Physical modelling of galaxy clusters using Einasto dark matter profiles

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ABSTRACT

We derive a model for Sunyaev–Zel’dovich data of galaxy clusters which uses an Einasto profile to model their dark matter component. This model is similar to the physical models for clusters previously used by the AMI consortium, which model the dark matter using a Navarro-Frenk-White (NFW) profile, but the Einasto profile provides an extra degree of freedom. We thus present a comparison between two physical models which differ only in the way they model dark matter: one which uses an NFW profile (PM I) and one that uses an Einasto profile (PM II). We illustrate the differences between the models by plotting physical properties of clusters as a function of cluster radius. We apply the models to real data from cluster A611 obtained with the Arcminute Microkelvin Imager (AMI), and find the mass estimates to be consistent with one another except in the case of when PM II is applied using an extreme value for the Einasto shape parameter. Finally we generate AMI simulations of clusters which are created and analysed with both models. From this we find that for 15 of the 16 simulations, PM II recovers the input masses more accurately than PM I. The Bayesian evidence values calculated from these analyses suggest that PM II provides a better fit to the data than PM I for 13 of the clusters, but the preference is statistically significant for only three of these according to the Jeffreys scale.

Key words: methods: data analysis – galaxies: clusters: general–cosmology: observations.

1 INTRODUCTION

Clusters of galaxies are the most massive gravitationally bound objects known in the Universe. It is thought that the dark matter component of these clusters contributes \approx 90\% to their total mass. Stars, gas and dust in galaxies, as well as a hot ionised intra-cluster medium (ICM) make up the remaining mass, with the latter being much the most massive baryonic component. The galaxies emit in the optical and infrared wavebands, the ICM emits in X-ray via thermal Bremsstrahlung, and interacts with cosmic microwave background (CMB) photons via inverse Compton scattering. This last effect is known as the Sunyaev–Zel’dovich (SZ) effect (Sunyaev and Zeldovich 1970).

It is this effect that the physical modelling of clusters described in Olamaie, Hobson, and Grainge (2012) (from here on referred to as MO12) and Olamaie, Hobson, and Grainge (2013) aims to predict; this physical modelling has been applied in for example Javid et al. I (2018) (from here on KJ18 I) and Javid et al. II (2018). The model presented in MO12 uses a Navarro-Frenk-White (NFW) profile (Navarro, Frenk, and White 1995) for the dark matter component of the galaxy cluster, which is derived from N-body simulations of galaxy clusters. Einasto (1965) derives an empirical profile for dark matter halos. Previous investigations comparing the two dark matter profiles using simulated data (see e.g. Dutton and Macciò 2014, Meneghetti et al. 2014, Klypin et al. 2016 and Sereno, Fedeli, and Moscardini 2016) have shown that the Einasto model provides a better fit. In particular, Sereno, Fedeli, and Moscardini (2016) showed that, for weak lensing analysis of clusters, the NFW profile can overestimate virial masses of very massive halos ($\geq 10^{15}M_{\text{Sun}}/h$ where $M_{\text{Sun}}$ is units of solar mass and $h$ is the reduced Hubble constant) by up to 10%.

It is these previous analyses which have motivated us to derive a physical galaxy cluster model for interferometric SZ data which uses the Einasto profile to model the dark matter component of the cluster. We then compare the parameter estimates and fits of the NFW and Einasto models for the cluster A611 with data obtained from the Arcminute Microkelvin Array (AMI) radio interferometer system (Zwart et al. 2008), and with simulations (see e.g. Grainge et al. 2002) created with both Einasto and NFW profiles.

Section 2 of this paper gives a brief overview of the theory...
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behind interferometry and the SZ effect. In Section 3 we derive the physical model for interferometric SZ data which uses the Einasto profile, and describe the Bayesian methodology for fitting a model to the data. Section 4 presents the results of our analysis, including the radial profiles of physical parameters using both the model presented here and the one derived in MO12 for a range of clusters. We also present the results of applying these models to the A611 cluster and the simulated cases using Bayesian analysis.

In this work a ‘concordance’ flat ΛCDM cosmology is assumed.

2 MEASURING THE SZ EFFECT WITH AN INTERFEROMETER

For a small field size, an interferometer samples from the two-dimensional complex visibility plane \( u \), also known as the \( u-v \) plane. At frequency \( v \), the samples correspond to the Fourier components of the sky brightness distribution \( I_v(u) \). \( I_v(u) \) is given by the weighted Fourier transform of the surface brightness \( I_\nu(x) \)

\[
I_v(u) = \int_{-\infty}^{\infty} A_v(x)I_\nu(x)e^{2\pi i u \cdot x} \, dx,
\]

where \( x \) is the position in the sky relative to the phase centre and \( A_v(x) \) is the primary beam of the antennas for a given frequency. The positions at which \( I_v(u) \) are sampled from is therefore determined by the physical orientation of the antennas. The change in CMB intensity \( \delta I \), due to the thermal SZ effect in a galaxy cluster is given by (see e.g. Birkinshaw 1999)

\[
\delta I_{\text{CMB}} = f_v \left| \frac{\partial B_v(T)}{\partial T} \right|_{T=T_{\text{CMB}}},
\]

where the last term is the derivative of the blackbody spectrum with respect to temperature evaluated at the absolute temperature of the CMB, which at present is \( T_{\text{CMB}} = 2.728 \) K (Fixsen et al. 1996). \( B_v(T) \) is the spectral radiance of blackbody radiation (given by Planck’s law). The function \( f_v \) expresses the spectral dependence of the SZ signal and is derived from the Kompaneets equation (Kompaneets 1957). Relativistic treatments of \( f_v \) have been considered in e.g. Rephaeli (1995), Itoh, Kohyama, and Nozawa (1998), Challinor and Lasenby (1998), Nozawa, Itoh, and Kohyama (1998), and Pointecouteau, Giard, and Barret (1998), by incorporating relativistic terms into the Kompaneets equation. Relativistic effects may be important in clusters where the ICM temperatures are high. Indeed Arnaud et al. (1994) and Markevitch et al. (1996) have shown that electrons in the ICM can reach energies above 10 keV. Challinor and Lasenby show that these effects lead to a small decrease in the SZ effect. However, Rephaeli argues that the non-relativistic treatment of Compton scattering adopted in Zeldovich and Sunyaev (1969) remains valid at frequencies well below the CMB peak value. For the observing frequencies of AMI (\( \approx 15 \) GHz), it can be assumed that this condition holds. Furthermore Rephaeli claims that for the unmodified Kompaneets equation to be valid, the optical depth of the cluster \( \tau \), must be sufficiently large to justify using a diffusion approximation for the scattering process. It is clear that at AMI observing frequencies \( h_\nu \ll m_e c^2 \), where \( h_\nu \) is the Planck constant, \( m_e \) is the mass of an electron and \( c \) is the speed of light; and so the photons scatter in the Thomson limit. In this limit the scattering rate is \( \propto \sigma_T n_e \), where \( \sigma_T \) is the Thomson scattering cross-section and \( n_e \) is the electron number density in the ICM. Thus the optical depth is given by

\[
\tau = \int n_e(r)\sigma_T \, dl,
\]

where \( \sigma_T \) is the Thomson scattering cross-section and \( n_e \) is the electron number density in the ICM.

