Passive Scalar Transport by Non-Smooth Incompressible Fluids: Mixing and Vanishing Viscosity

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To Moeke, who never got the opportunity to study.
Je wordt erg gemist.

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted, or is being concurrently submitted, for any degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Abstract

This thesis explores fundamental questions in fluid dynamics through rigorous mathematical analysis of the passive scalar transport model. Our investigation centers on the behaviour of fluid flows characterised by vector fields of lower regularity—a crucial feature in understanding turbulent dynamics. Through careful examination of these flows in various function spaces, particularly Sobolev spaces, we develop new analytical tools and insights into three key areas: well-posedness, regularity, and solution selection.

The first major contribution introduces a novel weak compactness technique that yields improved quantitative estimates for the transport equation. This approach leads to several significant advances, including enhanced classical mixing estimates with exponential lower bounds, propagation of mild logarithmic fractional regularity, and state-of-the-art weak stability estimates for transport along Sobolev vector fields. Most notably, we establish the first quantitative stability estimate for transport along vector fields with bounded variation, marking progress on the challenging $p = 1$ case of Bressan's conjecture.

Our second principal contribution extends to the analysis to the transport-diffusion equation, where we develop techniques beyond standard energy estimates. By combining mild solutions, weak convolution estimates, and maximal regularity methods, we establish new results under the Ladyzhenskaya-Prodi-Serrin integrability condition on the vector. These methods effectively capture the interplay between transport and diffusion on regularisation, leading to improved uniqueness and regularity results.

The final contribution challenges conventional approaches to solution selection through vanishing diffusion limits. Through explicit constructions, we demonstrate that the vanishing diffusion approach fails to consistently select physically meaningful solutions for the passive scalar transport model. Our results show that this method can produce solutions violating basic thermodynamic principles, including time-arrow reversal—a finding that questions traditional approaches to solution selection in fluid dynamics.

These contributions advance our understanding of irregular fluid flows while raising important questions about current mathematical frameworks in fluid mechanics. The thesis concludes by identifying critical open problems, particularly regarding the well-posedness of turbulent fluid flows and the development of alternative approaches to solution selection.

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Nomenclature

Acronyms

TE Transport equation, Definition 2.1.

TDE Transport-diffusion equation, Definition 3.1.

ODE Ordinary differential equation.

PDE Partial differential equation.

a.e. Almost everywhere, in Lebesgue measure.

Function spaces

L^p Lebesgue space.

L^p_{loc} Local Lebesgue space.

L^p_x Shorthand for $L^p(\mathbb{R}^d;\mathbb{R}^m)$ with the vector dimension $m \in \mathbb{N}$ clear from context.

$L^p_{x,t}$ Shorthand for $L^p(\mathbb{R}^d \times [0,T];\mathbb{R}^m)$ with the vector dimension $m \in \mathbb{N}$ clear from context.

$L^p_t L^q_x$ Shorthand for $L^p([0,T];L^q(\mathbb{R}^d;\mathbb{R}^m))$ with the vector dimension $m \in \mathbb{N}$ clear from context. Specific functions $f(x,t)$ will always be written with the time variable $t \in [0,T]$ last.

$L^p_t L^q_{\mathrm{loc}}$ Functions such that for each compact subset of space, the cutoff lies in $L^p_t L^q_x$.

C^∞ Infinitely differentiable functions

C^∞_c Infinitely differentiable functions with compact support.

C^0 Continuous functions.

C^0_{weak} Weakly continuous functions $\Omega \to Y$, from a domain Ω into a Banach space Y endowed with the weak topology, e.g. $C^0_{\text{weak}}([0,T];L^1(\mathbb{R}^d;\mathbb{R}))$.

$C^0_{\text{weak}-*}$ Weak-$*$ continuous functions $\Omega \to Y'$, from a domain Ω into the dual of a Banach space Y' endowed with the weak-$*$ topology, e.g. $C^0_{\text{weak}-*}([0,T];L^\infty(\mathbb{R}^d;\mathbb{R}))$.

