

A Sampling Theorem for Exact Identification of Continuous-time Nonlinear Dynamical Systems

Zhexuan Zeng, Zuogong Yue, Alexandre Mauroy, Jorge Gonçalves and Ye Yuan

Abstract—Low sampling frequency challenges the exact identification of continuous-time (CT) dynamical systems from sampled data, even when their models are identifiable. A necessary and sufficient condition is proposed, built from the Koopman operator, for the exact identification of CT systems from sampled data. This condition provides a Nyquist-Shannon-like critical frequency for the exact identification of CT nonlinear dynamical system which has Koopman invariant spaces. Firstly, it establishes a sufficient condition for a sampling frequency that permits a discretized sequence of samples to discover the underlying system. Secondly, it establishes a necessary condition for the sampling frequency to avoid system aliasing, which would render the underlying system distinguishable. Lastly, unlike the requirement in the Nyquist-Shannon sampling theorem, the CT signal associated with the system need not be band-limited. The theoretical criterion has been demonstrated on a number of simulated examples, including linear systems, nonlinear systems with equilibria, and limit cycles.

Index Terms—System identification, sampling period, Koopman operator.

I. INTRODUCTION

ONE of the central problems in science and engineering is to discover underlying physical laws from observations. Physical laws are invariably described as continuous-time (CT) dynamical systems, whereas the available measurements are inevitably discrete-time (DT) [1]. This leads to a key challenge: when the sampling frequency is not high enough to capture the dynamics, several CT dynamical systems may generate the same DT measurements, and as a result, the underlying CT dynamical system cannot be uniquely identified. Similar to the aliasing of signals, aliasing of systems (or we call it system aliasing) is defined to characterize such a many-to-one relationship between the CT systems and DT measurements [2].

The goal of this paper is to investigate problems related to system aliasing of nonlinear dynamical systems, i.e., we aim

to derive a sampling condition, under which the CT system can be determined uniquely from the sampled measurements. Based on the equivalence between the vector field and the infinitesimal generator of the Koopman operator [3], [4], our goal is recast in obtaining the generator uniquely from its Koopman operator [5]. Building on theorems of semigroup and operator logarithm, the sampling theorem that prevents system aliasing is proposed for the system whose Koopman operators admits a special invariant space, which is called valid invariant space. The sampling theorem for the case where the valid invariant space is finite-dimensional was initially proposed in [6]. This paper further extends this result to encompass the case of infinite-dimensional, valid invariant space. The main result shows that the critical sampling period that avoids system aliasing is associated with the imaginary part of the Koopman spectrum, i.e., the spectrum of the Koopman generator. Since this Koopman spectrum is typically unknown unless the system has been identified, the application of this sampling theorem is discussed by approximating the associated spectrum from fast-sampled data. Several numerical examples show that this approximation can be used to accurately estimate the critical sampling period.

There have been efforts in the identification of CT dynamical systems from DT measurements in the past decades; a continuous-time system identification (CONTSID) toolbox has been developed in Matlab [7]. The identification approaches of CT dynamical system include indirect methods [8] and direct methods [9]–[11]. For indirect methods, the CT system is obtained by the transformation from the identified DT dynamical system. However, performance of this transformation is highly dependent on the choice of the sampling period [11]. To overcome this shortcoming, the direct methods use pre-filters to provide smoothed differentiated data and identify the parameters of CT models directly, which outperforms the indirect methods [12]. These identification approaches implicitly assume that the sampling frequency is sufficiently high to permit the identification of a CT system from sampled data [10]. For linear systems, the lowest sampling frequency was successfully investigated in [2] to identify them without system aliasing. In this work, the proposed sampling theorem provides a natural extension from this known result to nonlinear systems whose associated Koopman operators admit the valid invariant space.

The problem and concepts related to system aliasing are presented in Section II. The preliminary knowledge of the Koopman operator and its spectrum are introduced in Section III. The sampling theorem and its explanations are presented

Manuscript submitted June 17, 2022. This work was supported by the National Natural Science Foundation of China under Grant 92167201. An abridged version of this paper was published on 61th IEEE Conference on Decision and Control, 2022.

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in Section IV. Then this theorem is further analyzed for systems whose Koopman operators have only point spectrum in Section V, and continuous/residual spectrum in Section VI. In Section VII, the application of this theorem is discussed by approximating the critical sampling period from fast-sampled data. Finally, the sampling theorem and its application are illustrated numerically through a number of dynamical systems in Section VIII.

A. Notations

The paper adopts the following notation: $\mathcal{L}(X)$ denotes all linear bounded operators from the Banach space X to itself, $\mathcal{D}(\cdot)$ denotes the domain of an operator, $\text{span}\{\cdot\}$ denotes the linear space spanned by the basis functions, ∇ denotes the gradient operator, $A|_X$ denotes the restriction of the operator A to the space X .

II. PROBLEM FORMULATION

A. A motivating example

We first present a motivating example to illustrate an interesting phenomenon that, the underlying physical systems cannot be identified from DT measurements when the sampling frequency is low.

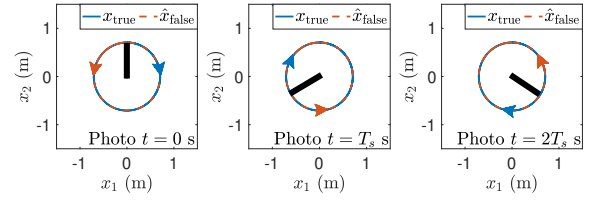
Suppose an extendable rod, whose length $r(t)$ satisfies $\dot{r} = ar, a \in \mathbb{R}$ is rotating clockwise around its fixed end with a constant angular velocity ω . We take a series of photos with the time interval T_s to infer the moving of the rod. From these photos, we can observe the position of the rotating end, represented by $x_1(t)$ and $x_2(t)$ in Cartesian coordinates. In essence, this problem can be regarded as the identification of a dynamical system that generates $x_1(t)$ and $x_2(t)$ from the data with sampling period T_s . The trajectories are generated by the following linear model:

$$\begin{aligned} \dot{x}_1 &= ax_1 + \omega x_2, \\ \dot{x}_2 &= -\omega x_1 + ax_2, \end{aligned} \quad (1)$$

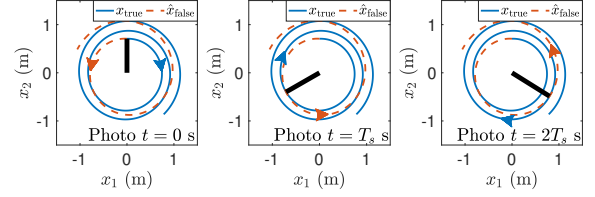
which are identifiable using identifiability tests in [14], [15], i.e., the parameters can be uniquely determined from $\mathbf{x}(t)$.

However, when the photo-taken frequency is not sufficiently high, it is difficult to correctly determine the movement. Fig. 1(a) and Fig. 1(b) show positions of the rod at three consecutive photo-taken moments with $\omega T_s = 4\pi/3$ when the length of the rod is constant ($a = 0$) or getting longer ($a = 0.1$), respectively. The black rectangle represents the rod with its fixed end at $(0, 0)$ and the blue line denotes the actual CT trajectory (clockwise rotation) of rotating end. From these DT observations, the orange dashed line more closely resembles an actual trajectory, rotating counterclockwise, than the blue line. In other words, these DT observations misleadingly suggest that the rod is rotating in the opposite direction to the actual movement, whose position perfectly matches that of the actual one at each moment captured by the camera.

From a system perspective, there are at least two different CT systems that produce the same DT measurements. As a result, although there exists many advanced identification algorithms [16], [17], one cannot identify the underlying system from these measurements.



(a) The rod $r(t)$ is constant by $\dot{r} = 0$ while rotating with ω rad/s.



(b) The rod $r(t)$ is getting longer by $\dot{r} = 0.1r$ while rotating with ω rad/s.

Fig. 1. The observation of extendable rod to illustrate system aliasing. The sampling is not high enough to capture the systems of (a) $\dot{r} = 0, \dot{\theta} = \omega$ and (b) $\dot{r} = 0.1r, \dot{\theta} = \omega$. For these dynamical systems, the outputs of true and false dynamical systems are represented by blue lines and orange dashed lines respectively.

To provide a sampling bound to avoid this aliasing phenomenon, the Nyquist-Shannon sampling theorem [13] is helpful when the generated CT signals are band-limited, as shown in Fig. 1(a). However, when they are not band-limited, as shown in Fig. 1(b), further research is needed for associated dynamical systems. Therefore, we shall define the concept of system aliasing and discuss the critical sampling frequency to avoid it in the later sections.

B. Definitions

Consider an autonomous nonlinear dynamical system described by the following ordinary differential equations:

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad (2)$$

where $\mathbf{x} \in \mathcal{M} \subseteq \mathbb{R}^n$ denotes the state vector and $\mathcal{M}$ denotes the state space. The vector field $f(\mathbf{x})$ describes the dynamical properties of the state. The flow induced by (2) is $S^t(\mathbf{x}) : \mathbb{R}^+ \times \mathcal{M} \rightarrow \mathcal{M}$, so that $t \mapsto S^t(\mathbf{x}_0)$ is a solution of (2) associated with initial state $\mathbf{x}_0$, i.e., $\mathbf{x}(t) = S^t(\mathbf{x}_0)$. We made the following assumptions about the dynamical system.

Assumption 1: The vector field is Lipschitz continuous.

Assumption 2: The DT flow S^{T_s} can be identified from data $\{\mathbf{x}_k, \mathbf{y}_k\}_{k=1}^K$ without any errors, where $\mathbf{y}_k = S^{T_s}(\mathbf{x}_k)$.

Assumption 1 guarantees that there exists one and only one solution $\mathbf{x}(t)$ with the initial state $\mathbf{x}_0 \in \mathcal{M}$ of the CT system. Assumption 2 holds when the data used for the identification are accurate and adequate in number as these data are direct numerical representations of S^{T_s} . It ensures that the failure of identification stems from the many-to-one relationship between the CT system and DT flow when the sampling period T_s is not sufficiently small, rather than from data issues that lead to an incorrect S^{T_s} . Now we define this phenomenon as system aliasing.

Definition 1 (System aliasing): Consider the vector field $f(\mathbf{x})$ and DT flow $S^{T_s}(\mathbf{x})$ of the dynamical system (2). If

there exists a function $\hat{f}(x) \neq f(x)$ for some $x \in \mathcal{M}$ that also generates the same DT flow $S^{T_s}(x)$ for all $x \in \mathcal{M}$, where $\hat{f} \in \mathcal{J}_f$ and $\mathcal{J}_f$ is the aliasing space of f , we call $\hat{f}$ a system alias of f with respect to $\mathcal{J}_f$.

Remark 1: This definition is a generalization of the definition of system aliasing in the linear case [2]. The aliasing space $\mathcal{J}_f$ denotes the functional space to which the vector field f belongs, which is determined by prior knowledge of this dynamical system. Hence, no system aliasing means that $f \in \mathcal{J}_f$ can be uniquely determined from the DT flow S^{T_s} as f is one and only one function in $\mathcal{J}_f$ that generates S^{T_s} . Furthermore, when the prior knowledge of the aliasing space is unavailable, the space of f that avoids system aliasing is discussed in Appendix II-C.

Accordingly, the concepts of critical sampling period / frequency bound are defined as follows.

Definition 2 (Critical sampling period / frequency bound): The critical sampling period T_γ of the CT dynamical system (2) is the supremum sampling period to avoid system alias, i.e., there is no system alias of the dynamical system if and only if $t < T_\gamma$. Equivalently, the lower bound of sampling frequency is $2\pi/T_\gamma$.

Remark 2: The lower bound of sampling frequency provides a criterion to collect data for system identification, which guarantees that the vector field can be uniquely identified. The identification result of CT dynamical system would be unreliable if the sampling frequency is lower than its bound. This is because multiple systems can produce the same sampled data, and we cannot determine which system is the actual one.

Our goal is to investigate the critical sampling period T_γ that avoids system aliasing of the dynamical system.