\[
f_v = X \coth(X/2) - 4,
\]

where

\[
X = \frac{h_\nu}{k_B T_{\text{CMB}}},
\]

Here \( k_B \) is the Boltzmann constant. Referring back to equation 2, \( y \) is the Comptonisation parameter which is the number of collisions multiplied by the mean fractional change in energy of the photons per collision, integrated along the line of sight. On average the electrons in the ICM transfer an energy \( k_B T_v(r)/m_e c^2 \) to the scattered CMB photons, where \( T_v(r) \) is the temperature of an electron in the ICM. In the Thomson scattering regime described above this leads to

\[
y = \frac{\sigma_T k_B}{m_e c^2} \int T_v(r)n_e(r) \, dl.
\]

If an ideal gas equation of state is assumed for the electron gas then in terms of the electron pressure \( P_e(r) \), the Comptonisation parameter is given by

\[
y = \frac{\sigma_T}{m_e c^2} \int P_e(r) \, dl.
\]

Combining equations 2, 4 and 7, we arrive at the following expression for \( \delta I_{\text{CMB}} \) in the non-relativistic limit

\[
\delta I_{\text{CMB}} = \frac{2\sigma_T^2 h_\nu^2 T_{\text{CMB}}^4 X^2}{h^2 k_e^4} \left[X \coth(X/2) - 4\right] \int P_e(r) \, dl.
\]

3 MODELLING INTERFEROMETRIC SZ DATA

We first discuss how to model \( \delta I_{\text{CMB}} \) arising from the SZ effect, starting from input parameters (see Section 3.2 for more on what input parameters are) which describe some physical properties of a cluster. Equation 1 can then be used to replicate the quantity measured by an interferometer. In Section 3.2 we discuss how to use Bayesian inference to perform parameter estimation and model selection for comparison between different models using the NFW and Einasto dark matter profiles.

3.1 Cluster models

We now consider two cases for modelling physical properties of galaxy clusters using interferometric SZ data, which we denote PM I and PM II. PM I is as described in MO12, but with the alteration in the mapping from \( r_{200} \) to \( r_{200} \) (defined below) described in KJ18. PM II follows the same calculational steps, but with an Einasto profile replacing the NFW one used for the dark matter component in PM I. Below we derive the relevant equations for the Einasto case and refer the reader to MO12 for the PM I equations.

The three input parameters required to calculate \( \delta I_{\text{CMB}} \) for either PM are \( M(r_{200}) \), which is the mass enclosed up to radius \( r_{200} \) from the cluster centre, \( f_{\text{gas}}(r_{200}) \) which is the fraction of the total mass attributed to the gas mass at radius \( r_{200} \), and \( z \), the redshift of the cluster. A fourth input parameter is required for the PM II which we call the Einasto parameter \( a_{\text{Ein}} \), which is also described below. Note that in general the radius \( r_d \) is the radius from the centre at which the average total enclosed mass density is \( A \) times
\( \rho_{\text{crit}}(z) \), the critical density at the \( z \) of the cluster. \( \rho_{\text{crit}}(z) \) is given by \( \rho_{\text{crit}}(z) = 3H(z)^2/8\pi G \) where \( H(z) \) is the Hubble parameter and \( G \) is Newton’s constant. Note further that the total mass out to \( r_A \) is given by

\[
M(r_A) = \frac{4\pi}{3} \rho_{\text{crit}}(z) r_A^3. \tag{9}
\]

Hence \( r_{200} \) can be calculated from \( M(r_{200}) \).

### 3.1.1 Dark matter profiles

Assuming an Einasto profile (Einasto 1965), the dark matter density profile for a cluster \( \rho_{\text{dm}, \text{PM II}} \) is given by

\[
\rho_{\text{dm}, \text{PM II}} = \rho_s \exp \left[ -\frac{2}{\alpha_{\text{Ein}}} \left( \frac{r}{r_s} - 1 \right) \right], \tag{10}
\]

where \( \alpha_{\text{Ein}} \) is a shape parameter, \( r_s \) is the scale radius where the logarithmic derivative of the density is \( -2 \) (analogue to \( r_s \) in the NFW model, but note that in general \( r_s \neq r_s \), and \( \rho_s \) is the density at this radius. The parameter \( \alpha_{\text{Ein}} \) controls the degree of curvature of the profile. The larger its value, the more rapidly the slope varies with respect to \( r \). In the limit that \( \alpha_{\text{Ein}} \to 0 \), the logarithmic derivative is \( -2 \) for all \( r \). For comparison we state the NFW dark matter profile used in PM I (Navarro, Frenk, and White 1995)

\[
\rho_{\text{dm}, \text{PM I}}(r) = \frac{\rho_s}{\left(1 + r^2/r_s^2\right)^{3/2}}, \tag{11}
\]

where \( \rho_s \) is a density normalisation constant, and \( r_s \) is another scale radius. It is tempting to assume that the Einasto profile is capable of providing a better fit due to the fact that the Einasto profile has an extra degree of freedom (three for the Einasto profile, two for the NFW), the shape parameter. However, Klypin et al. (2016) claims that this is not strictly true, as the Einasto profile was seen to give a better fit to simulated dark matter haloes even with \( \alpha_{\text{Ein}} \) fixed. The asymptotic values of the logarithmic slope for the two profiles are as follows: as \( r \to 0 \) then \( d \ln \rho_{\text{dm}, \text{PM I}}(r)/d \ln r \to -1 \) and \( d \ln \rho_{\text{dm}, \text{PM II}}(r)/d \ln r \to 0 \). As \( r \to \infty \) then \( d \ln \rho_{\text{dm}, \text{PM I}}(r)/d \ln r \to -3 \) and \( d \ln \rho_{\text{dm}, \text{PM II}}(r)/d \ln r \to -\infty \). The magnitude of \( \alpha_{\text{Ein}} \) determines how quickly the slope changes between the two asymptotic values. Throughout this work when we refer to the NFW or Einasto model, we really mean the physical model which uses the NFW or Einasto model when considering the dark matter density profile.

