Can Feedback Traders Rock the Markets?
A Logistic Tale of Persistence and Chaos

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Abstract

This paper introduces a nonlinear feedback trading model at high frequency. All price adjustment is endogenous, driven by asset return and volatility in the previous trading period. There is no stochastic uncertainty or asymmetric information. The dynamics of expected returns display stable or unstable behavior—including the possibility of turbulence and chaos—as a function of market liquidity (inverse price impact) and the concentration of investor beliefs, which is proportional to the intensity of positive feedback. The results highlight the complementary role of investor diversity and market liquidity in maintaining financial stability.

Keywords: Feedback Trading, Liquidity, Heterogeneity, Financial Stability, Chaos, Nonlinear Dynamics

JEL classification: G10, G12

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1   Introduction

Can feedback trading sustain positive expected returns on financial assets? According to the weak form of the efficient markets hypothesis (EMH), expected returns should be zero regardless of the trading frequency. The main objective of this paper is to investigate the key premise of EMH in a nonlinear (logistic) feedback trading model with no stochastic uncertainty. There are two key trading parameters: asset liquidity, which is inversely related to the price impact of net order flow, and the intensity of positive feedback trading, which is inversely related to the diversity of investor opinion.

The intensity of high-frequency trading reflects the relative strength of two feedback rules. First, investors display positive feedback, linked to "momentum" trading strategies: they buy/sell the risky financial asset if returns in the previous period were positive/negative. Second, investors also exhibit risk feedback: their asset demand falls/rises following relatively volatile/tranquil trading periods. Risk feedback amounts to a "buy low-sell high" trading strategy in a mean-variance world where risk and expected returns are positively related.

Positive feedback can destabilize financial markets while risk feedback has a stabilizing influence. Thus, the interaction of the two types of feedback matters for financial stability. I capture this interaction by assuming the intensities of positive and risk feedback to be inversely related. There are two possible justifications for this assumption. First, it is consistent with the findings of Cohen and Shin (2003) on the high-frequency US Treasury bond market. These authors report strong evidence of positive feedback in high-frequency trading on US Treasury bonds: (a) returns tend to be more positively autocorrelated when market conditions are more volatile; and (b) price declines/rises elicit asset sales/purchases, and such feedback trading is stronger in volatile market conditions. Arguably, this applies to positive feedback both during episodes of market euphoria—such as the dot.com bubble

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of 1999/2000—and during market turbulence. Risk feedback captures this trading pattern: sales pressure is growing in the previous period's volatility, given the intensity of feedback trading. Second, Persaud (2001, 2003) argues that momentum strategies may gather strength as investors' beliefs become more concentrated. Heterogeneity decreases during market euphoria as well as during crises: in both cases, the market becomes one-sided and risk considerations matter very little. To quote Alan Greenspan, during the Long Term Capital Management liquidity crisis "...everyone wanted out" (October 7, 1998, quoted in Longstaff (2004)). Thus, a useful way of capturing sharp falls in the diversity of investor opinion is to assume the relative intensity of momentum and risk feedback to be inversely related. In the sequel, I use the terms *heterogeneity / diversity* and *concentration / uniformity* of investor opinion interchangeably.

The main features of the model are as follows. At the start of each trading period, the risk-neutral market maker receives the net order flow and adjusts the asset price level from the previous period using a linear pricing rule. The return dynamics generated by the two feedback trading rules follow a quadratic logistic map which is parametric in asset liquidity and investor diversity. In turn, the logistic map has two fixed points, one of which is always zero and the other generically non-zero. If stable, the zero fixed point corresponding to zero expected returns is consistent with EMH. Similarly, if the non-zero fixed point is stable then it violates EMH. The fixed points' relative stability depends on the size of the two trading parameters.

There are three main findings. First, the zero fixed point is dynamically stable for sufficiently diverse investor opinion (i.e. for sufficiently low intensity of positive feedback) and sufficiently high asset liquidity (i.e., for sufficiently small price impact coefficients). For such combinations of the trading parameters the non-zero fixed point is dynamically unstable. Importantly, if investor diversity and/or asset liquidity decline, the relative stability of the two fixed points is *reversed*. Now the zero fixed point becomes unstable while

the non-zero fixed point is stable.

Further, the threshold level of investor diversity triggering the transition to dynamic instability increases in the asset's liquidity. Equivalently, the threshold level of positive feedback intensity at which the transition occurs is decreasing in asset liquidity. This property indicates that if liquidity is already low, then a small decline in investor diversity—equivalently, a small rise in positive feedback intensity—can destabilize the zero fixed point. Indeed, if price impact coefficient grows and asset liquidity falls below a certain point, both fixed points become unstable for any level of investor diversity. The dynamics of the logistic map then enter a range of period-doubling bifurcations. Eventually, as liquidity "dries up" equilibrium market dynamics become completely unstable and chaos emerges. A deterministic nonlinear dynamical system is chaotic if it is sensitively dependent on initial conditions and produces random-looking dynamic paths.

The relative stability of the two fixed points also depends on the returns distribution. If high-frequency returns are Gaussian, zero expected returns is a stable equilibrium for all feasible values of the trading parameters. In turn, non-Gaussian returns is a necessary, but not sufficient, condition for zero expected returns to be unstable. It should be stressed that the dynamic progression from stable equilibrium to chaos is deterministic as it is completely controlled by asset liquidity and investor diversity.

