# Instabilities in asymptotically AdS spacetimes


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# Summary


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In recent years, more and more efforts have been expended on the study of $n$-dimensional asymptotically anti-de Sitter spacetimes $(\mathcal{M}, g)$ as solutions to the Einstein vacuum equations

$$\mathrm{Ric}(g) = \frac{2}{n-2}\Lambda\, g \tag{EVE}$$

with negative cosmological constant $\Lambda$. This has been motivated mainly by the conjectured instability of these solutions. The author of this thesis joins these efforts with two contributions, which are themselves independent of each other.

In the first part, we are concerned with a superradiant instability for $n = 4$. For any cosmological constant $\Lambda = -3/\ell^2$ and any $\alpha < 9/4$, we find a Kerr-AdS spacetime $(\mathcal{M}, g_{\mathrm{KAdS}})$, in which the Klein-Gordon equation

$$\Box_g \psi + \frac{\alpha}{\ell^2}\psi = 0$$

has an exponentially growing mode solution satisfying a Dirichlet boundary condition at infinity. The spacetime violates the Hawking-Reall bound $r_+^2 > |a|\ell$. We obtain an analogous result for Neumann boundary conditions if $5/4 < \alpha < 9/4$. Moreover, in the Dirichlet case, one can prove that, for any Kerr-AdS spacetime violating the Hawking-Reall bound, there exists an open family of masses $\alpha$ such that the corresponding Klein-Gordon equation permits exponentially growing mode solutions. Our result provides the first rigorous construction of a superradiant instability for a negative cosmological constant.

In the second part, we study perturbations of five-dimensional Eguchi-Hanson-AdS spacetimes exhibiting biaxial Bianchi IX symmetry. Within this symmetry class, the system (EVE) is equivalent to a system of non-linear partial differential equations for the radius $r$ of the spheres, the Hawking mass $m$ and $B$, a quantity measuring the squashing of the spheres, which satisfies a non-linear wave equation. First we prove that the system is well-posed as an initial-boundary value problem around infinity $\mathcal{I}$ with $B$ satisfying a Dirichlet boundary condition. Second, we show that initial data in the biaxial Bianchi IX symmetry class around Eguchi-Hanson-AdS spacetimes cannot form horizons in the dynamical evolution.

Meinen Eltern gewidmet

# Declaration

This thesis is based on research conducted as a PhD student at the Department of Pure Mathematics and Mathematical Statistics in the Cambridge Centre for Analysis in the period between October 2014 and August 2017. None of the work is the outcome of a collaboration. Chapter 2 – without Section 2.5 – and the Appendices A and B are based on my publication:

> *Unstable mode solutions to the Klein-Gordon equation in Kerr-anti de Sitter spacetimes*, Communications in Mathematical Physics, March 2017, Volume 350, Issue 2, pp. 639-697

The contents of Section 2.5 and Chapter 3 are unpublished in any form at the time of submission. This dissertation has not been submitted for any other degree or qualification.

D. N. Dold
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# Acknowledgments

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I thank all my colleagues in mathematical relativity for helpful discussions throughout the years of my PhD work. I am grateful for all of my friends in Magdalene College and in the wider university for helping me to grow outside of the confinements of my discipline; among them, I am abundantly thankful for two in particular. First, I thank Sean Geddes, not only for all his tea and coffee that I drank during four years, but also for giving me food to eat when I got hungry during our long conversations that jumped from minutiae to poetry, or from plans for the future to the problem of divine foresight. Second, I thank Yanling Li for being who she is.

Finally, I would like to thank my parents, Bernhard and Ursula Dold, for their love from the first day of my life. This thesis is dedicated to them.

# Contents

# 1. Preface

Ἦτοι μὲν πρώτιστα Χάος γένετ[ο.]
In truth, at first Chaos came to be.

Hesiod, Theogony, v. 116

Every real object – as it presents itself to our consciousness – occupies a certain amount of space at a particular time. In other words, space and time are among the most basic concepts that can be derived from immediate sense perception and therefore need to be part of every natural philosophy. The most intuitive notions of space and time are beautifully woven into Hesiod's Theogony (around 700 BC).[1] His epic poem sings of the origin of the cosmos; natural forces appear as deities that ultimately give way to the Olympic gods. In this epos, at the beginning of time, Chaos, the gaping emptiness of space, came to be and thus marked out the room for the emergence of deities, gods and men. This space, Chaos, was needed, so everything could evolve. Time comes to be in two different guises: explicit and implicit. Personified time, Kronos, is a cruel god emasculating his father and devouring his children; it is the reality of time as a passing yet unceasing threat to our existence. Abstract time, however, did not need creation, nor any direct mentioning: it is always present as the poem begins (πρώτιστα) and moves forward, and as the inhabitants of space change. Time is the measure of change, as Aristotle would put it over four centuries later. Here, philosophically abstract space and abstract time are thought of as a continuum of copies of gaping emptiness, constituting the background for physical events.

In contrast, the theory of general relativity, formulated by Albert Einstein, has taught us that space and time should not be imagined as the canvas upon which physical events are painted, but rather as forming a dynamical continuum, a physical object called spacetime, interacting non-trivially with matter and its movements. The dynamics of spacetime is governed by the Einstein equations, a system of non-linear partial differential equations; a spacetime is an $(n+1)$-dimensional smooth manifold and is equipped with a Lorentzian metric[2], whose curvature is described by the Einstein equations. More precisely, the Einstein equations with cosmological constant $\Lambda$ are given by

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}(g)\, g + \Lambda\, g = 8\pi\, \mathbb{T}, \tag{1.0.1}$$

where $g$ is the metric of the spacetime, Ric is its Ricci curvature, $\mathrm{R}(g)$ its scalar curvature,

[1] Our presentation here is far from philological scholarship and merely abuses Hesiod's breathtaking imagery as a source for illustration.

[2] A semi-Riemannian metric of an $(n+1)$-dimensional manifold is Lorentzian if it has signature $(-1, 1, \dots, 1)$.

$\Lambda$ is the cosmological constant and $\mathbb{T}$ the energy-momentum tensor of the matter contained in the spacetime. Moreover, we have used geometrised units, where the speed of light and the gravitational constant both take a value of one. It is a remarkable property of the Einstein equations that they still exhibit non-trivial solutions in the absence of matter. In other words, curvature can curve itself.

Since its discovery, Einstein's theory has become a powerful predictive and theoretical tool in physics, from performing astrophysical computations to helping to formulate theories in high energy physics. However, to fully understand its implications, its limits, its internal structure and its relation to pre-scientific concepts more work and research by physicists, mathematicians and philosophers alike are still needed. The mathematician views (1.0.1) as an evolution problem: given an initial configuration of 'space', the Einstein equations yield its evolution over 'time'.[3] This point of view makes (1.0.1) accessible to the theory of partial differential equations, a mathematical subject that made significant progress between Einstein's days and ours. PDE theory establishes a robust framework in which conjectures and heuristics from the physics literature can be proven and open problems can be clarified. In particular, the mathematical study of Einstein's theory has recourse only to abstract mathematical tools and the Einstein equations; thus it focuses on understanding the internal structure of the classical theory, instead of prediction and modification, which is often the focus of physical research. However, reliable modification of existing theories presupposes a solid understanding of those theories, and here mathematical research can be an active part of the perennial effort to understand the structure of space and time.

A theoretical prediction from high energy physics, the AdS-CFT correspondence, has sparked recent interest in the physics community in asymptotically anti-de Sitter (asymptotically AdS) spacetimes, which are solutions to the Einstein equations for negative cosmological constant. This thesis is concerned with (1.0.1) for $\Lambda < 0$ for $\mathbb{T} = 0$, i. e. with the system of equations

$$\mathrm{Ric}(g) = \frac{2}{n-1}\Lambda\, g. \tag{1.0.2}$$

The Einstein vacuum equations allow for a well-posed initial-boundary value problem (IBVP) – see [Fri95]. In the language of the IBVP, stability of a spacetime means that if we perturb 'space' initially and the perturbation is sufficiently small, then 'at late times' the perturbed and unperturbed geometries are close in a suitable norm.[4] This thesis studies instability phenomena in two different types of asymptotically AdS spacetimes.

Since settling the question of stability for a given spacetime is very difficult, one has recourse to an intuition developed over the last two or three decades – see [CK94] and [DR13] – which links the problem of stability to the study of waves on a fixed black hole spacetime. Put differently, a prerequisite for treating the stability problem of the Einstein

[3] Here space and time are used more as a blurred short-hand than as having a specific content, and we shall refrain from trivialising what has been said before.

[4] Small perturbations always occur in nature. Hence a theoretical model without the stability property cannot produce observable predictions.

vacuum equations around a spacetime with metric $g$ is a robust understanding of solutions to (non-linear) wave equations

$$\Box_g \psi = F(\psi, \partial\psi), \tag{1.0.3}$$

where $\Box_g$ is the d'Alembertian associated with $g$. Much recent effort has been expended on understanding (1.0.3) for asymptotically AdS metrics $g$. In Chapter 2, we prove an instability result for solutions to the Klein-Gordon equations in the superradiant parameter range of the Kerr-AdS family, which is in sharp contrast to the known stability results in the complementary regime.[5]

In Chapter 3, we study the Einstein vacuum equations (1.0.2) in five dimensions directly, but restricted to biaxial Bianchi IX symmetry. Within this symmetry class, the system (1.0.2) is equivalent to a system of non-linear partial differential equations for the radius $r$ of the spheres, the Hawking mass $m$ and $B$, a quantity measuring the squashing of the spheres, which satisfies a non-linear wave equation. First, we prove that the system is well-posed as an initial-boundary value problem. Second, we show that initial data with negative mass cannot form horizons in the dynamical evolution.

The author of this thesis feels humbled that he has been given the opportunity to pursue research in mathematical general relativity during his time at the University of Cambridge, and that he could take however minor a place in the long line of thinkers questing after the nature of space and time.

[5] This part of the thesis has been published separately as [Dol17].

# 2. Unstable mode solutions to the Klein-Gordon equation in Kerr-AdS spacetimes

## 2.1. Introduction

### 2.1.1. The Klein-Gordon equation in asymptotically anti-de Sitter spacetimes

The Einstein vacuum equations

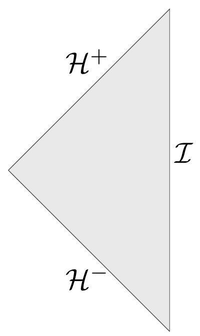
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.1.1}$$

with cosmological constant $\Lambda$ can be understood as a system of second-order partial differential equations for the metric tensor $g$ of a four-dimensional spacetime $(\mathcal{M}, g)$. Solutions with negative cosmological constant have drawn considerable attention in recent years, mainly due to the conjectured instability of these spacetimes. For more details, see [And06], [DH06a], [BR11], [DHS11b], [DHMS12], [HLSW15] and references therein.

In appropriate coordinates, (2.1.1) forms a system of non-linear wave equations. A first step in understanding the global dynamics of solutions to (2.1.1) – and thus eventually answering the question of stability – is the study of linear wave equations on a fixed background. For $\Lambda < 0$, efforts have focused on understanding the dynamics of the Klein-Gordon equation

$$\Box_g \psi + \frac{\alpha}{\ell^2} \psi = 0 \tag{2.1.2}$$

for an asymptotically AdS metric $g$ with cosmological constant $\Lambda = -3/\ell^2$ and a mass term $\alpha$ satisfying the Breitenlohner-Freedman bound $\alpha < 9/4$ [BF82], which is required for well-posedness of the equation – see [War12], [Hol11] and [Vas09]. The conformally coupled case $\alpha = 2$ encompasses scalar-type metric perturbations around an exact AdS spacetime [IW04].




The Penrose diagram of the exterior of the Kerr-AdS spacetime shows the timelike nature of null infinity.

For $g$ being the metric of an exact AdS spacetime, the massive wave equation (2.1.2) allows for time-periodic solutions due to the timelike nature of null and spacelike infinity $\mathcal{I}$; in particular, general solutions to (2.1.2), while remaining bounded, do not decay. The behaviour of solutions to (2.1.2) on black-hole spacetimes is very different. Given a Kerr-AdS spacetime with parameters $\ell$, $M$ and $a$ satisfying $|a| < \ell$, define the Hawking-Reall Killing vector field

$$K := T + \frac{a\Xi}{r_+^2 + a^2}\Phi,$$

where, using Boyer-Lindquist coordinates, $T = \partial_t$ and $\Phi = \partial_{\tilde{\varphi}}$; see Section 2.1.4 for definitions of $\Xi$ and $r_+$. The vector field $K$ is the (up to normalisation) unique Killing vector field that is null on the horizon $\mathcal{H}$ and non-spacelike in a neighbourhood of $\mathcal{H}$. It is globally timelike in the black hole exterior if the Hawking-Reall bound $r_+^2 > |a|\ell$ is satisfied. If the bound is violated, $K$ becomes non-timelike far away from the horizon.

In [HR99], Hawking and Reall use the existence of a globally causal $K$ for $r_+^2 > |a|\ell$ to argue towards the stability of these spacetimes. Indeed, uniform boundedness of solutions to (2.1.2) in the full regime $\alpha < 9/4$ was proved for $r_+^2 > |a|\ell$ in [Hol09] and [HW12]. Moreover, in [HS11a], it was shown that solutions with the fastest radial decay (Dirichlet conditions at infinity) in fact decay logarithmically in time[1] and [HS13] proves that this logarithmic bound is sharp.

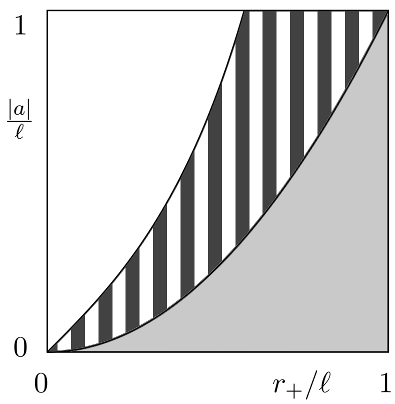



Figure 2.1.: For $r_+ < \ell$, the two shaded regions represent the set of admissible parameters for $|a|/\ell$ and $r_+/\ell$. Within the plain grey area (bottom right), the Hawking-Reall bound is satisfied, whereas it is violated in the striped (intermediate) domain.

For spacetimes violating the Hawking-Reall bound, the global behaviour of solutions to (2.1.2) has not been investigated rigorously, but it was argued in the physics literature – see [CD04], [CDLY04], [CDY06] and [DHS11b] – that at least for small black holes, i. e. for $|a| \ll \ell$ and $|a| \ll r_+$, instability of solutions to (2.1.2) is to be expected if $r_+^2 < |a|\ell$.[2] As, in this regime, there is no Killing vector field which is globally timelike in the black hole exterior, this parallels the situation of asymptotically flat Kerr spacetimes, where superradiance is present. For the present discussion, we will understand superradiance loosely as energy extraction from a rotating black hole. We will make this more precise in Lemma 2.1.1.

### 2.1.2. Unstable modes and superradiance in spacetimes with $\Lambda = 0$

The study of energy extraction from black holes in asymptotically flat spacetimes has a long history in the physics literature and two different, but related mechanisms have been proposed. On the one hand, Press and Teukolsky [PT72] suggested that the leakage of

[1] Slightly stronger restrictions on $\alpha$ and the spacetime parameters were imposed in [HS11a] for technical reasons, but the result is believed to hold in full generality by virtue of [HW12].

[2] Building on similar arguments, there are also investigations into unstable modes to the Teukolsky equation in the same parameter regimes; see [DS13] and [CDH$^+$14]. Periodic solutions to the Teukolsky equation are expected to produce resonator solutions; see [DHS11a] and [DSW15].

energy through the horizon of a rotating black hole could be used to create a black hole bomb by placing a mirror around it. Superradiance would increase the radiation pressure on the mirror over time until it finally breaks, setting free all the energy at once.[3] On the other hand, it was argued that energy could be extracted by the aid of massive waves acting as a natural mirror. This goes back to Zel'dovich [Zel71] and was explored further by Starobinsky in [Sta73]. Numerous heuristic and numerical studies on the superradiant behaviour of solutions to the Klein-Gordon equation followed, e. g. [DDR76], [ZE79], [Det80], [Dol07] and [Dol12]. These studies found exponentially growing solutions to the massive wave equation on Kerr spacetimes.

Remarkably, this instability is not present at the level of the massless wave equation

$$\Box_g \psi = 0,$$

see [DRSR14b], where boundedness and decay for such solutions is proved in the full subextremal range $|a| < M$. Even though energy can potentially leak out of the black hole, superradiance can be overcome here as the superradiant frequencies in Fourier spaces are not trapped. In particular, in the context of scattering [DRSR14a], a quantitative bound on the maximal superradiant amplification was shown.

In accordance with the above heuristic of massive waves acting as a natural mirror for a black hole bomb, this situation changes dramatically for the Klein-Gordon equation

$$\Box_g \psi - \mu^2 \psi = 0 \tag{2.1.3}$$

with scalar mass $\mu > 0$. A first rigorous construction of exponentially growing finite-energy solutions in Kerr spacetimes was given by Shlapentokh-Rothman [SR13]. The constructed solutions were modes. Mode solutions are solutions of the form

$$\psi(t, r, \vartheta, \tilde{\varphi}) = \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{e}^{\mathrm{i}m\tilde{\varphi}}\, S_{ml}(\cos\vartheta) R(r) \tag{2.1.4}$$

in Boyer-Lindquist coordinates $(t, r, \vartheta, \tilde{\varphi})$ for $\omega \in \mathbb{C}$, $m \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq |m|}$, where the smooth functions $S_{ml}$ and $R$ satisfy ordinary differential equations arising from the separability property of the wave equation in Boyer-Lindquist coordinates [Car69]. We call a mode unstable if it is exponentially growing in time, i. e. if $\operatorname{Im}\omega > 0$. Shlapentokh-Rothman showed that, for any given Kerr spacetime with $0 < |a| < M$, there is an open family of masses $\mu$ producing unstable modes with finite energy. The construction starts from proving existence of real modes and hence produces in particular periodic solutions. We will adopt this strategy.

[3] In the asymptotically AdS case, infinity could serve as such a mirror due to the timelike character of spacelike and null infinity.

### 2.1.3. Unstable modes and superradiance in Kerr-AdS spacetimes

Let us return to the Kerr-AdS case and connect the existence of unstable modes to superradiance. Recall that the energy-momentum tensor for the Klein-Gordon equation (2.1.2) is given by

$$\mathbb{T}_{\mu\nu} := \mathrm{Re}\,\left(\nabla_\mu\psi\overline{\nabla_\nu\psi}\right) - \frac{1}{2}g_{\mu\nu}\left(|\nabla\psi|^2 - \frac{\alpha}{\ell^2}|\psi|^2\right)$$

and that, for each vector field $X$, we obtain a current

$$J^X_\mu := \mathbb{T}_{\mu\nu}X^\nu.$$

While in Kerr spacetimes, $T = \partial_t$ (see Section 2.1.4) is the (up to normalisation) unique timelike Killing field at infinity, the family of vector fields $T + \lambda\Phi$ with $\Phi = \partial_{\tilde{\varphi}}$ is timelike near infinity in Kerr-AdS spacetimes if and only if

$$-\ell^{-2}\left(\ell + a\right) < \lambda < \ell^{-2}\left(\ell - a\right). \tag{2.1.5}$$

Hence, in this range of values for $\lambda$, the conserved current $J^{T+\lambda\Phi}_\mu$ encapsulates the energy density of the scalar field measured by different (rotating) observers at infinity. The vector field $T + \lambda\Phi$ becomes spacelike or null at the horizon.

Recall that the Hawking-Reall vector field $K$ is tangent to the null generators of the horizon $\mathcal{H}$. Therefore the energy density radiated through the horizon is measured by

$$\begin{aligned} J^{T+\lambda\Phi}_\mu K^\mu\Big|_{\mathcal{H}} &= \mathrm{Re}\,\left(\left(T\psi + \lambda\Phi\psi\right)\overline{K\psi}\right)\Big|_{\mathcal{H}} \\ &= \mathrm{Re}\,\left(\left(T\psi + \lambda\Phi\psi\right)\overline{\left(T\psi + \frac{a\Xi}{r_+^2 + a^2}\Phi\psi\right)}\right)\Bigg|_{\mathcal{H}} \end{aligned}$$

since $g(T + \lambda\Phi, K) = 0$ on the horizon. For mode solutions (2.1.4), this yields

$$J^{T+\lambda\Phi}_\mu K^\mu\Big|_{\mathcal{H}} = \left(|\omega|^2 - \mathrm{Re}\,(\omega)\frac{ma\Xi}{r_+^2 + a^2} + m\lambda\left(\frac{ma\Xi}{r_+^2 + a^2} - \mathrm{Re}\,(\omega)\right)\right)|\psi|^2\Big|_{\mathcal{H}}. \tag{2.1.6}$$

A non-trivial mode solution radiates energy away from the horizon if and only if the expression (2.1.6) is negative for all $\lambda$ in the range (2.1.5).

**Lemma 2.1.1.** *Let $r_+^2 < |a|\ell$. Let $\psi$ be a mode solution with $\omega(\varepsilon) = \omega_R(\varepsilon) + \mathrm{i}\varepsilon$ for sufficiently small $\varepsilon > 0$, $\omega_R(\varepsilon) \in \mathbb{R}$ and $\omega_R(0) = ma\Xi/(r_+^2 + a^2)$. If*

$$\omega_R(0)\frac{\partial\omega_R}{\partial\varepsilon}(0) < 0, \tag{2.1.7}$$

*then $J^{T+\lambda\Phi}_\mu K^\mu\Big|_{\mathcal{H}} < 0$ for sufficiently small $\varepsilon > 0$ and $\lambda$ in (2.1.5).*

*Proof.* Since $J^{T+\lambda\Phi}_\alpha K^\alpha = 0$ at the horizon for $\varepsilon = 0$, it suffices to differentiate (2.1.6) with respect to $\varepsilon$ and evaluate at $\varepsilon = 0$. We see that the derivative is negative if and only if

$$\omega_R(0)\frac{\partial \omega_R}{\partial \varepsilon}(0) - m\lambda \frac{\partial \omega_R}{\partial \varepsilon}(0) < 0.$$

This, however, can be easily checked to hold using (2.1.5) and $r_+^2 < |a|\ell$. □

*Remark* 2.1.2. If $r_+^2 > |a|\ell$, then $K$ induces an energy density at infinity and $J^K_\mu K^\mu \geq 0$, in accordance with the intuition of not being in the superradiant regime if the Hawking-Reall bound is satisfied.

We will show that our constructed growing mode solutions – as the modes of [SR13] – satisfy the assumptions of Lemma 2.1.1. This corroborates our interpretation that the unstable modes are a linear manifestation of the superradiant properties of Kerr-AdS spacetimes.

### 2.1.4. The Kerr-AdS family

Before stating our results, we introduce the Kerr-AdS family of spacetimes. For a more exhaustive presentation, we refer the reader to [HS11a]. Kerr-AdS spacetimes depend on three parameters $(\ell, M, a)$, where $\ell$ is related to the cosmological constant $\Lambda$ via $\Lambda = -3/\ell^2$. The parameter $M > 0$ relates to the mass of the black hole and $a$ to the agnular momentum. More precisely, the total energy $E$ and the angular momentum $J$ are

$$E = \frac{M}{\Xi^2} \quad \text{and} \quad J = \frac{aM}{\Xi^2}.$$

The parameter $a$ is assumed to satisfy $|a| < \ell$. This condition guarantees for the metric to be regular. Let

$$\Delta_-(r) := (r^2 + a^2)\left(1 + \frac{r^2}{\ell^2}\right) - 2Mr.$$

We require for the polynomial $\Delta_-$ to have two real roots, denoted by $r_- < r_+$, thus avoiding naked singularities. We can write

$$\Delta_-(r) = \ell^{-2}(r - r_+)(r^3 + r^2 r_+ + r(r_+^2 + a^2 + \ell^2) - a^2\ell^2 r_+^{-1}), \tag{2.1.8}$$

whence

$$\partial_r \Delta_-(r_+) = \frac{1}{\ell^2}(3r_+^3 + r_+ a^2 + r_+\ell^2 - a^2\ell^2 r_+^{-1}).$$

This expression imposes some restrictions on the range of $|a|$ in terms of $r_+$ as shown in the following

**Lemma 2.1.3.** *Let $(\ell, M, a)$ be spacetime parameters such that the corresponding Kerr-AdS* *spacetime does not contain a naked singularity. Then, if $r_+ < \ell$,*

$$a^2 < r_+^2 \frac{3\frac{r_+^2}{\ell^2} + 1}{1 - \frac{r_+^2}{\ell^2}}. \tag{2.1.9}$$

*If $r_+ \geq \ell$, $|a|$ can take any value in $[0, \ell)$.*

*Proof.* These statements follow from $\partial_r \Delta_-(r_+) > 0$, which is a necessary condition for $r_- < r_+$. Note also that $r_+ \geq \ell$ implies $r_+^2 \geq |a|\ell$. □

Therefore, under the restriction of Lemma 2.1.3, there is a bijection between Kerr-AdS spacetimes with parameters $(\ell, M, a)$ and spacetimes with parameters $(\ell, r_+, a)$. Henceforth we will use the shorthand notations $\mathcal{M}_{\mathrm{KAdS}}(\ell, M, a)$ and $\mathcal{M}_{\mathrm{KAdS}}(\ell, r_+, a)$ to denote Kerr-AdS spacetimes with parameters $(\ell, M, a)$ and $(\ell, r_+, a)$ respectively. The restriction of Lemma 2.1.3 can be seen in Figure 2.1.

Given $(\ell, M, a)$, a chart covering all of the domain of outer communication is given by Boyer-Lindquist coordinates $(t, r, \vartheta, \tilde{\varphi}) \in \mathbb{R} \times (r_+, \infty) \times S^2$. The metric in these coordinates is

$$g_{\mathrm{AdS}} = -\frac{\Delta_- - \Delta_\vartheta a^2 \sin^2\vartheta}{\Sigma}\,\mathrm{d}t^2 - 2\frac{\Delta_\vartheta(r^2+a^2) - \Delta_-}{\Xi\Sigma} a \sin^2\vartheta \,\mathrm{d}t\,\mathrm{d}\tilde{\varphi} + \frac{\Sigma}{\Delta_-}\,\mathrm{d}r^2 + \frac{\Sigma}{\Delta_\vartheta}\,\mathrm{d}\vartheta^2 + \frac{\Delta_\vartheta(r^2+a^2)^2 - \Delta_- a^2\sin^2\vartheta}{\Xi^2\Sigma}\sin^2\vartheta\,\mathrm{d}\tilde{\varphi}^2,$$

v where

$$\Sigma = r^2 + a^2\cos^2\vartheta, \qquad \Delta_\vartheta = 1 - \frac{a^2}{\ell^2}\cos^2\vartheta, \qquad \Xi = 1 - \frac{a^2}{\ell^2}.$$

Since Boyer-Lindquist coordinates break down at $r = r_+$, we introduce Kerr-AdS-star coordinates $(t^*, r, \vartheta, \varphi)$. These are related to Boyer-Lindquist coordinates by

$$t^* := t + A(r) \quad \text{and} \quad \varphi := \tilde{\varphi} + B(r),$$

where

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{2Mr}{\Delta_-(1 + r^2/\ell^2)} \quad \text{and} \quad \frac{\mathrm{d}B}{\mathrm{d}r} = \frac{a\Xi}{\Delta_-}.$$

In these coordinates, the metric extends smoothly through $r = r_+$. One sees that the boundary $r = r_+$ of the Boyer-Lindquist patch is null and we shall call it the event horizon $\mathcal{H}$.

Finally, we introduce the tortoise coordinate $r^*$ which is related to $r$ by

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{r^2 + a^2}{\Delta_-(r)}$$

with $r^*(+\infty) = \pi/2$. We will denote the derivative with respect to $r^*$ by $'$.

### 2.1.5. Statement of the results

The analysis in this paper yields two types of instability results:

A. Given a cosmological constant $\Lambda$ and a mass $\alpha$, there is a Kerr-AdS spacetime for this $\Lambda$ in which (2.1.2) has an exponentially growing solution.

B. Given a Kerr-AdS spacetime violating the Hawking-Reall bound, there is a range for the scalar mass such that, in this spacetime, (2.1.2) has an exponentially growing solution.

To make this more precise, recall that mode solutions are Fourier modes that take the form

$$\psi(t, r, \vartheta, \tilde{\varphi}) = \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{e}^{\mathrm{i}m\tilde{\varphi}}\, S_{ml}(\cos\vartheta) R(r)$$

in Boyer-Lindquist coordinates $(t, r, \vartheta, \tilde{\varphi})$ for $\omega \in \mathbb{C}$, $m \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq |m|}$. It should always be understood implictly that $S_{ml}$ additionally depends on $\omega$, and that $R$ depends on $\omega$ as well as $m$ and $l$. Define $u(r) := (r^2 + a^2)^{1/2} R(r)$. Use $\mathcal{S}_{\mathrm{mode}}(\alpha, \omega, m, l)$ to denote the set of all mode solutions with parameters $\omega, m, l$ to the Klein-Gordon equation with scalar mass $\alpha$. Set

$$\kappa^2 := 9/4 - \alpha. \tag{2.1.10}$$

We require that all mode solutions are smooth. For the $S_{ml}$ this is ensured automatically by the definition – see Section 2.2.1. Hence we only need to impose a regularity condition on the function $u$, given parameters $\ell$, $r_+$, $a$, $m$ and $\omega$, which will come from requiring regularity in ingoing Eddington-Finkelstein-type coordinates, that are regular on the future horizon.

**Definition 2.1.4** (Horizon regularity condition)**.** A smooth function $f : (r_+, \infty) \to \mathbb{C}$ satisfies the horizon regularity condition if $f(r) = (r - r_+)^{\xi} \varrho$ for a smooth function $\varrho$ as well as a constant

$$\xi := \mathrm{i} \frac{\Xi a m - (r_+^2 + a^2)\omega}{\partial_r \Delta_-(r_+)}. \tag{2.1.11}$$

*Remark* 2.1.5. The horizon is a trapped surface. Therefore the boundary condition at the horizon has to be chosen such that the group velocity of the wave is ingoing. The horizon regularity condition guarantees an infalling observer of finite energy; see [Sta73].

Henceforth we will only call a mode $\psi$ a mode solution to (2.1.2) if its radial part $R$ (and hence $u$) satisfies the horizon regularity condition. At infinity, we will study two different boundary conditions for $u$.

**Definition 2.1.6** (Dirichlet boundary condition)**.** Given a mass $\alpha < 9/4$ (i. e. a $\kappa > 0$), a smooth function $f : (r_+, \infty) \to \mathbb{C}$ satisfies the Dirichlet boundary condition if

$$r^{1/2-\kappa} f \to 0$$

as $r \to \infty$.

We say that a $\psi \in \mathcal{S}_{\text{mode}}(\alpha, \omega, m, l)$ satisfies Dirichlet boundary condition if its radial part $u$ satisfies the Dirichlet boundary condition.

Mode solutions satisfying these boundary conditions are analogous to the modes considered in [SR13].

We are able to show the following result.

**Theorem 2.1.7.** *Given a cosmological constant $\Lambda = -3/\ell^2$, a black hole radius $0 < r_+ < \ell$ and a scalar mass parameter $\alpha_0 \in (-\infty, 9/4)$, there are a spacetime parameter $a$ satisfying the regularity condition $|a| < \ell$, mode parameters $m$ and $l$ and a $\delta > 0$ such that there are a smooth curve*

$$(-\delta, \delta) \to \mathbb{R}^2,\ \varepsilon \mapsto (\alpha(\varepsilon), \omega_R(\varepsilon))$$

*with*

$$\alpha(0) = \alpha_0 \quad \text{and} \quad \omega_R(0) = \frac{\Xi a m}{r_+^2 + a^2} \tag{2.1.12}$$

*and corresponding mode solutions in $\mathcal{S}_{\text{mode}}(\alpha(\varepsilon), \omega_R(\varepsilon) + \mathrm{i}\varepsilon, m, l)$ satisfying the horizon regularity condition and Dirichlet boundary conditions.*

*For all $\varepsilon \in (0, \delta)$, these modes satisfy*

$$\frac{\partial \alpha}{\partial \varepsilon}(0) > 0 \quad \text{and} \quad \omega_R(0)\frac{\partial \omega_R}{\partial \varepsilon}(0) < 0.$$

*Remark* 2.1.8. The $u$ in the theorem has finite energy and hence the spacetime parameters of the theorem must violate the Hawking-Reall bound as explained in the previous sections; this is explained further in Lemma 2.2.18 and Remark 2.3.9. By Lemma 2.1.3, we know that the $a$ can be located anywhere in the range

$$\frac{r_+^2}{\ell} < |a| < r_+\sqrt{\frac{3\frac{r_+^2}{\ell^2} + 1}{1 - \frac{r_+^2}{\ell^2}}}.$$

We remark that our result does not restrict to small $|a|$. In fact, we can enforce $|a|$ to be as close to $\ell$ as we wish by choosing $r_+/\ell < 1$ large.

Lemma 2.1.1 implies that the constructed modes are superradiant and indicates that the instability is driven by energy leaking through the horizon.

Our next theorem builds on the first, but allows for the construction of an unstable superradiant mode with Dirichlet boundary conditions for each given $\alpha < 9/4$.

**Theorem 2.1.9.** *Let $\ell > 0$ and $\alpha < 9/4$. Then there is an $\mathcal{M}_{\mathrm{KAdS}}(\ell, r_+, a)$ and a superradiant $\psi \in \mathcal{S}_{\mathrm{mode}}(\alpha, \omega_R + \mathrm{i}\varepsilon, m, l)$ for an $\omega_R \in \mathbb{R}$ and $\varepsilon > 0$ satisfying the horizon regularity condition and the Dirichlet boundary condition at infinity.*

The methods used in our proof also show the following statement:

**Corollary 2.1.10.** *Let $\ell > 0$, $\alpha < 9/4$ and $0 < r_+ < \ell$. Then for all $\varepsilon > 0$ there is an $a$ with $|a| \in (r_+^2/\ell, r_+^2/\ell + \varepsilon)$ such that the Klein-Gordon equation with mass $\alpha$ has an exponentially growing mode solution in $\mathcal{M}_{\mathrm{KAdS}}(\ell, r_+, a)$.*

*Remark* 2.1.11. These results also apply to the massless wave equation, which is an important difference to the asymptotically flat case.

Furthermore, although this will not be pursued explicitly in this paper, one can also show the analogue of Shlapentokh-Rothman's result in our setting by only adapting the proof slightly.

**Theorem 2.1.12.** *Given a Kerr-AdS spacetime $\mathcal{M}_{\mathrm{KAdS}}(\ell, r_+, a)$ satisfying, $|a| < \ell$, $r_+ > 0$ and $r_+^2 < |a|\ell$ (and the restrictions of Lemma 2.1.3), there are mode parameters $m$ and $l$ as well as a $\delta > 0$ such that, for each $\varepsilon \in (-\delta, \delta)$, there is an open family of masses $\alpha(\varepsilon)$ and a mode solution in $\mathcal{S}_{\mathrm{mode}}(\alpha(\varepsilon), \omega_R(\varepsilon) + \mathrm{i}\varepsilon, m, l)$ satisfying Dirichlet boundary conditions with $\omega_R(0)$ as in (2.1.12).*

*Remark* 2.1.13.

1. Conversely, in the asymptotically flat Kerr case of [SR13], it is also possible to prove an analogue of Theorem 2.1.9 instead of only the analogue of Theorem 2.1.7, using our strategy explained in the next section.
2. To contrast our case to the asymptotically flat setting, we add three observations. First, in [SR13], the curve $\varepsilon \mapsto (\mu(\varepsilon), \omega_R(\varepsilon) + \mathrm{i}\varepsilon)$ must satisfy $\mu(0)^2 > \omega_R(0)^2$. There is no equivalent condition for Kerr-AdS spacetimes as the instability is not driven by the interplay of frequency and mass, but by the violation of the Hawking-Reall bound. Second, in both cases, $\partial\omega_R/\partial\varepsilon < 0$ for small $\varepsilon$, so $\omega_R(0)$ can be seen as the upper bound of the superradiant regime. Third, the result in Kerr holds for all $m \neq 0$, $l \geq |m|$. In contrast, our result is a statement about high azimuthal frequencies.

It is known – see [HW12] and references therein – that, for $0 < \kappa < 1$, i. e. $5/4 < \alpha < 9/4$, we also have well-posedness for different boundary conditions at infinity. This underlies the following

**Definition 2.1.14** (Neumann boundary condition)**.** Given a mass $5/4 < \alpha < 9/4$ (i. e. $0 < \kappa < 1$), a smooth function $f : (r_+, \infty) \to \mathbb{C}$ satisfies the Neumann boundary condition if

$$r^{1+2\kappa} \frac{\mathrm{d}}{\mathrm{d}r} \left( r^{\frac{1}{2}-\kappa} f \right) \to 0$$

as $r \to \infty$.

Using the techniques of twisted derivatives, introduced in [War12], we can prove versions of Theorems 2.1.7 and 2.1.9 for Neumann boundary conditions.

**Theorem 2.1.15.** *Given a cosmological constant $\Lambda = -3/\ell^2$, a black hole radius $0 < r_+ < \ell$ and a scalar mass parameter $\alpha_0 \in (5/4, 9/4)$, there are a spacetime parameter $a$ satisfying the regularity condition $|a| < \ell$, mode parameters $m$ and $l$ and a $\delta > 0$ such that there is a smooth curve*

$$(-\delta, \delta) \to \mathbb{R}^2, \; \varepsilon \mapsto (\alpha(\varepsilon), \omega_R(\varepsilon))$$

*with (2.1.12) and corresponding mode solutions in $\mathcal{S}_{\text{mode}}(\alpha(\varepsilon), \omega_R(\varepsilon) + \mathrm{i}\varepsilon, m, l)$ satisfying the horizon regularity condition and Neumann boundary conditions at infinity. If $\varepsilon \in (0, \delta)$, then the modes satisfy*

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\varepsilon}(0) > 0 \quad \text{and} \quad \omega_R(0)\frac{\partial \omega_R}{\partial \varepsilon}(0) < 0.$$

**Theorem 2.1.16.** *Let $\ell > 0$ and $5/4 < \alpha < 9/4$. Then there is an $\mathcal{M}_{\text{KAdS}}(\ell, r_+, a)$ violating the Hawking-Reall bound and a superradiant $\psi \in \mathcal{S}_{\text{mode}}(\alpha, \omega_R + \mathrm{i}\varepsilon, m, l)$ for an $\omega_R \in \mathbb{R}$ and $\varepsilon > 0$ satisfying Neumann boundary conditions.*

Let us conclude this discussion with a general remark on boundedness. From [HW12], we know that solutions to the Klein-Gordon equation with Dirichlet boundary conditions remain bounded for all $r_+^2 > |a|\ell$. A similar statement holds for Neumann boundary conditions under more restrictive assumptions on the parameters. For $r_+^2 = |a|\ell$, one can easily repeat the proof of the second theorem of [HS11a] to see that there are no periodic solutions. One can potentially also extend the decay result of [HS11a] to $r_+^2 = |a|\ell$. Our results do not rule out boundedness in the entire parameter range in which $r_+^2 < |a|\ell$ since we did not show that for *any* given Kerr-AdS spacetime and *any* $\alpha$, there are unstable mode solutions; they do, however, impose restrictions on the ranges of spacetime parameters and masses $\alpha$ in which boundedness could potentially hold. It is believed that, using more refined spectral estimates, our results can be shown to hold in the full regime $r_+^2 < |a|\ell$, but we will not pursue this further.

Having shown existence of unstable modes for two types of reflecting boundary conditions, it is natural to ask if this instability is present for non-reflecting boundary conditions. We can show that at least for optimally dissipative boundary conditions at infinity – as considered in [HLSW15] –, unstable modes are absent.

**Definition 2.1.17.** Let $0 < \kappa < 1$. A mode solution satisfies the optimally dissipative boundary condition at infinite if

$$\ell^2 \frac{\partial}{\partial t}\left(r^{3/2-\kappa}\psi\right) + r^{1+2\kappa}\frac{\partial}{\partial r}\left(r^{3/2-\kappa}\psi\right) \to 0$$

as $r \to \infty$.

In Section 2.5, we show the following

**Theorem 2.1.18.** *Let $\ell > 0$ and $\alpha \in (5/4, 9/4)$. Then there is no Kerr-AdS spacetime $\mathcal{M}_{\mathrm{KAdS}}(\ell, r_+, a)$ such that the Klein-Gordon equation with mass $\alpha$ has an exponentially growing mode solution $\psi$ satisfying the horizon regularity condition and the optimally dissipative boundary condition at infinity.*

### 2.1.6. Outline of the proof

The difficulty lies in the construction of the radial part $u$, for which we use the strategy of [SR13], which, as our present work shows, can be applied to more general settings than Kerr spacetimes. The technique contains two main steps.

I. Construct $u$ corresponding to a real frequency $\omega_0 \in \mathbb{R}$.

II. Obtain a mode solution corresponding to a complex $\omega$ with $\operatorname{Im}\omega > 0$ by varying spacetime and mode parameters.

We note that both steps are completely independent of each other, in particular step II does not rely on the method by which the periodic mode solution was constructed, but only requires existence of such a mode.

Let us first only deal with Dirichlet boundary conditions. To complete step I, $u$ needs to satisfy the radial ODE

$$u'' - (V - \omega_0^2)u = 0 \tag{2.1.13}$$

for the given boundary condition – see Section 2.2.1. Lemma 2.2.20 then already restricts $\omega_0$ to $\omega_+ := ma\Xi/(r_+^2 + a^2)$. It is important to note that the boundary value problem does not admit non-trivial solutions in general.

**Lemma 2.1.19.** *If $u$ satisfies the horizon regularity condition and the Dirichlet boundary condition for real $\omega_0$ and $V - \omega_0^2 \geq 0$, then $u = 0$.*

*Proof.* Define $Q(r) := \operatorname{Re}(u'\overline{u})$, note that $Q(r_+) = Q(\infty) = 0$ and integrate $\mathrm{d}Q/\mathrm{d}r$ as in Section 2.2.4. □

Hence, in a first step in Section 2.2.3, we will find spacetime and mode parameters such that $V - \omega_0^2 < 0$ on some subinterval of $(r_+, \infty)$ for given $\ell$ and $\alpha_0$ by a careful analysis of the shape of the potential $V$ in Lemma 2.2.13. This requires proving an asymptotic estimate for the eigenvalues of the modified oblate spheroidal harmonics (Lemma 2.2.12). The spacetime parameters will necessarily violate the Hawking-Reall bound.

The radial ODE is the Euler-Lagrange equation of the functional

$$\mathcal{L}_a(f) := \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2 + a^2} \left| \frac{\mathrm{d}f}{\mathrm{d}r} \right|^2 + (V - \omega^2)\frac{r^2 + a^2}{\Delta_-}|f|^2 \right) \mathrm{d}r. \tag{2.1.14}$$

The functional is not bounded below, so we need to impose a norm constraint, which we choose to be $\|f/r\|_{L^2(r_+,\infty)} = 1$. Then Lemma 2.3.3 gives a coercivity-type estimate. To carry out the direct method of the calculus of variations, we use the weighted Sobolev spaces that arise naturally from the functional – see Section 2.3.1. This setting of the minimisation problem then guarantees that the minimiser satisfies the correct boundary conditions. We remark that we will directly work with the functional (2.1.14) instead of regularising first at the horizon and then taking the limit, as in [SR13]. Then, in Lemma 2.3.7, we obtain an ODE

$$u'' - (V - \omega_0^2)u + \nu_a \frac{\Delta_-}{r^2 + a^2} \frac{u}{r^2} = 0$$

with a Lagrange multiplier $\nu_a \leq 0$ that depends continuously on the spacetime parameter $a$. By varying $a$, we find an $\hat{a}$ such that $\nu_{\hat{a}} = 0$ (Proposition 2.3.8) and hence a solution to the radial ODE.

To carry out step II, we need the asymptotic analysis of (2.1.13) that is worked out in Section 2.2.2. There are two branches that asymptote $r^{-1/2+\kappa}$ and $r^{-1/2-\kappa}$, respectively, at infinity. Let $h_1$ denote the branch with slow decay and $h_2$ the one with fast decay. Then

$$u(r, \alpha, \omega) = A(\alpha, \omega) h_1(r, \alpha, \omega) + B(\alpha, \omega) h_2(r, \alpha, \omega).$$

For the parameters from step I, $A(\alpha_0, \omega_0) = 0$. By varying $\omega$ and $\alpha$ simultaneously in Section 2.3.2, the implicit function theorem yields a curve

$$\varepsilon \mapsto (\omega_R(\varepsilon) + \mathrm{i}\varepsilon, \alpha(\varepsilon))$$

with $\omega_R(0) = \omega_0$ and $\alpha(0) = \alpha_0$ such that

$$A(\alpha(\varepsilon), \omega(\varepsilon)) = 0.$$

along the curve. As $\operatorname{Im} \omega(\varepsilon) > 0$ for $\varepsilon > 0$, these modes grow exponentially whilst satisfying Dirichlet boundary conditions. In Section 2.3.3, we show that

$$\omega_R(0) \frac{\partial \omega_R}{\partial \varepsilon}(0) < 0 \quad \text{and} \quad \frac{\partial \alpha}{\partial \varepsilon}(0) > 0, \tag{2.1.15}$$

which proves Theorem 2.1.7. A careful analysis of the domain of the implicit function theorem in Section 2.3.4 yields Theorem 2.1.9. Here, the analysis heavily exploits several continuity properties in the parameters. A difficulty is caused by $\hat{a}$ being defined as the infimum of an open set.

For Corollary 2.1.10, one observes that, by Lemma 2.2.13, once the Hawking-Reall bound is violated, one can always make the potential $V$ negative on some interval by choosing $|m|$ sufficiently large. This yields periodic modes for very small violation of the Hawking-Reall bound and hence growing modes by repeating the above argument.

The situation is more complicated if $u$ satisfies the Neumann boundary condition. Since, in this case, $u \sim r^{-1/2+\kappa}$ as $r \to \infty$, $\mathcal{L}_a$ is not well-defined and hence cannot be used to produce periodic modes. To carry out the construction of step I, we use twisted derivatives as introduced in [War12] and used extensively in [HW12]. To find the minimiser via the variational argument, we also need to modify our function spaces and use twisted weighted Sobolev spaces. All details are given in Section 2.4.1.

The main technical problems, however, arise in the second part of the argument. The underlying reason is that the proofs for step II rely severely on establishing monotonicity properties for the functional when varying $\alpha$. Since the twisting necessarily depends on $\alpha$, proving monotonicity in $\alpha$ is more involved and indeed the monotonicity properties shown in the Neumann case are weaker; nevertheless, the ideas introduced in Section 2.4.2 are sufficiently robust not only to construct the growing modes, but also to be applicable to showing (2.1.15) and to transition from Theorem 2.1.15 to Theorem 2.1.16. It is also in the Neumann case, where the independence of steps I and II – alluded to above – is exploited.

## 2.2. Preliminaries

### 2.2.1. The modified oblate spheroidal harmonics

Following [HS11a], we define the $L^2(\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\tilde{\varphi})$ operator $P$ acting on $H^1(S^2)$-complex valued functions as

$$\begin{aligned}-P(\omega,\ell,a)f = {}&\frac{1}{\sin\vartheta}\partial_\vartheta(\Delta_\vartheta\sin\vartheta\partial_\vartheta f) + \frac{\Xi^2}{\Delta_\vartheta}\frac{1}{\sin^2\vartheta}\partial^2_{\tilde{\varphi}}f\\ &+\Xi\frac{a^2\omega^2}{\Delta_\vartheta}\cos^2\vartheta f - 2\mathrm{i}a\omega\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2}\cos^2\vartheta\partial_{\tilde{\varphi}}f.\end{aligned}$$

For $\omega\in\mathbb{R}$, $P$ is self-adjoint. We also define

$$P_\alpha(\omega,\ell,a,\alpha) := \begin{cases} P(\omega,\ell,a) + \frac{\alpha}{\ell^2}a^2\sin^2\vartheta & \text{if } \alpha>0\\ P(\omega,\ell,a) - \frac{\alpha}{\ell^2}a^2\cos^2\vartheta & \text{if } \alpha\le 0.\end{cases}$$

For equivalent definitions in Kerr spacetime see [DR10] and also [FS04] for a more detailed discussion. From elliptic theory [cf. HS11a], we can make the following definitions: $P(\omega,\ell,a)$ has eigenvalues $\tilde{\lambda}_{ml}(\omega,\ell,a)$ with eigenfunctions $\mathrm{e}^{\mathrm{i}m\tilde{\varphi}}\,\tilde{S}_{ml}(\omega,\ell,a,\cos\vartheta)$; $P_\alpha(\omega,\ell,a,\alpha)$ has eigenvalues $\lambda(\omega,\ell,a,\alpha)$ with eigenfunctions $\mathrm{e}^{\mathrm{i}m\tilde{\varphi}}\,S_{ml}(\omega,\ell,a,\alpha,\cos\vartheta)$. The eigenfunctions form an orthonormal basis of $L^2(\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\tilde{\varphi})$. Below we will suppress $(\omega,\ell,a,\alpha)$ in the notation.