III. ANALYSIS OF SYSTEM ALIASING AND THE KOOPMAN OPERATOR

The analysis of system aliasing leverages the equivalent description between the finite-dimensional nonlinear system and infinite-dimensional linear system in the Koopman operator framework. The main steps for analyzing the system aliasing are illustrated in Fig. 2. First, the nonlinear dynamical system is described in a special invariant space $\mathcal{F}_e$, where the Koopman operator provides an equivalent representation of this system. Secondly, the system aliasing in the state space is expressed in terms of generator aliasing (see Appendix II-A) in this functional space $\mathcal{F}_e$. Finally, the necessary and sufficient condition of system aliasing and critical sampling period are derived using operator logarithm theorems. The details can be found in Appendix II. Here we introduce foundational concepts and properties of the Koopman operator.

A. Operator-theoretic approach to dynamical systems

Consider the system (2) in the state space $\mathcal{M} \subseteq \mathbb{R}^n$ and the generated flow S^t . Rather than acting on the state $x \in \mathcal{M}$, the Koopman operator $U^t : \mathcal{F} \rightarrow \mathcal{F}, t \geq 0$ acts on observable functions $g : \mathcal{M} \rightarrow \mathbb{C}$ belonging to the (Banach) space $\mathcal{F}$. The transformation $U^t g$ is defined by the evolution of g along the flow S^t , i.e.,

$$U^t g = g \circ S^t, \quad g \in \mathcal{F}, \quad (3)$$

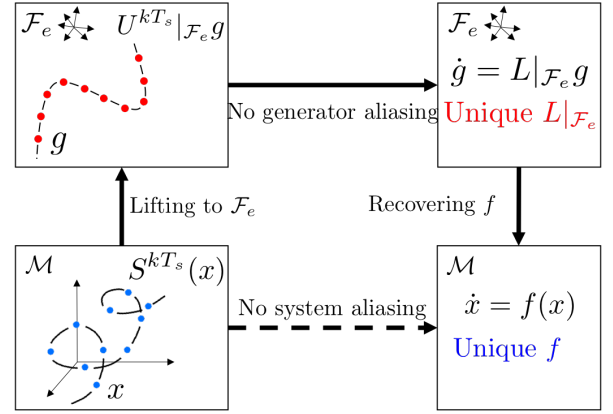


Fig. 2. The steps for analyzing the system aliasing of nonlinear dynamical systems whose Koopman operators admit a special invariant space $\mathcal{F}_e$. It consists of three steps: (1) lifting the DT flow S^{T_s} to $\mathcal{F}_e$, (2) obtaining the generator from the Koopman operator $U^{T_s}|_{\mathcal{F}_e}$ uniquely, and (3) recovering the vector field f .

where the space $\mathcal{F}$ is a Koopman invariant space, i.e.,

$$U^t g \in \mathcal{F}, \quad \forall t \geq 0, \forall g \in \mathcal{F}. \quad (4)$$

Since this operator is a linear operator, i.e.,

$$U^t(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 U^t g_1 + \alpha_2 U^t g_2, \quad \alpha_1, \alpha_2 \in \mathbb{R}, \quad (5)$$

its definition (3) has the following dynamical interpretation: the linear transformation of the observable function g , i.e., $U^t g(x)$, has the value of this function g at $S^t(x)$, i.e., $g(S^t(x))$. Specifically, if $g(x) = x_j$, where x_j ($1 \leq j \leq n$) is the j th component of x , then $U^t g(x(0))$ has the value that the initial state $x_j(0)$ flows after the lapse of the time t .

The infinitesimal generator of the Koopman operator is defined by

$$Lg = \lim_{t \rightarrow 0^+} \frac{1}{t} (U^t - I)g, \quad g \in \mathcal{D}(L), \quad (6)$$

which is also a linear operator. Based on this definition, Lg can be intuitively understood as “derivative” of $U^t g$, i.e., $\dot{g} = Lg$, where $\dot{g}$ denotes $\partial(g \circ S^t)/\partial t$ [5]. In particular, if the generator L acts on $g_j(x) = x_j, j = 1, \dots, n$, one actually obtains the vector field, i.e., $[Lg_1, \dots, Lg_n]^T = f$. This connection between the generator L and the vector field f stems from the formula:

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (S^t(x) - x) = f(x). \quad (7)$$

If the observable function g is continuously differentiable with compact support, there is a general relationship between the vector field f and the generator L of the corresponding Koopman operator as follows [18, Section 7.6]:

$$Lg = f \cdot \nabla g, \quad g \in \mathcal{D}(L). \quad (8)$$

Based on (3)-(8), the dynamical system (2), which may be nonlinear, can be equivalently represented by an infinite-dimensional linear system of observable functions $g \in \mathcal{F}$ as follows:

$$\begin{aligned} \dot{g} &= Lg, \\ U^{T_s} g &= g \circ S^{T_s}. \end{aligned} \quad (9)$$

where the observable function g and functional space $\mathcal{F}$ can be intuitively understood as the “state” and “state space” of this infinite-dimensional system, respectively. Hence, based on this equivalent description, nonlinear problems can be viewed as infinite-dimensional linear problems.

B. Spectrum of the Koopman operator

The spectrum $\sigma(\cdot)$ of the operator $B \in \mathcal{L}(X)$ is defined as the set of values $\lambda \in \mathbb{C}$ such that $B - \lambda I$ is not invertible, where I denotes identity operator. This set $\sigma(\cdot)$ is composed of three parts: point spectrum $\sigma_p(\cdot)$, continuous spectrum $\sigma_c(\cdot)$, and residual spectrum $\sigma_r(\cdot)$, i.e., $\sigma(\cdot) = \sigma_p(\cdot) \cup \sigma_c(\cdot) \cup \sigma_r(\cdot)$. In the following, these three different parts are introduced.

1) *Point spectrum*: The point spectrum $\sigma_p(B)$ of the operator $B \in \mathcal{L}(X)$ is the set of value $\lambda \in \mathbb{C}$ such that $B - \lambda I$ is not injective. When the Koopman operator admits the point spectrum, the eigenvalue and associated eigenfunction of the Koopman operator are defined as follows.

Definition 3 (Koopman eigenvalue and eigenfunctions): An eigenvalue of the Koopman operator is the value $\lambda_i \in \mathbb{C}$ such that:

$$U^t \phi_i = e^{\lambda_i t} \phi_i, \quad (10)$$

where $\phi_i \in \mathcal{F}$, $\phi_i \neq 0$ is the associated Koopman eigenfunction.

Following from (6), the Koopman eigenfunction ϕ_i is also an eigenfunction of the generator L , i.e.,

$$L\phi_i = \lim_{t \rightarrow 0^+} \frac{e^{\lambda_i t} - 1}{t} \phi_i = \lambda_i \phi_i. \quad (11)$$

The value λ_i is the eigenvalue of the generator L , and it belongs to its point spectrum $\sigma_p(L)$.

The set of eigenfunctions and eigenvalues of the Koopman operator have the property described as follows.

Property 1 (Multiplication of eigenfunctions [16]):

Consider the eigenfunctions $\phi_1 \in \mathcal{F}$ and $\phi_2 \in \mathcal{F}$ of the Koopman operator $U^t : \mathcal{F} \rightarrow \mathcal{F}$ with associated eigenvalues λ_1 and λ_2 . When $\phi_1^{c_1} \phi_2^{c_2}$ belongs to the space $\mathcal{F}$ with $c_1, c_2 \in \mathbb{N}$, the observable function $\phi_1^{c_1} \phi_2^{c_2}$ is also an eigenfunction of U^t with the associated eigenvalue $c_1 \lambda_1 + c_2 \lambda_2$, where $\phi_1^{c_1} \phi_2^{c_2}(\mathbf{x}) = (\phi_1(\mathbf{x}))^{c_1} (\phi_2(\mathbf{x}))^{c_2}$.

2) *Continuous spectrum*: The Koopman operator may admit continuous or residual spectrum if the system has a non-hyperbolic attractor [16]. The continuous spectrum of the operator $B \in \mathcal{L}(X)$ is defined as follows.

Definition 4 (Continuous spectrum [19, Page 12]): The continuous spectrum $\sigma_c(B)$ of the operator $B \in \mathcal{L}(X)$ is the set of values $\lambda \in \mathbb{C}$ such that the operator $B - \lambda I$ is injective, not surjective, and has a dense image, where I is the identity operator.

3) *Residual spectrum*: Here is the definition of residual spectrum of the operator $B \in \mathcal{L}(X)$.

Definition 5 (Residual spectrum [19, Page 12]): The residual spectrum $\sigma_r(B)$ of the operator $B \in \mathcal{L}(X)$ is the set of values $\lambda \in \mathbb{C}$ such that the operator $B - \lambda I$ is injective, not surjective, and does not have a dense image, where I is the identity operator.

Next, we shall propose sampling theorem that prevents the system aliasing for a class of nonlinear dynamical systems, where the critical sampling period T_γ is determined by the Koopman spectrum.

IV. THE SAMPLING THEOREM

Here we first define the valid invariant space $\mathcal{F}_e$. Noted that, in the following, the notations like $U^t|_{\mathcal{F}_e}$ and $L|_{\mathcal{F}_e}$ will be employed to emphasize the observable space where the Koopman operator U^t and its generator L are restricted, instead of U^t and L used in Section III for introducing their general properties.

Definition 6 (Valid invariant space $\mathcal{F}_e$): If a functional space $\mathcal{F}_e$ is the Banach space and satisfies:

- (Koopman invariance) The space is invariant under the Koopman operator, i.e., $U^t|_{\mathcal{F}_e} g \in \mathcal{F}_e, \forall g \in \mathcal{F}_e, \forall t \geq 0$;
- (Boundedness) The generator restricted to the space is bounded, i.e., $\|L|_{\mathcal{F}_e}\| < \infty$;
- (Independence) For $\forall \mathbf{x} \in \mathcal{M} \subseteq \mathbb{R}^n$, there exist n distinct functions $g_j \in \mathcal{F}_e, j = 1, 2, \dots, n$ such that the gradients $\{\nabla g_j(\mathbf{x})\}_{j=1}^n$ are linearly independent;

then $\mathcal{F}_e$ is a valid invariant space of the Koopman operator.

Then we assume the existence of $\mathcal{F}_e$ as follows.

Assumption 3: The Koopman operator of the dynamical system (2) has a valid invariant space $\mathcal{F}_e$.

Remark 3: Assumption 3 guarantees that the vector field f can be recovered from the restriction of the DT Koopman operator $U^{T_s}|_{\mathcal{F}_e}$. Specifically, the Koopman invariance of $\mathcal{F}_e$ guarantees that the Koopman operator is well-defined on this space. The boundedness of $L|_{\mathcal{F}_e}$ ensures that $L|_{\mathcal{F}_e}$ can be obtained from $U^{T_s}|_{\mathcal{F}_e}$ by the operator logarithm. Moreover, independent gradients are needed to recover the vector field f from the generator $L|_{\mathcal{F}_e}$ (see Appendix I). The systems that meet this assumption will be discussed in later sections, separately addressing cases where the valid invariant space $\mathcal{F}_e$ is finite-dimensional and infinite-dimensional.

Based on Definition 6, the “best” valid invariant space is denoted as $\mathcal{F}_e^\gamma$, which leads to the minimum of $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e}))|$, i.e.,

$$\mathcal{F}_e^\gamma \triangleq \arg \min_{\mathcal{F}_e} \{\max |\text{Im}(\sigma(L|_{\mathcal{F}_e}))|\}. \quad (12)$$

Then the critical sampling period of the system is determined by the Koopman spectrum $\sigma(L|_{\mathcal{F}_e^\gamma})$, which is present in a more clear way as follows:

Theorem 1 (The critical sampling period T_γ):

Considering the dynamical system that satisfies Assumptions 1-3, if data are equidistant sampled with sampling period T_s , the sampling frequency ω (rad/s) must satisfy $\omega > \omega_\gamma$ to uniquely identify the vector field f , where the frequency bound

$$\omega_\gamma = 2 \max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|. \quad (13)$$

Equivalently, the sampling period satisfy $T_s < T_\gamma$, where the critical sampling period T_γ (i.e., $2\pi/\omega$) is

$$T_\gamma = \frac{\pi}{\max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|}. \quad (14)$$

Proof: The proof will be given in Appendix II ■

Theorem 1 indicates that the frequency bound ω_γ is twice the maximum absolute value of the imaginary part of the Koopman spectrum, i.e., $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|$. It provides a Nyquist-Shannon-like critical frequency to ensure that the sampling frequency of data is sufficient for exact system identification. Moreover, it naturally extends the known result of linear systems, where ω_γ is determined by eigenvalues of the system matrix [2], to nonlinear systems, where it is determined by the Koopman spectrum. Since the Koopman spectrum in Theorem 1 may contain different parts, i.e., $\sigma(\cdot) = \sigma_p(\cdot) \cup \sigma_c(\cdot) \cup \sigma_r(\cdot)$, further analysis will follow in subsequent sections, addressing two scenarios: one where σ_c and σ_r are empty, and another where they are not.