Second, the model yields several testable properties on short-term returns persistence, proxied by the first-order autocorrelation coefficient. The benchmark implication of the Shiller (1984)-Sentana and Wadwani (1992) framework is that higher conditional volatility unambiguously increases return autocorrelation, consistent with the asymmetric GARCH models of Nelson (1991) and Glosten, Jagannathan and Runkle (1993). By contrast, in this paper persistence can take either sign depending on the interaction of the two trading parameters with the second and third conditional moments of returns. In particular, positive and risk feedback respectively introduce

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conditional volatility and skewness to the autocorrelation coefficient.

In tranquil market conditions when skewness is negligible, autocorrelation is always positive but independent of volatility, and its magnitude falls with investor diversity (rises with positive feedback intensity). In order for conditional volatility to affect persistence the returns distribution has to be asymmetric. Specifically, higher volatility induces more persistence (raising positive or lowering negative autocorrelation) only if conditional skewness is positive. That is more likely to be the case during financial bubbles associated with investor euphoria. Higher volatility induces less persistence *à la* Shiller-Sentana-Wadwani only if conditional skewness is negative, such as in the aftermath of a market crash. Moreover, the model predicts that bigger average order flow—a reasonable proxy of trading volume—induces more persistence if skewness is negative. Conversely, with positive skewness, bigger average order flow actually lowers persistence, as found by Campbell, Grossman and Wang (1993) for stock returns. These results suggest that distributional asymmetries may contribute to short term returns persistence over and above the current level of asset risk.

The paper combines research on non-fundamental-based investor behavior, driven from behavioral finance (see Barberis and Thaler (2003)) with research on the potential for instability and chaos arising from the interaction of financial market participants, driven by work on nonlinear dynamics. Recent applications of chaos to economics include Benhabib, Schmitt-Grohe and Uribe (2004), Brock et al. (1996), and Christiano and Harrison (1999). Models where asset prices are driven by non-fundamentals can be traced to Shiller (1984), Cutler, Poterba and Summers (1990) and Sentana and Wadhwani (1992). In the Sentana-Wadhwani model, positive feedback traders interact with smart-money traders, who trade based on expected fundamentals. In contrast, I assume both trading rules are based on non-fundamentals.[1]

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[1]On the relationship between heterogeneity of investor beliefs and speculative bubbles see Hong, Scheinkman and Xiong (2006), Scheinkman and Xiong (2003) and Shiller (2000).

On the empirical front, there is strong evidence that liquidity is time-varying especially during market stress; see Farmer et al. (2005). Liquidity risk is being integrated in asset pricing models (Acharya and Pedersen (2004), Cochrane (2001) and Engle and Lange (1997)) and financial risk management (Bangia et al. (2002)). The direction of causation from signed order flow to asset prices is well documented by Hasbrouck (1991) for the stock market and Evans and Lyons (2002) for foreign exchange. On the reverse causation, Watanabe (2002) finds that daily Japanese stock returns exhibit positive autocorrelation when volatility is low and negative autocorrelation when it is high. Also, returns tend to be more negatively autocorrelated after price declines than after price rises. Kim and Wei (2001) use panel data to show that Korean off-shore funds displayed less positive feedback than their on-shore counterparts during the Asian crises of 1997-98. Finally, Bohl and Siklos (2004) using daily stock index data report that feedback trading is more pronounced in emerging than in mature financial markets.

In the remainder of the paper, Section 2 presents the feedback trading rules; Section 3 derives returns autocorrelation; Section 4 studies the existence and stability of equilibrium market dynamics as fixed points of a parametric logistic map; Section 5 classifies fixed point stability in terms of asset liquidity (inverse price impact), investor diversity, and the conditional variance and skewness of returns; Section 6 shows that (*non-*) Gaussian returns are a sufficient (*necessary*) condition for financial (*in*)stability, and highlights the policy implications for financial regulators; and Section 7 concludes. All fixed point definitions are in the Appendix.

2   Asset demand and order flow

All trading on the single risky asset is at high frequency, so time intervals between trades are small but discretized; typically, sampling is at 5-minute frequency or higher. Letting period t denote the interval between times $t-1$

and t, the asset return in period t is $r_t \equiv \Delta \log P_t$, where $\log P_t$ is the log price level at time t. By the weak form of the EMH, the conditional expectation of returns h-periods-ahead is zero: $E_{t-1} r_{t+h} = 0$ for all $h \geq 0$. When expected returns are zero, actual and excess returns coincide. Defining excess returns in period t as $x_t \equiv r_t - E_{t-1} r_t$ implies $r_t = x_t$, $\sigma_{xt}^2 = var[r_t] = E_{t-1} x_t^2$, and $s_{xt} = skew[r_t] = E_{t-1} x_t^3$. Conditional variance and skewness then are the second and third centered moment of x_t.

Investors employ two feedback trading rules. Following *positive feedback*, they buy/sell the risky asset in period t if they observe positive/negative returns in $t-1$, a characteristic of momentum strategies. Following *risk feedback*, traders react to conditional volatility in period $t-1$ by buying/selling the asset at time t if they observe lower/higher volatility in period $t-1$, consistent with buy low-sell high strategies.