C^0_{b} Banach space of bounded, continuous functions with L^∞-norm.

$\mathscr{M}$ Banach space of signed Radon measures, the Banach space dual of C^0_b.

$W^{n,p}$ Sobolev space.

$W^{n,p}_{\text{loc}}$ Local Sobolev space.

H^n Hilbert Sobolev space, equal to $W^{n,2}$.

BV Banach space of bounded variation.

BV_{loc} Space of local bounded variation.

$W^{-n,p}$ Negative Sobolev space, the Banach space dual of $W^{n,p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$.

H^{-n} Negative Hilbert Sobolev space, the Hilbert space dual of H^n, equal to $W^{-n,2}$.

C^{-1} The Banach space dual of BV, containing the distributional derivatives of C^0_b.

$L^{p,\infty}$ Weak Lebesgue space [15, Section 1.3]

$L^{p,q}$ Lorentz space [15, Section 1.3].

$\mathscr{D}$ Distributional test functions $C^\infty_c(\mathbb{R}^m;\mathbb{R})$ with appropriate topology [58].

$\mathscr{D}'$ Space of distributions, the topological dual to $\mathscr{D}$ [58].

$\mathscr{S}$ Schwartz test functions on $\mathbb{R}^m$ with 'Schwartz' topology [58].

$\mathscr{S}'$ Space of Schwartz distributions, the topological dual of $\mathscr{S}$ [58].

$\dot{W}^{n,p}$ Homogeneous Sobolev quasi-norm $\nabla^n\phi \in L^p$ on distributions $\phi \in \mathscr{D}'$.

$\dot{H}^n$ Homogeneous Sobolev quasi-norm $\nabla^n\phi \in L^2$ on distributions $\phi \in \mathscr{D}'$, equal to $\dot{W}^{n,2}$.

$\dot{W}^{-n,p}$ Homogeneous negative Sobolev quasi-norm on $\mathscr{D}'$, the dual quasi-norm to $\dot{W}^{n,p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$.

$\dot{H}^{-n}$ Homogeneous negative Sobolev quasi-norm on $\mathscr{D}'$, the dual quasi-norm to $\dot{H}^n$, equal to $\dot{W}^{-n,2}$.

$W^{s,p}$ Fractional Sobolev-Slobodeckij space [40, Section 2].

H^s Fractional Hilbert Sobolev space [40, Section 3].

$H^{s,p}$ Fractional Bessel-potential Sobolev space [15, Section 6.2].

$\dot{H}^{s,p}$ Homogeneous Bessel-potential Sobolev space [15, Section 6.3].

$B^s_{p,q}$ Besov space [15, Section 6.2]

$\dot{B}^s_{p,q}$ Homogeneous Besov space [15, Section 6.3]

Algebraic conventions

d Spatial dimension.

x Spatial variables $(x_1,\dots,x_d) \in \mathbb{R}^d$.

t Time variable $t \in [0,T]$, with given positive time $T > 0$.

ρ Passive scalar field.

u d-dimensional divergence-free vector field.

X Flow map.

Other notation

$\mathbb{T}^d$ The d-dimensional torus, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

∇ Spatial gradient, $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d}\right)$.

Δ Spatial Laplacian, $\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$.

$\mathscr{F}$ The Fourier transform on $\mathbb{R}^d$.

$\nabla^\perp$ Two-dimensional perpendicular gradient $\left(-\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\right)$.

Chapter 1

Introduction

The mathematical analysis of turbulent fluid dynamics presents fundamental challenges that lie at the intersection of physics and mathematical theory. This thesis investigates these challenges through the passive scalar transport model in incompressible fluid mechanics. This partial differential equation (PDE) describes how a tracer—such as a dye—evolves as it follows the flow of a given vector field. Our investigation centers on the dynamics that emerge when the vector field exhibits lower regularity, as measured in Sobolev spaces and related function spaces. This focus on lower regularity directly addresses and is motivated by the mathematical structure of turbulent fluid flow.

