Complexity and Undescribability*

Nabil I. Al-Najjar
(Northwestern University)

Luca Anderlini
(Georgetown University)

Leonardo Felli
(London School of Economics)

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1. Motivation

In the well known case of Jacobellis v. Ohio,[1] Supreme Court Justice Potter Stewart argued that only "hard-core" pornography could be banned, but conceded:

> *"I shall not today attempt to further define the kind of materials I understand to be embraced within the shorthand definition; and perhaps I could never succeed in doing so," Stewart had said. "But I know it when I see it."* (Woodward and Armstrong, 1979, p. 94)

The describability problem faced by Justice Potter Stewart exemplifies well the type of circumstances we focus on. These are events, objects or activities that are well known and understood to all the parties involved but that are impossible to describe in an exhaustive manner.

An other example is the one of "academic tenure." Academic institutions routinely decide whether to grant tenure to junior faculty members. An ex-ante contingent tenure rule would spell out in advance a detailed set of conditions under which tenure would be granted as a function of a candidate's performance. Formulating such rule would entail considerable gains, such as reducing uncertainty, cutting down on the effort and resources spent in committee work, and reducing the potential for allegations of inequity, bias, etc. Despite this, to our knowledge no research-oriented department in the United States has set forth such rule. Instead, decisions are usually made using a lengthy case-by-case process that often suffers from the drawbacks mentioned above. The issue is that the underlying event, "the candidate has a tenurable vita," is well understood by most of us but inherently hard to describe ex-ante in its full details.

Finally, closer to home, when we first heard and learned of Grossman and Hart's key insight on the effects of contractual incompleteness we all anticipated the revolutionary effect of such an insight on our understanding of ownership rights and the inner working of some

*The results we prove in this chapter were presented in the panel discussion on the "Foundation of Incomplete Contracts" of the conference "Grossman and Hart at 25." We are grateful to the panel chair, Sönje Reiche, for very helpful comments.

[1] *Jacobellis v. Ohio*, 378 U. S. 184 (1964).

key economic institutions. However, had we been asked 25 years ago to describe in detail such an impact, the task would clearly have been impossibe.

It is exactly this type of events or actions that are well understood and yet impossible to describe that are the core of contractual incompleteness.

> *"A basic assumption of the model is that the production decisions are sufficiently complex that they cannot be specified completely in an initial contract between the firms. We have in mind a situation in which it is prohibitively difficult to think about and describe unambiguously in advance how all the potentially relevant aspects of the production allocation should be chosen as a function of the many states of the world.* (Grossman and Hart, 1986, p. 696)

Of course, the description above could refer to an event that is simply unforeseen, a surprise to all parties involved. However, as Oliver Hart clarifies, these fully unforeseen events cannot be the reason behind the contractual incompleteness assumed in Grossman and Hart (1986).

> *"[...] it is essential that, even though the agents are not capable of writing a contract that avoids hold-up problems, they are clever enough to understand (at least roughly) the consequences of their inability to do this."* (Hart, 1990, p. 699)

Here, we identify the necessary conditions for a model that admits complex events that are understood by the parties to the economic transaction, in the sense that they are able to assess the probability associated with these events and hence compute their expected utility, and yet cannot be described at an ex-ante stage in a non-trivial sense of the word. Indeed, consider the tenure example above, it is easy to write simple, clear-cut rules like "grant tenure if and only if the candidate publishes x or more papers in journal y." The problem is that such rule is too coarse to capture the subtle ways in which membership in the event "the candidate has a tenurable vita" is determined as a function of observable characteristics of a candidate's record.

In the model proposed below *all* rules are feasible, provided only they are *finite* in a well defined sense. Moreover, an event is complex if any feasible rule envisioned by the parties will leave a "positive measure" of exceptions. We label these events "undescribable".[2]

In Al-Najjar, Anderlini, and Felli (2006) we construct a model that allows for undescribable events and show in a simple co-insurance setting that risk-averse contracting parties facing such an event would optimally choose to write no-contract at all and face the uncertainty of the environment. Here, instead, we identify the necessary conditions that every model of undescribable events needs to satisfy. Needless to say the example in Al-Najjar, Anderlini, and Felli (2006) satisfies such conditions.

[2]The more commonly used english term would be "indescribable". However, we decided here to use the less common term "undescribable" to highlight the fact that the event we are referring to cannot be described by the parties involved *but* can be exactly foreseen by them. The only rationale we can provide for such a choice is that this is what you get when three non-native english speakers select the more suitable english term.

These necessary conditions correspond to two critical non-standard features of the type of models we propose. The first is the use of a probability distribution over states of nature that is finitely additive, but *fails* countable additivity.[3] The second is a state space that is a "small" subset of the set of all possible potential states (the set of all possible infinite strings of 0s and 1s).

In other words, we here show that the two key features of the model in Al-Najjar, Anderlini, and Felli (2006) are what it takes to get a formal notion of undescribable events. There is a sense in which a rejection of the non-standard ingredients that we highlight here is equivalent to saying that the formal notion of a event that is undescribable because it is "too complex" rather than because the parties do not have a sufficiently "rich language" is unattainable.

2. Ingredients

Our aim is to identify the properties that a model of undescribable events needs to satisfy. We start from a state space $\mathcal{S}$ with typical element s. The state space $\mathcal{S}$ needs to allow for a non-trivial event denoted $\mathcal{Z} \subset \mathcal{S}$ that is undescribable. In other words, all parties involved are able to foresee the event $\mathcal{Z}$ but are not able to describe such an event.

In order to define formally what can or cannot be described we start from the language available to the parties. Since we are aiming at an event that is *too complex* to describe rather than at a language that is too simple to handle the event we are interested in, we allow for a *rich enough language*. In other words, we only restrict parties to use finite descriptions. Formally, the language is then an algebra $\mathcal{A}$ of subsets of $\mathcal{S}$.

Let μ be a finitely additive probability measure defined on $(\mathcal{S}, \mathcal{A})$ and $\sigma(\mathcal{A})$ the sigma algebra generated by $\mathcal{A}$. Denote μ^* any extension of μ to the sigma-algebra $\sigma(\mathcal{A})$.

We now formally specify what we mean when we label the event $\mathcal{Z}$ as undescribable. In the first place all parties are able to foresee the event in question. We take this to mean that the parties can assess the probability of $\mathcal{Z}$. In other words, we restrict ourselves to events $\mathcal{Z} \in \sigma(\mathcal{A})$ for which $\mu^*(\mathcal{Z})$ is well defined.