If $\alpha\le 0$, $S_{ml}$ satisfies the angular ODE

$$\begin{aligned}&\frac{1}{\sin\vartheta}\partial_\vartheta\left(\Delta_\vartheta\sin\vartheta\partial_\vartheta S_{ml}(\cos\vartheta)\right) - \left(\frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{\sin^2\vartheta} - \frac{\Xi}{\Delta_\vartheta}a^2\omega^2\cos^2\vartheta\right.\\ &\qquad\left.-2ma\omega\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2}\cos^2\vartheta - \frac{\alpha}{\ell^2}a^2\cos^2\vartheta\right)S_{ml}(\cos\vartheta) + \lambda_{ml}S_{ml}(\cos\vartheta) = 0\end{aligned} \tag{2.2.1}$$

for $\lambda_{m\ell}(\omega, \alpha, a) \in \mathbb{C}$. If $\alpha > 0$, the angular ODE takes the form

$$\begin{aligned} &\frac{1}{\sin\vartheta}\partial_\vartheta\left(\Delta_\vartheta \sin\vartheta\partial_\vartheta S_{ml}(\cos\vartheta)\right) - \left(\frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{\sin^2\vartheta} - \frac{\Xi}{\Delta_\vartheta}a^2\omega^2\cos^2\vartheta \right.\\ &\qquad \left. - 2ma\omega\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2}\cos^2\vartheta + \frac{\alpha}{\ell^2}a^2\sin^2\vartheta\right)S_{ml}(\cos\vartheta) + \lambda_{ml}S_{ml}(\cos\vartheta) = 0. \end{aligned} \tag{2.2.2}$$

Using these modified oblate spheroidal harmonics, one obtains that, for fixed $m$ and $l$, $u := \sqrt{r^2+a^2}R$ satisfies the radial ODE

$$u''(r) + (\omega^2 - V(r))u(r) = 0, \tag{2.2.3}$$

with

$$\begin{aligned} V(r) &= V_+(r) + V_0(r) + V_\alpha(r) \\ V_+(r) &= -\Delta_-^2\frac{3r^2}{(r^2+a^2)^4} + \Delta_-\frac{5\frac{r^4}{\ell^2} + 3r^2\left(1+\frac{a^2}{\ell^2}\right) - 4Mr + a^2}{(r^2+a^2)^3} \\ V_0(r) &= \frac{\Delta_-(\lambda_{ml} + \omega^2 a^2) - \Xi^2 a^2 m^2 - 2m\omega a\Xi(\Delta_- - (r^2+a^2))}{(r^2+a^2)^2} \\ V_\alpha(r) &= -\frac{\alpha}{\ell^2}\frac{\Delta_-}{(r^2+a^2)^2}(r^2 + \Theta(\alpha)a^2). \end{aligned}$$

Here $\Theta(x) = 1$ if $x > 0$ and zero otherwise. We will use the shorthand $\tilde{V} := V - \omega^2$. Note that $V_0$ has an explicit dependence on $\omega$, whence $V$ depends on $\omega$ explicitly. Recall that $'$ denotes an $r^*$-derivative

$$\frac{\mathrm{d}}{\mathrm{d}r^*} = \frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}}{\mathrm{d}r}.$$

To indicate the dependence upon $a$, we will often write $V_a$ and $\tilde{V}_a$ for $V$ and $\tilde{V}$ respectively.

### 2.2.2. Local analysis of the radial ODE

To see which boundary conditions are appropriate for $u$, we perform a local analysis of the radial ODE near the horizon $r = r_+$ and at infinity, using the following theorem about regular singularities, which we cite from [Tes12], but it can also be found in [SR13] or [Olv74].

**Theorem 2.2.1.** *Consider the complex ODE*

$$\frac{\mathrm{d}^2H}{\mathrm{d}z^2} + f(z,\nu)\frac{\mathrm{d}H}{\mathrm{d}z} + g(z,\nu)H = 0. \tag{2.2.4}$$

*Suppose $f$ and $g$ are meromorphic and have poles of order (at most) one and two, respectively, at $z_0 \in \mathbb{C}$. Let $f_0(\nu)$ and $g_0(\nu)$ be the coefficients of pole of order one and two, respectively, in the Laurent expansions. Then we say that $z_0$ is a regular singularity. Let $s_1(\nu)$ and $s_2(\nu)$*

*be the two solutions of the indicial equation*

$$s(s-1)+f_0(\nu)s+g_0(\nu)=0$$

*with* $\mathrm{Re}\,(s_1)\leq \mathrm{Re}\,(s_2)$.

*If* $s_2(\nu)-s_1(\nu)\notin \mathbb{N}_0$, *a fundamental system of solutions is given by*

$$h_j(z,\nu)=(z-z_0)^{s_j(\nu)}\varrho_j(z,\nu),$$

*where the functions* $\varrho_j$ *are holomorphic and satisfy* $\varrho_j(z_0,\nu)=1$.

*If* $s_2(\nu)-s_1(\nu)=m\in\mathbb{N}_0$, *a fundamental system is given by*

$$\begin{aligned}h_1&=(z-z_0)^{s_1}\varrho_1+c\log(z)h_2\\ h_2&=(z-z_0)^{s_2}\varrho_2.\end{aligned}$$

*The constant* $c$ *may be zero unless* $m=0$.

*In both cases, the radius of convergence of the power series of* $\varrho_j$ *is at least equal to the minimum of the radii of convergence of the Laurent series of* $f$ *and* $g$.

#### The horizon

Adopting the notation of the previous section, and, after expressing the radial ODE (2.2.3) with $r$-derivatives, we have

$$f=\frac{\partial_r\Delta_-}{\Delta_-}-\frac{2r}{r^2+a^2},\qquad g=\frac{(r^2+a^2)^2}{\Delta_-^2}(\omega^2-V).$$

Thus we obtain

$$\begin{aligned}f_0&=\lim_{r\to r_+}(r-r_+)f=1\\ g_0&=\lim_{r\to r_+}(r-r_+)^2\frac{(r^2+a^2)^2}{\Delta_-^2}(\omega^2-V)=\lim_{r\to r_+}\frac{(r-r_+)^2}{\Delta_-^2}\left(\omega(r^2+a^2)-\Xi am\right)^2=-\xi^2\end{aligned}$$

with

$$\xi:=\mathrm{i}\frac{\Xi am-\omega(r_+^2+a^2)}{\partial_r\Delta_-(r_+)}$$

as $\partial_r\Delta_-(r_+)>0$. Thus, the indicial equation is solved by $s=\pm\xi$.

Therefore if $\xi\neq 0$, a local basis of solutions $u$ (or $R$) is given by

$$\{(\cdot-r_+)^{\xi}\varphi_1,(\cdot-r_+)^{-\xi}\varphi_2\}$$

for holomorphic functions $\varphi_i$ satisfying $\varphi_i(r_+)=1$. For $\xi=0$, a local basis is given by

$$\{\varphi_1,\varphi_1\left(1+c\log(\cdot-r_+)\right)\}$$

for $\varphi_1(r_+) = 1$ and some constant $c$, where $\kappa$ as in (2.1.10).

**Lemma 2.2.2.** *If $u$ extends smoothly to the horizon, then there is a smooth function $\varrho\colon [r_+, \infty) \to \mathbb{C}$ such that*

$$u = (\cdot - r_+)^{\xi} \varrho.$$

*Proof.* Boyer-Lindquist coordinates break down at the horizon, so we need to change to Kerr-star coordinates. Then the solution $\psi$ takes the form

$$\psi(t^*, r, \varphi^*, \vartheta) = \mathrm{e}^{-\mathrm{i}\omega(t^* - A(r))}\, \mathrm{e}^{im(\varphi^* - B(r))}\, S_{ml}(a\omega, \cos\vartheta) \frac{u(r)}{(r^2 + a^2)^{1/2}},$$

where

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{(r^2 + a^2)(1 + r^2/\ell^2) - \Delta_-}{\Delta_-(1 + r^2/\ell^2)}, \qquad \frac{\mathrm{d}B}{\mathrm{d}r} = \frac{a(1 - a^2/\ell^2)}{\Delta_-}.$$

Hence $u$ extends smoothly to the horizon if and only if there is a smooth function $f$ such that

$$u(r) = \mathrm{e}^{-\mathrm{i}(\omega A(r) - mB(r))}\, f(r).$$

Therefore the claim reduces to showing that

$$\varrho(r) := (r - r_+)^{-\xi}\, \mathrm{e}^{-\mathrm{i}(\omega A(r) - mB(r))}\, f(r)$$

is smooth. Since

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(-\mathrm{i}(\omega A(r) - mB(r))\right) = \frac{\xi}{r - r_+} + \mathcal{O}(1),$$

we have

$$\varrho(r) = \mathrm{e}^{-\xi \log(r - r_+)}\, \mathrm{e}^{\xi \log(r - r_+) + \mathcal{O}(r - r_+)}\, f(r),$$

which proves the claim. □

**Corollary 2.2.3.** *Assume $u$ satisfies the horizon regularity condition. Then a local basis of solutions to the ODE at the horizon is given by*

$$(\cdot - r_+)^{\xi} \varrho$$

*for a holomorphic function $\varrho$ defined around $r = r_+$.*

This asymptotic analysis at the horizon motivates the horizon regularity condition of Definition 2.1.4.

### Infinity

The radial ODE has a regular singularity at $r = \infty$. To analyse it using the Theorem 2.2.1, we rewrite equation (2.2.4) by introducing $x := 1/z$. This yields

$$\frac{\mathrm{d}^2 H}{\mathrm{d}x^2} + \left(\frac{2}{x} - \frac{f}{x^2}\right)\frac{\mathrm{d}H}{\mathrm{d}x} + \frac{g}{x^4}H = 0.$$

For the radial ODE, we have $x = 1/r$. We obtain

$$f(x=0) = 0, \qquad \lim_{x\to 0}\frac{f}{x} = 2, \qquad g(x=0) = 0, \qquad \lim_{x\to 0}\frac{g}{x} = 0, \qquad \lim_{x\to 0}\frac{g}{x^2} = \alpha - 2.$$

The indicial equation becomes

$$s^2 - s + \alpha - 2 = 0,$$

which is solved by $s_\pm = \frac{1}{2} \pm \sqrt{\frac{9}{4} - \alpha}$. Set

$$\mathcal{E} := \left\{\frac{9-k^2}{4} \,:\, k \in \mathbb{N}\right\}.$$

Then, for $\alpha \notin \mathcal{E}$, a local basis of solutions near infinity is given by

$$\{r^{-1/2+\sqrt{9/4-\alpha}}\varrho_1(r), r^{-1/2-\sqrt{9/4-\alpha}}\varrho_2(r)\}$$

with functions $\varrho_1, \varrho_2$, smooth at $\infty$ and satisfying $\varrho_1(\infty) = \varrho_2(\infty) = 1$. For $\alpha \in \mathcal{E}$, a local basis is given by

$$\left\{C_3 r^{-1/2-\kappa}\log\frac{1}{r} + r^{-1/2+\kappa}\varrho_2, r^{-1/2-\sqrt{9/4-\alpha}}\varrho_2(r)\right\}.$$

If $u$ extends smoothly to $r = r_+$ and we specify a boundary value $u(r_+)$, then the arguments of Section 2.2.2 show that $C_3$ has to be zero.

**Lemma 2.2.4.** *Let $u$ satisfy (2.2.3) on $(r_+, \infty)$ and extend smoothly to $r = r_+$, then, for large $r$, $u$ is a linear combination of*

$$h_1(r,\alpha,\omega,a) = r^{-1/2+\kappa}\varrho_1(r,\alpha,\omega,a)$$
$$h_2(r,\alpha,\omega,a) = r^{-1/2-\kappa}\varrho_2(r,\alpha,\omega,a)$$

*for functions $\varrho_1$ and $\varrho_2$ holomorphic at $r = \infty$ and satisfying $\varrho_1(\infty) = \varrho_2(\infty) = 1$.*

**Corollary 2.2.5.** *If $u$ satisfies the horizon regularity condition and the Neumann boundary condition at infinity, then, for $5/4 < \alpha < 9/4$,*

$$u = C_1 h_1$$

*for a constant* $C_1 \in \mathbb{C}$.

*If* $u$ *satisfies the horizon regularity condition and the Dirichlet boundary condition at infinity, then, for all* $\alpha < 9/4$,

$$u(r) = C_2 h_2$$

*for a constant* $C_2 \in \mathbb{C}$.

*Remark* 2.2.6. The asymptotics near infinity do not change if we add $\nu(r^2 + a^2)/(r^2\Delta_-)$ to $g$ as in Section 2.3.1.

#### Uniqueness of solutions and dependence on parameters

As one would expect, choosing a value of $u$ at $r = r_+$ determines the solution to the radial ODE uniquely, which is being made more precise in the following

**Lemma 2.2.7.** *Let* $C_0 \in \mathbb{C}$. *Then there is a unique classical solution to (2.2.3) on* $(r_+, \infty)$ *satisfying* $u(r_+) = C_0$ *and extending smoothly to* $r = r_+$.

*Proof.* There are constants $r_+ < r_0 < R_0 < \infty$ such that, by Corollary 2.2.3,

$$u = (\cdot - r_+)^\xi \varrho$$

in $(r_+, r_0)$ and, by the previous section, is linear combination of two branches on $(R_0, \infty)$. The boundary condition at the horizon hence defines $u$ uniquely on $(r_+, r_0)$. Moreover, the corresponding linear first-order ODE is Lipschitz continuous in $[r_0, R_0 + 1]$, so $u$ can be uniquely extended to this interval, but now $u$ is a solution to (2.2.3) on $[R_0, R_0 + 1]$ and is hence a unique linear combination of the branches near infinity. □

**Lemma 2.2.8.** *Let* $u_0$ *be a unique solution to (2.2.3) for a certain set of parameters* $(\alpha_0, \omega_0, a_0)$ *with fixed* $u(r_+)$ *satisfying either the Dirichlet or Neumann boundary condition. Let there be a neighbourhood of these parameters such that for all* $(\alpha, \omega, a)$ *in said neighbourhood, there is a unique solution* $u_{\alpha,\omega,a}$ *with the same boundary conditions. Fix an* $\hat{r} \in (r_+, \infty)$. *Then*

$$(\alpha, \omega, a) \mapsto u_{\alpha,\omega,a}(\hat{r})$$

*is smooth.*

*Proof.* It is known that an initial value problem

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = F(g, \lambda), \quad g(0) = g_0$$

for a smooth function $F$ with parameter $\lambda$ depends smoothly on initial data. By defining the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} g \\ \lambda \end{pmatrix} = \begin{pmatrix} F(g,\lambda) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} g(0) \\ \lambda(0) \end{pmatrix} = \begin{pmatrix} g_0 \\ \lambda_0 \end{pmatrix},$$

one deduces the smooth dependence on parameters. By dividing $(r_+,\infty)$ in three intervals $(r_+,r_0)$, $[r_0,R_0]$ and $(R_0,\infty)$ as in the previous proof, we obtain the claim □

Let $u$ be a solution to (2.2.3) that extends smoothly to the horizon. Fixing $u(r_+)$, we can uniquely define reflection and transmission coefficients $A(\alpha,\omega,a)$ and $B(\alpha,\omega,a)$ via

$$u(r,\alpha,\omega,a) = A(\alpha,\omega,a)h_1(r,\alpha,\omega,a) + B(\alpha,\omega,a)h_2(r,\alpha,\omega,a) \tag{2.2.5}$$

for large $r$. Here $h_1$ and $h_2$ are the local basis near infinity from Section 2.2.2. Let $W$ denote the Wronskian. Then

$$A = \frac{W(u,h_2)}{W(h_1,h_2)}$$

and similarly for $B$.

**Lemma 2.2.9.** *$A$ and $B$ are smooth in $\alpha$, $\omega$ and $a$.*

*Proof.* Note that $A$ and $B$ are independent of $r$ and apply Lemma 2.2.8. □

### 2.2.3. Detailed analysis of the potential

From the analysis in [HS11a] we know that the angular ODE has countably many simple eigenvalues $\lambda_{ml}$, labelled by $l = |m|, |m|+1, \dots$ for any given $m \in \mathbb{Z}$, and corresponding real-valued eigenfunction $S_{ml}$ if $\omega \in \mathbb{R}$. For later use, we need a bound from below which can be found in [HS11a]. We state it here for the sake of completeness.

**Lemma 2.2.10.** *Let $\omega \in \mathbb{R}$. If the Hawking-Reall bound $r_+^2 > |a|\ell$ is satisfied, the eigenvalues can be bounded below as*

$$\begin{aligned} \lambda_{ml} + a^2\omega^2 &\geq \Xi^2|m|(|m|+1) \\ \lambda_{ml} + a^2\omega^2 &\geq \Xi^2|m|(|m|+1) + a^2\omega_+^2 - C_{\ell,a}|m||\omega-\omega_+|, \end{aligned} \tag{2.2.6}$$

*where $C_{\ell,a} > 0$ depends on $\ell$ and $a$ only and*

$$\omega_+(\ell,r_+,a,m) := \frac{ma\Xi}{r_+^2+a^2}$$

We will also need an asymptotic upper bound on the ground state eigenvalue $\lambda_{mm}$. For $\alpha \leq 0$ and by the min-max principle, we know that

$$\lambda_{mm} = \min_{u\in U,\|u\|=1}\int_0^\pi \left( \left[ \Delta_\vartheta \left|\frac{\mathrm{d}u}{\mathrm{d}\vartheta}\right|^2 + \frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{\sin^2\vartheta}|u|^2 \right] - \frac{\Xi}{\Delta_\vartheta}a^2\omega^2\cos^2\vartheta|u|^2 \right.$$

$$- 2ma\omega\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2}\cos^2\vartheta|u|^2 - \frac{\alpha}{\ell^2}a^2\cos^2\vartheta|u|^2\Bigg)\sin\vartheta\,\mathrm{d}\vartheta$$

$$\leq \min_{u\in U,\,\|u\|=1}\int_0^\pi\left(\Delta_\vartheta\left|\frac{\mathrm{d}u}{\mathrm{d}\vartheta}\right|^2 + \frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{\sin^2\vartheta}|u|^2\right)\sin\vartheta\,\mathrm{d}\vartheta + (|\alpha| + 2|ma|\cdot|\omega-\omega_+|)\,\frac{a^2}{\ell^2}$$

for $U = \{(\sin\vartheta)^{|m|}\varrho(\vartheta) \;:\; \varrho \text{ analytic}\}$, which is the subspace of $L^2$ which contains all $S_{ml}$ – see appendix A.1. We obtain exactly the same estimate for $\alpha > 0$.

**Lemma 2.2.11.** *Let $n \in \mathbb{N}$. Then*

$$\int_0^\pi \sin^n\vartheta\,\mathrm{d}\vartheta = \sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \quad\text{and}\quad \int_0^\pi \sin^n\vartheta\cos^2\vartheta\,\mathrm{d}\vartheta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)},$$

*in particular*

$$\int_0^\pi \sin^n\vartheta\,\mathrm{d}\vartheta \sim \sqrt{2\pi}n^{-1/2} \quad\text{and}\quad \int_0^\pi \sin^n\vartheta\cos^2\vartheta\,\mathrm{d}\vartheta \sim \sqrt{2\pi}n^{-3/2}$$

*as $n \to \infty$.*

*Proof.* The exact identities can be proved by induction. For $n = 0$ and $n = 1$ use $\Gamma(1/2) = \sqrt{\pi}$ (from Euler's reflection formula) and the duplication formula

$$\Gamma(z/2)\,\Gamma((1+z)/2) = \sqrt{\pi}2^{-n}\Gamma(z)$$

for $z \neq 0, -1, -2, \ldots$. The induction step follows by integration by parts:

$$\int_0^\pi \sin^n\vartheta\,\mathrm{d}\vartheta = \int_0^\pi \sin^{n-1}\vartheta\sin\vartheta\,\mathrm{d}\vartheta = \int_0^\pi (n-1)\sin^{n-2}\vartheta(1-\sin^2\vartheta)\,\mathrm{d}\vartheta$$

The formula involving $\cos^2\vartheta$ follows immediately. For the asymptotics, we use the Stirling formula

$$\Gamma(z) \sim \sqrt{2\pi}z^{z-1/2},$$

yielding the result. □

Define

$$u_m := \left(\sqrt{\pi}\frac{\Gamma(|m|+1)}{\Gamma(|m|+3/2)}\right)^{-1/2}\sin^{|m|}\vartheta.$$

Then $u_m \in U$ and $\|u_m\|_{L^2((0,\pi);\sin\vartheta\,\mathrm{d}\vartheta)}$=1. As $1-\cos^2\vartheta \leq \Delta_\vartheta \leq 1$, we hence know that

$$\lambda_{mm} \leq \int_0^\pi\left(\left|\frac{\mathrm{d}u_m}{\mathrm{d}\vartheta}\right|^2 + \Xi^2\frac{m^2}{\sin^4\vartheta}|u_m|^2\right)\sin\vartheta\,\mathrm{d}\vartheta + (|\alpha| + 2|ma|\cdot|\omega-\omega_+|)\,\frac{a^2}{\ell^2}.$$

**Lemma 2.2.12.**

$$\lim_{m\to\infty}\frac{\lambda_{mm}}{m^2}\le \Xi^2$$

*Proof.* To prove the result, we compute

$$\begin{aligned}\frac{1}{m^2}\int_0^\pi\left(\left|\frac{\mathrm{d}u_m}{\mathrm{d}\vartheta}\right|^2+\Xi^2\frac{m^2}{\sin^4\vartheta}|u_m|^2\right)\sin\vartheta\,\mathrm{d}\vartheta = \pi^{-1/2}\frac{\Gamma\left(|m|+3/2\right)}{\Gamma\left(|m|+1\right)}\times&\\ \times\int_0^\pi\left(\cos^2\vartheta\sin^2\vartheta+\Xi^2\right)\sin^{2|m|-3}\,\mathrm{d}\vartheta&\\ \sim m^{1/2}m^{-3/2}+\Xi^2m^{1/2}m^{-1/2}&\end{aligned}$$

by the previous lemma. Hence $\lim_{m\to\infty}\lambda_{mm}/m^2\le\Xi^2$. □

**Lemma 2.2.13.** *Let $\ell>0$ and $\alpha<9/4$ be fixed and let $N,L>0$. Moreover assume the spacetime parameters $r_+$ and $a$ satisfy*

$$\frac{r_+^4-a^2\ell^2}{(r_+^2+a^2)^2}<-N. \tag{2.2.7}$$

*Then there is an $m_0>0$ such that for all mode parameters $|m|\ge m_0$ and $l=m$, we have*

$$V-\omega^2\le -N\frac{\Delta_-m^2\Xi^2}{(r^2+a^2)^2} \tag{2.2.8}$$

*on an interval $(R_1,R_2)$ of length $L$ at $\omega=\omega_+(\ell,r_+,a,m)$.*

*Proof.* Let us first rewrite the potential:

$$\begin{aligned}V-\omega^2 &= V_++V_\alpha\\ &\quad+\frac{\Delta_-}{(r^2+a^2)^2}\left(\Xi^2m^2-2\frac{a^2\Xi^2m^2}{r_+^2+a^2}+\frac{m^2a^4\Xi^2}{(r_+^2+a^2)^2}-\frac{m^2a^2\Xi^2}{\Delta_-}\frac{(r^2-r_+^2)^2}{(r_+^2+a^2)^2}\right)\\ &\quad+\frac{\Delta_-}{(r^2+a^2)^2}(\lambda-\Xi^2m^2)\\ &=V_++V_\alpha+\frac{\Delta_-}{(r^2+a^2)^2}(\lambda-\Xi^2m^2)+\frac{\Delta_-m^2\Xi^2}{(r^2+a^2)^2(r_+^2+a^2)^2}(r_+^4-a^2\ell^2)\\ &\quad+\frac{\Delta_-m^2\Xi^2}{(r^2+a^2)^2(r_+^2+a^2)^2}\frac{a^2}{\Delta_-}(r-r_+)[r(2r_+^2+a^2+\ell^2)-a^2\ell^2r_+^{-1}+r_+^3]\\ &=\frac{\Delta_-m^2\Xi^2}{(r^2+a^2)^2}\times\\ &\quad\Big[\frac{2r^2}{\Xi^2m^2\ell^2}+\frac{\Delta_-}{(r^2+a^2)^2}\frac{a^2}{m^2\Xi^2}+\frac{r^2-a^2}{(r^2+a^2)^2}\frac{2Mr}{m^2\Xi^2}-\frac{\alpha}{\ell^2}\frac{1}{m^2\Xi^2}\left(r^2+\Theta(\alpha)a^2\right)\\ &\quad+\left(\frac{\lambda}{\Xi^2m^2}-1\right)+\boxed{\frac{r_+^4-a^2\ell^2}{(r_+^2+a^2)^2}}\\ &\quad+\frac{a^2(r-r_+)}{\Delta_-}[r(2r_+^2+a^2+\ell^2)-a^2\ell^2r_+^{-1}+r_+^3]\Big]\end{aligned}\tag{2.2.9}$$

Note that

$$[r(2r_+^2 + a^2 + \ell^2) - a^2\ell^2 r_+^{-1} + r_+^3]|_{r=r_+} = \ell^2 \Delta_-'(r_+) > 0.$$

Moreover $\lambda - \Xi^2 m^2 > 0$ by (2.2.6) if the Hawking-Reall bound is satisfied. To obtain negativity, we will violate the Hawking-Reall bound in the boxed term. First we can choose $|m|$ large such that $\lambda/\Xi^2 m^2 - 1$ is bounded above by a sufficiently small constant, in virtue of Lemma 2.2.12. Since the term in the last line is decaying, we can find an $R_1$ such that the last term is bounded on $[R_1, R_1 + L]$. By making $|m|$ possibly larger, the terms of the first line are also bounded on the interval.

Let

$$\frac{r_+^4 - a^2\ell^2}{(r_+^2 + a^2)^2} \leq -N - \varepsilon_m$$

Now we choose $m$ sufficiently large such that

$$\frac{\lambda_{mm}}{\Xi^2 m^2} - 1 < \frac{\varepsilon_m}{3}.$$

There is an $R_1$ such that

$$\frac{a^2(r - r_+)}{\Delta_-}[r(2r_+^2 + a^2 + \ell^2) - a^2\ell^2 r_+^{-1} + r_+^3] < \frac{\varepsilon_m}{3}$$

for all $r \geq R_1$. Set $R_2 := R_1 + L$ and choose $m$ such that

$$\frac{1}{m^2}\left(\frac{2r^2}{\Xi^2\ell^2} + \frac{\Delta_-}{(r^2+a^2)^2}\frac{a^2}{\Xi^2} + \frac{r^2 - a^2}{(r^2+a^2)^2}\frac{2Mr}{\Xi^2} - \frac{\alpha}{\ell^2}\frac{r^2}{\Xi^2}\right) < \frac{\varepsilon_m}{3}$$

on $[R_1, R_2]$. Putting everything together, the lemma follows. □

*Remark* 2.2.14. The same proof yields the analogous negativity results for $V - \omega^2 + F$, where $F$ is any continuous function on $(r_+, \infty)$ that is independent of $m$. This will be used in Section 2.4.1.

Define the functional

$$\mathcal{L}_{\alpha,r_+,a}(f) := \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2}\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2 + \frac{(V-\omega^2)(r^2+a^2)}{\Delta_-}|f|^2 \right) \mathrm{d}r$$

on $C_0^\infty(r_+, \infty)$. We often suppress some of the indices and write $\mathcal{L}_a$ and $V_a$ in view of Section 2.3.1.

**Lemma 2.2.15.** *Choose $(r_+, a, \ell)$ and $(m, l)$ as in Lemma 2.2.13. Then there is a function $f \in C_0^\infty(r_+, \infty)$ such that*

$$\mathcal{L}_a(f) < 0.$$

*Proof.* We have the following estimate for the functional if $f$ is supported in $(R_1, R_2)$:

$$\mathcal{L}_a(f) \leq \int_{R_1}^{R_2} \left( \frac{\Delta_-(R_2)}{r_+^2 + a^2} \left| \frac{\mathrm{d}f}{\mathrm{d}r} \right|^2 - N \frac{m^2 \Xi^2}{R_2^2 + a^2} |f|^2 \right) \mathrm{d}r$$

Choose an $f$ such that $f$ is 1 on $[R_1 + L/4, R_2 - L/4]$ and 0 outside of $(R_1, R_2)$. Furthermore we require that

$$\left| \frac{\mathrm{d}f}{\mathrm{d}r} \right| \leq 2 \frac{4}{L}.$$

Hence

$$\mathcal{L}_a(f) \leq \frac{64 \Delta_-(R_2)}{(r_+^2 + a^2)L} - N \frac{m^2 \Xi^2 L}{2(R_2^2 + a^2)}.$$

If necessary, we can increase $m$ further to make the expression negative. □

*Remark* 2.2.16. Fix $\ell > 0$ and $0 < r_+ < \ell$. Choose $\alpha_0 < 9/4$ and $a = a_0$ such that (2.2.7) holds. Then there is a non-empty open interval $I \subseteq (-\infty, 9/4)$ with $\alpha_0 \in I$, a non-empty open Interval $I'$ around $a_0$ and an $m_0$ such that Lemma 2.2.15 holds for all $\alpha \in I$, $a \in I'$ and $|m| \geq m_0$.

We want to conclude this section by showing that $\mathcal{L}_a$ is always non-negative if the Hawking-Reall bound is satisfied. We borrow the following Hardy inequality from [HS11a].

**Lemma 2.2.17.** *For any $r_{\text{cut}} \geq r_+$, we have for a smooth function $f$ with $fr^{1/2} = o(1)$ at infinity that*

$$\frac{1}{4\ell^2} \int_{r_{\text{cut}}}^{\infty} |f|^2 \, \mathrm{d}r \leq \int_{r_{\text{cut}}}^{\infty} \frac{\Delta_-}{r^2 + a^2} \left| \frac{\mathrm{d}f}{\mathrm{d}r} \right|^2 \mathrm{d}r.$$

*Proof.* We include a proof for the sake of completeness. Integrating by parts and applying the Cauchy-Schwarz inequality yields

$$\int_{r_{\text{cut}}}^{\infty} \frac{\mathrm{d}}{\mathrm{d}r}(r - r_{\text{cut}}) |f|^2 \, \mathrm{d}r \leq 4 \int_{r_{\text{cut}}}^{\infty} (r - r_{\text{cut}})^2 \left| \frac{\mathrm{d}f}{\mathrm{d}r} \right|^2 \mathrm{d}r.$$

The lemma follows by estimating $r_{\text{cut}} \geq r_+$. □

Thus we can prove the

**Lemma 2.2.18.** *Let $r_+^2 \geq |a|\ell$. Then for all $f \in C_0^\infty(r_+, \infty)$,*

$$\mathcal{L}_a(f) > 0.$$

*Proof.* Noting that $r^2 \geq r_+^2 \geq a^2$, one deduces

$$V_+ + V_\alpha = \frac{(2-\alpha)\Delta_-}{(r^2 + a^2)^2} \frac{r^2}{\ell^2} + \frac{\Delta_-}{(r^2 + a^2)^4} \left( a^4 \Delta_- + (r^2 - a^2) 2Mr \right) \geq \frac{2 - \alpha}{\ell^2} \frac{\Delta_-}{(r^2 + a^2)^2} r^2$$

and one sees from (2.2.9) that

$$\tilde{V}_a > \frac{2-\alpha}{\ell^2}\frac{\Delta_-}{(r^2+a^2)^2}r^2 > -\frac{1}{4\ell^2}\frac{\Delta_-}{(r^2+a^2)^2}r^2.$$

Using Lemma 2.2.17, we conclude $\mathcal{L}_a(f) > 0$. □

An analogue of Lemma 2.2.15 can be proved for the twisted functional used in Section 2.4.1. For $0 < \kappa < 1$, define

$$\tilde{\mathcal{L}}_a(f) := \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} r^{-1+2\kappa} \left| \frac{\mathrm{d}}{\mathrm{d}r} \left( r^{\frac{1}{2}-\kappa} f \right) \right|^2 + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} |f|^2 \right) \mathrm{d}r$$

with $\tilde{V}_a^h$ as in Section 2.4.1.

**Lemma 2.2.19.** *Choose $(r_+, a, \ell)$ and $(m, l)$ as in Lemma 2.2.13. Then there is a function $f \in C_0^\infty(r_+, \infty)$ such that*

$$\tilde{\mathcal{L}}_a(f) < 0.$$

*Proof.* For $f \in C_0^\infty(r_+, \infty)$, $\mathcal{L}_a(f) = \tilde{\mathcal{L}}_a(f)$ by choice of $\tilde{V}_a^h$. □

### 2.2.4. Periodic mode solutions

**Lemma 2.2.20.** *Suppose we have a $\psi \in \mathcal{S}_{\mathrm{mod}}(\alpha, \omega, m, l)$ such that $\omega \in \mathbb{R}$. Then the following statements are true:*

*(i) We have $ma\Xi - (r_+^2 + a^2)\omega = 0$, i. e. that $\omega = \omega_+(\ell, r_+, a, m)$.*

*(ii) We have $am \neq 0$.*

*Proof.* We wish to show $(i)$. First let us only deal with the Dirichlet branch. Then $u$ is decaying at infinity. Define the microlocal energy current

$$Q_T := \mathrm{Im}\left(u'\overline{u}\right).$$

We have $Q_T(\infty) = 0$. Moreover

$$\frac{\mathrm{d}Q_T}{\mathrm{d}r} = \frac{\mathrm{d}r^*}{\mathrm{d}r}\mathrm{Im}\left(u''\overline{u} + |u'|^2\right) = 0$$

by the radial ODE. By Lemma 2.2.2, we obtain

$$u' = \frac{\mathrm{d}r}{\mathrm{d}r^*}\left(\frac{\xi}{r-r_+}u(r) + (r-r_+)^{\xi}\varrho'(r)\right) = \frac{\Delta_-}{r^2+a^2}\left(\frac{\xi}{r-r_+}u(r) + (r-r_+)^{\xi}\varrho'(r)\right)$$

and so

$$u'(r_+) = \mathrm{i}\frac{\Xi am - (r_+^2 + a^2)\omega}{r_+^2 + a^2}u(r_+).$$

We conclude

$$0 = (r_+^2 + a^2) Q_T(r_+) = (r_+^2 + a^2) \mathrm{Im} \left( \frac{\mathrm{d}u}{\mathrm{d}r^*}(r_+) \overline{u(r_+)} \right) = (am\Xi - (r_+^2 + a^2)\omega)|u(r_+)|^2. \tag{2.2.10}$$

If $u(r_+) = 0$, then $u$ vanishes identically by Lemma 2.2.7. Hence we conclude that

$$ma\Xi - (r_+^2 + a^2)\omega = 0.$$

For the Neumann branch of the solution we observe that

$$Q_T = \mathrm{Im} \left( r^{-\frac{1}{2}+\kappa} \frac{\mathrm{d}}{\mathrm{d}r^*} \left( r^{\frac{1}{2}-\kappa} u \right) \overline{u} \right).$$

From the boundary condition, we immediately get $Q_T(\infty) = 0$ as well and the rest follows as above.

Part (ii) follows immediately from $(r_+^2 + a^2)\omega = \Xi am$. □

## 2.3. Growing mode solutions satisfying Dirichlet boundary conditions

### 2.3.1. Existence of real mode solutions

We now fix $\ell > 0$, $\alpha < 9/4$ and $0 < r_+ < \ell$. Recall the variational functional

$$\mathcal{L}_a(f) := \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2 + a^2} \left| \frac{\mathrm{d}f}{\mathrm{d}r} \right|^2 + \tilde{V}_a \frac{r^2 + a^2}{\Delta_-} |f|^2 \right) \mathrm{d}r.$$

for $\omega = \omega_+ = am\Xi/(r_+^2 + a^2)$. Define

$$\mathcal{A} := \{a > 0 \,:\, \exists f \in C_0^\infty \,:\, \mathcal{L}_a(f) < 0\}.$$

By Lemma 2.2.15, there is an $m_0$ such that $\mathcal{A}$ is non-empty for all $|m| \geq m_0$ and $l = m$. Fix $m$ and $l$ henceforth.

If the bound $r_+^2 \geq |a|\ell$ is satisfied, then $\mathcal{L}_a(f) \geq 0$ for all compactly supported $f$ by Lemma 2.2.18. Hence $\mathcal{A}$ is bounded below by a strictly positive infimum. Moreover $\mathcal{A}$ is open as $a \mapsto \mathcal{L}_a(f)$ is continuous for any fixed $f$.

*Remark* 2.3.1. We restrict ourselves to $a > 0$, but we could have defined the set $\mathcal{A}$ to also include negative values of $a$.

Our aim is to show that $\mathcal{L}_a$ has a minimiser for $a \in \mathcal{A}$. We will apply the natural steps of the direct method of the calculus of variations. First, we will specify an appropriate function space, then we will show that the functional obeys a coercivity condition and that the functional is weakly lower semicontinuous. The existence of a minimiser follows by an application of compactness results.

For $U \subseteq (r_+, \infty)$ define the weighted norm

$$\|f\|^2_{\underline{L}^2(U)} = \int_U \frac{1}{r^2}|f|^2 \,\mathrm{d}r$$

and the space

$$\underline{L}^2(U) := \{f \text{ measurable } : \|f\|_{\underline{L}^2(U)} < \infty\}.$$

This is clearly a Hilbert space with the natural inner product $(\cdot,\cdot)_{\underline{L}^2(U)}$.

For $U \subseteq (r_+, \infty)$, we define the weighted Sobolev space $\underline{H}^1$ via the norm

$$\|f\|^2_{\underline{H}^1(U)} := \int_U \left(|f|^2 + r(r-r_+)\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2\right) \mathrm{d}r$$

Note that for $U \subseteq (r_+, \infty)$ compact, the $\underline{H}^1$ norm is equivalent to the standard Sobolev norm. As usual, let $\underline{H}^1_0(U)$ be the completion of $C_0^\infty(U)$ under $\|\cdot\|_{\underline{H}^1(U)}$.

**Lemma 2.3.2.** *Let $u \in \underline{H}^1_0(r_+, \infty)$. Then $u$ is also in $C(r_+ + 1, \infty)$ (after possibly changing it on a set of measure zero) and*

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} r^{1/2-\kappa} u(r) = 0$$

*for all $\kappa > 0$.*

*Proof.* Let $v \in C_0^\infty(r_+, \infty)$ and $r > r_+ + 1$. Then

$$\begin{aligned}
|v(r)| &\leq |v(r_+ + 1)| + \int_{r_+ + 1}^{r} \left|\partial_{r'} v(r')\right| \mathrm{d}r' \\
&\leq C \|v\|_{H^1(r_+ + 1, r_+ + 2)} + \left(\int_{r_+ + 1}^{\infty} \frac{1}{r'^2}\,\mathrm{d}r'\right)^{1/2} \left(\int_{r_+ + 1}^{\infty} r'^2 \left|\frac{\mathrm{d}v}{\mathrm{d}r'}\right|^2 \mathrm{d}r'\right)^{1/2} \\
&\leq C' \|v\|_{\underline{H}^1(r_+, \infty)}.
\end{aligned}$$

There is a sequence $(u_m)$ in $C_0^\infty$ such that $u_m \to u$ in $\underline{H}^1_0$ and pointwise almost everywhere. Choose an $R$ such that $(u_m)$ converges pointwise there. For any $\beta < 1/2$, we have

$$\begin{aligned}
\left|\lim_{r\to\infty} r^\beta u(r)\right| &\leq R^\beta \left|u - u_m\right|(R) + \int_R^\infty \left|\partial_r \left(r^\beta (u - u_m)\right)\right| \mathrm{d}r \\
&\leq R^\beta \left|u - u_m\right|(R) + C'' \|u - u_m\|_{\underline{H}^1}.
\end{aligned}$$

Therefore, the claim follows. □

To establish a coercivity-type inequality, we use the Hardy inequality of Lemma 2.2.17:

**Lemma 2.3.3.** *Let $a \in \mathcal{A}$ be fixed. There exist constants $r_+ < B_0 < B_1 < \infty$ and constants $C_0, C_1, C_2 > 0$, such that, for sufficiently large $m$, we have for all smooth functions $f$ with*

*$fr^{1/2} = o(1)$ at infinity that*

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} \left| \frac{\mathrm{d}f}{\mathrm{d}r} \right|^2 + C_0 1_{[B_0,B_1]^c} |f|^2 \right) \mathrm{d}r \leq C_1 \int_{B_0}^{B_1} |f|^2 \,\mathrm{d}r + C_2 \mathcal{L}_a(f).$$

*Here we can choose $C_2 = 1$ if $\alpha < 2$.*

*Remark* 2.3.4. Note that the dependence of the expression on $\alpha$ is via $\tilde{V}$ in $\mathcal{L}_a$. Recall from Section 2.2.1 that

$$\tilde{V} = V - \omega^2 = V_+ + V_0 + V_\alpha - \omega^2.$$

*Proof.* First, we have to study the potential again:

$$\begin{aligned}
(r_+^2+a^2)\frac{V_+}{\Delta_-}(r_+) &= \frac{3\frac{r_+^4}{\ell^2} + r_+^2\left(1+\frac{a^2}{\ell^2}\right) - a^2}{(r_+^2+a^2)^2} \\
(r_+^2+a^2)\frac{V_\alpha}{\Delta_-}(r_+) &= -\frac{\alpha}{\ell^2}\frac{1}{r_+^2+a^2}(r_+^2+\Theta(\alpha)a^2) \\
(r_+^2+a^2)\frac{V_0-\omega_+^2}{\Delta_-}(r_+) &= \frac{\lambda+\omega_+^2 a^2 - 2m\omega a\Xi}{r_+^2+a^2} \\
&\geq \frac{\Xi^2 m^2}{(r_+^2+a^2)^3} r_+^4
\end{aligned}$$

Thus for sufficiently large $|m|$, the expression is greater than zero. Furthermore, note the asymptotics

$$(r^2+a^2)\frac{\tilde{V}}{\Delta_-} \to \ell^{-2}(2-\alpha) \tag{2.3.1}$$

as $r \to \infty$.

We will deal with the cases $\alpha < 2$ and $\alpha \geq 2$ separately. First, let $\alpha < 2$. The function $\frac{r^2+a^2}{\Delta_-}\tilde{V}$ is only non-positive on an interval $[R_1, R_2]$. Choose constants such that $r_+ < B_1 < R_1 < R_2 < B_2 < \infty$. Set $C_0$ to be the minimum of $\frac{r^2+a^2}{\Delta_-}\tilde{V}$ on $(r_+,\infty)\backslash[B_1,B_2]$ and set $-C_1$ to be its minimum on $[B_1, B_2]$. This immediately yields the result.

Now let $\alpha \geq 2$. There exist $R_1, R_2$ such that $\frac{r^2+a^2}{\Delta_-}\tilde{V}$ is positive on $(r_+, R_1)$ and

$$\frac{r^2+a^2}{\Delta_-}\tilde{V} > -\frac{1}{4\ell^2}(1-\varepsilon)$$

on $(R_2, \infty)$ for an $\varepsilon > 0$ because of (2.3.1). Hence

$$\begin{aligned}
\int_{R_2}^{\infty} \frac{r^2+a^2}{\Delta_-}\tilde{V}|f|^2 \,\mathrm{d}r &> -\frac{1-\varepsilon/2}{4\ell^2}\int_{R_2}^{\infty} |f|^2\,\mathrm{d}r + \frac{\varepsilon}{8\ell^2}\int_{R_2}^{\infty}|f|^2\,\mathrm{d}r \\
&\geq -\left(1-\frac{\varepsilon}{2}\right)\int_{R_2}^{\infty}\frac{\Delta_-}{r^2+a^2}\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2 \mathrm{d}r + \frac{\varepsilon}{8\ell^2}\int_{R_2}^{\infty}|f|^2\,\mathrm{d}r
\end{aligned}$$

by Lemma 2.2.17. Choose $B_1, B_2$ as before. Let $C$ be the minimum of $\frac{r^2+a^2}{\Delta_-}\tilde{V}$ on $(r_+, B_1)$. Let $\varepsilon C_0/2$ be the minimum of $C$ and $\varepsilon/(8\ell^2)$. Moreover, set $-\varepsilon C_1/2$ to be the minimum of $\frac{r^2+a^2}{\Delta_-}\tilde{V}$ on $[B_1, B_2]$, we obtain

$$\int_{r_+}^{\infty}\left(\varepsilon\frac{\Delta_-}{r^2+a^2}\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2 + \varepsilon C_0 1_{[B_0,B_1]^c}|f|^2\right)\mathrm{d}r \leq \varepsilon C_1\int_{B_0}^{B_1}|f|^2\,\mathrm{d}r + \mathcal{L}_a(f)$$

and hence the inequality. □

**Lemma 2.3.5.** *The functional $\mathcal{L}_a$ is weakly lower semicontinuous in $\underline{H}^1(r_+,\infty)$ when restricted to functions of unit $\underline{L}^2$ norm.*

*Proof.* As the functional is convex in the derivative, the statement is standard and a proof can be extracted from [Eva10, §8]. We note that the boundedness from below comes from the norm constraint. The $r$ weight deals with $(r_+,\infty)$ having non-finite measure. □

**Lemma 2.3.6.** *Let $a \in \mathcal{A}$. Then there exists an $f_a \in \underline{H}^1_0(r_+,\infty)$ with unit $\underline{L}^2(r_+,\infty)$ norm such that $\mathcal{L}_a$ achieves its infimum over*

$$\{f \in \underline{H}^1_0(r_+,\infty)\ :\ \|f\|_{\underline{L}^2} = 1\}$$

*on $f_a$.*

*Proof.* By Lemma 2.3.3,

$$\int_{r_+}^{\infty}\left(\frac{\Delta_-}{r^2+a^2}\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2 + C_0 1_{[B_0,B_1]^c}|f|^2\right)\mathrm{d}r \leq C_1\int_{B_0}^{B_1}|f|^2\,\mathrm{d}r + C_2\mathcal{L}_a(f) \tag{2.3.2}$$

holds for all $f \in \underline{H}^1_0$. From this, it is evident that

$$\mathcal{L}_a(f) > -\infty$$

if $\|f\|_{\underline{L}^2} = 1$, whence

$$\nu_a = \inf\{\mathcal{L}_a(f)\ :\ f \in \underline{H}^1_0,\ \|f\|_{\underline{L}^2} = 1\} > -\infty.$$

We can choose a minimising sequence of functions of compact support by density. Thus let $\{f_{a,n}\}$ be a sequence of smooth functions, compactly supported in $(r_+,\infty)$ with $\|f_{a,n}\|_{\underline{L}^2} = 1$, such that

$$\mathcal{L}_a(f_{a,n}) \to \nu_a.$$

The bound (2.3.2) implies that $\|f_{a,n}\|_{\underline{H}^1}$ is uniformly bounded. Thus by the Banach-Alaoglu theorem, it has a weakly convergent subsequence in $\underline{H}^1_0(r_+,\infty)$. Recall a simple version of Rellich-Kondrachov: $H^1[a,b]$ embeds compactly into $L^2[a,b]$. Hence by the equivalence of norms, the subsequence has a strongly in $L^2$ convergent subsequence on compact subsets of $(r_+, \infty)$. Relabelling, we have a sequence $\{f_{a,n}\}$ that converges to $f_a$ weakly in $\underline{H}^1_0$ and strongly in $L^2$ on compact subsets of $(r_+, \infty)$. The space $\underline{H}^1_0$ is a linear (hence convex) subspace of $\underline{H}^1$ that is norm-closed. Every convex subset of a normed space that is norm-closed is weakly closed. Therefore, $f_a \in \underline{H}^1_0$.

We claim that $\|f_a\|_{\underline{L}^2} = 1$. We have

$$\begin{aligned}\Big|\|f_a\|_{\underline{L}^2} - 1\Big| &\leq \Big|\|f_a\|_{\underline{L}^2(r_++1/N,N)} - \|f_{a,n}\|_{\underline{L}^2(r_++1/N,N)}\Big| \\ &\quad + \Big|\|f_a\|_{\underline{L}^2(r_++1/N,N)^c} - \|f_{a,n}\|_{\underline{L}^2(r_++1/N,N)^c}\Big|\end{aligned}$$

Due to the $L^2$ convergence on compact subsets, the claim follows if

$$\lim_{N\to\infty}\lim_{n\to\infty}\|f_{a,n}\|_{\underline{L}^2((r_+,\infty)\setminus[r_++1/N,N]} = 0.$$

Suppose not. Then there is a $\varrho$ such that, for any $N$, there are infinitely many of the $f_{a,n}$ such that

$$\|f_{a,n}\|_{\underline{L}^2((r_+,\infty)\setminus[r_++1/N,N])} \geq \varrho > 0,$$

i. e. the norm must concentrate either near the horizon or near infinity. Suppose first that

$$\|f_{a,n}\|_{\underline{L}^2(r_+,r_++\delta)} \geq \varrho_1 > 0$$

for infinitely many $f_{a,n}$ and any $\delta > 0$. By (2.3.2), we have for $r \in (r_+, r_+ + 1)$:

$$\begin{aligned}|f_{a,n}(r)| &\leq \int_r^{r_++1}\left|\frac{\mathrm{d}f_{a,n}}{\mathrm{d}r'}\right|\mathrm{d}r' + \int_{r_++1}^{\infty}\left|\frac{\mathrm{d}f_{a,n}}{\mathrm{d}r'}\right|\mathrm{d}r' \\ &\leq \left(\int_r^{r_++1}\frac{1}{r'-r_+}\,\mathrm{d}r'\right)^{1/2}\left(\int_r^{r_++1}(r'-r_+)\left|\frac{\mathrm{d}f_{a,n}}{\mathrm{d}r'}\right|^2\mathrm{d}r'\right)^{1/2} \\ &\quad + \left(\int_{r_++1}^{\infty}\frac{1}{(r')^2}\,\mathrm{d}r'\right)^{1/2}\left(\int_{r_++1}^{\infty}r^2\left|\frac{\mathrm{d}f_{a,n}}{\mathrm{d}r'}\right|^2\mathrm{d}r'\right)^{1/2} \\ &\leq C\left(1 + \sqrt{\log\frac{1}{r-r_+}}\right)\end{aligned}$$

for a constant $C > 0$. Since $r \mapsto \sqrt{|\log(r - r_+)|}$ is integrable on compact subsets of $[r_+, \infty)$, we obtain $\|f_{a,n}\|_{\underline{L}^2(r_+,r_++\delta)} \to 0$ as $\delta \to 0$, a contradiction. Hence we only need to exclude the case that the norm is bounded away from zero for large $r$. Thus, suppose that

$$\|f_{a,n}\|_{\underline{L}^2(R_0,\infty)} \geq \varrho_2 > 0$$

for infinitely many $f_{a,n}$ and any $R_0 > 0$. However,

$$R_0\varrho_2 \leq \|f_{a,n}\|_{L^2(R_0,\infty)} \leq C'$$

for a constant $C' > 0$ by (2.3.2) and any $R_0$, a contradiction. This shows that

$$\|f_a\|_{\underline{L}^2} = 1.$$

By the infimum property, we have

$$\nu_a \leq \mathcal{L}_a(f_a).$$

By Lemma 2.3.5, we get

$$\mathcal{L}_a(f_a) \leq \liminf_{n\to\infty} \mathcal{L}_a(f_{a,n}).$$

As

$$\mathcal{L}_a(f_{a,n}) \to \nu_a,$$

the latter equals $\nu_a$. Thus the minimum is attained by $f_a$. □

We would like to derive the Euler-Lagrange equation corresponding to this minimiser.