Although the sampling bound is formulated in a complicated way, its implication can be explained intuitively. In fact, the imaginary part of the Koopman spectrum $\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))$ characterizes the oscillation frequency components of $g(\mathbf{x}(t))$ for all $g \in \mathcal{F}_e^\gamma$ [20], which will be proved in Appendix III. By this explanation of $|\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|$, Theorem 1 implies that the lower bound of the sampling frequency is determined by the highest oscillation frequency of the signal $g(\mathbf{x}(t))$ for all $g \in \mathcal{F}_e^\gamma$. It can be understood intuitively that one must sample more frequently to capture the oscillation of $g(\mathbf{x}(t))$ if its maximum oscillation frequency component is higher, where $g(\mathbf{x}(t))$ is closely related to the dynamical system.

Here we explain why $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|$ determines the critical sampling period. Through the equivalent description of the system by the Koopman operator, associated generated signals are evolutions of observable functions $g(\mathbf{x}(t))$ belonging to the valid invariant space $\mathcal{F}_e$. Then the critical sampling period is determined by the maximum oscillation frequency component of $g(\mathbf{x}(t))$ for all $g \in \mathcal{F}_e$, i.e., $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e}))|$. By the definition of the “best” valid invariant space $\mathcal{F}_e^\gamma$ (see (12)), it has the lowest $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|$ for all $\mathcal{F}_e$, which is able to provide a tight bound for the critical sampling period. Therefore, $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|$ determines the critical sampling period.

However, the spectral value $\max |\text{Im}(\sigma(L|_{\mathcal{F}_e^\gamma}))|$, which determines T_γ , requires knowledge of the dynamical system in advance. The prior knowledge is essential because the sampling bound depends on the oscillatory property of the system itself. In fact, this principle is the same as that of the Nyquist-Shannon sampling theorem, which also needs to know the maximum (Nyquist) frequency component of the signal in advance to recover it without signal aliasing. Even though, there exists various successful approaches to approximate the Koopman spectrum from data [21], [22]. Utilizing the approximation of $\sigma(L|_{\mathcal{F}_e^\gamma})$, the critical sampling period can be computed by Theorem 1, which no longer necessitates prior knowledge of the system. In later sections, we will further discuss the application of Theorem 1 and show that the critical sampling period can be accurately approximated from fast-sampled data through various numerical examples.

Remark 4 (Comparison with the Nyquist-Shannon theorem): The Nyquist-Shannon sampling theorem states that the sampling frequency must exceed twice the maximum frequency component of the signal, which is determined by (Fourier) frequency spectrum. Similarly, according to Theorem

1, the lower bound of sampling frequency is twice the highest oscillation frequency of $g(\mathbf{x}(t))$, $g \in \mathcal{F}_e^\gamma$, determined by the Koopman spectrum. However, the Nyquist-Shannon sampling theorem is limited by its applicability only to band-limited signals, whereas most signals generated by nonlinear systems do not satisfy this requirement. Theorem 1 allows the analysis of systems related to non-band-limited signals. This extension arises from the fundamental difference between the one-dimensional nature of the (Fourier) frequency spectrum and the two-dimensional nature of the Koopman spectrum, i.e., its real and imaginary part. In particular, the Koopman spectrum captures both the growth rate and the oscillation frequency separately [20], providing a richer perspective than the frequency spectrum. The frequency bound that avoids aliasing is only dependent on the oscillation frequency characterized by its imaginary part, which is still a bounded value for many types of non-band-limited signals. Furthermore, it is demonstrated in a subsequent work [23] that the Fourier spectrum of band-limited signal is characterized by the associated Koopman spectrum that restricted in a bounded interval on the imaginary axis. Hence, the applicability of Theorem 1 encompasses a broader range of systems than that of the Nyquist-Shannon sampling theorem, which will be further illustrated through examples in later sections.

V. SAMPLING THEOREM IN FINITE-DIMENSIONAL CASE

Based on Theorem 1, the critical sampling period relies on the existence of the valid invariant space $\mathcal{F}_e$ (see Assumption 3). To address this issue, we first analyze the formulation of finite-dimensional $\mathcal{F}_e$ and the associated dynamical systems. This scenario corresponds to the absence of continuous and residual parts $\sigma(L|_{\mathcal{F}_e})$, i.e., $\sigma(L|_{\mathcal{F}_e}) = \sigma_p(L|_{\mathcal{F}_e})$. Furthermore, it is crucial to emphasize that this finite-dimensional case encompasses not only linear systems but also a class of nonlinear systems. Subsequently, we proceed to illustrate Theorem 1 in this finite-dimensional case, and compare it with the Nyquist-Shannon sampling theorem through an example.

A. The existence of the finite-dimensional valid invariant space

When the Koopman operator $U^t : \mathcal{F} \rightarrow \mathcal{F}$ admits point spectrum, there are infinitely many Koopman eigenvalues and eigenfunctions $\{\phi_k\}_{k=1}^\infty$ (see Property 1). Consider n Koopman eigenfunctions whose gradients can be linearly independent vectors for all $\mathbf{x} \in \mathcal{M}$. We define the Koopman valid eigenspace $\mathcal{F}_n$ as follows, which is a finite-dimensional case of the valid invariant space $\mathcal{F}_e$.

Definition 7 (Koopman valid eigenspace $\mathcal{F}_n$): The Koopman valid eigenspace $\mathcal{F}_n$ is spanned by n eigenfunctions of the Koopman operator: $\mathcal{F}_n = \text{span}\{\phi_1, \dots, \phi_n\}$, where n is the dimension of the state space $\mathcal{M}$ and the gradients of these eigenfunctions $\{\nabla\phi_i\}_{i=1}^n$ are linearly independent vectors for all $\mathbf{x} \in \mathcal{M}$, i.e., the matrix $A_{ij} = \partial\phi_i/\partial x_j$ is invertible for all $\mathbf{x} \in \mathcal{M}$. We call $\{\phi_i\}_{i=1}^n$ a set of Koopman valid eigenfunctions.

Here we show that the Koopman valid eigenspace $\mathcal{F}_n$ is a finite-dimensional case of $\mathcal{F}_e$ by Definition 6. Firstly, the

invariance of $\mathcal{F}_n$ is guaranteed by the property of eigenfunctions, i.e., for $g \in \mathcal{F}_n$,

$$U^t|_{\mathcal{F}_n}g = U^t|_{\mathcal{F}_n}\left(\sum_{i=1}^n a_i\phi_i\right) = \sum_{i=1}^n a_ie^{\lambda_i t}\phi_i \in \mathcal{F}_n. \quad (15)$$

Secondly, the Koopman operator $U^t|_{\mathcal{F}_n}$ and its generator $L|_{\mathcal{F}_n}$ are both bounded because linear operators defined on the finite-dimensional space can be represented by a matrix. Thirdly, the independence of gradients is also required in the definition of $\mathcal{F}_n$.

Remark 5 (The existence of $\mathcal{F}_n$ and related systems):

The linear system is a special case of systems whose valid invariant space is finite-dimensional. In particular, its valid invariant eigenspace is $\mathcal{F}_n = \{g_1(\mathbf{x}) = x_1, \dots, g_n(\mathbf{x}) = x_n\}$, where x_k denotes the k th component of the state $\mathbf{x}$. For n -dimensional nonlinear dynamical systems with some transient and asymptotic behaviors, the Koopman spectrum is a lattice generated by linear combinations of integers of n principal eigenvalues, provided that the space of observable functions is well-chosen [24]. The eigenfunctions associated with these n distinct principal eigenvalues span the space $\mathcal{F}_n$ according to Definition 7 because the gradients of associated eigenfunctions are linearly independent vectors for all $\mathbf{x}$ in the basin of the attraction. It includes dynamical systems with an equilibrium, a limit cycle, or a quasi-periodic attractor. Then the critical sampling period can be analyzed using Koopman eigenvalues. However, if an eigenfunction is generated by the multiplication of other $n - 1$ eigenfunctions that already belong to $\mathcal{F}_n$ (see Property 1), the independence requirement on the gradients of eigenfunctions is not satisfied. In addition, in some cases (e.g., chaotic systems), eigenfunctions might not satisfy this independence condition. Then one might need to complete them with infinite-dimensional functional space associated with the continuous or residual spectrum.

B. Sampling theorem for the motivating example

Let us take the system of extendable rod described in Section II as a special example of finite-dimensional case to illustrate Theorem 1. We will show in this example that, when the system signal $\mathbf{x}(t)$ is band-limited, the critical sampling period which avoids system aliasing is consistent with the Nyquist-Shannon sampling bound for $\mathbf{x}(t)$. Moreover, unlike the Nyquist-Shannon sampling theorem, Theorem 1 does not require $\mathbf{x}(t)$ to be band-limited.

We compare the Nyquist-Shannon sampling theorem with Theorem 1 in two cases of the example described in Section II. Firstly, when the length of the rod is constant and the angular velocity is ω rad/s, the trajectory of $x_1(t)$ and $x_2(t)$ are periodic with the period $2\pi/\omega$ s. The lower bound of sampling frequency given by Nyquist-Shannon sampling theorem is 2ω rad/s. In the system perspective, since the system (1) is linear, we choose the space $\mathcal{F}_e^\gamma = \mathcal{F}_2 = \text{span}\{g_1(\mathbf{x}) = x_1, g_2(\mathbf{x}) = x_2\}$. Then the eigenvalues of the system matrix are Koopman eigenvalues, i.e., $\pm\omega i$. Accordingly, the lower bound of sampling frequency given by Theorem 1 is $2\max|\text{Im}(\sigma(L|_{\mathcal{F}_2}))| = 2\omega$, which is consistent with the bound provided by Nyquist-Shannon sampling theorem.

Secondly, when the length of rod satisfies $\dot{r} = a$ while rotating with the constant angular velocity ω rad/s, the Nyquist-Shannon sampling theorem is not applicable since the states of $x_1(t)$ and $x_2(t)$ are not band-limited signals. The eigenvalues of the system matrix in (1) are $a \pm \omega i$, and according to Theorem 1, the lower bound of sampling frequency remains at 2ω rad/s. The numerical analysis of this example will be further detailed in the experimental section.

As illustrated in this example, in the case of finite-dimensional $\mathcal{F}_e^\gamma$, the consistency between Theorem 1 and Nyquist-Shannon sampling theorem corresponds to purely complex Koopman eigenvalues. The associated signals are periodic band-limited, which can be represented by linear compositions of finite terms in their Fourier series. The example described in the subsequent section will further discuss the consistency regarding non-periodic band-limited signals, which corresponds to infinite-dimensional $\mathcal{F}_e^\gamma$.

VI. SAMPLING THEOREM IN INFINITE-DIMENSIONAL CASE

Many dynamical systems whose attractors are not hyperbolic (e.g., chaotic systems) have continuous spectrum [16]. The valid invariant space $\mathcal{F}_e$ of these systems needs to be infinite-dimensional because there is a lack of Koopman valid eigenfunctions (see Definition 7). However, the boundedness condition of $\mathcal{F}_e$ (see Definition 6) might be difficult to satisfy because the generator, which is essentially a derivative operator, is generally unbounded in an infinite-dimensional space. Hence, we will first discuss the existence of infinite-dimensional $\mathcal{F}_e$ and then provide an example to illustrate Theorem 1 in this case.

A. The existence of the infinite-dimensional valid invariant space

Here we present a necessary condition for the existence of an infinite-dimensional valid invariant space $\mathcal{F}_e$ as follows.

Proposition 1 (Infinite-dimensional valid invariant space): If the Koopman operator of the dynamical system (2) admits an infinite-dimensional valid invariant space $\mathcal{F}_e$, then for $\forall g \in \mathcal{F}_e, \forall t \geq 0$, we have $\|g \circ S^t\| < \infty$ and

$$\sup_{t \geq 0} \frac{\|f \cdot \nabla(g \circ S^t)\|}{\|g \circ S^t\|} < \infty. \quad (16)$$

Proof: It can be derived from the invariant requirement and the bounded requirement of the Definition 6. ■

Proposition 1 implies that the existence of $\mathcal{F}_e$ depends on the flow S^t of associated dynamical system. However, due to the unknown S^t , the multitude of functional spaces and the norm involved, it is difficult to derive an existence condition or formulate an infinite-dimensional valid invariant space $\mathcal{F}_e$ similarly to Definition 7. For this reason, we will consider a straightforward system that is frequently used for the analysis of continuous spectrum [24] to show the existence of an infinite-dimensional space $\mathcal{F}_e$.

