correspondence between quotient singularities of the form $\mathbb{C}^2/\Gamma$, for Γ a finite subgroup of $SU(2)$, and Dynkin diagrams of type A_n, $n \geq 1$, D_n, $n \geq 4$, E_6, E_7 and E_8. These diagrams appear in the classification of Lie groups, each corresponding to a simple, compact Lie group, and the above ones are precisely the diagrams containing no double or triple edges. This correspondence is summarised in the table below:

ADE Correspondence		
Subgroup of $SU(2)$, Γ	*Presentation*	*Dynkin Diagram*
Cyclic group, order n+1	$\langle x \vert x^n = 1\rangle$	A_n, $n \geq 1$
Binary dihedral group with $4(n-2)$ elements	$\langle x, y \vert x^{2n} = 1, x^n = y^2$ $, y^{-1}xy = x^{-1}\rangle$	D_n, $n \geq 4$
Binary tetrahedral group	$\langle x, y \vert (xy)^2 = x^3 = y^3\rangle$	E_6
Binary octahedral group	$\langle x, y \vert (xy)^2 = x^3 = y^4\rangle$	E_7
Binary icosahedral group	$\langle x, y \vert (xy)^2 = x^3 = y^5\rangle$	E_8

When we want to talk about the type of an orbifold singularity of a complex surface we will be referring to it by the Dynkin diagram, rather than by the corresponding subgroup of $SU(2)$ (e.g. if we consider a singularity given by the quotient of $\mathbb{C}^2$ by an action induced by the cyclic group of order 2, we will call such a singularity an A_1 singularity) and we will use Dynkin diagrams to compute the root lattices for orbifold $K3$ surfaces (see the next paragraph). For more details on Dynkin diagrams and their correspondence to ADE singularities, one can check [55, Chapter 2] or [34, Chapter 6.4.1].

If we consider the unique crepant resolution of $\mathbb{C}^2/\Gamma$, then the pre-image of the singular locus, $\pi^{-1}(0)$, consists precisely of a union of finitely many rational curves of self-intersection -2, which correspond to the vertices of the Dynkin diagram associated to G; any two such rational curves intersect transversely at a point whenever there is an edge between the corresponding vertices of the Dynkin diagram. In fact, if we denote by $\mathcal{M}$ the crepant resolution of $\mathbb{C}^2/\Gamma$, then the rational curves above form a basis for the homology group $H_2(\mathcal{M};\mathbb{Z})$ and the negative of the Cartan matrix of the associated Dynkin diagram represents the intersection form (i.e., a square matrix, whose size is given by the number of vertices, with entries a_{ij}, where $a_{ii} = -2$ and $a_{ij} = 1 = a_{ji}$ if and only if there is an edge between vertex i and vertex j) [34, Chapter 6.4.1]. Below is a table of the relevant diagrams from which we can derive the Cartan matrix as explained for ADE singularities:

Singularities and Dynkin Diagrams	
Singularity type	*Dynkin Diagram*
A_n, $n \geq 1$	
D_n, $n \geq 4$	
E_6	
E_7	
E_8	

Hence if we consider a complex orbifold, $\mathcal{X}$, with Γ_p a subgroup of $SU(2)$ for all p in $\mathcal{X}$ then the singular locus consists of isolated points and the above shows that the unique crepant resolution of $\mathcal{X}$, $\pi : X \to \mathcal{X}$ satisfies that $\pi^{-1}(\mathcal{X}_{sing}) = \cup(\text{rational } -2\text{-curves})$. We also note that we can view this process in reverse, starting with a complex surface X we blowdown rational curves to obtain an orbifold; i.e., locally if $\mathcal{M}$ is the unique crepant resolution of $\mathbb{C}^2/\Gamma$, then we can find rational -2-curves on $\mathcal{M}$ which we can blowdown to a point to obtain an orbifold singularity $\mathbb{C}^2/\Gamma$. This will form our basis for how we will construct orbifold $K3$ surfaces, but first we need to introduce some notation:

Remark 1.48. We shall use the following notation to denote the root lattice of an orbifold $K3$:

$$\mathcal{R}(\mathcal{X})^L = \bigoplus_{i=1}^{n} k_i Q_i \tag{1.33}$$

where n denotes the number of distinct types of singular points, Q_i denotes the different types of singular points, and k_i denotes the number of singular points of type Q_i. In this notation we have that $\sum_{i=1}^n k_i$ is equal to the total number of singular points. We shall say that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are orbifold $K3$ surfaces, they *have the same singular locus* if $\mathcal{R}(\mathcal{X}_1)^L = \mathcal{R}(\mathcal{X}_2)^L$ Now note that for an orbifold $K3$ surface, $\mathcal{X}$, with singularities of type $\{Q_{n_i}\}$, where $Q = A$, D, or E we have:

$$h^{1,1}(\mathcal{X}) = 20 - \sum n_i \tag{1.34}$$

or equivalently, in terms of the root lattice, $\mathcal{R}(\mathcal{X})^L = \bigoplus_{i=1}^n k_i Q_i$, we have:

$$h^{1,1}(\mathcal{X}) = 20 - \sum_{i=1}^{n} i k_i \tag{1.35}$$

which tells us that $sing(\mathcal{X}) = \sum_{i=1}^n i k_i \leq 19$.

Now, if we are given $\mathcal{R}(\mathcal{X})^L = \bigoplus_{i=1}^n k_i Q_i$ such that $\sum_{i=1}^n i k_i \leq 19$ and $Q_i \in \{A_n, D_m, E_k | n \geq 1, m \geq 4, k = 6, 7, 8\}$, then we can find an orbifold $K3$ surface whose root lattice is exactly $\mathcal{R}(\mathcal{X})^L$ in the following way:

Suppose, for simplicity, that we want an orbifold $K3$ surface whose singular locus consists of a single point with a singularity of type Q:

Start with a smooth $K3$ surface, X, whose *Picard number* (the rank over $\mathbb{Z}$ of $H^{1,1}(X) \cap H^2(X;\mathbb{Z})$) is at least equal to the number of vertices of the Dynkin diagram associated to Q and let the number of vertices be n. Further suppose that we can choose lattice elements, $v_1, \ldots, v_n$ in

$H^{1,1}(X) \cap H^2(X;\mathbb{Z})$, *the Picard lattice*, such that $v_i^2 = -2$ and $\langle v_i, v_j\rangle = 1$ whenever there is an edge in the Dynkin diagram between vertex i and j and 0 otherwise. Now consider $\pi : X \to \mathcal{X}$ to be the blowdown map which contracts the -2-curves associated to the v_i to a unique point i.e., if e_i are the chosen -2-curves, then $\pi^{-1}(0) = \cup_i e_i$. We get an orbifold $K3$, whose singular locus consists precisely of a single point of type Q.

Now, if we associate to the root lattice the union of Dynkin diagrams corresponding to each desired singularity, and we choose a $K3$ surface with Picard number greater than or equal to $\sum_{i=1}^n ik_i = N$ labeling all the vertices with numbers from 1 to N, and for which we can choose N vectors in the Picard lattice, $v_1, \ldots, v_N$ such that $v_i^2 = -2$ and $\langle v_i, v_j, \rangle = 1$ if there is an edge between vertex i and vertex j and 0 otherwise. Then let $\pi : X \to \mathcal{X}$ be the map contracting -2-curves associated to the v_i, e_i, to $\sum_{i=1}^n k_i$ points such that curves e_i and e_j are contracted to the same point if and only if $e_i \cap e_j \neq \emptyset$. Then we get an orbifold $K3$ surface, $\mathcal{X}$ such that $\pi^{-1}(\mathcal{X}_{sing}) = \cup_i^N e_i$ and which has the desired singular locus.

The condition $sing(\mathcal{X}) \leq 19$ ensures that we can find enough -2-curves on our $K3$ surface since the Picard number for a $K3$ surface is at most 19 ([32, p. 16, equation (3.3)]). The surjectivity of the period map guarantees that we can find a smooth $K3$ surface that satisfies all our requirements above; in particular, the Picard lattice contains the required root lattice ([32, Chapter 13, Section 1]). Hence, we can produce orbifold $K3$ surfaces of any prescribed singular locus whose singularity number is less than or equal to 19, provided that we start with a suitable smooth $K3$ surface initially.

Remark 1.49. Note that if $\pi : X \to \mathcal{X}$ is a crepant resolution of the orbifold $K3$ $\mathcal{X}$, then it induces an injective homomorphism $\pi^* : H^2(\mathcal{X};\mathbb{Z}) \to H^2(X;\mathbb{Z})$ with $H^2(X;\mathbb{Z})/\pi^* H^2(\mathcal{X};\mathbb{Z})$ torsion-free, since the cell complex underlying $\mathcal{X}$ is the same as that of X in dimensions different from 2, while in dimension 2, some of the cells of X, corresponding to -2-curves, are contracted to a point by π. Hence we can identify $I^2(\mathcal{X})$, which we think of as the image of $H^2(\mathcal{X};\mathbb{Z})$ in $H^2(X;\mathbb{Z})$ through a marking, with a primitive sublattice of L, the $K3$ lattice.

Theorem 1.50. *(Global Torelli Theorem for orbifold $K3$ surfaces, [50, p. 319]) Let $(\mathcal{X}, \kappa)$ and $(\mathcal{X}', \kappa')$ be two orbifold $K3$ surfaces with κ and κ' respective Kähler classes. Moreover, let $\pi : X \to \mathcal{X}$ and $\pi' : X' \to \mathcal{X}'$ be their minimal resolutions. Suppose that $h : I^2(\mathcal{X}') \to I^2(\mathcal{X})$ is an isometry such that $h_{\mathbb{C}}(H^{2,0}(\mathcal{X}')) = H^{2,0}(\mathcal{X})$, $h_{\mathbb{R}}(\kappa') = \kappa$ and h extends to an isometry $\overline{h} : H^2(X';\mathbb{Z}) \to H^2(X;\mathbb{Z})$. Then there is a unique isomorphism $\phi : \mathcal{X} \to \mathcal{X}'$ such that $\phi^* = h$.*

Now suppose that we have an orbifold $K3$ surface $\mathcal{X}$ with a crepant resolution $\pi : X \to \mathcal{X}$ such that X is a $K3$ surface. Then the resolution map, π, induces a map $\pi^* : H^2(\mathcal{X}; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ and hence any marking $\phi : H^2(X; \mathbb{Z}) \to L$ induces an embedding:

$$\phi\pi^* : H^2(\mathcal{X}; \mathbb{Z}) \hookrightarrow L \tag{1.36}$$

we call this a *marking for an orbifold $K3$ surface.* Let $(\kappa_I, \kappa_J, \kappa_K)$ denote the Kähler classes associated to a hyper-Kähler structure on $\mathcal{X}$ and let $\alpha = (x, y, z) = (\phi\pi_{\mathbb{R}}(\kappa_I), \phi\pi_{\mathbb{R}}(\kappa_J), \phi\pi_{\mathbb{R}}(\kappa_K))$ where $\phi\pi_{\mathbb{R}}$ denotes the $\mathbb{R}$-linear extension of $\phi\pi$. We then know from [55] that the set

$$\mathcal{D}_\alpha = \{d \in L | d^2 = -2, \langle d, x\rangle = \langle d, y\rangle = \langle d, z\rangle = 0\} \tag{1.37}$$

can be used to describe the singular locus of $\mathcal{X}$. The set $\mathcal{D}_\alpha$ is a subset of $R(\mathcal{X})$ consisting precisely of the classes of all the (-2)-curves contracted by π. Moreover, denote the sublattices generated by $\mathcal{R}(\mathcal{X})$ and $\mathcal{D}_\alpha$ by:

$$\mathcal{D}_\alpha^L = \{\delta = \sum_{finite} a_i\delta_i | a_i \in \mathbb{Z}, \delta_i \in \mathcal{D}_\alpha\} \tag{1.38}$$

$$\mathcal{R}(\mathcal{X})^L = \{\delta = \sum_{finite} a_i\delta_i | a_i \in \mathbb{Z}, \delta_i \in \phi\mathcal{R}(\mathcal{X})\} \tag{1.39}$$

where we denoted by $\phi\mathcal{R}(\mathcal{X})$ the image of the root system under the marking ϕ. Then we can easily see that $\mathcal{D}_\alpha^L = \mathcal{R}(\mathcal{X})^L$ so that $I^2(\mathcal{X})$ can be taken to be the orthogonal complement of the sublattice $\mathcal{D}_\alpha^L$. This has the advantage of making it clear what a basis for the root lattice is, namely $\mathcal{D}_\alpha$.

Example 1.51. Consider an orbifold $K3$ surface with 3 ordinary double points. The root lattice is a rank 3 lattice with bilinear form given by $\langle ., . \rangle = -2I_3$, where I_3 denotes the 3×3 identity matrix. We write $\mathcal{R}(\mathcal{X})^L = 3A_1$.

Now consider an orbifold $K3$ surface whose singular locus consists of a single point at which we have an A_3 singularity. The root lattice in this case is a rank 3 lattice with bilinear form given by:

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

We write $\mathcal{R}(\mathcal{X})^L = A_3$

Lastly, consider a $K3$ surface whose singular locus consists of 2 ordinary double points and another singular point which is an A_3 singularity. In this case, the root lattice is a rank 5 lattice with bilinear form given by:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

We write $\mathcal{R}(\mathcal{X})^L = 2A_1 \bigoplus A_3$

1.4 Twisted Connected Sum

We review, largely following [44], the Twisted Connected Sum construction which we shall generalise to orbifolds in the following chapter. The idea is to construct 6-manifolds X_i for $i = 1, 2$ carrying *asymptotically cylindrical metrics* with holonomy $SU(3)$ such that we can define a G_2-structure on $X_i \times S^1$. Then as long as they satisfy some appropriate "matching condition", the $X_i \times S^1$ can be "glued" to give a compact 7-fold M, admitting a family of "glued" G_2-structures ϕ_T which have "exponentially decaying" torsion (in some appropriate Banach norms) as $T \to \infty$, where T represents the diameter of M; this allows us to exploit some perturbative analysis to get a torsion-free G_2-structure on M. For us, the manifolds X_i will be *asymptotically cylindrical Calabi-Yau 3-folds*:

Definition 1.52. A complete Riemannian manifold (W, g) is called *asymptotically cylindrical* if there exists a compact manifold with boundary $U \subset W$, a closed Riemannian manifold (R, h), a diffeomorphism $\psi : R \to \partial U$ and a diffeomorphism $\Phi : R \times [0; \infty) \to W \setminus U$ such that if ∇ denotes the Levi-Civita connection of the cylindrical metric $g_\infty = dt^2 + h$, where t is the cylindrical coordinate on $R \times [0; \infty)$, then $||\nabla^k(\Omega^* g - g_\infty)|| = O(e^{-\delta t})$ for all $k \in \mathbb{N}_0$ and for some $\delta > 0$. We call the connected components of $R \times [0; \infty)$ the *cylindrical ends* and (R, h) is called the *cross section.*

We shall be interested in the case where R is the product of a $K3$ surface, S, with a circle S^1. Then, using the notation of the previous sections a product Ricci-flat Kähler metric on the half-cylinder $S \times S^1 \times [0; \infty)$ has Kähler form given by:

$$\omega_0 = dt \wedge d\theta + \kappa_I \tag{1.40}$$

and holomorphic volume form

$$\Omega_0 = (dt + id\theta) \wedge (\kappa_J + i\kappa_K) \tag{1.41}$$

where t is the standard coordinate on $[0;\infty)$ and θ is the standard coordinate on S^1. Then it can be shown ([44, Theorems 2.4 and 2.7]) that if $\overline{W}$ is a smooth compact simply connected Kähler 3-fold such that $H^1(\overline{W};\mathbb{R}) = 0$, and a $K3$ surface S in $\overline{W}$ is an anticanonical divisor with holomorphically trivial normal bundle $N_{S|W}$, then $W = \overline{W} \setminus S$ admits an asymptotically cylindrical Ricci-flat Kähler metric g, such that near S, the Kähler form of g and the holomorphic volume form are asymptotic to the product ones in **Equations (1.40) and (1.41)**. Moreover, the metric g will have holonomy $SU(3)$. We shall expand on this in **Chapter** 2.

Now a Riemannian product of a manifold W as above and S^1 yields a real 7-dimensional manifold with holonomy $SU(3)$. "Cutting off" the cylindrical end we obtain a compact 7-manifold with boundary given by the Riemannian product $S^1 \times S^1 \times S$, where S is the manifold underlying the Ricci-flat Kähler $K3$ surface above. We take the union of two such manifolds and identify their boundaries via an isometry to obtain a closed compact 7-fold carrying a G_2-structure with arbitrarily small torsion. The "twist" is that to avoid infinite fundamental group the isometry above needs to interchange the S^1 factors and the Kähler classes on the two $K3$ surfaces. Explicitly we have:

Definition 1.53. Let S_1 and S_2 be $K3$ surfaces with a hyper-Kähler metric. We say S_1 and S_2 *satisfy the Donaldson matching condition*, or simply the *matching condition*, if there exists a $\mathbb{Z}$-linear map $h : H^2(S_2;\mathbb{Z}) \to H^2(S_1;\mathbb{Z})$ that preserves the intersection form and its $\mathbb{R}$-linear extension satisfies:

$$\begin{aligned} h([\kappa_I^2]) &= [\kappa_J^1], \\ h([\kappa_J^2]) &= [\kappa_I^1], \\ h([\kappa_K^2]) &= -[\kappa_K^1] \end{aligned} \tag{1.42}$$

where the complex structures on S_j are given by I_j, J_j, K_j and the corresponding Kähler forms are κ_I^j, κ_J^j and κ_K^j. Moreover, the holomorphic volume form for I_j is given by $\kappa_J^j + i\kappa_K^j$.

It has been shown in [44] that if two $K3$ surfaces S_1 and S_2 satisfy the matching condition, then there exists an isomorphism of complex surfaces $f : S_{1J} \to S_2$ (here S_{1J} denotes the complex $K3$ surfaces defined by considering on S_1 the complex structure J_1 rather than I_1) such that $f^* = h$ and moreover, f is an isometry of hyper-Kähler manifolds such that its action on Kähler forms by pullback is given by:

$$\begin{aligned} f^*\kappa_I^2 &= \kappa_J^1, \\ f^*\kappa_J^2 &= \kappa_I^1, \\ f^*\kappa_K^2 &= -\kappa_K^1 \end{aligned} \tag{1.43}$$

Note that the $\mathbb{C}$-linear extension of h above converts the Hodge decomposition for S_2 into one for S_{1J}.

Now if we have two asymptotically cylindrical manifolds W_1 and W_2 as above with the $K3$ surfaces S_1 and S_2 satisfying the matching condition, the boundaries admit a product decomposition for the cylindrical end given by $\iota : S_i \times S^1 \times [0;\infty) \to W_i$ and we set for $i = 1, 2$

$$M_{i,T} = (W_i \setminus \iota(S_i \times S^1 \times (T+1, \infty))) \times S^1 \tag{1.44}$$

and use the following isometry to identify their boundaries:

$$\begin{gathered} F : S_1 \times S^1 \times S^1 \times (T, T+1) \to S_2 \times S^1 \times S^1 \times (T, T+1) \\ (y, \theta_1, \theta_2, T+t) \mapsto (f(y), \theta_2, \theta_1, T+1-t) \end{gathered} \tag{1.45}$$

Let

$$M_T = M_{1,T} \cup_F M_{2,T} \tag{1.46}$$

which we call the *Twisted Connected Sum of* $W_1 \times S^1$ *and* $W_2 \times S^1$. Note that, since the diffeomorphism class of M_T is independent of T, the subscript T indicates only that we equip M with the G_2-structure obtained after "cutting off the cylindrical end at T". In the above setup, it is shown in [44] that there exists a parallel G_2-structure on M as $T \to \infty$ and moreover, that M has finite fundamental group if W_1 and W_2 both have finite fundamental groups. In practice most examples of G_2-manifolds constructed this way are simply-connected. We say that *a G_2-manifold, M, is constructed from the pairs* $(\overline{W}_1, S_1)$ *and* $(\overline{W}_2, S_2)$. Examples of suitable pairs include those coming from Fano 3-folds which always admit a smooth $K3$ divisor. A further important class of examples, [45] arise from products $S \times \mathbb{CP}^1/(\rho \times \psi)$ where $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$ is any holomorphic involution with two distinct fixed points, and S is a $K3$ surface admitting a holomorphic non-symplectic involution, ρ. Non-symplectic involutions will be quite important at various points throughout the thesis, and as such we record their definition below:

Definition 1.54. Let S be a $K3$ surface and $\rho : S \to S$ be a holomorphic involution of S. We say that ρ is *non-symplectic* if for all $[\omega] \in H^{2,0}(S)$, then

$$\rho^*[\omega] = -[\omega] \tag{1.47}$$

Chapter 2

Orbifold Twisted Connected Sum

The main purpose of this chapter is to generalise the Twisted Connected Sum construction to orbifolds. We will look at the key theorems in the smooth setting and find generalisations to orbifolds. We also discuss the matching problem for orbifolds, the basic topology of the resulting G_2 orbifolds, as it was done in [45], and present a strategy for building examples of the non-symplectic type suggested and discussed in detail by Reidegeld in [55]. Most of the results are routine generalisations of the smooth equivalents; however, we will need to be more careful concerning results in the later half of the chapter, particularly when discussing the matching problem for orbifolds and their basic topological invariants.

2.1 Asymptotically Cylindrical Orbifolds

The main building blocks of the Twisted Connected Sum are Asymptotically Cylindrical (ACyl) 6-folds equipped with holonomy $SU(3)$ metrics and whose cross-section is given by a product of a $K3$ surface and a circle. Hence we need to obtain orbifold equivalents to glue together. The purpose of this section is to obtain a set of conditions on compact Kähler 3-orbifolds under which we get ACyl orbifolds after removing an orbifold $K3$ divisor. This is **Theorem 2.7** below, which is our main theorem for this section. To prove **Theorem 2.7** we will follow the proof of the analogous theorem in the smooth setting ([44, Theorem 2.4]). For this, we will need to prove an ACyl version of the Calabi Conjecture for orbifolds. To do this, we define weighted Sobolev and Hölder spaces in the context of ACyl orbifolds, $\mathcal{W}$.

Definition 2.1. Let $\delta > 0$, $t \in C^\infty(\mathcal{W})^{orb}$ coincide along the cylindrical end with the coordinate on the $\mathbb{R}_{>0}$ factor. Let $\mathcal{E} \to \mathcal{W}$ be an orbibundle. For $u \in C_0^\infty(\mathcal{E})^{orb}$ we define

$$||u||_{C_\delta^{k,\alpha}(\mathcal{E})^{orb}} = ||e^{-\delta t}u||_{C^{k,\alpha}(\mathcal{E})^{orb}} \tag{2.1}$$

and we let $C_\delta^{k,\alpha}(\mathcal{E})^{orb} = \left\{u \in C^{k,\alpha}(\mathcal{E})^{orb} : ||u||_{C_\delta^{k,\alpha}(\mathcal{E})^{orb}} \text{ is finite}\right\}$ so that $C_\delta^{k,\alpha}(\mathcal{E})^{orb}$ consists of sections whose derivatives of order $\leq k$ are exponentially decaying.

We set $C_\delta^\infty(\mathcal{E})^{orb} = \cap C_\delta^{k,\alpha}(\mathcal{E})^{orb}$. Similarly, the weighted Sobolev space, $e^{-\delta t}L_k^p(\mathcal{E})^{orb}$ is defined as the space of all functions $e^{-\delta t}v$ with $v \in L_k^p(\mathcal{E})^{orb}$. We also denote the $L^p(\mathcal{W})^{orb}$-norm of $u \in C_0^\infty(\mathcal{W})^{orb}$ by $||u||_p$.

Theorem 2.2 (ACyl Calabi Conjecture for orbifolds)**.** *Let $(\mathcal{W}, g, J)$ be an ACyl Kähler orbifold of complex dimension n with Kähler form ω. $0 < \lambda \ll 1$ and if $f \in C_\lambda^\infty(\mathcal{W})^{orb}$ satisfies*

$$\int_{\mathcal{W}} (e^f - 1)\omega^n = 0 \tag{2.2}$$

then there exists a unique $u \in C_\lambda^\infty(\mathcal{W})^{orb}$ such that $\omega + \frac{i}{2\pi}\partial\overline{\partial}u > 0$ and $(\omega + \frac{i}{2\pi}\partial\overline{\partial}u)^n = e^f\omega^n$

The proof of **Theorem 2.2**, will also closely follow the smooth version ([27, Theorem 4.1]). Now the proof will require proving a global Sobolev inequality on $\mathcal{W}$, which we prove using the following orbifolds version of the Hölder, Sobolev, and Poincaré inequalities:

Theorem 2.3 (Poincaré-Sobolev Inequality, [22])**.** *Let $(\mathcal{M}, g)$ be a closed orientable Riemannian orbifold and suppose that $u \in C^1(\mathcal{M})_0^{orb}$, i.e., u is continuously differentiable with compact support. Then for $1 \leq p < dim(\mathcal{M})$ there is a constant C depending only on $dim(\mathcal{M})$ and p such that*

$$||u||_{L^{p^*}(\mathcal{M})^{orb}} \leq C||\nabla u||_{L^p(\mathcal{M})^{orb}} \tag{2.3}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$

Theorem 2.4 (Poincaré Inequality, [22])**.** *Let $(\mathcal{M}, g)$ be a closed orientable Riemannian orbifold and suppose that $1 \leq p \leq \infty$. Let $\mathcal{U}$ be a bounded connected open subset of $\mathcal{M}$. Then there exists a constant C, depending only on $\mathcal{U}$ and p such that for every u in the Sobolev space $W^{1,p}(\mathcal{U})^{orb}$ we have:*

$$||u - u_{\mathcal{U}}||_{L^p(\mathcal{U})^{orb}} \leq C||\nabla u||_{L^p(\mathcal{U})^{orb}} \tag{2.4}$$

where $u_{\mathcal{U}} = \frac{1}{vol(\mathcal{U})}\int_{\mathcal{U}} udVol$ and $vol(\mathcal{U}) = \int_{\mathcal{U}} dVol$

Theorem 2.5 (Hölder Inequality)**.** *Let* $(\mathcal{M}, g)$ *be a Riemannian orbifold and suppose that* $1 \leq q, p \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for all* $f \in L^p(\mathcal{M})^{orb}$ *and* $g \in L^q(\mathcal{M})^{orb}$ *we have:*

$$||fg||_{L^1(\mathcal{M})^{orb}} \leq ||f||_{L^p(\mathcal{M})^{orb}} ||g||_{L^q(\mathcal{M})^{orb}} \tag{2.5}$$

We note that the Hölder Inequality is valid for any measure space and hence the version presented here follows from the standard Hölder Inequality one might meet in an undergraduate analysis course (e.g. [62, Theorem 8.6]).

Proposition 2.6. *Let* $\mathcal{W}$ *be an ACyl orbifold as above. Then for all* $\mu > 0$ *there exists a piecewise constant positive function* $\psi_\mu = O(e^{-2\mu t})$ *with* $\int_{\mathcal{W}} \psi_\mu dVol = 1$ *such that*

$$||e^{-\mu t}(u - \overline{u}_\mu)||_{2\sigma} \leq C||\nabla u||_2 \tag{2.6}$$

holds for all $\sigma \in \left[1, \frac{n}{n-2}\right]$ *and all* $u \in C^\infty(\mathcal{W})^{orb}$ *where* $\overline{u}_\mu = \int_{\overline{W}} u\psi_\mu dVol$*,* $dVol$ *is the volume form associated with the ACyl metric on* $\mathcal{W}$ *and* C *is a constant which depends only on* $\mathcal{W}$*,* μ *and* σ*.*

Proof. The proof in the smooth case ([27]) requires the use of the Sobolev and Poincaré inequalities to bound the norm of the difference between u and $\overline{u}_\mu$, the "asymptotic average of u", by the norm of the gradient of u, ∇u. Since both of these inequalities hold for orbifolds, then so will the bounds and the proof will follow without any additional work. □

Proof (of ***Theorem 2.2****).* The proof follows from the proof of a similar result for smooth ACyl manifolds in [27, Theorem 4.1]. We will capture the main ideas and note where there are additional steps due to moving to the orbifolds setting.