Moreover, the parties cannot describe the event $\mathcal{Z}$ given the language $\mathcal{A}$. We take an event $\mathcal{Z} \in \sigma(\mathcal{A})$ to be describable if either $\mathcal{Z} \in \mathcal{A}$ or $\mathcal{Z}$ can be approximated using the language $\mathcal{A}$. We obviously first need to specify what we mean by approximating an event in a probability space.

Definition 1. *Approximation: An event $\mathcal{Z} \in \sigma(\mathcal{A})$ can be approximated using the language $\mathcal{A}$ if for every real number $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ such that $\mu^*(\mathcal{Z} \triangle A) < \varepsilon$.*[4]

3. Results

Our main challenge here is that the most obvious model that comes to mind when attempting to formalize undescribable events is such that a general approximation result holds (Anderlini and Felli, 1994). Let first define what we mean by the term "approximation result."

[3] See also (Al-Najjar, 2009).

[4] Throughout the rest of the paper we use the standard notation $C \triangle D$ to indicate the symmetric difference between the two sets C and D. In other words we define $C \triangle D = [C - (C \cap D)] \cup [D - (D \cap C)]$.

Definition 2. *General Approximation Result: Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also μ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive). Let μ^* be any extension of μ to the sigma algebra $\sigma(\mathcal{A})$.*

We say that a general approximation result holds for the model $(\mathcal{S}, \mathcal{A}, \mu)$ if and only if every $\mathcal{Z} \in \sigma(\mathcal{A})$ can be approximated as in Definition 1 above.

Clearly, if the approximation result holds for a space $(\mathcal{S}, \mathcal{A}, \mu)$, then *no set* $\mathcal{Z} \in \sigma(\mathcal{A})$ will be undescribable. In other words, our aim here is to identify the properties of a model $(\mathcal{S}, \mu, \mathcal{A})$ for which the approximation result fails.

We first show that any model $(\mathcal{S}, \mu, \mathcal{A})$ that delivers a set $\mathcal{Z} \in \sigma(\mathcal{A})$ that cannot be approximated in the sense of Definition 1 above must involve a measure μ that fails to be countably additive.

Proposition 1. *Finitely Additive Measures: Let a space $(\mathcal{S}, \mathcal{A}, \mu)$ as in Definition 2 be given, and assume that μ is countably additive on $\mathcal{A}$.*

Then the approximation result holds for the space $(\mathcal{S}, \mathcal{A}, \mu)$.

The intuition behind Proposition 1 is not hard to outline. Roughly speaking, since μ is countably additive on the algebra $\mathcal{A}$ it has, by Carathéodory's Extension Theorem,[5] a *unique* countably additive extension μ^* to the sigma algebra $\sigma(\mathcal{A})$. Consider now a sequence of sets $\{A_n\}$ in $\mathcal{A}$ such that the symmetric difference $A_n \Delta \mathcal{Z} \downarrow \emptyset$. Then, by countable additivity $\mu^*(A_n \Delta \mathcal{Z})$ converges to 0 and hence the approximation result holds.

We next focus on an event $\mathcal{Z}$ for which the approximation result fails in a *strong sense*. We take this to mean that $\mathcal{Z}$ cannot be approximated at all ($\mathcal{Z}$ is *independent* of all $A \in \mathcal{A}$), and the approximation result fails *uniformly* over the entire state space $\mathcal{S}$. These two features of our model imply that not only μ must fail countable additivity, but it must fail to be countably additive in the *strongest* possible way.

Definition 3. *Strong Undescribabilty: We say that $\mathcal{Z} \subseteq \mathcal{S}$ cannot be approximated in a strong sense if and only if for every $A \in \mathcal{A}$ with $\mu(A) > 0$,*

$$0 \; < \; \mu(\mathcal{Z} | \, A) \; = \; \mu(\mathcal{Z}) \; < \; 1 \tag{1}$$

So, the density of $\mathcal{Z}$ is the same conditional on all finitely definable sets that have positive measure under μ and $\mathcal{Z}$ is a not trivial subset of $\mathcal{S}$: $\mathcal{Z} \neq \mathcal{S}$, and $\mathcal{Z} \neq \emptyset$.

In other words, $\mathcal{Z}$ cannot be approximated in a strong sense if, knowing that s belongs to any finitely definable subset of $\mathcal{S}$ does not help us to "predict" better whether it belongs to $\mathcal{Z}$ or to its complement.

We can now move to a complete characterization of the properties of a model $(\mathcal{S}, \mu, \mathcal{A})$ that allows for an event $\mathcal{Z} \in \sigma(\mathcal{A})$ that cannot be approximated in the strong sense of Definition 3 above. The following is a standard result that will enable us to derive this characterization.[6]

[5] See, for instance, Royden (1988, Ch. 12.2).

[6] Many of the results we quote and use in our arguments below are well known in the mathematical literature. A measure that fails countable additivity is known as a "charge." The most comprehensive reference of which we are aware in this field is Bhaskara Rao and Bhaskara Rao (1983).

Remark 1. <u>*Decomposition Theorem:*</u> *Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also μ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive).*

Then μ can be written in the form $\mu = \mu^{CA} + \mu^{FA}$, where μ^{CA} is a countably additive measure, and μ^{FA} is purely finitely additive in the sense that there does not exist a non-zero countably additive measure ν on $(\mathcal{S}, \mathcal{A})$ such that $\nu \leq \mu^{FA}$.

Moreover, the decomposition of μ into $\mu^{CA} + \mu^{FA}$ is unique.[7]

We are now ready to prove that the strong failure of the approximation result as in Definition 3, corresponds to a μ that is purely finitely additive in the sense of Remark 1.[8]

Proposition 2. <u>*Pure Finite Additivity:*</u> *Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also μ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$.*

Assume now that there exists a set $\mathcal{Z} \in \sigma(\mathcal{A})$ that cannot be approximated in the strong sense of Definition 3. Then the unique decomposition of μ into $\mu^{FA} + \mu^{CA}$ (as in Remark 1) is such that $\mu^{FA} = \mu$, and μ^{CA} is identically equal to zero.