Here is a dynamical system whose Koopman operator admits a continuous spectrum:

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\theta} &= r, \end{aligned} \quad (17)$$

where the action-angle variables $(r, \theta) \in [0, r_0] \times \mathbb{S}^1$ and $r_0 \geq 0$. The flow S^t , $t \geq 0$ is given as follows:

$$\begin{aligned} S_1^t(r, \theta) &= r, \\ S_2^t(r, \theta) &= \theta + rt. \end{aligned} \quad (18)$$

For this dynamical system, the generator of the Koopman operator has an eigenvalue 0 with associated eigenfunction $g(r, \theta) = r$ and continuous spectrum [24].

Then we will prove that the (infinite-dimensional) Banach space $\mathcal{F}_e = \{g(r, \theta) = h(r)e^{i\theta} : h(r) \in L^2[0, r_0]\}$ is a valid invariant space of the Koopman operator by Definition 6. Firstly, we show the invariance under the Koopman operator. For $\forall g(r, \theta) = h(r)e^{i\theta}$ and $t \geq 0$, we have

$$U^t g(r, \theta) = g(S^t(r, \theta)) = h(r)e^{irt}e^{i\theta}. \quad (19)$$

Since $h(r)e^{irt} \in L^2[0, r_0]$, we have $U^t g \in \mathcal{F}_e$ for $\forall g \in \mathcal{F}_e$. Then we demonstrate the boundedness of $L|_{\mathcal{F}_e}$. Consider the norm

$$\|g\| = \int_0^{r_0} \int_{\mathbb{S}^1} |g(r, \theta)|^2 d\theta dr. \quad (20)$$

We have

$$\begin{aligned} \|L|_{\mathcal{F}_e}\| &= \sup_{g \in \mathcal{F}_e} \frac{\|f \cdot \nabla g\|}{\|g\|} = \sup_{g \in \mathcal{F}_e} \frac{\|irg\|}{\|g\|} \\ &\leq \sup_{g \in \mathcal{F}_e} \frac{\|ir\| \|g\|}{\|g\|} = \int_0^{r_0} \int_{\mathbb{S}^1} r^2 d\theta dr = \frac{2\pi}{3} r_0^3 \\ &\leq \infty. \end{aligned} \quad (21)$$

Finally, we illustrate that there exist at least 2 independent gradients ∇g for $(r, \theta) \in [0, r_0] \times \mathbb{S}^1$, i.e., there exists $g_1(r, \theta) = h_1(r)e^{i\theta}$, $g_2(r, \theta) = h_2(r)e^{i\theta} \in \mathcal{F}_e$ such that the matrix A is invertible, where

$$A = \begin{bmatrix} \frac{dh_1(r)}{dr} e^{i\theta} & ih_1(r)e^{i\theta} \\ \frac{dh_2(r)}{dr} e^{i\theta} & ih_2(r)e^{i\theta} \end{bmatrix}. \quad (22)$$

To satisfy this requirement, the observable functions can be $g_1(r, \theta) = e^{i\theta}$ and $g_2(r, \theta) = re^{i\theta}$. Therefore, the space $\mathcal{F}_e = \{g(r, \theta) = h(r)e^{i\theta} : h(r) \in L^2[0, r_0]\}$ is an infinite-dimensional valid invariant space of the Koopman operator.

B. Sampling theorem for the system with infinite-dimensional $\mathcal{F}_e$

In the infinite-dimensional valid invariant space $\mathcal{F}_e$ case, the analysis of critical sampling period requires the consideration of the continuous or residual spectrum, rather than directly computing the eigenvalues. In this section, we will take the example with continuous spectrum in Section VI-A to show that the critical sampling period T_γ of this example is consistent with the Nyquist-Shannon sampling bound. In particular, this Nyquist-Shannon bound guarantees the reconstruction of all signals generated by the system.

We first analyze the frequency bound by Nyquist-Shannon sampling theorem. Consider the initial condition of $r(0) \in [0, r_0]$ and $\theta(0) \in \mathbb{S}^1$, where $r_0 \geq 0$. The signals $x_1(t)$, $x_2(t)$ of dynamical system (17) is given in Cartesian coordinate:

$$\begin{aligned} x_1(t) &= r(t) \cos(\theta(t)) = r(0) \cos(\theta(0) + r(0)t), \\ x_2(t) &= r(t) \sin(\theta(t)) = r(0) \sin(\theta(0) + r(0)t). \end{aligned} \quad (23)$$

Hence, the frequency of these signals depends on the initial condition of $r(0)$. In particular, the lower bound of sampling frequency provided by Nyquist-Shannon sampling theorem is $2r(0)$ rad/s for the states with initial condition of $r(0) \in [0, r_0]$. To reconstruct all signals generated within this space without aliasing, according to the Nyquist-Shannon sampling theorem, the sampling frequency must exceed $2r_0$ rad/s.

Here we analyze the lower bound of sampling frequency by Theorem 1. Consider the valid invariant space

$$\mathcal{F}_e = \{g(r, \theta) = h(r)e^{i\theta} : h(r) \in L^2[0, r_0]\}, \quad (24)$$

which has been proved in Section VI-A. In the following, we will prove that the Koopman spectrum $\sigma(L|_{\mathcal{F}_e})$ is $i[0, r_0]$. Based on the definition of the resolvent set $\rho(L|_{\mathcal{F}_e})$ [25, Page 239], the operator $\lambda I - L|_{\mathcal{F}_e} : \mathcal{F}_e \rightarrow \mathcal{F}_e$ is bijective for $\forall \lambda \in \rho(L|_{\mathcal{F}_e})$, and the spectrum is $\sigma(L|_{\mathcal{F}_e}) = \mathbb{C}/\rho(L|_{\mathcal{F}_e})$. If the operator $\lambda I - L|_{\mathcal{F}_e} : \mathcal{F}_e \rightarrow \mathcal{F}_e$ is injective, i.e.,

$$(\lambda I - L|_{\mathcal{F}_e})g = (\lambda - ir)g = 0 \Rightarrow g = 0, \quad r \in [0, r_0], \quad (25)$$

then we have $\lambda \in \mathbb{C}/i[0, r_0]$. Similarly, the sufficiency can be proved, i.e., the operator $(\lambda I - L|_{\mathcal{F}_e})$ is injective for $\forall \lambda \in \mathbb{C}/i[0, r_0]$. Moreover, the operator $(\lambda I - L|_{\mathcal{F}_e})$ is also surjective for $\forall \lambda \in \mathbb{C}/i[0, r_0]$. This is because, for $\forall \lambda \in \mathbb{C}/i[0, r_0]$ and $\forall g \in \mathcal{F}_e$, there exists an observable function

$$\hat{g} = g/(\lambda - ir) \in \mathcal{F}_e, \quad (26)$$

such that $(\lambda I - L|_{\mathcal{F}_e})\hat{g} = g$. Therefore, the resolvent set $\rho(L|_{\mathcal{F}_e})$ and the spectrum $\sigma(L|_{\mathcal{F}_e})$ are

$$\begin{aligned} \rho(L|_{\mathcal{F}_e}) &= \mathbb{C}/i[0, r_0], \\ \sigma(L|_{\mathcal{F}_e}) &= i[0, r_0]. \end{aligned} \quad (27)$$

Noted that the continuous spectrum is $i[-r_0, 0]$ if we consider the valid invariant space $\mathcal{F}_e = \{g(r, \theta) = h(r)e^{-i\theta} : h(r) \in L^2[0, r_0]\}$.

According to Theorem 1, the lower bound of sampling frequency is $2r_0$ rad/s. It is consistent with the bound provided by Nyquist-Shannon sampling theorem, which guarantees the reconstruction of all $x(t)$ generated by this system.

VII. THE APPLICATION OF THE CRITICAL SAMPLING PERIOD T_γ

Although Theorem 1 provides the critical sampling period T_γ for exact identification based on the Koopman spectrum $\sigma(L|_{\mathcal{F}_e^\gamma})$, this Koopman spectrum is unknown in advance as it requires some prior knowledge from the exact system. In this section, we introduce the application of Theorem 1 by approximating $\sigma(L|_{\mathcal{F}_e^\gamma})$ from fast-sampled data.

The implementation of applying Theorem 1 in practice includes two steps. Firstly, one builds a prototype and uses fast-sampling analog-to-digital converter (ADC) to collect its data. From these fast-sampled data, we can approximate the critical sampling period T_γ by Theorem 1. Secondly, advised by this bound T_γ , the final product can then use a lower (and cheaper) sampling ADC. In the following, we will specifically introduce the first step of this implementation, i.e., approximating T_γ from fast-sampled data.

The main idea of approximating the critical sampling period T_γ from data is described as follows. First of all, we consider time-delay functions as a set of basis functions of observable space. Then we approximate $\sigma(L|_{\mathcal{F}_e^\gamma})$ from fast-sampled data by a variant of a standard Arnoldi method [21]. Finally, we compute the critical sampling period T_γ by the result of Theorem 1. In the following, we will show the corresponding steps of this approach.

1) *Collect data*: Consider the state $\mathbf{x} \in \mathbb{R}^n$ of the system. The collected data is represented by a matrix

$$G = [\mathbf{x}(0), \mathbf{x}(T_s), \dots, \mathbf{x}(KT_s)] \in \mathbb{R}^{n \times (K+1)}, \quad (28)$$

where $T_s \geq 0$ denotes the sampling period. The variable K represents the number of data points, and it is assumed to be sufficiently large. We sample these data by fast-sampling ADC, such that the sampling period T_s is much less than the critical sampling period T_γ and the system aliasing does not exist. This is to guarantee that the Koopman spectrum can be accurately approximated from these data.

2) *Lift data to observable space*: While the Koopman invariant space may possess infinite dimensions, for practical computational feasibility, we need to approximate the Koopman operator on a finite-dimensional observable space. Here we consider a set of time-delay functions $g_{l,j} = U^{jT_s} g_l$, where $g_l(\mathbf{x}) = x_l$, $l = 1, \dots, n$, and x_l denotes the l th component of $\mathbf{x}$. Denote vector-valued functions $\mathbf{g}_j = [g_{1,j}, \dots, g_{n,j}]^T$. We lift the data $\{\mathbf{x}(kT_s)\}_{k=0}^K$ to the space spanned by time-delay functions, i.e., $\text{span}\{\mathbf{g}_0, \dots, \mathbf{g}_{K-1}\}$, yielding the matrices:

$$\begin{aligned} G_1 &= \begin{pmatrix} x_1(0) & \dots & x_1((K-1)T_s) \\ \vdots & \ddots & \vdots \\ x_n(0) & \dots & x_n((K-1)T_s) \end{pmatrix}, \\ G_2 &= \begin{pmatrix} x_1(T_s) & \dots & x_1(KT_s) \\ \vdots & \ddots & \vdots \\ x_n(T_s) & \dots & x_n(KT_s) \end{pmatrix}. \end{aligned} \quad (29)$$

Remark 6 (Time-delay functions): There are two reasons for choosing time-delay functions as observable functions to approximate $\hat{T}_\gamma$. Firstly, these time-delay functions inherently span an approximate invariant space of the Koopman operator by the definition $U^{T_s} g_{l,j} = g_{l,j+1}$ [26]. Secondly, these functions enable a tight estimation of the critical sampling period T_γ . Specifically, Theorem 1 indicates that T_γ is determined by the maximum absolute value of the imaginary part of the Koopman spectrum, denoted by $\max |\text{Im}(L|_{\mathcal{F}_e^\gamma})|$. Given the definition of $\mathcal{F}_e^\gamma$ (see (12)), the value $\max |\text{Im}(L|_{\mathcal{F}_e^\gamma})|$ achieves the minimum of $\max |\text{Im}(L|_{\mathcal{F}_e})|$ among all possible $\mathcal{F}_e$. In fact, this quantity $\max |\text{Im}(L|_{\mathcal{F}_e})|$ characterizes the highest oscillation frequency component within the evolution $U^t g(\mathbf{x}(0))$ of all observable functions $g \in \mathcal{F}_e$ (see Appendix III). This highest frequency component is typically not lower than that of trajectories for well-chosen $\mathcal{F}_e$. Given that time-delay functions are observable functions, the highest oscillation frequency of $U^t g(\mathbf{x}(0))$ is identical to that of the system's trajectories because $U^t g_{l,j}(\mathbf{x}(0)) = x_l(t)$, where $l = 1, \dots, n$. Hence, the space of time-delay functions provides a tight estimation for the minimum of $\max |\text{Im}(L|_{\mathcal{F}_e})|$, which is

useful to approximate T_γ . In fact, this approach is closely related to the Hankel-DMD method as it is also created by time-delay functions, whose spectral approximation has been proved to converge to the Koopman eigenvalues for ergodic systems and systems with ergodic attractors [22].