The idea is to show there exists a solution $u \in C^{k+2,\alpha}_\lambda$ for any $k \in \mathbb{N}$ and $\alpha \in (0;1)$. For this we consider $0 < \lambda \leq \sqrt{\lambda_1(\mathcal{X})}$ where $\lambda_1(\mathcal{X})$ is the first eigenvalue of the Laplacian on the cross-section $\mathcal{X}$ and define

$$X = \left\{u \in C^{k+2,\alpha}_\lambda(\mathcal{W})^{orb} | \omega_u = \omega + \frac{i}{2\pi}\partial\overline{\partial}u > 0\right\} \tag{2.7}$$

$$Y = \left\{f \in C^{k,\alpha}_\lambda(\mathcal{W})^{orb} | \int_{\mathcal{W}} (e^f - 1)\omega^n = 0\right\} \tag{2.8}$$

Note that X is open and consider the Monge-Ampère operator F given by $(\omega + \frac{i}{2\pi}\partial\overline{\partial}u)^n = e^{F(u)}\omega^n$. In the smooth case, the operator induces a map $F : X \to Y$; in the orbifold case we need to check this map is well-defined:

Now, for any orbifold chart (U, Γ, ϕ), $u|_{\mathcal{U}} \in C_{\lambda}^{k+2,\alpha}(U)_{\Gamma}$ and $\omega|_{\mathcal{U}}$ are Γ-invariant so the Monge-Ampère holds if and only if $F(u)$ is Γ-invariant, meaning that $F(u) \in C_{\lambda}^{k,\alpha}(U)_{\Gamma}$. Moreover, $F(u)$ will also be compatible with chart injections since both u and ω are. Thus F is indeed a well-defined map.

Now given f as in the statement of the theorem, the idea is to consider for $\tau \in [0;1]$ the family of equations $F(u_\tau) = f_\tau$ for $u_\tau \in X$ and $f_\tau = ln(1 + \tau(e^f - 1)) \in Y$ and to apply the continuity method by showing the set of all τ for which a solution u_τ exists is both open and closed, since we know the trivial solution $u_0 = 0$ exists. The proof involves using the Moser iteration trick along regions of the neck of the form $\{t < T\}$ and $\{t > T\}$ for some fixed $T > 0$, together with the Poincaré and Hölder inequalities and **Proposition (2.6)** to show this. The crucial difference to note in the case of orbifolds is that the cylindrical end is of the form $\mathcal{X} \times [0;\infty)$ and hence the singular locus is of the form $\mathcal{X}_{sing} \times [0;\infty)$. Now to apply the Moser iteration trick we want smoothing functions along the neck regions $\{t < T\}$ and $\{t > T\}$ which are Γ_p invariant around every $p \in \mathcal{X}_{sing} \times [0;T)$ and every $p \in \mathcal{X}_{sing} \times (T;\infty)$ respectively. We obtain these by choosing smoothing functions, which may not be locally Γ_p invariant, say $\rho_{t<T}$ and $\rho_{t>T}$. Now for every orbifold chart, $(U \times U_{S^1} \times [0;T), \varphi \times \theta \times t, \Gamma)$, we replace the local expression of $\rho_{t<T}$ with an average of the form:

$$\overline{\rho}_{t<T}(x, \theta, t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \rho_{t<T}(\gamma.x, \theta, t) \tag{2.9}$$

where (U_{S^1}, θ) is a coordinate chart on S^1 and $([0;T), t)$ is a coordinate chart on $[0;\infty)$. We can then use a partition of unity to glue the local expressions together in a way which will be compatible with chart injections. Similarly, we can obtain the required smoothing functions on the $\{t > T\}$ region. Hence, the rest of the details of the proof follow. $\square$

We now state the main theorem of this section:

Theorem 2.7. *Let $\overline{\mathcal{W}}$ be a compact complex Kähler 3-orbifold with ω' the Kähler form on $\overline{\mathcal{W}}$. Suppose that an orbifold K3 surface $\mathcal{X}$ in $\overline{\mathcal{W}}$ is an anticanonical Weil divisor which has holomorphically trivial normal orbibundle $N_{\mathcal{X}|\overline{\mathcal{W}}}$. Suppose further that $\overline{\mathcal{W}}$ is simply connected ($\pi_1(\overline{\mathcal{W}}) = 1$) and that $\mathcal{W} = \overline{\mathcal{W}} \setminus \mathcal{X}$ has finite fundamental group, $\pi_1(\mathcal{W})$. Then $\mathcal{W}$ admits a Ricci-flat Kähler metric, g, with holonomy $SU(3)$ and a nowhere vanishing holomorphic $(3,0)$-form. Moreover, the Kähler form of g and the holomorphic volume form can be written near $\mathcal{X}$ as:*

$$\omega_g|_{\mathcal{U}\setminus\mathcal{X}} = \omega_0 + d\psi \tag{2.10}$$

$$\Omega_g|_{\mathcal{U}\setminus\mathcal{X}} = \Omega_0 + d\Psi \tag{2.11}$$

where we identify ω_0 and Ω_0 with the product Ricci-flat Kähler metric on the half cylinder $\mathcal{X} \times S^1 \times [0;\infty)$ via $z = e^{-t-i\theta}$ for θ the standard coordinate on S^1 and t the standard coordinate on $[0;\infty)$ and the Kähler class $[\kappa_I] \in H^{1,1}(\mathcal{X})$ is $[\omega'|_{\mathcal{X}}]$. The 1-form, ψ, and the 2-form, Ψ, are smooth and satisfy $||\nabla^k\psi|| = O(e^{-\lambda t})$ and $||\nabla^k\Psi|| = O(e^{-\lambda t})$ where ∇ is the Levi-Civita connection of the metric corresponding to the Kähler form ω_0, $k \in \mathbb{N}_0$ and λ is any positive real number satisfying $\lambda < min\{1, \sqrt{\lambda_1(\mathcal{X})}\}$ where $\lambda_1(\mathcal{X})$ is the first positive eigenvalue of the Laplacian on $\mathcal{X}$.

This result is similar to the corresponding result in [44] and [45]. The differences between the orbifold case and the smooth case are once again minimal, with the orbifold proof being a routine generalisation. We will include the relevant details below, noting that with **Theorem 2.2** proven, the contents of **Theorem 2.7** that require proof are the statements on the asymptotic behaviour of the Ricci-flat Kähler form and the holomorphic volume form we obtain and the statement on the existence of an ACyl structure on $\mathcal{W}$. On top of this, the setup of the theorem gives us some restrictions on the singular locus of $\mathcal{X}$ and $\overline{\mathcal{W}}$ which we capture below:

Remark 2.8. Note that the holomorphic triviality of the normal bundle, implies that if $p \in \mathcal{X}$ is a singular point with isotropy group $\Gamma^{\mathcal{X}}_p$, $\iota : \mathcal{X} \to \overline{\mathcal{W}}$ is the embedding of $\mathcal{X}$ into $\overline{\mathcal{W}}$ and $\Gamma^{\overline{\mathcal{W}}}_{\iota(p)}$ is the isotropy group of $\iota(p)$ (which we can think of as the isotropy group of p in $\overline{\mathcal{W}}$), then in fact we have that $\Gamma^{\mathcal{X}}_p \cong \Gamma^{\overline{\mathcal{W}}}_{\iota(p)}$. In general $\Gamma^{\mathcal{X}}_p$ is only a subgroup of $\Gamma^{\overline{\mathcal{W}}}_{\iota(p)}$ so that $\overline{\mathcal{W}}$ can be "more singular" over $\iota(\mathcal{X})$ (for example, $\mathcal{X}$ could be a smooth manifold, but $\overline{\mathcal{W}}_{sing} \cap \iota(\mathcal{X}) \neq \emptyset$). This motivates the following definition.

Definition 2.9. Let $\mathcal{X}$ be a suborbifold of the orbifold $\overline{\mathcal{W}}$. We say that $\mathcal{X}$ *shares its singularities with* $\overline{\mathcal{W}}$ if for all $p \in \mathcal{X}_{sing}$ we have that $\Gamma^{\mathcal{X}}_p \cong \Gamma^{\overline{\mathcal{W}}}_{\iota(p)}$. Note that if we view a manifold M as an orbifold with $M_{sing} = \emptyset$, we let X be an embedded submanifold of M which we also view as a suborbifold with $X_{sing} = \emptyset$, then X vacuously shares its singularities with M.

*Proof (of **Theorem 2.7**).* Let s denote the defining section for the divisor $\mathcal{X}$. Then by the orbifold adjunction formula ([13, Section 4]), the anticanonical orbibundle of $\overline{\mathcal{W}}$ restricts to a holomorphically trivial bundle on $\mathcal{X}$. Moreover, the section s defines a holomorphic function on $\mathcal{U}$, τ, with $\tau^{-1}(0) = \mathcal{X}$.

Since $\pi_1(\overline{\mathcal{W}}) = 1$, then the Dolbeault cohomology $H^{0,1}(\overline{\mathcal{W}}) = 0$, as a consequence of the Hodge Decomposition for orbifolds [8]. Hence, we can extend $\frac{1}{\tau}$ to a meromorphic function on $\overline{\mathcal{W}}$ with only a pole of order 1 along $\mathcal{X}$. Hence τ extends to a $K3$ fibration of $\overline{\mathcal{W}}$, denoted by $\tau : \overline{\mathcal{W}} \to \mathbb{CP}^1$, such that $\mathcal{X}$ is the fibre over 0. Moreover, the anticanonical orbibundle of $\overline{\mathcal{W}}$ is the pullback via τ of a holomorphic line bundle of degree 1 over $\mathbb{CP}^1$ so that there is a section s_1 with $s = s_1 \circ \tau$. Now note that by the Tubular Neighbourhood Theorem for orbifolds ([17]), there is a neighbourhood $\mathcal{U}$ of $\mathcal{X}$ in $\overline{\mathcal{W}}$ such that $\mathcal{U} \cong N_{\mathcal{X}|\overline{\mathcal{W}}}$. Moreover, since the normal bundle is holomorphically trivial we get a diffeomorphism of the underlying real orbifolds:

$$F : \mathcal{U} \to \{|z| < 1\} \times \mathcal{X} \tag{2.12}$$

such that $F^{-1}(\{0\} \times \mathcal{X})$ corresponds to $\mathcal{X} \subset \mathcal{U}$ so we can set $\Phi = F|_{\mathcal{U}\setminus\mathcal{X}} : \mathcal{U} \setminus \mathcal{X} \to \{0 < |z| < 1\} \times \mathcal{X}$ as the required diffeomorphism.

Moreover, note that via a change of coordinates $z = e^{-t-i\theta}$ we can identify $\{0 < |z| < 1\} \times \mathcal{X}$ with $S^1 \times [0;\infty) \times \mathcal{X}$. With this change of coordinates, the metric with Kähler form:

$$\omega_0 = dt \wedge d\theta + \kappa_I \tag{2.13}$$

where κ_I denotes the Kähler form of $\mathcal{X}$, is holomorphic with respect to the complex structure on $\{|z| < 1\} \times \mathcal{X}$.

Let u_0 be a smooth function with $\overline{supp}(u_0) \subset \mathcal{U}$ such that $\left(\omega' + \frac{i}{2\pi}\partial\overline{\partial}u_0\right)$ is the Calabi-Yau metric in the Kähler class $[\omega'_{\mathcal{X}}]$. Using a smooth cutoff function, $\beta : \overline{\mathcal{W}} \to [0;1]$ with $\mathcal{X} \subset \overline{supp}(\beta) \subset \mathcal{U}$, then we may choose u_0 such that $\left(\omega' + \frac{i}{2\pi}\partial\overline{\partial}u_0\right)$ is positive on any fibre of τ in $\overline{\mathcal{W}}$. Again, note that this is possible using a similar argument to the one related to applying the Moser iteration trick to orbifolds.

Now we can apply **Theorem 2.2** to conclude that there exists a unique exponentially decaying solution, u, for the complex Monge-Ampere equation. We set:

$$\omega_g = \omega + \frac{i}{2\pi}\partial\overline{\partial}u \tag{2.14}$$

and then we can show using a routine generalisation of the arguments in [44] that ω_g and its corresponding holomorphic volume form, Ω_g, have the required asymptotic behaviour.

□

2.2 Existence Theorem

Just as in the case of the regular Twisted Connected Sum, given two orbifold 3-folds satisfying the conditions of **Theorem 2.7**, $\mathcal{W}_1$ and $\mathcal{W}_2$, their boundaries admit a product decomposition of their cylindrical ends, $\iota_i : \mathcal{X}_i \times S^1 \times [0;\infty) \to \mathcal{W}_i$, where $\mathcal{X}_i$ is the anti-canonical orbifold $K3$ surface. We set:

$$\mathcal{M}_{i,T} = (\mathcal{W}_i \setminus \iota_i(\mathcal{X}_i \times S^1 \times (T+1;\infty))) \times S^1 \tag{2.15}$$

for $i = 1,2$. Moreover, if $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy the *orbifold matching condition*, then by **Proposition 2.18**, we can consider the following isometry to glue the boundaries of $\mathcal{M}_1$ and $\mathcal{M}_2$:

$$\begin{aligned} F : \mathcal{X}_1 \times S^1 \times S^1 \times (T, T+1) &\to \mathcal{X}_2 \times S^1 \times S^1 \times (T, T+1) \\ (y, \theta_1, \theta_2, T+t) &\mapsto (f(y), \theta_2, \theta_1, T+1-t) \end{aligned} \tag{2.16}$$

where $f : \mathcal{X}_1 \to \mathcal{X}_2$ is the isometry from **Proposition 2.18**. We set:

$$\mathcal{M}_T = \mathcal{M}_{1,T} \cup \mathcal{M}_{2,T} \tag{2.17}$$

We call $\mathcal{M}_T$, the *orbifold Twisted Connected Sum of* $\mathcal{W}_1 \times S^1$ *and* $\mathcal{W}_2 \times S^1$. Note that, just as with the smooth Twisted Connected Sum, the subscript T, indicates only the resulting orbifold G_2-structure equipped to $\mathcal{M}_T$, since the diffeomorphism class of $\mathcal{M}_T$ is independent of T.

Just as in the smooth case, the G_2-structure on $\mathcal{W} \times S^1$ is given by the 3-form $\phi \in \Omega^3_+(\mathcal{W} \times S^1)$, expressed as:

$$\phi = \omega \wedge d\theta' + Im(\Omega) \tag{2.18}$$

where ω and Ω are the Kähler form and holomorphic volume form, respectively, on $\mathcal{W}$ and θ' is the coordinate on the S^1 factor. The Hodge dual of ϕ can be expressed as:

$$\star_\phi \phi = \frac{1}{2}\omega \wedge \omega - Re(\Omega) \wedge d\theta' \tag{2.19}$$

Now let $\alpha : \mathbb{R} \to [0;1]$ be a cutoff function such that $\alpha(t) = 0$ for $t \leq 0$ and $\alpha(t) = 1$ for $t \geq 1$, ψ and Ψ be as in **Theorem 2.7**, and set

$$\omega_T = \omega + d(\alpha(t - T + 1)\psi) \tag{2.20}$$

$$\Omega_T = \Omega + d(\alpha(t - T + 1)\Psi) \tag{2.21}$$

Then, since ψ and Ψ are exponentially decaying to 0, we have that

$$\phi_T = \omega_T \wedge d\theta' + Im(\Omega_T) \tag{2.22}$$

gives a well-defined G_2-structure on $\mathcal{W} \times S^1$ for every $T \geq T_0 > 1$, for some large enough T_0, for which we have $d\phi_T = 0$ by construction, but in general we do have torsion, $\star_{\phi_T}\phi_T \neq 0$. We measure the torsion of ϕ_T using the 4-form:

$$\widetilde{\Theta}(\phi_T) = \star_{\phi_T}\phi_T - \frac{1}{2}\omega_T \wedge \omega_T - Re(\Omega_T) \wedge d\theta' \tag{2.23}$$

so that $d\widetilde{\Theta}(\phi_T) = d(\star_{\phi_T}\phi_T)$. Since the cutoff function, α, is fixed, from **Theorem 2.7** we get the following:

Lemma 2.10. *For any $\epsilon > 0$,*

$$||\widetilde{\Theta}(\phi_T)||_{L^p_k(\mathcal{W}\times S^1)^{orb}} < C_{p,k,\epsilon}e^{-(\lambda-\epsilon)T} \tag{2.24}$$

*where $C_{p,k,\epsilon}$ is a constant independent of T, $p > 1$, $k \in \mathbb{N}_0$, the exponent $\lambda > 0$ for a given $\mathcal{W}$ is defined as in **Theorem 2.7** and the norms are taken in the metric induced by ϕ_T on $\mathcal{W} \times S^1$*

Note that near the boundary of $\mathcal{M}_{i,T}$, the G_2-structure is equivalent to the torsion-free G_2-structure on the cylinder $\mathcal{X}_i \times S^1 \times S^1 \times \mathbb{R}_{>0}$, $\phi_{\mathcal{X}_i}$, given by

$$\phi_{\mathcal{X}_i} = \kappa^i_I \wedge d\theta_1 + \kappa^i_J \wedge d\theta_2 + \kappa^i_K \wedge dt + d\theta_1 \wedge d\theta_2 \wedge dt \tag{2.25}$$

where $(\kappa^i_I, \kappa^i_J, \kappa^i_K)$ is the hyper-Kähler structure on $\mathcal{X}_i$, θ_1 is the coordinate on the first circle factor, arising from the asymptotically cylindrical structure on $\mathcal{W}_i$, θ_2 is the coordinate on the second circle factor, the one we take a global product with to get $\mathcal{M}_{i,T}$ and t is the standard coordinate on $\mathbb{R}_{>0}$.

Now we have equipped $\mathcal{M}$ with a family of closed G_2-structures ϕ_T having arbitrarily small $\star_{\phi_T}\phi_T$ as $T \to \infty$. We wish to solve the system:

$$d\phi = 0 \tag{2.26}$$

$$d \star_\phi \phi = 0 \tag{2.27}$$

for which we will search for solutions of the form $\phi_T + d\eta$, where η is some unknown 2-form. It is also useful to define a map $\Theta : \Omega^3_+(\mathcal{M})^{orb} \to \Omega^4(\mathcal{M})^{orb}$ taking a G_2-structure ϕ to its Hodge dual with respect to the metric it induces on $\mathcal{M}$, $\Theta(\phi) = \star_\phi\phi$. In [34, Section 10.3], the torsion-free condition

$d\Theta(\phi + d\eta) = 0$ has been reduced to an elliptic equation in η using the local expression:

$$\Theta(\phi + d\eta) = \Theta(\phi) + (d\Theta)(\phi)d\eta + R(d\eta) \tag{2.28}$$

where the second order remainder $R(d\eta)$ satisfies the estimate:

$$||dR(\xi_1) - dR(\xi_2)||_{L^p} < (C_1 + C_2||d\Theta(\phi_T)||_{L^p_1})(||\xi_1||_{L^p_1} + ||\xi_2||_{L^p_1})||\xi_1 - \xi_2||_{L^p_1} \tag{2.29}$$

for $p > 7$, C_1, C_2 constants independent of ϕ_T or the manifold. The result requires Stokes' Theorem, which holds for orbifolds, and a few direct calculations, which can be rewritten verbatim for orbifolds to give the following generalisation:

Proposition 2.11 (cf. [34, Theorem 10.3.7])**.** *Let $\mathcal{M}'$ be a compact Riemannian 7-orbifold whose metric comes from a closed G_2-structure, ϕ'. Then there exist constants $\rho > 0$ and $\widetilde{\rho} > 0$ independent of $\mathcal{M}'$ and ϕ' such that if a 4-form, $\widetilde{\Theta}$, on $\mathcal{M}'$ satisfies $d\widetilde{\Theta} = d\Theta(\phi')$ and $||\widetilde{\Theta}||_{C^0} < \widetilde{\rho}$, then for any 2-form η on $\mathcal{M}'$ with $||\eta||_{C^1} < \rho$ the equation:*

$$(dd^* + d^*d)\eta + \star d((1 + \frac{1}{3}\langle d\eta, \phi'\rangle)\widetilde{\Theta}) - \star dR(d\eta) = 0 \tag{2.30}$$

implies that $d\Theta(\phi' + d\eta) = 0$, where $\langle ., . \rangle$ denotes the inner product induced on the fibres of the tangent bundle and any other associated bundles. Hence $\phi' + d\eta$ gives a well-defined torsion-free G_2-structure on $\mathcal{M}'$

Given our previous estimate in **Lemma 2.10**, the condition on the sup-norm of $\widetilde{\Theta}(\phi_T)$ is satisfied for sufficiently large T so that we can solve **Equation (2.30)** for an unknown η, by putting $\phi' = \phi$ and $\widetilde{\Theta} = \widetilde{\Theta}(\phi_T)$ on the compact 7-orbifold $\mathcal{M}' = \mathcal{M}_T$. In fact, we will generalise a result due to Joyce ([34, Theorem 11.6.1]) to conclude the existence of torsion-free G_2-structures on $\mathcal{M}$:

Theorem 2.12. *Let λ, μ, ν be positive constants. Then there exist positive constants T_0 and C such that whenever $T \geq T_0$ the following is true:*

Let $\mathcal{M}_T$ be a compact 7-orbifold and ϕ_T a closed G_2-structure on $\mathcal{M}_T$. Let ψ_T be a 3-form on $\mathcal{M}_T$ such that $d^*\psi = d^*\phi$ and

- $||\psi||_{L^2} \leq \lambda e^{-4T}$
- $||\psi||_{C^0} \leq \lambda e^{-T/2}$
- $||\psi||_{L^{14}} \leq \lambda$
- *the injectivity radius, of the induced metric, g_T, $\delta(g_T)$, satisfies $\delta(g_T) \geq \mu e^{-T}$*
- *the Riemann curvature $R(g_T)$ satisfies $R(g_T) \leq \nu e^{2T}$*

Then there exists a torsion-free G_2-structure on $\mathcal{M}_T$, $\widetilde{\phi}$, such that $||\widetilde{\phi} - \phi||_{C^0} \leq Ce^{-T/2}$ for some positive constant C.

Note that by **Lemma 2.10**, the form $\star_{\phi_T}\widetilde{\Theta}(\phi_T)$ satisfies the norm estimates above. Moreover, in the case of the Twisted Connected Sum the curvature and injectivity radius are determined by those of the $\overline{\mathcal{W}}_i$ since the singular locus of the "neck", $\mathcal{X}_i \times S^1 \times S^1 \times [0; 2T]$, can be expressed as a direct product of the singular locus of the cross-section and $[0; 2T]$. Hence, they are constant along the "neck", where the gluing is performed. Away from the "neck", on the $\mathcal{W}_i \setminus \mathcal{U}_i$ part, they are trivially independent of the "neck" coordinates, in particular of T. As such, we can use **Theorem 2.12** to obtain a torsion-free G_2-structure on $\mathcal{M}_T$. All that is left is to prove **Theorem 2.12** in the case of orbifolds as stated. This is done by following the arguments in [34, Sections 11.7 and 11.8] without any significant differences from the smooth case. We will record the main steps of the proof below. To start, we will need a few results from analysis that routinely generalise to orbifolds, the first of which is the Sobolev Embedding Theorem and a version of Schauder estimates we will need (a proof in the smooth case can be found in [34, Theorem 6.4.8, p. 251].

Theorem 2.13 (Sobolev Embedding Theorem)**.** *Suppose $\mathcal{M}$ is a compact Riemannian n-orbifold, k, l integers with $k \geq l \geq 0$, q, r, real numbers with $q, r \geq 1$ and $\alpha \in (0; 1)$. If*

$$\frac{1}{q} \leq \frac{1}{r} + \frac{k-l}{n} \tag{2.31}$$

then $L^q_k(\mathcal{M})^{orb}$ is continuously embedded in $L^r_l(\mathcal{M})^{orb}$ by inclusion. If

$$\frac{1}{q} \leq \frac{k-l-\alpha}{n} \tag{2.32}$$

then $L^q_k(\mathcal{M})^{orb}$ is continuously embedded in $C^{l,\alpha}(\mathcal{M})^{orb}$ by inclusion.

Theorem 2.14 (Schauder Estimates)**.** *Suppose $\mathcal{M}$ is a compact Riemannian orbifold, $\mathcal{V}$, $\mathcal{W}$ vector orbibundles over $\mathcal{M}$ of the same dimension and P a linear, elliptic differential operator of order k from $\mathcal{V}$ to $\mathcal{W}$. Let $\alpha \in (0;1)$ and $l \geq 0$ be an integer. Suppose that the coefficients of P are in $(C^{l,\alpha})^{orb}$ and let $P(v) = w$ for some $v \in C^{k,\alpha}(\mathcal{V})^{orb}$ and $w \in C^{l,\alpha}(\mathcal{W})^{orb}$. Then $v \in C^{k+l,\alpha}(\mathcal{V})^{orb}$ and:*

$$||v||_{C^{k+l,\alpha}} \leq C(||w||_{C^{l,\alpha}} + ||v||_{C^0}) \tag{2.33}$$

for some constant independent of v and w.

We also require the following result about linear elliptic operators, the proof of which can be found in [34] in the smooth case:

Proposition 2.15. *Let $\mathcal{V}$, $\mathcal{W}$ be vector orbibundles over a compact Riemannian orbifold $\mathcal{M}$ equipped with metrics in the fibres and let P be a linear, elliptic differential operator of order k from $\mathcal{V}$ to $\mathcal{W}$. Let $l \geq 0$ and $\alpha \in (0;1)$. Then there is a constant $D > 0$ such that if $v \in C^{k+l,\alpha}(\mathcal{V})^{orb}$ and $v \perp ker(P)$, then*

$$||v||_{C^{k+l,\alpha}} \leq D||P(v)||_{C^{l,\alpha}} \tag{2.34}$$

Similarly, if $p > 1$ and $l \geq 0$ is an integer, then there exists a constant $D > 0$ such that $v \in L^p_{k+l}(\mathcal{V})^{orb}$ and $v \perp ker(P)$, then:

$$||v||_{L^p_{k+l}} \leq D||P(v)||_{L^p_l} \tag{2.35}$$

The first thing we will prove is the following:

Theorem 2.16. *Let μ, ν and T be positive constants, and suppose $\mathcal{M}$ is a complete Riemannian 7-orbifold, whose injectivity radius, $\delta(g)$, and Riemann curvature, $R(g)$ satisfy $\delta(g) \geq \mu e^{-T}$ and $R(g) \leq \nu e^{2T}$. Then there exist constants C_1 and C_2, depending only on μ and ν, such that if $\chi \in L^{14}_1(\Lambda^3 T^*\mathcal{M}) \cap L^2(\Lambda^3 T^*\mathcal{M})$, then the following hold:*

$$||\nabla\chi||_{L^{14}} \leq C_1(||d\chi||_{L^{14}} + ||d^*\chi||_{L^{14}} + e^{4T}||\chi||_{L^2}) \tag{2.36}$$

$$||\chi||_{C^0} \leq C_2(e^{-\frac{1}{2}T}||\nabla\chi||_{L^{14}} + e^{\frac{7}{2}T}||\chi||_{L^2}) \tag{2.37}$$

Proof. Using **Theorem 2.13** we note that L^{14}_1 embeds in C^0 in dimension 7, so we can follow the proofs in [7, Lemma 2.22] and [4, Theorem 10.3] to get the following "Euclidean" inequality:

If B_2 and B_3 denote Euclidean balls of radius 2 and 3 respectively about the origin in $\mathbb{R}^7/\Gamma$, then there exists $F_1, F_2, F_3 > 0$ such that if g is a Riemannian metric on B_3, $||g - g_{Eucl}||_{C^{1,\frac{1}{2}}} \leq F_1$, and $\chi \in L^{14}_1(\Lambda^3 T^* B_3)$ then the following hold:

$$||\chi|_{B_2}||_{C^0} \leq F_2(||\nabla\chi||_{L^{14}} + ||\chi||_{L^2}) \tag{2.38}$$

$$\int_{B_2} |\nabla\chi|^{14} dV_g \leq \int_{B_3} (|d\chi|^{14} + |d^*\chi|^{14}) dV_g + F_3 \left(\int_{B_3} |\chi|^2 dV_g \right)^7 \tag{2.39}$$

where dV_g denotes the volume form associated to g and g_{Eucl} denotes the standard metric on $\mathbb{R}^7$.