Referring back to equation 10, the ratio \( r_{200}/r_s \) is defined as the concentration parameter \( c_{200} \). Dutton and Macciò (2014) determines an analytical form for \( c_{200} \) as a function of total mass and redshift for Einasto profiles based on simulations similar to those described in Macciò et al. (2007) and Macciò, Dutton, and van den Bosch (2008)

\[
\log_{10}(c_{200}) = j(z) + k(z) \log_{10} \left( \frac{M(r_{200})}{10^{17} h^{-1} M_{\odot}} \right), \tag{12}
\]

where \( j(z) = 0.459 + 0.518 \exp(-0.49 z^{1.303}) \) and \( k(z) = -0.13 + 0.029 z \). The fitting is said to be accurate in the redshift range [0, 5]. To calculate \( \rho_s \) we must make the assumption asserted in MO12, that the total mass enclosed at \( r_{200} \) is approximately equal to the enclosed dark matter mass. That is

\[
M(r_{200}) = M_{\text{dm}}(r_{200}) + M_g(r_{200}) \approx M_{\text{dm}}(r_{200}). \tag{13}
\]

---

**Figure 1.** Logarithmic dark matter density profiles as a function of log cluster radius using NFW and Einasto models. Three values of the Einasto profile are used: 0.05, 0.2, and 2.0. The additional input parameters used to generate these profiles are: \( z = 0.15, M(r_{200}) = 1 \times 10^{15} M_{\odot} \) and \( f_{\text{gas}}(r_{200}) = 0.12 \).
due to the mass enclosed within that radius, i.e. that
\[
\frac{dP_e}{dr} = -\rho_g(r) \frac{GM(r)}{r^2},
\]  
(16)
where \(\rho_g(r)\) is the gas density and \(P_e(r)\) is the gas pressure. Note that when evaluating this equation, we assume the mass approximation stated in Section 3.1.1. As in PM I we follow Nagai, Kravtsov, and Vikhlinin (2007) and assume a generalised-NFW (GNFW) profile to parameterise the electron pressure as a function of radius from the cluster centre,
\[
P_e(r) = \frac{P_{ei}}{\left(\frac{r}{r_p}\right)^a \left(1 + \left(\frac{r}{r_p}\right)^a\right)^{(b-c)/a}},
\]  
(17)
where \(P_{ei}\) is the pressure normalisation constant and \(r_p\) is another characteristic radius, defined by \(r_p = r_{500}/c_{500}\). The parameters \(a, b,\) and \(c\) characterise the slope of the pressure profile at \(r \approx r_p\), \(r \gg r_p\) and \(r \ll r_p\) respectively. The slope parameters are taken to be \(a = 1.0510, b = 5.4905\) and \(c = 0.3081\). These ‘universal’ values were taken from Arnaud et al. (2010) and are the best fit GNFW slope parameters derived from the REXCESS sub-sample (observed with XMM-Newton, Böhringer et al. 2007, as described in Section 5 of Arnaud et al. We also take the Arnaud et al. value of \(c_{500}\) which is 1.177. Note that in MO12, KJ18 I and KJ18 II slightly different values derived for the standard self-similar case (Appendix B of Arnaud et al.) were used \((a = 1.0620, b = 5.4807, c = 0.3292\) and \(c_{500} = 1.156\). It was shown in Olamaie, Hobson, and Grainge (2013) that PM I is not affected by which of these two sets of parameters is used. The analytical function used to convert from \(r_{200}\) to \(r_{500}\) in KJ18 I is specific to the NFW dark matter profile case and so is not applicable in PM II. We have not found an analytic fitting function for the conversion in the case of an Einasto dark matter profile and so we obtain \(r_{500}\) iteratively as described in Appendix B.

We can relate the gas pressure \(P_g(r)\), to the electron pressure through the relation
\[
\rho_g P_g(r) = \mu_e P_e(r),
\]  
(18)
where \(\mu_e\) is the mean gas mass per electron and \(\mu_g\) is the mean mass per gas particle. Mason and Myers (2000) state that for a plasma with the cosmic helium mass fraction \(C_{He} = 0.24\) and the solar abundance values in Anders and Grevesse (1989), then \(\mu_e = 1.146\) and \(\mu_g = 0.592\) in units of proton mass.

Incorporating equations 14 and 18 into 16 gives the gas density
\[
\rho_g(r) = \frac{\mu_e P_{ei}}{4\pi \mu_g G \rho_{g,2} r_p^3} \left(1/\alpha_{Ein}\right) \exp\left(2/\alpha_{Ein}\right)\left(\alpha_{Ein}/2\right)^{3/\alpha_{Ein}}
\]  
\[
\frac{1}{\gamma\left(1/\alpha_{Ein}\right)\left(2/\alpha_{Ein}\right)^{2} \left(r / r_p\right)^{\alpha_{Ein}}}
\]  
\[
\times \left[\left(\frac{r}{r_p}\right)^{-c} \left[1 + \left(\frac{r}{r_p}\right)^a\right] \left[\frac{b}{\gamma}\right] \left(\frac{r}{r_p}\right)^{b+c}\right].
\]  
(19)
Note that even though an analytical expression for \(\rho_g\) exists, within this model there is no such equivalent for the total gas mass as
\[
M_g(r) = \int_0^r 4\pi \rho_g(r') r'^2 dr'.
\]  
(20)
must be integrated numerically. Hence \(f_{gas}(r) = M_g(r)/M(r)\) does not have a closed form solution. Nevertheless, we can use equations 19 and 20 to determine \(P_{ei}\) since we know \(M(r_{200})\), \(f_{gas}(r_{200})\) and \(r_{200}\). Evaluating equations 19 and 20 at \(r_{200}\) and solving for \(P_{ei}\) gives the following expression
\[
P_{ei} = \frac{\mu_e}{\mu_g} \left(\frac{G \rho_{g,2} r_p^3}{500}\right) \left[\frac{\exp\left(2/\alpha_{Ein}\right)\left(\alpha_{Ein}/2\right)^{3/\alpha_{Ein}}}{\gamma\left(1/\alpha_{Ein}\right)\left(2/\alpha_{Ein}\right)^{2} \left(r / r_p\right)^{\alpha_{Ein}}}
\]  
\[
\times \int_0^{r_{200}} r'^3 \left[\left(\frac{r'}{r_p}\right)^{-c} \left[1 + \left(\frac{r'}{r_p}\right)^a\right] \left[\frac{b}{\gamma}\right] \left(\frac{r'}{r_p}\right)^{b+c}\right]
\]  
\[
\times \left[1 + \left(\frac{r'}{r_p}\right)^a\right] \left[\frac{b}{\gamma}\right] \left(\frac{r'}{r_p}\right)^{b+c}\right] \right] dr',
\]  
(21)
which must be evaluated numerically. Once \(P_{ei}\) has been calculated, the Comptonisation parameter can be calculated using equation 7 which in turn can be used to calculate the surface brightness using equation 8. Finally this can be Fourier transformed as to get the quantity comparable to what an interferometer measures, so that the physical model can be used to analyse data obtained with AMI.

When \(r\) is calculated at different radii from the cluster centre it is assumed that the cluster is spherically symmetric, but the dark matter and gas pressure profiles used up to this point already assume this. To summarise, the four main assumptions of the physical models are:

- The cluster is spherically symmetric.
- The cluster is in hydrostatic equilibrium up to radius \(r_{200}\).
- The gas mass fraction, \(f_{gas}(r)\) is much less than unity up to radius \(r_{200}\), so that the total mass density at this radius is given by \(\rho(r_{200}) \approx \rho_{dm}(r_{200})\).
- The cluster gas is assumed to be an ideal gas.