3) *Approximate the Koopman spectrum*: By the definition of the Koopman operator U^{T_s} , we have

$$U^{T_s}[\mathbf{g}_0, \dots, \mathbf{g}_{K-1}] = [\mathbf{g}_1, \dots, \mathbf{g}_K]. \quad (30)$$

Suppose that the functional space $\text{span}\{\mathbf{g}_0, \dots, \mathbf{g}_{K-1}\}$ is a Koopman invariant space for some $K \geq 0$ as stated in [21]. Then the Koopman operator can be realized as the companion matrix $\hat{U}$, i.e., $[\mathbf{g}_1, \dots, \mathbf{g}_K] = [\mathbf{g}_0, \dots, \mathbf{g}_{K-1}]\hat{U}$, where

$$\hat{U} = \begin{pmatrix} 0 & & & c_1 \\ 1 & 0 & & c_2 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 & c_{K-1} \\ & & & 1 & c_K \end{pmatrix}. \quad (31)$$

In order to guarantee the robustness for the computation of eigenvalues of $\hat{U}$, we use a variant of a standard Arnoldi method to solve this problem. Specifically, we use a singular value decomposition of G_1 , i.e., $G_1 = U_{\text{svd}} \Sigma_{\text{svd}} W_{\text{svd}}^T$, where $U_{\text{svd}} \in \mathbb{R}^{n \times n}$, $W_{\text{svd}} \in \mathbb{R}^{K \times K}$ are orthogonal matrices and $\Sigma_{\text{svd}} \in \mathbb{R}^{n \times K}$. Based on the derivation in [21], we obtain the matrix $C = U_{\text{svd}}^T G_2 W_{\text{svd}} \Sigma_{\text{svd}}^{-1}$, which is related to $\hat{U}$ via a similarity transformation. Then the eigenvalues of C are employed to approximate some eigenvalues of the Koopman operator.

4) *Compute the sampling bound*: Based on the exponential relationship between the spectrum of the Koopman operator and its generator [25, Lemma 3.13], the spectrum of the generator is approximated by

$$\hat{\sigma}(L|_{\mathcal{F}_e^\gamma}) = \{\log(\lambda)/T_s : \lambda \in \sigma(C)\}. \quad (32)$$

According to Theorem 1, the critical sampling period is computed by

$$\hat{T}_\gamma = \pi / \max |\text{Im}(\hat{\sigma}(L|_{\mathcal{F}_e^\gamma}))|. \quad (33)$$

Subsequent numerical experiments will demonstrate that this approach can be used to approximate the critical sampling period with high accuracy.

Remark 7 (The initial condition): For certain systems, the approximation of $\hat{T}_\gamma$ remains unaffected by the initial condition $\mathbf{x}(0)$, as generated signals with different $\mathbf{x}(0)$ share the identical highest oscillation frequency. However, for dynamical systems like (17), the oscillatory property of system's trajectories $\mathbf{x}(t)$ depends on specific choice of $\mathbf{x}(0)$. In this case, the initial condition $\mathbf{x}(0)$ has a significant impact on the result of $\hat{T}_\gamma$ because $\hat{T}_\gamma$ is determined by the highest oscillation frequency of all related signals. Hence, the final result of $\hat{T}_\gamma$ is the minimum of all results approximated from time-series initiated from diverse $\mathbf{x}(0) \in \mathcal{M}$.

VIII. NUMERICAL EXPERIMENTS

In this section, various types of dynamical systems are used to numerically illustrate the effectiveness of Theorem 1 and to approximate the critical sampling period from data. We will first introduce the experimental setup and then show the numerical results.

To illustrate Theorem 1 numerically, we first identify the system from time-series of various sampling periods T_s by the Koopman operator-based identification approach. We choose this method for identification because it does not require the knowledge of time derivatives of the states, which is efficient to identify systems from measurements collected at low sampling frequency [5]. After each identification experiment, we compute the identification error and we plot it with respect to the sampling periods. Then we compare the error curve with the theoretical bound T_γ to determine whether T_γ is a critical sampling period, beyond which the system aliasing occurs. In particular, if T_γ is a critical bound, the identified vector field $\hat{f}$ is expected to be totally different from the correct f once the sampling period $T_s \geq T_\gamma$, i.e., the error exhibits a marked increase. Even though, the data generated from the identified system match the collected data at the same time points.

A. Experimental setup

Here we introduce the Koopman operator-based identification approach [5]. Consider the state vector $\mathbf{x} \in \mathbb{R}^n$ and the following dynamical system:

$$\dot{\mathbf{x}}_k = \mathbf{f}_k(\mathbf{x}) = \sum_{j=1}^N w_j^k g_j(\mathbf{x}), \quad k = 1, \dots, n, \quad (34)$$

where $g_j(\mathbf{x})$ are assumed to be monomial functions and coefficients w_j^k are to be identified.

When implementing the Koopman operator-based identification method in practice, a finite-dimensional observable space is considered to obtain the projection of the Koopman operator and the approximation of its generator. Based on these approximations, the vector field can be recovered through (8). In recent years, this Koopman operator framework has found a variety of applications across the extensive field of control, such as synthetic biology [27], nanometric positioning [28], building energy modeling [29], and the reconstruction of dynamics in power systems [30].

Specifically, consider an N -dimensional observable space $\mathcal{F}_m = \text{span}\{g_1, g_2, \dots, g_N\}$, where m denotes the total degree of the monomial basis function, i.e.,

$$g_j(\mathbf{x}) \in \{x_1^{s_1} \dots x_n^{s_n} | (s_1, \dots, s_n) \in \mathbb{N}^n : s_1 + \dots + s_n \leq m\}. \quad (35)$$

The snapshot data $\{\mathbf{x}_l, \mathbf{y}_l\}_{l=1}^K$ of sampling period T_s are lifted to $\mathcal{F}_m$. This yields the matrices $[X_{\text{lift}}]_{lj} = g_j(\mathbf{x}_l)$, $[Y_{\text{lift}}]_{lj} = g_j(\mathbf{y}_l)$. The matrix representation of the generator $\hat{L}$ and CT Koopman operator $\hat{U}^t, t \geq 0$ are approximated by matrix logarithm and exponential function:

$$\hat{L} = \frac{1}{T_s} \text{Log}(X_{\text{lift}}^\dagger Y_{\text{lift}}), \quad (36)$$

$$\hat{U}^t = \exp(\hat{L}t). \quad (37)$$

Denote q as the index of the function $g_q(\mathbf{x}) = x_k$. Following from (8), the parameters of the vector field are estimated as

$$\hat{w}_j^k = [\hat{L}]_{jq}, \quad (38)$$

where $[\hat{L}]_{jq}$ denotes the element at the j th row and q th column of $\hat{L}$. Further details on the implementation and convergence of this approach can be found in [5], [32]. Based on the approximation of $\hat{U}^t$, the state is predicted as

$$\hat{\mathbf{x}}_k(t) = [g_1(\mathbf{x}(0)), \dots, g_N(\mathbf{x}(0))] \hat{U}^t[:, q], \quad (39)$$

where $\hat{U}^t[:, q]$ denotes the q th column of the matrix $\hat{U}^t$.

In order to illustrate the effect of the sampling frequency on the identification, we collect time-series at various sampling periods from the related system. At each sampling period, the data are noise-free and sufficient in number to ensure that the identification error is caused by the system aliasing rather than data issues. Then we identify the system and compute the normalized root-mean-square error (NRMSE), i.e.,

$$\text{RMSE} = \sqrt{\frac{1}{nN} \sum_{k=1}^n \sum_{j=1}^N (\hat{w}_j^k - w_j^k)^2}, \quad (40)$$

$$\text{NRMSE} = \text{RMSE}/|w|, \quad (41)$$

where $|w|$ is the average value of all nonzero coefficients $|w_j^k|$ of associated dynamical systems, i.e., $|w| = \sum_{k=1}^n \sum_{j=1}^N |w_j^k| / \|\mathbf{w}\|_0$, $\mathbf{w} = [w_1^1, \dots, w_N^1, \dots, w_N^n]$.

B. Experimental results

Consider several dynamical systems (described below). Table I shows the data details and the critical sampling period analyzed by Theorem 1. This analysis can be found in Appendix IV. For each system, we set 1000 trajectories, 30 snapshots for each trajectory, which is the same setup with [33].

a) *The linear system.* We first illustrate Theorem 1 on the linear system (1) with parameters $\omega = 3, a = 0.1$. We select the observable space $\mathcal{F}_m$ with $m = 1$ for system identification. Fig. 3(a) shows the NRMSE (see (41)) at different sampling periods and the comparison with the critical sampling period. The blue line and the red dashed line represent the NRMSE curve and critical sampling period $\pi/3$ s of the linear system. It shows that the NRMSE undergoes a significant change as T_s approaches the red dashed line, which implies that the identified vector field is fundamentally different from the correct one. Besides, Fig. 3(b)-Fig. 3(d) shows the simulated trajectories of the correct system (blue lines), identified system (orange lines), and the collected data (black triangles) when sampling periods are 0.5s, 1.1s, 2.8s. It shows that the data generated by the identified system accurately match the collected data even though the identified vector field is incorrect. Consequently, this numerical result is in agreement with Theorem 1 that T_γ is a critical bound to avoid system aliasing of this system.

b) *The nonlinear system with real Koopman eigenvalues.* In order to illustrate Theorem 1 in a special case, where the critical sampling period is $T_\gamma \rightarrow \infty$, we consider the nonlinear system with real Koopman eigenvalues.

TABLE I

THE DETAILED INFORMATION OF DATA AND THE CRITICAL SAMPLING PERIOD GIVEN BY THEOREM 1 OF DYNAMICAL SYSTEMS

Dynamical system	Initial states	Total number of data pairs	Initial state for simulation	Critical sampling period [s]
a) Linear system	$\mathbf{x} \in [-1, 1]^2$	30×1000	$(0.5, 0.5)$	$\pi/3$
b) Real Koopman eigenvalues	$\mathbf{x} \in [-1, 1]^2$	30×1000	$(0.5, 0.5)$	∞
c) Fixed point	$\mathbf{x} \in [-1, 1]^2$	30×1000	$(0.5, 0.5)$	$\pi/3$
d) Limit cycle	$\mathbf{x} \in [-1.5, 1.5]^2$	30×1000	$(0.5, 0.5)$	$\pi/3$
e) Nonpolynomial vector field	$\mathbf{x} \in [-1, 1]^2$	30×1000	$(0.5, 0.5)$	$\pi/4$
f) Continuous spectrum	$r \in [0, 4], \theta \in \mathbb{S}^1$	30×1000	$(3.9898, 0)$	$\pi/4$

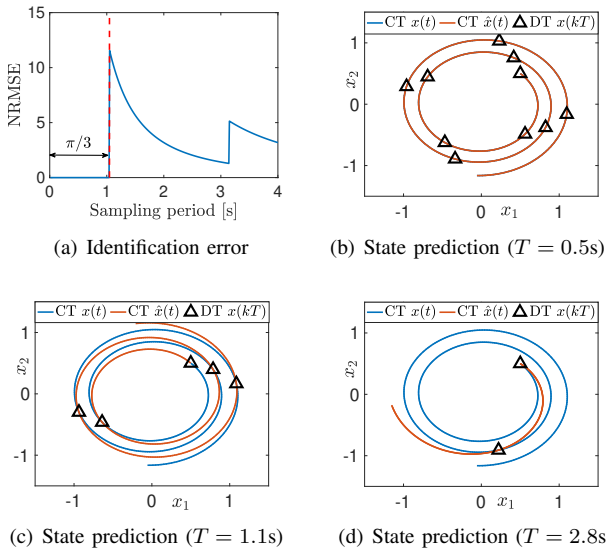


Fig. 3. Identification errors of the linear system (1) with $\alpha = 0.1, \omega = 3$ (a), predictions by vector fields identified from data with sampling period $T = 0.5s$ (b), $T = 1.1s$ (c), and $T = 2.8s$ (d). The parameter of the observable space is $m = 1$ for prediction.