Moreover, following [39, p. 124] we can conclude that there exist coordinates on all balls of a given radius such that for each $\alpha \in (0;1)$, the $C^{0,\alpha}$ norm of the metric is bounded in terms of α. Firstly, note that rescaling the metric to $e^{2T}g$ we can, without loss of generality, assume that $T = 0$. Since $\delta(g) \geq \mu$ and $R(g) \leq \nu$, about every point $p \in \mathcal{M}$, we can find a coordinate system $\Psi_p : B_3 \to \mathcal{M}$ such that $||L^{-2}\Psi_p^* g - g_{Eucl}||_{C^{1,\frac{1}{2}}} \leq F_1$, where F_1 is as above and $L > 0$ is a constant depending only on μ, ν and F_1. Hence, following the argument in [34, Proposition 11.7.2 and Theorem G_1] we can find the desired constants.

□

By noting that standard results from Hodge Theory and the Cauchy-Schwarz inequality generalise to orbifolds, we can obtain the following theorem, which is an orbifold analogue of [34, Theorem $G2$]:

Theorem 2.17 (*cf.* [34, Theorem $G2$]). *Let λ, C_1 and C_2 be positive constants, then there exist constants k, K such that whenever $0 < e^{-T} < k$ the following is true:*

Let $\mathcal{M}$ be a 7-orbifold and ϕ the G_2 form associated to a closed G_2-structure on $\mathcal{M}$. Suppose ψ is a 3-form on $\mathcal{M}$ with $d^*\psi = d^*\phi$ and

- $||\psi||_{L^2} \leq \lambda e^{-4T}$
- $||\psi||_{C^0} \leq \lambda e^{-T/2}$
- $||\psi||_{L^{14}} \leq \lambda$
- $||\nabla\chi||_{L^{14}} \leq C_1(||d\chi||_{L^{14}} + ||d^*\chi||_{L^{14}} + e^{4T}||\chi||_{L^2})$
- $||\chi||_{C^0} \leq C_2(e^{-\frac{1}{2}T}||\nabla\chi||_{L^{14}} + e^{\frac{7}{2}T}||\chi||_{L^2})$

*Then if R is as in **Proposition 2.11**, we have that there exists a smooth* 2 *form* $\eta \in C^{\infty}(\Lambda^2 T^*\mathcal{M})$ *with* $||d\eta||_{C^0} \leq Ke^{-\frac{1}{2}T}$ *and* $f \in C^{\infty}(\mathcal{M})$ *such that:*

$$(dd^* + d^*d)\eta = d^*\psi + d^*(f\psi) + \star dR(d\eta) \tag{2.40}$$

$$f\phi = \frac{7}{3}\pi_1(d\eta) \tag{2.41}$$

where $\pi_1 : \Omega^3(\mathcal{M}) \to \Omega^3_1(\mathcal{M})$*, the space of forms which are a "function-multiple" of the* G_2 *form,* ϕ*, defined by*

$$\Omega^3_1(\mathcal{M}) = \{f\phi | f \in C^{\infty}(\mathcal{M})\} \tag{2.42}$$

Hence now the proof of **Theorem 2.12** follows straightforwardly:

*Proof (of **Theorem 2.12**).* Using **Theorem 2.16** and **Theorem 2.17**, we can define $\widetilde{\phi} = \phi + d\eta$, which gives a well-defined torsion-free G_2-structure on $\mathcal{M}$, exponentially close in the C^0 norm to ϕ.

□

2.3 Matching Orbifold $K3$ Surfaces

In this section we will deal with the matching problem for orbifold $K3$ surfaces. The singularity number of a $K3$ orbifold (see **Section** 1.3) plays a crucial role in the matching theorem we prove. Suppose that $(\mathcal{X}, \kappa_I)$ is an orbifold Ricci-flat Kähler $K3$ surface. Then, by analogy with the smooth story, $\mathcal{X}$ is hyper-Kähler with two complex structures J and K, with κ_J and κ_K the corresponding Kähler forms. Denote by $(\mathcal{X}_J, \kappa_J)$ the orbifold $K3$ surface we obtain by considering $\mathcal{X}$ equipped with the complex structure J as above.

Definition 2.18. Let $(\mathcal{X}, \kappa_I)$ and $(\mathcal{X}', \kappa'_I)$ be two orbifold Ricci-flat Kähler $K3$ surfaces with minimal resolutions $\pi : X \to \mathcal{X}$ and $\pi' : X' \to \mathcal{X}'$. We say that these satisfy the *matching condition* if there is a choice of holomorphic $(2,0)$-forms $\kappa_J + i\kappa_K$, $\kappa'_J + i\kappa'_K$ and an isometry $h : I^2(\mathcal{X}') \to I^2(\mathcal{X})$, extending to an isometry $\overline{h} : H^2(X'; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ such that $h_{\mathbb{R}}([\kappa'_I]) = [\kappa_J]$, $h_{\mathbb{R}}([\kappa'_J]) = [\kappa_I]$ and $h_{\mathbb{R}}([\kappa'_K]) = -[\kappa_K]$

For hyper-Kähler $K3$ orbifolds, $\mathcal{X}, \mathcal{X}'$, satisfying the matching condition, the lattice isometry above is in fact induced by an isometry of hyper-Kähler orbifolds $f : \mathcal{X}_J \to \mathcal{X}'$. This is the content of the next proposition which is an orbifold analogue of [44, Proposition 4.20]; the proof will also be analogous to the smooth one.

Proposition 2.19. *Suppose that* $(\mathcal{X};\kappa_I,\kappa_J,\kappa_K)$ *and* $(\mathcal{X}';\kappa'_I,\kappa'_J,\kappa'_K)$ *satisfy the matching condition. Then there is an isomorphism of complex orbifolds* $f:\mathcal{X}_J\to\mathcal{X}'$ *such that* $f^*=h$*. Moreover,* f *is an isometry of hyper-Kähler orbifolds, with the pullback action on Kähler forms given by:*

$$\begin{aligned} f^*\kappa'_I &= \kappa_J,\\ f^*\kappa'_J &= \kappa_I, \\ f^*\kappa'_K &= -\kappa_K \end{aligned} \tag{2.43}$$

Proof. Note that we have:

$$\begin{aligned} H^{2,0}(\mathcal{X}') &= \mathbb{C}[\kappa'_J+\kappa'_K],\\ H^{2,0}(\mathcal{X}_J) &= \mathbb{C}[\kappa_I - i\kappa_K] \end{aligned} \tag{2.44}$$

so $h_{\mathbb{C}}(H^{2,0}(\mathcal{X}')) = H^{2,0}(\mathcal{X})$. By the matching condition h maps $[\kappa'_I]$ to $[\kappa_J]$ and it extends to an isometry $\overline{h}: H^2(X';\mathbb{Z})\to H^2(X_J;\mathbb{Z})$ so that by **Theorem 1.50**, h arises as the pullback of a unique isomorphism $f:\mathcal{X}_J\to\mathcal{X}'$. To see the last part we note that a Kähler class uniquely determines a Ricci-flat Kähler metric on orbifold $K3$ surfaces so that the image $f^*\kappa'_I$ has to be the Ricci-flat κ_J on $\mathcal{X}_J$ and hence f is an isometry with the required pullback action. □

Because for two orbifold $K3$ surfaces to satisfy the matching condition we require the lattice isometry to extend to a lattice isometry of the resolutions, we can view the problem of matching orbifold $K3$ surfaces as a problem of matching their resolutions, as long as we are careful with how the root lattice is embedded into the $K3$ lattice. We can informally view this as follows:

We first primitively embed the root lattice into the $K3$ lattice and perform the matching of the two smooth resolutions in the orthogonal complement of the root lattice. This defines a smooth hyper-Kähler $K3$ surface for which we know how the root lattice embeds into L. We then blowdown the -2-curves corresponding to the generators of the root lattice and we have 2 orbifold $K3$'s with the correct singular locus which satisfy the matching condition.

Theorem 2.20 ([51]). *A primitive embedding of an even unimodular lattice* N *of signature* (t_+,t_-) *into an even unimodular lattice* E *of signature* (l_+,l_-) *exists provided* $t_+\le l_+$*,* $t_-\le l_-$ *and at least one of the following conditions holds:*

- *$2rk(N) \leq rk(E)$*
- *$rk(N) + l(N) < rk(E)$, where $l(N)$ is the minimum number of generators of the discriminant group N^*/N.*

If moreover $t_+ < l_+$, $t_- < l_-$ and one of the following two inequalities holds, $rk(N) + l(N) \leq rk(E) - 2$ or $2rk(N) \leq rk(E) - 2$, then a primitive embedding $N \to E$ is unique, up to isometry of E.

Now due to the embedding result of Nikulin we can deduce that if $sing(\mathcal{X}) \leq 9$, then $\mathcal{R}(\mathcal{X})^L$ primitively embeds into the even unimodular lattice, $(-E_8) \bigoplus (-E_8) \bigoplus H$, of signature $(1, 17)$. On the other hand, if $sing(\mathcal{X}) \leq 5$ then we can primitively embed $\mathcal{R}(\mathcal{X})^L$ into $-E_8 \bigoplus H$. In each case, the orthogonal complement, $I^2(\mathcal{X})$, primitively embeds into L as a rank $22 - sing(\mathcal{X})$ sublattice and contains at least two copies of the hyperbolic lattice, H. We now wish to investigate the main class of examples considered in this thesis, namely, $K3$ surfaces arising as anticanonical divisors in Fano 3-folds. For this we make the following definition:

Definition 2.21. Let $\mathcal{V}$ be a compact complex n-dimensional orbifold. We say that $\mathcal{V}$ is a *Fano n-fold* if the orbifold first Chern class is positive:

$$c_1(\mathcal{V})^{orb} > 0 \tag{2.45}$$

This is equivalent to the orbifold anticanonical bundle $K_{\mathcal{V}}^{-1}$ being ample.

Hence we let $\mathcal{V}$ be a Fano 3-orbifold. This case will follow in analogy with [45, Section 5] which we will refer to throughout the coming discussion and proof as the *smooth case*. Suppose that $\mathcal{X}$ arises as an anticanonical divisor on a Fano 3-orbifold $\mathcal{V}$ and let $\iota : \mathcal{X} \to \mathcal{V}$ denote the inclusion map. Then ι induces a primitive embedding $H^2(\mathcal{V}; \mathbb{Z}) \to H^2(\mathcal{X}; \mathbb{Z})$, by the Lefschetz Hyperplane Theorem ([47]) and since $\mathcal{V}$, up to homotopy equivalence, is obtained from $\mathcal{X}$ by adding cells of real dimension at least 3. Choose a marking, ϕ, for X the crepant resolution of $\mathcal{X}$. We then have primitive embeddings:

$$L(\mathcal{V}) \subset I^2(\mathcal{X}) \subset L \tag{2.46}$$

where $L(\mathcal{V}) = \phi(\iota^* H^2(\mathcal{V}; \mathbb{Z}))$. In analogy with the smooth case, we call orbifold $K3$ surfaces which contain a primitively embedded non-degenerate lattice $L(\mathcal{V})$ containing a Kähler class *ample $L(\mathcal{V})$-polarised $K3$ orbifolds.* In fact, the matching problem for orbifolds in this case is equivalent to the matching problem of "ample $L(\mathcal{V}), I^2(\mathcal{X})$-polarised (smooth) $K3$ surfaces" meaning that the $K3$ surfaces we are interested in are ample $I^2(\mathcal{X})$-polarised and $I^2(\mathcal{X})$ also contains a primitively embedded lattice, $L(\mathcal{V})$, which itself contains a Kähler class, as in **Equation (2.46)**.

Theorem 2.22. *Suppose that we have two pairs* $(\mathcal{V}_j, \mathcal{X}_j)$ *such that* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *have the same singular locus and are anticanonical orbifold* $K3$ *divisors of the Fano* 3*-folds* $\mathcal{V}_1$ *and* $\mathcal{V}_2$*. Let* L_j *denote the image of* $\iota^* H^2(\mathcal{V}_j; \mathbb{Z})$ *in the* $K3$ *lattice. Moreover, suppose that for each* j*, the* $K3$ *orbifolds arising from deformations of the pair* $(\mathcal{V}_j, \mathcal{X}_j)$ *form a Zariski open set in the moduli space of* L_j*-polarised* $K3$ *orbifolds. Suppose further that one of the following holds:*

- $sing(\mathcal{X}_j) \leq 9$ *and* $rkL_j = 1$ *for each* j
- $sing(\mathcal{X}_j) \leq 4$*,* $rkL_1 \leq 5$ *and* $rkL_2 = 1$*.*

Then $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *satisfy the matching condition.*

Proof. Most of the details of the proof are the same as those in ([45, Theorem 5.3]), since, as mentioned before, we try to primitively embed the polarising lattices into the orthogonal complement of the root lattice to obtain markings for the $K3$ surfaces which we can use to define the matching. The relevant distinction is that we need to embed both the L_j and the root lattices, $\mathcal{R}(\mathcal{X}_j)^L$, into L. Denote the direct sum decomposition of L as

$$L = (-E_8)_1 \oplus (-E_8)_2 \oplus H_1 \oplus H_2 \oplus H_3 \tag{2.47}$$

so that we can distinguish the different factors of $-E_8$ and H.

For the first case, we embed the root lattice into $(-E_8)_1 \bigoplus (-E_8)_2 \bigoplus H_3$ and then we have primitive embeddings of L_j into H_j, $j = 1, 2$.

For the second case we embed $\mathcal{R}(\mathcal{X}_j)^L$ into $(-E_8)_1 \bigoplus H_1$, L_2 into H_3, and L_1 into $(-E_8)_2 \bigoplus H_2$. In each case, we get the required primitive embeddings from **Theorem 2.20** and the setup has the required orthogonality properties needed for similar arguments to those in [45, Section 5] to go through. □

Remark 2.23. The first case of the theorem above covers all examples where the Fano variety is a weighted projective space and the $K3$ is a hypersurface in that weighted projective space, such that the $K3$ has "small" singularity number. To be a little bit more quantitative, this case covers matchings between 35 of the 95 families of $K3$ orbifolds which arise as hypersurfaces in weighted projective spaces ([20, Section 13.3]).

The second case is designed to cover examples where we match a hypersurface in a weighted projective space with a codimension 2 $K3$ orbifold in weighted $\mathbb{CP}^4$ with appropriately chosen weights. Note that such matchings are possible if and only if both $K3$ divisors have the same singular locus. Nonetheless, the restriction on the singular number of the $K3$ factors and the rank of the polarizing lattice make it quite difficult to find examples as designed for this case.

For more details see **Chapter** 4 below.

2.4 Topology of orbifold Twisted Connected Sums

We wish to find the orbifold fundamental group of $\mathcal{M}_T$, based on the orbifold fundamental groups of $\mathcal{W}_1$ and $\mathcal{W}_2$, just as in the case of the smooth Twisted Connected Sum. This will be useful for us later, once we have established a theorem linking the holonomy of an orbifold to its orbifold fundamental group (see **Chapter** 3, **Theorem 3.2** below). We also want similar results for the Betti numbers of the constructed orbifold. Towards this end, we first note that the Mayer-Vietoris exact sequence applies directly to $\mathcal{M}$ (as it is valid for all topological spaces) to allow the computation of its first Betti number, just as in the smooth case and secondly that the de Rham theorem (the fact that the de Rham cohomology is isomorphic to the singular cohomology with real coefficients) also holds for orbifolds ([17, Theorem 3.4.4]). Hence we get the following:

Theorem 2.24. *Let $\mathcal{M}_T$, $\mathcal{W}_1$ and $\mathcal{W}_2$ be as above. Then we have:*

$$b^1(\mathcal{M}_T) = b^1(\mathcal{W}_1) + b^1(\mathcal{W}_2) \tag{2.48}$$

$$\pi_1(\mathcal{M}_T) = \pi_1(\mathcal{W}_1) \times \pi_1(\mathcal{W}_2) \tag{2.49}$$

$$\pi_1^{orb}(\mathcal{M}_T) = \pi_1^{orb}(\mathcal{W}_1) \times \pi_1^{orb}(\mathcal{W}_2) \tag{2.50}$$

Proof. For the first part, note that applying the Mayer-Vietoris exact sequence to the decomposition $\mathcal{M}_T = \mathcal{M}_{1,T} \cup \mathcal{M}_{1,T}$ gives the result. Similarly, applying the regular Seifert-van Kampen Theorem and its orbifold version, **Theorem 1.24**, to the same decomposition yields the last part. □

Remark 2.25. In view of **Theorem 2.24**, we note that as long as $\pi_1^{orb}(\mathcal{W}_1)$ and $\pi_1^{orb}(\mathcal{W}_2)$ are finite, then so will $\pi_1^{orb}(\mathcal{M}_T)$. This observation will be useful in helping us conclude that the holonomy of the induced metric is exactly G_2 once we prove **Theorem 3.2** in the coming chapter.

For the rest of this section we ignore orbifold structures and view orbifolds purely as topological spaces. This allows us to compute various Betti and Hodge numbers of our orbifolds by applying the techniques of [45, section 2]. The only difference from the smooth case will be the appearance of correction terms due to the singularities on the $K3$ factors. We first present a result on the Hodge numbers of $\overline{\mathcal{W}}$:

Proposition 2.26. *Let $\overline{\mathcal{W}}$ be a 3-fold satisfying the hypothesis of* ***Theorem 2.7****. Then we have*

$$h^{1,0}(\overline{\mathcal{W}}) = h^{2,0}(\overline{\mathcal{W}}) = h^{3,0}(\overline{\mathcal{W}}) = 0 \tag{2.51}$$

Proof. Follows exactly as the proof of a similar result in [45, Proposition 2.2] □

We now aim to compute some topological invariants of G_2 orbifolds, $\mathcal{M}$, constructed via the orbifold Twisted Connected Sum from pairs $(\overline{\mathcal{W}_i}, \mathcal{X}_i)$, $i = 1, 2$, as was done in [45, Section 2]. The embeddings $\iota_i : \mathcal{X}_i \to \overline{\mathcal{W}_i}$ induce homomorphisms:

$$\iota_i : H^2(\mathcal{W}_i; \mathbb{R}) \to H^2(\mathcal{X}_i; \mathbb{R}) \tag{2.52}$$

where we note that $\mathcal{X}_i$ is viewed as part of the cross-section of $\mathcal{W}_i$. Set

$$X = \iota_1(H^2(\mathcal{W}_1; \mathbb{R})) \cap f^* \iota_2(H^2(\mathcal{W}_2; \mathbb{R})) \tag{2.53}$$

$$d_i = dim(ker \iota_i) \tag{2.54}$$

$$n = dim(X) \tag{2.55}$$

$$m = dim(H^2(\mathcal{X}_1; \mathbb{R})) = dim(H^2(\mathcal{X}_2; \mathbb{R})) \tag{2.56}$$

where f^* is the pullback action via the orbifold hyper-Kähler isometry we get from **Proposition 2.19**. Note in fact that if $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy the matching condition, then we must have $dim(H^2(\mathcal{X}_1; \mathbb{R})) = dim(H^2(\mathcal{X}_2; \mathbb{R}))$ so the definition of m makes sense. Moreover, we see that the restriction of the intersection form on $H^2(\mathcal{X}_1; \mathbb{R})$ to $\iota_1(H^2(\mathcal{W}_1; \mathbb{R}))$ and $f^*\iota_2(H^2(\mathcal{W}_2; \mathbb{R}))$ is non-degenerate with positive index 1. Just like in [45], X is negative-definite and we have uniquely defined subspaces X_i such that we get direct sum decompositions:

$$\iota_1(H^2(\mathcal{W}_1; \mathbb{R})) = X \oplus X_1 \tag{2.57}$$

$$f^*\iota_2(H^2(\mathcal{W}_2; \mathbb{R})) = X \oplus X_2 \tag{2.58}$$

Lemma 2.27. *Let $\overline{\mathcal{W}_i}$, $\mathcal{W}_i$, $\mathcal{X}_i$, $\mathcal{M}$, d_i, X_i, X, n and m be as above. Then we have:*

$$b^2(\mathcal{M}) = n + d_1 + d_2 \tag{2.59}$$

Moreover, if $b^2(\mathcal{W}_1) - d_1 + b^2(\mathcal{W}_2) - d_2 \leq m$ and X_1 is orthogonal to X_2 with respect to the intersection form on $H^2(\mathcal{X}_1; \mathbb{R})$ then

$$b^3(\mathcal{M}) = b^3(\overline{\mathcal{W}_1}) + b^3(\overline{\mathcal{W}_2}) + b^2(\mathcal{M}) - 2n + 1 + m \tag{2.60}$$

Proof. Since the normal bundle of $\mathcal{X}_i$ is trivial, it has a tubular neighbourhood, $\mathcal{U}_i$ in $\overline{\mathcal{W}_i}$ such that $\mathcal{U}_i$ is homotopy equivalent to $\mathcal{X}_i \times S^1$ so that using the Mayer-Vietoris sequence and the Künneth formula we get:

$$b^2(\overline{\mathcal{W}_i}) = b^2(\mathcal{W}_i) + 1 \tag{2.61}$$

$$b^3(\mathcal{W}_i) = b^3(\overline{\mathcal{W}_i}) - b^2(\mathcal{W}_i) + m - d_i \tag{2.62}$$

Just like in [45], to get $b^2(\mathcal{M})$ we use the Mayer-Vietoris sequence for

$$\mathcal{M} = (\mathcal{W}_1 \times S^1) \cup_F (\mathcal{W}_2 \times S^1) \tag{2.63}$$

where F is the gluing isometry defined in **Section** 2.2 above. The rest of the proof follows just like in [45], the only difference being that we replace instances of $H^2(K3; \mathbb{R})$ which has $dim = 22$ with $H^2(\mathcal{X}_i; \mathbb{R})$ which has $dim = m < 22$ leading to "orbifold corrections" in the formula for $b^3(\mathcal{M})$ which amount to adding $m - 22$ to the corresponding formula in [45].

□

Remark 2.28. In fact, as remarked in **section** 1.3, $m = 22 - sing(\mathcal{X}_1) = 22 - sing(\mathcal{X}_2)$ so we can rewrite the above as:

$$b^3(\mathcal{M}) = b^3(\overline{\mathcal{W}_1}) + b^3(\overline{\mathcal{W}_2}) + b^2(\mathcal{M}) - 2n + 1 + 22 - sing(\mathcal{X}_1) \tag{2.64}$$

which is similar to the expression in [45], the difference being the correction term, $sing(\mathcal{X}_1)$, arising due to the singularities on the orbifold $K3$ factors.

2.5 Examples of non-symplectic type

We end this section by describing a class of examples initially described by Kovalev and Lee in [45]. These rely on $K3$ surfaces with *non-symplectic involutions*. We recall that a holomorphic involution, ρ, of a $K3$ surface X, is called *non-symplectic* when $\rho^*(\omega) = -\omega$ for all $\omega \in H^{2,0}(X)$.

Non-symplectic involutions have been studied and completely classified by Nikulin in [51]. We will review his work for completeness:

Denote by L^ρ the set of classes in the $K3$ lattice, L, which is fixed by ρ^*. L^ρ is a primitive sublattice of the Picard lattice, $H^{1,1}(X) \cap H^2(X;\mathbb{Z})$. We call L^ρ the *invariant lattice* of ρ and it is a primitive non-degenerate sublattice of L of signature $(1,t)$ and rank $r = 1+t$. The orthogonal complement of L^ρ is the -1-eigenspace of ρ^* and moreover, we see that the discriminant group of L^ρ, $(L^\rho)^*/L^\rho$ is isomorphic to $(\mathbb{Z}_2)^a$ so if $l(L^\rho)$ denotes the minimal number of generators of the discriminant group of L^ρ, then we have $l(L^\rho) = a$. Lastly, we define:

$$\delta(L^\rho) = \begin{cases} 0, & t^2 \in \mathbb{Z}, \forall t \in (L^\rho)^* \\ 1, & \text{otherwise} \end{cases} \tag{2.65}$$

where the quadratic form on $(L^\rho)^*$ is the one taking values in $\mathbb{Q}$ induced by the bilinear form of L. We then have:

Proposition 2.29 ([51])**.** *The triple of (r,a,δ) defined above determine ρ^* and L^ρ uniquely, up to isometries of L.*

We shall denote by $L(r,a,\delta)$ the invariant lattice defined by the triple (r,a,δ). Nikulin classified all the possible triples which can appear; there are 75 possibilities and the ranges of values are $1 \le r \le 20$ and $0 \le a \le 11$. Moreover, any triple satisfies $r - a \ge 0$. Now given a triple, we can use the global Torelli theorem and the surjectivity of the period map to obtain a smooth $K3$ surface X with a non-symplectic involution ρ such that $L^\rho \cong L(r,a,\delta)$. Denote by X^ρ the fixed locus of ρ. Then we have the following proposition:

Proposition 2.30 ([51])**.** *Let X be a smooth $K3$ surface and ρ a non-symplectic involution whose invariant lattice has triple of invariants (r,a,δ). Denote by X^ρ the fixed locus of ρ, then we have:*

- *$X^\rho = \emptyset$ if $(r,a,\delta) = (10,10,0)$*
- *X^ρ is a disjoint union of elliptic curves if $(r,a,\delta) = (10,8,0)$*
- *otherwise,*
$$X^\rho = c_g + e_1 + \cdots + e_k \text{ all disjoint} \tag{2.66}$$
where $g = \frac{22-r-a}{2}$, $k = \frac{r-a}{2}$, c_g is a curve of genus g and $e_i \cong \mathbb{CP}^1$.

Now the suitable 3-folds are constructed from quotients of the form $(X \times \mathbb{CP}^1)/(\rho \times \psi)$, where $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$ is an involution fixing two distinct points, by resolving the singularities.

We wish to use orbifold $K3$ surfaces with non-symplectic holomorphic involutions, $(\mathcal{X}, \rho)$, to obtain suitable 3-folds obtained in a similar fashion to the smooth case i.e., from quotients of $(\mathcal{X} \times \mathbb{CP}^1)/(\rho \times \psi)$, with ψ as above. One way to do this, as pointed out by Frank Reidegeld in his Habilitation thesis, [55, Chapter 5.2. p. $71 - 74$], is by using an orbifold $K3$ surface, $\mathcal{X}$, such that $\mathcal{X}_{sing} \cap \mathcal{X}^{\rho} = \emptyset$, where, as before, $\mathcal{X}^{\rho}$ denotes the fixed point set of ρ, a non-symplectic holomorphic involution of $\mathcal{X}$. The condition that the singular locus of $\mathcal{X}$ is not fixed by ρ allows us to resolve the singularities of $(\mathcal{X} \times \mathbb{CP}^1)/(\rho \times \psi)$ which arise due to the fixed locus of $\rho \times \psi$, while leaving those that $\mathcal{X}$ started with unchanged. We want to resolve the singularities arising from taking the quotient by $\rho \times \psi$. This is because the resulting $K3$ divisors have a singular locus of complex dimension 1 while the matching results we have apply only for $K3$ orbifolds with 0-dimensional singular locus.

An initial observation we can derive from this set up, is that any orbifold $K3$ surface which admits a non-symplectic involution has singular locus given by:

$$\mathcal{R}(\mathcal{X})^L = \bigoplus_{i=1}^{n} 2k_i Q_i \tag{2.67}$$

where, as before, n denotes the number of distinct types of singularities, Q_i denotes the different types of singular points, and k_i denotes the number of singular points of type Q_i. This is because ρ needs to map a singular point of a certain type to a distinct singular point of the same type. Otherwise, either ρ maps singular points of one type to singular points of a different type, in which case ρ will not be holomorphic, or it fixes singular points; both cases are situations we want to avoid. Hence, the number of singular points of a given type on our orbifold $K3$ surface has to be even.