### 3.1.3 Additional cluster parameters

As stated in Section 2 of MO12, the radial profile of the electron number density is given by \(n_e(r) = \rho_g(r)/\mu_e\). Using the ideal gas assumption, the electron temperature \(T_e\) is given by
\[
T_e(r) = \left(\frac{4\pi \mu_e G \rho_{g,2} r_p^3}{k_B} \right) \left[\frac{1}{\alpha_{Ein}} \exp\left(2/\alpha_{Ein}\right)\left(\alpha_{Ein}/2\right)^{3/\alpha_{Ein}}\right]
\]  
\[
\times \left[1 + \left(\frac{r}{r_p}\right)^a\right] \left[\frac{b}{\gamma}\right] \left(\frac{r}{r_p}\right)^{b+c}\right]^{-1},
\]  
(22)
which also equals \(T_g(r)\).

The gas mass can be determined numerically from equation 20,
\[
M_g(r) = \left(\frac{\mu_e}{\mu_g} \left(\frac{G \rho_{g,2}}{500}\right) \left[\frac{1}{\alpha_{Ein}} \exp\left(2/\alpha_{Ein}\right)\left(\alpha_{Ein}/2\right)^{3/\alpha_{Ein}}\right] r_{200}\right)
\]  
\[
\times \int_0^{r_{200}} r'^3 \left[\left(\frac{r'}{r_p}\right)^{b+c}\right] \left[\frac{b}{\gamma}\right] \left(\frac{r'}{r_p}\right)^{b+c}\right] \right] dr',
\]  
(23)
which can also be used to determine \(f_{gas}(r)\).
3.2 Bayesian inference

The analysis of AMI data carried out in Section 4.2 is done using Bayesian inference. We now give a summary of this in the context of both parameter estimation and model comparison.

3.2.1 Parameter estimation

Given a model $M$ and data $D$, one can obtain probability distributions of the input parameters (also known as sampling parameters or model parameters) $\theta$ conditioned on $M$ and $D$ using Bayes’ theorem:

$$Pr(\theta|D, M) = \frac{Pr(D|\theta, M)Pr(\theta|M)}{Pr(D|M)},$$

(24)

where $Pr(\theta|D, M) \equiv \mathcal{P}(\theta)$ is the posterior distribution of the model parameter set, $Pr(D|\theta, M) \equiv \mathcal{L}(\theta)$ is the likelihood function for the data, $Pr(\theta|M) \equiv \pi(\theta)$ is the prior probability distribution for the model parameter set, and $Pr(D|M) \equiv Z$ is the Bayesian evidence of the data given a model $M$. The evidence can be interpreted as the factor required to normalise the posterior over the model parameter space:

$$Z(D) = \int \mathcal{L}(\theta) \pi(\theta) d\theta,$$

(25)

where the integral is carried out over the $N$-dimensional parameter space. For the models using AMI data considered here, the input parameter set can be split into two subsets, (which are assumed to be independent of one another): cluster parameters, $\theta_{cl}$ and radio-source or ‘nuisance’ parameters, $\theta_{ns}$. The set of cluster parameters is $\alpha_{Ein}, M(r_{200}), f_{gas}(r_{200}), z, x_c$, and $y_c$ (where the former only appears for PM II). $x_c$ and $y_c$ are the cluster centre offsets from the interferometer pointing centre, measured in arcseconds. The cluster prior probability distributions are given in Section 3.2.3. For more details on the radio-source modelling, please refer to Section 5.2 of Feroz et al. (2009) (from here on FF09) for more information on the likelihood function and covariance matrix used in the AMI analysis, we refer the reader to Hobson and Maisinger (2002) and Sections 5.3 of FF09 and 3.2.3 of KJ18 I.

3.2.2 Model comparison

The nested sampling algorithm, MultiNest (Feroz, Hobson, and Bridges 2009) calculates $Z(D)$ by making use of a transformation of the $N$-dimensional evidence integral into a one-dimensional integral. The algorithm also generates samples from $\mathcal{P}(\theta)$ as a by-product, meaning that it is suitable for both the parameter estimation and model comparison aspects of this work. Comparing probability in a Bayesian way can be done as follows. The probability of a model $M$, conditioned on $D$ can also be calculated using Bayes’ theorem

$$Pr(M|D) = \frac{Pr(D|M)Pr(M)}{Pr(D)},$$

(26)

Hence for two models $M_1$ and $M_2$, the ratio of the probability of the models conditioned on the same dataset is given by

$$\frac{Pr(M_1|D)}{Pr(M_2|D)} = \frac{Pr(D|M_1)Pr(M_1)}{Pr(D|M_2)Pr(M_2)},$$

(27)

where $Pr(M_2)/Pr(M_1)$ is the a-priori probability ratio of the models. We set this to one, i.e. place no bias towards a particular model before performing the analysis. Hence the ratio of the probabilities of the models given the data is equal to the ratio of the evidence values obtained from the respective models (for brevity we define $Z_i = Pr(D|M_i)$). The evidence is simply the average of the likelihood function over the parameter space, weighted by the prior distribution. This means that the evidence is larger for a model if more of its parameter space is likely and smaller for a model with large areas in its parameter space having low likelihood values. A larger parameter space, either in the form of higher dimensionality or a wider domain results in a lower evidence value all other things being equal. Hence the evidence ‘punishes’ more complex models over basic (lower dimensionality / smaller input parameter space domains) ones which give an equally good fit to the data. Thus the evidence automatically implements Occam’s razor: when you have two competing theories that make exactly the same predictions, the simpler one is the preferred. Jeffreys (1961) provides a scale for interpreting the ratio of evidences as a means of performing model comparison (see Table 1). A value of $\ln(Z_1/Z_2)$ above 5.0 (less than $-5.0$) presents strong evidence in favour of $M_1$ ($M_2$). Values $2.5 < \ln(Z_1/Z_2) < 5.0$ (or $-5.0 < \ln(Z_1/Z_2) < -2.5$) present moderate evidence in favour of $M_1$ ($M_2$). Values $1 \leq \ln(Z_1/Z_2) < 2.5$ ($-2.5 < \ln(Z_1/Z_2) \leq -1$) present weak evidence in favour of $M_1$ ($M_2$). Finally, values $-1.0 < \ln(Z_1/Z_2) < 1.0$ require more information to come to a conclusion over preference between $M_1$ and $M_2$.

3.2.3 Prior probability distributions

For both PM I and PM II we adopt the following approach (excluding any mention of $\alpha_{Ein}$ in the former case). Following FF09, the cluster parameters are assumed to be independent of one another, so that

$$\pi(\theta_{cl}) = \pi(\alpha_{Ein})\pi(M(r_{200}))\pi(f_{gas}(r_{200}))\pi(z)\pi(x_c)\pi(y_c).$$

(28)

Table 2 lists the type of prior used for each cluster parameter and the probability distribution parameters. The values used for $z$ and $\alpha_{Ein}$ will be specified on a case by case basis in Section 4.2.