$$\begin{aligned}\dot{x}_1 &= -x_1, \\ \dot{x}_2 &= x_1^2 - x_2.\end{aligned}\quad (42)$$

To minimize the impact of other factors such as the choice of observable space on the identification error, we select the Koopman invariant space, i.e., $\text{span}\{g_1(\mathbf{x}) = x_1, g_2(\mathbf{x}) = x_2, g_3(\mathbf{x}) = x_1^2\}$. Fig. 4(a) shows the NRMSE of the identified result. The blue line denotes the NRMSE respect to increasing sampling periods, showing that NRMSE remains small as the sampling period grows. This implies that it is efficient to identify this system at a very low sampling rate, which is in agreement with the theoretical result $T_\gamma \rightarrow \infty$ by Theorem 1. When the sampling periods are 0.5s, 1.1s and 2.8s, the trajectory $\hat{\mathbf{x}}(t)$ simulated by the identified dynamical system and the actual trajectory $\mathbf{x}(t)$ are visualized in Fig. 4(b)-Fig. 4(d). Specifically, the orange and blue line represent the simulation of $\hat{\mathbf{x}}(t)$ and $\mathbf{x}(t)$, respectively. In this case, the true trajectory $\mathbf{x}(t)$ is always the "simplest" one that matches measurements, i.e., without oscillation between consecutive measurements, which is the reason why this system can be identified without aliasing, regardless of the sampling period.

c) *The nonlinear system with a fixed point:* Then we consider a nonlinear system with a fixed point $(0, 0)$, whose

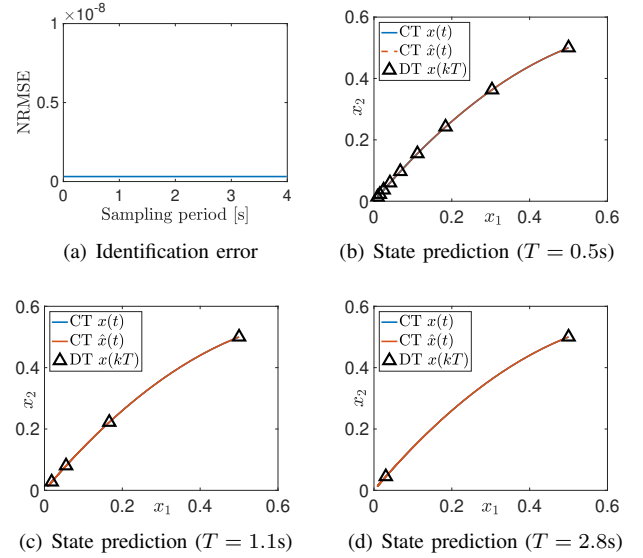


Fig. 4. Identification errors with growing sampling periods (a), and prediction by vector field identified from data with sampling period $T = 0.5s$ (b), $T = 1.1s$ (c), and $T = 2.8s$ (d) of nonlinear system (42) whose Koopman eigenvalue are real.

Koopman operator has non-zero imaginary part of $\sigma(L|_{\mathcal{F}_\gamma^\gamma})$.

$$\begin{aligned}\dot{x}_1 &= -3x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= 3x_1 - x_2(x_1^2 + x_2^2).\end{aligned}\quad (43)$$

Since the Koopman invariant space of nonlinear system (43) is unknown, we choose three observable spaces $\mathcal{F}_m$ with $m = 7, 10, 13$ for the identification of this system to make the curves of NRMSE convincing. Fig. 5(a) shows that the $\text{NRMSE}^{1/4}$ of the nonlinear system (43) exhibits a peak as the sampling period nears the red dashed line, where the red dashed line represents the critical sampling period T_γ of this system. And the generated data from identified system match the collected data as in Fig. 5(b)-Fig. 5(d). These results means that T_γ is a critical bound for identification, above which the identification is not reliable anymore. Hence, it is in agreement with Theorem 1 that T_γ is a critical bound to avoid aliasing.

d) *The nonlinear system with a limit cycle.* Besides, consider a different type of nonlinear system, i.e., nonlinear system with a limit cycle at $r = \sqrt{x_1^2 + x_2^2} = 1$.

$$\begin{aligned}\dot{x}_1 &= 3x_2 - x_1(x_1^2 + x_2^2 - 1), \\ \dot{x}_2 &= -3x_1 - x_2(x_1^2 + x_2^2 - 1).\end{aligned}\quad (44)$$

Three observable space $\mathcal{F}_m, m = 7, 10, 13$ are used for the identification in the Koopman-based approach. Fig. 6(a) shows

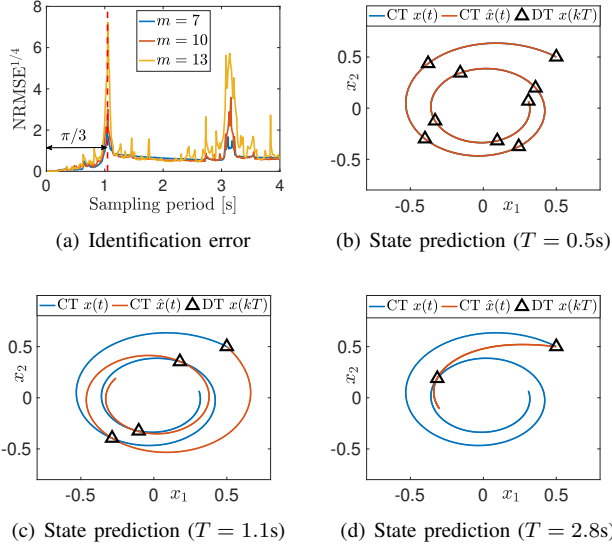


Fig. 5. Identification errors with growing sampling periods (a), and prediction by vector field identified from data with sampling period $T = 0.5$ s (b), $T = 1.1$ s (c), and $T = 2.8$ s (d) of nonlinear system with fixed point (43). The parameter of the observable space is $m = 13$ for prediction.

the $\text{NRMSE}^{1/4}$ of the identified vector fields and the critical sampling period $\pi/3$ s represented by the red dashed line. It is observed that, similar to previous experiments, the curve of $\text{NRMSE}^{1/4}$ changes significantly as the sampling period tends to ($m = 7, 10$) or a little earlier ($m = 13$) than $\pi/3$ s. In addition, the generated data match the correct data as in Fig. 6(b)-Fig. 6(d). Hence, it is consistent with Theorem 1.

Furthermore, this numerical result ($m = 13$) also suggests that it may require higher sampling frequency when more basis functions are considered in the Koopman operator-based identification method. This can be explained in theoretical and intuitive perspective. Theoretically, the computation of the matrix logarithm needs the sampling frequency satisfying $\omega > 2 \max |\text{Im}(\sigma(\hat{L}))|$ (rad/s), where $\hat{L}$ is approximated matrix representation of $L|_{\mathcal{F}_m}$. When we consider more basis functions, the space $\mathcal{F}_m$ may encompass eigenfunctions which are power of other eigenfunctions, with multiple of associated Koopman eigenvalues (see Property 1). This can result in an increase in $\max |\text{Im}(\sigma(\hat{L}))|$. Consequently, the computation of the matrix logarithm requires higher sampling frequency, and the pronounced rise in the curve of $\text{NRMSE}^{1/4}$ may occur before the line denoting the critical sampling period T_γ . Intuitively, the Koopman operator-based identification method is equivalent to identifying a high-dimensional system whose states are selected functions. The functions of higher degrees may exhibit higher oscillation frequencies, necessitating increased sampling rates to accurately capture the dynamics.

e) *The nonlinear system with nonpolynomial vector field.* Since the above systems all have polynomial vector fields, here we illustrate Theorem 1 on the system with nonpolynomial

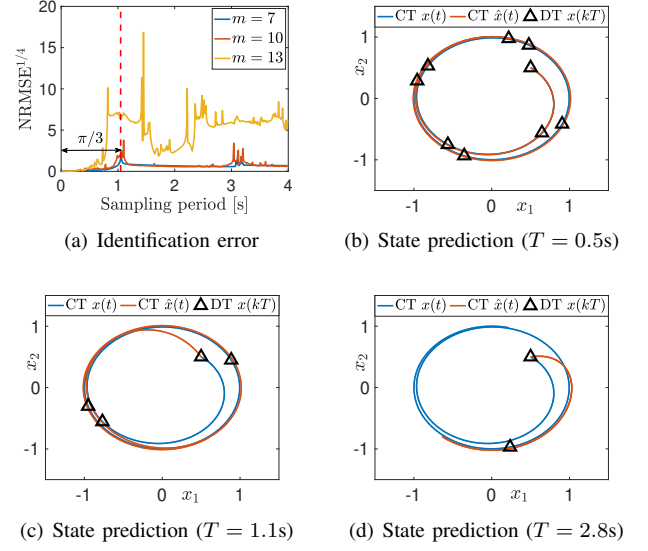


Fig. 6. Identification errors with growing sampling periods (a), and prediction by vector field identified from data with sampling period $T = 0.5$ s (b), $T = 1.1$ s (c), and $T = 2.8$ s (d) of nonlinear system with limit cycle (44). The parameter of the observable space is $m = 13$ for prediction.

vector field.

$$\begin{aligned} \dot{x}_1 &= -x_1 + 4 \frac{x_2}{1 + x_2^2}, \\ \dot{x}_2 &= -x_2 - 4 \frac{x_1}{1 + x_2^2}. \end{aligned} \quad (45)$$

Since the vector field is nonpolynomial, the basis functions of the observable space are set by $g(x) \in \{x_1^{s_1} x_2^{s_2} | (s_1, s_2) \in \mathbb{N}^2, s_1 + s_2 \leq m\}$ and

$$h(x) = \frac{x_l}{1 + x_k^p}, \quad k, l \in \{1, 2\}, \quad p \in \{1, \dots, P\}. \quad (46)$$

Four observable spaces, i.e., $\{m = 1, P = 2\}$, $\{m = 1, P = 3\}$, $\{m = 2, P = 2\}$, $\{m = 2, P = 3\}$, are chosen to identify the nonlinear system (45) with growing sampling period. The critical sampling period of this system is represented by the red dashed line in Fig. 7(a). It shows that the NRMSE curve of the nonlinear system in (45) exhibits a sharp increase and reaches a peak when the sampling period approaches T_γ . Fig. 7(b)-Fig. 7(d) also show that, even through the identified system is incorrect, the error between data collected from true system and that of identified system is small at the corresponding sampling period. Hence, it illustrates Theorem 1 that T_γ is a critical sampling period that prevents aliasing.

f) *The dynamical system whose Koopman operator has continuous spectrum.* In the previous examples, the Koopman operators all admit point spectrum. Here we consider a system with continuous spectrum to illustrate Theorem 1.

$$\begin{aligned} \dot{x}_1 &= -x_2 \sqrt{x_1^2 + x_2^2}, \\ \dot{x}_2 &= x_1 \sqrt{x_1^2 + x_2^2}. \end{aligned} \quad (47)$$

Since the vector field is also nonpolynomial, the basis functions of the observable space are given by $g(x) = x_k$, $k = 1, 2$ and $h(x) = x_1^{s_1} x_2^{s_2} \sqrt{x_1^2 + x_2^2}$, where $(s_1, s_2) \in \mathbb{N}^2, s_1 + s_2 \leq$

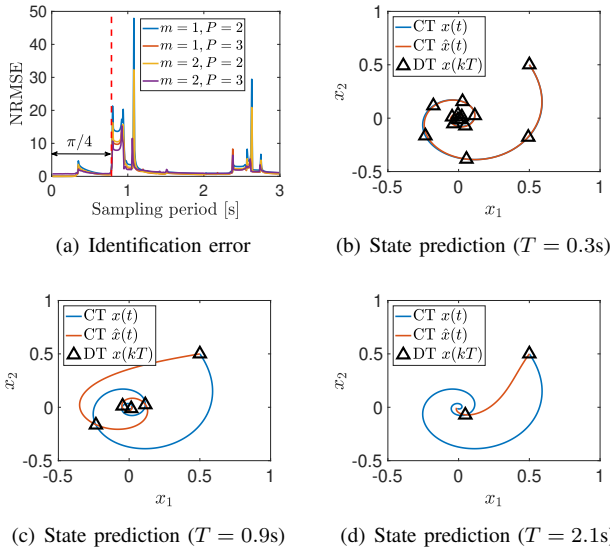


Fig. 7. Identification errors with growing sampling periods (a), and prediction by vector field identified from data with sampling period $T = 0.3s$ (b), $T = 0.9s$ (c), and $T = 2.1s$ (d) of nonlinear system with nonpolynomial vector field (45). The parameters of the observable space are $m = 1, P = 2$ for prediction.

