Spaces of diffeomorphisms and embeddings via algebraic K-theory

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Declaration

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To Juan Álvaro and Patricia.

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Abstract

This thesis comprises two papers that study the homotopy type of spaces of automorphisms and embeddings of high-dimensional manifolds via algebraic K-theory.

In the first paper, presented in Chapter 1, we show that the mapping class group is not an h-cobordism invariant of high-dimensional manifolds by exhibiting h-cobordant manifolds whose mapping class groups have different cardinalities. To do so, we introduce a moduli space of "h-block" bundles and compare it to the moduli space of ordinary block bundles.

In the second paper, spanning Chapters 2 and 3, we establish a pseudoisotopy result for embedding spaces. We describe, within a range of homotopical degrees, the difference between spaces of block and ordinary embeddings in terms of relative algebraic K-theory; this is analogous to a theorem of Weiss and Williams for spaces of automorphisms. We use our result to provide a full description of the homotopy type—localised away from 2 and in the aforementioned degree range—of the space of long knots of codimension at least 3. This analysis involves a detailed study of certain geometric involutions in algebraic K-theory spaces that can be of independent interest.

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Chapter 0

Introduction

The present thesis consists of two papers that explore various questions in the subject of *high-dimensional manifold theory.* Each paper includes its own introduction, so this preface instead provides a brief and accessible overview to the field, highlighting the key results that will be relevant to the thesis.

0.1 A synopsis of high-dimensional manifold theory

A central question in algebraic and geometric topology is to understand the *moduli space* $\mathcal{M}_d$ of all closed, smooth manifolds of dimension d. A point-set model for $\mathcal{M}_d$ consists of the collection of all subsets M of $\mathbb{R}^\infty := \bigcup_{n\geq 0} \mathbb{R}^n$ which happen to be a closed smooth d-dimensional submanifold, equipped with the finest topology that makes the map

$$\mathrm{Im} \ : \coprod_{M \in \mathcal{M}_d} \mathrm{Emb}(M, \mathbb{R}^\infty) \longrightarrow \mathcal{M}_d, \quad \varphi \longmapsto \mathrm{Im}\ \varphi$$

continuous; here the embedding space $\mathrm{Emb}(M, \mathbb{R}^\infty)$ is topologised with the usual C^∞-Whitney topology, i.e., as a subspace of the space of smooth maps $C^\infty(M, \mathbb{R}^\infty)$.

For our purposes, $\mathcal{M}_d$ is more conveniently modelled as the geometric realisation of the simplicial set $\mathcal{M}_{d,\bullet}$ whose set of p-simplices consists of all subsets $W \subset \mathbb{R}^\infty \times \Delta^p$ for which the projection map $W \to \Delta^p$ is a *smooth d-dimensional fibre bundle*—a fibre bundle over Δ^p with fibre some (closed) smooth d-manifold M^d, and whose transition functions take values in the topological group $\mathrm{Diff}(M)$ of diffeomorphisms of M. By this definition, $\mathcal{M}_d$ becomes the space responsible for classifying such smooth fibre bundles, in the sense that for every (reasonable) space B, there is a bijection

$$\left\{\begin{array}{c}\text{continuous maps}\\ B\longrightarrow\mathcal{M}_d\end{array}\right\}/\text{htpy.}\longleftrightarrow\left\{\begin{array}{c}\text{smooth } d\text{-dimensional}\\ \text{fibre bundles over } B\end{array}\right\}/\text{iso.}$$

The space $\mathcal{M}_d$ decomposes as a disjoint union of path components

$$\mathcal{M}_d=\coprod_{M\text{ up to diffeo.}}\mathcal{M}_d(M),$$

indexed over the diffeomorphism classes of closed d-manifolds M, and where $\mathcal{M}_d(M)$ denotes the moduli space of manifolds diffeomorphic to M, i.e., it classifies those smooth fibre bundles whose fibre is diffeomorphic to M (or *smooth M-bundles* for short). As such, it is necessarily (a model for) the classifying space $B\mathrm{Diff}(M)$ of the diffeomorphism group of M. The homotopy type of this last space is very rich: on the one hand, its cohomology ring encodes all characteristic classes of smooth M-bundles, many of which play an important role in algebraic geometry. On the other hand, its homotopy groups, which coincide with those of $\mathrm{Diff}(M)$ upon shifting by one, detect non-trivial isotopy classes of families of diffeomorphisms of M.

Two guiding questions in manifold theory are:

- What is $\pi_0\mathcal{M}_d$? Or, when are two d-manifolds diffeomorphic?
- Given M, what is the homotopy type of $\mathcal{M}_d(M)\simeq B\mathrm{Diff}(M)$?

To answer these kind of questions, it is traditional to consider the following series of simpler variants of $\mathcal{M}_d$ that approximate its homotopy type:

$$\mathcal{M}_d\xrightarrow{①}\widetilde{\mathcal{M}}_d\xrightarrow{②}\widetilde{\mathcal{M}}^h_d\xrightarrow{③}h\mathcal{M}_d. \tag{0.1.1}$$

Let us briefly explain each of these moduli spaces:

- $\widetilde{\mathcal{M}}_d$ is the *block moduli space* of d-manifolds, defined as the geometric realisation of a simplicial set $\widetilde{\mathcal{M}}_{d,\bullet}$. A p-simplex of this simplicial set is a *smooth d-dimensional block bundle* over Δ^p, meaning a $(d+p)$-dimensional submanifold $W\subset\mathbb{R}^\infty\times\Delta^p$ that is transverse to $\mathbb{R}^\infty\times\sigma$, for every face $\sigma\subset\Delta^p$, and which is diffeomorphic to $M\times\Delta^p$, for some closed d-manifold M, by a strata-preserving diffeomorphism—one that restricts to a diffeomorphism $W_\sigma:=W\cap(\mathbb{R}^\infty\times\sigma)\cong M\times\sigma$ for every face $\sigma\subset\Delta^p$ (cf. [ERW14, Defn. 2.2] for details). Since a smooth fibre bundle over Δ^p is, in particular, a block bundle over Δ^p, we obtain the map ①.

By a similar reasoning as before, the component $\widetilde{\mathcal{M}}_d(M)$ classifying block bundles with fibre M serves as a model for the classifying space of the simplicial group $\widetilde{\mathrm{Diff}}(M)_\bullet$ of *block diffeomorphisms* of M, whose p-simplices are strata-preserving diffeomorphisms of $M \times \Delta^p$. It follows that

$$\widetilde{\mathcal{M}}_d \simeq \coprod_{M \text{ up to diffeo.}} B\widetilde{\mathrm{Diff}}(M).$$

- $\widetilde{\mathcal{M}}^h_d$ is the *h-block moduli space* of d-manifolds. This is a new object, related to $\widetilde{\mathcal{M}}_d$, which we construct in Chapter 1 and that plays a significant role in it. A p-simplex in the corresponding simplicial set $\widetilde{\mathcal{M}}^h_{d,\bullet}$ is a $(d+p)$-dimensional submanifold $W \subset \mathbb{R}^\infty \times \Delta^p$ that is transverse to $\mathbb{R}^\infty \times \sigma$ for every face $\sigma \subset \Delta^p$ and for which the inclusions of strata $W_\sigma \hookrightarrow W_\xi$ are homotopy equivalences for every face inclusion $\sigma \subset \xi \subset \Delta^p$ (cf. Definitions 1.3.1 and 1.3.4). A p-simplex in $\widetilde{\mathcal{M}}_{d,\bullet}$ clearly satisfies both of these conditions, giving the inclusion ②.

  So the 0-simplices in $\widetilde{\mathcal{M}}^h_d$ are all closed d-manifolds in $\mathbb{R}^\infty$ (just like $\mathcal{M}_{d,0}$ and $\widetilde{\mathcal{M}}_{d,0}$), its 1-simplices are *h-cobordisms*—cobordisms $W : M_0 \rightsquigarrow M_1$ where the inclusions $M_0 \hookrightarrow W$ and $M_1 \hookrightarrow W$ are both homotopy equivalences—while higher simplices generalise this condition. If $B\widetilde{\mathrm{Diff}}^h(M)$ denotes the path component of a d-manifold M in $\widetilde{\mathcal{M}}^h_d$, then

$$\widetilde{\mathcal{M}}^h_d = \coprod_{M \text{ up to } h\text{-cob.}} B\widetilde{\mathrm{Diff}}^h(M).$$

  Note that $B\widetilde{\mathrm{Diff}}^h(M)$ is not defined as the classifying space of some automorphism group $\widetilde{\mathrm{Diff}}^h(M)$; we will still justify this piece of notation in Section B.1.

- $h\mathcal{M}_d$ is the *moduli space* of d-dimensional *Poincaré complexes*—CW complexes satisfying d-dimensional Poincaré duality—a p-simplex of which is a fibration[1] $E \to \Delta^p$ whose fibre has the homotopy type of a Poincaré complex of dimension d.

  Given a Poincaré complex X, the path component $h\mathcal{M}_d(X)$ is the moduli space of *all* spaces homotopy equivalent to X, i.e. it is equivalent to the classifying space $Bh\mathrm{Aut}(X)$ of the topological monoid of self homotopy equivalences of X. Thus

$$h\mathcal{M}_d \simeq \coprod_{X \text{ up to htpy eq.}} Bh\mathrm{Aut}(X).$$

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[1] As before, one can arrange the fibrations to lie inside $\mathbb{R}^\infty \times \Delta^p$ for $h\mathcal{M}_d$ to be a set.

Among these moduli spaces, $h\mathcal{M}_d$ is the only one defined purely in terms of homotopy theory and can therefore be hoped to be understood in those terms. To study $\mathcal{M}_d$, we proceed iteratively: given a map $\iota : \mathcal{M} \to \mathcal{M}'$ between two of these moduli spaces and where $\mathcal{M}'$ is understood, we analyse the homotopy fibre $\mathrm{hofib}_X(\iota)$ at a point X in each path component of $\mathcal{M}'$. Applying this strategy to the maps ①, ②, and ③, we work step by step from right to left in (0.1.1) until reaching $\mathcal{M}_d$, our ultimate goal. This approach turns out to be successful only in *high dimensions*—that is, for $d \geq 5$—which is the setting of this thesis. In what follows, we survey how to understand the homotopy fibres of each of the maps above.

0.1.1 Surgery theory

The theory of *surgery* originated from the influential work of Kervaire and Milnor [KM63] on homotopy spheres and was later developed into a systematic framework by Browder, Novikov, Sullivan, and Wall, among others, making it a highly prolific field during the 60s and 70s [Bro72, Wal99, LM24]. Their work, which will be the main focus of this section, established surgery as a fundamental tool in geometric topology. In the following decades, the theory remained central to the field, in part due to Ranicki's algebraic reformulation of the theory [Ran92].

Surgery theory provides a framework to address questions such as:

1. Given a d-dimensional Poincaré complex X, is there a d-manifold N homotopy equivalent to X?
2. Given a homotopy equivalence $f : N \xrightarrow{\sim} M$ between d-manifolds, is it homotopic to a diffeomorphism?

The classical theory did not typically address these questions at a "space level", but developments by Quinn [Qui70] and others eventually made this possible. In this more refined framework, the focus shifts to the *h-block surgery structure space* of a Poincaré complex X, defined as

$$\widetilde{\mathcal{S}}^h(X) := \mathrm{hofib}_X\left(③ : \widetilde{\mathcal{M}}^h_d \longrightarrow h\mathcal{M}_d\right).$$

By definition, X is homotopy equivalent to a d-manifold if and only if $\widetilde{\mathcal{S}}^h(X)$ is non-empty. A point in this space corresponds to a pair (N, f) where N is a closed d-manifold and $f : N \xrightarrow{\sim} X$ is a homotopy equivalence. A path between two such points, (N_0, f_0) and (N_1, f_1), is given by an h-cobordism $W : N_0 \overset{h}{\rightsquigarrow} N_1$ and a homotopy equivalence of triples $(F, f_0, f_1) : (W, N_0, N_1) \xrightarrow{\sim} (X \times I, X \times 0, X \times 1)$, with higher simplices defined similarly.

When $X = M^d$ is already a manifold, the fundamental theorem of surgery theory by Browder–Novikov–Sullivan–Wall provides the homotopy fibre sequence

$$\widetilde{\mathcal{S}}^h(M) \longrightarrow \mathcal{N}(M) \longrightarrow \mathcal{L}^h(\mathbb{Z}[\pi_1 M]). \tag{0.1.2}$$

The space of *normal invariants* $\mathcal{N}(M)$ consists of degree-one normal maps to M—that is, pairs (ξ, F) where ξ is a stable vector bundle over M and $F : \nu_N^s \to \xi$ is a stable bundle map from the stable normal bundle of some manifold N to ξ, such that the underlying map $f : N \to M$ has degree one. By an instance of the Pontryagin–Thom construction, $\mathcal{N}(M)$ is equivalent to the mapping space $\mathrm{Map}(M, G/O)$, where G/O is the homotopy fibre of the J-homomorphism $J : BO \to BG$. The homotopy groups of $BO = \mathrm{colim}_n\, BO(n)$ are well-understood by *Bott periodicity*, whilst those of BG are essentially the stable homotopy groups of spheres, so $\mathcal{N}(M)$ is often computable.

The extra normal data on a degree-one map f allows one to measure how far it is from being a homotopy equivalence through embedded spheres in M with trivial normal bundles, which can then be used to "perform surgery" on f. After surgery, the failure of f to actually be an equivalence is captured by the *h-decorated quadratic L-theory space* $\mathcal{L}^h(\mathbb{Z}[\pi_1 M])$ in the shape of a "quadratic form" over $\mathbb{Z}[\pi_1 M]$. The upshot is that $\mathcal{N}(M)$ is entirely homotopy-theoretic, while $\mathcal{L}^h(\mathbb{Z}[\pi_1 M])$ is an algebraic invariant that is also often computable. As a result, the homotopy type of the structure space $\widetilde{\mathcal{S}}^h(M)$ is relatively accessible via (0.1.2).

However, the analysis of $\widetilde{\mathcal{S}}^h(M)$ does not directly address Question 2. Instead, one should look at the homotopy fibre at M of the composition ③ ∘ ②,

$$\mathrm{hofib}_M(③ \circ ② : \widetilde{\mathcal{M}}_d \longrightarrow h\mathcal{M}_d).$$

Its 0-simplices coincide with those of $\widetilde{\mathcal{S}}^h(M)$, while a 1-simplex between two such 0-simplices, (N_0, f_0) and (N_1, f_1), is given by a cobordism $W : N_0 \rightsquigarrow N_1$ that is diffeomorphic to $N_0 \times \Delta^1 \cong N_0 \times [0,1]$ (relative to N_0), together with a homotopy equivalence of triples $(F, f_0, f_1) : (W, N_0, N_1) \xrightarrow{\sim} (M \times I, M \times 0, M \times 1)$. In other words, (N_0, f_0) and (N_1, f_1) lie in the same path component of $\mathrm{hofib}_M(③ \circ ②)$ if and only if there exists a diffeomorphism $\phi : N_0 \cong N_1$ such that f_0 is homotopic to $f_1 \circ \phi$. Thus, a homotopy equivalence $f : N \xrightarrow{\sim} M$ is homotopic to a diffeomorphism if and only if (N, f) lies in the same path component of $\mathrm{hofib}_M(③ \circ ②)$ as (M, Id_M)—this was Question 2.

A version of the fibre sequence (0.1.2) of $\widetilde{\mathcal{S}}^h(M)$ also holds for $\mathrm{hofib}_M(\textcircled{3} \circ \textcircled{2})$—or rather, for a certain collection, $\widetilde{\mathcal{S}}^s(M)$, of path components of this space—making its homotopy type understandable too. In the next section, we explain which path components of $\mathrm{hofib}_M(\textcircled{3} \circ \textcircled{2})$ constitute $\widetilde{\mathcal{S}}^s(M)$, the *s-block structure space* of M.

0.1.2 Whitehead torsion

Given a homotopy equivalence $f : X \to Y$ between *finite* connected[2] CW complexes, its *Whitehead torsion* $\tau(f)$ is an algebraic invariant of the homotopy class of f that detects whether f is homotopic to a finite composition of "elementary collapses and expansions" of cells. It lives[3] in the *Whitehead group* $\mathrm{Wh}(\pi_1 X)$ of the fundamental group of X, where for a group π,

$$\mathrm{Wh}(\pi) := GL(\mathbb{Z}[\pi])^{\mathrm{ab}} / \langle \pm g : g \in \pi \rangle,$$

a "reduced" version of the first algebraic K-theory group $K_1(\mathbb{Z}[\pi]) = GL(\mathbb{Z}[\pi])^{\mathrm{ab}}$. Here, $GL(R) = \mathrm{colim}_n\, GL_n(R)$ denotes the infinite general linear group of a ring R.

A homotopy equivalence is said to be *simple* if its Whitehead torsion vanishes. Examples include homeomorphisms—surprisingly, it took many years until Chapman [Cha74] eventually proved this—and homotopy equivalences between simply-connected spaces, since the Whitehead group $\mathrm{Wh}(\{e\})$ of the trivial group is trivial (a result known as Whitehead's Lemma). The study of simple homotopy equivalences, known as *simple homotopy theory*, is a central part of high-dimensional manifold theory, as we will see later in this introduction and in Chapter 1 of the present thesis.

Going back to surgery, the s-block structure space of a Poincaré complex X,

$$\widetilde{\mathcal{S}}^s(X) \subset \mathrm{hofib}_X(\textcircled{3} \circ \textcircled{2} : \widetilde{\mathcal{M}}_d \to h\mathcal{M}_d),$$

consists of those path components of the homotopy fibre which are represented by pairs (N, f) where the homotopy equivalence $f : N \xrightarrow{\sim} X$ is simple[4]—thus the decoration "s". There is a map $\widetilde{\mathcal{S}}^s(M) \to \widetilde{\mathcal{S}}^h(M)$ whose homotopy fibre is, up to subtleties with path components, that of the map $\textcircled{2} : \widetilde{\mathcal{M}}_d \hookrightarrow \widetilde{\mathcal{M}}^h_d$ by unraveling the definitions. Understanding this difference naturally leads to the study of h-cobordisms.

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[2] This is mostly for simplicity. See the relevant discussion in Section 1.2.1 for details.

[3] It is a more usual convention for $\tau(f)$ to be an element of the Whitehead group of the codomain Y, but for our purposes, it will be more convenient to place it in the domain X. Of course, both conventions are interchangeable through the identification $f_* : \pi_1 X \cong \pi_1 Y$.

[4] To answer our previous Question 2, it suffices to consider $\widetilde{\mathcal{S}}^s(M)$ instead of $\mathrm{hofib}_M(\textcircled{3} \circ \textcircled{2})$: a homotopy equivalence that is homotopic to a diffeomorphism is necessarily simple.

0.1.3 h-Cobordisms

Recall that a cobordism $W : M \rightsquigarrow M'$ is an h-cobordism if both of the inclusions $M \hookrightarrow W \hookleftarrow M'$ are homotopy equivalences. The analysis of the homotopy fibre of ② : $\widetilde{\mathcal{M}}_d \hookrightarrow \widetilde{\mathcal{M}}^h_d$ at a d-manifold M involves understanding questions such as:

- When is an h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$ starting at M diffeomorphic (or h-cobordant) to the product $M \times [0,1]$?
- How many h-cobordisms starting at M are there?

The *s-cobordism theorem* of Mazur, Barden, and Stallings [Maz63, Bar64] provides a complete answer to these questions. Namely, it establishes a bijection

$$h\mathrm{Cob}(M) \longleftrightarrow \mathrm{Wh}(\pi_1 M),$$

where $h\mathrm{Cob}(M)$ denotes the set of diffeomorphism classes of h-cobordisms starting at M. The map from left to right sends (the class of) $W : M \overset{h}{\rightsquigarrow} M'$ to the Whitehead torsion $\tau(M \hookrightarrow W)$ of the inclusion of M in W; the inverse map is more involved.

This remarkable result plays a key role in the classification of high-dimensional manifolds, as it reduces the problem of determining whether two manifolds are diffeomorphic to finding an h-cobordism between them that represents the zero element in the Whitehead group—a task that is often significantly more feasible. For example, since $\mathrm{Wh}(\{e\}) = 0$, any h-cobordism between simply-connected manifolds[5] is diffeomorphic to a product. This is Smale's celebrated *h-cobordism theorem* [Sma62].

A consequence of the s-cobordism theorem is that an h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$ is diffeomorphic to $M \times [0,1]$ if and only if the inclusion $M \hookrightarrow W$ (and hence $M' \hookrightarrow W$, by the duality formula (1.2.4)) is a *simple* homotopy equivalence. Building on this idea, we can explicitly relate the moduli spaces $\widetilde{\mathcal{M}}_d$ and $\widetilde{\mathcal{M}}^h_d$: the set of p-simplices $\widetilde{\mathcal{M}}_{d,p}$ in the block moduli space consists of precisely those p-simplices W in $\widetilde{\mathcal{M}}^h_{d,p}$ for which the strata inclusions $W_\sigma \overset{\sim}{\hookrightarrow} W_\xi$ are simple for all $\sigma \subset \xi \subset \Delta^p$ (cf. Proposition 1.3.6).

In Theorem B of Chapter 1, we use this characterisation of ② : $\widetilde{\mathcal{M}}_d \hookrightarrow \widetilde{\mathcal{M}}^h_d$ to fully describe the homotopy type of hofib_M(②). Its homotopy groups turn out to be given by certain group homology of C_2 with coefficients in $\mathrm{Wh}(\pi_1 M)$. As mentioned before, this also describes the difference between the structure spaces $\widetilde{\mathcal{S}}^h(M)$ and $\widetilde{\mathcal{S}}^s(M)$. (This difference was already well-known at the level of homotopy groups [Ran81, Prop.

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[5] In the smooth setting, it is essential that the manifolds have dimension at least 5: Donaldson [Don87] showed that there exists an h-cobordism between closed smooth simply-connected 4-manifolds that is *not* diffeomorphic to a product.

1.10.1]: our main contribution in this direction lies in introducing the h-block moduli space $\widetilde{\mathcal{M}}^h_d$ and identifying $\widetilde{\mathcal{S}}^h(M)$ as the homotopy fibre of ③ at M.)

Continuing the recurring theme in this introduction, it is enlightening to consider the previous questions at the level of spaces. To this end, consider the moduli space $H(M)$ of h-cobordisms starting at M, which is the geometric realisation of a simplicial set $H(M)_\bullet$. A p-simplex of it is, roughly, a smooth fibre bundle over Δ^p with fibre an h-cobordism starting at M (cf. Section 3.1.2 for more details). Thus, it follows that

$$\pi_0 H(M) = h\mathrm{Cob}(M) \cong \mathrm{Wh}(\pi_1 M).$$

So, what are the higher homotopy groups of this space? For this, it suffices to study its loop space $\Omega H(M)$ based at $M \times [0,1]$. By holonomy considerations, such a loop is completely determined by an automorphism of the product h-cobordism $M \times [0,1]$, i.e. a diffeomorphism of $M \times [0,1]$ that is the identity on $M \times 0$; this is known as a *concordance* of M. The topological group of such diffeomorphisms,

$$C(M) := \mathrm{Diff}_{M\times 0}(M \times [0,1]),$$

is thus equivalent to $\Omega H(M)$. In other words, the classifying space $BC(M)$ of the group of concordances of M is equivalent to the path component of $M \times [0,1]$ in $H(M)$.

The study of the space $C(M)$ of concordances (or equivalently $H(M)$) is known as *pseudoisotopy theory.* In the next section, we shall explain how this theory is closely related to the analysis of the homotopy fibre of our last remaining map ① : $\mathcal{M}_d \to \widetilde{\mathcal{M}}_d$.

0.1.4 Pseudoisotopy theory

The homotopy fibre of ① : $\mathcal{M}_d \to \widetilde{\mathcal{M}}_d$ at a d-manifold M is

$$\mathrm{hofib}_M\,(①) = \mathrm{hofib}_M(B\mathrm{Diff}(M) \to B\widetilde{\mathrm{Diff}}(M)) =: \widetilde{\mathrm{Diff}}/\mathrm{Diff}(M).$$

In other words, this space measures the difference between the topological groups $\mathrm{Diff}(M)$ and $\widetilde{\mathrm{Diff}}(M)$ of ordinary and block diffeomorphisms.

Let us start with this difference in the lowest homotopy group. The *mapping class group* of M is $\Gamma(M) := \pi_0\mathrm{Diff}(M)$, the group of isotopy classes of diffeomorphisms of M. The block mapping class group $\widetilde{\Gamma}(M) := \pi_0\widetilde{\mathrm{Diff}}(M)$ is a quotient of $\Gamma(M)$, where two diffeomorphisms ϕ_0 and ϕ_1 are further identified if there exists a diffeomorphism Φ of $M \times \Delta^1 \cong M \times [0,1]$ that restricts to ϕ_i on $M \times \{i\}$ for $i = 0, 1$; such diffeomorphism Φ is called a *pseudoisotopy* from ϕ_0 to ϕ_1, and one says that ϕ_0 and ϕ_1 are *pseudoisotopic.*

So a concordance of M, i.e. a point in $C(M)$, is simply a pseudoisotopy starting at the identity. Restricting a concordance to $M \times \{1\}$ defines a group homomorphism

$$\mathrm{res}_{M\times\{1\}} : C(M) \to \mathrm{Diff}(M),$$

which satisfies

$$\mathrm{Im}\Big(\mathrm{res}_{M\times\{1\}} : \pi_0 C(M) \to \Gamma(M)\Big) = \ker\Big(\Gamma(M) \twoheadrightarrow \widetilde{\Gamma}(M)\Big).$$

Thus, we see that $\pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M))$ and $\pi_0 C(M)$ are closely related. For $d \geq 7$, Hatcher–Wagoner [HW73] and Igusa [Igu82] computed $\pi_0 C(M)$ in terms of $\pi_2 M$ and the first and second (reduced) algebraic K-theory groups of $\pi_1 M$. Since $\pi_0 C(M) \cong \pi_1 H(M)$, this computation serves as a higher analogue of the classical s-cobordism theorem.

A similar relation holds for higher homotopy groups: continuing the line of thought begun above, Hatcher [Hat78] constructed a spectral sequence

$$E^1_{s,t} = \pi_t(C(M \times I^s)) \implies \pi_{s+t+1}(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)).$$

This was later subsumed by the celebrated Weiss–Williams theorem [WW88], which fully describes the homotopy type of $\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)$ within a range of degrees in terms of certain "reduced algebraic K-theory of M". We now make this a bit more precise in the next section.

0.1.5 Waldhausen's algebraic K-theory and the parametrised h-cobordism theorem

It had long been anticipated the existence of a variant of algebraic K-theory "for a space M" that captures more about the homotopy type of M than just its fundamental group. This idea was made precise in Waldhausen's seminal work [Wal85], which later gave rise to the active field of *higher algebra* (or, as he termed it, "brave new algebra").

The *algebraic K-theory space* $K(R)$ of an associative ring R can be defined as the group completion

$$K(R) := \Omega B\Big(\coprod_{[P]} BGL(P)\Big),$$

where the coproduct is indexed over all isomorphism classes $[P]$ of finitely generated projective R-modules. The algebraic K-theory groups of R are the homotopy groups of this space, denoted $K_*(R)$. Waldhausen successfully extended this definition to a broader notion of "associative ring" R—namely, an $\mathbb{E}_1$-ring spectrum. A particularly important case to us arises when $R = \mathbb{S}[\Omega X] := \Sigma^\infty_+ \Omega X$, the suspension spectrum of the loop space of a based space X, where the ring multiplication is induced by loop concatenation. Notably, this ring spectrum recovers the discrete ring $\mathbb{Z}[\pi_1 X]$ on π_0. The K-theory space of this ring, denoted $A(X) := K(\mathbb{S}[\Omega X])$, defines Waldhausen's *$A$-theory* functor $A(-)$. There is a fibre sequence

$$\Omega^\infty \Sigma^\infty_+ X := \operatorname*{colim}_n \Omega^n \Sigma^n (X_+) \longrightarrow A(X) \longrightarrow \mathrm{Wh}^{\mathsf{Diff}}(X),$$

where the homotopy groups of the left hand term are the stable homotopy groups of X. The base space $\mathrm{Wh}^{\mathsf{Diff}}(X)$ is known as the *smooth Whitehead space* of X, which we shall think of as the "reduced" algebraic K-theory of X in the previous sense.

The homotopy groups of $\mathrm{Wh}^{\mathsf{Diff}}(M)$ have geometric significance and connect back to previous sections. Namely, its π_1 is the Whitehead group $\mathrm{Wh}(\pi_1 M) \cong h\mathrm{Cob}(M)$, while its π_2 is Hatcher–Wagoner's obstruction group computing $\pi_0 C(M)$ for $d \geq 7$. Thus, these computations can be summarised as

$$\pi_1 \mathrm{Wh}^{\mathsf{Diff}}(M) = \mathrm{Wh}(\pi_1 M) \cong \pi_0 H(M), \qquad \pi_2 \mathrm{Wh}^{\mathsf{Diff}}(M) \cong \pi_1 H(M).$$

These foreshadowed the celebrated *stable parametrised h-cobordism theorem* of Waldhausen, Jahren, and Rognes [WJR13], who constructed a map

$$H(M) \longrightarrow \Omega \mathrm{Wh}^{\mathsf{Diff}}(M)$$

that is highly-connected—the precise connectivity is known as the *concordance stable range* of M, which by a deep theorem of Igusa [Igu88] is roughly at least $\dim M/3$.

This remarkable result reduces pseudoisotopy theory, within the concordance stable range, to a series of computations in algebraic K-theory which, while intricate, become significantly more feasible, particularly over $\mathbb{Q}$; in Section 2.1.1.2 we explain the usual way to deal with these. Combined with knowledge of $B\widetilde{\mathrm{Diff}}(M)$ from surgery theory, these computations yield, in theory, a complete description of the homotopy type of $B\mathrm{Diff}(M)$ within the concordance stable range and up to extensions. In practice, there are not so many complete computations available other than the case when M is a disc or a sphere, or when M is aspherical [FH78].

Nevertheless, this is only one side of the story. Though scarce, computations of $B\mathrm{Diff}(M)$ outside the concordance stable range do exist—most notably those of Krannich, Kupers, and Randal-Williams for $M = D^d$ [KRW24, KRW21] (see also

Watanabe [Wat18, Wat20]). The techniques used in these works differ fundamentally from those in surgery/pseudoisotopy theory, which is still the main focus of this thesis.

0.2 Summary of the thesis

The thesis is based on two preprints, listed below:

[ME22] S. Muñoz-Echániz, *Mapping class groups of h-cobordant manifolds*, 2022, arXiv:2210.06573.

[ME23] S. Muñoz-Echániz, *A Weiss–Williams theorem for spaces of embeddings and the homotopy type of spaces of long knots*, 2023, arXiv:2311.05541.

The thesis consists of three chapters, all centred around the themes of h-cobordisms and pseudoisotopy theory. Chapter 1 presents [ME22], while our second paper [ME23] is divided between Chapters 2 and 3. The following sections provide a brief overview of each chapter, with further details available in the respective chapter introductions.

0.2.1 Chapter 1: Mapping class groups of h-cobordant manifolds

Given an h-cobordism $W : L \overset{h}{\rightsquigarrow} M$, this chapter studies the difference between the homotopy types of the diffeomorphism groups of L and M. The main theorem (Theorem A) constructs an infinite family of such h-cobordisms for which the mapping class groups $\Gamma(L)$ and $\Gamma(M)$ are not isomorphic. For instance:

Theorem. *There is an* 11*-dimensional manifold* M h*-cobordant to the lens space* $L = L^{11}_7(1:2:3:4:5:6)$ *such that* $\Gamma(L) \ncong \Gamma(M)$*. In fact, both groups are finite and their cardinalities have different* 3*-adic valuations.*

All the other examples of Theorem A are higher-dimensional versions of the one above. A key ingredient in the proof is the comparison between the block moduli spaces $\widetilde{\mathcal{M}}_d$ and $\widetilde{\mathcal{M}}^h_d$ established in Theorem B, which allows us to show that the block mapping class groups $\widetilde{\Gamma}(L)$ and $\widetilde{\Gamma}(M)$ are also not isomorphic.

0.2.2 Chapter 2: A Weiss–Williams theorem for spaces of embeddings

Embedding spaces are both a central object of study in geometric topology and a key tool for understanding diffeomorphism groups via the *isotopy extension theorem.*

Motivated by this principle, we establish a pseudoisotopy result for embedding spaces, analogous to the previously mentioned theorem of Weiss and Williams [WW88] for diffeomorphism groups. Specifically, we describe the difference between spaces of ordinary and block embeddings within a certain range of degrees in terms of relative algebraic K-theory. This range, known as the *concordance embedding stable range*, has recently been shown by Goodwillie, Krannich, and Kupers [GKK23] to be significantly larger than the usual concordance stable range.

Both our result and Weiss–Williams' theorem stem from the same general phenomenon in *orthogonal calculus* [Wei95], the framework in which our work is formulated. As a consequence of our main theorem, we also establish a splitting result for embedding spaces (Theorem 2.4.2) analogous to a theorem of Burghelea–Lashof [BL82].

0.2.3 Chapter 3: On the homotopy type of spaces of long knots

Let d and p be positive integers with $d-p\geq 3$. We use the results from the previous chapter to provide a complete description of the homotopy type of the *space of long knots*—compactly supported embeddings of $\mathbb{R}^p$ into $\mathbb{R}^d$—in the concordance embedding stable range and localised away from the prime 2. On homotopy groups:

Theorem. *(cf. Theorem D and Corollary E) Let* $p\leq d-3$. *For* $*\leq 2d-p-6$,

$$\pi_*\left(\mathrm{Emb}_c(\mathbb{R}^p,\mathbb{R}^d)\right)\left[\tfrac{1}{2}\right]\cong A_*\oplus B_*,$$

where $B_*:=\pi_{*+p}\left(\mathrm{hofib}(G(d-p)/O(d-p)\to G/O)\right)\left[\frac{1}{2}\right]$ *and* $G(n):=h\mathrm{Aut}(S^{n-1})$.

The groups A_* *are the homotopy groups of an infinite wedge of Thom spectra, whose homotopy type is largely accessible but often treated on a case-by-case basis. For example, if* $p=10$ *and* $d=13$:

$*$	0	1	2	3	4	5	6	7	8	9	10	≥11
A_*	0	0	0	$\mathbb{Z}[\frac{1}{2}]$	0	0	$\mathbb{Z}/9$	$\mathbb{Z}[\frac{1}{2}]$	0	0	$\mathbb{Z}/3\oplus\mathbb{Z}/9\oplus\mathbb{Z}/5$	...

Most of this chapter is dedicated to analysing a certain geometric involution in the algebraic K-theory space of a stably parallelisable manifold (cf. Theorem 3.1.13 and Corollary 3.1.17). This forms the technical core of [ME23] and is not specific to the case of long knots, so it may be of independent interest.

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Chapter 1

Mapping class groups of h-cobordant manifolds

1.1 Introduction

1.1.1 The main result

Automorphism groups of manifolds have been subject to extensive research in algebraic and geometric topology. Inspired by the study of how different h-cobordant manifolds can be (see e.g. [JK15, JK18]), in the present paper we investigate the question of how automorphism groups of manifolds can vary within a fixed h-cobordism class. Namely: given an h-cobordism W^{d+1} between (closed) smooth[1] manifolds M and M' of dimension $d \geq 0$, how different can the homotopy types of the diffeomorphism groups $\mathrm{Diff}(M)$ and $\mathrm{Diff}(M')$ be?

Certain analogues of this question have led to invariance-type results. Dwyer and Szczarba [DS83, Cor. 2] proved that when $d \neq 4$, the rational homotopy type of the identity component $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ does not change as M^d varies within a fixed homeomorphism class of smooth manifolds. Krannich [Kra19, Thm. A] gave another instance of such a result, showing that when $d = 2k \geq 6$ and M^d is closed, oriented and simply connected, the rational homology of $B\mathrm{Diff}^+(M)$ in a range is insensitive to replacing M by $M\#\Sigma$, for Σ any homotopy d-sphere.

Our main result is, however, that the homotopy types of the diffeomorphism groups of h-cobordant manifolds can indeed be different in general. Let $\Gamma(M)$ denote the

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[1] We will work in the smooth setting for notational preference, but all of the results in this paper are equally valid for the topological and PL categories. See Remarks 1.4.11 and 1.5.4 for modified arguments when $CAT = \mathrm{Top}$ and PL.

mapping class group of M—the group of isotopy classes of diffeomorphisms of M, i.e., $\Gamma(M) := \pi_0(\mathrm{Diff}(M))$. The *block mapping class group* $\widetilde{\Gamma}(M)$ is the quotient of $\Gamma(M)$ by the normal subgroup of those classes of diffeomorphisms which are pseudoisotopic to the identity.

Theorem A. *In each dimension $d = 12k - 1 \geq 0$, there exists a d-manifold M^d (see Theorem 1.4.6) h-cobordant to the lens space $L = L_7^{12k-1}(r_1 : \cdots : r_{6k})$, where*

$$r_1 = \cdots = r_k = 1, \qquad r_{k+1} = \cdots = r_{2k} = 2, \quad \ldots \quad r_{5k+1} = \cdots = r_{6k} = 6 \mod 7,$$

such that

(i) $\widetilde{\Gamma}(L)$ and $\widetilde{\Gamma}(M)$ are finite groups with cardinalities of different 3-adic valuations,

(ii) $\Gamma(L)$ and $\Gamma(M)$ are finite groups with cardinalities of different 3-adic valuations.

Remark 1.1.1. For an oriented connected manifold M, there are orientation preserving mapping class groups $\Gamma^+(M)$ and $\widetilde{\Gamma}^+(M)$, which have index one or two inside the whole mapping class groups $\Gamma(M)$ and $\widetilde{\Gamma}(M)$, respectively. Therefore, the conclusions of Theorem A also hold for $\widetilde{\Gamma}^+(-)$ and $\Gamma^+(-)$.

Remark 1.1.2. Theorem A(i) is the best possible result in the following sense: let $\widetilde{\mathrm{Diff}}(M)$ denote the (geometric realisation of the) semi-simplicial group of *block diffeomorphisms* of M (cf. [BLR75, p. 20] or [ERW14, Defn. 2.1]), whose p-simplices consist of diffeomorphisms $\phi : M \times \Delta^p \xrightarrow{\cong} M \times \Delta^p$ which are face-preserving (i.e. for every face $\sigma \subset \Delta^p$, ϕ restricts to a diffeomorphism of $M \times \sigma$). Then we have $\widetilde{\Gamma}(M) = \pi_0(\widetilde{\mathrm{Diff}}((M))$. The restriction map $\rho_M : \widetilde{\mathrm{Diff}}(W) \longrightarrow \widetilde{\mathrm{Diff}}(M)$ is a fibration with fibre $\widetilde{\mathrm{Diff}}_M(W)$, the subspace of block diffeomorphisms of W which fix pointwise a neighbourhood of $M \subset W$. By the s-cobordism theorem (see Theorem 1.2.1 below), there exists some h-cobordism $-W$ from M' to M such that $W \cup_{M'} -W \cong M \times I$ and $-W \cup_M W \cong M' \times I$. Then the group homomorphisms

$$\begin{aligned} \mathrm{Id}_W \cup_{M'} - : \widetilde{C}(M') := \widetilde{\mathrm{Diff}}_{M' \times \{0\}}(M' \times I) &\longrightarrow \widetilde{\mathrm{Diff}}_M(W), \\ \mathrm{Id}_{-W} \cup_M - : \widetilde{\mathrm{Diff}}_M(W) &\longrightarrow \widetilde{\mathrm{Diff}}_{M'}(-W \cup_M W) \cong \widetilde{C}(M') \end{aligned}$$

are easily seen to be homotopy inverse to each other. But the group $\widetilde{C}(M')$ of *block concordances* of M' is contractible (cf. [BLR75, Lem. 2.1]), and therefore ρ_M induces an equivalence onto the components that it hits, and similarly for $\rho_{M'}$. In other words, the classifying spaces $B\widetilde{\mathrm{Diff}}(M)$ and $B\widetilde{\mathrm{Diff}}(M')$ share the same universal cover, so

$$\pi_i(\widetilde{\mathrm{Diff}}(M)) \cong \pi_i(\widetilde{\mathrm{Diff}}(M')), \quad i \geq 1.$$

The upshot is that the homotopy types of $\widetilde{\mathrm{Diff}}(M)$ and $\widetilde{\mathrm{Diff}}(M')$ can at most differ by their sets of path-components, and Theorem A(i) provides an example showcasing this phenomenon.

1.1.2 Moduli spaces of h- and s-block bundles

Recall that for $d \geq 5$, the *Whitehead group* $\mathrm{Wh}(M)$ of a compact d-manifold M (see Section 1.2.1) classifies isomorphism classes of h-cobordisms starting at M. This group has an involution denoted $\tau \mapsto \overline{\tau}$ which, roughly speaking and up to a factor of $(-1)^d$, corresponds to reversing the direction of an h-cobordism (see (1.2.4)).

In Section 1.3 we will introduce the h- and *s-block moduli spaces*, $\widetilde{\mathcal{M}}^h$ and $\widetilde{\mathcal{M}}^s$ respectively, whose vertices (as semi-simplicial sets) are the smooth closed d-manifolds, for some fixed integer $d \geq 0$. A path in the former (resp. latter) space between d-manifolds M and M' is exactly an h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$ (resp. an s-cobordism $W : M \overset{s}{\rightsquigarrow} M'$, i.e., an h-cobordism with vanishing Whitehead torsion (see Section 1.2.2)). The s-block moduli space $\widetilde{\mathcal{M}}^s$ is, somewhat in disguise, a well-known object; in Proposition 1.3.6 we identify the path-component of M^d in $\widetilde{\mathcal{M}}^s$ with $B\widetilde{\mathrm{Diff}}(M)$, the classifying space for the group of block diffeomorphisms of M.

The second main result we state arises as part of the proof of Theorem A, but may be of independent interest: there is a natural inclusion $\widetilde{\mathcal{M}}^s \hookrightarrow \widetilde{\mathcal{M}}^h$ which forgets the simpleness condition. We identify the homotopy fibre of this inclusion (i.e. the homotopical difference between the h- and s-block moduli spaces) as a certain infinite loop space.

Theorem B. *Let M be a closed d-dimensional manifold, and let $C_2 := \{e, t\}$ act on the Whitehead group $\mathrm{Wh}(M)$ by $t \cdot \tau := (-1)^{d-1}\overline{\tau}$. Write $H\mathrm{Wh}(M)$ for the Eilenberg–MacLane spectrum associated to $\mathrm{Wh}(M)$, and let $H\mathrm{Wh}(M)_{hC_2} := H\mathrm{Wh}(M) \wedge_{C_2} (EC_2)_+$ stand for the homotopy C_2-orbits of $H\mathrm{Wh}(M)$. For $d \geq 5$, there is a homotopy cartesian square*

$$\begin{tikzcd}
\Omega^\infty(H\mathrm{Wh}(M)_{hC_2}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{M}}^s \arrow[d, hook] \\
\{M^d\} \arrow[r] & \widetilde{\mathcal{M}}^h,
\end{tikzcd}$$

where the lower horizontal map is the inclusion of M as a point in $\widetilde{\mathcal{M}}^h$.

As we will explain in Section 1.3.3, this result is intimately tied to the *Rothenberg exact sequence* [Ran81, Prop. 1.10.1].

Structure of Chapter 1

Section 1.2 serves as a reminder to the reader of the *s-cobordism theorem* and some of the properties of *Whitehead torsion*.

In Section 1.3 we prove Theorem B. The proof boils down to arguing that certain simplicial abelian group $F_\bullet^{\mathsf{alg}}(A)$ corresponds to the spectrum HA_{hC_2} under the Dold–Kan correspondence (see Theorem 1.3.10).

Section 1.4 deals with part (i) of Theorem A, which is proved in Theorem 1.4.6. We analyse the lower degree part of the homotopy long exact sequence associated to the homotopy pullback square of Theorem B. The proof of Theorem A(ii) builds on part (i) and pseudoisotopy theory, and comprises Section 1.5.

Appendix A is an algebraic K-theory computation required for Sections 1.4 and 1.5. Appendix B explores the connection between Theorem B and the theory of Weiss–Williams [WW88].

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1.2 Notation and recollections

All manifolds will be assumed to be compact and smooth (possibly with corners).

1.2.1 Whitehead Torsion

Let π be a group and $w : \pi \to C_2 = \{\pm 1\}$ be a homomorphism. The *Whitehead group* of (π, w) [Mil66, §6] is the abelian group

$$\mathrm{Wh}(\pi, w) := GL(\mathbb{Z}\pi)^{\mathrm{ab}}/(\pm\pi)$$

equipped with the following involution: the anti-involution on $\mathbb{Z}\pi$ given by

$$a = \sum_{g \in \pi} a_g \cdot g \longmapsto \overline{a} := \sum_{g \in \pi} w(g) a_g \cdot g^{-1}, \quad a_g \in \mathbb{Z},$$

induces an involution on $\mathrm{Wh}(\pi, w)$ by sending an element represented by a matrix $\tau = (\tau_{ij})$ to its conjugate transpose $\overline{\tau} := (\overline{\tau}_{ji})$. We will refer to this involution as the *algebraic involution* on $\mathrm{Wh}(\pi, w)$. We will write $\mathrm{Wh}(\pi)$ for $\mathrm{Wh}(\pi, w)$ if w is the trivial homomorphism, or if we are simply disregarding this involution. If X is a finite CW-complex with a choice of basepoint in each of its connected components, the *Whitehead group* of X is

$$\mathrm{Wh}(X) := \bigoplus_{X_j \in \pi_0(X)} \mathrm{Wh}(\pi_1(X_j)).$$

If $X = M$ is moreover a manifold, the algebraic involution on $\mathrm{Wh}(M)$ is that induced by $w = w_1(M) \in H^1(M; \mathbb{Z}/2)$, the first Stiefel–Whitney class of M. Since every inner automorphism of a group π induces the identity on $\mathrm{Wh}(\pi)$ [Mil66, Lem. 6.1], the Whitehead group $\mathrm{Wh}(X)$ does not depend (up to canonical isomorphism) on the choice of basepoint in each path component of X; we shall ignore basepoints from now on.

Given a homotopy equivalence between finite pointed CW-complexes $f : X \xrightarrow{\simeq} Y$, we will denote by $\tau(f) \in \mathrm{Wh}(X)$ its (*Whitehead*) *torsion* [Mil66, §7]. It only depends on f up to homotopy [Mil66, Lem. 7.7]. Let us collect a few properties of the Whitehead torsion $\tau(-)$ that we will use throughout the paper:

- *Composition rule*: $\tau(-)$ is a crossed homomorphism in the sense that if $f : X \xrightarrow{\simeq} Y$ and $g : Y \xrightarrow{\simeq} Z$ are homotopy equivalences, then [Mil66, Lem. 7.8]

  $$\tau(g \circ f) = \tau(f) + f_*^{-1}\tau(g), \tag{1.2.1}$$

  where $f_* : \mathrm{Wh}(X) \xrightarrow{\cong} \mathrm{Wh}(Y)$ is the natural isomorphism induced by $\pi_1(f) : \pi_1(X) \xrightarrow{\cong} \pi_1(Y)$.

- *Inclusion-exclusion principle*: if $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$, where X_0, X_1, Y_0, Y_1, $X_{01} := X_0 \cap X_1$ and $Y_{01} := Y_0 \cap Y_1$ are all finite CW-complexes, and

  $$f_0 : X_0 \xrightarrow{\simeq} Y_0, \quad f_1 : X_1 \xrightarrow{\simeq} Y_1, \quad f_{01} = f_0 \cap f_1 : X_{01} \xrightarrow{\simeq} Y_{01},$$

are homotopy equivalences, then the torsion of the homotopy equivalence $f = f_0 \cup f_1 : X \xrightarrow{\simeq} Y$ is [Coh73, Thm. 23.1]

$$\tau(f) = (i_0)_*\tau(f_0) + (i_1)_*\tau(f_1) - (i_{01})_*\tau(f_{01}) \in \mathrm{Wh}(X), \tag{1.2.2}$$

where $i_0 : X_0 \hookrightarrow X$, $i_1 : X_1 \hookrightarrow X$ and $i_{01} : X_{01} \hookrightarrow X$ are the inclusions.

- *Product rule*: $\tau(-)$ is multiplicative with respect to the Euler characteristic in the sense that for any homotopy equivalence $f : X \xrightarrow{\simeq} Y$ and any finite connected CW-complex K with basepoint $* \in K$ [Coh73, Thm. 23.2],

  $$\tau(f \times \mathrm{id}_K) = \chi(K) \cdot i_*\tau(f) \in \mathrm{Wh}(X \times K), \tag{1.2.3}$$

  where $i : X \cong X \times \{*\} \hookrightarrow X \times K$ is the inclusion.

A homotopy equivalence f as above is said to be *simple*, and denoted $f : X \xrightarrow{\simeq_s} Y$, if $\tau(f) = 0$. We will write $s\mathrm{Aut}(X) \subset h\mathrm{Aut}(X)$ for the topological submonoid (see (1.2.1)) of simple homotopy automorphisms of X.

1.2.2 The s-cobordism theorem.

Let M^d be a smooth compact manifold of dimension d. A *cobordism from M rel ∂M* is a triple $(W; M, M')$, also written as $W : M \rightsquigarrow M'$, consisting of a $(d+1)$-manifold W^{d+1} with boundary

$$\partial W \cong M \cup M' \cup (\partial M \times [0,1])$$

so that $M \cap (\partial M \times [0,1]) = \partial M \times \{0\}$ and $M' \cap (\partial M \times [0,1]) = \partial M \times \{1\}$ (in particular $\partial M' = \partial M$). Cobordisms are often accompanied with an additional data of *collars*, i.e., open neighbourhoods of M and M' in W diffeomorphic to $M \times [0, \epsilon)$ and $M' \times (1-\epsilon, 1]$ (rel $\partial M \times I$) for some small $\epsilon > 0$, but the choice of such is contractible. If $\partial M = \emptyset$, this coincides with the usual notion of a cobordism between closed manifolds. Such a cobordism is called an *h-cobordism* if the inclusions $i_M : (M, \partial M) \to (W, \partial M \times I)$ and $i_{M'} : (M', \partial M') \to (W, \partial M \times I)$ are homotopy equivalences of pairs. In such case we will write $W : M \overset{h}{\rightsquigarrow} M'$ to emphasise that W is an h-cobordism from M to M'. The *torsion* of W with respect to M is

$$\tau(W, M) := \tau(i_M) \in \mathrm{Wh}(M).$$

If $\tau(W,M)=0$, such an h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$ is said to be *simple* (or an *s-cobordism*), and denoted $W : M \overset{s}{\rightsquigarrow} M'$. This definition is independent of the direction of W since the torsion of an h-cobordism satisfies the *duality formula* [Mil66, §10]

$$\tau(W,M') = (-1)^d (h^W)_* \overline{\tau(W,M)}. \tag{1.2.4}$$

Here $h^W : M \simeq M'$ is the *natural homotopy equivalence*

$$h^W : M \xhookrightarrow[\simeq]{i_M} W \xrightarrow[\simeq]{r_{M'}} M', \tag{1.2.5}$$

where $r_{M'}$ is some homotopy inverse to $i_{M'}$ (so h^W is only well-defined up to homotopy).

Due to the composition rule (1.2.1), the torsion of an h-cobordism is nearly additive with respect to composition: namely if $W : M \overset{h}{\rightsquigarrow} M'$ and $W' : M' \overset{h}{\rightsquigarrow} M''$ are h-cobordisms, we write $W' \circ W : M \overset{h}{\rightsquigarrow} M''$ for the h-cobordism $W \cup_{M'} W'$, which can be made smooth by pasting along collars. Then

$$\tau(W' \circ W, M) = \tau(W,M) + (h^W)_*^{-1} \tau(W',M'). \tag{1.2.6}$$

Let $h\mathrm{Cob}_\partial(M)$ denote the set of h-cobordisms rel boundary starting at M, up to diffeomorphism rel M. We will use the following a great deal [Maz63, Bar64].

Theorem 1.2.1 (s-Cobordism Theorem rel boundary)**.** *If $d = \dim M \geq 5$, then there is a bijection*

$$h\mathrm{Cob}_\partial(M) \longleftrightarrow \mathrm{Wh}(M), \qquad (W : M \overset{h}{\rightsquigarrow} M') \longmapsto \tau(W,M).$$

1.3 The block moduli spaces of manifolds

As explained in the introduction, we now present the h- and s-block moduli spaces of manifolds, in which a path, i.e., a 1-simplex, is an h- or s-cobordism, respectively. To describe what higher-dimensional simplices should be we give the next definition, which is inspired by [HLLRW21, §2].

Definition 1.3.1. *Fix once and for all some small $\epsilon > 0$. A compact smooth manifold with corners $W^{d+p} \subset \mathbb{R}^\infty \times \Delta^p$ is said to be* ***stratified over*** *Δ^p if:*

(i) W is a closed manifold if $p = 0$,

(ii) W is transverse to $\mathbb{R}^\infty \times \sigma$ for every proper face $\sigma \subset \Delta^p$ and $W_\sigma := W \cap (\mathbb{R}^\infty \times \sigma)$ is a $(d + \dim \sigma)$-dimensional manifold stratified over $\Delta^{\dim \sigma} \cong \sigma$,

(iii) W satisfies the ϵ-collaring conditions of [HLLRW21, Defn. 2.3.1(ii)].

We will write $W^{d+p} \Rightarrow \Delta^p$ for such a manifold. A map $f : W \to V$ between manifolds stratified over Δ^p is said to be ***face-preserving****, and denoted $f : W \to_\Delta V$, if for every face $\sigma \subset \Delta^p$ we have $f(W_\sigma) \subset V_\sigma$ and f satisfies the collaring conditions of [HLLRW21, Defn. 2.3.1(iii)] (roughly, f must be the product $f_\sigma \times \mathrm{Id}$ in the ϵ-neighbourhood of the strata W_σ, where $f_\sigma := f\,|_{W_\sigma}$). If moreover f_σ is a homotopy equivalence, simple homotopy equivalence or diffeomorphism for all $\sigma \subset \Delta^p$, we will write $f : W \xrightarrow{\spadesuit}_\Delta V$ for $\spadesuit = \simeq_h,\ \simeq_s$ or $\cong$, respectively.*

Notation 1.3.2. *Let $\Lambda_i^p \subset \Delta^p$ denote the i-th horn of Δ^p ($i = 0, \dots, p$).*

- *If $0 \le i_0 < \cdots < i_r \le p$, we write $\langle i_0, \dots, i_r\rangle \subset \Delta^p$ for the face spanned by the vertices $i_0, \dots, i_r \in \Delta^p$.*
- *If W is stratified over Δ^p, we will often write $\partial_i W$ for $W_{\langle 0,\dots,\widehat{i},\dots,p\rangle}$ and W_i for $W_{\langle i\rangle}$. For instance, $\langle 0 \dots, \widehat{i}, \dots, p\rangle \equiv \partial_i\Delta^p \subset \Delta^p$.*
- *If $K \subset \Delta^p$ is a simplicial sub-complex, we will write W_K for $W \cap (\mathbb{R}^\infty \times K)$. In the particular case that $K = \Lambda_i^p$, we set $\Lambda_i(W) := W_{\Lambda_i^p}$. For instance, if $\sigma \subset \Delta^p$ is some face, $\Lambda_i(\sigma)$ denotes the i-th horn of σ ($i = 0, \dots, \dim\sigma$).*
- *If $f : W \longrightarrow_\Delta V$ is face-preserving, we will write $\partial_i f$ for $f_{\partial_i\Delta^p} = f\,|_{\partial_i W}$.*

Example 1.3.3. A cobordism $W^{d+1} : M \rightsquigarrow M'$ between closed manifolds M and M' is always diffeomorphic to a manifold $W' \subset \mathbb{R}^\infty \times \Delta^1$ stratified over Δ^1 with $W'_0 \cong M$ and $W'_1 \cong M'$.

Definition 1.3.4. *Fix some integer $d \ge 0$. The* ***h-block moduli space of d-manifolds*** *is the semi-simplicial set $\widetilde{\mathcal{M}}^h_\bullet$ with p-simplices*

$$\widetilde{\mathcal{M}}^h_p := \left\{ \begin{array}{c} W^{d+p} \\ \Downarrow \\ \Delta^p \end{array} : \exists\ f : W \xrightarrow{\simeq_h}_\Delta W_0 \times \Delta^p \right\}, \tag{1.3.1}$$

and with face maps given by restriction to face-strata

$$\partial_i : \widetilde{\mathcal{M}}^h_p \longrightarrow \widetilde{\mathcal{M}}^h_{p-1}, \qquad \begin{array}{c} W^{d+p} \\ \Downarrow \\ \Delta^p \end{array} \longmapsto \begin{array}{c} \partial_i W^{d+p} \\ \Downarrow \\ \partial_i\Delta^p \cong \Delta^{p-1}, \end{array} \qquad i = 0, \dots, p.$$

*The s-**block moduli space of d-manifolds** $\widetilde{\mathcal{M}}^s_\bullet$ is its simple analogue, where $\simeq_h$ in (1.3.1) is replaced by $\simeq_s$, and has a natural inclusion $\widetilde{\mathcal{M}}^s_\bullet \hookrightarrow \widetilde{\mathcal{M}}^h_\bullet$. We will let $\widetilde{\mathcal{M}}^h$ and $\widetilde{\mathcal{M}}^s$ denote the geometric realisations $|\widetilde{\mathcal{M}}^h_\bullet|$ and $|\widetilde{\mathcal{M}}^s_\bullet|$, respectively.*

Remark 1.3.5. Let M be a closed d-manifold and $\widetilde{\mathcal{M}}^s_\bullet(M)$ denote the path-component of M in $\widetilde{\mathcal{M}}^s_\bullet$. Our definition of $\widetilde{\mathcal{M}}^s_\bullet(M)$ differs from that of $\mathcal{M}(M)_\bullet$ in [HLLRW21] in that our condition (ii) in Definition 1.3.1 is stronger than condition (i) of Definition 2.3.1 loc. cit.; there, it is only required that W be transverse to $\mathbb{R}^\infty \times \sigma$ for faces of the form $\sigma = \partial_i \Delta^p$. As noted right after [HLLRW21, Defn. 2.3.1], their defined $\mathcal{M}(M)_\bullet$ is Kan, and our stronger requirement does not affect this Kan condition as any proper subface of Δ^p that is not of the form $\partial_i \Delta^p$ is already a subface of any horn Λ^p_j. Thus, $\widetilde{\mathcal{M}}^s_\bullet$ (and similarly $\widetilde{\mathcal{M}}^h_\bullet$) is Kan.

The next two subsections are devoted to prove Theorem B. But first, we study the s-block moduli space $\widetilde{\mathcal{M}}^s$ more closely. We recall that the classifying space $B\widetilde{\mathrm{Diff}}(M)$ for the simplicial group of block diffeomorphisms has a semi-simplicial model (see e.g. [ERW14]) in which the p-simplices are

$$B\widetilde{\mathrm{Diff}}(M)_p = \left\{ \begin{matrix} W^{d+p} \\ \Downarrow \\ \Delta^p \end{matrix} \quad : \exists\, \phi : W \xrightarrow{\cong}_{\Delta} M \times \Delta^p \right\},$$

and therefore there is a forgetful inclusion $B\widetilde{\mathrm{Diff}}(M) \hookrightarrow \widetilde{\mathcal{M}}^s$.

Proposition 1.3.6. *For $d \geq 5$, there is a decomposition of $\widetilde{\mathcal{M}}^s$ into connected components*

$$\widetilde{\mathcal{M}}^s = \bigsqcup_{\substack{[M^d] \text{ up} \\ \text{to s-cob.}}} B\widetilde{\mathrm{Diff}}(M) = \bigsqcup_{\substack{[M^d] \text{ up} \\ \text{to diffeo.}}} B\widetilde{\mathrm{Diff}}(M). \tag{1.3.2}$$

In order to prove this proposition, we will need the following simple observation.

Lemma 1.3.7. *Let $W^{d+p} \Rightarrow \Delta^p$ represent a p-simplex in $\widetilde{\mathcal{M}}^s_\bullet$. For every face inclusion $\xi \subset \sigma \subset \Delta^p$, the map $W_\xi \hookrightarrow W_\sigma$ is a simple homotopy equivalence. In particular if $p = 1$, W is an s-cobordism from W_0 to W_1.*

Proof. Let $f : W \xrightarrow{\simeq_s}_{\Delta} W_0 \times \Delta^p$ be some face-preserving simple homotopy equivalence. The inclusion $W_\xi \hookrightarrow W_\sigma$ is homotopic to a composition of simple maps

$$W_\xi \xrightarrow[f_\xi]{\simeq_s} W_0 \times \xi \xhookrightarrow[\text{by (1.2.3)}]{\simeq_s} W_0 \times \sigma \xrightarrow[f_\sigma^{-1}]{\simeq_s} W_\sigma,$$

where f_σ^{-1} is any homotopy inverse to f_σ. Therefore it is also simple. □

Proof of Proposition 1.3.6. For a closed manifold M^d, let $\widetilde{\mathcal{M}}^s_\bullet(M)$ denote the path-component of M in $\widetilde{\mathcal{M}}^s_\bullet$. We only have to argue that $\widetilde{\mathcal{M}}^s(M) \subset B\widetilde{\mathrm{Diff}}(M)$, which is the following claim when $r = -1$.

Claim. *Let $W \in \widetilde{\mathcal{M}}^s_p(M)$ and suppose that for some $-1 \le r \le p-1$, there exist face-preserving diffeomorphisms*

$$\phi_i : \partial_i W \xrightarrow{\cong}_\Delta M \times \Delta^{p-1}, \quad 0 \le i \le r,$$

such that $\partial_i \phi_j = \partial_{j-1}\phi_i$ for $0 \le i < j \le r$. Then there exists some face-preserving diffeomorphism $\phi : W \xrightarrow{\cong}_\Delta M \times \Delta^p$ such that $\partial_i \phi = \phi_i$ for $0 \le i \le r$. In particular $W \in B\widetilde{\mathrm{Diff}}(M)_p$.