Intuitively, if the countably additive component of μ is not identically equal to zero then, from Proposition 1, we can approximate, at least in part, any event in the sigma algebra $\sigma(\mathcal{A})$. This contradicts the presence of a set like $\mathcal{Z}$ that cannot be approximated in the strong sense of Definition 3 above.

Once we know that μ is purely finitely additive, it is easy to see that there cannot be a state s in $\mathcal{S}$ that has point mass. So, another necessary feature of a model that delivers events $\mathcal{Z}$ that are undescribable in a strong way is that the measure μ is "diffuse" in a well defined sense.[9]

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[7]The proof of this claim can be found for instance in Bhaskara Rao and Bhaskara Rao (1983, Theorem 10.2.1). Notice that the standard name for a purely finitely additive measure like μ^{FA} is that of a "pure charge."

[8]A probability measure that is finitely additive and fails countable additivity, in the terminology of Bhaskara Rao and Bhaskara Rao (1983), is known as a *pure charge.* These measures do address the frustrations associated with countable additivity, as clearly and strongly stated by De Finetti:

"Suppose we are given a countable partition into events E_i, and let us put ourselves into the subjectivistic position. An individual wishes to evaluate the $p_i = \mathbf{P}(E_i)$; he is free to choose them as he pleases, except that, if he wants to be coherent, he must be careful not to inadvertently violate the conditions of coherence.

Someone tells him that in order to be coherent he can choose the p_i in any way he likes, so long as the sum = 1 (it is the same thing as in the finite case, anyway!).

The same thing?!!! You must be joking, the other will answer. In the finite case, this condition allowed me to choose the probabilities to be all equal, or slightly different, or very different; in short, I could express any opinion whatsoever. Here, on the other hand, the content *of my judgements enter in the picture: I am allowed to express them only if they are unbalanced to the extent illustrated [above]. Otherwise, even if I think they are equally probable [...] I am obliged to pick 'at random' a convergent series which, however I choose it, is in absolute contrast to what I think. If not, you call me* incoherent! *In leaving the finite domain, is it I who has ceased to understand anything, or is it you who has gone mad?"* (De Finetti, 1974, p.123)

[9]We refrain from using the term "non-atomic" here since a whole host of technical problems arise if one attempts to define this term in a general way for a measure μ that fails countable additivity. Bhaskara Rao and Bhaskara Rao (1983, Ch. 5) devote an entire chapter to the subject.

Proposition 3. *Diffuse Probabilities: Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also μ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$.*

Then if μ is purely finitely additive as in Proposition 2 there cannot be a state in $\mathcal{S}$ that has point mass in the following sense. There exists no $s \in \mathcal{S}$ and $\varepsilon > 0$ such that $s \in A$ implies $\mu(A) \geq \varepsilon$ for every $A \in \mathcal{A}$.[10]

Let $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, the set of all infinite strings of 0s and 1s, be the ambient set of $\mathcal{S}$, $\mathcal{S} \subset \mathcal{C}$. Of course $\mathcal{C}$ has the cardinality of the continuum. We show next that the state space $\mathcal{S}$ of a model $(\mathcal{S}, \mu, \mathcal{A})$ that delivers complex undescribable events is "small" relative to $\mathcal{C}$. In other words, the fact that $\mathcal{S}$ must be a "small" subset of $\mathcal{C}$, is also a consequence of the fact that the model admits a set $\mathcal{Z}$ that is undescribable in a strong sense.

Clearly if $\mathcal{S} \subset \mathcal{C}$ then each state of nature s_n is an infinite sequence $\{s_n^1, \ldots, s_n^i, \ldots\}$ of 0's and 1's where $s_n^i \in \{0,1\}$ indicate the value of the i-th digit of s_n. Define also

$$A(i,j) \quad = \quad \{s_n \in \mathcal{S} \quad \text{such that} \quad s_n^i \, = j\} \tag{2}$$

so that $A(i,j)$ is the set of states that have the i-th digit equal to $j \in \{0,1\}$. These are the elementary statements of the underlying language $\mathcal{A}$. We are now ready to define the finitely definable subsets of $\mathcal{S}$. These are the sets that can be described in the available language.

Definition 4. *Finitely Definable Sets: Consider the* algebra *of subsets of $\mathcal{S}$ generated by the collection of sets of the type $A(i,j)$ defined in (2). Let this algebra be denoted by $\mathcal{A}$. We refer to any $A \in \mathcal{A}$ as a finitely definable set.*

Elements of $\mathcal{A}$ can be obtained by complements and/or finite intersections and/or finite unions of the sets $A(i,j)$. Hence every element of $\mathcal{A}$ can be defined by finitely many elementary statements about the digits of the states of nature that it contains.

Let λ denote the "uniform" distribution on $\mathcal{C}$. By this we mean the (unique, countably additive) probability distribution on $\mathcal{C}$ obtained as the product distribution on each of the digits and under which $\lambda(A(i,0)) = \lambda(A(i,1)) = 1/2$ for every feature i. Note that this may be viewed as the translation of the Lebesgue measure on $\mathcal{C}$.[11]

Proposition 4. *Zero Lebesgue Measure: Let $\mathcal{S}$ be any subset of $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, and let $\mathcal{A}$ be the algebra of finitely definable sets of Definition 4.*

[10] Obviously, if $\{s\} \in \mathcal{A}$, then Proposition 3 tells us that it cannot be that $\mu(s) > 0$.

[11] To see this, embed the interval $[0,1]$ in the real line as a subset of $\mathcal{C}$, denoted $\mathcal{C}_1$, by identifying each point in $[0,1]$ with its binary expansion. This assignment is unique except for a countable number of points in $[0,1]$ that have two possible binary expansions. For these points, we choose a unique point in $\mathcal{C}$. Then the restriction of λ to $\mathcal{C}$ coincides with the Lebesgue measure on $[0,1]$.

The measure λ is defined formally in Definition A.1 in the Appendix. Remarks A.4 and A.5 formalize the relationship between $\mathcal{C}$ and the interval $[0,1]$ that we have just sketched out.

Suppose that μ is such that the space $(\mathcal{S}, \mathcal{A}, \mu)$ admits a set $\mathcal{Z} \in \sigma(\mathcal{A})$ that is undescribale in the strong sense of Definition 3 above. Assume also that $\mu(A) > 0$ for every $A \in \mathcal{A}$.[12] *Then $\lambda(\mathcal{S}) = 0$.*

Broadly speaking, Proposition 4 is a consequence of the fact that μ must be purely finitely additive, which in turn of course is a consequence of strong undescribability.