1.1 Historical Context and Motivation

The Navier-Stokes Existence and Smoothness Problem

The study of such irregular dynamics has a rich history in fluid mechanics, historically necessitating increasingly general function spaces for their analysis. A cornerstone problem in this field remains the existence and smoothness of solutions to the Navier-Stokes equations [47]. These PDEs, which express momentum conservation in incompressible fluids, incorporate viscous strain through Newton's law of viscosity [14, 103].

The mathematical solvability of these equations, despite their fundamental role in scientific modeling, remains an open question. The core challenge lies in determining whether the velocity field representing the fluid flow maintains sufficient smoothness to meaningfully satisfy the governing equations. This difficulty led to Leray's groundbreaking introduction of "weak solutions", which satisfy the equations in an integral rather than pointwise sense [78].

This mathematical framework aligns with physical reality: the derivation of fluid models fundamentally relies on forces acting over integral volume elements rather than on individual particles. Thus, there exists no inherent physical requirement for an incompressible fluid to admit a classical pointwise solution. This mathematical challenge extends beyond mere technicality. The failure to obtain classical solutions correlates with physical singularities, including infinite velocities and pressures [99, 98]. Conversely, weak solutions in more general function spaces often lack sufficient mathematical structure to ensure deterministic behaviour [27, 31, 7]. Rather, the appropriate mathematical description depends on the regularity class of the fluid's turbulent dynamics.

Turbulence and Kolmogorov 1941 Theory

Therefore, understanding the regularity and structure of turbulent fluid flows remains central to both mathematical analysis and physics. In the case of incompressible Newtonian fluids governed by the Navier-Stokes equations, turbulent dynamics exhibit universal characteristics in the regime of "fully developed turbulence". When fluid energy overcomes viscous friction—quantified by the Reynolds number—a self-similar cascade of eddies emerges, transferring energy through progressively smaller length scales until the fluid's kinetic energy ultimately dissipates as heat [104, 50].

Kolmogorov's seminal prediction [65], later verified experimentally [93], revealed this phenomenon's remarkable independence from viscosity. This discovery carries profound implications, as the equations contain no inherent mechanism for kinetic energy dissipation without viscosity—a phenomenon termed anomalous dissipation of kinetic energy.

Mathematical analysis has proven that solutions to the inviscid Euler equations—the inviscid analogue of Navier-Stokes equations—should conserve kinetic energy when the vector field represents a classical solution. This led to Onsager's influential speculation about the minimum regularity required for kinetic energy conservation [90]. Later work made these predictions rigorous through Besov function spaces [33], aligning precisely with Kolmogorov's $\frac{4}{5}$ scaling law prediction for velocity regularity derived from self-similar analysis. These predictions and results fundamentally motivate the investigation of fluid flow with lower regularity.

Passive Scalar Transport as a Foundational Model

Fluid dynamics fundamentally concerns the flow, or transport, of quantities such as mass, energy, and momentum. The passive scalar transport model represents the simplest such system, describing a quantity that flows without interacting with the fluid through internal forces like pressure or material stresses. This model has proven invaluable in two distinct ways: illuminating various aspects of turbulence, including anomalous dissipation, through solvable statistical models [80, 66], and providing a mathematical foundation for understanding more complex fluid models through its deterministic dynamics.

For a passive scalar $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and a given vector field $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, the inviscid passive scalar transport equation takes the form:

$$\frac{\partial \rho}{\partial t}(x,t) + \sum_{i=1}^{d} u_i(x,t) \frac{\partial \rho}{\partial x_i}(x,t) = 0,$$

subject to the incompressibility condition:

$$\sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i}(x,t) = 0.$$

It is the study of these equations that forms the body of this thesis.