Relative asset demand is assumed to be different for each rule. I define the *intensity* of positive and risk feedback trading to be f and $1-f$ respectively, where $0 \leq f \leq 1$. The two trading rules have equal intensity iff $f = 0.5$.[2] As argued in Section 1, the inverse relationship between positive and risk feedback captures the time-varying diversity of investor opinion. As investor diversity declines, positive feedback gathers momentum and risk considerations matter relatively less. By contrast, with sufficient heterogeneity of investor beliefs positive feedback becomes less intense and risk considerations gain in importance.

Let the net order flow from positive feedback be $\omega_t^+ = f\gamma x_{t-1}$. The constant feedback intensity f is *decreasing* in investor diversity (increasing in the uniformity of investor opinion), and γ is a scalar mapping actual returns in period $t-1$ to asset units demanded at time t. The sign of ω_t^+ depends on last period's return. In contrast, Sentana and Wadhwani

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[2] Kim and Wei (2001) also model trading intensity, proxied by time-varying portfolio weights on individual stocks. An alternative interpretation of trading intensity, motivated from game theory, is that there are two types of trader in the market at any time, a proportion $f\%$ of positive feedback and $(1-f)\%$ of risk feedback traders.

(1992) allow both positive and negative feedback by assuming $\gamma > 0$ or $\gamma < 0$, respectively. Similarly, let the net order flow from risk feedback be $\omega_t^- = -(1 - f)\gamma x_{t-1}^2 < 0$. In this case, the intensity of risk feedback trading $1 - f$ is *increasing* in investor diversity. Risk feedback corresponds to the risk component of smart money traders (driven by mean-variance) in the Shiller-Sentana-Wadwani framework. By generating less/more asset demand as conditional volatility x_{t-1}^2 grows/declines, it acts as an automatic stabilizer on the market.

The aggregate order flow to the market maker at time t is given by

$$\omega_t \equiv \omega_t^+ + \omega_t^- = f\gamma x_{t-1} - (1 - f)\gamma x_{t-1}^2 \ , \tag{1}$$

where the sign of ω_t depends on the previous period's trend and risk—proxied by the level and square of returns at $t - 1$—and the diversity of investor opinion, captured by $1 - f$. It follows that the expected net order flow for period t is always negative

$$E_{t-1}\omega_t = -\lambda(1 - f)\gamma\sigma_x^2 < 0 \tag{2}$$

At time t, the risk-neutral market maker receives ω_t and adjusts the asset price from period $t - 1$ to t using a linear pricing rule[3]

$$x_t = r_t = \lambda\,\omega_t, \ \lambda > 0 \tag{3}$$

The price impact of a unit change in order flow measures market depth. An asset's liquidity is then proxied simply by inverse price impact, $1/\lambda$. Substituting equation (1) into (3) yields

$$x_t = \lambda f\gamma x_{t-1} - \lambda(1 - f)\gamma x_{t-1}^2 \tag{4}$$

As $\lambda > 0$ always, the risky asset's return in period t is positive (negative) iff $\omega_t > (<)0$.

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[3]Holden and Subrahmanyam (1996) list the full set of assumptions for the pricing rule to be linear.

3 Short term returns persistence

As returns dependence is limited to adjacent time periods, the first autocorrelation coefficient of x_t captures short term persistence. From equation (4), the first autocovariance of x_{t+1} is

$$
\begin{aligned}
cov(x_{t+1}, x_t) &= cov[\lambda f \gamma x_t - \lambda(1-f)\gamma x_t^2, x_t] \\
&= \lambda f \gamma \sigma_{xt}^2 - \lambda(1-f)\gamma s_{xt}
\end{aligned}
$$

σ_{xt}^2 and s_{xt} denote conditional volatility and skewness of returns at time t. Dividing through by σ_{xt}^2 yields the first-order autocorrelation

$$
\rho_{1t}(x_{t+1}, x_t) = \lambda f \gamma - \lambda(1-f)\gamma \frac{s_{xt}}{\sigma_{xt}^2} \; , \tag{5}
$$

where $-1 \leq \rho_{1t} \leq 1$.[4] Note that the first term is always positive and its magnitude increases with f, the concentration of investor opinion. Therefore, $\rho_{1t} = \lambda f \gamma > 0$ iff $s_{xt} = 0$. The second term is *negative* (positive) iff $s_{xt} > 0$ ($s_{xt} < 0$). Higher volatility will then generate more positive, or less negative autocorrelation: $\partial \rho_{1t}/\partial \sigma_{xt}^2 > 0$ when $s_{xt} > 0$.

This comparative static result differs from Sentana and Wadhwani (1992). There, positive feedback dominates in periods of high volatility and generates negative autocorrelation, while negative feedback ($\gamma < 0$) prevails at low volatility and induces positive autocorrelation. In this paper I rule out the possibility of negative feedback to focus on the convenient analytical properties of the logistic map in Section 4. Also, note that $\partial \rho_{1t}/\partial \sigma_{xt}^2 > 0$ when $s_x > 0$, consistent with Cohen and Shin's (2003) finding that short term persistence in the US Treasury bond market is stronger when volatility is higher.