Proceed by induction on $p \ge 0$. The statement is vacuous when $p = 0$, and it holds by Lemma 1.3.7 and the s-cobordism theorem when $p = 1$. Suppose that the claim is true for $p - 1 \ge 0$. By the induction hypothesis, we can find diffeomorphisms $\phi_i : \partial_i W \xrightarrow{\cong}_\Delta M \times \Delta^{p-1}$ for $0 \le i \le p-1$ such that $\partial_i \phi_j = \partial_{j-1}\phi_i$ for $0 \le i < j \le p-1$. By pasting these together, we obtain a (face-preserving) diffeomorphism $\Lambda_p(\phi) : \Lambda_p(W) \xrightarrow{\cong}_\Delta M \times \Lambda^p_p$. Now by Lemma 1.3.7 and the inclusion-exclusion principle (1.2.2), the inclusion $\Lambda_p(W) \hookrightarrow W$ is a simple homotopy equivalence. Unbending the corners of $\Lambda_p(W)$, the s-cobordism theorem for manifolds with boundary (Theorem 1.2.1) provides a face-preserving diffeomorphism $\phi : W \xrightarrow{\cong}_\Delta M \times \Delta^p$ extending $\Lambda_p(\phi)$, as required. ☐

By analogy to (1.3.2), we define $B\widetilde{\mathrm{Diff}}^h(M)$ to be the path-component of M in $\widetilde{\mathcal{M}}^h$, and so obtain a decomposition

$$\widetilde{\mathcal{M}}^h = \coprod_{\substack{[M^d] \text{ up} \\ \text{to } h\text{-cob.}}} B\widetilde{\mathrm{Diff}}^h(M). \tag{1.3.3}$$

Remark 1.3.8. The semi-simplicial sets $B\widetilde{\mathrm{Diff}}(M)_\bullet$ and $B\widetilde{\mathrm{Diff}}^h(M)_\bullet$ have M as their natural basepoint. If M and M' are s-cobordant, i.e. diffeomorphic (resp. h-cobordant), then $B\widetilde{\mathrm{Diff}}(M)$ and $B\widetilde{\mathrm{Diff}}(M')$ (resp. $B\widetilde{\mathrm{Diff}}^h(M)$ and $B\widetilde{\mathrm{Diff}}^h(M')$) are the same semi-simplicial set but equipped with different basepoints.

Digression 1.3.9. Let $G : \mathsf{sSet}_* \to \mathsf{sGrp}$ denote the *Kan simplicial loop space functor* [Kan58, §10 and 11]. As we will see in Remark 1.3.25, the semi-simplicial set $B\widetilde{\mathrm{Diff}}^h(M)_\bullet$ can be upgraded to a simplicial set. The simplicial group $\widetilde{\mathrm{Diff}}^h(M) := GB\widetilde{\mathrm{Diff}}^h(M)$ has been studied in previous literature under different names [WW88, Appendix 5]. More precisely, if $d \ge 5$ then $\widetilde{\mathrm{Diff}}^h(M^d)$ is weakly equivalent to $\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})$, the space of block diffeomorphisms of $M \times \mathbb{R}$ *bounded* in the $\mathbb{R}$-direction. This will be proved in Proposition B.1 of Appendix B.1. With this in mind, the computation of the homotopy groups of $\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})/\widetilde{\mathrm{Diff}}(M)$ in [WW88, Cor. 5.5] agrees with Theorem B.

1.3.1 A simplicial model for $H(-)_{hC_2}$

Let A be a $\mathbb{Z}[C_2]$-module, i.e., an abelian group equipped with a $\mathbb{Z}$-linear involution $a \mapsto a^* := t \cdot a$, where $t \in C_2$ denotes the generator. Write HA for the *Eilenberg–MacLane spectrum* associated to A, and let $HA_{hC_2} := HA \wedge_{C_2} (EC_2)_+$ denote the homotopy C_2-orbits of HA. In this section we present a simplicial model

$$F_\bullet^{\mathsf{alg}}(-) : \mathsf{Mod}_{\mathbb{Z}[C_2]} \longrightarrow \mathsf{sAb}$$

for the functor $H(-)_{hC_2} : \mathsf{Mod}_{\mathbb{Z}[C_2]} \longrightarrow H\mathbb{Z}\text{-}\mathsf{Mod}$ in the following sense.

Theorem 1.3.10. *Let Ω^∞-Top denote the category of infinite loop spaces. There is a natural equivalence*

$$|F_\bullet^{\mathsf{alg}}(-)| \simeq \Omega^\infty(H(-)_{hC_2}) : \mathsf{Mod}_{\mathbb{Z}[C_2]} \longrightarrow \Omega^\infty\text{-}\mathsf{Top},$$

i.e., there is a zig-zag of natural weak equivalences connecting the left and the right hand functors.

1.3.1.1 The simplicial abelian group $F_\bullet^{\mathsf{alg}}(A)$

We now define $F_\bullet^{\mathsf{alg}}(A)$ as an *algebraic analogue* of the semi-simplicial set $F_\bullet(M)$ (see (1.3.17) and Proposition 1.3.24) when $A = \mathrm{Wh}(M)$ with the C_2-action $t \cdot \tau := (-1)^{d-1}\overline{\tau}$ of Theorem B. We will need some preliminaries first.

A *simplicial sub-complex* of Δ^p is a collection[2] K of non-empty subsets σ of $[p] = \{0, \dots, p\}$ such that if $\xi \subset \sigma$ and $\sigma \in K$, then $\xi \in K$ too. The *realisation* of a subset $\sigma \subset [p]$ is the subspace

$$|\sigma| := \{(t_0, \dots, t_p) \in \Delta^p : t_i = 0 \text{ if } i \notin \sigma\} \subset \Delta^p.$$

Then, the *realisation* of a simplicial sub-complex K of Δ^p is the subspace

$$|K| := \bigcup_{\sigma \in K} |\sigma| \subset \Delta^p.$$

---

[2] We allow the empty collection $\emptyset = \{\ \}$.

Since $|K| = |K'|$ if and only if $K = K'$, we will often identify a simplicial sub-complex K of Δ^p with its realisation $|K|$.

Let $\mathsf{SubComp}_p$ denote the poset of simplicial complexes of Δ^p, ordered by inclusion of collections. The assignment $[p] \mapsto \mathsf{SubComp}_p$ assembles into a cosimplicial poset $\mathsf{SubComp}_\bullet$ in the obvious way: given an order-preserving arrow $a : [p] \to [q]$, we set

$$a : \mathsf{SubComp}_p \longrightarrow \mathsf{SubComp}_q, \quad K \mapsto a(K) := \{a(\sigma) \subset [q] : \sigma \in K\}.$$

On realisations, we have $|a(K)| = a(|K|) \subset \Delta^q$, where the second a now denotes the map $\Delta^p \to \Delta^q$ coming from the cosimplicial space $\Delta^\bullet$. So, for instance, the i-th coface map $\partial^i : \mathsf{SubComp}_{p-1} \to \mathsf{SubComp}_p$ identifies Δ^{p-1} with the i-th face of Δ^p, i.e., $\partial^i \Delta^{p-1} := \partial_i \Delta^p$.

More important to us will be the subposet $\mathsf{SubComp}_p^{\simeq *} \subset \mathsf{SubComp}_p$ consisting of those sub-complexes whose realisation is contractible (in particular non-empty). The assignment $[p] \mapsto \mathsf{SubComp}_p^{\simeq *}$ now assembles only into a *semi*-cosimplicial poset, for the coface maps $\partial^i : \mathsf{SubComp}_{p-1} \to \mathsf{SubComp}_p$ send $\mathsf{SubComp}_{p-1}^{\simeq *}$ to $\mathsf{SubComp}_p^{\simeq *}$, but the codegeneracies fail to do so: for instance, the codegeneracy $s^0 : \mathsf{SubComp}_3 \to \mathsf{SubComp}_2$ sends the contractible sub-complex of Δ^3 cosisting of the edges $\{0, 2\}$, $\{2, 3\}$ and $\{1, 3\}$ (and all its subsets) to $\partial \Delta^2$.

Seen as a category, $\mathsf{SubComp}_p^{\simeq *}$ admits all pushouts, for the pushout of contractible simplicial complexes is also contractible. For an (abelian) group A (seen as a one-object groupoid), write $\mathrm{Fun}(\mathsf{SubComp}_p^{\simeq *}, A)$ for the set of functors $\tau : \mathsf{SubComp}_p^{\simeq *} \to A$ and $\mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A) \subset \mathrm{Fun}(\mathsf{SubComp}_p^{\simeq *}, A)$ for the subset consisting of those functors τ such that for any pushout square

$$\begin{tikzcd} K_{01} \arrow[r] \arrow[d] & K_1 \arrow[d] \\ K_0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & K \end{tikzcd} \tag{1.3.4}$$

in $\mathsf{SubComp}_p^{\simeq *}$:

$$\tau(K_{01} \to K_1) = \tau(K_0 \to K) \in A. \tag{1.3.5}$$

Equivalently, by taking the transpose of (1.3.4),

$$\tau(K_{01} \to K_0) = \tau(K_1 \to K).$$

Notation 1.3.11. *If $K, L \in \mathsf{SubComp}_p^{\simeq *}$ with $K \leq L$, there is a unique arrow $K \to L$. We will write $\tau(L, K) \in A$ for $\tau(K \to L)$ resembling the notation for h-cobordisms.*

Note that $\tau(K, K) = 0$ *for every* K. *We will sometimes refer to* $\tau(L, K)$ *as a* ***torsion element***.

Lemma 1.3.12. *Let* $\tau : \mathsf{SubComp}_p^{\simeq *} \to A$ *be a functor. Then* $\tau \in \mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$ *if and only if for every diagram in* $\mathsf{SubComp}_p^{\simeq *}$ *of the form*

$$\begin{tikzcd}
K_{01} \arrow[r] \arrow[d] & K_1 \arrow[d] \arrow[ddr, bend left] & \\
K_0 \arrow[r] \arrow[drr, bend right] & K \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] & \\
& & L,
\end{tikzcd} \tag{1.3.6}$$

the functor τ *satisfies the inclusion-exclusion principle (compare to (1.2.2)):*

$$\tau(L, K) = \tau(L, K_0) + \tau(L, K_1) - \tau(L, K_{01}). \tag{1.3.7}$$

Proof. If $\tau \in \mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$, then

$$\begin{aligned}
\tau(L, K_0) + \tau(L, K_1) - \tau(L, K_{01}) &= \tau(L, K) + \tau(K, K_0) + \tau(L, K_1) \\
&\qquad - \tau(L, K_1) - \tau(K_1, K_{01}) \\
&= \tau(L, K),
\end{aligned}$$

where the second line follows from (1.3.5). Conversely (1.3.5) follows from the inclusion-exclusion principle (1.3.7) applied to the diagram (1.3.6) with $L = K$, noting that $\tau(K, K) = 0$. □

Observe that both $\mathrm{Fun}(\mathsf{SubComp}_p^{\simeq *}, A)$ and $\mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$ are abelian groups under morphism-wise addition. Therefore, $\mathrm{Fun}(\mathsf{SubComp}_\bullet^{\simeq *}, A)$ defines a semi-simplicial abelian group whose i-th face map is $\partial_i^{\mathrm{Fun}} := \mathrm{Fun}(\partial^i, A)$. These face maps clearly descend to $\mathrm{Fun}^{\square}(\mathsf{SubComp}_\bullet^{\simeq *}, A)$, and we will write $\partial_i^{\square} \equiv \partial_i^{\mathrm{Fun}} |_{\mathrm{Fun}^{\square}}$ for their restriction. We will now construct a system of degeneracies $s_i^{\square}$ for $\mathrm{Fun}^{\square}(\mathsf{SubComp}_\bullet^{\simeq *}, A)$ compatible with the $\partial_i^{\square}$'s which makes it into a simplicial abelian group—this will be handy when invoking the Dold–Kan correspondence later in Section 1.3.1.2 (see also Remark 1.3.23).

Remark 1.3.13. One might consider the analogous abelian groups $\mathrm{Fun}(\mathsf{SubComp}_p, A)$ and $\mathrm{Fun}^{\square}(\mathsf{SubComp}_p, A)$. Then, $\mathrm{Fun}(\mathsf{SubComp}_\bullet, A)$ defines an actual simplicial abelian group whose i-th face and degeneracy maps are $\partial_i^{\mathrm{Fun}}$ and $s_i^{\mathrm{Fun}} = \mathrm{Fun}(s^i, A)$, respectively. However, $\mathrm{Fun}^{\square}(\mathsf{SubComp}_\bullet, A)$ is only semi-simplicial: for any non-zero

$\tau \in \mathrm{Fun}^{\square}(\mathsf{SubComp}_1, A)$, $s_1^{\mathrm{Fun}}\tau$ does not satisfy (1.3.5) for $K = \Lambda^2_2$, $K_0 = \partial_0\Delta^2$, $K_1 = \partial_1\Delta^2$ and $K_{01} = \langle 2\rangle$.

Let $\mathsf{Face}_\bullet$ denote the sub-*cosimplicial* poset of $\mathsf{SubComp}_\bullet$ consisting of those sub-complexes of $\Delta^\bullet$ which are *faces*—those collections K for which there exists some $\sigma \subset [\bullet]$ such that $\xi \in K$ for every $\xi \subset \sigma$; thus, the empty sub-complex $\emptyset \subset \Delta^\bullet$ is *not* a face. Note that $\mathsf{Face}_\bullet \subset \mathsf{SubComp}^{\simeq *}_\bullet$ and write ι for the inclusion. The following result says that a functor in $\mathrm{Fun}^{\square}(\mathsf{SubComp}^{\simeq *}_\bullet, A)$ is completely determined by the torsion elements corresponding to face inclusions.

Proposition 1.3.14. *There is a natural isomorphism of semi-simplicial abelian groups*

$$\iota^* : \mathrm{Fun}^{\square}(\mathsf{SubComp}^{\simeq *}_\bullet, A) \cong \mathrm{Fun}(\mathsf{Face}_\bullet, A) : \iota_!.$$

Therefore, $\mathrm{Fun}^{\square}(\mathsf{SubComp}^{\simeq *}_\bullet, A)$ *upgrades to a simplicial abelian group with degeneracy maps*

$$s_i^{\square} := \iota_! \circ \mathrm{Fun}(s^i, A) \circ \iota^* : \mathrm{Fun}^{\square}(\mathsf{SubComp}^{\simeq *}_\bullet, A) \to \mathrm{Fun}^{\square}(\mathsf{SubComp}^{\simeq *}_{\bullet+1}, A).$$

Proof. We begin with the definition of the map $\iota_!$. For $\tau \in \mathrm{Fun}(\mathsf{Face}_p, A)$, let us first define $\iota_!\tau(L, K)$ for any inclusion of sub-complexes $K \subset L \subset \Delta^p$ in $\mathsf{SubComp}^{\simeq *}_p$. Setting

$$\iota_!\tau(L, K) := \iota_!\tau(\Delta^p, K) - \iota_!\tau(\Delta^p, L), \tag{1.3.8}$$

it suffices to specify $\iota_!\tau(\Delta^p, K)$ for any $K \in \mathsf{SubComp}^{\simeq *}_p$. Note that, as defined in (1.3.8), $\iota_!\tau$ is immediately a functor $\mathsf{SubComp}^{\simeq *}_p \to A$ (even for arbitrary values of $\iota_!\tau(\Delta^p, K)$ for $K \subsetneq \Delta^p$), since if $K \subset L \subset M$ are sub-complexes of Δ^p, then

$$\begin{aligned}\iota_!\tau(M, K) &= \iota_!\tau(\Delta^p, K) - \iota_!\tau(\Delta^p, M)\\ &= \iota_!\tau(\Delta^p, K) - \iota_!\tau(\Delta^p, L) + \iota_!\tau(\Delta^p, L) - \iota_!\tau(\Delta^p, M)\\ &= \iota_!\tau(L, K) + \iota_!\tau(M, L).\end{aligned}$$

We now define $\iota_!\tau(\Delta^p, K)$ for *any*[3] sub-complex K of Δ^p as

$$\iota_!\tau(\Delta^p, K) := \sum_{\sigma \in K} T(\Delta^p, \sigma), \quad \text{where} \quad T(\Delta^p, \sigma) := \sum_{\emptyset \neq \xi \subseteq \sigma} (-1)^{\dim\sigma + \dim\xi}\tau(\Delta^p, \xi). \tag{1.3.9}$$

Here $\dim\sigma$ denotes the dimension of the face σ (one less than the cardinality of σ viewed as a subset of $[p]$). We now establish the following properties of the functor $\iota_!\tau$:

---

[3] Not only contractible ones!

(i) $\iota_!\tau$ satisfies the inclusion-exclusion principle (1.3.7) and hence, by Lemma 1.3.12, $\iota_!\tau$ restricted to $\mathsf{SubComp}_p^{\simeq *}$ is an element of $\mathrm{Fun}^\square(\mathsf{SubComp}_p^{\simeq *}, A)$.

(ii) $\iota^*\iota_!\tau \equiv \tau$ as functors $\mathsf{Face}_p \to A$.

(iii) For every $0 \leq i \leq p$, $\partial_i(\iota_!\tau) \equiv \iota_!(\partial_i\tau)$ as functors $\mathsf{SubComp}_{p-1}^{\simeq *} \to A$.

To establish property (i), note that, in light of (1.3.8), it suffices to show that $\iota_!\tau$ satisfies the inclusion-exclusion for diagrams (1.3.6) where $L = \Delta^p$. This follows by a simple check:

$$\begin{aligned}
\iota_!\tau(\Delta^p, K) &= \sum_{\sigma\in K} T(\Delta^p,\sigma) \\
&= \sum_{\sigma\in K_0} T(\Delta^p,\sigma) + \sum_{\sigma\in K\setminus K_0} T(\Delta^p,\sigma) \\
&= \sum_{\sigma\in K_0} T(\Delta^p,\sigma) + \sum_{\sigma\in K_1\setminus K_{01}} T(\Delta^p,\sigma) \\
&= \sum_{\sigma\in K_0} T(\Delta^p,\sigma) + \sum_{\sigma\in K_1} T(\Delta^p,\sigma) - \sum_{\sigma\in K_{01}} T(\Delta^p,\sigma) \\
&= \iota_!\tau(\Delta^p, K_0) + \iota_!\tau(\Delta^p, K_1) - \iota_!\tau(\Delta^p, K_{01}).
\end{aligned}$$

To prove (ii), it suffices yet again to show that $\iota_!\tau(\Delta^p,\sigma) = \tau(\Delta^p,\sigma)$ for every face $\sigma \in \mathsf{Face}_p$. Given a linear combination X of $\tau(\Delta^p,\xi)$'s for $\xi \in \mathsf{Face}_p$, let $X_{(\eta)}$ denote the coefficient of $\tau(\Delta^p,\eta)$ in X. We now count such coefficient $\iota_!\tau(\Delta^p,\sigma)_{(\eta)}$ for any face $\eta \in \mathsf{Face}_p$. Clearly $\iota_!\tau(\Delta^p,\sigma)_{(\eta)} = 0$ if η is not contained in σ, so assume that $\eta \subset \sigma$. Then,

$$\begin{aligned}
\iota_!\tau(\Delta^p,\sigma)_{(\eta)} &= \sum_{\xi\in\sigma} T(\Delta^p,\xi)_{(\eta)} = \sum_{\eta\subset\xi\in\sigma} T(\Delta^p,\xi)_{(\eta)} = \sum_{\eta\subset\xi\in\sigma} (-1)^{\dim\xi+\dim\eta} \\
&= \sum_{i=0}^{\dim\sigma-\dim\eta} (-1)^i \binom{\dim\sigma-\dim\eta}{i} = \begin{cases} 1, & \dim\eta = \dim\sigma, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $\dim\eta = \dim\sigma$ if and only if $\eta = \sigma$, it follows that $\iota_!\tau(\Delta^p,\sigma) = \tau(\Delta^p,\sigma)$, as claimed.

For (iii), again we only need to show that both functors $\partial_i(\iota_!\tau)$ and $\iota_!(\partial_i\tau)$ agree on arrows in $\mathsf{SubComp}_{p-1}^{\simeq *}$ of the form $K \to \Delta^{p-1}$. Then,

$$\begin{aligned}
\iota_!(\partial_i\tau)(\Delta^{p-1}, K) &= \sum_{\sigma\in K}\sum_{\emptyset\neq\xi\subset\sigma}(-1)^{\dim\sigma+\dim\xi}\partial_i\tau(\Delta^{p-1},\xi)\\
&= \sum_{\sigma\in K}\sum_{\emptyset\neq\xi\subset\sigma}(-1)^{\dim\sigma+\dim\xi}\tau(\partial^i\Delta^{p-1},\partial^i\xi)\\
&= \sum_{\sigma\in K}\sum_{\emptyset\neq\xi\subset\sigma}(-1)^{\dim\sigma+\dim\xi}\Big(\tau(\Delta^p,\partial^i\xi)-\tau(\Delta^p,\partial^i\Delta^{p-1})\Big)\\
&= \iota_!\tau(\Delta^p,\partial^i K) - \left(\sum_{\sigma\in K}\sum_{\emptyset\neq\xi\subset\sigma}(-1)^{\dim\sigma+\dim\xi}\right)\cdot\tau(\Delta^p,\partial^i\Delta^{p-1})
\end{aligned}$$

Now by a similar consideration as in (ii), the sum $\sum_{\emptyset\neq\xi\subset\sigma}(-1)^{\dim\sigma+\dim\xi}$ is $(-1)^{\dim\sigma}$ (as we are not summing over $\xi=\emptyset$). Thus, using this and property (ii), we see that

$$\iota_!(\partial_i\tau)(\Delta^{p-1},K) = \iota_!\tau(\Delta^p,\partial^i K) - \chi_K\cdot\iota_!\tau(\Delta^p,\partial^i\Delta^{p-1}), \tag{1.3.10}$$

where χ_K stands for the Euler characteristic of K. As K *is contractible* (so $\chi_K = 1$), we obtain

$$\iota_!(\partial_i\tau)(\Delta^{p-1},K) = \iota_!\tau(\partial^i\Delta^{p-1},\partial^i K) = \partial_i(\iota_!\tau)(\Delta^{p-1},K),$$

as desired.

By definition, $\iota_! : \mathrm{Fun}(\mathsf{Face}_p, A) \to \mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$ is clearly a group homomorphism, so, by (iii), we have successfully constructed a morphism of semi-simplicial abelian groups

$$\iota_! : \mathrm{Fun}(\mathsf{Face}_\bullet, A) \longrightarrow \mathrm{Fun}^{\square}(\mathsf{SubComp}_\bullet^{\simeq *}, A),$$

and by (ii), we also have that $\iota_!$ is a right inverse of ι^*. To finish the proof of the proposition, it only remains to show that ι^* is injective.

To this end, suppose that $\tau \in \mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$ is a functor such that $\iota^*\tau \equiv 0$. We show that $\tau(\Delta^p, K) = 0$ for every $K \in \mathsf{SubComp}_p^{\simeq *}$ (and hence $\tau \equiv 0$), by induction on the *dimension* of K, that is, the maximal dimension of a face of Δ^p contained in K. This is clear if $\dim K = 0$, so assume the claim holds for every $K' \in \mathsf{SubComp}_p^{\simeq *}$ of dimension $\leq j-1$, and let $K \in \mathsf{SubComp}_p^{\simeq *}$ be of dimension $j \geq 1$. Suppose that there exists a j-dimensional face $\sigma \subset K$ and some $0 \leq i \leq j$ such that $\partial_i\sigma$ is not contained in any other j-dimensional face σ' of K. Then, consider the sub-complex

$K' := K \setminus (\operatorname{Int}\sigma \cup \operatorname{Int}\partial_i\sigma)$ and the diagram

$$\begin{tikzcd}
\Lambda_i(\sigma) \arrow[r] \arrow[d] & K' \arrow[d] \arrow[ddr, bend left] & \\
\sigma \arrow[r] \arrow[drr, bend right] & K \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] & \\
 & & \Delta^p.
\end{tikzcd}$$

Since $\Lambda_i(\sigma) \to \sigma$ is an equivalence and the square is a pushout, it follows that $K' \in \mathsf{SubComp}_p^{\simeq *}$. Then, by induction on the number of j-dimensional faces of K, we may assume that $\tau(\Delta^p, K') = 0$, and hence, by the inclusion-exclusion principle (1.3.7) for the above diagram,

$$\tau(\Delta^p, K) = \tau(\Delta^p, \sigma) + \tau(\Delta^p, K') - \tau(\Delta^p, \Lambda_i(\sigma)) = 0 + 0 - 0 = 0.$$

The penultimate equality follows from our assumption $\iota^*\tau \equiv 0$ and from our induction hypothesis as $\Lambda_i(\sigma)$ is $(j-1)$-dimensional. But such σ and $0 \leq i \leq j$ must always exist. For if it did not, then

$$M := \bigcup_{\sigma \in K\colon\ \dim\sigma = j} \sigma$$

would be a non-empty, closed PL-manifold of dimension j, and hence $H_j(M; \mathbb{Z}/2) \neq 0$. But since K is obtained from M by attaching faces of dimension $\leq j-1$, then $H_j(M; \mathbb{Z}/2) \hookrightarrow H_j(K; \mathbb{Z}/2)$, leading to a contradiction as K is contractible. This concludes the proof of the proposition. □

Remark 1.3.15. As part of the proof, we see that, for each $p \geq 0$, the group homomorphism $\iota_! : \mathrm{Fun}(\mathsf{Face}_p, A) \to \mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$ actually factors through $\mathrm{Fun}^{\square}(\mathsf{SubComp}_p, A)$. However,

$$\iota_! : \mathrm{Fun}(\mathsf{Face}_\bullet, A) \longrightarrow \mathrm{Fun}^{\square}(\mathsf{SubComp}_\bullet, A)$$

does *not* assemble to a morphism of semi-simplicial abelian groups by (1.3.10); surprisingly, though, if $\mathsf{SubComp}_\bullet^{\chi=1}$ denotes the sub-poset of those sub-complexes with Euler characteristic one, then

$$\iota_! : \mathrm{Fun}(\mathsf{Face}_\bullet, A) \longrightarrow \mathrm{Fun}^{\square}(\mathsf{SubComp}_\bullet^{\chi=1}, A)$$

is indeed a semi-simplicial isomorphism that factors the one of Proposition 1.3.14.

A functor $\tau \in \mathrm{Fun}(\mathsf{SubComp}_p^{\simeq *}, A)$ will be said to satisfy *face-horn duality* for σ if

$$\tau(\sigma, \partial_i \sigma) = (-1)^{\dim \sigma} \tau^*(\sigma, \Lambda_i(\sigma)), \quad i = 0, \dots, \dim \sigma. \tag{1.3.11}$$

Here $\tau^*(L,K)$ stands for $(\tau(L,K))^*$. Write $D_p(A) \subset \mathrm{Fun}(\mathsf{SubComp}_p^{\simeq *}, A)$ for the subgroup of functors that satisfy face-horn duality for every face $\sigma \subset \Delta^p$, and let $D_\bullet(A) \subset \mathrm{Fun}(\mathsf{SubComp}_\bullet^{\simeq *}, A)$ denote the corresponding semi-simplicial abelian group.

Remark 1.3.16. If $\tau \subset D_p(A) \cap \mathrm{Fun}^\square(\mathsf{SubComp}_p^{\simeq *}, A)$, then τ satisfies more general sorts of dualities. For instance, if $\sigma \subset \Delta^p$ is a face and $0 \leq i < j \leq \dim \sigma$, then

$$\tau(\sigma, \partial_i \sigma \cup \partial_j \sigma) = (-1)^{\dim \sigma} \tau^*(\sigma, \partial \sigma \setminus \mathrm{Int}(\partial_i \sigma \cup \partial_j \sigma)).$$

This follows from the inclusion-exclusion principle (1.3.7) of Lemma 1.3.12 applied to $K = \partial_i \sigma \cup \partial_j \sigma$ and $\Lambda_i(\sigma) = (\partial \sigma \setminus \mathrm{Int}(\partial_i \sigma \cup \partial_j \sigma)) \cup \partial_j \sigma$, and $L = \sigma$. By induction, one can generalise this duality to any proper collection of faces $\partial_I \sigma := \bigcup_{i \in I} \partial_i \sigma$, $I \subsetneq \{0, \dots, \dim \sigma\}$:

$$\tau(\sigma, \partial_I \sigma) = (-1)^{\dim \sigma} \tau^*(\sigma, \partial_J \sigma), \quad J := \{0, \dots, \dim \sigma\} \setminus I. \tag{1.3.12}$$

Even more generally, if $K \in \mathsf{SubComp}_p^{\simeq *}$ is a union of k-dimensional faces and $Q \subset \partial K$ is a contractible sub-complex which is a union of $(k-1)$-dimensional faces, then

$$\tau(K, Q) = (-1)^k \tau^*(K, \overline{\partial K \setminus Q}). \tag{1.3.13}$$

We now give a simple inductive criterion to check if a functor satisfies all face-horn dualities.

Lemma 1.3.17. *Let $\tau \in \mathrm{Fun}^\square(\mathsf{SubComp}_p^{\simeq *}, A)$ satisfy face-horn duality for all $\sigma \subsetneq \Delta^p$ and for the 0-th face-horn of Δ^p:*

$$\tau(\Delta^p, \partial_0 \Delta^p) = (-1)^p \tau^*(\Delta^p, \Lambda_0^p).$$

Then τ satisfies face-horn duality for Δ^p too, i.e., $\tau \in D_p(A)$.

Proof. For $i \in \{1,\ldots,p\}$ denote Λ^p_{0i} for $\partial_{\{1,\ldots,\widehat{i},\ldots,p\}}\Delta^p$, and consider the two pushout diagrams in $\mathsf{SubComp}_p^{\simeq *}$

$$\begin{tikzcd}
\Lambda_0(\partial_i\Delta^p) \arrow[r] \arrow[d] & \Lambda^p_{0i} \arrow[d] \arrow[ddr, bend left] & \\
\partial_i\Delta^p \arrow[r] \arrow[drr, bend right] & \Lambda^p_0 \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & \Delta^p,
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda_{i-1}(\partial_0\Delta^p) \arrow[r] \arrow[d] & \partial_0\Delta^p \arrow[d] \arrow[ddr, bend left] & \\
\Lambda^p_{0i} \arrow[r] \arrow[drr, bend right] & \Lambda^p_i \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & \Delta^p.
\end{tikzcd}$$

Note that $\Lambda_{i-1}(\partial_0\Delta^p)$ is the union of the codimension one sub-faces of $\partial_0\Delta^p = \langle 1,\ldots,p\rangle \subset \Delta^p$ that contain the i-th vertex of Δ^p. We check directly that τ satisfies duality for the i-th face-horn using the inclusion-exclusion principle (1.3.7).

$$\begin{aligned}
\tau(\Delta^p,\partial_i\Delta^p) &= \tau(\Delta^p,\Lambda^p_0) + \tau(\Lambda^p_0,\partial_i\Delta^p)\\
&= (-1)^p\tau^*(\Delta^p,\partial_0\Delta^p) + \tau(\Lambda^p_{0i},\Lambda_0(\partial_i\Delta^p))\\
&= (-1)^p\tau^*(\Delta^p,\Lambda^p_i) + (-1)^p\tau^*(\Lambda^p_i,\partial_0\Delta^p) + (-1)^{p-1}\tau^*(\Lambda^p_{0i},\Lambda_{i-1}(\partial_0\Delta^p))\\
&= (-1)^p\tau^*(\Delta^p,\Lambda^p_i).
\end{aligned}$$

In the third line we have used (1.3.13) for $K = \Lambda^p_{0i}$ and $Q = \Lambda_0(\partial_i\Delta^p)$. $\square$

Finally, we write $Z_p(A) \subset \mathrm{Fun}(\mathsf{SubComp}_{p+1}^{\simeq *}, A)$ for the subgroup of functors τ such that

$$\tau(L,K) = 0, \quad \forall\, K \subset L \subset \partial_0\Delta^{p+1} = \langle 1,\ldots,p+1\rangle.$$

The assignment $[p] \mapsto Z_p(A)$ defines a semi-simplicial abelian group $Z_\bullet(A)$ whose i-th face map is the restriction of $\partial_{i+1}^{\mathrm{Fun}}$ to $Z_p(A)$.

Definition 1.3.18. *The simplicial abelian group $F_\bullet^{\mathsf{alg}}(A) \subset \mathrm{Fun}(\mathsf{SubComp}_{\bullet+1}^{\simeq *}, A)$ has as p-simplices*

$$F_p^{\mathsf{alg}}(A) := Z_p(A) \cap D_{p+1}(A) \cap \mathrm{Fun}^{\square}(\mathsf{SubComp}_{p+1}^{\simeq *}, A),$$

as face maps $\delta_i : F_p^{\mathsf{alg}}(A) \to F_{p-1}^{\mathsf{alg}}(A)$ the restriction to $F_p^{\mathsf{alg}}(A)$ of $\partial_{i+1}^{\mathrm{Fun}}$, and as degeneracy maps $s_i : F_p^{\mathsf{alg}}(A) \to F_{p+1}^{\mathsf{alg}}(A)$ the restriction to $F_p^{\mathsf{alg}}(A)$ of the map $s_{i+1}^{\square}$ from Proposition 1.3.14.

Lemma 1.3.19. *$F_\bullet^{\mathsf{alg}}(A)$ as defined above is a simplicial abelian group.*

Proof. The only non-trivial thing to check is that s_i sends $D_p(A)$ into $D_{p+1}(A)$, so let $\tau \in D_p(A) \cap \mathrm{Fun}^{\square}(\mathsf{SubComp}_p^{\simeq *}, A)$. Without loss of generality assume $i = 0$, and by the induction hypothesis and the simplicial identities, it suffices to check that $s_0\tau$ satisfies face-horn duality for the top face Δ^{p+1}. By Lemma 1.3.17, just checking this for the 0-th face-horn of Δ^{p+1} will suffice. Since $s_0\tau$ satisfies inclusion-exclusion,

$$\begin{aligned}
s_0\tau(\Delta^{p+1}, \Lambda_0^{p+1}) &= \sum_{k=1}^{p+1}(-1)^{k-1} \sum_{0<j_1<\cdots<j_k\leq p+1} s_0\tau\left(\Delta^{p+1}, \bigcap_{r=1}^{k} \partial_{j_r}\Delta^{p+1}\right) \\
&= \sum_{k=1}^{p+1}(-1)^{k-1}\Bigg\{ \sum_{1<j_1<\cdots<j_k} \tau(\Delta^p, \langle 0, \ldots, \widehat{j_1-1}, \ldots, \widehat{j_k-1}, \ldots, p+1\rangle) \\
&\qquad\qquad + \sum_{1=j_1<j_2<\cdots<j_k} \tau(\Delta^p, \langle 0, \ldots, \widehat{j_2-1}, \ldots, \widehat{j_k-1}, \ldots, p+1\rangle)\Bigg\} \\
&= (-1)^p \underbrace{\sum_{1<j_1<\cdots<j_{p+1}\leq p+1}}_{=\emptyset} \tau(\Delta^p, \langle 0, \ldots, \widehat{j_1-1}, \ldots, \widehat{j_k-1}, \ldots, p+1\rangle) = 0.
\end{aligned}$$

In the other hand, $s_0\tau(\Delta^{p+1}, \partial_0\Delta^{p+1}) = \tau(\Delta^p, \Delta^p) = 0 = (-1)^{p+1}(s_0\tau(\Delta^{p+1}, \Lambda_0^{p+1}))^*$, as required. ☐

1.3.1.2 Proof of Theorem 1.3.10

Recall that the *Dold–Kan correspondence* [GJ99, §III.2, Cor. 2.3] establishes an equivalence of categories

$$N : \mathsf{sAb} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Ch}_{\geq 0}(\mathbb{Z}) : \Gamma, \tag{1.3.14}$$

where N is the *normalised Moore complex* functor, given for a simplicial group $G = (G_\bullet, \delta_\bullet)$ by

$$(NG)_n := \bigcap_{i=1}^{n} \ker\left(\delta_i : G_n \to G_{n-1}\right), \quad d_n = \delta_0 \mid_{(NG)_n} : (NG)_n \longrightarrow (NG)_{n-1}.$$

Under (1.3.14), we will identify $F_\bullet^{\mathsf{alg}}(A)$ with the (connective) chain complex $\mathbb{A}_{hC_2}$ given by

$$\ldots \xrightarrow{1-t} A \xrightarrow{1+t} A \xrightarrow{1-t} A \longrightarrow 0 = (\mathbb{A}_{hC_2})_{-1}.$$

Proposition 1.3.20. *The map* $\psi_\bullet^A : (NF^{\mathsf{alg}}(A)_\bullet, d_\bullet) \longrightarrow \mathbb{A}_{hC_2}$ *given by*

$$\psi_n^A : NF^{\mathsf{alg}}(A)_n \longrightarrow (\mathbb{A}_{hC_2})_n = A, \quad \tau \longmapsto \tau(\Delta^{n+1}, \langle 0\rangle), \tag{1.3.15}$$

is a quasi-isomorphism of chain complexes. In particular,

$$\pi_n(F^{\mathsf{alg}}_\bullet(A)) \cong H_n(NF^{\mathsf{alg}}(A)) \cong H_n(C_2;A) = \begin{cases} \frac{A}{\{b-b^* \mid b\in A\}}, & n=0,\\ \frac{\{a\in A \mid a=(-1)^{n+1}a^*\}}{\{b+(-1)^{n+1}b^* \mid b\in A\}}, & n\geq 1. \end{cases}$$

Proof. First we verify that $\psi_\bullet = \psi^A_\bullet$ is a chain map. Let $\tau \in NF^{\mathsf{alg}}(A)_n$ and write

$$a := \psi_n(\tau) = \tau(\Delta^{n+1}, \langle 0\rangle), \quad b := \psi_{n-1}(d_n\tau) = \tau(\partial_1\Delta^{n+1}, \langle 0\rangle), \quad c := \tau(\Delta^{n+1}, \partial_1\Delta^{n+1}).$$

Noting that $\tau(\Lambda^{n+1}_1, \langle 0\rangle) = 0$ by inclusion-exclusion, applying τ to the diagram in $\mathsf{SubComp}^{\simeq *}_{n+1}$

$$\begin{tikzcd}
\langle 0\rangle \arrow[r] \arrow[d] & \Lambda^{n+1}_1 \arrow[d] \\
\partial_1\Delta^{n+1} \arrow[r] & \Delta^{n+1}
\end{tikzcd}$$

and duality of τ, we obtain

$$(-1)^{n+1}c^* = a = b + c \implies a + (-1)^n a^* = b + c + (-1)^n \left((-1)^{n+1}c^*\right)^* = b,$$

i.e., $d_n(\psi_n(\tau)) = \psi_{n-1}(d_n\tau)$.

We now have to show that the map

$$\psi_* : H_n(NF^{\mathsf{alg}}(A)) \longrightarrow H_n(C_2;A)$$

is an isomorphism for $n \geq 0$.

Claim. *For $n > 0$, there is a bijection*

$$\tau_{(-)} : \{a \in A \mid a = (-1)^{n+1}a^*\} \longleftrightarrow \bigcap_{i=0}^{n} \ker(\delta_i : F^{\mathsf{alg}}_n(A) \longrightarrow F^{\mathsf{alg}}_{n-1}(A)) : \psi_n,$$
$$a \longmapsto \tau_a$$
$$\tau(\Delta^{n+1}, \langle 0\rangle) \longleftarrow\!\shortmid \tau$$

where $\tau_a \in F^{\mathsf{alg}}_n(A)$ is the functor given by

$$\tau_a(L,K) = \begin{cases} a, & \textit{if } K \subsetneq L = \Delta^{n+1},\\ 0, & \textit{otherwise.} \end{cases}$$

Proof of Claim. Note that the condition $a = (-1)^{n+1}a^*$ is exactly the face-horn duality for Δ^{n+1}, so τ_a is indeed an element of $F_n^{\mathsf{alg}}(A)$. Also observe that $\psi_n(\tau_a) = a$, so we only need to show that $\tau_{(-)}$ is surjective. Let τ be a cycle in $F_n^{\mathsf{alg}}(A)$, and set $a := \tau(\Delta^{n+1}, \langle 0\rangle)$; we check that $\tau = \tau_a$. By the functoriality relation $\tau(L, K) = \tau(\Delta^{n+1}, K) - \tau(\Delta^{n+1}, L)$, we may assume that $L = \Delta^{n+1}$, and by Proposition 1.3.14 that $K = \sigma \in \mathsf{Face}_{n+1}$. Let i be such that $\sigma \subset \partial_i\Delta^{n+1}$. As $\partial_i\tau = 0$,

$$\tau(\Delta^{n+1}, \sigma) = \tau(\Delta^{n+1}, \partial_i\Delta^{n+1}) + \tau(\partial_i\Delta^{n+1}, \sigma) = \tau(\Delta^{n+1}, \partial_i\Delta^{n+1}),$$

so it is enough to show that $\tau(\Delta^{n+1}, \partial_i\Delta^{n+1}) = a$ for $i = 0, \ldots, n+1$. If $i \neq 0$, this follows from the definition of $a := \tau(\Delta^{n+1}, \langle 0\rangle)$. Applying τ to the diagram in $\mathsf{SubComp}_{n+1}^{\simeq *}$

$$\begin{tikzcd}
\langle 2, \ldots, n+1\rangle \arrow[r] \arrow[d] & \partial_1\Delta^{n+1} \arrow[d] \\
\partial_0\Delta^{n+1} \arrow[r] & \Delta^{n+1}
\end{tikzcd}$$

and noting that $\partial_0\tau = 0$, we obtain $\tau(\Delta^{n+1}, \partial_0\Delta^{n+1}) = a$, as required. Also by definition τ_a is a cycle in $F_\bullet^{\mathsf{alg}}(A)$ and $\tau_a(\Delta^{n+1}, \langle 0\rangle) = a$, so the claim follows. □

The previous claim shows that ψ_* is surjective when $n > 0$. But this is also the case when $n = 0$, as

$$\psi_0 : NF_0^{\mathsf{alg}}(A) = F_0^{\mathsf{alg}}(A) \longrightarrow A,$$

is an isomorphism: namely, for $a \in A$, the functor $\tau : \mathsf{SubComp}_1^{\simeq *} \to A$ given by $\tau(\Delta^1, \langle 0\rangle) := a$ and $\tau(\Delta^1, \langle 1\rangle) := -a^*$ (and zero otherwise) is clearly in $F_0^{\mathsf{alg}}(A)$ and sent to a under ψ_0. Conversely, if $\tau \in F^{\mathsf{alg}}(A)_0$ and $\tau(\Delta^1, \langle 0\rangle) = 0$, duality then forces τ to be zero itself.

For injectivity of ψ_*, let $\tau \in NF^{\mathsf{alg}}(A)_n$ be a cycle such that $\psi_*[\tau] = 0$, i.e., $\tau(\Delta^{n+1}, \langle 0\rangle) = b + (-1)^{n+1}b^*$ for some $b \in A$. It is not difficult to see that there exists a functor $T \in \mathrm{Fun}^{\square}(\mathsf{SubComp}_{n+2}^{\simeq *}, A)$ with

$$\partial_i T = \begin{cases} \tau, & i = 1, \\ 0, & i \neq 1, \end{cases} \qquad T(\Delta^{n+2}, \partial_i\Delta^{n+2}) := \begin{cases} -b, & i = 1, \\ (-1)^{n+1}b^*, & i \neq 1, \end{cases} \qquad (0 \leq i \leq n+2).$$

By construction, T satisfies face-horn duality for any face $\sigma \neq \Delta^{n+1}$ and for the first face-horn $(\partial_1\Delta^{n+2}, \Lambda_1^{n+2})$. Therefore by Lemma 1.3.17 it satisfies all face-horn dualities. Then T is clearly an element of $NF^{\mathsf{alg}}(A)_{n+1}$ bounding τ, so $[\tau] = 0$ in $H_n(NF^{\mathsf{alg}}(A)_\bullet)$. This finishes the proof. □

Remark 1.3.21. It is not difficult to see that $\psi^A_\bullet : NF^{\mathsf{alg}}(A) \xrightarrow{\cong} \mathbb{A}_{hC_2}$ is in fact an isomorphism of chain complexes. An element $\tau \in NF^{\mathsf{alg}}(A)_n$ is completely determined by $\delta_0\tau (= \partial_1\tau)$ and $b := \tau(\Delta^{n+1}, \langle 0\rangle) \in A$ using functoriality and duality. By the claim in the proof of Proposition 1.3.20, $\delta_0\tau = \tau_a$ for some $a \in A$ with $a = (-1)^n a^*$. Face-horn duality for $(\partial_1\Delta^{n+1}, \Lambda^{n+1}_1)$ yields

$$a + (-1)^{n+1} b^* = b \implies a = b + (-1)^n b^*,$$

so $\delta_0\tau = \tau_a$ is completely determined by $b = \tau(\Delta^{n+1}, \langle 0\rangle)$. As we will not need this fact, we leave it to the reader to check that $\psi^A_\bullet$ is indeed surjective.

Before moving on to the proof of Theorem 1.3.10, we still need some categorical background. For the rest of the section, we shall adopt the conventions of [Sch99]—for instance, our category Ω^∞-Top of infinite loop spaces (a.k.a. connective spectra) is modelled by the model category of Γ-spaces thereof. For $G = \{e\}$ or C_2 and C a category, the category of G-objects in C is $\mathsf{C}^G := \mathrm{Fun}(G, \mathsf{C})$. Observe that there are natural isomorphisms of categories $\mathsf{Mod}_{\mathbb{Z}[G]} \cong \mathsf{Ab}^G$ and $H\mathbb{Z}[G]$-$\mathsf{Mod} \cong (H\mathbb{Z}$-$\mathsf{Mod})^G$. There is an inclusion of categories $\mathsf{Mod}_{\mathbb{Z}[G]} \hookrightarrow \mathsf{sMod}_{\mathbb{Z}[G]}$ sending a $\mathbb{Z}[G]$-module M to the *constant* simplicial $\mathbb{Z}[G]$-module on M, denoted by $\underline{M} = \underline{M}_\bullet$. By [Sch99, p. 332], the Eilenberg–MacLane functor $H : \mathsf{Mod}_{\mathbb{Z}[G]} \to H\mathbb{Z}[G]$-$\mathsf{Mod}$ upgrades to a functor

$$H : \mathsf{sMod}_{\mathbb{Z}[G]} \longrightarrow H\mathbb{Z}[G]\text{-}\mathsf{Mod}$$

such that, for $M_\bullet \in \mathsf{sMod}_{\mathbb{Z}[G]}$, the underlying infinite loop space of $HM_\bullet$ is the realisation $|M_\bullet|$.

Given a simplicial $\mathbb{Z}[C_2]$-module $M_\bullet$, we will write $(M_\bullet)_{hC_2}$ for the simplicial abelian group

$$(M_\bullet)_{hC_2} := \mathrm{Diag}(M_\bullet \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[E_\bullet C_2]) : [p] \longmapsto M_p \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}[C_2^{\times(p+1)}].$$

Now, given a C_2-spectrum X, we will write X_{hC_2} for the spectrum $X \wedge_{C_2} (EC_2)_+$. The following result says that both meanings of $(-)_{hC_2}$ are intertwined by the Eilenberg–MacLane functor.

Lemma 1.3.22. *For $M_\bullet \in \mathsf{sMod}_{\mathbb{Z}[C_2]}$, there is a natural equivalence of spectra*

$$(HM_\bullet)_{hC_2} \xrightarrow{\simeq} H((M_\bullet)_{hC_2}).$$

Proof. According to [Sch99, Lem. 4.2], there is a natural equivalence

$$HM_\bullet \wedge_{H\mathbb{Z}[C_2]} H(\mathbb{Z}[E_\bullet C_2]) \xrightarrow{\simeq} H((M_\bullet)_{hC_2}),$$

where we are using that $H(\mathbb{Z}[E_\bullet C_2])$ is a cofibrant $H\mathbb{Z}[C_2]$-module. But for a simplicial set $X_\bullet$, $H(\mathbb{Z}[X_\bullet])$ is the free $H\mathbb{Z}$-module on $X_\bullet$, i.e., it is (equivalent to) $H\mathbb{Z} \wedge |X_\bullet|_+$. Thus

$$\begin{aligned} HM_\bullet \wedge_{H\mathbb{Z}[C_2]} H(\mathbb{Z}[E_\bullet C_2]) &\cong \operatorname{colim}_{C_2} HM_\bullet \wedge_{H\mathbb{Z}} H(\mathbb{Z}[E_\bullet C_2]) \\ &\simeq \operatorname{colim}_{C_2} HM_\bullet \wedge_{H\mathbb{Z}} (H\mathbb{Z} \wedge (EC_2)_+) \\ &\simeq HM_\bullet \wedge (EC_2)_+ \\ &=: (HM_\bullet)_{hC_2}. \end{aligned}$$

In the second and third equivalences, we are implicitly using that the C_2-spectra we are taking the colimit of are free, thus the colimit coincides with the homotopy colimit. □

Finally, we will denote $C : \mathsf{sAb} \to \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ for the functor sending a simplicial abelian group $(M_\bullet, \delta_\bullet)$ to the chain complex $(CM_\bullet, d_\bullet)$

$$CM_n := M_n, \quad d_n = \sum_{i=0}^{n} (-1)^i \delta_i : M_n \longrightarrow M_{n-1}.$$

The normalised chain complex $NM_\bullet$ is a sub-complex of $CM_\bullet$, and in fact the inclusion $NM_\bullet \hookrightarrow CM_\bullet$ is a split quasi-isomorphism [GJ99, §III.2, Thm. 2.1 & Thm. 2.4].

Proof of Theorem 1.3.10. Our goal is to identify $F_\bullet^{\mathsf{alg}}(A)$ with $\underline{A}_{hC_2}$, functorially in $A \in \mathsf{Mod}_{\mathbb{Z}[C_2]}$. To do so, we first compare $N\underline{A}_{hC_2}$ and $\mathbb{A}_{hC_2}$.

Write $M_\bullet \overset{\epsilon}{\twoheadrightarrow} \mathbb{Z}$ for the minimal resolution of $\mathbb{Z}$ by free $\mathbb{Z}[C_2]$-modules

$$\ldots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{1+t} \mathbb{Z}[C_2] \xrightarrow{1-t} \mathbb{Z}[C_2] \overset{\epsilon}{\twoheadrightarrow} \mathbb{Z},$$

where ϵ sets $t = 1$. The chain complex $C(\mathbb{Z}[E_\bullet C_2])$ together with the augmentation map $\epsilon : C(\mathbb{Z}[E_\bullet C_2])_0 = \mathbb{Z}[C_2] \to \mathbb{Z}$ provides another such resolution (also known as the *canonical resolution* of $\mathbb{Z}$ by free $\mathbb{Z}[C_2]$-modules). Therefore, there is a map $C(\mathbb{Z}[E_\bullet C_2]) \to M_\bullet$ of resolutions of $\mathbb{Z}$ which, upon applying $A \otimes_{\mathbb{Z}[C_2]} (-)$, provides a quasi-isomorphism of chain complexes $C\underline{A}_{hC_2} \xrightarrow{\simeq} \mathbb{A}_{hC_2}$. We thus obtain the desired quasi-isomorphism of chain complexes

$$\phi_\bullet^A : N\underline{A}_{hC_2} \xhookrightarrow{\simeq} C\underline{A}_{hC_2} \xrightarrow{\simeq} \mathbb{A}_{hC_2},$$

which is clearly functorial in A.

The Dold–Kan correspondence (1.3.14) can be upgraded to a Quillen equivalence of model categories [Qui67, §II.4] (for the projective model structure on chain complexes), so there is a zig-zag of equivalences (functorial in $A \in \mathsf{Mod}_{\mathbb{Z}[C_2]}$)

$$F^{\mathsf{alg}}_\bullet(A) \xrightarrow[\sim]{(\psi^A_\bullet)^\vee} \Gamma\mathbb{A}_{hC_2} \xleftarrow[\sim]{(\phi^A_\bullet)^\vee} \underline{A}_{hC_2}. \tag{1.3.16}$$

Each map in the zig-zag is an equivalence since Γ preserves equivalences (Γ is a right Quillen functor and every object in $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$ is fibrant) and because N and Γ are inverse to each other (as the Dold–Kan correspondence (1.3.14) is an actual equivalence of categories). Applying geometric realisation to (1.3.16) and noting Lemma 1.3.22, there results a zig-zag of infinite loop spaces

$$|F^{\mathsf{alg}}_\bullet(A)| \xrightarrow{\simeq} |\Gamma\mathbb{A}_{hC_2}| \xleftarrow{\simeq} |\underline{A}_{hC_2}| = \Omega^\infty(H(\underline{A}_{hC_2})) \xleftarrow{\simeq} \Omega^\infty((HA)_{hC_2}),$$

which once again is functorial in A. This finishes the proof. □

Remark 1.3.23. A key step in the proof is the identification of $F^{\mathsf{alg}}_\bullet(A)$ with $\Gamma\mathbb{A}_{hC_2}$ in (1.3.16), via the (homotopical) Dold–Kan correspondence. Crucially, this relies on the fact that $F^{\mathsf{alg}}_\bullet(A)$ is a simplicial abelian group (cf. Lemma 1.3.19), rather than just semi-simplicial, as we are unaware of a generalisation of Dold–Kan to the semi-simplicial setting. This hopefully clarifies the need for Proposition 1.3.14 and Lemma 1.3.19.

1.3.2 Proof of Theorem B

Let $F_\bullet(M)$ denote the simplicial homotopy fibre

$$F_\bullet(M) := \operatorname{holim}\Big(\{M^d\} \longrightarrow \widetilde{\mathcal{M}}^h_\bullet \hookleftarrow \widetilde{\mathcal{M}}^s_\bullet \Big). \tag{1.3.17}$$

It has as p-simplices

$$F_p(M) = \Big\{ W \in \widetilde{\mathcal{M}}^h_{p+1} : W_0 = M, \quad \partial_0 W \in \widetilde{\mathcal{M}}^s_p \Big\},$$

and as face maps

$$\delta_i : F_p(M) \longrightarrow F_{p-1}(M), \quad W \longmapsto \partial_{i+1} W = W_{\langle 0,\dots,\widehat{i+1},\dots,p+1\rangle}, \quad i = 0, \dots, p.$$

Recall that C_2 acts on $\mathrm{Wh}(M)$ by $t \cdot \tau := (-1)^{d-1}\overline{\tau}$. Our task is to find an equivalence $|F_\bullet(M)| \simeq \Omega^\infty(H\mathrm{Wh}(M)_{hC_2})$, and by Theorem 1.3.10 we already know that the latter space is equivalent to $|F_\bullet^{\mathsf{alg}}(\mathrm{Wh}(M))|$. Therefore, in order to show Theorem B, it suffices to establish an equivalence of semi-simplicial sets between $F_\bullet(M)$ and $F_\bullet^{\mathsf{alg}}(\mathrm{Wh}(M))$.

Let $W^{d+p+1} \in F_p(M)$. We may find some face preserving homotopy equivalence $f : W \xrightarrow{\simeq_h}_\Delta M \times \Delta^{p+1}$ with $f_0 = \mathrm{Id}_M : W_0 = M \to M$. With such a homotopy equivalence we can identify $\pi_1(W_K)$ with $\pi_1(M)$ for every subcomplex $K \in \mathsf{SubComp}_p^{\simeq *}$ so that for $K \subset L$, the identifications $\pi_1(W_K) \cong \pi_1(M)$ and $\pi_1(W_L) \cong \pi_1(M)$ are compatible with the induced isomorphism $\pi_1(W_K \hookrightarrow W_L) : \pi_1(W_K) \cong \pi_1(W_L)$. Any other choice of such a homotopy equivalence f' will not change this identification, as f and f' are homotopic rel $W_0 = M$ (to see this, note that it is equivalent to showing that a homotopy equivalence $f : M \times \Delta^p \xrightarrow{\simeq} M \times \Delta^p$ with $f_0 = \mathrm{Id}_M$ is homotopic rel $M \times \Delta^p_0$ to the identity. This in part follows from the fact that it is block homotopic rel $M \times \Delta^p_0$ to the identity by an *Alexander trick*-like argument similar to [BLR75, Lem. 2.1] and that $h\mathrm{Aut}(M)_\bullet \simeq \widetilde{h\mathrm{Aut}}(M)_\bullet$). Once we have made such an identification of fundamental groups, we can define the functor $\tau_W \in \mathrm{Fun}(\mathsf{SubComp}_p^{\simeq *}, \mathrm{Wh}(M))$

$$\tau_W(L, K) := \tau(W_K \xhookrightarrow{\simeq} W_L) \in \mathrm{Wh}(M).$$

The composition rule (1.2.1) of the Whitehead torsion and the inclusion-exclusion principle (1.2.2) guarantees that τ_W satisfies (1.3.7) and therefore, by Lemma 1.3.12, τ_W is an element of $\mathrm{Fun}^\square(\mathsf{SubComp}_p^{\simeq *}, \mathrm{Wh}(M))$. In fact, by the definition of the $F_\bullet(M)$, the functor τ_W is a p-simplex in this space (remember that τ^* in (1.3.12) should now be replaced by $(-1)^{d-1}\overline{\tau}$).

Proposition 1.3.24. *For $d = \dim M \geq 5$, there is an equivalence of semi-simplicial sets*

$$\tau_{(-)} : F_\bullet(M) \xrightarrow{\simeq} F_\bullet^{\mathsf{alg}}(\mathrm{Wh}(M)), \quad W \longmapsto \tau_W.$$

Together with Theorem 1.3.10, this will prove Theorem B. In particular, by Proposition 1.3.20,

$$\pi_n(F_\bullet(M)) \cong H_n(C_2; \mathrm{Wh}(M)) \cong \begin{cases} \dfrac{\mathrm{Wh}(M)}{\{b + (-1)^d \overline{b} \mid b \in \mathrm{Wh}(M)\}}, & n = 0, \\[2ex] \dfrac{\{a \in \mathrm{Wh}(M) \mid a = (-1)^{d+n}\overline{a}\}}{\{b + (-1)^{d+n}\overline{b} \mid b \in \mathrm{Wh}(M)\}}, & n \geq 1. \end{cases} \tag{1.3.18}$$

Proof of Proposition 1.3.24. We need to prove that $\tau_{(-)}$ induces isomorphisms in homotopy groups. To prove surjectivity, let $a \in \mathrm{Wh}(M)$ be such that $a = (-1)^{d+n}\overline{a}$

and let $W^{d+n+1} : M \times \Lambda_0^{n+1} \overset{h}{\rightsquigarrow} W_{\langle 1,\dots,n+1\rangle}$ be an h-cobordism rel boundary with $\tau(W, M \times \Lambda_0^{n+1}) = a$. The manifold W admits a stratified structure over Δ^{n+1} with 0-th horn $\Lambda_0(W) = M \times \Lambda_0^{n+1}$ and 0-th face $W_{\langle 1,\dots,n+1\rangle}$. It is not difficult to see that $\tau_W = \tau_a$ by Lemma 1.3.14 and noting that τ_W satisfies face-horn dualities for all faces (in particular Δ^{n+1}).

For injectivity, let $W \in F_n(M)$ be a cycle such that $[\tau_W] = 0$ in $\pi_n(F_\bullet^{\mathsf{alg}}(\mathrm{Wh}(M))$. By the first step in the proof of Proposition 1.3.20, $\tau_W = \tau_{b+(-1)^{d+n}\overline{b}}$ for some $b \in \mathrm{Wh}(M)$. Let $V : W \overset{h}{\rightsquigarrow} W'$ be an h-cobordism rel boundary with torsion $\tau(V, W) = -b$ (after having identified $\pi_1(W)$ with $\pi_1(M)$ appropriately. Then $\pi_1(V)$ and $\pi_1(W')$ get identified with $\pi_1(M)$ too). We claim W' is (face-preservingly) diffeomorphic to $M \times \Delta^{n+1}$, i.e., we have to show that $\tau(W', \Lambda_0(W')) = 0$ by the s-cobordism theorem. Since $\Lambda_0(W') = \Lambda_0(W) = M \times \Lambda_0^{n+1}$,

$$\tau(W, \Lambda_0(W)) + \tau(V, W) = \tau(W', \Lambda_0(W')) + \tau(V, W')$$

which, by duality, yields

$$\tau(W', \Lambda_0(W')) = b + (-1)^{d+n}\overline{b} - b - (-1)^{d+n+1}\overline{(-b)} = 0.$$

Let $\phi : W' \cong M \times \Delta^{n+1}$ be a diffeomorphism fixing $\Lambda_0(W') = M \times \Lambda_0^{n+1}$, and consider $V' := M_\phi \circ V$, where $M_\phi : W' \overset{s}{\rightsquigarrow} M \times \Delta^{n+1}$ denotes the *mapping cylinder*[4] of ϕ. Then using the canonical diffeomorphism rel boundary $\Delta^{n+1} \cong \Lambda_1^{n+2}$, the manifold V' admits a stratified structure over Δ^{n+2} with $\partial_1 V' = W$ and $\Lambda_1(V') = M \times \Lambda_1^{n+2}$. Therefore V' provides a null-homotopy of W in $F_\bullet(M)$, as desired. □

Remark 1.3.25. The semi-simplicial sets $\widetilde{\mathcal{M}}_\bullet^h$ and $\widetilde{\mathcal{M}}_\bullet^s$ admit compatible systems of degeneracies that make them into simplicial objects. Namely, the i-th degeneracy map $s_i : \widetilde{\mathcal{M}}_p^{h/s} \to \widetilde{\mathcal{M}}_{p+1}^{h/s}$ sends a p-simplex $W^{d+p} \Rightarrow \Delta^p$ to the pullback $W\ {}_{\mathrm{pr}}\!\times_{s^i} \Delta^{p+1}$, where $\mathrm{pr} : W \to \Delta^p$ is the composition $W \subset \mathbb{R}^\infty \times \Delta^p \twoheadrightarrow \Delta^p$, and $s^i : \Delta^{p+1} \to \Delta^p$ is the linear i-th codegeneracy map. The pullback $W\ {}_{\mathrm{pr}}\!\times_{s^i} \Delta^{p+1}$ is regarded as a manifold stratified over Δ^{p+1} under the inclusion

$$W\ {}_{\mathrm{pr}}\!\times_{s^i} \Delta^{p+1} \hookrightarrow \mathbb{R}^\infty \times \Delta^{p+1}, \quad ((w,x),y) \longmapsto (w,y),$$

for $(w,x) \in W \subset \mathbb{R}^\infty \times \Delta^p$ and $y \in \Delta^{p+1}$ such that $x = s^i(y)$. The semi-simplicial homotopy fibre $F_\bullet(M)$ thus inherits a simplicial structure which agrees with that

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[4] This should really be the mapping cylinder *with collars* (see Definition 1.3.1(*iii*)). Namely, given a diffeomorphism (possibly rel boundary) $\phi : A \cong B$, we define M_ϕ by $A \times [0,1/2] \cup_{\phi\times\{1/2\}} B \times [1/2,1]$.

of $F_\bullet^{\mathsf{alg}}(\mathrm{Wh}(M))$, i.e., the map $\tau_{(-)}$ of Proposition 1.3.24 becomes an equivalence of simplicial sets.