**Lemma 2.3.7.** *The minimiser $f_a$ satisfies*

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} \frac{\mathrm{d}f_a}{\mathrm{d}r} \frac{\mathrm{d}\psi}{\mathrm{d}r} + \tilde{V}_a \frac{r^2+a^2}{\Delta_-} f_a \psi \right) \mathrm{d}r = -\nu_a \int_{r_+}^{\infty} \frac{f_a}{r^2} \psi \,\mathrm{d}r \tag{2.3.3}$$

*for all $\psi \in \underline{H}^1_0(r_+, \infty)$.*

*Proof.* The proof can be extracted from [Eva10]. The analogous proof for twisted derivatives is given for Lemma 2.4.6. □

**Proposition 2.3.8.** *There is an $\hat{a}$ and a corresponding non-zero function $f_{\hat{a}} \in C^\infty(r_+, \infty)$ such that*

$$\frac{\Delta_-}{r^2+\hat{a}^2} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+\hat{a}^2} \frac{\mathrm{d}f_{\hat{a}}}{\mathrm{d}r} \right) - \tilde{V}_{\hat{a}} f_{\hat{a}} = 0$$

*and $f_{\hat{a}}$ satisfies the horizon regularity condition and the Dirichlet boundary condition at infinity.*

*Proof.* First we would like to show that $\nu_a$ is continuous in $a$. We will use the notation $\Delta_-^a$ to denote the $\Delta_-$ corresponding to $a$. Given $a_1$ and $a_2$, we have

$$\begin{aligned}
\nu_{a_1} &= \mathcal{L}_{a_1}(f_{a_1}) \\
&= \int_{r_+}^{\infty} \left( \frac{\Delta_-^{a_2}}{r^2+a_2^2} \left| \frac{\mathrm{d}f_{a_1}}{\mathrm{d}r} \right|^2 + \tilde{V}_{a_2} \frac{r^2+a_2^2}{\Delta_-^{a_2}} |f_{a_1}|^2 \right) \mathrm{d}r \\
&\quad + \int_{r_+}^{\infty} \left[ \left( \frac{\Delta_-^{a_1}}{r^2+a_1^2} - \frac{\Delta_-^{a_2}}{r^2+a_2^2} \right) \left| \frac{\mathrm{d}f_{a_1}}{\mathrm{d}r} \right|^2 + \left( \tilde{V}_{a_1} \frac{r^2+a_1^2}{\Delta_-^{a_1}} - \tilde{V}_{a_2} \frac{r^2+a_2^2}{\Delta_-^{a_2}} \right) |f_{a_1}|^2 \right] \mathrm{d}r.
\end{aligned}$$

Due to the continuity of $\mathcal{L}_a(f)$ in $a$, the first line is greater or equal than $\nu_{a_2}$ if $a_1$ is sufficiently close to $a_2$. Since the coefficients in the second line are continuously differentiable in $a$, we can use the mean value theorem to obtain

$$\nu_{a_1} \geq \nu_{a_2} - C|a_1 - a_2| \int_{r_+}^{\infty} \left( (r - r_+) \left| \frac{\mathrm{d} f_{a_1}}{\mathrm{d} r} \right|^2 + |f_{a_1}|^2 \right) \mathrm{d} r$$

for some constant $C > 0$. We obtain an analogous inequality reversing the rôles of $a_1$ and $a_2$. Using (2.3.2) and $\|f_a\|_{\underline{L}^2} = 1$ yields

$$|\nu_{a_1} - \nu_{a_2}| \leq C|a_1 - a_2| \int_{r_+}^{\infty} \left( (r - r_+) \left| \frac{\mathrm{d} f_{a_1}}{\mathrm{d} r} \right|^2 + |f_{a_1}|^2 \right) \mathrm{d} r \leq C' |a_1 - a_2|.$$

Since $\mathcal{A} \neq \emptyset$, we set

$$\hat{a} := \inf \mathcal{A}.$$

As stated in the introduction to this section, $\mathcal{A}$ is open, so $\hat{a} \notin \mathcal{A}$. By continuity of $\nu_a$, this implies that $\nu_{\hat{a}} = 0$.

Now choose a sequence $a_n \to \hat{a}$ and corresponding minimisers $f_{a_n} \in \underline{H}_0^1$ satisfying $\|f_{a_n}\|_{L^2} = 1$. Then, as in the proof of Lemma 2.3.6, by Lemma 2.3.3, $f_{a_n}$ is bounded in $\underline{H}^1$ and there is a subsequence (also denoted $(a_n)$) such that $f_{a_n} \to f_{\hat{a}}$ weakly in $\underline{H}^1$ and strongly in $L^2$ on compact subsets for a $f_{\hat{a}} \in \underline{H}_0^1$. Again by Lemma 2.3.3 and the strong $L^2$ convergence on compact subsets, we see that $f_{\hat{a}}$ is non-zero. Moreover, we have sufficient decay towards infinity by Lemma 2.3.2. Hence we get the desired asymptotics.

From the weak convergence of $(f_{a_n})$, Lemma 2.3.7 yields that $f_{\hat{a}}$ satisfies

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2 + \hat{a}^2} \frac{\mathrm{d} f_{\hat{a}}}{\mathrm{d} r} \frac{\mathrm{d} \psi}{\mathrm{d} r} + \tilde{V}_{\hat{a}} \frac{r^2 + \hat{a}^2}{\Delta_-} f_{\hat{a}} \psi \right) \mathrm{d} r = 0$$

for all $\psi \in \underline{H}_0^1(r_+, \infty)$. Since this is the weak formulation of an elliptic problem with smooth coefficients, we readily obtain $f_{\hat{a}} \in C^\infty$ and

$$\frac{\Delta_-}{r^2 + \hat{a}^2} \frac{\mathrm{d}}{\mathrm{d} r} \left( \frac{\Delta_-}{r^2 + \hat{a}^2} \frac{\mathrm{d} f_{\hat{a}}}{\mathrm{d} r} \right) - \tilde{V}_{\hat{a}} f_{\hat{a}} = 0.$$

It remains to check the boundary condition at the horizon. The lower semi-continuity of convex functionals with respect to weak convergence implies that

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2 + a_n^2} \left| \frac{\mathrm{d} f_{a_n}}{\mathrm{d} r} \right|^2 + \tilde{V}_{a_n} \frac{r^2 + a_n^2}{\Delta_-} |f_{a_n}|^2 \right) \mathrm{d} r \leq \nu_{a_n},$$

whence

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2 + \hat{a}^2} \left| \frac{\mathrm{d} f_{\hat{a}}}{\mathrm{d} r} \right|^2 + \tilde{V}_{\hat{a}} \frac{r^2 + \hat{a}^2}{\Delta_-} |f_{\hat{a}}|^2 \right) \mathrm{d} r \leq 0.$$

Hence

$$\int_{r_+}^{\infty} \frac{\Delta_-}{r^2+\hat{a}^2} \left| \frac{\mathrm{d}f_{\hat{a}}}{\mathrm{d}r} \right|^2 \mathrm{d}r < \infty. \tag{2.3.4}$$

Near $r_+$, the local theory (Theorem 2.2.1) implies that there exist constants $A$, $B$ and non-zero analytic functions $\varphi_i$ such that

$$f_{\hat{a}} = A\varphi_1 + B(\log(r-r_+)\varphi_2 + \varphi_3).$$

If $B \neq 0$, then

$$\int_{r_+}^{\infty} \frac{\Delta_-}{r^2+\hat{a}^2} \left| \frac{\mathrm{d}f_{\hat{a}}}{\mathrm{d}r} \right|^2 \mathrm{d}r = \infty,$$

whence $B = 0$. Hence $f_{\hat{a}}$ satisfies the horizon regularity condition. □

*Remark* 2.3.9. From Lemma 2.2.18, we conclude that $|\hat{a}| > r_+^2/\ell$. An alternative way to see this comes from [HS11a]. There it is shown directly that, if the Hawking-Reall bound is satisfied, there are no periodic solutions. One can easily see that the proof generalises to the case when the Hawking-Reall bound is saturated. Thus we obtain $|\hat{a}| > r_+^2/\ell$.

**Corollary 2.3.10.** *Assume our choice of parameters, $a = \hat{a}$ and $\omega = \omega_+$. Let $C_0 \in \mathbb{C}$. Then the radial ODE (2.2.3) has a unique solution satisfying $u(r_+) = C_0$ and the Dirichlet boundary condition at infinity.*

### 2.3.2. Perturbing the Dirichlet modes into the complex plane

We have shown that, for given $\ell > 0$ and $\alpha < 9/4$, there exists a real mode solution in a Kerr-AdS spacetime with parameters $(\ell, r_+, \hat{a})$ and $\omega = \omega_R(0) := \Xi\hat{a}m/(r_+^2 + \hat{a}^2)$. Henceforth, we shall denote the chosen $\hat{a}$ simply by $a$. Now we wish to vary $\omega$ and $\alpha$, keeping all the other parameters constant. Keeping $u(r_+, \omega, \alpha)$ fixed, satisfying $|u|(r_+, \omega, \alpha) = 1$, the local theory yields a unique solution to the radial ODE of the form

$$u(r, \alpha, \omega) = A(\alpha, \omega)h_1(r, \alpha, \omega) + B(\alpha, \omega)h_2(r, \alpha, \omega) \tag{2.3.5}$$

for large $r$, cf. Lemma 2.2.7 and (2.2.5). The functions $A$ and $B$ are smooth in $\omega$ and $\alpha$. Finding a mode solution is equivalent to finding a zero of $A$. We already have $A(\alpha(0), \omega_R(0)) = 0$. Write $A = A_R + \mathrm{i}A_I$. Recall

$$Q_T(r) = \mathrm{Im}\,(u'\overline{u})$$

and that

$$\frac{\mathrm{d}Q_T}{\mathrm{d}r}(r) = \frac{r^2+a^2}{\Delta_-}\mathrm{Im}\,\left(V - \omega^2\right)|u(r)|^2$$

$$Q_T(r_+) = \frac{\Xi am - \omega_R(r_+^2 + a^2)}{r_+^2 + a^2},$$

where we have used $|u(r_+)| = 1$. Therefore,

$$Q_T(r) = \frac{\Xi am - (r_+^2 + a^2)\omega_R}{r_+^2 + a^2} + \int_{r_+}^{r} \frac{(r')^2 + a^2}{\Delta_-} \left(\operatorname{Im} \tilde{V}\right) |u|^2 \, \mathrm{d}r'.$$

Differentiating at $\omega_R = \omega_R(0)$ and $\alpha = \alpha_0$ with respect to $\omega_R$ and $\alpha$ yields

$$\frac{\partial Q_T}{\partial \omega_R}\Big|_{\alpha=\alpha_0,\,\omega=\omega_+} = -1 \tag{2.3.6}$$

$$\frac{\partial Q_T}{\partial \alpha}\Big|_{\alpha=\alpha_0,\,\omega=\omega_+} = 0 \tag{2.3.7}$$

for all $r \in (r_+, \infty)$. In a similar way, for large $r$, using (2.3.5), we have

$$\begin{aligned} Q_T(r) = |A|^2 \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(\frac{\mathrm{d}h_1}{\mathrm{d}r}\overline{h_1}\right) + \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(A\frac{\mathrm{d}h_1}{\mathrm{d}r}\overline{Bh_2}\right) \\ + \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(B\frac{\mathrm{d}h_2}{\mathrm{d}r}\overline{Ah_1}\right) + |B|^2 \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(\frac{\mathrm{d}h_2}{\mathrm{d}r}\overline{h_2}\right). \end{aligned} \tag{2.3.8}$$

Upon differentiation at $\omega_R = \omega_R(0)$ and $\alpha = \alpha_0$ with respect to $\omega_R$ and $\alpha$, the first term vanishes for all $r$. Therefore, combining (2.3.8) with (2.3.6) and (2.3.7), we obtain

$$\begin{aligned} -1 = \frac{\partial}{\partial \omega_R}\Bigg( & \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(A\frac{\mathrm{d}h_1}{\mathrm{d}r}\overline{Bh_2}\right) \\ & + \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(B\frac{\mathrm{d}h_2}{\mathrm{d}r}\overline{Ah_1}\right) + |B|^2 \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(\frac{\mathrm{d}h_2}{\mathrm{d}r}\overline{h_2}\right)\Bigg)\Bigg|_{\alpha=\alpha_0,\,\omega=\omega_+} \\ 0 = \frac{\partial}{\partial \alpha}\Bigg( & \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(A\frac{\mathrm{d}h_1}{\mathrm{d}r}\overline{Bh_2}\right) \\ & + \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(B\frac{\mathrm{d}h_2}{\mathrm{d}r}\overline{Ah_1}\right) + |B|^2 \frac{\Delta_-}{r^2 + a^2} \operatorname{Im}\left(\frac{\mathrm{d}h_2}{\mathrm{d}r}\overline{h_2}\right)\Bigg)\Bigg|_{\alpha=\alpha_0,\,\omega=\omega_+} \end{aligned}$$

for all $r$ sufficiently large. Note that these equations imply that the right hand sides are constant. Evaluating in the limit $r \to \infty$ yields

$$-1 = \frac{2}{\ell^2}\left(\frac{9}{4} - \alpha\right)^{1/2} \operatorname{Im}\left(\frac{\partial A}{\partial \omega_R}\overline{B}\right) = \frac{2}{\ell^2}\left(\frac{9}{4} - \alpha\right)^{1/2} \left(\frac{\partial A_I}{\partial \omega_R} B_R - \frac{\partial A_R}{\partial \omega_R} B_I\right) \tag{2.3.9}$$

$$0 = \frac{2}{\ell^2}\left(\frac{9}{4} - \alpha\right)^{1/2} \operatorname{Im}\left(\frac{\partial A}{\partial \alpha}\overline{B}\right) = \frac{2}{\ell^2}\left(\frac{9}{4} - \alpha\right)^{1/2} \left(\frac{\partial A_I}{\partial \alpha} B_R - \frac{\partial A_R}{\partial \alpha} B_I\right) \tag{2.3.10}$$

To extend the coefficient $A(\alpha, \omega_R) = 0$ to complex $\omega$, we want to appeal to the implicit function theorem establishing

$$\det \begin{pmatrix} \frac{\partial A_R}{\partial \omega_R} & \frac{\partial A_R}{\partial \alpha} \\ \frac{\partial A_I}{\partial \omega_R} & \frac{\partial A_I}{\partial \alpha} \end{pmatrix} \neq 0.$$

From equations (2.3.9) and (2.3.10) we see that this holds if

$$\frac{\partial A}{\partial \alpha}(\alpha(0), \omega_R(0)) \neq 0.$$

This is true indeed:

**Lemma 2.3.11.**

$$\frac{\partial A}{\partial \alpha}(\alpha(0), \omega_R(0)) \neq 0.$$

*Proof.* Suppose $\partial A / \partial \alpha = 0$. Then we have

$$\begin{aligned} \frac{\partial u}{\partial \alpha}(r, \omega_R(0), \alpha(0)) = & \frac{\partial B}{\partial \alpha}(\omega_R(0), \alpha(0)) h_2(r, \omega_R(0), \alpha(0)) \\ & + B(\alpha(0), \omega_R(0)) \frac{\partial h_2}{\partial \alpha}(r, \omega_R(0), \alpha(0)). \end{aligned}$$

Thus $\partial u / \partial \alpha$ is polynomially decreasing at infinity as $r^{-1/2-\sqrt{9/4-\alpha(0)}}$ and extends smoothly to $r = r_+$. Defining the derivative $u_\alpha := \partial u / \partial \alpha$, we get from the radial ODE

$$\frac{\Delta_-}{r^2+a^2} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+a^2} \frac{\mathrm{d}u_\alpha}{\mathrm{d}r} \right) - \tilde{V} u_\alpha = \left[ \frac{\Delta_-}{(r^2+a^2)^2} \frac{\partial \lambda}{\partial \alpha} - \frac{1}{\ell^2} \frac{\Delta_-}{(r^2+a^2)^2} (r^2 + \Theta(\alpha) a^2) \right] u.$$

Multiplying by $\overline{u}$ and integrating by parts, we obtain at $\omega_R(0)$ and $\alpha(0)$

$$\int_{r_+}^\infty \frac{\Delta_-}{(r^2+a^2)^2} \left( \frac{\partial \lambda}{\partial \alpha} - \frac{1}{\ell^2}(r^2 + \Theta(\alpha) a^2) \right) |u|^2 \, \mathrm{d}r = 0. \tag{2.3.11}$$

Now the two cases $\alpha \leq 0$ and $0 < \alpha < 9/4$ have to be treated separately. If $\alpha \leq 0$, then Proposition A.1.3 readily gives $\partial \lambda / \partial \alpha < 0$, so that $u$ would vanish identically.

For $\alpha > 0$, we need to use the formula for $\partial \lambda / \partial \alpha$ from Proposition A.1.3. Together with (2.3.11), this yields

$$\int_{r_+}^\infty \frac{\Delta_-}{(r^2+a^2)^2} \int_0^\pi \frac{1}{\ell^2} \left( -r^2 - a^2 \cos^2 \vartheta \right) |S|^2 \sin \vartheta |u|^2 \, \mathrm{d}\vartheta \, \mathrm{d}r = 0,$$

whence we get the same contradiction. ∎

### 2.3.3. Behaviour for small $\varepsilon > 0$ for Dirichlet boundary conditions

From the analysis of the previous section, we have a family of mode solutions $u(r, \varepsilon)$ to the radial ODE parameters $(\omega(\varepsilon), m, l, \alpha(\varepsilon))$, where

$$\omega(\varepsilon) = \omega_R(\varepsilon) + \mathrm{i}\varepsilon.$$

The mode $u$ satisfies the horizon regularity condition and the Dirichlet boundary condition at infinity. This proves the first part of Theorem 2.1.7. To prove the second part, we would like to study the behaviour of $\omega(\varepsilon)$ and $\alpha(\varepsilon)$ for small $\varepsilon > 0$.

To obtain the following statements, we potentially need to make $|m|$ even larger than in the previous sections.

**Proposition 2.3.12.** *If $|m|$ is sufficiently large, we have*

$$\omega_R(0)\frac{\partial \omega_R}{\partial \varepsilon}(0) < 0.$$

*Proof.* Define

$$\tilde{Q}_T := \mathrm{Im}\, \left(u'\overline{\omega u}\right).$$

Let $\varepsilon > 0$. We have

$$\tilde{Q}_T(r_+) = \mathrm{Im}\, \left(\frac{\xi}{r_+^2 + a^2}\overline{\omega}\right) |u|^2(r_+) = 0$$

since $\xi$ has a positive real part (see (2.1.11)), $u \sim (r - r_+)^\xi$ and hence $|u|(r_+) = 0$. Moreover, $\tilde{Q}_T(r) \to 0$ as $r \to \infty$ since $u \sim r^{-1/2-\kappa}$. Furthermore, using the radial ODE, one computes

$$\frac{\mathrm{d}\tilde{Q}_T}{\mathrm{d}r} = -\varepsilon\frac{\Delta_-}{r^2 + a^2}\left|\frac{\mathrm{d}u}{\mathrm{d}r}\right|^2 + \frac{r^2 + a^2}{\Delta_-}\mathrm{Im}\, \left((V_a - \omega^2)\overline{\omega}\right)|u|^2.$$

Hence

$$\int_{r_+}^{\infty} \left(\varepsilon\frac{\Delta_-}{r^2 + a^2}\left|\frac{\mathrm{d}u}{\mathrm{d}r}\right|^2 - \frac{r^2 + a^2}{\Delta_-}\mathrm{Im}\, \left((V_a - \omega^2)\overline{\omega}\right)|u|^2\right) \mathrm{d}r = 0 \tag{2.3.12}$$

with

$$\begin{aligned}-\mathrm{Im}\left((V_a - \omega^2)\overline{\omega}\right) = \frac{\varepsilon}{(r^2 + a^2)^2}\Big(&V_+(r^2 + a^2)^2 + |\omega|^2(r^2 + a^2)^2 - \Xi^2 a^2 m^2 \\ &- \frac{\alpha}{\ell^2}\Delta_-(r^2 + a^2\Theta(\alpha))\Big) - \frac{\Delta_-}{(r^2 + a^2)^2}\mathrm{Im}\left((\lambda + a^2\omega^2)\overline{\omega}\right).\end{aligned}$$

From Proposition A.1.2, we know that $-\mathrm{Im}\,(\lambda\overline{\omega}) > 0$. Hence

$$\begin{aligned}-\mathrm{Im}\,((V_a-\omega^2)\overline{\omega}) > \frac{\varepsilon}{(r^2+a^2)^2}\Big(&V_+(r^2+a^2)^2+|\omega|^2(r^2+a^2)^2-\Xi^2a^2m^2\\ &-\frac{\alpha}{\ell^2}\Delta_-(r^2+a^2\Theta(\alpha))\Big)-\frac{\Delta_-}{(r^2+a^2)^2}a^2\varepsilon|\omega|^2.\end{aligned} \tag{2.3.13}$$

We set

$$K(r) := |\omega|^2(r^2+a^2)^2-\Xi^2a^2m^2-\Delta_-a^2|\omega|^2.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}r}K(r) = |\omega|^2\left(4\left(1-\frac{a^2}{\ell^2}\right)r^3+2a^2M+2a^2\left(1-\frac{a^2}{\ell^2}\right)r\right) > 0. \tag{2.3.14}$$

As already used in Section 2.3.1, there is an $R > r_+$ such that, for $r \geq R$,

$$V_+ + V_\alpha > -\frac{1}{4\ell^2}\frac{\Delta_-}{r^2+a^2}.$$

By an application of Lemma 2.2.17, we conclude

$$\begin{aligned}\int_R^\infty \left(\varepsilon\frac{\Delta_-}{r^2+a^2}\left|\frac{\mathrm{d}u}{\mathrm{d}r}\right|^2-\frac{r^2+a^2}{\Delta_-}\mathrm{Im}\,\left((V_a-\omega^2)\overline{\omega}\right)|u|^2\right)\mathrm{d}r\\ > \int_R^\infty \frac{\varepsilon}{(r^2+a^2)^2}K(r)|u|^2\,\mathrm{d}r.\end{aligned} \tag{2.3.15}$$

For the sake of contradiction, suppose $K(r_+) \geq 0$. Then, by (2.3.14), $K > 0$ on $(r_+,\infty)$, whence we obtain strict positivity for (2.3.15). As

$$|\omega(0)|^2 = \frac{m^2a^2\Xi^2}{(r_+^2+a^2)^2}$$

and as, for fixed $r_+$, $\hat{a}$ is bounded away from zero for all $m$,

$$|\omega(0)|^2 \geq Cm^2.$$

Since $\varepsilon \mapsto \omega(\varepsilon)$ is continuous, $|\omega|^2$ scales as $m^2$ for small $\varepsilon$, so $\mathrm{d}K/\mathrm{d}r$ can be chosen as large as possible at $r = r_+$, in particular, it can be used to overcome the potentially non-positive derivative of the remaining terms of the right hand side of (2.3.13) on $(r_+,R)$. Then, (2.3.12) implies $u = 0$, a contradiction.

Hence $K(r_+) < 0$ which is equivalent to

$$\omega_R(\varepsilon)^2+\varepsilon^2 < \left(\frac{\Xi am}{r_+^2+a^2}\right)^2.$$

This in turn is equivalent to the claim. □

In the following, we will fix an $|m| \geq m_0$ such that Proposition 2.3.12 holds.

*Remark* 2.3.13. The choice of $m$ could have been made right at the beginning as the choice of $m_0$ in Lemma 2.2.13 is independent of the largeness required for Proposition 2.3.12.

The next proposition shows that the mass $\alpha$ is at first increasing along the curve obtained by the implicit function theorem. The proof requires a technical lemma which is given at the end of this section.

**Proposition 2.3.14.** *Let $\alpha(0) < 9/4$. Then*

$$\frac{\partial \alpha}{\partial \varepsilon}(0) > 0.$$

*Proof.* Define

$$u_\varepsilon = \frac{\partial u}{\partial \varepsilon}.$$

Then

$$\frac{\partial}{\partial r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\partial u_\varepsilon}{\partial r}\right) - \frac{(V-\omega^2)(r^2+a^2)}{\Delta_-}u_\varepsilon = \frac{\partial}{\partial \varepsilon}\left(\frac{r^2+a^2}{\Delta_-}\tilde{V}\right)u.$$

We would like to multiply this equation by $\overline{u}$ and then integrate by parts, Using $u \sim r^{-1/2-\kappa(\varepsilon)}$ for all $\varepsilon$ by Section 2.3.2, $u_\varepsilon \sim \log r(r-r_+)^{-1/2-\kappa}$ and hence satisfies the Dirichlet boundary condition at infinity.

At $r = r_+$, $u_\varepsilon$ does not satisfy the horizon regularity condition. Thus we have to compute the boundary term created by the integration by parts. We know that

$$f(r,\varepsilon) := \exp\left(-\mathrm{i}\frac{\Xi am - (r_+^2+a^2)\omega(\varepsilon)}{\partial_r \Delta_-}\log(r-r_+)\right)u(r,\varepsilon)$$

is smooth, whence

$$\begin{aligned}\frac{\partial f}{\partial \varepsilon} &= \mathrm{i}(r_+^2+a^2)\frac{\log(r-r_+)}{\partial_r\Delta_-(r_+)}\left(\frac{\partial \omega_R}{\partial \varepsilon}+\mathrm{i}\right)f(r,\varepsilon)\\ &\quad + \exp\left(-\mathrm{i}\frac{\Xi am - (r_+^2+a^2)\omega(\varepsilon)}{\partial_r\Delta_-(r_+)}\log(r-r_+)\right)u_\varepsilon\end{aligned}$$

and

$$u_\varepsilon(r,0) = \frac{r_+^2+a^2}{\partial_r\Delta_-(r_+)}\left(1-\mathrm{i}\frac{\partial\omega_R}{\partial\varepsilon}\right)\log(r-r_+)u + \frac{\partial f}{\partial \varepsilon}(r,0).$$

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}r}\right)\overline{u} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}r}\overline{u}\right) - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}u_\varepsilon\frac{\mathrm{d}\overline{u}}{\mathrm{d}r}\right)\\ &\quad + u_\varepsilon\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}r}\right)\end{aligned}$$

and

$$\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}r}\overline{u} - u_\varepsilon\frac{\mathrm{d}\overline{u}}{\mathrm{d}r} = \frac{r_+^2+a^2}{\partial_r\Delta_-(r_+)}\left(1-\mathrm{i}\frac{\partial\omega_R}{\partial\varepsilon}\right)\frac{1}{r-r_+}|u|^2 + \frac{r_+^2+a^2}{\partial_r\Delta_-(r_+)}\left(1-\mathrm{i}\frac{\partial\omega_R}{\partial\varepsilon}\right)\log(r-r_+)\mathrm{Im}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\overline{u}\right).$$

We note that

$$\frac{\Delta_-}{r^2+a^2}\log(r-r_+)\mathrm{Im}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\overline{u}\right)$$

is zero at $r=r_+$. Thus evaluating the radial ODE at $\varepsilon=0$, multiplying it by $\overline{u}$, taking real parts and integrating by parts yields

$$-|u(r_+)|^2 = \int_{r_+}^{\infty}\frac{r^2+a^2}{\Delta_-}\mathrm{Re}\left(\frac{\partial\tilde{V}}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0}|u|^2\,\mathrm{d}r \tag{2.3.16}$$

For $\alpha\neq 0$, the derivative is given by

$$\begin{aligned}\mathrm{Re}\left(\frac{\partial\tilde{V}}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0} &= \frac{\Delta_-}{(r^2+a^2)^2}\left[\mathrm{Re}\left(\frac{\partial\lambda}{\partial\varepsilon}\right)+2a^2\omega_R(0)\frac{\partial\omega_R}{\partial\varepsilon}(0)\right]\\ &\quad-\frac{\Delta_-}{(r^2+a^2)^2}2ma\Xi\frac{\partial\omega_R}{\partial\varepsilon}(0)-2\omega_R(0)\frac{\partial\omega_R}{\partial\varepsilon}(0)\frac{r^2-r_+^2}{r^2+a^2}\\ &\quad-\frac{1}{\ell^2}\frac{\partial\alpha}{\partial\varepsilon}\frac{\Delta_-}{(r^2+a^2)^2}(r^2+\Theta(\alpha)a^2)\\ &= \frac{\Delta_-}{(r^2+a^2)^2}\left[\mathrm{Re}\left(\frac{\partial\lambda}{\partial\varepsilon}-2r_+^2\omega_R(0)\frac{\partial\omega_R}{\partial\varepsilon}(0)\right)-\frac{1}{\ell^2}\frac{\partial\alpha}{\partial\varepsilon}(r^2+\Theta(\alpha)a^2)\right]\\ &\quad-2\omega_R(0)\frac{\partial\omega_R}{\partial\varepsilon}(0)\frac{r^2-r_+^2}{r^2+a^2}.\end{aligned} \tag{2.3.17}$$

Noting that $\int_0^\pi|S|^2\sin\vartheta\,\mathrm{d}\vartheta=1$ and using Lemma 2.3.15 to eliminate the dependence on $\lambda$ and then Proposition 2.3.12, we conclude that $\partial\alpha/\partial\varepsilon(0)$ needs to be positive to make the integrand of (2.3.16) negative. The restriction to $\alpha\neq0$ can by removed by continuity from the left of the reflection and transmission coefficients $A$ and $B$. □

**Lemma 2.3.15.** *At $\varepsilon=0$, for $\alpha\leq 0$,*

$$\int_0^\pi\left(2\left[\Xi a^2+(r_+^2+a^2)\frac{a^2}{\ell^2}\right]\frac{\cos^2\vartheta}{\Delta_\vartheta}\omega_R\frac{\partial\omega_R}{\partial\varepsilon}+\frac{a^2}{\ell^2}\cos^2\vartheta\frac{\partial\alpha}{\partial\varepsilon}+\mathrm{Re}\left(\frac{\partial\lambda}{\partial\varepsilon}\right)\right)|S|^2\sin\vartheta\,\mathrm{d}\vartheta=0$$

*and, for $\alpha>0$,*

$$\int_0^\pi\left(2\left[\Xi a^2+(r_+^2+a^2)\frac{a^2}{\ell^2}\right]\frac{\cos^2\vartheta}{\Delta_\vartheta}\omega_R\frac{\partial\omega_R}{\partial\varepsilon}-\frac{a^2}{\ell^2}\sin^2\vartheta\frac{\partial\alpha}{\partial\varepsilon}+\mathrm{Re}\left(\frac{\partial\lambda}{\partial\varepsilon}\right)\right)|S|^2\sin\vartheta\,\mathrm{d}\vartheta=0.$$

*Proof.* Let $\alpha\geq0$. Set $S_\varepsilon:=\partial S/\partial\varepsilon$. Then, differentiating the angular ODE with respect to

$\varepsilon$, evaluating at $\varepsilon = 0$, multiplying by $\overline{S}$, taking the real part and integrating by part yields the claimed identity. An analogous computation yields the result for $\alpha > 0$. $\square$

### 2.3.4. A continuity argument

We now deduce Theorem 2.1.9 from Theorem 2.1.7. In this section, we fix $\ell > 0$ and $\alpha_0 < 9/4$. In the previous sections, we have produced a curve $\varepsilon \mapsto \alpha_0(\varepsilon)$ of masses with $\partial\alpha_0(0)/\partial\varepsilon > 0$. This means that the constructed mode solutions will solve a radial ODE with a different scalar mass. This section formalises the intuitive idea of "following up" the curves $\varepsilon \mapsto \alpha(\varepsilon)$ starting at an $\alpha$ close to $\alpha_0$ until one "hits" the desired mass, which is made possible by $\partial\alpha(0)/\partial\varepsilon > 0$. The proof consists simply in establishing necessary continuity and carefully choosing neighbourhoods. This can be divided into two independent steps.

1. We show that the function mapping $\alpha$ to the corresponding $\hat{a}$ is left-continuous.
2. We show that, for $\alpha$ and corresponding $\hat{a}$ sufficiently close to $\alpha_0$ and the corresponding $\hat{a}_0$, the implicit function theorem guarantees a curve, starting at $\alpha$ and the corresponding $\hat{a}$ and real frequency $\omega_+$, which exists "long enough" to "hit" $\alpha_0$.

Note that, for any $f \in C_0^\infty$, $(\alpha, r_+, a) \mapsto \mathcal{L}_{\alpha,r_+,a}(f)$ defines a continuous function. For a given $f \in C_0^\infty$, define the family of sets

$$\mathcal{A}_{\alpha,r_+}(f) := \{a > 0 \,:\, \mathcal{L}_{\alpha,r_+,a}(f) < 0\}$$

and

$$\begin{aligned}\mathcal{A}_{\alpha,r_+} :=& \bigcup_{f\in C_0^\infty} \mathcal{A}_{\alpha,r_+}(f)\\ =&\{a > 0 \,:\, \exists f \in C_0^\infty \,:\, \mathcal{L}_{\alpha,r_+,a}(f) < 0\}.\end{aligned}$$

*Remark* 2.3.16. $\mathcal{A}_{\alpha,r_+}$ corresponds to the set $\mathcal{A}$ from Section 2.3.1.

Define the function

$$\Phi : (-\infty, 9/4) \times (0,\infty) \to (0,\infty),\ \Phi(\alpha, r_+) := \inf \mathcal{A}_{\alpha,r_+}$$

if $\mathcal{A}_{\alpha,r_+} \neq \emptyset$.

**Lemma 2.3.17.** *Let $0 < r_+ < \ell$. Then there is an interval $I \subseteq (-\infty, 9/4)$ with $\alpha_0 \in I$ and an $m_0$ such that $\Phi(\cdot, r_+)$ is well-defined for all $\alpha \in I$ and $|m| \geq m_0$.*

*Proof.* The set $\mathcal{A}_{\alpha,r_+}$ non-empty, open and bounded away from zero for all $\alpha \in I \subseteq (-\infty, 9/4)$ by Remark 2.2.16, whence $\Phi(\cdot, r_+)$ is well-defined. $\square$

We shall fix $r_+$ now. Moreover, we shall fix $m_0$ such that Lemma 2.3.17 holds and an $m \geq m_0 > 0$.

**Lemma 2.3.18.** *The function $\Phi(\cdot, r_+)$ is non-increasing in $\alpha \in I$.*

*Proof.* Suppose $\Phi(\cdot, r_+)$ was not non-increasing. Then

$$\inf \mathcal{A}_{\alpha,r_+} < \inf \mathcal{A}_{\alpha',r_+}$$

for some $\alpha < \alpha'$. Hence there is an $a > 0$ with

$$\inf \mathcal{A}_{\alpha,r_+} < a < \inf \mathcal{A}_{\alpha',r_+}$$

and an $f \in C_0^\infty$ such that

$$\mathcal{L}_{\alpha,r_+,a}(f) < 0 \leq \mathcal{L}_{\alpha',r_+,a}(f).$$

This contradicts that $\mathcal{L}_{\alpha,r_+,a}(f) > \mathcal{L}_{\alpha',r_+,a}(f)$ for all $f$ and $a$ if $\alpha < \alpha'$. □

**Lemma 2.3.19.** *The function $\Phi(\cdot, r_+)$ is left-continuous at $\alpha_0$, i. e.*

$$\lim_{\alpha \uparrow \alpha_0} \Phi(\alpha, r_+) = \Phi(\alpha_0, r_+).$$

*Proof.* Suppose $\Phi(\cdot, r_+)$ was not left-continuous at $\alpha_0$. Then there is an $\varepsilon > 0$ such that, for all $\delta > 0$, there is an $\alpha < \alpha_0$ with

$$\alpha_0 - \alpha < \delta$$

and

$$\Phi(\alpha, r_+) - \Phi(\alpha_0, r_+) \geq \varepsilon.$$

Then there is an $a$ between $\Phi(\alpha_0, r_+)$ and $\Phi(\alpha, r_+)$ such that there is an $f$ with $a \in \mathcal{A}_{\alpha_0,r_+}(f)$, but, for each $\delta$, there is an $\alpha$ with $a \notin \mathcal{A}_{\alpha,r_+}(f)$. Since $\mathcal{L}_{\alpha_0,r_+,a}(f) < 0$ and due to the continuity of $\mathcal{L}_\cdot(f)$, there is a $\delta > 0$ such that for all $\alpha_0 - \alpha < \delta$, we have $\mathcal{L}_{\alpha,r_+,a}(f) < 0$, i. e. $a \in \mathcal{A}_{\alpha,r_+}(f)$, a contradiction. □

For $\alpha \in I$, we define

$$\Omega_R(\alpha) := \frac{m\Phi(\alpha, r_+)\left(1 - \frac{\Phi(\alpha,r_+)^2}{\ell^2}\right)}{r_+^2 + \Phi(\alpha, r_+)^2}. \tag{2.3.18}$$

As shown, this is left-continuous at $\alpha_0$.

Now we turn to the second step. Recall that, for all $\alpha \in I$, there is a periodic Dirichlet mode with frequency $\omega = \Omega_R(\alpha) \in \mathbb{R}$ in a Kerr-AdS spacetime with parameters $(\ell, r_+, \Phi(\alpha, r_+))$ by Proposition 2.3.8. Using Section 2.3.2, we can find unstable Dirichlet mode solutions with frequency $\omega = \omega_R + \mathrm{i}\omega_I = \omega_R(\varepsilon) + \mathrm{i}\varepsilon$ (where $\omega_R(0) = \Omega_R(\alpha)$) to the Klein-Gordon equation with mass $\alpha(\varepsilon)$ (where $\alpha(0) = \alpha_0$). As by Lemma 2.3.17 the results of Section 2.3.3

hold, we know that

$$\frac{\partial \alpha}{\partial \varepsilon}(0) > 0, \qquad \frac{\partial \omega_R}{\partial \varepsilon}(0) < 0. \tag{2.3.19}$$

In this section, $B_\varrho(x)$ will denote an open $\ell^\infty$ ball of radius $\varrho$ centred around $x \in \mathbb{R}^4$, i. e.

$$B_\varrho(x) := \left\{ y \in \mathbb{R}^4 \;:\; \max_{j=1,\dots,4} |x_j - y_j| < \varrho \right\}.$$

We view the column vectors $(\alpha, \omega_R, \omega_I, a)^t$ as points in $\mathbb{R}^4$.

Consider

$$D := \det \begin{pmatrix} \frac{\partial A_R}{\partial \omega_R} & \frac{\partial A_R}{\partial \alpha} \\ \frac{\partial A_I}{\partial \omega_R} & \frac{\partial A_I}{\partial \alpha} \end{pmatrix}.$$

It was shown in Section 2.3.2 that $D(\alpha, \Omega_R(\alpha), 0, \Phi(\alpha, r_+)) \neq 0$ for all $\alpha \in I$.

From Lemma 2.2.9, we know that $A$ is smooth in $\alpha$, $\omega$ and $a$. Hence there is an $L > 0$ such that $D \neq 0$ in $B_L(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$ and such that, for all values $(\alpha, \Omega_R(\alpha), 0, \Phi(\alpha, r_+)) \in B_L(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$, we have $\alpha \in I$.

Hence in this neighbourhood, the vector field

$$W := -\begin{pmatrix} \frac{\partial A_R}{\partial \alpha} & \frac{\partial A_R}{\partial \omega_R} & 0 & 0 \\ \frac{\partial A_I}{\partial \alpha} & \frac{\partial A_I}{\partial \omega_R} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial A_R}{\partial \omega_I} \\ \frac{\partial A_I}{\partial \omega_I} \\ 1 \\ 0 \end{pmatrix}$$

is well-defined. It is this vector field whose integral curves describe the solutions given by the implicit function theorem as applied in Section 2.3.2. In particular, solving the ODE

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\alpha(\varepsilon), \omega_R(\varepsilon), \omega_I(\varepsilon), a(\varepsilon))^t = W(\alpha(\varepsilon), \omega_R(\varepsilon), \omega_I(\varepsilon), a(\varepsilon))$$

with initial conditions $(\alpha, \Omega_R(\alpha), 0, \Phi(\alpha, r_+))$ $(\alpha \in I)$ gives the previously introduced $\alpha(\varepsilon)$ and $\omega_R(\varepsilon)$.

Set $W := (W^\alpha, W^{\omega_R}, W^{\omega_I}, W^a)^t$. By (2.3.19),

$$W^\alpha(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+)) > 0$$

and

$$W^{\omega_R}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+)) < 0.$$

Let $\delta > 0$. Again by smoothness of $A$, there are $\varrho > 0$ and $L' \leq L$ such that

$$\|W - W(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))\|_\infty < \delta \;\text{ and }\; W^\alpha \geq \varrho,\; W^{\omega_R} \leq -\varrho$$

in $B_{L'}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$.

We now study integral curves of $W$ in $B_{L'}(\alpha_0, \omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$. Let

$$\tau \mapsto \gamma(\tau, p)$$

be the integral curve of $W$ with $\gamma(0, p) = p \in B_{L'}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$. Define the map

$$\begin{aligned} &T : B_{2L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+)) \to \mathbb{R}, \\ &T(p) = \inf \left\{ \tau > 0 \, : \, \gamma(\tau, p) \in \overline{B_{2L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))}^c \right\}. \end{aligned}$$

The set $\left\{ \tau > 0 \, : \, \gamma(\tau, p) \in \overline{B_{2L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))}^c \right\}$ is non-empty since

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \omega_I(\tau) = 1. \tag{2.3.20}$$

Therefore $T$ is well-defined.

**Lemma 2.3.20.** *$T$ is continuous.*

*Proof.* For this proof use the abbreviation $B := B_{2L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$. Let $p_0 \in B$, $\tau_0 := T(p_0) > 0$. Let $0 < \varepsilon < \tau_0$ such that $\gamma(\tau, p_0) \in \overline{B}^c$ for $\tau_0 + \varepsilon \leq \tau \leq \tau_0 + 2\varepsilon$, which exists by (2.3.20). Define

$$d_1 := \min\{\mathrm{dist}(\gamma(\tau, p_0), \partial B) \, : \, 0 \leq \tau \leq \tau_0 - \varepsilon\}$$

We claim that $d_1 > 0$. Suppose not. Then there is a $\tau' \in (0, \tau_0 - \varepsilon]$ such that $\mathrm{dist}(\gamma(\tau', p_0), \partial B) = 0$. Since

$$W^\alpha \geq \varrho, \quad W^{\omega_R} \leq -\varrho, \quad W^{\omega_I} = 1, \quad W^a = 0, \tag{2.3.21}$$

whence $W$ is not parallel to any side of the boundary $\partial B$ of the $\ell^\infty$ ball, this would imply that $\gamma(\tau, p_0) \in \overline{B}^c$ for a range of $\tau$'s in a small neighbourhood of $\tau'$. This, however, contradicts $\tau_0 = T(p_0)$. Hence $d_1 > 0$.

Furthermore define

$$d_2 := \min\{\mathrm{dist}(\gamma(\tau, p_0), \partial B) \, : \, \tau_0 + \varepsilon \leq \tau \leq \tau_0 + 2\varepsilon\}.$$

Using (2.3.21), we can see again that $d_2 > 0$. Set $d := \min(d_1, d_2)$.

Set

$$G := \{\gamma(\tau, p_0) \, : \, \tau \in [0, \tau_0 + 2\varepsilon] \backslash [\tau_0 - \varepsilon, \tau_0 + \varepsilon]\}$$

Since the solutions of linear ODEs depend continuously on the initial data, there is a $\delta > 0$ such that, for all $p \in B_\delta(p_0)$, $\gamma(\tau, p)$ is in a $d/2$-neighbourhood of $G$ for all $\tau \in [0, \tau_0 + 2\varepsilon] \backslash (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$. Thus for all $p \in B_\delta(p_0)$, $T(p) > \tau_0 - \varepsilon$ and $T(p) < \tau_0 + \varepsilon$, i. e.

$|T(p) - T(p_0)| < \varepsilon$. ☐

Hence there exists a $T_0 \geq 0$ such that

$$T(p) \geq T_0$$

for all $p \in \overline{B_{L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))}$. As continuous functions attain their minimum on compact sets, $T_0$ can be chosen to be positive. This shows that all integral curves of $W$ starting in $B_{L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$ exist for $0 \leq \tau \leq T_0$ and remain in $B_{L'}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$.

We can prove the following

**Lemma 2.3.21.** *Given* $(\alpha(0), \omega_R(0), \omega_I(0), a(0))^t \in B_{L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+))$ *with*

$$\alpha_0 - T_0\varrho \leq \alpha(0) < \alpha_0,$$

*let*

$$s \mapsto (\alpha(s), \omega_R(s), \omega_I(s), a(s))^t$$

*be the integral curve starting at* $(\alpha(0), \omega_R(0), \omega_I(0), a(0))^t$. *Then there is a* $\tau \in (0, T_0]$ *such that* $\alpha(\tau) = \alpha_0$.

*Proof.* The ODE yields

$$\begin{aligned}\alpha(T_0) &= \alpha(0) + \int_0^{T_0} W^\alpha(\alpha(s), \omega_R(s), \omega_I(s), a(s))\,\mathrm{d}s \\ &\geq \alpha(0) + T_0\varrho.\end{aligned}$$

If $\alpha_0 - T_0\varrho \leq \alpha(0) < \alpha_0$, then $\alpha(T_0) \geq \alpha_0$ and, by the intermediate value theorem, there is a $\tau \in (0, T_0]$ such that $\alpha(\tau) = \alpha_0$. ☐

The function $\Phi(\cdot, r_+)$ induces the curve

$$\Gamma : \ \alpha \mapsto (\alpha, \Omega_R(\alpha), 0, \Phi(\alpha, r_+))^t$$

for $\alpha \in I$; it is continuous on the left at $\alpha_0$. The result of the previous section says that along this curve, the implicit function theorem produces parameter curves that correspond to superradiant modes; these parameter curves are exactly the integral curves of $W$ starting on a point of $\Gamma$. Since $\Gamma$ is left-continuous,

$$\Gamma \cap B_{L'/3}(\alpha_0, \Omega_R(\alpha_0), 0, \Phi(\alpha_0, r_+)) \cap \{\alpha_0 - T_0\varrho \leq \alpha(0) < \alpha_0\} \neq \emptyset.$$

This shows Theorem 2.1.9

## 2.4. Growing mode solutions satisfying Neumann boundary conditions

### 2.4.1. Existence of real mode solutions

In this section, we will construct growing mode solutions satisfying Neumann boundary conditions. Every result has a counterpart in Section 2.3. In the following, whenever proofs will be short in detail, the reader can extract those from Section 2.3. The two novel techniques in this section are the use of twisted derivatives with appropriately modified Sobolev spaces and a new Hardy inequality (Lemma 2.4.2).

Fix $\ell > 0$ and $r_+$. In this section, we look at the range $5/4 < \alpha < 9/4$, i. e. $0 < \kappa < 1$, for Neumann boundary conditions.

To treat the Neumann case variationally, we need to modify the functional, so it becomes finite for Neumann modes. We achieve this by conjugating the derivatives by a power of r; more precisely, we consider the twisted derivative $h\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}\cdot\right)$, where $h = r^{-1/2+\kappa}$. This "kills off" the highest order term of the Neumann branch. Moreover, squaring the twisted derivative term does not introduce any "mixed terms" in $f$ and its derivative; it only produces a zeroth order term that also makes the potential finite.

Thus introduce the twisted variational functional

$$\tilde{\mathcal{L}}_a(f) := \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} h^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1} f\right) \right|^2 + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} |f|^2 \right) \mathrm{d}r,$$

where $\tilde{V}_a^h$ as in appendix B.1, i. e.

$$\tilde{V}_a^h = \tilde{V}_a + \left(\frac{1}{2} - \kappa\right) \frac{\Delta_-}{r^2+a^2} r^{\frac{1}{2}-\kappa} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2} r^{-\frac{3}{2}+\kappa}\right).$$

By Lemma B.1.2, $\tilde{V}_a^h = \mathcal{O}(1)$ and $\tilde{V}_a^h$ is positive near infinity for sufficiently large $|m|$, which shall be assumed henceforth. Moreover, $\tilde{V}_a^h$ is chosen such that the twisted variational problem leads to the same Euler-Lagrange equation as the untwisted one.

For $U \subseteq (r_+, \infty)$, we define the twisted Sobolev norm

$$\|f\|^2_{\underline{H}^1_\kappa(U)} := \int_U \left( \frac{1}{r^2}|f|^2 + r(r-r_+)h^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1} f\right) \right|^2 \right) \mathrm{d}r.$$

Note that for $U \subseteq (r_+, \infty)$ compact, the $\underline{H}^1_\kappa$ norm is equivalent to the standard Sobolev norm. For $U = (r_+, \infty)$, let $\underline{H}^1_\kappa(U)$ be the completion of functions of the form

$$f(r) = r^{-\frac{1}{2}+\kappa} g(r) \tag{2.4.1}$$

under $\|\cdot\|_{\underline{H}^1_\kappa(U)}$, where $(x \mapsto g(1/x)) \in C_0^\infty[0, 1/r_+)$. Henceforth, we will sometimes refer to such a function $g$ as being "compactly supported around infinity".