Short term persistence is constrained by requiring that $-1 \leq \rho_{1t} \leq 1$ in equation (5). Expressed in terms of the concentration (uniformity) of investor

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[4]Note that conditional kurtosis would also enter in expression (5) if conditional skewness affected trading decisions. For evidence of nonlinear dependence in stock returns see LeBaron (1992) and Campbell, Lo and MacKinlay (1997).

investor opinion, proxied by f, the inequalities become

$$0 \le f_L = \frac{s_{xt}\lambda\gamma - \sigma_{xt}^2}{\lambda\gamma(s_{xt} + \sigma_{xt}^2)} \le f \le \frac{s_{xt}\lambda\gamma + \sigma_{xt}^2}{\lambda\gamma(s_{xt} + \sigma_{xt}^2)} = f^U \le 1 \;, \qquad (6)$$

where $f_L < f^U$ for all $\sigma_x \ne 0$. Assuming that $s_{xt} + \sigma_{xt}^2 > 0$ to combine conditional volatility with all positive and some negative skewness values $(s_{xt} > -\sigma_{xt}^2)$ suggests $f^U \le 1$ iff $\lambda\gamma \ge 1$. The price impact coefficient λ is then bounded below by $1/\gamma$. Similarly, $f_L \ge 0$ iff $s_{xt}\lambda\gamma \ge \sigma_{xt}^2$. Combining the two inequalities yields

$$
\begin{aligned}
\lambda &\ge \max\{\frac{\sigma_{xt}^2}{s_{xt}\gamma}, \frac{1}{\gamma}\} \ge 0 \\[2mm]
0 &\le \frac{1}{\lambda} \le \min\{\frac{s_{xt}\gamma}{\sigma_{xt}^2}, \gamma\}
\end{aligned}
\qquad (7)
$$

Inequalities (7) yield testable implications for the admissible range of asset liquidity in terms of the second and third conditional moments of returns; discussing these is beyond the scope of the present paper.

Equation (5) can also be used to assess the empirical finding of Campbell, Grossman and Wang (1993) that stock returns persistence is negatively related to trading volume. Proxying the latter by net order flow, substitute equation (2) into (5)

$$
\begin{aligned}
\rho_{1t} &= \lambda f\gamma - \lambda\left[(1-f)\gamma\sigma_{xt}^2\right]\frac{s_{xt}}{\sigma_{xt}^4} = \\[2mm]
&= \lambda f\gamma - \frac{\lambda s_{xt}}{\sigma_{xt}^4}[E_{t-1}\omega_t]
\end{aligned}
\qquad (8)
$$

Hence, average order flow and autocorrelation are negatively related iff skewness is positive. In contrast, if skewness is negative then higher average order flow actually *contributes* to short term returns persistence.

4 Equilibrium expected returns

4.1 Fixed point existence

I propose solving first-order difference equation (4) as a parametric logistic map $h_{f,\lambda}(\cdot)$ from x_t to x_{t+1}

$$
\begin{aligned}
x_{t+1} &= h_{f,\lambda}(x_t) \equiv px_t(1 - qx_t) \\
p &= f\lambda\gamma > 0 \ , \ q = \frac{1-f}{f} > 0
\end{aligned}
\tag{9}
$$

Given $\gamma > 0$, logistic parameter p is monotonically decreasing in asset liquidity $1/\lambda$ and investor diversity $1 - f$. In turn, logistic parameter q reflects the feedback rules' relative strength; $\partial q/\partial p > 0$ and $q = 1$ iff $f = 0.5$.

The logistic map has two fixed points, defined as $x = h_{f,\lambda}(x_t) = x_{t+1}$

$$
\begin{aligned}
x^1 &= 0 \\
x^2 &= \frac{p-1}{pq} = \frac{f\lambda\gamma - 1}{\lambda\gamma(1-f)}
\end{aligned}
\tag{10}
$$

Note that $f \neq 1$ is required for x^2 and q to be finite, so not all market activity can be driven by positive feedback trading—some risk feedback must exist at any time. Equations (10) also imply $\frac{\partial x^2}{\partial \lambda} = \frac{1}{\lambda^2\gamma(1-f)} > 0$ always, while $\frac{\partial x^2}{\partial f} = \frac{\lambda\gamma-1}{\lambda\gamma(1-f)^2} > 0$ iff $\lambda\gamma > 1$. Further, check that

$$
h_{f,\lambda}\left(\frac{1}{q} = \frac{f}{1-f}\right) = x^1
\tag{11}
$$

and

$$
h_{f,\lambda}\left(\frac{1}{pq} = \frac{1}{\lambda\gamma(1-f)}\right) = x^2 \ ,
\tag{12}
$$

so $x^{1\prime} = \frac{f}{1-f} > 0$ and $x^{2\prime} = \frac{1}{\lambda\gamma(1-f)} > 0$ map onto fixed points x^1 and x^2, respectively after one iteration of $h_{f,\lambda}$. The two *eventually fixed* points of the logistic function are important in the classification of dynamic stability in Section 5. Setting $\gamma = 1$, without loss of generality, Figure 1 below illustrates the behavior of $x_{t+1} = h_{f,\lambda}(x_t)$ for different f and λ combinations.[5] Fixed points x^1 and x^2 lie at the intersections of $h_{f,\lambda}$ with the 45º line.

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[5] The analysis can accommodate $\gamma \neq 1$ without changing the essence of the results.