1.3.3 Relation to the Rothenberg exact sequence

The purpose of this section is to derive a consequence of Theorem B in a different direction to Theorem A. The reader may want to skip it on first reading.

For a finite group G and a naïve G-spectrum X, we will denote by X^{tG} the *Tate construction* of X [ACD89, Defn. 2.2], i.e., the homotopy cofibre of the *norm map* (cf. [WW89, Prop. 2.4]) $N : X_{hG} \to X^{hG}$, where $X_{hG} := X \wedge_G (EG)_+$ are the homotopy G-orbits of X as before, and $X^{hG} := F(\Sigma^\infty_+ EG, X)^G$ denotes the homotopy G-fixed points of X. Here $F(-,-)^G$ is the G-equivariant mapping spectrum. When $X = HA$ for some $\mathbb{Z}[G]$-module A, $\pi_*^s((HA)_{hG}) = H_*(G; A)$ whilst $\pi_*^s((HA)^{hG}) = H^{-*}(G; A)$, and the norm map in degree zero $A_G \to A^G$ is multiplication by the norm element $N = \sum_{g \in G} g \in \mathbb{Z}[G]$. Therefore when $G = C_2$ and $A = \mathrm{Wh}(M)$,

$$\widehat{H}^*(C_2; \mathrm{Wh}(M)) := \pi_*^s(H\mathrm{Wh}(M)^{tC_2}) = \begin{cases} H_{*-1}(C_2; \mathrm{Wh}(M)), & * \geq 2, \\ \frac{\left\{a \in \mathrm{Wh}(M) \ \mid \ a = (-1)^d \overline{a}\right\}}{\left\{b + (-1)^d \overline{b} \ \mid \ b \in \mathrm{Wh}(M)\right\}} \subset H_0(C_2; \mathrm{Wh}(M)), & * = 1, \\ \frac{\left\{a \in \mathrm{Wh}(M) \ \mid \ a = (-1)^{d-1} \overline{a}\right\}}{\left\{b + (-1)^{d-1} \overline{b} \ \mid \ b \in \mathrm{Wh}(M)\right\}} \subset H^0(C_2; \mathrm{Wh}(M)), & * = 0, \\ H^{-*}(C_2; \mathrm{Wh}(M)), & * \leq -1. \end{cases}$$

Let $\mathbb{L}^{h/s}(M)$ denote the quadratic ordinary/simple *L-theory spectrum* of M^d [Ran92, §13]. In this section we establish a spacified version of the positive-degree part of the *Rothenberg exact sequence* for quadratic L-theory [Ran81, Prop. 1.10.1].

Proposition 1.3.26. *For $d \geq 5$, there is an equivalence of spaces*

$$\Omega^{\infty+d+1} \operatorname{hofib}\left(\mathbb{L}^s(M) \to \mathbb{L}^h(M)\right) \simeq \Omega^{\infty+1}(H\mathrm{Wh}(M)^{tC_2}). \tag{1.3.19}$$

Proof. Let $\widetilde{\mathcal{S}}_\bullet^{h/s}(M)$ denote the ordinary/simple *block structure spaces* of M [Qui70]. Roughly speaking, a p-simplex in the space $\widetilde{\mathcal{S}}_\bullet^{h/s}(M)$ is a pair (W^{d+p}, f) consisting of a manifold W stratified over Δ^p and a face-preserving homotopy equivalence $f : W \xrightarrow{\simeq_{h/s}}_\Delta$

$M \times \Delta^p$. *Surgery theory* establishes a diagram of fibration sequences [Qui70, §3]

$$\begin{tikzcd}
\Omega^{\infty+d+1}\mathbb{L}^s(M) \arrow[r] \arrow[d] & \widetilde{\mathcal{S}}^s(M) \arrow[r] \arrow[d] & (G/O)^{M_+}_* \arrow[d, equal] \\
\Omega^{\infty+d+1}\mathbb{L}^h(M) \arrow[r] & \widetilde{\mathcal{S}}^h(M) \arrow[r] & (G/O)^{M_+}_*.
\end{tikzcd}$$

Taking homotopy fibres we obtain an equivalence

$$\Omega^{\infty+d+1}\operatorname{hofib}\left(\mathbb{L}^s(M) \to \mathbb{L}^h(M)\right) \simeq \operatorname{hofib}\left(\widetilde{\mathcal{S}}^s(M) \longrightarrow \widetilde{\mathcal{S}}^h(M)\right). \tag{1.3.20}$$

On the other hand, it is not difficult to see that there is another diagram of fibration sequences

$$\begin{tikzcd}
s\mathrm{Aut}(M) \arrow[r] \arrow[d, hook, "\substack{\text{incl. of}\\ \text{cpts.}}"] & \widetilde{\mathcal{S}}^s(M) \arrow[r, "u"] \arrow[d] & \widetilde{\mathcal{M}}^s \arrow[d] \\
h\mathrm{Aut}(M) \arrow[r] & \widetilde{\mathcal{S}}^h(M) \arrow[r, "u"] & \widetilde{\mathcal{M}}^h,
\end{tikzcd}$$

where u is the (geometric realisation of the) forgetful map sending a p-simplex (W^{d+p}, f) in $\widetilde{\mathcal{S}}^{h/s}_\bullet(M)$ to $W \in \widetilde{\mathcal{M}}^{h/s}_p$. Taking again homotopy fibres (in the basepoint components corresponding to M and Id_M), we get a map $\overline{u} : \operatorname{hofib}(\widetilde{\mathcal{S}}^s(X) \longrightarrow \widetilde{\mathcal{S}}^h(X)) \to |F_\bullet(M)| \simeq \Omega^\infty(H\mathrm{Wh}(M)_{hC_2})$ which is an equivalence onto the components that are hit. For each $[a] \in H_0(C_2; \mathrm{Wh}(M))$, write $\Omega^\infty_{[a]}(H\mathrm{Wh}(M)_{hC_2}) \subset \Omega^\infty(H\mathrm{Wh}(M)_{hC_2})$ for the connected component corresponding to $[a]$. There is hence a chain of equivalences

$$\begin{aligned}
\Omega^{\infty+d+1}\operatorname{hofib}\left(\mathbb{L}^s(M) \to \mathbb{L}^h(M)\right) &\overset{(1.3.20)}{\simeq} \operatorname{hofib}\left(\widetilde{\mathcal{S}}^s(X) \to \widetilde{\mathcal{S}}^h(X)\right) \\
&\simeq \coprod_{[a]\in \mathrm{Im}(\pi_0(\overline{u}))} \Omega^\infty_{[a]}(H\mathrm{Wh}(M)_{hC_2}).
\end{aligned}$$

To establish (1.3.19), it remains to argue that $\mathrm{Im}(\pi_0(\overline{u})) = \widehat{H}^1(C_2; \mathrm{Wh}(M))$. Choose a 0-simplex in $\operatorname{hofib}(\widetilde{\mathcal{S}}^s_\bullet(X) \longrightarrow \widetilde{\mathcal{S}}^h_\bullet(X))$, that is, a 1-simplex $(W, f) \in \widetilde{\mathcal{S}}^h_1(M)$ such that $W_0 = M$, $f_0 = \mathrm{Id}_M$ and $f_1 : W_1 \xrightarrow{\simeq_s} M \times \{1\}$ is a simple homotopy equivalence. In particular, $W : M \overset{h}{\rightsquigarrow} W_1$ is an h-cobordism starting at M. By definition of $\psi_\bullet^{\mathrm{Wh}(M)}$ (see (1.3.15)),

$$\pi_0(\overline{u}) : \pi_0\left(\operatorname{hofib}\left(\widetilde{\mathcal{S}}^s(X) \longrightarrow \widetilde{\mathcal{S}}^h(X)\right)\right) \longrightarrow H_0(C_2, \mathrm{Wh}(M)), \quad [W, f] \longmapsto [\tau(W, M)].$$

Now $0 = \tau(f_0) = (i_M)^{-1}_*\tau(f) + \tau(W, M)$ and, by duality, $0 = (h^W_*)^{-1}\tau(f_1) = (i_M)^{-1}_*\tau(f) + (-1)^d\overline{\tau}(W, M)$. Putting these two together we obtain $\tau(W, M) =$

$(-1)^d\overline{\tau}(W,M)$, so it follows that $\mathrm{Im}(\pi_0(\overline{u})) \subset \widehat{H}^1(C_2;\mathrm{Wh}(M))$. For the other inclusion, let $W \in F_0(M)$ be such that $\tau(W,M) = (-1)^d\overline{\tau}(W,M)$, and pick some face-preserving homotopy equivalence $f : W \xrightarrow{\simeq_h}_{\Delta} M \times I$. By post-composing f with $f_0^{-1} \times I$, for some f_0^{-1} homotopy inverse to f_0, we may assume that $f_0 = \mathrm{Id}_M$. Then

$$\tau(f_1) = (i_{W_1})_*^{-1}\tau(f) + (-1)^d h_*^W \overline{\tau}(W,M) = (i_{W_1})_*^{-1}\tau(f) + h_*^W\tau(W,M) = h_*^W\tau(f_0) = 0,$$

so $f_1 : W_1 \simeq_s M \times \{1\}$ is a simple homotopy equivalence and thus (W,f) represents a 0-simplex in $\mathrm{hofib}(\widetilde{\mathcal{S}}^s_\bullet(X) \longrightarrow \widetilde{\mathcal{S}}^h_\bullet(X))$. This concludes the proof of Propostion 1.3.26. $\square$

Remark 1.3.27 (Speculative). The equivalence (1.3.19) can presumably be upgraded to one of infinite loop spaces. The argument should be similar to that of Theorem B by replacing the block moduli spaces $\widetilde{\mathcal{M}}^{h/s}_\bullet$ with the L-theory semi-simplicial sets $\mathsf{L}^{h/s}_\bullet$ as defined in [Qui70, §2]. More generally, an equivalence of spectra $H\mathrm{Wh}(M)^{tC_2} \simeq \Sigma^{-d}\,\mathrm{hofib}\left(\mathbb{L}^s(M) \to \mathbb{L}^h(M)\right)$ should hold.

1.4 Proof of Theorem A(i)

By analysing the lower-degree portion of the long exact sequence of homotopy groups associated to the homotopy pullback of Theorem B, we propose a general strategy to prove Theorem A(i) (see Proposition 1.4.5). We then present an example of an h-cobordism $W : L \overset{h}{\rightsquigarrow} M$, where L is as in the statement of Theorem A, which satisfies the conditions of the proposed strategy. All throughout let M^d denote a closed smooth manifold of dimension $d \geq 5$.

1.4.1 A general strategy

From the homotopy cartesian square of the homotopy fibre $F_\bullet(M)$ (see (1.3.17)) we obtain an associated long exact sequence of homotopy groups

$$\begin{aligned}\ldots \longrightarrow \pi_n(F_\bullet(M)) \longrightarrow \pi_n(\widetilde{\mathcal{M}}^s,\{M\}) \longrightarrow \pi_n(\widetilde{\mathcal{M}}^h,\{M\}) \longrightarrow \ldots \\ \ldots \longrightarrow \pi_1(\widetilde{\mathcal{M}}^h,\{M\}) \xrightarrow{\partial} \pi_0(F_\bullet(M)) \longrightarrow \pi_0(\widetilde{\mathcal{M}}^s) \longrightarrow \pi_0(\widetilde{\mathcal{M}}^h).\end{aligned} \tag{1.4.1}$$

For $n \geq 1$, the boundary map $\partial : \pi_n(\widetilde{\mathcal{M}}^h,\{M\}) \to \pi_{n-1}(F_\bullet(M))$ sends an n-cycle $W^{d+n} \in \widetilde{\mathcal{M}}^h_n$ based at M to W as an $(n-1)$-cycle in $F_\bullet(M)$. So the image of the lowest-degree boundary map $\partial : \pi_1(\widetilde{\mathcal{M}}^h, \{M\}) \to \pi_0(F_\bullet(M))$ consists of those classes represented by h-cobordisms $W : M \overset{h}{\rightsquigarrow} M$.

Definition 1.4.1. *An h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$ is said to be* ***inertial*** *[JK18, Defn. 2.1] if M' is diffeomorphic to M. The set of inertial h-cobordisms starting at M (up to diffeomorphism rel M) is denoted by $I(M) \subset h\mathrm{Cob}(M) \cong \mathrm{Wh}(M)$.*

Example 1.4.2. Given an h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$, denote $\overline{W} : M' \overset{h}{\rightsquigarrow} M$ for W with the reversed cobordism direction. The *double* $D(W) := \overline{W} \circ W = W \cup_{M'} \overline{W}$ [Mil66, p. 400] is an inertial h-cobordism $M \overset{h}{\rightsquigarrow} M$ with torsion (see (1.2.4) and (1.2.6))

$$\tau(D(W), M) = \tau(W, M) + (-1)^d \overline{\tau}(W, M).$$

The subgroup of *double* h-cobordisms of M^d,

$$\mathcal{D}(M) := \{\sigma + (-1)^d \overline{\sigma} : \sigma \in \mathrm{Wh}(M)\} \subset \mathrm{Wh}(M), \tag{1.4.2}$$

is therefore a subset of $I(M)$ too. Also observe from (1.3.18) that

$$H_0(C_2; \mathrm{Wh}(M)) = \mathrm{Wh}(M)/\mathcal{D}(M).$$

Lemma 1.4.3. *Let $\frac{I(M)}{\mathcal{D}(M)}$ denote the image of $I(M)$ under $\mathrm{Wh}(M) \twoheadrightarrow H_0(C_2; \mathrm{Wh}(M))$. Under the isomorphism $\pi_0(F_\bullet(M)) \cong H_0(C_2; \mathrm{Wh}(M))$ established in (1.3.18),*

$$\mathrm{Im}\ \left\{\partial : \pi_1(\widetilde{\mathcal{M}}^h, \{M\}) \to \pi_0(F_\bullet(M)) \cong \frac{\mathrm{Wh}(M)}{\mathcal{D}(M)}\right\} = \frac{I(M)}{\mathcal{D}(M)}.$$

Proof. The inclusion ($\subset$) is immediate. Conversely if $W : M \rightsquigarrow M'$ is an inertial h-cobordism with $\phi : M \cong M'$, let $W' : M \rightsquigarrow M$ denote the h-cobordism $M_{\phi^{-1}} \circ W$. Recall that the isomorphism $\pi_0(F_\bullet(M)) \cong H_0(C_2; \mathrm{Wh}(M))$ sends the class represented by W to that of its torsion $\tau(W, M) = \tau(W', M)$. As W' represents a class in $\pi_1(\widetilde{\mathcal{M}}^h, \{M\})$, we are done. □

Recall from Proposition 1.3.6 and (1.3.3) that $B\widetilde{\mathrm{Diff}}(M)$ and $B\widetilde{\mathrm{Diff}}^h(M)$ are the connected components of $\widetilde{\mathcal{M}}^s$ and $\widetilde{\mathcal{M}}^h$, respectively, which contain M^d as basepoint. We will denote $\widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(M) \subset F_\bullet(M)$ for the union of connected components corresponding to $\frac{I(M)}{\mathcal{D}(M)}$. By exactness of (1.4.1), these are exactly the components of $F_\bullet(M)$ that map to $B\widetilde{\mathrm{Diff}}(M) \subset \widetilde{\mathcal{M}}^s$. We thus obtain a fibration sequence

$$\widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(M) \longrightarrow B\widetilde{\mathrm{Diff}}(M) \longrightarrow B\widetilde{\mathrm{Diff}}^h(M). \tag{1.4.3}$$

For the remaining of the section, let $W^{d+1} : L^d \overset{h}{\rightsquigarrow} M^d$ be some h-cobordism with torsion $\tau := \tau(W, L) \in \mathrm{Wh}(L)$. The homotopy long exact sequences of the fibration (1.4.3) for L and M yield the diagram

$$\begin{tikzcd}
\pi_2(B\widetilde{\mathrm{Diff}}^h(L)) \arrow[r, "\partial"] \arrow[d, "\text{base pt. change}"', "\cong"] \arrow[dr, phantom, "(\dagger)"] & H_1(C_2; \mathrm{Wh}(L)) \arrow[r] \arrow[d, "h_*^W"', "\cong"] & \pi_1(B\widetilde{\mathrm{Diff}}(L)) \arrow[r] & \pi_1(B\widetilde{\mathrm{Diff}}^h(L)) \arrow[r, two heads, "\partial"] \arrow[d, "\text{base pt. change}"', "\cong"] & \frac{I(L)}{\mathcal{D}(L)} \\
\pi_2(B\widetilde{\mathrm{Diff}}^h(M)) \arrow[r, "\partial"] & H_1(C_2; \mathrm{Wh}(M)) \arrow[r] & \pi_1(B\widetilde{\mathrm{Diff}}(M)) \arrow[r] & \pi_1(B\widetilde{\mathrm{Diff}}^h(M)) \arrow[r, two heads, "\partial"] & \frac{I(M)}{\mathcal{D}(M)},
\end{tikzcd} \tag{1.4.4}$$

where we have used the isomorphism $\pi_n(\widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(-)) \cong H_n(C_2; \mathrm{Wh}(-))$ for $n \geq 1$ from Proposition 1.3.24. We are trying to compare the middle terms of the two extensions above, since

$$\pi_1(B\widetilde{\mathrm{Diff}}(-)) \cong \pi_0(\widetilde{\mathrm{Diff}}(-)) =: \widetilde{\Gamma}(-).$$

We first study the left part of the extensions in (1.4.4).

Proposition 1.4.4. *For $n \geq 2$, the following square commutes:*

$$\begin{tikzcd}
\pi_n(B\widetilde{\mathrm{Diff}}^h(L)) \arrow[r, "\partial"] \arrow[d, "\text{base pt. change}"', "\cong"] & H_{n-1}(C_2; \mathrm{Wh}(L)) \arrow[d, "h_*^W"', "\cong"] \\
\pi_n(B\widetilde{\mathrm{Diff}}^h(M)) \arrow[r, "\partial"] & H_{n-1}(C_2; \mathrm{Wh}(M)).
\end{tikzcd}$$

In particular, the square decorated by $(\dagger)$ in (1.4.4) commutes.

Proof. The basepoint change map sends an n-cycle $V^{d+n} \in B\widetilde{\mathrm{Diff}}^h(L)_n$ to the manifold (see Figure 1.1)

$$W_\# V := V \cup_{L \times \partial\Delta^n} (W \times \partial\Delta^n).$$

The union is made along the boundary $\partial V = L \times \partial\Delta^n$. The manifold $W_\# V$ is naturally stratified over Δ^n, and clearly represents an n-cycle in $B\widetilde{\mathrm{Diff}}^h(M)$. As mentioned before, the boundary map $\partial : \pi_n(B\widetilde{\mathrm{Diff}}^h(L)) \longrightarrow H_{n-1}(C_2; \mathrm{Wh}(L))$ sends $[V]$ to the class represented by $\tau(V, V_0) = \tau(V, L)$ in $H_{n-1}(C_2; \mathrm{Wh}(L))$. We thus need to show that

$$\tau(W_\# V, M) \equiv h_*^W \tau(V, L) \mod \{\sigma + (-1)^{d+n-1}\overline{\sigma} : \sigma \in \mathrm{Wh}(L)\}.$$

We compute $\tau(W_\# V, M)$ directly. For any subspaces $A, B \subset W_\# V$ with $A \subset B$, write i_A^B for the inclusion. If $P := V \cup_{V_0 = L} W$ (see Figure 1.1), we can factor the inclusion

$i_M^{W_\# V} : M = M \times \{0\} \hookrightarrow W_\# V$ as

$$\begin{tikzcd}
M \arrow[r, hook, "\simeq"] \arrow[d, hook, "\simeq"] & W_\# V \\
W \arrow[r, hook, "\simeq"] & P. \arrow[u, hook, "\simeq"']
\end{tikzcd}$$

We compute the torsion of these three maps using the inclusion-exclusion principle (1.2.2):

$$\begin{aligned}
\tau(W, M) &= (-1)^d h_*^W \overline{\tau}, \\
\tau(P, W) &= (i_L^W)_* \tau(V, L) + (i_W^W)_* \tau(W, W) - (i_L^W)_* \tau(L, L) \\
&= (i_L^W)_* \tau(V, L), \\
\tau(W_\# V, P) &= (i_V^P)_* \tau(W_\# V, V) + (i_W^P)_* \tau(W, W) - (i_L^P)_* \tau(W, L) \\
&= (i_V^P)_* (i_{\partial V}^V)_* \tau(W \times \partial\Delta^n, L \times \partial\Delta^n) - (i_L^P)_* \tau \\
&= \chi(\partial\Delta^n) \cdot (i_L^P)_* \tau - (i_L^P)_* \tau \\
&= (-1)^{n-1} (i_L^P)_* \tau.
\end{aligned}$$

In the penultimate line we have used that $i_V^P \circ i_{\partial V}^V \circ i_{L \times 0}^{L \times \partial\Delta^n} = i_L^P$ and the product rule (1.2.3) of $\tau(-)$, for which we need the condition $n \geq 2$ for $\partial\Delta^n$ to be connected. By the composition rule (1.2.1), we get

$$\begin{aligned}
\tau(W_\# V, M) &= (-1)^d h_*^W \overline{\tau} + (i_M^W)_*^{-1} (i_L^W)_* \tau(V, L) + (-1)^{n-1} (i_M^P)_*^{-1} (i_L^P)_* \tau \\
&= (-1)^d h_*^W \overline{\tau} + h_*^W \tau(V, L) + (-1)^{n-1} h_*^W \tau \\
&= h_*^W \tau(V, L) + (-1)^{n-1} \left(h_*^W \tau + (-1)^{d+n-1} \overline{h_*^W \tau} \right),
\end{aligned}$$

where in the second line we have used the commutative diagram

$$\begin{tikzcd}
M \arrow[r, hook, "\simeq"] \arrow[dr, hook, "\simeq"'] & W \arrow[d, hook, "\simeq"] & L \arrow[l, hook', "\simeq"'] \arrow[dl, hook', "\simeq"] \\
& P, &
\end{tikzcd}$$

so $(i_M^P)_*^{-1} (i_L^P)_* = (i_M^W)_*^{-1} (i_L^W)_* = h_*^W$. This finishes the proof. □

In the next section we will focus on the task of finding an example of $W : L \overset{h}{\rightsquigarrow} M$ for which $\widetilde{\Gamma}(L) \neq \widetilde{\Gamma}(M)$ as in Theorem A(i). By the previous result, we should make the right hand sides of the two extensions in (1.4.4) differ. In order to do so, we will use the following.

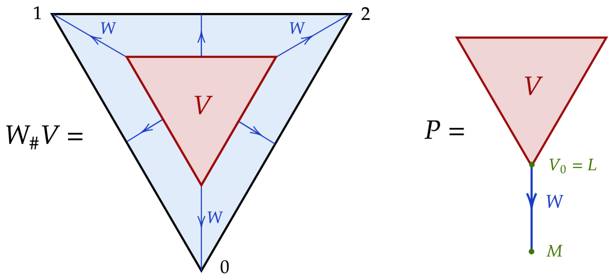

Figure 1.1 Illustration of $W_{\#}V$ and $P := V \cup_{L\times\{0\}} W$ when $n = 2$.

Proposition 1.4.5. *Let* $W : L \overset{h}{\rightsquigarrow} M$ *be such that*

$\textcircled{\scriptsize I}$ $\frac{\mathcal{I}(L)}{\mathcal{D}(L)} = 0$ *but* $\frac{\mathcal{I}(M)}{\mathcal{D}(M)} \neq 0$, $\qquad$ $\textcircled{\scriptsize II}$ $\pi_1(B\widetilde{\mathrm{Diff}}(L))$ *is finite.*

Then $\frac{\mathcal{I}(M)}{\mathcal{D}(M)}$ *is finite of cardinality* $N > 1$ *and*

$$|\pi_1(B\widetilde{\mathrm{Diff}}(L))| = N \cdot |\pi_1(B\widetilde{\mathrm{Diff}}(M))| < \infty.$$

In particular, $\widetilde{\Gamma}(L) \neq \widetilde{\Gamma}(M)$.

Proof. If $\frac{\mathcal{I}(L)}{\mathcal{D}(L)} = 0$ then $\pi_1(B\widetilde{\mathrm{Diff}}(L))$ surjects onto $\pi_1(B\widetilde{\mathrm{Diff}}^h(L))$, and hence the latter is finite too. As $\pi_1(B\widetilde{\mathrm{Diff}}^h(M)) \cong \pi_1(B\widetilde{\mathrm{Diff}}^h(L))$ surjects onto $\frac{\mathcal{I}(M)}{\mathcal{D}(M)}$, this is also finite, say of cardinality $N > 1$ since $\frac{\mathcal{I}(M)}{\mathcal{D}(M)} \neq 0$. Write

$$K := \left\{ x \in \pi_1(B\widetilde{\mathrm{Diff}}^h(M)) \cong \pi_1(B\widetilde{\mathrm{Diff}}^h(L)) : \partial x = 0 \in \frac{\mathcal{I}(M)}{\mathcal{D}(M)} \right\}.$$

We now have two extensions of finite groups

$$\begin{tikzcd}
0 \arrow[r] & H_1(C_2;\mathrm{Wh}(L))/\mathrm{Im}\ \partial \arrow[r] \arrow[d, "\cong"] & \pi_1(B\widetilde{\mathrm{Diff}}(L)) \arrow[r, two heads] & \pi_1(B\widetilde{\mathrm{Diff}}^h(L)) \arrow[r] \arrow[d, phantom, "\cup"] & 1 \\
0 \arrow[r] & H_1(C_2;\mathrm{Wh}(M))/\mathrm{Im}\ \partial \arrow[r] & \pi_1(B\widetilde{\mathrm{Diff}}(M)) \arrow[r, two heads] & K \arrow[r] & 1,
\end{tikzcd}$$

where the left vertical isomorphism is a consequence of Proposition 1.4.4. Even though ∂ is a crossed homomorphism, we still have that $N \cdot |K| = |\pi_1(B\widetilde{\mathrm{Diff}}^h(M))| = |\pi_1(B\widetilde{\mathrm{Diff}}^h(L))|$, and therefore the result follows. □

1.4.2 The candidate $W : L \overset{h}{\rightsquigarrow} M$

Let $L_p^{2n-1}(r_1 : \cdots : r_n)$ denote the linear lens space with fundamental group C_p and weights $r_1, \ldots, r_n \mod p$, i.e., the quotient of the sphere S^{2n-1} by the free (left)

C_p-action given by

$$t \cdot (z_1, \dots, z_n) := (\zeta^{r_1} z_1, \dots, \zeta^{r_n} z_n), \qquad (z_1, \dots, z_n) \in S^{2n-1} \subset \mathbb{C}^n,$$

where $t \in C_p$ is the generator and $\zeta = \exp(2\pi \mathrm{i}/p)$. We identify $\pi_1(L_p^{2n-1}(r_1 : \cdots : r_n))$ with C_p by sending the homotopy class represented by the loop

$$[0,1] \longrightarrow L_p^{2n-1}(r_1 : r_2 : \cdots : r_n), \quad s \longmapsto [\zeta^{s \cdot r_1}, 0, \dots, 0]$$

to $t \in C_p$. The goal of this section is to prove

Theorem 1.4.6. *Let L be the lens space $L_7^{12k-1}(r_1 : \cdots : r_{6k})$ with*

$$r_1 = \cdots = r_k = 1, \qquad r_{k+1} = \cdots = r_{2k} = 2, \quad \dots \quad r_{5k+1} = \cdots = r_{6k} = 6 \mod 7.$$

The element $u := 2 + 2t - t^3 - t^4 - t^5$ is a unit in $\mathbb{Z}[C_7]$ with inverse $u^{-1} = 1 - 2t + 3t^2 - 3t^3 + 3t^4 - 2t^5 + t^6$, and hence represents an element of $\mathrm{Wh}(L)$. Then the h-cobordism $W : L \overset{h}{\rightsquigarrow} M$ with torsion $\tau(W, L) = u$ satisfies conditions Ⓘ and Ⓘⓘ of Proposition 1.4.5 with $N = \left|\frac{I(M)}{\mathcal{D}(M)}\right| = 3$. In particular

$$|\pi_1(B\widetilde{\mathrm{Diff}}(L))| = 3 \cdot |\pi_1(B\widetilde{\mathrm{Diff}}(M))| < \infty,$$

and Theorem A(i) holds.

The proof of Theorem 1.4.6 will be established in Propositions 1.4.7 and 1.4.10 below.

Proposition 1.4.7. *The h-cobordism $W : L \overset{h}{\rightsquigarrow} M$ of Theorem 1.4.6 satisfies condition Ⓘ of Proposition 1.4.5. In fact,*

$$\left|\frac{I(M)}{\mathcal{D}(M)}\right| = 3.$$

Proof. The algebraic involution $\bar{\cdot} : \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi)$ is trivial when π is a finite abelian group [Bas74, Prop. 4.2]. Therefore by (1.4.2), the subgroups of double h-cobordisms $\mathcal{D}(L)$ and $\mathcal{D}(M)$ are trivial since L and M are odd-dimensional and orientable (so their first Stiefel–Whitney classes vanish), and $\pi_1(L) \cong \pi_1(M) \cong C_7$ is certainly finite abelian. It thus suffices to show that $I(L) = 0$ and $|I(M)| = 3$.

The first assertion follows from [Mil66, Cor. 12.12]. We now prove that $I(M) \neq 0$, i.e., we construct non-trivial inertial h-cobordisms starting at M. For a diffeomorphism

$\phi \in \mathrm{Diff}(L)$, write V_ϕ for the inertial h-cobordism

$$V_\phi := W \circ M_{\phi^{-1}} \circ \overline{W} : M \overset{h}{\rightsquigarrow} L \overset{s}{\rightsquigarrow} L \overset{h}{\rightsquigarrow} M,$$

i.e., $V_\phi = \overline{W} \cup_{\phi^{-1}} W \in I(M)$. Observe that $h^{\overline{W}} : M \xrightarrow{\simeq} L$ is homotopy inverse to $h^W : L \xrightarrow{\simeq} M$ because $\overline{W} \circ W = D(W)$ and $W \circ \overline{W} = D(\overline{W})$ are h-cobordant rel boundary to the trivial h-cobordisms $L \times I$ and $M \times I$, respectively (see Figure 1.2). So $h^{\overline{W} \circ W} = h^{\overline{W}} \circ h^W$ is homotopic to $h^{L \times I} = \mathrm{Id}_L$ (and similarly $h^W \circ h^{\overline{W}} \simeq \mathrm{Id}_M$). The h-cobordism V_ϕ then has torsion

$$\begin{aligned}
\tau(V_\phi, M) &= \tau(\overline{W}, M) + (h^{\overline{W}})_*^{-1} \tau(W \circ M_{\phi^{-1}}, L) \\
&= (-1)^{12k-1} h_*^W \overline{u} + h_*^W \phi_* u \\
&= h_*^W (\phi_* u - u),
\end{aligned}$$

where we have used that $\overline{u} = u$ by the triviality of the algebraic involution. Therefore, if we are able to find self-diffeomorphisms ϕ of L for which $\phi_* u \neq u$ in $\mathrm{Wh}(L)$, then we will have achieved our task.

Claim. *There are orientation-preserving self-diffeomorphisms $\phi_i : L \xrightarrow{\cong} L$ for $i \in (\mathbb{Z}/7)^\times$ such that $\pi_1(\phi_i) : t \mapsto t^i$.*

Proof of Claim. In fact by the main theorem of [HJ83] (see (1.4.5) below), the natural map $\pi_0(\mathrm{Diff}(L)) \to \pi_0(s\mathrm{Aut}(L))$ is surjective, so it will suffice to find simple homotopy automorphisms $f_i \in s\mathrm{Aut}(L)$ for each $i \in (\mathbb{Z}/7)^\times$ such that $\pi_1(f_i) : t \mapsto t^i$. By [Olu53, Thm. II.b & Thm. V], the natural map $\gamma : \pi_0(h\mathrm{Aut}(L)) \to \mathrm{Aut}(\pi_1 L) \cong (\mathbb{Z}/7)^\times$ is injective with image

$$\mathrm{Im}\,\gamma = \{i \in \mathbb{Z}/7 : i^{6k} \equiv \pm 1 \mod 7\}.$$

The classes $[f]$ sent to $i \in (\mathbb{Z}/7)^\times$ with $i^n \equiv +1 \mod 7$ (resp. $i^n \equiv -1 \mod 7$) are orientation-preserving (resp. orientation-reversing). But if $i \in (\mathbb{Z}/7)^\times$, $i^6 \equiv 1 \mod 7$ and so $i^{6k} \equiv 1 \mod 7$ too. Therefore for each $i \in (\mathbb{Z}/7)^\times$, there exists some (orientation-preserving) homotopy automorphism $f_i \in h\mathrm{Aut}(L)$ such that $\pi_1(f_i) : t \mapsto t^i$. By [Mil66, Lem. 12.5], f_i is a simple homotopy automorphism if and only if

$$(f_i)_* \Delta(L) = \Delta(L) \in \mathbb{Q}[C_7]/\sim,$$

where $\Delta(L)$ denotes the *R-torsion* of L [Mil66, Lem. 12.4]. Here, two elements $x, y \in \mathbb{Q}[C_7]$ are related $x \sim y$ if and only if there exists some $g \in C_7$ such that

$x = \pm g \cdot y$. Recall also [Mil66, p. 406] that the R-torsion of L is

$$\Delta(L) = \prod_{j=1}^{6k}(t^{r_j} - 1) = \prod_{j=1}^{6}(t^j - 1)^k,$$

and so

$$(f_i)_*\Delta(L) = (f_i)_*\left(\prod_{j=1}^{6}(t^j - 1)^k\right) = \prod_{i=1}^{6}(t^{i\cdot j} - 1)^k = \Delta(L), \quad i \in (\mathbb{Z}/7)^\times.$$

Therefore f_i is a simple homotopy equivalence for $i \in (\mathbb{Z}/7)^\times$, as claimed. $\square$

Now it is easily checked that $(\phi_6)_*u = u$, so $(\phi_2)_*u = (\phi_5)_*u$ and $(\phi_3)_*u = (\phi_4)_*u$. On the other hand, the three non-trivial units

$$u = 2 + 2t - t^3 - t^4 - t^5, \quad (\phi_2)_*u = 2 + 2t^2 - t^6 - t - t^3, \quad (\phi_3)_*u = 2 + 2t^3 - t^2 - t^5 - t$$

represent different elements in $\mathrm{Wh}(L)$ (for instance, the difference between the powers of t of the terms whose coefficient is 2 in the three units is different mod 7). By our previous argument, the h-cobordisms $V_{\phi_1} = M \times I$, V_{ϕ_2} and V_{ϕ_3} are pairwise non-diffeomorphic inertial h-cobordisms starting at M, so $|I(M)| \geq 3$. Note that $V_{\phi_i} \cong V_{\phi_{7-i}}$ for $i = 1, \ldots 6$. Conversely, suppose that $V : M \overset{h}{\rightsquigarrow} M$ is an inertial h-cobordism (by possibly post-composing with a mapping cylinder, we may assume that the target of W is M itself). Then the inertial h-cobordism $\overline{W} \circ V \circ W : L \overset{h}{\rightsquigarrow} L$ must be trivial as $I(L) = 0$, and $(h^{\overline{W}\circ V\circ W})_*^{-1} = (h^W)_*^{-1}(h^V)_*^{-1}h_*^W = (\phi_i)_*$ for some $i \in (\mathbb{Z}/7)^\times$ because $\mathrm{Aut}(\pi_1 L) \cong (\mathbb{Z}/7)^\times$. Using $\tau(\overline{W} \circ V \circ W, L) = 0$ we get that

$$\tau(V, M) = (h^V)_*^{-1}h_*^W u - h_*^W u = h_*^W((\phi_i)_*u - u) = \tau(V_{\phi_i}, M),$$

i.e. V is diffeomorphic to V_{ϕ_i} rel M. Hence, $|I(M)| = 3$ and this finishes the proof. $\square$

Remark 1.4.8. This is an example of how badly-behaved the set of inertial h-cobordisms of a manifold may be. For instance in the case at hand, it is not an h-cobordism invariant. It is also *not* a subgroup of $\mathrm{Wh}(M) \cong \mathrm{Wh}(C_7) \cong \mathbb{Z}^2$ (see [Bas64, §7] and [Ste78, pp. 202–205]) because $I(M)$ is a finite subset with cardinality strictly greater than 1. See [Hau80, Rmk. 6.2] for more instances of this phenomenon.

Warning 1.4.9. The main theorem of [KS99] states that $I(M) = 0$ if M is a fake lens space, that is, the orbit space of a free (possibly non-linear) action of a finite cyclic group on a sphere. This clearly contradicts Proposition 1.4.7, but we believe that the

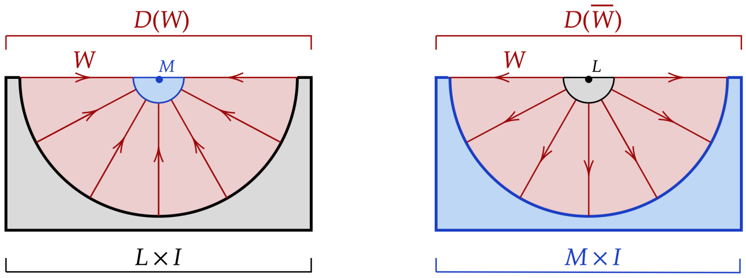

Figure 1.2 h-cobordisms rel boundary $D(W) \overset{h}{\rightsquigarrow} L \times I$ and $D(\overline{W}) \overset{h}{\rightsquigarrow} M \times I$.

proof of the result in [KS99] is fallacious: following a trail of references [KS99, Claim 2], [KS92, Prop. 3.2] and [Kwa86, p. 353], it is eventually stated that if $\tau \in \mathrm{Wh}(M)$ is the torsion of some inertial h-cobordism of M, then $2\tau = 0$. This supposedly follows from the proof of [Mil66, Prop. 12.8], but in the statement of this result it is required that the h-cobordism between special manifolds be compatible with the given identifications of the fundamental groups, i.e., that $h^W_* : \mathrm{Wh}(M) \to \mathrm{Wh}(M')$ has been trivialised beforehand. This requirement does not hold in the case of [Kwa86, p. 353], and is exactly what we exploit in the proof of Proposition 1.4.7.

We now deal with $\textcircled{II}$. Since the natural map $\pi_0(\mathrm{Diff}(L)) \to \pi_0(\widetilde{\mathrm{Diff}}(L)) \cong \pi_1(B\widetilde{\mathrm{Diff}}(L))$ is surjective, it suffices to show

Proposition 1.4.10. *The mapping class group $\Gamma(L) = \pi_0(\mathrm{Diff}(L))$ of L is finite. In particular, $W : L \overset{h}{\rightsquigarrow} M$ satisfies condition $\textcircled{II}$ of Proposition 1.4.5.*

Proof. According to the main result of [HJ83], the mapping class group of L fits into an extension of groups

$$0 \longrightarrow Q \oplus H \longrightarrow \pi_0(\mathrm{Diff}(L)) \longrightarrow \pi_0(s\mathrm{Aut}(L)) \longrightarrow 0. \tag{1.4.5}$$

The group H is the image of $[\Sigma L_+, \mathrm{Top}/O]_*$ in $[\Sigma L_+, G/O]_*$. By [HS76, Thm. 1.1], the group Q also appears in an exact sequence

$$L^s_{2n+2}(\mathbb{Z}[C_7]) \longrightarrow H_0(C_2; \pi_0(C(L))) \longrightarrow Q \longrightarrow L^s_{2n+1}(\mathbb{Z}[C_7]), \tag{1.4.6}$$

where $\pi_0(C(L))$ is the group of (isotopy classes) of pseudoisotopies of L which, by a computation of Hatcher–Wagoner [HW73] and corrections of Igusa [Igu82], it fits in yet another exact sequence

$$\mathrm{Wh}^+_1(C_7; \mathbb{F}_2) \longrightarrow \pi_0(C(L)) \longrightarrow \mathrm{Wh}_2(C_7) \longrightarrow 0 \tag{1.4.7}$$

as long as $\dim L \geq 7$; this is our case. In (1.4.6) and (1.4.7), for an abelian group with involution π,

- $L_*^s(\mathbb{Z}\pi)$ are the simple quadratic L-groups of the group ring $\mathbb{Z}\pi$,
- $\mathrm{Wh}_2(\pi)$ is the abelian group (with involution) defined as the cokernel of the map arising from algebraic K-theory (see (B.1))

$$\pi_2^s((B\pi)_+) \longrightarrow K_2(\mathbb{Z}\pi) \longrightarrow \mathrm{Wh}_2(\pi) \longrightarrow 0, \tag{1.4.8}$$

- $\mathrm{Wh}_1^+(\pi; \mathbb{F}_2) := H_0(C_2; \mathbb{F}_2[\pi])$ with a certain involution.

We now show that each of the groups in the extension (1.4.5) are finite. As mentioned in the proof of Proposition 1.4.7, $\pi_0(s\mathrm{Aut}(L)) \subset \pi_0(h\mathrm{Aut}(L)) \subset (\mathbb{Z}/7)^\times$, so it is definitely finite.

By Proposition A.1, $K_2(\mathbb{Z}[C_7])$ is finite, and hence so is $\mathrm{Wh}_2(C_7)$ by (1.4.8). Since $\mathbb{F}_2[C_7]$ is finite, $\mathrm{Wh}_1^+(C_7; \mathbb{F}_2)$ is so too; thus $\pi_0(C(L))$ is finite. Moreover, the (simple) L-theory of $\mathbb{Z}\pi$ for finite groups π of odd order is zero in odd degrees [Bak75, Thm. 1]. Therefore $L_{2n+1}^s(\mathbb{Z}[C_7]) = 0$, and thus Q is finite by (1.4.6).

The finiteness of H is a consequence of the following two observations: firstly that the infinite loop space Top/O has finite homotopy groups in every degree (see [KS77, Thm. 5.5]). Secondly that if X is a (pointed) finite CW-complex and Y a (pointed) space with finite homotopy groups in every degree, then the set $[X, Y]_*$ of (pointed) maps from X to Y up to homotopy is finite—indeed, this follows easily by induction on the skeleta $\{X_k\}_{k\geq 0}$ of X by considering the cofibre sequences

$$X_{k-1} \hookrightarrow X_k \twoheadrightarrow \textstyle\bigvee_{i\in I_k} S^k,$$

where I_k is a finite set (because X is a finite CW-complex). Hence $[\Sigma L_+, \mathrm{Top}/O]_*$ is finite, and thus so is H. This finishes the proof. □

Remark 1.4.11. For $CAT = \mathrm{Top}$ or PL, the group H of (1.4.5) should be replaced by the image of $[\Sigma L_+, \mathrm{Top}/CAT]_*$ in $[\Sigma L_+, G/CAT]$, which readily vanishes for $CAT = \mathrm{Top}$ and is seen to be finite too for $CAT = PL$ (see e.g. [Bru68]), so the same argument in the proof of Proposition 1.4.10 goes through.

1.5 Proof of Theorem A(ii)

In this section we finish the proof of Theorem A using the candidate $W : L \overset{h}{\rightsquigarrow} M$ of Theorem 1.4.6. We would hope that $\mathrm{Diff}(L)$ and $\widetilde{\mathrm{Diff}}(L)$ differ as much as $\mathrm{Diff}(M)$ and $\widetilde{\mathrm{Diff}}(M)$ do, so that the difference of block mapping class groups established in Theorem 1.4.6 carries over to the Diff-level. This is the case in some range and *up to extensions* [WW88, Thm. A].

Theorem 1.5.1 (Weiss–Williams)**.** *Let M^d be compact a smooth d-manifold. There exists a map*

$$\Phi^s : \widetilde{\mathrm{Diff}}/\mathrm{Diff}(M) \longrightarrow \Omega^\infty \left(\mathbf{H}^s_{\mathrm{Diff}}(M)_{hC_2}\right)$$

which is $(\phi_M + 1)$*-connected, where* ϕ_M *denotes the concordance stable range of* M *(which by Igusa's theorem [Igu88] is at least* $\min(\frac{d-4}{3}, \frac{d-7}{2})$*).*

Remark 1.5.2. The C_2-spectrum $\mathbf{H}^s_{\mathrm{Diff}}(M)$, known as the (smooth) *s-cobordism spectrum of* M, is the 1-connective cover of the (non-connective smooth) *h-cobordism spectrum* $\mathbf{H}_{\mathrm{Diff}}(M)$. This latter spectrum is roughly built out of deloopings of spaces of h-cobordisms (cf. [WW88, Lem. 1.12]) and its infinite loop space $\mathcal{H}_{\mathrm{Diff}}(M)$, the *space of stable h-cobordisms*, coincides with that of $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$ by the stable parametrised h-cobordism theorem of Waldhausen–Jahren–Rognes [WJR13]. Here $\mathbf{Wh}^{\mathrm{Diff}}(M)$ stands for the *smooth Whitehead spectrum* of M (see Section B.2). Moreover, the negative homotopy groups of these two spectra abstractly coincide (cf. [WW88, Cor. 5.6]), which lead Weiss and Williams to rename $\mathbf{H}_{\mathrm{Diff}}(M)$ by $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$ (though conjecturally true, this was not fully justified).

The only property we will use about $\mathbf{H}(-)$ is that its homotopy groups (ignoring the involution) are invariants of the homotopy type of $(-)$, as those of $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(-)$ are. In particular, if $W : L \overset{h}{\rightsquigarrow} M$ is an h-cobordism, there is an isomorphism of groups

$$\pi^s_*(\mathbf{H}_{\mathrm{Diff}}(L)) \cong \pi^s_*(\mathbf{H}_{\mathrm{Diff}}(M)). \tag{1.5.1}$$

We will not need to analyse the involutions in $\mathbf{H}(L)$ and $\mathbf{H}(M)$, but one can show that these two C_2-spectra are equivalent (in fact, the homotopy type of $\mathbf{H}(-)$ is nearly an invariant of the tangential homotopy type of $(-)$; see Section 3.1 for more details on the involutions in $\mathbf{H}_{\mathrm{Diff}}(M)$ and $\mathbf{Wh}^{\mathrm{Diff}}(M)$). It is also not difficult to see that the involution on $\pi_0(\mathbf{H}_{\mathrm{Diff}}(M)) \cong \mathrm{Wh}(M)$ corresponds to the rule $\tau \mapsto (-1)^{d-1}\overline{\tau}$ (see Corollary 3.1.9), which fits well with Theorem B. We expand on the relation between Theorems B and 1.5.1 in Section B.2.

Now since $d = 12k - 1 \geq 11$ (so $\phi_M + 1 \geq 2$), it follows from Theorem 1.5.1 that $\pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(L)) \cong \pi_1^s(\mathbf{H}^s_{\mathrm{Diff}}(L)_{hC_2})$. As $\mathbf{H}^s_{\mathrm{Diff}}(L)$ is 1-connective, its homotopy fixed point spectral sequence (cf. [BK72]) then yields isomorphisms

$$\begin{aligned} \pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(L)) &\cong H_0(C_2; \pi_1^s(\mathbf{H}_{\mathrm{Diff}}(L))), \\ \pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)) &\cong H_0(C_2; \pi_1^s(\mathbf{H}_{\mathrm{Diff}}(M))), \end{aligned} \tag{1.5.2}$$

for potentially different C_2-actions on $\pi_1^s(\mathbf{H}_{\mathrm{Diff}}(L)) \cong \pi_1^s(\mathbf{H}_{\mathrm{Diff}}(M))$. Consider the extensions

$$\begin{tikzcd} \pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(L)) \arrow[r, "\partial"] & \pi_0(\mathrm{Diff}(L)) \arrow[r, two heads] & \pi_0(\widetilde{\mathrm{Diff}}(L)) \arrow[r] & 0, \\ \pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)) \arrow[r, "\partial"] & \pi_0(\mathrm{Diff}(M)) \arrow[r, two heads] & \pi_0(\widetilde{\mathrm{Diff}}(M)) \arrow[r] & 0. \end{tikzcd} \tag{1.5.3}$$

We know from Theorem 1.4.6 that $|\pi_0(\widetilde{\mathrm{Diff}}(L)| = 3 \cdot |\pi_0(\widetilde{\mathrm{Diff}}(M))|$, so in order to prove Theorem A(ii) it suffices to establish the next result.

Proposition 1.5.3. *The groups* $\pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(L))$ *and* $\pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M))$ *are finite and their cardinality is not divisible by* 3*. Together with Theorem 1.4.6, it follows that the* 3*-adic valuations of* $|\Gamma(L)|$ *and* $|\Gamma(M)|$ *differ. This proves Theorem A*(ii)*.*

Proof. Given (1.5.1) and (1.5.2), we need only verify the first claim for the abelian group $\pi_1^s(\mathbf{H}_{\mathrm{Diff}}(L)) \cong \pi_1(\mathcal{H}_{\mathrm{Diff}}(L))$. Since $\dim L = d \geq 11$, it follows from Igusa's lower bound on the concordance stable range that $\pi_1(\mathcal{H}_{\mathrm{Diff}}(L)) = \pi_0(C(L))$, which fits in the exact sequence (1.4.7).

We have already argued in the proof of Proposition 1.4.10 that both of the groups $\mathrm{Wh}_2(C_7)$ and $\mathrm{Wh}_1^+(C_7; \mathbb{F}_2)$ in the extension are finite. Both summands are moreover 3-locally trivial ($\mathrm{Wh}_1^+(C_7; \mathbb{F}_2)$ is 2-tosion, and $\mathrm{Wh}_2(C_7)$ is a quotient of $K_2(\mathbb{Z}[C_7])$, which has no 3-torsion by Proposition A.1). Any quotient of this group (e.g. those of (1.5.2)) will have this same property, so the result follows. The proof of Theorem A is now complete. □

Remark 1.5.4. For CAT = Top or PL, the h-cobordism spectrum $\mathbf{H}_{\mathrm{Diff}}(L)$ should be replaced by its topological version $\mathbf{H}_{\mathrm{Top}}(L)$ (this is in fact the one that appears originally in [WW88]). To argue that $\pi_1^s(\mathbf{H}_{\mathrm{Top}}(L)) \cong \pi_2^s(\mathbf{Wh}^{\mathrm{Top}}(L))$ is finite and 3-local as in the previous proof, we consider the diagram of cofibre sequences of spectra (see (B.1) and (B.2))

$$\begin{tikzcd}
\mathbb{S} \wedge L_+ \arrow[r, "\iota"] \arrow[d] & \mathbf{A}(L) \arrow[r, two heads] \arrow[d, equal] & \mathbf{Wh}^{\mathrm{Diff}}(L) \arrow[d] \\
\mathbf{A}(*) \wedge L_+ \arrow[r, "\alpha"] \arrow[d, two heads] & \mathbf{A}(L) \arrow[r, two heads] \arrow[d] & \mathbf{Wh}^{\mathrm{Top}}(L) \arrow[d, two heads] \\
\mathbf{Wh}^{\mathrm{Diff}}(*) \wedge L_+ \arrow[r] & * \arrow[r] & \Sigma\mathbf{Wh}^{\mathrm{Diff}}(*) \wedge L_+.
\end{tikzcd}$$

Then $\pi_2^s(\mathbf{Wh}^{\mathrm{Top}}(L))$ will be finite and 3-locally trivial if the group $\pi_2^s(\Sigma\mathbf{Wh}^{\mathrm{Diff}}(*) \wedge L_+) \cong \pi_1^s(\mathbf{Wh}^{\mathrm{Diff}}(*) \wedge L_+)$ is. This in turn follows from the Atiyah–Hirzebruch spectral sequence, as $\mathbf{Wh}^{\mathrm{Diff}}(*) \simeq \mathbf{Wh}^{\mathrm{Diff}}(D^5)$ by the homotopy invariance of the Whitehead spectrum, and because the latter is 1-connective by the s-cobordism theorem (in fact it is 2-connective by Cerf's pseudoisotopy theorem).

Remark 1.5.5 (Another possible example). Theorem A may also holds for the lens space $L = L_5^{8k-1}(r_1 : \cdots : r_{4k})$, where

$$r_1 = \cdots = r_k = 1, \qquad r_{k+1} = \cdots = r_{2k} = 2, \quad \ldots \quad r_{3k+1} = \cdots = r_{4k} = 4 \mod 5,$$

and the h-cobordism $W : L \overset{h}{\rightsquigarrow} M$ with $\tau(W, L) = [1 - t - t^4] \in \mathrm{Wh}(C_5)$. The argument for part (i) is exactly analogous to that of Subsection 1.4.2, but part (ii) is trickier. The inertia set $I(M)$ will have size two (instead of three), and the group $\mathrm{Wh}_1^+(C_5; \mathbb{F}_2)$ does have 2-torsion. The alternative then is to show directly that the map ∂ in (1.5.3) is injective by identifying $\pi_1(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(L))$ with the cobordism group $\pi_0(\mathcal{B}(L))$ of [HJ83, p.1]. However, this argument does rely on the claim made in the proof of [HJ83, Sublemma 4.2] that a certain map $H_0(C_2; \mathrm{Wh}_2(C_5)) \to L_{8k-1}^{\mathrm{St}}(C_5)$ is injective when inverting the prime 2. We do not know how to prove this, nor have we found a reference that does.

A Appendix: An algebraic K-theory computation

The aim of this section is to prove the following.

Proposition A.1. *For p a prime, $K_2(\mathbb{Z}[C_p])$ is finite. Moreover when $p = 7$, its 3-torsion part vanishes:*

$$K_2(\mathbb{Z}[C_7])_{(3)} = 0.$$

Remark A.2. The author would like to thank John Nicholson for making him aware of the paper [ZTC19, Thm. 2.7] which, taken together with the computation in [ZXDS21, Thm. 1.1], easily implies Proposition A.1 when $p = 7$; this is the only case needed in the proof of Theorem A. By the time we became aware of this fact, we had already come up with an alternative proof, which we believe to be a nice application of a celebrated result of Land–Tamme. For this reason, we still present our original proof below, but the pragmatic reader may wish to skip this section.

The main ingredient of this computation is the main theorem of Land–Tamme [LT19]: Given a Milnor square of ring (spectra)

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & B',
\end{tikzcd}$$

i.e., a pullback square of ring spectra with $\pi_0(B) \to \pi_0(B')$ surjective, they functorially associate a connective ring spectrum $\mathcal{R}$ for which there is a Mayer–Vietoris sequence for algebraic K-theory

$$\dots \longrightarrow K_{i+1}(\mathcal{R}) \longrightarrow K_i(A) \longrightarrow K_i(A') \oplus K_i(B) \longrightarrow K_i(\mathcal{R}) \longrightarrow \dots \tag{A.1}$$

for every $i \in \mathbb{Z}$. Moreover, there is an equivalence of spectra $\mathcal{R} \to A' \otimes_A B$ (but *not* of $\mathbb{E}_1$-rings in general) and a map of $\mathbb{E}_1$-rings $\mathcal{R} \to B'$. For p a prime, the pullback square we will consider is

$$\begin{tikzcd}
\mathbb{Z}[C_p] \cong \mathbb{Z}[t]/(1-t^p) \arrow[r] \arrow[d, "t=1"] & \mathbb{Z}(\zeta_p) \cong \mathbb{Z}[t]/(1+t+\dots+t^{p-1}) \arrow[d, "t=1"] \\
\mathbb{Z} \arrow[r, "\mathrm{mod}\ p"] & \mathbb{Z}/p,
\end{tikzcd} \tag{A.2}$$

or rather that induced by applying the Eilenberg–MacLane functor $H(-)$ to (A.2). A straight-forward computation of $\mathrm{Tor}_i^{\mathbb{Z}[C_p]}(\mathbb{Z}, \mathbb{Z}(\zeta_p))$ shows that

$$\pi_i^s(\mathcal{R}) \cong \begin{cases} \mathbb{Z}/p, & i = 2k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the natural map $\mathcal{R} \to H\mathbb{Z}/p$ is an isomorphism on π_0 and a $\mathbb{Z}[1/p]$-equivalence of connective $\mathbb{E}_1$-rings. Therefore by [LT19, Lem. 2.4], it induces an isomorphism of localised K-theory $K_*(\mathcal{R}) \otimes \mathbb{Z}[1/p] \cong K_*(\mathbb{Z}/p) \otimes \mathbb{Z}[1/p]$. A portion of the exact sequence (A.1) localised away from p thus reads

$$\Big(K_3(\mathbb{Z}) \oplus K_3(\mathbb{Z}(\zeta_p)) \to K_3(\mathbb{Z}/p) \to K_2(\mathbb{Z}[C_p]) \to K_2(\mathbb{Z}) \oplus K_2(\mathbb{Z}(\zeta_p))\Big) \otimes \mathbb{Z}\Big[\tfrac{1}{p}\Big]. \tag{A.3}$$

We first analyse the (3-adic part of the) map $K_3(\mathbb{Z}) \to K_3(\mathbb{Z}/p)$ for $p \neq 3$.

Lemma A.3. *The map $K_3(\mathbb{Z})_{(3)} \to K_3(\mathbb{Z}/p)_{(3)}$ is injective for $p \neq 3$.*

Proof. According to [Qui76, Claim 4], for every integer $k \geq 1$ and odd prime $\ell \neq p$, the composition $\pi_{4k-1}^s \to K_{4k-1}(\mathbb{Z}) \to K_{4k-1}(\mathbb{Z}/p)$ is injective on $\mathrm{Im}(J : \pi_{4k-1}(O) \to \pi_{4k-1}^s)_{(\ell)}$: indeed, Diagram 4 loc. cit. is the commutative diagram

$$\begin{tikzcd}
\pi_{4k-1}^s \arrow[r] \arrow[d, "-e"'] & K_{4k-1}(\mathbb{Z}/p) \cong \mathbb{Z}/(p^{2k}-1) \arrow[d, hook, "\theta"'] \\
\mathbb{Q}/a_k\mathbb{Z} \arrow[r] & \mathbb{Q}/\mathbb{Z}[\frac{1}{p}],
\end{tikzcd}$$

where e denotes Adams' invariant, which is injective on $\mathrm{Im}\, J$, θ is injective with image the unique subgroup of order $p^{2k} - 1$, and a_k is 1 or 2 depending on whether k is odd or even, respectively. The lower horizontal map is the natural one, which is injective on ℓ-torsion if $2p$ does not divide ℓ.

For $k = 1$, the image of the J-homomorphism is the whole of $\pi_3^s \cong \mathbb{Z}/24$, $K_3(\mathbb{Z}/p) \cong \mathbb{Z}/(p^2 - 1)$ by [Qui72, Thm. 8(i)], and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ by [LS76]. Noting that $3 \mid p^2 - 1$ if $p \neq 3$ is prime, the result readily follows from the previous claim when $\ell = 3$. ☐

Proof of Proposition A.1. It is well known that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ [Mil71, Cor. 10.2], $K_3(\mathbb{Z}/p) \cong \mathbb{Z}/(p^2 - 1)$ and K_2 of the ring of integers of a number field is finite [Qui73, Thm. 1], [Bor74, Prop. 12.2] (in particular $K_2(\mathbb{Z}(\zeta_p))$ is). A very similar argument to [LT19, Lem. 2.4] replacing the Serre class of Λ-local abelian groups with the Serre class of finitely generated abelian groups shows that the map $K_3(\mathcal{R}) \to K_3(\mathbb{Z}/p)$ is an equivalence mod this Serre class, so as $K_3(\mathbb{Z}/p) \cong \mathbb{Z}/(p^2-1)$ is finitely generated, so is $K_3(\mathcal{R})$. In fact since $K_3(\mathcal{R})$ is finitely generated and $K_3(\mathcal{R}) \otimes \mathbb{Z}[1/p] \cong K_3(\mathbb{Z}/p) \otimes \mathbb{Z}[1/p] \cong \mathbb{Z}/(p^2-1)$ is finite, $K_3(\mathcal{R})$ is finite too. It follows from (A.3) that $K_2(\mathbb{Z}[C_p])$ is finite for every p.