4 RESULTS

4.1 Cluster parameter profiles

We first present the results of using the Einasto model in the profiling of cluster dark matter for a range of different cluster input parameters, along with the equivalent results from PM I. We consider two input masses, $M(r_{200}) = 1 \times 10^{14} M_{\odot}$ and $M(r_{200}) = 1 \times 10^{15} M_{\odot}$, which roughly span the range of galaxy cluster masses. We use $z$-values of 0.15 and 0.9, and take $f_{gas}(r_{200}) = 0.12$ following Komatsu et al. (2011), and consider $\alpha_{Ein}$ values of 0.05, 0.2, and 2.0 – see Figure 1. We note that the same $r$ range ($-2.0 \leq \log_{10}(r) (\text{Mpc}) \leq 2.0$) is considered for each cluster, and thus even though each parameter profile is self-similar in $r$ with respect to mass and redshift, they are different for each cluster over the range of $r$ considered here.

4.1.1 Dark matter mass profiles

Figure 2 shows the dark matter mass profiles. The Einasto profiles are calculated using equation 14 and the NFW profile from the equivalent relation given in MO12 (equation 5). Note that even though the notation in these equations corresponds to the total mass, this is in fact just the dark matter mass as we have used the approximation
shows the gas density profiles. The Einasto profiles are calculated, along with that obtained with the NFW profile. We first compare the posterior distributions for the input parameters (except those with δ-function priors). The means and standard deviations of the four analyses are given in Table 3. As in Section 4.1, αEin = 0.05 and αEin = 0.2 show similar results to PM I. αEin = 2 gives a different estimate for M(r200), and its posterior distribution is shown in Figure 6 along with that obtained with the NFW profile. These posterior distributions are plotted using GetDist1, and the contours on the two-dimensional plots represent the 95% and 68% mean confidence intervals. The mean mass estimates are within one combined standard deviation away from each other. However, as seen in Table 3 the value of ln(ZEin/ZNFW) imply that ‘no model is favoured by the data’ according to the Jeffreys scale.

4.2.2 Simulated AMI data

Sereno, Fedeli, and Moscardini (2016) study the errors associated with fitting NFW profiles to Einasto dark matter halos and vice versa for weak lensing studies. We conduct similar work in the context of simulated SZ observations. The simulations were carried out using the in-house AMI simulation package Profile, which has been used in various forms in e.g. Grainge et al. (2002), Olamaie, Hobson, and Grainge (2013) and KJ18 I. As before we consider Einasto profiles with the αEin values 0.05, 0.2, and 2.0 plus an NFW profile. Each with M(r200) = 1 × 10^{14} M_{Sun} or M(r200) = 1 × 10^{15} M_{Sun}, z = 0.15 or z = 0.90 and f_{gas}(r200) = 0.12. These 16 simulations, were analysed as in Section 4.2.1. Note for all of these simulations no radio-sources, primordial CMB or confusion noise were included, and instrumental noise was set to a negligible level. Table C1 in Appendix C summarises the input and output values of the 16 simulations. The first column gives the model used to simulate the cluster, with the following two columns giving the mass and z input values. For each simulation, we analysed the data using two models, one using the NFW profile and one using an Einasto profile. For data simulated using an NFW profile, when analysing the data with an Einasto profile we used αEin = 0.2. For data simulated using an Einasto profile, when analysing the data with an Einasto profile we set αEin equal to the value used as the input for the simulation.

In all but one of the simulations (NFW simulated with $M(r_{200}) = 1 \times 10^{14} M_{\text{Sun}}$ and $z = 0.9$), the Einasto posterior

### Table 1

Jeffreys scale for assessing model preferability based on the log of the evidence ratio of two models $M_1$ and $M_2$.

<table>
<thead>
<tr>
<th>Interpretation</th>
<th>Probability of favoured model</th>
</tr>
</thead>
<tbody>
<tr>
<td>better data are needed</td>
<td>$\leq 1.0$</td>
</tr>
<tr>
<td>weak evidence in favour of $M_1$</td>
<td>$\leq 2.5$</td>
</tr>
<tr>
<td>moderate evidence in favour of $M_1$</td>
<td>$\leq 5.0$</td>
</tr>
<tr>
<td>strong evidence in favour of $M_1$</td>
<td>$&gt; 5.0$</td>
</tr>
</tbody>
</table>

### Table 2

Cluster parameter prior distributions, where the normal distributions are parameterised by their mean and standard deviations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_c$</td>
<td>$N(0^\circ, 60^\circ)$</td>
</tr>
<tr>
<td>$y_c$</td>
<td>$N(0^\circ, 60^\circ)$</td>
</tr>
<tr>
<td>$z$</td>
<td>$\delta(z)$</td>
</tr>
<tr>
<td>$M(r_{200})$</td>
<td>$\delta[\log(0.5 \times 10^{14} M_{\text{Sun}}), \log(50 \times 10^{14} M_{\text{Sun}})]$</td>
</tr>
<tr>
<td>$f_{\text{gas}}(r_{200})$</td>
<td>$\delta(0.12, 0.02)$</td>
</tr>
<tr>
<td>$\alpha_{\text{Ein}}$</td>
<td>$\delta(0.05, 0.05)$</td>
</tr>
</tbody>
</table>

### Table 3

Cluster mass estimates from radio-sources and that the cluster is close to the $T_X-T_{SZ}$ relation for clusters in hydrostatic equilibrium. We first compare the posterior distributions for the input parameters (except those with δ-function priors). The means and standard deviations of the four analyses are given in Table 3. As in Section 4.1, $\alpha_{\text{Ein}} = 0.05$ and $\alpha_{\text{Ein}} = 0.2$ show similar results to PM I. $\alpha_{\text{Ein}} = 2$ gives a different estimate for $M(r_{200})$, and its posterior distribution is shown in Figure 6 along with that obtained with the NFW profile. These posterior distributions are plotted using GetDist1, and the contours on the two-dimensional plots represent the 95% and 68% mean confidence intervals. The mean mass estimates are within one combined standard deviation away from each other. However, as seen in Table 3 the value of $\ln(Z_{\text{Ein}}/Z_{\text{NFW}})$ imply that ‘no model is favoured by the data’ according to the Jeffreys scale.

4.2.1 Analysis of real AMI observations of A611

Following MO12 we conduct Bayesian analysis on data from observations with AMI of the cluster A611 at $z = 0.288$, which has been studied through its X-ray emission, strong lensing, weak lensing and SZ effect (see Schmidt and Allen 2007, Donnarumma et al. 2011, Romano et al. 2010 and Rumsey et al. 2016 respectively). These studies suggest that there is no significant contamination from radio-sources and that the cluster is close to the $T_X-T_{SZ}$ relation for clusters in hydrostatic equilibrium.