FIGURE 1 HERE

Clockwise from the top left, in Panel A investor diversity is set at $1 - f = 0.8$ and asset liquidity at $\lambda = 1.6$. In the population view of feedback trading, 20% of market participants are positive feedback traders and 80% are risk feedback traders. For these trading parameters $p = 0.60$ and $q = 1$, hence $x^2 = -0.53$ and $x^{2\prime} = 0.78$. In Panel B $f = 0.5$ and $\lambda = 2$, so 50% of investors are positive feedback traders. Then $p = q = 1$ and the two fixed points collapse to $x^1 = x^2 = 0$. In Panel C $f = 0.1$ and $\lambda = 15$, implying $p = 1.5$ and $x^2 = 0.04$. Lastly, in Panel D $f = 0.2$ and $\lambda = 20$, hence $p = 4$ and $x^2 = 0.19$. Figure 1 indicates that the location of the non-zero fixed point is sensitive to the trading parameters.

4.2   Fixed point stability

A fixed point is stable (*unstable*) if the absolute value of the slope of $h_{f,\lambda}$ at that fixed point is smaller (*greater*) than one; see Appendix. At this point, note that expected returns must be zero—defined as a stable zero fixed point corresponding to $E_{t-}x_t = 0$—in order to be consistent with EMH. In other words, asset returns must be (linearly) unpredictable at any forecast horizon. It follows that fixed point $x^1 = 0$ always satisfies $E_{t-}x_t = 0$. However, the second fixed point is generically non-zero: $x^2 < (>)0$ iff $p = f\lambda\gamma < (>)1$. It thus violates $E_{t-1}x_t = 0$ unless $p = 1$.

To determine the range of asset liquidity and investor diversity such that x^1 is stable and x^2 unstable, differentiate equation (9) with respect to x

$$h'_{f,\lambda}(x) = p - 2pqx = f\lambda\gamma - 2(1-f)\lambda\gamma x \tag{13}$$

The absolute value of (13) at x^1 and x^2 is

$$\mid h'_{f,\lambda}(x^1 = 0) \mid = \mid f\lambda\gamma \mid = \mid p \mid \tag{14}$$
$$\mid h'_{f,\lambda}(x^2 \neq 0) \mid = \mid 2 - f\lambda\gamma \mid = \mid 2 - p \mid,$$

which depends only on p. Note that for $\gamma = 1$, any $\lambda > 1$ implies $p > 1$ for some $0 < f < 1$. Equations (14) suggest that in order for $\mid f\lambda\gamma \mid < 1$ the value of f cannot exceed $1/\lambda\gamma$ given $\lambda > 0$—similarly, the price impact measure cannot exceed $1/f\gamma$ given $f \in (0,1)$. Thus, for any level of asset liquidity, a more diverse investor opinion has a stabilizing influence on the zero fixed point while lower diversity is destabilizing, and *vice versa* for the non-zero fixed point.

As $h'_{f,\lambda}$ is smoothly decreasing in x given λ and f, the solutions of $\mid h'_{f,\lambda}(x) \mid = 1$ constitute bounds for the stable range of either fixed point. The bounds are

$$x_{\min} = \frac{f\lambda\gamma - 1}{2\lambda\gamma(1-f)} \ , \ x^{\max} = \frac{f\lambda\gamma + 1}{2\lambda\gamma(1-f)} \tag{15}$$

Comparing (10) and (15) indicates $x_{\min} = x^2/2$. If $x^2 < 0$ then $x_{\min} > x^2$, so x^2 is unstable. Conversely, if $x^2 > 0$ then $x_{\min} < x^2$ and x^2 is stable.

5 A taxonomy of fixed point dynamics

For notational convenience, I illustrate the route from stability to chaos numerically in terms of f. Market dynamics change as positive feedback intensity—growing in the concentration of investor opinion—and asset liquidity affect logistic parameter p. Fixing the liquidity parameter at $1/\lambda$, positive feedback intensity is increased from $f = 0.001$ to 0.999 in steps of size 0.001. Starting at $x_0 = 0.00001$, corresponding to zero expected returns, Figure 2 below plots the evolution of the fixed point(s) of $h_{f,\lambda}(x)$ as f varies. In Panels A, B and C the price impact coefficient is respectively fixed at $\lambda = 4$, 5 and 6

FIGURE 2 HERE

Figure 2 suggests the following taxonomy for the dynamic stability of $h_{f,\lambda}(x)$ in terms of trading parameters λ and f:

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(i) When $p \in (0, 1)$ clearly $\mid f\lambda\gamma \mid < 1$ and $\mid 2 - f\lambda\gamma \mid > 1$. Therefore, $x^1 = 0$ is a stable (attracting) fixed point while $x^2 \neq 0$ is unstable (repelling). The stable set of x^1 is the open interval bounded from below by x^2, which is negative, and from above by its eventually fixed point $x^{2\prime}$, which is positive: $W(x^1) = \left(\frac{f\lambda\gamma - 1}{\lambda\gamma(1-f)} , \frac{1}{\lambda\gamma(1-f)} \right)$. Then, paths of x_t starting from any point in $W(x^1)$ will converge to x^1 after finitely many iterations of $h_{f,\lambda}(x_t)$. In contrast, the stable set of x^2 includes only itself and $x^{2\prime}$, that is $W(x^2) = \{x^2, x^{2\prime}\}$. Paths of x_t at *any* point other than x^2 or $x^{2\prime}$ will diverge to infinity (the stable set of infnity is the remainder of the real line).