Intuitively, it is easiest to think of Proposition 4 in the following way. Suppose that we equip the set $\mathcal{C}$ with the algebra of finitely definable sets $\mathcal{A}$ and we place a *finitely* additive measure, say ν, on this pair. Then, by a theorem of Kolmogorov we know that ν must necessarily be *countably additive* as well.[13] It is then clear that we could not have our state space $\mathcal{S}$ equal to $\mathcal{C}$, since to deliver strong undescribablity we need a measure that is purely finitely additive, as Proposition 2 above shows.

Could it then be that $\mathcal{S}$ contains at least a subset of $\mathcal{C}$ which, conditional on say the first m "features" (digits of the binary expansion of an elementary state) being equal to a given sequence of 0s and 1s, contains all elements of $\mathcal{C}$ (a whole "cylinder")? The answer to this question is no. Roughly speaking, we could then apply the same theorem to this subset of $\mathcal{C}$ to obtain at least a "portion" of ν that is countably additive. But this is impossible if the measure is to be purely finitely additive, as Proposition 2 asserts that it must be if we are to obtain strong undescribability. It follows that the state space of a model of undescribable events must have a state space that is a "small" subset of $\mathcal{C}$ as in Proposition 4.

4. Conclusions

In Al-Najjar, Anderlini, and Felli (2006) we showed that it is possible to construct a contracting environment in which some events have the following properties. Their probabilities and consequences are understood by all concerned, and all agents involved use this information to compute expected utilities arising from any possible finite ex-ante contract. Yet these events are undescribable in the sense that any attempt to describe them in a finite ex-ante agreement must fail. The contracting parties cannot describe these events to any degree that will improve their expected utilities relative to an agreement that ignores them altogether. This is so notwithstanding the fact that the contracting parties' language can in fact distinguish between any two states.

We have shown here that two key properties of the model presented in Al-Najjar, Anderlini, and Felli (2006) are *necessary* to capture the intuitive notion of an undescribable event. In particular, we proved that the probability measure that capture the parties' ability to predict the undescribable event need to be finitely additive and fail countable additivity. Moreover, the state space of any model that exhibits a strong failure of the approximation result needs to be small in a well defined sense.

[12]Note that we make the assumption that $\mu(A) > 0$ for every $A \in \mathcal{A}$ purely for the sake of simplicity. Without it we would need to take care separately of any possible "superfluous" portion of $\mathcal{C}$. By this we mean that, for instance, μ could assign a mass of zero to the set of all states in $\mathcal{S}$ that have, say, feature 1 equal to 1. In this case it is possible that this entire cylinder in $\mathcal{C}$ is included in $\mathcal{S}$. Since this part of μ is identically equal to 0, it would be purely finitely additive in the sense of Remark 1 since *both* its countably additive component and its purely finitely additive components are identically equal to 0.

[13]See for instance Billingsley (1995, Theorem 2.3) or Doob (1994, Theorem V.6).

In our view, it is this subtle but intuitive notion of undescribable event that is at the foundation of the seminal insight of Grossman and Hart (1986) on the theory of ownership and institutions.

Appendix

Proof of Proposition 1: Since μ is countably additive on $\mathcal{A}$ by Catahéodory's Extension Theorem there exists a unique extension μ^* of μ to $\sigma(\mathcal{A})$. Since $\mathcal{Z} \in \sigma(\mathcal{A})$, we must then have that $\mu^*(\mathcal{Z})$ is equal to the outer measure of $\mathcal{Z}$ induced by μ. In other words it must be that

$$\mu^*(\mathcal{Z}) \ = \ \inf \sum_n \mu(O_n) \ , \tag{A.1}$$

where the infimum extends over all finite and infinite sequences $\{O_n\}$ that satisfy

$$O_n \in \mathcal{A} \ \ \forall n \quad \text{and} \quad \mathcal{Z} \subseteq \bigcup_n O_n \ . \tag{A.2}$$

Hence, for any real number $\xi > 0$ there exists a sequence $\{O_n\}$ satisfying (A.2) and

$$\sum_n \mu(O_n) \ - \ \mu^*(\mathcal{Z}) \ < \ \xi \ . \tag{A.3}$$

Since the first term in (A.3) is a convergent series, for any real number $\eta > 0$ there exists a finite m such that

$$\sum_n \mu(O_n) \ - \ \sum_{n=1}^{m} \mu(O_n) \ < \ \eta \ , \tag{A.4}$$

Notice next that (A.3) implies that

$$\mu^*(\bigcup_n O_n) \ - \ \mu^*(\mathcal{Z}) \ < \ \xi \ . \tag{A.5}$$

Since the sequence $\{O_n\}$ satisfies (A.2), the inequality in (A.5) implies

$$\mu^*(\mathcal{Z} \triangle \bigcup_n O_n) \ < \ \xi \ . \tag{A.6}$$

From (A.4) we deduce

$$\sum_{n>m} \mu(O_n) \ < \ \eta \ , \tag{A.7}$$

and hence

$$\mu^*(\bigcup_{n>m} O_n) \ < \ \eta \ . \tag{A.8}$$

From this it follows immediately that

$$\mu^*(\bigcup_n O_n \triangle \bigcup_{n=m}^{m} O_n) \ < \ \eta \ . \tag{A.9}$$

It is straightforward to verify that the operator $\mu^*(\cdot \triangle \cdot)$ is in fact a pseudo-metric on the sigma algebra of sets $\sigma(\mathcal{A})$. Hence it satisfies the triangular inequality. Therefore

$$\mu^*(\mathcal{Z} \triangle \bigcup_{n=1}^{m} O_n) \ \leq \ \mu^*(\mathcal{Z} \triangle \bigcup_n O_n) \ + \ \mu^*(\bigcup_n O_n \triangle \bigcup_{n=m}^{m} O_n) \ . \tag{A.10}$$

Using (A.6) and (A.9), (A.10) yields

$$\mu^*(\,\mathcal{Z} \triangle \bigcup_{n=1}^{m} O_n\,) \;\leq\; \xi + \eta\,. \tag{A.11}$$

Finally, since ξ and η are both arbitrary, and the finite union $\bigcup_{n=1}^{m} O_n$ is clearly an element of $\mathcal{A}$, (A.11) is obviously enough to prove the claim. ■

We will use the following result in the proof of Proposition 2 below. We state it here without proof for completeness. For the proof see Bhaskara Rao and Bhaskara Rao (1983, Theorem 10.3.1).