4.2 Bayesian analysis of AMI data

We now focus our attention on applying the PM II to real and simulated AMI data, to compare the parameter estimates and Bayesian evidences with those obtained from the PM I.

mean mass value was closer to the input value than the corresponding NFW value. In 11 out of 16 cases the Einasto profile recovers the input mass to within 10% (interestingly, it does so for all the NFW simulated clusters). However, in only two of 16 cases does the Einasto model recover the input value within three standard deviations. This suggests that other systematic sources of error are causing the offsets in estimates such as the sampling error. Higson et al. (2017) finds that the posterior distribution errors associated with sampling are in general underestimated by nested sampling algorithms, if bootstrap sampling techniques are not applied. Looking at the individual evidence values for both Einasto and NFW models, the value is considerably lower for the high mass simulations, \( \ln(\frac{Z_{\text{low mass}}}{Z_{\text{high mass}}}) \approx 3000 \) suggests the models fit the low mass datasets much better when averaged over the (same) parameter sampling spaces. It is crucial to note that when comparing evidences calculated from different datasets (specifically their ratio), we are not looking at two different datasets does give a measure of the relative goodness of fit of the datasets to the model. Nevertheless for the same model, the evidence ratio between two different datasets does give a measure of the relative goodness of fit of the datasets to the model. Looking at the evidence ratios between the Einasto and NFW models for a given simulation, more

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Dark matter mass profiles as a function of log cluster radius using NFW and Einasto models. Values of \( \alpha_{\text{Ein}} = 0.05, 0.2, \) and 2.0 are used as inputs. Top row has \( z = 0.15, \) bottom row has \( z = 0.9. \) Left column has \( M(r_{200}) = 1 \times 10^{14} M_{\odot}, \) right column has \( M(r_{200}) = 1 \times 10^{15} M_{\odot}. \)

<table>
<thead>
<tr>
<th>Model</th>
<th>( x_c ) (arcsec)</th>
<th>( y_c ) (arcsec)</th>
<th>( M(r_{200}) \times 10^{14} M_{\odot} )</th>
<th>( f_{\text{max}}(r_{200}) )</th>
<th>( \ln(\frac{Z}{D}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFW</td>
<td>24.7 ± 12.4</td>
<td>13.9 ± 11.5</td>
<td>7.84 ± 1.24</td>
<td>0.129 ± 0.020</td>
<td>3.862944 ± 0.25</td>
</tr>
<tr>
<td>( \alpha_{\text{Ein}} = 0.05 )</td>
<td>22.7 ± 12.5</td>
<td>13.1 ± 12.6</td>
<td>7.45 ± 1.24</td>
<td>0.130 ± 0.019</td>
<td>3.862921 ± 0.25</td>
</tr>
<tr>
<td>( \alpha_{\text{Ein}} = 0.2 )</td>
<td>25.5 ± 12.8</td>
<td>14.9 ± 13.0</td>
<td>7.67 ± 1.27</td>
<td>0.127 ± 0.017</td>
<td>3.862967 ± 0.24</td>
</tr>
<tr>
<td>( \alpha_{\text{Ein}} = 2.0 )</td>
<td>24.3 ± 12.4</td>
<td>14.3 ± 13.2</td>
<td>6.17 ± 1.12</td>
<td>0.130 ± 0.017</td>
<td>3.862924 ± 0.24</td>
</tr>
</tbody>
</table>

**Table 3.** Marginalised posterior distribution mean values and standard deviations of physical model input parameters and Bayesian evidences associated with each model, applied to real A611 data.
data is needed to come to a conclusive decision over model preference in 10 of the simulations. Three simulations lead to ‘substantial preference’ in favour of the Einasto model (\(\ln(\frac{Z_{\text{Ein}}}{Z_{\text{NFW}}}) \geq 5\)). In two of these cases (\(\alpha_{\text{Ein}} = 0.2\) with \(M(r_{200}) = 1 \times 10^{15} M_{\odot}\) and \(z = 0.9\), and \(\alpha_{\text{Ein}} = 2.0\) with \(M(r_{200}) = 1 \times 10^{15} M_{\odot}\) and \(z = 0.9\)) the posteriors show reasonable constraints in both the Einasto and NFW analyses (Figure 7 shows posterior distributions for \(\alpha_{\text{Ein}} = 2.0\) with \(M(r_{200}) = 1 \times 10^{15} M_{\odot}\) and \(z = 0.9\)), with the former giving better estimates of mass and \(f_{\text{Ein}}(r_{200})\). The third case however (\(\alpha_{\text{Ein}} = 2.0\) simulated with \(M(r_{200}) = 1 \times 10^{15} M_{\odot}\) and \(z = 0.9\)) leads to low estimates of \(f_{\text{Ein}}(r_{200})\) in both cases (Figure 8), and a very high mass estimate in the case of the NFW model. The two cases where the NFW model is preferred over the Einasto also produce posteriors similar to those in Figure 8.

Finally, we tried running the Bayesian analysis on eight of the Einasto simulated clusters with uniform analysis priors on \(\alpha_{\text{Ein}}\). These clusters corresponded to the simulations with input values of either \(\alpha_{\text{Ein}} = 0.2\) or \(\alpha_{\text{Ein}} = 2.0\). For the former value of \(\alpha_{\text{Ein}}\) we assigned the uniform prior \(\mathcal{U}[0.05, 0.35]\) and \(\mathcal{U}[0.5, 3.5]\) for the latter. For two of these simulations the posterior distributions did not show much degeneracy between any of the input parameters, including \(\alpha_{\text{Ein}}\). Both of these clusters had \(\alpha_{\text{Ein}} = 2.0, M(r_{200}) = 1 \times 10^{15} M_{\odot}\) and \(z = 0.15\) or \(z = 0.9\) as inputs. Their posterior distributions are shown in Figure 9. Both posteriors give a mean value for the shape parameter within one standard deviation of the input value (2.01 ± 0.54 and 2.39 ± 0.40), but looking at the distributions they are not sharply peaked, meaning the errors on the estimates are quite large. Nevertheless these simulations do show the Einasto profile is capable of recovering some information about \(\alpha_{\text{Ein}}\), in contrast to the efforts in MO12 to recover \(c_{200}\) which led to large \(c_{200} = M(r_{200})\) degeneracies (although \(c_{200}\) relates to the scale of the dark matter profile, not its shape).

5 CONCLUSIONS

Based on the physical model introduced in Olamaie, Hobson, and Grainge (2012) (PM I) which uses an NFW profile (Navarro, Frenk, and White 1995) to model the dark matter content of galaxy clusters, we derive a new physical model (PM II) which models the dark matter with an Einasto profile...
The Einasto profile has an additional degree of freedom compared to the NFW profile, which dictates the shape of the dark matter density as a function of radius. For different values of $\alpha_{\text{Ein}}$ we have investigated the profiles of several physical properties of a cluster, namely the dark matter density, dark matter mass, gas density, gas mass and gas temperature. We have also provided the equivalent profiles in the NFW case. From this we found the following.