(ii) At $p = 1$, the absolute value of $h_{f,\lambda}$ at the fixed points equals one ($\mid h'_{f,\lambda}(x^1) \mid = \mid h_{f,\lambda}(x^2) \mid = 1$) and the logistic map displays a transcritical bifurcation. From (10), $x^1 = x^2 = 0$ and the two fixed points coincide as shown in Figure 1, Panel B. The positive feedback intensity corresponding to each value of λ constitutes a *stability threshold*. These are located at $f_{\min} = 0.25$, 0.20 and 0.17 when $\lambda = 4$, 5 and 6, and highlighted with dashed vertical lines in Figure 2. The stable set of the single (zero) fixed point is $W(x = 0) = [0, x^{1\prime}]$.

(iii) When $p \in (1, 3)$, $\mid f\lambda \mid > 1$ and $0 < \mid 2 - f\lambda \mid < 1$ so the stability properties are reversed. In this logistic parameter range, x^1 becomes unstable and x^2 stable. The stable set of x^1 contains only itself and its eventually fixed point, $W(x^1) = \{0, x^{1\prime}\}$, while the stable set of x^2 is the open interval $W(x^2) = (0, x^{2\prime})$. In the range $p \geq 1$ the stable set of infinity includes the intervals $(-\infty, x^1 = 0)$ and $(x^{2\prime}, \infty)$. Figure 2 clearly shows that the steady-state path of x_t exhibits a discontinuity at approximately $f = 0.68$, 0.55 and 0.47 in Panels A, B and C respectively. The jump from the unstable zero fixed point to the stable non-zero fixed point is located at $x^2 = 1.38$, 0.77 and 0.55% when $\lambda = 4$, 5 and 6. Note that the steady-state path encounters the non-zero fixed point at $p \simeq 2.74$ and $\mid h'_{f,\lambda}(x^2) \mid \simeq 0.74 < 1$ in all three cases, so the dynamics are within the attracting range of x^2.

(iv) At $p = 3$, $\mid f\lambda \mid = 3$ and $\mid 2 - f\lambda \mid = 1$. Similar to (ii) above, the intensity of positive feedback for each value of λ is an *instability threshold*, located at $f^{\max} = 0.75$, 0.60 and 0.50 when $\lambda = 4$, 5 and 6 respectively and highlighted with dashed vertical lines in Figure 2.

(v) Finally, if $p \in (3, \infty)$ then $\mid f\lambda \mid > 1$ and $\mid 2 - f\lambda \mid > 1$, so *both* fixed points are unstable. At $p \simeq 3.3$, occuring at about $f = 0.83$, 0.65 and 0.54 in Panels A, B and C, the steady-state path displays a cascade of period-doubling bifurcations (every period-2 attracting orbit splits into a period-4 attracting orbit and a period-2 repelling orbit). When the logistic paramer is in the interval $3.45 < p \leq 4$ the dynamics begin to change very rapidly. For $p > 3.6$ it can be shown that there are periodic points of all orders if a period-3 periodic point exists.[6]

Finally, beyond $p > 4$ the dynamics of $h_{f,\lambda}$ become chaotic; the orbits of x_t are period-k for all $k > 0$ and display sensitive dependence to initial conditions; see Appendix. It should be stressed that all market dynamics are deterministic, i.e. endogenous. Tambakis (2006) extends the present model with a stationary stochastic process driving the price impact coefficient.

6 Financial (in)stability

6.1 Gaussian versus non-Gaussian returns

The distributional properties of returns impact upon fixed point stability. If returns are standard normal, $x_t \sim N(0,1)$ so that $\sigma_x = 1$ and $s_x = 0$, autocorrelation inequalities (6) reduce to

$$-\frac{1}{\lambda\gamma} \leq f \leq \frac{1}{\lambda\gamma} \tag{16}$$

Given $f \in (0,1)$, the relevant inequality is $f\lambda\gamma \leq 1$ so investor uniformity (positive feedback intensity) is bounded above by $1/\lambda\gamma$. From equation (14),

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[6]The proof lies beyond the scope of this paper, for a sketch see Holmgren (1997). The seminal paper on chaos in low-dimensional non-linear systems is Li and Yorke (1975).

the two fixed points are on opposite sides of one: $\mid h'_{f,\lambda}(x^1) \mid \leq 1$ and $\mid h'_{f,\lambda}(x^2) \mid \geq 1$. The zero fixed point is then stable while the non-zero fixed point is unstable. Hence, if returns are standard normal there is no feasible combination of the trading parameters such that the non-zero fixed point is stable. Put differently, $x_t \sim N(0,1)$ is a *sufficient* condition for zero expected returns to be a stable steady state.

These observations suggest that the unstable logistic range of Section 5 ($1 < p < 3$, shown in Figure 1, Panel C) cannot arise unless returns are non-Gaussian. The presence of skewness and/or excess kurtosis (fat tails) is a *necessary* condition for $p > 1$. It is clearly not a sufficient condition, as one can always select f and λ so that $f\lambda\gamma < 1$ regardless of the returns distribution. Thus, zero expected returns may be unstable *only if* the returns pdf is non-Gaussian.

Further, setting $\gamma = 1$ and defining $(p_{\min}, p^{\max})$ to be an open interval for p implies $\lambda > p_{\min}/f$. As $f < 1$, it follows $p_{\min}$ is the minimum feasible price impact. Note well that the converse does not hold: the upper bound on liquidity ($\lambda < p^{\max}/f$) is infinite as f can be arbitrarily close to zero.