Let now $p = 7$ so that $K_3(\mathbb{Z}/7) \cong \mathbb{Z}/48$, and hence by Lemma A.3, the map $\mathbb{Z}/3 \cong K_3(\mathbb{Z})_{(3)} \to K_3(\mathbb{Z}/7)_{(3)} \cong \mathbb{Z}/3$ is an isomorphism. Now $K_2(\mathbb{Z}(\zeta_7)) = \mathbb{Z}/2$ [ZXDS21, Thm. 1.1], and localising (A.3) at the prime $3 (\neq p = 7)$ we get that $K_2(\mathbb{Z}[C_7])_{(3)} = 0$. □

B Appendix: Connections to Weiss–Williams I

B.1 The group of h-block diffeomorphisms $\widetilde{\mathrm{Diff}}^h(M)$

Recall that $\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_\bullet$ denotes the semi-simplicial group of block diffeomorphisms of $M \times \mathbb{R}$ *bounded in the* $\mathbb{R}$*-direction*—a p-simplex consists of a face-preserving diffeomorphism $\phi : M \times \mathbb{R} \times \Delta^p \xrightarrow{\cong}_\Delta M \times \mathbb{R} \times \Delta^p$ such that there exists some positive constant $K > 0$ with $|\mathrm{pr}_\mathbb{R}\phi(x,t,v) - t| < K$ for all $(x,t,v) \in M \times \mathbb{R} \times \Delta^p$. In this section we prove the following:

Proposition B.1. *For $d = \dim M \geq 5$, there is a zig-zag of weak equivalences of Kan semi-simplicial sets*

$$\Omega B\widetilde{\mathrm{Diff}}^h(M)_\bullet \xleftarrow[\simeq]{\mathcal{R}_\bullet} \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet \xhookrightarrow{\simeq} \widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_\bullet.$$

In particular, there are homotopy equivalences

$$\widetilde{\mathrm{Diff}}^h(M) := |GB\widetilde{\mathrm{Diff}}^h(M)_\bullet| \simeq |\Omega B\widetilde{\mathrm{Diff}}^h(M)_\bullet| \simeq \widetilde{\mathrm{Diff}}^b(M \times \mathbb{R}).$$

Let us explain the new notation. The simplicial loop space $\Omega B\widetilde{\mathrm{Diff}}^h(M)_\bullet$ has as p-simplices those $(p+1)$-simplices $W \Rightarrow \Delta^{p+1}$ of $B\widetilde{\mathrm{Diff}}^h(M)_\bullet$ with $W_0 = M$ and $\partial_0 W = M \times \Delta^p$. The sub-semi-simplicial set $\widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet \subset \widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_\bullet$ has as p-simplices those bounded diffeomorphisms $\phi : M \times \mathbb{R} \times \Delta^p \xrightarrow{\cong}_\Delta M \times \mathbb{R} \times \Delta^p$ with

$$\phi(M \times (1/2, \infty) \times \Delta^p) \subset M \times (1/2, \infty) \times \Delta^p.$$

The map $\mathcal{R}_\bullet$ sends a diffeomorphism $\phi \in \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_p$ to the region in $M \times \mathbb{R} \times \Delta^p$ enclosed by $M \times \{0\} \times \Delta^p$ and $\phi(M \times \{1\} \times \Delta^p)$, seen as a $(p+1)$-simplex in $B\widetilde{\mathrm{Diff}}^h(M)_\bullet$.

More precisely, if we denote this region by R_ϕ, then

$$\mathcal{R}_p(\phi) := (R_\phi \cup_{\phi^{-1}} M \times \Delta^p) \,/\, \sim, \quad (x,0,v) \sim (x,0,w), \quad \forall\, v,w \in \Delta^p,\ x \in M,$$

where $\phi^{-1} : \phi(M \times \{1\} \times \Delta^p) \xrightarrow{\cong} M \times \Delta^p$ (see Figure 1.3). The manifold $\mathcal{R}_p(\phi)^{d+p+1}$ is stratified over Δ^{p+1} with $\mathcal{R}_p(\phi)_0 = [M \times \{0\} \times \Delta^p] \cong M$ and $\partial_0 \mathcal{R}(\phi) = [\phi(M \times \{1\} \times \Delta^p)] = M \times \Delta^p$, so it constitutes a p-simplex in $\Omega B \widetilde{\mathrm{Diff}}^h(M)_\bullet$. Clearly $\mathcal{R}_\bullet$ is a semi-simplicial map.

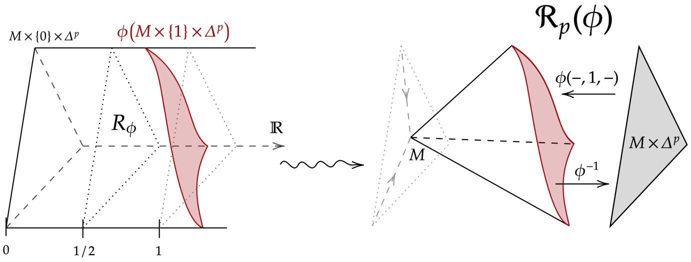

Figure 1.3 The map $\mathcal{R}_\bullet$ with $p = 2$ and $\dim M = 0$.

We have to argue that both of the maps in the zig-zag of Proposition B.1 are equivalences. We begin with the inclusion.

Lemma B.2. *The inclusion* $\widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet \xhookrightarrow{\simeq} \widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_\bullet$ *is a weak equivalence.*

Proof. For a smooth function $\rho : \Delta^p \to \mathbb{R}$, let T_ρ denote the bounded diffeomorphism

$$T_\rho : M \times \mathbb{R} \times \Delta^p \xrightarrow{\cong} M \times \mathbb{R} \times \Delta^p, \quad (x,t,v) \longmapsto (x, t + \rho(v), v).$$

We first show that if $\phi \in \widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_p$ with $\partial_i \phi \in \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_{p-1}$ for all $i = 0, \ldots, p$, then there exists some $\psi \in \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_p$ with $\partial_i \psi = \partial_i \phi$ for $i = 0, \ldots, p$ (simplicially) homotopic to ϕ in $(\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_\bullet, \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet)$. So let ϕ be such a diffeomorphism and set

$$t_- := 1/2 - \min \{ \mathrm{pr}_{\mathbb{R}}(\phi(x, 1/2, v) : x \in M,\ v \in \Delta^p \}.$$

As ϕ is continuous, there exists some $\delta > 0$ such that for a δ-neighbourhood $B_\delta(\partial \Delta^p)$ of $\partial \Delta^p \subset \Delta^p$,

$$\phi(M \times (1/2, \infty) \times B_\delta(\partial \Delta^p)) \subset M \times (1/2, \infty) \times \Delta^p.$$

Let $\rho : \Delta^p \to \mathbb{R}_{\geq 0}$ be a smooth cut-off function such that

$$\rho\,|_{B_{\delta/2}(\partial\Delta^p)} \equiv 0, \qquad \rho\,|_{\Delta^p \setminus B_\delta(\partial\Delta^p)} \equiv t_-.$$

Then $\psi := T_\rho \circ \phi \in \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_p$ is as required. Moreover, the diffeomorphism

$$T_{(-)\cdot\rho} \circ \phi : (M \times \mathbb{R} \times \Delta^p) \times I \xrightarrow{\cong} (M \times \mathbb{R} \times \Delta^p) \times I, \quad (x,t,v,s) \longmapsto T_{s\cdot\rho}(\phi(x,t,v))$$

provides the required simplicial homotopy between ϕ and ψ.

It follows easily from the previous claim that $\pi_p(\widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet) \to \pi_p(\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})_\bullet)$ is an isomorphism for all $p \geq 0$. □

Lemma B.3. *The map $\mathcal{R}_\bullet$ is a weak equivalence.*

Proof. There is a map of fibration sequences

$$\begin{tikzcd}
\Omega B\widetilde{\mathrm{Diff}}(M)_\bullet \arrow[d] & \widetilde{\mathrm{Diff}}(M)_\bullet \arrow[l, "M_{(-)}", "\simeq"'] \arrow[d, "\times \mathrm{Id}_{\mathbb{R}}"] & \\
\Omega B\widetilde{\mathrm{Diff}}^h(M)_\bullet \arrow[d] & \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet \arrow[l, "\mathcal{R}_\bullet"'] \arrow[d] & \\
\widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(M)_\bullet & \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})/\widetilde{\mathrm{Diff}}(M)_\bullet \arrow[l, "{[\mathcal{R}_\bullet]}"'] \arrow[r, hook, "\simeq"] & \widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})/\widetilde{\mathrm{Diff}}(M)_\bullet.
\end{tikzcd}$$

The map $M_{(-)}$ is the mapping cylinder construction, so it is an equivalence. In [WW88, Cor. 5.5] it is shown that the map

$$\pi_*([\mathcal{R}_\bullet]) : \pi_*(\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})/\widetilde{\mathrm{Diff}}(M)) \longrightarrow H_*(C_2; \mathrm{Wh}(M))$$

is injective if $* = 0$ and an isomorphism if $* \geq 1$. Clearly the image of $\pi_0([\mathcal{R}_\bullet])$ lies inside $\frac{I(M)}{\mathcal{D}(M)} \cong \pi_0(\widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(M))$, as $\mathcal{R}_0(\phi)$ is an inertial h-cobordism for any $\phi \in \mathrm{Diff}^b(M \times \mathbb{R})$. By the five lemma, $\pi_*(\mathcal{R}_\bullet)$ is an isomorphism for $* \geq 1$, and $\pi_0(\mathcal{R}_\bullet)$ is injective (note that $\frac{I(M)}{\mathcal{D}(M)}$ is just a set, but this does not cause any difficulties in the argument).

It remains to show that $\pi_0(\mathcal{R}_\bullet)$ is surjective. We do this by an *Eilenberg swindle*-like argument as in [WW88, Cor. 5.5]: namely given an inertial h-cobordism $W \in \Omega B\widetilde{\mathrm{Diff}}^h(M)_0$, fix two trivialisations (rel the left ends) $W \cup -W \cong M \times [0,1]$ and $-W \cup W \cong M \times [0,1]$. Then there are two different ways of identifying the Eilenberg swindle

$$S(W) := \cdots \cup W \cup -W \cup W \cup \cdots = \cdots \cup -W \cup W \cup -W \cup \ldots$$

with $M \times \mathbb{R} = \bigcup_{i \in \mathbb{Z}} M \times [i, i+1]$ (in a bounded way). After shifting by an integer, these two identifications give rise to a bounded diffeomorphism $\phi \in \widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_0$ such that $\mathcal{R}_0(\phi)$ is diffeomorphic to $M \times [0, 2] \cup_{M \times \{2\}} W$ (see Figure 1.4). The homotopy

$$t \in [0,1] \longmapsto M \times [0, 1-2t] \cup_{M \times \{1-2t\}} W$$

provides a 1-simplex in $\Omega B \widetilde{\mathrm{Diff}}^h(M)_\bullet$ between $\mathcal{R}_0(\phi)$ and W, so $\pi_0(\mathcal{R}_\bullet)([\phi]) = [W]$ as required. We also obtain that $\pi_0(\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})/\widetilde{\mathrm{Diff}}(M)) \cong \frac{\mathcal{I}(M)}{\mathcal{D}(M)} \subset H_0(C_2; \mathrm{Wh}(M))$, which very slightly improves [WW88, Cor. 5.5]. $\square$

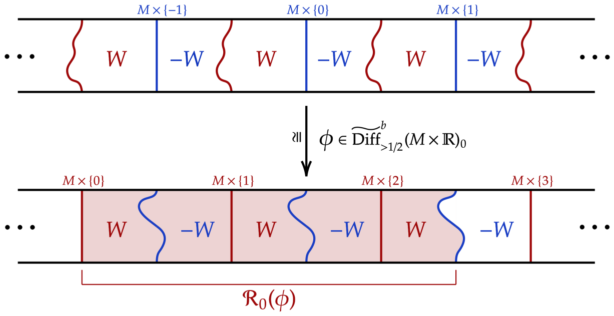

Figure 1.4 Geometric Eilenberg swindle.

Proof of Proposition B.1. Every term in the zig-zag is a Kan complex; the (simplicial) loop space of a Kan complex is Kan, and both $\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R}^\infty)_\bullet$ and $\widetilde{\mathrm{Diff}}^b_{>1/2}(M \times \mathbb{R})_\bullet$ are Kan (here we use the collaring condition on face-preserving maps of Definition 1.3.1). Moreover, the simplicial loop space and the Kan loop group are weakly equivalent functors. Since geometric realisations of weak equivalences between Kan complexes are homotopy equivalences, the homotopy equivalence in the second line of the statement follows. $\square$

B.2 The Whitehead spectrum and Theorem B in the context of Weiss–Williams I

For each space X, there exists a spectrum $\mathbf{Wh}^{\mathrm{Diff}}(X)$, the *non-connective smooth Whitehead spectrum* of X, which recovers the Whitehead group of X,

$$\pi_1^s(\mathbf{Wh}^{\mathrm{Diff}}(X)) = \mathrm{Wh}(X).$$

It is defined to fit in a split[5] cofibre sequence of spectra

$$\Sigma^\infty_+ X \xrightarrow{\iota} \mathbf{A}(X) \twoheadrightarrow \mathbf{Wh}^{\mathrm{Diff}}(X), \tag{B.1}$$

where $\mathbf{A}(-)$ denotes Walhausen's non-connective A-theory spectrum [Wal85, WJR13]. The map ι is the composition of the unit map of A-theory $\Sigma^\infty_+ X = \mathbb{S} \wedge X_+ \to \mathbf{A}(*) \wedge X_+$ and the *assembly map* $\alpha : \mathbf{A}(*) \wedge X_+ \to \mathbf{A}(X)$. The topological and piecewise linear versions of the Whitehead spectrum of a space X coincide, and are denoted, slightly abusively, by $\mathbf{Wh}^{\mathrm{Top}}(X)$. Explicitly, it fits in a similar cofibre sequence of spectra

$$\mathbf{A}(*) \wedge X_+ \xrightarrow{\alpha} \mathbf{A}(X) \twoheadrightarrow \mathbf{Wh}^{\mathrm{Top}}(X). \tag{B.2}$$

The Whitehead spectrum is an invariant of the homotopy type of X, for ι and α are.

With this in mind, let us explain the relation of Theorem B to the work of [WW88]. Following the trend of the paper, define the *connective (smooth) h-cobordism spectrum* to be

$$\mathbf{H}^h_{\mathrm{Diff}}(M) := \tau_{\geq 0}\mathbf{H}_{\mathrm{Diff}}(M),$$

the 0-connective cover of the non-connective version $\mathbf{H}_{\mathrm{Diff}}(M)$. By [WW88, Cor. 5.6] and Corollary 3.1.9, it fits in a C_2-equivariant fibration sequence of spectra

$$\mathbf{H}^s_{\mathrm{Diff}}(M) \longrightarrow \mathbf{H}^h_{\mathrm{Diff}}(M) \longrightarrow H\mathrm{Wh}(M), \tag{B.3}$$

where C_2 acts on $\mathrm{Wh}(M)$ as in Theorem B. In [WW88, Thm. B & C] there is established the outer solid square of the homotopy commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M) \arrow[rr, "\Phi^s", "\approx"'] \arrow[d, hook] & & \Omega^\infty\left(\mathbf{H}^s_{\mathrm{Diff}}(M)_{hC_2}\right) \arrow[d] \\
\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R})/\mathrm{Diff}(M) \arrow[r, phantom, "\overset{\text{Prop. B.1}}{\simeq}"] \arrow[d, hook] & \widetilde{\mathrm{Diff}}^h/\mathrm{Diff}(M) \arrow[r, dashed, "\Phi^h", "\approx_0"'] & \Omega^\infty\left(\mathbf{H}^h_{\mathrm{Diff}}(M)_{hC_2}\right) \arrow[d] \\
\widetilde{\mathrm{Diff}}^b(M \times \mathbb{R}^\infty)/\mathrm{Diff}(M) \arrow[r, phantom, "\simeq"] & \mathrm{Diff}^b(M \times \mathbb{R}^\infty)/\mathrm{Diff}(M) \arrow[r, "\Phi", "\approx_0"'] & \Omega^\infty\left(\mathbf{H}_{\mathrm{Diff}}(M)_{hC_2}\right)
\end{tikzcd}$$

and proved to be homotopy cartesian. The decoration $\approx$ stands for $(\phi_M + 1)$-connected, and $\approx_0$ for $(\phi_M + 1)$-connected onto the components that are hit, where we recall that

---

[5] The splitting is provided by the composition of the Dennis trace map $\mathrm{Tr} : \mathbf{A}(X) \to \Sigma^\infty_+ LX$ postcomposed with the evaluation map $\Sigma^\infty_+ LX \to \Sigma^\infty_+ X$. Note that non-connective K-theory is the universal localising invariant in the sense of [BGT13], so the usual Dennis trace map from *connective* K-theory to topological Hochshild homology indeed factors through non-connective K-theory.

ϕ_M is the concordance stable range for M (see Theorem 1.5.1). The existence of the dashed arrow Φ^h is analogous to that of Φ^s (in a similar notation as in [WW88, §4], replace the filtration of $X := \mathrm{Diff}^b(M \times \mathbb{R}^\infty)$ by $\Sigma\mathrm{Filt}_i(X) := \mathrm{Diff}^b(M \times \mathbb{R}^{i+1})$). The connectivity of Φ^h can be deduced from that of Φ^s and Φ.

Now observe that the composition

$$\Phi^{h/s} : \widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(M) \xhookrightarrow{\simeq_0} |F_\bullet(M)| \overset{\text{Thm. B}}{\simeq} \Omega^\infty\left(H\mathrm{Wh}(M)_{hC_2}\right)$$

provides a filler in the diagram of fibre sequences

$$\begin{tikzcd}
\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M) \arrow[r, "\Phi^s", "\approx"'] \arrow[d] & \Omega^\infty\left(\mathbf{H}^s_{\mathrm{Diff}}(M)_{hC_2}\right) \arrow[d] \\
\widetilde{\mathrm{Diff}}^h/\mathrm{Diff}(M) \arrow[r, "\Phi^h", "\approx_0"'] \arrow[d] & \Omega^\infty\left(\mathbf{H}^h_{\mathrm{Diff}}(M)_{hC_2}\right) \arrow[d] \\
\widetilde{\mathrm{Diff}}^h/\widetilde{\mathrm{Diff}}(M) \arrow[r, dashed, "\Phi^{h/s}", "\simeq_0"'] & \Omega^\infty\left(H\mathrm{Wh}(M)_{hC_2}\right),
\end{tikzcd}$$

where the right vertical sequence is obtained from (B.3) by applying the functor $\Omega^\infty((-)_{hC_2})$. The lower subsquare ought to be homotopy commutative, but we do not have a proof of this claim.

Chapter 2

A Weiss–Williams theorem for spaces of embeddings

2.1 Introduction

The classical approach to study the homotopy type of the diffeomorphism group $\mathrm{Diff}_\partial(M)$ of a compact, possibly with boundary, high-dimensional manifold M^d (i.e. $d \geq 5$) is based on the so called *surgery-pseudoisotopy* program, which focuses on the homotopy fibre sequence

$$(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \longrightarrow B\mathrm{Diff}_\partial(M) \xrightarrow{i} B\widetilde{\mathrm{Diff}}_\partial(M). \tag{2.1.1}$$

The right-hand term is the classifying space of the simplicial group $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$ of *block diffeomorphisms* of M (cf. Definition 2.2.7), an approximation to the ordinary diffeomorphism group of M that closely resembles the behaviour of the topological monoid $h\mathrm{Aut}_\partial(M)$ of homotopy automorphisms of M; for instance, one of the defining properties of $\widetilde{\mathrm{Diff}}_\partial(-)$ is that it satisfies a natural equivalence $\widetilde{\mathrm{Diff}}_\partial(M \times I) \simeq \Omega\widetilde{\mathrm{Diff}}_\partial(M)$, which also holds for $h\mathrm{Aut}_\partial(-)$ (but is very much not true for $\mathrm{Diff}_\partial(-)$). *Surgery theory*, as developed by Browder, Novikov, Ranicki, Sullivan, Wall, et al., roughly studies the difference between $\widetilde{\mathrm{Diff}}_\partial(M)$ and $h\mathrm{Aut}_\partial(M)$ in terms of the mapping space $(G/O)_*^{(M,\partial M)}$ and the L-theory of the group ring $\mathbb{Z}[\pi_1(M)]$, making the homotopy type of $\widetilde{\mathrm{Diff}}_\partial(M)$ theoretically accessible via homotopy theory and L-theory.

It is in understanding the homotopy type of $(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M)$, the homotopy fibre of the map i, where *pseudoisotopy theory* [Igu88, HW73] comes into play. Originally, this theory was concerned with the study of the topological group $C(M)$ of *concordances* or *pseudoisotopies* of M consisting of diffeomorphisms of $M \times I$ that restrict to the identity on a neighbourhood of $M \times \{0\} \cup \partial M \times I$; in other words, pseudoisotopies of M are precisely the automorphisms of the trivial h-cobordism starting at M. The spaces $C(M)$ and $\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)$ are intimately related, as was first made precise by Hatcher [Hat78, Prop. 2.1] through a spectral sequence. Hatcher's realisation eventually evolved into the following celebrated theorem of Weiss and Williams [WW88, Thm. A], which will be of central importance all throughout this paper.

Theorem 2.1.1 (Weiss–Williams)**.** *Let M^d be a compact smooth d-manifold. There exists a map*

$$\Phi^{\mathrm{Diff}} : (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \longrightarrow \Omega^\infty\left(\mathbf{H}^s(M)_{hC_2}\right)$$

which is $(\phi(d)+1)$-connected, where $\phi(d)$ denotes the concordance stable range of dimension d (which by Igusa's theorem [Igu88] is at least $\min(\frac{d-4}{3}, \frac{d-7}{2})$*).*

The C_2-spectrum $\mathbf{H}^s(M)$ is the 1-connective cover of a spectrum $\mathbf{H}(M)$ which is roughly built out of deloopings of spaces of smooth h-cobordisms (cf. Notation 2.2.5 and Section 3.1.2), and whose involution corresponds (up to a minus sign) to "reversing the direction of an h-cobordism". This latter spectrum has a close connection to algebraic K-theory: $\Omega^\infty \mathbf{H}(M)$ is equivalent to the (smooth) *stable h-cobordism space* $\mathcal{H}(M)$ (cf. Remark 2.3.10), which fits in a fibre sequence of spaces [WJR13]

$$\mathcal{H}(M) \longrightarrow Q_+M := \Omega^\infty\Sigma^\infty_+ M \xrightarrow{\alpha} A(M), \tag{2.1.2}$$

where $A(M)$ denotes Waldhausen's A-theory space of M (cf. [Wal85]). This sequence is natural in codimension-zero embeddings, and the map α is (naturally) a split injection. Even though the homotopy type of $A(M)$ and its involutions are difficult to understand in general, much can be said when M is homotopy equivalent to a point [Rog02, Rog03, BM19] or the circle [Hes09], or when working rationally [BF86].

2.1.1 The surgery-pseudoisotopy program for spaces of embeddings

The homotopy type of embedding spaces is intrinsically tied to that of diffeomorphism groups as seen via the *isotopy extension theorem.* More precisely, let M^d be as before and let $\iota : P \hookrightarrow M$ be a compact submanifold that meets ∂M transversely. Write $\mathrm{Emb}_{\partial_0}(P, M)$ for the space of smooth embeddings of P into M which agree with ι in a neighbourhood of $\partial_0 P := P \cap \partial M$ and send $\partial P - \partial_0 P$ to the interior of M. Then there is a homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \longrightarrow B\mathrm{Diff}_\partial(M-\nu P) \longrightarrow B\mathrm{Diff}_\partial(M), \tag{2.1.3}$$

where νP is a small tubular neighbourhood of the standard embedding $\iota : P \hookrightarrow M$, and the subscript $\langle\iota\rangle$ stands for the collection of components of $\mathrm{Emb}_{\partial_0}(P,M)$ that are hit by the restriction map $\iota^* : \mathrm{Diff}_\partial(M) \to \mathrm{Emb}_{\partial_0}(P,M)$ that sends a diffeomorphism ϕ to $\phi \circ \iota$. In this sense, embedding spaces are the corresponding "relative analogues" of diffeomorphism groups, and often their homotopy types become easier to study.

In this paper we would like to advertise a direct approach for studying the homotopy type of embedding spaces (in a range of degrees) which is analogous to the one for diffeomorphism groups previously surveyed. As before, one first analyses the space of block embeddings $\widetilde{\mathrm{Emb}}_{\partial_0}(P,M)$ via relative surgery methods; the main result in this direction is due to Browder–Casson–Sullivan–Wall (cf. [GKW01, Thm. 2.2.1]), and asserts that, as long as the codimension of $P \subset M$ is at least 3, then the space of block embeddings is the homotopy pullback of a diagram involving so called *Poincaré block embeddings* and *immersions* and ordinary block immersions. Due to the Smale–Hirsch immersion theorem, all the ingredients that come into the mix are accessible through homotopy theory and thus, up to extensions, so are block embeddings.

Following the same strategy as for the classical surgery-pseudoisotopy program, it remains to understand the difference between ordinary and block embeddings, i.e. the homotopy fibre

$$\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M) := \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)), \tag{2.1.4}$$

by means of pseudoisotopy theory. This space also fits in another homotopy fibre sequence

$$\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M) \longrightarrow (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M-\nu P) \xrightarrow{\mu} (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \tag{2.1.5}$$

obtained as the fibre of the map from (2.1.3) to its block analogue (see (2.3.4)). What was previously fulfilled by Theorem 2.1.1 for the pseudoisotopy part of diffeomorphism groups seems to be missing in the case of embedding spaces; the best result known in this direction is *Morlet's lemma of disjunction* [BLR75, Thm. 3.1] which, in that reformulation, determines the connectivity of the map μ.

Our first main result fills in this gap in the surgery-pseudoisotopy program for embedding spaces, and describes the homotopy type of $\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M)$ in a range outside of the connectivity of μ.

Theorem C. *There exists a map*

$$\Phi^{\mathrm{Emb}} : \mathrm{Emb}^{(\sim)}_{\partial}(P, M) \longrightarrow \Omega^{\infty} \left(\mathbf{CE}(P, M)_{hC_2}\right)$$

which is $\phi_{C\mathrm{Emb}}(d, p)$-connected if the handle dimension p of P relative to $\partial_0 P$ satisfies $p \leq d - 3$. Here $\phi_{C\mathrm{Emb}}$ is the concordance embedding stable range (see (2.1.6)) and

$$\mathbf{CE}(P, M) := \mathrm{hofib}(\mathbf{H}(M - \nu P) \to \mathbf{H}(M)).$$

Remark 2.1.2. The involutions in the h-cobordism spectra involved in the statement of Theorem C are exactly those of Theorem 2.1.1, which naturally arise from Weiss' orthogonal calculus (see Sections 2.2.1 and C.1.2). When M is stably parallelisable (and localising away from 2), we relate these involutions to well-known algebraic ones in Theorem 3.1.13 and Corollary 3.1.17 (cf. Notation 3.1.1 for conventions). See also Corollary 3.1.9 for the effect of these involutions on $\pi_0^s(\mathbf{H}(M)) = \mathrm{Wh}(\pi_1(M))$ in terms of Milnor's involution [Mil66].

Remark 2.1.3 (Topological version of Theorem C)*.* As stated, Theorem C only applies to smooth embeddings, but an analogous statement in $CAT = \mathrm{Top}$ also holds after some adjustments: Firstly, the smooth embedding spaces should be replaced by the spaces of locally flat topological embeddings, and the h-cobordism spectra by their topological analogues. Moreover, the result is only valid when P has geometric codimension zero in M; this is because a topological analogue of Proposition 2.3.3 cannot be true, as we explain in Remark 2.3.4. One should bear in mind, however, that the bound (2.1.7) is a priori only valid for smoothable topological manifolds. See also Remarks 2.3.6, 2.3.2 and D.5 for modified arguments in the topological setting.

There are two remarkable features of Theorem C that make this result specially suitable for computational purposes. Let us comment on these now.

2.1.1.1 The concordance embedding stable range

Fix $\iota : P \hookrightarrow M$ as before. A *concordance embedding* of P into M is an embedding $\varphi : P \times I \hookrightarrow M \times I$ such that

(*a*) $\varphi^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0, 1$ and

(b) φ agrees with the inclusion $\iota \times \mathrm{Id}_I$ on a neighbourhood of $P \times \{0\} \cup \partial_0 P \times I$.

We denote by $C\mathrm{Emb}(P, M)$ the space of all such embeddings, topologised as a subspace of $\mathrm{Emb}(P \times I, M \times I)$. There are stabilisation maps

$$\Sigma : C\mathrm{Emb}(P, M) \to C\mathrm{Emb}(P \times I, M \times I)$$

given by taking the product of an embedding with I and unbending corners appropriately (cf. [GKK23, Fig. 1]), and the *concordance embedding stable range* of the pair (M, P), denoted $\phi_{C\mathrm{Emb}}(M, P)$, is the largest integer k such that all the stabilisations in

$$C\mathrm{Emb}(P, M) \xrightarrow{\Sigma} C\mathrm{Emb}(P \times I, M \times I) \xrightarrow{\Sigma} C\mathrm{Emb}(P \times I^2, M \times I^2) \xrightarrow{\Sigma} \dots$$

are k-connected. Then, the *concordance stable range* for a tuple (d, p) is

$$\phi_{C\mathrm{Emb}}(d, p) := \min \left\{ \phi_{C\mathrm{Emb}}(M, P) : \dim M = d \text{ and } h\text{-}\dim(P, \partial_0 P) = p \right\}. \qquad (2.1.6)$$

Goodwillie–Krannich–Kupers [GKK23] have recently shown that if $p \leq d - 3$, then

$$\phi_{C\mathrm{Emb}}(d, p) \geq 2d - p - 5, \qquad (2.1.7)$$

which is far beyond Igusa's lower bound for the concordance stable range $\phi(d)$. In the concordance stable range $\phi(d)$, Theorem C is a consequence of Theorem 2.1.1 and the isotopy extension sequence (2.1.5), so our main contribution is improving the connectivity of the map Φ^{Emb} to the concordance embedding stable range $\phi_{C\mathrm{Emb}}(d, p)$. We will also see in Remark 3.2.6 that the lower bound of (2.1.7) is the best one could do (this was already apparent in [GKK23]).

Remark 2.1.4. In fact, our proof will show that the map Φ^{Emb} of Theorem C is $\phi_{C\mathrm{Emb}}(M, P)$-connected under the codimension assumption $p \leq d - 3$.

2.1.1.2 Relative algebraic K-theory via trace methods

Given a map of spaces $Y \to X$, let $A(Y \to X)$ denote the homotopy fibre of the induced map $A(Y) \to A(X)$. By (2.1.2), there is an equivalence of spaces

$$\Omega A(M - P \to M) \simeq \Omega^\infty \mathbf{CE}(P, M) \times \Omega^2 Q(M/M - P).$$

The codimension assumption on the embedding $\iota : P \subset M$ in the statement of Theorem C guarantees that the inclusion $M - P \to M$ is 2-connected. This can be used to our advantage, as firstly it ensures that the spectrum $\mathbf{CE}(P, M)$ is connective (see Lemma 2.3.12), but more importantly that, via *trace methods*, the homotopy type of $A(M - P \to M)$ is far more accessible than that of $A(M - P)$ and $A(M)$ on their own.

Trace methods are concerned with the study of *topological cyclic homology* [BHM93], denoted $TC(-)$, and related invariants as an approximation to algebraic K-theory. This is something sensible to do by the seminal work of Dundas–Goodwillie–McCarthy [DM94, Dun97, DGM12], who showed that the cyclotomic trace map provides an equivalence of relative theories $A(Y \to X) \simeq TC(Y \to X)$, so long as $Y \to X$ is 2-connected. The treatment of TC by Nikolaus–Scholze [NS18] provides even further computational control of this invariant. In the cases we are concerned with (spherical group rings), the homotopy type of TC was fully described by Bökstedt–Hsiang–Madsen [BHM93] in terms of the stable homotopy of the free loop space $L(-) := \mathrm{Map}(S^1, -)$ together with its natural S^1-action and cyclotomic structure. When working over the field of rational numbers, this whole story simplifies even further by Goodwillie's isomorphism [Goo86]

$$\pi_*(A(Y \to X)) \otimes \mathbb{Q} \cong HC_*(\Omega X, \Omega Y; \mathbb{Q}) \cong H_*^{S^1}(LX, LY; \mathbb{Q}), \qquad (2.1.8)$$

where HC_* denotes Connes' cyclic homology, and $H_*^{S^1}$ stands for the S^1-equivariant homology.

So as we have just seen, the homotopy type of (the infinite loop space of) the connective spectrum $\mathbf{CE}(P, M)$ of Theorem C is pretty accessible in general. However, one still needs to deal with the involution appearing in the statement in order to apply the result, which is a rather technical task that, so far, had only been carried out rationally by Bustamante–Farrell–Jiang [BFJ20] (they relate this involution to one on the right hand side of (2.1.8)). Integrally, one has to proceed with more care; our analysis in Section 3.1 deals with this issue localised away from 2 and when M is stably parallelisable.

2.1.2 The homotopy type of spaces of long knots

The homology and homotopy of *spaces of long knots* $\mathrm{Emb}_\partial(D^p, D^d)$ has been subject to extensive reasearch in recent years, especially through the lens of embedding calculus and its relation to the little disks operad and graph complexes. See for instance Volić [Vol06], Watanabe [Wat07], Sinha [Sin09], Budney and Cohen [Bud08, BC09] for when $p = 1$ and $d = 3, 4$ mainly, or more modern treatments as in Arone–Turchin [AT14, AT15], Dwyer–Hess [DH12], Boavida de Brito–Weiss [BdBW18], at last culminating in the work of Fresse–Turchin–Willwacher [FTW17] where a complete description of $\pi_*(\mathrm{Emb}_\partial(D^p, D^d)) \otimes \mathbb{Q}$ is given in terms of the homology of the *hairy graph complex*. See also Boavida de Brito–Horel [BdBH21] for some torsion computations in the homotopy groups of spaces of long knots when $p = 1$.

The second main result of this paper is a full description of the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d - p \geq 3$ roughly in the concordance embedding stable range (2.1.6) and localised away from 2. This is done by analysing the homotopy fibre sequence

$$\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \longrightarrow \mathrm{Emb}_\partial(D^p, D^d) \longrightarrow \widetilde{\mathrm{Emb}}_\partial(D^p, D^d) \tag{2.1.9}$$

following the surgery-pseudoisotopy program for embedding spaces surveyed in Section 2.1.1, a crucial step of which is Theorem C. Given a finite dimensional virtual G-representation ρ over $\mathbb{R}$, denote by $\mathbb{S}^\rho$ the representation sphere spectrum associated to it; we will consider its homotopy orbit spectrum $\mathbb{S}^\rho_{hG}$, which is equivalent to the Thom spectrum of the associated virtual vector bundle $EG \times_G \rho \to BG$. Let ψ_m denote the real m-dimensional representation of the dihedral group D_m (seen as a subgroup of the symmetric group Σ_m) given by permuting the factors of $\mathbb{R}^m$, and let $\sigma : C_2 = \{\pm 1\} \hookrightarrow \mathbb{R}^\times$ be the sign representation (also regarded as a D_m-representation by restriction along the determinant $D_m \hookrightarrow O(2) \overset{\det}{\to} \{\pm 1\} = C_2$).

Theorem D. *For $p \leq d - 3$ and $d \geq 5$, consider the virtual D_m-representations*

$$\rho_m := (d+1)(\sigma - 1) + \psi_m \otimes (d - p - 3 + \sigma).$$

Then the homotopy fibre sequence (2.1.9), upon localising away from 2 and taking $(\phi_{C\mathrm{Emb}}(d,p) - 1)$-th Postnikov sections, takes the form

$$\textstyle\prod_{m \geq 2} \Omega^\infty\Big(\mathbb{S}^{\rho_m}_{hD_m}\Big) \longrightarrow \mathrm{Emb}_\partial(D^p, D^d) \longrightarrow \Omega^p \,\mathrm{hofib}\,\Big(G(d-p)/O(d-p) \to G/O\Big). \tag{2.1.10}$$

The resulting sequence is split if $p \geq 2$, and splits after being looped once if $p = 1$.

Remark 2.1.5. (*i*) The spaces $G(n)/O(n)$ and G/O appearing in (2.1.10) denote the homotopy fibres of the natural maps $BO(n) \to BG(n)$ and $BO \to BG$, respectively, where $G(n)$ is the topological grouplike monoid of self-homotopy equivalences of S^{n-1}, and G is the homotopy colimit of the suspension maps $G(n) \to G(n+1)$. Understanding the homotopy groups of these spaces roughly amounts to understanding unstable and stable homotopy groups of spheres.

(*ii*) The space $\mathrm{Emb}_\partial(D^p, D^d)$ is an $\mathbb{E}_p$-algebra, and so it can indeed be localised. When $d-p \geq 3$, it is exactly $(2d-3p-4)$-connected by work of Budney [Bud08, Prop. 3.9]. So when $2d-3p-4<0$, by *localising* $\mathrm{Emb}_\partial(D^p, D^d)$ we really mean localising each of its connected components one at a time (and *not* localising the abelian group $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))$ directly). The same applies to the outer terms in (2.1.9).
(*iii*) When $p=1$ and $d=4$ (i.e. the lowest dimensional case of interest if $d-p\geq 3$), Theorem D holds after looping both (2.1.9) and (2.1.10); see Remark 3.2.1. Using this to study the homotopy groups of $\mathrm{Emb}_\partial(D^1, D^4)$, however, yields weaker results than the ones in [Bud08, Prop. 3.9].

For A an abelian group and ℓ a prime, write $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$, i.e. the localisation of A at the prime ℓ. We will compute some of the homotopy groups of the fibre in (2.1.10) in Section 3.2.2, and hence deduce new torsion information about the homotopy groups of $\mathrm{Emb}_\partial(D^p, D^d)$ in high-dimensions (i.e. $d \geq 5$).

Corollary E (Propositions 3.2.5 & 3.2.7)**.** *For $d-p\geq 3$ and ℓ an odd prime, there are isomorphisms*

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)} \oplus \bigoplus_{m\geq 2} \pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}$$

in degrees $* \leq \phi_{C\mathrm{Emb}}(d,p)-1$. When $m\geq 2$ and $\ell \nmid 2m$,

- *if d is even and p is even, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 3,5,7,\ldots \\ 0, & \textit{otherwise.} \end{cases}$$

- *if d is odd and p is odd, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 2,4,6,\ldots \\ 0, & \textit{otherwise.} \end{cases}$$

- *if d is even and p is odd, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 5,9,13,\ldots \\ 0, & \textit{otherwise.} \end{cases}$$

- *if d is odd and p is even, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 3, 7, 11, \dots \\ 0, & \textit{otherwise.} \end{cases}$$

If ℓ divides m, the computation of $\pi_*^s(\mathbb{S}^{\rho_m}_{D_m})_{(\ell)}$ must be treated case by case. When $m = \ell = 3$ and $d - p = 3$, the first few such homotopy groups are given in Table 3.1 (cf. Proposition 3.2.7 for notation).

Rationally, this computation roughly recovers the homology of the 0- and 1-loop order part of the hairy graph complex appearing in [FTW17, Eq. 2] (see Remark 3.2.6 for more details).

Structure of Chapters 2 and 3

Sections 2.2 and 2.3 will be devoted to the proof of Theorem C. We start by briefly reviewing Weiss' theory of orthogonal calculus and then, in Section 2.2.2, we present the orthogonal functors that will play a role in the proof of Theorem C. In doing this, we will have to carefully describe the topology on spaces of bounded diffeomorphisms in such a way that we can employ the machinery of orthogonal calculus in this setting. After reducing to the codimension zero case in Section 2.3.1, we use the results in the preceding section to define the map Φ^{Emb} in Section 2.3.3 and analyse its connectivity in 2.3.4.

As a consequence of Theorem C, in Section 2.4 we establish a splitting result (Theorem 2.4.2) for embedding spaces of manifolds containing interval factors reminiscent of work of Burghelea–Lashof [BL82, Cor. E].

Section 3.1 deals with the analysis of the C_2-spectra involved in the statements of Theorems 2.1.1 and C. The main results in this direction are Theorem 3.1.13 and Corollary 3.1.17, where the involutions on these spectra are expressed (up to homotopy) in terms of the standard involution in algebraic K-theory.

Section 3.2 is devoted to Theorem D, whose proof is a formal consequence of the results in the preceding sections. We then draw some conclusions on the homotopy groups of spaces of long knots in Section 3.2.2.

Appendix C deals with some subtleties regarding the definition of the first derivative of an orthogonal functor as an $O(1)$-spectrum, and with a technical argument in the proof of Proposition 2.2.2.

In Appendix D we explore certain aspects related to spaces of bounded diffeomorphisms and embeddings. Namely in Section D.1 we show that the topological models for these spaces introduced in Section 2.2.2 coincide (up to weak equivalence) with the simplicial ones of Definition 2.2.7. In Section D.2 we give a "moduli space of manifolds" description for the classifying space of the bounded diffeomorphism group.

In Appendix E we show that the h-cobordism stabilisation map anti-commutes with the involutions in these spaces. This is analogous to a result of Hatcher [Hat78, Appendix I, Lem.] and Burghelea–Lashof [BL82, Cor. A7] for spaces of concordance diffeomorphisms.

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2.2 Orthogonal calculus and spaces of bounded diffeomorphisms

Much of the proof of Theorem 2.1.1 in [WW88] is an application of Weiss' *orthogonal calculus* but in disguise, as this theory was not yet formalised at the time. In this section we briefly review the main aspects of this theory and develop some necessary tools required for the proof of Theorem C.

2.2.1 A quick tour through orthogonal calculus

Weiss' *orthogonal calculus* [Wei95] is a calculus of functors useful to understand objects of geometric flavour. It studies *continuous* functors from the category $\mathcal{J}$ of real finite-dimensional inner product vector spaces and linear isometries to the category of (compactly generated weakly Hausdorff) spaces Top. Such a functor $F : \mathcal{J} \to \mathsf{Top}$ is said to be *continuous* if the evaluation map

$$\mathrm{mor}_{\mathcal{J}}(U, V) \times F(U) \longrightarrow F(V)$$

is continuous for all $U, V \in \mathcal{J}$. Here $\mathrm{mor}_{\mathcal{J}}(U, V)$ denotes the *Stiefel manifold* of linear isometries from U to V, so that $\mathcal{J}$ is enriched over Top. We will work in a slightly different setup, where Top is replaced by the category Top_* of pointed spaces and $\mathcal{J}$ is replaced by the pointed topological category $\mathcal{J}_0$ with the same objects and with

$$\mathrm{mor}_{\mathcal{J}_0}(U, V) := \mathrm{mor}_{\mathcal{J}}(U, V)_+,$$

as morphism spaces. Similarly, a functor $F : \mathcal{J}_0 \to \mathsf{Top}_*$ is *continuous* if the evaluation map

$$\mathrm{mor}_{\mathcal{J}_0}(U, V) \wedge F(U) \longrightarrow F(V)$$

is continuous for all $U, V \in \mathcal{J}_0$. Such a functor $F(-)$ is also sometimes called an *orthogonal functor.*

The machinery of orthogonal calculus associates to each such orthogonal functor $F(-)$ a sequence of (naïve) $O(k)$-spectra $\Theta F^{(k)}$ for $k \geq 1$—the *derivatives of* F—which fit in a tower

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & T_2F(-) \arrow[d] & \Omega^\infty\left((S^{2\cdot(-)} \wedge \Theta F^{(2)})_{hO(2)}\right) \arrow[l] \\
 & T_1F(-) \arrow[d] & \Omega^\infty\left((S^{1\cdot(-)} \wedge \Theta F^{(1)})_{hO(1)}\right) \arrow[l] \\
F(-) \arrow[r, "\eta_0"'] \arrow[ur] \arrow[uur] \arrow[uuur] & T_0F(-) \arrow[r, equal] & F(\mathbb{R}^\infty)
\end{tikzcd} \tag{2.2.1}$$

of orthogonal functors—the *Taylor tower.* Here

- $S^{k\cdot V}$ is the one point compactification of $k \cdot V := \mathbb{R}^k \otimes V$, which is acted upon by $O(k)$ in the $\mathbb{R}^k$ component, and diagonally on the smash $S^{k\cdot V} \wedge \Theta F^{(k)}$,

- the right hand horizontal maps—the *layers*—indicate the inclusions of the homotopy fibres of the subsequent vertical maps between the *stages* $T_kF(-)$ of the tower,

- the zeroth stage $T_0F(-)$ is given by $T_0F(V) := \mathrm{hocolim}_k\, F(V \oplus \mathbb{R}^k)$, and thus admits a canonical equivalence from the constant orthogonal functor with *value at infinity* $F(\mathbb{R}^\infty) := \mathrm{hocolim}_k\, F(\mathbb{R}^k)$. The map $\eta_0 : F(V) \to T_0F(V)$ is simply the inclusion map.

In the proof of Theorem C we will analyse the Taylor tower (2.2.1) only up to the first layer, so we shall now describe the spectrum $\Theta F^{(1)}$ in detail. For $V \in \mathcal{J}_0$, consider

$$F^{(1)}(V) := \operatorname{hofib}(F(V) \to F(V \oplus \mathbb{R})),$$

the homotopy fibre of the map induced by the standard inclusion $V \to V \oplus \mathbb{R}$. These spaces inherit an $O(1)$-action by declaring $-1 \in O(1)$ to act on V and $V \oplus \mathbb{R}$ by -1 on *all* coordinates. There are $O(1)$-equivariant maps

$$s_V : S^1 \wedge F^{(1)}(V) \longrightarrow F^{(1)}(V \oplus \mathbb{R}), \tag{2.2.2}$$

given, roughly, by performing a 180º rotation of $V \oplus \mathbb{R}^2$ about the 2-plane $0 \oplus \mathbb{R}^2$; we give an explicit model of this map in Section C.1.1. As notation suggests in (2.2.2), $O(1)$ acts trivially on the suspension coordinate. (In general, we adopt the convention that S^n denotes the n-sphere with the trivial $O(1)$-action.) Then, the $O(1)$-spectrum $\Theta F^{(1)}$ has $F^{(1)}(\mathbb{R}^n)$ as its n-th space, and $s_{\mathbb{R}^n}$ as the structure map.

Remark 2.2.1. This is not quite the $O(1)$-action described in [Wei95, Prop. 3.1]; rather, it more closely follows the convention adopted in [WW88, p. 601]. We justify our convention choice in Section C.1.2.

For $V \in \mathcal{J}_0$, let $S(V)$ denote the unit sphere of V, seen as an unbased $O(1)$-space by the antipodal action. The following proposition will be the main ingredient for the construction of the map Φ^{Emb} of Theorem C.

Proposition 2.2.2. *Let $F : \mathcal{J}_0 \to \mathsf{Top}_*$ be an orthogonal functor. For each $n \geq 0$, there are maps*

$$\Phi^F_n : \operatorname{hofib}(F(0) \to F(\mathbb{R}^n)) \xrightarrow{\eta_1} \operatorname{hofib}(T_1F(0) \to T_1F(\mathbb{R}^n)) \simeq \Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \tag{2.2.3}$$

giving rise to a map of homotopy fibre sequences

$$\begin{tikzcd}
\operatorname{hofib}(F(0) \to F(\mathbb{R}^n)) \arrow[r] \arrow[d, "\Phi^F_n"] & \operatorname{hofib}(F(0) \to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "\Phi^F_{n+1}"] & \operatorname{hofib}(F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1})) =: \Theta F^{(1)}_n \arrow[d, "\text{stab.}"] \\
\Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(S(\mathbb{R}^{n+1})_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(\Sigma^n \Theta F^{(1)}),
\end{tikzcd}$$

*where the vertical map "*stab.*" is $\Theta F^{(1)}_n \hookrightarrow \operatorname{hocolim}_k \Omega^k(\Theta F^{(1)}_{n+k})$. Letting $n \to \infty$ in (2.2.3), we get*

$$\Phi^F_\infty : \operatorname{hofib}(F(0) \to F(\mathbb{R}^\infty)) \longrightarrow \Omega^\infty(EO(1)_+ \wedge_{O(1)} \Theta F^{(1)}) =: \Omega^\infty(\Theta F^{(1)}_{hO(1)}). \tag{2.2.4}$$

Proof. We first need to argue that there is a natural weak equivalence between $\mathrm{hofib}(T_1F(0) \to T_1F(V))$ and $\Omega^\infty(S(V)_+ \wedge_{O(1)} \Theta F^{(1)})$ for $V \in \mathcal{J}_0$. Observe that there is an $O(1)$-equivariant homotopy cofibre sequence

$$S(V)_+ \longrightarrow *_+ \longrightarrow \Sigma S(V) \cong S^{V\cdot\sigma},$$

where Σ here stands for the unreduced suspension. For X an $O(1)$-spectrum we thus obtain a fibre sequence

$$\begin{tikzcd}
\Omega^\infty\left((S(V)_+ \wedge X)_{hO(1)}\right) \simeq \Omega^\infty(S(V)_+ \wedge_{O(1)} X) \arrow[r] & \Omega^\infty(X_{hO(1)}) \arrow[d] \\
& \Omega^\infty\left((S^{V\cdot\sigma} \wedge X)_{hO(1)}\right),
\end{tikzcd}$$

where the first equivalence follows since the pointed $O(1)$-action on $S(V)_+$ is free. The above sequence when $X = \Theta F^{(1)}$ is the left vertical fibration in the following commutative diagram of fibre sequences:

$$\begin{tikzcd}
\Omega^\infty\left(S(V)_+ \wedge_{O(1)} \Theta F^{(1)}\right) \arrow[r, "\simeq"] \arrow[d] & \mathrm{hofib}(T_1F(0) \to T_1F(V)) \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^\infty(\Theta F^{(1)}_{hO(1)}) \arrow[r] \arrow[d] & T_1F(0) \arrow[r] \arrow[d] & T_0F(0) = F(\mathbb{R}^\infty) \arrow[d, "\wr"] \\
\Omega^\infty\left((S^{V\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\right) \arrow[r] & T_1F(V) \arrow[r] & T_0F(V) = F(\mathbb{R}^\infty).
\end{tikzcd}$$

This gives rise to the required weak equivalence, which is visibly natural in $V \in \mathcal{J}_0$.

Clearly because of this naturality, the left square of the diagram in the statement commutes, so it remains to show that the right one is homotopy commutative too. A similar claim (without proof) appears in [Wei95, Ex. 10.1], so we provide our own proof. However, as it is rather technical, we defer it to Section C.2. □

2.2.2 The orthogonal functors of bounded diffeomorphisms

All throughout, let $\iota : P \hookrightarrow M$ be as in the statement of Theorem C. In this section we present the orthogonal functors that will play a role in the proof of Theorem C. These are built out of spaces of bounded diffeomorphisms, for which we will present point-set topological models that agree up to weak equivalence with the more classical simplicial ones.

Let $V \in \mathcal{J}$ be an inner product finite-dimensional real vector space with associated norm $\|-\|_V$, and let Q and Q' be smooth (possibly non-compact) manifolds equipped with proper maps $\pi : Q \to V$ and $\pi' : Q' \to V$. For $t \geq 0$, a smooth map $f : Q \to Q'$ is said to be *t-bounded* if the set $\{\|\pi'(f(q)) - \pi(q)\|_V : q \in Q\} \subset \mathbb{R}$ is bounded by t. More generally, f is *bounded* if it is t-bounded for some $t \geq 0$. If $Q = N \times V$ for some compact manifold N, π will be assumed to be the projection to V.

Definition 2.2.3. *Let $V \in \mathcal{J}$. The* ***space of bounded diffeomorphisms*** *of $M \times V$ relative to $\partial M \times V$ is*

$$\mathrm{Diff}^b_\partial(M \times V) := \{(t, \phi) \in [0, \infty) \times \mathrm{Diff}_\partial(M \times V) : \phi \textit{ is } t\textit{-bounded}\},$$

endowed with the subspace topology inherited from the product $[0, \infty) \times \mathrm{Diff}_\partial(M \times V)$*. Here* $\mathrm{Diff}_\partial(M \times V)$ *is endowed with the Whitney C^∞-topology. It is a group-like topological monoid under the rule*

$$(t, \phi) \cdot (t', \phi') := (t + t', \phi \circ \phi').$$

Example 2.2.4. Orthogonal calculus was largely inspired by the work of Weiss–Williams in [WW88], as can be seen in [WW88, Digr. 3.8]. For $U \in \mathcal{J}_0$, let $B\mathrm{Diff}^b_\partial(M \times U)$ be the classifying space of the group-like topological monoid just introduced. Denote by $B(-)$ the orthogonal functor given by $B(U) := B\mathrm{Diff}^b_\partial(M \times U)$ and, for $i : U \to V$ a morphism in $\mathcal{J}_0$, write $V = U \oplus U^\perp$ and let $B(i)$ be induced by the monoid homomorphism sending $(t, \phi) \in \mathrm{Diff}^b_\partial(M \times U)$ to $(t, \phi \oplus \mathrm{Id}_{U^\perp}) \in \mathrm{Diff}^b_\partial(M \times U \oplus U^\perp)$. Then

$$\Phi^B_\infty : \mathrm{Diff}^b_\partial(M \times \mathbb{R}^\infty)/\mathrm{Diff}_\partial(M) \longrightarrow \Omega^\infty(\Theta B^{(1)})$$

should be[1] the map from [WW88, Thm. C], and Proposition 2.2.2 recovers [WW88, Prop. 3.1] in this case.

Notation 2.2.5. *In the remainder of Section 2.2, we will denote by $E(-)$ and $B(-)$ the orthogonal functors given on objects by*

$$E(V) := B\mathrm{Diff}^b_\partial((M - \nu P) \times V), \qquad B(V) := B\mathrm{Diff}^b_\partial(M \times V), \tag{2.2.5}$$

where νP is an open tubular neighbourhood of the embedding $\iota : P \subset M$, and on morphisms as in Example 2.2.4. There is a natural transformation $E(-) \to B(-)$

---

[1] Though Φ^B_∞ is not visibly the same map as the one appearing in loc. cit., they share the same formal properties by Proposition 2.2.2.

given by extending a diffeomorphism by the identity on $\nu P \times (-)$*, and the orthogonal functor* $F(-) := \mathrm{hofib}(E(-) \to B(-))$ *will play an especially important role in the proof of Theorem C. We will often use the following notation for the derivatives of these functors:*

$$\mathbf{CE}(P, M) := \Theta F^{(1)}, \qquad \mathbf{H}(M - \nu P) := \Theta E^{(1)}, \qquad \mathbf{H}(M) := \Theta B^{(1)}. \tag{2.2.6}$$

Thus, $\mathbf{CE}(P, M)$ *is, by definition, the homotopy fibre of the map* $\mathbf{H}(M - \nu P) \to \mathbf{H}(M)$.

Remark 2.2.6. Let us comment on the notation in (2.2.6). Write $H(M)$ for the space of smooth h-cobordisms starting at M (cf. Section 3.1.2). We will see in Remark 2.3.10 that there is an equivalence (of spaces)

$$\Omega^\infty \mathbf{H}(M) \simeq \mathcal{H}(M) := \underset{k}{\mathrm{hocolim}}\ H(M \times D^k), \tag{2.2.7}$$

where the colimit on the right hand side—the *stable h-cobordism space*—is induced by the h-cobordism stabilisation maps of Appendix E. The equivalence (2.2.7) is natural with respect to codimension-zero embeddings, provided we restrict to basepoint components. It should also be natural when considering all components, but establishing this seems more tedious. See Remark 2.3.10 for further discussion of this naturality.

By the stable parametrised h-cobordism theorem of Waldhausen–Jahren–Rognes [WJR13], the infinite loop space of the desuspension of the *smooth Whitehead spectrum* $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$ is also equivalent to $\mathcal{H}(M)$ (as ordinary spaces). Moreover, Weiss and Williams showed in [WW88, Cor. 5.6] that the spectra $\Theta B^{(1)} = \mathbf{H}(M)$ and $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$ also share the same negative homotopy groups, which led them to rename the former as the latter. This, though conjecturally true, was not fully justified since no equivalence between these two spectra was given.

We hope that the homotopy fibre sequence $C\mathrm{Emb}(P, M) \to H(M - \nu P) \to H(M)$ and Proposition 2.3.11 together explain why we denote $\Theta F^{(1)}$ by $\mathbf{CE}(P, M)$.

Spaces of bounded diffeomorphisms are usually defined as the geometric realisation of certain simplicial groups/sets. Before we recall these simplicial models in Definition 2.2.7 below, let us fix some notation first. For a subset $S \subset \mathbb{R}^{p+1}$ and $\epsilon > 0$, let $B_\epsilon(S) \subset \mathbb{R}^{p+1}$ denote the open ϵ-ball around S. For $0 < \epsilon \leq 1/2$ and for any face $\sigma \subset \Delta^p$, we fix radial identifications $\rho_\sigma : \partial\sigma(\epsilon) := B_\epsilon(\partial\sigma) \cap \sigma \cong \partial\sigma \times [0, \epsilon)$; let us first do it for $\sigma = \Delta^p$. Given $x = (t_0, \ldots, t_p) \in \partial\Delta^p(\epsilon)$, let $j \in [p]$ be such that $t_j \leq t_i$ for every $i \in [p]$. Note that since x cannot be the barycenter $b_p = (\frac{1}{p+1}, \ldots, \frac{1}{p+1})$ of Δ^p (since this lies at distance greater than $1/2 \geq \epsilon$ from $\partial\Delta^p$), we must have that t_j is strictly smaller than $\frac{1}{p+1}$. Then set

$$\rho_p : \partial\Delta^p(\epsilon) \xrightarrow{\cong} \partial\Delta^p \times [0,\epsilon), \quad x = (t_0, \dots, t_p) \mapsto \left(x - \frac{t_j}{\frac{1}{p+1} - t_j}(b_p - x), d(x, \partial\Delta^p) \right),$$

where $j = j(x)$ is as above, and $d(x, \partial\Delta^p)$ stands for the (Euclidean) distance between x and $\partial\Delta^p$. For a general face $\sigma \subset \Delta^p$, fix the standard order-preserving identification $\eta_\sigma : \sigma \cong \Delta^{|\sigma|}$; then the radial identification $\rho_\sigma : \partial\sigma(\epsilon) \cong \partial\sigma \times [0,\epsilon)$ is

$$\rho_\sigma : \partial\sigma(\epsilon) \xrightarrow{\eta_\sigma} \partial\Delta^{|\sigma|}(\epsilon) \xrightarrow{\rho_{|\sigma|}} \partial\Delta^{|\sigma|} \times [0,\epsilon) \xrightarrow{\eta_\sigma^{-1} \times \mathrm{Id}_{[0,\epsilon)}} \partial\sigma \times [0,\epsilon).$$

We will say that a continuous map $f : X \times \Delta^p \to Y \times \Delta^p$ over Δ^p (i.e., such that $\mathrm{proj}_{\Delta^p} = \mathrm{proj}_{\Delta^p} \circ f$) satisfies the *$\epsilon$-collaring condition* if for every face $\sigma \subset \Delta^p$,

$$f \mid_{X \times \partial\sigma(\epsilon)} \equiv f \mid_{X \times \partial\sigma} \times \mathrm{Id}_{[0,\epsilon)}$$

under the identifications $\rho_\sigma : \partial\sigma(\epsilon) \cong \partial\sigma \times [0,\epsilon)$ fixed above.

Definition 2.2.7. *Let $V \in \mathcal{J}$. The **semi-simplicial group** $\mathrm{Diff}^b_\partial(M \times V)_\bullet$ **of bounded diffeomorphisms** of $M \times V$ relative to $\partial M \times V$ has as p-simplices the set of diffeomorphisms of $\Delta^p \times M \times V$ over Δ^p which are bounded (with respect to V), that are the identity in a neighbourhood of $\Delta^p \times \partial M \times V$, and that satisfy the ϵ-collaring condition for some $0 < \epsilon \leq 1/2$. Face maps are determined by the coface maps of the cosimplicial space $\Delta^\bullet$. If we relax the condition on diffeomorphisms to be over Δ^p to only face-preserving (i.e. diffeomorphisms that send $\sigma \times M \times V$ to itself for every face $\sigma \subset \Delta^p$), we obtain the semi-simplicial group $\widetilde{\mathrm{Diff}}^b_\partial(M \times V)_\bullet$ of **bounded block diffeomorphisms** of $M \times V$.*

Warning 2.2.8. One could have defined the orthogonal functor $B(-)$ of Notation 2.2.5, for instance, to be

$$\mathcal{J}_0 \longrightarrow \mathsf{Top}_*, \quad U \longmapsto B|\mathrm{Diff}^b_\partial(M \times U)_\bullet|.$$

This latter rule, however, does not give rise to a continuous functor in the sense of orthogonal calculus, i.e., it is not enriched over Top_*. A way to fix this is to replace Top_* by sSet_*, $\mathcal{J}_0$ by a category $\mathcal{J}_0^\Delta$ enriched now over sSet_*, and doing orthogonal calculus for sSet_*-enriched functors $\mathcal{J}_0^\Delta \to \mathsf{sSet}_*$. This is morally the point of view taken by Weiss and Williams in [WW88], but orthogonal calculus for simplicially enriched functors has not yet been carried out rigorously, so we prefer to not pursue this approach.

The simplicial models of Definition 2.2.7 are more convenient to work with than the point-set topological ones of Definition 2.2.3. Moreover, we will need some results of [WW88] that are stated in the simplicial setting, so we will have to argue that both models share the same weak homotopy type.

Proposition D.1. *There is a zig-zag of weak equivalences of semi-simplicial group-like monoids*

$$\mathrm{Diff}^b_\partial(M \times V)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M \times V)).$$

In particular, there is a zig-zag of weak equivalences of group-like topological monoids connecting $|\mathrm{Diff}^b_\partial(M \times V)_\bullet|$ *and* $\mathrm{Diff}^b_\partial(M \times V)$*.*

We defer the proof of this proposition to Section D.1 in the appendix.