- Of the three values of $\alpha_{\text{Ein}}$ considered, $\alpha_{\text{Ein}} = 0.2$ gave the most similar profile to that given by the NFW model (as discussed in Dutton and Macciò 2014), with the main discrepancy between the two arising in the peak amplitude of the gas temperature.
- $\alpha_{\text{Ein}} = 2.0$ showed the most convergent behaviour in $M_{\text{Ein}}(r)$ at high $r$, but the most divergent in $M_g(r)$ in the same limit.
- The gas temperature profiles were somewhat different for the $\alpha_{\text{Ein}}$ values considered here. This suggests that if one can carefully measure the temperature profile of a cluster, then one could infer $\alpha_{\text{Ein}}$ and use this in the model presented here (though one has to be aware of cooling flow and merger activity).

Next we applied Bayesian analysis to real and simulated AMI data-sets using PM I and PM II, to compare the models’ parameter estimates and fits to the data. Using real data from cluster A611 we found the following.

- The $\alpha_{\text{Ein}} = 0.05$ and $\alpha_{\text{Ein}} = 0.2$ models gave very similar results to the NFW model; the $\alpha_{\text{Ein}} = 2$ model however underestimates $M(r_{200})$ relative to the other three models.
- The Bayesian evidence values calculated from these four analyses were roughly equal, suggesting no model provided a statistically significant fit relative to the others.

Simulating clusters with either NFW or Einasto dark matter profiles, which were then ‘observed’ by AMI, we found the following.

- For 15 out of 16 clusters, the Einasto model recovered the input mass better than the NFW model. The only cluster where this was not the case (NFW simulated with $M(r_{200}) = 1 \times 10^{15}M_{\odot}$ and $z = 0.9$), the posterior distributions do not show good constraints on the sampling parameters, and so the parameter estimates should not be used.

**Figure 4.** Gas mass profiles as a function of log cluster radius using NFW and Einasto models. Values of $\alpha_{\text{Ein}} = 0.05, 0.2,$ and $2.0$ are used as inputs. Top row has $z = 0.15$, bottom row has $z = 0.9$. Left column has $M(r_{200}) = 1 \times 10^{14}M_{\odot}$, right column has $M(r_{200}) = 1 \times 10^{15}M_{\odot}$.
The evidence values of both Einasto and NFW models are considerably lower for the high mass simulations.

Considering the evidence ratios between the Einasto and NFW models for a given simulation, more data is needed to come to a conclusive decision over model preference in 10 of the cases. However according to the Jeffreys scale (Jeffreys 1961), three of simulations gave ‘substantial’ preference towards the Einasto model; and in two of these cases the NFW analysis did not constrain the sampling parameters as well as the Einasto analysis. In the third case neither analysis constrained the parameters well.

The two clusters where the evidence ratio was in favour of the NFW model also showed poor posterior distribution constraints.

When allowing $\alpha_{\text{Ein}}$ to vary in the analysis, in two out of eight of the Einasto simulations used the posterior distributions showed some constraints on the value of $\alpha_{\text{Ein}}$ which gave estimates close to the input values.

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**References**

Anders E., Grevesse N., 1989, GeCoA, 53, 197

APPENDIX A: EINAUSTO MASS INTEGRAL

From equations 10 and 14 we have that

\[
M(r) = \int_0^r 4\pi r'^2 \rho_{\text{Ein},\text{PH}(r')} \, dr'
= 4\pi \rho_{\text{Ein}} \exp\left(2/\alpha_{\text{Ein}}\right) \int_0^r r'^2 \exp\left[-2\alpha_{\text{Ein}}\left(r'/r\right)^{\alpha_{\text{Ein}}}\right] \, dr'.
\]

(A1)

Using the substitution

\[
u = \frac{2/\alpha_{\text{Ein}}}{r^{\alpha_{\text{Ein}}}} \Rightarrow du = \frac{3 \times 2/\alpha_{\text{Ein}}}{r^{\alpha_{\text{Ein}}}} \, dr,
\]

(A2)

then equation A1 becomes

\[
M(r) = \frac{4\pi \rho_{\text{Ein}} \exp\left(2/\alpha_{\text{Ein}}\right) 3/\alpha_{\text{Ein}}}{3 \times 2/\alpha_{\text{Ein}}} \int_0^r \frac{3/\alpha_{\text{Ein}}}{r^{\alpha_{\text{Ein}}}} \exp\left(-9\alpha_{\text{Ein}}/3\right) \, du.
\]

(A3)

Finally, using the substitution \( t = \rho_{\text{Ein}} \), so that \( dt = \rho_{\text{Ein}}^2 \, du \), then the integral in equation A3 (ignoring the constant factor) becomes

\[
\frac{3}{\alpha_{\text{Ein}}} \int_0^{\theta_{\text{Ein}}/3} u^{\alpha_{\text{Ein}}/3} e^{-u/3} \, du = \frac{3}{\alpha_{\text{Ein}}} \int_0^{e^{\alpha_{\text{Ein}}/3}} t^{\alpha_{\text{Ein}}-3} e^{-t} \, dt
= \gamma \left[ \frac{3}{\alpha_{\text{Ein}}} \frac{2}{\alpha_{\text{Ein}}} \left( \frac{r}{r-2} \right)^{\alpha_{\text{Ein}}} \right],
\]

(A4)

where the last equality follows from the definition of the incomplete lower Gamma function \( \gamma \), and including the constant factor in equation A3 leads to the result

\[
M(r) = \frac{4\pi \rho_{\text{Ein}} \exp\left(2/\alpha_{\text{Ein}}\right)}{3} \frac{3/\alpha_{\text{Ein}}}{\alpha_{\text{Ein}}} \gamma \left[ \frac{3}{\alpha_{\text{Ein}}} \frac{2}{\alpha_{\text{Ein}}} \left( \frac{r}{r-2} \right)^{\alpha_{\text{Ein}}} \right].
\]

(A5)

APPENDIX B: DETERMINING \( r_{500} \) ITERATIVELY

Evaluating equations 9 and 14 at \( r_{500} \) and equating we get

\[
\frac{4\pi}{3} \frac{500\rho_{\text{crit}}(z)}{\alpha_{\text{Ein}}} \frac{3}{\alpha_{\text{Ein}}} = 4\pi \rho_{\text{Ein}} \exp\left(2/\alpha_{\text{Ein}}\right) \left[ \frac{3}{\alpha_{\text{Ein}}} \frac{2}{\alpha_{\text{Ein}}} \left( \frac{r_{500}}{r-2} \right)^{\alpha_{\text{Ein}}} \right].
\]
Figure 7. Posterior distributions for cluster simulated with $\alpha_{Ein} = 2.0$, $M(r_{200}) = 1 \times 10^{15} M_{\odot}$ and $z = 0.9$, modelled with: (a) Einasto dark matter profile, and (b) NFW dark matter profile.