We now describe in some detail Reidegeld's work towards obtaining non-symplectic examples and then complete the examples by computing the orbifold fundamental group in **Section** 3.2. In **Section** 1.3, we outlined how one might obtain orbifold $K3$ surfaces with a prescribed singular locus starting from a smooth $K3$ surface with a large enough Picard number. We can slightly modify this procedure by starting with an orbifold $K3$ surface with a high singularity number, resolving its singularities to obtain a smooth $K3$ surface that has a high Picard number and then blowing down -2-curves as appropriate to obtain our desired singularities. Moreover, Reidegeld proved in [55, Chapter 5.2] that if we start with a smooth or an orbifold $K3$ surface which admits a pair of commuting involutions, then we can obtain two $K3$ surfaces (smooth or not, respectively) which satisfy the matching condition and can be used to construct suitable 3-folds.

To see how this is done, suppose that X is a hyper-Kähler $K3$ surface, ρ_i for $i = 1, 2$ are a pair of commuting holomorphic involutions with invariant lattices L_i, $\phi_i : H^2(X;\mathbb{Z}) \to L$ for $i = 1, 2$ are two markings for the $K3$ surfaces such that $\phi_1(L_1) \perp \phi_2(L_2)$ and $\phi_1(L_1) \oplus \phi_2(L_2)$ is primitively embedded into L. Let the hyper-Kähler structure of X be given by Kähler forms, ω_j with $j = 1, 2, 3$, and denote their classes by $x_j = \phi_1[\omega_j]$. We want ρ_1 to be non-symplectic with respect to $[\omega_1]$ and ρ_2 to be non-symplectic with respect to $[\omega_2]$ so that we have:

$$\rho_1^* x_1 = x_1,\ \rho_1^* x_2 = -x_2,\ \rho_1^* x_3 = -x_3 \tag{2.68}$$

$$\rho_2^* x_1 = -x_1,\ \rho_2^* x_2 = x_2,\ \rho_2^* x_3 = -x_3 \tag{2.69}$$

Then we can define another hyper-Kähler $K3$ surface X', for which we have $\phi_2([\omega'_j]) = y_j$, where the images of the Kähler classes, y_j, are given by:

$$\begin{aligned} y_1 &= x_2 \\ y_2 &= x_1 \\ y_3 &= -x_3 \end{aligned} \tag{2.70}$$

which admits $\rho' = \rho_2$ as a holomorphic non-symplectic involution, and the matching can be defined using the two initial markings we started with; explicitly, the matching map is $h : H^2(X';\mathbb{Z}) \to H^2(X;\mathbb{Z})$ defined by $h = \phi_1^{-1} \circ \phi_2$. Hence, we have produced two hyper-Kähler $K3$ surfaces, X and X', both equipped with a non-symplectic involution, ρ and ρ' respectively, and satisfying the matching condition.

Hence, we can use X and X' to obtain suitable 3-folds for the Twisted Connected Sum and then glue together those 3-folds to obtain a G_2-orbifold. All that we need now is an orbifold $K3$ surface with large singularity number and two non-commuting involutions. Reidegeld has managed to find such an orbifold $K3$ surface through lattice considerations; i.e. by appropriately choosing the classes x_i in L. He constructs an orbifold $K3$ surface, $\widetilde{\mathcal{X}}$ with 2 E_8 singularities, where the two commuting involutions are given by ψ_i, and act on the $K3$ lattice, $L = H_1 \oplus H_2 \oplus H_3 \oplus (-E_8)_1 \oplus (-E_8)_2$, as follows:

- ψ_i acts as the identity on H_i
- ψ_i acts as -1 on $H_{3-i} \oplus H_3$
- ψ_i interchanges $(-E_8)_1$ with $(-E_8)_2$

The invariants, (r_i, a_i, δ_i), for the fixed lattices of these involutions are $(r_i, a_i, \delta_i) = (10, 8, 0)$. Reidegeld then proves that the fixed locus of the non-symplectic involutions does not intersect the two E_8 singularities of $\widetilde{\mathcal{X}}$. Hence $\widetilde{\mathcal{X}}$ is suitable for the Twisted Connected Sum. This produces a G_2 orbifold, $\mathcal{M}$, with singular locus given by 2 S^3's and isotropy given by E_8. In fact, this method always produces a G_2 orbifold whose singular locus is given along some copies of S^3, with various isotropies. All in all, this gives the following proposition:

Proposition 2.31. *Let $\widetilde{\mathcal{X}}$ be the orbifold $K3$ surface constructed by Reidegeld in [55]. Then we can produce a $K3$ orbifold, $\mathcal{X}$, starting from $\widetilde{\mathcal{X}}$ via an appropriate finite sequence of blowups and blowdowns whose singular locus is given by:*

$$\mathcal{R}(\mathcal{X})^L = \bigoplus_{i=1}^{n} 2k_i Q_i \tag{2.71}$$

as long as the Q_i and k_i are chosen such that $sing(\mathcal{X}) \leq 18$. Furthermore, $\mathcal{X}$ admits two involutions ψ_i for $i = 1, 2$, as above, such that we can use $\mathcal{X}$ to construct a G_2 orbifold $\mathcal{M}$, for which $Hol(\mathcal{M})^{orb} = G_2$ and which has singular locus given by $\sum_{i=1}^{n} 2k_i Q_i$ copies of S^3, with isotropies given by the Q_i's.

Remark *2.32.* Computing the orbifold fundamental group in the following section will complete the statement related to the holonomy group of the constructed orbifold, $\mathcal{M}$.

Chapter 3

Orbifold G_2 Holonomy

In the smooth setting, one can easily tell if a compact G_2-manifold has holonomy group G_2 by computing the fundamental group. In this chapter, we prove that, for compact orbifolds, a similar result holds under mild assumptions on the codimension of the singular locus, as long as one computes the orbifold fundamental group. We use our theorem to compute the holonomy of various global quotients found in the literature. We also show that if M is a G_2-manifold and Γ is a finite non-trivial group of isometries, then the holonomy of the quotient M/Γ is G_2 if and only if the holonomy of M is G_2.

3.1 Main Result

In the case of compact manifolds, we can exploit a result due to Joyce, which is commonly referred to as Joyce's Lemma, to determine whether the holonomy group of a torsion-free G_2 metric on a 7-manifold, M, is exactly G_2. We record this result below:

Lemma 3.1 (Joyce, [34], Proposition 10.2.2)**.** *Let M be compact and $\phi \in \Omega^3(M)$ a torsion-free G_2-structure on M. Then $Hol(g_\phi) = G_2$ if and only if the fundamental group, $\pi_1(M)$, is finite.*

Note first that if we have a Riemannian orbifold $(\mathcal{M}, g)$, then we define the holonomy of the orbifold metric $Hol(g)^{orb} = Hol(\mathcal{M})^{orb} = Hol(\mathcal{M}_{reg}) = Hol(g|_{\mathcal{M}_{reg}})$. Now, if $\mathcal{M}$ is a compact orbifold, then $\mathcal{M}_{reg} = \mathcal{M} \setminus \mathcal{M}_{sing}$ will no longer be compact, but it is important to note that to apply Berger's Theorem we only require $\mathcal{M}_{reg}$ to be simply connected.

Moreover, to expand on the previous discussion in **Section** 1.2 related to the dimension of the singular locus of an orbifold, $\mathcal{M}$, $\mathcal{M}_{sing}$ is a finite disjoint union of connected components of strata $\Sigma_\Gamma = \{p \in \mathcal{M} | \Gamma_p \cong \Gamma\}$ ([17]). Hence we can define its codimension to be:

$$codim(\mathcal{M}_{sing}) = min\{codim((\Sigma_\Gamma)_i) | (\Sigma_\Gamma)_i \text{ is a connected components} \\ \text{of the non-empty stratum } \Sigma_\Gamma\}$$

Note that this is in fact just $n - n_{sing}$, where n is the dimension of the orbifold $\mathcal{M}$ and n_{sing} is the dimension of the singular locus, as defined previously by **Equation (1.6)**. Hence, we can prove the following generalisation to orbifolds:

Theorem 3.2. *Let $\mathcal{M}$ be a compact orbifold with $codim(\mathcal{M}_{sing}) \geq 3$ and $\phi \in \Omega^3(\mathcal{M})^{orb}$ a torsion-free G_2-structure on $\mathcal{M}$. Then $Hol(g_\phi) = G_2$ if and only if the orbifold fundamental group, $\pi_1^{orb}(\mathcal{M})$, is finite.*

Towards this goal, we have the following important results:

Theorem 3.3 ([46, Theorem 6.26])**.** *Let X, N be smooth manifolds and $f : N \to X$ be a continuous map. Then f is homotopic to a smooth map $F : N \to X$.*

Theorem 3.4 ([46, Thm 6.34])**.** *Let X, N be smooth manifolds, $V \subset X$ an embedded submanifold. Every smooth map $g : N \to X$ is homotopic to a smooth map $\widetilde{g} : N \to X$ transverse to V. If N is a manifold with boundary, then g and $\widetilde{g}$ can be chosen to be homotopic relative to the ∂N.*

Theorem 3.5 ([46, Theorem 6.29])**.** *Let X, N be smooth manifolds, $F, G : N \to X$ smooth maps. If F and G are homotopic, then they are smoothly homotopic, meaning that the homotopy $H : N \times [0;1] \to X$ can be taken to be a smooth map. If F and G are homotopic relative to some closed subset $A \subset N$, then they are smoothly homotopic relative to A.*

Proposition 3.6 ([25, Theorem 2.3, p. 146])**.** *Let X be a smooth, connected real manifold without boundary, $V \subset X$ a closed submanifold, x a point of $X \setminus V$ and $\iota : X \setminus V \to X$ is the inclusion map and the codimension of V is at least 3, then the map induced by ι:*

$$\iota_* : \pi_1(X \setminus V, x) \to \pi_1(X, x) \tag{3.1}$$

is an isomorphism.

Proof. Let $g : S^1 \to X$ be a continuous map. Then by **Theorem 3.3**, g is homotopic to a smooth map $S^1 \to X$, which we shall also denote by g. Now by **Theorem 3.4**, g is also homotopic to a smooth map $S^1 \to X$ that is transverse to V, which we shall also denote by g. Then we have $\forall p \in g^{-1}(V)$:

$$Im(D_p g) + T_{g(p)}V = T_{g(p)}X \tag{3.2}$$

Now, since $Codim(V) \geq 3$ and $Codim(g(S^1)) = n-1$, then clearly $V \cap g(S^1) = \emptyset$ so in particular any loop $\gamma \subset X$ is homotopic to a loop, $\gamma' \subset X \setminus V$ so indeed the map induced by the inclusion is surjective.

Now suppose g_0,$g_1 : S^1 \to X$ are continuous and homotopic via $G : S^1 \times [0;1] \to X$ with $G(-,0) = g_0$ and $G(-,1) = g_1$. Just as above we may assume that g_0 and g_1 are smooth maps and, moreover, by **Theorem 3.5** we may assume that G is smooth. By the previous part we can find (smooth) homotopies such that $g_0 \cap V = \emptyset$ and $g_1 \cap V = \emptyset$. Then using **Theorem 3.4** we know that G is homotopic to a smooth map $S^1 \times [0;1] \to X$ that is transverse to V and which we shall denote also by G. Hence as before we have $Codim(g(S^1 \times [0;1])) = n-2$ and $Codim(V) \geq 3$ so we must have $g(S^1 \times [0;1]) \cap V = \emptyset$ so indeed we have that g_0 and g_1 are homotopic in $X \setminus V$ hence if γ_1 and γ_2 are loops in $X \setminus V$ such that $\iota^*[\gamma_1] = \iota^*[\gamma_2]$, then $\gamma_1 \simeq \gamma_2$ in $X \setminus V$, hence the map induced by the inclusion is injective. □

We can generalise **Proposition 3.6** to orbifolds to obtain:

Proposition 3.7. *Let $\mathcal{M}$ be a n-dimensional real orbifold. If the codimension of $\mathcal{M}_{sing}$ is at least* 3 *then, $\pi_1(\mathcal{M}_{reg}) \cong \pi_1(\mathcal{M})$ as topological spaces.*

Proof. Follows directly from repeated application of **Proposition 3.6** on each connected component of $\mathcal{M}_{sing}$ that we have $\pi_1(\mathcal{M}_{reg}) \cong \pi_1(\mathcal{M})$ □

Remark 3.8. If $\mathcal{M}$ is a complex orbifold, then $\mathcal{M}_{sing}$ has complex codimension at least 2, so it has real codimension at least 4, so $\pi_1(\mathcal{M}_{reg}) \cong \pi_1(\mathcal{M})$

Before we go on to prove **Theorem 3.2**, we also note the following theorem:

Theorem 3.9 ([12])**.** *Let $\mathcal{M}$ be a compact Riemannian orbifold with non-negative Ricci curvature and let $\widehat{\mathcal{M}}$ denote its universal orbifold cover. Then $\widehat{\mathcal{M}} = N \times \mathbb{R}^l$ where N is a compact orbifold and $l \geq 0$. Also, there is a short exact sequence:*

$$1 \to F \to \pi_1^{orb}(\mathcal{M}) \to C \to 1 \tag{3.3}$$

where F is a finite group and C is a discrete cocompact group of isometries acting on $\mathbb{R}^l$, that is C is a crystallographic group.

Recall from our previous discussion in **Section** 1.2 that the orbifold universal cover is orbifold simply connected ($\pi_1^{orb}(\widehat{\mathcal{M}}) = \{1\}$) and so it is also simply connected as a topological space $\pi_1(\widehat{\mathcal{M}}) = \{1\}$. Moreover, as mentioned in [12], we have that the isometry group of $\widehat{\mathcal{M}}$ splits:

$$Isom(\widehat{\mathcal{M}}) = Isom(N) \times Isom(\mathbb{R}^l) \tag{3.4}$$

so that we have the following corollary:

Corollary 3.9.1. *Let $(\mathcal{M}, g)$ be a compact Riemannian orbifold. If g is Ricci-flat, then $\mathcal{M}$ admits a finite cover isometric to $N \times T^l$, where N is a compact Riemannian orbifold that is orbifold simply connected and T^l is a flat torus.*

Hence we can now prove **Theorem 3.2**:

*Proof (of **Theorem 3.2**).* $\mathcal{M}$ is a compact G_2 orbifold so g_ϕ is Ricci-flat so by **Theorem 3.9** and **Corollary 3.9.1**, there is a finite cover isometric to $N \times T^l$, where T^l is a flat torus and N is a compact orbifold simply connected Riemannian orbifold, so that

$$\pi_1^{orb}(\mathcal{M}) \cong F \ltimes \mathbb{Z}^l \tag{3.5}$$

where F is a finite group so indeed $\pi_1^{orb}(\mathcal{M})$ is finite if and only if $l = 0$.

Now since ϕ is a torsion-free G_2-structure and N is orbifold simply connected (and hence simply connected as a topological space), the metric, g, induced on N_{reg} satisfies $Hol(N)^{orb} \subset G_2$. Therefore $Hol^0(N)^{orb}$ is either $\{1\}$, $SU(2)$, $SU(3)$, or G_2. Moreover, since N_{reg} is simply connected by **Proposition 3.7**, then by Berger's Theorem we have that $l = 7$ when $Hol^0(N)^{orb} = \{1\}$, $l = 3$ when $Hol^0(N)^{orb} = SU(2)$, $l = 1$ when $Hol^0(N)^{orb} = SU(3)$ and $l = 0$ when $Hol^0(N)^{orb} = G_2$. Thus, overall we get that $Hol(g_\phi)^{orb} = G_2$ if and only if $\pi_1^{orb}(\mathcal{M})$ is finite. $\square$

Remark 3.10. For orbifolds constructed via a Twisted Connected Sum, we can see that the singular locus will consist of disjoint copies of S^1, $S^1 \times \{\text{Complex Curve}\}$, or $S^1 \times S^1 \times [-T; T]$.

To see this, denote by $(\mathcal{W}, \mathcal{X})$ the pair of an ACyl 3-dimensional complex orbifold, $\mathcal{W}$, and an orbifold $K3$ surface, $\mathcal{X}$, so that the cylindrical end of $\mathcal{W}$ is given by $\mathcal{X} \times S^1 \times [0; \infty)$. Then the singular locus of $\mathcal{W}$ can only consist of isolated points away from the cylindrical end, complex curves away from the cylindrical end, or shared singularities with $\mathcal{X}$, which are isolated points of $\mathcal{X}$; hence in $\mathcal{W}$, the isolated orbifold singularities of $\mathcal{X}$ become orbifold singularities along $\{p\} \times S^1 \times [0; \infty)$ for $p \in \mathcal{X}_{sing}$. The Twisted Connected Sum glues together 7-orbifold of the form $\mathcal{W} \times S^1$ so that the obtained G_2-orbifolds will have the claimed singular locus, by noting that along the neck region components of the singular locus of one pair are glued to components of the singular locus of the other with the same isotropy.

This then shows that the singular locus has codimension > 3 so **Theorem 3.2** applies. Recalling the observation in **Remark 2.25**, that **Theorem 2.24** implies that, for the constructed orbifold $\mathcal{M}_T$, the orbifold fundamental group $\pi_1^{orb}(\mathcal{M}_T)$ is finite if and only if the orbifold fundamental groups of the two ACyl complex 3-orbifold, $\mathcal{W}_1$ and $\mathcal{W}_2$, are finite. Hence, similarly to the smooth construction, we can reduce the question of irreducibility of orbifold Twisted Connected Sums to a question about the sizes the orbifold fundamental groups of the "building blocks".

3.2 Global Quotients and Non-Symplectic Examples

We first note that by a *global quotient*, we mean an orbifold of the form $\mathcal{M} = M/\Gamma$, where M is a Riemannian manifold and Γ is a finite group of isometries. We begin this section by applying **Theorem 3.2** to a few cases in the literature.

Example 3.11. The construction of compact G_2-manifolds, due to Joyce and Karigiannis [37], starts with a compact 7-manifold, M, admitting a torsion-free G_2-structure and an involution ι preserving the G_2-structure, but whose holonomy group can be a proper subgroup of G_2. They then consider the quotient of this manifold by the group generated by the involution, $\mathcal{M} = M/\langle\iota\rangle$, which will be a G_2 orbifold, whose singular locus $\mathcal{M}_{sing}$ is an associative submanifold of M and the singular points are locally modeled on $\mathbb{R}^3 \times (\mathbb{R}^4/\langle\pm 1\rangle)$ (in fact, whenever we quotient a G_2-manifold by an involution preserving the G_2 3-form, the result will be an orbifold whose singular locus will be a closed embedded 3-fold in M, [35, Prop. 12.3.7]). They then resolve the singularities by gluing families of Eguchi-Hanson spaces parametrised by a non-zero, closed, and coclosed 1-form on the singular locus to obtain new examples of G_2-manifolds.

In fact, their construction also works if we consider more general orbifolds, $\mathcal{M}$, (i.e., not global quotients), as long as the singular locus of $\mathcal{M}$

is locally modeled on $\mathbb{R}^3 \times (\mathbb{R}^4/\Gamma)$ where Γ is a finite group of isometries preserving the G_2 3-form.

We will be interested in $\mathcal{M}$, the intermediate orbifold they obtain and in the cases where $\mathcal{M} = M/\Gamma$, with $\Gamma \cong \mathbb{Z}_2^n$, $n = 1, 2$, is a group of involutions preserving the G_2 3-form and M is a torsion-free G_2-manifold such that the singular locus has the above local model. Note that this orbifold is a quotient orbifold so we have a short exact sequence of groups ([3, p. 27]):

$$1 \to \pi_1(M) \to \pi_1^{orb}(\mathcal{M}) \to \Gamma \to 1 \tag{3.6}$$

There are two main types of examples considered in [37]:

(a) $M = T^3 \times X$ where the 3-torus, T^3, carries a flat metric, X is a $K3$ surface and $\Gamma \cong \mathbb{Z}_2^2$

(b) $M = Y \times S^1$, where Y is a Calabi-Yau 3-fold admitting an anti-holomorphic involution τ, $S^1 = \{z \in \mathbb{C} |\ |z| = 1\}$ is the unit circle, and $\Gamma = \langle \iota \rangle \cong \mathbb{Z}_2$, where $\iota : Y \times S^1 \to Y \times S^1$ is given by $\iota(y, z) = (\tau(y), -z)$

For the examples in (a), note that $\pi_1(M) = \mathbb{Z}^3$ so $\pi_1^{orb}(\mathcal{M})$ is infinite by the above exact sequence; moreover, note that $Hol(M) = SU(2)$, since M is equipped with the product metric and the 3-torus is flat. Similarly, for the examples in (b) we have that $\pi_1(M) = \mathbb{Z}$ so that $\pi_1^{orb}(\mathcal{M})$ is also infinite, and $Hol(M) = SU(3)$.

Theorem 3.2 tells us that in both cases, $Hol(\mathcal{M})^{orb}$ is not G_2. We can also see this directly by looking at the generators or the fundamental group. We see that for (b) we have, in fact, $\pi_1^{orb}(Y \times S^1) = \mathbb{Z} \times \mathbb{Z}_2$. Similarly, we could compute the orbifold fundamental group of the examples in (a), if we are given the action of $\mathbb{Z}_2^2$ on $T^3 \times X$.

Now suppose that we have a compact manifold M and a group, Γ, of isometries preserving the G_2 3-form on M and such that the fixed locus consists only of a 3-dimensional submanifold and set $\mathcal{M} = M/\Gamma$. If $Hol(M) = G_2$, then we know that $\pi_1(M)$ is finite, and therefore by viewing $\pi_1(M)$ as a normal subgroup of $\pi_1^{orb}(\mathcal{M})$, we have $\pi_1^{orb}(\mathcal{M})/\pi_1(M) \cong \Gamma$. Thus, we see that $\pi_1^{orb}(\mathcal{M})$ is finite and hence $Hol(\mathcal{M}) = G_2$.

We can thus ask, for what kinds of finite groups is a global quotient a full holonomy orbifold. To answer this question, we need to investigate what kind of singular loci we can obtain when quotienting a G_2-manifold by a finite group of diffeomorphisms preserving the G_2 3-form; more concretely we only need to find out what are the possible codimensions we get for various finite subgroups of G_2. Toward this, note that Joyce proved the following proposition:

Proposition 3.12 ([34], Prop. 11.1.3)**.** *Let* $\Gamma \subset G_2$ *be a finite subgroup and let*

$$V^{\Gamma} = \{v \in \mathbb{R}^7 | \gamma(v) = v, \textit{ for all } \gamma \in \Gamma\} \tag{3.7}$$

be the subspace of Γ*-invariant vectors in* $\mathbb{R}^7$*. Then* V^{Γ} *is one of the following:*

a) $V^{\Gamma} = \mathbb{R}^7$

b) $dim(V^{\Gamma}) = 3$ *and* $\mathbb{R}^7/\Gamma \cong \mathbb{R}^3 \times (\mathbb{C}^2/\Gamma)$

c) $dim(V^{\Gamma}) = 1$ *and* $\mathbb{R}^7/\Gamma \cong \mathbb{R} \times (\mathbb{C}^3/\Gamma)$

d) $V^{\Gamma} = \{0\}$

The above proposition implies that if $\gamma \neq id_{\mathbb{R}^7} \in G_2$, then the subspace of vectors fixed by γ has codimension ≥ 3. Now suppose that M is a G_2-manifold with full holonomy and let $\Gamma \neq \{id_M\}$ be a finite group of diffeomorphisms of M such that $\gamma^*\phi = \phi$ for all $\gamma \in \Gamma$, where ϕ is the G_2 3-form of M and denote by $\mathcal{M}$ the quotient of M by Γ, $\mathcal{M} = M/\Gamma$. Firstly, since all the diffeomorphisms in Γ preserve ϕ, then $\mathcal{M}$ is a G_2 orbifold. Secondly, note that $\pi_1(M)$ is finite so by the previous discussion $\pi_1^{orb}(\mathcal{M})$ is also finite. Finally, note that, by the previous proposition, if $M_{fix} = \{p \in M | \gamma(p) = p, \text{ for all } \gamma \in \Gamma\}$, then $codim(M_{fix}) = codim(\mathcal{M}_{sing}) \geq 3$. Hence we can apply **Theorem 3.7** to conclude that $\mathcal{M}$ is an orbifold with full holonomy.

Conversely, if $Hol(M) \neq G_2$, then we have that $\pi_1(M)$ is infinite, so by the short exact sequence for a global quotient ([3, p. 27]):

$$1 \to \pi_1(M) \to \pi_1^{orb}(\mathcal{M}) \to \Gamma \to 1 \tag{3.8}$$

we have that $\pi_1^{orb}(\mathcal{M})$ is also infinite; in fact, it is an extension of $\pi_1(M)$ by Γ. Hence, again by **Theorem 3.7**, $Hol(\mathcal{M})^{orb} \neq G_2$. All in all, we have proved the following:

Corollary 3.12.1. *Let* M *be a compact* G_2*-manifold. Let* $\Gamma \neq \{id_M\}$ *be a finite group of diffeomorphisms of* M*, such that for all* $g \in \Gamma$*,* $g^*\phi = \phi$*, where* ϕ *is the* G_2 *3-form of* M*. Then* $\mathcal{M} = M/\Gamma$ *is a* G_2*-orbifold and we have that* $Hol(\mathcal{M})^{orb} = G_2$ *if and only if* $Hol(M) = G_2$

This shows that in terms of global quotients of G_2-manifolds with full G_2 holonomy, then we will always get an orbifold with full G_2 holonomy (as the resulting singular locus will have codimension ≥ 3), as long as the group we are quotienting by is not trivial, hence answering our initial question.

Example 3.13. In [35, Section 12.3.3], Joyce considers a compact G_2-manifold with full holonomy obtained by resolving the torus quotient T^7/Γ, where Γ is a group of involutions of T^7 generated by

$$\alpha(x_1,\ldots,x_7)=(x_1,x_2,x_3,-x_4,-x_5,-x_6,-x_7) \tag{3.9}$$

$$\beta(x_1,\ldots,x_7)=(x_1,-x_2,-x_3,x_4,x_5,\tfrac{1}{2}-x_6,-x_7) \tag{3.10}$$

$$\gamma(x_1,\ldots,x_7)=(-x_1,x_2,-x_3,x_4,\tfrac{1}{2}-x_5,x_6,\tfrac{1}{2}-x_7) \tag{3.11}$$

where $(x_1,\ldots,x_n)$ denote the standard coordinates on T^7. He then considers the involution $\sigma : T^7 \to T^7$ defined by

$$\sigma(x_1,\ldots,x_7)=(x_1,x_2,x_3,\tfrac{1}{2}-x_4,-x_5,-x_6,-x_7) \tag{3.12}$$

and notes that σ commutes with Γ so it gives a well-defined involution of T^7/Γ, to be denoted by σ still. He then argues that the resolution of T^7/Γ, can be done in such a way that we get a compact G_2-manifold M, with G_2 3-form φ, such that $Hol(M)=G_2$ and σ lifts to an involution $\widehat{\sigma} : M \to M$ with $\widehat{\sigma}^*(\varphi)=\varphi$. The fixed points of $\widehat{\sigma}$ are two disjoint copies of T^3. Hence we conclude by **Proposition 3.12.1** that $\mathcal{M}=M/\langle\widehat{\sigma}\rangle$ is a G_2-orbifold with $Hol(\mathcal{M})^{orb}=G_2$.