Remark A.1: *Fix an $\mathcal{S}$, and an algebra of subsets $\mathcal{A}$. Let also μ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive).*

Then μ is purely finitely additive if and only if for every countably additive measure ν on $(\mathcal{S}, \mathcal{A})$, every $A \in \mathcal{A}$, and every $\eta > 0$ there exists a set $M \in \mathcal{A}$ such that $M \subseteq A$

$$\nu(M) \;<\; \eta \quad \text{and} \quad \mu(A) \;-\; \mu(M) \;<\; \eta\,. \tag{A.12}$$

Remark A.2: *Let any set $\mathcal{S}$ be given, and $\mathcal{A}$ an algebra of subsets of $\mathcal{S}$. Let also μ be a finitely additive probability measure on $(\mathcal{S}, \mathcal{A})$ (not necessarily countably additive), and consider its (unique decomposition) into $\mu^{CA} + \mu^{FA}$ as in Remark 1.*

Then for every $\eta > 0$ there exists a set $B \in \mathcal{A}$ such that

$$\mu^{CA}(B) \;>\; \mu^{CA}(\mathcal{S}) - \eta \quad \text{and} \quad \mu^{FA}(B) \;<\; \eta\,. \tag{A.13}$$

Proof: The claim is a straightforward consequence of Remark A.1.

Since μ^{CA} is countably additive and μ^{FA} is purely finitely additive, in Remark A.1 we can set $\mu = \mu^{FA}$ and $\nu = \mu^{CA}$. Hence, setting $A = \mathcal{S}$, Remark A.1 now tells us that for every $\eta > 0$ there exists a set $M \in \mathcal{A}$ such that

$$\mu^{CA}(M) \;<\; \eta \quad \text{and} \quad \mu^{FA}(\mathcal{S}) \;-\; \mu^{FA}(M) \;<\; \eta\,. \tag{A.14}$$

Next, set $B = \overline{M}$. We then note that $\mu^{CA}(M) = \mu^{CA}(\mathcal{S}) - \mu^{CA}(B)$ and $\mu^{FA}(M) = \mu^{FA}(\mathcal{S}) - \mu^{FA}(B)$. Substituting these equalities in (A.14) now immediately yields that for every $\eta > 0$ there exists a set $B \in \mathcal{A}$ such that

$$\mu^{CA}(\mathcal{S}) - \mu^{CA}(B) < \eta \quad \text{and} \quad \mu^{FA}(\mathcal{S}) - \mu^{FA}(\mathcal{S}) + \mu^{FA}(B) \;<\; \eta\,. \tag{A.15}$$

Rearranging (A.15) then immediately gives the result. ■

Proof of Proposition 2: Use Remark 1 to write $\mu = \mu^{CA} + \mu^{FA}$. From Proposition 1 we know that $\mu^{CA}(\mathcal{S}) < 1$. Assume now that the Proposition is false. Then it must also be the case that $\mu^{CA}(\mathcal{S}) > 0$.

Using Remark A.2 we know that for every $\eta > 0$ there exists a set $B \in \mathcal{A}$ such that

$$\mu^{CA}(B) \;>\; \mu^{CA}(\mathcal{S}) - \eta \quad \text{and} \quad \mu^{FA}(B) \;<\; \eta\,. \tag{A.16}$$

Since by assumption $\mu(\mathcal{Z}) \in (0, 1)$, we can choose η in (A.16) to satisfy

$$\eta \;<\; \mu^{CA}(\mathcal{S})\, \frac{\mu(\mathcal{Z}) - \mu(\mathcal{Z})^2}{2 + \mu(\mathcal{Z}) + \mu(\mathcal{Z})^2}\,. \tag{A.17}$$

Notice next that since we know that $\mu^{CA}(\mathcal{S}) > 0$ (the contradiction hypothesis), and by assumption $\mu(\mathcal{Z}) \in (0, 1)$, the inequalities in (A.16) and (A.17) guarantee that $\mu(B) \geq \mu^{CA}(B) > 0$. Therefore, we can define the restrictions of μ and μ^{CA} to $B \in \mathcal{A}$ as $\mu_B = \mu/\mu(B)$ and $\mu_B^{CA} = \mu^{CA}/\mu^{CA}(B)$. Further, define μ_B^{FA} to be identically equal to 0 if $\mu^{FA}(B) = 0$, and $\mu_B^{FA} = \mu^{FA}/\mu^{FA}(B)$ if $\mu^{FA}(B) > 0$. Therefore,

$$\mu_B \;=\; \alpha\, \mu_B^{CA} \;+\; (1 \;-\; \alpha)\, \mu_B^{FA}\,, \tag{A.18}$$

where $\alpha = \mu^{CA}(B)/\mu(B)$. Notice that, since $\mu^{CA}(\mathcal{S}) \geq \mu^{CA}(B)$, we can use (A.16) and (A.17) to conclude that

$$\alpha \;=\; \frac{\mu^{CA}(B)}{\mu^{CA}(B) + \mu^{FA}(B)} \;>\; \frac{\mu^{CA}(\mathcal{S}) - \eta}{\mu^{CA}(\mathcal{S}) + \eta} \;>\; \frac{1 + \mu(\mathcal{Z})^2}{1 + \mu(\mathcal{Z})} \,. \tag{A.19}$$

Next, define $\mathcal{Z}_B = \mathcal{Z} \cap B$, and notice that since $\mathcal{Z}$ is strongly undescribable then so is $\mathcal{Z}_B$ with respect to the restriction μ_B. In other words whenever $A \in \mathcal{A}$ and $A \subseteq B$ we must have that $\mu_B(\mathcal{Z}_B|A) = \mu_B(\mathcal{Z}_B)$, with the latter, of course, also equal to $\mu(\mathcal{Z})$.