2.3 Proof of Theorem C

We now prove Theorem C. Section 2.3.1 will first reduce it to the case when P is a codimension zero submanifold of M. Some necessary preliminaries will be presented in Section 2.3.2. Finally the map Φ^{Emb} of Theorem C and its connectivity will be analysed in Sections 2.3.3 and 2.3.4.

Before we move on to the next section, let us record a disjunction result for concordance embeddings known as *Hudson's concordance-implies-isotopy theorem* [Hud70, Thm. 2.1, Addendum 2.1.2].

Theorem 2.3.1 (Hudson)**.** *The space* $C\mathrm{Emb}(P, M)$ *is connected if* $p \leq d - 3$*. Equivalently, the natural map* $\pi_0(\mathrm{Emb}_{\partial_0}(P, M)) \to \pi_0(\widetilde{\mathrm{Emb}}_{\partial_0}(P, M))$ *is an isomorphism.*

Remark 2.3.2. Hudson's theorem also holds in the PL setting (cf. [Hud70, Thm. 1.5]).

2.3.1 Reduction to geometric codimension zero embeddings

Let $\iota : P \hookrightarrow M$ be as in the statement of Theorem C. It will be convenient to be able to assume that $P \subset M$ is a codimension zero submanifold (though of handle codimension at least 3). The following result deals with this technicality, and shows that the difference between block and ordinary *smooth* embeddings is insensitive to the geometric codimension.

Proposition 2.3.3. *Let M^d be a compact smooth Riemannian manifold and $\iota : P^p \hookrightarrow M^d$ a neat submanifold that is closed as a subspace. Let $\overline{\nu}P$ be the closed disk bundle of the normal bundle ν_ι of the embedding ι, and let $\hat{\iota} : \overline{\nu}P \hookrightarrow M$ be the induced embedding. Then the square*

$$\begin{tikzcd}
\mathrm{Emb}_{\partial_0,\hat{\iota}}(\overline{\nu}P, M) \arrow[r, "\mathrm{res}_P"] \arrow[d, hook] & \mathrm{Emb}_{\partial_0,\iota}(P, M) \arrow[d, hook] \\
\widetilde{\mathrm{Emb}}_{\partial_0,\hat{\iota}}(\overline{\nu}P, M) \arrow[r, "\widetilde{\mathrm{res}}_P"] & \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P, M).
\end{tikzcd} \tag{2.3.1}$$

is homotopy cartesian. Here the subscripts ι or $\hat{\iota}$ in the embedding spaces stand for the path component consisting of embeddings isotopic to ι or $\hat{\iota}$ (relative to $\partial_0 P$).

Equivalently, by taking vertical homotopy fibres in (2.3.1) and noting Hudson's Theorem 2.3.1 and that res_P *and* $\widetilde{\mathrm{res}}_P$ *are surjective, there is a weak equivalence*

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P, M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(P, M)) \simeq \mathrm{hofib}_{\hat{\iota}}(\mathrm{Emb}_{\partial_0}(\overline{\nu}P, M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(\overline{\nu}P, M)).$$

Proof. We will show that the horizontal homotopy fibre of the vertical inclusions in (2.3.1) can be identified, up to equivalence, with the identity map of the topological group $\mathrm{Aut}_{\partial_0}(\nu_\iota)$ of bundle automorphisms of ν_ι which are standard near $\partial_0 P$. In particular, the total homotopy fibre of (2.3.1) will be weakly contractible.

We first deal with the top horizontal homotopy fibre. Consider the fibration

$$E := \left\{ \begin{tikzcd} \nu_\iota \arrow[r, hook, "G"] \arrow[d] & \tau_M \arrow[d] \\ P \arrow[r, hook, "\varphi"] & M \end{tikzcd} \;\middle|\; \begin{array}{l} \varphi \in \mathrm{Emb}_{\partial_0,\iota}(P, M), \\ G \in \mathrm{BunInj}_{\partial_0}(\nu_\iota, \tau_M), \\ D\varphi \oplus G : \tau_P \oplus \nu_\iota \cong \varphi^*\tau_M. \end{array} \right\} \xrightarrow{r} \mathrm{Emb}_{\partial_0,\iota}(P, M), \quad (G, \varphi) \mapsto \varphi.$$

Taking derivatives at the zero section of $\overline{\nu}P$ defines a map $D : \mathrm{Emb}_{\partial_0,\hat{\iota}}(\overline{\nu}P, M) \to E$ over $\mathrm{Emb}_{\partial_0,\iota}(P, M)$. A homotopy inverse $E \to \mathrm{Emb}_{\partial_0,\hat{\iota}}(\overline{\nu}P, M)$ to D can be defined using the exponential map. Therefore the homotopy fibre of res_P is equivalent to the fibre of r (observe that r is a fibration). Now $\iota^*\tau_M$ is already identified with $\tau_P \oplus \nu_\iota$, so the fibre $F := r^{-1}(\iota)$ can be described as the subspace of bundle automorphisms of $\tau_P \oplus \nu_\iota$ over P which are the identity on the tangent summand τ_P (and near $\partial_0 P$). As the space of bundle maps $\nu_\iota \to \tau_P$ over P is contractible, it follows that the inclusion $\mathrm{Aut}_{\partial_0}(\nu_\iota) \hookrightarrow F$ is a homotopy equivalence.

The argument for the bottom map of (2.3.1) is similar but trickier; we work with the simplicial model of block embeddings of Definition 2.2.7. First let ξ and π be vector bundles over spaces B and B', respectively, and fix some bundle map $I : \xi \to \pi$. For any closed subset $\partial_0 \subset B$, let $\widetilde{\mathrm{BunMap}}_{\partial_0}(\xi, \pi)_\bullet$ denote the semi-simplicial set whose

n-simplices consist of bundle maps $G : \Delta^n \times \xi \to \tau_{\Delta^n} \boxplus \pi := (\tau_{\Delta^n} \times B') \oplus (\Delta^n \times \pi)$ such that

- G agrees with $\mathbf{0}_{\Delta^n} \boxplus I$ near $\Delta^n \times \partial_0$, where $\mathbf{0}_{\Delta^n} : \epsilon^0_{\Delta^n} \cong \Delta^n \to \tau_{\Delta^n}$ is the inclusion as the zero section, and
- for every face $\sigma \subset \Delta^n$, $G(\sigma \times \xi) \subset \tau_\sigma \boxplus \pi \subset \tau_{\Delta^n} \boxplus \pi$.

Given a map $i : B \to B'$ which agrees with the underlying map of I on $\partial_0 \subset B$, let $\widetilde{\mathrm{BunMap}}_{\partial_0}(\xi, \pi; i)_\bullet$ the semi-simplicial subset consisting of those bundle maps G whose underlying map on the base spaces $\Delta^n \times B \to \Delta^n \times B'$ is $\mathrm{Id}_{\Delta^n} \times i$. Let $\widetilde{\mathrm{BunInj}}_{\partial_0}(\xi, \pi)_\bullet$ and $\widetilde{\mathrm{BunInj}}_{\partial_0}(\xi, \pi; i)_\bullet$ be the semi-simplicial subsets of those bundle maps that are fibrewise injective. Then again, taking derivatives at the zero section of $\Delta^\bullet \times \overline{\nu}P$ yields a simplicial map $\widetilde{D}_\bullet$ from $\widetilde{\mathrm{Emb}}_{\partial_0,\hat{\iota}}(\overline{\nu}P, M)_\bullet$ to a semi-simplicial set $\widetilde{E}_\bullet$ whose n-simplices are

$$\widetilde{E}_n := \left\{ \begin{tikzcd} \Delta^n \times \nu_\iota \arrow[r, hook, "G"] \arrow[d] & \tau_{\Delta^n} \boxplus \tau_M \arrow[d] \\ \Delta^n \times P \arrow[r, hook, "\varphi"] & \Delta^n \times M \end{tikzcd} \;\middle|\; \begin{array}{c} \varphi \in \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P, M)_n, \\ G \in \widetilde{\mathrm{BunInj}}_{\partial_0}(\nu_\iota, \tau_M)_n, \\ D\varphi \oplus G : \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota) \cong \varphi^*(\tau_{\Delta^n} \boxplus \tau_M). \end{array} \right\},$$

and whose face maps are given by restriction to face strata. The map $\widetilde{r}_\bullet : \widetilde{E}_\bullet \to \widetilde{\mathrm{Emb}}_{\partial_0}(P, M)_\bullet$ given by $\widetilde{r}(G, \varphi) := \varphi$ is now a Kan fibration, and $\widetilde{r}_\bullet \circ \widetilde{D}_\bullet = \widetilde{\mathrm{res}}_P$. By a similar argument as in the previous case, the homotopy fibre of $\widetilde{\mathrm{res}}_P$ is equivalent to the fibre of $\widetilde{r}_\bullet$. Using the canonical identification

$$(\mathrm{Id}_{\Delta^n} \times \iota)^*(\tau_{\Delta^n} \boxplus \tau_M) = \tau_{\Delta^n} \boxplus \iota^*\tau_M \cong \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota),$$

the fibre $\widetilde{F}_\bullet := \widetilde{r}_\bullet^{-1}(\iota)$ is isomorphic to the semi-simplicial subset of $\widetilde{\mathrm{BunInj}}_{\partial_0}(\nu_\iota, \iota^*\tau_M; \mathrm{Id}_P)_\bullet$ of bundle maps

$$G = G_{\Delta^n} \oplus G_\tau \oplus G_\nu : \Delta^n \times \nu_\iota \longrightarrow \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota) = (\tau_{\Delta^n} \times P) \oplus (\Delta^n \times \tau_P) \oplus (\Delta^n \times \nu_\iota)$$

for which G_ν is an isomorphism. Thus it follows that

$$\widetilde{F}_\bullet = \widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet \times \mathrm{Aut}_{\partial_0}(\nu_\iota)_\bullet,$$

where the boundary condition on $\widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet$ forces bundle maps to be zero near $\Delta^\bullet \times \partial_0 P$. Clearly $\widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet$ is weakly contractible: indeed given an n-cycle G in this semi-simplicial set, a nullhomotopy of G is roughly given by regarding Δ^{n+1} as $(\Delta^n \times [0,1], \Delta^n \times \{0\})$ and applying $t \cdot G$ on $\Delta^n \times \{t\}$, for $0 \leq t \leq 1$. Therefore $|\widetilde{F}_\bullet| \simeq |\mathrm{Aut}_{\partial_0}(\nu_\iota)_\bullet| = \mathrm{Aut}_{\partial_0}(\nu_\iota)$, as required. □

Remark 2.3.4. Proposition 2.3.3 is false in the topological setting. First of all, a locally flat embedding $\iota : P^p \hookrightarrow M^d$ does not always admit a normal microbundle (cf. [RS67]; they do admit one stably though [Hir66, Thm. B]). But even if it did, the statement would still not hold in general: consider the case $(M, P) = (D^d, D^p)$ for $p \leq d - 3$. Then both $\mathrm{Emb}^{\mathrm{Top}}_\partial(D^p, D^d)$ and $\widetilde{\mathrm{Emb}}^{\mathrm{Top}}_\partial(D^p, D^d)$ are contractible by the Alexander trick. However, using the topological version of Theorem C (cf. Remark 2.1.3), we will see in Remark 3.2.3 that the homotopy fibre of the map

$$\mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d) \longrightarrow \widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(D^d \times D^{d-p}, D^d)$$

is not contractible. In particular, the topological analogue of the square (2.3.1) cannot possibly be homotopy cartesian in this case.

2.3.2 Last ingredients

From now on, let $\iota : P^d \hookrightarrow M^d$ be a codimension zero closed embedding that meets ∂M transversely in $\partial_0 P$, and denote by p the handle dimension of P relative to $\partial_0 P$; we will write $\overline{M - P}$ instead of the isotopy equivalent manifold $M - \nu P$ to emphasise that P has codimension zero in M. It suffices to prove Theorem C in this case by Proposition 2.3.3. We now present the last necessary prelimary results.

2.3.2.1 Parametrised isotopy extension theorem

The *parametrised isotopy extension theorem* states that for $\varphi_t : P \hookrightarrow M$ any continuous family of embeddings parametrised by $t \in \Delta^k$ (with P *compact*), there exists a continuous family of diffeomorphisms $\{\phi_t\}_{t \in \Delta^k}$ of M (which are the identity away from a compact set of M) such that $\phi_0 = \mathrm{Id}_M$ and $\phi_t(\varphi_0(x)) = \varphi_t(x)$ for all $(x,t) \in P \times \Delta^k$. Moreover, if $K \subset \Delta^k$ is some contractible subcomplex containing the 0-th vertex and $\{\phi'_t\}_{t \in K}$ is another continuous family of diffeomorphisms of M parametrised by K such that $\phi'_0 = \mathrm{Id}_M$ and $\phi'_t(\varphi_0(x)) = \varphi_t(x)$ for all $(x,t) \in P \times K$, then we can arrange $\{\phi_t\}_{t \in \Delta^k}$ as above to agree with $\{\phi'_t\}_{t \in K}$ on K. A consequence of this fact due to Palais [Pal60] (see [Lim64] for a simple proof) is that the restriction map $\mathrm{Diff}_\partial(M) \to \mathrm{Emb}_{\partial_0}(P, M)$ is a locally trivial fibre bundle with $\mathrm{Diff}_\partial(\overline{M - P})$ as fibre. Such a fibration can be delooped to the homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \longrightarrow B\mathrm{Diff}_\partial(\overline{M-P}) \longrightarrow B\mathrm{Diff}_\partial(M), \tag{2.3.2}$$

where the subscript $\langle\iota\rangle$ stands for the union of all the components in $\mathrm{Emb}_{\partial_0}(P,M)$ that contain embeddings of the form $\phi\circ\iota$ for $\phi\in\mathrm{Diff}_\partial(M)$. By replacing P and M in (2.3.2) by $P\times I$ and $M\times I$, and modifying the boundary conditions, we get a similar homotopy fibre sequence

$$C\mathrm{Emb}(P,M) \longrightarrow BC(\overline{M-P}) \longrightarrow BC(M). \tag{2.3.3}$$

Note that $C\mathrm{Emb}(M,P)$ is connected by Hudson's Theorem 2.3.1. Finally, there is a block analogue of (2.3.2).

Proposition 2.3.5. *There is a homotopy fibre sequence*

$$\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M) \longrightarrow B\widetilde{\mathrm{Diff}}_\partial(\overline{M-P}) \longrightarrow B\widetilde{\mathrm{Diff}}_\partial(M). \tag{2.3.4}$$

Proof. There is a right action of the simplicial group $\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$ on $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$; we will write $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$ for the simplicial set of (level-wise) cosets of this right action. The geometric realisation $|\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet|$ of this simplicial set is homotopy equivalent to the homotopy fibre of the right map of (2.3.4), so it suffices to show that the action map

$$a:\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet \longrightarrow \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)_\bullet, \quad [\phi]\longmapsto \phi\circ\iota$$

is an isomorphism. It is visibly injective, for if $\phi\circ\iota=\psi\circ\iota$ for $\phi,\psi\in\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$, then $\psi^{-1}\circ\phi\in\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$ and hence $[\psi]=[\psi\circ\psi^{-1}\circ\phi]=[\phi]$ in $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$.

For surjectivity, let φ be some k-simplex in $\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)_\bullet$. Then there exists some $\phi\in\widetilde{\mathrm{Diff}}_\partial(M)_k$ for which φ and $\phi\circ\iota$ lie in the same component in $\widetilde{\mathrm{Emb}}_{\partial_0}(P,M)_\bullet$. Then $\varphi':=\phi^{-1}\circ\varphi\in\widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M)_k$ and, in fact, we can arrange that its restriction to the zero-th vertex φ'_0 is ι by rechoosing ϕ (if necessary) using the isotopy extension theorem. Then applying the isotopy extension theorem to φ' restricted to each of the faces that contains the 0-th vertex, inductively on the dimension of the face, we obtain some $\Phi'\in\widetilde{\mathrm{Diff}}_\partial(M)_k$ such that $\Phi'\mid_{P\times\Delta^k}\equiv\varphi'$. Then $\Phi:=\phi\circ\Phi'\in\widetilde{\mathrm{Diff}}_\partial(M)_k$ is such that $\Phi\mid_{P\times\Delta^k}\equiv\varphi$, as desired. ☐

Remark 2.3.6. There also exists a topological version of the isotopy extension theorem [EK71, Cor. 1.4]. The same proof as above also works in the topological setting.

Remark 2.3.7 (Speculative). Weiss and Williams point out in [WW88, §1] that an analogue of the (parametrised) isotopy extension theorem in the bounded setting does not hold (see [Hir76, Ch. 8, Ex. 9] for a counterexample in codimension 2). However, we believe that a weaker version of the theorem should still hold: namely, for $V \in \mathcal{J}_0$ define the bounded embedding space $\mathrm{Emb}^b_{\partial_0}(P \times V, M \times V)$ as in Definition 2.2.3. Then, there should be a homotopy fibre sequence

$$\mathrm{Emb}^b_{\partial_0,\langle\iota\rangle}(P \times V, M \times V) \to E(V) = B\mathrm{Diff}^b_\partial(\overline{M-P} \times V) \to B(V) = B\mathrm{Diff}^b_\partial(M \times V),$$

where $E(-)$ and $B(-)$ are as in Notation 2.2.5. We will not give a proof of this claim, as it seems rather technical and we will not need it for the argument of Theorem C. The reader may however find it useful to think of the orthogonal functor $F(-) := \mathrm{hofib}(E(-) \to B(-))$ as $\mathrm{Emb}^b_{\partial_0,\langle\iota\rangle}(P \times (-), M \times (-))$.

2.3.2.2 Alexander trick-like equivalences

For $V \in \mathcal{J}_0$, let $D(V) \subset V$ denote the correspoding closed unit disk (so that $D^k = D(\mathbb{R}^k)$). The following is proved in Propositions 1.8, 1.10 and 1.12 of [WW88]. Even though we state it for the orthogonal functor $B(-)$ of Notation 2.2.5, it of course holds for $E(-)$ too.

Proposition 2.3.8. *For $V \in \mathcal{J}_0$, the Alexander trick-like map*

$$\mathrm{alex} : C(M \times D(V)) \xrightarrow{\sim} \Omega^{V\oplus\mathbb{R}} B^{(1)}(V) = \Omega^{V\oplus\mathbb{R}} \Big(\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)\Big)$$

is a weak equivalence. Moreover, there is a homotopy commutative diagram

$$\begin{tikzcd}
C(M \times D(V)) \arrow[r, "\Sigma"] \arrow[d, "\wr"', "\mathrm{alex}"] & C(M \times D(V) \times D^1) \arrow[r, equal, "\sim"] & C(M \times D(V \oplus \mathbb{R})) \arrow[d, "\wr"', "\mathrm{alex}"] \\
\Omega^{V\oplus\mathbb{R}} B^{(1)}(V) \arrow[rr, "s_V^\vee"] & & \Omega^{V\oplus\mathbb{R}^2} B^{(1)}(V \oplus \mathbb{R}),
\end{tikzcd} \qquad (2.3.5)$$

where Σ denotes the usual concordance stabilisation map and $s_V^\vee$ is the adjoint of the structure map (2.2.2) for the orthogonal spectrum $\Theta B^{(1)} = \mathbf{H}(M)$.

We will describe the map "alex" below, but let us first make a few remarks about this statement and its consequences.

Remark 2.3.9. Both the domain and codomain of the map "alex" of Proposition 2.3.8 are group-like $\mathbb{E}_1$-spaces; the former by composition of concordance diffeomorphisms, and the latter by the loop space structure induced by $\Omega^{\mathbb{R}}(-)$. In Section 3.1.2, we construct a (non-connected) delooping of this map (see (3.1.4)).

It seems likely that the homotopy commutative square (2.3.5) can also be delooped in a similar manner. Proving this, however, is quite technical and we will not need it in any case. What we will need instead is the observation that if M is replaced by $M \times I$ in Proposition 2.3.8, the whole statement can be delooped once with respect to the $\mathbb{E}_1$-structures induced by *stacking in the I-direction*. This is straightforward to check from the proofs in [WW88].

Remark 2.3.10. It follows from this proposition that there is a natural[2] equivalence

$$\Omega^{\infty+1}\mathbf{H}(M) \simeq \mathcal{C}(M) := \operatorname*{hocolim}_{k} C(M \times D^k).$$

As pointed out in Remark 2.3.9, this equivalence can be delooped once if we replace M by $M \times I$. Moreover, the (non-equivariant) homotopy types of both $\mathbf{H}(-)$ and $\mathcal{C}(-)$ are invariant under crossing with I, namely, there are natural equivalences $\mathbf{H}(M \times I) \simeq \mathbf{H}(M)$ (by Lemma 3.1.15 below) and $\mathcal{C}(M \times I) \simeq \mathcal{C}(M)$ (by definition). By this line of reasoning, we obtain natural equivalences

$$\Omega^\infty_0\mathbf{H}(M) \simeq \Omega^\infty_0\mathbf{H}(M \times I) \simeq B\mathcal{C}(M \times I) \simeq B\mathcal{C}(M)$$

By [Vog85, Prop. 2.1], $B\mathcal{C}(M)$ is also naturally equivalent to the basepoint component of the space of stable h-cobordisms $\mathcal{H}(M)$ of Remark 2.2.6, and by [WW88, Cor. 5.6] and the s-cobordism theorem, the groups $\pi_0^s(\mathbf{H}(M))$ and $\pi_0(\mathcal{H}(M))$ are both isomorphic to the Whitehead group $\mathrm{Wh}(\pi_1 M)$. Since there is an (non-natural) equivalence *of spaces* $\Omega^\infty X \simeq \Omega^\infty_0 X \times \pi_0^s(X)$, we obtain the promised equivalence (2.2.7)

$$\Omega^\infty\mathbf{H}(M) \simeq \mathcal{H}(M).$$

This, of course, ought to be an equivalence of infinite loop spaces, but that seems to be more difficult to see. Making the above equivalence natural for codimension-zero embeddings requires establishing the (non-connected) delooped analogues of (2.3.5), using the delooped Alexander trick-like maps (3.1.4) constructed in Section 3.1.2 (which satisfy this naturality by construction). However, this is a rather tedious task that we do not undertake.

---

[2] Natural for codimension-zero embeddings.

Proof of Proposition 2.3.8. This is proved in Propositions 1.8, 1.10 and Lemma 1.12 of [WW88]. Let us just explain how the Alexander trick-like map

$$\mathrm{alex} : C(M \times D(V)) \longrightarrow \Omega^{V\oplus\mathbb{R}} \left(\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)\right)$$

is defined: given a concordance diffeomorphism $\phi : M \times D(V) \times I \cong M \times D(V) \times I$, extend it by $\phi\mid_{M\times D(V)\times\{1\}} \times \mathrm{Id}_{[1,+\infty)}$ on $M \times D(V) \times [1, +\infty)$ and by the identity elsewhere to obtain a bounded self-diffeomorphism $\widehat{\phi}$ of $M \times V \oplus \mathbb{R}$; then shift it along $V \oplus \mathbb{R}$ to obtain a $(V \oplus \mathbb{R})$-fold loop in $\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)$. We refer to loc. cit. for the rest of the proofs. □

Taking fibres of Proposition 2.3.8 for $E(-)$ and $B(-)$ yields the first part of the analogous result for $F(-)$.

Proposition 2.3.11. *For $V \in \mathcal{J}_0$, there are weak equivalences*

$$\mathrm{alex} : \Omega C\mathrm{Emb}(P \times D(V), M \times D(V)) \xrightarrow{\sim} \Omega^{1+V} F^{(1)}(V),$$

making the following diagram commute up to homotopy:

$$\begin{tikzcd}
\Omega C\mathrm{Emb}(P \times D(V), M \times D(V)) \arrow[r, "\Sigma"] \arrow[d, "\wr"', "\mathrm{alex}"] & \Omega C\mathrm{Emb}(P \times D(V \oplus \mathbb{R}), M \times D(V \oplus \mathbb{R})) \arrow[d, "\wr"', "\mathrm{alex}"] \\
\Omega^{1+V} F^{(1)}(V) \arrow[r, "s_V^\vee"] & \Omega^{1+V\oplus\mathbb{R}} F^{(1)}(V \oplus \mathbb{R}),
\end{tikzcd} \tag{2.3.6}$$

where Σ is the concordance embedding stabilisation map of Section 2.1.1.1. Moreover, if $p \leq d - 3$, there is a natural equivalence

$$\Omega^\infty(\mathbf{CE}(P, M)) := \Omega^\infty(\Theta F^{(1)}) \simeq \mathcal{CE}\mathrm{mb}(P, M) := \operatorname*{hocolim}_k C\mathrm{Emb}(P \times D^k, M \times D^k). \tag{2.3.7}$$

To establish (2.3.7), we will need the following result, which was suggested to us by Manuel Krannich.

Lemma 2.3.12. *For $p \leq d - 3$ and $n \geq 0$, the space $\Theta F^{(1)}_n = F^{(1)}(\mathbb{R}^n)$ is n-connected.*

Proof. It suffices to show that the map $\Theta E^{(1)}_n \to \Theta B^{(1)}_n$, call it λ, is such that $\pi_*(\lambda)$ is

(a) surjective if $* = n + 1$, (b) injective if $* = 0$, (c) an isomorphism if $1 \leq * \leq n$.

For (a), observe that $\Omega^{n+1}\lambda$ is, up to equivalence, the natural map of concordance spaces $C(\overline{M-P}\times D^n)\to C(M\times D^n)$ by Proposition 2.3.8. By exactness of

$$\pi_0(C(\overline{M-P}\times D^n))\xrightarrow{\pi_{n+1}(\lambda)}\pi_0(C(M\times D^n))\longrightarrow\pi_0(C\mathrm{Emb}(P\times D^n,M\times D^n))=*,$$

where the equality on the right is the statement of Hudson's Theorem 2.3.1, it follows that $\pi_{n+1}(\lambda)$ is surjective.

For (b) and (c), consider the commutative diagram

$$\begin{tikzcd}
\Theta E_n^{(1)} \arrow[r, "\lambda"] \arrow[d, "\text{stab.}"] & \Theta B_n^{(1)} \arrow[d, "\text{stab.}"] \\
\Omega^\infty(\Sigma^n\Theta E^{(1)}) \arrow[r, "\eta"] & \Omega^\infty(\Sigma^n\Theta B^{(1)}).
\end{tikzcd}$$

We claim that the map of (non-connective) spectra $\mathbf{H}(\overline{M-P})\to\mathbf{H}(M)$ underlying η is an isomorphism in π_*^s for $*\leq 0$: indeed, the inclusion $\overline{M-P}\hookrightarrow M$ is 2-connected and $\pi_*^s(\mathbf{H}(-))$ for $*\leq 0$ (see (2.3.9)) only depends on $\pi_1(-)$ by [WW88, Cor. 5.6]. So η itself satisfies (b) and (c). Now by [WW88, Cor. 5.8], both vertical maps are injective in π_0 and isomorphisms in $\pi_{1\leq*\leq n}$. Claims (b) and (c) now follow. □

Proof of Proposition 2.3.11. It remains to deloop the natural equivalence

$$\Omega^{\infty+1}(\mathbf{CE}(P,M))\simeq\Omega\mathcal{CE}\mathrm{mb}(P,M)$$

obtained from the squares (2.3.6), so to yield (2.3.7). We do this as in Remark 2.3.10.

First observe that both $\Omega^\infty\mathbf{CE}(P,M)$ and $\mathcal{CE}\mathrm{mb}(P,M)$ are connected under the codimension assumption—the former by Lemma 2.3.12 and the latter by Hudson's Theorem 2.3.1. Just like in Remark 2.3.9, the homotopy commutative square (2.3.6) can be delooped if we replace $\iota: P\hookrightarrow M$ by $\iota\times\mathrm{Id}_I: P\times I\hookrightarrow M\times I$. Finally, observe that there are natural equivalences $\mathbf{CE}(P\times I,M\times I)\simeq\mathbf{CE}(P,M)$ (by Lemma 3.1.15) and $\mathcal{CE}\mathrm{mb}(P\times I,M\times I)\simeq\mathcal{CE}\mathrm{mb}(P,M)$ (by definition). We thus obtain the desired chain of natural equivalences

$$\Omega^\infty\mathbf{CE}(P,M)\simeq\Omega^\infty\mathbf{CE}(P\times I,M\times I)\simeq\mathcal{CE}\mathrm{mb}(P\times I,M\times I)\simeq\mathcal{CE}\mathrm{mb}(P,M).$$

□

2.3.3 The map Φ^{Emb} of Theorem C

Recall the map Φ^F_∞ of Proposition 2.2.2 for $F(-)$. Noting that $F(0) \simeq \mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M)$ by the isotopy extension sequence (2.3.2), this map, up to equivalence, takes the form

$$\Phi^F_\infty : \mathrm{hofib}(\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \to F(\mathbb{R}^\infty)) \longrightarrow \Omega^\infty(\mathbf{CE}(P,M)_{hC_2}).$$

To obtain Φ^{Emb}, we need to replace $F(\mathbb{R}^\infty)$ by $\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)$ above (and deal with some path-components considerations). This turns out to be possible by a principle similar to that of [WW88, Rem. 3.5].

Proposition 2.3.13. *If $p \leq d-3$, then there is a (zig-zag of) weak equivalence*

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) \simeq \mathrm{hofib}(F(0) \to F(\mathbb{R}^\infty)).$$

Proof. First observe that because $C\mathrm{Emb}(P,M)$ is connected (by Hudson's Theorem 2.3.1), we have that

$$\begin{aligned}\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) &= \mathrm{hofib}(\mathrm{Emb}_{\partial_0,\iota}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M)) \\ &= \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)).\end{aligned}$$

Therefore, it suffices to construct a homotopy commutative diagram

$$\begin{tikzcd}
F(0) \arrow[r] & F(\mathbb{R}^\infty) \arrow[r, "i", "\sim"'] & \widetilde{F}(\mathbb{R}^\infty) \\
\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[u, "\wr"', "(2.3.2)"] \arrow[rr, hook] & & \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M). \arrow[u, "\wr"', "j"]
\end{tikzcd} \tag{2.3.8}$$

To that end, let $\mathcal{J}_0^\delta$ denote the underlying ordinary category of the topological category $\mathcal{J}_0$, and write $\widetilde{E}(-)$ and $\widetilde{B}(-)$ for the functors $\mathcal{J}_0^\delta \to \mathsf{Top}_*$ given by

$$\widetilde{E}(V) := B|\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V)_\bullet|, \qquad \widetilde{B}(V) := B|\widetilde{\mathrm{Diff}}^b_\partial(M \times V)_\bullet|.$$

Set $\widetilde{F}(-) := \mathrm{hofib}(\widetilde{E}(-) \to \widetilde{B}(-))$. Then the map i of (2.3.8) arises as the map on homotopy fibres in

$$\begin{tikzcd}
F(\mathbb{R}^\infty) \arrow[r] \arrow[d, dashed, "i"] & E(\mathbb{R}^\infty) = B\mathrm{Diff}^b_\partial(\overline{M-P} \times \mathbb{R}^\infty) \arrow[r] \arrow[d, hook, "\wr"] & B(\mathbb{R}^\infty) = B\mathrm{Diff}^b_\partial(M \times \mathbb{R}^\infty) \arrow[d, hook, "\wr"] \\
\widetilde{F}(\mathbb{R}^\infty) \arrow[r] & \widetilde{E}(\mathbb{R}^\infty) = B\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times \mathbb{R}^\infty) \arrow[r] & \widetilde{B}(\mathbb{R}^\infty) = B\widetilde{\mathrm{Diff}}^b_\partial(M \times \mathbb{R}^\infty).
\end{tikzcd}$$

The middle and right vertical maps are equivalences by [WW88, Thm. B], so i is too by the five lemma.

The map j of (2.3.8) arises as the map on homotopy fibres in

$$\begin{tikzcd}
\widetilde{F}(\mathbb{R}^\infty) \arrow[r] & \widetilde{E}(\mathbb{R}^\infty) \arrow[r] \arrow[rd, phantom, "(\dagger)"] & \widetilde{B}(\mathbb{R}^\infty) \\
\widetilde{F}(0) \simeq \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[u, dashed, "j"] \arrow[r] & \widetilde{E}(0) = B\widetilde{\mathrm{Diff}}_\partial(\overline{M-P}) \arrow[u] \arrow[r] & \widetilde{B}(0) = B\widetilde{\mathrm{Diff}}_\partial(M). \arrow[u]
\end{tikzcd}$$

Then the square (2.3.8) is the homotopy fibre of the map between the similar (strictly commutative) squares associated to $E(-)$ and $B(-)$, and so it is homotopy commutative by construction.

It remains to show that j is an equivalence or, equivalently, that the square $(\dagger)$ is homotopy cartesian. Write $\widetilde{F}^{(1)}(V) := \mathrm{hofib}(\widetilde{F}(V) \to \widetilde{F}(V \oplus \mathbb{R}))$, and similarly for $\widetilde{E}^{(1)}(V)$ and $\widetilde{B}^{(1)}(V)$. In other words,

$$\widetilde{E}^{(1)}(V) := \frac{\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V \oplus \mathbb{R})}{\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V)}, \qquad \widetilde{B}^{(1)}(V) := \frac{\widetilde{\mathrm{Diff}}^b_\partial(M \times V \oplus \mathbb{R})}{\widetilde{\mathrm{Diff}}^b_\partial(M \times V)}.$$

For a group π and an integer $j \leq 1$, set $\kappa_j(\pi) := \pi_j^s(\mathbf{Wh}^{\mathrm{Diff}}(B\pi))$. More explicitly,

$$\kappa_j(\pi) = \begin{cases} \mathrm{Wh}_1(\pi), & j = 1, \\ \widetilde{K}_0(\mathbb{Z}\pi), & j = 0, \\ K_j(\mathbb{Z}\pi), & j \leq -1. \end{cases} \tag{2.3.9}$$

It was shown in [WW88, Cor. 5.5] (see also [AP83]) that, for a certain C_2-action on $\kappa_j(\pi)$, there are maps for $n \geq 0$

$$\beta : \pi_*(\widetilde{E}^{(1)}(\mathbb{R}^n)) \longrightarrow H_*(C_2; \kappa_{1-n}(\pi_1(M-P))),$$
$$\beta : \pi_*(\widetilde{B}^{(1)}(\mathbb{R}^n)) \longrightarrow H_*(C_2; \kappa_{1-n}(\pi_1(M))),$$

which are injective if $* = 0$ and isomorphisms if $* \geq 1$. Moreover, it is not difficult to see from its proof that these are compatible, in the sense that the square

$$\begin{tikzcd}
\pi_*(\widetilde{E}^{(1)}(\mathbb{R}^n)) \arrow[r] \arrow[d, "\beta"] & \pi_*(\widetilde{B}^{(1)}(\mathbb{R}^n)) \arrow[d, "\beta"] \\
H_*(C_2; \kappa_{1-n}(\pi_1(M-P))) \arrow[r, "\cong"] & H_*(C_2; \kappa_{1-n}(\pi_1(M)))
\end{tikzcd}$$

is commutative. The lower horizontal map is an isomorphism because the fundamental groups of $M-P$ and M can be identified under the obvious inclusion by the assumption that $p \leq d-3$. Hence, as $\widetilde{F}^{(1)}(V) \to \widetilde{E}^{(1)}(V) \to \widetilde{B}^{(1)}(V)$ is a homotopy fibre sequence for all V, it follows that $\widetilde{F}^{(1)}(\mathbb{R}^n)$ is weakly contractible for all $n \geq 0$. Using the homotopy fibre sequences

$$\operatorname{hofib}(\widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^n)) \longrightarrow \operatorname{hofib}(\widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^{n+1})) \longrightarrow \widetilde{F}^{(1)}(\mathbb{R}^n) \simeq *$$

for $n \geq 0$, we must have by induction that $\operatorname{hofib}(j : \widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^\infty))$ is contractible, i.e., that j is a weak equivalence, as required. $\square$

Definition 2.3.14. *The map* Φ^{Emb} *of Theorem C is the zig-zag*

$$\begin{aligned} \operatorname{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) &\simeq \operatorname{hofib}(F(0) \to F(\mathbb{R}^\infty)) && \textit{by Prop. 2.3.13,} \\ &\xrightarrow{\Phi^F_\infty} \Omega^\infty(\Theta F^{(1)}_{hO(1)}) = \Omega^\infty(\mathbf{CE}(P,M)_{hC_2}) && \textit{by (2.2.4) and (2.2.6).} \end{aligned}$$

2.3.4 Connectivity of the map Φ^{Emb}

In this section we show that the map Φ^{Emb} just defined is $\phi_{C\mathrm{Emb}}(M,P)$-connected, at last establishing Theorem C (modulo the proof of Proposition D.1). The connectivity of Φ^{Emb} is that of Φ^F_∞; we show by induction on $n \geq 0$ that the maps Φ^F_n of Proposition 2.2.2 are at least $\phi_{C\mathrm{Emb}}(M,P)$-connected. Note that this is clear for $n=0$, as both the domain and codomain are contractible.

Suppose now that Φ^F_n is $\phi_{C\mathrm{Emb}}(M,P)$-connected for some $n \geq 0$. To show that Φ^F_{n+1} has this connectivity, it suffices to show that the map stab. : $\Theta F^{(1)}_n \to \Omega^\infty(\Sigma^n \Theta F^{(1)})$ of Proposition 2.2.2 is $(\phi_{C\mathrm{Emb}}(M,P)+n)$-connected. But $\Theta F^{(1)}_n$ is n-connected by Lemma 2.3.12 and $\Sigma^n \Theta F^{(1)}$ is $(n+1)$-connective. So it suffices to show that $\Omega^{n+1}(\text{stab.})$ is $(\phi_{C\mathrm{Emb}}(M,P)-1)$-connected. This follows from the homotopy commutative diagram

$$\begin{tikzcd} \Omega C\mathrm{Emb}(P \times D^n, M \times D^n) \arrow[r, hook] \arrow[d, "\wr"', "\mathrm{alex}"] & \Omega \mathcal{CE}\mathrm{mb}(P,M) \arrow[d, "\wr"', "(2.3.7)"] \\ \Omega^{n+1}\Theta F^{(1)}_n \arrow[r, "\Omega^{n+1}(\mathrm{stab.})"] & \Omega^{\infty+1}(\Theta F^{(1)}). \end{tikzcd}$$

By definition, the connectivity of the top horizontal map is

$$\phi_{C\mathrm{Emb}}(M \times D^n, P \times D^n) - 1 \geq \phi_{C\mathrm{Emb}}(M,P) - 1.$$

One obtains the above homotopy commutative diagram from stacking together squares of the form (2.3.6) with $V = \mathbb{R}^k$ for $k \geq n$. This concludes the proof of Theorem C. $\square$

2.4 A splitting result for embedding spaces

In this section we derive, as a consequence of Theorem C, a general splitting result[3] for embedding spaces of manifolds with interval factors. This will then be used for the splitting part of Theorem D. All throughout, let $\iota : P^p \subset M^d$ be as in the statement of Theorem C.

For $D(-)$ any of $\mathrm{Diff}^b_\partial(- \times V)$ with $V \in \mathcal{J}_0$, $\widetilde{\mathrm{Diff}}_\partial(-)$ or $C(-)$, there are *graphing maps*

$$\Gamma : \Omega D(M) \longrightarrow D(M \times I)$$

given (roughly) by regarding a 1-parameter family of automorphisms of M as an automorphism of $M \times I$ itself. These are natural with respect to codimension zero embeddings. Moreover, these maps can be delooped as, up to homotopy, they intertwine the (group-like) $\mathbb{E}_1$-structures of concatenating loops for the domain, and stacking automorphisms in the I-direction for the codomain. There are similar maps

$$\Gamma : \Omega E(P, M) \longrightarrow E(P \times I, M \times I) \tag{2.4.1}$$

for $E(-,-)$ denoting any of $\mathrm{Emb}_{\partial_0}(-,-)$, $\widetilde{\mathrm{Emb}}_{\partial_0}(-,-)$, $\mathrm{Emb}^{(\sim)}_{\partial_0}(-,-)$ (see (2.1.4)) or $C\mathrm{Emb}(-,-)$. In what follows, we will write Γ for any map of this same nature.

Remark 2.4.1. Most of the functors $D(-)$ and $E(-,-)$ above either admit a point-set topological model or a simplicial model. In the first case, the graphing maps just introduced are really zig-zags of maps

$$\Omega E(P, M) \xleftarrow{\ \sim\ } \Omega^{\mathsf{col,sm}} E(P, M) \xrightarrow{\ \Gamma\ } E(P \times I, M \times I),$$

where, for X a pointed (Fréchet) manifold, here $\Omega^{\mathsf{col,sm}} X$ stands for the space of smooth loops $\gamma : S^1 \to X$ which are *collared* in the sense that there exists some neighbourhood of $1 \in S^1$ which is sent by γ to the basepoint in X. The inclusion $\Omega^{\mathsf{col,sm}} E(P, M) \hookrightarrow \Omega E(P, M)$ is an equivalence by smooth approximation of continuous functions.

---

[3] This should be compared to the analogous result of Burghelea and Lashof [BL82, Cor. E].

In the simplicial case, the graphing maps Γ are the geometric realisations of the simplicial maps

$$\Gamma_\bullet : (\Omega E(P, M))_\bullet \longrightarrow E(P \times I, M \times I)_\bullet$$

that send a q-simplex in $(\Omega E(P, M))_\bullet$ (seen as a $(q+1)$-simplex $g \in E(P, M)_\bullet$ whose 0-th face and vertex are the basepoint $* \in E(P, M)_\bullet$) to the q-simplex in $E(P \times I, M \times I)_\bullet$ obtained from g by expanding out the 0-th vertex of Δ^{q+1} to a q-dimensional simplex (i.e. regarding Δ^{q+1} as $(\Delta^q \times I, \Delta^q \times \{0\})$).

In the cases when $D(-)$ or $E(-,-)$ admit both models, one verifies that these two graphing maps agree up to homotopy. We ignore both of these technicalities in most of what follows.

Theorem 2.4.2. *Let I and J both denote closed intervals. For $p \leq d-3$, $N := \phi_{\mathrm{CEmb}}(d+1, p+1) - 1$ and $N' := \phi_{\mathrm{CEmb}}(d+2, p+2) - 1$, there are equivalences away from* 2

$$\Omega\tau_{\leq N} \operatorname{Emb}_{\partial_0}(P \times I, M \times I) \simeq_{[\frac{1}{2}]} \Omega\tau_{\leq N} \left(\widetilde{\operatorname{Emb}}_{\partial_0}(P \times I, M \times I) \times \operatorname{Emb}^{(\sim)}_{\partial_0}(P \times I, M \times I)\right),$$

$$\tau_{\leq N'} \operatorname{Emb}_{\partial_0}(P \times I \times J, M \times I \times J) \simeq_{[\frac{1}{2}]} \tau_{\leq N'} \begin{pmatrix} \widetilde{\operatorname{Emb}}_{\partial_0}(P \times I \times J, M \times I \times J) \\ \times \\ \operatorname{Emb}^{(\sim)}_{\partial_0}(P \times I \times J, M \times I \times J) \end{pmatrix}.$$

Proof. Suppose given a map of fibration sequences

$$\begin{tikzcd}
F' \arrow[r] & E' \arrow[r, "p'"] & B' \\
F \arrow[u, "f\simeq *"] \arrow[r] & E \arrow[u, "e"] \arrow[r, "p"] & B \arrow[u, "b", "\wr"']
\end{tikzcd}$$

such that f is nullhomotopic and b is an equivalence. If $\delta : \Omega B \to F$ (and similarly for δ') denotes the connecting map, then it follows that $\delta' \circ \Omega b \simeq f \circ \delta \simeq *$, and thus Ωb lifts, up to homotopy, to a map $\tilde{\sigma} : \Omega B \to \Omega E'$. Then for $(\Omega b)^{-1}$ any homotopy inverse to Ωb, the map $\sigma := \tilde{\sigma} \circ (\Omega b)^{-1} : \Omega B' \to \Omega E'$ is a homotopy section of the fibration $\Omega F' \to \Omega E' \to \Omega B'$ and so provides a splitting $\Omega E' \simeq \Omega B' \times \Omega F'$.

For now, let us focus solely on the first equivalence of the statement. By the previous argument, it suffices to show that the leftmost vertical graphing map in the diagram of fibration sequences

$$\begin{tikzcd}
\mathrm{Emb}^{(\sim)}_{\partial_0}(P \times I, M \times I) \arrow[r] & \mathrm{Emb}_{\partial_0}(P \times I, M \times I) \arrow[r] & \widetilde{\mathrm{Emb}}_{\partial_0}(P \times I, M \times I) \\
\Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) \arrow[r] \arrow[u, "\Gamma"] & \Omega\,\mathrm{Emb}_{\partial_0}(P, M) \arrow[r] \arrow[u, "\Gamma"] & \Omega\widetilde{\mathrm{Emb}}_{\partial_0}(P, M). \arrow[u, "\Gamma", "\wr"']
\end{tikzcd}$$

is nullhomotopic after localising away from 2 and taking $(\phi_{C\mathrm{Emb}}(d+1, p+1) - 1)$-th Postnikov sections.

By Proposition 2.3.3, we may assume that $\dim P = \dim M = d$. Then, similar to Notation 2.2.5, let $\Omega F(-)$ and $FI(-)$ denote the orthogonal functors given by

$$\begin{aligned}
\Omega F(V) &:= \Omega\,\mathrm{hofib}(B\mathrm{Diff}^b_\partial(\overline{M - P} \times V) \to B\mathrm{Diff}^b_\partial(M \times V)), \\
FI(V) &:= \mathrm{hofib}(B\mathrm{Diff}^b_\partial(\overline{M - P} \times I \times V) \to B\mathrm{Diff}^b_\partial(M \times I \times V)).
\end{aligned}$$

By taking fibres of the (delooped) graphing maps introduced at the beginning of the section, we obtain a natural transformation $\Gamma : \Omega F(-) \to FI(-)$ of orthogonal functors, giving rise to a map of $O(1)$-spectra $\Gamma : \Theta(\Omega F)^{(1)} \longrightarrow \Theta FI^{(1)}$ and a commutative diagram

$$\begin{tikzcd}
\mathrm{Emb}^{(\sim)}_{\partial_0}(P \times I, M \times I) \arrow[r, "\Phi^{FI}_\infty"] & \Omega^\infty(\Theta FI^{(1)}_{hO(1)}) \arrow[r, "\mathrm{Trf}_{O(1)}"] & \Omega^\infty \Theta FI^{(1)} \\
\Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) \arrow[r, "\Phi^{\Omega F}_\infty"] \arrow[u, "\Gamma"] & \Omega^\infty(\Theta(\Omega F)^{(1)}_{hO(1)}) \arrow[r, "\mathrm{Trf}_{O(1)}"] \arrow[u, "\Gamma"] & \Omega^\infty \Theta(\Omega F)^{(1)}, \arrow[u, "\Gamma"]
\end{tikzcd} \tag{2.4.2}$$

where $\mathrm{Trf}_{O(1)}$ is the $O(1)$-*transfer map.* This map is injective in the homotopy category of infinite loop spaces at odd primes (it splits the quotient map $X \to X_{hC_2}$). Therefore, since the map Φ^{FI}_∞ is $\phi_{C\mathrm{Emb}}(d+1, p+1)$-connected by Section 2.3.4 (and thus becomes an equivalence after taking $(\phi_{C\mathrm{Emb}}(d+1, p+1) - 1)$-th Postnikov sections), it will suffice to show that the rightmost vertical map in (2.4.2) is nullhomotopic. By (2.3.7), we have that $\Omega^\infty \Theta(\Omega F)^{(1)} \simeq \Omega\mathcal{CE}\mathrm{mb}(P, M)$ and $\Omega^\infty \Theta FI^{(1)} \simeq \mathcal{CE}\mathrm{mb}(P \times I, M \times I)$ and, under these equivalences, the right vertical map in (2.4.2) then becomes the graphing map

$$\Gamma : \Omega\mathcal{CE}\mathrm{mb}(P, M) \longrightarrow \mathcal{CE}\mathrm{mb}(P \times I, M \times I). \tag{2.4.3}$$

This is because both the concordance stabilisation map and the Alexander trick-like map of Proposition 2.3.11 that give rise to the previous equivalences commute on the nose with the graphing maps, i.e., the following diagrams commute:

$$\begin{tikzcd}
\Omega C\mathrm{Emb}(P \times D^{n+1}, M \times D^{n+1}) \arrow[r, "\Gamma"] & C\mathrm{Emb}(P \times I \times D^{n+1}, M \times I \times D^{n+1}) \\
\Omega C\mathrm{Emb}(P \times D^{n}, M \times D^{n}) \arrow[r, "\Gamma"] \arrow[u, "\Sigma"] & C\mathrm{Emb}(P \times I \times D^{n}, M \times I \times D^{n}), \arrow[u, "\Sigma"]
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^n \Theta(\Omega F)^{(1)}_n \arrow[r, "\Gamma"] & \Omega^n \Theta F I^{(1)}_n \\
\Omega C\mathrm{Emb}(P \times D^{n}, M \times D^{n}) \arrow[r, "\Gamma"] \arrow[u, "\mathrm{alex}", "\wr"'] & C\mathrm{Emb}(P \times I \times D^{n}, M \times I \times D^{n}). \arrow[u, "\mathrm{alex}", "\wr"']
\end{tikzcd}$$

So in order to show that (2.4.3) is nullhomotopic, it suffices to argue that it is so unstably, i.e. that the graphing maps

$$\Gamma : \Omega C\mathrm{Emb}(P \times D^n, M \times D^n) \longrightarrow C\mathrm{Emb}(P \times I \times D^n, M \times I \times D^n)$$

are nullhomotopic for all $n \geq 0$. Replacing $M \times D^n$ by M, we may assume $n = 0$. This claim is a consequence of the following trick, due to Oscar Randal-Williams: there is a “U-shaped graphing map”

$$U\Gamma : \Omega C\mathrm{Emb}(P, M) \longrightarrow C\mathrm{Emb}(P \times I, M \times I),$$

which is homotopic to the standard Γ by pulling down the U-shape to the base of the concordance. This homotopy is illustrated in Figure 2.1, where we replace $C\mathrm{Emb}(-,-)$ by standard concordances $C(-)$ because it is easier to depict, but the idea is the same. Observe that, throughout the homotopy, there are no issues about smoothness in the upper corners because the concordances are equal to the identity near these. Here we are explicitly using the collared condition imposed by the functor $\Omega^{\mathsf{col,sm}}(-)$ (see Remark 2.4.1).

But clearly $U\Gamma$ factors through the path space $\mathrm{Map}(I, C\mathrm{Emb}(P, M))$, and hence there is a homotopy commutative diagram

$$\begin{tikzcd}
C\mathrm{Emb}(P \times I, M \times I) \arrow[r, equal] & C\mathrm{Emb}(P \times I, M \times I) & & \\
 & & \mathrm{Map}(I, C\mathrm{Emb}(P, M)) \arrow[lu, "U\Gamma"'] \arrow[r, "\mathrm{ev}_0", "\sim"'] & C\mathrm{Emb}(P, M), \\
\Omega C\mathrm{Emb}(P, M) \arrow[r, equal] \arrow[uu, "\Gamma"] & \Omega C\mathrm{Emb}(P, M) \arrow[uu, "U\Gamma"] \arrow[ru, hook] \arrow[rru, "\mathrm{ev}_0 = *"'] & &
\end{tikzcd}$$

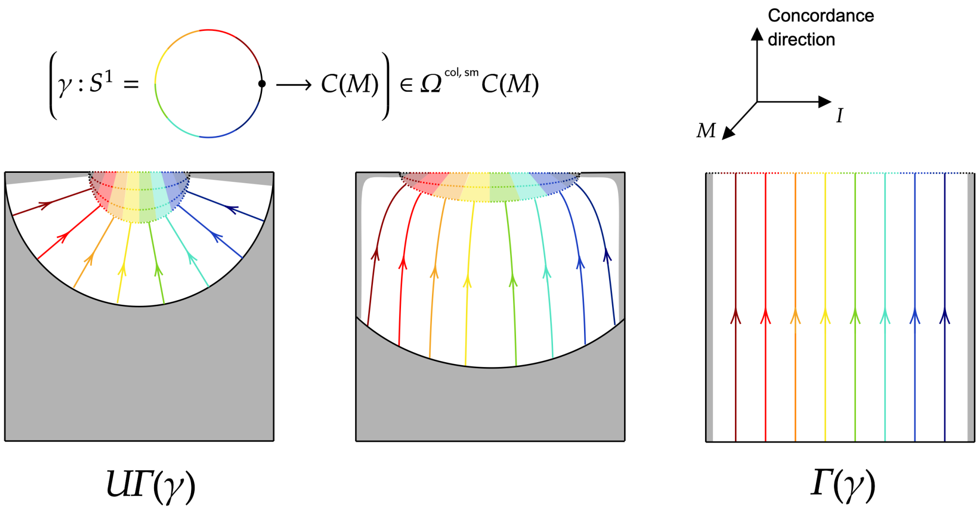

Figure 2.1 Images of $\gamma \in \Omega C(M)$ under the graphing maps Γ and $U\Gamma$, and the homotopy between them. The concordances are equal to the identity on grey shaded regions.

which exhibits the leftmost vertical map as nullhomotopic, as desired. This establishes the first equivalence of the statement.

Observe now that for $E(-,-)$ any of the mapping spaces involved in the proof up until now, the space $E(P \times J, M \times J)$ is a group-like topological monoid with respect to stacking in the J-direction. Then replacing (M, P) by $(M \times J, P \times J)$, one checks that each of the steps in the previous argument can be delooped with respect to this $\mathbb{E}_1$-structure. This results in getting rid of the loopings in the first equivalence of the statement, thus yielding the second one. This finishes the proof of the theorem. □

The following result will be used to establish the splitting part of Theorem D.

Corollary 2.4.3. *For $2 \leq p \leq d-3$ and $N := \phi_{C\mathrm{Emb}}(d,p) - 1$, there is an equivalence away from 2*

$$\tau_{\leq N} \operatorname{Emb}_\partial(D^p, D^d) \simeq_{[\frac{1}{2}]} \tau_{\leq N} \left(\widetilde{\operatorname{Emb}}_\partial(D^p, D^d) \times \operatorname{Emb}_\partial^{(\sim)}(D^p, D^d) \right).$$

For $p = 1$, such equivalence exists only after looping, i.e.,

$$\Omega\tau_{\leq N} \operatorname{Emb}_\partial(D^1, D^d) \simeq_{[\frac{1}{2}]} \Omega\tau_{\leq N} \left(\widetilde{\operatorname{Emb}}_\partial(D^1, D^d) \times \operatorname{Emb}_\partial^{(\sim)}(D^1, D^d) \right).$$

Proof. When $p \geq 2$, set $(M, P) = (D^{d-2}, D^{p-2})$ in the second equivalence of Theorem 2.4.2. For $p = 1$, set $(M, P) = (D^{d-1}, D^0)$ in the first equivalence of the same theorem. □

C Appendix: Orthogonal calculus technicalities

Throughout, let $F : \mathcal{J}_0 \to \mathsf{Top}_*$ be an orthogonal functor.

C.1 The $O(1)$-spectrum $\Theta F^{(1)}$

We present an explicit model for the structure maps (2.2.2) of the $O(1)$-spectrum $\Theta F^{(1)}$, and compare our convention for its $O(1)$-action with that of [Wei95, Prop. 3.1].

C.1.1 An explicit model of the (pre-)spectrum

For V, we now describe an explicit model for the structure map (2.2.2)

$$s_V : S^1 \wedge F^{(1)}(V) \longrightarrow F^{(1)}(V \oplus \mathbb{R}),$$

where $F^{(1)}(V) := \mathrm{hofib}(F(V) \to F(V \oplus \underline{\mathbb{R}}))$; here $\underline{\mathbb{R}}$ simply stands for a copy of $\mathbb{R}$ that we underline to distinguish it from the one appearing in the codomain of s_V. Our model for the homotopy fibre of a map $X \to Y$ of pointed spaces is the standard one, i.e., the subspace $\{(x, \gamma) \in X \times Y^{[0,1]} : \gamma(0) = x,\ \gamma(1) = *\}$.

The evident commutative diagram in $\mathcal{J}_0$

$$\begin{tikzcd}
V \arrow[r, "v \mapsto (v,0)"] \arrow[d, "\begin{smallmatrix} v \\ \downarrow \\ (v,0) \end{smallmatrix}"'] & V \oplus \mathbb{R} \arrow[d, "\begin{smallmatrix} (v,t) \\ \downarrow \\ (v,t,0) \end{smallmatrix}"] \\
V \oplus \underline{\mathbb{R}} \arrow[r, "(v,\underline{t}) \mapsto (v,0,\underline{t})"'] & V \oplus \mathbb{R} \oplus \underline{\mathbb{R}}
\end{tikzcd}$$

induces a commutative diagram of based spaces

$$\begin{tikzcd}
F^{(1)}(V) \arrow[r, dashed] \arrow[d, "i"] \arrow[dr, "\alpha"] & F^{(1)}(V \oplus \mathbb{R}) \arrow[d] \\
F(V) \arrow[r, "h"] \arrow[d, "\beta"] & F(V \oplus \mathbb{R}) \arrow[d, "\beta'"] \\
F(V \oplus \underline{\mathbb{R}}) \arrow[r, "h'"] & F(V \oplus \mathbb{R} \oplus \underline{\mathbb{R}}).
\end{tikzcd}$$

Our task is to define a loop of dashed arrows (based at the constant map)—we will try to do so by providing a loop of null-homotopies $\beta'\alpha \sim *$ (we will fail, but just slightly).

Note that, since βi is null-homotopic via $\widetilde{H}^{(t)}(x,\gamma) = \gamma(t)$, we get a null-homotopy $H_0 := h'\widetilde{H} : \beta'\alpha \sim *$. For each $\theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$, let θ_* denote the automorphism of $F(V \oplus \mathbb{R} \oplus \underline{\mathbb{R}})$ induced by the rotation of the plane $0 \oplus \mathbb{R} \oplus \underline{\mathbb{R}}$ with angle θ. Then note that by functoriality of F, we have that $\theta_*\beta'h = \beta'h$. Thus, the maps $H_\theta := \theta_* H_0 : F^{(1)}(V) \times I \to F(V \oplus \mathbb{R} \oplus \underline{\mathbb{R}})$ provide a path of null-homotopies $\beta'\alpha \sim *$.

As foreshadowed, the null-homotopies $H_{\frac{-\pi}{2}} := (\frac{-\pi}{2})_* h'\widetilde{H}$ and $H_{\frac{\pi}{2}} := (\frac{\pi}{2})_* h'\widetilde{H}$ are distinct (as one can check). The crucial point then is that both are of the form $\beta'G$ for some null-homotopy $G : \alpha \sim *$: indeed, if $\phi : F(V \oplus \underline{\mathbb{R}}) \cong F(V \oplus \mathbb{R})$ denotes the map induced by identifying $\underline{\mathbb{R}}$ with $\mathbb{R}$, then

$$(\tfrac{-\pi}{2})_* h' = \beta'(-1_{\mathbb{R}})_*\phi, \qquad (\tfrac{\pi}{2})_* h' = \beta'\phi,$$

where $(-1_{\mathbb{R}})_*$ is the automorphism of $F(V \oplus \mathbb{R})$ induced by $(v,t) \mapsto (v,-t)$. By noting that $\phi\beta = h$ and that $(-1_{\mathbb{R}})_* h = h$, one easily verifies that $G_{\frac{-\pi}{2}} := (-1_{\mathbb{R}})\phi\widetilde{H}$ and $G_{\frac{\pi}{2}} := \phi\widetilde{H}$ indeed provide the desired null-homotopies $\alpha \sim *$.

Finally, we note that the null-homotopies $G_{\frac{\pm\pi}{2}}$ give rise to canonical null-homotopies of the maps $\sigma_{\frac{\pm\pi}{2}} : F^{(1)}(V) \to F^{(1)}(V \oplus \mathbb{R})$ induced by $H_{\frac{\pm\pi}{2}}$. All together, we obtain a loop of maps $F^{(1)}(V) \to F^{(1)}(V \oplus \mathbb{R})$ that is adjoint to the structure map σ_V of (2.2.2).

C.1.2 The $O(1)$-action

We have defined the first derivative spectrum $\Theta F^{(1)}$ of F to be the (naïve) $O(1)$-spectrum with n-th $O(1)$-space $\Theta F^{(1)}_n = F^{(1)}(\mathbb{R}^n)$ and with structure maps as defined just above. Recall that

$$F^{(1)}(V) = \operatorname{hofib}(F(V) \longrightarrow F(V \oplus \underline{\mathbb{R}}))$$

is an $O(1)$-space by declaring $-1 \in O(1)$ to act on V and $V \oplus \underline{\mathbb{R}}$ by -1 on *all* coordinates. A straightforward verification shows that, under this convention, the map

$$s_V : S^1 \wedge F^{(1)}(V) \longrightarrow F^{(1)}(V \oplus \mathbb{R})$$

is indeed $O(1)$-equivariant, where $O(1)$ acts trivially on S^1. The key point in this verification is that if R is a (2×2)-matrix (e.g. a rotation of the plane), then $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)R$ and $R\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ agree on the subspace $\mathbb{R} \oplus 0 \subset \mathbb{R} \oplus \underline{\mathbb{R}}$ even though maybe not equal.