1.2 Thesis Structure and Main Results

Chapter 2: Quantitative Estimate For Transport along Sobolev Vector Fields

The solvability of the passive scalar transport equation forms the foundation of Chapter 2, beginning with a comprehensive literature review in Section 2.1. While vector fields of class C^1 permit classical solutions through the Cauchy approach of characteristics [9], vector fields with lower regularity necessitate weak (or integral) solutions, as classical solutions cease to exist.

The groundbreaking work of DiPerna and Lions [41] established well-posedness of such integral solutions for vector fields in the Sobolev class $W^{1,p}$. Their framework has sparked active research into stability and mixing bounds, with Bressan's mixing conjecture [22] remaining an important open problem.

Contributing to this, in Section 2.2 we present a novel weak compactness technique for improving such transport equation estimates. By establishing uniform decay of the DiPerna-Lions commutator in their well-posedness theory [41], we achieve several significant advances:

1. Improved classical mixing estimates, with exponential lower bounds on the mixing scale for all initial data (Theorems 2.12 and 2.13)

2. Propagation of mild 'logarithmic' fractional regularity of the passive scalar (Theorem 2.14)

3. State-of-the-art weak stability estimates for transport by Sobolev vector fields (Theorems 2.15 and 2.16)

Meanwhile, Section 2.3 extends our weak compactness technique to address the challenging $p = 1$ case of Bressan's conjecture, culminating in Theorem 2.22—the first quantitative stability estimate for transport along vector fields with bounded variation.

We conclude this chapter with Section 2.4, examining open problems regarding optimal constants and optimality of our results, particularly concerning logarithmic stability in the Sobolev setting and tetration stability in the bounded-variation setting, compared with linear stability in the Lipschitz case.

In summary, this chapter advances our understanding of mixing bounds and weak stability for solutions transported by Sobolev vector fields in the non-classical sub-Lipschitz regime.

Chapter 3: Improved Regularity and Well-posedness of the Transport-Diffusion Equation

Our investigation extends now to less regular divergence-free vector fields, specifically velocity fields $u(x,t)$ in the space $L^p_t L^q_x$. Under these conditions, solutions exhibit intriguing phenomena, including non-uniqueness and "perfect unmixing," where non-zero solutions emerge from zero initial data, [39].

In Chapter 3 we explore how isotropic diffusion restore uniqueness and regularity of weak solutions for these more general vector fields $u(x,t)$. The transport-diffusion equation, with diffusion parameter $\kappa > 0$, is given by:

$$\frac{\partial \rho}{\partial t}(x,t) + \sum_{i=1}^{d} u_i(x,t)\frac{\partial \rho}{\partial x_i}(x,t) - \kappa \sum_{i=1}^{d} \frac{\partial^2 \rho}{\partial x_i^2}(x,t) = 0. \tag{1.1}$$

Section 3.1 reviews standard energy estimates, which fail to capture regularity and uniqueness in broader function spaces. We address this gap in the literature by introducing:

- Mild solutions
- Weak convolution estimates
- Maximal regularity techniques for the heat equation

Applying these techniques in Section 3.2, we demonstrate how the Ladyzhenskaya-Prodi-Serrin condition captures the competing effects of transport and diffusion on regularisation. Our analysis yields three key results:

1. A condition for weak solutions to lie in the classical energy class (Theorem 3.10)

2. Improved uniqueness when $u(x,t)$ satisfies the Ladyzhenskaya-Prodi-Serrin criterion (Theorem 3.11)

3. Enhanced regularity of weak solutions when $u(x,t)$ satisfies the Ladyzhenskaya-Prodi-Serrin criterion (Theorem 3.13)

Finally, in Section 3.3 we discuss open questions on regularity and potential extensions to more general vector fields. In summary, this chapter extends standard energy estimates by introducing new techniques to analyse the regularity in the transport-diffusion equation.

Chapter 4: Vanishing Diffusion Limit and Solution Selection

The final chapter examines solution selection for the inviscid passive scalar transport equation when it admits non-unique weak solutions. We analyse whether the vanishing diffusion limit as $\kappa \to 0$ can restore uniqueness and eliminate non-physical solutions.