Clearly, μ_B^{CA} is countably additive. Applying Proposition 1, for every real number $\xi > 0$ there exists $Q_\xi \in \mathcal{A}$ such that

$$\mu_B^{CA}(\mathcal{Z}_B|\overline{Q}_\xi) \;<\; \xi \quad \text{and} \quad \left|\, \mu_B^{CA}(Q_\xi) \;-\; \mu_B^{CA}(\mathcal{Z}_B) \right| \;<\; \xi \,. \tag{A.20}$$

Therefore

$$\begin{aligned} \mu_B(\mathcal{Z}_B|\overline{Q}_\xi) \;&=\; \frac{\alpha\, \mu_B^{CA}(\mathcal{Z}_B|\overline{Q}_\xi)\, \mu_B^{CA}(\overline{Q}_\xi) + (1-\alpha)\, \mu_B^{FA}(\mathcal{Z}_B|\overline{Q}_\xi)\, \mu_B^{FA}(\overline{Q}_\xi)}{\alpha\, \mu_B^{CA}(\overline{Q}_\xi) + (1-\alpha)\, \mu_B^{CA}(\overline{Q}_\xi)} \\ &<\; \frac{\xi + (1-\alpha)\, \mu_B^{FA}(\mathcal{Z}_B|\overline{Q}_\xi)\, \mu_B^{FA}(\overline{Q}_\xi)}{\alpha\, \mu_B^{CA}(\overline{Q}_\xi) + (1-\alpha)\, \mu_B^{CA}(\overline{Q}_\xi)} \\ &<\; \frac{\xi + 1 - \alpha}{\alpha\, \mu_B^{CA}(\overline{Q}_\xi)} \,. \end{aligned} \tag{A.21}$$

In other words, using the fact that $\mathcal{Z}$ is strongly undescribable with respect to μ_B, we can now write

$$\alpha\, \mu_B(\mathcal{Z}_B)(1 \,-\, \mu_B^{CA}(Q_\xi)) \;<\; \xi \,+\, 1 \,-\, \alpha \,. \tag{A.22}$$

Since $\mu_B(\mathcal{Z}_B) \geq \alpha\, \mu_B^{CA}(\mathcal{Z}_B)$, we use (A.20) to re-write (A.22) as

$$\alpha \;<\; \frac{1 + \xi + \mu_B(\mathcal{Z}_B)^2}{1 + \mu_B(\mathcal{Z}_B)\,(1-\xi)} \,. \tag{A.23}$$

Since $\mu_B(\mathcal{Z}_B) = \mu(\mathcal{Z})$, for ξ sufficiently small (A.23) implies that

$$\alpha \;<\; \frac{1 + \mu(\mathcal{Z})^2}{1 + \mu(\mathcal{Z})} \,. \tag{A.24}$$

However, since (A.24) directly contradicts (A.19) this is clearly enough to prove our claim. ■

Proof of Proposition 3: Since μ is purely finitely additive, from Remark 1 we know that μ^{CA} is identically equal to 0. Hence from Theorem 10.2.2 of Bhaskara Rao and Bhaskara Rao (1983) we can conclude that

$$0 \;=\; \inf \left\{ \sum_n \mu(A_n) \right\} \,, \tag{A.25}$$

where the infimum extends over all (finite or infinite) sequences of disjoint sets $\{A_n\}$ such that $A_n \in \mathcal{A}$ for every n, and $\bigcup_n A_n = \mathcal{S}$.

Suppose, by way of contradiction, that the statement of the Proposition is false. Then there exists an $s \in \mathcal{S}$ such that $\mu(A) \geq \varepsilon$ whenever A contains s. Since for any sequence $\{A_n\}$ as above we must have that $s \in A_n$ for some n, this implies that the infimum in (A.25) is at least ε. This contradiction is enough to establish the result. ■

We will use the following result in the proof of Lemma A.1 below. We state it here without proof for completeness. For the proof see Billingsley (1995, Theorem 2.3) or Doob (1994, Theorem V.6).

Remark A.3: *Consider the set* $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, *and any subset* $\mathcal{S}$ *of* $\mathcal{C}$. *Assume that* $\mathcal{S}$ *is equipped with the algebra* $\mathcal{A}$ *of finitely definable sets, and equip* $\mathcal{C}$ *with the algebra* $\tilde{\mathcal{A}}$ *corresponding to the algebra of finitely definable sets as follows.*

For each $c \in \mathcal{C}$ *let* $\{c^i\}_{i\in\mathbb{N}}$ *be the sequence of digits in* $\{0,1\}$ *that define* c, *and for every* $i \in \mathbb{N}$ *and* $j \in \{0,1\}$ *let*

$$\tilde{A}(i,j) \;=\; \{c \in \mathcal{C} \quad \text{such that} \quad c^i \;=\; j\}\,, \tag{A.26}$$

and let $\tilde{\mathcal{A}}$ *be the algebra of subsets of* $\mathcal{C}$ *generated by the collection of sets of the type* $\tilde{A}(i,j)$. *Notice that in this way, using (2), we obviously have that for every* $\tilde{A} \in \tilde{\mathcal{A}}$ *it must be that* $\tilde{A} \cap \mathcal{S} = A \in \mathcal{A}$.

Let μ *be any finitely additive measure on* $(\mathcal{A},\mathcal{S})$ *(not necessarily countably additive). Then there exists a unique countably additive measure* $\tilde{\mu}$ *on* $(\sigma(\tilde{\mathcal{A}}),\mathcal{C})$ *that satisfies* $\tilde{\mu}(\tilde{A}) = \mu(A)$ *whenever* $\tilde{A} \cap \mathcal{S} = A$.[14]

Lemma A.1: *Let any* $\mathcal{S} \subset \mathcal{C}$ *be given, and consider a purely finitely additive measure* μ *on* $(\mathcal{S},\mathcal{A})$. *Let* $\tilde{\mu}$ *be the extension of* μ *to* $(\sigma(\tilde{\mathcal{A}}),\mathcal{C})$ *as in Remark A.3 above.*

Then, for every real number $\varepsilon > 0$ *there exists* $\tilde{A}_\varepsilon \in \sigma(\tilde{\mathcal{A}})$ *such that* $\mathcal{S} \subseteq \tilde{A}_\varepsilon$ *and* $\tilde{\mu}(\tilde{A}_\varepsilon) < \varepsilon$.