In [Wei95, Prop. 3.1], $O(1)$ instead acts on $F^{(1)}(V) = \operatorname{hofib}(F(V) \to F(V \oplus \mathbb{R}))$ by declaring the action of $-1 \in O(1)$ on V to be trivial and by -1 on the $\mathbb{R}$-summand of $V \oplus \mathbb{R}$. If we write $\underline{F}^{(1)}(V)$ for this $O(1)$-space, then the maps

$$s_V : S^{\sigma} \wedge \underline{F}^{(1)}(V) \longrightarrow \underline{F}^{(1)}(V \oplus \mathbb{R}) \quad \text{and} \quad \underline{F}^{(1)}(V) \to F(V) \tag{C.1}$$

are $O(1)$-equivariant, where σ stands for the (1-dimensional) sign $O(1)$-representation and S^{σ} for its associated representation sphere. The corresponding (sequential) spectrum, call it $\underline{\Theta}F^{(1)}$, is *not* a naïve $O(1)$-spectrum in the usual sense anymore, as $O(1)$ acts non-trivially on the suspension coordinates. To solve this issue, Weiss introduces in [Wei95, p. 17] the $O(1)$-spectrum $\Theta^{\#}F^{(1)}$ with n-th $O(1)$-space

$$\Theta^{\#}F^{(1)}_n := \Omega^{\infty\cdot\sigma}(S^n \wedge \underline{\Theta}F^{(1)}) = \operatorname*{hocolim}_{k} \Omega^{k\cdot\sigma}(S^n \wedge \underline{F}^{(1)}(\mathbb{R}^k)), \tag{C.2}$$

where $\Omega^{k\cdot\sigma}(-) := \mathrm{Map}_*(S^{k\cdot\sigma}, -)$ and $O(1)$ acts by conjugation on this mapping space.

In order to relate the $O(1)$-spectra $\Theta F^{(1)}$ and $\Theta^{\#}F^{(1)}$, we observe that F can be naturally upgraded to a functor $\underline{F} : \mathcal{J}_0^{O(1)} \to \mathsf{Top}_*^{O(1)}$ enriched over Top_*, where $\mathcal{J}_0^{O(1)} := \mathrm{Fun}(O(1), \mathcal{J}_0)$ is regarded as the pointed topological category of inner product finite-dimensional $O(1)$-representations. We likewise define for $V \in \mathcal{J}_0^{O(1)}$ the $O(1)$-space $\underline{F}^{(1)}(V) := \mathrm{hofib}(\underline{F}(V) \to \underline{F}(V \oplus \sigma))$. Now tensoring such an $O(1)$-representation with the sign representation σ gives a self-isomorphism of $\mathcal{J}_0^{O(1)}$ denoted by $- \cdot \sigma$. One could stabilise $\underline{F}^{(1)}(-)$ with respect to $\mathbb{R}$ as in (C.1), or with the sign representation σ, giving rise to maps

$$s_{a,b} : S^{a\cdot\sigma+b} \wedge \underline{F}^{(1)}(V) \longrightarrow \underline{F}^{(1)}(V \oplus \mathbb{R}^{a,b}), \quad a, b \geq 0, \quad V \in \mathcal{J}_0^{O(1)},$$

where $S^{a\cdot\sigma+b} := S^{a\cdot\sigma} \wedge S^b$ and $\mathbb{R}^{a,b} := \mathbb{R}^a \oplus b \cdot \sigma$. We then obtain a zig-zag of maps of $O(1)$-spectra

$$\begin{tikzcd}
\Theta^{\#}F^{(1)} := \operatorname*{hocolim}_{a\geq 0} \mathbb{S}^{-a\cdot\sigma} \wedge \underline{F}^{(1)}(\mathbb{R}^a) \arrow[dr, hook, "b=0", "\sim"'] & & \operatorname*{hocolim}_{b\geq 0} \mathbb{S}^{-b} \wedge \underline{F}^{(1)}(b\cdot\sigma) =: \Theta F^{(1)} \arrow[dl, hook', "a=0"', "\sim"] \\
& \operatorname*{hocolim}_{a,b\geq 0} \mathbb{S}^{-a\cdot\sigma-b} \wedge \underline{F}^{(1)}(\mathbb{R}^{a,b}), &
\end{tikzcd} \tag{C.3}$$

where the maps in the colimit of the middle spectrum are induced by $s_{1,0}$ and $s_{0,1}$. Non-equivariantly, both of the maps in the zig-zag are equivalences by Fubini's theorem. This establishes the desired $O(1)$-equivariant equivalence[4] $\Theta F^{(1)} \simeq \Theta^{\#}F^{(1)}$.

---

[4] In the sense of Borel (see Notation 3.1.2(*i*)).

C.2 Completion of the Proof of Proposition 2.2.2

Recall that, so far, we have constructed the horizontal fibre sequences in the diagram

$$\begin{tikzcd}
\operatorname{hofib}(F(0) \to F(\mathbb{R}^n)) \arrow[r] \arrow[d, "\Phi_n^F"] & \operatorname{hofib}(F(0) \to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "\Phi_{n+1}^F"] & \operatorname{hofib}(F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1})) =: \Theta F_n^{(1)} \arrow[d, "\text{stab.}"] \\
\Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(S(\mathbb{R}^{n+1})_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(\Sigma^n \Theta F^{(1)}),
\end{tikzcd}$$

and argued that the left subsquare is homotopy commutative. Here

$$\Phi_n^F : \operatorname{hofib}(F(0) \to F(\mathbb{R}^n)) \xrightarrow{\eta_1} \operatorname{hofib}(T_1F(0) \to T_1F(\mathbb{R}^n)) \simeq \Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)})$$

is the map (2.2.3) and "stab." is $\Theta F_n^{(1)} \hookrightarrow \operatorname{hocolim}_k \Omega^k(\Theta F_{n+k}^{(1)})$. It remains to verify the following:

Claim. *The right subsquare in the diagram above is homotopy commutative.*

Proof. Without loss of generality, we may replace F by $\operatorname{hofib}(\eta_0 : F \to T_0F)$ so that T_1F becomes homogeneous of degree 1 (see [Wei95, Defn. 7.1]). Because of the $O(1)$-equivalence $\eta_1 : \Theta F^{(1)} \simeq \Theta(T_1F)^{(1)}$ proved in [Wei95, Thm. 6.3(bis)], we may also replace F by T_1F so that F itself is homogeneus of degree 1. Then it was shown in [Wei95, Thm. 7.3] that there is a natural zig-zag of equivalences between $F(V)$ and $\Omega^\infty((S^{V\cdot\sigma} \wedge \Theta^{\#}F^{(1)})_{hO(1)})$ for $V \in \mathcal{J}_0$, where $\Theta^{\#}F^{(1)}$ is the $O(1)$-spectrum of (C.2). But we have seen in Section C.1.2 that there is a zig-zag (C.3) of $O(1)$-equivariant equivalences between $\Theta F^{(1)}$ and $\Theta^{\#}F^{(1)}$. Therefore, we obtain a zig-zag of equivalences between $\operatorname{hofib}(F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1}))$ and the homotopy fibre of the map $i_n : \Omega^\infty((S^{n\cdot\sigma} \wedge \Theta F^{(1)}_{hO(1)}) \to \Omega^\infty((S^{(n+1)\cdot\sigma} \wedge \Theta F^{(1)}_{hO(1)})$. This is the zig-zag showing up in the top part of the diagram

$$\begin{tikzcd}
F^{(1)}(\mathbb{R}^n) \arrow[d, hook, "\text{stab.}"] & \cdot \arrow[l, "\sim"'] \arrow[r, "\sim"] & \operatorname{hofib}\begin{pmatrix} \Omega^\infty((S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}) \\ \downarrow i_n \\ \Omega^\infty((S^{(n+1)\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}) \end{pmatrix} \arrow[d] \\
\Omega^\infty(\Sigma^n \Theta F^{(1)}) \arrow[urr, dashed, "\widehat{\lambda}", "\simeq"'] \arrow[rr, "\lambda"] \arrow[drr, bend right, "\simeq *"] & & \Omega^\infty\Big((S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\Big) \arrow[d, "i_n"] \\
& & \Omega^\infty\Big((S^{(n+1)\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\Big)
\end{tikzcd} \tag{C.4}$$

where λ is defined as follows: given two copies S^n_i of the n-sphere for $i = 0, 1$ (with the trivial $O(1)$-action), there is an $O(1)$-map $a : S^n_0 \vee S^n_1 \to S^{n\cdot\sigma}$ that sends $v \in S^n_i$ to $(-1)^i v \in S^{n\cdot\sigma}$. Then

$$\Sigma^n\Theta F^{(1)} \xleftarrow{\sim} S^n \wedge \Theta F^{(1)} \xleftarrow{\sim} ((S^n_0 \vee S^n_1) \wedge \Theta F^{(1)})_{hO(1)} \xrightarrow{(a\wedge\mathrm{Id})_{hO(1)}} (S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}$$

gives rise to (the zig-zag) λ upon taking infinite loop spaces. Identifying $S^{(n+1)\cdot\sigma}$ as the homotopy cofibre of the map a (really $S^{n+1} \vee S^{n+1}$ is the cofibre of the inclusion $S^{n\cdot\sigma} \to S^{(n+1)\cdot\sigma}$), we see that λ lifts, uniquely up to homotopy, to the weak equivalence $\widehat{\lambda}$. As the top zig-zag of (C.4) in place of the map "stab." in the diagram from the statement would make its right square commutive by construction, we must show that the upper triangle of (C.4) is homotopy commutative. For this, it suffices to argue that the bigger square that contains it commutes up to homotopy.

One verifies from the proofs of [Wei95, Prop. 7.2 & Thm. 7.3] and the construction of (C.3) that the top-right composition of the square in (C.4) becomes, up to homotopy, the top-right composition of

$$\begin{tikzcd}
F^{(1)}(\mathbb{R}^n) \overset{(!)}{=} \underline{F}^{(1)}_{n,0} \arrow[r] \arrow[d, hook, "k=0"] & (\underline{F}^{(1)}_{n,0})_{hO(1)} \arrow[r, hook, "k=0", "\sim"'] & \left(\operatorname*{hocolim}_k \Omega^k(\underline{F}^{(1)}_{n,k})\right)_{hO(1)} & \left(\Omega^\infty(S^{n\cdot\sigma} \wedge \Theta F^{(1)})\right)_{hO(1)} \arrow[l, "s_{n,0}", "\sim"'] \arrow[d, hook] \\
\operatorname*{hocolim}_k \Omega^k(\underline{F}^{(1)}_{n,k}) \arrow[urr] \arrow[d, equal, "(!)"] & \Omega^\infty(S^{n\cdot\sigma} \wedge \Theta F^{(1)}) \arrow[l, "\simeq"'] \arrow[urr] \arrow[rr, "q"] \arrow[d, equal, "(!)"] \arrow[drr, phantom, "(*)"] & & \Omega^\infty\left((S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\right) \\
\Omega^\infty(\Sigma^n\Theta F^{(1)}) & \Omega^\infty(S^n \wedge \Theta F^{(1)}) \arrow[l, "\sim"'] & \Omega^\infty\left(((S^n \vee S^n) \wedge \Theta F^{(1)})_{hO(1)}\right). \arrow[l, "\simeq"'] \arrow[ur, "(a\wedge\mathrm{Id})_{hO(1)}"] &
\end{tikzcd}$$

Here $\underline{F}^{(1)}_{a,b} = \underline{F}^{(1)}(\mathbb{R}^{a,b}) = \underline{F}^{(1)}(\mathbb{R}^a \oplus b \cdot \sigma)$ and the decoration (!) indicates that the equality is not (supposed to be) $O(1)$-equivariant. All subtriangles and squares, except possibly the bottom right one labelled by $(*)$, are visibly commutative. Moreover the left-bottom composition is the composition $\lambda \circ \text{stab.}$ of the square in (C.4) by definition. The bottom right subsquare $(*)$ is commutative because the following is for every

$O(1)$-spectrum X:

$$\begin{tikzcd}
S^{n\cdot\sigma}\wedge X \arrow[rrr, "q"] \arrow[d, equal, "(!)"] \arrow[dr, "\cong"] & & & (S^{n\cdot\sigma}\wedge X)_{hO(1)} \\
S^{n}\wedge X \arrow[dr, "\cong"] & (S_0^{n\cdot\sigma}\vee S_1^{n\cdot\sigma})\wedge_{O(1)} X & ((S_0^{n\cdot\sigma}\vee S_1^{n\cdot\sigma})\wedge X)_{hO(1)} \arrow[l, "\sim"'] \arrow[ur, "(c\wedge \mathrm{Id})_{hO(1)}"] & \\
& (S_0^{n}\vee S_1^{n})\wedge_{O(1)} X \arrow[u, "b\wedge\mathrm{Id}", "\cong"'] & ((S_0^{n}\vee S_1^{n})\wedge X)_{hO(1)} \arrow[l, "\sim"'] \arrow[u, "b\wedge\mathrm{Id}", "\cong"'] \arrow[uur, bend right, "(a\wedge \mathrm{Id})_{hO(1)}"'] &
\end{tikzcd}$$

Here the $O(1)$-equivariant isomorphism $b : S_0^n \vee S_1^n \to S_0^{n\cdot\sigma} \vee S_1^{n\cdot\sigma}$ sends $v \in \mathbb{R}_i^{n\cdot\sigma} \subset S_i^{n\cdot\sigma}$ to $(-1)^i v \in \mathbb{R}_i^{n\cdot\sigma} \subset S_i^{n\cdot\sigma}$, and $c : S_0^{n\cdot\sigma} \vee S_1^{n\cdot\sigma} \to S^{n\cdot\sigma}$ is the identity in each wedge summand. This finishes the proof of the claim, and thus of Proposition 2.2.2. □

Convention C.1. *In the body of Chapter 2, $F^{(1)}(V)$ stands for $\underline{F}^{(1)}(V\cdot\sigma) := \mathrm{hofib}(\underline{F}(V\cdot\sigma) \to \underline{F}((V\oplus\mathbb{R})\cdot\sigma))$ in the notation of Section C.1.2, unless we explictly say otherwise. This way (2.2.2) is $O(1)$-equivariant.*

D Appendix: Bounded geometry

Throughout, M^d denotes a smooth compact d-manifold (possibly with boundary).

D.1 Models for bounded diffeomorphisms and embeddings

Let N be a (possibly non-compact) smooth manifold and fix some smooth embedding $\iota : M \hookrightarrow N$. For $V \in \mathcal{J}_0$, we will write $\mathrm{Emb}_\partial(M\times V, N\times V)$ for the space of smooth embeddings of M into N that agree with $\iota\times\mathrm{Id}_V$ on some neighbourhood of the boundary $\partial M\times V$, endowed with the Whitney weak C^∞-topology. Following Definition 2.2.3, the *space of bounded embeddings* of $M\times V$ into $N\times V$ relative to $\partial M\times V$ is the subspace of $[0,+\infty)\times\mathrm{Emb}_\partial(M\times V, N\times V)$ given by

$$\mathrm{Emb}^b_\partial(M\times V, N\times V) := \{(t,\varphi)\in[0,+\infty)\times\mathrm{Emb}_\partial(M\times V, N\times V) : \varphi \text{ is } t\text{-bounded}\}.$$

Define similarly its simplicial version $\mathrm{Emb}^b_\partial(M\times V, N\times V)_\bullet$ as in Definition 2.2.7. In this section we prove

Proposition D.1. *There is a zig-zag of weak equivalences of semi-simplicial group-like monoids*

$$\mathrm{Diff}^b_\partial(M\times V)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M\times V)).$$

Similarly, there is a zig-zag of weak equivalences of semi-simplicial sets

$$\mathrm{Emb}^b_\partial(M\times V, N\times V)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Sing}_\bullet(\mathrm{Emb}^b_\partial(M\times V, N\times V)).$$

We will only deal with the first part of the statement, as the proof for the embedding case is completely analogous. Let us first introduce some notation. Given a topological space X, let $\mathrm{Sing}^{\mathsf{col}}_\bullet(X)$ be the sub-simplicial set of $\mathrm{Sing}_\bullet(X)$ consisting of those singular simplices that satisfy the ϵ-collaring condition of Section 2.2.2 for some $0<\epsilon<1/2$. Denote by $\mathrm{Sing}^{\mathsf{col},b}_\bullet(\mathrm{Diff}_\partial(M\times V))$ the sub-simplicial group of $\mathrm{Sing}^{\mathsf{col}}_\bullet(\mathrm{Diff}_\partial(M\times V))$ consisting of those j-simplices which are adjoint to a bounded map $\Delta^j\times M\times V\to M\times V$. Then, there is a zig-zag of maps of simplicial group-like monoids

$$\begin{tikzcd}
\mathrm{Diff}^b_\partial(M\times V)_\bullet \arrow[r, hook, "①"] & \mathrm{Sing}^{\mathsf{col},b}_\bullet(\mathrm{Diff}_\partial(M\times V)) & \mathrm{Sing}^{\mathsf{col}}_\bullet(\mathrm{Diff}^b_\partial(M\times V)) \arrow[l, hook', "②"'] \arrow[d, hook, "③"] \\
 & & \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M\times V)),
\end{tikzcd} \tag{D.1}$$

where the map ② forgets the explicit bounding constant of a simplex. We will show that all the maps in (D.1) are weak equivalences. We start with ③.

Lemma D.2. *The inclusion* $i:\mathrm{Sing}^{\mathsf{col}}_\bullet(X)\hookrightarrow\mathrm{Sing}_\bullet(X)$ *is a weak equivalence for every topological space* X.

Proof. We show that the relative homotopy groups $\pi_j(\mathrm{Sing}_\bullet(X),\mathrm{Sing}^{\mathsf{col}}_\bullet(X))$ vanish for all $j\geq 0$. Indeed, a homotopy class $x\in\pi_j(\mathrm{Sing}_\bullet(X),\mathrm{Sing}^{\mathsf{col}}_\bullet(X))$ corresponds, by the Yoneda Lemma, to a singular j-simplex $g:\Delta^j\to X$ which satisfies the ϵ-collaring condition for all faces $\sigma\subset\partial\Delta^j$ and some $\epsilon>0$. Now fix some identification $\Delta^j\cong\Delta^j\cup_{\partial\Delta^j}(\partial\Delta^j\times[0,\epsilon])$, and consider the singular j-simplex

$$\overline{g}=g\cup(g\,|_{\partial_{\Delta^j}}\circ\mathrm{proj}_{\partial\Delta^j}):\Delta^j\cong\Delta^j\cup_{\partial\Delta^j}(\partial\Delta^j\times[0,\epsilon])\longrightarrow X.$$

By construction $\overline{g}$ now satisfies the δ-collaring conditions for all faces $\sigma\subset\Delta^j$ and some $0<\delta\leq\epsilon$, so the corresponding relative homotopy class $\overline{x}$ is trivial. But clearly g and $\overline{g}$ are homotopic relative to the boundary by shrinking the added collar, and hence $x=\overline{x}=0$ in $\pi_j(\mathrm{Sing}_\bullet(X),\mathrm{Sing}^{\mathsf{col}}_\bullet(X))$, as claimed. □

Remark D.3. The inclusion $i:\mathrm{Sing}^{\mathsf{col}}_\bullet(X)\hookrightarrow\mathrm{Sing}_\bullet(X)$ is in fact a simplicial homotopy equivalence; a homotopy inverse is constructed by induction on the skeleta of Δ^j. We will not need this though.

Lemma D.4. *The map* ② $: \mathrm{Sing}^{\mathsf{col}}_\bullet(\mathrm{Diff}^b_\partial(M \times V)) \to \mathrm{Sing}^{\mathsf{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ *of (D.1) is a weak equivalence.*

Proof. We again show that the relative homotopy groups π_j(②) vanish for all $j \geq 0$. Such a homotopy x class can be represented, for some $\epsilon > 0$, by an ϵ-collared singular j-simplex $g : \Delta^j \to \mathrm{Diff}_\partial(M \times V)$, adjoint to a map $g^\vee : \Delta^j \times M \times V \to M \times V$ bounded by some $K \geq 0$, together with a continuous ϵ-collared map $r : \partial\Delta^j \to [0, \infty)$ such that $g(s)$ is $r(s)$-bounded for all $s \in \partial\Delta^j$. To show that x is trivial, we need to extend r to a continuous δ-collared map $R : \Delta^j \to [0, \infty)$, for some $0 < \delta \leq \epsilon$, such that $g(s)$ is $R(s)$-bounded for all $s \in \Delta^j$. Fix some identification $\Delta^j \cong (\partial\Delta^j \times [0, \epsilon]) \cup_{\partial\Delta^j \times \{\epsilon\}} \Delta^j$; then $R \mid_{\partial\Delta^j \times [0,\epsilon/2]} \equiv r \circ \mathrm{proj}_{\partial\Delta^j}$ whilst $R \mid_{\partial\Delta^j \times [\epsilon/2,\epsilon]}$ is a linear interpolation along $[\epsilon/2, \epsilon]$ between r and the constant map $c_K : \partial\Delta^j \times \{\epsilon\} \to [0, \infty)$ with value $K \geq 0$. Finally set R to be constant of value K in the inner $\Delta^j \subset (\partial\Delta^j \times [0, \epsilon]) \cup_{\partial\Delta^j \times \{\epsilon\}} \Delta^j$. Then R is as required, and hence the relative homotopy class $x \in \pi_j$(②) is trivial. ☐

Remark D.5. For CAT = Top, the map ① of (D.1) is an equality and thus, at this point, Proposition D.1 is established in the topological case.

Proof of Proposition D.1. It remains to show that ① is a weak equivalence, i.e., that the relative homotopy groups π_k(①) vanish for all $k \geq 0$. This is clear for $k = 0$ by definition. Such a homotopy class in π_k(①) is represented by a bounded homeomorphism

$$\overline{g} = (\mathrm{proj}_{\Delta^k}, g) : \Delta^k \times M \times V \longrightarrow \Delta^k \times M \times V$$

which is collared in the simplex direction and such that $g \mid_{\partial\Delta^k \times M \times V}$ is smooth. Therefore g is smooth on (a neighbourhood of) $\partial\Delta^k \times M \times V$. We need to smooth g outside of such neighbourhood in the Δ^k-direction and preserving boundedness. For $r \in \Delta^k$, we will write $g_r \in \mathrm{Diff}_\partial(M \times V)$ for $g \mid_{\{r\} \times M \times V}$.

Standard smoothing techniques [Mun66, §4] (see also [Kup19, Prop. 6.4.2] or [Lur09, Prop. 1]) can be used to prove the following: given nested compact subsets $L \subset K \subset \Delta^k \times M \times V$ with $L \subset \mathrm{int}\, K$ and any arbitrarily small $\epsilon > 0$, there exists a homotopy $H : I \times \Delta^k \times M \times V \to M \times V$ from g to some map $g' : \Delta^k \times M \times V \to M \times V$ satisfying that:

(*i*) H remains fixed on $\Delta^k \times M \times V - \mathrm{int}\, K$. In particular g and g' agree there.

(*ii*) g' is smooth on L. Moreover if g was already smooth on some (open neighbourhood of a) closed subset $\partial\Delta^k \times M \times V \subset F \subset \Delta^k \times M \times V$, the homotopy H remains fixed on F.

(*iii*) For each $t \in I$ and $r \in \Delta^k$, the map $H_r^t = H\mid_{\{t\}\times\{r\}\times M\times V}\colon M \times V \to M \times V$ is smooth.

(*iv*) H^t remains arbitrarily close to g for all $t \in I$. Consequently, if g is bounded by some $C \geq 0$, then for every $(r,t) \in I \times \Delta^k$ the map $H_r^t : M \times V \to M \times V$ is bounded by $C + \epsilon$, and is a diffeomorphism (as diffeomorphisms of compact manifolds are open in the space of smooth self-maps).

With this in mind, we construct a homotopy in $\mathrm{Sing}_\bullet^{\mathsf{col},b}(\mathrm{Diff}_\partial(M \times V))$ from the k-simplex $\overline{g} = (\mathrm{proj}_{\Delta^k}, g)$ to some $\overline{h} \in \mathrm{Diff}_\partial^b(M\times V)_k$ (relative to $\partial\Delta^k \times M \times V$), as follows: without loss of generality assume $V = \mathbb{R}^n$. Also for $v \in \mathbb{R}^n$ and $\delta > 0$, let $C_\delta(v) \subset \mathbb{R}^n$ denote the cube of side length 2δ and centered at v (i.e. $C_\delta(v) := v + [-\delta,\delta]^n$). Fix an $\epsilon > 0$ (e.g. $\epsilon = 1$). Then for each $v \in 3\mathbb{Z}^n \subset \mathbb{R}^n$, choose a homotopy as above starting from g with $(K, L) = (C_1(v), C_{2/3}(v))$, and perform all of these at the same time[5] to obtain some $g' : \Delta^k \times M \times V \to M \times V$. Now apply the same process to g' on $(K, L) = (C_1(v), C_{2/3}(v))$ for each $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$ with $v_1 \equiv 1 \mod 3$ and $v_i \equiv 0 \mod 3$ for $2 \leq i \leq n$, keeping in mind that, by condition (*ii*) above, the homotopies keep fixed the parts that have been smoothed in the previous step. Continue this process in a similar fashion. After 3^n steps, we will obtain a smooth $(C + 3^n \cdot \epsilon)$-bounded map $h : \Delta^k \times M \times V \to M \times V$ such that $\overline{h} := (\mathrm{proj}_{\Delta^k}, h)$ represents the required k-simplex of $\mathrm{Diff}_\partial^b(M \times V)_\bullet$. This means that the relative homotopy class $[\overline{g}] \in \pi_k(①)$ is trivial, as was to be shown. ☐

D.2 A moduli space model for classifying spaces of bounded diffeomorphism groups

Fix an embedding $\iota : M \hookrightarrow \mathbb{R}^m \subset \mathbb{R}^\infty$. Recall that the classifying space $B\mathrm{Diff}_\partial(M)$ of the diffeomorphism group of M admits a model as the moduli space of all d-manifolds $N^d \subset \mathbb{R}^\infty$ with $\partial N = \partial M$ which are diffeomorphic to M relative to the boundary. In this section we give an analogous description of the classifying space $B\mathrm{Diff}_\partial^b(M \times V)$, for any real finite-dimensional inner product vector space $V \in \mathcal{J}_0$.

Proposition D.6. *Set* $\mathrm{Emb}_\partial^b(M \times V, \mathbb{R}^\infty \times V) := \mathrm{colim}_n\ \mathrm{Emb}_\partial^b(M \times V, \mathbb{R}^n \times V)$*, and let* $\mathrm{Diff}_\partial^b(M \times V)$ *act on it by precomposition. Then there is an equivalence*

$$B\mathrm{Diff}_\partial^b(M \times V) \simeq \mathrm{Emb}_\partial^b(M \times V, \mathbb{R}^\infty \times V)/\mathrm{Diff}_\partial^b(M \times V).$$

---

[5] This can be done as, by condition (*i*), the supports of such homotopies are disjoint by construction.

In other words, $B\mathrm{Diff}^b_\partial(M \times V)$ is (equivalent to) the moduli space of all submanifolds in $\mathbb{R}^\infty \times V$ with boundary $\partial M \times V$ which are diffeomorphic to $M \times V$ boundedly in V and relative to $\partial M \times V$.

Proof. By Proposition D.1, we have that $B\mathrm{Diff}^b_\partial(M \times V) \simeq B|\mathrm{Diff}^b_\partial(M \times V)_\bullet| \simeq |B\mathrm{Diff}^b_\partial(M \times V)_\bullet|$ and

$$\frac{\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)}{\mathrm{Diff}^b_\partial(M \times V)} \simeq \frac{|\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet|}{|\mathrm{Diff}^b_\partial(M \times V)_\bullet|} \simeq \left|\frac{\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet}{\mathrm{Diff}^b_\partial(M \times V)_\bullet}\right|.$$

As the simplicial action of $\mathrm{Diff}^b_\partial(M \times V)_\bullet$ on $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ is visibly free, we only need to show that $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ is weakly contractible by [GJ99, Cor. 2.6]. To that end, let

$$\varphi = (\mathrm{proj}_{\Delta^k}, \varphi_n, \varphi_V) : \Delta^k \times M \times V \hookrightarrow \Delta^k \times \mathbb{R}^n \times V, \quad n \geq m,$$

represent some homotopy class in $\pi_k(\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet)$ for some $k \geq 0$. We will show that $[\varphi] = [\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V]$ by constructing a simplicial map $H : \Delta^1_\bullet \to \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ such that, under the Yoneda isomorphism, $\partial_0 H = \varphi$ and $\partial_1 H = \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$. The map H will be given by (a modification of) the usual straight-line homotopy between φ and $\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$.

Let us fix some notation. Pick some open collar $c : [0,1) \times \partial M \hookrightarrow M$ of the boundary of M. We can arrange the embedding $\iota : M \hookrightarrow \mathbb{R}^m$ to be such that

(*i*) $\iota \equiv (\mathrm{Id}_{[0,1)]} \times i) \circ c^{-1} |_{c([0,1) \times \partial M)}$ for some embedding $i : \partial M \hookrightarrow \mathbb{R}^{m-1}$, and

(*ii*) $\iota(M \setminus c((0,1] \times \partial M)) \subset [1, +\infty) \times \mathbb{R}^{m-1}$.

From now on we will supress ι and c from the notation, i.e., we canonically identify M (resp. $[0,1) \times \partial M$) with its image under ι (resp. c). Choose some increasing smooth function $\alpha : [0,1] \to [0,1]$ for which there exists some $0 < \delta$ with $\alpha |_{[0,\delta]} \equiv 0$, $\alpha |_{[1-\delta,1]} \equiv 1$ and $0 < \alpha(t) < 1$ for $\delta < t < 1 - \delta$ (this δ is required for the collaring condition right before Definition 2.2.7). Now by the collaring condition,

$$\text{there exists some } 0 < \epsilon < 1 \text{ such that } \varphi \equiv \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V \text{ on } \Delta^k \times [0,\epsilon) \times \partial M \times V. \tag{D.2}$$

Finally, fix some smooth function $\rho : M \to [0,1]$ such that

$$\rho |_{[0,\epsilon/2] \times \partial M} \equiv 0 \quad \text{and} \quad \rho |_{M \setminus [0,\epsilon) \times \partial M} \equiv 1.$$

Then for $t \in [0,1]$, consider the map

$$H_t : \Delta^k \times M \times V \longrightarrow \Delta^k \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{|V|} \times V \subset \Delta^k \times \mathbb{R}^\infty \times V,$$

$$(r,x,v) \longmapsto \begin{pmatrix} r \\ \alpha(t)\cdot x + (1-\alpha(t))\cdot \varphi_n(r,x,v) \\ \rho(x)\alpha(t)(1-\alpha(t))\cdot x \\ \rho(x)\alpha(t)(1-\alpha(t))\cdot v \\ \alpha(t)\cdot v + (1-\alpha(t))\cdot \varphi_V(t,x,v) \end{pmatrix} \tag{D.3}$$

Here $x \in M \subset \mathbb{R}^m \subset \mathbb{R}^n$ and $\mathbb{R}^{|V|}$ is a Euclidean space of the same dimension as V, treated as a copy of V.

Claim. *Let $C \geq 0$ be the bound of φ on the V-coordinate. Then the map H_t is a C-bounded embedding for $t \in [0,1]$. Moreover, H_t agrees with $\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$ on $\Delta^k \times [0,\epsilon/2] \times \partial M \times V$.*

Proof of Claim. Indeed H_t is bounded by $C \geq 0$ (in the V-coordinate) as

$$\begin{aligned} \|(\alpha(t)\cdot v + (1-\alpha(t))\cdot\varphi_V(t,x,v)) - v\|_V &= (1-\alpha(t))\cdot\|\varphi_V(t,x,v) - v\|_V \\ &\leq (1-\alpha(t))\cdot C \leq C. \end{aligned}$$

To see that H_t is an embedding, suppose that $H_t(r,x,v) = H_t(r',x',v')$. Clearly then $r = r'$ by the first coordinate in (D.3). Note that $H_t = \varphi$ if $t \leq \delta$ and $H_t = \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$ if $t \geq 1-\delta$. As both are embeddings, we may assume that $\delta < t < 1-\delta$ so that $\alpha(t)(1-\alpha(t)) \neq 0$. To show that $x = x'$ we consider three cases:

- If $x, x' \in [0,\epsilon] \times \partial M$, then by (D.2), the equation on the second coordinate of (D.3) yields $x = x'$.
- If $x, x' \notin [0,\epsilon] \times \partial M$, then $\rho(x) = \rho(x') = 1$ and thus the third coordinate equation yields $x = x'$.
- If $x \in [0,\epsilon] \times \partial M$ but $x' \notin [0,\epsilon] \times \partial M$, then the third coordinate equation becomes $\rho(x)\cdot x = x' \in \mathbb{R}^n$. On the first coordinate of $[0,+\infty) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$, this implies, by items (*i*) and (*ii*) above, that $\rho(x) > 1$ which is a contradiction.

In all cases $x = x'$. Then the equation on the fourth coordinate of (D.3) implies that $v = v'$, as required.

Finally, the last part of the claim again follows from (D.2) and the nature of ρ. $\square$

The family of C-bounded embeddings $\{H_t\}_{t\in[0,1]}$ gives rise to the required simplicial map H. This finishes the proof of the proposition. $\square$

Chapter 3

On the homotopy type of spaces of long knots

This chapter is devoted to Theorem D. Its proof requires a detailed analysis of the involution featuring in Theorem C, which occupies a substantial part of the chapter. The main results in this direction are Theorem 3.1.13 and Corollary 3.1.17. We then establish Theorem D and deduce some consequences for the homotopy groups of spaces of long knots (cf. Propositions 3.2.5 and 3.2.7).

3.1 Involutions in algebraic K-theory

The aim of this section is to explore the involutions of the C_2-spectra involved in the statements of Theorems 2.1.1 and C, and to express them in terms of simpler and more computable involutions coming from algebraic K-theory—the main result in this direction is Theorem 3.1.13, which is further simplified by Proposition 3.1.22 in the case of a suspension. This will then be used in Section 3.2 to study the case $(M, P) = (D^d, D^p)$. As we will shortly see in Section 3.1.1, it will be significantly helpful to invert the prime 2 in the analysis of these involutions. Let us now introduce the notation that will be relevant in this section.

Notation 3.1.1. *(i) For M a compact (smooth) manifold and $\iota : P \subset M$ a compact submanifold, recall from Notation 2.2.5 the definitions of the C_2-spectra $\mathbf{H}(M)$, the **h-cobordism spectrum** of M, and $\mathbf{CE}(P, M)$, the **concordance embedding spectrum** of $\iota : P \hookrightarrow M$. We refer to their involutions by τ_{WW}, for Weiss–Williams.*

(ii) Given a space X and a spherical fibration ξ over X equipped with a section, Vogell defined in [Vog85, p. 300] an involution τ_ξ on[1] *$\mathbf{A}(X)$ by means of Spanier–Whitehead duality with respect to the Thom spectrum of ξ; we will write $\mathbf{A}(X;\xi)$ for the corresponding C_2-spectrum. When $\xi = \epsilon := X \times S^0$ is the trivial 0-dimensional sphere bundle, τ_ϵ is the identity on the first summand of the splitting*

$$\mathbf{A}(X) \simeq \Sigma^\infty_+ X \vee \mathbf{Wh}^{\mathrm{Diff}}(X), \tag{3.1.1}$$

and thus τ_ϵ descends to an involution on $\mathbf{Wh}^{\mathrm{Diff}}(X)$; call it τ_ϵ too. We will refer to τ_ϵ as the ***canonical involution*** *of K-theory, and sometimes write $\mathbf{A}(X)$ and $\mathbf{Wh}^{\mathrm{Diff}}(X)$ for $\mathbf{A}(X;\epsilon)$ and $\mathbf{Wh}^{\mathrm{Diff}}(X;\epsilon)$. We will recall a construction of τ_ϵ in terms of Spanier–Whitehead duality in Section 3.1.4.*

3.1.1 Homotopy involutions

A *homotopy involution* τ on a space or infinite loop space or spectrum X is a self-map $\tau : X \to X$ whose square τ^2 is homotopic to the identity Id_X. In this section we explain why, in the stable setting and once the prime 2 is inverted, an involution carries the same amount of information as its underlying homotopy involution. This will be very useful when comparing the C_2-spectra $\mathbf{H}(M)$ and $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon)$ (see Proposition 3.1.17). Let us fix some notation first.

Notation 3.1.2. *(i) Let C denote any of Top_*, Ω^∞-Top or Sp, and let $X, X' \in \mathsf{C}$ be equipped with homotopy involutions τ and τ', respectively. A map $f : X \to X'$ will be said to be* ***homotopy C_2-equivariant****, or* ***C_2-equivariant up to homotopy****, if $f\tau \simeq \tau' f$. If X and X' can be connected by a zig-zag of homotopy C_2-equivariant weak equivalences, we will say that X and X' are* ***homotopy C_2-equivariantly equivalent*** *and write*

$$X \approx X'.$$

A ***C_2-equivariant equivalence*** *will always mean a zig-zag of weak equivalences which are C_2-equivariant.*

(ii) An ***H-group*** *(X,μ) is a group-like $\mathbb{A}_3$-space (i.e. a homotopy associative H-space such that $\pi_0(X)$ is a group with respect to μ). Given H-spaces (X,μ) and (X',μ'), a based map $f : X \to X'$ will be said to be* ***monoidal up to homotopy****, or simply an*

---

[1] Vogell defined τ_ξ on the A-theory space $A(X)$, but this involution can be upgraded to $\mathbf{A}(X)$ by specifying it on the Waldhausen category of retractive spaces over X "with ξ-duality" and appealing to the definition of algebraic K-theory via the $S_\bullet$-construction.

*H-**map**, if the following diagram is homotopy commutative:*

$$\begin{tikzcd}
X \times X \arrow[r, "f\times f"] \arrow[d, "\mu"] & X' \times X' \arrow[d, "\mu'"] \\
X \arrow[r, "f"] & X'.
\end{tikzcd}$$

*An **equivalence of H-groups** will mean a zig-zag of H-maps that are additionally weak equivalences. In practice, all H-groups we will consider are actually $\mathbb{E}_1$-groups (i.e. group-like $\mathbb{E}_1$-spaces), and all H-maps can be upgraded to $\mathbb{E}_1$-maps even though we will not need this.*
(iii) Given a C_2-object X in an appropriate category, X_{hC_2} stands for $\operatorname{hocolim}_{BC_2} X$.

In the cases of interest to us and once the prime 2 is inverted, taking homotopy C_2-orbits with respect to a homotopy involution turns out to make sense.

Proposition 3.1.3. *Let X denote a spectrum or infinite loop space (a.k.a. connective spectrum), and let τ be a homotopy involution on X. Suppose that multiplication by two is invertible on X, i.e. $2 : X \xrightarrow{\sim} X$ is an equivalence, and define*

$$E(X,\tau) := \operatorname{hocolim}\Big(X \xrightarrow{\frac{1+\tau}{2}} X \xrightarrow{\frac{1+\tau}{2}} \dots \Big),$$

where $\frac{1+\tau}{2}$ really stands for the zig-zag $X \xrightarrow{1+\tau} X \xleftarrow[\sim]{2} X$. Then if τ is an actual involution on X, there is a natural equivalence away from two

$$X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau).$$

Proof. Let us assume that X is a C_2-spectrum (the other case is completely analogous). We also assume that 2 is inverted. Observe now that as $t \cdot \frac{1+t}{2} = \frac{1+t}{2}$ in $\mathbb{Z}[C_2]$, then the following commutes up to homotopy

$$\begin{tikzcd}
X \arrow[r, "\frac{1+\tau}{2}"] \arrow[dr, "q"'] & X \arrow[r, "\frac{1+\tau}{2}"] \arrow[d, "q"] & \dots \arrow[dl, "q"] \\
 & X_{hC_2}, &
\end{tikzcd}$$

where $q : X \to X_{hC_2} = \operatorname{hocolim}_{BC_2} X$ is the map on colimits induced by the inclusion of categories $\{*\} \hookrightarrow BC_2$. We thus obtain a map $\eta_{(X,\tau)} : E(X,\tau) \to X_{hC_2}$. The homotopy orbits spectral sequence for X, together with the assumption that 2 is inverted, gives a natural isomorphism $\pi_*(X_{hC_2}) \cong H_0(C_2;\pi_*(X)) \cong \pi_*(X)_{C_2}$. Also by definition, we have that $\pi_*(E(X,\tau)) \cong \mathrm{Im}(\frac{1+\tau}{2} : \pi_*(X) \to \pi_*(X))$. Under these identifications, the map $\pi_*(\eta_{(X,\tau)})$ is the natural isomorphism (away from 2) sending an element $\beta = \frac{1+\tau}{2}\alpha \in \pi_*(E(X,\tau))$ to $[\beta] = [\alpha] \in \pi_*(X)_{C_2}$. So $\eta_{(X,\tau)}$ is the desired equivalence $X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau)$. □

Corollary 3.1.4. *Let X and X' be C_2-spectra and let the prime 2 be inverted.*

(i) If there is a homotopy C_2-equivariant equivalence $X \approx X'$, then there is an equivalence of spectra

$$X_{hC_2} \simeq_{[\frac{1}{2}]} X'_{hC_2}.$$

*(ii) If there is only a homotopy C_2-equivariant equivalence $\Omega^\infty X \approx \Omega^\infty X'$ of **H-spaces**, then we still have an equivalence of spaces*

$$\Omega^\infty(X_{hC_2}) \simeq_{[\frac{1}{2}]} \Omega^\infty(X'_{hC_2}).$$

Proof. Let us only deal with (ii) (as (i) is analogous and easier). Assume without loss of generality that the equivalence $\Omega^\infty X \approx \Omega^\infty X'$ of H-spaces is induced by a single homotopy C_2-equivariant H-map $g : \Omega^\infty X \xrightarrow{\sim} \Omega^\infty X'$. Then, the diagram *of spaces*

$$\begin{tikzcd}
\Omega^\infty X \arrow[r, "\frac{1+\tau}{2}"] \arrow[d, "g", "\wr"'] & \Omega^\infty X \arrow[r, "\frac{1+\tau}{2}"] \arrow[d, "g", "\wr"'] & \dots \\
\Omega^\infty X' \arrow[r, "\frac{1+\tau'}{2}"] & \Omega^\infty X' \arrow[r, "\frac{1+\tau'}{2}"] & \dots
\end{tikzcd}$$

commutes up to homotopy, where τ and τ' are the involutions of X and X', respectively. Since the fogetful map from infinite loop spaces to spaces preserves directed colimits, the diagram above (upon taking horizontal colimits) induces an equivalence of spaces

$$E(\Omega^\infty X, \tau) \simeq E(\Omega^\infty X', \tau').$$

The claim now follows from Proposition 3.1.3 and because the natural map $(\Omega^\infty X)_{hC_2} \to \Omega^\infty(X_{hC_2})$ is an equivalence away from 2 (this is a consequence of the homotopy orbits spectral sequence). □

Remark 3.1.5. The upshot of part (ii) of the previous corollary is that, given a C_2-spectrum X that is local away from 2, the homotopy type of $\Omega^\infty(X_{hC_2})$ *as a space* is completely determined by the homotopy type of the space $\Omega^\infty X$ and the homotopy classes of the maps $\tau : \Omega^\infty X \to \Omega^\infty X$ and $+ : \Omega^\infty X \times \Omega^\infty X \to \Omega^\infty X$.

The following result, though unrelated to what has been discussed so far in this section, will be useful later on. Given an $\mathbb{E}_1$-space X, we will write X^{op} for X equipped with the opposite $\mathbb{E}_1$-structure. An *anti-involution* τ on an $\mathbb{E}_1$-space X is an $\mathbb{E}_1$-map $\tau : X \to X^{\mathrm{op}}$ whose square equals the identity of X (noting that $(X^{\mathrm{op}})^{\mathrm{op}} \cong X$). Up to equivalence, there is a standard way of delooping such an anti-involution.

Lemma 3.1.6. *Let X be an $\mathbb{E}_1$-space. There is a natural equivalence*

$$\iota : B(X^{\mathrm{op}}) \simeq BX.$$

such that, for any anti-involution τ on X, the composition

$$\overline{B}\tau : BX \xrightarrow{B\tau} B(X^{\mathrm{op}}) \overset{\iota}{\simeq} BX$$

is an involution on BX.

Proof. For each $k \geq 0$, the map $\mathbb{E}_1(k) \to \pi_0(\mathbb{E}_1(k))$ is an equivalence, and hence there is a natural zig-zag of equivalences of $\mathbb{E}_1$-algebras

$$B(\pi_0(\mathbb{E}_1), \mathbb{E}_1, X) \xleftarrow{\sim} B(\mathbb{E}_1, \mathbb{E}_1, X) \xrightarrow{\sim} X.$$

But the $\mathbb{E}_1$-structure on the left hand side factors through the associative operad $\mathcal{A}ss := \pi_0(\mathbb{E}_1)$, so for simplicity, we may assume that X is strictly associative. The equivalence ι is then induced on the realisation of the nerve $N_\bullet X$ by the maps

$$X^q \times \Delta^q \longrightarrow X^q \times \Delta^q, \quad (x_1, \dots, x_q, r) \longmapsto (x_q, \dots, x_1, \Phi_q(r)),$$

where $\Phi_q : \Delta^q \cong \Delta^q$ is the linear homeomorphism induced by reversing the order of the vertices. It is easy to check that the map $\overline{B}\tau$ indeed defines an involution on BX. ☐

3.1.2 From the h-cobordism spectrum to spaces of h-cobordisms

All throughout this section, assume that $d = \dim M \geq 5$; this condition will not be a problem later, as all of the results in this section will be used only once our original manifold M has been stabilised sufficiently many times.

We now recall Vogell's model for spaces of h-cobordisms (cf. [Vog85, p. 296]). A *partition* of a manifold M^d is a triple (W, F, V), where W is a codimension zero submanifold of $M \times [-1, 1]$, V is the closure of the complement of W and $F^d := W \cap V$. For technical reasons, we require F to be standard near $\partial M \times [-1, 1]$, and that it intersects it in $\partial M \times \{0\}$. Let $H(M)_\bullet$ denote the simplicial set a p-simplex of which is a (locally trivial smooth) family of partitions of M parametrised by Δ^p such that W is an h-cobordism from $M \times \{-1\} \times \Delta^p$ to F. Set $H(M) := |H(M)_\bullet|$ and write $H^s(M) \subset H(M)$ for the connected component containing the trivial partition $* = (M \times [-1,0], M \times \{0\}, M \times [0,1])$. There is a canonical involution ι_H given by turning upside down partitions. Namely

$$\iota_H : H(M) \longrightarrow H(M), \quad \rho = (W, F, V) \longrightarrow \rho^* := (V^*, F^*, W^*),$$

where W^*, F^* and V^* are respectively the images of W, F and V under the reflection $r = \mathrm{Id}_M \times -1$. For the smooth case, we will also need a small variant of this h-cobordism space, denoted $H_{\mathsf{col}}(M)$, a point of which consists of a partition $\rho = (W, F, V) \in H(M)$ together with a bicollar of F for W and V which is standard near $\partial M \times [-1,1]$. The forgetful map $H_{\mathsf{col}}(M) \to H(M)$ is a weak equivalence by the contractibility of the space of collars. In this section we construct a homotopy C_2-equivariant $(\phi(d)+1)$-connected map

$$\mathrm{alex} : H(M) \longrightarrow \Omega^\infty \mathbf{H}(M) \tag{3.1.2}$$

which generalises the map $B(\mathrm{alex})$ of Proposition 2.3.8. We first recall an important construction.

3.1.2.1 The geometric Eilenberg swindle

An h-cobordism $W : M \overset{h}{\rightsquigarrow} M'$ induces a unique (up to contractible choice) bounded diffeomorphism

$$ES_W : M \times \mathbb{R} \cong M' \times \mathbb{R} \tag{3.1.3}$$

as follows: choose embeddings $i_r : W \hookrightarrow M \times I$ rel $M \times \{0\}$ and $i_\ell : W \hookrightarrow M' \times I$ rel $M' \times \{1\}$ (these are essentially partitions for M and M', which amount to no homotopical information by the contractibility of the space of collars). Write $V_r := \overline{M \times I - i_r(W)}$ and $V_\ell := \overline{M' \times I - i_\ell(W)}$; both of these manifolds are h-cobordisms $M' \overset{h}{\rightsquigarrow} M$, and in fact they are diffeomorphic relative to *both* ends since

$$V_\ell \cong V_\ell \cup_M M \times I \cong V_\ell \cup_M W \cup_{M'} V_r \cong M' \times I \cup_{M'} V_r \cong V_r \quad \text{rel } M' \sqcup M.$$

Then the *Eilenberg swindle* diffeomorphism ES_W is given by the composition

$$ES_W : M \times \mathbb{R} = \cdots \cup_{M'} V_r \cup_M W \cup_{M'} V_r \cup_M \ldots \cong \cdots \cup_{M'} V_\ell \cup_M W \cup_{M'} V_\ell \cup_M \cdots = M' \times \mathbb{R}.$$

Clearly, by construction, ES_W is bounded by 1 and unique up to contractible choice.

3.1.2.2 The map (3.1.2)

Fix an embedding $M^d \subset \mathbb{R}^N \subset \mathbb{R}^\infty$ and recall that $B\mathrm{Diff}_\partial(M)$ admits a model as the moduli space of manifolds embedded in $\mathbb{R}^\infty$ which are abstractly diffeomorphic to M relative to the boundary ∂M. Similarly $B\mathrm{Diff}^b_\partial(M \times \mathbb{R})$ is the moduli space of manifolds embedded in $\mathbb{R}^\infty \times \mathbb{R}$ which are abstractly diffeomorphic to $M \times \mathbb{R} \subset \mathbb{R}^\infty \times \mathbb{R}$ *boundedly* with respect to the $\mathbb{R}$-direction and relative to the boundary $\partial M \times \mathbb{R}$ (this is proved in Appendix D.2). For the remaining of this section, we will denote by $\underline{\mathbb{R}}$ the bounded direction, i.e., the last coordinate in $\mathbb{R}^\infty \times \mathbb{R} =: \mathbb{R}^\infty \times \underline{\mathbb{R}}$. There is a natural map $- \times \underline{\mathbb{R}} : B\mathrm{Diff}_\partial(M) \hookrightarrow B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$ given by sending a manifold $N \subset \mathbb{R}^\infty$ to $N \times \underline{\mathbb{R}} \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$. In fact, in light of (3.1.3), this map extends to

$$- \times \underline{\mathbb{R}} : \coprod_{[M']} B\mathrm{Diff}_\partial(M') \longrightarrow B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}}), \quad N \longmapsto N \times \underline{\mathbb{R}},$$

where the coproduct in the domain runs over all diffeomorphism classes of manifolds M' with boundary ∂M that are h-cobordant to M rel ∂M. The map $\times\underline{\mathbb{R}}$ is the value of the morphism $0 \to \underline{\mathbb{R}}$ in $\mathcal{J}_0$ under

$$\widehat{B} : \mathcal{J}_0 \longrightarrow \mathsf{Top}_*, \quad \widehat{B}(V) := \begin{cases} \coprod_{[M']} B\mathrm{Diff}_\partial(M'), & V = 0, \\ B(V) = B\mathrm{Diff}^b_\partial(M \times V), & \text{otherwise.} \end{cases}$$

Clearly $\widehat{B}$ is an orthogonal functor and we will write $\widehat{\mathbf{H}}(M)$ for its first derivative, which is canonically equivalent to $\mathbf{H}(M)$. The Alexander trick-like map (3.1.2) will factor through $\widehat{\mathbf{H}}(M)_0 \hookrightarrow \Omega^\infty \widehat{\mathbf{H}}(M) \simeq \Omega^\infty \mathbf{H}(M)$.

Suppose we are given some partition $\rho = (W, F, V) \in H(M)$ of $M \times \underline{[-1,1]} \subset \mathbb{R}^N \times \underline{\mathbb{R}}$. Then W is an h-cobordism from M to F rel boundary, and so the manifold $F \subset \mathbb{R}^N \times \mathbb{R} \subset \mathbb{R}^\infty$ gives rise to a point in $B\mathrm{Diff}_\partial(F) \subset \widehat{B}(0)$; more precisely, the image of the embedding

$$i_\rho : F \subset M \times I \subset \mathbb{R}^N \times \underline{\mathbb{R}} \cong \mathbb{R}^{N+1} \subset \mathbb{R}^\infty,$$

is a point in $B\mathrm{Diff}_\partial(F)$, where the isomorphism $\mathbb{R}^N \times \underline{\mathbb{R}} \cong \mathbb{R}^{N+1}$ identifies $\underline{\mathbb{R}}$ with the last coordinate in $\mathbb{R}^{N+1}$. We now construct a point in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$ by *extending* W

towards infinity. Consider the embedding of $F \times [0,1]$ into $\mathbb{R}^N \times \mathbb{R} \times \underline{\mathbb{R}}$ given by

$$R : F \times [0,1] \hookrightarrow \mathbb{R}^N \times \mathbb{R} \times \underline{\mathbb{R}}, \quad (x,t) \longmapsto \underline{e} + (\mathrm{Id}_{\mathbb{R}^N} \times Q_{-\pi t/2})(x - \underline{e}),$$

where $Q_\theta : \mathbb{R} \times \underline{\mathbb{R}} \cong \mathbb{R} \times \underline{\mathbb{R}}$ is the rotation matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$, and $\underline{e}$ denotes the unit length vector in $\underline{\mathbb{R}}$. Write $r := R\,|_{F\times\{1\}} \colon F \hookrightarrow \mathbb{R}^{N+1} \times \underline{\{1\}}$, and consider

$$S : F \times [1,+\infty) \hookrightarrow \mathbb{R}^{N+1} \times \underline{\mathbb{R}}, \quad (x,t) \longmapsto \begin{cases} r(x) + (1-t)\cdot e_{N+1} + (t-1)\cdot \underline{e}, & t \in [1,2], \\ r(x) - e_{N+1} + (t-1)\cdot \underline{e}, & t \geq 2. \end{cases}$$

Finally consider the region $D \subset \mathbb{R} \times \underline{\mathbb{R}}$ given by tuples (u,v) with

$$u \geq 0, \quad u \leq 2 - v, \quad \text{and if } 0 \leq u \leq 1, \text{ then } u \leq (1-(v-1)^2))^{1/2}.$$

Then we define a topological manifold $\widehat{a}(\rho) \subset \mathbb{R}^{N+1} \times \underline{\mathbb{R}}$, depicted in Figure 3.1, by

$$\widehat{a}(\rho) := M \times \{0\} \times (-\infty \times -1] \cup W \cup R(F \times [0,1]) \cup S(F \times [1,+\infty)) \cup \partial M \times D.$$

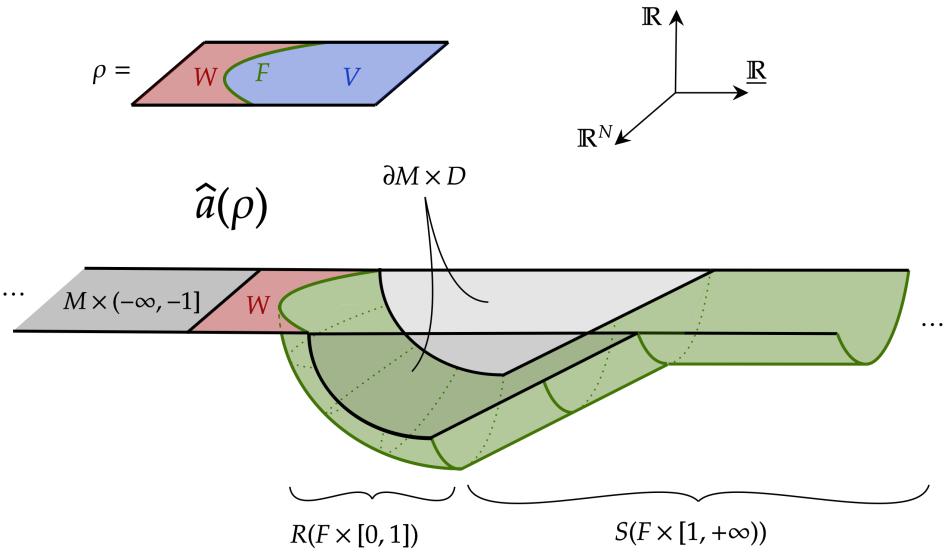

Figure 3.1 Depiction of the topological manifold $\widehat{a}(\rho)$ for $\rho \in H^s(M)$.

Now if ρ is a collared partition, i.e. a point in $H^s_{\mathsf{col}}(M)$, one can use the collar of F to smooth out the corners of the topological manifold $\widehat{a}(\rho)$, and thus obtain a smooth manifold $\widetilde{a}(\rho) \subset \mathbb{R}^{N+1} \times \underline{\mathbb{R}}$ with the same boundary as $M \times \underline{\mathbb{R}}$, and which is boundedly diffeomorphic to $M \times \mathbb{R}$ relative to the boundary by a one-sided Eilenberg swindle argument. This construction can be done simplex-wise in $H_{\mathsf{col}}(M)_\bullet \simeq H(M)_\bullet$, and so up to weak equivalence gives rise to an Alexander trick-like map

$$\mathrm{alex} : H(M) \xrightarrow{\sim} \widehat{\mathbf{H}}(M)_0 := \mathrm{hofib}\left(\coprod_{[M']} B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})\right), \tag{3.1.4}$$

$$\rho = (W, F, V) \longmapsto \left(i_\rho(F),\ \gamma_W : [-\infty, \infty] \ni t \mapsto \begin{cases} M \times \underline{\mathbb{R}}, & t = -\infty, \\ \widetilde{a}(\rho) - t \cdot \underline{e}, & -\infty < t < +\infty, \\ i_\rho(F) \times \underline{\mathbb{R}}, & t = +\infty. \end{cases} \right),$$

where we can regard the path γ_W as a 1-simplex in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})_\bullet$ from $M \times \underline{\mathbb{R}}$ to $F \times \underline{\mathbb{R}}$. Then (3.1.2) is

$$\mathrm{alex} : H(M) \xrightarrow[\sim]{(3.1.4)} \widehat{\mathbf{H}}(M)_0 \longhookrightarrow \Omega^\infty \widehat{\mathbf{H}}(M) \simeq \Omega^\infty \mathbf{H}(M)$$

Proposition 3.1.7. *The map (3.1.4) is indeed an equivalence. Therefore, (3.1.2) is* $(\phi(d) + 1)$*-connected.*

Proof. Noting the equivalence $H^s(M) \simeq BC(M)$ (cf. [Vog85, Prop. 2.1]), the map (3.1.4) is, up to homotopy, a (non-connected) delooping of the Alexander trick-like equivalence $C(M) \simeq \Omega(\mathrm{Diff}^b_\partial(M \times \mathbb{R})/\mathrm{Diff}(M))$ of [WW88, Prop. 1.10], and therefore it is an equivalence on basepoint components.

Given a diffeomorphism class $[M']$ of manifolds h-cobordant to M (rel boundary), denote by $H(M, M')$ the collection of path components in $H(M)$ consisting of (collared) partitions $\rho = (W, F, V)$ with $F \in [M']$. A choice of basepoint $\rho_0 = (W_0, F_0, V_0) \in H(M, M')$, a bicollar $c_0 : F_0 \times [-\epsilon, \epsilon] \hookrightarrow M \times [-1, 1]$ and a diffeomorphism $\phi_0 : M' \cong F_0$ gives rise to an equivalence $H(M', M') \xrightarrow{\sim} H(M, M')$ which sends a partition $\rho' = (W', F', V')$ of $M' \times [-1, 1]$ to the partition of $M \times [-1, 1]$ whose F-part is the image of F' under

$$M' \times [-1, 1] \xrightarrow{\phi_0 \times \cdot \epsilon} F_0 \times [-\epsilon, \epsilon] \overset{c_0}{\hookrightarrow} M \times [-1, 1].$$

By the s-cobordism theorem, a homotopy inverse $H(M, M') \xrightarrow{\sim} H(M', M')$ is given by the same kind of map for a choice of basepoint $\rho_0' = (W_0', F_0', V_0') \in H(M', M')$ such that $\phi_0(W_0')$ is an h-cobordism starting at F_0 with the same torsion as V_0 (the inverse of W_0).

Observe also that the choices (ρ_0, c_0) and ϕ_0 above give a preferred path in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}}) = B\mathrm{Diff}^b_\partial(M' \times \underline{\mathbb{R}})$ from $M \times \underline{\mathbb{R}}$ to $M' \times \underline{\mathbb{R}}$; namely, it is the composition of γ_{W_0}, as defined in (3.1.4), with the mapping cylinder of $\phi_0 \times \mathrm{Id}_{\underline{\mathbb{R}}}$. These preferred paths give rise to the "change of basepoint" equivalences in the right column of the homotopy commutative diagram

$$\begin{tikzcd}
H(M) \arrow[rr, "(3.1.4)"] \arrow[d, equal] & & \mathrm{hofib}_{M\times\underline{\mathbb{R}}}\left(\coprod_{[M']} B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M\times\underline{\mathbb{R}})\right) \arrow[d, phantom, "\wr" rotate=90] \\
\coprod_{[M']} H(M,M') \arrow[d, phantom, "\wr" rotate=90] & & \coprod_{[M']} \mathrm{hofib}_{M'\times\underline{\mathbb{R}}}\left(B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M'\times\underline{\mathbb{R}})\right) \arrow[d, equal] \\
\coprod_{[M']} H(M',M') \arrow[rr, "\coprod_{[M']}\mathrm{alex}"] & & \coprod_{[M']} \mathbf{H}(M')_0,
\end{tikzcd}$$

where alex : $H(M',M') \to \mathbf{H}(M')_0 \subset \widehat{\mathbf{H}}(M')_0$ is the restriction to $H(M',M') \subset H(M')$ of the map (3.1.4) for $M = M'$. Moreover if $d \geq 5$, this map is an isomorphism in π_0 (in fact it is an equivalence by the argument above); indeed, the inverse

$$\pi_0\left(\mathrm{Diff}^b_\partial(M'\times\underline{\mathbb{R}})/\mathrm{Diff}_\partial(M')\right) \longrightarrow \pi_0(H(M',M')) \subset \mathrm{Wh}(M)$$

sends the coset $[\phi]$ of some bounded diffeomorphism $\phi \in \mathrm{Diff}^b_\partial(M'\times\underline{\mathbb{R}})$ (say bounded by 1/2 for simplicity) to a partition of $M' \times [-1,1]$ whose W-part is the h-cobordism obtained as the region in $M'\times\underline{\mathbb{R}}$ between $M'\times\{0\}$ and $\phi(M'\times\{1\})$ (cf. [WW88, Cor. 5.4]). It follows that the lower horizontal map, and hence (3.1.4), is an isomorphism in π_0. This proves the first claim.