To end this section, we compute the orbifold fundamental group of the examples constructed by Reidegeld in [55], as promised in **Section** 2.5. In the notation of **Proposition 2.31**, let $\mathcal{X}$ be an orbifold $K3$ surface with singular locus $\mathcal{R}(\mathcal{X})^L=\bigoplus_{i=1}^n 2k_iQ_i$, $\psi : \mathcal{X}\to\mathcal{X}$ a holomorphic non-symplectic involution of $\mathcal{X}$ and $\rho : \mathbb{CP}^1\to\mathbb{CP}^1$ a holomorphic involution of $\mathbb{CP}^1$ fixing two distinct points p_1 and p_2. Let $\mathcal{Z}=(\mathcal{X}\times\mathbb{CP}^1)/(\psi\times\rho)$. The fixed locus of ψ is given by $\mathcal{X}^\psi=c_1+\cdots+c_l$, with c_i a complex curve for all $i=1,\ldots,l$ and write $P_{ij}=c_i\times\{p_j\}\subset\mathcal{X}\times\mathbb{CP}^1$. Denote by $\widetilde{\mathcal{W}}$ the blow-up of $\mathcal{X}\times\mathbb{CP}^1$ along the P_{ij}'s and note that the involution $\psi\times\rho$ of $\mathcal{X}\times\mathbb{CP}^1$ induces an involution $\widetilde{\psi\times\rho}$ on $\widetilde{\mathcal{W}}$. If we denote by $\pi : \mathcal{X}\times\mathbb{CP}^1\to\mathcal{Z}$ the quotient map and we define $P'_{ij}=\pi(P_{ij})$ and $\overline{\mathcal{W}}$ the blow up of $\mathcal{Z}$ along the P'_{ij}, then we get that $\overline{\mathcal{W}}$ is isomorphic (this time as complex orbifolds) to the quotient of $\widetilde{\mathcal{W}}$ by $\widetilde{\psi\times\rho}$, just as in [45, Section 4, p. 11]. Lastly, let G_i denote the group associated to Q_i by the ADE correspondence as in **Section** 1.3. We prove the following:

Lemma 3.14. *We have that* $\pi_1^{orb}(\overline{\mathcal{W}})=\mathbb{Z}_2\times_{i=1}^n 2k_iG_i$ *and* $\pi_1^{orb}(\mathcal{W})=\mathbb{Z}_2\times_{i=1}^n 2k_iG_i$*, where* $\mathcal{W}=\overline{\mathcal{W}}\setminus\mathcal{D}$*,* $\mathcal{D}$ *is the inverse image of* $\mathcal{D}'$ *in* $\overline{\mathcal{W}}$*, and* $\mathcal{D}'$ *is the image of* $\mathcal{X}\times\{p\}$ *in* $\mathcal{Z}$ *for* $p\in\mathbb{CP}^1$ *a point which is not fixed by* ρ*.*

Proof. Note that by repeatedly applying Seifert-van Kampen Theorem to $\mathcal{X}$, we get

$$\pi_1^{orb}(\mathcal{X}) = \times_{i=1}^{n} 2k_i G_i \tag{3.13}$$

so that $\pi_1^{orb}(\mathcal{X} \times \mathbb{CP}^1) = \times_{i=1}^{n} 2k_i G_i$.

Note that the arguments in [45, Lemma 4.2] still apply, and show that $\widetilde{\mathcal{W}}$, $\overline{\mathcal{W}}$, $\mathcal{W}$, and $\mathcal{D}$ are simply connected. Moreover, $\pi_1^{orb}(\widetilde{\mathcal{W}}) \cong \pi_1^{orb}(\mathcal{X} \times \mathbb{CP}^1)$. Hence, by considering the generators of $\pi_1^{orb}(\widetilde{\mathcal{W}})$, we see that

$$\pi_1^{orb}(\overline{\mathcal{W}}) \cong \mathbb{Z}_2 \times \pi_1^{orb}(\widetilde{\mathcal{W}}) \tag{3.14}$$

giving the stated result.

If γ is a $\mathcal{H}_{\overline{\mathcal{W}}}$-loop in $\mathcal{W}$ around $\mathcal{D}$, then there exists an element of the form $(g_0, g_{1,1}, \dots, g_{1,2k_1}, \dots, g_{n,2k_n}) \in \mathbb{Z}_2 \times_{i=1}^{n} 2k_i G_i$ such that the $\mathcal{H}_{\overline{\mathcal{W}}}$-loop obtained by the induced action on γ is contractible since $\mathcal{W}$ is simply connected. This shows $\pi_1^{orb}(\mathcal{W}) \cong \pi_1^{orb}(\overline{\mathcal{W}})$ giving the second result. □

Corollary 3.14.1. *If $\mathcal{M}$ is the G_2 orbifold constructed using $K3$ orbifolds with non-symplectic involutions, as in* ***Section*** *2.5, then $Hol(\mathcal{M})^{orb} = G_2$.*

Proof. Note that by **Lemma 3.14**, the building blocks obtained by Reidegeld, $\mathcal{W}_i$, have finite fundamental group. We then have $\pi_1^{orb}(\mathcal{M}) = \pi_1^{orb}(\mathcal{W}_1) \times \pi_1^{orb}(\mathcal{W}_2)$ by **Theorem 2.24** and hence is finite. Thus by **Theorem 3.2**, $Hol(\mathcal{M})^{orb} = G_2$. □

Chapter 4

New Examples of G_2-orbifolds

In this chapter, we exploit the orbifold Twisted Connected Sum to construct new examples of G_2 orbifolds by using weighted projective spaces. We find two classes of examples: those that have orbifold singularities along the neck of the Twisted Connected Sum, and those for which the singular locus is away from the neck region. We find 35 examples of G_2 orbifolds of the first class and 26 examples of the second kind. Note that, this doesn't exhaust all the possible examples that could, in principle, be constructed from weighted projective spaces, by investigating the matching problem for orbifolds in more detail.

4.1 Weighted Projective Spaces

This section will be dedicated to providing foundational results on *weighted projective spaces*, which will be an important class of examples for our orbifold Twisted Connected Sums. Most of the content of this section can be found in [24] and was originally introduced and proved in [20]. We denote by $\mathbb{CP}^n(a_0, \dots, a_n)$ the *weighted projective space with weights* $(a_0, \dots, a_n)$, for some positive integers $a_0, \dots, a_n$; that is the quotient $\mathbb{C}^{n+1}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts via:

$$\lambda.(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n) \tag{4.1}$$

In this setting affine pieces, $x_i \neq 0$, are isomorphic to $\mathbb{C}^n/\mathbb{Z}_{a_i}$, with the action given by:

$$z_j \mapsto \epsilon^{a_j} z_j \tag{4.2}$$

where ϵ is a primitive a_i-th root of unity and $j \neq i$. One important aspect of weighted projective spaces is that not all tuples $(a_0, \ldots, a_n)$ give us distinct spaces:

Lemma 4.1 ([20], 1.3.1)**.** *$\mathbb{CP}^n(a_0, \ldots, a_n) \simeq \mathbb{CP}^n(b_0, \ldots, b_n)$ for some tuple $\{b_0, \ldots, b_n\}$ such that*

$$hcf(b_0, \ldots, \widehat{b_i}, \ldots, b_n) = 1 \tag{4.3}$$

for each i.

For example, $\mathbb{CP}^3(2,5,5) \simeq \mathbb{CP}^3(2,1,1)$. We shall say that a weighted projective space, $\mathbb{CP}^n(a_0, \ldots, a_n)$ is *well-formed* if and only if the following holds $hcf(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for all i.

Definition 4.2. Let $r > 0$, $a_1, \ldots, a_n \in \mathbb{Z}$ and $x_1, \ldots, x_n$ be coordinates on $\mathbb{C}^n$. Suppose $\mathbb{Z}_r$ acts on $\mathbb{C}^n$ via:

$$x_i \mapsto \epsilon^{a_i} x_i \tag{4.4}$$

for all i, where ϵ is a fixed r-th root of unity. If X is a variety, we say that $Q \in X$ is a *(quotient) singularity of type* $\frac{1}{r}(a_1, \ldots, a_n)$ if a neighbourhood of Q in X is analytically isomorphic to a neighbourhood of 0 in $\mathbb{C}^n/\mathbb{Z}_r$.

Let $P_i = [0, \ldots, 1, \ldots, 0] \in \mathbb{CP}^n(a_0, \ldots, a_n)$ where the 1 is in the i-th position and $\mathcal{M} = \mathbb{CP}^n(a_0, \ldots, a_n)$ is well-formed. We call P_i a vertex and the union of all the coordinate hyperplanes $P_0 \ldots \widehat{P_i} \ldots P_n$ the fundamental simplex. All singularities of $\mathbb{CP}^n(a_0, \ldots, a_n)$ occur on the fundamental simplex; the vertices are singularities of type $\frac{1}{a_i}(a_0, \ldots, \widehat{a_i}, \ldots, a_n)$, which are not necessarily isolated. The generic point P on the edge P_iP_j has an analytic neighbourhood $P \in U$ which is isomorphic to $(0, Q) \in \mathbb{C} \times Y$ where $Q \in Y$ is a singularity of type $\frac{1}{h_{i,j}}(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n)$, where $h_{i,j} = hcf(a_i, a_j)$. Similar results hold for points on higher dimensional strata. Note that $codim(\mathcal{M}_{sing}) \geq 2$.

Definition 4.3. Let $f_1, \ldots, f_c$ be homogeneous polynomials in the graded polynomial ring $S(a_0, \ldots, a_n) = \mathbb{C}[x_0, \ldots, x_n]$ with $deg(x_i) = a_i$ and having degrees $deg(f_i) = d_i$. We say that

$$X = \cap_{i=1}^{c} \{P | f_i(P) = 0\} \subset \mathbb{CP}^n(a_0, \ldots, a_n) \tag{4.5}$$

is a **weighted complete intersection of multidegree** $\{d_i | i = 1, \ldots, c\}$. In this case, we denote $X = X_{d_1,\ldots,d_c}$. We have $dim X = n - c$. We say that $X_d \subset \mathbb{CP}^n(a_0, \ldots, a_n)$ is a linear cone if $d = a_i$ for some i and in this case we have that $X_d \simeq \mathbb{CP}^{n-1}(a_0, \ldots, \widehat{a_i}, \ldots, a_n)$.

We adopt the convention that we write general polynomials of a given weighted homogeneous degree without the nonzero coefficients. For example the defining polynomial of X_3 in $\mathbb{CP}^2(1, 1, 1)$ will be written as:

$$f = x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + xyz \tag{4.6}$$

All weighted projective spaces and weighted complete intersections are orbifolds, possibly without singular points. Now, let $\mathcal{M} = \mathbb{CP}^n(a_0, \ldots, a_n)$ be well-formed. We say that the weighted complete intersection $X_{d_1,\ldots,d_c}$ is *well-formed* if and only if for all $\mu \in \{1, \ldots, c\}$, the highest common factor of any $(n-1-c+\mu)$ of the $\{a_i\}$ divides at least μ of the $\{d_j\}$. For example, if we have a hypersurface of degree d, $X_d \subset \mathcal{M}$, then X_d is well-formed if and only if $hcf(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$. We note ([24, Section 6]) that well-formedness means that if $X = X_{d_1,\ldots,d_c}$, then

$$codim_X(X \cap (\mathbb{CP}^n(a_0, \ldots, a_n))_{sing}) \geq 2 \tag{4.7}$$

In particular if X is a well-formed curve in $\mathbb{CP}^n(a_0, \ldots, a_n)$ and X is smooth, then $X \cap (\mathbb{CP}^n(a_0, \ldots, a_n))_{sing} = \emptyset$.

Definition 4.4. Let X be a subvariety of $\mathbb{CP}^n(a_0, \ldots, a_n)$, $p : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n(a_0, \ldots, a_n)$ be the canonical projection. The *affine cone*, C_X, over X is the completion of $p^{-1}(X)$ in $\mathbb{C}^{n+1}$. We say X is *quasismooth of codimension c* if and only if its affine cone is smooth of codimension $c - 1$ outside its vertex 0.

We note that for curves in $\mathbb{CP}^n(a_0, \ldots, a_n)$ we have the following theorem:

Theorem 4.5 ([24, Theorem 12.1])**.** *A curve in* $\mathbb{CP}^n(a_0, \ldots, a_n)$ *is smooth if and only if it is quasismooth.*

Theorem 4.6 (Adjunction Formula, [20, Theorem 3.3.4])**.** *Let* $X_{d_1,\ldots,d_c} \subset \mathbb{CP}^n(a_0, \ldots, a_n)$ *be well-formed and quasismooth. Define the amplitude,* α*, to be* $\alpha = \sum d_i - \sum a_j$*, then* $K_X \simeq \mathcal{O}_X(\alpha)$*.*

We have that if X is a well-formed quasismooth weighted projective complete intersection of dimension 2, then the following are equivalent ([20, Section 3.4.3]):

- X is an orbifold $K3$ surface
- $K_X \simeq \mathcal{O}_X$
- $\alpha = \sum d_i - \sum a_j = 0$

In particular, if X is a well-formed, quasismooth hypersurface of degree d in $\mathbb{CP}^3(a_0, a_1, a_2, a_3)$, then X is a $K3$ orbifold if and only if $d = \sum_{i=0}^{3} a_i$. These have been classified by Reid into 95 distinct families ([54, Section 4.5]).

Consider $\mathcal{M} = \mathbb{CP}^4(a_0, a_1, a_2, a_3)$ be well-formed, then $\mathcal{M} \simeq X_1 \subset \mathbb{CP}^5(a_0, a_1, a_2, a_3, 1)$ so by the adjunction formula $K_{\mathcal{M}} \simeq \mathcal{O}_{\mathcal{M}}(-\sum_{i=0}^{3} a_i)$. Hence every weighted projective space of dimension 3 is Fano. For hypersurfaces we have the following result:

Theorem 4.7 ([20, Section 4])**.** *Let $X_d \subset \mathbb{CP}^n(a_0, \ldots, a_n)$ be a well-formed quasismooth weighted projective hypersurface with defining equation f and amplitude $\alpha = d - \sum a_i$. Then the Hodge structure is given by:*

$$h^{i,j}(X) = \begin{cases} 0 & \textit{if } i+j \neq n-1 \textit{ and } i \neq j \\ 1 & \textit{if } i+j \neq n-1 \textit{ and } i = j \\ dim_{\mathbb{C}} \left(\frac{S(a_0,\ldots,a_n)}{\theta_f} \right)_{jd+\alpha} & \textit{if } i+j = n-1 \textit{ and } i \neq j \\ dim_{\mathbb{C}} \left(\frac{S(a_0,\ldots,a_n)}{\theta_f} \right)_{jd+\alpha+1} & \textit{if } i+j = n-1 \textit{ and } i = j \end{cases} \tag{4.8}$$

where $\theta_f = (\partial f/\partial x_i)_{i=0,\ldots,n}$ is the Jacobian ideal of f.

When X is a well-formed quasismooth weighted hypersurface of dimension 3, the Euler number is given by:

$$e(X) = 2(1 - h^{1,2}(X)) \tag{4.9}$$

4.2 Suitable Fano 3-folds from $\mathbb{CP}^3(a_0, \ldots, a_3)$

In this section we use the ideas in [44] and [45] to construct G_2 orbifolds from pairs of Fano 3-folds obtained from weighted projective spaces. We first illustrate the principles via an explicit example and then discuss the general class of examples and we finish with a discussion of the examples that can be produced by this method.

Example 4.8. Consider the weighted projective space $V = \mathbb{CP}^3(2, 2, 1, 1)$. This is a Fano 3-fold with singular locus:

$$V_{sing} = \{[x_1 : x_2 : 0 : 0] | [x_1 : x_2] \in \mathbb{CP}^1\} \tag{4.10}$$

with $\Gamma_p \cong \mathbb{Z}_2$ for all $p \in V_{sing}$. On V an algebraic hypersurface of weighted degree $d = 6$ has vanishing first Chern class and it is simply connected so it will be a $K3$ surface (as previously discussed). Moreover, any such hypersurface will share its singularities with V as long as the defining polynomial is transverse (this is equivalent to the hypersurface being well-formed and quasismooth [34, Section 6.7.2]).

For example, let $f_\pm(x_1, x_2, y_1, y_2) = x_1^3 \pm x_2^3 + g(y_1, y_2)$ and let $\mathcal{X}_{f_\pm} = \mathbb{V}(f_\pm)$ be the vanishing locus of $f_\pm$, where the weighted homogeneous coordinates on V are $[x_1 : x_2 : y_1 : y_2]$ and g is any polynomial of degree 6 such that $g(0,0) = 0$ and if $(y_1, y_2) \neq (0,0)$ we have that $g(y_1, y_2) = 0$ and $dg(y_1, y_2) = 0$ share no solutions (for example $g(y_1, y_2) = y_1^6 + y_2^6$ so $dg(y_1, y_2) = 6y_1^5 dy_1 + 6y_2^5 dy_2 = 0$ if and only if $(y_1, y_2) = (0,0)$). Then $(\mathcal{X}_{f_+})_{sing} = \{[1; \xi_3^+] | \xi_3^+ \text{ is a third root of } -1\}$ with $\Gamma_p^{\mathcal{X}_{f_+}} \cong \mathbb{Z}_2$ for all $p \in (\mathcal{X}_{f_+})_{sing}$. We also have that $(\mathcal{X}_{f_-})_{sing} = \{[1; \xi_3^-] | \xi_3^- \text{ is a third root of } 1\}$ with $\Gamma_p^{\mathcal{X}_{f_-}} \cong \mathbb{Z}_2$ for all $p \in (\mathcal{X}_{f_-})_{sing}$. Hence $\mathcal{X}_{f_\pm}$ are orbifold $K3$ surfaces whose singular locus consists of 3 points of type A_1.

Now let $C = \mathcal{X}_{f_+} \cap \mathcal{X}_{f_-}$, so that C is a connected smooth curve. Let $\mathcal{X}_{\lambda:\mu} = \mathbb{V}(\lambda f_+ + \mu f_-)$, for $[\lambda : \mu] \in \mathbb{CP}^1$ so that $\mathcal{X}_{1:0} = \mathcal{X}_{f_+}$ and $\mathcal{X}_{0:1} = \mathcal{X}_{f_-}$. Let $\overline{\mathcal{W}} = Bl_C(V)$ be the blow up of V at C and $\sigma : \overline{\mathcal{W}} \to V$ denote the blow up map. Then we have $\overline{\mathcal{W}}_{sing} = \{\sigma^{-1}(p) | p \in V_{sing}\}$, all points having isotropy groups $\mathbb{Z}_2$, since σ is an isomorphism away from C. Now if $\overline{\mathcal{X}}_{\lambda:\mu}$ is the proper transform of $\mathcal{X}_{\lambda:\mu}$, then it has holomorphically trivial normal bundle in $\overline{\mathcal{W}}$ and satisfies all the requirements of **Theorem 2.7** so that $\mathcal{W} = \overline{\mathcal{W}} \setminus \overline{\mathcal{X}}_{\lambda:\mu}$ admits an asymptotically cylindrical Ricci-flat Kähler metric whose Kähler form and holomorphic volume form can be expressed as in **Equations (2.10) and (2.11)** near $\overline{\mathcal{X}}_{\lambda:\mu}$. We denote the pair $(\mathcal{W}, \overline{\mathcal{X}})$.

Moreover, we can consider two pairs $(\mathcal{W}_j, \overline{\mathcal{X}}_j)$, $j = 1, 2$. Since $sing(\mathcal{X}_j) = 3$ and $rkH^2(\mathcal{W}_j; \mathbb{Z}) = 1$ we can apply **Theorem 2.22** to conclude that we can obtain a G_2-orbifold from the pairs $(\mathcal{W}_j, \overline{\mathcal{X}}_j)$ whose singular locus is topologically given by two copies of $\mathbb{CP}^1 \times S^1 \setminus \{3 \text{ points}\}$ with isotropy $\mathbb{Z}_2$, along the compact pieces, and three copies of $S^1 \times S^1 \times [0; 1]$ with isotropy $\mathbb{Z}_2$, along the neck so $codim(\mathcal{M}_{sing}) = 4$. Moreover, by the Seifert-van Kampen Theorem for the orbifold fundamental group we see that $\pi_1^{orb}(\mathcal{W}_j) = \mathbb{Z}_2^2$ so in fact by **Theorem 2.24** $\pi_1^{orb}(\mathcal{M})$ is finite so by **Theorem 3.2** we have obtained a G_2 orbifold with $Hol(\mathcal{M}) = G_2$.

Now with this example in mind, let's consider how this can be generalised to obtain further examples of G_2-orbifolds. First note that we shall consider *generic choices of hypersurfaces in* $\mathbb{CP}^n(a_0, \dots, a_n)$, which means that if $\mathcal{X}_d$ is an algebraic hyper surface defined by the vanishing of some homogeneous weighted degree d polynomial, f, then the coefficients of f can be chosen outside of a set of measure zero.

In [54] we have a list of $K3$ surfaces which arise as hypersurfaces of certain algebraic degrees, d, in certain weighted projective spaces, $V = \mathbb{CP}^3(a_0, \ldots, a_3)$, call these $\mathcal{X}_d$. Note that $d = \sum_{i=0}^{3} a_i$ so once we chose a weighted projective space, the degree of the corresponding hypersurface $K3$ is fixed by the orbifold adjunction formula.

A generic $\mathcal{X}_d$ will share their singularities with the weighted projective space, and two generic choices will have the same singular locus. Moreover, their intersection will be a smooth connected curve which will not pass through the singular locus of $\mathcal{X}_d$. Hence we can consider a pencil of orbifold $K3$ surfaces defined by our initial $K3$ surfaces; explicitly if $\mathcal{X}_d = \mathbb{V}(f)$ and $\widetilde{\mathcal{X}}_d = \mathbb{V}(\widetilde{f})$ are our initial generic choices, then they define the pencil $X_{\lambda;\mu} = \mathbb{V}(\lambda f + \mu \widetilde{f})$, for $[\lambda; \mu] \in \mathbb{CP}^1$. Let $C = \mathcal{X}_d \cap \widetilde{\mathcal{X}}_d$ and we consider the blow-up of V at C, $\overline{\mathcal{W}} = Bl_C(V)$. If $\overline{\mathcal{X}}_{\lambda;\mu}$ denotes the proper transform of the pencil, then a generic element of this pencil will have the same singular locus as $\mathcal{X}_d$, which are shared with $\overline{\mathcal{W}}$, and holomorphically trivial normal bundle. As such, if we denote the generic element of the pencil by $\overline{\mathcal{X}}$, then the pair $(\overline{\mathcal{W}}, \overline{\mathcal{X}})$ satisfies the hypothesis of **Theorem 2.7**. The singular locus of $\mathcal{W} = \overline{\mathcal{W}} \setminus \overline{\mathcal{X}}$ will depend on the singular locus of V and $\mathcal{X}_d$ and will topologically be given as $S^1 \times (V_{sing} \setminus \{p_1, \ldots, p_n\})$ and $\{p_1, \ldots, p_n\} \times S^1 \times [0; \infty)$ where $p_1, \ldots, p_n$ denotes the singular locus of $\mathcal{X}_d$. Moreover, the isotropy at each point of the singular locus is inherited by the isotropy on V or on $\mathcal{X}_d$.

We now have 95 distinct families of pairs $(\mathcal{W}, \mathcal{X})$ which correspond to the families of hypersurface orbifold $K3$'s found by Reid in [54]. Any two such families satisfy that $rkH^2(\mathcal{W}; \mathbb{Z}) = 1$. If moreover, the $K3$ factor $\mathcal{X}$, has $sing(\mathcal{X}) \leq 9$, and we find two families (not necessarily distinct), such that the $K3$ factors have the same singular locus, then by applying **Theorem 2.22** the two families can be used to construct a compact G_2 orbifold. Moreover, these will always have finite orbifold fundamental group, which can be computed using Seifert-van Kampen for orbifolds. The orbifold fundamental group will depend on the isotropy groups of the various singular points of V, but it will be in general a finite direct product of the isotropy groups. So the G_2 orbifolds we can construct in all have full holonomy.

Of the 95 families not all satisfy $sing(\mathcal{X}) \leq 9$. In fact, only 35 satisfy this condition. We summarise in the table below these 35 families:

Suitable Pairs with $sing(\mathcal{X}_d) \leq 9$

Nr	$\mathcal{X}_d \subset \mathbb{CP}^3(a_0, \ldots, a_3)$	*Singular locus of* $K3$
1.	$\mathcal{X}_5 \subset \mathbb{CP}^3(1,1,1,2)$	A_1
2.	$\mathcal{X}_6 \subset \mathbb{CP}^3(1,1,1,3)$	Smooth
3.	$\mathcal{X}_6 \subset \mathbb{CP}^3(1,1,2,2)$	$3 \times A_1$
4.	$\mathcal{X}_7 \subset \mathbb{CP}^3(1,1,2,3)$	A_1, A_2
5.	$\mathcal{X}_8 \subset \mathbb{CP}^3(1,1,2,4)$	$2 \times A_1$
6.	$\mathcal{X}_8 \subset \mathbb{CP}^3(1,2,2,3)$	$4 \times A_1$, A_2
7.	$\mathcal{X}_9 \subset \mathbb{CP}^3(1,1,3,4)$	A_3
8.	$\mathcal{X}_9 \subset \mathbb{CP}^3(1,2,3,3)$	A_1, $3 \times A_2$
9.	$\mathcal{X}_{10} \subset \mathbb{CP}^3(1,1,3,5)$	A_2
10.	$\mathcal{X}_{10} \subset \mathbb{CP}^3(1,2,2,5)$	$5 \times A_1$
11.	$\mathcal{X}_{10} \subset \mathbb{CP}^3(1,2,3,4)$	$2 \times A_1$, A_2, A_3
12.	$\mathcal{X}_{11} \subset \mathbb{CP}^3(1,2,3,5)$	A_1, A_2, A_4
13.	$\mathcal{X}_{12} \subset \mathbb{CP}^3(1,1,4,6)$	A_1
14.	$\mathcal{X}_{12} \subset \mathbb{CP}^3(1,2,3,6)$	$2 \times A_1$, $2 \times A_2$
15.	$\mathcal{X}_{12} \subset \mathbb{CP}^3(1,2,4,5)$	$3 \times A_1$, A_4
16.	$\mathcal{X}_{12} \subset \mathbb{CP}^3(1,3,4,4)$	$3 \times A_3$
17.	$\mathcal{X}_{13} \subset \mathbb{CP}^3(1,3,4,5)$	A_2, A_3, A_4
18.	$\mathcal{X}_{14} \subset \mathbb{CP}^3(1,2,4,7)$	$3 \times A_1$, A_3
19.	$\mathcal{X}_{14} \subset \mathbb{CP}^3(2,2,3,7)$	$7 \times A_1$, A_2
20.	$\mathcal{X}_{15} \subset \mathbb{CP}^3(1,2,5,7)$	A_1, A_6
21.	$\mathcal{X}_{15} \subset \mathbb{CP}^3(1,3,4,7)$	A_3, A_6
22.	$\mathcal{X}_{15} \subset \mathbb{CP}^3(1,3,5,6)$	$2 \times A_2$, A_5
23.	$\mathcal{X}_{16} \subset \mathbb{CP}^3(1,2,5,8)$	$2 \times A_1$, A_4
24.	$\mathcal{X}_{16} \subset \mathbb{CP}^3(1,3,4,8)$	A_2, $2 \times A_3$
25.	$\mathcal{X}_{18} \subset \mathbb{CP}^3(1,2,6,9)$	$3 \times A_1$, A_2
26.	$\mathcal{X}_{18} \subset \mathbb{CP}^3(1,3,5,9)$	$2 \times A_2$, A_4
27.	$\mathcal{X}_{20} \subset \mathbb{CP}^3(1,4,5,10)$	A_1, $2 \times A_4$
28.	$\mathcal{X}_{21} \subset \mathbb{CP}^3(1,3,7,10)$	A_9
29.	$\mathcal{X}_{22} \subset \mathbb{CP}^3(1,3,7,11)$	A_2, A_6
30.	$\mathcal{X}_{22} \subset \mathbb{CP}^3(1,4,6,11)$	A_1, A_3, A_5
31.	$\mathcal{X}_{24} \subset \mathbb{CP}^3(1,3,8,12)$	$2 \times A_2$, A_3
32.	$\mathcal{X}_{28} \subset \mathbb{CP}^3(1,4,9,14)$	A_1, A_8
33.	$\mathcal{X}_{30} \subset \mathbb{CP}^3(1,4,10,15)$	A_1, A_3, A_4
34.	$\mathcal{X}_{36} \subset \mathbb{CP}^3(1,5,12,18)$	A_4, A_5
35.	$\mathcal{X}_{42} \subset \mathbb{CP}^3(1,6,14,21)$	A_1, A_2, A_6

If we want to get a G_2 orbifold from any of these families we need the $K3$ factors to have the same singular locus. Hence, for any $i = 1, \ldots, 35$ we can glue together two copies of the family i to get 35 distinct G_2 orbifolds. In fact, if $i, j = 1, \ldots, 35$ we will denote by $\mathcal{M}_{i,j}$ the G_2 orbifold obtained from gluing pair number i with pair number j. We can easily see these are distinct by considering the singular locus:

First, we consider the singular locus of the $K3$ factors which determines the singular locus along the neck region of the constructed orbifold. Then, if two constructed orbifolds have the same singular locus along the neck regions (for example, both $\mathcal{M}_{1,1}$ and $\mathcal{M}_{13,13}$ have singular locus along the neck region, topologically given by $S^1 \times S^1 \times [0;1]$ with isotropy $\mathbb{Z}_2$), we examine the singular locus of the compact pieces. Since each family uses a different weighted projective space as the 3-fold, even if the $K3$ divisors have the same singular locus, they will have distinct singular loci outside of the $K3$ factors and hence will give distinct G_2 orbifolds. $\mathcal{M}_{1,1}$ is smooth outside of the neck region, while $\mathcal{M}_{13,13}$ is not, its singular locus is topologically given by 2 copies of $S^1 \times (\mathbb{CP}^3(1,1,4,6)_{sing} \setminus \{p_1\})$ showing the constructed G_2-orbifolds are distinct.