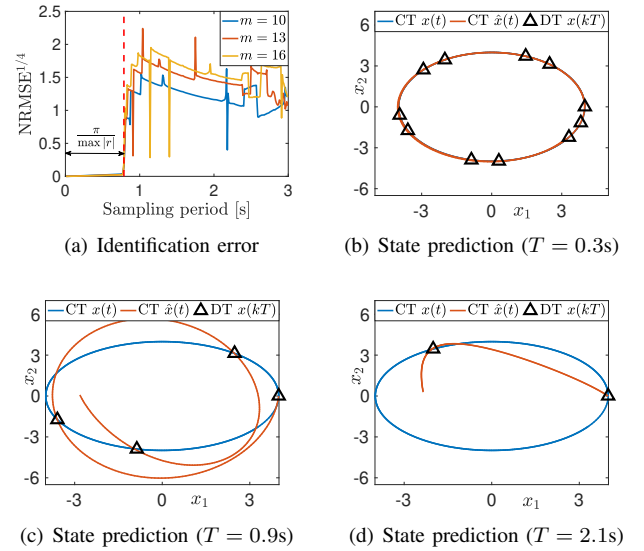


Fig. 8. Identification errors with growing sampling periods (a), and prediction by vector field identified from data with sampling period $T = 0.3s$ (b), $T = 0.9s$ (c), and $T = 2.1s$ (d) of the dynamical system (47) with continuous spectrum of the Koopman operator. The parameters of the observable space are $m = 16$ for prediction.

m . Three observable spaces, i.e., $m = 10, 13, 16$, are chosen to identify the dynamical system (47).

Fig. 8(a) shows changes in $NRMSE^{1/4}$ of the identified vector field as the sampling period increases. The red dashed line represents the critical sampling period. It shows that the curve of $NRMSE^{1/4}$ rises sharply as T_s approaches the red dashed line, which represents the critical sampling period. Besides, the trajectory $\hat{x}(t)$ simulated by the identified dynamical systems and the true trajectory $x(t)$ are visualized in Fig. 8(b)-Fig. 8(d) when the sampling periods are 0.3s, 0.9s and 2.1s for dynamical systems (47). The blue line represents the true trajectory $x(t)$ with the initial condition $(\max|r|, 0)$, where $\max|r| = 3.9898$ is the maximum of $|r(0)|$ in the collected data, and the orange lines represent the simulations $\hat{x}(t)$ of identified systems. The black triangles indicate the state at sampling times. It shows that $\hat{x}(t)$ matches $x(t)$ at sampling points no matter whether the system is accurately identified or not. Hence, it is in agreement with Theorem 1 that T_γ is a critical bound for identification.

For all examples that have bounded T_γ , there is a common phenomenon that the error curves reach a peak when the sampling period T_s approaches T_γ . Subsequently, the curves decline briefly before rising again as in Fig. 3(a)-Fig. 7(a). This can be explained by the simulation results. Let us take the Fig. 5(b)-Fig. 5(d) as an example for explanation. The directions of trajectories $\hat{x}(t)$ and $x(t)$ at certain times t reflect the directions of identified vector field and the actual vector field at certain states $\hat{x}(t)$ and $x(t)$, respectively. In the simulation results Fig. 5(b)-Fig. 5(d), it shows that the direction of the identified vector field at certain $x \in \mathcal{M}$ appears to rotate as the sampling period increases to find the “simplest” trajectory that matches the subsequent data. In particular, at the initial state $x = (0.5, 0.5)$, Fig. 5(b) shows that the direction of identified vector field matches the truth because trajectories

$\hat{x}(t)$ and $x(t)$ are almost the same, leading to small NRMSE. When the sampling period exceeds the critical sampling period T_γ a little, i.e., the sampling period is 1.1s, the directions of the blue line and the orange line are almost opposite to each other at $x = (0.5, 0.5)$ in Fig. 5(c), which indicates that the identified vector field $\hat{f}(x)$ significantly deviates from the true vector field. It leads to a peak of NRMSE curve. After that, the direction of identified vector field $\hat{f}$ at $x = (0.5, 0.5)$ continues to rotate, and the difference between $\hat{f}$ and f decreases as illustrated in Fig. 5(d). Consequently, the NRMSE curve exhibits a downward trend.

Although these results show that there is a peak in the NRMSE curve when the sampling period is T_γ , it is also evident that the curves increase gradually when the sampling period T_s is less than T_γ , as shown in Fig. 5(a)-7(a). Hence, here we show that the frequency domain information of $\hat{x}(t)$, simulated from the identified vector field, are little affected by this identification error when $T_s < T_\gamma$. For these systems, discrete Fourier transform is applied to the true trajectory $x(t)$ and the predicted trajectory $\hat{x}(t)$ with the initial condition $x(0) = (0.5, 0.5)$. We use a fast Fourier transform algorithm with the sampling frequency 100 Hz and 500 samples. The spectral error of $x_k(t)$ is computed by $\|\hat{P}_k - P_k\|_2^2$, where $\hat{P}_k$ and P_k are the discrete Fourier transforms of $\hat{x}_k(t)$ and $x_k(t)$ respectively. The spectral error, which varies with the sampling period, is shown in Fig. 9 for these systems. The blue lines and orange lines represent the spectral error of states $x_1(t), x_2(t)$, and the red dashed lines represent T_γ of associated dynamical system. It shows that there is little loss to reconstruct $x(t)$ when the sampling period is lower than T_γ . However, it fails to reconstruct $x(t)$ when the sampling period exceeds the bound. Hence, when $T_s < T_\gamma$, the identification error does not indicate system aliasing as the identified system still provides a close approximation of the correct one. This

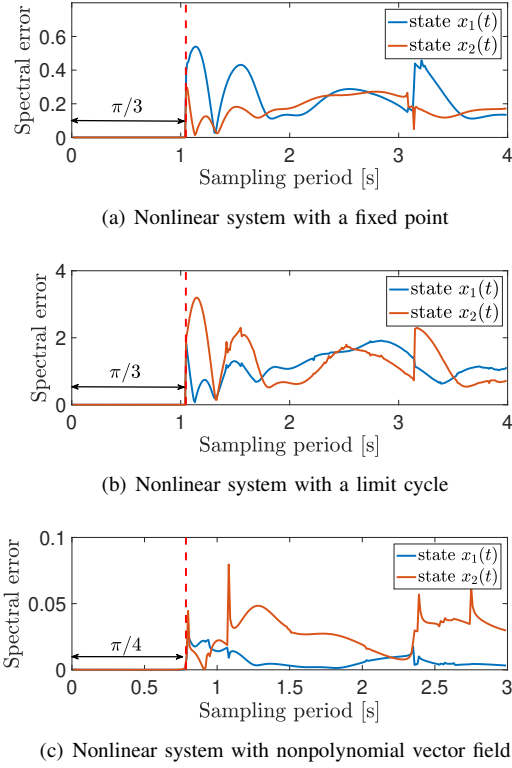


Fig. 9. Spectral error of predicted states $x_1(t)$ and $x_2(t)$ with growing sampling periods of nonlinear system with a fixed point (a), limit cycle (b), and nonlinear system whose vector field is nonpolynomial (c). The parameter of observable space is $m = 1$ for the linear system, $m = 13$ for nonlinear systems, and $m = 1, P = 2$ for the dynamical system with nonpolynomial vector field.

also implies that the critical sampling period with respect to system aliasing may also be a criterion to avoid aliasing of associated signal $x(t)$ indirectly.

In addition, to illustrate the efficiency of the approximation method introduced in Section VII, we approximate the critical sampling bounds $\hat{T}_\gamma$ from fast-sampled data by this approach for all these dynamical systems. Since the relative error of the approximation is influenced by the initial state $x(0)$ of the signal (see Remark 7), we collect sufficient time-series with various $x(0) \in \mathcal{M}$ for approximations. All data are collected at a sampling period of $T_s = 0.01$. Then we select the minimum value among the approximations as the final result of $\hat{T}_\gamma$. The detailed information of the specific time-series and approximation results are provided in Table II, showing that the critical sampling period can be accurately approximated.

IX. CONCLUSIONS

This paper considers the system aliasing of nonlinear dynamical systems caused by low sampling frequency. It proposes a necessary and sufficient condition on the critical sampling period that avoids system aliasing. When the Koopman operator admit only point spectrum, the critical sampling period of the related system can be analytically computed from the imaginary part of the corresponding Koopman eigenvalues. When the Koopman operator has non-empty continuous or residual spectrum, the critical sampling period is analyzed

by imaginary part of the whole spectrum of the infinitesimal generator. In practice, this critical sampling period can be approximated from fast-sampled data. Through examples of different types of dynamics, such as those with equilibria and limit cycles, the critical sampling period offers a fundamental limit of performance for any system identification algorithms, which could be essential for experimental design.

X. ACKNOWLEDGEMENT

We wish to thank Prof. Yiwei Zhang (Huazhong University of Science and Technology) and Wenchao Li (Peking University) for discussion.

APPENDIX I

THE VALID INVARIANT SPACE $\mathcal{F}_e$

Proposition 2 (Conditions to recover f): Consider the Koopman invariant space $\mathcal{F}$, the restriction of the Koopman operator $U^t|_{\mathcal{F}}$ and its generator $L|_{\mathcal{F}}$. The vector field f can be recovered from $L|_{\mathcal{F}}$ if and only if there exist at least n functions $g_i \in \mathcal{F}$ such that the gradients $\{\nabla g_i\}_{i=1}^n$ are linearly independent vectors for $\forall x \in \mathcal{M} \subseteq \mathbb{R}^n$.

Proof: Sufficiency. For $\forall x \in \mathcal{M}$, consider n observable functions $g_i \in \mathcal{F}$ that have n linearly independent vectors $\{\nabla g_i\}_{i=1}^n$. Then we have n equations based on (8):

$$\sum_{j=1}^n f_j \partial g_i / \partial x_j = L|_{\mathcal{F}_e} g_i, \quad (48)$$

where $i = 1, \dots, n$. Indeed, we have $A[f_1(x), \dots, f_n(x)]^T = [L|_{\mathcal{F}} g_1(x), \dots, L|_{\mathcal{F}} g_n(x)]^T$, where $A_{ij} = \partial g_i / \partial x_j(x)$ is an invertible matrix for $\forall x \in \mathcal{M}$. Accordingly, the vector field $f(x) = [f_1(x), \dots, f_n(x)]$ can be recovered.

Necessity. Since the vector field $f = [f_1, \dots, f_n]$ can be recovered from $L|_{\mathcal{F}}$, it follows from the equations $f \cdot \nabla g_i = L|_{\mathcal{F}} g_i, i = 1, \dots, N$ that the rank of the matrix $A_{ij} = \partial g_i / \partial x_j(x)$ is n , i.e., there is at least n linearly independent vectors $\{\nabla g_i(x)\}_{i=1}^n$ for $g_i \in \mathcal{F}, \forall x \in \mathcal{M}$. ■

APPENDIX II

THE SAMPLING THEOREM

In Appendix II-A, we restate the system aliasing in Koopman operator framework. Then we provide lemmas in Appendix II-B and prove Theorem 1 in Appendix II-C.

A. Equivalent description of system aliasing in $\mathcal{F}_e$

Proposition 3 (System aliasing and generator aliasing): For the dynamical system (2) that has valid invariant spaces $\mathcal{F}_e$ (see Definition 7), the following assertions are equivalent:

- No system aliasing, i.e., there is only one vector field $f \in \mathcal{J}_f$ that generates the DT flow $S^{T_s}(x)$ for $\forall x \in \mathcal{M}$.
- No generator aliasing, i.e., there is only one generator $L|_{\mathcal{F}_e} \in \mathcal{J}_L$ that generates the Koopman operator $U^{T_s}|_{\mathcal{F}_e}$ for at least one $\mathcal{F}_e$, where $\mathcal{J}_L$ is the space of $L|_{\mathcal{F}_e}$ that determined by $f \in \mathcal{J}_f$ through (8).