**Lemma 2.4.1.** *Let $f \in \underline{H}^1_\kappa(r_+,\infty)$, then $f$ is also in $C(r_+ + 1,\infty)$ and $r^{1/2-\kappa}f(r)$ is bounded.*

*Proof.* The existence of a continuous version follows as in Lemma 2.3.2. Then, there exists a sequence $(f_n) \in C^\infty$ as in the definition such that $f_n \to f$ in $\underline{H}^1_\kappa$. Let $\tilde{R} > R > r_+$ and let $f_n(R)$ converge to $f(R)$:

$$\begin{aligned}\left|\tilde{R}^{1/2-\kappa}(f_n(\tilde{R}) - f(\tilde{R}))\right| &\leq \left|R^{1/2-\kappa}(f_n(R) - f(R))\right| + \int_R^{\tilde{R}} \left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{1/2-\kappa}\left(f_n - f\right)\right)\right| \mathrm{d}r \\ &\leq \left|R^{1/2-\kappa}(f_n(R) - f(R))\right| \\ &\quad + \left(\int_R^\infty r(r-r_+)r^{-1+2\kappa}\left|r^{-1/2+\kappa}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{1/2-\kappa}\left(f_n - f\right)\right]\right|^2 \mathrm{d}r\right)^{1/2} \times \\ &\quad \times \left(\int_R^\infty \frac{1}{r(r-r_+)r^{-1+2\kappa}}\,\mathrm{d}r\right)^{1/2}\end{aligned}$$

Hence $r^{1/2-\kappa}\left(f_n(r) - f(r)\right)$ converges uniformly for all $r \geq R$. Hence we even have convergence at $r = \infty$. Since $\lim_{r\to\infty} r^{1/2-\kappa}f_n(r) \neq \infty$ for all $n$, we obtain the result. □

As in Section 2.3.1, choose mode parameters such that the conditions for Lemma 2.2.19 are satisfied. Let

$$\mathcal{A} := \{a > 0 \,:\, \exists\,(x \mapsto g(1/x)) \in C_0^\infty[0,1/r_+) :\, \tilde{\mathcal{L}}_a(r^{-1/2+\kappa}g) < 0\}. \tag{2.4.2}$$

Note that $\mathcal{A}$ is non-empty, open and bounded below.

**Lemma 2.4.2.** *For $r_{\mathrm{cut}} > r_+ + 1$, $0 < \kappa < 1$ and a smooth function $f$ with $fr^{1/2-\kappa} = \mathcal{O}(1)$ at infinity, we have that*

$$\begin{aligned}\int_{r_{\mathrm{cut}}}^\infty \frac{|f|^2}{r^2}\,\mathrm{d}r \leq{}& \frac{1}{1-\kappa}r_{\mathrm{cut}}^{-1}\left(\frac{c}{2}\int_{r_{\mathrm{cut}}-1}^{r_{\mathrm{cut}}} |f|^2\,\mathrm{d}r + \frac{1}{2c}\int_{r_{\mathrm{cut}}-1}^{r_{\mathrm{cut}}}\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2 \mathrm{d}r\right) \\ &+ \frac{1}{(1-\kappa)^2}\int_{r_{\mathrm{cut}}}^\infty r^{-1+2\kappa}\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{1/2-\kappa}f\right)\right|^2 \mathrm{d}r\end{aligned}$$

*for any $c > 0$ sufficiently large.*

*Proof.* We compute:

$$\begin{aligned}\int_{r_{\mathrm{cut}}}^\infty \frac{|f|^2}{r^2}\,\mathrm{d}r ={}& \frac{1}{2(1-\kappa)}r_{\mathrm{cut}}^{-1}|f|^2(r_{\mathrm{cut}}) \\ &+ \frac{1}{1-\kappa}\int_{r_{\mathrm{cut}}}^\infty r^{2\kappa-2}\mathrm{Re}\left(r^{1/2-\kappa}f\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{1/2-\kappa}\overline{f}\right)\right)\mathrm{d}r.\end{aligned}$$

Using the Cauchy-Schwarz inequality, one easily sees that

$$\int_{r_{\mathrm{cut}}}^\infty \frac{|f|^2}{r^2}\,\mathrm{d}r \leq \frac{1}{1-\kappa}r_{\mathrm{cut}}^{-1}|f|^2(r_{\mathrm{cut}})$$

$$+\frac{1}{(1-\kappa)^2}\int_{r_{\mathrm{cut}}}^{\infty} r^{-1+2\kappa}\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{1/2-\kappa}f\right)\right|^2 \mathrm{d}r.$$

Let $\chi \geq 0$ be a smooth function of compact support with $\chi(r_{\mathrm{cut}}) = 1$ and $\chi = 0$ for $r \leq r_{\mathrm{cut}} - 1$. Then

$$\begin{aligned}|f|^2(r_{\mathrm{cut}}) &\leq |f|^2(r_{\mathrm{cut}})\chi(r_{\mathrm{cut}})\\ &\leq \frac{c}{2}\int_{r_{\mathrm{cut}}-1}^{r_{\mathrm{cut}}} |f|^2\,\mathrm{d}r + \frac{1}{2c}\int_{r_{\mathrm{cut}}-1}^{r_{\mathrm{cut}}}\left|\frac{\mathrm{d}f}{\mathrm{d}r}\right|^2 \mathrm{d}r\end{aligned}$$

for any $c > 0$ sufficiently large. □

**Lemma 2.4.3.** *There exist constants $r_+ < B_0 < B_1 < \infty$ and $C_0, C_1 > 0$, such that, for large enough $m$, we have for all smooth functions $f$ (for which the following integrals are defined) that*

$$\int_{r_+}^{\infty}\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}f\right)\right|^2 + C_0 1_{[B_0,B_1]^c}\frac{|f|^2}{r^2}\right)\mathrm{d}r \leq C_1\int_{B_0}^{B_1}|f|^2\,\mathrm{d}r + 2\tilde{\mathcal{L}}_a(f).$$

*for all $a \in \mathcal{A}$.*

*Proof.* The proof follows the strategy of Lemma 2.3.3. The analysis of the potential goes through as in Section 2.3.1 as the twisting part of $\tilde{V}_a^h$ does not depend on $m$ and has the right asymptotics. Thus we know that there is an $R_1$ such that

$$\frac{r^2+a^2}{\Delta_-}\tilde{V}_a^h > 0$$

on $(r_+, R_1)$. Moreover, there is an $R_2 > R_1$ such that

$$\frac{r^2+a^2}{\Delta_-}\tilde{V}_a^h > -\frac{C}{2r^2} \quad \text{and} \quad \frac{C}{(1-\kappa)^2} < \frac{\Delta_-}{r^2+a^2}$$

for $r \geq R_2$. Hence

$$\int_{R_2}^{\infty}\frac{r^2+a^2}{\Delta_-}\tilde{V}_a^h|f|^2\,\mathrm{d}r \geq -\int_{R_2-1}^{\infty}\frac{\Delta_-}{r^2+a^2}r^{-1+2\kappa}\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{1/2-\kappa}f\right)\right|^2\mathrm{d}r - C'\int_{R_2-1}^{R_2}|f|^2\,\mathrm{d}r$$

for some large constant $C' > 0$ by the Hardy inequality of Lemma 2.4.2. Choosing $B_0$,$B_1$, $C_0$ and $C_1$ appropriately (as in the proof of Lemma 2.3.3), we obtain the inequality. □

**Lemma 2.4.4.** *The functional $\tilde{\mathcal{L}}_a$ is weakly lower semicontinuous in $\underline{H}^1_\kappa(r_+,\infty)$.*

*Proof.* See the comments to Lemma 2.3.5. □

**Lemma 2.4.5.** *Let $a \in \mathcal{A}$. There exists an $f_a \in \underline{H}^1_\kappa(r_+,\infty)$ with norm $\|f_a\|_{\underline{L}^2(r_+,\infty)} = 1$ such that $\tilde{\mathcal{L}}_a$ achieves its infimum in*

$$\underline{H}^1_\kappa(r_+,\infty) \cap \{\|f\|_{\underline{L}^2(r_+,\infty)} = 1\}$$

*on* $f_a$.

*Proof.* The proof is similar to the one in Section 2.3.1. We obtain a minimising sequence $(f_{a,n})$ that converges weakly in $\underline{H}^1_\kappa$ and strongly in $L^2$ on compact subsets of $(r_+,\infty)$. In analogy to the Dirichlet, the $f_{a,n}$ are can be taken from a dense subset and can be chosen to be of the form $f_{a,n}=r^{-1/2+\kappa}g_{a,n}$ for $g_{a,n}$ smooth and compactly supported around infinity.

We will show that the norm is conserved. Suppose not. Then, for any $N$, there are infinitely many of the $f_{a,n}$ such that

$$\|f_{a,n}\|_{\underline{L}^2((r_+,\infty)\setminus[r_++1/N,N])}\geq\varrho>0.$$

Suppose

$$\|f_{a,n}\|_{\underline{L}^2(r_+,r_++\delta)}\geq\varrho_1>0$$

for infinitely many $f_{a,n}$ and any $\delta>0$. Because of the $L^2$ convergence on compact subsets, there is an $R$ such that $f_{a,n}(R)\to f_a(R)$ as $n\to\infty$, in particular $f_{a,n}(R)$ is bounded for all $n$. By Lemma 2.4.3, we have for $r\in(r_+,R)$:

$$\begin{aligned}|r^{1/2-\kappa}f_{a,n}(r)|&\leq\int_r^R\left|\frac{\mathrm{d}}{\mathrm{d}r'}\left(r'^{1/2-\kappa}f_{a,n}\right)\right|\mathrm{d}r'+R^{1/2-\kappa}f_{a,n}(R)\\&\leq\left(\int_r^R\frac{1}{r'-r_+}\,\mathrm{d}r'\right)^{1/2}\times\\&\qquad\times\left(\int_r^R(r'-r_+)r'^{-1+2\kappa}\left|\frac{\mathrm{d}}{\mathrm{d}r'}\left(r'^{1/2-\kappa}f_{a,n}\right)\right|^2\mathrm{d}r'\right)^{1/2}\\&\qquad+R^{1/2-\kappa}f_{a,n}(R)\\&\leq C\left(1+\sqrt{\log\frac{R-r_+}{r-r_+}}\right)\end{aligned}$$

for a constant $C>0$. Since $r\mapsto\sqrt{|\log(r-r_+)|}$ is integrable on compact subsets of $[r_+,\infty)$, we obtain $\|f_{a,n}\|_{\underline{L}^2(r_+,r_++\delta)}\to0$ as $\delta\to0$, a contradiction.

Hence we only need to exclude the case that the norm is bounded away from zero for large $r$. Thus, suppose that

$$\|f_{a,n}\|_{\underline{L}^2(R_0,\infty)}\geq\varrho_2>0$$

for infinitely many $f_{a,n}$ and any $R_0>0$. Since $f_{a,n}(r_+)=0$, we have

$$\begin{aligned}r^{\frac12-\kappa}|f_{a,n}|(r)&\leq\int_{r_+}^r\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r'^{\frac12-\kappa}f_{a,n}\right)\right|\mathrm{d}r'\\&\leq\left(\int_{r_+}^\infty\frac{1}{r^{1+2\kappa}}\,\mathrm{d}r\right)^{1/2}\left(\int_{r_+}^\infty r^{1+2\kappa}\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac12-\kappa}f_{a,n}\right)\right|\mathrm{d}r\right)^{1/2},\end{aligned}$$

which is uniformly bounded for all $n$. Hence

$$\int_{R_0}^{\infty} \frac{|f_{a,n}|^2}{r^2} \,\mathrm{d}r \leq C' \int_{R_0}^{\infty} r^{-3+2\kappa} \,\mathrm{d}r \to 0$$

as $R_0 \to \infty$, a contradiction.

As in the proof of Lemma 2.3.6, we have

$$\nu_a \leq \mathcal{L}_a(f_a) \leq \liminf_{n\to\infty} \mathcal{L}_a(f_{a,n}) = \nu_a$$

and the rest follows. □

We would like to derive the Euler-Lagrange equation corresponding to this minimiser.

**Lemma 2.4.6.** *The minimiser $f_a$ satisfies*

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1} f_a\right) \frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}\psi\right) + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} f_a \psi \right) \mathrm{d}r = -\nu_a \int_{r_+}^{\infty} \frac{f_a}{r^2} \psi \,\mathrm{d}r \quad (2.4.3)$$

*for all $\psi \in \underline{H}^1_\kappa(r_+, \infty)$.*

The proof of Lemma 2.4.6 can be found in appendix B.2.

**Proposition 2.4.7.** *There is an $\hat{a}$ and a corresponding non-zero function $f_{\hat{a}} \in C^\infty(r_+, \infty)$ such that*

$$\frac{\Delta_-}{r^2+\hat{a}^2} \frac{\mathrm{d}}{\mathrm{d}r}\left( \frac{\Delta_-}{r^2+\hat{a}^2} \frac{\mathrm{d}f_{\hat{a}}}{\mathrm{d}r} \right) - \tilde{V}_{\hat{a}} f_{\hat{a}} = 0$$

*and $f_a$ satisfies the horizon regularity condition and the Neumann boundary condition at infinity.*

*Proof.* As in Proposition 2.3.8, we find an $f_{\hat{a}} \in \underline{H}^1_\kappa$ such that

$$\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+\hat{a}^2} r^{-1+2\kappa} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa} f_{\hat{a}}\right) \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa}\psi\right) + \tilde{V}_{\hat{a}}^h \frac{r^2+\hat{a}^2}{\Delta_-} f_{\hat{a}} \psi \right) \mathrm{d}r = 0.$$

Choosing $\psi(r) = r^{-\frac{1}{2}+\kappa} g(r)$ with $g$ having compact support around infinity and integrating by parts, we obtain

$$\frac{\Delta_-}{r^2+\hat{a}^2} r^{-1+2\kappa} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa} f_{\hat{a}}\right) g \to 0$$

as $r = \infty$ for all $g$ as in (2.4.1). This yields the asymptotics.

Moreover, as in the proof of Proposition 2.3.8, we retrieve the ODE. The boundary condition at the horizon follows analogously to Section 2.3.1. □

### 2.4.2. Perturbing the Neumann modes into the complex plane

In Section 2.4.1, we constructed real mode solutions for $5/4 < \alpha < 9/4$ satisfying Neumann conditions. For the growing radial parts, we proceed as in Section 2.3.1 with the difference that here, finding a mode solution is equivalent to finding a zero of $B$. The present case is considerably more difficult than the Dirichlet case. A first manifest difference is the asymmetry in the definitions of Dirichlet and Neumann boundary conditions since a Dirichlet mode has more decay than required by Definition 2.1.6. This means that if a function satisfies the Dirichlet boundary condition for a mass $\alpha_1$, it also does so for every $\alpha_2$ sufficiently close to $\alpha_1$. As Definition 2.1.14 is tighter, this is not true in the Neumann case. Another difficulty stems from twisting as the dependence of the equations on $\alpha$ becomes more complicated.

We have already chosen $B(\alpha(0), \omega_R(0)) = 0$. Recall from Section 2.2.4 that

$$Q_T = \operatorname{Im}\left(r^{-\frac{1}{2}+\kappa}\frac{\mathrm{d}}{\mathrm{d}r^*}\left(r^{\frac{1}{2}-\kappa}u\right)\overline{u}\right), \qquad Q_T(r_+) = \frac{\Xi am - \omega_R(r_+^2+a^2)}{r_+^2+a^2}|u(r_+)|^2.$$

Hence, analogously to Section 2.3.2, the problem reduces to showing that

$$\frac{\partial B}{\partial \alpha}(\alpha(0), \omega_R(0)) \neq 0.$$

Again, for the sake of contradiction, suppose that this is not the case. Then, near infinity, we have

$$\begin{aligned} u_\alpha(r, \alpha(0), \omega_R(0)) = {} & \frac{\partial A}{\partial \alpha}(\alpha(0), \omega_R(0)) h_1(r, \alpha(0), \omega_R(0)) \\ & + A(\alpha(0), \omega_R(0)) \frac{\partial h_1}{\partial \alpha}(r, \alpha(0), \omega_R(0)). \end{aligned}$$

By the horizon regularity condition, $u \sim (r - r_+)^\xi$ near the horizon, $u_\alpha$ is smooth at $r = r_+$. However, $u_\alpha$ does not satisfy the Neumann condition at infinity as the second term behaves as $r^{-1/2+\kappa}\log r$.

Let $f : (r_+, \infty) \to \mathbb{C}$ be $C^1$ and piecewise $C^2$. Then the function

$$v(r) := u_\alpha(r) - \frac{\partial \kappa}{\partial \alpha} f(r) u(r) = u_\alpha(r) + \frac{1}{2\kappa} f(r) u(r),$$

does satisfy the Neumann boundary condition if, for large $r$, $f(r) = \log r + \mathcal{O}(r^{-\gamma})$, where $\gamma > 0$.

From the radial ODE, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}u_\alpha}{\mathrm{d}r}\right) - \frac{r^2+a^2}{\Delta_-}\tilde{V}_a u_\alpha = \frac{1}{r^2+a^2}\left[\frac{\partial \lambda}{\partial \alpha} - \frac{1}{\ell^2}(r^2+a^2)\right]u.$$

Lemma B.1.1 yields a twisted version

$$\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\alpha}{h}\right)\right) - \tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u_\alpha = \frac{1}{r^2+a^2}\left[\frac{\partial \lambda}{\partial \alpha} - \frac{1}{\ell^2}(r^2+a^2)\right]u. \tag{2.4.4}$$

We will use the previous twisting, i. e. $h = r^{-1/2+\kappa}$.

For the second term of $v$, we compute:

$$\begin{aligned}\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(f\frac{u}{h}\right)\right) &= h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\right)(h^{-1}u)\\ &\quad + 2h^{-1}\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}(h^{-1}u)\\ &\quad + h^{-1}f\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}(h^{-1}u)\right)\end{aligned}$$

We add this to the equation to (2.4.4) multiplied by $2\kappa$. Then we multiply the resulting equation by $\overline{u}$ and integrate by parts, noting that $v$ satisfies the Neumann boundary condition. Hence we obtain

$$\begin{aligned}0 &= \int_{r_+}^{\infty}\frac{2\kappa}{r^2+a^2}\left(\frac{\partial\lambda}{\partial\alpha}-\frac{a^2}{\ell^2}\right)|u|^2\,\mathrm{d}r\\ &\quad - \int_{r_+}^{\infty}\left(\frac{2\kappa}{\ell^2}\frac{r^2}{r^2+a^2}|u|^2-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\right)\left|h^{-1}u\right|^2\right)\mathrm{d}r\\ &\quad - \int_{r_+}^{\infty} f\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2+\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}|u|^2\right)\mathrm{d}r\\ &\quad - \int_{r_+}^{\infty} f\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2\,\mathrm{d}r.\end{aligned} \tag{2.4.5}$$

Our aim is to show that the right hand side of (2.4.5) is negative, which yields the desired contradiction.

From Section 2.3.2, we already know that

$$\int_{r_+}^{\infty}\frac{2\kappa}{r^2+a^2}\left(\frac{\partial\lambda}{\partial\alpha}-\frac{a^2}{\ell^2}\right)|u|^2\,\mathrm{d}r = -\int_{r_+}^{\infty}\frac{2\kappa}{r^2+a^2}\int_0^{\pi}\frac{a^2}{\ell^2}\cos^2\vartheta|S|^2\sin\vartheta|u|^2\,\mathrm{d}\vartheta\,\mathrm{d}r$$

has the right sign. We set

$$f(r) := \begin{cases}\log r+\frac{1}{2\kappa}\frac{R^{2\kappa}}{r^{2\kappa}}, & r\geq R\\ \log R+\frac{1}{2\kappa}, & r<R\end{cases} \tag{2.4.6}$$

for an $R>r_++1$ to be determined. Note that $f$ is continuously differentiable. For $r>R$,

$$\frac{\mathrm{d}f}{\mathrm{d}r}(r)=\frac{1}{r}\left(1-\frac{R^{2\kappa}}{r^{2\kappa}}\right)>0,$$

whence $f$ is monotonic.

First, we choose $R$ sufficiently large such that $\tilde{V}_a^h>0$ for $r>R$ according to Lemma B.1.2, whence

$$\int_{r_+}^{\infty} f\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2+\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}|u|^2\right)\mathrm{d}r$$

$$= \left(\log R + \frac{1}{2\kappa}\right) \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} h^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(h^{-1}u\right) \right|^2 + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} |u|^2 \right) \mathrm{d}r$$
$$+ \int_R^{\infty} \left( \log \frac{r}{R} + \frac{1}{2\kappa} \left( \frac{R^{2\kappa}}{r^{2\kappa}} - 1 \right) \right) \left( \frac{\Delta_-}{r^2+a^2} h^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(h^{-1}u\right) \right|^2 + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} |u|^2 \right) \mathrm{d}r,$$

which is non-negative since the first integral with the constant coefficient is zero and the second integral is positive. For $r > R$, one easily computes

$$\frac{2\kappa}{\ell^2} \frac{r^2}{r^2+a^2} - h^{-2} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}f}{\mathrm{d}r} \right) = - \frac{2\kappa a^2/\ell^2 + (2\kappa-2)}{r^2} - 2 \frac{R^{2\kappa}}{r^{2\kappa}} \frac{1}{r^2} + \mathcal{O}(r^{-3})$$

Therefore, there is a $C_1 > 0$ such that

$$\int_R^{\infty} \left( \frac{2\kappa}{\ell^2} \frac{r^2}{r^2+a^2} |u|^2 - h^{-2} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}f}{\mathrm{d}r} \right) \right) |u|^2 \, \mathrm{d}r > -C_1 \int_R^{\infty} \frac{|u|^2}{r^2} \, \mathrm{d}r.$$

We can prove the following Hardy inequality:

**Lemma 2.4.8.** *Let $u$ satisfy the Neumann boundary condition at infinity and let $\beta > 0$. Then*

$$\int_R^{\infty} \frac{1}{r^{1+\beta}} \left| r^{\frac{1}{2}-\kappa} u \right|^2 \mathrm{d}r \leq \lim_{r\to\infty} \frac{2\beta}{R^\beta} \left| r^{\frac{1}{2}-\kappa} u(r) \right|^2$$
$$+ 4\beta^2 \int_R^{\infty} r^{2-2\kappa-\beta} \left( 1 - \left( \frac{r}{R} \right)^\beta \right)^2 h^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{u}{h} \right) \right|^2 \mathrm{d}r.$$

*Proof.* We compute:

$$\int_R^{\infty} \frac{1}{r^{1+\beta}} \left| r^{\frac{1}{2}-\kappa} u \right|^2 \mathrm{d}r = \int_R^{\infty} \partial_r \left( -\frac{\beta}{r^\beta} + \frac{\beta}{R^\beta} \right) \left| r^{\frac{1}{2}-\kappa} u \right|^2 \mathrm{d}r$$
$$\leq \frac{\beta}{R^\beta} \left| r^{\frac{1}{2}-\kappa} u \right|^2 (\infty) + \int_R^{\infty} \frac{1}{2} \frac{1}{r^{1+\beta}} \left| r^{\frac{1}{2}-\kappa} u \right|^2 \mathrm{d}r$$
$$+ 2\beta^2 \int_R^{\infty} r^{1+\beta} \left( r^{-\beta} - R^{-\beta} \right)^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left( r^{\frac{1}{2}-\kappa} u \right) \right|^2 \mathrm{d}r,$$

yielding the result. □

Since

$$C_1 \lim_{r\to\infty} \frac{2\beta}{R^\beta} \left| r^{\frac{1}{2}-\kappa} u(r) \right|^2 \to 0$$

as $R \to \infty$, by choosing $R$ possibly larger, we obtain

$$C_1 \frac{2\beta}{R^\beta} \lim_{r\to\infty} \left| r^{\frac{1}{2}-\kappa} u(r) \right|^2 < - \int_{r_+}^{\infty} \frac{2\kappa}{r^2+a^2} \left( \frac{\partial\lambda}{\partial\alpha} - \frac{a^2}{\ell^2} \right) |u|^2 \, \mathrm{d}r.$$

Since $\tilde{V}_a^h = \mathcal{O}(1)$, we need to show that

$$C_1 \int_R^\infty \frac{|u|^2}{r^2} \,\mathrm{d}r < \int_R^\infty \left( \log \frac{r}{R} + \frac{1}{2\kappa} \left( \frac{R^{2\kappa}}{r^{2\kappa}} - 1 \right) \right) \frac{\Delta_-}{r^2 + a^2} h^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left( h^{-1} u \right) \right|^2 \mathrm{d}r. \tag{2.4.7}$$

We will deal with the two cases $0 < \kappa \leq 1/2$ and $1/2 < \kappa < 1$ separately. Let us first consider $0 < \kappa \leq 1/2$. We choose $\beta = 2\kappa$. Note that in this case

$$\int_R^\infty \frac{|u|^2}{r^2} \,\mathrm{d}r \leq \int_R^\infty \frac{1}{r^{1+2\kappa}} \left| r^{1/2-\kappa} u \right|^2 \mathrm{d}r.$$

**Lemma 2.4.9.** *Let $C > 0$. There is an $R$ such that, for all $r > R$,*

$$C r^{2-4\kappa} \left( 1 - \left( \frac{r}{R} \right)^{2\kappa} \right)^2 \leq r^2 \left( \log \frac{r}{R} + \frac{1}{2\kappa} \left( \left( \frac{R}{r} \right)^{2\kappa} - 1 \right) \right).$$

*Proof.* It suffices to show that

$$C r^{-4\kappa} \left( 1 - \left( \frac{r}{R} \right)^{2\kappa} \right)^2 \leq \log \frac{r}{R} + \frac{1}{2\kappa} \left( \left( \frac{R}{r} \right)^{2\kappa} - 1 \right).$$

As this holds at $r = R$, it suffices to show the statement for the derivatives. Substituting $x := r^{2\kappa}$, we need to show

$$0 \leq x^2 - (R^{2\kappa} + 4\kappa C R^{-2\kappa}) x + 4\kappa C = (x - R^{2\kappa})(x - 4\kappa C R^{-2\kappa}).$$

Therefore, the result holds if $R^{4\kappa} > 4\kappa C$. □

This lemma immediately yields (2.4.7) for $0 < \kappa \leq 1/2$. Let us now turn to $1/2 < \kappa < 1$. Here we choose $\beta = 2 - 2\kappa$.

**Lemma 2.4.10.** *Let $C > 0$ and $1/2 < \kappa < 1$. There is an $R$ such that, for all $r > R$,*

$$C \left( 1 - \left( \frac{r}{R} \right)^{2-2\kappa} \right)^2 \leq r^2 \left( \log \frac{r}{R} + \frac{1}{2\kappa} \left( \left( \frac{R}{r} \right)^{2\kappa} - 1 \right) \right).$$

*Proof.* As equality holds for $r = R$, it suffices to consider the derivatives, i. e. we would like to establish

$$\begin{aligned} 2r \left( \log \frac{r}{R} + \frac{1}{2\kappa} \left( \left( \frac{R}{r} \right)^{2\kappa} \right) \right) + r^2 \left( \frac{1}{r} - \left( \frac{R}{r} \right)^{2\kappa} \frac{1}{r} \right) & \\ - 2C \left( \left( \frac{r}{R} \right)^{2-2\kappa} - 1 \right) (2 - 2\kappa) R^{-2+2\kappa} r^{1-2\kappa} &\geq 0. \end{aligned}$$

This again holds for $r = R$, so, after dividing the inequality by $r$, it suffices to prove the corresponding inequality for the derivatives, i. e.

$$\frac{1}{r}\left(2-2(1-\kappa)\left(\frac{R}{r}\right)^{2\kappa}\right)-4C\kappa(2-2\kappa)\frac{1}{R^2}\left(\frac{r}{R}\right)^{-1-2\kappa}$$
$$+2C(2-2\kappa)(4\kappa-2)R^{-4-4\kappa}r^{1-4\kappa}\geq 0.$$

The last term on the left hand side is always positive. Thus the left hand side is greater than

$$\frac{\kappa}{r}-4C\kappa(2-2\kappa)\frac{1}{R}\left(\frac{R}{r}\right)^{2\kappa}\geq\left(\frac{1}{2}-4C\frac{1}{R}\right)\frac{1}{r},$$

which is positive for sufficiently large $R$. □

Therefore, for both ranges of $\kappa$, the right hand side of (2.4.5) is bounded above by

$$-\int_{r_+}^{R}\frac{1}{\ell^2}\frac{r^2}{r^2+a^2}|u|^2\,\mathrm{d}r-2\int_{R}^{\infty}\left(\log\frac{r}{R}+\frac{1}{2\kappa}\left(\frac{R^{2\kappa}}{r^{2\kappa}}-1\right)\right)\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}|u|^2\,\mathrm{d}r$$
$$-\int_{r_+}^{\infty}f\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2\,\mathrm{d}r<0$$

for non-trivial $u$, a contradiction. Thus we have shown the following

**Lemma 2.4.11.**

$$\frac{\partial B}{\partial\alpha}(\alpha(0),\omega_R(0))\neq 0.$$

### 2.4.3. Behaviour for small $\varepsilon>0$ for Neumann boundary conditions

The main new idea of this section can be found in the proof of Proposition 2.4.13, where the insights of Section 2.4.2 are essential to overcome the difficulties outlined at the beginning of the previous section.

**Proposition 2.4.12.** *For sufficiently large $|m|$,*

$$\omega_R(0)\frac{\partial\omega_R}{\partial\varepsilon}(0)<0.$$

*Proof.* We define an appropriate modified microlocal energy current

$$\tilde{Q}_T:=\mathrm{Im}\left(r^{-\frac{1}{2}+\kappa}\left(r^{\frac{1}{2}-\kappa}\right)'\,\overline{\omega u}\right).$$

Let $\varepsilon>0$, then $\tilde{Q}_T(r_+)=\tilde{Q}_T(\infty)=0$. This yields

$$\int_{r_+}^{\infty}\left(\varepsilon\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2-\frac{r^2+a^2}{\Delta_-}\mathrm{Im}\,(\tilde{V}_a^h\overline{\omega})|u|^2\right)\mathrm{d}r=0. \tag{2.4.8}$$

Similarly to Section 2.3.3, we obtain

$$
\begin{aligned}
-\mathrm{Im}\left(\tilde{V}_a^h\overline{\omega}\right) > {} & \frac{\varepsilon}{(r^2+a^2)^2}\left(K(r)+V_+(r^2+a^2)^2-\frac{\alpha}{\ell^2}\Delta_-(r^2+a^2)\right) \\
& +\varepsilon\left(\frac{1}{2}-\kappa\right)\frac{\Delta_-}{r^2+a^2}r^{1/2-\kappa}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}r^{-3/2+\kappa}\right)
\end{aligned}
\tag{2.4.9}
$$

with the additional term due to the twisting. Again

$$
K(r)=|\omega|^2(r^2+a^2)^2-\Xi^2a^2m^2-\Delta_-a^2|\omega|^2.
$$

Recall from Section 2.3.3 that

$$
\frac{\mathrm{d}K}{\mathrm{d}r}(r)=|\omega|^2\left(4\left(1-\frac{a^2}{\ell^2}\right)r^3+2a^2M+2a^2\left(1-\frac{a^2}{\ell^2}\right)\right)>0. \tag{2.4.10}
$$

By Lemma B.1.2, there is an $R>r_+$ such that

$$
\frac{r^2+a^2}{\Delta_-}\left|V_++V_\alpha+\left(\frac{1}{2}-\kappa\right)\frac{\Delta_-}{r^2+a^2}r^{1/2-\kappa}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}r^{-3/2+\kappa}\right)\right|<\frac{C}{2r^2}
$$

for any $C>0$. Thus, by an application of Lemma 2.4.2 as in the proof of Lemma 2.4.3, we have

$$
\begin{aligned}
\int_R^\infty & \left(\varepsilon\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2-\frac{r^2+a^2}{\Delta_-}\mathrm{Im}\,(\tilde{V}_a^h\overline{\omega})|u|^2\right)\mathrm{d}r \\
& > \int_R^\infty\frac{\varepsilon}{(r^2+a^2)^2}K(r)|u|^2-\int_{R-1}^R\varepsilon C'|u|^2\,\mathrm{d}r
\end{aligned}
\tag{2.4.11}
$$

for sufficiently large $R$ and a large constant $C'>0$.

For the sake of contradiction, suppose that $K(r_+)\geq 0$. By (2.4.10), this means that $K>0$ on $(r_+,\infty)$. Since $\varepsilon\mapsto\omega(\varepsilon)$ is continuous and

$$
|\omega(0)|^2\geq Cm^2,
$$

$|\omega|^2$ scales as $m^2$, so $\mathrm{d}K/\mathrm{d}r$ can be chosen to be as large as possible by increasing $m^2$, in particular, it can be used to overcome the potentially negative derivative of the remaining terms of the right hand side of (2.4.9) on $(r_+,R)$ and in (2.4.11) on $(R-1,R)$. Using (2.4.8) and (2.4.11), we conclude $u=0$, a contradiction. □

From now on we fix $m$ – see Remark 2.3.13.

**Proposition 2.4.13.**

$$
\frac{\partial\alpha}{\partial\varepsilon}(0)>0.
$$

*Proof.* The proof proceeds as in Section 2.3.3, adapting the idea used already in Section 2.4.2.

Set $u_\varepsilon := \partial u/\partial\varepsilon$. For an $f$ as in (2.4.6) with an $R$ to be determined,

$$v(r) := u_\varepsilon(r) - \frac{\partial\kappa}{\partial\varepsilon} f(r)u(r) = u_\varepsilon(r) + \frac{1}{2\kappa}\frac{\partial\alpha}{\partial\varepsilon} f(r)u(r)$$

satisfies the Neumann boundary condition at infinity. As $fu$ extends smoothly to the horizon, the behaviour of $v$ at $r = r_+$ is dominated by $u_\varepsilon$. Using the $h$ of Lemma B.1.2 yields the ODE

$$\begin{aligned}
&h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v}{h}\right)\right) - \frac{r^2+a^2}{\Delta_-}\tilde{V}_a^h v\\
&\quad= \frac{r^2+a^2}{\Delta_-}\frac{\partial\tilde{V}_a}{\partial\varepsilon}u + \frac{1}{2\kappa}\frac{\partial\alpha}{\partial\varepsilon}h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\right)\left(h^{-1}u\right)\\
&\qquad+ 2\frac{1}{2\kappa}\frac{\partial\alpha}{\partial\varepsilon}h^{-1}\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right) + \frac{1}{2\kappa}\frac{\partial\alpha}{\partial\varepsilon}fh^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right).
\end{aligned}$$

Observe that as in the proof of Proposition 2.3.14

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u_\varepsilon\right)\right)h^{-1}\overline{u} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u_\varepsilon\right)\overline{u} - u_\varepsilon\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}\overline{u}\right)\right]\right)\\
&\quad+ u_\varepsilon h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}\overline{u}\right)\right).
\end{aligned}$$

This yields

$$\begin{aligned}
-2\kappa|u(r_+)|^2 &= \int_{r_+}^\infty\left(\frac{r^2+a^2}{\Delta_-}2\kappa\mathrm{Re}\left(\frac{\partial\tilde{V}}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0} + \frac{\partial\alpha}{\partial\varepsilon}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\right)\left|h^{-1}u\right|^2\right)\mathrm{d}r\\
&\quad- \frac{\partial\alpha}{\partial\varepsilon}\int_{r_+}^\infty f\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2 + \tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}|u|^2\right)\mathrm{d}r\\
&\quad- \frac{\partial\alpha}{\partial\varepsilon}\int_{r_+}^\infty f\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2\mathrm{d}r.
\end{aligned} \tag{2.4.12}$$

The expression for $\mathrm{Re}\left(\frac{\partial\tilde{V}}{\partial\varepsilon}\right)\Big|_{\varepsilon=0}$ can be taken from (2.3.17). First, one can eliminate the explicit $\lambda$ dependence via Lemma 2.3.15 and one obtains a lower bound on terms involving $\omega_R(0)$ on the right hand side of (2.3.17) using Proposition 2.4.12. Now suppose for the sake of contradiction that $\partial\alpha/\partial\varepsilon \le 0$. Then $\mathrm{Re}\left(\frac{\partial\tilde{V}}{\partial\varepsilon}\right)\Big|_{\varepsilon=0}$ is as before in the proof of Proposition 2.3.14. It follows immediately from Section 2.4.2 that the remaining terms on the right hand side of (2.4.12) are positive, a contradiction. □

### 2.4.4. The continuity argument for Neumann boundary conditions

To apply the continuity argument to the Neumann case, we need to take the two steps outlined in the introduction to Section 2.3.4. The second step merely relied on continuity properties of $A$ and the monotonicity properties of $\omega(\varepsilon)$ and $\alpha(\varepsilon)$ established in Sections 2.3.3

and 2.4.3, respectively; in particular, it did not rely directly on properties of the functional. Hence this part of the argument can be carried out almost verbatim. Therefore, we only need to deal with the first step here.

We make the analogous definitions for $\mathcal{A}_{\alpha,r_+}$ and $\Phi$ as in Section 2.3.4. For $f = r^{-1/2+\kappa}g$, $x \mapsto g(1/x) \in C_0^\infty[0, 1/r_+)$, we define

$$\mathcal{A}_{\alpha,r_+}(f) := \{a > 0 \,:\, \tilde{\mathcal{L}}_{\alpha,r_+,a}(f) < 0\}$$

and

$$\begin{aligned}\mathcal{A}_{\alpha,r_+} :=& \bigcup_{g\in C_0^\infty[0,1/r_+)} \mathcal{A}_{\alpha,r_+}(r \mapsto r^{-1/2+\kappa}g(1/r)))\\ =&\{a > 0 \,:\, \exists (x \mapsto g(1/x)) \in C_0^\infty[0, 1/r_+) \,:\, \tilde{\mathcal{L}}_{\alpha,r_+,a}(r^{-1/2+\kappa}g) < 0\}.\end{aligned}$$

Moreover, we define

$$\Phi \,:\, (5/4, 9/4) \times (0, \infty) \to (0, \infty),\ \Phi(\alpha, r_+) := \inf \mathcal{A}_{\alpha,r_+}$$

if $\mathcal{A}_{\alpha,r_+} \neq \emptyset$. Instead of showing monotonicity for $\Phi$, we will define a left-continuous function $\Psi$ that can play the rôle of $\Phi$ in the continuity argument. For each $\alpha \in (5/4, 9/4)$, there will be a value $\Phi(\alpha, r_+)$ for $a$ such that there is a real mode solution satisfying the Neumann boundary condition for $\alpha$ and this $a$. The function $\Psi(\cdot, r_+)$ will essentially look like $\Phi(\cdot, r_+)$, but will be modified on potential jump points to achieve left-continuity. The arguments of Sections 2.4.2 and 2.4.3 (which depend only on the existence of a Neumann mode solution) can be repeated for $\Psi(\alpha, r_+)$ instead of $\Phi(\alpha, r_+)$, thus we can substitute $\Phi$ by $\Psi$ in the remainder of the proof of Section 2.3.4.

**Lemma 2.4.14.** *There is a left-continuous function $\Psi(\cdot, r_+)$ such that there is a real mode solution satisfying the Neumann boundary condition for each $5/4 < \alpha < 9/4$ and each $a = \Psi(\alpha, r_+)$.*

To prove this lemma, we need a monotonicity result about the twisted functional. Note that for a fixed $g$ (where $(x \mapsto g(1/x)) \in C_0^\infty[0, 1/r_+)$), the function

$$(\alpha, a) \mapsto \tilde{\mathcal{L}}_{\alpha,r_+,a}(r^{-1/2+\kappa}g) \tag{2.4.13}$$

is continuous.

**Lemma 2.4.15.** *Let $5/4 < \kappa_0 < 9/4$. Fix all spacetime parameters. Let $u_0 := r^{-1/2+\kappa_0}g_0$ be a solution to the radial ODE at $\kappa_0$. Define $u(r, \kappa) := r^{-1/2+\kappa}g_0$. Then*

$$\frac{\partial}{\partial\kappa}\tilde{\mathcal{L}}_{\alpha,r_+,a}(u(r, \kappa))\bigg|_{\kappa=\kappa_0} > 0.$$

*Proof.* We start from the identity

$$h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u}{h}\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u=\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}u}{\mathrm{d}r}\right)-\tilde{V}_a\frac{r^2+a^2}{\Delta_-}u$$

where we always take $h=r^{-1/2+\kappa}$. Set $u_\kappa:=\partial u/\partial\kappa$. Let $f$ be as in (2.4.6) with an $R>r_++1$ to be determined and set $v:=u_\kappa-fu$. Then we have

$$\begin{aligned}
&\frac{\partial}{\partial\kappa}\left(h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u}{h}\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u\right)\\
&=\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}u_\kappa}{\mathrm{d}r}\right)-\tilde{V}_a\frac{r^2+a^2}{\Delta_-}u_\kappa-\frac{\partial\tilde{V}_a}{\partial\kappa}\frac{r^2+a^2}{\Delta_-}u\\
&=h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\kappa}{h}\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u_\kappa-\frac{\partial\tilde{V}_a}{\partial\kappa}\frac{r^2+a^2}{\Delta_-}u\\
&=h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v}{h}\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}v-\frac{\partial\tilde{V}_a}{\partial\kappa}\frac{r^2+a^2}{\Delta_-}u\\
&\quad+h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}fu\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}fu.
\end{aligned}$$

Multiplying by $\overline{u}$, integrating over $(r_+,\infty)$, integrating by parts as in Section 2.4.2 and evaluating at $\kappa=\kappa_0$ yields:

$$\begin{aligned}
&\int_{r_+}^{\infty}\frac{\partial}{\partial\kappa}\left(h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u}{h}\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u\right)\overline{u}\,\mathrm{d}r\bigg|_{\kappa=\kappa_0}\\
=&2\kappa\int_{r_+}^{\infty}\frac{1}{r^2+a^2}\left(\frac{\partial\lambda}{\partial\alpha}-\frac{a^2}{\ell^2}\right)|u|^2\,\mathrm{d}r\bigg|_{\kappa=\kappa_0}\\
&-\int_{r_+}^{\infty}\left(\frac{2\kappa}{\ell^2}\frac{r^2}{r^2+a^2}|u|^2-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}f}{\mathrm{d}r}\right)\left|h^{-1}u\right|^2\right)\,\mathrm{d}r\bigg|_{\kappa=\kappa_0}\\
&-\int_{r_+}^{\infty}f\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2+\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}|u|^2\right)\,\mathrm{d}r\bigg|_{\kappa=\kappa_0}\\
&-\int_{r_+}^{\infty}f\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2\,\mathrm{d}r,\bigg|_{\kappa=\kappa_0}.
\end{aligned}\tag{2.4.14}$$

By repeating the proof of Section 2.4.2, one shows that the right hand side is negative.

For the left hand side, we compute:

$$\begin{aligned}
&\frac{\partial}{\partial\kappa}\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2+\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}|u|^2\right)\\
=&\frac{\partial^2}{\partial r\partial\kappa}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)h^{-1}\overline{u}\right)\\
&\quad-\frac{\partial}{\partial\kappa}\left(h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u}{h}\right)\right)\overline{u}-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u\overline{u}\right)\\
=&-\frac{\partial}{\partial\kappa}\left(h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u}{h}\right)\right)-\tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}u\right)\overline{u}
\end{aligned}$$

$$
\begin{aligned}
&- \left( h^{-1} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{u}{h} \right) \right) - \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} u \right) \frac{\partial \overline{u}}{\partial \kappa} \\
&+ \frac{\partial^2}{\partial r \partial \kappa} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}}{\mathrm{d}r} \left( h^{-1} u \right) h^{-1} \overline{u} \right)
\end{aligned}
$$

Again we have

$$
\left( h^{-1} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{u}{h} \right) \right) - \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} u \right) \Big|_{\kappa=\kappa_0} = 0.
$$

Moreover,

$$
\frac{\partial}{\partial \kappa} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}}{\mathrm{d}r} \left( h^{-1} u \right) h^{-1} \overline{u} \right) \Big|_{\kappa=\kappa_0} \sim r^{1+2\kappa_0} \frac{\mathrm{d}}{\mathrm{d}r} \left( r^{1/2-\kappa_0} u_0 \right) \log r.
$$

Therefore,

$$
\frac{\partial}{\partial \kappa} \tilde{\mathcal{L}}_{\alpha, r_+, a}(u(r,\kappa)) \Big|_{\kappa=\kappa_0} = - \int_{r_+}^{\infty} \frac{\partial}{\partial \kappa} \left( h^{-1} \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{\Delta_-}{r^2+a^2} h^2 \frac{\mathrm{d}}{\mathrm{d}r} \left( \frac{u}{h} \right) \right) - \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} u \right) \overline{u} \, \mathrm{d}r,
$$

whence positivity by (**??**. □

**Corollary 2.4.16.** *For all $\alpha \in (5/4, 9/4)$, there is a $\delta > 0$ such that $\Phi(\cdot, r_+)$ is decreasing in $[\alpha, \alpha + \delta)$*

*Proof.* Lemma 2.4.15 shows that monotonicity is an open property. □

Now we can prove the main lemma.

*Proof of Lemma 2.4.14.* Let $\varepsilon > 0$. By Corollary 2.4.16, there is an indexing set $X$ and disjoint half-open intervals $I_\beta$, $\beta \in X$, containing their left endpoints, such that one has $\cup_{\beta \in X} I_\beta = [5/4 + \varepsilon, 9/4)$ and $\Phi(\cdot, r_+)\big|_{I_\beta}$ is decreasing for all $\beta \in X$. For $\alpha \in \cup_{\beta \in X} \mathring{I}_\beta$, set

$$
\Psi(\alpha, r_+) := \Phi(\alpha, r_+).
$$

Let $\alpha_0 \in \partial I_{\beta_1} \cap I_{\beta_2}$. Choose a sequence $(\alpha_k) \subseteq I_{\beta_1}$ such that $\alpha_k \to \alpha_0$. As the sequence $(\Phi(\alpha_k, r_+))$ is monotonically decreasing and bounded below, it is convergent. We set $a_0 := \lim_k \Phi(\alpha_k, r_+)$. Let $f_{\alpha_k}$ be the unique solution to the radial ODE with parameters $\alpha_k$, $a_k = \Phi(\alpha_k, r_+)$ and $\omega_k = \Omega_R(\alpha_k)$ – see definition (2.3.18). Let $f_{\alpha_0}$ be the unique solution corresponding to the parameters $\alpha_0$ and $a_0$. Since all $f_{\alpha_k}$ satisfy the Neumann boundary condition, continuity of the reflection and transmission coefficients yields that $f_{\alpha_0}$ satisfies the Neumann boundary condition as well. We set

$$
\Psi(\alpha_0, r_+) := a_0.
$$

As we can repeat this construction for all $\varepsilon > 0$ and all jump points $\alpha_0$, we obtain a function $\Psi(\cdot, r_+)$ defined in $(5/4, 9/4)$, whose values correspond to parameters $a$ with periodic mode solutions.

Since we have left-continuity at the jump points by construction, it remains to show that $\Psi(\cdot, r_+)$ is left-continuous in $\alpha \in \cup_{\beta\in X} \mathring{I}_\beta$, which can be proved as Lemma 2.3.19: Suppose not. Then there is an $\varepsilon > 0$ such that, for all $\delta > 0$, there is an $\alpha' < \alpha$ with

$$\alpha - \alpha' < \delta$$

and

$$\Psi(\alpha', r_+) - \Psi(\alpha, r_+) \geq \varepsilon.$$

Then there is an $a$ between $\Psi(\alpha, r_+)$ and $\Psi(\alpha', r_+)$ such that there is an $g$ with $a \in \mathcal{A}_{\alpha,r_+}(r^{-1/2+\kappa}g)$, but, for each $\delta$, there is an $\alpha'$ with $a \notin \mathcal{A}_{\alpha',r_+}(r^{-1/2+\kappa}g)$. Therefore, since $\tilde{\mathcal{L}}_{\alpha,r_+,a}(r^{-1/2+\kappa}g) < 0$ and due to the continuity (2.4.13), there is a $\delta > 0$ such that for all $\alpha - \alpha' < \delta$, we have $\tilde{\mathcal{L}}_{\alpha',r_+,a}(r^{-1/2+\kappa}g) < 0$, i. e. $a \in \mathcal{A}_{\alpha',r_+}(r^{-1/2+\kappa}g)$, a contradiction. $\square$

## 2.5. Optimally dissipative boundary conditions

This section is devoted to the proof of Theorem 2.1.18, i. e. we want to show that there are no non-trivial exponentially growing mode solutions

$$\psi = \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{e}^{\mathrm{i}m\tilde{\varphi}}\, S_{ml}(\cos\vartheta)\frac{u}{(r^2+a^2)^{1/2}}$$

satisfying the optimally dissipative boundary condition

$$\ell^2 \frac{\partial}{\partial t}\left(r^{3/2-\kappa}\psi\right) + r^{1+2\kappa}\frac{\partial}{\partial r}\left(r^{3/2-\kappa}\psi\right) \to 0$$

as $r \to \infty$, studied in [HLSW15]. In terms of reflection and transmission coefficients, this imposes

$$B = -\mathrm{i}\frac{\omega\ell^2}{2\kappa}A$$

on the modes.