So far we have 35 distinct examples given as $\mathcal{M}_{i,i}$ for $i = 1, \ldots, 35$. Further examining the singular locus of the orbifold $K3$ factors we can identify one further G_2 orbifold we can construct, namely $\mathcal{M}_{1,13}$, giving a total of 36 distinct families which can be constructed. We summarise this in the following result:

Proposition 4.9. *With the notation above, for any* $i = 1, \ldots, 35$ *we can construct a* G_2 *orbifold with full holonomy of the form* $\mathcal{M}_{i,i}$. *For any* $i, j = 1, \ldots, 35$ *distinct, the two orbifolds* $\mathcal{M}_{i,i}$ *and* $\mathcal{M}_{j,j}$ *are distinct as they have distinct singular loci. Moreover, we can construct an additional* G_2 *orbifold with full holonomy,* $\mathcal{M}_{1,13}$, *distinct from all the other constructed so far giving a total of* 36 *distinct* G_2 *orbifolds of the form* $\mathcal{M}_{i,j}$

An interesting case is family number 2, where the $K3$ factor is smooth. This family can be glued to any smooth pair, (W, X) where W is an ACyl 3-fold and X is an anticanonical $K3$ divisor, to give an orbifold $K3$ whose singular locus is topologically given by a copy of S^1 with isotropy $\mathbb{Z}_3$. This is a consequence of [45, **Theorems** 5.3 **and** 5.7] applied in our case. The two theorems together, roughly, say that we can find a matching between the smooth $K3$ divisors, if the ranks $r_1 = rkH^2(V_1;\mathbb{Z})$ and $r_2 = rkH^2(V_2;\mathbb{Z})$ satisfy $r_1 + r_2 \leq 11$, where V_i are the Fano 3-folds underlying the pairs (W_i, X_i) used in the Twisted Connected Sum. For us, $r_1 = 1$ and smooth

Fano 3-fold, V_2, satisfies $dim(H^2(V_2;\mathbb{Z})) \leq 10$, so in this case we are guaranteed that a matching between the smooth $K3$ divisors exists.

These examples can not be distinguished purely by considering their singular locus; one needs to examine other topological invariants, e.g. Betti numbers, to distinguish examples of this type. Nonetheless, these examples rely on the theory of matching smooth $K3$ surfaces and only require the use of **Theorem 2.12** to conclude that a G_2-orbifold can be constructed, denoted as $\mathcal{M}_{2,(W,X)}$. Moreover, their orbifold fundamental group will be given as $\pi_1^{orb}(\mathcal{M}_{2,(W,X)}) = \mathbb{Z}_3 \times \pi_1(W)$.

Note that if $V' = \mathbb{CP}^n(a_0, \dots, a_n)$, then we have ([24]):

$$-K_{V'}^n = \frac{(a_0 + \dots + a_n)^n}{a_0 \dots a_n} \tag{4.11}$$

so that if $V = \mathbb{CP}^3(1,1,1,3)$, then $K_V^3 = 72$. Moreover, we have $b^2(V) = 1$ so that we can apply results from [44] to conclude the following:

Proposition 4.10. *In the notation above, let V_2 be a smooth Fano 3-fold producing a pair (W, X) and consider the constructed G_2 orbifold $\mathcal{M}_{2,(W,X)}$. Then*

$$b^2(\mathcal{M}_{2,(W,X)}) = 0 \tag{4.12}$$

$$b^3(\mathcal{M}_{2,(W,X)}) = b^3(V_2) - K_{V_2}^3 + 99 \tag{4.13}$$

Proof. Note that $b^2(V) = 1$, $b^3(V) = 0$ and $-K_V^3 = 72$ and we have that ([44, Theorem 8.57]):

$$b^3(\mathcal{M}_{2,(W,X)}) = b^3(V) - K_V^3 + b^3(V_2) - K_{V_2}^3 + 27 \tag{4.14}$$

Note that for the proof of the above equality we only need to view our orbifolds as topological spaces, and since the $K3$ factor is smooth, then we get the same equality for $b^3(\mathcal{M}_{2,(W,X)})$ as in the smooth case. □

A complete classification of non-singular Fano 3-folds into 105 distinct families is given in [49]. The table below (reproduced from [45, Appendix]) shows the possible values of $b^2(V)$, $b^3(V) - K_V^3$ and $b^3(\mathcal{M}_{2,(W,X)})$ where (W, X) is the suitable pair we obtain from one of the Fano 3-fold families. Note that there are less than 105 entries in the second or third columns below, as some of the families have the same value of $b^3(V) - K_V^3$. Moreover, note that all the entries in the third column are odd; this is a feature of the chosen matching isometry between the $K3$ divisors in [44, 45] The dots denote consecutive integers of the same parity:

$b^2(V)$	$b^3(V) - K_V^3$	$b^3(\mathcal{M}_{2,(W,X)})$
1	22, 24, 26, 30, 34, 36, 40, 46, 50, 54, 64, 104	121, 123, 125, 129, 133, 135, 139, 145, 149, 153, 163, 203
2	22,. . . , 34, 38, 40, 42, 46, 48, 54, 56, 62	121,. . . , 133, 137, 139, 141, 145, 147, 153, 155, 161
3	20,. . . , 52	119,. . . , 151
4	26, 30,. . . , 46	125, 129,. . . , 145
5	28, 36	127, 135
6	30	129
7	24	123
8	18	117
9	12	111
10	6	105

Now by inspecting the third column above we can summarise the discussion in the following proposition:

Proposition 4.11. *With the notation above, if (W, X) is a suitable pair obtained from a smooth Fano 3-fold, V, then we can obtain at least 25 distinct families of G_2 orbifolds, $\mathcal{M}_{2,(W,X)}$, with full holonomy, whose singular locus is topologically given by S^1 with pointwise isotropy $\mathbb{Z}_3$ and which satisfy:*

$$b^2(\mathcal{M}_{2,(W,X)}) = 0 \tag{4.15}$$

$$105 \leq b^3(\mathcal{M}_{2,(W,X)}) \leq 203 \tag{4.16}$$

Remark *4.12.* The number of families of the form $\mathcal{M}_{2,(W,X)}$, where (W, X) comes from a smooth Fano 3-fold is greater than 25. This is because of the choice in matching in [44, 45] as previously mentioned. Moreover, if (W_1, X_1) is the family produced by a smooth Fano 3-fold, V_1, with $b^2(V_1) = 1$ and $b^3(V_1) - K_{V_1}^3 = 22$ and (W_2, X_2) is the family produced by a smooth Fano 3-fold, V_2, with $b^2(V_2) = 2$ and $b^3(V_2) - K_{V_2}^3 = 22$, then $b^3(\mathcal{M}_{2,(W_1,X_1)}) = b^3(\mathcal{M}_{2,(W_2,X_2)})$ so we can not distinguish these families by their Betti numbers. We would need to look at different topological invariants to help distinguish these 2 G_2-orbifolds.

The issue with the remaining 60 families comes from matching the orbifold $K3$ factors, since their singularity numbers are too large so we can not use **Theorem 2.22**. These families should be treated on a case-by-case to see if they are suitable for the construction of G_2 orbifolds.

Chapter 5

Associative 3-folds

In this chapter, we will investigate associative 3-folds in product G_2-manifolds of the form $X \times T^3$, where X is a hyper-Kähler $K3$ surface. We will also investigate associative suborbifolds in a class of global quotients of the form $(X \times T^3)/\mathbb{Z}_2^2$. Our main focus will be on classifying, up to isometry of the ambient G_2-manifold/orbifold, a special class of associative 3-folds, possibly admitting orbifold singularities. The defining condition for this class of associatives is that the derivative of the projection to the torus factor has constant rank. We will require results related to hyper-Kähler structures, and Nikulin's results on non-symplectic involutions of $K3$ surfaces as discussed in **Chapters** 1 **and** 2, respectively.

5.1 G_2-orbifolds of the $(K3 \times T^3)/\mathbb{Z}_2^2$ type

In this section we consider global quotient orbifolds of the type $(X \times T^3)/\Gamma$, where X is a $K3$ surface carrying a hyper-Kähler structure, (I, J, K), T^3 is a flat 3-torus where we denote the standard coordinates on it by $(x_1, x_2, x_3) + \mathbb{Z}^3$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$, and Γ is a group isomorphic to a product of two cyclic groups, $\Gamma \cong \mathbb{Z}_2^2$, generated by two isometries α,$\beta : X \times T^3 \to X \times T^3$. The action of α and β is as follows.

If we denote by $(\omega_1, \omega_2, \omega_3)$ the Kähler triple on X corresponding to the complex structures (I, J, K), and by (dx^1, dx^2, dx^3) the standard basis of one-forms on T^3, induced via $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, then we want the action of α on X to be holomorphic non-symplectic with respect to I, the action of β on X to be holomorphic non-symplectic with respect to K, and we extend them to T^3 to act as $\alpha((x_1, x_2, x_3) + \mathbb{Z}^3) = (x_1, -x_2, -x_3) + \mathbb{Z}^3$, $\beta((x_1, x_2, x_3) + \mathbb{Z}^3) = (-x_1, -x_2, x_3) + \mathbb{Z}^3$. Note that the action on the T^3

factor corresponds to the following action on 1-forms on T^3:

$$\alpha^* dx^i = \begin{cases} dx^i, & i = 1 \\ -dx^i, & i = 2, 3 \end{cases} \tag{5.1}$$

$$\beta^* dx^i = \begin{cases} dx^i, & i = 3 \\ -dx^i, & i = 1, 2 \end{cases} \tag{5.2}$$

$K3$ surfaces admitting such a pair of isometries have been studied by Reidegeld in [56] and used to construct G_2 orbifolds of the non-symplectic type in [55]. Reidegeld found 320 examples of pairs of commuting non-symplectic involutions on hyper-Kähler $K3$ surfaces, which can all be extended diagonally to commuting involutions on $X \times T^3$. Our main interest is in finding and classifying a special class of associative suborbifolds in $(X \times T^3)/\Gamma$. Throughout this section we will refer to the isometries α and β as non-symplectic, although this is a slight misnomer, as only their action on the $K3$ factor can be seen as non-symplectic.

Before stating and proving our main result we review the G_2 geometry of the quotient $(X \times T^3)/\mathbb{Z}_2^2$, we introduce associative submanifolds and associative suborbifolds, and briefly discuss an example of associative suborbifolds found in [1] which will motivate our main result.

Let $M = X \times T^3$ and $\mathcal{M} = M/\Gamma$ where $\Gamma = \langle \alpha, \beta \rangle \cong \mathbb{Z}_2^2$. We equip M with the torsion-free product G_2-structure:

$$\varphi = \sum_{i=1}^{3} \omega_i \wedge dx^i - dx^1 \wedge dx^2 \wedge dx^3 \tag{5.3}$$

Note that by the conditions imposed on α and β, their action on the hyper-Kähler triple $(\omega_1, \omega_2, \omega_3)$ is:

$$\alpha^* \omega_i = \begin{cases} \omega_i, & i = 1 \\ -\omega_i, & i = 2, 3 \end{cases} \tag{5.4}$$

$$\beta^* \omega_i = \begin{cases} \omega_i, & i = 3 \\ -\omega_i, & i = 1, 2 \end{cases} \tag{5.5}$$

such that we clearly have $\alpha^*\varphi = \beta^*\varphi = \varphi$ so that the G_2-structure descends to a well-defined torsion-free G_2-structure on the quotient making $\mathcal{M}$ into a torsion-free G_2 orbifold. We shall denote by

$$\pi_T : M \to T^3 \tag{5.6}$$

and

$$\pi_X : M \to X \tag{5.7}$$

the standard projections onto the T^3 and $K3$ factors respectively. We shall further denote by

$$\widehat{\pi}_T : \mathcal{M} \to T^3/\Gamma \tag{5.8}$$

and

$$\widehat{\pi}_X : \mathcal{M} \to X/\Gamma \tag{5.9}$$

the maps induced on the quotient.

Noting Nikulin's classification of non-symplectic involutions, [51], in terms of lattice invariants (r, a, δ) we know that, if $(r, a, \delta) \neq (10, 10, 0)$, then the singular locus of $\mathcal{M}$ is non-empty and has codimension $4 > 3$, since the fixed loci of α and β consist of products of complex curves on the $K3$ factors and circles on the T^3 factor. Thus we can investigate the holonomy by computing the orbifold fundamental group and using **Theorem 3.2**. To do this, note that for a global quotient we have a short exact sequence:

$$1 \to \pi_1(M) \to \pi_1^{orb}(\mathcal{M}) \to \Gamma \to 1 \tag{5.10}$$

where $\pi_1(M) \cong \mathbb{Z}^3$ and $\Gamma \cong \mathbb{Z}_2^2$ so we have $\pi_1^{orb}(\mathcal{M}) \cong \mathbb{Z}^3 \rtimes \mathbb{Z}_2^2$. This is infinite so we know that the holonomy of $\mathcal{M}$ is a proper subgroup of G_2; in fact, due to Berger's Theorem and the proof of **Theorem 3.2** we see that the holonomy of $\mathcal{M}$ is exactly $SU(2)$.

We now turn our attention to a special class of suborbifolds called *associative*, which we recall here:

Definition 5.1. Let $\mathcal{M}$ be a torsion-free G_2 orbifold with G_2 3-form φ. Let $\mathcal{A} \subset \mathcal{M}$ be a 3-dimensional suborbifold and $\iota : \mathcal{A} \to \mathcal{M}$ denote the inclusion map. We say that $\mathcal{A}$ is *associative* if $\iota^*\varphi = dVol_{\mathcal{A}}$ where $dVol_{\mathcal{A}}$ is the volume form of $\mathcal{A}$ equipped with the pull-back metric. Such a suborbifold is volume minimising in its homology class.

We are now ready to state our main result:

Main Theorem. *Let $(X; I, J, K)$ be a hyper-Kähler $K3$ surface admitting two commuting involutions α and β. Let α be holomorphic non-symplectic for the complex structure I and β holomorphic non-symplectic for the complex structure K. Moreover, suppose that the invariants of α and β satisfy $(r_\alpha, a_\alpha, \delta_\alpha), (r_\beta, a_\beta, \delta_\beta) \neq (10, 10, 0), (10, 8, 0)$. Extend α and β to $M = X \times T^3$ to involutions acting on the product such that on T^3 they act as in **Equations (5.1) and (5.2)**. Equip M with the standard product G_2*

3-form, defined in ***Equation (5.3)****, and let* $\mathcal{M} = M/\langle \alpha, \beta \rangle$ *denote the resulting* G_2 *orbifold.*

Then the singular locus of $\mathcal{M}$ *contains an associative suborbifold of the form* $(\Sigma \times \gamma)/\langle \alpha, \beta \rangle$*, where* $\Sigma \subset X$ *is a complex curve and* $\gamma \subset T^3$ *is an embedded circle.*

Conversely, suppose that $\mathcal{A}$ *is a compact connected associative suborbifold in* $\mathcal{M}$ *such that the derivative of the projection map* $\widehat{\pi}_T : \mathcal{M} \to T^3/\langle \alpha, \beta \rangle$ *restricted to* $\mathcal{A}$*,* $d\widehat{\pi}_T|_{\mathcal{A}}$*, has constant rank* 1*. Then there exists a surface* $\Sigma \subset X$ *holomorphic with respect to some complex structure in the sphere of complex structures generated by* (I, J, K) *on* X*, and an embedded circle* $\gamma \subset T^3$ *such that* $\mathcal{A}$ *is isometric to* $(\Sigma \times \gamma)/G$*, where* G *is a subgroup of* $\langle \alpha, \beta \rangle$*. Moreover, the isometry* $\mathcal{A} \to (\Sigma \times \gamma)/G$ *is the restriction of an isometry* $\mathcal{M} \to \mathcal{M}$ *induced from an isometry of* T^3*.*

We end this section by briefly commenting on the increasing interest in associative submanifolds/suborbifolds, both in pure mathematics and high-energy physics. In pure mathematics, authors such as Joyce ([36]), Doan and Walpuski ([19]) are trying to build "counting invariants" of G_2-manifolds/orbifolds based on weighted counts of associatives similar to the Gromov-Witten invariants in symplectic geometry. In high energy physics, calibrated submanifolds/suborbifolds are known as "supersymmetric cycles", and the interest stems from a link between counting associative 3-folds and "M-theory compactifications", especially in the orbifold case ([1]).

In the following section we review some elements of the work of Acharya *et al.* in [1]. This will serve to motivate our main results; however, the details will not be particularly useful for our work.

5.2 An example of Acharya *et al.*

In this section we discuss one particular example treated in detail by Acharya *et al.* in [1]. Their motivation was to choose two specific involutions as above on an elliptically fibred $K3$ surface so that they can exhibit infinitely many, homologically distinct, associative suborbifolds in a prescribed quotient of the form $(X \times T^3)/\Gamma$ as in the previous section. For us, this example helps motivate the main result, although our interest will be in exhibiting associative suborbifolds in G_2 orbifolds of this type and classifying them. We will adopt the notation of the previous section.

In [1], Acharya and co-authors choose a specific family of elliptically fibred hyper-Kähler $K3$ surfaces, $(X; I, J, K)$. These $K3$ surfaces are given by a complete intersection in the product of a weighted projective space with a regular $\mathbb{CP}^2$, namely $\mathbb{CP}^2(3,2,1) \times \mathbb{CP}^2$. If we denote by $[\mu : z_1 : z_2]$ homogeneous coordinates in $\mathbb{CP}^2$ and by $[y : x : w]$ weighted homogeneous coordinates on $\mathbb{CP}^2(3,2,1)$, then the family of $K3$ surfaces they consider is given by:

$$\begin{cases} y^2 = x^3 + xw^4 f_4(z_1, z_2) + w^6 g_6(z_1, z_2) \\ \mu^2 = z_1 z_2 \end{cases} \tag{5.11}$$

where f_4 and g_6 are homogeneous polynomials in z_1 and z_2 of degrees 4 and 6, respectively. Following in the footsteps of Acharya *et al.*, we will tacitly use Poincaré duality to switch between the homology and cohomology lattices of the $K3$ surfaces we consider.

Moreover, this family of $K3$ surfaces admits a pair of commuting non-symplectic involutions, α, β, with invariants $(r, a, \delta) = (10, 8, 0)$, which they give as restrictions to X of explicit involutions of the ambient product space, $\mathbb{CP}^2(3,2,1) \times \mathbb{CP}^2$. Then they show that if

$$L = (-E_8) \oplus (-E_8') \oplus H_1 \oplus H_2 \oplus H_3 \tag{5.12}$$

denotes the $K3$ lattice, then α^* interchanges the two E_8 factors, acts as the identity on H_1 and $-id$ on H_2, and H_3, and β^* acts as the identity on H_3, $-id$ on H_1, and H_2 and $\beta^* = -\alpha^*$ on the $(-E_8) \oplus (-E_8')$ factors. For involutions with $(r, a, \delta) = (10, 8, 0)$, their fixed locus consists of two elliptic curves. The product, $\alpha\beta$, of the chosen involutions is then shown to have invariants $(r, a, \delta) = (2, 0, 0)$ so its fixed locus consists of a surface of genus 10 and one rational curve. The authors then find that the singular locus of $\mathcal{M}$ consists of 16 disjoint copies of T^3, 4 disjoint copies of S^3, and 4 disjoint copies of 3-orbifolds $(\Sigma_{10} \times S^1)/\alpha$ where Σ_{10} is a surface of genus 10 fixed by $\alpha\beta$.

Denoting by $(-E_8^{\pm}) = \{\gamma \pm \gamma' | \gamma \in (-E_8), \gamma' = \alpha^*\gamma \in (-E_8')\}$ the two diagonals in $(-E_8) \oplus (-E_8')$, and $\{e_1^i, e_2^i\}$ the standard basis of H_i, the authors identify the homology class of the zero section of the elliptic fibration for (X, I), a distinguished section picking a point serving as the origin of each fibre, by $\sigma_0^1 = e_1^1 - e_2^1$ and the homology class of a generic fibre $F^1 = e_2^1$. Then for every element $\gamma \in (-E_8)$ they define

$$\sigma_\gamma^1 = \sigma_0^1 + 2nF^1 + \gamma + \alpha^*\gamma \tag{5.13}$$

where $\gamma^2 = -2n$, $n \in \mathbb{N}$. They then show $(\sigma_\gamma^1)^2 = -2$ and $\langle F^1, \sigma_\gamma^1 \rangle = 1$ which implies the existence of a rational curve $\Sigma_\gamma^1 \subset X$ which is a section of the elliptic fibration. Moreover, Σ^1_γ is preserved by Γ so that it descends to a well-defined suborbifold of $\mathcal{M}$, namely

$$C^1_\gamma = (\Sigma^1_\gamma \times S^1_1)/\Gamma \tag{5.14}$$

where S^1_i denote the three circle factors in T^3, whose volume form is dx^i. They then prove that C_γ is homeomorphic to S^3 and remark that Σ^1_γ is calibrated by ω_1 so C^1_γ is associative. The authors then argue that (X, K) is also elliptically fibred so that by interchanging α with β and swapping 1 with 3 above they construct another family of associatives,

$$C^3_\gamma = (\Sigma^3_\gamma \times S^1_3)/\Gamma \tag{5.15}$$

where now Σ^3_γ are calibrated by ω_3 and are sections of the elliptic fibration for (X, K). Lastly, they argue that (X, J) is also elliptically fibred so that there is another associative,

$$\widehat{C} = (\widehat{\Sigma} \times S^1_2)/\Gamma \tag{5.16}$$

where $\widehat{\Sigma}$ is calibrated by ω_2 and is a section of an elliptic fibration whose fibre is given by e^2_2. This is how Acharya *et al.* construct infinitely many associatives representing distinct homology classes in $H_3(\mathcal{M}; \mathbb{Z})$. Thus, their findings, which motivate our work, can be summarised in the following theorem:

Theorem 5.2 (*cf.* Acharya *et al.*, [1])**.** *Let $\mathcal{M} = (X \times T^3)/\Gamma$ be the compact G_2 orbifold described above, where X is a generic $K3$ surface in the family defined by* ***Equations (5.11)****. Then there exist two infinite families of associative suborbifolds, C^i_γ with $i = 1, 3$ and a distinguished associative suborbifold $\widehat{C}_\gamma$, all representing distinct homology classes in $H_3(\mathcal{M}; \mathbb{Z})$.*

One thing to note here is that if $\pi_T : M \to T^3$ denotes the standard projection on the 3-torus factor, $p : M \to \mathcal{M}$ the quotient map, and $p_T : T^3 \to T^3/\Gamma$ its restriction to the T^3 factor, then π_T descends to a well-defined map $\widehat{\pi}_T : \mathcal{M} \to T^3/\Gamma$ satisfying $p_T \circ \pi_T = \widehat{\pi}_T \circ p$. Moreover, $d\widehat{\pi}_T|_{C^i_\gamma}$ for $i = 1, 3$ and $d\widehat{\pi}_T|_{\widehat{C}}$ have constant rank 1 for all γ, since $d\pi_{\Sigma^i_\gamma \times S^1_i}$ for $i = 1, 3$ and $d\pi_{\widehat{\Sigma}^i_\gamma \times S^1_2}$ have constant rank 1 for all γ. This motivates the following definition:

Definition 5.3. If A is a compact connected associative submanifold of $X \times T^3$ such that $d\pi_T|_A$ has constant rank we shall say that A is a *constant rank associative submanifold*. Similarly, we define *constant rank associative suborbifolds* of $(X \times T^3)/\langle \alpha, \beta \rangle$ to be compact connected associative suborbifolds, $\mathcal{A}$, such that $d\widehat{\pi}_T|_A$ has constant rank.

5.3 Constant Rank 1 Associatives

Motivated by the work of Acharya *et al.*, we are naturally led to ask if any associative suborbifold for which the derivative of the torus projection has constant rank 1 splits as a product of a complex curve in the $K3$ factor and a circle in the torus, or is a quotient of such a product. The answer turns out to be that this is indeed the case; in fact, this is true even at the level of the covering manifolds, which we capture in **Proposition 5.8** below. Before we present the proposition, we include the statements of the Constant Rank Theorem and a couple of other auxiliary results we will need.

Theorem 5.4 (Constant Rank Theorem, [46, Theorem 4.12])**.** *Let M^m and N^n be smooth manifolds and $f : M \to N$ a smooth map such that Df_p has constant rank r. Then for every $p \in M$, there exists a chart (ϕ, U) around p and a chart (ψ, V) around $f(p)$ such that f is given locally by projection onto the first r coordinates i.e.,*

$$\psi \circ f \circ \phi^{-1}(u_1, \ldots, u_m) = (u_1, \ldots, u_r, 0, \ldots, 0) \tag{5.17}$$

Theorem 5.5 ([46, Theorem 10.35])**.** *Let E and E' be vector bundles over a smooth manifold M and let $F : E \to E'$ be a smooth bundle homomorphism over M. Define $ker(F) = \cup_{p \in M} ker(F|_{E_p})$. Then $ker(F)$ is a smooth subbundle of E if and only if F has constant rank.*

The last result we will need is the following theorem which we will use immediately to prove an auxiliary lemma that will help us factor $\pi_T|_A$ into a submersion of A onto S^1 and an immersion of S^1 into T^3.

Theorem 5.6 ([46, Theorem 4.30])**.** *Let M and N be smooth manifolds, and $f : M \to N$ a smooth surjective submersion. If P is a smooth manifold with or without boundary and $F : M \to P$ is a smooth map that is constant on the fibres of f, then there exists a unique smooth map, $\widetilde{F} : N \to P$ such that $\widetilde{F} \circ f = F$.*

The following "factorisation" result seems to be a fairly straightforward exercise in differential topology, however we have not been able to find an appropriate reference for it and hence decided to include a proof:

Lemma 5.7. *Let M be a compact connected m-dimensional manifold, N a n-dimensional manifold, and $f : M \to N$ a smooth map such that $D_p f$ has constant rank r for all $p \in M$. Then there exists a compact r-dimensional smooth manifold Y, a submersion $p : M \to Y$, and an immersion $\iota : Y \to N$ such that $f = \iota \circ p$.*

Proof. We begin by considering the quotient map $p : M \to M/\sim$, where $\sim$ is the equivalence relation on M defined by $x \sim z$ if and only if $f(x) = f(z) = y$ and x, z belong to the same connected component of $f^{-1}(y)$. The quotient map here is clearly surjective and the quotient space is clearly locally Euclidean of dimension r as a consequence of the Constant Rank Theorem and the fact that different connected components of the pre-image correspond to different points in $M/\sim$. We need to show that the quotient space is Hausdorff, then we know that it is a compact and connected r-manifold without boundary, Y. By noting that the quotient map is a submersion, due to the Constant Rank Theorem, and that, by definition, f is constant on the fibres of p, we can apply **Theorem 5.6**, to get a unique smooth map $\iota : Y \to N$ such that $f = \iota \circ p$. Using the chain rule, we conclude that $d\iota$ is injective so it is an immersion of Y into N.