The second claim is a consequence of the fact that $\widehat{\mathbf{H}}(M)_0 \hookrightarrow \Omega^\infty\widehat{\mathbf{H}}(M)$ is $(\phi(d)+1)$-connected: indeed this map is $\phi(d)$-connected upon looping once by [WW88, Lem. 1.12], and is an isomorphism in π_0 by the analysis above and [WW88, Prop. 1.8 & Cor. 5.3]. $\square$

Proposition 3.1.8. *The map (3.1.4) is C_2-equivariant up to homotopy. Therefore, so is (3.1.2).*

Proof. We will give an argument only in the topological setting; in the smooth setting one works with $H_{\mathsf{col}}(M)$ instead to smooth out corners, and uses smooth approximations of the continuous functions that will appear in proof below. We will however state the argument in the smooth setting to simplify notation. We will also assume at any point in the argument where it is necessary that a partition (W,F,V) is equipped with some bicollar of F in W and V. We adopt the convention that $\pm\infty + r \equiv \pm\infty$ for any real number $r \in \mathbb{R}$.

The Weiss–Williams involution on $\widehat{\mathbf{H}}(M)_0$ is induced by the identity on $\coprod_{[M']} B\mathrm{Diff}_\partial(M')$ and the involution $U \mapsto U^* = (\mathrm{Id}_{\mathbb{R}^\infty}\times(-1)_{\underline{\mathbb{R}}})(U)$ on $B\mathrm{Diff}^b_\partial(M\times\underline{\mathbb{R}})$.

Then for $\rho = (W, F, V) \in H^s(M)$,

$$\tau_{WW} \circ \mathrm{alex}(W, F, V) = (i_\rho(F), \gamma_W^*), \qquad \mathrm{alex} \circ \iota_H(W, F, V) = (i_{\rho^*}(F^*), \gamma_{V^*}),$$

where $\gamma_W^*(t) := (\gamma_W(t))^*$. We have depicted the paths γ_W^* and γ_{V^*} in Figure 3.2. We need to find a path $\eta : [-1, 1] \to B\mathrm{Diff}_\partial(M)$ from $i_\rho^*(F^*)$ to $i_\rho(F)$ and a homotopy $\{H_s(-)\}_{-1\leq s\leq 1}$ from $\gamma_{V^*}(-)$ to $\gamma_W^*(-)$ such that $H_s(-\infty) = M \times \mathbb{R}$ and $H_s(+\infty) = \eta(s) \times \mathbb{R}$ for all $s \in [-1, 1]$.

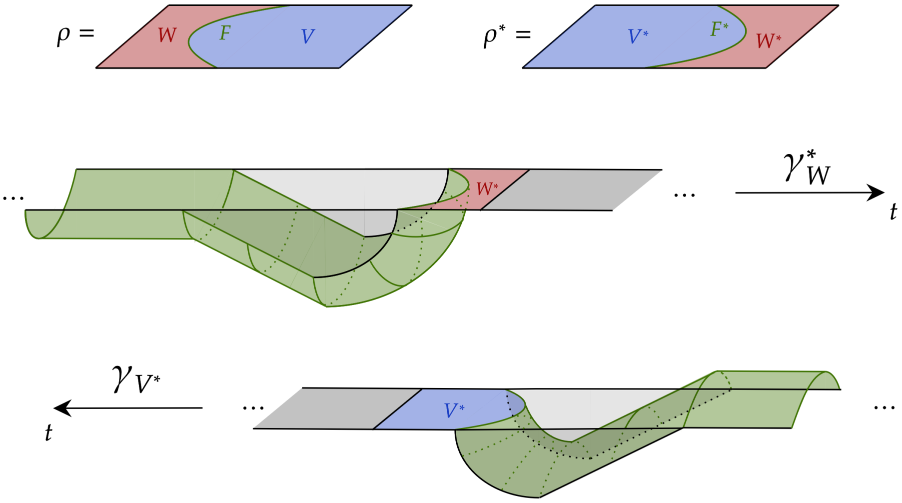

Figure 3.2 Paths γ_W^* and γ_{V^*} in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$. The arrow indicates the direction of the path as time increases.

For η, we use the last two coordinates in $\mathbb{R}^{N+2}$ to do a half rotation of that plane. More explictly, $\eta(s) := (\mathrm{Id}_{\mathbb{R}^N} \times Q_{\pi\cdot(s+1)/2})(i_\rho(F))$ where $Q_\theta : \mathbb{R}^2 \cong \mathbb{R}^2$ is as before.

View $\mathbb{R}^\infty$ as $\mathbb{R}^\infty \times \underline{\{0\}} \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$ and write $N := \bigcup_{s\in[-1,1]} \eta(s) + s \cdot \underline{e}$. For $X \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$, write $X\,|_{[a,b]}$ for $X \cap (\mathbb{R}^\infty \times \underline{[a, b]})$. Then consider the compact manifold

$$U_\rho := \Big(\widehat{a}(\rho^*)\,|_{[-1,2]} - 3 \cdot \underline{e}\Big) \cup N \cup \Big((\widehat{a}(\rho)\,|_{[-1,2]})^* + 3 \cdot \underline{e}\Big)$$

depicted in Figure 3.3. Using the contractibility of $\mathrm{Emb}_\partial(F^* \times \underline{[-3, 3]}, \mathbb{R}^\infty \times \underline{[-3, 3]})$, we obtain a path from $\overline{U_\rho \setminus (V^* \cup W^*)}$ (the green part in Figure 3.3) to a scaled (in the $\underline{\mathbb{R}}$-direction) version of the bicollar of F^* in W^* and V^*. Rescaling this bicollar back to normal whilst dragging V^* and W^* in the process, we obtain a path ψ from U_ρ to $M \times \underline{[-4, 4]} = M \times \underline{[-4, -1]} \cup V^* \cup W^* \cup M \times \underline{[1, 4]}$ in the moduli space of manifolds inside $\mathbb{R}^{N+2} \times \underline{[-4, 4]}$ which are diffeomorphic to $M \times \underline{[-4, 4]}$ relative to its boundary $M \times \underline{\{-4\}} \cup \partial M \times \underline{[-4, 4]} \cup M \times \underline{\{4\}}$.

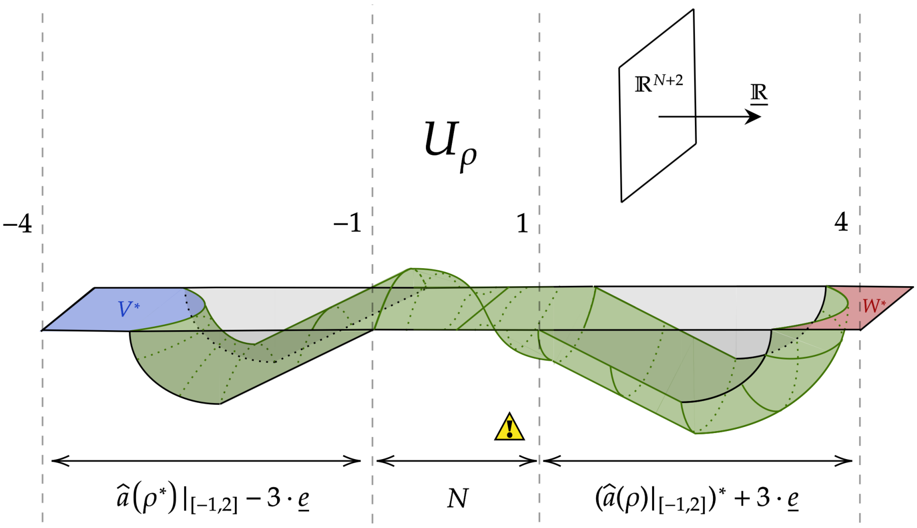

Figure 3.3 Depiction of the manifold U_ρ. Proceed with caution: the part of the picture corresponding to N takes place in an extra dimension that we are unable to depict accurately.

We now describe the homotopy $\{H_s(-)\}_{-1\leq s\leq 1}$. Fix some homeomorphism $l : [-1,1] \cong [-\infty,+\infty]$, and assume that the path ψ from $M \times \underline{[-4,4]}$ to U_ρ just described is parametrised by $[-\infty,+\infty]$. Then $H_s(-)$ is the concatenation of two paths $H_s^{(1)}(-)$ and $H_s^{(2)}(-)$ in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$: the path $H_s^{(1)}(-)$ performs $\psi(-)$ on $M \times \underline{[-4,4]} + l(s)\cdot\underline{e} \subset M \times \underline{\mathbb{R}}$ (if $s = \pm 1$, $H_s^{(1)}(-)$ is constant on $M \times \underline{\mathbb{R}}$). The path $H_s^{(2)}(-)$ starts at $H_s^{(1)}(+\infty)$, and sends $H_s^{(1)}(+\infty)\,|_{(-\infty,l(s)+s]}$ and $H_s^{(1)}(+\infty)\,|_{[l(s)+s,+\infty)}$ towards $\mathbb{R}^{N+2} \times \underline{\pm\infty}$, respectively, extending by $H_s^{(1)}(+\infty)\,|_{l(s)+s}$ times an interval of diverging length. The resulting paths $H_{\pm 1}(-) = H_{\pm 1}^{(1)}(-)\cdot H_{\pm 1}^{(2)}(-)$ are reparametrisations of γ_{V^*} and γ_{W^*} (the reparametrisations only depend on our choice of homeomorphism $l : [-1,1] \cong [-\infty,+\infty]$ and the parametrisation of the path ψ). Thus η and H give rise to the required homotopy $\tau_{WW} \circ \mathrm{alex} \simeq \mathrm{alex} \circ \iota_H$. ☐

Corollary 3.1.9. *The Weiss-Williams involution on* $\pi_0^s(\mathbf{H}(M)) \cong \pi_0(H(M)) \cong \mathrm{Wh}(\pi_1 M)$ *corresponds to the rule* $\kappa \mapsto (-1)^{d-1}\overline{\kappa}$, *where* $\overline{(-)}$ *is Milnor's involution [Mil66] on* $\mathrm{Wh}(\pi_1(M))$ *(see Warning 3.1.21).*

Proof. The isomorphism $\pi_0(H(M)) \cong \mathrm{Wh}(\pi_1 M)$ sends the class $[\rho]$ of a partition $\rho = (W,F,V)$ to the Whitehead torsion $\tau(W,M)$ of W with respect to M. The claim follows from Proposition 3.1.8, the *duality formula* of [Mil66, §10] and the fact that $\tau(V,F)$ is roughly $-\tau(W,M)$ (cf. [Mil66, Lem. 7.8]). ☐

Recall that $H^s(M) \simeq BC(M)$. Now if $P \subset M$ is a codimension zero embedding and $p \leq d-3$ (in the notation of Theorem C), then $C\mathrm{Emb}(P,M) \simeq \mathrm{hofib}(H^s(\overline{M-P}) \to$

$H^s(M)$) by the isotopy extension sequence (2.3.3), and therefore $C\mathrm{Emb}(P,M)$ inherits an involution ι_H up to weak equivalence. The map (3.1.4) is functorial with respect to codimension zero embeddings, so the following diagram is commutative:

$$\begin{tikzcd}
H^s(\overline{M-P}) \arrow[r, "\mathrm{alex}"] \arrow[d] & \mathbf{H}(\overline{M-P})_0 \arrow[r, hook] \arrow[d] & \Omega^\infty(\mathbf{H}(\overline{M-P})) \arrow[d] \\
H^s(M) \arrow[r, "\mathrm{alex}"] & \mathbf{H}(M)_0 \arrow[r, hook] & \Omega^\infty(\mathbf{H}(M)).
\end{tikzcd} \tag{3.1.5}$$

Corollary 3.1.10. *If* $\dim P = d$*, the vertical homotopy fibre of the horizontal compositions in (3.1.5) gives a map*

$$\mathrm{alex} : C\mathrm{Emb}(P,M) \longrightarrow \Omega^\infty(\mathbf{CE}(P,M))$$

which is $\phi_{C\mathrm{Emb}}(d,p)$*-connected and* C_2*-equivariant up to homotopy.*

Proof. The connectivity of this map is the content of Proposition 2.3.11. It is homotopy C_2-equivariant since both alex : $H(M) \to \Omega^\infty(\mathbf{H}(M))$ and alex : $H(\overline{M-P}) \to \Omega^\infty(\mathbf{H}(\overline{M-P}))$ are by Proposition 3.1.8. □

Warning 3.1.11. There is a canonical involution ι_C in the concordance space $C(M)$ given by turning upside down a concordance and precomposing by the inverse of the top diffeomorphism (see e.g. [Vog85, p. 296]). The restriction map $C(M) \to C\mathrm{Emb}(P,M)$ is *not* C_2-equivariant with respect to ι_C and ι_H—rather, it is anti-equivariant. This may seem to contradict [Vog85, Prop. 2.2], but what Vogell really proves there is that there is a homotopy C_2-equivariant equivalence $C(M) \approx \Omega^\sigma H(M) := \mathrm{Map}_*(S^\sigma, H(M))$, where we recall S^σ stands for the representation sphere of the 1-dimensional sign representation σ. This is due to an extra flip in the loop component that he introduces at the end of the proof of the proposition.

3.1.3 From h-cobordism spaces back to A-theory

Given a spherical fibration ξ over M equipped with a section, fibrewise smashing a retractive space over M with ξ gives rise to a functor $-\cdot\xi : \mathbf{A}(M) \to \mathbf{A}(M)$ which, by [Vog85, Prop. 2.5], makes the following diagram homotopy commutative:

$$\begin{tikzcd}
\mathbf{A}(M) \arrow[r, "\tau_\epsilon"] \arrow[dr, "\tau_\xi"'] & \mathbf{A}(M) \arrow[d, "-\cdot\xi"] \\
 & \mathbf{A}(M),
\end{tikzcd} \tag{3.1.6}$$

where $\epsilon := \epsilon^0 = M \times S^0$ is the trivial 0-dimensional sphere bundle over M. When $\xi = \epsilon^d = M \times S^d$ is the trivial d-spherical fibration, the functor $- \cdot \xi$ corresponds to $\Sigma^d_M(-) : \mathbf{A}(M) \to \mathbf{A}(M)$, the d-fold *fibrewise suspension over* M (cf. [Vog85, p. 281]). By the *additivity theorem* of [Wal85, Prop. 1.6.2] applied to the Waldhausen category of retractive spaces over M, it follows that Σ^d_M acts (up to homotopy) as $(-1)^d$ on $\mathbf{A}(M)$, and thus by (3.1.6)

$$\xi = M \times S^d \implies \mathbf{A}(M;\xi) \approx S^{d\cdot(\sigma-1)} \wedge \mathbf{A}(M;\epsilon). \tag{3.1.7}$$

Vogell also introduced [Vog85, p. 299] a model for the homotopy fibre sequence

$$\mathcal{H}(M) \longrightarrow Q_+M \longrightarrow A(M) := \Omega^\infty \mathbf{A}(M) \tag{3.1.8}$$

of the parametrised h-cobordism theorem of Waldhausen–Jahren–Rognes [WJR13] (see Remark 3.1.12), and equipped each of the terms in the sequence with compatible homotopy involutions. The one on $\mathcal{H}(M)$ is compatible with ι_H on $H(M) \subset \mathcal{H}(M)$.

Remark 3.1.12. Vogell's work precedes (by more than 25 years) that of Waldhausen–Jahren–Rognes, so let us explain how both fit together. In [Vog85, p. 299], Vogell presents a commutative square with compatible homotopy involutions in each of the terms, and this square is equivalent to the one considered by Waldhausen [Wal82, p. 8] in his "manifold approach" paper. This latter square is, up to equivalence, of the form

$$\begin{tikzcd}
\mathcal{H}(M) \arrow[r] \arrow[d] & Q \arrow[d] \\
* \arrow[r] & A,
\end{tikzcd}$$

for some spaces Q and A, and the goal of that paper was to argue that (i) the square becomes homotopy cartesian upon plus-constructing the vertical right map, and (ii) that $A^+ \simeq A(M)$; these are, respectively, Propositions 5.5 and 5.4 in [Wal82]. While (ii) was fully proved there, only an outline of the argument for (i) was provided, with forward references to a preliminary version of [WJR13].

As stated at the very end of page 22 in [WJR13], the square considered by Waldhausen (after plus-constructing the vertical right arrow) is equivalent to the homotopy cartesian square of [WJR13, Prop. 1.4.8], which in turn is equivalent to one giving rise to the fibre sequence (3.1.8).

All of the above takes place in the PL-setting, but as explained in [WJR13, pp. 15–16], these arguments also deal with the remaining categories Top and Diff.

Going back to Vogell's work, his homotopy involution[2] $\mathcal{T}$ on $A(M)$ is further showed in [Vog85, Cor. 2.10] to agree up to equivalence with the involution τ_ξ when ξ is the d-spherical fibration associated to the once stabilised tangent bundle $TM \oplus \epsilon^1$ of M^d.

The upshot of this discussion then is that when M^d is stably parallelisable, the Weiss–Williams involution is compatible with $(-1)^d \tau_\epsilon$ in the following sense.

Theorem 3.1.13. *If M is stably parallelisable, then there is an equivalence away from two*

$$\Omega^\infty\Big(\mathbf{H}(M)_{hC_2}\Big) \simeq_{[\frac{1}{2}]} \Omega^\infty\Big(\big(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon)\big)_{hC_2}\Big).$$

Remark 3.1.14. Though it probably is, we do not claim the equivalence above is one of infinite loop spaces.

We will need the a few preliminary results for the proof of Theorem 3.1.13.

Lemma 3.1.15. *For each $k \geq 0$, there is a natural C_2-equivariant equivalence of spectra*

$$e_k : \mathbf{H}(M \times I^k) \simeq S^{k\cdot(\sigma-1)} \wedge \mathbf{H}(M). \tag{3.1.9}$$

such that the following square is homotopy commutative:

$$\begin{tikzcd}
\mathbf{H}(M \times I^k) \arrow[r, dash, "\sim", "e_1"'] \arrow[d, dash, "\wr"', "e_k"] & S^{\sigma-1} \wedge \mathbf{H}(M \times I^{k-1}) \arrow[d, dash, "\wr"', "S^{\sigma-1}\wedge e_{k-1}"] \\
S^{k\cdot(\sigma-1)} \wedge \mathbf{H}(M) \arrow[r, equal] & S^{k\cdot(\sigma-1)} \wedge \mathbf{H}(M).
\end{tikzcd}$$

Proof. Set $e_0 = \mathrm{Id}_{\mathbf{H}(M)}$. By inductively defining e_k to fit in the commutative square above, we may assume that $k = 1$. Recall $\mathbb{R}^{a,b} := \mathbb{R}^a \oplus b \cdot \sigma$, and for any orthogonal functor $F(-)$ let $C_2 = O(1)$ act on $F(\mathbb{R}^{a,b})$ by the induced action. Finally let $B(-) := B\mathrm{Diff}^b_\partial(M \times (-))$ and $BI(-) := B\mathrm{Diff}^b_\partial(M \times I \times (-))$. The Alexander trick-like map of [WW88, Prop. 1.5] is a C_2-equivariant map

$$\mathrm{alex} : \mathrm{Diff}^b_\partial(M \times I \times \mathbb{R}^{a,b}) \xrightarrow{\sim} \Omega\mathrm{Diff}^b_\partial(M \times \mathbb{R}^{a+1,b})$$

which, upon delooping, gives rise to a C_2-equivariant equivalence on basepoint components $B(\mathrm{alex}) : BI(\mathbb{R}^{a,b}) \simeq_0 \Omega B(\mathbb{R}^{a+1,b})$. Writing Ξ for the C_2-spectrum whose n-th space is $B^{(1)}(\mathbb{R}^{1,n+1})$ and with stabilisation maps $s_{0,1} : S^1 \wedge B^{(1)}(\mathbb{R}^{1,n}) \to B^{(1)}(\mathbb{R}^{1,n+1})$,

---

[2] Vogell refers to $\mathcal{T}$ as a *weak* involution in the sense that it is a homotopy involution when restricted to "any compactum" (cf. [Vog85, Lem. 2.4]) or, in better words, to each stage of the colimit in [Vog85, p. 299] modelling $A(M)$.

we obtain a C_2-equivariant equivalence of spectra

$$B(\text{alex}) : \mathbf{H}(M \times I) := \Theta(BI)^{(1)} \xrightarrow{\sim} \Xi := \{B^{(1)}(\mathbb{R}^{1,n+1})\}_{n\geq 0}. \tag{3.1.10}$$

But now the stabilisation map $s_{1,0} : S^\sigma \wedge B^{(1)}(\mathbb{R}^{0,n}) \to B^{(1)}(\mathbb{R}^{1,n}) = \Xi_{n-1}$ induces another C_2-equivariant equivalence of spectra

$$S^{\sigma-1} \wedge \mathbf{H}(M) \xrightarrow{\sim} \Xi.$$

Composing these two equivalences gives the one in the statement. □

Vogell introduced in [Vog85, p. 298] the *lower* and *upper stabilisation maps* $\Sigma_\ell, \Sigma_u : H(M) \to H(M \times I)$. Roughly, the former sends a partition $\rho = (W, F, V)$ to $(U(W), W \cup_F W, \overline{M \times I \times I \setminus U(W)})$, where $U(W)$ is obtained from W by bending $W \times I$ into a U-shape, whilst Σ_u does the same to V instead of W (see Figure 3.6 for a pictorial representation of Σ_ℓ). We will only be interested in the lower stabilisation Σ_ℓ, which we will denote by Σ for simplicity. Here's how it interacts with the h-cobordism involution ι_H.

Lemma 3.1.16. *Let $+_I$ stand for the "stacking in the I-direction" $\mathbb{E}_1$-algebra structure*[3] *in $H(M \times I)$. Then if J denotes another copy of I:*

(a) $\iota_H\Sigma +_I \Sigma\iota_H \simeq * : H(M) \to H(M\times I)$, *(b)* $\iota_H\Sigma^2 \simeq \Sigma^2\iota_H : H(M) \to H(M\times I\times J)$.

Proof. We defer the proof of (a) to Lemma E.1 in Appendix E as it is a bit technical. Note that $+_I$ and $\Sigma : H(M \times I) \to H(M \times I \times J)$ are compatible in the sense that

$$\Sigma(\rho +_I \rho') = \Sigma\rho +_I \Sigma\rho', \quad \rho, \rho' \in H(M \times I).$$

Then (b) follows from

$$\iota_H\Sigma^2 \simeq \iota_H\Sigma^2 +_I \Sigma(\iota_H\Sigma +_I \Sigma\iota_H) \simeq (\iota_H\Sigma +_I \Sigma\iota_H)\Sigma +_I \Sigma^2\iota_H \simeq \Sigma^2\iota_H.$$

□

Proof of Theorem 3.1.13. We may assume without loss of generality that $\dim M \geq 5$, for if not replace it by $M \times J^{2k}$ for $k \geq 3$. The effect this has on both sides of the

---

[3] The technical assumption we imposed on a partition $\rho = (W, F, V) \in H(M \times I)$ so that the intersection of F with $\partial(M \times I) \times [-1, 1]$ is standard and happens exactly at $\partial(M \times I) \times \{0\}$ makes $+_I$ well-defined.

equivalence in the statement is rather mild: as there is a homotopy C_2-equivariant equivalence $S^2 \approx S^{2\sigma}$, it follows by Lemma 3.1.15 and Corollary 3.1.4(i) that there are equivalences of spectra

$$\mathbf{H}(M \times J^{2k})_{hC_2} \simeq_{[\frac{1}{2}]} \mathbf{H}(M)_{hC_2},$$
$$\Big(S^{(d+2k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M \times J^{2k}; \epsilon)\Big)_{hC_2} \simeq_{[\frac{1}{2}]} \Big(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M; \epsilon)\Big)_{hC_2}.$$

In the second equivalence we also use that $\mathbf{Wh}^{\mathrm{Diff}}(-)$ and τ_ϵ are homotopy invariants of $(-)$.

Let $k \geq 0$ and let ξ denote the $(d+k)$-spherical fibration corresponding to the stable tangent bundle $T(M \times I^k) \oplus \epsilon^1$. If M (and hence $M \times I^k$) is stably parallelisable, the involution $(-1)^{d+k}\tau_\epsilon$ is homotopic to τ_ξ by (3.1.7), and by [Vog85, Prop. Cor. 2.10] it agrees with $\mathcal{T}$ (and hence extends ι_H). All in all, we obtain a zig-zag of homotopy C_2-equivariant maps

$$\begin{tikzcd}
\Omega^\infty \mathbf{H}(M \times I^k) \arrow[d, "\wr", "(3.1.9)"'] & H(M \times I^k) \arrow[l, "\mathrm{alex}"', "(3.1.2)"] \arrow[r, "(\dagger)"] & \Omega^\infty\Big(S^{(d+k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M \times I^k)\Big) \arrow[d, "\wr"] \\
\Omega^\infty(S^{k\cdot(\sigma-1)} \wedge \mathbf{H}(M)) & & \Omega^\infty\Big(S^{(d+k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)\Big)
\end{tikzcd} \tag{3.1.11}$$

Every space involved in (3.1.11) is an $\mathbb{E}_1$-group if $k \geq 1$: both $H(M \times I^k)$ and $\Omega^\infty(\mathbf{H}(M \times I^k))$ by stacking in the first of the I^k-coordinates, and the others by their own infinite loop structures.

Claim 1. *All of the maps in (3.1.11) are H-maps if $k \geq 1$.*

Proof. The $\mathbb{E}_1$-algebra structure $+_I$ on $\Omega^\infty\mathbf{H}(M \times I^k)$ is equivalent to any of the other ones coming from its infinite loop structure. As (3.1.9) and the right vertical equivalence are infinite loop maps, they are in particular H-maps. As for (3.1.2), it is an $\mathbb{E}_1$-map since (3.1.4) : $H(M \times I^k) \to \widehat{\mathbf{H}}(M \times I^k)_0$ is (by construction).

Non equivariantly, the map $(\dagger)$ is the composition $(\dagger) : H(M \times I^k) \hookrightarrow \mathcal{H}(M \times I^k) \simeq \Omega^{\infty+1}\mathbf{Wh}^{\mathrm{Diff}}(M \times I^k)$, where the last equivalence is the stable parametrised h-cobordism theorem of Waldhausen–Jahren–Rognes [WJR13, Thm. 0.1]. As communicated to us in private by Bjørn Jahren and John Rognes, such equivalence is only stated to hold in the category of spaces (and not of infinite loop spaces, though it should definitely also hold there). We now explain why this equivalence is one of H-groups (which is the general consensus, but we couldn't find it written down anywhere): again assume $k = 1$. One reduces to the PL-case as in [WJR13, pp. 15–16]. Then if X is a simplicial set such that $|X| \simeq M$, the equivalence in the PL-setting is induced by a zig-zag of equivalences of simplicial sets (cf. the left vertical column of [WJR13, Eq. (0.4)])

$$\mathcal{H}(M \times I)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} s\mathcal{C}^h(X \times I),$$

where $s\mathcal{C}^h(X \times I)$ is the category (seen as a simplicial set by taking its nerve) of acyclic cofibrations $X \times I \hookrightarrow Y$ together with simple maps over $X \times I$. One can verify that it makes sense to stack in the I-direction in each of the simplicial sets involved in the zig-zag, and that the maps between them respect this monoidal structure. Let $\mu_0 : s\mathcal{C}^h(X \times I) \times s\mathcal{C}^h(X \times I) \to s\mathcal{C}^h(X \times I)$ stand for this monoidal structure, given explicitly by $\mu_0(Y, Z) := Y{}_{X\times 1}\cup_{X\times 0} Z$. There is another monoidal structure μ_1 induced by the pushout along $X \times I$, i.e. $\mu_1(Y, Z) := Y \cup_{X\times I} Z$. Sliding gives a homotopy between μ_0 and μ_1: intuitively, the maps

$$\mu_t : s\mathcal{C}^h(X\times I)\times s\mathcal{C}^h(X\times I) \longrightarrow s\mathcal{C}^h(X\times I), \quad (Y, Z) \longmapsto Y{}_{X\times[1-t,1]}\cup_{X\times[0,t]} Z, \quad t \in [0, 1],$$

constitute the homotopy. More precisely, consider the simplicial category $s\widetilde{\mathcal{C}}^h_\bullet(X)$ [WJR13, Defn. 3.1.1] whose objects in simplicial degree q consist of commutative diagrams

$$\begin{tikzcd}
X \times \Delta^q_\bullet \arrow[rr, hook, "i", "\sim"'] \arrow[dr, "\mathrm{pr}"] & & Y \arrow[dl, "\pi"] \\
& \Delta^q_\bullet, &
\end{tikzcd}$$

where i is an acyclic cofibration and π is a Serre fibration. The inclusion $s\mathcal{C}^h(X) \hookrightarrow s\widetilde{\mathcal{C}}^h_\bullet(X)$ as the 0-simplices is a homotopy equivalence by [WJR13, Cor. 3.5.2], where a simplicial category is seen as a bisimplicial set by taking its nerve, and a bisimplicial set as an ordinary simplicial set by taking its totalisation. The monoidal structures μ_t make perfectly good sense in $s\widetilde{\mathcal{C}}^h_\bullet(X \times I)$, and one can indeed define a simplicial homotopy between μ_0 and μ_1 in this setting resembling the idea above.

But by Proposition 3.1.1, and Theorems 3.1.7 and 3.3.1 of [Wal85] (see also [WJR13, p. 5]), for any simplicial set T, there is a zig-zag of equivalences connecting $|s\mathcal{C}^h(T)|$ and $\Omega\mathrm{Wh}^{PL}(T) := \Omega^{\infty+1}(\mathbf{Wh}^{PL}(T))$ which is monoidal up to homotopy with respect to μ_1 in the domain and the loop structure on the looped Whitehead space. It hence follows that (†) is indeed a zig-zag of H-maps. □

Claim 2. *For each $k \geq 1$, the diagram*

$$\begin{tikzcd}
\Omega^\infty(S^{2k\cdot(\sigma-1)} \wedge \mathbf{H}(M)) \arrow[d, "\wr"'] & H(M \times I^{2k}) \arrow[l] \arrow[r] \arrow[d, "\Sigma^2"] & \Omega^\infty(S^{(d+2k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \arrow[d, "\wr"] \\
\Omega^\infty(S^{(2k+2)\cdot(\sigma-1)} \wedge \mathbf{H}(M)) & H(M \times I^{2k+2}) \arrow[l] \arrow[r] & \Omega^\infty(S^{(d+2k+2)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M))
\end{tikzcd} \tag{3.1.12}$$

is homotopy commutative, where the rows are the zig-zags (3.1.11) and the vertical external maps are induced by the homotopy C_2-equivariant equivalence $\mathbb{S}^0 \approx \mathbb{S}^{2\cdot(\sigma-1)}$.

Proof. The right hand square is clearly commutative, for both right horizontal maps factor through $\mathcal{H}(M)$. For the commutativity of the left one, by Lemma 3.1.15, it suffices to argue that the square

$$\begin{tikzcd}
H(M \times I^{2k}) \arrow[r, "\mathrm{alex}"] \arrow[d, "\Sigma"] & \Omega^\infty \mathbf{H}(M \times I^{2k}) \arrow[d, "e_1"', "\wr"] \\
H(M \times I^{2k+1}) \arrow[r, "\mathrm{alex}"] & \Omega^\infty \mathbf{H}(M \times I^{2k+1})
\end{tikzcd} \tag{3.1.13}$$

homotopy commutes (non-equivariantly[4]). Recall that the map e_1, non-equivariantly, is induced by the zig-zags

$$\mathbf{H}(M \times I^{2k})_n \xrightarrow{s^\vee} \Omega\mathbf{H}(M \times I^{2k})_{n+1} \xleftarrow[\sim]{\mathrm{alex}} \mathbf{H}(M \times I^{2k+1})_n,$$

where $s^\vee$ is the (adjoint to the) structure map of $\mathbf{H}(M \times I^{2k})$. Since (3.1.13) is a diagram of $\mathbb{E}_1$-groups by stacking in the first of the I^{2k}-coordinates, it suffices to provide a homotopy for the diagram

$$\begin{tikzcd}
H^s(M \times I^{2k}) \arrow[rr, "(3.1.4)"] \arrow[d, "\Sigma"] & & \mathbf{H}(M \times I^{2k})_0 \arrow[d, "s^\vee"] \\
H^s(M \times I^{2k+1}) \arrow[r, "(3.1.4)"] & \mathbf{H}(M \times I^{2k+1})_0 \arrow[r, "\mathrm{alex}"] & \Omega\mathbf{H}(M \times I^{2k})_1.
\end{tikzcd}$$

As in Remark 2.3.9, such homotopy is obtained by delooping (with respect to stacking in the second of the I^{2k}-coordinates) the diagram (2.3.5) of Proposition 2.3.8. □

Clearly Σ^2 is an H-map and also homotopy C_2-equivariant (with respect to ι_H) by Lemma 3.1.16(b). Therefore by Claim 1, all of the maps involved in (3.1.12) are

---

[4] Homotopy C_2-equivariant maps that are homotopic (as ordinary maps) induce the same arrow in the homotopy category of C_2-spaces.

H-maps and homotopy C_2-equivariant. Taking the homotopy colimit as $k \to \infty$, we obtain a homotopy C_2-equivariant zig-zag

$$\Omega^\infty \mathbf{H}(M) \xleftarrow{\approx} \operatorname*{hocolim}_k H(M \times I^{2k}) \xrightarrow{\approx} \Omega^\infty(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \tag{3.1.14}$$

of H-maps. The connectivity of, say, the upper horizontal maps in (3.1.12) is $\phi(d+2k) \gtrsim (d+2k)/3$ by Igusa's theorem and, as this lower bound increases linearly with k, the horizontal maps in (3.1.14) are indeed equivalences. The equivalence in the statement now follows by Corollary 3.1.4(ii) applied to (3.1.14), and because taking homotopy C_2-orbits commutes up to equivalence with $\Omega^\infty(-)$ if 2 is inverted (as in Corollary 3.1.4). The proof of Theorem 3.1.13 is now complete. ☐

Corollary 3.1.17. *If M^d is stably parallelisable and $P \subset M^d$ is a codimension zero submanifold with $p \leq d-3$ (in the notation of Theorem C), then there is an equivalence away from two*

$$\Omega^\infty\Big(\mathbf{CE}(P,M)_{hC_2}\Big) \simeq_{[\frac{1}{2}]} \Omega^\infty\Big(\big(S^{d\cdot(\sigma-1)-2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M, M-P;\epsilon)\big)_{hC_2}\Big),$$

where $\mathbf{Wh}^{\mathrm{Diff}}(M, M-P;\epsilon)$ stands for the homotopy cofibre of $\mathbf{Wh}^{\mathrm{Diff}}(M-P;\epsilon) \to \mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon)$.

Proof. Note that $\overline{M-P}$ is stably parallelisable because M is. The zig-zag (3.1.14) is functorial with respect to codimension zero embeddings of stably parallelisable manifolds, and hence taking homotopy fibres in the map from (3.1.14) with M replaced by $\overline{M-P}$ to (3.1.14) itself, we obtain another homotopy C_2-equivariant zig-zag of equivalences

$$\Omega^\infty \mathbf{CE}(P,M) \xleftarrow{\approx} \operatorname*{hocolim}_k C\mathrm{Emb}(P \times I^{2k}, M \times I^{2k}) \xrightarrow{\approx} \Omega^\infty(S^{d\cdot(\sigma-1)-2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M, M-P)).$$

The same line of reasoning as before yields the desired result. ☐

3.1.4 The canonical involution in algebraic K-theory

We now define the canonical involution τ_ϵ on $A(X)$, for X based, and relate it to an involution in the model of A-theory via "spaces of matrices with values in the ring up to homotopy" $Q_+\Omega X$ [Wal85, §2.2]. Throughout, let $G := GX$ denote the topological monoid of Moore loops on X, and write $\mathbb{S}[G]$ for the $\mathbb{E}_1$-ring spectrum $\mathbb{S} \wedge G_+$.

We will work over the ∞-category $\mathsf{Mod}_{\mathbb{S}[G]}$ of right $\mathbb{S}[G]$-module spectra; we will also write ${}_{\mathbb{S}[G]}\mathsf{Mod}$ for the ∞-category of left $\mathbb{S}[G]$-modules. Then for $m \geq 1$, if $\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ denotes the homotopy invertible components of the mapping space $\mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \oplus^m \mathbb{S}[G])$, Waldhausen showed in [Wal85, Thm. 2.2.1] that for X connected, there is a natural equivalence

$$A(X) \simeq \mathbb{Z} \times \underset{m}{\mathrm{hocolim}}\; B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^+. \tag{3.1.15}$$

In order to define τ_ϵ, we will introduce compatible anti-involutions on $\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ defined in terms of Spanier–Whitehead duality. As $\mathbb{S}[G]$ is not commutative, this duality really arises as an instance of a *duality in the symmetric closed bicategory* $\mathsf{Bimod}_{\mathbb{S}}$ of bimodule spectra, in the sense of May–Sigurdsson [MS06, §16.4]. This duality coincides with the one considered by Vogell in [Vog85, §1].

Remark 3.1.18. We can safely import the duality theory of May–Sigurdsson [MS06]: even though it is developed only 2-categorically, and $\mathsf{Bimod}_{\mathbb{S}}$ (a.k.a. the Morita category of the sphere spectrum, cf. [Hau17]) is an $(\infty, 2)$-category, the arguments that rely on duality only involve the homotopy 2-category $\mathrm{Ho}_2(\mathsf{Bimod}_{\mathbb{S}})$, which is *symmetric closed* in the sense of [MS06, Defn. 16.2.1 & 16.3.1].

First observe that a right $\mathbb{S}[G]$-module M can always be regarded as a left $\mathbb{S}[G]$-module by

$$\mathbb{S}[G] \otimes M \xrightarrow{\text{swap}} M \otimes \mathbb{S}[G] \xrightarrow{\mathrm{Id}_M \otimes \mathrm{inv}} M \otimes \mathbb{S}[G^{\mathrm{op}}] = M \otimes \mathbb{S}[G] \xrightarrow{\text{act}} M,$$

where "inv" stands for inversion in the monoid G—write M_ℓ for this left $\mathbb{S}[G]$-module. Here $\otimes = \otimes_{\mathbb{S}}$ stands for the usual smash product of spectra. Note also that

$${}_{\mathbb{S}[G]}\mathsf{Mod}(M_\ell, M_\ell) \simeq \mathsf{Mod}_{\mathbb{S}[G]}(M, M)$$

as $\mathbb{E}_1$-algebras. If $\nu : \mathbb{S} \to \mathbb{S}[G]_\ell$ denotes the unit, consider the map of spectra

$$\eta_1 : \mathbb{S} \simeq \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \mathbb{S} \xrightarrow{1 \otimes \nu} \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \mathbb{S}[G]_\ell$$

and the map of $(\mathbb{S}[G], \mathbb{S}[G])$-bimodules

$$I_1 : \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \xrightarrow{\mathrm{inv} \otimes 1} \mathbb{S}[G] \otimes \mathbb{S}[G] \xrightarrow{\text{act}} \mathbb{S}[G].$$

Then (η_1, I_1) make $(\mathbb{S}[G], \mathbb{S}[G]_\ell)$ a dual pair [MS06, Defn. 16.4.1]. More generally, the map of spectra

$$\eta_m : \mathbb{S} \xrightarrow{\bigoplus_{i,j} \delta_{ij}\eta_1} \oplus_{i,j=1}^m \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \mathbb{S}[G]_\ell \cong \oplus_{j=1}^m \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \Big(\oplus_{i=1}^m \mathbb{S}[G]\Big)_\ell$$

together with the map of $(\mathbb{S}[G], \mathbb{S}[G])$-bimodules

$$I_m : \Big(\oplus_{i=1}^m \mathbb{S}[G]\Big)_\ell \otimes \oplus_{j=1}^m \mathbb{S}[G] \cong \oplus_{i,j=1}^m \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \xrightarrow{\bigoplus_{i,j} \delta_{ij} I_1} \mathbb{S}[G],$$

exhibit $(\oplus^m \mathbb{S}[G])_\ell$ as a *right* dual to $\oplus^m \mathbb{S}[G]$. Therefore as in [MS06, Prop. 16.4.9], I_m induces an equivalence of left $\mathbb{S}[G]$-modules $\widetilde{I}_m : (\oplus^m \mathbb{S}[G])_\ell \simeq D_r(\oplus^m \mathbb{S}[G]) := \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \mathbb{S}[G])$, where the right hand side stands for the right $\mathbb{S}[G]$-linear mapping spectrum.

With (3.1.15) and Lemma 3.1.6 in mind, the involution τ_ϵ on $A(X;\epsilon)$ is then induced by the map of $\mathbb{E}_1$-algebras

$$\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \xrightarrow{D_r} {}_G\mathrm{Aut}(D_r(\oplus^m \mathbb{S}[G]))^{\mathrm{op}} \overset{\widetilde{I}_\#}{\simeq} {}_G\mathrm{Aut}((\oplus^m \mathbb{S}[G])_\ell)^{\mathrm{op}} \simeq \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}}, \tag{3.1.16}$$

where $\widetilde{I}_\#$ stands for conjugation with the equivalence $\widetilde{I}_m$. It will be convenient to think of (3.1.16) in the following way: let $GL_m(Q_+G)$ denote the union of path components in $(Q_+G)^{m\times m}$ in the image of $\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ under the natural equivalence

$$\begin{aligned} u : \mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \oplus^m \mathbb{S}[G]) &\xrightarrow{\sim} \mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \prod^m \mathbb{S}[G]) \\ &\simeq \mathsf{Sp}(\mathbb{S}, \mathbb{S}[G])^{m\times m} \simeq (Q_+G)^{m\times m}, \end{aligned}$$

where $\mathsf{Sp} \simeq \mathsf{Mod}_{\mathbb{S}}$ stands for the ∞-category of spectra. So $u : \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \simeq GL_m(Q_+G)$ and, just as in standard linear algebra, under this equivalence the anti-involution (3.1.16) corresponds to the rule that sends a matrix A to its conjugate transpose $A^\dagger$ (conjugate with respect to inversion of G in $\mathbb{S}[G]$). More precisely:

Proposition 3.1.19. *Write* $\mathrm{End}_G(\oplus^m \mathbb{S}[G]) := \mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \oplus^m \mathbb{S}[G])$ *with the action of the cyclic group* C_m *by conjugation with the permutation automorphisms of* $\oplus^m \mathbb{S}[G]$. *Let* C_m *act similarly on* $(Q_+G)^{m\times m}$ *by conjugation. Then the following* *square is commutative in the homotopy category of C_m-spaces:*

$$\begin{tikzcd}
\mathrm{End}_G(\oplus^m \mathbb{S}[G]) \arrow[r, "(3.1.16)", "\sim"'] \arrow[d, "u", "\wr"'] & \mathrm{End}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "u", "\wr"'] \\
(Q_+G)^{m\times m} \arrow[r, "\dagger", "\sim"'] & (Q_+G)^{m\times m}.
\end{tikzcd} \tag{3.1.17}$$

Remark 3.1.20. Passing to the homotopy invertible components in (3.1.17), we obtain the following commutative square in the homotopy category of C_m-spaces

$$\begin{tikzcd}
\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[r, "(3.1.16)", "\sim"'] \arrow[d, "u", "\wr"'] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "u", "\wr"'] \\
GL_m(Q_+G) \arrow[r, "\dagger", "\sim"'] & GL_m(Q_+G).
\end{tikzcd}$$

We have suppressed the $(-)^{\mathrm{op}}$ in the codomain of the map (3.1.16) in the previous squares to emphasise that such squares take place in the homotopy category of C_m-spaces, and not that of $\mathbb{E}_1$-spaces.

Proof of Proposition 3.1.19. First note that all the maps involved in (3.1.17) are indeed C_m-maps: the only one that is not obviously so is (3.1.16), but this follows from the observation that I_m is C_m-equivariant for the diagonal action on the domain and the trivial action on the target. Note also that the C_m-action on $(Q_+G)^{m\times m}$ restricts to a cofree C_m-action on each of the right C_m-cosets of the diagonal subspace, and hence $(Q_+G)^{m\times m} = \prod^m \mathrm{coInd}_e^{C_m} Q_+G$ as a C_m-space. Thus, in order to show that (3.1.17) commutes in the homotopy category of C_m-spaces, it suffices to prove that it commutes in the homotopy category of spaces after postcomposing it with the map

$$(Q_+G)^{m\times m} = \prod^m \mathrm{coInd}_e^{C_m} Q_+G \longrightarrow \prod^m Q_+G$$

that records the first column of a matrix.

Now given an endomorphism h of $\oplus^m \mathbb{S}[G]$, the $(m\times m)$-matrix $u(h) = (h_{ij}) \in GL_m(Q_+G)$ has components

$$h_{ij} : \mathbb{S} \xrightarrow{\nu} \mathbb{S}[G] \xrightarrow{\mathrm{inc}_j} \bigoplus_{k=1}^m \mathbb{S}[G] \xrightarrow{h} \bigoplus_{k=1}^m \mathbb{S}[G] \xrightarrow{\mathrm{pr}_i} \mathbb{S}[G].$$

Slightly abusing the notation, we will write τ_ϵ to mean (3.1.16). Then we must only check that $\tau_\epsilon(h)_{ij}$ is homotopic to $\overline{h}_{ji}$, coherently in h (and for $j=1$, though it is still true for all j of course). Observe now that $(-)_\ell : \mathsf{Mod}_{\mathbb{S}[G]} \to {}_{\mathbb{S}[G]}\mathsf{Mod}$ is a functor over

Sp, and hence the last equivalence in (3.1.16) happens over the automorphism space of $\oplus^m\mathbb{S}[G]$ as a regular spectrum. Consequently, $\tau_\epsilon(h)_{ij}$ is by definition the top horizontal composition in the diagram of spectra

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\nu"] \arrow[dr, "\nu"'] & \mathbb{S}[G]_\ell \arrow[d, "\wr"', "\mathrm{inv}"] \arrow[rr, "\mathrm{inc}_j"] & & \bigoplus_{k=1}^m \mathbb{S}[G]_\ell \arrow[rr, "\tau_\epsilon(h)"] \arrow[dd, "\wr"', "\widetilde{I}_m"] & & \bigoplus_{k=1}^m \mathbb{S}[G]_\ell \arrow[rr, "\mathrm{pr}_i"] \arrow[dd, "\wr"', "\widetilde{I}_m"] & & \mathbb{S}[G]_\ell \\
& \mathbb{S}[G] \arrow[d, phantom, sloped, "\cong"] & (*_1) & & (*_2) & & (*_3) & \mathbb{S}[G] \arrow[u, "\mathrm{inv}", "\wr"'] \arrow[d, phantom, sloped, "\cong"] \\
& \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G],\mathbb{S}[G]) \arrow[rr, "\mathrm{pr}_j^*"] & & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G],\mathbb{S}[G]) \arrow[rr, "h^*"] & & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G],\mathbb{S}[G]) \arrow[rr, "\mathrm{inc}_i^*"] & & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G],\mathbb{S}[G]).
\end{tikzcd}$$

On the other hand, one recognises the composite that goes through the bottom row to be $\overline{h}_{ji}$. Note that the left triangle is commutative, and that the square $(*_2)$ is too by definition of $\tau_\epsilon(h)$. Moreover, $(*_1)$ and $(*_3)$ do not depend on h; we must then argue that $(*_1)$ and $(*_3)$ are commutative up to homotopy.

For the commutativity of $(*_1)$, first observe that the left vertical composite equivalence of $(*_1)$ coincides up to homotopy with the equivalence of left $\mathbb{S}[G]$-modules $\widetilde{I}_1 : \mathbb{S}[G]_\ell \simeq \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G],\mathbb{S}[G])$. This is because, under the usual tensor-hom adjunction, both maps represent the same element in

$$\pi_0\left({}_{\mathbb{S}[G]}\mathsf{Mod}\Big(\mathbb{S}[G]_\ell, \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G],\mathbb{S}[G])\Big)\right) \cong \pi_0\Big({}_{\mathbb{S}[G]}\mathsf{Mod}_{\mathbb{S}[G]}\Big(\mathbb{S}[G]_\ell \otimes \mathbb{S}[G], \mathbb{S}[G]\Big)\Big)$$

by definition of I_1. But now $(*_1)$, with $\widetilde{I}_1$ in place of the left vertical composite, commutes up to homotopy as both composites represent the same element in

$$\pi_0\left({}_{\mathbb{S}[G]}\mathsf{Mod}\Big(\mathbb{S}[G]_\ell, \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G],\mathbb{S}[G])\Big)\right) \cong \pi_0\Big({}_{\mathbb{S}[G]}\mathsf{Mod}_{\mathbb{S}[G]}\Big(\mathbb{S}[G]_\ell \otimes (\oplus^m\mathbb{S}[G]), \mathbb{S}[G]\Big)\Big)$$

simply because the following diagram commutes by definition of I_1 and I_m:

$$\begin{tikzcd}
\mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[r, "\mathrm{inc}_j\otimes 1"] \arrow[d, "1\otimes\mathrm{pr}_j"] & \bigoplus^m \mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[d, "I_m"] \\
\mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[r, "I_1"] & \mathbb{S}[G].
\end{tikzcd}$$

Finally the commutativity of $(*_3)$ follows by a similar reasoning using that

$$\begin{tikzcd}
\bigoplus^m \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[r, "\mathrm{pr}_i\otimes 1"] \arrow[d, "1\otimes\mathrm{inc}_i"] & \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[d, "I_1"] \\
\bigoplus^m \mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[r, "I_m"] & \mathbb{S}[G]
\end{tikzcd}$$

is also commutative by definition. □

Warning 3.1.21. The canonical involution τ_ϵ induces an involution on the Whitehead group

$$\mathrm{Wh}(X) := \pi_1^s(\mathbf{Wh}^{\mathrm{Diff}}(X)) \cong GL(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}/(\pm\pi_1(X)).$$

In the foundational paper [Mil66], Milnor also defined an involution $\mathrm{Wh}(X) \ni \kappa \mapsto \overline{\kappa}$ induced by sending a matrix in $GL(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}$ to its conjugate transpose (conjugate with respect to inversion in $\pi_1(X)$). This is an actual homomorphism because of the abelianisation present in the general linear group of $\mathbb{Z}[\pi_1(X)]$. It is worth being aware that these two involutions on $\mathrm{Wh}(X)$ are only the same after introducing a minus sign, i.e.

$$\tau_\epsilon(\kappa) = -\overline{\kappa}. \tag{3.1.18}$$

This does *not* contradict the commutativity of (3.1.17). On the contrary, this extra minus sign is the result of having to deloop the anti-involution (3.1.16) in the sense of Lemma 3.1.6 in order to obtain τ_ϵ.

3.1.5 A-theory of a suspension

In this section we focus our attention on the homotopy type of $A(X;\epsilon)$ when X is the suspension ΣY of a connected based space Y. By a theorem of Carlsson–Cohen–Goodwillie–Hsiang[5] [CCGH87, Thm. 3], in such cases there is an equivalence of infinite loop spaces

$$\theta : \prod_{m\geq 1} Q(Y^{\wedge m}_{hC_m}) \xrightarrow{\sim} \Omega\widetilde{A}(\Sigma Y), \tag{3.1.19}$$

where $\widetilde{A}(-) := \mathrm{hofib}(A(-) \to A(*))$ and C_m acts on $Y^{\wedge m}$ by cyclic permutation of the factors. In this section, we argue that (3.1.19) can be upgraded to be C_2-equivariant up to homotopy.

Proposition 3.1.22. *Let Y be a connected, based C_2-space. There is an equivalence of spectra*

$$\boldsymbol{\theta} : \bigvee_{m\geq 1} \Sigma^{\infty+\sigma}\Big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\Big) \xrightarrow{\sim} \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon)$$

---

[5] As pointed out in [BCC+96, p. 543], the proof in [CCGH87] has a serious flaw around p. 71. This issue was fixed in [BCC+96, Cor. 4.15], and in particular the map θ of (3.1.19) constructed in [CCGH87, §1] is still an equivalence. We are indebted to Tom Goodwillie for his help in clearing out this matter and for carefully explaining to us another more general principle for which (3.1.19) holds—namely, it is the observation that if F is a functor (from based spaces to based spaces, say) whose m-th derivative spectrum is of the form $X \mapsto X^{\wedge m}_{hC_m}$ for every $m \geq 1$, then its Taylor tower must split globally. This is indeed the case for the functor $F(-) := \Omega \circ A \circ \Sigma(-)$.

that is C_2-equivariant up to homotopy, and whose underlying (non-equivariant) equivalence induces (3.1.19). Here $\Sigma^\sigma Y := S^\sigma \wedge Y$ and $D_m \subset \Sigma_m$ acts on $Y^{\wedge m}$ by

$$g \cdot (y_1 \wedge \cdots \wedge y_m) := g \cdot y_{g(1)} \wedge \cdots \wedge g \cdot y_{g(m)}, \quad g \in D_m, \quad y_i \in Y,$$

where Y is now seen as a based D_m-space (on which C_m acts trivially). Finally $C_2 = D_m/C_m$ acts on $(ED_m)_+ \wedge_{C_m} Y^{\wedge m}$ by its residual diagonal action.

Remark 3.1.23. The C_2-space $\Sigma^\sigma Y$ induces an involution on $\mathbf{A}(\Sigma^\sigma Y)$ which commutes with the canonical involution τ_ϵ described in the previous section, by naturality of its construction. Therefore its composite gives the involution on $\widetilde{\mathbf{A}}(\Sigma^\sigma Y; \epsilon)$ appearing in the statement of Proposition 3.1.22. Alternatively, we can allow X in the previous section to mean a C_2-space (e.g. $\Sigma^\sigma Y$), and agree that $\mathrm{inv} : G = GX \to G^{\mathrm{op}}$ there stands for inversion in the monoid G followed by the C_2-action on X.

In practice, we will apply Proposition 3.1.22 to the case when $Y = S^\sigma \wedge Z$ for some trivial C_2-space Z, as then $\Sigma^\sigma Y \simeq S^{2\sigma} \wedge Z \approx S^2 \wedge Z$ because of the homotopy C_2-equivariant equivalence $S^{2\sigma} \approx S^2$. In such case, as $\mathbf{A}(-;\epsilon)$ is a homotopy functor, Proposition 3.1.22 provides a simple description of the homotopy C_2-equivariant homotopy type of $\widetilde{\mathbf{A}}(\Sigma^2 Z; \epsilon) \approx \widetilde{\mathbf{A}}(\Sigma^{2\sigma} Z; \epsilon)$. This, together with Corollary 3.1.4, can then be used to analyse the homotopy type of $\widetilde{\mathbf{A}}(\Sigma^2 Z; \epsilon)_{hC_2}$ away from 2.

We will need the following observation for the proof of Proposition 3.1.22.

Lemma 3.1.24. *Let X be a based, connected C_m-space. The equivalence*

$$\iota : B((C_m \ltimes \Omega X)^{\mathrm{op}}) \simeq B(C_m \ltimes \Omega X)$$

of Lemma 3.1.6 coincides up to equivalence with the delooping of the inversion map

$$\mathrm{inv} : (C_m \ltimes \Omega X)^{\mathrm{op}} \longrightarrow C_m \ltimes \Omega X, \quad (s^i, \gamma) \mapsto (s^{-i}, s^i \cdot \overline{\gamma}),$$

where $\overline{\gamma}$ stands for the loop γ with the reversed orientation.

Proof. Given a topological monoid M equipped with a C_m-action, it is well-known (see e.g. [AM04, §II, Thm. 1.12]) that the classifying space of the semi-direct product $C_m \ltimes M$ is equivalent to $EC_m \times_{C_m} BM$. On the simplicial level, this equivalence is given by

$$\begin{aligned} \beta : B_\bullet(C_m \ltimes M) &\xrightarrow{\sim} E_\bullet C_m \times_{C_m} B_\bullet M, \\ \big((s^{i_1}, m_1), \dots, (s^{i_q}, m_q)\big) &\longmapsto \big[(e, s^{i_1}, \dots, s^{i_q}), (s^{i_1} \cdot m_1, s^{i_1+i_2} \cdot m_2, \dots, s^{i_1+\cdots+i_q} \cdot m_q)\big]. \end{aligned}$$

Now, for simplicity, we may assume that $\Omega(-)$ stands for the Moore loop space, so that $C_m \ltimes \Omega X$ is strictly associative (a *Moore loop* is a pair (γ, t) where $t \geq 0$ and $\gamma : [0,t] \to Y$ is a map with $\gamma(0) = \gamma(t) = *$; multiplication of Moore loops is given by concatenation of loops and addition of its lengths). Also recall that there is an identification $\Delta^q \cong \{\mathbf{v} = (v_1, \ldots, v_q) : 0 \leq v_1 \leq \cdots \leq v_q \leq 1\}$; under this identification, the self-isomorphism $\Phi_q : \Delta^q \cong \Delta^q$ in the proof of Lemma 3.1.6 becomes the rule that sends a partition $\mathbf{v} = (0 \leq v_1 \leq \cdots \leq v_q \leq 1)$ to $1-\mathbf{v} := (0 \leq 1-v_q \leq \cdots \leq 1-v_1 \leq 1)$. There is a C_m-equivariant map

$$\xi : B\Omega X \longrightarrow X, \quad \Big[(\gamma_1, t_1), \ldots, (\gamma_q, t_q), \mathbf{v}\Big] \longmapsto \gamma_q \cdot \gamma_{q-1} \cdot \ldots \cdot \gamma_1 \left(\sum_{i=1}^{q} t_i v_i\right)$$

that is an equivalence if X is connected. This map satisfies the property that

$$\xi\left(\Big[(\overline{\gamma}_1, t_1), \ldots, (\overline{\gamma}_q, t_q), \mathbf{v}\Big]\right) = \xi\left(\Big[(\gamma_q, t_q), \ldots, (\gamma_1, t_1), 1-\mathbf{v}\Big]\right).$$

With all of this in mind, one verifies that the following diagram commutes up to homotopy

$$\begin{tikzcd}
B((C_m \ltimes \Omega X)^{\mathrm{op}}) \arrow[r, "\iota", "\sim"'] \arrow[d, "B(\mathrm{inv})"] & B(C_m \ltimes \Omega X) \arrow[r, "\beta", "\sim"'] & EC_m \times_{C_m} B\Omega X \arrow[d, "EC_m \times_{C_m} \xi", "\wr"'] \\
B(C_m \ltimes \Omega X) \arrow[r, "\beta", "\sim"'] & EC_m \times_{C_m} B\Omega X \arrow[r, "EC_m \times_{C_m} \xi", "\sim"'] & EC_m \times_{C_m} X.
\end{tikzcd}$$

□

Proof of Proposition 3.1.22. In the notation of the previous section, we let $X = \Sigma^\sigma Y$ now, so that $G = GX = \Omega^\sigma \Sigma^\sigma Y$ as a monoid with anti-involution (i.e. "inv" now means inversion in the monoid G followed by the C_2-action on $X = \Sigma^\sigma Y$). For each $m \geq 1$, let us write $\varrho : GL_m(Q_+ G) \simeq \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ for u^{-1} (meaning u as a wrong way equivalence); it should be thought of as given by the rule that sends a matrix $h = (h_{ij}) \in \mathrm{Map}(\mathbb{S}, \mathbb{S}[G])^{m \times m} \cong \mathrm{Map}_G(\mathbb{S}[G], \mathbb{S}[G])^{m \times m}$ to

$$\varrho(h) : \textstyle\bigoplus_{j=1}^m \mathbb{S}[G] \xrightarrow{\bigoplus_j \left(\bigoplus_i h_{ij}\right)} \bigoplus_{i=1}^m \mathbb{S}[G].$$

The map θ of (3.1.19) is constructed in several steps in [CCGH87, §1], each of which we now upgrade to the homotopy C_2-equivariant setting. As these homotopy C_2-actions will get mixed up with strict C_m-actions, it will be more convenient and clear, at least throughout the first few steps, to avoid speaking about “homotopy equivariance” and rather regard a homotopy involution as what it is, i.e. a map whose square happens to be homotopic to the identity.

Step 1. For each $m \geq 2$, consider the D_m-equivariant map of spaces

$$\widetilde{\theta}_{m,1} : Y^{\times m} \longrightarrow GL_m(Q_+G), \quad (y_1, \dots, y_m) \longmapsto \begin{pmatrix} 1 & y_1 - 1 & & & & \\ & 1 & y_2 - 1 & & & \\ & & 1 & \ddots & & \\ & & & & \ddots & \\ & & & & 1 & y_{m-1} - 1 \\ y_m - 1 & & & & & 1 \end{pmatrix}$$

where, if $D_m := \langle s, r \mid s^m = r^2 = rsrs = e \rangle$, the notation is as follows:

- A point $y \in Y$ is identified in G with the path $\eta(y) := (t \mapsto t \wedge y) \in G$, which is itself identified with a point in $\{1\} \times QG \subset QS^0 \times QG \simeq Q_+G$. Then $y - 1$ is the corresponding point in $\{0\} \times QG \subset Q_+G$. Here the n-th component of QS^0 has been fixed a basepoint $n \in QS^0$.
- The action of D_m on $(y_1, \dots, y_m) \in Y^{\times m}$ is given by

  $$s \cdot (y_1, \dots, y_m) := (y_m, y_1, \dots, y_{m-1}), \quad r \cdot (y_1, \dots, y_m) := (y^*_{m-1}, y^*_{m-2}, \dots, y^*_1, y^*_m),$$

  where $y \mapsto y^*$ denotes the C_2-action on Y.
- Let $S, R \in GL_m(\mathbb{Z})$ be the permutation matrices that send the i-th unit vector e_i to e_{i+1} and e_{m+1-i} (with subindexes taken modulo m), respectively. Then D_m acts on $A \in GL_m(Q_+G)$ by

  $$s \cdot A := SAS^{-1}, \qquad r \cdot A := RA^{\dagger}R.$$

  In other words, r acts by transposition along the “$x = y$”-axis together with conjugation on G.