Using the microlocal energy current

$$Q_T(r) = \mathrm{Im}\,\left(u'\overline{u}\right),$$

we know that

$$\frac{\Xi a m - (r_+^2+a^2)\omega_R}{r_+^2+a^2} + \int_{r_+}^{\infty} \frac{r^2+a^2}{\Delta_-}\mathrm{Im}\,\left(V_a^h - \omega^2\right)|u|^2\,\mathrm{d}r = \frac{2\kappa}{\ell^2}\mathrm{Im}\,\left(A\overline{B}\right) = \omega_R|A|^2 \quad (2.5.1)$$

for a mode $\psi$ satisfying the optimally dissipative boundary condition; the calculation is the same as in in Section 2.3.2.

Let us assume now that $\psi$ is an exponentially growing mode solution. We will show that

$u = 0$. Since $u$ solves the radial ODE,

$$-\mathrm{i}\,\omega|A|^2 + \int_{r_+}^{\infty}\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2 + \frac{r^2+a^2}{\Delta_-}\tilde{V}_a^h|u|^2\right)\mathrm{d}r = 0$$

or, in terms of real and imaginary parts:

$$\begin{aligned}\omega_I|A|^2 + \int_{r_+}^{\infty}\left(\frac{\Delta_-}{r^2+a^2}h^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}u\right)\right|^2 + \frac{r^2+a^2}{\Delta_-}\mathrm{Re}\left(\tilde{V}_a^h\right)|u|^2\right)\mathrm{d}r = 0\\ -\omega_R|A|^2 + \int_{r_+}^{\infty}\frac{r^2+a^2}{\Delta_-}\mathrm{Im}\left(\tilde{V}_a^h\right)|u|^2\,\mathrm{d}r = 0.\end{aligned} \tag{2.5.2}$$

Comparing with (2.5.1), we see that

$$\omega_R = \frac{\Xi a m}{r_+^2+a^2}.$$

We compute

$$\begin{aligned}\mathrm{Im}\left(V_a^h-\omega^2\right) &= \frac{\Delta_-}{(r^2+a^2)^2}\mathrm{Im}\,\lambda + \frac{\Delta_-}{(r^2+a^2)^2}2\omega_R\omega_I a^2 - 2ma\Xi\omega_I\frac{\Delta_- - (r^2+a^2)}{(r^2+a^2)^2} - 2\omega_R\omega_I\\ &= \frac{\Delta_-}{(r^2+a^2)^2}\mathrm{Im}\,\lambda\\ &\quad + \frac{2\omega_R\omega_I}{(r^2+a^2)^2}\left[a^2\Delta_- - (r_+^2+a^2)\Delta_- + (r^2+a^2)(r_+^2+a^2) - (r^2+a^2)^2\right]\\ &= \frac{\Delta_-}{(r^2+a^2)^2}\mathrm{Im}\,\lambda + \frac{2\omega_R\omega_I}{(r^2+a^2)^2}\left[-r_+^2\Delta_- - (r^2-r_+^2)(r^2+a^2)\right].\end{aligned}$$

Multiplying the angular ODE for $\omega$ by $\overline{S}$, multiplying the angular ODE for $\overline{\omega}$ by $S$, integrating by parts and subtracting the two, we see

$$\lambda - \overline{\lambda} = -\int_0^{\pi}\left(\frac{\Xi}{\Delta_\vartheta}a^2(\omega^2-\overline{\omega}^2)\cos^2\vartheta + 2ma(\omega-\overline{\omega})\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2}\cos^2\vartheta\right)|S|^2\sin\vartheta\,\mathrm{d}\vartheta,$$

whence

$$\mathrm{Im}\,\lambda = -2\omega_R\omega_I\int_0^{\pi}\left(\frac{\Xi}{\Delta_\vartheta}a^2 + \frac{r_+^2+a^2}{\Delta_\vartheta}\frac{a^2}{\ell^2}\right)|S|^2\cos^2\vartheta\,\sin\vartheta\,\mathrm{d}\vartheta.$$

If $\omega_I > 0$, there is a constant $C > 0$ such that

$$\frac{1}{\omega_R}\mathrm{Im}\left(V_a^h-\omega^2\right) < -C\frac{\Delta_-}{(r^2+a^2)^2}\omega_I,$$

which contradicts (2.5.2) unless $u = 0$.

# 3. Absence of horizons in perturbations of Eguchi-Hanson-AdS spacetimes

## 3.1. Introduction

### 3.1.1. The Einstein vacuum equations with negative cosmological constant

The Einstein vacuum equations in $n$ dimensions ($n > 2$)

$$\mathrm{Ric}(g) = \frac{2}{n-2}\Lambda g \tag{3.1.1}$$

with cosmological constant $\Lambda$ can be understood as a system of second-order partial differential equations for the metric tensor $g$ of an $n$-dimensional spacetime $(\mathcal{M}, g)$. Solutions with negative cosmological constant $\Lambda = -(n-1)(n-2)/(2\ell^2) < 0$ have drawn considerable attention in recent years, mainly due to the conjectured instability of these spacetimes.

The system (3.1.1) is of hyperbolic nature, and studying the dynamic evolution of initial data is very difficult in general, leading us to take recourse to settings with high degrees of symmetry. In particular, it is desirable to reduce the dimension of the dynamical problem to the simplest case of $1 + 1$ dimensions. This approach has a longer history for $\Lambda = 0$. There, for $n = 4$, the only symmetry group achieving the reduction to a $(1 + 1)$-dimensional problem whilst consistent with the spacetime being asymptotically flat is spherical symmetry. However, the well-known Birkhoff theorem prevents any dynamical consideration since such a four-dimensional spacetime is necessarily static, embedding locally into a subset of a member of the Schwarzschild family.

To study spherically symmetric gravitational dynamics in four dimensions, one can follow the approach of the seminal work by Christodoulou and couple gravity to matter. In a sequence of papers – see his own survey article [Chr99] for references – he initiated the rigorous analysis of spherically symmetric gravitational collapse for $\Lambda = 0$ by studying the Einstein-scalar field system. The model of a real massless scalar field was chosen because, on the one hand, this matter model does not develop singularities in the absence of gravity and, on the other hand, its wave-like character resembles the character of general gravitational perturbations of Minkowski space. Christodoulou's work led to a complete understanding of weak and strong cosmic censorship for this model. His approach has later been extended to other matter models; see [Kom13] for a systematic overview and references.

Christodoulou's approach was adapted to the context of $\Lambda < 0$ by Holzegel and Smulevici in [HS11b] and by Holzegel and Warnick in [HW13], who show well-posedness of the Einstein-

Klein-Gordon system with the scalar field satisfying various reflecting boundary conditions at infinity. The work [HS11c] shows stability of Schwarzschild-AdS in this symmetry class for Dirichlet boundary conditions. A recent breakthrough has been achieved by Moschidis in [Mos17a] and [Mos17b]; in his work, he shows instability of exact anti-de Sitter space as a solution to the Einstein-null dust system in spherical symmetry with an inner mirror.

Another possibility of evading the restrictions of Birkhoff's Theorem is to study (3.1.1) in higher dimensions. Working in five dimensions and imposing biaxial Bianchi IX symmetry, a symmetry corresponding to a subgroup of $SO(4)$, still reduces the system to 1+1 dimensions and introduces a dynamical variable $B$, not dissimilar to the scalar field in the coupled system. This model was introduced by Bizón, Chmaj and Schmidt. In [BCS05], they initiated the study of gravitational collapse for $\Lambda = 0$ in this symmetry class by numerical computations; investigations along those lines were continued in [DH06b] and [BCS06].

The study of this system in the realm of negative $\Lambda$ has been initiated by Dafermos and Holzegel in 2006. In [DH06a] – now mostly cited for the conjecture of the instability of exact AdS space –, our Corollary 3.1.21 has been put forward without rigorous proof. Back then, the problem of proving local well-posedness for the system in biaxial Bianchi IX symmetry was not solved, thus no extension principle sufficiently strong was available. The present paper can be seen as a completion of [DH06a], building on the insight into problems in asymptotically locally AdS spacetimes obtained over the past decade; for an overview of this work, see [HW13], [EK14] and references therein.

In five dimensions and for $\Lambda < 0$, (3.1.1) has many static solutions which are asymptotically locally AdS. A spacetime is asymptotically locally AdS if the asymptotics of the metric towards conformal infinity $\mathcal{I}$ is modelled after AdS space, but $\mathcal{I}$ need not be $\mathbb{R} \times S^3$ topologically. Prominent examples of such static solutions are exact $\mathrm{AdS}_5$ space with spherical conformal infinity[1] and the AdS soliton of [HM98b] with toric $\mathcal{I}$. Eguchi-Hanson-AdS spacetimes form another such family with $\mathcal{I} \cong \mathbb{R} \times (S^3/\mathbb{Z}_n)$ for $n \geq 3$.[2]

### 3.1.2. Spaces of Eguchi-Hanson type

We introduce four-dimensional Riemannian manifolds modifying Eguchi-Hanson space to the asymptotically locally AdS context. These will serve as initial data for the five-dimensional Einstein vacuum equations

$$\mathrm{Ric}(g) = \frac{2}{3}\Lambda g \tag{3.1.2}$$

via the local well-posedness Theorem 3.1.7. Our data also exhibit an $SU(2)\times U(1)$ symmetry, thus giving rise to spacetimes with biaxial Bianchi IX symmetry. Then Eguchi-Hanson-AdS spacetimes form particular examples of the spacetimes thus obtained.

[1] Numerical studies within the biaxial Bianchi IX symmetry class for perturbations of $\mathrm{AdS}_5$ were carried out recently in [BR17].

[2] The space $S^3/\mathbb{Z}_n$ is defined in the usual way as the lense space $L(n,1)$.

**Definition 3.1.1.** We say that an initial data set $(\mathcal{S}, \overline{g}, K)$ to (3.1.2) exhibiting $SU(2)\times U(1)$ symmetry is of Eguchi-Hanson type if

$$\mathcal{S} = (a, \infty) \times \left(S^3/\mathbb{Z}_n\right)$$

for fixed $a > 0$ and $n \geq 3$ satisfying

$$\frac{n^2}{4} = 1 + \frac{a^2}{\ell^2}$$

(in particular $a > \ell$), and if

$$\overline{g} = \frac{1}{A}\mathrm{d}\mathfrak{r}^2 + \gamma \quad \text{ with } \quad \gamma = \frac{1}{4}r^2\,\mathrm{e}^{2B}\left(\sigma_1^2 + \sigma_2^2\right) + \frac{1}{4}r^2\,\mathrm{e}^{-4B}\,\sigma_3^2,$$

where $\mathfrak{r} \in (a, \infty)$ and $(\sigma_1, \sigma_2, \sigma_3)$ is a basis of left-invariant one-forms on $SU(2)$ (see below) and

$$A,\, r,\, B :\, (a, \infty) \to \mathbb{R}$$

are smooth functions such that the following conditions hold:

(i) Around the centre $\mathfrak{r} = a$, the functions satisfy the regularity conditions

$$\begin{aligned} A =& \frac{n^2}{a}(\mathfrak{r} - a) + \mathcal{O}\left((\mathfrak{r} - a)^2\right) \\ r =& 2^{1/3}a^{5/6}(\mathfrak{r} - a)^{1/6} + \mathcal{O}\left((\mathfrak{r} - a)^{7/6}\right) \\ B + \log r =& \log a + \mathcal{O}((\mathfrak{r} - a)). \end{aligned}$$

(ii) The function $A$ is non-zero on $(a, \infty)$. Moreover

$$A = \frac{\mathfrak{r}^2}{\ell^2} + 1 + o(1)$$

as $\mathfrak{r} \to \infty$.

(iii) The function $r$ is the radius of the topological 3-spheres at $\varrho$. Moreover

$$r = \mathfrak{r} + \mathcal{O}(1)$$

as $\mathfrak{r} \to \infty$.

(iv) For an $R > a$, we have

$$\int_R^\infty \left(\mathfrak{r}^3 B^2 + \mathfrak{r}^7\left(\partial_\mathfrak{r} B\right)^2\right)\mathrm{d}\mathfrak{r} < C \quad \text{and} \quad \sup_{\mathfrak{r}\in(R,\infty)} |\mathfrak{r}^3\partial_\mathfrak{r} B| < C.$$

We require that

$$M := \lim_{\mathfrak{r}\to\infty}\left(\frac{r^2}{2}\left[1 + \frac{r^2}{12}\left(\left(\mathrm{tr}_\gamma K\right)^2 - H^2\right)\right] + \frac{r^4}{2\ell^2}\right),$$

where $H$ is the mean curvature of the symmetry orbits, is finite; we call $M$ the mass of $(\mathcal{S}, \overline{g}, K)$ at infinity.

*Remark* 3.1.2. 1. The left-invariant one-forms satisfy

$$\mathrm{d}\sigma_1 + \sigma_2 \wedge \sigma_3 = 0, \ \mathrm{d}\sigma_2 + \sigma_3 \wedge \sigma_1 = 0, \ \mathrm{d}\sigma_3 + \sigma_1 \wedge \sigma_2 = 0.$$

One can choose Euler angles $(\vartheta, \varphi, \psi)$, $0 < \vartheta < \pi$, $0 \leq \varphi < 2\pi$, $0 \leq \psi < 4\pi$ on $SU(2)$ such that

$$\sigma_1 = \sin\vartheta\, \sin\psi\, \mathrm{d}\varphi + \cos\psi\, \mathrm{d}\vartheta, \quad \sigma_2 = \sin\vartheta\, \cos\psi\, \mathrm{d}\varphi - \sin\psi\, \mathrm{d}\vartheta, \quad \sigma_3 = \cos\vartheta\, \mathrm{d}\varphi + \mathrm{d}\psi.$$

In terms of the left-invariant one-forms, the Minkowski metric on $\mathbb{R}^5$ is given by

$$g_{\mathrm{Mink}} = -\mathrm{d}t^2 + \mathrm{d}r^2 + \frac{1}{4} r^2 \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right).$$

The Euler angles $(\vartheta, \varphi, \psi)$ parametrise the 3-sphere away from the poles. By restricting $\psi$ to have period $4\pi/n$, we obtain coordinates on $S^3/\mathbb{Z}_n$.

2. By identity (C.1.1), the notion of mass at infinity is consistent with the renormalised Hawking mass introduced in Definition 3.1.4.
3. The triple $(\mathcal{S}, \overline{g}, K)$ is asymptotically locally AdS, consistent with Definition 3.1.5.

Prima facie it seems as if $\overline{g}$ had a singularity at $\mathfrak{r} = a$. However, one should compare this situation to that of spherical symmetry in spherical coordinates. This intuition is made more precise in the following

**Proposition 3.1.3.** *Let $(\mathcal{S}, \overline{g})$ be of Eguchi-Hanson type. Then there is a $b > 0$ such that for $\varrho \in (a, b)$, $(\mathcal{S} \cap \{\varrho \leq b\}, \overline{g}|_{\{\varrho \leq b\}})$ has topology $\mathbb{R}^2 \times S^2$ and can be smoothly compactified by adding a 2-sphere at $\varrho = a$. The resulting manifold is smooth and has no boundary.*

*Proof.* Define

$$z := \frac{4\sqrt{a}}{n} (\varrho - a)^{1/2}.$$

To leading order, we have

$$r^2\, \mathrm{e}^{-4B} \sim \frac{r^6}{a^4} \sim 4a\, (\mathfrak{r} - a) = \frac{1}{4} n^2 z^2$$

around $\mathfrak{r} = a$. Thus the metric becomes

$$\overline{g} \sim \frac{1}{4} \left(\mathrm{d}z^2 + n^2 z^2 \left(\mathrm{d}\psi + \cos\vartheta \mathrm{d}\varphi\right)^2\right) + \frac{\mathfrak{r}^2}{4} \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right)$$

to leading order. For fixed $(\vartheta, \varphi)$, the restriction on the range of $\psi$ (see Remark 3.1.2) guarantees that the metric can be extended smoothly to $\mathfrak{r} = a$. By adding an $S^2$ at $\mathfrak{r} = a$, we obtain a manifold without boundary that has local topology $\mathbb{R}^2 \times S^2$. □

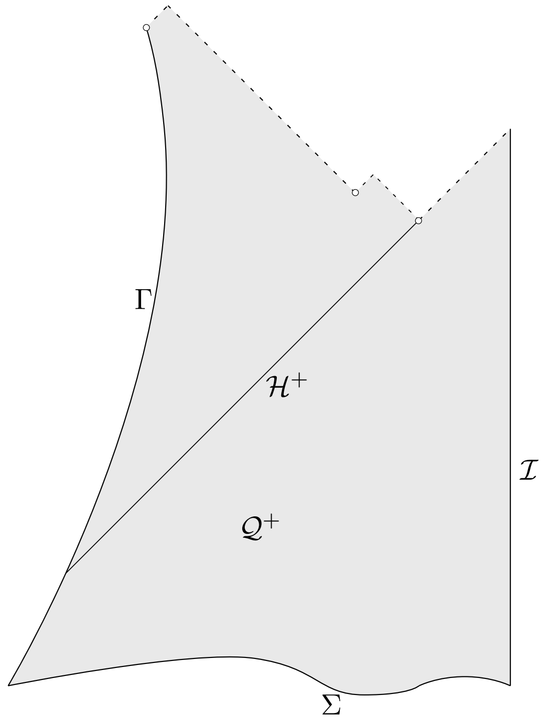



Figure 3.1.: A general Penrose diagram of the future maximal development of initial data of Eguchi-Hanson type via Theorem 3.1.7

Given Eguchi-Hanson-type initial data, the Einstein vacuum equations (3.1.2) are well-posed in the biaxial Bianchi IX symmetry class, which is the first fundamental result of this chapter of the thesis.

**Definition 3.1.4.** Let $(\mathcal{M}, g)$ be a five-dimensional spacetime. Then $(\mathcal{M}, g)$ exhibits a biaxial Bianchi IX symmetry if, topologically,

$$\mathcal{M} = \mathcal{Q} \times \left(S^3/\Gamma\right)$$

for $\mathcal{Q}$ a two-dimensional manifold (possibly with boundary) and a discrete group $\Gamma \in \{\emptyset, \mathbb{Z}_2, \mathbb{Z}_3, \ldots\}$ such that

$$g = h + \frac{1}{4} r^2 \left(\mathrm{e}^{2B}(\sigma_1^2 + \sigma_2^2) + \mathrm{e}^{-4B}\, \sigma_3^2\right). \tag{3.1.3}$$

Here $h$ is a Lorentzian metric on $\mathcal{Q}$, and $r$ and $B$ are smooth real-valued functions on $\mathcal{Q}$. The value $r(q)$ is the area radius of the squashed sphere through $q \in \mathcal{Q}$, i. e.

$$2\pi^2 r^3 = \mathrm{vol}\left(S_q^3\right),$$

where $S_q^3$ is the sphere at $q$. In this symmetry class, we introduce the renormalised Hawking mass (henceforth referred to as Hawking mass)

$$m:\ \mathcal{Q}\to\mathbb{R}$$

by

$$m=\frac{r^2}{2}\left(1-g\left(\nabla r,\nabla r\right)\right)+\frac{r^4}{2\ell^2}.$$

**Definition 3.1.5.** A spacetime $(\mathcal{M},g)$ exhibiting biaxial Bianchi IX symmetry is asymptotically locally AdS with radius $\ell$ and conformal infinity $\mathcal{I}$ if it is conformally equivalent to a manifold $(\tilde{\mathcal{M}},\tilde{g})$ with boundary $\mathcal{I}:=\partial\tilde{\mathcal{M}}$ such that

(i) Conformal infinity $\mathcal{I}$ has topology $\mathbb{R}\times(S^3/\Gamma)$.

(ii) The inverse $\tilde{r}:=r^{-1}$ is a boundary defining function for $\mathcal{I}$, i. e. $\tilde{r}=0$ and $\mathrm{d}\tilde{r}\neq 0$ on $\mathcal{I}$.

(iii) The rescaled metric $r^{-2}g$ is a smooth metric on a neighbourhood of $\mathcal{I}$ in $\tilde{\mathcal{M}}$.

(iv) For small $\tilde{r}>0$, there exist coordinates $(t,\tilde{r})$ on $\mathcal{Q}$ such that, locally,

$$h\left(\frac{1}{\tilde{r}^2}\partial_{\tilde{r}},\frac{1}{\tilde{r}^2}\partial_{\tilde{r}}\right)=\ell^2\tilde{r}^2+\mathcal{O}\left(\tilde{r}^4\right).$$

in a neighbourhood of $\mathcal{I}$.

(v) The quantity $B$ satisfies a Dirichlet boundary condition, i. e. $B=0$ on $\mathcal{I}$.

*Remark* 3.1.6. If $B=0$, a Birkhoff-type theorem implies that the spacetime is a member of the five-dimensional Schwarzschild-AdS family. This can be seen explicitly and directly from the system of equations for $\Omega$ and $r$ in Theorem C.2.1: If we impose asymptotically AdS boundary conditions at infinity and integrate the equations, we obtain a spacetime that is a member of the Schwarzschild-AdS family.

Then the following theorem can be proven, whose sketch of proof is contained in Section 3.1.4:

**Theorem 3.1.7.** *Let $(\mathcal{S},\overline{g},K)$ be an initial data set with mass $M$ at infinity such that $(\mathcal{S},\overline{g},K)$ is of Eguchi-Hanson type. Then there is a $T>0$ and a manifold $\mathcal{M}:=(-T,T)\times\mathcal{S}$ equipped with a metric $g$ exhibiting biaxial Bianchi IX symmetry such that $(\mathcal{M},g)$ is asymptoticall locally AdS, $g$ solves* (3.1.2) *and $\{0\}\times\mathcal{S}$ has induced metric $\overline{g}$ and second fundamental form $K$. Moreover, $(\mathcal{M},g)$ is the unique asymptotically locally AdS solution to* (3.1.2) *with initial data $(\mathcal{S},\overline{g},K)$.*

*Remark* 3.1.8.
1. The local well-posedness theorem for an initial data set $(\mathcal{S},\overline{g},K)$ yields the existence of a unique maximal development in the sense of [HS11b].
2. The proof of the local well-posedness theorem in this thesis proceeds along the lines of [HW13]. Well-posedness of the Einstein vacuum equations for $\Lambda<0$ in four dimensions without symmetry assumptions was shown by Friedrich in [Fri95], and a recent generalisation to higher dimensions by Enciso and Kamran is also available; see [EK14]. In particular, Theorem 3.1.7 follows from their work. However the theorem

as stated is too general for an extension principle (Section 3.3.2); so to exploit the monotonicity of the Hawking mass (see Section 3.3.3), a local well-posedness result in norms propagated by the mass is required.

The explicit examples behind this well-posedness theorem are Eguchi-Hanson-AdS spacetimes, constructed in [CM06]. They form a family of solutions $(\mathcal{M}_{\mathrm{EH},a}, g_{\mathrm{EH},a})$ to (3.1.1) in five dimensions. For fixed $\Lambda = -6/\ell^2 < 0$, they form a one-parameter family of static spacetimes exhibiting biaxial Bianchi IX symmetry. If we define

$$r = \mathfrak{r}\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)^{1/6}, \qquad \Omega^2 = 1 + \frac{\mathfrak{r}^2}{\ell^2}, \qquad B = -\frac{1}{6}\log\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)$$

and choose coordinates such that

$$h = -\frac{1}{2}\Omega^2\left(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u\right),$$

with

$$\mathrm{d}u = \mathrm{d}t - \frac{1}{\left(1 + \frac{\mathfrak{r}^2}{\ell^2}\right)\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)^{1/2}}\mathrm{d}\mathfrak{r}, \qquad \mathrm{d}v = \mathrm{d}t + \frac{1}{\left(1 + \frac{\mathfrak{r}^2}{\ell^2}\right)\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)^{1/2}}\mathrm{d}\mathfrak{r},$$

the metric takes the form

$$\begin{aligned} g_{\mathrm{EH},a} = &-\left(1 + \frac{\mathfrak{r}^2}{\ell^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1 + \frac{\mathfrak{r}^2}{\ell^2}\right)\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)}\mathrm{d}\varrho^2 \\ &+ \frac{1}{4}\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)\mathfrak{r}^2\left(\mathrm{d}\psi + \cos\vartheta\,\mathrm{d}\varphi\right)^2 + \frac{\mathfrak{r}^2}{4}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \end{aligned}$$

in $(t, \mathfrak{r}, \vartheta, \varphi, \psi)$ variables with $\mathfrak{r} \in (a, \infty)$. In the limit $\ell \to \infty$, the metric $g_{\mathrm{EH},a}$ restricted to hypersurfaces of constant $t$ yields the Riemannian Eguchi-Hanson metric, which was first presented in [EH78].

We immediately note:

**Proposition 3.1.9.** *Let $(\mathcal{M}_{\mathrm{EH},a}, g_{\mathrm{EH},a})$ be an Eguchi-Hanson-AdS spacetime. Then*

$$M_{\mathrm{EH},a} := \lim_{\mathfrak{r}\to\infty} m = -\frac{5}{6}\frac{a^4}{\ell^2}$$

*is negative. At the centre $\mathfrak{r} = a$, the Hawking mass is ill-defined, tending to $-\infty$.*

*Proof.* First note that

$$\begin{aligned} r_u =& \frac{\partial\mathfrak{r}}{\partial u}\partial_{\mathfrak{r}} r = -\frac{\left(1 + \frac{\mathfrak{r}^2}{\ell^2}\right)\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)^{1/2}}{2}\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)^{-5/6}\left(1 + \frac{1}{3}\frac{a^4}{\mathfrak{r}^4}\right) \\ r_v =& \frac{\partial\mathfrak{r}}{\partial v}\partial_{\mathfrak{r}} r = +\frac{\left(1 + \frac{\mathfrak{r}^2}{\ell^2}\right)\left(1 - \frac{a^4}{\mathfrak{r}^4}\right)^{1/2}}{2}\left(1 - \frac{a^4}{\varrho^4}\right)^{-5/6}\left(1 + \frac{1}{3}\frac{a^4}{\mathfrak{r}^4}\right). \end{aligned}$$

Thus,

$$4\frac{r_u r_v}{\Omega^2} = -\left(1+\frac{\mathfrak{r}^2}{\ell^2}\right)\left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{-2/3}\left(1+\frac{1}{3}\frac{a^4}{\mathfrak{r}^4}\right)^2.$$

Therefore

$$\begin{aligned} m =& \frac{\mathfrak{r}^4}{2\ell^2}\left[\left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{2/3} - \left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{-1/3}\left(1+\frac{1}{3}\frac{a^4}{\mathfrak{r}^4}\right)^2\right] \\ &+ \frac{\mathfrak{r}^2}{2}\left[\left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{1/3} - \left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{-1/3}\left(1+\frac{1}{3}\frac{a^4}{\mathfrak{r}^4}\right)^2\right]. \end{aligned}$$

The terms in brackets are of order $\mathfrak{r}^{-4}$. Therefore the mass is obtained from the leading order term of the first expression:

$$\frac{\mathfrak{r}^4}{2\ell^2}\left[\left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{2/3} - \left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{-1/3}\left(1+\frac{1}{3}\frac{a^4}{\mathfrak{r}^4}\right)^2\right] = \frac{\mathfrak{r}^2}{2\ell^2}\left(1-\frac{a^4}{\mathfrak{r}^4}\right)^{-1/3}\left[-\frac{5}{3}\frac{a^4}{\ell^2} - \frac{1}{9}\frac{a^8}{\mathfrak{r}^8}\right].$$

This yields the value for $M_{\mathrm{EH},a}$. □

### 3.1.3. The significance of Eguchi-Hanson-AdS spacetimes

The main motivation that sparked recent interest in asymptotically locally AdS solutions to the Einstein vacuum equations within the physics community is a putative connection between spacetimes of this form and conformal field theories defined on their respective boundaries: the AdS-CFT correspondence. It is of interest to understand what the positivity of gravitational energy means in the conformal field theory and thus 'ground states', lowest energy configurations classically allowed, deserve consideration – see [Mal99], [GKP98], [Wit98], [AGM$^+$00] and [GSW02] for more details and references on the issue of gravitational energy in this context.

A ground state depends heavily on the topology at infinity. If the spacetime is asymptotically AdS, this ground state is exact anti-de Sitter space with vanishing mass – see [BGH84]. For asymptotically locally AdS spacetimes with toroidal topology at infinity, the works [HM98a], [HM98b] and [GSW02] lend support to the conjecture that the so-called AdS soliton is the ground state in a suitable class of spacetimes.

The article [CM06] was motivated by searching for a spacetime that asymptotically approaches $\mathrm{AdS}_5/\Gamma$, where $\Gamma$ is any freely acting discrete group of isometries, but has energy less than that of $\mathrm{AdS}_5/\Gamma$. This led to the Eguchi-Hanson-AdS solution in five dimensions. These spacetimes have also been conjectured in [CM06] to have minimal mass among asymptotically locally AdS spacetimes with topology $\mathrm{AdS}_5/\mathbb{Z}_n$ at infinity:

**Conjecture 3.1.10.** *Let $(\mathcal{S}, \overline{g}, K)$ be of Eguchi-Hanson type with $\mathcal{S} = (a,\infty)\times(S^3/\mathbb{Z}_n)$, then*

$$M \geq M_{\mathrm{EH},a}$$

*with equality if and only if the data agree with those induced by the Eguchi-Hanson-AdS spacetime with parameter $a$.*

In a neighbourhood of Eguchi-Hanson-AdS solutions, this was indeed shown to be true:

**Theorem 3.1.11** ([DH06a], see also [CM06])**.** *Given any $a > 0$, assume initial data $(\mathcal{S}, \overline{g}, K)$ of Eguchi-Hanson type with $\mathcal{S} = (a, \infty) \times (S^3/\mathbb{Z}_n)$ which are a sufficiently small, but non-zero perturbation of the data induced by the Eguchi-Hanson-AdS spacetime with parameter $a$, then the mass $M$ at infinity satisfies*

$$M_{\mathrm{EH},a} < M < 0. \tag{3.1.4}$$

Motivated by the static uniqueness theorem for exact AdS space, one conjectures – see [DH06a]:

**Conjecture 3.1.12.** *There are no static, globally regular asymptotically locally AdS solutions to (3.1.2) with topology $S^3/\mathbb{Z}_n$ with mass $M$ satisfying (3.1.4).*

Thus, fixing $a$, the Eguchi-Hanson-AdS spacetime satisfying (3.1.4) can be seen as the ground state in the biaxial Bianchi IX symmetry class. There is a folklore statement that such ground states would be stable under gravitational perturbations. However, in contrast, the present work, paired with the above conjectures, heuristically hints at an instability: Perturbing an Eguchi-Hanson-AdS spacetime slightly increases its mass at infinity, whilst remaining negative; therefore, by Corollary 3.1.21, the future maximal development cannot contain a black hole, but by Conjecture 3.1.12, there is no static end state for the perturbation, which intimates that a first singularity forms, emanating from the centre. Therefore, such perturbations are potential candidates for examples of the formation of naked singularities.

It is interesting to note that a dual situation is found for perturbations of $\mathrm{AdS}_3$, as investigated in [BJ13]. There, small perturbations of three-dimensional AdS space were studied numerically as solutions to the Einstein-scalar field system. The parallel to our case is that in three dimensions, there exists a mass threshold below which no black holes can form. In contrast, while the numerical computations of [BJ13] suggest turbulence which cannot be terminated by a black hole formation, they provide evidence that small perturbations remain globally regular in time since the turbulence is too weak.

Finally, studying five-dimensional static spacetimes for various values of $\Lambda$ or, more precisely, classifying their four-dimensional Riemannian counterparts is still an active field of research in geometry. It is known that there are exactly four complete non-singular four-dimensional Ricci flat Riemannian spaces: Euclidean space, Eguchi-Hanson space, self-dual Taub-NUT space and Taub-Bolt space. See [Gib05] for further details. Moreover, Eguchi-Hanson space has been used in geometric gluing constructions; see [Biq13] and [BK17]. For more results in this realm, both classical and recent, see [BGPP78], [LeB88], [EH79], [BK17] and references therein.

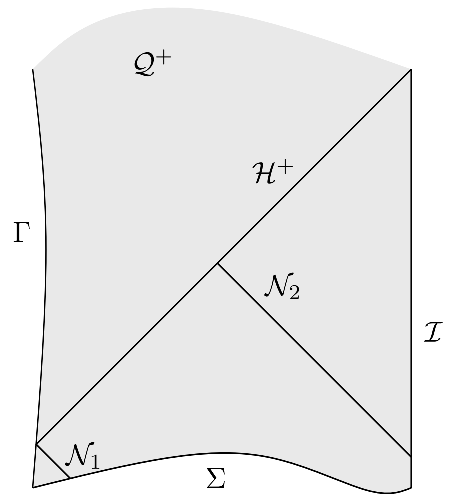



Figure 3.2.: We can achieve that the initial data slice touches null infinity and does not reach $\Gamma$ by moving from a slice such as $\mathcal{N}_1$ to $\mathcal{N}_2$.

### 3.1.4. The main results

In biaxial Bianchi IX symmetry, the system (3.1.1) is equivalent to a system of non-linear partial differential equations for the radius $r$ of the spheres, the Hawking mass $m$ and the quantity $B$. In this work, we first prove that the system is well-posed as an initial-boundary value problem and then show that initial data sufficiently close to the Eguchi-Hanson-AdS metric restricted to a spacelike hypersurface cannot form horizons in the dynamical evolution.

#### Well-posedness in biaxial Bianchi IX symmetry

Given Eguchi-Hanson-type initial data, the Einstein vacuum equations (3.1.2) are well-posed:

*Proof of Theorem 3.1.7.* To construct the solution, one splits up $\mathcal{S}$ in four regions and solves the Einstein vacuum equations over each region. Afterwards, the solutions are patched together using domain of dependence arguments.

First, by a soft argument, one proves that we can always solve the Einstein vacuum equations around the centre of $\mathcal{S}$: By Proposition 3.1.3, the local topology around $\varrho = a$ is $\mathbb{R}^2 \times S^2$ and we can choose a finite cover $(U_i)_i$ of $U := \{|x| \leq 1 : x \in \mathbb{R}^2\}$. Let $V_i \subseteq D(U_i) \cap (\mathbb{R} \times \mathcal{S})$ be the domain of the local solution $(V_i, g_{U_i})$ over $U_i$. By patching the solutions together, we obtain a solution $((-T,T) \times U), g_U)$. Since $(\mathcal{S}, \overline{g})$ has biaxial Bianchi IX symmetry, the evolution exhibits the same symmetry by standard arguments. Therefore, we can change back to coordinates adapted to biaxial Bianchi IX symmetry in $(-T,T) \times U$.

Second, we solve (3.1.2) for over $\{R_1 \leq \varrho \leq R_2\}$ for $0 < R_1, R_2 < \infty$. This is standard for each such choice of $R_1$ and $R_2$. Third, we solve (3.1.2) of $\{R_2 \leq \varrho \leq \infty\}$ for $R_2$ sufficiently large. The fourth region is a triangular region where the past boundary is the outer future

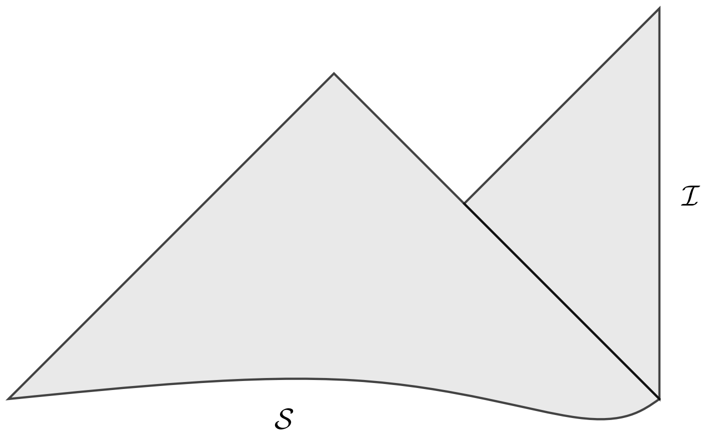



Figure 3.3.: The third and fourth region of the proof of Theorem 3.1.7 are shown in this Penrose diagram.

boundary of the solution in the third region and the outer boundary is null infinity. See Figure 3.3. The proof of existence in this region is the most difficult and will be the content of this chapter.

The proof for the existence in the third region can be extracted from the proof given for the forth region. In particular, the same norms on the metric quantities and the Hawking mass can be used, and the analogous estimates can be proved. The difference is that the equations have to be rewritten in terms of a timelike and a spacelike coordinate instead of the double null coordinates.

Uniqueness of the solution thus obtained can be shown using the arguments for geometric uniqueness from [HS11b]. □

*Remark* 3.1.13. The local well-posedness theorem for an initial data set $(\mathcal{S}, \overline{g}, K)$ yields the existence of a unique maximal development in the causal future of $\mathcal{S} \cup \mathcal{I}$, where null infinity $\mathcal{I}$ is timelike since the resulting spacetime is asymptotically AdS.

Proving local well-posedness in the context of negative cosmological constant around infinity has been achieved for the four-dimensional Einstein-Klein-Gordon system in [HS11b] and [HW13]. Several differences arise in the present context, which are outlined in Section 3.1.5. However, we can follow the general strategy of [HW13]. We first define the triangle

$$\Delta_{\delta,u_0} := \{(u,v) \in \mathbb{R}^2 \,:\, u_0 \leq v \leq u_0 + \delta,\, v < u \leq u_0 + \delta\}$$

and the conformal boundary

$$\mathcal{I} := \overline{\Delta_{\delta,u_0}} \backslash \Delta_{\delta,u_0}.$$

See Figure 3.4 for a visualisation. Our dynamical variables are

$$(\tilde{r}, m, B) : \Delta_{\delta,u_0} \to \mathbb{R}_+ \times \mathbb{R}^2.$$

We treat these as defining auxiliary variables

$$r := \frac{1}{\tilde{r}}, \qquad 1-\mu := 1 - \frac{2m}{r^2} + \frac{r^2}{\ell^2}, \qquad \Omega^2 := -\frac{4r^4 \tilde{r}_u \tilde{r}_v}{1-\mu}.$$

The general discussion of Appendix C.2 show that the correct notion of a solution to the Einstein vacuum equations in the triangular region is encapsulated in the following

**Definition 3.1.14.** A weak solution to the Einstein vacuum equations in $\Delta_{\delta,u_0}$ is a triple

$$(\tilde{r}, m, B) \in C^1_{\text{loc}}(\Delta_{\delta,u_0}) \cap W^{1,1}_{\text{loc}}(\Delta_{\delta,u_0}) \cap H^1_{\text{loc}}(\Delta_{\delta,u_0})$$

such that $\tilde{r}_{uu}, B_u, m_u \in C^0_{\text{loc}}$ and the equations

$$\tilde{r}_{uv} = \Omega^2 \tilde{r}^3 \left(-1 + \frac{1}{3}R - 2m\tilde{r}^2\right) \tag{3.1.5}$$

$$\partial_u m = -\frac{\tilde{r}_u}{\tilde{r}^3}\left(1 - \frac{2}{3}R\right) + \frac{4}{\Omega^2}\frac{\tilde{r}_v}{\tilde{r}^5}(B_u)^2 \tag{3.1.6}$$

$$\partial_v m = -\frac{\tilde{r}_v}{\tilde{r}^3}\left(1 - \frac{2}{3}R\right) + \frac{4}{\Omega^2}\frac{\tilde{r}_u}{\tilde{r}^5}(B_v)^2 \tag{3.1.7}$$

$$B_{uv} = \frac{3}{2}\frac{\tilde{r}_u}{\tilde{r}}B_v + \frac{3}{2}\frac{\tilde{r}_v}{\tilde{r}}B_u - \frac{1}{3}\Omega^2\tilde{r}^2\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right), \tag{3.1.8}$$

where

$$R = 2\,\mathrm{e}^{-2B} - \frac{1}{2}\,\mathrm{e}^{-8B},$$

are satisfied in the interior of $\Delta_{\delta,u_0}$ in a weak sense.

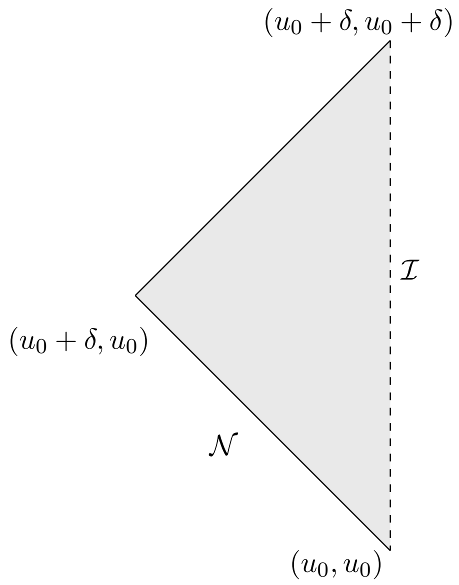



Figure 3.4.: The triangular domain $\Delta_{\delta,u_0}$ of local existence

**Definition 3.1.15.** A classical solution to the Einstein equations is a weak solution with

$$(\tilde{r}, m, B) \in C^2_{\text{loc}}(\Delta_{\delta,u_0}) \cap C^1_{\text{loc}}(\Delta_{\delta,u_0}) \cap C^2_{\text{loc}}(\Delta_{\delta,u_0})$$

such that (3.1.5), (3.1.6), (3.1.7) and (3.1.8) are satisfied classically

*Remark* 3.1.16. From Theorem C.2.1 and its proof, one sees that a classical solution in the sense of Definition 3.1.15 yields a $C^0$ curvature, such that (3.1.2) holds classically.

Equations (3.1.5), (3.1.8) and (3.1.7) will be treated as the dynamical equations, whereas we will see that (3.1.6) can be treated as a constraint equation that is propagated.

The local well-posedness result shown in this chapter is the following

**Theorem 3.1.17.** *Let $(\overline{\tilde{r}}, M, \overline{B})$ be a free data set in the sense of Definition 3.2.1 on $\mathcal{N} = (u_0, u_1]$. Then there is a $\delta > 0$ such that there exists a unique weak solution $(\tilde{r}, m, B)$ of the Einstein equations in the triangle $\Delta_{\delta,u_0}$ such that*

1. *$\tilde{r}$ satisfies the boundary condition*

$$\tilde{r}\big|_{\mathcal{I}} = 0$$

2. *$B$ satisfies the boundary condition*

$$B\big|_{\mathcal{I}} = 0$$

   *in a weak sense*

3. *$\tilde{r}$ and $B$ agree with $\overline{\tilde{r}}$ and $\overline{B}$ when restricted to $\mathcal{N}$.*

We can also obtain classical solutions from sufficiently regular initial data.

**Theorem 3.1.18.** *Let $(\tilde{r}, M, B)$ be a free data set giving rise to $H^2$ initial data in the sense of Definition 3.2.1. Then the solution of Theorem 3.1.17 is a classical solution.*

We conclude this section with a remark about the Hawking mass. The mass is a dynamical variable and does not have to be conserved at infinity a priori. However, a geometric version of conservation holds:

**Proposition 3.1.19.** *Let $(\tilde{r}, m, B)$ be a classical solution. Set*

$$\begin{aligned} \mathcal{T} &:= \frac{1}{\Omega^2}\left(r_v \partial_u - r_u \partial_v\right) \\ \mathcal{R} &:= \frac{1}{\Omega^2}\left(r_v \partial_u + r_u \partial_v\right) \end{aligned}$$

*Then $\mathcal{T}$ and $\mathcal{R}$ are invariant under a change of the $(u, v)$ coordinates that preserves the form of the metric and*

$$\mathcal{T} m\Big|_{\mathcal{I}} = 0.$$

*Proof.* From (C.2.6) and (C.2.7), we find

$$\mathcal{T} m = -\frac{4}{\Omega^4} r^3 \left[r_v^2 (B_u)^2 - r_u^2 (B_v)^2\right] = -4r^3 \left(\mathcal{T} B\right)\left(\mathcal{R} B\right).$$

If $B$ is at least $H^2$ around infinity, the result of Appendix D applies and we conclude that the limit of the right hand side vanishes as $r \to \infty$. □

### Absence of horizons

The main novel result of this chapter consists in showing that for initial data of Eguchi-Hanson type with negative mass, a Penrose diagram such as Figure 3.5 cannot arise.

**Theorem 3.1.20.** *Let $(\mathcal{S}, \overline{g}, K)$ be of Eguchi-Hanson type with negative mass $M < 0$ at infinity. Then there is no future horizon in the maximal development, i. e. the causal past of null infinity is empty.*

**Corollary 3.1.21.** *Small perturbations of Eguchi-Hanson-AdS spacetimes do not contain future horizons.*

It is important to stress that the absence of a horizon is a stronger statement than the absence of trapping – shown in Proposition 3.3.2 – for a horizon concerns the causal past of null infinity and hence the global geometry of the spacetime, whereas trapping is a local phenomenon.

Combining Theorem 3.1.20 with the arguments in Section 3.3.1 leaves us with the following dichotomy: either the future development of Eguchi-Hanson-type data with negative mass contains a first singularity in $\overline{\Gamma}\backslash\Gamma$, where $\Gamma$ is the centre – see Figure 3.6 –, or no first singularities form at all.

In virtue of the properties and conjectures described in the next section, our result, restricted to perturbations of Eguchi-Hanson-AdS spacetimes, can corroborate the conjecture put forward in in [DH06a]:

**Conjecture 3.1.22.** *Small perturbations of Eguchi-Hanson-AdS spacetimes have a Penrose diagram as depicted in Figure 3.6. Moreover, $\mathcal{I}$ is future incomplete.*

In contrast, in a comparable context where no horizons can form, the work [BJ13] allows for growth of perturbations and global existence of the solution without the formation of a naked singularity. A similar behaviour is observed in [DHMS12]. Thus the dynamics is very complicated and the question of the formation of naked singularities remains open.

## 3.1.5. Outline of the proofs

### Theorems 3.1.17 and 3.1.18

The proof follows the strategy of [HW13] and adopts the structure of its exposition; however, twisting is not required in our context since we are only treating Dirichlet boundary conditions, which allows for streamlining some parts of the proof. The crucial step is contained in Section 3.2.2. There we define suitable function spaces for $(\tilde{r}, m, B)$ and set up the system (3.1.5), (3.1.7) and (3.1.8) as a fixed point problem. In particular, the correct weights for the $H^1$-based function spaces for $B$, which differ from [HW13], had to be found. A map

$$\Phi : (\tilde{r}, m, B) \mapsto (\tilde{r}', m', B')$$

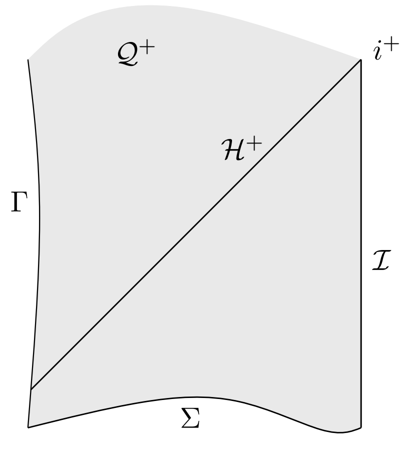



Figure 3.5.: The Penrose diagram depicts the situation of a horizon. The endpoint of $\mathcal{I}$, denoted by $i^+$, is either in $\mathcal{I}$ or its completion; the latter case corresponds to future complete $\mathcal{I}$, the former one to an incomplete $\mathcal{I}$. We will show the absence of a horizon, that is the impossibility of either of those two cases.

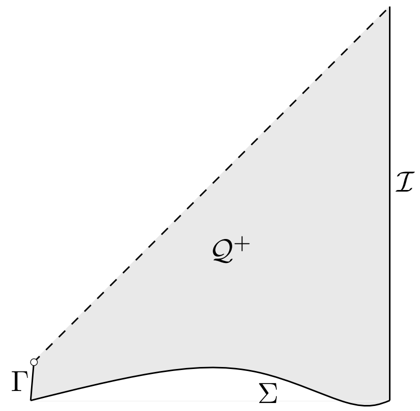



Figure 3.6.: The Penrose diagram depicts the formation of a first singularity emanating from the central wordline $\Gamma$. The singularity is denoted by the circled dot; the dotted line is not part of the spacetime.

is defined and shown to be a contraction map in Sections 3.2.3 and 3.2.4, so by Banach's fixed point theorem, the system has a unique weak solution in the function spaces (Theorem 3.2.5). The constraint equation (3.1.6) holds as a consequence, which is shown in 3.2.5. Achieving higher regularity is a simple matter of commuting with $T := \partial_u + \partial_v$ (see Section 3.2.6).

To define the contraction map $\Phi$, we need to show well-posedness for the inhomogeneous wave equation governing the evolution of $B'$ in $\Delta_{\delta,u_0}$. One could appeal to the work by [War12] and use Warnick's well-posedness theorem. However, [War12] does not deal with initial data on a null hypersurface, so the problem at hand would need to be reduced to the setting of spacelike initial data. Moreover, [War12] uses twisting, which is not necessary here. Therefore, a direct elementary proof of well-posedness is given in Appendix D, which also allows us to use the function spaces of Section 3.2.2.

The reader wishing to understand the estimates on $B'$ in the fixed point argument without delving into the coupled non-linear system can extract all the ideas from the simpler estimates of Appendix D.

**Theorem 3.1.20**

From the local well-posedness theorem (Theorem 3.1.7), we obtain the existence of a maximal development of Eguchi-Hanson-type data, with $B$ satisfying a Dirichlet boundary condition at infinity. The global geometry of spacetimes arising from such data is described in Section 3.3.1. We also prove in that section that the spacetime is either globally regular without a horizon, or forms a horizon, or evolves into a first singularity at the centre. We proceed to show that no horizons can form in the dynamical evolution.