6.2 Implications for financial regulators

For comparison purposes, fix the concentration of investor opinion to f and consider a negative liquidity shock (positive shock to the price impact coefficient). Figure 2, Panels A-C then indicate that the threshold level of investor concentration at which the zero expected returns become unstable increases in asset liquidity. Hence, trading in more liquid assets is stable for a *wider* range of investor diversity than trading in less liquid assets. It also follows that, if an asset suddenly becomes illiquid for exogenous reasons (such that $1/\lambda$ falls), then investor opinion must become more diverse (so $1 - f$ should rise) if zero expected returns are to remain stable. Equivalently, the intensity of risk feedback must increase to compensate for the instability

induced by the adverse liquidity movement. In the game-theoretic "population" interpretation of investor diversity, there must be more risk feedback traders to exert a compensating stabilizing influence on the dynamics. The converse situation is also instructive. If herding behavior grips the market and investor diversity drops sharply, then asset liquidity needs to increase (the price impact coefficient to fall) for zero expected returns to remain stable. This compensating function of asset liquidity and investor diversity is apparent from comparing Figure 2, Panels A and C.

The progression of the two trading parameters from stability to instability and chaos has clear implications for policy makers charged with maintaining financial stability. Given the level of asset liquidity, if investor diversity gets sufficiently low the market dynamics will become unstable and progressively turbulent and chaotic. The analysis also suggests that, depending on the starting level of parameter values, very small perturbations in asset liquidity and/or investor diversity can cause large jumps in expected returns. Given pervasive uncertainty regarding investor diversity and positive feedback intensity—indeed, both could be said to be intractable, especially at high-frequency—the model suggests policy makers' responsibility should be to ensure adequate liquidity, i.e. preventing the price impact coefficient from blowing up. The critical role of liquidity provision and stable market-making is supported by recent arguments that asset liquidity can evaporate in turbulent periods of market stress. Persaud (2001, 2003) and Taleb (1997) have referred to such crisis episodes as liquidity black holes.

7   Concluding remarks

In this paper I presented a nonlinear model of feedback trading at high frequency. The dynamics of asset returns displayed stable or unstable behavior—including the possibility of market turbulence and chaos—depending on the diversity (heterogeneity) of investor opinion and the level of asset liquidity.

There was no stochastic uncertainty or asymmetric information and all price adjustment was deterministic, driven by last previous trading period's actual return and volatility. Potential applications of this framework include simulating market dynamics using a stochastic liquidity parameter to explore the relative likelihood of financial turbulence and chaos. Also, the comparative static results relating autocorrelation to returns' second and third moments suggest the need to test for conditional skewness in high-frequency data. Assessing the presence of time-varying return asymmetries seems important for understanding non-fundamental trading patterns. These extensions are the subject of current research.

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APPENDIX

Fixed point existence and stability

Let $h : I \to I$ be a continuous function on I, where $I = [a, b]$ is a closed interval on $\mathbb{R}$. Then point c is a *fixed* point of h if $h(c) = c$.

The point p is a *periodic* point of h with period k if $h^k(p) = p$. Then p is also a fixed point of h^k, the k-iteration of h.

The point p is *eventually periodic* with period k if there exists N such that $f^{n+k}(p) = f^n(p)$ for all $n \geq N$.

Fixed point c of continuous function h is stable (*unstable*) if all points in the neighborhood of c approach (*leave*) the neighborhood under repeated iteration of h. Let $h : I \to I$ be a continuously differentiable function, denote its derivative at $a \in I$ as $h'(a)$, and let p be a periodic point of h with (prime) period k. If $\mid (h^k)'(p) \mid < 1$, then p is a stable (or attracting) fixed or periodic point of h. If $\mid (h^k)'(p) \mid > 1$ then p is an unstable (or repelling) fixed or periodic point.

If $\mid (h^k)'(p) \mid \neq 1$ then p is a *hyperbolic* periodic point of h^k, while if $\mid (h^k)'(p) \mid = 1$ then p is *non-hyperbolic*. Non-hyperbolic periodic points do

not have predictable behavior in any local neighborhood.

Let p be a periodic point of h with period k. The point x is *forward asymptotic* to p if the sequence $\{x, h^k(x), h^{2k}(x), h^{3k}(x), ...\}$ converges to p, that is if $\lim_{n \to \infty} h^{nk}(x) = p$. The *stable set* of p, denoted $W(p)$, is the set of all points which are forward asymptotic to p.

Bifurcations and chaos

Let $f_c(x)$ be a parametric family of functions. There is a *bifurcation* at c_0 if there exists an $\varepsilon > 0$ such that, if a and b satisfy $c_0 - \varepsilon < \alpha < c_0$ and $c_0 < b < c_0 + \varepsilon$, then the dynamics of $f_a(x)$ are different from the dynamics of $f_b(x)$.

Transcritical bifurcations occur when two hyperbolic periodic points merge into a single non-hyperbolic fixed point.

Period-doubling bifurcations occur when a period-$2k$ periodic orbit is added to the orbit of a period-k periodic point. There are two more types of bifurcation, *saddle-point* and *pitchfork*, which do not appear in the logistic map.