We also define $\widetilde{\theta}_{1,1} : Y \to GL_1(Q_+G) = (Q_+G)^{\times}$ by sending $y \in Y$ to $y \in \{1\} \times QG \subset Q_+G$. We note that $\widetilde{\theta}_{m,1}(y_1, \dots, y_m)$ is homotopy invertible by choosing a path from each of the y_i's to the basepoint $* \in Y$. Then for $m \geq 1$, define $\theta_{m,1}$ as the composite

$$\theta_{m,1} : Y^{\times m} \xrightarrow{\widetilde{\theta}_{m,1}} GL_m(Q_+G) \xrightarrow[\sim]{\varrho} \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]).$$

By construction, $\theta_{m,1}$ is a C_m-map as $\widetilde{\theta}_{m,1}$ and u are. Recall that $s \in C_m \subset D_m$ acts on $\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ by conjugation with $S \in \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$, which we denote by $S_{\#}$.

Step 2. Recall that the free $\mathbb{E}_1$-algebra on a based, connected space X is naturally equivalent to $\Omega\Sigma X$. Therefore, we can extend $\theta_{m,1}$ to a C_m-equivariant $\mathbb{E}_1$-map

$$\theta_{m,2} : \Omega\Sigma(Y^{\times m}) \longrightarrow \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]).$$

For any based space X, let us write $\sigma : \Omega X \to (\Omega X)^{\mathrm{op}}$ for inversion in ΩX (i.e. reversing the loop direction). Given a C_m-space X, we will denote $X^{\mathrm{op}_{C_m}}$ for X with the opposite C_m-action (i.e. that in which s acts by s^{-1}, which is a valid left action as C_m is abelian). If X additionally has an $\mathbb{E}_1$-structure, we will write $X^{\mathrm{op},\mathrm{op}_{C_m}}$ for X with both the opposite $\mathbb{E}_1$-structure and the opposite C_m-action. Then the square of $\mathbb{E}_1$-maps

$$\begin{tikzcd}
\Omega\Sigma(Y^{\times m}) \arrow[r, "\theta_{m,2}"] \arrow[d, "\sigma\circ r"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "R_{\#}\circ\tau_\epsilon"] \\
\Omega\Sigma(Y^{\times m})^{\mathrm{op},\mathrm{op}_{C_m}} \arrow[r, "\theta_{m,2}^{\mathrm{op},\mathrm{op}_{C_m}}"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op},\mathrm{op}_{C_m}}
\end{tikzcd} \tag{3.1.20}$$

commutes in the homotopy category of $\mathbb{E}_1$-spaces with a C_m-action. Here $r : \Omega\Sigma(Y^{\times m}) \to \Omega\Sigma(Y^{\times m})^{\mathrm{op}_{C_m}}$ is induced by the action of $r \in D_m$ on $Y^{\times m}$ together with the flip of the suspension coordinate, and τ_ϵ really stands for (3.1.16). To see this, consider the diagram

$$\begin{tikzcd}
Y^{\times m} \arrow[rr, "\widetilde{\theta}_{m,1}"] \arrow[dr, "r"] \arrow[dd, "\sigma\circ r\circ\eta"'] & & GL_m(Q_+G) \arrow[r, "\varrho", "\sim"'] \arrow[d, "r=R_{\#}\circ\dagger"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[dd, "R_{\#}\circ\tau_\epsilon"] \\
& (Y^{\times m})^{\mathrm{op}_{C_m}} \arrow[r, "\widetilde{\theta}_{m,1}^{\mathrm{op}_{C_m}}"] \arrow[dl, "\eta^{\mathrm{op}_{C_m}}"'] & GL_m(Q_+G)^{\mathrm{op}_{C_m}} \arrow[dr, "\varrho^{\mathrm{op}_{C_m}}", "\sim"'] & \\
\Omega\Sigma(Y^{\times m})^{\mathrm{op}_{C_m}} \arrow[rrr, "\theta_{m,2}^{\mathrm{op}_{C_m}}"] & & & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}_{C_m}}
\end{tikzcd} \tag{3.1.21}$$

of C_m-spaces. By definition, $\theta_{m,2}$ is the $\mathbb{E}_1$-map induced from the top horizontal composite in (3.1.21). Thus, in order to show that (3.1.20) homotopy commutes as C_m-equivariant $\mathbb{E}_1$-maps, it suffices to show that the outer square of (3.1.21) commutes in the homotopy category of C_m-spaces. But each of its subsquares/triangles commute in this category: indeed the lower subsquare does so by definition of $\theta_{m,2}$ (after applying $(-)^{\mathrm{op}_{C_m}}$), the left subtriangle and the upper subsquare too by an easy check, and the right subsquare by Proposition 3.1.19 and the observation that $R_{\#} \circ u = u^{\mathrm{op}_{C_m}} \circ R_{\#}$.

Step 3. The C_m-equivariant $\mathbb{E}_1$-map $\theta_{m,2}$ gives rise to an $\mathbb{E}_1$-map

$$\theta_{m,3} : C_m \ltimes \Omega\Sigma(Y^{\times m}) \xrightarrow{C_m \ltimes \theta_{m,2}} C_m \ltimes \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \xrightarrow{\ \mu\ } \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]),$$

where $\mu(s^i, h) := S^i h$. Observe that μ is indeed an $\mathbb{E}_1$-map as

$$\mu((s^i, h) \cdot (s^j, h')) = \mu(s^{i+j}, S^{-j} h S^j h') = S^i h S^j h' = \mu(s^i, h)\mu(s^j, h').$$

Now from the homotopy commutativity of (3.1.20), it immediately follows that the left subsquare in

$$\begin{tikzcd}
C_m \ltimes \Omega\Sigma(Y^{\times m}) \arrow[r, "C_m \ltimes \theta_{m,2}"] \arrow[rr, bend left, "\theta_{m,3}"] \arrow[d, "C_m \ltimes (\sigma \circ r)"] & C_m \ltimes \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[r, "\mu"] \arrow[d, "C_m \ltimes (R_{\#} \circ \tau_\epsilon)"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "R_{\#} \circ \tau_\epsilon"] \\
C_m \ltimes \Omega\Sigma(Y^{\times m})^{\mathrm{op}, \mathrm{op}_{C_m}} \arrow[r, "C_m \ltimes \theta_{m,2}^{\mathrm{op}, \mathrm{op}_{C_m}}"] & C_m \ltimes \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}, \mathrm{op}_{C_m}} \arrow[r, "\mu^{\mathrm{op}}"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}}
\end{tikzcd} \tag{3.1.22}$$

commutes in the homotopy category of $\mathbb{E}_1$-spaces. Here $\mu^{\mathrm{op}}(s^i, h) := h S^i$, and since $\tau_\epsilon(S) = S^\dagger = S^{-1}$ and $RS = S^{-1} R$, it easily follows that the right subsquare also commutes as $\mathbb{E}_1$-maps. So the outer square of (3.1.22) commutes in the homotopy category of $\mathbb{E}_1$-spaces.

But given an $\mathbb{E}_1$-space X equipped with a C_m-action, there is an isomorphism of $\mathbb{E}_1$-spaces

$$\alpha : C_m \ltimes X^{\mathrm{op}, \mathrm{op}_{C_m}} \xrightarrow{\cong} (C_m \ltimes X)^{\mathrm{op}}, \quad (s^i, x) \longmapsto (s^i, s^{-i} \cdot x).$$

Under this identification, the lower horizontal composite of (3.1.22) becomes $\theta^{\mathrm{op}}_{m,3}$, and hence

$$\begin{tikzcd}
C_m \ltimes \Omega\Sigma(Y^{\times m}) \arrow[r, "\theta_{m,3}"] \arrow[d, "\alpha \circ (C_m \ltimes (\sigma \circ r))"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "R_{\#} \circ \tau_\epsilon"] \\
(C_m \ltimes \Omega\Sigma(Y^{\times m}))^{\mathrm{op}} \arrow[r, "\theta^{\mathrm{op}}_{m,3}"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}}
\end{tikzcd} \tag{3.1.23}$$

is commutative in the homotopy category of $\mathbb{E}_1$-spaces.

Step 4. We wish to deloop (3.1.23), viewing the vertical maps as anti-involutions of their respective domains, and appealing to Lemma 3.1.6 to do so. But by Lemma 3.1.24, the delooping of the anti-involution $\alpha \circ (C_m \ltimes (\sigma \circ r))$ is homotopic to the delooping of the involution $\mathrm{inv} \circ \alpha \circ (C_m \ltimes (\sigma \circ r))$, where inv stands for inversion in the $\mathbb{E}_1$-space $C_m \ltimes \Omega\Sigma(Y^{\times m})$. It is given explictly by $\mathrm{inv}(s^i, \gamma) := (s^{-i}, s^i \cdot \sigma(\gamma))$, and hence we see that

$$\mathrm{inv} \circ \alpha \circ (C_m \ltimes (\sigma \circ r)) : (s^i, \gamma) \longmapsto (s^{-i}, r \cdot \gamma).$$

We denote this map simply by $\mathrm{inv} \ltimes r$. From now on we treat r as the action map on the D_m-space $\Sigma^\sigma(Y^{\times m})$, where σ is seen as a D_m-representation on which C_m acts trivially. Putting this together, the delooped version of (3.1.23) yields a homotopy commutative square of spaces

$$\begin{tikzcd}
B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \arrow[r, "B(\theta_{m,3})"] \arrow[d, "B(\mathrm{inv}\ltimes r)"] & B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "\overline{B}(R_\# \circ \tau_\epsilon)"] \\
B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \arrow[r, "B(\theta_{m,3})"] & B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]),
\end{tikzcd}$$

where the notation $\overline{B}(-)$ stands for delooping in the sense of Lemma 3.1.6.

To simplify the terms in this last diagram, first observe that as $Y^{\times m}$ is connected, we have

$$B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \simeq ED_m \times_{C_m} B\Omega\Sigma^\sigma(Y^{\times m}) \simeq ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}).$$

The inversion on C_m coincides with the residual $C_2 = D_m/C_m$-action on C_m by conjugation, which explains why we chose to write ED_m instead of EC_m. As for the right hand side, note that $R_\#$ is an inner automorphism of $\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$, and hence it induces a map homotopic to the identity on the classifying space level [AM04, §II, Thm. 1.9]. But delooping is functorial, so $\overline{B}(R_\# \circ \tau_\epsilon)$ and $\overline{B}(\tau_\epsilon) =: \tau_\epsilon$ are homotopic involutions on $B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$. All together, we obtain a homotopy C_2-equivariant map

$$\theta_{m,4} : ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}) \xrightarrow{B(\theta_{m,3})^+} B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^+ \subset \mathbb{Z} \times B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^+ \to A(\Sigma^\sigma Y; \epsilon)$$

where the last map is the passage to the colimit as $m \to \infty$ (see (3.1.15)).

Step 5. The following diagram commutes up to homotopy:

$$\begin{tikzcd}
\theta_{m,4} : ED_m/C_m \simeq BC_m \arrow[r, "Bj"] \arrow[d] & B\mathrm{Aut}_G(\oplus^m \mathbb{S}) \arrow[r] \arrow[d] & A(*; \epsilon) \arrow[d] \\
\theta_{m,4} : ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}) \arrow[r] & B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[r] & A(\Sigma^\sigma Y; \epsilon),
\end{tikzcd}$$

where $j : C_m \to \mathrm{Aut}_G(\oplus^m \mathbb{S})$ is the inclusion of the permutation automorphisms. As $A(-;\epsilon) = \Omega^\infty \mathbf{A}(-;\epsilon)$, we can adjoin the $\Omega^\infty(-)$ to get a similar homotopy commutative diagram of spectra. Then passing to vertical cofibres and noting that C_m acts trivially on the suspension coordinate of $\Sigma^\sigma(Y^{\times m})$, we get a homotopy C_2-equivariant map of spectra

$$\theta_{m,5} : \Sigma^{\infty+\sigma}\Big((ED_m)_+ \wedge_{C_m} Y^{\times m}\Big) \longrightarrow \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon).$$

Step 6. Now by [CCGH87, Lem. 1.4] (see also [CC87, Lem. 2.4]), the obvious projection $(ED_m)_+ \wedge_{C_m} Y^{\times m} \to (ED_m)_+ \wedge_{C_m} Y^{\wedge m}$ has a stable section $\Sigma^\infty\Big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\Big) \to \Sigma^\infty\Big((ED_m)_+ \wedge_{C_m} Y^{\times m}\Big)$ that is D_m/C_m-equivariant. This observation gives rise to a homotopy C_2-equivariant map of spectra

$$\boldsymbol{\theta}_{m,6} : \Sigma^{\infty+\sigma}\Big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\Big) \longrightarrow \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon).$$

Finally set $\boldsymbol{\theta}$ to be

$$\boldsymbol{\theta} : \bigvee_{m\geq 1} \Sigma^{\infty+\sigma}\Big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\Big) \xrightarrow{\bigvee_{m\geq 1}(\boldsymbol{\theta}_{m,6})} \bigvee_{m\geq 1} \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon) \twoheadrightarrow \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon).$$

This map is homotopy C_2-equivariant by construction, and non-equivariantly yields (3.1.19) after applying $\Omega^{\infty+\sigma}(-)$. This latter map is an equivalence of infinite loop spaces by [CCGH87, Thm. 1.6], and as both the domain and codomain of $\boldsymbol{\theta}$ are 1-connective, it follows that $\boldsymbol{\theta}$ is itself an equivalence of spectra. This concludes the proof of Proposition 3.1.22. □

Corollary 3.1.25. *Let Y be a connected, based C_2-space. Denote by $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(-)$ the homotopy fibre* $\mathrm{hofib}(\mathbf{Wh}^{\mathrm{Diff}}(-) \to \mathbf{Wh}^{\mathrm{Diff}}(*))$*. Then there is an equivalence of spectra*

$$\boldsymbol{\theta} : \bigvee_{m\geq 2} \Sigma^{\infty+\sigma}\Big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\Big) \xrightarrow{\sim} \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma Y;\epsilon)$$

that is C_2-equivariant up to homotopy.

Proof. It is clear from the construction that the map

$$\Sigma^\infty(\Sigma^\sigma Y) \simeq \Sigma^{\infty+\sigma}\Big((ED_1)_+ \wedge_{C_1} Y^{\wedge 1}\Big) \xrightarrow{\boldsymbol{\theta}_{1,6}} \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon)$$

is the (reduced version of the) usual inclusion of the stable homotopy into A-theory. Thus its cofibre is $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma Y;\epsilon)$, and the claim follows immediately. □

Remark 3.1.26. As the reader may have noticed by now, the last two sections are a tiny bit technical, and one may wonder if there could be alternative approaches to deal with them. Such an approach that may come to mind is to use trace methods to analyse $\widetilde{A}(\Sigma^\sigma Y;\epsilon)$, since it coincides (non-equivariantly) with the reduced TC of $\mathbb{S}[\Omega\Sigma Y]$ (as Y is connected). In fact, recent developments have been made towards a (genuine) C_2-equivariant version of topological cyclic homology for ring spectra with anti-involutions, commonly known as *real topological cyclic homology* (cf. [Hø16, HM16, DMP21]). This approach has two caveats:

- A *real cyclotomic trace* map does not yet exist (at the time of writing). The construction of such a map was supposed to appear in [HM16], but it never saw the light in the end. This is, nevertheless, current work in progress by Harpaz–Nikolaus–Shah [HNS21, p. 24].
- Even though much is known about the p-complete homotopy type of the TC of spherical group rings (cf. [BCC$^+$96] or [NS18, §4.3]), the analysis of its integral homotopy type does not seem to be present in the literature.

For these two reasons, we preferred to proceed as we have.

3.2 The homotopy type of spaces of long knots

This section is devoted to Theorem D, which describes the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$ for $p \leq d-3$ and $d \geq 5$, localised at odd primes and up to the concordance embedding stable range $\phi_{C\mathrm{Emb}}(d,p)$. After its proof, which will not take too much effort given the results in the preceeding sections, we will draw some conclusions on the homotopy groups of spaces of long knots. For convenience let us recall the statement of Theorem D. Recall that ψ_m stands for the real m-dimensional permutation representation of the dihedral group D_m and σ for the sign representation, regarded as a D_m-representation by restricting along the determinant $D_m \hookrightarrow O(2) \overset{\det}{\to} \{\pm 1\} = C_2$.

Theorem (Theorem D). *For $p \leq d-3$ and $d \geq 5$, consider the virtual D_m-representations*

$$\rho_m := (d+1)(\sigma - 1) + \psi_m \otimes (d-p-3+\sigma).$$

Then the homotopy fibre sequence (3.2.1), upon localising away from 2 and taking $(\phi_{C\mathrm{Emb}}(d,p)-1)$-th Postnikov sections, takes the form

$$\textstyle\prod_{m\geq 2} \Omega^\infty\Big(\mathbb{S}^{\rho_m}_{hD_m}\Big) \longrightarrow \mathrm{Emb}_\partial(D^p, D^d) \longrightarrow \Omega^p\,\mathrm{hofib}\Big(G(d-p)/O(d-p) \to G/O\Big).$$

The resulting sequence is split if $p \geq 2$*, and splits after being looped once if* $p = 1$*.*

Recall from Remark 2.1.5(*ii*) what we mean by localising the spaces $\mathrm{Emb}_\partial(D^p, D^d)$ and $\widetilde{\mathrm{Emb}}_\partial(D^p, D^d)$.

3.2.1 Proof of Theorem D

Recall from (2.1.4) that $\mathrm{Emb}_\partial^{(\sim)}(P, M)$ denotes $\mathrm{hofib}_\iota(\mathrm{Emb}_\partial(P, M) \to \widetilde{\mathrm{Emb}}_\partial(P, M))$ when ι is clear from the context. We saw in Corollary 2.4.3 that the fibration sequence

$$\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \longrightarrow \mathrm{Emb}_\partial(D^p, D^d) \longrightarrow \widetilde{\mathrm{Emb}}_\partial(D^p, D^d), \tag{3.2.1}$$

upon localising at odd primes and taking $(\phi_{C\mathrm{Emb}}(d, p) - 1)$-th Postnikov sections, is split for $2 \leq p \leq d - 3$, and splits for $p = 1$ after looping once. So we need to describe the exterior terms of (3.2.1) after inverting 2.

For the block embeddings, the graphing map

$$\Gamma : \Omega^p \widetilde{\mathrm{Emb}}(*, D^{d-p}) \xrightarrow{\sim} \widetilde{\mathrm{Emb}}_\partial(D^p, D^{d-p} \times D^p) \cong \widetilde{\mathrm{Emb}}_\partial(D^p, D^d) \tag{3.2.2}$$

of (2.4.1) is an equivalence by inspection. Then by [GKW01, Thm. 2.2.1] and the example right after it, when $d - p \geq 3$ and $d \geq 5$ (see Remark 3.2.1 below), it follows that

$$\begin{aligned} \widetilde{\mathrm{Emb}}_\partial(D^p, D^d) &\simeq \Omega^p \, \mathrm{hofib}(O/O(d-p) \to G/G(d-p)) \\ &\simeq \Omega^p \, \mathrm{hofib}(G(d-p)/O(d-p) \to G/O), \end{aligned} \tag{3.2.3}$$

yielding the base of (3.2.1).

Remark 3.2.1. The equivalence (3.2.3) is only valid if $d - p \geq 3$ and $d \geq 5$. As pointed out right after [GKW01, Thm. 2.2.1], the second condition is not that important. For instance in the case $p = 1$ and $d = 4$, it follows directly from (3.2.2) and (3.2.3) that

$$\Omega \widetilde{\mathrm{Emb}}_\partial(D^1, D^4) \simeq \widetilde{\mathrm{Emb}}_\partial(D^2, D^5) \simeq \Omega^2 \, \mathrm{hofib}(G(3)/O(3) \to G/O).$$

The codimension condition $d - p \geq 3$, however, is essential.

For the fibre of (3.2.1), we know by Theorem C that for $N = \phi_{C\mathrm{Emb}}(d, p) - 1$, there is an equivalence

$$\tau_{\leq N} \, \mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \simeq \tau_{\leq N} \, \Omega^\infty \left(\mathbf{CE}(D^p, D^d)_{hC_2} \right).$$

We now use Corollaries 3.1.17 and 3.1.25 to describe the right hand side of the equivalence above.

Proposition 3.2.2. *For ρ_m as in Theorem D, there is an equivalence*

$$\Omega^\infty\left(\mathbf{CE}(D^p, D^d)_{hC_2}\right) \simeq_{[\frac{1}{2}]} \prod_{m\geq 2} \Omega^\infty\left(\mathbb{S}^{\rho_m}_{hD_m}\right).$$

Proof. First observe that there is a homotopy C_2-equivariant equivalence $S^2 \approx S^{2\sigma}$. So by Corollary 3.1.25, for each $n \geq 2$ there is a homotopy C_2-equivariant equivalence of spectra

$$\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^n;\epsilon) \approx \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma S^{n-2+\sigma};\epsilon) \approx \bigvee_{m\geq 2} \Sigma^{\infty+\sigma}\left(S^{\psi_m\otimes(n-2+\sigma)}_{hC_m}\right)$$

Using this for $n = d-p-1 \geq 2$, we obtain a chain of equivalences

$$\begin{aligned}\Omega^\infty\left(\mathbf{CE}(D^p, D^d)_{hC_2}\right) &\simeq_{[\frac{1}{2}]} \Omega^\infty\left((S^{d\cdot(\sigma-1)-1} \wedge \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^{d-p-1}))_{hC_2}\right)\\ &\simeq_{[\frac{1}{2}]} \Omega^\infty\Big(\Big(\bigvee_{m\geq 2} \mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathbb{S}^{\psi_m\otimes(n-2+\sigma)}_{hC_m}\Big)_{hC_2}\Big).\\ &= \prod_{m\geq 2} \Omega^\infty\left(\mathbb{S}^{\rho_m}_{hD_m}\right).\end{aligned}$$

The first equivalence follows from Corollary 3.1.17, together with the observation that $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^{d-p-1}) \simeq \Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(D^p, S^{d-p-1};\epsilon)$ as both τ_ϵ and $\mathbf{Wh}^{\mathrm{Diff}}(-)$ are homotopy invariants of $(-)$. The second equivalence is a consequence of the previous argument and Corollary 3.1.4. This establishes the desired equivalence. □

All together, this concludes the proof of Theorem D. □

Remark 3.2.3 (Topological version of Theorem D). The space $\mathrm{Emb}^{\mathrm{Top}}_\partial(D^p, D^d)$ of topological long knots is contractible (for all $p \leq d$) by the Alexander trick. We could still be interested in the homotopy type of the space $\mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d)$ of thickened topological long knots with $p \leq d-3$, and one can get a description of it localised away from 2 and up to the concordance embedding stable range, similar to the one in Theorem D—let us explain how. As before, we have a homotopy fibre sequence

$$\mathrm{Emb}^{\mathrm{Top},(\sim)}_{\partial_0}(D^p \times D^{d-p}, D^d) \longrightarrow \mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d) \longrightarrow \widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d)$$

which, upon localising at odd primes and taking $(\phi_{C\mathrm{Emb}}(d,p)-1)$-th Postnikov sections, is split for $2 \leq p \leq d-3$, and splits for $p=1$ after looping once. So we should describe the side terms.

For the block embeddings, consider the space

$$B\widetilde{\mathrm{Top}}(q) := \operatorname{holim} \left(\begin{array}{ccc} & & B\mathrm{Top} \\ & & \downarrow \\ BG(q) & \longrightarrow & BG \end{array} \right)$$

which is responsible for the classification of topological block normal bundles if $q \geq 3$ (cf. [RS68a, §2], [RS68b] and [Wal99, §11]). As $\widetilde{\mathrm{Emb}}_\partial^{\mathrm{Top}}(D^p, D^d)$ is contractible by the Alexander trick, it then follows that

$$\widetilde{\mathrm{Emb}}_{\partial_0}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d) \simeq \mathrm{Map}_\partial(D^p, \widetilde{\mathrm{Top}}(d-p)) = \Omega^p \widetilde{\mathrm{Top}}(d-p).$$

As for the pseudoisotopy embeddings, the topological version of Theorem C (see Remark 2.1.3) tells us that for $N = \phi_{C\mathrm{Emb}(d,p)} - 1$, there is an equivalence

$$\tau_{\leq N} \operatorname{Emb}_{\partial_0}^{\mathrm{Top},(\sim)}(D^p \times D^{d-p}, D^d)) \simeq \tau_{\leq N}\, \Omega^\infty \left(\mathbf{CE}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d)_{hC_2} \right), \tag{3.2.4}$$

where $\mathbf{CE}^{\mathrm{Top}}(P, M)$ stands for the first orthogonal derivative of the topological analogue of $F(-)$ (as in Corollary 3.1.17, this is a C_2-spectrum whose infinite loop space is equivalent to that of $\Sigma^{-2}\mathbf{Wh}^{\mathrm{Top}}(M, M-P; \epsilon)$). Noting that there is fibre sequence of C_2-spectra $\mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon) \to \mathbf{Wh}^{\mathrm{Top}}(M;\epsilon) \to \Sigma\mathbf{Wh}^{\mathrm{Diff}}(*;\epsilon) \wedge M_+$, one verifies that the infinite loop space in the right hand side of (3.2.4) fits in a fibre sequence away from 2

$$\textstyle\prod_{m\geq 2} \Omega^\infty \Big(\mathbb{S}^{\rho_m}_{hD_m} \Big) \longrightarrow \Omega^\infty \left(\mathbf{CE}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d)_{hC_2} \right) \longrightarrow \Omega^\infty \Big((S^{d\cdot\sigma - p - 2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(*;\epsilon))_{hC_2} \Big).$$

In particular, it is easy to check (e.g. rationally) that the left hand side of (3.2.4) is not contractible (at least if d is sufficiently large). This was claimed in Remark 2.3.4.

3.2.2 On the homotopy groups of spaces of long knots

We can get plenty of information about the homotopy groups of $\mathrm{Emb}_\partial(D^p, D^d)$ from Theorem D. First observe that by Morlet's lemma of disjunction [BLR75, Thm. 3.1] (and Proposition 2.3.3 to reduce to the codimension zero case), the pseudoisotopy embedding space $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ is at least $(2(d-p-2)-1)$-connected. So by (3.2.3), it follows that

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d)) \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O)), \quad * < 2(d-p-2). \tag{3.2.5}$$

Remark 3.2.4. This should be compared to work of Budney [Bud08, Prop. 3.9]. The main result there is the computation of the first non-trivial homotopy group of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d-p \geq 3$, which lies in degree $2d-3p-3$, together with a geometric interpretation of the generators. From our point of view, he shows that $\mathrm{hofib}(G(d-p)/O(d-p) \to G/O)$ is exactly $(2d-2p-4)$-connected, which follows by work of Haefliger (see Section 3, Equation 4.11 and Corollary 6.6 of [Hae66]), and computes $\pi_{2d-2p-3}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))$. Moreover, it is stated in [Bud08, Prop. 3.9(1)] that the graphing map

$$\Gamma : \pi_*(\Omega \,\mathrm{Emb}_\partial(D^{p-1}, D^{d-1})) \longrightarrow \pi_*(\mathrm{Emb}_\partial(D^p, D^d))$$

is surjective for $* \leq 2d-2p-5$. From (3.2.3) and the fact that $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ is $(2d-2p-5)$-connected, we see that it is in fact an isomorphism.

Recall that $\phi_{C\mathrm{Emb}}(d,p) \geq 2d-p-5$ by work of Goodwillie–Krannich–Kupers [GKK23], and so the space $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ has interesting homotopy in degrees from $2d-2p-4$ up to that range that we can understand by Theorem D. For any odd prime ℓ there are isomorphisms in degrees $* \leq \phi_{C\mathrm{Emb}}(d,p) - 1$

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)} \oplus \bigoplus_{m \geq 2} \pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)},$$

and if $\phi = \phi_{C\mathrm{Emb}}(d,p)$, there is also an exact sequence of abelian groups

$$\textstyle\bigoplus_{m\geq 2} \pi_\phi^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \longrightarrow \pi_\phi(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \twoheadrightarrow \pi_{\phi+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)},$$

where $A_{(\ell)}$ denotes $A \otimes \mathbb{Z}_{(\ell)}$, for A an abelian group. It remains to understand the groups $\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}$, which are easier to study when ℓ is coprime to m.

Proposition 3.2.5. *Let $m \geq 2$ and $d-p \geq 3$. For $\ell \nmid 2m$ a prime,*

- *if d is even and p is even, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 3, 5, 7, \ldots \\ 0, & \textit{otherwise.} \end{cases}$$

- *if d is odd and p is odd, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 2, 4, 6, \ldots \\ 0, & \textit{otherwise.} \end{cases}$$

- *if d is even and p is odd, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 5, 9, 13, \ldots \\ 0, & \textit{otherwise.} \end{cases}$$

- *if d is odd and p is even, then*

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 3, 7, 11, \ldots \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 3.2.6 (Rational homotopy of spaces of long knots). The rational homology and homotopy of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d - p \geq 3$ has been extensively studied in recent years (see e.g. [Tur10, AT14, AT15]) through the lens of embedding calculus and its relation to the little disks operads and their formality, finally culminating in the work of Fresse–Turchin–Willwacher [FTW17]. There they compute the rational homotopy groups of $\overline{\mathrm{Emb}}_\partial(D^p, D^d) := \mathrm{hofib}_\iota(\mathrm{Emb}_\partial(D^p, D^d) \to \mathrm{Imm}_\partial(D^p, D^d))$ as the homology of the *hairy graph complex* (shifted appropriately). Observationally, our results correspond to the 0- and 1-loop order parts of this graph complex up to degree $\phi_{C\mathrm{Emb}}(d, p) \geq 2d-p-5$, where higher loop orders are still not seen. More precisely, the 0-loop part corresponds to the rational homotopy of $G/G(d-p)$, the lowest summand (i.e. $m = 1$ when $d-p$ is even) of the 1-loop part appears as that of $O/O(d-p)$, and the higher summands of the 1-loop part come from the rational homotopy of the spectra $\mathbb{S}^{\rho_m}_{hD_m}$ for $m \geq 2$, which we just computed. It is worth noting that:

- The first non-trivial rational homotopy group of $\mathrm{Emb}_\partial(D^p, D^d)$ coming from the 2-loop part of the hairy graph complex lies in degree $2d-p-4$ when both d and p are odd (cf. [FTW17, Eq. 3]). Therefore, the lower bound $\phi_{C\mathrm{Emb}}(d, p) \geq 2d-p-5$ on the concordance stable range of Goodwillie–Krannich–Kupers is quite sharp.
- The 1-loop part of the hairy graph complex seems to be completely generated by the spectra $\mathbb{S}^{\rho_m}_{hD_m}$ for $m \geq 2$ in all degrees outside of the concordance embedding stable range. In other words, the computations in [FTW17] give evidence for the

existence of a rational left splitting of the Weiss–Williams map

$$\Phi^{\mathrm{Emb}} : \mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \longrightarrow \prod_{m\geq 2} \Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m}).$$

To investigate this, one should first understand the attachment of the second orthogonal derivative $\Theta F^{(2)}$ of the orthogonal functor $F(U) := \mathrm{Emb}^b_\partial(D^p \times U, D^d \times U)$ to its Taylor tower (2.2.1). It would also be interesting to understand the integral picture.

Proof of Proposition 3.2.5. When ℓ is coprime to $2m$, we have that

$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong H_0(D_m; \pi_*^s(\mathbb{S}^{\rho_m}))_{(\ell)}$$

by the homotopy fixed point spectral sequence, because the higher group homology of D_m is $2m$-torsion. Let t and r denote the generators of D_m with $t^m = e$ and $rtr = t^{-1}$ such that

$$\begin{aligned}\psi_m(t) &: \mathbb{R}^m \ni (a_1, \dots, a_m) \longmapsto (a_m, a_1, \dots, a_{m-1}),\\ \psi_m(r) &: \mathbb{R}^m \ni (a_1, \dots, a_m) \longmapsto (a_m, a_{m-1}, \dots, a_1).\end{aligned}$$

Then t and r act on the group $\pi_*^s(\mathbb{S}^{(d+1)(\sigma-1)+\psi_m\otimes(d-p-3+\sigma)})$ by $(-1)^{\epsilon_t}$ and $(-1)^{\epsilon_r}$, respectively, where

$$\epsilon_t = (m-1)(d-p-2), \qquad \begin{aligned}\epsilon_r &= \underbrace{d+1}_{(1)} + \underbrace{\frac{1}{2}m(m-1)(d-p-3)}_{(2)} + \underbrace{m + \frac{1}{2}m(m-1)}_{(3)}\\ &\equiv d+1+m+\frac{1}{2}m(m-1)(d-p-2) \mod 2.\end{aligned}$$

The terms (1), (2) and (3) are the contributions coming, respectively, from the summands $(d+1)(\sigma-1)$, $\psi_m \otimes (d-p-3)$ and $\psi_m \otimes \sigma$ of ρ_m. One then readily verifies that the groups $H_0(D_m; \pi_*^s(\mathbb{S}^{\rho_m}))$ are given by the fomulae in the statement. □

A bit more interesting are the homotopy groups $\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}$ when ℓ is odd but divides m. We treat the case when $\ell = m = 3$, which hopefully serves as a sample computation for other cases.

Proposition 3.2.7. *The first few homotopy groups $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ of the spectrum $\mathbb{S}^{\rho_3}_{hD_3}$, localised at 3 and when $d-p=3$, are given in Table 3.1. Equally coloured groups in this table correspond to the same case depending on whether certain differentials* *in Figure 3.4 vanish or not. Entries containing "?" correspond to potentially more complicated answers that do not conveniently fit in the table.*

Table 3.1 $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ for $d-p=3$ for low values of $* \geq 3$.

$*$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
p even	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/9$	0	0	0	$\mathbb{Z}/9$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/27$	0	0	$\mathbb{Z}/3$ $\mathbb{Z}/3 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3^4$ $\mathbb{Z}/3^5$	0	?	?	$\mathbb{Z}/9$	$\mathbb{Z}/3$	0
p odd	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/9$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/9 \oplus \mathbb{Z}/3$

Remark 3.2.8. Since the time of writing this article, more extended computations of the groups $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ for $d-p=3$ have become available in the Master's thesis of Andrés Morán Lamas [Lam24]. For instance, he computes that in the "p even" case, the groups in degrees $*=20$ and 21 are both $\mathbb{Z}/3$. He also provides an extended version of Table 3.1 in [Lam24, Tables 3.3 & 3.4], computing most of the groups up to degree $* \leq 39$. We are grateful to him for his enthusiasm in this particular computation.

To prove Proposition 3.2.7, we will need to understand the cohomology of $\mathbb{S}^{\rho_3}_{hC_3}$ as a module over the Steenrod algebra $\mathcal{A}_3$, which we recall is generated by the Steenrod powers P^k and the Bockstein operation β.

Lemma 3.2.9. *The spectrum cohomology of $\mathbb{S}^{\rho_3}_{hC_3}$ is given by*

$$H^*(\mathbb{S}^{\rho_3}_{hC_3};\mathbb{F}_3) \cong \mathbb{F}_3\langle u\rangle \otimes_{\mathbb{F}_3} \mathbb{F}_3[\alpha,s]/(\alpha^2), \qquad |\alpha|=1, \quad |s|=2, \quad |u|=3(d-p-2),$$

with

$$P^k(u\alpha^i s^j) = \left(\sum_{r=0}^{k}\binom{d-p-2}{r}\binom{j}{k-r}\right) u\alpha^i s^{j+2k}, \qquad \beta(u\alpha^i s^j) = \begin{cases} 0, & i=0,\\ -us^{j+1}, & i=1.\end{cases}$$

Moreover $C_2 = D_3/C_3$ acts on $H^*(\mathbb{S}^{\rho_3}_{hC_3};\mathbb{F}_3)$ by $u\alpha^i s^j \mapsto (-1)^{p+i+j}u\alpha^i s^j$.

Proof. The key observation to carry out this calculation is that the C_3-representation $\psi_3 \mid_{C_3}$ decomposes as $1+\theta$, where θ is the 2-dimensional representation pulled back from the standard complex $U(1) \cong SO(2)$-representation on $\mathbb{C} \cong \mathbb{R}^2$. In particular, the associated vector bundle of the representation $\psi_m \otimes (d-p-3+\sigma) \mid_{C_m}$ is orientable;

write $u \in H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3)$ for the corresponding Thom class. The $\mathbb{F}_3$-cohomology of BC_3 is $\mathbb{F}_3[\alpha, s]/(\alpha^2)$ with $|\alpha| = 1$, $|s| = 2$ and

$$\beta(\alpha) = s, \qquad P^k(\alpha^i s^j) = \binom{j}{k}\alpha^i s^{j+2k}.$$

So it remains to understand the action of $\mathcal{A}_3$ on u. Clearly $\beta(u) = 0$, as if $\beta(u) = nu\alpha$ for some $n \in \mathbb{F}_3$, then $0 = \beta^2(u) = n^2 u\alpha^2 - nus$ and hence $n = 0$. For u_θ the Thom class of θ, $P^1(u_\theta) = u_\theta^3 = u_\theta c_1(\theta)^2 = u_\theta s^2$. It then easily follows that $P^k(u) = \binom{d-p-2}{k}us^{2k}$.

The residual D_3/C_3-action on the $\mathbb{F}_3$-cohomology of BC_3 sends s to $-s$, and hence α to $-\alpha$. This action also switches the orientation of the vector bundle associated to θ, and hence that of $\psi_3 = 1 + \theta$. So that of $\psi_3 \otimes \sigma$ does not change then, as ψ_3 is odd-dimensional. All in all, this means that u is sent by the C_2-action to $(-1)^\epsilon u$ with $\epsilon = (d+1) + (d-p-3) \equiv p \mod 2$, where the first term is the contribution of $\mathbb{S}^{(d+1)(\sigma-1)}$. This establishes the claim. □

Proof of Proposition 3.2.7. Consider the $\mathcal{A}_3$-submodules of $H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3)$ given by

$$J_0 := \langle u\alpha^i s^j : i + j \equiv 0 \mod 2\rangle, \qquad J_1 := \langle u\alpha^i s^j : i + j \equiv 1 \mod 2\rangle.$$

Then $H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3) = J_0 \oplus J_1$ as $\mathcal{A}_3$-modules, and J_p, where p here is taken mod 2, is the $(+1)$-eigenspace of the residual $C_2 = D_3/C_3$-action. Therefore $H^*(\mathbb{S}^{\rho_3}_{hD_3}; \mathbb{F}_3) = J_p$ as an $\mathcal{A}_3$-module, and the Adams spectral sequence of J_p then converges to the stable homotopy of $\mathbb{S}^{\rho_3}_{hD_3} = (\mathbb{S}^{\rho_3}_{hC_3})_{hC_2}$. The software [CCBFY22] computes the E_2-page of this spectral sequence, from which we determine some of the first homotopy groups of $\mathbb{S}^{\rho_3}_{hD_3}$. We illustrate the case $d - p = 3$ in Figure 3.4.

Apart from standard arguments exploiting the Leibniz rule and multiplicative structures of the differentials in Figure 3.4, we can appeal to some more refined tricks that allow us to solve the first few possible non-zero differentials in the case when p is even. Denote by $\widehat{\theta}$ the D_3-representation pulled back from the standard $O(2)$-representation on $\mathbb{R}^2$. Observe that $\widehat{\theta} \mid_{C_3} \equiv \theta$ in the notation of Lemma 3.2.9. Write $\underline{S}(\widehat{\theta} \otimes \sigma \mid_{C_3})$ for the unit sphere bundle of the associated vector bundle $ED_3 \times_{C_3} (\widehat{\theta} \otimes \sigma)$. As it is an S^1-bundle over BC_3, $\underline{S}(\widehat{\theta} \otimes \sigma \mid_{C_3})$ must itself be a $K(\pi, 1)$, and in fact it must be homotopy equivalent to S^1 because $q : \underline{S}(\widehat{\theta} \otimes \sigma \mid_{C_3}) \to BC_3$ does not admit a section (its euler class is $s \in H^2(BC_3; \mathbb{F}_3) = \mathbb{F}[\alpha, s]/(\alpha^2)$). Moreover the homology class represented by q is the dual of α, and hence the residual $C_2 = D_3/C_3$-action on $\underline{S}(\widehat{\theta} \otimes \sigma \mid_{C_3}) \simeq S^1$ must have degree -1. So there is an equivalence of unbased spaces

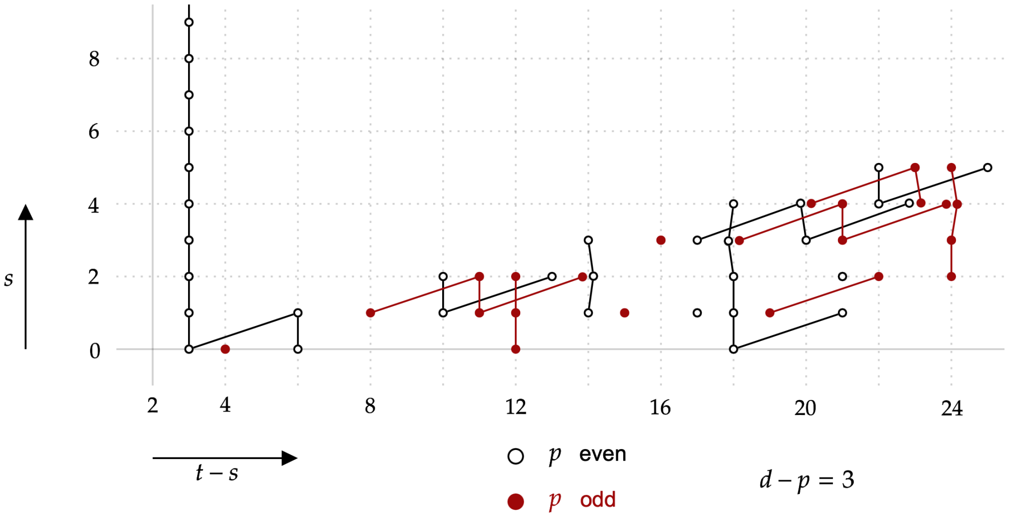

Figure 3.4 E_2-page of the Adams spectral sequence at the prime 3 for $\mathbb{S}^{\rho_3}_{hD_3}$ when $d-p=3$. Here s denotes the degree in the Adams filtration and $t-s$ is the total degree.

$\underline{S}(\widehat{\theta}\otimes\sigma\,|_{C_3})\simeq S^{\sigma}$ which is C_2-equivariant up to homotopy. We thus get a cofibration

$$S^{\sigma}_{+}\simeq\underline{S}(\widehat{\theta}\otimes\sigma\,|_{C_3})_{+}\xrightarrow{q}(BC_3)_{+}\longrightarrow\!\!\!\!\rightarrow \mathrm{Th}(\widehat{\theta}\otimes\sigma\,|_{C_3})\simeq S^{-\sigma}\wedge\mathrm{Th}(\psi_3\otimes\sigma\,|_{C_3})$$

which is C_2-equivariant up to homotopy (see Notation 3.1.2(i)). By equipping both S^{σ} and BC_3 with distinguished basepoints which are fixed under the respective involutions and which match under q, we can get rid of the added basepoints and yield a homotopy cofibre sequence of C_2-spectra

$$\mathbb{S}^{(d+1)(\sigma-1)+2\sigma}\xrightarrow{q}S^{(d+1)(\sigma-1)+\sigma}\wedge\Sigma^{\infty}BC_3\longrightarrow\!\!\!\!\rightarrow\mathbb{S}^{\rho_3}_{hC_3}.$$

Then, upon inverting 2 and taking homotopy C_2-orbits in the sequence above, we obtain equivalences of spectra

$$\mathbb{S}^{\rho_3}_{hD_3}\simeq_{[\frac{1}{2}]}\begin{cases}(\Sigma^{\infty+1}BC_3)_{hC_2}\simeq_{[\frac{1}{2}]}\Sigma^{\infty+1}BD_3, & d\text{ even (so }p\text{ odd)},\\ \mathrm{hocofib}(q_{hC_2}:\mathbb{S}^2\to(S^{\sigma}\wedge\Sigma^{\infty}BC_3)_{hC_2}), & d\text{ odd (so }p\text{ even)}.\end{cases}\tag{3.2.6}$$

Note that the second equivalence in the "d even" case really only holds after inverting 2: by definition, there is an equivalence $(\Sigma^{\infty}_{+}BC_3)_{hC_2}\simeq\Sigma^{\infty}_{+}BD_3$—we need to get rid of the "+"s. Now $\Sigma^{\infty}_{+}BC_3\simeq\mathbb{S}\oplus\Sigma^{\infty}BC_3$ is a C_2-equivariant equivalence, and $\mathbb{S}_{hC_2}=\Sigma^{\infty}_{+}BC_2\simeq\mathbb{S}\oplus\Sigma^{\infty}BC_2$; the first summand $\mathbb{S}$ cancels with that of $\Sigma^{\infty}_{+}BD_3\simeq\mathbb{S}\oplus\Sigma^{\infty}BD_3$, so a copy of $\Sigma^{\infty}BC_2$ remains, which is contractible only upon inverting 2.

Now by the Kahn–Priddy theorem [KP78] at the prime 3, the transfer-like map $\Sigma^{\infty+1}BD_3 \to \tau_{>1}\mathbb{S}^1$ is split surjective on homotopy groups localised at 3, and hence by (3.2.6), the group $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3})_{(3)}$ split surjects onto $\pi^s_{*-1} \otimes \mathbb{Z}_{(3)}$ for $* > 1$ when p is odd. We will use this fact together with knowledge of π_*^s to determine the differentials of the red spectral sequence in Figure 3.4. For convenience, let us reillustrate a different portion of it in Figure 3.5.

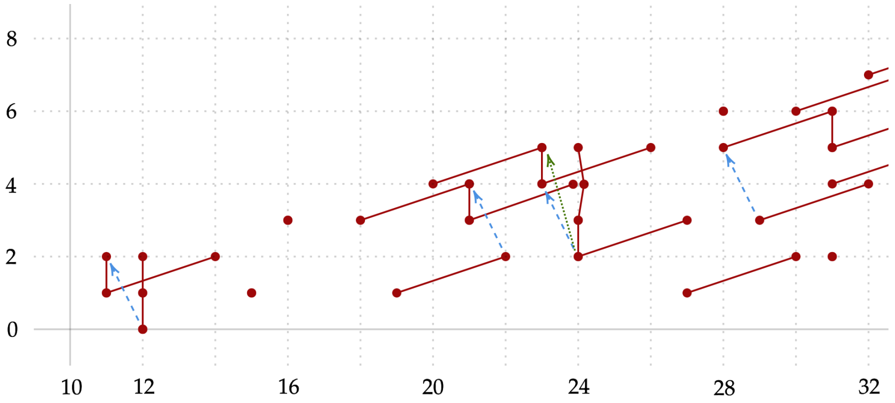

Figure 3.5 E_2-page of Adams spectral sequence at the prime 3 for $\mathbb{S}^{\rho_3}_{hD_3}$ when $d - p = 3$ and p is odd. Some of the d_2 (dashed blue) and d_3 (dotted green) differentials that will be analysed are depicted.

The first possible non-zero differential goes from $(12, 0)$ to $(11, 2)$. Depending on its (non-)vanishing, we must have that either

$$(a) \quad \pi_*^s(\mathbb{S}^{\rho_3}_{hD_3})_{(3)} \cong \begin{cases} \mathbb{Z}/9, & * = 11, \\ \mathbb{Z}/27, & * = 12, \end{cases} \qquad \text{or} \qquad (b) \quad \pi_*^s(\mathbb{S}^{\rho_3}_{hD_3})_{(3)} \cong \begin{cases} \mathbb{Z}/3, & * = 11, \\ \mathbb{Z}/9, & * = 12. \end{cases}$$

Since $\pi^s_{10} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3$, we must rule out possibility (a) as $\mathbb{Z}/9$ does not split surject onto $\mathbb{Z}/3$. In other words, the d_2 differential in Figure 3.4 from $(12, 0)$ to $(11, 2)$ is non-zero and (b) holds.

The d_2 differential from $(22, 2)$ to $(21, 4)$ must be non-zero, and hence so will that from $(19, 1)$ to $(18, 3)$ by the multiplicative structure. Indeed if such differential was zero, it would follow that $\pi^s_{21}(\mathbb{S}^{\rho_3}_{hD_3})_{(3)} \cong \mathbb{Z}/9$, which does not split surject onto $\pi^s_{20} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3$.

The d_2 differential from $(24, 2)$ to $(23, 4)$ must also be non-zero (and hence that from $(24, 3)$ to $(23, 5)$) by a similar reason; indeed if it were trivial, then $\pi^s_{23}(\mathbb{S}^{\rho_3}_{hC_3})_{(3)}$ would be isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/81$ or $\mathbb{Z}/3 \oplus \mathbb{Z}/27$ (depending on whether the d_3 differential from $(24, 2)$ to $(24, 5)$ vanishes or not), either of which do not split surject onto $\pi^s_{23} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3 \oplus \mathbb{Z}/9$.

The arguments we just made above, together with standard ones exploiting the Leibniz rule and the multiplicative structure of the differentials in the spectral sequence in Figure 3.4, establish the homotopy groups appearing in Table 3.1. ☐

One can keep up using the Kahn–Priddy theorem and go quite far up determining all possible non-zero differentials when p is odd; we leave it as a fun exercise to the eager reader. One would also hope that (3.2.6) could be used in the case when p is even, but this approach has somehow been inconclusive for us (at least for the first non-zero differentials that cannot be ruled out by elementary means).

E Appendix: The h-cobordism stabilisation map

The (lower) stabilisation map $\Sigma : H(M) \to H(M \times I)$ of [Vog85, p. 298] is depicted in Figure 3.6 below.

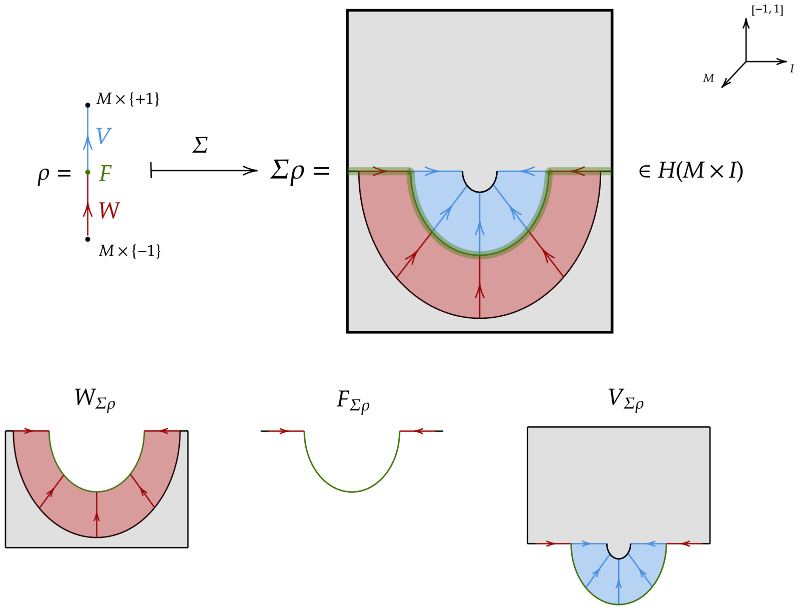

Figure 3.6 The h-cobordism (lower) stabilisation map $\Sigma_\ell = \Sigma : H(M) \to H(M \times I)$, sending $\rho = (W, F, V)$ to $\Sigma\rho := (W_{\Sigma\rho}, F_{\Sigma\rho}, V_{\Sigma\rho})$. A grey shaded region of shape S represents a manifold of the form $M \times S$.

Recall that the h-cobordism space $H(M \times I)$ is an $\mathbb{E}_1$-space under stacking in the I-direction, denoted $+_I$. In this section we argue that Σ anticommutes with the h-cobordism involution ι_H in the following sense:

Lemma E.1 (Lemma 3.1.16(*a*)). *The map $\iota_H \Sigma +_I \Sigma \iota_H : H(M) \to H(M \times I)$ is null-homotopic, i.e. it is homotopic to the constant map at the trivial partition $* \in H(M \times I)$.*

Proof. We describe the null-homotopy in the topological setting; the smooth case is very similar, but one has to be slightly careful with issues regarding corners (which can be overcome by working with the collared version $H_{\mathsf{col}}(M \times I)$ of the h-cobordism space). It will be convenient to work with yet another (upgraded) version of the h-cobordism space: let $\overline{H}_{\mathsf{col}}(M)_\bullet$ denote the simplicial set in which a q-simplex consists of a pair (ρ, ϕ) with $\rho := (W, F, V) \in H_{\mathsf{col}}(M)_q$ and a diffeomorphism $\phi : V \cup_{M \times \Delta^q} W \cong F \times [-1, 1] \times \Delta^q$ over Δ^q which fixes pointwise (a neighbourhood of) $\partial(F \times [-1, 1]) \times \Delta^q$. There is a Kan fibration

$$\mathrm{Diff}_\partial(M \times [-1,1])_\bullet \xrightarrow{j} \overline{H}_{\mathsf{col}}(M)_\bullet \xrightarrow{p} H_{\mathsf{col}}(M)_\bullet,$$

where $p(\rho, \phi) := \rho$. The inclusion j admits a (left) section up to homotopy $s : \overline{H}_{\mathsf{col}}(M)_\bullet \to \mathrm{Diff}_\partial(M \times [-1,1])_\bullet$ given roughly by applying ϕ^{-1} on the collar of F in $M \times [-1,1] = W \cup_F V$ and then canonically identifying $W \cup_F V \cup_M W \cup_F V$ with $M \times [-1,1]$. This yields an equivalence

$$(p, s) : \overline{H}_{\mathsf{col}}(M)_\bullet \xrightarrow{\sim} H_{\mathsf{col}}(M)_\bullet \times \mathrm{Diff}_\partial(M \times [-1,1])_\bullet.$$

But now the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\overline{H}_{\mathsf{col}}(M)_\bullet \arrow[d, "\wr"] \arrow[dr, "f_1"] & \\
H_{\mathsf{col}}(M)_\bullet \times \mathrm{Diff}_\partial(M \times [-1,1])_\bullet \arrow[r, "f_2"] \arrow[d, two heads, shift right, "\mathrm{pr}_1"'] & H(M \times I)_\bullet \\
H_{\mathsf{col}}(M)_\bullet \arrow[u, shift right, "i"'] \arrow[ur, "f_3"] \arrow[d, "u", "\wr"'] & \\
H(M)_\bullet \arrow[uur, "f_4=\iota_H\Sigma+_I\Sigma\iota_H"'] &
\end{tikzcd}$$

where u is the map that forgets the collaring data, and all the horizontal maps (strictly) factor through the bottom horizontal map $f_4 := \iota_H\Sigma +_I \Sigma\iota_H$. Therefore, in order to show that f_4 is null-homotopic, it suffices to show that f_1 is so: indeed this would imply that f_2 is null-homotopic. But $f_2 = f_3 \circ \mathrm{pr}_1$, so $f_3 \simeq f_3 \circ \mathrm{pr}_1 \circ i \simeq *$ too. This in turn would imply that f_4 is null-homotopic, as we aim to prove.

We therefore need to describe a null-homotopy of f_1. We will just describe a path (or rather, a 1-simplex) between $f_4(\rho, \phi) = (\iota_H\Sigma +_I \Sigma\iota_H)(\rho)$, for a fixed 0-simplex $(\rho = (W, F, V), \phi) \in \overline{H}_{\mathsf{col}}(M)_0$, and the trivial partition $* \in H(M \times I)_0$—an exactly analogous argument yields an actual simplicial null-homotopy. This path is depicted in Figures 3.7, 3.8, 3.9 and 3.10. The green shaded regions in each picture represent the F-part of a partion, i.e., the intersection of the two h-cobordisms making up the partition, which is a $(d+1)$-manifold embedded in $M \times I \times [-1,1]$. The partition $\rho = (W, F, V) \in H(M)_0$ is as depicted in Figure 3.6.

Firstly, the path between the partition $P_0 = \iota_H\Sigma(\rho) +_I \Sigma\iota_H(\rho)$ and P_1 is obtained from rescaling (and slightly shifting inwards). But as $W \cup_F V$ is canonically[6] identified with $M \times [-1,1]$ as part of the data of ρ, the outer bent regions of the form $(W \cup_F V) \times I$ added to P_1 in order to obtain P_2 are canonically identified with $M \times [-1,1] \times I$, and therefore $P_1 = P_2$ in $H(M \times I)$. See Figure 3.7.

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[6] *Canonically* in the sense that it does not depend in any other choice than the one of ρ.

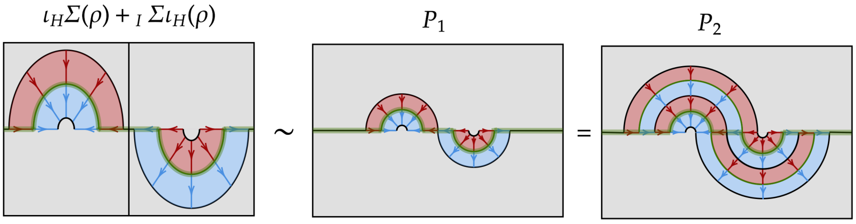

Figure 3.7 Path in $H(M \times I)$ between $\iota_H\Sigma(\rho) + \Sigma\iota_H(\rho)$ and P_2.

Unbending and straightening the region of the form $(W \cup_F V \cup_M W \cup_F V) \times I \equiv M \times [-1,1] \times I$ in P_2, we get the path to the partition P_3 of Figure 3.8 (this step is not strictly necessary, but convenient for depiction).

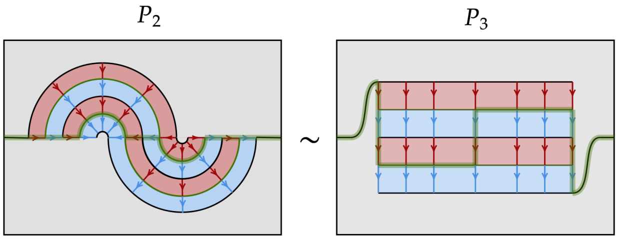

Figure 3.8 Path in $H(M \times I)$ between P_2 and P_3.

We now use the diffeomorphism $\phi : V \cup_M \cup W \cong F \times [-1,1]$ (rel. $\partial(F \times [-1,1])$) to carry out the path depicted in the lower part of Figure 3.9 locally in the circled sub-rectangle of P_3. This yields the path of Figure 3.9 between P_3 and P_4.

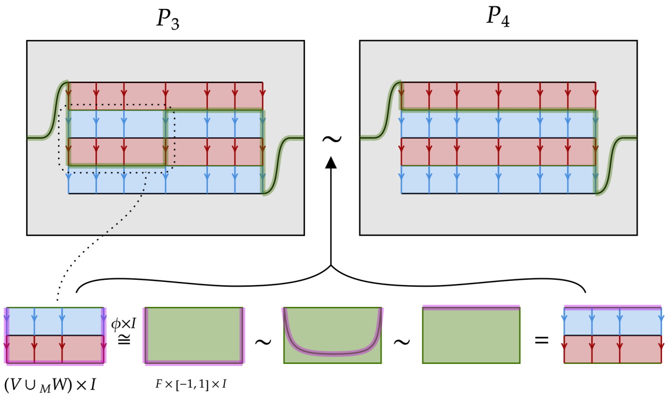

Figure 3.9 Path in $H(M \times I)$ between P_3 and P_4. The green shaded region in the lower part of the figure represents $F \times [-1,1] \times I$. The purple shaded region there represents the F-part of the partition (which used to be green, but is purple momentarily).

Retracting the region of the form $(W \cup_F V \cup_M W \cup_F V) \times I$ in P_4 onto its midpoint $(W \cup_F V \cup_M W \cup_F V) \times \{1/2\}$ yields the path of Figure 3.10 between P_4 and P_6

(passing through P_5). But now as $W \cup_F V \cup_M W \cup_F V \equiv M \times [-1, 3]$, the F-part of the partition P_6 is of the form $M \times \gamma$, for the 1-dimensional submanifold $\gamma \subset [-1, 1] \times I$ depicted in P_6. By straightening γ to the submanifold $\{0\} \times I \subset [-1, 1] \times I$, we finally get the path of Figure 3.10 between P_6 and the trivial partition $P_7 = * \in H(M \times I)$.

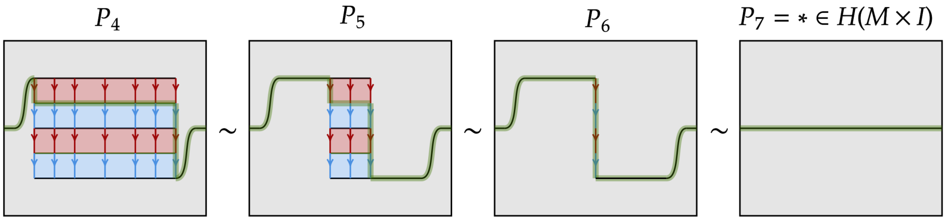

Figure 3.10 Path in $H(M \times I)$ between P_4 and P_7.

This process depends continuously on $(\rho, \phi) \in \overline{H}_{\mathsf{col}}(M)$ (i.e. can be set up as a homotopy of simplicial maps $\overline{H}_{\mathsf{col}}(M)_\bullet \to H(M \times I)_\bullet$), and therefore gives the required null-homotopy. □

Remark E.2. This result should be compared with [Hat78, Appendix I, Lem.] (or [BL82, Cor. A7]), where it is shown that the concordance stabilisation map $\Sigma : C(M) \to C(M \times I)$ anti-commutes (in the same sense of Lemma 3.1.16) with the concordance involution ι_C of Warning 3.1.11.

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