PROOF: Since μ is purely finitely additive, appealing again to Theorem 10.2.2 of Bhaskara Rao and Bhaskara Rao (1983) we can conclude that

$$0 \;=\; \inf\left\{\sum_n \mu(A_n)\right\}\,, \tag{A.27}$$

where the infimum extends over all (finite or infinite) sequences of disjoint sets $\{A_n\}$ such that $A_n \in \mathcal{A}$ for every n, and $\bigcup_n A_n = \mathcal{S}$. Hence, for every $\varepsilon > 0$ there exists a sequence of disjoint sets $\{A_{n,\varepsilon}\}$ such that $A_{n,\varepsilon} \in \mathcal{A}$ for every n, $\bigcup_n A_{n,\varepsilon} = \mathcal{S}$ and

$$\sum_n \mu(A_{n,\epsilon}) \;<\; \varepsilon\,. \tag{A.28}$$

Consider any sequence $\{A_{n,\varepsilon}\}$ as in (A.28) and the sequence $\{\tilde{A}_{n,\varepsilon}\}$ of subsets of $\mathcal{C}$ corresponding to it in the sense of Remark A.3, so that $\tilde{A}_{n,\varepsilon} \cap \mathcal{S} = A_{n,\varepsilon}$ for every n. Let $\tilde{A}_\varepsilon = \bigcup_n \tilde{A}_{n,\varepsilon}$. Clearly, $\tilde{A}_\varepsilon \in \sigma(\tilde{\mathcal{A}})$.

Notice next that $\bigcup_n A_{n,\varepsilon} = \mathcal{S} \cap \bigcup_n \tilde{A}_{n,\varepsilon}$. Hence $\mathcal{S} = \tilde{A}_\varepsilon \cap \mathcal{S}$, and therefore $\mathcal{S} \subseteq \tilde{A}_\varepsilon$. Since $\tilde{\mu}$ is countably additive we now have that $\tilde{\mu}(\tilde{A}_\varepsilon) = \sum_n \tilde{\mu}(\tilde{A}_{n,\varepsilon})$. Since by construction we must have that $\tilde{\mu}(\tilde{A}_{n,\varepsilon}) = \mu(A_{n,\varepsilon})$ for every n we also know that

$$\tilde{\mu}(\tilde{A}_\varepsilon) \;=\; \sum_n \tilde{\mu}(\tilde{A}_{n,\varepsilon}) \;=\; \sum_n \mu(A_{n,\varepsilon})\,. \tag{A.29}$$

Using (A.28), and (A.29) it is immediate that $\tilde{\mu}(\tilde{A}_\varepsilon) < \varepsilon$, as required. ■

Lemma A.2: *Let any* $\mathcal{S} \subset \mathcal{C}$ *be given, and consider a purely finitely additive measure* μ *on* $(\mathcal{S},\mathcal{A})$. *Let* $\tilde{\mu}$ *be the extension of* μ *to* $(\tilde{\mathcal{A}},\mathcal{C})$ *as in Remark A.3 above.*

Then, there exists $\tilde{\mathcal{S}} \in \sigma(\tilde{\mathcal{A}})$ *such that* $\mathcal{S} \subseteq \tilde{\mathcal{S}}$ *and* $\tilde{\mu}(\tilde{\mathcal{S}}) = 0$.

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[14] With a slight abuse of language we refer to $\tilde{\mu}$ as the *extension* of μ to $(\sigma(\tilde{\mathcal{A}}),\mathcal{C})$.

PROOF: From Lemma A.1 we know that, given any sequence $\varepsilon_m \to 0$ we can construct a corresponding sequence of sets $\{\tilde{A}_{\varepsilon_m}\}$ such that $\mathcal{S} \subseteq \tilde{A}_{\varepsilon_m}$, $\tilde{\mu}(\tilde{A}_{\varepsilon_m}) < \varepsilon_m$, and $\tilde{A}_{\varepsilon_m} \in \sigma(\tilde{\mathcal{A}})$ for every m. To prove the claim it is then sufficient to set $\tilde{\mathcal{S}} = \bigcap_m \tilde{A}_{\varepsilon_m}$ and to notice that it must be the case that $\tilde{\mathcal{S}} \in \sigma(\tilde{\mathcal{A}})$. ■

Remark A.4: *Each element c of $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ can be interpreted as the binary expansion of a real number r in the interval $[0,1]$ by taking the elements of the sequence c to be the digits of the binary expansion of r following a "0" and the "decimal" point.*

This map assign a unique real in $[0,1]$ to each element of $\mathcal{C}$ except for those that are of the form $\{c_1, \ldots, c_m, 1, 0, \ldots, 0, \ldots\}$ and $\{c_1, \ldots, c_m, 0, 1, \ldots, 1, \ldots\}$ which obviously correspond to the same real number r. Notice that there are countably many such pairs of elements of $\mathcal{C}$.

In what follows we will denote by $\mathcal{C}_0$ the set of elements of $\mathcal{C}$ that are of the form $\{c_1, \ldots, c_m, 1, 0, \ldots, 0, \ldots\}$, excluding $\{0, \ldots, 0, \ldots\}$, and by $\mathcal{C}_1$ the remainder of $\mathcal{C}$ so that $\mathcal{C}_1 = \mathcal{C} - \mathcal{C}_0$.

From what we have just stated, it is clear that we can assign a unique real in $[0,1]$ to each element of $\mathcal{C}_1$ and a unique element of $\mathcal{C}_1$ to every real in $[0,1]$.

Finally, notice that if we define the sigma algebra $\sigma(\tilde{\mathcal{A}}_1)$ of subsets of $\mathcal{C}_1$ as consisting of the collection of sets $\tilde{A} \cap \mathcal{C}_1$ for every $\tilde{A} \in \sigma(\tilde{\mathcal{A}})$ we obtain that $\sigma(\tilde{\mathcal{A}}_1)$ contains all the half-open intervals in $[0,1]$ of the form $(a,b]$ where a and b are reals in $[0,1]$.

Remark A.5: *Consider the sigma algebra $\sigma(\tilde{\mathcal{A}}_0)$ of subsets of $\mathcal{C}$ consisting of the collection of sets $\tilde{A} \cap \mathcal{C}_0$ for every $\tilde{A} \in \sigma(\tilde{\mathcal{A}})$. Consider also the sigma algebra $\sigma(\tilde{\mathcal{A}}_1)$ of Remark A.4.*

Then $\sigma(\tilde{\mathcal{A}}) = \sigma(\tilde{\mathcal{A}}_0) \cup \sigma(\tilde{\mathcal{A}}_1)$.