The only thing left to do is to check that $M/\sim$ is indeed Hausdorff, and, hence, a smooth r-manifold. We do this by showing that the equivalence relation

$$R = \{(x, z) \in M \times M | x \sim z\} \tag{5.18}$$

is closed in $M \times M$. First note that M is metrizable as a compact connected manifold and we pick a metric d on M. Let $(\mathcal{K}, d_H)$ be the set of non-empty compact subsets of M equipped with the Hausdorff metric

$$d_H(K, K') = max\{\sup_{x \in K} d(x, K'), \sup_{x' \in K'} d(K, x')\} \tag{5.19}$$

Note that, since M is compact, then $(\mathcal{K}, d_H)$ is also compact ([6, Proposition 2.2]). Now take a convergent sequence $(x_n, z_n) \to (x, z) \in M \times M$ with $(x_n, z_n) \in R$ for all $n \in \mathbb{N}$. We write $y_n = f(x_n) = f(z_n) \in N$ and note that by continuity of f we have that $y_n \to y = f(x) = f(z)$. For each n, since $(x_n, z_n) \in R$, let C_n be the connected component of the fibre $f^{-1}(y_n)$, containing x_n and z_n so that C_n is a non-empty, compact, connected subset of M.

By compactness of $(\mathcal{K}, d_H)$ (so that we pass to a subsequence if necessary), we have that $C_n \to K$ in the Hausdorff metric, for some non-empty, compact $K \subset M$. Moreover, since the subspace of connected, compact subsets of M is closed in $(\mathcal{K}, d_H)$, we know that K is also connected. Let $p \in K$. Since $d_H(C_n, K) \to 0$, there exists $p_n \in C_n$ such that $d(p_n, p) \to 0$. Now, for all n, $p_n \in f^{-1}(y_n)$ so that by continuity of f we see that $f(p) = y$. This shows that $K \subset f^{-1}(y)$, and since K is connected, this means that K is a subset of a connected component of $f^{-1}(y)$. Lastly, note that $x_n, z_n \in C_n$, so that, since $d(x_n, x), d(z_n, z) \to 0$ and $d_H(C_n, K) \to 0$, we see $x, z \in K$. Hence, x and z are in the same connected component of $f^{-1}(y)$ showing that R is indeed closed in $M \times M$ and thus $M/\sim$ is indeed Hausdorff, thus completing the proof.

□

Before proving an immediate corollary of the above lemma, we quickly review the relevant notation. Let $M = X \times T^3$ be a product of a hyper-Kähler $K3$ surface with associated complex structures (I, J, K) and associated Kähler forms $(\omega_1, \omega_2, \omega_3)$, and a flat 3-torus with standard basis of 1-forms, induced via $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, denoted by (dx^1, dx^2, dx^3). Let α, β be two commuting involutions of X such that α is holomorphic non-symplectic with respect to I and β is holomorphic non-symplectic with respect to K and we extend them as in **Section** 5.1 to isometric involutions of M. We equip M with the standard product torsion-free G_2-structure, φ. Let $\mathcal{M} = M/\langle\alpha, \beta\rangle$ and π_T and $\widehat{\pi}_T$ be as in **Section** 5.2. We will further use the Constant Rank Theorem to understand the local picture of $\pi_T(A)$, and **Theorem 5.5** will be used when we prove **Proposition 5.8** in the following setting $M = A$, $E = TA$, $E' = (\pi_T|_A)^*TT^3$ and $F = d\pi_T|_A$.

Corollary 5.7.1. *Let A be a compact connected associative submanifold of M such that the derivative of the projection onto the torus factor, $d\pi_T|_A$ has constant rank* 1. *Then there exists a submersion $p : A \to S^1$ and an immersion $\iota : S^1 \to T^3$ such that $\iota \circ p = \pi_T|_A$*

Proof. By **Lemma 5.7** we know that there exists a compact connected smooth 1-manifold, Y, a submersion $p : A \to Y$ and an immersion $\iota : Y \to T^3$. Note that by the classification of 1-manifolds, we know that Y is diffeomorphic to a circle, S^1, hence giving the result □

We are finally ready to state and prove the final auxiliary result we need for our main theorem, which is a smooth analogue of our main theorem. In fact, the following proposition is best understood as a kind of converse to the fact that if we choose a complex curve in X, Σ, and an appropriate circle γ in T^3, then the product $\Sigma \times \gamma$ will be a compact connected associative submanifold of M. The surface Σ is a complex curve in X with respect to a complex structure in the S^2-family of complex structures defined by the hyper-Kähler structure on X. We will identify a complex structure on X with its Kähler form in what follows, noting that the metric is fixed.

Proposition 5.8. *Let A be a compact connected associative submanifold of M such that the derivative of the projection map onto the torus factor,*

$d\pi_T|_A$, has constant rank equal to 1. Then A is isometric to a product $\Sigma \times \gamma$, where Σ is a surface in X holomorphic with respect to some complex structure ω_v in the sphere of complex structures on X, and γ is a circle in T^3. Moreover, this isometry is the restriction of an isometry $M \to M$ of the form $id_X \times f$ where $f : T^3 \to T^3$ is an isometry.

Remark 5.9. At this point, it would be natural to ask if one can find an example of a compact connected submanifold $A \subset M$ such that $d\pi_T|_A$ has constant rank 1, but such that A is not a product $\Sigma \times \gamma$?. The answer is yes, which shows that the associativity hypothesis on A is crucial and can not be omitted, since in light of **Proposition 5.8**, if $d\pi_T|_A$ has constant rank 1 and is not a product, then A will not be associative.

We let $(X, \omega_1, \omega_2, \omega_3)$ be a generic elliptically fibred hyper-Kähler $K3$ surface from the family considered by Acharya *et al.* as in **Section** 5.2. In particular, this means that there exists a holomorphic fibration $\pi : X \to S^2$, whose generic fibre, is an ω_1-holomorphic copy of T^2. Let $\Delta \subset S^2$ be the finite set of points over which the fibres degenerate, so that over $S^2 \setminus \Delta$, the fibration is a smooth T^2-bundle, and note that Δ, generically, consists of 12 points.

Now, pick a loop $\gamma : S^1 \to S^2$, such that γ avoids the discriminant locus, $\gamma(S^1) \cap \Delta = \emptyset$, and γ has winding number ± 1 around a point of Δ. Let $(x_1, x_2, x_3) + \mathbb{Z}^3$ denote standard local coordinates on T^3 with $(x_1, x_2, x_3) \in \mathbb{R}^3$, and for fixed x_1^0, x_3^0 define

$$A = \{(t, (x_1^0, x_2, x_3^0) + \mathbb{Z}^3) | x_2 + \mathbb{Z} \in S^1, t \in \pi^{-1}(\gamma(x_2 + \mathbb{Z}))\} \subset X \times T^3 \quad (5.20)$$

Note that for each x_2, topologically, the fibre $\pi^{-1}(\gamma(x_2 + \mathbb{Z}))$ is a 2-torus, smoothly varying with x_2 so that $A \to S^1$ is a smooth fibre bundle. In fact, we see that A is the total space of the pullback of the fibration along γ. Hence, we see that A is homeomorphic to a mapping torus $(T^2 \times [0; 1])/\sim$, where $\sim$ is the equivalence relation given by $(x, 0) \sim (f(x), 1)$ for some homeomorphism $f : T^2 \to T^2$. Moreover, note that for all $p \in A$,

$$d\pi_T|_A(T_pA) = Span(\partial_{x_2}) \quad (5.21)$$

so indeed $d\pi_T|_A$ has constant rank 1. Compactness and connectedness are clear, and we can see by our choice of γ that f is not homotopic to the identity, since for a loop around a singular fibre, the homotopy class of f depends on the Kodaira classification of the singularity, and is not the class of the identity map ([42]). This shows that A is a non-trivial fibre bundle so A is clearly not homeomorphic to a product $T^2 \times \gamma$; in fact, since f is not homotopic to the identity, we know that A is not homotopy equivalent to $T^2 \times \gamma$. We can easily see this using the long exact sequence of a fibre bundle, for which we have a short exact sequence:

$$1 \to \pi_1(T^2) \to \pi_1(A) \to \pi_1(S^1) \to 1 \tag{5.22}$$

so that the fundamental group of A is a semidirect product, $\pi_1(A) \cong \pi_1(T^2) \times_{f_*} \mathbb{Z}$. This is a product, i.e. $\mathbb{Z}^3$ if and only if $f_* : \pi_1(T^2) \to \pi_1(T^2)$ is the identity map, i.e., f is homotopic to the identity of T^2.

We can also see directly that A is not associative. Note that if we denote by $\{dx^1, dx^2, dx^3\}$ the standard basis of 1-forms on T^3, then the pullbacks of dx^1 and dx^3 to A vanish. Moreover, the fibres, $F_{\gamma(x_2)} = \pi^{-1}(\gamma(x_2))$ are ω_1-holomorphic curves so we have that for all $t \in F_{\gamma(x_2)}$,

$$\omega_2|_{T_t F_{\gamma(x_2)}} = \omega_3|_{T_t F_{\gamma(x_2)}} = 0 \tag{5.23}$$

Hence the pullback of φ to A is given by

$$\varphi|_A = (\omega_2 \wedge dx^2)|_A = 0 \tag{5.24}$$

but the volume form of A is clearly non-zero.

*Proof (of **Proposition 5.8**).* Note that, since $d\pi_T|_A$ has constant rank 1 then $ker(d\pi_T|A) \subset TA$ is a rank 2 subbundle of the tangent bundle of A, by **Theorem 5.5**. We show that if $\pi_X|_A : A \to X$ denotes the restriction to A of the projection onto the $K3$ factor, then $ker(d\pi_X|_A)$ is a rank 1 subbundle of TA such that $TA = ker(d\pi_X|_A) \oplus ker(d\pi_T|_A)$. To do this we exploit the cross-product structure induced from the G_2 3-form.

Since A is associative, the cross-product induced by the G_2 3-form,

$$g_\varphi(u \times v, w) = \varphi(u, v, w) \tag{5.25}$$

where g_φ is the metric induced by φ and $u, v, w \in T_{(p,x)}(X \times T^3) \cong T_pX \oplus T_xT^3$, $(p, x) \in X \times T^3$ is such that if $u, v \in T_qA$, then $u \times v \in T_qA$ for all $q \in A$. Moreover, we note that, by direct computation using φ given by **Equation (5.3)**, for all $(p, x) \in X \times T^3$, if $v = v_X + v_T \in T_{(p,x)}X \times T^3$ with $v_X \in T_pX$ and $v_T \in T_xT^3$, and $w = w_X + w_T \in T_{(p,x)}X \times T^3$ with $w_X \in T_pX$ and $w_T \in T_xT^3$ then:

$$\begin{aligned} v_X \times w_X &\in T_xT^3 \\ v_T \times w_T &\in T_xT^3 \\ v_X \times w_T &\in T_pX \end{aligned} \tag{5.26}$$

Now if $u = u_T + u_X \in ker(d\pi_T|_A)_{(p,x)}$, then $u_T = 0$ so if $\{u, v\} \in ker(d\pi_T|_A)_{(p,x)}$ is an orthonormal basis, then $w = u \times v = u_X \times v_X \neq 0 \in T_xT^3$ so $\{u, v, w\}$ is an orthonormal basis of $T_{(p,x)}A$ and moreover, $d\pi_X|_A(w) = 0$ so $w \in ker(d\pi_X|A)_{(p,x)}$. Since $ker(d\pi_X|_A) \cap ker(d\pi_T|_A) = \{0\}$ this shows that $d\pi_X|_A$ also has constant rank equal to 2 and hence $ker(d\pi_X|_A) \subset TA$ is a rank 1 subbundle of TA and we get the claimed splitting

$$TA = ker(d\pi_X|_A) \oplus ker(d\pi_T|_A) \tag{5.27}$$

Moreover, U, V are vector fields on A taking values in $ker(d\pi_T|_A)$, if and only if for any $f \in C^\infty(T^3)$ we have that

$$U(f \circ \pi_T|_A) = 0 = V(f \circ \pi_T|_A) \tag{5.28}$$

since the pullback, $f \circ \pi_T|_A$, is constant along the fibres of $\pi_T|_A$. This implies that for any function $f \in C^\infty(T^3)$, by expanding the definition of the bracket, we have

$$[U, V]\,(f \circ \pi_T|_A) = 0 \tag{5.29}$$

so $[U, V] \in ker(d\pi_T|_A)$. Hence $ker(d\pi_T|_A)$ is integrable and by Frobenius's Integrability Theorem there exists a foliation of A whose leaves are surfaces tangent to $ker(d\pi_T|_A)$. We also see that A admits a foliation by 1 manifolds by noting that $ker(d\pi_X|_A)$ is trivially locally integrable as a real line bundle. Furthermore, the leaves of the two foliations are transverse since the tangent bundle splits as a direct sum.

Now, our goal is to move from this "local product structure" to a global one. We do this by investigating the two local foliations in more detail and exploiting the associativity of A. In fact, we will immediately see that only the foliations arising from $ker(d\pi_T|_A)$ needs to be investigated, since the local foliations arising from the kernel of $d\pi_X|_A$ can be taken to be restrictions of a global foliation of A. In what follows, we will denote

$$\begin{aligned} W &= ker(d\pi_T|_A) \\ V &= ker(d\pi_X|_A) \end{aligned} \tag{5.30}$$

We now apply **Corollary 5.7.1** to conclude that we can factor $\pi_T|_A$ into a submersion $p : A \to S^1$ and an immersion $\iota : S^1 \to T^3$ such that $\pi_T|_A = \iota \circ p$. We know that an injective immersion of a compact space, S^1, into a Hausdorff space, T^3, is an embedding, so that the only obstruction to $\pi_T(A) = \iota(S^1)$ being an embedded circle is the injectivity of ι. We will show a bit later that ι is injective.

One immediate consequence of this factorisation, is the fact that V is orientable and hence trivial. This is because for all $a \in A$, since $d\iota_a$ is injective, we have that:

$$ker(dp)_a = ker(d\pi_T|_A)_a \tag{5.31}$$

so that its restriction to V, $dp|_V : V \to p^*TS^1$ is a fibrewise injective map between real line bundles so it is a vector bundle isomorphism, noting that p is a submersion. We know TS^1 is orientable and trivial so then the same is true for V. This means that V admits a global section; since A is compact, this means that the resulting vector field admits integral curves defined for all time ([5, Chapter 5, Section 35]), so indeed the local foliations arising from V can be taken to be the restrictions of the integral curves of the trivialising section, i.e. V is globally integrable over A.

Let $y \in \pi_T(A) \subset T^3$ be a point and define $\Sigma_y = A \cap (X \times \{y\})$. This is a submanifold of A; to see this note that since $d\pi_T|_A$ has rank 1, we know from the Constant Rank Theorem that there exist charts about every point p of A, $\phi : U \subset A \to \mathbb{R}^3$ and $\psi : \mathcal{U} \subset T^3 \to \mathbb{R}^3$, about its image $\pi_T(p)$, such that we have

$$\psi \circ \pi_T|_A \circ \phi^{-1}(u_1, u_2, u_3) = (0, 0, u_3) \tag{5.32}$$

In these coordinates we have $\Sigma_y \cap U = (\pi_T|_A)^{-1}(y) \cap U = \phi^{-1}(\{u_3 = \text{constant}\})$, giving that Σ_y is indeed a (compact) 2 dimensional submanifold of A. Moreover, we see that the tangent space of Σ_y at a point p is precisely given by the fibre of W over p, W_p.

In a trivialization around y, we have that W admits pointwise orthonormal local sections $\{e_1, e_2\}$, which we complete to a pointwise orthonormal local frame of TA, $\{e_1, e_2, e_3\}$ with $e_3 = e_1 \times e_2$, where $\times$ denotes the cross product structure induced from the G_2 3-form φ and we note that e_3 is a local section of V, and hence, up to a sign choice, it is globally defined since V is a trivial real line bundle. To see this, note that the induced cross product on the tangent spaces of A satisfies that if $e_1, e_2 \neq 0$ are tangent vectors to X, then their cross-product, $e_1 \times e_2 \neq 0$, is tangent to T^3. Pointwise, we let

$$v_y = (dx^1(e_3), dx^2(e_3), dx^3(e_3)) \in S^2 \subset \mathbb{R}^3 \tag{5.33}$$

Moreover, we define, for $i = 1, 2, 3$,

$$\begin{aligned} v_y^i &= dx^i(e_3) \\ \omega_{v_y} &= \sum_{i=1}^{3} v_y^i \omega_i \end{aligned} \tag{5.34}$$

so that we have the following:

$$\varphi(e_3, e_1, e_2) = \sum_{i=1}^{3} dx^i(e_3)\omega_i(e_1, e_2) = \omega_{v_y}(e_1, e_2) \tag{5.35}$$

which together with the calibration condition for A, $\varphi|_{TA} = dVol_A$ gives us that

$$\omega_{v_y}(e_1, e_2) = 1 \tag{5.36}$$

Now, let Y be any vector field tangent to Σ_y so Y is a section of W and note that in the Levi-Civita connection on A, which is the restriction of the Levi-Civita product connection on $X \times T^3$ to A, we have that:

$$\nabla dx^i = 0 \tag{5.37}$$

since T^3 is flat. From the Leibniz rule for the induced Levi-Civita on 1-forms, we then have:

$$Y(v_y^i) = Y(dx^i(e_3)) = (\nabla_Y dx^i)(e_3) + (dx^i)(\nabla_Y e_3) = (dx^i)(\nabla_Y e_3) \tag{5.38}$$

and then using metric compatibility of the Levi-Civita connection we see that:

$$\langle \nabla_Y e_3, e_3 \rangle = \frac{1}{2} Y(\langle e_3, e_3 \rangle) = 0 \tag{5.39}$$

since e_3 is a unit vector field. Thus $\nabla_Y e_3 \in W$ and hence $dx^i(\nabla_Y e_3) = 0$. Hence, this gives us that for any vector field, Y, on Σ_y, we have $Y(v_y^i) = 0$ so that v_y^i are constant on Σ_y.

Hence we see that Σ_y is a ω_{v_y}-holomorphic curve in X. In other words, we have a well-defined map taking a point $y \in \pi_T(A)$ to a complex structure in the sphere of complex structures on X. We now show that this map is constant so that the G_2 3-form and A determine a preferred complex structure on X, in which the fibre over every point, Σ_y, is a holomorphic curve in X. Let

$$a_i(y) = \int_{\Sigma_y} \omega_i \tag{5.40}$$

and let $I \subset \pi_T(A)$ be the image of a small closed interval parametrised by u_3 as above, centered on y with endpoints y_1, y_2, and set $B_y = A \cap (X \times I)$. Note that by a similar application of the Constant Rank Theorem as above, we see that B_y is a submanifold with boundary, $\partial B = \Sigma_{y_1} \cup \Sigma_{y_2}$. Then by Stokes's theorem we have:

$$0 = \int_B d\omega_i = \int_{\partial B} \omega_i = a_i(y_2) - a_i(y_1) \tag{5.41}$$

so that, by connectedness of A, a_i are constant on $\pi_T(A)$. Writing v_y^i for the $i-th$ coordinate of v_y we see that

$$\omega_i(e_1, e_2) = \varphi(\partial_{x_i}, e_1, e_2) = \langle \partial_{x_i}, e_3 \rangle = dx^i(e_3) = v_y^i = v_y^i \omega_{v_y}(e_1, e_2) \tag{5.42}$$

which we can summarise as $\omega_i|_{T\Sigma_y} = v_y^i \omega_{v_y}|_{T\Sigma_y}$. Hence, integrating along Σ_y, and noting that v_y is constant along Σ_y and that Σ_y is calibrated by ω_{v_y}, we see that:

$$a = \int_{\Sigma_y} v_y \omega_{v_y} = Vol(\Sigma_y) v_y \tag{5.43}$$

where $a = (a_1, a_2, a_3)$. Noting that $||v_y|| = 1$ and taking norms above, we see that $Vol(\Sigma_y)$ is constant, and so v_y is also constant as a map $\pi_T(A) \to S^2$. This shows that indeed, the map taking a point of $\pi_T(A)$ to the sphere of complex structures on X is constant.

Recall that since the line bundle V is trivial, e_3 is a globally defined section of V. We identify $TT^3 \cong T^3 \times \mathbb{R}^3$ and consider the vector field

$$y \in T^3 \to (y, v_y) \in T^3 \times \mathbb{R}^3 \tag{5.44}$$

which we will also denote by v_y and note that, it is constant in each fibre under the product identification of TT^3. Hence, since T^3 is flat we have that $\nabla^{T^3} v_y = 0$ where ∇^{T^3} denotes the Levi-Civita connection on T^3. Pulling back the tangent bundle of T^3 to A, we get a bundle

$$\pi_T|_A^* TT^3 \to A \tag{5.45}$$

for which we can identify $d\pi_T|_A(e_3)$ with the pullback of v_y, $v_y \circ \pi_T|_A$. Hence $d\pi_T|_A(e_3)$ is covariantly constant as a section of $\pi_T|_A^* TT^3$. Moreover, once we establish the injectivity of ι, we have that along the image curve $\gamma = \pi_T(A) \subset T^3$

$$\nabla_{\dot{\gamma}}^{T^3} v_y = 0 \tag{5.46}$$

i.e. the constant vector field $v_y \in \Gamma(TT^3)$ is tangent to $\pi_T(A)$.

We now argue by contradiction that $\iota : S^1 \to T^3$, with $\iota(S^1) = \pi_T(A)$, is injective. Suppose that there exist two distinct points $x, x' \in S^1$ such that

$\iota(x) = \iota(x') = y_0$. This implies that there are at least two distinct points $q, q' \in A$ such that $\pi_T(q) = \pi_T(q') = y_0$, since the submersion $p : A \to S^1$ is surjective, so we can choose q and q' to be in the pre-images of x and x' respectively, and we have that $\pi_T(q) = \iota \circ p(q) = \iota \circ p(q') = \pi_T(q')$. Moreover, note that q and q' lie in distinct connected components of $\pi_T|_A^{-1}(y_0)$. Hence, by the Constant Rank Theorem we can choose small enough neighbourhoods of q and q' in A, U_q and $U_{q'}$ respectively, such that $\pi_T(U_q) = \{(0, 0, u_3)\} = C_q \subset T^3$, $\pi_T(U_{q'}) = \{(0, 0, u'_3)\} = C_{q'} \subset T^3$, $C_q \neq C_{q'}$.

Note that $y_0 \in C_q \cap C_{q'}$ and moreover, that in this parametrisation we have that the tangent spaces to the embedded arcs C_q and $C_{q'}$ are spanned by $v_y(q)$ and $v_y(q')$ as previously defined. In particular, at y_0, since v_y is a constant map, we have that:

$$T_{y_0}C_q = Span(v_y(q)) = Span(v_y(q')) = T_{y_0}C_{q'} \tag{5.47}$$

which means that at y_0, the intersection of the two arcs has to be tangential.

Denote C_q and $C_{q'}$ by C_1 and C_2 respectively. Let $\gamma_i : (-\epsilon; \epsilon) \to T^3$ be arc-length parametrisations of C_i with $\gamma_i(0) = y_0$, so that we have that γ_i satisfy the first-order differential equations:

$$\begin{aligned} \dot{\gamma}_i(s) &= v_y(\gamma_i(s)) \\ \gamma_i(0) &= y_0 \end{aligned} \tag{5.48}$$

for $s \in (-\epsilon; \epsilon)$ the arc-length parameter. Now note that v_y is constant so then both γ_1 and γ_2 satisfy the same differential equation and, hence, by Picard-Lindelöf we know that the solution of the above differential equation is unique so we have in fact $\gamma_1 = \gamma_2$. This means that two arcs C_1 and C_2 are in fact the same, $C_1 = C_2$, thus giving a contradiction. Hence, indeed $\pi_T(A)$ is an embedded circle in T^3 so let $\gamma = \pi_T(A)$, and note that $\pi_T|_A$ is a proper surjective submersion of A onto γ.

Fix $y_0 \in \gamma$, and set $\Sigma = \Sigma_{y_0}$ then Σ is holomorphic with respect to ω_v where $v = v_y$ as above. Hence by the Ehresmann fibration theorem ([21]), since $\pi_T|_A$ is a proper surjective submersion, we see that $\pi_T|_A : A \to \gamma$ is a smooth fibre bundle over the circle γ, with fibre Σ. To show triviality of this fibre bundle we look at the flow of the unit vector field trivialising $T\gamma$:

Let $h : S^1 \to \gamma$ be a diffeomorphism such that $h(0) = y_0$. For $\theta \in \mathbb{R} \pmod{2\pi}$ define the rotation by θ map, $R_\theta(e^{it}) = e^{i(t+\theta)}$, and set

$$\tau_\theta = h \circ R_\theta \circ h^{-1} : \gamma \to \gamma \tag{5.49}$$

Let ∂_t be the unit speed tangent vector field on S^1 and denote by ∂_θ its pushforward under h, $\partial_\theta = dh(\partial_t)$ at $t = h^{-1}(y)$ and note that τ_θ is the time θ flow of ∂_θ. As previously discussed, γ is tangent to the constant vector field on T^3, v, so we take h such that ∂_θ is the restriction of v to γ; this means that rotations in the angular parameter on γ correspond to translations by v in the T^3.

Since $ker(d\pi_T|_A) = W$ and $TA = W \oplus V$ and $rank(d\pi_T|_A) = 1$, the restriction $d\pi_T|_V : V \to (\pi_T|_A)^*T\gamma$ is a vector bundle isomorphism so that there is a unique smooth vector field ξ on A with values in V such that

$$d\pi_T|_A(\xi) = \partial_\theta \tag{5.50}$$

As previously discussed, since A is compact, the flow of ξ is complete ([5, Chapter 5, Section 35]). Let $\Phi_\theta : A \to A$ denote the time θ flow of ξ. We now show that this flow projects to rotation by θ in γ and that it preserves the projection onto the $K3$ factor, X, so that its integral curves are circles.

Because, $\xi \in V = ker(d\pi_X|_A)$ we have that for all $\theta \in \mathbb{R}$:

$$\pi_X \circ \Phi_\theta = \pi_X \tag{5.51}$$

Now define the map $F_\theta : A \to \gamma$ by $F_\theta = \pi_T \circ \Phi_\theta$ and differentiate in θ and use the chain rule:

$$\frac{d}{d\theta}F_\theta(p) = d\pi_T(\xi(\Phi_\theta(p))) = \partial_\theta(F_\theta(p)) \tag{5.52}$$

and note that we also have $F_0 = \pi_T$. By uniqueness of flows of vector fields on S^1, we see that $F_\theta = \tau_\theta \circ \pi_T$ giving us for all $\theta \in \mathbb{R}$:

$$\pi_T \circ \Phi_\theta = \tau_\theta \circ \pi_T \tag{5.53}$$

From the above we see that if $p \in A$, then we have that $\Phi_{2\pi}$ preserves every fibre setwise since for all $\theta \in \mathbb{R}$:

$$(\pi_X(\Phi_\theta(p)), \pi_T(\Phi_\theta(p))) = (\pi_X(p), \tau_\theta(\pi_T(p))) \tag{5.54}$$

hence we conclude that $\Phi_{2\pi}(p) = p$ for all $p \in A$ so that the monodromy of the fibre bundle is trivial. We now construct the global trivialization as follows:

Let $F : \Sigma \times S^1 \to A$ be the map defined by $F(p,\theta) = \Phi_\theta(p)$. This has a smooth inverse given by $F^{-1}(p) = (\Phi_{-\theta}(p), \theta)$ where $\theta = h^{-1}(\pi_T(p))$ so that F is a diffeomorphism. Composing with the map $id_\Sigma \times h : \Sigma \times S^1 \to \Sigma \times \gamma$ we get the desired diffeomorphism $\widetilde{F} : \Sigma \times \gamma \to A$.

All that remains is to identify $\widetilde{F}$ with the restriction of an isometry of M. To do this we let $Y = (0, v)$ be the constant vector field on $M = X \times T^3$.