Proving the absence of horizons will take the structure of an argument by contradiction (Section 3.3.3). Suppose that the Penrose diagram of the spacetime looks like Figure 3.2. By soft arguments relying on the well-posedness result, we show that one can always find a null hypersurface such as $\mathcal{N}_2$ that does not intersect the initial hypersurface, but reaches from the horizon to null infinity. In Section 3.3.2, an extension principle is shown for triangular regions around null infinity – such as the one enclosed by $\mathcal{H}^+$, $\mathcal{N}_2$ and $\mathcal{I}$ –, which permits to extend the solutions to the future to a strictly larger triangle, provided uniform bounds hold in the triangular region. The proof of the extension principle uses the local existence result. We then proceed (Section 3.3.3) to establish that those quantities can indeed be bounded only in terms of their values on $\mathcal{N}_2$, where they hold by compactness of $\mathcal{N}_2$ and the local well-posedness result. Thus we can extend the solution along $\mathcal{I}$ beyond $\mathcal{H}^+$, which is a contradiction.

## 3.2. Local well-posedness

This section contains proof of local well-posedness of the Einstein vacuum equations around null infinity. As pointed out before, the exposition parallels that of [HW13]. This will allow the read familiar with that argument to gain quick access to the problem at hand.

### 3.2.1. Initial data

**Definition 3.2.1.** Let $\mathcal{N} = (u_0, u_1]$. A triple $(\tilde{\bar{r}}, M, \overline{B}) \in C^2(\mathcal{N}) \times \mathbb{R} \times C^1(\mathcal{N})$ is a free data set if the following hold:

1. $\tilde{\bar{r}} > 0$ and $\tilde{\bar{r}}_u > 0$ in $\mathcal{N}$, as well as $\lim_{u\to u_0} \tilde{\bar{r}}(u) = 0$, $\lim_{u\to u_0} \tilde{\bar{r}}_u = 1/2$ and $\lim_{u\to u_0} \tilde{\bar{r}}_{uu} = 0$.
2. There is a constant $C_0$ such that

$$\int_{u_0}^{u_1} \frac{1}{(u-u_0)^3}\left[\overline{B}^2 + (\overline{B}_u)^2\right] \mathrm{d}u < C_0$$

$$\sup_{\mathcal{N}} \left|\tilde{\bar{r}}^{-2}\overline{B}\right| + \sup_{\mathcal{N}} \left|\tilde{\bar{r}}^{-1}\partial_u \overline{B}\right| < C_0.$$

From this, we obtain a complete initial data set

$$(\tilde{\bar{r}}, \overline{B}, \overline{m}, \tilde{\bar{r}}_v) \in C^2(\mathcal{N}) \times C^1(\mathcal{N}) \times C^1(\mathcal{N}) \times C^1(\mathcal{N}).$$

First, we integrate

$$\partial_u \overline{m} = -\frac{\tilde{\bar{r}}_u}{\tilde{\bar{r}}^3}\left(1 - \frac{2}{3}\overline{R}\right) - \frac{1}{\tilde{\bar{r}}\tilde{\bar{r}}_u}\left(1 - 2\overline{m}\tilde{\bar{r}}^2 + \frac{1}{\ell^2\tilde{\bar{r}}^2}\right)(\overline{B}_u)^2 \tag{3.2.1}$$

with boundary condition

$$\lim_{u\to u_0} \overline{m} = M$$

for the Hawking mass $M$ at infinity. Note here that

$$1-\frac{2}{3}\overline{R}=8\overline{B}^2+\mathcal{O}(\overline{B}^3),$$

whence the first term on the right hand side of (3.2.1) is regular as $\tilde{r}\to 0$ due to the bounds on $\overline{B}$. The function $\overline{\tilde{r}_v}$ is obtained from solving the ODE

$$\frac{\partial_u\overline{\tilde{r}_v}}{\overline{\tilde{r}_v}}=-\frac{4\overline{\tilde{r}}_u}{\overline{\tilde{r}}-2\overline{m}\overline{\tilde{r}}^3+\frac{1}{\ell^2\overline{\tilde{r}}}}\left(-1+\frac{\overline{R}}{3}-2\overline{m}\overline{\tilde{r}}^2\right)$$

with boundary condition

$$\lim_{u\to u_0}\overline{\tilde{r}_v}=-\frac{1}{2}.$$

In the proof, we require additional weights, which can be obtained by potentially truncating the initial data ray in accordance with the following lemma.

**Lemma 3.2.2.** *Let $(\overline{\tilde{r}},M,B)$ a free data set. Then, for any $0<s<1$ and $\varepsilon>0$, there is a $\delta>0$ such that the following bounds hold on the truncated ray $\mathcal{N}_\delta:=\mathcal{N}\cap\{u\le u_0+\delta\}$:*

$$\|\overline{\tilde{r}}\|_{C^0}+\left\|\overline{\tilde{r}}_u-\frac{1}{2}\right\|_{C^0}+\left\|\overline{\tilde{r}_v}+\frac{1}{2}\right\|_{C^0}+\|\overline{\tilde{r}}_{uu}\|_{C^0}<\varepsilon \tag{3.2.2}$$

$$\int_{u_0}^{u_0+\delta}\left[\left(\overline{B}_u\right)^2+\overline{B}^2\right]\overline{\tilde{r}}^{-3}\,\mathrm{d}u<\varepsilon \tag{3.2.3}$$

$$\left|\overline{\tilde{r}}^{-1+s/4}\overline{B}_u\right|<\varepsilon \tag{3.2.4}$$

$$\|\overline{m}-M\|_{C^0}<\varepsilon \tag{3.2.5}$$

$$\left\|\overline{\tilde{r}}^{1+s}\overline{m}_u\right\|_{C^0}<\varepsilon. \tag{3.2.6}$$

*Moreover, for $\delta>0$ sufficiently small,*

$$1-\mu\ge C\frac{r^2}{\ell^2}$$

*on $\mathcal{N}_\delta$ for a $c>0$.*

*Proof.* The function $\overline{\tilde{r}}$ is $C^2$, so the (3.2.2) follows immediately. Expression (3.2.3) follows from the continuity of the integral together with $\overline{\tilde{r}}(u)\le Cu$ for some $C>0$ as a consequence of (3.2.2). Introducing an additional smallness factor $\overline{\tilde{r}}^{s/4}$ yields (3.2.4). Inequality (3.2.5) comes from the continuity of $\overline{m}$. To obtain (3.2.6), we multiply (3.2.1) by $\overline{\tilde{r}}^{1+s}$. This yields that the first term on the right hand side is bounded by $\overline{\tilde{r}}^{2+s}$ up to a constant factor and the second one by $\overline{\tilde{r}}^{s/2}$ up to a constant. □

*Remark* 3.2.3. The $s$ is introduced for technical reasons to achieve $\varepsilon$-smallness.

### 3.2.2. Function spaces and the contraction map

We denote by $C^{1+}_{\tilde r}(\Delta_{\delta,u_0})$ the space of all positive functions that are $C^1$ and agree with $\overline{\tilde r}$ on $\mathcal{N}$ and are such that the $uu$- and $uv$-derivatives both exist and are continuous. We define the following metric

$$\begin{aligned} d_{\tilde r}(\tilde r_1, \tilde r_2) :=& \left\| \log \frac{\tilde r_1}{\tilde r_2} \right\|_{C^0} + \left\| \log |(\tilde r_1)_u| - \log |(\tilde r_2)_u| \right\|_{C^0} + \left\| \log |(-\tilde r_1)_v| - \log |(-\tilde r_2)_v| \right\|_{C^0} \\ &+ \left\| \frac{T(\tilde r_1)}{\varrho} - \frac{T(\tilde r_2)}{\varrho} \right\|_{C^0} + \|(\tilde r_1)_{uv} - (\tilde r_2)_{uv}\|_{C^0} + \|(\tilde r_1)_{uu} - (\tilde r_2)_{uu}\|_{C^0} \,, \end{aligned}$$

where $\varrho = (u-v)/2$ and $T = \partial_u + \partial_v$.

We denote by $C^{0+}_m(\Delta_{\delta,u_0})$ the space of all $C^0$ functions that agree with $\overline{m}$ on $\mathcal{N}$ and are such that the $u$-derivative exists and is continuous. We define

$$d_m(m_1, m_2) := \|m_1 - m_2\|_{C^0} + \left\| \varrho^{1+s} \left( (m_1)_u - (m_2)_u \right) \right\|_{C^0}$$

for $0 < s < 1$.

We denote by $C^{0+}_B(\Delta_{\delta,u_0})$ the space of all functions that are continuously differentiable in $u$, agree with $\overline{B}$ on $\mathcal{N}$ and are continuous in $v$ with values in $\underline{H}^1_u(v)$ and continuous in $u$ with values in $\underline{H}^1_v(u)$. Here the spaces $\underline{H}^1_v(u)$ and $\underline{H}^1_u(v)$ are equipped with the norms

$$\begin{aligned} \|g\|^2_{\underline{H}^1_v(u)} &:= \int_{v_0}^{u} \left[ g_v(u,v')^2 + g(u,v')^2 \right] \varrho(u,v')^{-3} \, \mathrm{d}v' \\ \|g\|^2_{\underline{H}^1_u(v)} &:= \int_{v}^{u_0+\delta} \left[ g_u(u',v)^2 + g(u',v)^2 \right] \varrho(u',v)^{-3} \, \mathrm{d}u'. \end{aligned}$$

We define the metric

$$d_B(B_1, B_2) := \|B_1 - B_2\|_{C^0 H^1} + \left\| \varrho^{-2} (B_1 - B_2) \right\|_{C^0} + \left\| \varrho^{-1+s/4} (\partial_u B_1 - \partial_u B_2) \right\|_{C^0} ,$$

where

$$\|g\|^2_{C^0 \underline{H}^1} := \sup_{u \in \Delta_{\delta,u_0}} \|g\|^2_{\underline{H}^1(u)} + \sup_{v \in \Delta_{\delta,u_0}} \|g\|^2_{\underline{H}^1(v)}$$

We define

$$X := C^{1+}_{\tilde r}(\Delta_{\delta,u_0}) \times C^{0+}_m(\Delta_{\delta,u_0}) \times C^{0+}_B(\Delta_{\delta,u_0})$$

as well as

$$d((\tilde r_1, m_1, B_1), (\tilde r_2, m_2, B_2)) := d_{\tilde r}(\tilde r_1, \tilde r_2) + d_m(m_1, m_2) + d_B(B_1, B_2),$$

which makes $(X, d)$ into a metric space. Define the map

$$\Phi : (X, d) \to (X, d), \ \Phi(\tilde r, m, B) = (\tilde r', m', B')$$

by

$$\begin{aligned}\tilde{r}'(u,v) =&\tilde{r}(u,u_0)-\tilde{r}(v,u_0)\\&+\int_v^u\int_{u_0}^v\left(\Omega(u',v')^2\tilde{r}(u',v')^3\left(-1+\frac{1}{3}R(u',v')+2m(u',v')\tilde{r}(u',v')^2\right)\right)\mathrm{d}v'\mathrm{d}u'\\m'(u,v) =&m(u,u_0)+\int_{u_0}^v\left(-\frac{\tilde{r}_v}{\tilde{r}^3}\left(1-\frac{2}{3}R'\right)+\frac{4}{\Omega^2}\frac{\tilde{r}_u}{\tilde{r}^5}(B_v')^2\right)\mathrm{d}v'.\end{aligned}$$

Note that on the right hand side of the equation of $m'$, $B'$ appears and not $B$. The function $B'$ is defined as the unique $H^1$ solution to

$$\partial_u(r^3B_v')+\partial_v(r^3B_u')=-\frac{2}{3}\Omega^2r\left(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}\right) \tag{3.2.7}$$

or, equivalently,

$$B_{uv}'=\frac{3}{2}\frac{\tilde{r}_u}{\tilde{r}}B_v'+\frac{3}{2}\frac{\tilde{r}_v}{\tilde{r}}B_u'-\frac{1}{3}\Omega^2\tilde{r}^2\left(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}\right)$$

satisfying the Dirichlet boundary condition at infinity. Existence and uniqueness are obtained from Proposition D.6.1.

Let $\mathcal{B}_b$ be the ball of radius $b$ centred around $((u-v)/2),M,0)$.

**Proposition 3.2.4.** *There is a $\delta>0$ such that $\Phi:\mathcal{B}_b\to\mathcal{B}_b$ is a contraction map from $\mathcal{B}_b$ into itself.*

*Proof.* The proof is the content of Sections 3.2.3 and 3.2.4. □

**Theorem 3.2.5.** *There is a $\delta>0$ such that $\Phi$ has a fixed point. This fixed point is a weak solution to the Einstein vacuum equations.*

*Proof.* An application of the contraction mapping theorem yields the fixed point. Together with Section 3.2.5, the rest follows immediately. □

### 3.2.3. Map $\mathcal{B}_b\to\mathcal{B}_b$

We have to show that $\Phi$ maps $\mathcal{B}_b$ into itself. Thus we assume $(\tilde{r},m,B)\in\mathcal{B}_b$ and want to show $(\tilde{r}',m',B')\in\mathcal{B}_b$. We will denote by $C_b$ a constant depending only on $b$ and the initial data. We use the algebra of constants where $C_b\cdot C_b=C_b$.

#### Radial bound

We first check that the contraction map satisfies the right boundary conditions. It suffices to show that the integrand is integrable in $v$. We have that

$$\left|\tilde{r}^2-2m\tilde{r}^4+\frac{1}{\ell^2}\right|\geq\frac{1}{2\ell^2}$$

for $\delta$ small, depending on $M$ and $b$. Thus one obtains

$$\left|\frac{\Omega^2}{r^2}\right| = \left|\frac{4\tilde{r}_u\tilde{r}_v}{\tilde{r}^2 - 2m\tilde{r}^4 + 1/\ell^2}\right| \leq C_b.$$

Moreover,

$$\begin{aligned} |R| \leq & C_b\left(1+|B|\right) \leq C_b \\ |m|\tilde{r}^2 \leq & \left(|M|+b\right)\tilde{r}^2 \leq C_b\tilde{r}^2 \end{aligned}$$

Therefore, the absolute value of the integrand satisfies

$$\left|\Omega^2\tilde{r}^3\left(-1+R+2m\tilde{r}^2\right)\right| \leq C_b\tilde{r}.$$

Hence it is integrable in $v$ and therefore $\tilde{r}'$ is zero on the boundary. Moreover, $\tilde{r}'(u,u_0) = \bar{\tilde{r}}(u)$.

We now turn to estimating $\tilde{r}'_{uu}$. We have

$$\tilde{r}'_{uu} = \tilde{r}_{uu}(u,u_0) + \int_{u_0}^{v} \partial_u\left(\Omega^2\tilde{r}^3\left(-1+R+2m\tilde{r}^2\right)\right)\,\mathrm{d}v'.$$

From the choice of $\delta$, we have

$$\left|\bar{\tilde{r}}_{uu}\right| < \varepsilon.$$

Moreover

$$\left|\partial_u\left(-1+\frac{1}{3}R+2m\tilde{r}^2\right)\right| \leq C_b\tilde{r}^{1-s/4} + C_b\tilde{r}^{1-s} + C_b\tilde{r} \leq C_b\tilde{r}^{1-s/4}.$$

and

$$\left|\partial_u\left(\frac{\Omega^2}{r^3}\right)\right| = \left|\partial_u\left(\frac{-4r\tilde{r}_u\tilde{r}_v}{1-\mu}\right)\right| \leq C_b\tilde{r}.$$

We have

$$\int_{u_0}^{v}\tilde{r}\,\mathrm{d}v' \leq C_b\int_{u_0}^{v}\varrho\,\mathrm{d}v' = C_b\left([v-u_0]^2-[u-v]^2\right) \leq C_b\delta^2.$$

Therefore

$$|\partial_u\left(\Omega^2\tilde{r}^3\left(-1+R+2m\tilde{r}^2\right)\right)| \leq C_b\tilde{r},$$

whence

$$|\tilde{r}'_{uu}(u,v)| \leq \varepsilon + C_b\delta^2.$$

Analogously, we obtain

$$|\tilde{r}'_{uv}| \leq C_b \delta.$$

The lower derivatives are

$$\begin{aligned} 1 - \varepsilon\delta - C_b\delta^2 &\leq + 2\tilde{r}'_u \leq 1 + \varepsilon\delta + C_b\delta^2 \\ 1 - \varepsilon\delta - C_b\delta^2 &\leq - 2\tilde{r}'_v \leq 1 + \varepsilon\delta + C_b\delta^2 \end{aligned}$$

since $\tilde{r}' = 0$ on the boundary and hence $T\tilde{r}' = 0$. Therefore

$$\left(1 - \varepsilon\delta - C_b\delta^2\right) \frac{u - v}{2} \leq \tilde{r}' \leq \left(1 + \varepsilon\delta + C_b\delta^2\right) \frac{u - v}{2}.$$

which yields

$$\left|\log \frac{\tilde{r}'}{\varrho}\right| + |\log 2\overline{\tilde{r}}_u| + |\log(-2\overline{\tilde{r}}_v)| < C_b\delta.$$

The missing estimate comes from

$$T(\tilde{r}')(u, v) = \int_v^u \left(\tilde{r}'_{uu}(u', v) + \tilde{r}'_{uv}(u', v)\right) \mathrm{d}u',$$

where we have used $T\tilde{r}'(v, v) = 0$. From the estimates above, we obtain

$$\left|T(\tilde{r}')\right| \leq C_b \left(\varepsilon + \delta^2 + \delta\right) \varrho.$$

**Bound for** $B'$

We compute that

$$\begin{aligned} &\partial_u \left(r^3(B'_v)^2 + r^3 B'^2\right) + \partial_v \left(r^3(B'_u)^2 + r^3 B'^2\right) \\ =&\frac{3}{\tilde{r}^4} T(\tilde{r}) B'_u B'_v - \frac{3}{\tilde{r}^4} T(\tilde{r}) B'^2 - \frac{2}{3}\Omega^2 r \left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right) T(B') + 2r^3 T(B') B' \end{aligned}$$

holds in a weak sense. Integrating[3] the identity over the domain

$$\left\{(u', v') \in \Delta_{\delta, u_0} \,:\, u' \leq u,\, v' \leq v\right\}$$

yields on the left hand side

$$\begin{aligned} &\int_{u_0}^{v} \int_{v'}^{u} \left[\partial_v \left(r^3(B'_u)^2 + r^3(B')^2\right) + \partial_u \left(r^3(B'_v)^2 + r^3(B')^2\right)\right] \mathrm{d}u' \mathrm{d}v' \\ =&\int_0^\delta r^3 \left((B'_u)^2 - (B'_v)^2\right)(u_0 + t, u_0 + t)\, \mathrm{d}t - \int_{u_0}^u r^3 \left((B')^2 + (B'_u)^2\right)(u', u_0)\, \mathrm{d}u' \end{aligned}$$

[3] All integrals are to be viewed as Lebesgue integrals in $\mathbb{R}^2$ and not as integrals in a geometric sense in $\mathbb{R}^{1+1}$. This means that we use the standard divergence theorem in $\mathbb{R}^2$.

$$
\begin{aligned}
&+\int_v^u r^3\left((B')^2+(B'_u)^2\right)(u',v)\,\mathrm{d}u' + \int_{u_0}^v r^3\left((B')^2+(B'_v)^2\right)(u,v')\,\mathrm{d}v' \\
=&\int_v^u r^3\left((B')^2+(B'_u)^2\right)(u',v)\,\mathrm{d}u' + \int_{u_0}^v r^3\left((B')^2+(B'_v)^2\right)(u,v')\,\mathrm{d}v' \\
&-\int_{u_0}^u r^3\left((B')^2+(B'_u)^2\right)(u',u_0)\,\mathrm{d}u'.
\end{aligned}
$$

We estimate each term on the right hand side separately.

$$
\begin{aligned}
\left|\int_\Delta \frac{3}{\tilde r^4}T(\tilde r)B'_uB'_v\,\mathrm{d}u'\mathrm{d}v'\right| \leq & C_b\int_{\Delta_{\delta,u_0}}\frac{1}{\varrho^3}|B'_uB'_v|\,\mathrm{d}u'\mathrm{d}v' \\
\leq & C_b\int_{\Delta_{\delta,u_0}}\frac{(B'_u)^2}{\varrho^3}\,\mathrm{d}u'\mathrm{d}v' + C_b\int_{\Delta_{\delta,u_0}}\frac{(B'_v)^2}{\varrho^3}\mathrm{d}u'\mathrm{d}v' \\
\leq & C_b\delta\left\|B'\right\|^2_{C^0\underline{H}^1}
\end{aligned}
$$

Similarly we can estimate the second term. The third term is estimated as follows

$$
\begin{aligned}
\left|\int_{\Delta_{\delta,u_0}}\frac{2}{3}\Omega^2 r\left(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}\right)T(B')\,\mathrm{d}u'\mathrm{d}v'\right| \leq & C_b\int_{\Delta_{\delta,u_0}} r^3T(B')^2\,\mathrm{d}u'\mathrm{d}v' \\
& + C_b\int_{\Delta_{\delta,u_0}}\frac{\Omega^4}{r^4}r^3\left(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}\right)^2\,\mathrm{d}u'\mathrm{d}v' \\
\leq & C_b\delta\left\|B'\right\|^2_{C^0\underline{H}^1} + C_b\delta^3
\end{aligned}
$$

since

$$
\int_{\Delta_{\delta,u_0}}\frac{\Omega^4}{r^4}r^3\left(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}\right)^2\,\mathrm{d}u'\mathrm{d}v' \leq C_b\int_{\Delta_{\delta,u_0}}\varrho\,\mathrm{d}u'\mathrm{d}v' \leq C_b\delta^3,
$$

where we have used the pointwise bound on $B$. The fourth term can be dealt with as follows:

$$
\begin{aligned}
\left|\int_{\Delta_{\delta,u_0}} 2r^3T(B')B\,\mathrm{d}u'\mathrm{d}v'\right| \leq & C_b\int_{\Delta_{\delta,u_0}}\frac{(B'_u)^2}{\varrho^3}\,\mathrm{d}u'\mathrm{d}v' + C_b\int_{\Delta_{\delta,u_0}}\frac{(B'_v)^2}{\varrho^3}\mathrm{d}u'\mathrm{d}v' \\
& + C_b\int_{\Delta_{\delta,u_0}}\frac{(B')^2}{\varrho^3}\mathrm{d}u'\mathrm{d}v' \\
\leq & C_b\delta\left\|B'\right\|^2_{C^0\underline{H}^1}.
\end{aligned}
$$

We conclude that

$$
\left\|B'\right\|_{C^0H^1} \leq \frac{C_b\delta^{3/2}+\varepsilon}{(1-C_b\delta)^{1/2}} \leq \varepsilon + C_b\delta.
$$

The pointwise bound on $B'$ follows from

$$
|B'(u,v)| = \left|\int_v^u B'_u\,\mathrm{d}u'\right| \leq C_b\varrho^2\left\|B'\right\|_{C^0\underline{H}^1}
$$

since $B' = 0$ on the boundary. It remains to show that we can retrieve the bound on the first derivative. Consider the transport equation

$$\partial_v \left( r^{3/2} B'_u \right) = -\frac{3}{2} r^{1/2} r_u B'_v - \frac{1}{3} \Omega^2 r^{-1/2} \left( \mathrm{e}^{-2B} - \mathrm{e}^{-8B} \right).$$

Integrating from $\mathcal{N}$, we obtain

$$\begin{aligned}\left| r(u,v)^{3/2} B'_u(u,v) \right| \leq & r(u,u_0)^{1/2} r(u,u_0) |B'_u(u,u_0)| + C_b \int_{u_0}^{v} \tilde{r}^{-5/2} |B'_v| \, \mathrm{d}v' \\ & + \frac{1}{3} \left( \int_{u_0}^{v} \frac{\Omega^2}{r^2} \, \mathrm{d}v' \right)^{1/2} \left( \int_{u_0}^{v} \Omega^2 r \left| \mathrm{e}^{-2B} - \mathrm{e}^{-8B} \right| \, \mathrm{d}v' \right)^{1/2},\end{aligned}$$

where we have used $r_u = -\tilde{r}_u / \tilde{r}^2$ and $r(u,v') \leq r(u,v)$ for $v' \leq v$. We estimate

$$\begin{aligned}\int_{u_0}^{v} \tilde{r}^{-5/2} |B'_v| \, \mathrm{d}v' \leq & \left( \int_{u_0}^{v} \tilde{r}^{-3} |B'_v|^2 \, \mathrm{d}v' \right)^{1/2} \left( \int_{u_0}^{v} \tilde{r}^{-2} \, \mathrm{d}v' \right)^{1/2} \\ \leq & C_b \left\| B' \right\|_{C^0 \underline{H}^1} \left( \int_{u_0}^{v} \varrho^{-2} \right)^{1/2} \\ \leq & C_b r(u,v)^{-1/2} \left\| B' \right\|_{C^0 \underline{H}^1}\end{aligned}$$

since

$$\int_{u_0}^{v} \varrho^{-2} \, \mathrm{d}v' = 2 \left( \varrho(u,v)^{-1} - \varrho(u,u_0)^{-1} \right) \leq 2 \varrho(u,v)^{-1}.$$

After estimating the remaining terms

$$\begin{aligned}& \int_{u_0}^{v} \frac{\Omega^2}{r"} \, \mathrm{d}v' \leq C_b \delta \\ & \int_{u_0}^{v} \Omega^2 r \left| \mathrm{e}^{-2B} - \mathrm{e}^{-8B} \right|^2 \, \mathrm{d}v' \leq C_b \left\| B \right\|_{C^0 \underline{H}^1} \leq C_b,\end{aligned}$$

we conclude

$$\left| r(u,v)^{1-s/4} B'_u(u,v) \right| \leq \varepsilon + C_b \left\| B' \right\|_{C^0 \underline{H}^1} \varrho^{s/4} + C_b \delta^{1+s/4}.$$

### Bound on the mass

We have

$$\begin{aligned}\left| m'(u,v) - M \right| \leq & \left| m(u,u_0) - M \right| + \left| \int_{u_0}^{v} \left( -\frac{\tilde{r}_v}{\tilde{r}^3} \left( 1 - \frac{2}{3} R' \right) + \frac{4}{\Omega^2} \frac{\tilde{r}_u}{\tilde{r}^5} (B'_v)^2 \right) \mathrm{d}v \right| \\ \leq & \varepsilon + C_b \left\| B' \right\|_{C^0 \underline{H}^1} \\ \leq & \varepsilon + C_b \delta.\end{aligned}$$

Taking a $u$-derivative yields

$$\partial_u m'(u,v) = \partial_u \overline{m}(u) + \int_{u_0}^{v} I(u,v')\,\mathrm{d}v'.$$

The full integrand is given by

$$\begin{aligned}
I(u,v') =& -\frac{\tilde r_{uv}}{\tilde r^3}\left(1-\frac{2}{3}R'\right) + 3\frac{\tilde r_u \tilde r_v}{\tilde r^4}\left(1-\frac{2}{3}R'\right) - \frac{8}{3}\frac{\tilde r_v}{\tilde r^3}\left(\mathrm{e}^{-2B'}-\mathrm{e}^{-8B'}\right)B'_u \\
&- \frac{2\tilde r \tilde r_u - 2m_u\tilde r^4 - 8m\tilde r^3 \tilde r_u}{\tilde r^3 \tilde r_v}\left(B'_v\right)^2 + 3\frac{\tilde r^2 - 2m\tilde r^4 + \frac{1}{\ell^2}}{\tilde r^4}\frac{\tilde r_u}{\tilde r_v}\left(B'_v\right)^2 \\
&+ \frac{\tilde r^2 - 2m\tilde r^4 + \frac{1}{\ell^2}}{\tilde r^3 \tilde r_v^2}\tilde r_{uv}\left(B'_v\right)^2 - 2\frac{\tilde r - 2m\tilde r^4 + \frac{1}{\ell^2}}{\tilde r^3 \tilde r_v}B'_v B'_{uv}.
\end{aligned}$$

The first term can be bounded by $C_b \left\|B'\right\|^2_{C^0\underline{H}^1}$, the second one by

$$\frac{C_b}{\tilde r(u,v)}\left\|B'\right\|^2_{C^0\underline{H}^1}$$

and all the other terms correspond to one of the two cases. Therefore

$$\left|\varrho^{1+s}\partial_u m'\right| \le C_b\varepsilon + C_b\delta^{3+s} + C_b\delta^{2+s}.$$

### 3.2.4. Contraction property

Let

$$D := d((\tilde r_1, m_1, B_1), (\tilde r_2, m_2, B_2)).$$

#### Radial bound

Let us first turn to $(\tilde r'_1)_{uv} - (\tilde r'_2)_{uv}$. To bound the integral, we need to estimate

$$\begin{aligned}
\frac{\Omega_1^2}{r_1^3} - \frac{\Omega_2^3}{r_2^2} =& -4\frac{(\tilde r_1)_u(\tilde r_1)_v - (\tilde r_2)_u(\tilde r_2)_v}{\tilde r_1 - 2m_1\tilde r_1^3 + \frac{1}{\ell^2 \tilde r_1}} \\
&+ 4(\tilde r_2)_u(\tilde r_2)_v\left[\frac{1}{\tilde r_1 - 2m_1\tilde r_1^3 + \frac{1}{\ell^2\tilde r_1}} - \frac{1}{\tilde r_2 - 2m_2\tilde r_2^3 + \frac{1}{\ell^2\tilde r_2}}\right].
\end{aligned}$$

The first term on the right hand side is easily estimated by

$$\begin{aligned}
\left|-4\frac{(\tilde r_1)_u(\tilde r_1)_v - (\tilde r_2)_u(\tilde r_2)_v}{\tilde r_1 - 2m_1\tilde r_1^3 + \frac{1}{\ell^2\tilde r_1}}\right| \le& C_b\varrho\left(\left|(\tilde r_1)_u - (\tilde r_2)_v\right| + \left|(\tilde r_1)_v - (\tilde r_2)_v\right|\right) \\
\le& C_b\varrho\left|\log\frac{(\tilde r_1)_u}{(\tilde r_2)_u}\right| + C_b\varrho\left|\log\frac{(\tilde r_1)_v}{(\tilde r_2)_v}\right| \\
\le& C_b\varrho d_{\tilde r}\left(\tilde r_1, \tilde r_2\right).
\end{aligned}$$

Note that

$$\left|\frac{\tilde r_1-\tilde r_2}{\varrho}\right|=\left|\frac{\tilde r_1}{\varrho}\left(\frac{\tilde r_2}{\varrho}-1\right)\right|\le C_b\left|\frac{\tilde r_1}{\varrho}\right|\left|\log\frac{\tilde r_1}{\tilde r_2}\right|\le C_b d_{\tilde r}\left(\tilde r_1,\tilde r_2\right)$$

and therefore

$$\begin{aligned}\left|\frac{1}{\tilde r_1-2m_1\tilde r_1^3+\frac{1}{\ell^2\tilde r_1}}-\frac{1}{\tilde r_2-2m_2\tilde r_2^3+\frac{1}{\ell^2\tilde r_2}}\right| \le & C_b\varrho^2\Big(\varrho d_{\tilde r}(\tilde r_1,\tilde r_2)+\varrho^2 d_m(m_1,m_2)\\ &+\varrho^2 d_{\tilde r}(\tilde r_1,\tilde r_2)+\frac{1}{\varrho}d_{\tilde r}(\tilde r_1,\tilde r_2)\Big)\\ \le & C_b\varrho^4 d_m(m_1,m_2)+C_b\varrho d_{\tilde r}(\tilde r_1,\tilde r_2).\end{aligned}$$

We also need to estimate

$$\begin{aligned}\left|\frac{\Omega_2^2}{r_2^3}\left(\frac{1}{3}(R_1-R_2)+2\left(m_1\tilde r_1^2-m_2\tilde r_2^2\right)\right)\right| \le & C_b\varrho^3 d_B(B_1,B_2)+C_b\varrho^3 d_m(m_1,m_2)\\ &+C_b\varrho^3 d_{\tilde r}(\tilde r_1,\tilde r_2).\end{aligned}$$

Therefore

$$\left|(\tilde r_1')_{uv}-(\tilde r_2')_{uv}\right|\le C_b\delta\cdot D.$$

We also readily obtain

$$\left|(\tilde r_1')_u-(\tilde r_2')_u\right|+\left|(\tilde r_1')_v-(\tilde r_2')_v\right|\le C_b\delta^2\cdot D$$

and

$$\left|\log\frac{\tilde r_2'}{\tilde r_1'}\right|=\left|\log\left(1+\frac{\tilde r_2'-\tilde r_1'}{\tilde r_1'}\right)\right|\le C_b\frac{1}{\varrho}\left|\tilde r_1'-\tilde r_2'\right|\le C_b\delta^2\cdot D.$$

Now we turn to $(\tilde r_1')_{uu}-(\tilde r_2')_{uu}$. We have

$$\left|\partial_u\left(\frac{\Omega_1^2}{r_1^3}-\frac{\Omega_2^2}{r_2^3}\right)\right|\le C_b d_{\tilde r}(\tilde r_1,\tilde r_2)+C_b\varrho^4 d_m(m_1,m_2),$$

whence

$$\begin{aligned}\left|\partial_u\left[\left(\frac{\Omega_1^2}{r_1^3}-\frac{\Omega_2^2}{r_2^3}\right)\left(-1+\frac{1}{3}R_1+2m_1\tilde r_1^2\right)\right]\right| \le & C_b d_{\tilde r}(\tilde r_1,\tilde r_2)+C_b\varrho^4 d_m(m_1,m_2)\\ &+C_b\left(\varrho d_{\tilde r}(\tilde r_1,\tilde r_2)+\varrho^4 d_m(m_1,m_2)\right)\varrho^{1-s/4}\\ \le & C_b D.\end{aligned}$$

Moreover,

$$\left|\partial_u\left(\frac{1}{3}(R_1-R_2)+2(m_1\tilde r_1^2-m_2\tilde r_2^2\right)\right|\le C_b\varrho^{3-s/4}d_B(B_1,B_2)+C_b\varrho^2 d_m(m_1,m_2)$$

$$
\begin{aligned}
&+ C_b \varrho^2 d_{\tilde{r}}(\tilde{r}_1, \tilde{r}_2) \\
&\leq C_b \varrho^2 \cdot D.
\end{aligned}
$$

Thus

$$\left| \partial_u \left[ \frac{\Omega_1^2}{r_1^3} \left( \frac{1}{3}(R_1 - R_2) + 2(m_1 \tilde{r}_1^2 - m_2 \tilde{r}_2^2 \right) \right] \right| \leq C_b \varrho^3 \cdot D.$$

Therefore

$$\left| (\tilde{r}_1')_{uu} - (\tilde{r}_2')_{uu} \right| \leq C_b \delta \cdot D.$$

It remains to estimate $(T(\tilde{r}_1') - T(\tilde{r}_2'))/\varrho$. Since, as above,

$$T(\tilde{r}_1' - \tilde{r}_2')(u,v) = \int_v^u \left( (\tilde{r}_1' - \tilde{r}_2')_{uu}(u',v) + (\tilde{r}_1' - \tilde{r}_2')_{uv}(u',v) \right) \mathrm{d}u',$$

we conclude

$$\left| T(\tilde{r}_1' - \tilde{r}_2') \right| \leq C_b \delta \varrho \cdot D.$$

#### Bound for $B_1' - B_2'$

Due to the absence of twisting, this section differs significantly from the analogous part of [HW13]. As above, we need to begin with some term manipulations. To ease the notation, we set

$$
\begin{aligned}
A :=& B_1' - B_2' \\
N_i :=& \frac{\Omega_i^2}{3 r_i} \left( \mathrm{e}^{-2B_i} - \mathrm{e}^{-8B_i} \right)
\end{aligned}
$$

for $i = 1, 2$. Therefore

$$
\begin{aligned}
&\partial_v \left( r_1^3 ((B_1')_u - (B_2')_u)^2 + r_1^3 (B_1' - B_2')^2 \right) + \partial_u \left( r_1^3 ((B_1')_v - (B_2')_v)^2 + r_1^3 (B_1' - B_2')^2 \right) \\
=& 3 r_1^2 \left( (r_1)_v A_u^2 + (r_1)_u A_v^2 \right) + 2 r_1^3 T(A) A_{uv} + 3 r_1^2 T(r_1) A^2 + 2 r_1^3 A T(A) \\
=& - 3 r_1^3 T(r_1) A_u A_v + 3 r_1^2 T(r_1) A^2 + 2 r_1^3 A T(A) - 2 r_1^3 \left[ N_1 - N_2 \right] T(A) \\
&- 3 r_1^3 \left( \frac{(r_1)_u}{r_1} - \frac{(r_2)_u}{r_2} \right) T(A) (B_2')_v - 3 r_1^3 \left( \frac{(r_1)_v}{r_1} - \frac{(r_2)_v}{r_2} \right) T(A) (B_2')_u.
\end{aligned}
$$

Integrating the identity yields on the left hand side

$$
\begin{aligned}
&\int_{u_0}^{v} \int_{v'}^{u} \left[ \partial_v \left( r_1^3 ((B_1')_u - (B_2')_u)^2 + r_1^3 (B_1' - B_2')^2 \right) \right. \\
&\qquad \left. + \partial_u \left( r_1^3 ((B_1')_v - (B_2')_v)^2 + r_1^3 (B_1' - B_2')^2 \right) \right] \mathrm{d}u' \mathrm{d}v' \\
=& \int_0^{\delta} r_1^3 \left( A_u^2 - A_v^2 \right) (u_0 + t, u_0 + t) \, \mathrm{d}t - \int_{u_0}^{u} r_1^3 \left( A^2 + A_u^2 \right) (u', u_0) \, \mathrm{d}u'
\end{aligned}
$$

$$
\begin{aligned}
&+\int_v^u r_1^3\left(A^2+A_u^2\right)(u',v)\,\mathrm{d}u' + \int_{u_0}^v r_1^3\left(A^2+A_v^2\right)(u,v')\,\mathrm{d}v' \\
=&\int_v^u r_1^3\left(A^2+A_u^2\right)(u',v)\,\mathrm{d}u' + \int_{u_0}^v r_1^3\left(A^2+A_v^2\right)(u,v')\,\mathrm{d}v' \\
&-\int_{u_0}^u r_1^3\left(A^2+A_u^2\right)(u',u_0)\,\mathrm{d}u'.
\end{aligned}
$$

Turning to the right hand side, we can immediately estimate the first three terms by

$$
\int_{\Delta_{\delta,u_0}} \left[\left|3r_1^3T(r_1)A_uA_v\right| + \left|3r_1^2T(r_1)A^2\right| + \left|2r_1^3AT(A)\right|\right]\,\mathrm{d}u'\mathrm{d}v' \leq C_b\delta\,\|A\|^2_{C^0\underline{H}^1}\,.
$$

For the fourth term, note that

$$
N_1-N_2=\frac{\Omega_1^2}{3r_1^2}\left(\mathrm{e}^{-2B_1}-\mathrm{e}^{-8B_1}-\mathrm{e}^{-2B_2}+\mathrm{e}^{-8B_2}\right)+\frac{1}{3}\left(\frac{\Omega_1^2}{r_1^2}-\frac{\Omega_2^2}{r_2^2}\right)\left(\mathrm{e}^{-2B_2}-\mathrm{e}^{-8B_2}\right).
$$

Therefore

$$
\begin{aligned}
&\int_{\Delta_{\delta,u_0}} \left|2\left(N_1-N_2\right)T(A)\right|\,\mathrm{d}u'\mathrm{d}v' \\
\leq& C_b\delta\,\|A\|^2_{C_0\underline{H}^1} \\
&+C_b\int_{\Delta_{\delta,u_0}}\frac{\Omega_1^4}{r_1^4}r_1^3\left(\mathrm{e}^{-2B_1}-\mathrm{e}^{-8B_1}-\mathrm{e}^{-2B_2}+\mathrm{e}^{-8B_2}\right)^2\,\mathrm{d}u'\mathrm{d}v' \\
&+C_b\int_{\Delta_{\delta,u_0}}\left(\frac{\Omega_1^2}{r_1^2}-\frac{\Omega_2^2}{r_2^2}\right)^2 r_1^3\left(\mathrm{e}^{-2B_2}-\mathrm{e}^{-8B_2}\right)^2\,\mathrm{d}u'\mathrm{d}v' \\
\leq& C_b\delta\,\|A\|^2_{C_0\underline{H}^1}+C_b\int_{\Delta_{\delta,u_0}}\varrho\left(\frac{\Omega_1^2}{r_1^2}-\frac{\Omega_2^2}{r_2^2}\right)^2\,\mathrm{d}u'\mathrm{d}v' \\
\leq& C_b\delta\,\|A\|^2_{C_0\underline{H}^1}+C_b\delta^4D^2.
\end{aligned}
$$

The pointwise bound on $B_1'-B_2'$ can be retrieved directly via a Sobolev inequality. It remains to show the pointwise bound on the derivative. We have the transport equation

$$
\begin{aligned}
&\partial_v\left(r_1^{3/2}A_u\right) \\
=&-\partial_v\left((r_1^{3/2}-r_2^{3/2})(B_2')_u\right) \\
&-\frac{3}{2}r_1^{1/2}(r_1)_uA_v-\frac{3}{2}\left(r_1^{1/2}(r_1)_u-r_2^{1/2}(r_2)_u\right)(B_2')_v \\
&-\frac{1}{3}\Omega_1^2r_1^{-1/2}\left(\mathrm{e}^{-2B_1}-\mathrm{e}^{-8B_1}-\mathrm{e}^{-2B_2}+\mathrm{e}^{-8B_2}\right) \\
&-\frac{1}{3}\left(\Omega_1^2r_1^{-1/2}-\Omega_2^2r_2^{-1/2}\right)\left(\mathrm{e}^{-2B_2}-\mathrm{e}^{-8B_2}\right).
\end{aligned}
$$

After integrating in $v$ and multiplying by $r_1^{-1/2-s/4}$, we can estimate all terms on the right hand side. The first term yields

$$\left|r_1^{-1/2-s/4}\left(r_1^{3/2}-r_2^{3/2}\right)\right|\left(B_2'\right)_u(u,v)\leq C_b\left|\varrho^{-1+s/4}\partial_u B_2'\right| d_{\tilde{r}}(\tilde{r}_1,\tilde{r}_2)\leq C_b\cdot D,$$

the second term

$$\frac{3}{2}r_1^{-1/2-s/4}\int_{u_0}^v r_1^{1/2}\left|(r_1)_u A_v\right| \mathrm{d}v'\leq C_b\delta^{s/4}\left\|A\right\|_{C^0\underline{H}^1},$$

the third term

$$\frac{3}{2}r_1^{-1/2-s/4}\int_{u_0}^v \frac{1}{r_1^{1/2}}\left(r_1^{1/2}(r_1)_u-r_2^{1/2}(r_2)_u\right)(B_2')_v\,\mathrm{d}v'\leq C_b\delta^{s/4}\left\|B'\right\|_{C^0\underline{H}^1}\cdot D,$$

the fourth term

$$\frac{1}{3}r_1^{-1/2-s/4}\int_{u_0}^v \Omega_1^2 r_1^{-1/2}\left(\mathrm{e}^{-2B_1}-\mathrm{e}^{-8B_1}-\mathrm{e}^{-2B_2}+\mathrm{e}^{-8B_2}\right)\mathrm{d}v'\leq C_b\delta^{1+s/4}\left\|A\right\|_{C^0\underline{H}^1}$$

and the fifth term

$$\frac{1}{3}r_1^{-1/2-s/4}\int_{u_0}^v \left(\Omega_1^2 r_1^{-1/2}-\Omega_2^2 r_2^{-1/2}\right)\left(\mathrm{e}^{-2B_2}-\mathrm{e}^{-8B_2}\right)\mathrm{d}v'\leq C_b\delta^{1+s/4}\left\|B_2'\right\|_{C^0\underline{H}^1}D.$$

#### Bound on the mass

Let us first note some useful inequalities:

$$\begin{aligned}
\left|\frac{(\tilde{r}_1)_v}{\tilde{r}_1^3}-\frac{(\tilde{r}_2)_v}{\tilde{r}_2^3}\right| \leq & C_b\varrho^{-6}\left(\left|(\tilde{r}_1)_v(\tilde{r}_2^3-\tilde{r}_1^3)\right|+\left|\tilde{r}_2^3((\tilde{r}_1)_v-(\tilde{r}_2)_v)\right|\right)\\
\leq & C_b\varrho^{-3}d_{\tilde{r}}(\tilde{r}_1,\tilde{r}_2)\\
4\left|\frac{(\tilde{r}_1)_u}{\Omega_1^2\tilde{r}_1^5}-\frac{(\tilde{r}_2)_u}{\Omega_2^2\tilde{r}_2^5}\right| = & \left|\frac{1-\mu_1}{\tilde{r}_1(\tilde{r}_2)_v}-\frac{1-\mu_2}{\tilde{r}_2(\tilde{r}_2)_v}\right|\\
\leq & C_b\varrho^{-3}\left(d_{\tilde{r}}(\tilde{r}_1,\tilde{r}_2)+d_m(m_1,m_2)\right)
\end{aligned}$$

From these estimates we deduce

$$\left|m_1'(u,v)-m_2'(u,v)\right|\leq C_b\left\|B_1'\right\|^2_{C^0\underline{H}^1}\left(d_{\tilde{r}}(\tilde{r}_1,\tilde{r}_2)+d_m(m_1,m_2)\right)+C_b\left\|B_1'-B_2'\right\|^2_{C^0\underline{H}^1}.$$

The estimate for $\partial_u m_1'-\partial_u m_2'$ can be derived analogously to above. We will refrain from writing out this straightforward, yet tedious calculation.

### 3.2.5. Propagation of constraints

It needs to be shown that in addition to the above equations, the constraint equation

$$\partial_u m = r r_u\left(1-\frac{2}{3}R\right)-\frac{4}{\Omega^2}r^3 r_v(B_u)^2$$

is satisfied as well. First, consider the equation

$$\partial_v m = \frac{2}{\tilde{r}_v}\tilde{r}^4 \left(r^{3/2}B_v\right)^2 m - \left[-rr_v\left(1-\frac{2}{3}R\right) + \left(1+\frac{r^2}{\ell^2}\right)\frac{\tilde{r}^2}{\tilde{r}_u}\left(r^{3/2}B_v\right)^2\right]. \tag{3.2.8}$$

or schematically,

$$\partial_v m = \alpha m + \beta. \tag{3.2.9}$$

With $(\tilde{r}, B, m) \in \mathcal{B}_b$, we have $r \in C^1$, $m, \partial_u m, r_{uv}, B, B_u \in C^0$ and $B_v \in C^0_u L^2_v$. Moreover, we have

$$\partial_u\left(r^{3/2}B_v\right) = -\frac{3}{2}r^{1/2}r_v B_u - \frac{\Omega^2}{3r^{1/2}}\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right),$$

where the left hand side is in $C^0$. Therefore, integrating (3.2.9) yields

$$m(u,v) = \exp\left(\int_{u_0}^{v}\alpha(u,v')\,\mathrm{d}v'\right)\left[m(u,u_0) + \int_{u_0}^{v}\beta(u,v')\,\mathrm{e}^{-\int_{u_0}^{v'}\alpha(u,u'')\,\mathrm{d}v''}\,\mathrm{d}v'\right].$$

Differentiating the identity (3.2.9) for $m$ by $u$ shows that $m_{uv} = m_{vu}$ is in $C^0_u L^1_v$.

Now, we write the constraint as a transport equation in $v$. Define

$$\chi := -\partial_u m + rr_u\left(1-\frac{2}{3}R\right) - \frac{4}{\Omega^2}r^3 r_v\left(B_u\right)^2.$$

We have

$$\begin{aligned}\partial_v\chi =& -\partial_v\partial_u m + r_u r_v\left(1-\frac{2}{3}R\right) + rr_{uv}\left(1-\frac{2}{3}R\right) - \frac{2}{3}rr_u\partial_v R - 4\partial_v\left(\frac{r_v}{\Omega^2}\right)r^3(B_u)^2\\ &- \frac{12(r_v)^2}{\Omega^2}r^2(B_u)^2 - \frac{8}{\Omega^2}r^3 r_v B_u B_{uv}\\ =&0.\end{aligned}$$

Since $\chi$ is zero initially, we conclude that $\chi = 0$ in $\Delta_{\delta,u_0}$.

### 3.2.6. Higher regularity

In this section, we shall show that by imposing higher regularity on the initial data, the fixed point argument above yields a classical solution to the Einstein vacuum equations.