Figure 8. Posterior distributions for cluster simulated with $\alpha_{Ein} = 2.0$, $M(r_{200}) = 1 \times 10^{14} M_{\odot}$ and $z = 0.9$, modelled with: (a) Einasto dark matter profile, and (b) NFW dark matter profile.

If we let $R = r_{500}/r_{-2}$, then we can determine $r_{500}$ by solving the following for $R$:

$$R^3 = \frac{\gamma}{\rho_{\text{crit}}(z) 500} \left( \frac{\alpha_{Ein}}{\rho_{\text{Ein}}} \right)^2 \frac{\exp \left( \frac{2}{\alpha_{Ein}} \right)}{\alpha_{Ein}} = 0$$
by some iterative root finding method e.g. Newton-Raphson. We use the starting point $R_0 = \frac{2r_{200}}{3}$ which usually results in the algorithm converging in $O(10)$ iterations.

We now show that equation B2 only has one solution for a given $r_{-2}$. We start by considering both sides of equation B1 as two different functions, and ignore constant terms for simplicity (this does not affect the truth of the final result), i.e. we consider the two functions

$$f(r_{500}) = r_{500}^{3}, \quad g(r_{500}) = \gamma \left[ \frac{3}{\alpha_{\text{Ein}}}, \frac{2}{\alpha_{\text{Ein}}} \frac{r_{500}}{r_{-2}} \right].$$

(B3)

We first note that $f(0) = g(0) = 0$, and differentiate both functions with respect to $r_{500}$

$$\frac{df}{dr_{500}} \propto r_{500}^{-2}, \quad \frac{dg}{dr_{500}} \propto r_{500}^{2} \exp \left[ \frac{2}{\alpha_{\text{Ein}}} \left( \frac{r_{500}}{r_{-2}} \right)^{\alpha_{\text{Ein}}} - 1 \right].$$

(B4)

Setting these two derivatives equal to each other yields one solution at $r_{500} = r_{-2}$ for all $\alpha_{\text{Ein}} \neq 0$, meaning the derivatives only intersect once. Furthermore $\frac{df}{dr_{500}}$ tends to zero for large $r_{500}$ whilst $\frac{dg}{dr_{500}}$ is a monotonically increasing function, meaning the former must be larger before the two intersect. This coupled with the fact that $f(0) = g(0) = 0$ means that $g(r_{500}) > f(r_{500})$ until some point (which has to be after the derivatives intersect) when the two intersect, after which $f(r_{500}) > g(r_{500})$ as $g(r_{500})$ flattens off. This proves that equation B2 only has one root and that equation B1 only has one solution in $r_{500}$ for fixed $r_{-2}$.
APPENDIX C: SIMULATIONS RESULTS TABLE

Table C1: Input and output values of simulations using NFW and Einasto dark matter profiles. The first column is what dark matter profile was used to simulate the cluster. Input $M(r_{200})$ and Input $z$ are the input values used to create the simulation for the given model. Ein out $M(r_{200})$ is the mean and standard deviation of the posterior distribution obtained inferred using an Einasto profile to model the cluster. Ein ln($Z$) is the log Bayesian evidence corresponding to the inference. NFW... is as before but using an NFW profile in the modelling. ln($Z_{Ein}/Z_{NFW}$) is the log ratio of the two evidence obtained.

<table>
<thead>
<tr>
<th>Model</th>
<th>Input $M(r_{200})$ ($\times 10^{14} M_{\odot}$)</th>
<th>Input $z$</th>
<th>Ein out $M(r_{200})$ ($\times 10^{14} M_{\odot}$)</th>
<th>NFW out $M(r_{200})$ ($\times 10^{14} M_{\odot}$)</th>
<th>Ein ln($Z$)</th>
<th>NFW ln($Z$)</th>
<th>ln($Z_{Ein}/Z_{NFW}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{Ein} = 0.2$</td>
<td>1</td>
<td>0.15</td>
<td>1.05 ± 0.01</td>
<td>1.08 ± 0.01</td>
<td>47104.4 ± 0.4</td>
<td>47104.6 ± 0.4</td>
<td>-0.2</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 2.0$</td>
<td>1</td>
<td>0.15</td>
<td>1.05 ± 0.01</td>
<td>1.48 ± 0.01</td>
<td>47181.4 ± 0.4</td>
<td>47180.7 ± 0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 0.05$</td>
<td>1</td>
<td>0.15</td>
<td>1.05 ± 0.01</td>
<td>0.97 ± 0.01</td>
<td>47175.0 ± 0.4</td>
<td>47175.3 ± 0.4</td>
<td>-0.3</td>
</tr>
<tr>
<td>NFW</td>
<td>1</td>
<td>0.15</td>
<td>1.03 ± 0.01</td>
<td>1.05 ± 0.01</td>
<td>47064.3 ± 0.4</td>
<td>47063.6 ± 0.4</td>
<td>0.7</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 0.2$</td>
<td>1</td>
<td>0.9</td>
<td>1.10 ± 0.01</td>
<td>1.21 ± 0.01</td>
<td>47173.2 ± 0.5</td>
<td>47174.5 ± 0.5</td>
<td>-1.3</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 2.0$</td>
<td>1</td>
<td>0.9</td>
<td>1.16 ± 0.01</td>
<td>1.57 ± 0.01</td>
<td>47100.8 ± 0.5</td>
<td>47095.1 ± 0.5</td>
<td>5.7</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 0.05$</td>
<td>1</td>
<td>0.9</td>
<td>1.02 ± 0.01</td>
<td>1.18 ± 0.01</td>
<td>47094.2 ± 0.4</td>
<td>47094.8 ± 0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>NFW</td>
<td>1</td>
<td>0.9</td>
<td>0.95 ± 0.01</td>
<td>1.05 ± 0.01</td>
<td>47105.2 ± 0.4</td>
<td>47106.7 ± 0.4</td>
<td>-1.5</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 0.2$</td>
<td>10</td>
<td>0.15</td>
<td>10.23 ± 0.02</td>
<td>10.33 ± 0.01</td>
<td>46814.8 ± 0.5</td>
<td>46815.0 ± 0.5</td>
<td>-0.2</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 2.0$</td>
<td>10</td>
<td>0.15</td>
<td>10.18 ± 0.01</td>
<td>15.06 ± 0.01</td>
<td>46640.7 ± 0.5</td>
<td>46638.3 ± 0.6</td>
<td>2.4</td>
</tr>
<tr>
<td>$\alpha_{Ein} = 0.05$</td>
<td>10</td>
<td>0.15</td>
<td>10.21 ± 0.02</td>
<td>9.61 ± 0.01</td>
<td>46844.5 ± 0.5</td>
<td>46844.9 ± 0.5</td>
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<td>46873.9 ± 0.5</td>
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<td>46829.7 ± 0.6</td>
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<td>12.26 ± 0.02</td>
<td>46833.8 ± 0.6</td>
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<tr>
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<td>46926.4 ± 0.6</td>
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<td>-0.2</td>
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