Along A, this lies in $\ker(d\pi_X) = V$ and satisfies $d\pi_T(Y) = v$. By our choice of diffeomorphism $h : S^1 \to \gamma$, we have $\partial_\theta = v$ along γ so that $Y|_A$ and ξ are both horizontal lifts of the same vector field living in the same line subbundle, V. Hence, by the linear isomorphism $d\pi_T|_V$ they must coincide:

$$\xi = Y|_A \tag{5.55}$$

Now, let $\Psi_\theta = id_X \times R_\theta$ be the isometric flow in M whose integral curves are circles in T^3 parallel to γ. Then we must have that $\Phi_\theta = \Psi_\theta|_A$ and hence

$$F(p, \theta) = \Psi_\theta(p) \tag{5.56}$$

for $(p, \theta) \in \Sigma \times S^1$ so that the global trivialization, $\widetilde{F}$, is the restriction of the global isometry Ψ_θ of M to A (up to composing with the chosen diffeomorphism h). $\square$

Now armed with the above proposition, we are ready to prove our main theorem, which we restate below:

Theorem 5.10. *Let $(X; I, J, K)$ be a hyper-Kähler $K3$ surface admitting two commuting involutions α and β. Let α be holomorphic non-symplectic for the complex structure I and β holomorphic non-symplectic for the complex structure K. Moreover, suppose that the invariants of α and β satisfy $(r_\alpha, a_\alpha, \delta_\alpha), (r_\beta, a_\beta, \delta_\beta) \neq (10, 10, 0), (10, 8, 0)$. Extend α and β to $M = X \times T^3$ to involutions acting on the product such that on T^3 they act as in* ***Equations (5.1) and (5.2)****. Equip M with the standard product G_2 3-form, defined in* ***Equation (5.3)****, and let $\mathcal{M} = M/\langle\alpha, \beta\rangle$ denote the resulting G_2 orbifold.*

Then the singular locus of $\mathcal{M}$ contains an associative suborbifold of the form $(\Sigma \times \gamma)/\langle\alpha, \beta\rangle$, where $\Sigma \subset X$ is a complex curve and $\gamma \subset T^3$ is an embedded circle.

Moreover, suppose that $\mathcal{A}$ is a compact connected associative suborbifold in $\mathcal{M}$ such that the derivative of the projection map $\widehat{\pi}_T : \mathcal{M} \to T^3/\langle\alpha, \beta\rangle$ restricted to $\mathcal{A}$, $d\widehat{\pi}_T|_{\mathcal{A}}$, has constant rank 1. Then there exists a surface $\Sigma \subset X$ holomorphic with respect to some complex structure in the sphere of complex structures generated by (I, J, K) on X, and an embedded circle $\gamma \subset T^3$ such that $\mathcal{A}$ is isometric to $(\Sigma \times \gamma)/G$, where G is a subgroup of $\langle\alpha, \beta\rangle$. Moreover, the isometry $\mathcal{A} \to (\Sigma \times \gamma)/G$ is the restriction of an isometry $\mathcal{M} \to \mathcal{M}$ induced from an isometry of T^3.

Proof. First note that if σ_α denotes the fixed point set of α, then $\beta(\sigma_\alpha) = \sigma_\alpha$ since α and β commute so that β is a homeomorphism of the fixed point set of α. Moreover, β is a homeomorphism of each connected component of the fixed point set of α onto another connected component. Now due to Nikulin's results ([51]) we know the fixed point set of α consists of a surface of genus $g = \frac{1}{2}(22 - r_\alpha - a_\alpha)$, Σ_g, and $k = \frac{1}{2}(r_\alpha - a_\alpha)$ disjoint rational curves.

Suppose first that $r_\alpha + a_\alpha \neq 22$, so that $g > 0$. This means that $\beta(\Sigma_g) = \Sigma_g$. Moreover, Σ_g is a holomorphic surface with respect to the complex structure I so that ω_1 calibrates Σ_g. Hence, if γ_1 denotes the circle corresponding to the x_1 coordinate and $\iota : \Sigma_g \times \gamma_1 \to M$ then

$$\iota^* \varphi = \omega_1 \wedge dx^1 = dVol_{\Sigma_g \times \gamma_1} \tag{5.57}$$

Hence $\Sigma_g \times \gamma_1$ is associative in M. Moreover, it is fixed setwise by β and α (in fact, pointwise by α) so it descends to a well-defined associative, $(\Sigma_g \times \gamma_1)/\langle \alpha, \beta \rangle$ in the quotient, $\mathcal{M}$.

If $r_\alpha + a_\alpha = 22$, then the fixed locus of α consists of $k+1$ rational curves. If $k = 0$, so that the only possible invariants of α are $(11, 11, 1)$, then the above proof also works, since β fixes Σ_0 setwise. If $k > 0$, then β may swap Σ_0 with another rational curve, E_1. In such a case, we consider the associative submanifold given by $(\Sigma_0 \cup E_1) \times \gamma_1$. This then descends to a well-defined associative $((\Sigma_0 \cup E_1) \times \gamma_1)/\langle \alpha, \beta \rangle$.

Now suppose that $\mathcal{A}$ is a compact associative suborbifold of $\mathcal{M}$ such that $d\widehat{\pi}_T|_{\mathcal{A}}$ has constant rank 1. Let $p : M \to \mathcal{M}$ denote the projection and let A be a connected component of the pre-image of $\mathcal{A}$ through the quotient projection, $A \subset p^{-1}(\mathcal{A})$. Since α and β preserve the G_2 3-form φ, then A is also an associative submanifold. Note that by definition $d\pi_T|_A$ has constant rank 1 so that **Proposition 5.8** implies that A is a product of $\Sigma \subset X$, a holomorphic curve with respect to some complex structure in the sphere of complex structures on X, and a circle $\gamma \subset T^3$.

Now let

$$G = \{g \in \langle \alpha, \beta \rangle | g(A) = A\} \tag{5.58}$$

be the stabilizer of A. Since every $g \in G$ acts by isometries on M and preserves φ and the diffeomorphism $A \to \Sigma \times \gamma$ is the restriction of an isometry of M, it follows that $(\Sigma \times \gamma)/G$ is a well-defined associative suborbifold of $\mathcal{M}$ and we have that

$$\mathcal{A} \cong A/G \cong (\Sigma \times \gamma)/G \tag{5.59}$$

$\square$

5.4 Classifying Constant Rank Associative 3-folds

In light of **Theorem 5.10** we are naturally led to ask about which other constant rank associatives can occur. To make this more precise, suppose that $rank(d\pi_T|_A) = r$ is constant. We have shown in the previous section that for $r = 1$, there exist associative 3-folds and that any such 3-fold is diffeomorphic to a product $\Sigma \times \gamma$ where Σ is a surface in X and γ is an embedded circle in T^3. Our question then becomes, for the other possibilities of r, namely $r = 0, 2$, or 3 can we find associative 3-folds, A, such that $rank(d\pi_T|_A) = r$ is constant? Moreover, if we can find such associative 3-folds, what can we say about them? The answer to these questions is presented in the following proposition:

Proposition 5.11. *Let $M = X \times T^3$ and $A \subset M$ be a compact connected associative such that $rank(d\pi_T|_A) = r$ is constant. Then r is either 1 or 3. Moreover, if $r = 3$, then $A = \{x_0\} \times T^3$ for some $x_0 \in X$, and so $\pi_T|_A$ is a diffeomorphism.*

Remark 5.12. Note that, as previously mentioned, the case $r = 1$ is already covered by **Proposition 5.8** so that **Proposition 5.11** together with **Proposition 5.8** provide a full description of the class of associative submanifolds such that $rank(d\pi_T|_A)$ is constant.

*Proof (of **Proposition 5.11**).* We need to show that for $r = 0, 2$ there exist no compact connected constant rank associative submanifolds and we need to show that for $r = 3$, we have that $\pi_T|_A$ is a diffeomorphism. For this, we recall **Equations 5.26** which we summarise as follows:

$$\begin{aligned} TX \times TX &\to TT^3 \\ TX \times TT^3 &\to TX \\ TT^3 \times TT^3 &\to TT^3 \end{aligned} \tag{5.60}$$

Let $p \in A$ and consider first the case $r = 0$. In this case $T_pA = ker(d\pi_T|_A)_p$ so if e_1, e_2 are two distinct non-zero vectors in T_pA, then $e_1, e_2 \in T_pX$ so that $e_1 \times e_2 \in TT^3$, but T_pA is stable under $\times$ so we have that $e_1 \times e_2 = 0$. Since both e_1 and e_2 are non-zero this is a contradiction.

Similarly, if $r = 2$ we have that $dim(ker(d\pi_T|_A)) = 1$. Let $Span\{e\} = ker(d\pi_T|_A)$. Since the rank is 2 we can choose $e_1, e_2 \in T_pA$ such that $t_1 = d\pi_T|_A(e_1)$ and $t_2 = d\pi_T|_A(e_2)$ are linearly independent. Then we have that for $i = 1, 2$:

$$e_i = a_i e + t_i \tag{5.61}$$

where $a_i \in \mathbb{R}$. Computing the cross-product we see:

$$e_1 \times e_2 = (a_1 e + t_1) \times (a_2 e + t_2) = a_1 e \times t_2 + a_2 t_1 \times e + t_1 \times t_2 \tag{5.62}$$

Note that $e \times t_i \in TX$ so that

$$d\pi_T|_A(e_1 \times e_2) = t_1 \times t_2 \tag{5.63}$$

Since t_1 and t_2 are linearly independent in $d\pi_T|_A(T_P A)$ we have that $t_3 = t_1 \times t_2 \neq 0$ and that $\{t_1, t_2, t_3\}$ is a linearly independent set, contradicting that $rank(d\pi_T|_A) = 2$.

The last case to consider is the case where $r = 3$. In this case $\pi_T|_A$ is a local diffeomorphism. Since A is compact, then $\pi_T|_A$ is a proper map. Moreover, $\pi_T|_A$ is surjective and a local homeomorphism, and T^3 is compact so that [30, Lemma 2] tells us that $\pi_T|_A$ is a covering map. The only thing left to understand, in this case, is what possibilities exist for the degree of $\pi_T|_A$. It turns out that, by exploiting the cross product once again, the only possibility for the degree is 1, so that $\pi_T|_A$ is a diffeomorphism and the associative is given by $\{x_0\} \times T^3$. To see this, let $p \in A$ and note that if $u = x_u + t_u$, $v = x_v + t_v \in T_{\pi_X(p)}X \oplus T_{\pi_T(p)}T^3$, then we have that

$$d\pi_T|_A(u \times v) = t_u \times t_v + \Lambda(x_u, x_v) \tag{5.64}$$

where Λ is the linear map characterised by

$$\langle \Lambda(u, v), s\rangle = \varphi(u, v, s) \tag{5.65}$$

for all $s \in T_{\pi_T(p)}T^3$, $u, v \in T_{\pi_X(p)}X$. Note that if we choose an orthonormal basis, $\{E_1, E_2, E_3\}$, of $T_{\pi_T(p)}T^3$, then we have that

$$\Lambda(u, v) = \sum_{i=1}^{3} \omega_i(u, v) E_i \tag{5.66}$$

Now, let $\{e_1, e_2, e_3\}$ be an oriented orthonormal basis of $T_p A$ and we decompose for all $i = 1, 2, 3$:

$$e_i = x_i + t_i \tag{5.67}$$

where $x_i \in T_{\pi_X(p)}T^3$ and $t_i \in T_{\pi_T(p)}T^3$. Moreover, note that because A is associative, then we have $e_3 = e_1 \times e_2$. This gives us that

$$t_3 = d\pi_T|_A(e_3) = t_1 \times t_2 + \Lambda(x_1, x_2) \tag{5.68}$$

Now, by definition of the cross product, for any $w = x_w + t_w \in T_pA$, we have that

$$\langle e_3, w\rangle = \langle e_1 \times e_2, w\rangle = \varphi(e_1, e_2, w) \tag{5.69}$$

We now compute both sides independently. The left-hand side gives us that

$$\langle e_3, w\rangle = \langle t_3, t_w\rangle + \langle x_3, x_w\rangle \tag{5.70}$$

while the right-hand side is equivalent to

$$\varphi(e_1, e_2, w) = \langle \Lambda(x_1, x_2), t_w\rangle + \langle t_1 \times t_2, t_w\rangle = \langle t_3, t_w\rangle \tag{5.71}$$

where for the first equality we plugged in the expression for φ and for the last equality we used **Equation 5.68**.

This means that for all $w \in T_pA$, we have that $\langle x_3, x_w\rangle = 0$, so for $w = e_3$, we have $||x_3||^2 = 0$ which gives us that $x_3 = 0$. Repeating the argument above by cyclically permuting the indices $\{1, 2, 3\}$ we get that $x_1 = x_2 = x_3 = 0$. This shows that $d\pi_X|_A = 0$ which means that $\pi_X|_A$ is locally constant. Since A is connected, this implies that $\pi_X|_A$ is a constant map, so there exists $x_0 \in X$ such that $\pi_X(A) = \{x_0\}$. This shows that indeed, if $rank(d\pi_T|_A) = 3$, then $A = \{x_0\} \times T^3$ and $\pi_T|_A$ is a diffeomorphism.

□

We end this section with a few remarks on the statement and the proof of **Theorem 5.10**

Remark 5.13. In the first part of the proof of **Theorem 5.10**, namely the one on the singular locus of $\mathcal{M}$, the roles of α and β can be interchanged. In other words, our proof doesn't depend on having chosen α: we could just as well have used β or $\alpha\beta$, provided we also select the circle factor in T^3 corresponding to the complex structure on X for which the fixed locus is holomorphic. This substitution would yield a topologically distinct associative suborbifold, due to Nikulin's classification ([51]), as long as

$$r_\alpha + a_\alpha \neq r_\beta + a_\beta \tag{5.72}$$

Then the genus, g_α, of the fixed surface for α and the genus, g_β, of the fixed surface for β will be distinct giving us topologically distinct associatives.

Another immediate consequence of this is that the 3-dimensional connected components of the singular locus of $\mathcal{M}$ which satisfy the "constant rank 1 torus projection" condition will be quotients of $\Sigma \times \gamma$, where Σ is a complex curve with genus at most 10 and γ an embedded circle, by investigating the possibilities for g_α given Nikulin's classification.

Remark 5.14. Our final remark is on the exclusion of the cases $(r, a, \delta) = (10, 10, 0)$ and $(r, a, \delta) = (10, 8, 0)$, the cases where the fixed loci of α and β are either empty or consist of two elliptic curves respectively. Excluding the case $(r, a, \delta) = (10, 10, 0)$ is genuinely needed to get a well-defined G_2 orbifold. If for α $(r, a, \delta) = (10, 10, 0)$ then the quotient of X by α would be an Enriques surface, E. We know that such a complex surface does not have trivial canonical bundle, $c_1(E) \neq 0$, and thus can not admit a hyper-Kähler structure ([9, Chapter VIII, Section 15]). Hence its quotient by β, E/β will also not admit a hyper-Kähler structure. Now $X/\langle\alpha, \beta\rangle$ is isomorphic to E/β so that it does not admit a hyper-Kähler structure. Hence the quotient of the product, $\mathcal{M} = (X \times T^3)/\langle\alpha, \beta\rangle$ will not be a G_2 orbifold. Hence we need to exclude the possibility of any of the two involutions being fixed point free.

We omit full details for the $(r, a, \delta) = (10, 8, 0)$ case, for brevity, as the existence of the relevant associative suborbifolds has already been established by Acharya *et al.*. Our arguments would go through with additional casework which we briefly outline below; without loss of generality suppose that the invariants for α are $(10, 8, 0)$.

Let $E_1 \cup E_2$ be the two elliptic curves fixed by α. Since α and β commute, β fixes $E_1 \cup E_2$, and we have two cases:

1) β fixes E_1 setwise; in this case we can consider $E_1 \times \gamma_1$, just as in the proof of **Theorem 5.10**, and the same computation as before shows $E_1 \times \gamma_1$ is associative.

2) β swaps E_1 and E_2; in this case we need to consider $(E_1 \cup E_2) \times \gamma_1$. The rest of the proof still follows in this case giving us an associative suborbifold.

Concerning local isotropy groups of the singularities in $\mathcal{M}$ for the constant rank 1 associative connected components, the second case above together with the case where the invariants of one of the involutions satisfy $r + a = 22$ and $r - a > 0$ are rather special since we have that the isotropy group is $\mathbb{Z}_2$ at every point of $((E_1 \cup E_2) \times \gamma_1)/\langle\alpha, \beta\rangle$. This is because β doesn't fix any points of $E_1 \cup E_2$. However, in the first case, and all the other cases considered in the proof of **Theorem 5.10**, β can also have fixed points when restricted to the fixed locus of α. At such fixed points of β, the local isotropy in $\mathcal{M}$ will be $\mathbb{Z}_2^2$.

Concluding Remarks and Open Questions

Prior to this thesis, the construction of compact G_2-orbifolds via Twisted Connected Sum was not analysed in detail. While the possibility of extending the construction to orbifolds has been considered in the literature by Frank Reidegeld in [55], various details remained unchecked; **Chapter** 2 above addresses this issue.

Moreover, the problem of irreducibility for G_2 orbifolds remained largely ignored. The work we have done helps bridge this gap by providing a criterion similar to that introduced by Dominic Joyce in [33]. We have also checked the irreducibility of many examples of compact G_2-orbifolds found in the literature. Nonetheless, one particular aspect of **Theorem 3.2** remains rather unsatisfactory, namely, that the holonomy of the orbifold is defined in terms of the smooth locus.

Recall, the holonomy group of the orbifold $\mathcal{M}$ is defined to be the holonomy of the smooth locus of $\mathcal{M}$:

$$Hol(\mathcal{M}) = Hol(\mathcal{M} \setminus \mathcal{M}_{sing}) \tag{5.73}$$

However, all of the other machinery used in this thesis for orbifolds did not require restricting to the smooth locus. It would be desirable to have a notion of holonomy which does not ignore the singular locus. Section 1.3 above contains one possible avenue to do this using $\mathcal{H}_{\mathcal{M}}$-loops. We have already used $\mathcal{H}_{\mathcal{M}}$-loops to define the orbifold fundamental group, and it would be quite natural to use them to define holonomy. We sketch how such a construction would work:

Let g be a Riemannian metric on $\mathcal{M}$. Consider an $\mathcal{H}_{\mathcal{M}}$-loop, (h_i, c_i). We say (h_i, c_i) is *smooth* if each c_i is in fact a smooth path. For each $i = 1, \ldots, n$, we have parallel transport maps P_{c_i}, sending $v \in T_{c_i(t_{i-1})}U_i \mapsto s_i(1) \in T_{c_i(t_i)}U_i$, where $s_i : [0;1] \to TU_i$ is the unique section satisfying

$s_i(0) = v \in T_{c_i(t_{i-1})}U_i$ and $(\nabla_i)_{\dot{c}_i} s = 0$.

Now each diffeomorphism h_i satisfies $h_i c_i(t_i) = c_{i+1}(t_i)$ so that its differential maps isomorphically $(dh_i)_{c_i(t_i)} : T_{c_i(t_i)}U_i \to T_{c_{i+1}(t_i)}U_{i+1}$. Hence we get a well-defined linear map, $P_{(h_i,c_i)} : T_pU_0 \to T_pU_0$, where, $p = c_1(0)$, given by

$$P_{(h_i,c_i)} = (dh_n)_{c_n(t_n)} P_{c_n} (dh_{n-1})_{c_{n-1}(t_{n-1})} \dots P_{c_2} (dh_1)_{c_1(t_1)} P_{c_1} \tag{5.74}$$

which we shall call the *parallel transport map for an $\mathcal{H}_{\mathcal{M}}$-loop.*

Definition 5.15. We define the *orbifold Riemannian holonomy group* based at p, to be

$$Hol_p^{orb}(g) = \{P_{(h_i,c_i)} | (h_i, c_i) \text{ is an } \mathcal{H}_{\mathcal{M}} - \text{loop based at } p\} \tag{5.75}$$

and the *restricted orbifold Riemannian holonomy group* based at p to be

$$\begin{aligned} Hol_p^{orb,0}(g) = \{P_{(h_i,c_i)} | &(h_i, c_i) \text{ is an } \mathcal{H}_{\mathcal{M}} - \text{loop based at } p, \\ &(h_i, c_i) \text{ is homotopic to a trivial loop,} \\ &(1, c_0), \text{ where } c_0 \text{ is a constant loop}\} \end{aligned}$$

In order for this definition to make sense, we need to show that $Hol_p^{orb}(g)$ is indeed a group:

Proposition 5.16. *$Hol_p^{orb}(g)$ forms a group under composition of linear maps.*

Proof. Let $C_0 : [0;1] \to U_{\mathcal{M}}$ be the constant loop based at p, $C_0(t) = p, \forall t \in [0;1]$. Then the identity element of $Hol_p^{orb}(g)$ is given by, $P_{(1,C_0)}$ (where we write 1 for the identity map of some neighbourhood of p) since:

$$P_{(1,C_0)}(v) = P_{C_0}(v) = v \tag{5.76}$$

for all $v \in T_pU_1$.

Note that since an $\mathcal{H}_{\mathcal{M}}$-loop is made of various paths glued together, it suffices to consider an $\mathcal{H}_{\mathcal{M}}$-loop of the form (h, c), with $c : [0;1] \to U$ and $hc(1) = c(0) = p$. This makes everything notationally cleaner and what happens with inverses will be more transparent. For such a $\mathcal{H}_{\mathcal{M}}$-loop we have that the parallel transport map takes the rather easy form:

$$P_{(h,c)} = (dh)_{c(1)} P_c \tag{5.77}$$

which has inverse given by:

$$\begin{aligned} P_{(h,c)}^{-1} &= P_c^{-1}((dh)_{c(1)})^{-1} \\ &= P_{c^{-1}}(d(h^{-1}))_{c(0)} \\ &= P_{c^{-1}}(d(h^{-1}))_p \end{aligned} \tag{5.78}$$

where $c^{-1}(t) = c(1-t)$. Now if we can find a $\mathcal{H}_{\mathcal{M}}$-loop (h'_i, c'_i) for which $P_{(h'_i,c'_i)} = P^{-1}_{(h,c)}$ we are done.

The problem with **Equation (5.78)** is that it first maps a tangent vector in T_pU to one in $T_{c(1)}U$ and then transports it back to T_pU, which is exactly the reverse of what our definition of parallel transport should do. However, this can be fixed; slightly abusing notation and allowing the sequence in the definition of an $\mathcal{H}_{\mathcal{M}}$-loop to "start" at -1, consider the following $\mathcal{H}_{\mathcal{M}}$-loop:

- The sequence will be $t_{-1} = -1 < t_0 = 0 < t_1 = 1 < t_2 = 2$
- $c'_0 \coloneqq C_0 : [-1;0] \to U$, such that $c_0(t) = p$, $\forall t \in [-1;0]$
- $c'_1 \coloneqq c^{-1} : [0;1] \to U$, such that $c'_1(t) = c(1-t)$
- $h'_0 = h^{-1}$ and $h'_1 = 1$

Note that with the above data we have that $h_0c'_0(0) = c'_1(0)$ and $h_1c'_1(1) = c'_1(1) = p = c_0(-1)$. Moreover, the parallel transport map for (h', c') takes the form:

$$\begin{aligned} P_{(h',c')} &= (dh'_1)_{c'_1(1)} P_{c'_1} (dh'_0)_{c'_0(0)} P_{c'_0} \\ &= (d1)_p P_{c^{-1}} (d(h^{-1}))_p id_{T_pU} \\ &= P_{c^{-1}}(d(h^{-1}))_p \end{aligned} \tag{5.79}$$

which is exactly the same as **Equation (5.78)**.

□

Moreover, note that for any two $\mathcal{H}_{\mathcal{M}}$-loops, (h_i, c_i) and (h'_i, c'_i) we have that $P_{(h_i,c_i)}P_{(h'_i,c'_i)} = P_{(h_i,c_i)\star(h'_i,c'_i)}$, where $(h_i, c_i)\star(h'_i, c'_i)$ is the $\mathcal{H}_{\mathcal{M}}$-loop obtained by concatenating the two loops. We also have the following property, which is analogous to how holonomy behaves for manifolds:

Proposition 5.17. *$Hol_p^{orb,0}(g)$ is a normal subgroup of $Hol_p^{orb}(g)$. Moreover, there exists a surjective group homomorphism:*

$$\pi : \pi_1^{orb}(\mathcal{M}, p) \to Hol_p^{orb}(g)/Hol_p^{orb,0}(g) \tag{5.80}$$

In particular, if $\mathcal{M}$ is orbifold simply-connected (i.e $\pi_1^{orb}(\mathcal{M}) = \{1\}$), then $Hol_p^{orb}(g) \cong Hol_p^{orb,0}(g)$

Proof. To see that $Hol_p^{orb,0}(g)$ is a normal subgroup, it suffices to note that for any contractible (i.e. homotopic to the trivial loop $(1, c_0)$, where c_0 is a constant loop) $\mathcal{H}_{\mathcal{M}}$-loop, say (h_i, c_i), and any other other $\mathcal{H}_{\mathcal{M}}$-loop, say (g_i, α_i), we have that $(g_i, \alpha_i) \star (h_i, c_i) \star (g_i, \alpha_i)^{-1}$ is also contractible, where $\star$ denotes loop concatenation and $(g_i, \alpha_i)^{-1}$ denotes the inverse loop to (g_i, α_i) as defined in the proof of **Proposition 5.16**, proving the first part of the claim and moreover the quotient $Hol_p^{orb}(g)/Hol_p^{orb,0}(g)$ is well-defined. Now we define the following map:

$$\begin{aligned} \pi :& \pi_1^{orb}(\mathcal{M}, p) \to Hol_p^{orb}(g)/Hol_p^{orb,0}(g) \\ & [(h_i, c_i)] \mapsto (h_i, c_i) Hol_p^{orb,0}(g) \end{aligned}$$

Now this is clearly surjective, and to see that it is a group homomorphism, note that we have

$$\begin{aligned} \pi([(h_i, c_i)].[(h_i', c_i')]) &= \pi([(h_i, c_i) \star (h_i', c_i')]) \\ &= P_{(h_i,c_i)\star(h_i',c_i')} Hol_p^{orb,0}(g) \\ &= (P_{(h_i,c_i)} P_{(h_i',c_i')}) Hol_p^{orb,0}(g) \\ &= (P_{(h_i,c_i)} Hol_p^{orb,0}(g)).(P_{(h_i',c_i')} Hol_p^{orb,0}(g)) \\ &= \pi([(h_i, c_i)]).\pi([(h_i', c_i')]) \end{aligned}$$

Now the last part of the claim follows trivially from the above. □

We expect that this notion of holonomy to be an extension of the usual one for manifolds; moreover, we expect orbifold holonomy, as defined above, to also satisfy a classification similar to Berger's list. We record this in the following conjecture:

Conjecture 1. *Let (M, g) be a smooth Riemannian manifold, which we view as a Riemannian orbifold for which all local isotropy groups are trivial. Let $p \in M$, then*

$$Hol_p^{orb}(g) \cong Hol_p(g) \tag{5.81}$$

where the right hand side above refers to the usual holonomy group of a Riemannian metric. We expect $Hol_p^{orb}(g)$ to satisfy a Berger-type classification; moreover, if $Hol(\mathcal{M})$ denotes the holonomy as defined before in this thesis (as the holonomy of the smooth part), we expect it to be a subgroup of $Hol_p^{orb}(g)$.

We also constructed examples by exploiting weighted projective spaces. There are quite a few avenues to explore in this direction. The first one is to expand on the found matching criterion so that further families of weighted projective spaces can be used in the construction. In fact, we conjecture the following:

Conjecture 2. *All of the* 95 *families of weighted projective spaces admitting a* $K3$ *hypersurface, found by Reid in [54], can be used to construct orbifold Twisted Connected Sums.*

A further class of examples that we can get from weighted projective spaces are those where the orbifold $K3$ is given by the intersection of two hypersurfaces in $\mathbb{CP}^4(a_0, \ldots, a_4)$. Most of these examples are also not suited for the use of **Theorem 2.22** and have to be treated on a case-by-case basis, or require sophisticated analysis of lattice embeddings to obtain the required matching. We expect a reasonable supply of such examples; in particular, if it were possible to refine **Theorem 2.22** to allow the case $sing(\mathcal{X}_j) \leq 9$ and $rkL_j \leq 5$, then we would expect to see at least 20 more families of examples.

Finally we investigated compact connected associative 3-folds in the product G_2-manifold, $X \times T^3$, with X a hyper-Kähler $K3$ surface. We found that such 3-folds satisfying an additional hypothesis, namely that the derivative of the restriction of the torus projection, $\pi_T : X \times T^3 \to T^3$, has constant rank, fall into exactly two classes; those which are given by a product of a complex curve, $\Sigma \subset X$, and an appropriately chosen embedded circle, $\gamma \subset T^3$, and those which are given by $\{x_0\} \times T^3$ for $x_0 \in X$.

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