#### Construction of initial data

Here we will derive conditions on the free data set $(\bar{\tilde{r}}, M, \overline{B})$ to give rise to a suitable full data set $(\bar{\tilde{r}}, \overline{B}, \overline{m}, \overline{\tilde{r}_v}, \overline{T\tilde{r}}, \overline{TB}, \overline{Tm}, \overline{T\tilde{r}_v})$, where $T = \partial_u + \partial_v$. From above we can already define

$$\overline{T\tilde{r}} := \bar{\tilde{r}}_u + \overline{\tilde{r}_v}.$$

We will assume that there exists a constant $B_1$ such that

$$\partial_u \overline{B} = \frac{1}{2} B_1 \overline{\tilde{r}}^3 + B_R,$$

where $B_R = \mathcal{O}\left(\overline{\tilde{r}}^4\right)$. We assume moreover that

$$\overline{\partial_v B} = -\frac{1}{2} B_1 \overline{\tilde{r}}^3 + \tilde{B}_R,$$

where $\tilde{B}_R$ is defined via the transport equation

$$\partial_u \left(\overline{\tilde{r}}^{-3/2} \overline{\partial_v B}\right) = \frac{3}{2} \overline{\tilde{r}}^{-5/2} \overline{\tilde{r}_v B_u} - \frac{1}{2} \overline{\Omega}^2 \overline{\tilde{r}}^{1/2} \left(\mathrm{e}^{-2\overline{B}} - \mathrm{e}^{-8\overline{B}}\right). \tag{3.2.10}$$

Note that upon integrating in $u$, the right hand side will reproduce the leading order term

$$\frac{3}{4} B_1 \int_{u_0}^{u} \overline{\tilde{r}_v \tilde{r}}^{1/2} \, \mathrm{d}u' \sim -\frac{3}{8} B_1 \int_{u_0}^{u} 2^{-1/2} u^{1/2} \, \mathrm{d}u' \sim -\frac{1}{2} B_1 \tilde{r}^{3/2}.$$

This suffices to define

$$\overline{TB} := \partial_u \overline{B} + \overline{\partial_v B}.$$

We can also define

$$\overline{\partial_v m} := -\frac{\overline{\tilde{r}_v}}{\overline{\tilde{r}}^3} \left(1 - \frac{2}{3} \overline{R}\right) + \frac{4}{\overline{\Omega}^2} \frac{\overline{\tilde{r}}_u}{\overline{\tilde{r}}^5} \left(\overline{\partial_v B}\right)^2.$$

and

$$\overline{Tm} := \partial_u \overline{m} + \overline{\partial_v m}.$$

Setting

$$\overline{\alpha} := -\frac{4 \overline{\tilde{r}}_u}{1 - \overline{\mu}} \frac{1}{\overline{\tilde{r}}} \left(-1 + \frac{1}{3} \overline{R} - 2 \overline{m} \overline{\tilde{r}}^2\right),$$

we define $\overline{\tilde{r}_{vv}}$ by integrating

$$\partial_u \left(\overline{\tilde{r}_{vv}}\right) = \overline{\alpha} \overline{\tilde{r}_{vv}} + \overline{\alpha}_v \overline{\tilde{r}_v}.$$

We note that $\overline{\alpha}_v$ can be expressed merely in terms of quantities already defined. We set

$$\overline{T \tilde{r}_v} := \left(\overline{\tilde{r}_v}\right)_u + \overline{\tilde{r}_{vv}}.$$

**Definition 3.2.6.** A free data set $(\overline{\tilde{r}}, \overline{B})$ gives rise to $H^2$-initial data if we can construct $(\overline{T\tilde{r}}, \overline{TB}, \overline{Tm})$ as above and if, for any $0 < s < 1$, there is a $C > 0$ such that the following bounds hold on the initial data ray $\mathcal{N}$:

$$\begin{aligned}
\|\overline{T\tilde{r}}\|_{C^0} + \left\|\left(\overline{T\tilde{r}}\right)_u\right\|_{C^0} + \|\overline{T\tilde{r}_v}\|_{C^0} + \left\|\left(\overline{T\tilde{r}}\right)_{uu}\right\|_{C^0} <& C \\
\int_{u_0}^{u_1} \bar{\tilde{r}}^{-3} \left[\left(\partial_u\left(\overline{TB}\right)\right)^2 + \left(\overline{TB}\right)^2\right] \mathrm{d}u <& C \\
\left\|\bar{\tilde{r}}^{-2}\overline{TB}\right\|_{C^0} <& C \\
\left\|\bar{\tilde{r}}^{-1+s/4}\partial_u\left(\overline{TB}\right)\right\|_{C^0} <& C \\
\left\|\overline{Tm}\right\|_{C^0} <& C \\
\left\|\bar{\tilde{r}}\partial_u\left(\overline{Tm}\right)\right\|_{C^0} <& C
\end{aligned}$$

**Lemma 3.2.7.** *For free data giving rise to $H^2$-initial data, we can obtain $C < \delta$ for any $\delta > 0$ by truncating the initial data ray.*

*Proof.* The proof proceeds verbatim as that of Lemma 3.2.2. □

### Propagation of regularity

**Definition 3.2.8.** The commuted ball $\mathcal{B}_b^1$ is the subset of $\mathcal{B}_b$ such that $T\tilde{r}_v$, $T\tilde{r}_{uv}$, $T\tilde{r}_{uu}$, $Tm$, $(Tm)_u$, $TB$, $TB_u$ are $C^0$ satisfying

$$\begin{aligned}
\left\|\frac{TT\tilde{r}}{\varrho}\right\|_{C^0} + \|T\tilde{r}_{uv}\|_{C^0} + \|T\tilde{r}_{uu}\|_{C^0} <& b \\
\|Tm\|_{C^0} + \left\|\varrho^{1+s}\partial_u\left(Tm\right)\right\|_{C^0} <& b \\
\|TB\|_{C^0H^1} + \left\|\varrho^{-2}TB\right\|_{C^0} + \left\|\varrho^{-1+s/4}TB\right\|_{C^0} <& b.
\end{aligned}$$

Moreover, $\tilde{r}$, $T\tilde{r}$, $m$, $B$ and $TB$ are required to agree with the initial data at $u = u_0$.

*Remark* 3.2.9. Morally and intuitively (but not actually rightly and correctly because of $T\tilde{r}_u$ and $T\tilde{r}_v$), elements of $\mathcal{B}_b^1$ should have $(T\tilde{r}, Tm, TB)$ bounded with respect to $d$. The above definition makes this idea more precise. We would like to justify this schematically here. Let $F$ be continuously differentiable function of $(\tilde{r}, \tilde{r}_u, \tilde{r}_v, T\tilde{r}, \tilde{r}_{uv}, \tilde{r}_{uu}, m, m_u, B, B_u)$ and assume that $F$ can be written as a product of functions of only one variable. We see that

$$\begin{aligned}
TF =& T\tilde{r}\frac{\partial F}{\partial \tilde{r}} + T\tilde{r}_u\frac{\partial F}{\partial \tilde{r}_u} + T\tilde{r}_v\frac{\partial F}{\partial \tilde{r}_v} + TT\tilde{r}\frac{\partial F}{\partial (T\tilde{r})} + T\tilde{r}_{uv}\frac{\partial F}{\partial \tilde{r}_{uv}} + T\tilde{r}_{uu}\frac{\partial F}{\partial \tilde{r}_{uu}} \\
&+ Tm\frac{\partial F}{\partial m} + Tm_u\frac{\partial F}{\partial m_u} + TB\frac{\partial F}{\partial B} + TB_u\frac{\partial F}{\partial B_u}.
\end{aligned}$$

Inspecting the arguments of Sections 3.2.3 and 3.2.4, we see that – to repeat the arguments for $TF$ – we will need that the terms involving differentiation with respect to $\tilde{r}$, $T\tilde{r}$, $m_u$, $B$ and $B_u$ will require smallness, whereas boundedness will suffice for the other term. Assuming that $F$ is indeed a product, Definition 3.2.8 guarantees exactly this. We observe that for estimates on $T\tilde{r}_u$ and $T\tilde{r}_v$, one obtains a bound on $T\tilde{r}_u = \tilde{r}_{uu} + \tilde{r}_{uv}$ immediately from the estimates for elements of $\mathcal{B}_b$. For $T\tilde{r}_v$, we get a bound by integrating from the boundary:

$$T\tilde{r}_v(u,v) = T\tilde{r}_v(v,v) + \int_v^u T\tilde{r}_{uv}\,\mathrm{d}u' = -T\tilde{r}_u(v,v) + \int_v^u T\tilde{r}_{uv}\,\mathrm{d}u'$$

**Proposition 3.2.10.** *Suppose that* $(\overline{\tilde{r}}, M, \overline{B}$ *give rise to an* $H^2$*-initial data set. Then* $\Phi : \mathcal{B}_b \to \mathcal{B}_b$ *maps* $\mathcal{B}_b^1$ *into itself for* $\delta$ *sufficiently small.*

*Proof.* First, let us turn to $\tilde{r}'$. To compute $T\tilde{r}'$, we remark that

$$\begin{aligned}
&\frac{1}{h}\left(\int_{v+h}^{u+h}\int_{u_0}^{v+h} G(u',v')\,\mathrm{d}v'\mathrm{d}u' - \int_v^u\int_{u_0}^{v} G(u',v')\,\mathrm{d}v'\mathrm{d}u'\right)\\
=&\frac{1}{h}\left(\int_{v+h}^{u+h}\int_{u_0+h}^{v+h} G(u',v')\,\mathrm{d}v'\mathrm{d}u' - \int_v^u\int_{u_0}^{v} G(u',v')\,\mathrm{d}v'\mathrm{d}u'\right)\\
&+\frac{1}{h}\int_{v+h}^{u+h}\int_{u_0}^{u_0+h} G(u',v')\,\mathrm{d}v'\mathrm{d}u'\\
\to&\int_v^u\int_{u_0}^{v} G(u',v')\,\mathrm{d}v'\mathrm{d}u' + \int_v^u T(G(u',v'))\,\mathrm{d}u'
\end{aligned}$$

as $h \to 0$ for a suitable function $G$. Therefore,

$$\begin{aligned}
T\tilde{r}' =&\overline{\tilde{r}}_u(u) - \overline{\tilde{r}}_v(v)\\
&+\int_v^u \left(\Omega(u',v')^2\tilde{r}(u',v')^3\left(-1+\frac{1}{3}R(u',v') + 2m(u',v')\tilde{r}(u',v')^2\right)\right)\mathrm{d}u'\\
&+\int_v^u\int_{u_0}^{v} T\left(\Omega(u',v')^2\tilde{r}(u',v')^3\left(-1+\frac{1}{3}R(u',v') + 2m(u',v')\tilde{r}(u',v')^2\right)\right)\mathrm{d}v'\mathrm{d}u'\\
=&\overline{T\tilde{r}}(u) - \overline{T\tilde{r}}(v)\\
&+\int_v^u\int_{u_0}^{v} T\left(\Omega(u',v')^2\tilde{r}(u',v')^3\left(-1+\frac{1}{3}R(u',v') + 2m(u',v')\tilde{r}(u',v')^2\right)\right)\mathrm{d}v'\mathrm{d}u'.
\end{aligned}$$

In the light of Remark 3.2.9, a repetition of the arguments above yields smallness for sufficiently small $\delta$.

We shall now deal with $TB'$. From the local theory of the wave equation in Appendix D.5, we know that $B' \in H^2_{\mathrm{loc}}$. Therefore, we can commute the wave equation with $T$:

$$\begin{aligned}
TB'_{uv} =&\frac{3}{2}\frac{\tilde{r}_u}{\tilde{r}}TB'_v + \frac{3}{2}\frac{\tilde{r}_v}{\tilde{r}}TB'_u + \frac{3}{2}T\left(\frac{\tilde{r}_u}{\tilde{r}}\right)B'_v + \frac{3}{2}T\left(\frac{\tilde{r}_v}{\tilde{r}}\right)B'_u\\
&-\frac{\tilde{r}^3}{3}T\left(\Omega^2\tilde{r}^{-1}\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right)\right)
\end{aligned}$$

Again, one computes

$$\begin{aligned}
&\partial_u\left(\tilde{r}^{-3}\left(TB'_v\right)^2 + \tilde{r}^{-3}\left(TB'\right)^2\right) + \partial_v\left(\tilde{r}^{-3}\left(TB'_u\right)^2 + \tilde{r}^{-3}\left(TB'\right)^2\right)\\
=&\frac{3}{\tilde{r}^4}\left(T\tilde{r}\right)\left(TB'_u\right)\left(TB'_v\right) - \frac{3}{\tilde{r}^4}\left(T\tilde{r}\right)\left(TB'\right)^2 + 2\tilde{r}^{-3}\left(TB'\right)\left(TTB'\right)\\
&-\frac{2}{\tilde{r}^3}\left(TTB'\right)\left[\frac{1}{3}T\left(\Omega^2\tilde{r}^2\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right)\right) - \frac{3}{2}T\left(\frac{\tilde{r}_u}{\tilde{r}}\right)B'_v - \frac{3}{2}T\left(\frac{\tilde{r}_v}{\tilde{r}}\right)B'_u\right].
\end{aligned}$$

The rest follows as before once we have established that

$$\begin{aligned}\left|T\left(\frac{\tilde r_u}{\tilde r}\right)\right| &< C_b\\ \left|T\left(\frac{\tilde r_v}{\tilde r}\right)\right| &< C_b.\end{aligned}$$

It suffices to prove only the statement involving the derivative in $u$ since

$$T\left(\frac{\tilde r_u}{\tilde r}\right)+T\left(\frac{\tilde r_v}{\tilde r}\right)=\frac{TT\tilde r}{\tilde r}-\left(\frac{T\tilde r}{\tilde r}\right)^2$$

is bounded. We can write

$$T\left(\frac{\tilde r_u}{\tilde r}\right)=\frac{1}{\tilde r\varrho}\left(\varrho T\tilde r_u-\frac{1}{2}T\tilde r\right)-\frac{T\tilde r}{\tilde r}\left(\frac{\tilde r_u}{\tilde r}-\frac{1}{2\varrho}\right).$$

Clearly, the second term on the right hand side is bounded. For the first term, we observe that

$$\left|\partial_v\left(\varrho T\tilde r_u-\frac{1}{2}T\tilde r\right)\right|=\left|\varrho T\tilde r_{uv}-\frac{1}{2}TT\tilde r\right|\leq C_b\varrho.$$

After integrating in $u$, we see that the first term is bounded too.

For $Tm'$, we have

$$Tm'=\overline{Tm}(u)+\int_{u_0}^{v}T\left(-\frac{\tilde r_v}{\tilde r^3}\left(1-\frac{2}{3}R'\right)+\frac{4}{\Omega^2}\frac{\tilde r_u}{\tilde r^5}(B_v')^2\right)\mathrm{d}v'$$

and proceed as before. □

## 3.3. The global problem

This section is devoted to studying the global dynamics arising from Eguchi-Hanson-type initial data. The existence of a maximal development is guaranteed by Theorem 3.1.7 and Remark 3.1.13. In Section 3.3.1, we specify our choice of coordinates on the orbits of the $SU(2)\times U(1)$ action and derive some geometric properties; here, we follow the exposition of [Daf04a] and [Daf04b] mutatis mutandis. Proving that the existence of a horizon would be contradictory is the content of Sections 3.3.2 and 3.3.3.

### 3.3.1. Global biaxial Bianchi IX symmetry

Let $(\mathcal{S},\overline{g},K)$ be of Eguchi-Hanson type with negative mass $M$ at infinity. Then, by Theorem 3.1.7 and Remark 3.1.13, there is a unique maximal forward development $(\mathcal{M}^+,g)$ which is asymptotically locally AdS. There is a projection map $\pi:\mathcal{M}^+\to\mathcal{Q}^+$ onto a two-dimensional manifold with boundary $\mathcal{Q}^+$ such that every $q\in\mathcal{Q}^+$ represents an orbit under the $SU(2)\times U(1)$ symmetry. The manifold $\mathcal{Q}^+$ can be embedded smoothly into $(\mathbb{R}^2,g_{\mathrm{Mink}})$ and its boundary consists of a one-dimensional curve $\Sigma$ (initial hypersurface) and a one-dimensional curve $\Gamma$ (central worldline, where $r = 0$). Choosing standard null coordinates $(u, v)$ on $\mathbb{R}^{1+1}$, $\mathcal{Q}^+$ shall be endowed with a metric

$$h = -\frac{1}{2}\Omega^2(u,v)\left(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u\right).$$

We choose $u$ such that the curves of constant $u$ are outgoing and such that $u$ as well as $v$ are increasing to the future along $\Gamma$. A coordinate chart $(u', v')$ preserves these assumptions if and only if

$$\frac{\partial u'}{\partial u} > 0, \; \frac{\partial v'}{\partial v} > 0, \; \frac{\partial u'}{\partial v} = \frac{\partial v'}{\partial u} = 0. \tag{3.3.1}$$

With respect to $h$, $\Sigma$ is spacelike and $\Gamma$ timelike. Conformal infinity $\mathcal{I} \subseteq \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$ is defined as follows: Set

$$\mathcal{U} := \left\{ u : \sup_{(u,v)\in\mathcal{Q}^+} r(u,v) = \infty \right\}.$$

For each $u \in \mathcal{U}$, there is a unique $v^*(u)$ such that

$$(u, v^*(u)) \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+.$$

Note here that the closure is always taken with respect to the topology of $\mathbb{R}^2$. Now define null infinity as

$$\mathcal{I} := \bigcup_{u\in\mathcal{U}} (u, v^*(u)).$$

Since the spacetime is asymptotically locally AdS, null infinity $\mathcal{I}$ is timelike. We have

$$\mathcal{Q}^+ = D^+\left(\Sigma \cup \mathcal{I}\right),$$

i. e. $\mathcal{Q}^+$ is in the future domain of dependence of $\Sigma$ and $\mathcal{I}$. By a simple change of coordinates satisfying (3.3.1), we achieve that $u = v$ on $\mathcal{I}$. Note that in general, we cannot achieve that both $\mathcal{I}$ and $\Gamma$ are straightened out in this way. We know that $B$ extends continuously to $\mathcal{I}$ and vanishes there. Moreover, we know that the Hawking mass $m$ extends continuously to $\mathcal{I}$ and equals a constant value $M < 0$.

**Lemma 3.3.1.** *The following hold:*

*(i)* $r$ *is unbounded on* $\Sigma$*.*

*(ii)* $m \to M < 0$ *as* $r \to \infty$*.*

*(iii)* $r_v > 0$ *for points in* $\Sigma$ *with large* $r$ *and*

$$\frac{r_v}{\Omega^2} \to c_0 > 0$$

*as* $r \to \infty$*.*

*Proof.* The radius $r$ is unbounded on $\Sigma$ by the definition of Eguchi-Hanson-type data. By $M < 0$, we immediately obtain that $m < 0$ on around $\mathcal{I}$. For the Hawking mass to be finite at infinity,

$$-4\frac{r_u r_v}{\Omega^2} = -\frac{r^2}{\ell^2} + \mathcal{O}(r)$$

as $r \to \infty$. Moreover, by the above choice of $(u, v)$

$$\frac{r_u}{r_v} \to -1$$

as $r \to \infty$. The conformal factor $\Omega^2$ grows as $r^2$ since the spacetime is asymptotically locally AdS. Therefore, we deduce that $r_v/\Omega^2$ is positive and finite as $r \to \infty$. □

**Proposition 3.3.2.** *The above manifold $\mathcal{M}^+$ does not have any trapped or marginally trapped surfaces, i. e.*

$$r_u < 0 \text{ and } r_v > 0$$

*globally in $\mathcal{Q}^+$.*

*Proof.* Define the set

$$\mathcal{A} := \{(u, v) \in \mathcal{Q}^+ \, : \, m(u, v) < 0\}.$$

Evidently, $\mathcal{A}$ is open. From Lemma 3.3.1, we know that the points of $\Sigma$ with $r$ sufficiently large are contained in $\mathcal{A}$. Call this set $U_1$. Moreover, since $M$ is negative and $m$ is continuous, $\mathcal{A}$ contains a neighbourhood of $\mathcal{I}$. Call this neighbourhood $U_2$. Let $\mathcal{C}$ be the connected component of $\mathcal{A}$ containing $U_1$ and $U_2$. Clearly, $\mathcal{C}$ is open as well. In $\mathcal{C}$, we have

$$0 > m = \frac{r^2}{2}\left(1 + 4\frac{r_u r_v}{\Omega^2}\right) + \frac{r^4}{2\ell^2}, \tag{3.3.2}$$

from which we conclude that $r_u r_v < 0$ wherever $r$ is finite because $\mathcal{C}$ is connected. Since there is a point on $\Sigma$, contained in $\mathcal{C}$, where $r_v > 0$, we have that $r_u < 0$ and $r_v > 0$ in $\mathcal{C}$. From (C.2.6) and (C.2.7), we obtain $\partial_u m \leq 0$ and $\partial_v m \geq 0$ in $\mathcal{C}$, thus also in $\overline{\mathcal{C}}$. Therefore, $m < 0$ in $\overline{\mathcal{C}}$. Hence $\mathcal{C}$ is open and closed. We conclude that $\mathcal{C} = \mathcal{Q}^+$ and the statement follows from (3.3.2). □

*Remark* 3.3.3. The absence of anti-trapped surfaces can also be guaranteed by fixing the sign of $r_u$ on $\Sigma$ and $\mathcal{I}$ and then using (C.2.1).

This fact already allows us to prove a weak geometric statement about the potential singularities that can arise in the time evolution.

**Definition 3.3.4.** Let $p \in \overline{\mathcal{Q}^+}$. The indecomposable past subset $J^-(p) \cap \mathcal{Q}^+$ is said to be eventually compactly generated if there exists a compact subset $X \subseteq \mathcal{Q}^+$ such that

$$J^-(p) \subseteq D^+(X) \cup J^-(X).$$

Here we denote by $J^-(S)$ the causal past of a subset $S$.

**Definition 3.3.5.** A point $p \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$ is a first singularity if $J^-(p) \cap \mathcal{Q}^+$ is eventually compactly generated and if any eventually compactly generated indecomposable past proper subset of $J^-(p) \cap \mathcal{Q}^+$ is of the form $J^-(q)$ for a $q \in \mathcal{Q}^+$.

**Lemma 3.3.6.** *Let $p \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$ be a first singularity. Then*

$$p \in \overline{\Gamma} \backslash \Gamma.$$

*Proof.* Suppose $p \notin \overline{\Gamma}$. Since the compact set $X$ of Definition 3.3.5 has to be wholly contained in $\mathcal{Q}^+$, we know that $p \notin \mathcal{I}$. In particular, $p$ is the future endpoint of a rectangle, whose remainder is completely contained in the interior of $\mathcal{Q}^+$; see Figure 3.7. By Proposition 3.3.2, $r_u < 0$ and $r_v > 0$ in this rectangle. Therefore, we can apply the standard extension principle away from infinity and the central worldline – in a manner as e. g. in [Daf04a] – to conclude that $p \in \mathcal{Q}^+$, a contradiction. □

**Theorem 3.3.7.** *If $\mathcal{Q}^+ \backslash J^-(\mathcal{I}) \neq \emptyset$, then there is a null curve $\mathcal{H}^+ \subseteq \mathcal{Q}^+$ such that*

$$\mathcal{H}^+ = \overline{J^-(\mathcal{I})} \backslash \left( I^-(\mathcal{I}) \cup \overline{\mathcal{I}} \right). \tag{3.3.3}$$

Note that $I^-(S)$ denotes the chronological past of a subset $S$.

*Proof.* The horizon is given by

$$\mathcal{H}^+ = \mathring{J}^-(\mathcal{I}) \cap \mathcal{Q}^+.$$

Let $p$ be the future endpoint of the horizon. Since $\mathcal{Q}^+ \backslash J^-(\mathcal{I}) \neq \emptyset$, $p \notin \overline{\Gamma} \backslash \Gamma$. If $p \in \Gamma$, there is an open neighbourhood $U$ of $p$ such that $U \cap \mathcal{Q}^+ \neq \emptyset$. Therefore $p$ would not be the future endpoint of the horizon. If $p \notin \overline{\Gamma} \cup \overline{\mathcal{I}}$, then $p$ is a first singularity and we have a contradiction to Lemma 3.3.6. Therefore $p \in \overline{\mathcal{I}}$. □

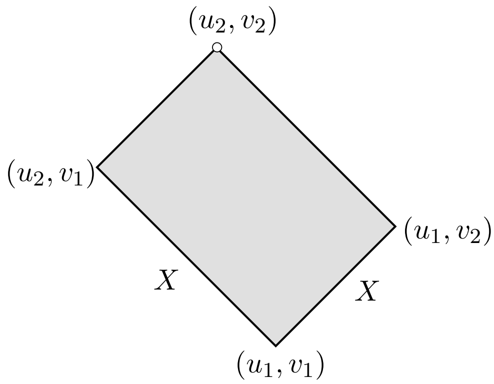


Figure 3.7.: The extension property in the interior

Sections 3.3.2 and 3.3.3 are devoted to showing Theorem 3.1.20.

## 3.3.2. An extension principle

We formulate and prove an extension principle tailored to extending a solution beyond a supposed horizon.

**Theorem 3.3.8.** *Let $(\tilde r, m, B)$ be a classical solution to the Einstein equations (3.1.5), (3.1.6), (3.1.7) and (3.1.8) in the punctured triangle $\Delta := \Delta_{d,u_0} \backslash \{(u_0+d, u_0+d)\}$. Let $\mathcal{I}_{\text{aux}} := \overline{\Delta_{d,u_0}} \backslash \{\Delta_{d,u_0} \cup \{(u_0+d, u_0+d)\}\}$. Assume that*

$$\tilde r\Big|_{\mathcal{I}_{\text{aux}}} = 0,\ m\Big|_{\mathcal{I}_{\text{aux}}} = M < 0,\ B\Big|_{\mathcal{I}_{\text{aux}}} = 0$$

*and that*

$$\lim_{v \to u_0+d} \tilde r(u_0+d, v) = 0.$$

*Suppose that*

$$\tilde r_u > 0 \text{ and } \tilde r_v < 0$$

*in $\Delta$, that*

$$\inf_{\mathcal{I}_{\text{aux}}} \tilde r_u > 0 \text{ and } \sup_{\mathcal{I}_{\text{aux}}} \tilde r_v < 0$$

*and that there is a $C > 0$ such that*

$$\sup_{u_0 \le v \le u_0+d} \int_v^{u_0+d} \tilde r^{-3} \left(B^2 + (B_u)^2\right) \mathrm{d}u' + \sup_{\Delta} \left|\tilde r^{-1} \partial_u B\right| < C.$$

*Then there is a $\delta > 0$ such that the solution $(\tilde r, m, B)$ can be extended to the strictly larger triangle $\Delta_{d+\delta,u_0}$.*

*Proof.* Extending beyond the domain of existence means using the local well-posedness result to extend the solution further into the future. We will first need to make sure that on each constant $v$-slice, the function $\tilde r$ satisfies the correct boundary conditions. Reformulating equation (3.1.5), we obtain

$$\tilde r_{uv} = -\frac{4\tilde r_u \tilde r_v}{\tilde r^2 + 2|m|\tilde r^4 + \ell^{-2}} \tilde r \left(-1 + \frac{1}{3}R - 2m\tilde r^2\right).$$

Therefore $\tilde r_{uv} = 0$ on $\mathcal{I}_{\text{aux}}$. Using a coordinate change, we want to fix the value of $\tilde r_u$. By the assumptions and since $\tilde r_u = -\tilde r_v$ on $\mathcal{I}_{\text{aux}}$, $|\tilde r_u|, |\tilde r_v| \ge c > 0$. Then

$$\frac{\mathrm{d}u'}{\mathrm{d}u}(u,v) = 2\tilde r_u(u,u),\ \frac{\mathrm{d}v'}{\mathrm{d}v} = 2\tilde r_v(v,v),$$

where $u' = u$ and $v' = v$ on $\mathcal{I}$, defines a regular change of coordinates that preserves the biaxial Bianchi IX symmetry. Moreover, in $(u', v')$ coordinates,

$$\tilde r\Big|_{\mathcal{I}_{\text{aux}}} = 0,\ \tilde r_{u'}\Big|_{\mathcal{I}_{\text{aux}}} = \frac{1}{2},\ \tilde r_{u'v'}\Big|_{\mathcal{I}_{\text{aux}}} = 0. \tag{3.3.4}$$

Hence we can assume without loss of generality that (3.3.4) already holds in the original

$(u, v)$ coordinates.

To increase the domain of existence, we also need initial $v$-slices of increased length. This can be achieved by an application of the standard local existence theorem away from infinity in double null coordinates whose proof proceeds by the same methods as for $\Lambda = 0$, which is standard by now. Prescribing data on the slice $(u_0, u_0 + d + \delta']$ of constant $v = u_0$ (for a $\delta' > 0$), we infer that for every $\varepsilon > 0$, there is a $\delta' > 0$ such that the solution can be extended to the set

$$\Delta_\varepsilon := \Delta \cup \left(\Delta_{d+\delta',u_0} \cap \{v \le u_0 + d - \varepsilon\}\right).$$

For each constant $v$-ray in $\Delta_\varepsilon$, we have a initial data set, whose functions have norms uniformly bounded by $2C$. Note that the condition $1 - \mu > r^2/\ell^2$ holds everywhere because $M < 0$.

Therefore, by the local existence theorem, there is a $\delta^*$ independent of $\varepsilon$ such that each slice of constant $v$ in $\Delta_\varepsilon$ yields a solution in a triangular domain of size $\delta^*$. Now we choose $\varepsilon = \delta^*/2$ and see that the solution $(\tilde{r}, m, B)$ extends to a strictly larger triangle $\Delta_{d+\delta,u_0}$, where $\delta = \varepsilon$. □

The proof above yields another version of the extension principle that we formulate separately for the sake of clarity.

**Corollary 3.3.9.** *Suppose the assumptions of Theorem 3.3.8 hold. Moreover, let us assume that the classical solution on $\Delta$ has an extension to the extended initial data slice $\tilde{\mathcal{N}} = (u_0, u_0 + d + \varepsilon]$. Then there is a $\delta > 0$ such that the solution $(\tilde{r}, m, B)$ can be extended to $\Delta_{d+\delta,u_0}$ such that it agree on $\tilde{\mathcal{N}} \cup \Delta_{d+u_0,u_0}$ with the given values. Furthermore, the extension is unique for sufficiently small $\delta > 0$*

### 3.3.3. A priori estimates

In this section, we first establish what was described through Figure 3.2 in Section **??** as the soft argument. This is the content of Lemma 3.3.10. The remainder of the section contains the argument by contradiction, using the extension principle in form of Corollary 3.3.9.

**Lemma 3.3.10.** *Let $\mathcal{Q}^+$ be as in Theorem 3.1.20. Set $\Delta_u^d := \Delta_{d,u} \backslash \{(u + d, u + d)\}$ and $\mathcal{N}_u^d := \{v = u\} \cap \Delta_u^d$. Then for any $\Delta_{u_1}^{d_1} \subseteq \mathcal{Q}^+$, there is a $u_0 \ge u_1$ and $d_0 := d_1 - (u_0 - u_1)$ such that*

*(i) $r \ge r_0 > 0$ in $\Delta_{u_0}^{d_0}$.*

*(ii) There are $q_1, q_2 > 0$ such that*

$$q_1 \le \frac{r_v}{\Omega^2} \le q_2 \tag{3.3.5}$$

*in $\Delta_{u_0}^{d}$. The constants $q_1$ and $q_2$ depend on the choice of $r_0$.*

*Proof.* If $\Delta_{u_1}^{d_1}$ touches $\Gamma$, then by moving the initial slice of constant $v$ to the future – as depicted in Figure 3.2 –, we achieve that $r \geq r_0 > 0$ since $r_v > 0$ globally by Proposition 3.3.2. Fixing $u_0$ and $d_0$, choose $r_0$ maximal such that $r \geq r_0$ holds.

By assumption, the bound on $r_v/\Omega^2$ holds on $\Sigma$. Set

$$\mathcal{R} := \{r = r_0\} \cap \{u \leq u_0 + d_0\} \cap \mathcal{Q}^+.$$

The set $\mathcal{R}$ is closed and touches $\{u = u_0 + d_0\}$ and $\Sigma$. The continuous function $r_v/\Omega^2$ is positive in $\mathcal{Q}^+$ by Proposition 3.3.2. Therefore the bound on $r_v/\Omega^2$ holds in $\mathcal{R}$. We will show that (3.3.5) holds in the causal future of

$$S := \mathcal{R} \cup (\Sigma \cap \{r \geq r_0\})$$

such that the constants $q_1$ and $q_2$ depend on the values of $r_v/\Omega^2$ on $S$. We rewrite the constraint equation (C.2.2) as

$$\partial_v\left(\frac{r_v}{\Omega^2}\right) = \left(-\frac{4r^3 r_u}{\Omega^2}(B_v)^2\right)\left(-\frac{2}{r^2(1-\mu)}\right)\frac{r_v}{\Omega^2}. \tag{3.3.6}$$

Given $(u,v) \in J^+(S)$, there is a $(u',v') \in S$ such that $(u,v)$ and $(u',v')$ are connected by a null curve. We integrate (3.3.6) along a ray of constant $u$ to find

$$\begin{aligned}\left(\frac{r_v}{\Omega^2}\right)(u,v) &= \exp\left(\int_{v'}^{v} \frac{4r^3 r_u}{\Omega^2}(B_v)^2 \frac{2}{r^2(1-\mu)}\,\mathrm{d}v''\right)\cdot\left(\frac{r_v}{\Omega^2}\right)(u',v')\\ &\geq \exp\left(-\frac{2}{r_0^2}\int_{u_0}^{v} \frac{-4r^3 r_u}{\Omega^2}(B_v)^2\,\mathrm{d}v''\right)\cdot\left(\frac{r_v}{\Omega^2}\right)(u',v')\\ &\geq \exp\left(-\frac{2}{r_0^2}\left(m(u,v) - m(u',v')\right)\right)\cdot\left(\frac{r_v}{\Omega^2}\right)(u',v')\end{aligned}$$

For the first inequality, we have used $1 - \mu > 1$ and $r \geq r_0$. For the second inequality, we have used (3.1.7) and have dropped a non-negative term. Therefore, we obtain

$$\mathrm{e}^{-\frac{2}{r_0^2}[M - m(u_0+d,u_0)]}\left.\frac{r_v}{\Omega^2}\right|_{(u',v')} \leq \left.\frac{r_v}{\Omega^2}\right|_{(u,v)} \leq \left.\frac{r_v}{\Omega^2}\right|_{(u',v')}. \tag{3.3.7}$$

This yields (3.3.5). □

*Remark* 3.3.11. A bound of the form (3.3.7) can always be achieved, independently of the exact value of $M$.

Now assume for the sake of contradiction that $\mathcal{Q}^+$ possesses a horizon $\mathcal{H}^+$. According to Lemma 3.3.10, we find a $\Delta := \Delta_{u_0}^{d_0}$ such that $(u_0+d, u_0) \in \mathcal{H}^+$ and such that the conclusions of the lemma hold. In particular, the constants and bounds will be fixed henceforth. Again, let $\mathcal{I}_{\text{aux}} := \overline{\Delta_{d,u_0}} \backslash \{\Delta_{d,u_0} \cup \{(u_0+d, u_0+d)\}\}$. Let $\mathcal{N} := \mathcal{N}_{u_0}^{d_0}$. We always have that

$$\tilde{r}\Big|_{\mathcal{I}_{\text{aux}}} = 0,\; m\Big|_{\mathcal{I}_{\text{aux}}} = M < 0,\; B\Big|_{\mathcal{I}_{\text{aux}}} = 0,$$

and that

$$r_u < 0 \text{ and } r_v > 0.$$

We will show that all the assumptions of the extension principle hold. Let us first turn to estimating the norms of $B$. The mass achieves its minimum at $(u_0+d, u_0)$ and its maximum on $\mathcal{I}_{\text{aux}}$. Therefore, upon integration over constant $v$, we obtain

$$\int_v^{u_1} \left(-rr_u\left(1-\frac{2}{3}R\right)+\frac{4}{\Omega^2}r^3r_v(B_u)^2\right)\,\mathrm{d}u = M - m(u_0+d,v) \leq M - m(u_1,u_0).$$

Note that we have

$$1-\frac{2}{3}R \geq \min\{B^2/2, 1\}.$$

We need to estimate the coefficients in the integral. From

$$\tilde{r}_u = \frac{1}{4r^2\frac{r_v}{\Omega^2}}(1-\mu)$$

and (3.3.6), we obtain

$$\frac{1}{4\ell^2}\left(\max_{u_0\leq u\leq u_1}\frac{r_v}{\Omega^2}\bigg|_{(u,u_0)}\right)^{-1} \leq \tilde{r}_u \leq \mathrm{e}^{\frac{2}{r_0^2}[M-m(u_0+d,u_0)]}\left(\min_{u_0\leq u\leq u_1}\frac{r_v}{\Omega^2}\bigg|_{(u,u_0)}\right)^{-1}\times \tag{3.3.8}$$
$$\times\frac{\tilde{r}^2}{4}\left(1+\frac{2|M|}{r^2}+\frac{r^2}{\ell^2}\right)$$

and see that $\tilde{r}_u$ is uniformly bounded above and below by a constant depending only on data on $\mathcal{N}$. Therefore

$$C_1(u-v) \leq \tilde{r} \leq C_2(u-v)$$

and

$$\lim_{v\to u_0+d}\tilde{r}(u_0+d,v) = 0. \tag{3.3.9}$$

Furthermore,

$$-r_u\bigg|_{(u,v)} = \frac{1+\frac{2|m|}{r^2}+\frac{r^2}{\ell^2}}{4\frac{r_v}{\Omega^2}} \geq \frac{r^2}{4\ell^2}\frac{1}{\max_{u_0\leq u\leq u_1}\frac{r_v}{\Omega^2}\bigg|_{(u,u_0)}}.$$

Therefore there is a constant $C_u$ depending only on values of $\tilde{r}$, $\tilde{r}_v$ and $m$ on $v=u_0$ such that

$$\int_v^{u_1} r^3\left(\min\{B^2,1\}+(B_u)^2\right)\,\mathrm{d}u < C_u \tag{3.3.10}$$

uniformly in $\Delta$.

Thus,

$$|B(u,v)| \leq \left(\int_v^u \frac{1}{r^3}\,\mathrm{d}u'\right)^{1/2}\left(\int_v^u r^3 B_u^2\,\mathrm{d}u'\right)^{1/2} \leq \frac{C_2^{1/2}}{2} C_u^{1/2}(u-v)^2.$$

It follows that

$$\tilde{r}^{-2}|B| \leq C_{\text{pointwise}} \tag{3.3.11}$$

uniformly. Together with (3.3.10), this yields

$$\int_u^{u_1} r^3\left(B^2 + (B_u)^2\right)\,\mathrm{d}u < C'_u. \tag{3.3.12}$$

In a similar way, one also obtains

$$\int_{u_0}^v r^3\left(B^2 + (B_v)^2\right)\,\mathrm{d}v < C'_v$$

for $v \in [u_0, u_1)$ from integrating $\partial_v m$ and then deriving a bound on $r_u/\Omega^2$ as (3.3.7). Here one uses that

$$\left.\frac{r_u}{\Omega^2}\right|_{\mathcal{I}} = -\left.\frac{r_v}{\Omega^2}\right|_{\mathcal{I}}.$$

Using the wave equation for $B$ in the form

$$\partial_v\left(r^{3/2}B_u\right) = -\frac{3}{2}r^{1/2}r_u B_v - \frac{\Omega^2}{3r^{1/2}}\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right) \tag{3.3.13}$$

yields

$$\begin{aligned}\left|r(u,v)^{3/2}B_u(u,v)\right| \leq & r(u,u_0)^{3/2}|B_u(u,u_0)| + C\int_{u_0}^v r(u,v')^{5/2}|B_v(u,v')|\,\mathrm{d}v' \\ & + \frac{1}{3}\left(\int_{u_0}^v \frac{\Omega^2}{r^2}\,\mathrm{d}v'\right)^{1/2}\left(\int_{u_0}^v \Omega^2 r\left|\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right|^2\,\mathrm{d}v'\right)^{1/2}\end{aligned}$$

Since $\Omega^2/r^2$ is bounded by virtue of the bounds established above, the third term is easily seen to be bounded. The second term is estimated as

$$\begin{aligned}\int_{u_0}^v r^{5/2}|B_v|\,\mathrm{d}v' \leq & \left(\int_{u_0}^v r^3(B_v)^2\,\mathrm{d}v'\right)^{1/2}\left(\int_{u_0}^v r^2\mathrm{d}v'\right)^{1/2} \\ \leq & C\left(\int_{u_0}^v \frac{\partial_v r}{(-\tilde{r}_v)}\,\mathrm{d}v'\right)^{1/2} \\ \leq & Cr(u,v)^{1/2}.\end{aligned}$$

Therefore,

$$|rB_u| \leq r(u, u_0)|B_u(u, u_0)| + C,$$

where $C$ depends on values on $\mathcal{N}$ and on $C_1$, $C_2$,, $C'_v$. Since $B$ is a classical solution up to and including the horizon, there is an $\alpha > 0$ such that

$$B = \varrho^\alpha \left(a_0(t) + a_1(t)\varrho + o(\varrho)\right) \tag{3.3.14}$$

for smoothly differentiable functions $a_0$ and $a_1$ of $t = (u+v)/2$. From above, the asymptotics of $r$, $r_u$, $r_v$ and $\Omega^2$ are known as we approach the boundary. Inserting (3.3.14) into (3.3.13), we obtain $\alpha = 4$. Therefore, $rB_u$ is bounded on $\mathcal{N}$ and we have established the desired pointwise bound on $rB_u$ in $\Delta$.

An application of the extension principle (Theorem 3.3.8) yields Theorem 3.1.20 if it also holds true that

$$\inf_{\mathcal{I}_{\text{aux}}} \tilde{r}_u > 0 \text{ and } \sup_{\mathcal{I}_{\text{aux}}} \tilde{r}_u < 0.$$

This has been established already in (3.3.8), thus finishing the proof of Theorem 3.1.20.

# A. More ODE theory

## A.1. The angular ODE

Assume throughout the section that $m \neq 0$. Recall equations (2.2.1) and (2.2.2). We will only give details for $\alpha \leq 0$. The other case can be treated analogously. Define $x := \cos\vartheta$. Then the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_\vartheta(1-x^2)\frac{\mathrm{d}S}{\mathrm{d}x}\right) - \left(\frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{1-x^2} - \left(\frac{\Xi}{\Delta_\vartheta}a^2\omega^2 - 2ma\omega\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2} - \frac{\alpha}{\ell^2}a^2\right)x^2\right)S + \lambda S = 0.$$

Set

$$K(x) := \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-\frac{a^2}{\ell^2}x^2\right)\left(1-x^2\right)\right) = 4\frac{a^2}{\ell^2}x^3 - 2x\left(1+\frac{a^2}{\ell^2}\right).$$

Using the language of Theorem 2.2.1, we see that at $\pm 1$, we have

$$f_0 = 1, \qquad g_0 = -m^2/4.$$

Thus for $m \neq 0, 1$, we have two zeros which do not differ by an integer. Then we know that solutions are linear combinations of $(x \mp 1)^{-|m|/2}$ and $(x \mp 1)^{|m|/2}$ near $\pm 1$.

**Proposition A.1.1.** *Suppose that for some fixed $\omega_0, \alpha_0 \in \mathbb{R}$, we have an eigenvalue $\lambda_0$. Then, for $\kappa$ sufficiently close to $\kappa_0$, we can uniquely find a complex analytic function $\lambda(\omega, \alpha)$ of eigenvalues for the angular ODE with parameter $(\omega, \alpha) \in \mathbb{C} \times \mathbb{R}$ such that $\lambda_0 = \lambda(\omega, \alpha)$.*

*Proof.* We can use the proof in [SR13]. If $S$ is an eigenfunction, we clearly must have

$$S \sim (1 \mp x)^{|m|/2}$$

as $x \to \pm 1$. For any $\omega, \alpha$ and $\lambda$, we can uniquely define a solution $S(\vartheta, \omega, \alpha, \lambda)$ by requiring that

$$S(x, \omega, \alpha, \lambda)(1+x)^{-|m|/2}$$

is holomorphic at $x = -1$ and

$$\left(S(\cdot, \omega, \alpha, \lambda)(1+\cdot)^{-|m|/2}\right)(x = -1) = 1.$$

Then we have holomorphic functions $F(\omega,\alpha,\lambda)$ and $G(\omega,\alpha,\lambda)$ such that

$$S(x,\omega,\alpha,\lambda) \sim F(\omega,\alpha,\lambda)(1-x)^{-|m|/2} + G(\omega,\alpha,\lambda)(1-x)^{|m|/2}$$

as $x \to 1$. Since $\lambda_0$ is an eigenvalue, we have $F(\omega_0,\alpha_0,\lambda_0) = 0$. We want to appeal to the implicit function theorem and define our function $\lambda(\omega,\alpha)$ uniquely near $(\omega_0,\alpha_0)$. Suppose (for the sake of contradiction) that

$$\frac{\partial F}{\partial \lambda}(\omega_0,\alpha_0,\lambda_0) = 0.$$

Set $S_\lambda := \partial S/\partial\lambda$. Since $\partial F/\partial\lambda = 0$, $S_\lambda$ satisfies the boundary conditions of eigenfunctions. Moreover, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_\vartheta(1-x^2)\frac{\mathrm{d}S_\lambda}{\mathrm{d}x}\right) - \left(\frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{1-x^2} - \left(\frac{\Xi}{\Delta_\vartheta}a^2\omega_0^2 - 2ma\omega_0\frac{\Xi}{\Delta_\vartheta}\frac{a^2}{\ell^2} - \frac{\alpha}{\ell^2}a^2\right)x^2\right)S_\lambda \\ + \lambda_0 S = -S.\end{aligned}$$

Multiplying both sides by $\overline{S}$, integrating over $(0,\pi)$ with measure $\sin\vartheta\,\mathrm{d}\vartheta$, integrating by parts and using that $\overline{S}$ satisfies the angular ODE implies

$$\int_0^\pi |S|^2 \sin\vartheta\,\mathrm{d}\vartheta = 0,$$

which is a contradiction. The proof for $\alpha < 0$ proceeds similarly. □

**Proposition A.1.2.** *If $\omega_I > 0$, then*

$$-\mathrm{Im}\,(\lambda\overline{\omega}) > 0.$$

*Proof.* Let $\alpha \le 0$. Multiplying the ODE by $\overline{\omega S}$, integrating by parts and taking imaginary parts gives

$$\begin{aligned}-\int_0^\pi \mathrm{Im}\,(\lambda\overline{\omega})\sin\vartheta\,\mathrm{d}\vartheta = \int_0^\pi \omega_I\left(\Delta_\vartheta\left|\frac{\mathrm{d}S}{\mathrm{d}\vartheta}\right|^2 + \left[\frac{\Xi^2}{\Delta_\vartheta}\frac{m^2}{\sin^2\vartheta} - \frac{\alpha}{\ell^2}a^2\cos^2\vartheta\right]|S|^2\right)\sin\vartheta\,\mathrm{d}\vartheta \\ + \int_0^\pi \frac{\Xi}{\Delta_\vartheta}\cos^2\vartheta\,\mathrm{Im}\,(a^2\omega^2\overline{\omega})|S|^2\sin\vartheta\,\mathrm{d}\vartheta,\end{aligned}$$

which is positive for $\omega_I > 0$. For $\alpha > 0$, the proof proceeds almost verbatim. □

**Proposition A.1.3.** *When $\omega$ is real, we have*

$$\frac{\partial\lambda}{\partial\alpha} = -\frac{a^2}{\ell^2}\int_0^\pi \cos^2\vartheta|S|^2\sin\vartheta\,\mathrm{d}\vartheta$$

*for $\alpha \le 0$ and*

$$\frac{\partial\lambda}{\partial\alpha} = \frac{a^2}{\ell^2}\int_0^\pi \sin^2\vartheta|S|^2\sin\vartheta\,\mathrm{d}\vartheta$$

*for $\alpha > 0$.*

*Proof.* Let $S_\alpha := \partial S/\partial\alpha$. First, let $\alpha \leq 0$. First one differentiates (2.2.1) with respect to $\alpha$, then multiplies by $\overline{S}$ and then integrates by part. Since $\omega \in \mathbb{R}$, $\overline{S}$ satisfies the angular ODE, which yields the result. Similarly we obtain the result for $\alpha > 0$. $\square$

# B. Twisting

## B.1. Twisted derivatives and the modified potential

To deal with the slow decay or even growth of modes satisfying the Neumann boundary condition, we need to use renormalised derivatives

$$h\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}\cdot\right)$$

with a sufficiently regular function $h$. Defining the modified potential

$$\tilde{V}_a^h := \tilde{V} - \frac{1}{h}\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}h}{\mathrm{d}r}\right),$$

we obtain a twisted expression for the radial ODE:

**Lemma B.1.1.** *For all $f \in C^1$ that are piecewise $C^2$,*

$$h^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}h^2\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1}f\right)\right) - \tilde{V}_a^h\frac{r^2+a^2}{\Delta_-}f = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}f}{\mathrm{d}r}\right) - \tilde{V}_a\frac{r^2+a^2}{\Delta_-}f.$$

By virtue of twisting, the modified potential can be chosen to be positive for large $r$:

**Lemma B.1.2.** *Let $h := r^{-1/2+\kappa}$. If $|m|$ is sufficiently large, then there is an $R > r_+$ such that $\tilde{V}_a^h > 0$ for $r > R$. The choice of $R$ is independent of $a$ and $\alpha$. Moreover $\tilde{V}_a^h = \mathcal{O}(1)$ as $r \to \infty$.*

*Proof.* We look at the asymptotic behaviour of the different parts of $\tilde{V}_a$:

$$\begin{aligned}
V_0 - \omega^2 &\sim \frac{1}{\ell^2}\left(\lambda + a^2\omega^2 - 2ma\omega\Xi\right) > \frac{1}{\ell^2}m^2\Xi^2 > 0\\
V_+ &= \frac{2\Delta_-}{(r^2+a^2)^2}\frac{r^2}{\ell^2} + \frac{\Delta_-}{(r^2+a^2)^4}\left(a^4\Delta_- + (r^2-a^2)2Mr\right)\\
&\sim \frac{2\Delta_-}{(r^2+a^2)^2}\frac{r^2}{\ell^2} + \frac{a^4\Delta_-^2}{(r^2+a^2)^4}
\end{aligned}$$

One easily computes that

$$v(r) := \frac{2-\alpha}{\ell^2}\frac{\Delta_-}{(r^2+a^2)^2}r^2 - h^{-1}\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta_-}{r^2+a^2}\frac{\mathrm{d}h}{\mathrm{d}r}\right) = \mathcal{O}(1),$$

which yields the result. □

## B.2. The twisted Euler-Lagrange equation

We give here the derivation of the weak twisted Euler-Lagrange equation.