Let D be a subset of a metric space with metric d. Function $h : D \to D$ then exhibits *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that, for any $x \in D$ and any $\varepsilon > 0$, there is a $y \in D$ and some number $n \in \mathbb{N}$ such that $d[x, y] < \varepsilon$ and $d[h^n(x), h^n(y)] > \delta$. The function $h : D \to D$ is *chaotic* if it satisfies the following three conditions:

(a) the periodic points of h are dense in D,

(b) h is topologically transitive, and

(c) h displays sensitive dependence on initial conditions.
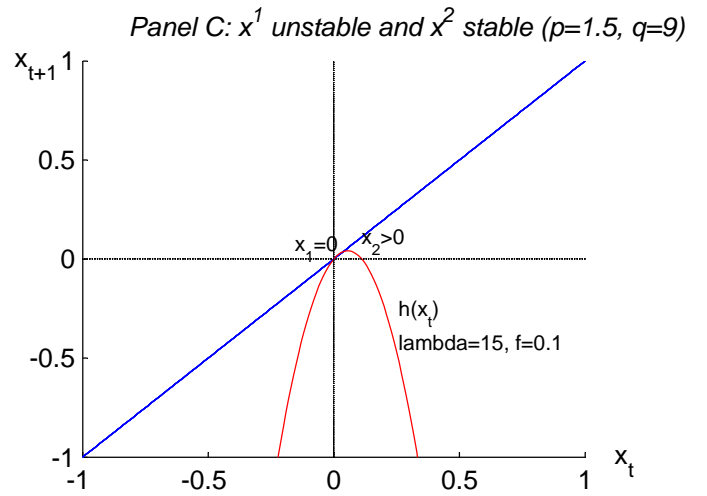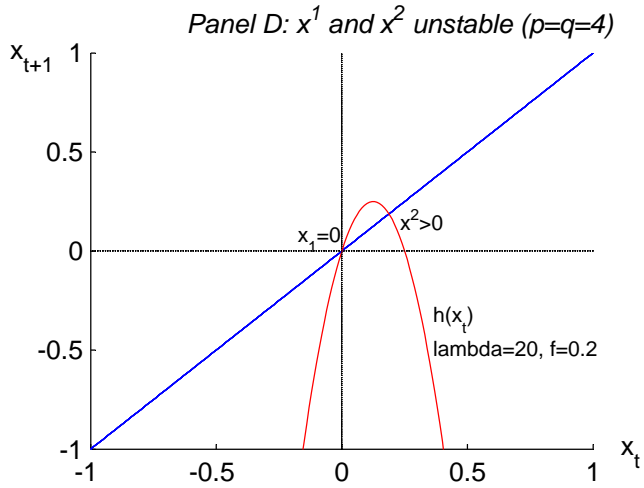
FIGURE 1: THE LOGISTIC MAP

Panel A: x^1 stable and x^2 unstable

x_{t+1}

$x_1=0$

$h(x_t)$

lambda=1.2, f=0.5

$x^2<0$

x_t

Panel B: The transcritical bifurcation (p=q=1)

x_{t+1}

$h(x_t)$

lambda=2, f=0.5

$x_1=x_2=0$

x_t

Panel D: x^1 and x^2 unstable (p=q=4)

x_{t+1}

$x_1=0$

$x^2>0$

$h(x_t)$

lambda=20, f=0.2

x_t

Panel C: x^1 unstable and x^2 stable (p=1.5, q=9)
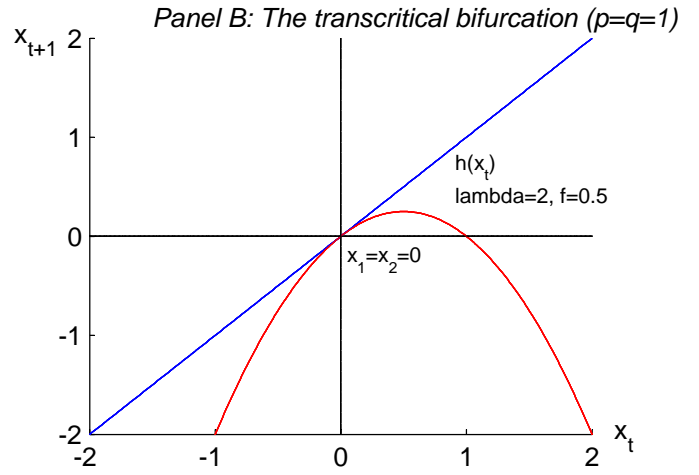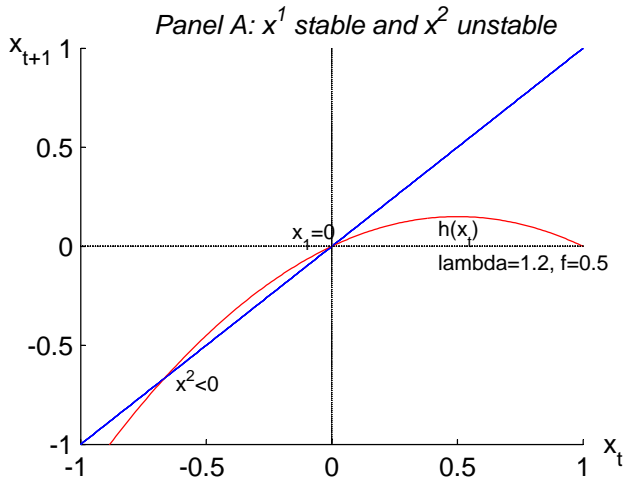
x_{t+1}

$x_1=0$ $x_2>0$

$h(x_t)$

lambda=15, f=0.1

x_t

FIGURE 2: POSITIVE FEEDBACK TRADING AND MARKET DYNAMICS