PROOF: Since $\mathcal{C}_0$ is a countable set it is enough to notice that every singleton set is already contained in $\sigma(\tilde{\mathcal{A}})$. Hence $\sigma(\tilde{\mathcal{A}}_0)$ consists of all subsets of $\mathcal{C}_0$. The assertion is then immediate from the definition of $\sigma(\tilde{\mathcal{A}}_0)$ and $\sigma(\tilde{\mathcal{A}}_1)$. The details are omitted. ■

Definition A.1: *Recall that from Remark A.5 we know that $\sigma(\tilde{\mathcal{A}}) = \sigma(\tilde{\mathcal{A}}_0) \cup \sigma(\tilde{\mathcal{A}}_1)$. The Lebesgue measure λ on $\mathcal{C}$ is then defined as follows.*

For every $\tilde{A}$ in $\sigma(\tilde{\mathcal{A}})$, set $\lambda(\tilde{A}) = 0$ if $\tilde{A} \in \sigma(\tilde{\mathcal{A}}_0)$, and $\lambda(\tilde{A}) = \mathcal{L}(\tilde{A})$ if $\tilde{A} \in \sigma(\tilde{\mathcal{A}}_1)$ where $\mathcal{L}$ is the Lebesgue measure on the real interval $[0,1]$ defined in the standard way.

Finally, as is standard, we take λ to be the completion *of the measure we have just defined in the sense that it is defined and is equal to zero on all subsets of all measurable sets that have zero measure.*[15]

Lemma A.3: *Let $\tilde{\mu}$ be the extension of μ to $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$ as in Remark A.3, and assume that μ is such that $\mu(A) > 0$ for every $A \in \mathcal{A}$.*

Then $\text{supp}(\tilde{\mu}) = \mathcal{C}$, *where* $\text{supp}(\cdot)$ *indicates the support of a given measure.*

PROOF: Suppose not. Then there is a non-empty open set O in $\mathcal{C}$ such that $\tilde{\mu}(O) = 0$. (We take O to be open in the product topology generated by the discrete topology on each coordinate of the elements of $\{0,1\}^{\mathbb{N}}$.)

We will show that for every open set O we can find an $\tilde{A} \in \tilde{\mathcal{A}}$ that is contained in O. Since $\tilde{\mu}(\tilde{A}) = \mu(\tilde{A} \cap \mathcal{S})$ and the latter is, by assumption, positive this yields a contradiction and hence is sufficient to prove the claim.

Assume by way of contradiction that we can find a non-empty open $O \subseteq \mathcal{C}$ such that $\tilde{A} \nsubseteq O$ for every $\tilde{A} \in \tilde{\mathcal{A}}$.

Fix $c \in O$ and consider the nested sequence of sets $\{\tilde{A}_n\}$ where for every n, $\tilde{A}_n \in \tilde{\mathcal{A}}$ is the set (the "cylinder") of all those $\hat{c}$s that have the first n digits equal to the first n digits of c.

By our contradiction hypothesis it must be that $\tilde{A}_n \nsubseteq O$ for every n. Hence, for every n we must be able to find a $\hat{c}_n \in \tilde{A}_n$ and $\hat{c}_n \notin O$.

Clearly, the sequence $\{\hat{c}_n\}$ converges to c. But since $\hat{c}_n \notin O$ for every n, and $c \in O$, this contradicts the fact that O is open. ■

[15] See for instance Billingsley (1995, p. 45).

Proof of Proposition 4: Let $\tilde{\mu}$ be the extension of μ to $(\sigma(\tilde{\mathcal{A}}), \mathcal{C})$ as in Remark A.3 and λ be the Lebesgue measure on $\mathcal{C}$ as in Definition A.1.

By Lemma A.2 we know that there exists a set $\tilde{\mathcal{S}} \in \sigma(\tilde{A})$ such that $\mathcal{S} \subseteq \tilde{\mathcal{S}}$ and $\tilde{\mu}(\tilde{\mathcal{S}}) = 0$, and By Lemma A.3 we know that $\text{supp}(\tilde{\mu}) = \mathcal{C}$.

Since λ is, by definition, *complete* in the sense that it assigns measure zero to all subsets of any set in $\sigma(\tilde{\mathcal{A}})$ that have λ-measure zero, it is enough to show that $\lambda(\tilde{\mathcal{S}}) = 0$.[16] We proceed by contradiction. Hence suppose that $\lambda(\tilde{\mathcal{S}}) > 0$.

By the "Lebesgue Decomposition Theorem,"[17] we know that $\tilde{\mu}$ can be (uniquely) written as $\tilde{\mu} = \tilde{\mu}^C + \tilde{\mu}^S$ where $\tilde{\mu}^C$ is absolutely continuous with respect to λ, and $\tilde{\mu}^S$ is singular with respect to λ.

Let $Q^S = \text{supp}(\tilde{\mu}^S)$ and $Q^C = \text{supp}(\tilde{\mu}^C)$. Since $\text{supp}(\tilde{\mu}) = \mathcal{C}$, we must have that $\mathcal{C} = Q^S \cup Q^C$. Hence $\tilde{\mathcal{S}} = [\tilde{\mathcal{S}} \cap Q^S] \cup [\tilde{\mathcal{S}} \cap Q^C]$.

Notice that, since $\tilde{\mu}^S$ is singular with respect to λ, we immediately know that $\lambda(\tilde{\mathcal{S}} \cap Q^S) = 0$. Hence, by our contradiction hypothesis it must be that $\lambda(\tilde{\mathcal{S}} \cap Q^C) > 0$.

Now let f be the Radon-Nikodym derivative of $\tilde{\mu}^C$ with respect to λ, which of course we know exists because $\tilde{\mu}^C$ is absolutely continuous with respect to λ. Notice that it must be the case that $f > 0$ except for a set of λ-measure zero on $\tilde{S} \cap Q^C$. Hence $\lambda(\tilde{S} \cap Q^C) > 0$ implies that

$$\tilde{\mu}^C(\tilde{\mathcal{S}} \cap Q^C) \;=\; \int_{\tilde{\mathcal{S}} \cap Q^C} f \, \mathrm{d}\,\lambda \;>\; 0\,. \tag{A.30}$$

However, since $\tilde{\mu} = \tilde{\mu}^C + \tilde{\mu}^S$ and $\tilde{\mu}(\tilde{\mathcal{S}}) = 0$, we must obviously have that $\tilde{\mu}^C(\tilde{\mathcal{S}} \cap Q^C) = 0$. This contradiction is sufficient to prove the claim. ■

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[16] See footnote 15 above.

[17] See for instance Royden (1988, Theorem 11.24).

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