*Proof of Lemma 2.4.6.* The following proof can be extracted from [Eva10]. We give the extension to twisted derivatives here for the sake of completeness. The minimiser $f_a$ is a minimiser of the functional

$$\tilde{\mathcal{L}}_a(f) := \int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} r^{-1+2\kappa} \left| \frac{\mathrm{d}}{\mathrm{d}r} \left( r^{\frac{1}{2}-\kappa} f \right) \right|^2 + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} |f|^2 \right) \mathrm{d}r$$

under the constraint

$$\mathcal{J}(f) = 0,$$

where

$$\mathcal{J}(f) = \int_{r_+}^{\infty} G(r,f)\,\mathrm{d}r, \qquad G(r,f) = \frac{1}{r^2}\left(|f|^2 - r_+\right).$$

Moreover, define $g(r,f) := 2f/r^2$. Fix $\psi_1 \in \underline{H}^1_\kappa(r_+,\infty)$. We assume in a first step that $g(r,f_a)$ is not identically zero almost everywhere on $(r_+,\infty)$. Then we can find a $\psi_2 \in \underline{H}^1_\kappa(r_+,\infty)$ such that

$$\int_{r_+}^{\infty} g(r,f_a)\psi_2(r)\,\mathrm{d}r \neq 0.$$

Define $j(\tau,\sigma) := \mathcal{J}(f_a+\tau\psi_1+\sigma\psi_2)$ for $\tau,\sigma \in \mathbb{R}$. Clearly, $j(0,0)-0$. Since $\frac{\partial g(r,f_a+\tau\psi_1+\sigma\psi_2)}{\partial\tau}\psi_1$ and $\frac{\partial g(r,f_a+\tau\psi_1+\sigma\psi_2)}{\partial\tau}\psi_2$ are integrable on $(r_+,\infty)$, $j$ is in $C^1$. In particular, we have

$$\frac{\partial j}{\partial\sigma}(0,0) = \int_{r_+}^{\infty} g(r,f_a)\psi_2(r)\,\mathrm{d}r \neq 0.$$

By the Implicit Function Theorem, there is a $\kappa : \mathbb{R} \to \mathbb{R}$ such that $\kappa(0) = 0$ and

$$j(\tau,\kappa(\tau)) = 0.$$

In other words, the function $f_a + \chi(\tau)$, where

$$\chi(\tau) := \tau\psi_1 + \kappa(\tau)\psi_2, \tag{B.2.1}$$

satisfies the integral constraint. Thus, setting $i(\tau) := \tilde{\mathcal{L}}_a(f_a+\chi(\tau))$, we obtain $i'(0) = 0$. Note here that $i$ is differentiable in $\tau$ since $f_a \in \underline{H}^1_\kappa(r_+,\infty)$. We have

$$\left.\frac{\mathrm{d}i}{\mathrm{d}\tau}\right|_{\tau=0} = 2\int_{r_+}^{\infty} \left( \frac{\Delta_-}{r^2+a^2} r^{-1+2\kappa} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa} f_a\right) \left( \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa}\psi_1\right) + \kappa'(0) \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa}\psi_2\right) \right) \right.$$

$$+ \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} f_a(\psi_1 + \kappa'(0)\psi_2)\Big) \,\mathrm{d}r.$$

From (B.2.1), we deduce

$$\kappa'(0) = -\frac{\int_{r_+}^\infty g(r,f_a)\psi_1 \,\mathrm{d}r}{\int_{r_+}^\infty g(r,f_a)\psi_2 \,\mathrm{d}r}.$$

Setting

$$\lambda := 2\frac{\int_{r_+}^\infty \left(\frac{\Delta_-}{r^2+a^2} r^{-1+2\kappa} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa} f_a\right) \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa}\psi_2\right) + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} f_a\psi_2\right) \mathrm{d}r}{\int_{r_+}^\infty g(r,f_a)\psi_2 \,\mathrm{d}r}$$

yields that

$$\int_{r_+}^\infty \left(\frac{\Delta_-}{r^2+a^2} r^{-1+2\kappa} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa} f_a\right) \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\frac{1}{2}-\kappa}\psi_1\right) + \tilde{V}_a^h \frac{r^2+a^2}{\Delta_-} f_a\psi_1\right) \mathrm{d}r = \lambda \int_{r_+}^\infty \frac{f_a}{r^2}\psi \,\mathrm{d}r$$

for all $\psi \in \underline{H}^1_\kappa(r_+,\infty)$. We have $f_a \in \underline{H}^1_\kappa(r_+,\infty)$, whence $\lambda = -\nu_a$.

It remains to deal with the case $g(r,f_a) = 0$ a. e. This, however, would yield that $f = 0$ in contradiction to the norm constraint. $\square$

# C. Biaxial Bianchi IX symmetry

## C.1. $SU(2)$ and Euler angles

To introduce Eguchi-Hanson space and the biaxial Bianchi IX symmetry class, we need to understand $SU(2)$, the space of linear maps $\mathbb{C}^2 \to \mathbb{C}^2$ which are unitary and have unit determinant. As a set

$$SU(2) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \,:\, z, w \in \mathbb{C},\, |z|^2 + |w|^2 = 1 \right\}.$$

Henceforth, we shall identify each matrix $A \in SU(2)$ uniquely by a pair $(z, w) \in \mathbb{C}^2$. Decomposing $z = \hat{x}_1 + \mathrm{i}\hat{y}_1$, $w = \hat{x}_2 + \mathrm{i}\hat{y}_2$ immediately yields an isomorphism from $SU(2)$ to $S^3 \subseteq \mathbb{R}^4$. This isomorphism allows us to introduce coordinates on $S^3$. Since $|z|^2 + |w|^2 = 1$, there are $\vartheta \in [0, \pi]$ and $\gamma, \delta \in [0, 2\pi]$ such that

$$z = \cos\left(\frac{\vartheta}{2}\right) \mathrm{e}^{-\mathrm{i}\gamma} \quad \text{and} \quad w = \sin\left(\frac{\vartheta}{2}\right) \mathrm{e}^{-\mathrm{i}\delta}.$$

We will write

$$\gamma = \frac{\psi + \varphi}{2} \quad \text{and} \quad \delta = \frac{\psi - \varphi}{2}$$

for $0 \leq \varphi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. Note that this choice of coordinates $(\vartheta, \varphi, \psi)$ breaks down when either $z$ or $w$ vanish. The angles $(\vartheta, \varphi, \psi)$ are the Euler angles on $S^3$. Following [Biq13] in defining

$$\begin{aligned}
\hat{\alpha}_1 :=& \hat{x}_1 \,\mathrm{d}\hat{x}_2 - \hat{x}_2 \,\mathrm{d}\hat{x}_1 + \hat{x}_3 \,\mathrm{d}\hat{x}_4 - \hat{x}_4 \,\mathrm{d}\hat{x}_3 \\
\hat{\alpha}_2 :=& \hat{x}_1 \,\mathrm{d}\hat{x}_3 - \hat{x}_3 \,\mathrm{d}\hat{x}_1 + \hat{x}_4 \,\mathrm{d}\hat{x}_2 - \hat{x}_2 \,\mathrm{d}\hat{x}_4 \\
\hat{\alpha}_3 :=& \hat{x}_1 \,\mathrm{d}\hat{x}_4 - \hat{x}_4 \,\mathrm{d}\hat{x}_1 + \hat{x}_2 \,\mathrm{d}\hat{x}_3 - \hat{x}_3 \,\mathrm{d}\hat{x}_2,
\end{aligned}$$

we can easily calculate identities with the left-invariant one-forms:

$$\begin{aligned}
\sigma_1 =& 2\hat{\alpha}_2 \\
\sigma_2 =& 2\hat{\alpha}_3 \\
\sigma_3 =& -2\hat{\alpha}_1,
\end{aligned}$$

where

$$\begin{aligned}\sigma_1 &= \sin\vartheta\,\sin\psi\,\mathrm{d}\varphi + \cos\psi\,\mathrm{d}\vartheta\\ \sigma_2 &= \sin\vartheta\,\cos\psi\,\mathrm{d}\varphi - \sin\psi\,\mathrm{d}\vartheta\\ \sigma_3 &= \cos\vartheta\,\mathrm{d}\varphi + \mathrm{d}\psi.\end{aligned}$$

The Euler angles induce coordinates $(r,\vartheta,\varphi,\psi)$ $(r>0)$ on $\mathbb{R}^4\backslash\{0\}$. We set

$$\begin{aligned}\alpha_1 &:= \frac{1}{r^2}\left(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1 + x_3\,\mathrm{d}x_4 - x_4\,\mathrm{d}x_3\right)\\ \alpha_2 &:= \frac{1}{r^2}\left(x_1\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_1 + x_4\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_4\right)\\ \alpha_3 &:= \frac{1}{r^2}\left(x_1\,\mathrm{d}x_4 - x_4\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_2\right)\end{aligned}$$

in analogy to the definitions above and immediately observe that $\alpha_i = \hat{\alpha}_i$ for $i=1,2,3$. Then one can easily see that the flat metric on $\mathbb{R}^4$ takes the form

$$\begin{aligned}g_{\text{flat}} &= \mathrm{d}r^2 + r^2\left(\alpha_1^2+\alpha_2^2+\alpha_3^2\right)\\ &= \mathrm{d}r^2 + \frac{1}{4}r^2\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right).\end{aligned}$$

Let us denote by $TS^2$ the tangent bundle of $S^2$, which can be viewed as

$$TS^2 = \left\{(x,y)\in\mathbb{R}^3\times\mathbb{R}^3 \,:\, |x|^2=1,\ x\cdot y=0\right\}.$$

We will denote by $STS^2$ the unit tangent bundle, which is given by

$$STS^2 = \left\{(x,y)\in\mathbb{R}^3\times\mathbb{R}^3 \,:\, |x|^2=|y|^2=1,\ x\cdot y=0\right\}.$$

**Theorem C.1.1.** *There is a differentiable surjective map $P: SU(2)\to STS^2$. Moreover, $SU(2)$ is a double cover (in fact the universal cover) of $STS^2$ via $P$.*

*Proof.* We will define $P$ from $SU(2)\backslash\{z=0 \text{ or } w=0\}$ onto $STS^2$ with the poles removed. This suffices to define $P$ taking into account the usual issue with spherical-type coordinates.

Let $A\in SU(2)$ be represented by $(z,w)$. Set

$$\begin{aligned}X &:= \overline{z}w + z\overline{w}\\ Y &:= -\mathrm{i}\left(\overline{z}w - z\overline{w}\right)\\ Z &:= z\overline{z} - w\overline{w}.\end{aligned}$$

We see

$$\begin{aligned}X &= \sin\vartheta\,\cos\varphi\\ Y &= \sin\vartheta\,\sin\varphi\end{aligned}$$

$$Z = \cos\vartheta.$$

Therefore $(X, Y, Z) \in S^2 \subseteq \mathbb{R}^3$. Now we use $\psi$ as a polar coordinate on the tangent plane. Set

$$\begin{aligned} X' &= \cos\vartheta\,\cos\varphi\,\cos\psi - \sin\varphi\,\sin\psi \\ Y' &= \cos\vartheta\,\cos\psi + \cos\varphi\,\sin\psi \\ Z' &= -\sin\vartheta\,\cos\psi. \end{aligned}$$

Set $\psi(z, w) = ((X, Y, Z), (X', Y', Z'))$. Since $\psi$ has period $4\pi$, $P$ covers $STS^2$ twice. □

*Remark* C.1.2. One can show that $STS^2$, $SO(3)$ and $\mathbb{R}P^3$ are diffeomorphic.

This statement can be paraphrased as follows: $\mathbb{R}^4\backslash\{0\}$ can be thought of as $TS^2$ with the zero vectors of each tangent plane removed and the tangent plane being 'attached twice', or, more concretely, $\mathbb{R}_+ \times (S^3/\mathbb{Z}_2) \cong (\mathbb{R}^4\backslash\{0\})/\mathbb{Z}_2$ – where we have taken the quotient with respect to the antipodal map induced by $\mathbb{Z}_2$ – is diffeomorphic to the tangent bundle of $S^2$ with the zero vectors removed.

### C.1.1. Formulae related to the renormalised Hawking mass

In this section, we collect some useful calculations and identities related to the renormalised Hawking mass. Let us first start generally with an $n$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with Levi-Civita connection $\nabla$ and a spacelike hypersurface $(\mathcal{N}, \overline{g})$ with induced second fundamental form $K$. Let $n$ be the timelike normal on $\mathcal{N}$. Let $(\Sigma, \gamma)$ be a compact three-dimensional submanifold of $\mathcal{N}$, separating $\mathcal{N}$ into an inside and an outside. Let $\nu$ be the unit normal pointing outside. Set $l_\pm := n \pm \nu$. For $X$ and $Y$ tangent to $\Sigma$, we define the symmetric null second forms

$$\chi_\pm(X, Y) := g\left(\nabla_X l_\pm, Y\right).$$

Clearly,

$$\chi_\pm = \frac{1}{2}\mathcal{L}_{l_\pm} g.$$

The associated null expansion scalars are defined by

$$\vartheta_\pm := \mathrm{tr}_\gamma \chi_\pm = \gamma^{AB}\left(\chi_\pm\right)_{AB} = \gamma^{AB}\nabla_A\left(l_\pm\right)_B.$$

Let $H$ be the mean curvature of $\Sigma$. Then we immediately obtain

$$\vartheta_\pm = \mathrm{tr}_\gamma K \pm H.$$

Let us now assume that $\mathcal{N}$ is foliated by topological 3-spheres $\Sigma_r$, where

$$4\pi^2 r = \mathrm{Vol}\left(\Sigma_r\right) = \int_{\Sigma_r} \mathrm{d}\mu_\gamma = \int_{S^3} \sqrt{\gamma}\, \mathrm{d}\mu_{S^3}.$$

We compute

$$\mathcal{L}_{l_\pm}\left(\mathrm{Vol}\left(\Sigma_r\right)\right) = \int_{S^3} \frac{1}{2\sqrt{\gamma}} \mathcal{L}_{l_\pm}\left(\det \gamma\right) \mathrm{d}\mu_{S^3} = \int_{S^3} \frac{1}{2}\gamma^{AB}\left(\mathcal{L}_{l_\pm}\gamma\right)_{AB} \sqrt{\gamma}\, \mathrm{d}\mu_{S^3} = \int_{\Sigma_r} \vartheta_\pm \, \mathrm{d}\mu_{\Sigma_r}.$$

Assuming that $\vartheta_\pm$ is constant on $\Sigma_r$, we obtain

$$l_\pm r = \frac{r}{3}\mathrm{tr}_\gamma \chi_\pm.$$

Therefore,

$$g(\nabla r, \nabla r) = -\frac{r^2}{12}\left(\mathrm{tr}_\gamma \chi_+\right)\left(\mathrm{tr}_\gamma \chi_-\right) = -\frac{r^2}{12}\left(\left(\mathrm{tr}_\gamma K\right)^2 - H^2\right). \tag{C.1.1}$$

## C.2. The Einstein vacuum equations reduced by biaxial Bianchi IX symmetry

We quote the following result from [DH06a] and do not present a derivation:

**Theorem C.2.1.** *Let $(\mathcal{M}, g)$ exhibit a biaxial Bianchi IX symmetry. Then the Einstein vacuum equations (3.1.2) for $\Lambda = -6/\ell^2 < 0$, understood as a classical system of partial differential equations, are equivalent to the system of two constraint equations*

$$\partial_u\left(\frac{r_u}{\Omega^2}\right) = -\frac{2r}{\Omega^2}\left(B_u\right)^2 \tag{C.2.1}$$

$$\partial_v\left(\frac{r_v}{\Omega^2}\right) = -\frac{2r}{\Omega^2}\left(B_v\right)^2 \tag{C.2.2}$$

*and four evolution equations*

$$r_{uv} = -\frac{\Omega^2 R}{3r} - \frac{2r_u r_v}{r} - \frac{\Omega^2 r}{\ell^2} \tag{C.2.3}$$

$$\left(\log \Omega\right)_{uv} = \frac{\Omega^2 R}{2r^2} + \frac{3}{r^2} r_u r_v - 3B_u B_v + \frac{\Omega^2}{2\ell^2} \tag{C.2.4}$$

$$B_{uv} = -\frac{3}{2}\frac{r_u}{r}B_v - \frac{3}{2}\frac{r_v}{r}B_u - \frac{\Omega^2}{3r^2}\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right), \tag{C.2.5}$$

*where*

$$R = 2\,\mathrm{e}^{-2B} - \frac{1}{2}\,\mathrm{e}^{-8B}$$

*is the scalar curvature of the group orbits.*

*Proof.* One needs to compute the components of the Ricci curvature. The constraints are the $uu$ and $vv$ components. The equation (C.2.4) comes from the $uv$ component. The equations (C.2.3) and (C.2.5) are the content of the remaining components. $\square$

Now our aim is to reformulate the equations of Theorem C.2.1 in terms of the renormalised Hawking mass. The derivatives of the mass are given by

$$\partial_u m = r r_u \left(1 - \frac{2}{3}R\right) - \frac{4}{\Omega^2} r^3 r_v (B_u)^2 \tag{C.2.6}$$

$$\partial_v m = r r_v \left(1 - \frac{2}{3}R\right) - \frac{4}{\Omega^2} r^3 r_u (B_v)^2. \tag{C.2.7}$$

Since $x \mapsto 2\,\mathrm{e}^{-2x} - \mathrm{e}^{-8x}/2$ is positive for $|x| \to \infty$ and has no extremum, we conclude that $\partial_u m \leq 0$ and $\partial_v m \geq 0$ if $r_u \leq 0$ and $r_v \geq 0$.

**Proposition C.2.2.** *Assume that (C.2.3), (C.2.5), (C.2.6) and (C.2.7) hold. Moreover, define $\Omega$ via (**??**). Then the constraints (C.2.1) and (C.2.2) hold. If the right hand side of (C.2.3) can be differentiated in $u$, then also (C.2.4) holds.*

*Proof.* The proof is a calculation. We obtain

$$\frac{4r_u}{\Omega^2} = -\frac{1}{r_v} + \frac{2m}{r^2 r_v} - \frac{r^2}{\ell^2 r_v}$$

from (**??**). This yields

$$\begin{aligned}
\partial_u\left(\frac{r_u}{\Omega^2}\right) &= -\frac{r_u}{r_v}\frac{r_{uv}}{\Omega^2} + \frac{1}{2}\frac{m_u}{r^2 r_v} - \frac{m r_u}{r^3 r_v} - \frac{1}{2}\frac{r}{\ell^2}\frac{r_u}{r_v} \\
&= \frac{r_u}{r_v}\left(\frac{1}{2r} - \frac{R}{3r} + \frac{R}{3r} - \frac{1}{2r} + \frac{m}{r^3} - \frac{r}{2\ell^2} + \frac{r}{\ell^2} - \frac{1}{2}\frac{r}{\ell^2} - \frac{m}{r^3}\right) - \frac{2}{\Omega^2} r\left(B_u\right)^2 \\
&= -\frac{2}{\Omega^2} r\left(B_u\right)^2.
\end{aligned}$$

The second constraint equation is obtained analogously. To obtain the equation for $(\log\Omega)_{uv}$, we multiply (C.2.1) by $\Omega^2$ and differentiate with respect to $u$:

$$-2r_u\left(\log\Omega\right)_{uv} + r_{uuv} - 2r_{uv}\frac{\Omega_u}{\Omega} = -2r_v\left(B_u\right)^2 - 4rB_uB_{uv}$$

Using (C.2.3), we obtain

$$r_{uuv} - 2r_{uv}\frac{\Omega_u}{\Omega} = \frac{\Omega^2}{r^2}r_u R + \frac{4\Omega^2}{3r}B_u\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right) + \frac{\Omega^2 r_u}{\ell^2} + 6\frac{(r_u)^2}{r^2}r_v + 4r_v(B_u)^2.$$

Applying (C.2.5), the desired equation follows. $\square$

# D. Well-posedness of the wave equation

## D.1. Function spaces and weak formulation

In this section, we prove well-posedness of the wave equation

$$\partial_u \left(h(u,v)^{-3}\psi_v\right) + \partial_v \left(h(u,v)^{-3}\psi_u\right) = f \tag{D.1.1}$$

for a fixed $h \in C^\infty\left(\overline{\Delta_{\delta,u_0}}\right)$, or equivalently

$$\psi_{uv} = \frac{3}{2}\frac{h_u}{h}\psi_v + \frac{3}{2}\frac{h_v}{h}\psi_u + \frac{h^3}{2}f$$

in $\Delta_{\delta,u_0}$ as an initial-boundary value problem (IBVP).

We shall use the spaces $\underline{H}^1_v(u)$, $\underline{H}^1_u(v)$ and $C^0\underline{H}^1$ as in Section 3.2.2. Denote by $C^0\underline{H}^1_0(\Delta_{\delta,u_0})$ the closure of $C^\infty_0(\Delta_{\delta,u_0})$ under the $C^0\underline{H}^1$ norm. Recall that only the null boundaries are part of the set $\Delta_{\delta',u_0}$. In addition, we also define

$$\begin{aligned}
\|g\|^2_{\underline{L}^2_u(v)} &:= \int_v^{u_0+d} \varrho^{-3} g(u',v)^2 \,\mathrm{d}u' \\
\|g\|^2_{\underline{L}^2_v(u)} &:= \int_{u_0}^{u} \varrho^{-3} g(u,v')^2 \,\mathrm{d}v'.
\end{aligned}$$

Furthermore,

$$\|g\|^2_{C^0\underline{L}^2(\Delta_{\delta,u_0})} := \sup_{u_0\le u\le u_0+\delta} \|g\|^2_{\underline{L}^2_v(u)} + \sup_{u_0\le v\le u_0+\delta} \|g\|^2_{\underline{L}^2_u(v)}.$$

We will also use

$$\begin{aligned}
\|g\|^2_{\underline{L}^2(\Delta_{\delta,u_0})} &:= \int_{\Delta_{\delta,u_0}} \varrho^{-3} g^2 \,\mathrm{d}u\mathrm{d}v \\
\|g\|^2_{\underline{H}^1(\Delta_{\delta,u_0})} &:= \|g\|^2_{\underline{L}^1(\Delta_{\delta,u_0})} + \|g\|^2_{\underline{H}^1(\Delta_{\delta,u_0})}.
\end{aligned}$$

Evidently,

$$\begin{aligned}
\|g\|^2_{\underline{L}^2(\Delta_{\delta,u_0})} &\le \delta \|g\|^2_{C^0\underline{L}^2(\Delta_{\delta,u_0})} \\
\|g\|^2_{\underline{H}^1(\Delta_{\delta,u_0})} &\le \delta \|g\|^2_{C^0\underline{H}^1(\Delta_{\delta,u_0})}.
\end{aligned}$$

**Definition D.1.1.** We say that $\psi \in C^0\underline{H}^1_0(\Delta_{\delta,u_0})$ is a weak solution of the Dirichlet IBVP (D.1.1) for $f \in L^2(\Delta_{\delta,u_0})$ in $\Delta_{\delta,u_0}$ with initial datum $\psi_0 \in \left(\underline{H}^1_u\right)_0(u_0)$ if

1. For all $\varphi \in C^0 \underline{H}_0^1(\Delta_{\delta,u_0})$ and all $0 \le s \le \delta$, we have

$$\begin{aligned}
&\int_{\Delta_{s,u_0}} h^{-3}\left(\psi_u \varphi_v + \psi_v \varphi_u\right) \mathrm{d}u\mathrm{d}v \\
&\quad + \int_{u_0}^{u_0+s} h^{-3}\psi_u(u_0+s,v)\varphi(u_0+s,v)\,\mathrm{d}v - \int_{u_0}^{u_0+s} h^{-3}\psi_v(u,u_0)\varphi(u,u_0)\,\mathrm{d}u \\
&= -\int_{\Delta_{s,u_0}} fw\,\mathrm{d}u\mathrm{d}v. \qquad \text{(D.1.2)}
\end{aligned}$$

2. $\psi$ satisfies the initial condition $\psi_0$ on the initial data slice $\mathcal{N} = \Delta_{\delta,u_0} \cap \{v = u_0\}$.

Henceforth we shall make some assumptions on the smooth function $h$. We will assume that there are $C_1, C_2, C_3 > 0$ such that

$$C_1 \varrho \le h \le C_2 \varrho \quad \text{and} \quad \frac{T(h)}{h} \le C_3. \qquad \text{(D.1.3)}$$

Recall that $T = \partial_u + \partial_v$.

## D.2. A priori estimates

For $k \in \mathbb{N}$, we define

$$D_k := \Delta_{\delta-1/k,u_0+1/k} \cup \{(u,v) \in \Delta_{\delta,u_0} \,:\, u = v = u_0 + 1/k\}.$$

**Lemma D.2.1.** *Let $\psi_0' \in C_0^\infty(\mathcal{N})$ and $f' \in C_0^\infty(\Delta_{\delta,u_0})$. Then for $k$ sufficiently large, there is a unique function $\psi^{(k)} \in C^\infty(D_k)$ satisfying*

$$\begin{cases}
\partial_u\left(h^{-3}\psi_v^{(k)}\right) + \partial_v\left(h^{-3}\psi_u^{(k)}\right) = f' & \text{in } D_k \\
\psi^{(k)} = \psi_0' & \text{on } v = u_0 \\
\psi^{(k)} = 0 & \text{on } u = v = u_0 + 1/k.
\end{cases} \qquad \text{(D.2.1)}$$

*and*

$$\left\|\psi^{(k)}\right\|_{C^0H^1} < C\left(\left\|\psi_0'\right\|^2_{\underline{H}^1_u(u_0)} + \left\|\varrho^{3/2} f'\right\|^2_{L^2\left(\Delta_{\delta,u_0}\right)}\right), \qquad \text{(D.2.2)}$$

*where the constant $C$ only depends on $\delta$.*

*Proof.* The finite linear problem (D.2.1) always has a solution for $k$ sufficiently large, by standard theory of the wave equation in 1+1 dimensions.

Now, we compute

$$\begin{aligned}
&\partial_u\left(h^{-3}\left(\psi^{(k)}\right)^2 + h^{-3}\left(\psi_v^{(k)}\right)^2\right) + \partial_v\left(h^{-3}\left(\psi^{(k)}\right)^2 + h^{-3}\left(\psi_u^{(k)}\right)^2\right) \\
=&2h^{-3}\psi^{(k)}T\left(\psi^{(k)}\right) + fT\left(\psi^{(k)}\right) + \frac{3}{h^4}T(h)\psi_u^{(k)}\psi_v^{(k)} - \frac{3}{h^4}T(h)\left(\psi^{(k)}\right)^2.
\end{aligned}$$

Integrating the left hand side over the domain

$$\{(u', v') \in \Delta_{\delta,u_0} \,:\, u' \leq u,\, v' \leq v\}$$

yields

$$\begin{aligned}
&\int_{u_0}^{v}\int_{v'}^{u}\left[\partial_u\left(h^{-3}\left(\psi^{(k)}\right)^2+h^{-3}\left(\psi_v^{(k)}\right)^2\right)+\partial_v\left(h^{-3}\left(\psi^{(k)}\right)^2+h^{-3}\left(\psi_u^{(k)}\right)^2\right)\right]\mathrm{d}u'\mathrm{d}v'\\
=&\frac{1}{2}\int_{u=v=u_0+1/k} h^{-3}\left(\left(\psi_v^{(k)}\right)^2-\left(\psi_u^{(k)}\right)^2\right)\mathrm{d}t-\int_{u_0}^{u}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_u^{(k)}\right)^2\right)\mathrm{d}u'\\
&+\int_{v}^{u}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_u^{(k)}\right)^2\right)\mathrm{d}u'+\int_{u_0}^{v}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_v^{(k)}\right)^2\right)\mathrm{d}v',
\end{aligned}$$

where

$$t=\frac{u+v}{2}.$$

The first term vanishes due to our boundary conditions. The right hand side yields

$$\begin{aligned}
&\int_{u_0}^{v}\int_{v'}^{u}\left[2h^{-3}\psi^{(k)}T(\psi^{(k)})+fT(\psi^{(k)}+\frac{3}{h^4}T(h)\psi_u^{(k)}\psi_v^{(k)}-\frac{3}{h^4}T(h)\psi^2\right]\mathrm{d}u'\mathrm{d}v'\\
\leq&(2+3C_3)\int_{u_0}^{u}\int_{u_0}^{\min(u',v)}h^{-3}\left[(\psi^{(k)})^2+(\psi_v^{(k)})^2\right]\mathrm{d}v'\mathrm{d}u'\\
&+(2+3C_3)\int_{u_0}^{v}\int_{v'}^{u}h^{-3}\left[(\psi^{(k)})^2+(\psi_u^{(k)})^2\right]\mathrm{d}u'\mathrm{d}v'\\
&+\int_{u_0}^{v}\int_{u_0}^{v'}h^3f^2\,\mathrm{d}v'\mathrm{d}u'.
\end{aligned}$$

Putting everything together, we obtain the inequality

$$\int_{v}^{u}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_u^{(k)}\right)^2\right)\mathrm{d}u'+\int_{u_0}^{v}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_v^{(k)}\right)^2\right)\mathrm{d}v' \tag{D.2.3}$$

$$\leq C_0+(2+3C_3)\int_{u_0}^{u}\int_{u_0}^{\min(u',v)}h^{-3}\left[\left(\psi^{(k)}\right)^2+\left(\psi_v^{(k)}\right)^2\right]\mathrm{d}v'\mathrm{d}u' \tag{D.2.4}$$

$$+(2+3C_3)\int_{u_0}^{v}\int_{v'}^{u}h^{-3}\left[\left(\psi^{(k)}\right)^2+\left(\psi_u^{(k)}\right)^2\right]\mathrm{d}u'\mathrm{d}v', \tag{D.2.5}$$

where

$$C_0:=\int_{u_0}^{u_1}h^{-3}\left(\left(\psi^{(k)}\right)^2+(\psi_u^{(k)})^2\right)\mathrm{d}u'+\int_{u_0}^{u_1}\int_{v'}^{u_1}h^3f^2\,\mathrm{d}u'\mathrm{d}v'$$

is a constant. We claim that

$$\int_{v}^{u}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_u^{(k)}\right)^2\right)\mathrm{d}u'+\int_{u_0}^{v}h^{-3}\left(\left(\psi^{(k)}\right)^2+\left(\psi_v^{(k)}\right)^2\right)\mathrm{d}v' \tag{D.2.6}$$

$$\leq 2C_0\exp\left(16+24C_3\left[(u-u_0)+(v-u_0)\right]\right) \tag{D.2.7}$$

for all $(u,v) \in \Delta_{\delta,u_0}$. We will prove this by bootstrap. Thus consider the set

$$S := \{(u,v) \in \Delta_{\delta,u_0} \,:\, \text{(D.2.3) holds}\}$$

This set is evidently closed. Moreover, it is non-empty since $(u_0,u_0) \in S$. We we can show that $S$ is open, we know that $S = \Delta_{\delta,u_0}$. Substituting the bootstrap assumption (D.2.6) into (D.2.3) yields

$$\begin{aligned}&\int_v^u h^{-3}\left(\left(\psi^{(k)}\right)^2 + \left(\psi_u^{(k)}\right)^2\right) \mathrm{d}u' + \int_{u_0}^v h^{-3}\left(\left(\psi^{(k)}\right)^2 + \left(\psi_v^{(k)}\right)^2\right) \mathrm{d}v'\\ \leq&\frac{3}{2} C_0 \, \mathrm{e}^{16+24C_3[(u-u_0)+(v-u_0)]}\,.\end{aligned}$$

Therefore, $S$ is open. The result (D.2.2) follows no by recalling the assumptions (D.1.3). $\square$

## D.3. Existence and regularity

**Theorem D.3.1.** *Assume that $h$ satisfies (D.1.3). For all $\psi_0 \in \underline{H}^1_u(u_0)$ and all $f$ such that $\left((u,v) \mapsto \varrho(u,v)^{3/2} f(u,v)\right) \in L^2(\Delta_{\delta,u_0})$, the wave equation has a weak solution $\psi$. The constructed $\psi$ satisfies*

$$\|\psi\|_{C^0\underline{H}^1(\Delta_{\delta,u_0})} \leq C_\delta \left(\|\psi_0\|^2_{\underline{H}^1_u(u_0)} + \left\|\varrho^{3/2} f\right\|^2_{L^2(\Delta_{\delta,u_0})}\right)$$

*as well as*

$$\left\|\varrho^{-2}\psi\right\|_{C^0(\Delta_{\delta,u_0})} \leq C_\delta \|\psi\|_{C^0\underline{H}^1(\Delta_{\delta,u_0})}\,.$$

*If moreover, the initial datum $\psi_0$ also satisfies*

$$\left\|\varrho^{-1}\partial_u\psi_0\right\|_{C^0(\mathcal{N})} < \infty$$

*for an $\varepsilon > 0$, then*

$$\left\|\varrho^{-1}\partial_u\psi\right\|_{C^0(\Delta_{\delta,u_0})} \leq C_\delta \left(\|\psi_0\|_{\underline{H}^1_u(u_0)} + \left\|\varrho^{-1}\partial_u\psi_0\right\|_{C^0(\mathcal{N})} + \left\|\varrho^{3/2} f\right\|_{L^2(\Delta_{\delta,u_0})}\right). \quad \text{(D.3.1)}$$

*Proof.* This existence proof follows the approach of [War12]. For each $n \in \mathbb{N}$, pick $\psi_0^{(n)} \in C_0^\infty(\mathcal{N} \cap D_k)$ and $f^{(n)} \in C_0^\infty(D_n)$ such that $\psi_0^{(n)} \to \psi_0$ in $\underline{H}^1_u(u_0)$ and $f^{(n)} \to f$ in $C^0\underline{H}^1(\Delta_{\delta,u_0})$ as $n \to \infty$. Let $\psi^{(n)}$ be the solution of the finite problem (D.2.1) on $D_n$ associated to $\left(\psi_0^{(n)}, f^{(n)}\right)$. Define $\overline{\psi^{(n)}}$ to be $\psi^{(n)}$ in $D_n$ and 0 in $\Delta_{\delta,u_0}\backslash D_n$. Similarly, we define $\overline{\partial_u\psi^{(n)}}$ and $\overline{\partial_v\psi^{(n)}}$ by continuing $\partial_u\psi^{(n)}$ and $\partial_v\psi^{(n)}$ by 0. From above, we know that

$$\sup_{u_0\leq v\leq u_0+\delta} \left\|\overline{\psi^{(n)}}\right\|_{\underline{L}^2_u(v)} \leq C$$

$$\begin{aligned}\sup_{u_0 \le u \le u_0+\delta} \left\| \overline{\psi^{(n)}} \right\|_{\underline{L}^2_v(u)} &\le C \\ \sup_{u_0 \le v \le u_0+\delta} \left\| \overline{\partial_u \psi^{(n)}} \right\|_{\underline{L}^2_u(v)} &\le C \\ \sup_{u_0 \le u \le u_0+\delta} \left\| \overline{\partial_v \psi^{(n)}} \right\|_{\underline{L}^2_v(u)} &\le C\end{aligned}$$

uniformly. By weak compactness and passing to a subsequence, there are $\psi$, $\psi_1$ and $\psi_2$ such that

$$\begin{aligned}\overline{\psi^{(n)}} &\rightharpoonup \psi \text{ in } C^0\underline{L}^2 \\ \overline{\partial_u \psi^{(n)}} &\rightharpoonup \psi_1 \text{ in } \underline{L}^2_u \\ \overline{\partial_v \psi^{(n)}} &\rightharpoonup \psi_2 \text{ in } \underline{L}^2_v\end{aligned}$$

weakly. By integrating against a compactly supported function, one easily sees that $\partial_u \psi = \psi_1$ and $\partial_v \psi = \psi_2$. Therefore $\psi \in C^0\underline{H}^1(\Delta_{\delta,u_0})$ and even $\psi \in C^0\underline{H}^1(\Delta_{\delta,u_0})$ since linear subspaces are weakly closed. By weak convergence $\psi$ also satisfies the wave equation in a weak sense.

Each $\overline{\psi^{(n)}}$ agrees with $\psi^{(n)}$ on $\mathcal{N}$. The convergence $\psi_0^{(n)} \to \psi_0$ is pointwise almost everywhere. By the Rellich-Kondrachov Theorem, the sequence $\left(\overline{\psi^{(n)}}\right)$ converges almost everywhere on $\mathcal{N}$. Therefore $\overline{\psi^{(n)}} \to \psi_0$ pointwise almost everywhere on $\mathcal{N}$ and the initial condition is satisfied.

The $C^0\underline{H}^1$ estimate follows from combining the above $\underline{L}^2_u$ and $\underline{L}^2_v$ estimates and the fact that $\psi \in C^0\underline{H}^1(\Delta_{\delta,u_0})$.

Let us turn to the addendum about higher regularity. Here we simply choose a sequence $\psi_0^{(n)}$ such that in addition

$$\left\| \varrho^{-1} \partial_u \left( \psi_0^{(n)} - \psi_0 \right) \right\|_{C^0(\mathcal{N})} \to 0.$$

Then one can estimate the corresponding $\psi^{(n)}$ from the transport equation

$$\partial_v \left( h^{-3/2} \psi_u^{(n)} \right) = \frac{3}{2} h_u h^{-5/2} \psi_v^{(n)} + \frac{1}{2} \varrho^{3/2} f^{(n)}.$$

Integrating from $\mathcal{N}$, using the assumptions on $h$ and the fact that $h_u$ is bounded since it extends smoothly to the boundary yields

$$\left| \varrho(u,v)^{-3/2} \psi_u^{(n)} \right| \le C \left( \varrho(u,u_0)^{-3/2} |\psi_u^{(n)}(u,u_0) + \int_{u_0}^{v} \varrho^{-5/2} |\psi_v^{(n)}| \, \mathrm{d}v' + \int_{u_0}^{v} \varrho f \, \mathrm{d}v' \right)$$

for a $C > 0$. Since

$$\int_{u_0}^{v} \varrho^{-5/2} |\psi_v^{(n)}| \, \mathrm{d}v' \le C \left\| \psi^{(n)} \right\|_{C^0\underline{H}^1} \varrho(u,v)^{-1/2},$$

we obtain

$$\varrho^{-1}|\psi_u^{(n)}(u,v)| \leq C\left(\left\|\varrho^{-1}\partial_u\psi_0^{(n)}\right\|_{C^0\underline{H}^1} + \left\|\psi^{(n)}\right\|_{C^0\underline{H}^1} + \delta^{1/2}\left\|\varrho^{3/2}f\right\|_{L^2}\right).$$

The inequality for $\psi$ follows after passing to the limit. □

## D.4. Uniqueness

**Theorem D.4.1.** *Assume that (D.1.3) holds. Let $\psi \in C^0\underline{H}_0^1(\Delta_{\delta,u_0})$ be a weak solution. Then $\psi$ is unique.*

*Proof.* The standard trick from [Eva10] cannot be applied right away to this setting since we work in double null coordinates; the following proof modifies the argument to apply to the case at hand.

It suffices to show uniqueness for $\psi_0 = 0$ and $f = 0$. Let $\psi$ be a weak solution. Fix any $s \in [0,\delta)$. Define

$$\varphi(u,v) := \begin{cases} \int_0^{s+u_0-u}\psi(u+\tau, v+\tau)\,\mathrm{d}\tau & \text{for } (u,v) \in \Delta_{s,u_0} \\ 0 & \text{for } (u,v) \in \Delta_{\delta,u_0}\backslash\Delta_{s,u_0}. \end{cases}$$

Evidently $\varphi \in C^0\underline{H}_0^1(\Delta_{\delta,u_0})$, satisfying $T\varphi = -\psi$ and $\varphi = 0$ for $u = u_0 + s$. In particular,

$$\int_{\Delta_{s,u_0}} h^{-3}\left(\psi_u\varphi_v + \psi_v\varphi_u\right)\mathrm{d}u\mathrm{d}v = 0 \tag{D.4.1}$$

from (D.1.2). We rewrite

$$\begin{aligned}\psi_u\varphi_v + \psi_v\varphi_u =& T\psi T\varphi - (\psi_u\varphi_u + \psi_v\varphi_v) \\ =& -\frac{1}{2}T\left(\psi^2\right) + \frac{1}{2}T\left(\varphi_u^2 + \varphi_v^2\right).\end{aligned}$$

Note here that $(T\varphi)_u$ and $(T\varphi)_v$ are well-defined.

Recalling that $\psi = 0$ on $\mathcal{N}$, we obtain

$$\begin{aligned}&\int_{u_0}^{u_0+s} h^{-3}\psi(u_0+s,v')^2\,\mathrm{d}v' + 3\int_{\Delta_{s,u_0}} h^{-4}T(h)\left(\psi^2 - \varphi_u^2 - \varphi_v^2\right)\mathrm{d}u\mathrm{d}v \\ =& \int_{u_0}^{u_0+s} h^{-3}\left(\varphi_u(u_0+s,v')^2 + \varphi_v(u_0+s,v')^2\right)\mathrm{d}v' \\ &- \int_{u_0}^{u_0+s} h^{-3}\left(\varphi_u(u',u_0)^2 + \varphi_v(u',u_0)^2\right)\mathrm{d}u'\end{aligned}$$

from (D.4.1) because the boundary integral over the portion of $\mathcal{I}$ vanishes. On the slice $\{u = u_0 + s\}$, we have $\varphi_v = 0$. Since $T\varphi = -\psi$, this implies $\varphi_u = -\psi$ on $\{u = u_0 + s\}$.

Therefore,

$$\int_{u_0}^{u_0+s} h^{-3}\left(\varphi_u^2+\varphi_v^2\right)\,\mathrm{d}u' + 3\int_{\Delta_{s,u_0}} h^{-4}T(h)\psi^2 = 3\int_{\Delta_{s,u_0}} h^{-4}T(h)(\varphi_u^2+\varphi_v^2)\,\mathrm{d}u'\mathrm{d}v'$$

and

$$\begin{aligned}&\int_{u_0}^{u_0+s} h^{-3}\left(\varphi_u(u',u_0)^2+\varphi_v(u',u_0)^2\right)\,\mathrm{d}u' \\ \leq &C\int_{\Delta_{s,u_0}} h^{-3}\left(\varphi_u(u',v')^2+\varphi_v(u',v')^2\right)\,\mathrm{d}u'\mathrm{d}v'\end{aligned} \tag{D.4.2}$$

for a $C>0$. Now let us define

$$\Phi(u,v) := \int_0^{v-u_0} \psi(u-\tau, v-\tau)\,\mathrm{d}\tau.$$

The relationship between $\varphi$ and $\Phi$ is expressed by

$$\varphi(u,v) = \Phi(u_0+s, v-u+u_0+s) - \Phi(u,v).$$

Moreover,

$$\begin{aligned}(\partial_u\varphi)(u,u_0) =& -(\partial_u\Phi)(u_0+s, 2u_0-u+s)\\ (\partial_v\varphi)(u,u_0) =& (\partial_v\Phi)(u_0+s, 2u_0-u+s)\end{aligned}$$

for $(u,v)\in\Delta_{s,u_0}$. Thus (D.4.2) becomes

$$\begin{aligned}&\int_{u_0}^{u_0+s} h^{-3}\left(\Phi_u\left(u_0+s,v'\right)^2+\Phi_v\left(u_0+s,v'\right)^2\right)\,\mathrm{d}v' \\ \leq &6sC_3\int_{u_0}^{u_0+s} h^{-3}\left(\Phi_u\left(u_0+s,v'\right)^2+\Phi_v\left(u_0+s,v'\right)^2\right)\,\mathrm{d}v' \\ &+6C_3\int_0^s\int_{u_0}^{u_0+\tau} h^{-3}\left(\Phi_u\left(u_0+s,v'\right)^2+\Phi_v\left(u_0+s,v'\right)^2\right)\,\mathrm{d}v'.\end{aligned}$$

Let $\delta'\in(0,\delta]$ such that

$$1-C\delta' \geq \frac{1}{2}.$$

We conclude that for all $s\in[0,\delta']$,

$$\begin{aligned}&\int_{u_0}^{u_0+s} \varrho^{-3}\left(\Phi_u\left(u_0+s,v'\right)^2+\Phi_v\left(u_0+s,v'\right)^2\right)\,\mathrm{d}v' \\ \leq &C\int_0^s\int_{u_0}^{u_0+\tau} \varrho^{-3}\left(\Phi_u\left(u_0+s,v'\right)^2+\Phi_v\left(u_0+s,v'\right)^2\right)\,\mathrm{d}v'.\end{aligned}$$

By Grönwall's inequality, this implies

$$\Phi_u = \Phi_v = 0$$

almost everywhere in $\Delta_{\delta',u_0}$, in particular $T\Phi = 0$ almost everywhere. As

$$\begin{aligned}
\Phi_u(u,u_0) &= \int_0^{v-u_0} \left(\partial_u \psi\right)(u-\tau, v-\tau)\,\mathrm{d}\tau \\
\Phi_v(u,u_0) &= \int_0^{v-u_0} \left(\partial_v \psi\right)(u-\tau, v-\tau)\,\mathrm{d}\tau,
\end{aligned}$$

we conclude $T\psi = 0$ almost everywhere in $\Delta_{\delta',u_0}$, whence $\psi = 0$ by integrating from $\mathcal{N}$. Now one immediately obtains that $\psi = 0$ in

$$\Delta_{s,u_0} \cap \{v \leq u_0 + \delta'\}.$$

Iterating the argument, yields $\psi = 0$ in $\Delta_{\delta,u_0}$. □

## D.5. Higher regularity

**Definition D.5.1.** We say that the IBVP has $H^2$ initial datum $\psi_0 \in \left(\underline{H}^1_u\right)_0(u_0)$ if

$$\partial_u \psi_0 = \frac{1}{2}\Psi \varrho^3 + \psi_R,$$

where $\Psi \in \mathbb{R}$ is a constant and $\psi_R = \mathcal{O}\left(\varrho^4\right)$.

This allows us to define

$$\left(\partial_v \psi\right)_0 = -\frac{1}{2}\Psi\varrho^3 + \tilde{\psi}_R$$

via the transport equation

$$\partial_u \left(\varrho^{-3/2}\left(\partial_v\psi\right)_0\right) = -\frac{3}{4}\varrho^{-5/2}\partial_u\psi_0 + \frac{1}{2}\varrho^{3/2} f.$$

We will impose the additional assumption

$$\left|T\left(\frac{h_u}{h}\right)\right| + \left|T\left(\frac{h_v}{h}\right)\right| \leq C_4. \tag{D.5.1}$$

We have the following a priori result:

**Lemma D.5.2.** *Assume that assumptions (D.1.3) and (D.5.1) hold. Then under the assumptions of Lemma D.2.1, we have*

$$\left\|T\psi^{(k)}\right\|_{C^0\underline{H}^1} < C\left(\left\|T\psi_0'\right\|^2_{\underline{H}^1_u(u_0)} + \left\|\varrho^{3/2}\left(Tf'\right)\right\|^2_{L^2\left(\Delta_{\delta,u_0}\right)}\right),$$

*where $C$ only depends on $\delta$.*

*Proof.* Commuting the wave equation with $T$ yields

$$(T\psi)_{uv} = \frac{3}{2}\frac{h_u}{h}T\psi_v + \frac{3}{2}\frac{h_v}{h}T\psi_u + \frac{h^3}{2}Tf + \frac{3}{2}T\left(\frac{h_u}{h}\right)\psi_v + \frac{3}{2}T\left(\frac{h_v}{h}\right)\psi_u + \frac{3}{2}h^2Thf.$$

Therefore,

$$\begin{aligned}
&\partial_u\left(h^{-3}\left(T\psi\right)^2 + h^{-3}\left(\partial_v T\psi\right)^2\right) + \partial_v\left(h^{-3}\left(T\psi\right)^2 + h^{-3}\left(\partial_u T\psi\right)^2\right)\\
=&2h^{-3}\left(T\psi\right)\left(TT\psi\right) - 3h^{-4}Th\left(T\psi\right)^2 + 3h^{-4}ThT\psi_u T\psi_v\\
&+ 2h^{-3}\left(TT\psi\right)\left[\frac{1}{2}h^3Tf + \frac{3}{2}Thh^2f + \frac{3}{2}T\left(\frac{h_u}{h}\right)\psi_v + \frac{3}{2}T\left(\frac{h_v}{h}\right)\psi_u\right].
\end{aligned}$$

We obtain the estimate as in the proof of Lemma D.2.1. □

Combining all of this with the previous results, one obtains the

**Theorem D.5.3.** *Given $H^2$ initial data there is a unique classical solution for the wave equation.*

*Proof.* It is easy to deduce an estimate for $\|T\psi\|_{C^0\underline{H}^1}$. Boundedness of $\|\partial_u\psi\|_{C^0\underline{H}^1}$ has been established already above. This yields boundedness for $\left\|\varrho^{-1+\varepsilon}\partial_v\psi\right\|_{C^0}$. The wave equation being satisfied, one deduces that $\psi_{uv} = \psi_{vu} \in C^0_{\text{loc}}$, which means that $\psi$ is a classical solution. □

## D.6. Application to the contraction map

**Proposition D.6.1.** *Let $(\tilde{r}, m, B) \in \mathcal{B}_b$. Then there is a unique weak solution to (3.2.7) in $C^0\underline{H}^1\left(\Delta_{\delta,u_0}\right)$ such that (D.3.1) holds.*

*Proof.* The initial data satisfy the assumptions of Theorem D.3.1 and the right hand side of (3.2.7) satisfies the conditions for $f$. Moreover, (D.1.3) holds for $\tilde{r}$, but $\tilde{r}$ is not smooth. Since clearly $\left(C^{1+}_{\tilde{r}}\left(\Delta_{\delta,u_0}\right), d_{\tilde{r}}\right)$ is the completion of $C^\infty$ under $d_{\tilde{r}}$, there is a sequence of smooth function $(\tilde{r}_n)$ converging to $\tilde{r}$ with respect to $d_{\tilde{r}}$ such that $(\tilde{r}_n, m, B) \in \mathcal{B}_b$, and such that (D.1.3) holds for each $\tilde{r}_n$. Each $\tilde{r}_n$ gives rise to a unique $B'_n$. Repeating the computation of Section 3.2.4 shows existence and regularity. For uniqueness, we note that the proof of Theorem D.4.1 does not require smoothness for $\tilde{r}$. □

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