Proof: a) $\rightarrow$ b): When there is only one vector field $f \in \mathcal{J}_f$ that generates S^{T_s} , i.e., there is no system alias, the uniqueness of $L|_{\mathcal{F}_e}$ is guaranteed through $L|_{\mathcal{F}_e} = f \cdot \nabla|_{\mathcal{F}_e}$.

TABLE II
APPROXIMATION RESULT OF THE CRITICAL SAMPLING PERIOD

Dynamical system	Initial condition	Theoretical bound [s]	Approximated bound [s]	Relative error
a) Linear system	(0.5, 0.5)	1.0472	1.0471	7.4588×10^{-5}
b) Real Koopman eigenvalues	(0.5, 0.5)	∞	∞	—
c) Fixed point	(0.5, 0.5)	1.0472	1.0476	3.7098×10^{-4}
d) Limit cycle	(0.5, 0.5)	1.0472	1.0469	2.4553×10^{-4}
e) Nonpolynomial vector field	(0.1, 0.1)	0.7854	0.7879	3.1000×10^{-3}
f) Continuous spectrum	(3.9898, 0)	0.7854	0.7872	2.7595×10^{-4}

$b) \rightarrow a)$: When there is only one generator $L|_{\mathcal{F}_e} \in \mathcal{J}_L$ that generates $U^{T_s}|_{\mathcal{F}_e}$, i.e., there is no generator alias, the vector field f can be recovered uniquely from $L|_{\mathcal{F}_e}$ based on Proposition 2.

B. Lemmas for proving Theorem 1

Based on Proposition 3, we need to obtain the generator $L|_{\mathcal{F}_e}$ uniquely from $U^{T_s}|_{\mathcal{F}_e}$ in at least one $\mathcal{F}_e$. Here we first introduce several useful lemmas for proving Theorem 1.

Since the generator $L|_{\mathcal{F}_e}$ is bounded, the Koopman semigroup $U^t|_{\mathcal{F}_e}$ and its generator $L|_{\mathcal{F}_e}$ have the exponential relationship defined as follows:

Lemma 1 (Exponential formulas [34, Section 11.8]): The semigroup $U^t|_{\mathcal{F}_e} \in \mathcal{L}(\mathcal{F}_e)$ is an exponential function of its generator of $L|_{\mathcal{F}_e}$, i.e., $U^t|_{\mathcal{F}_e} = \exp(L|_{\mathcal{F}_e} t)$. For the bounded generator $L|_{\mathcal{F}_e}$, the exponential of the operator is defined by:

$$\exp(L|_{\mathcal{F}_e} t) = \sum_{k=0}^{\infty} \frac{t^k L|_{\mathcal{F}_e}^k}{k!}. \quad (49)$$

By Lemma 1, the following equation between DT Koopman operator $U^{T_s}|_{\mathcal{F}_e}$ and its generator $L|_{\mathcal{F}_e}$ holds:

$$U^{T_s}|_{\mathcal{F}_e} = \exp(L|_{\mathcal{F}_e} T_s). \quad (50)$$

However, there are several (in fact infinitely many) solutions of $L|_{\mathcal{F}_e}$ by operator logarithm, which is described as follows.

Lemma 2 (Many-valued function [34, Section 5.4]): Let $B \in \mathcal{L}(X)$, I denotes the identity operator and I_k denotes the idempotent, i.e., $\exp(2\pi i I_k) = I$,

$$\log(\exp B) = B + 2\pi i \sum_k n_k I_k, \quad (51)$$

where the n_k are integers.

In order to guarantee the uniqueness of $L|_{\mathcal{F}_e}$, the essential theorem of principal logarithm is as follows.

Lemma 3 (Principal Logarithm [35, Theorem 2]): Let $Y \in \mathcal{L}(\mathcal{F}_e)$ whose logarithm is well defined. There is a unique principal logarithm of Y , denoted as $B = \text{Log}(Y)$, where the spectrum of B lies in the strip $\mathcal{G}(B) = \{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$, i.e., $\sigma(B) \subset \mathcal{G}(B)$.

C. Proof of Theorem 1

Proof: It follows from Proposition 3 and Lemma 1 that no system aliasing equivalently requires $L|_{\mathcal{F}_e}$ to be the only element in the set $G(L|_{\mathcal{F}_e}, T_s, \mathcal{J}_L)$ for at least one $\mathcal{F}_e$, where $G(L|_{\mathcal{F}_e}, T_s, \mathcal{J}_L) := \{\tilde{L} \in \mathcal{J}_L : \exp(L|_{\mathcal{F}_e} T_s) = \exp(\tilde{L} T_s)\}$.

When there is no prior knowledge of the aliasing space, by Lemma 2-3, we require $T_s L|_{\mathcal{F}_e}$ to be the principal logarithm of the Koopman operator $U^{T_s}|_{\mathcal{F}_e}$, i.e.,

$$L|_{\mathcal{F}_e} = \frac{1}{T_s} \text{Log}(U^{T_s}|_{\mathcal{F}_e}). \quad (52)$$

It follows from Lemma 3 that the generator $L|_{\mathcal{F}_e}$ satisfies $\sigma(L|_{\mathcal{F}_e}) \subset \mathcal{G}(T_s)$ for at least one $\mathcal{F}_e$, where

$$\mathcal{G}(T_s) := \{z \in \mathbb{C} : -\pi/T_s < \text{Im}(z) < \pi/T_s\}. \quad (53)$$

Hence, the sampling period satisfy $T_s < T_\gamma$, where the critical sampling period T_γ is

$$T_\gamma = \frac{\pi}{\max |\text{Im}(\sigma(L|_{\mathcal{F}_e}))|}, \quad (54)$$

and

$$\mathcal{F}_e^\gamma \triangleq \arg \min_{\mathcal{F}_e} \{\max |\text{Im}(\sigma(L|_{\mathcal{F}_e}))|\}. \quad (55)$$

Actually, as long as this sampling condition is satisfied, **there is one and only one operator in the space $\mathcal{J}_L = \{\tilde{L} \in \mathcal{L}(\mathcal{F}_e^\gamma) : -\pi/T_s < \text{Im}(\sigma(\tilde{L})) < \pi/T_s\}$ that generates $U^{T_s}|_{\mathcal{F}_e}$. Correspondingly, if prior knowledge of the space to which the vector field f belongs is unavailable, we reconstruct f without aliasing in the space defined by $\mathcal{J}_L$ using (8). In this space, the true f is the unique function that generates DT flow S^{T_s} .**

APPENDIX III THE KOOPMAN SPECTRUM

Here we show that, the real and imaginary parts of the spectrum $\sigma(L|_{\mathcal{F}_e^\gamma})$ characterize limited growth rate and oscillation frequency components of $g(\mathbf{x}(t))$, $g \in \mathcal{F}_e^\gamma$, respectively.

Based on the definition of the Koopman operator, we have

$$g(\mathbf{x}(t)) = g(S^t(\mathbf{x}(0))) = U^t|_{\mathcal{F}_e^\gamma} g(\mathbf{x}(0)) \quad (56)$$

It follows from Lemma 1 that

$$g(\mathbf{x}(t)) = e^{L|_{\mathcal{F}_e^\gamma} t} g(\mathbf{x}(0)) \quad (57)$$

Then we have

$$g(\mathbf{x}(t)) = U^t|_{\mathcal{F}_e^\gamma} g(\mathbf{x}(0)) = \int_{\sigma(L|_{\mathcal{F}_e^\gamma})} e^{\lambda t} E(d\lambda) g(\mathbf{x}(0)), \quad (58)$$

where E is the resolution of the identity for $L|_{\mathcal{F}_e^\gamma}$ [36, Page 898]. Hence the oscillation and growth of amplitude are characterized by the real and imaginary parts of $\lambda \in \sigma(L|_{\mathcal{F}_e^\gamma})$, respectively. Moreover, the spectral radius $r_\sigma(L|_{\mathcal{F}_e^\gamma})$ satisfies [25, Page 241] $r_\sigma(L|_{\mathcal{F}_e^\gamma}) \leq \|L|_{\mathcal{F}_e^\gamma}\|$, where $r_\sigma(L|_{\mathcal{F}_e^\gamma}) := \sup\{|\lambda| : \lambda \in \sigma(L|_{\mathcal{F}_e^\gamma})\}$. Hence, the real and imaginary parts of $\sigma(L|_{\mathcal{F}_e^\gamma})$ are bounded by the definition of $\mathcal{F}_e^\gamma$.

APPENDIX IV

THE CRITICAL SAMPLING PERIOD T_γ

The critical sampling periods T_γ of dynamical systems in Section VIII are analyzed as follows.

A. Linear system

Since the system is linear, the valid invariant space is $\mathcal{F}_2 = \text{span}\{g_1(\mathbf{x}) = x_1, g_2(\mathbf{x}) = x_2\}$. Restricted in this space, the generator of the system can be represented by the following system matrix:

$$\begin{pmatrix} 0.1 & 3 \\ -3 & 0.1 \end{pmatrix} \quad (59)$$

Thus the eigenvalues are determined by $(\lambda - 0.1)^2 + 3^2 = 0$, leading to $\lambda_{1,2} = 0.1 \pm 3i$. According to Theorem 1, the critical sampling period is $\pi / \max |\text{Im}(\lambda(L|_{\mathcal{F}_2}))| = \pi/3$ s.

B. Real Koopman eigenvalues

The Koopman operator of the system (42) has a finite-dimensional invariant space, i.e., $\text{span}\{g_1(\mathbf{x}) = x_1, g_2(\mathbf{x}) = x_2, g_3(\mathbf{x}) = x_1^2\}$, which is a valid invariant space. Restricted in this space, the generator is represented by a matrix:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad (60)$$

The eigenvalues of the generator are -1 and -2 , which are all real values. According to Theorem 1, the lower bound of sampling frequency is 0 and the critical sampling period is ∞ .

C. Nonlinear system with fixed point

For nonlinear systems with fixed point x^* , i.e., $f(x^*) = 0$, the principal Koopman eigenvalues λ_i are the eigenvalues of the Jacobian matrix of the vector field f at the fixed point x^* [16]. Therefore, eigenvalues of the generator of the system (43) are eigenvalues of the Jacobian matrix at $(0, 0)$:

$$J = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}. \quad (61)$$

It leads to the principal Koopman eigenvalues $\pm 3i$. According to Theorem 1, the lower bound of sampling frequency is 6 rad/s and the critical sampling period is $\pi/3$ s.

D. Nonlinear system with a limit cycle

For dynamical systems that admit a limit cycle, the Floquet exponents are principal Koopman eigenvalues [16]. The Koopman principal eigenvalues of system (44) are calculated as follows. In polar coordinates, i.e., through the transformation $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$, we obtain the following dynamical system

$$\begin{aligned} \dot{r} &= -r^3 + r, \\ \dot{\theta} &= -3. \end{aligned} \quad (62)$$

Since the limit cycle is at $r = 1$, the eigenvalue (Floquet exponent) is -2 . Other principal eigenvalues are $\pm 3i$ with associated eigenfunctions $e^{\mp i\theta}$. According to Theorem 1, the critical sampling period is $\pi/3$ s.

E. Non-polynomial

Since the dynamical system (45) has the fixed point $x^* = (0, 0)$, the principal Koopman eigenvalues λ_i are the eigenvalues of the Jacobian matrix of the vector field f at $(0, 0)$ [16]. Therefore, the principal eigenvalues of the generator are the eigenvalues of the matrix:

$$J = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}. \quad (63)$$

It leads to the principal eigenvalues $-1 \pm 4i$. Therefore, the critical sampling period is $\pi/4$ s based on Theorem 1.

F. Continuous spectrum

Through the coordinate transformation $x_1 = r \cos \theta, x_2 = r \sin \theta$, the system (47) can be expressed in polar coordinate:

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\theta} &= r, \end{aligned} \quad (64)$$

where $r \in [0, \max |r|]$, $\theta \in \mathbb{S}^1$ and $\max |r|$ denotes the maximum absolute value of $r(0)$ in the data. According to the analysis in Section VI-B, the continuous spectrum is $\sigma(L|_{\mathcal{F}_e}) = i[0, \max |r|]$. Based on Theorem 1, the critical sampling period of dynamical system (47) is $\pi / \max |r|$ s.

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