This approach succeeds in other fluid models (discussed in Section 4.1), and in contrast we demonstrate two striking results for the passive scalar transport model:

- Uniqueness fails in the vanishing diffusion limit

- The resulting solution can exhibit highly non-physical behaviour, including the reversal of the arrow of time introduced with diffusion

Section 4.2 presents these findings in two main theorems:

1. Theorem 4.10: Establishes the existence of vector fields for which the vanishing diffusion limit yields non-unique solutions, even though the resulting solutions satisfy the 'physical' renormalisation condition

2. Theorem 4.13: Demonstrates an example for which the vanishing diffusion limit selects a unique solution that perfectly mixes and subsequently unmixes back to its initial state, violating thermodynamically permissible behaviour

This raises several open questions regarding vanishing diffusion and viscosity in incompressible fluid models, which we discuss in Section 4.3. In summary, Chapter 4 demonstrates the failure of vanishing diffusion to addressing the physical solution selection problem in fluid dynamics, and highlights the need for novel approaches.

Chapter 2

Quantitative Estimates for Transport along Sobolev Vector Fields

2.1 Introduction

This chapter presents novel stability and mixing estimates for the transport of a passive scalar by a Sobolev vector field. In Section 2.1, we introduce the transport problem, providing a comprehensive overview of existing literature regarding the well-posedness and regularity of this passive scalar model. Our discussion emphasises the dependence of these properties on the regularity of the advecting incompressible vector field.

We explore both classical and modern approaches to the transport equation. The classical Cauchy-Lipschitz method, which utilises characteristics or trajectories, is contrasted with more contemporary weak solution theory. This comparison highlights the critical issues discussed in this thesis, namely non-uniqueness, propagation of regularity, and mixing.

Section 2.2 introduces our primary contribution: quantitative weak compactness arguments to improve non-quantitative convergence results. When applied to the transport equation, this approach provides new quantitative stability estimates, with a variety of applications, for the transport within the desired class of vector fields. We present several key theorems:

- Theorems 2.8 and 2.11: Quantitative decay of the DiPerna-Lions commutator, crucial for well-posedness of transport along Sobolev vector fields.
- Theorems 2.12 and 2.13: First bounds on possible mixing rates for all solutions in the DiPerna-Lions class, addressing both 'geometric' and 'functional' mixing scales.

- Theorem 2.14: Propagation of mild 'logarithmic' fractional regularity of the passive scalar.
- Theorems 2.15 and 2.16: State-of-the-art weak stability estimates for transport by Sobolev vector fields.

Finally, we extend our methods to address a significant gap in the literature: the absence of quantitative stability or mixing estimates for transport along vector fields in the Sobolev space $W^{1,1}$. The main contribution of section 2.3 is Theorem 2.22, the first quantitative estimate in this setting. This result yields an exponential tetration rate for stability and mixing, contrasting sharply with previous conjectures. The optimality of this rate remains an open question, which we discuss along with other unresolved issues in Section 2.4.

The results in this section have been developed in collaboration with Ayman Said and reported in [59].

2.1.1 The Transport Problem

We begin by introducing the transport problem. Working on the whole d-dimensional space $x \in \mathbb{R}^d$, consider a time-dependent vector field $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ on the time interval $t \in [0,T]$.

The trajectory along the vector field $u(x,t)$ starting at $x_0 \in \mathbb{R}^d$ is the solution $X(t) \in \mathbb{R}^d$ to the following ODE (ordinary differential equation),

$$\begin{aligned} \frac{d}{dt}X(t) &= u(X(t),t), \text{ for } t \in [0,T], \\ X(0) &= x_0. \end{aligned} \tag{2.1}$$

The flow map $X(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is the collection of all trajectories and so satisfies the 'ODE'

$$\begin{aligned} \frac{d}{dt}X(x,t) &= u(X(x,t),t), \text{ for } t \in [0,T], \\ X(x,0) &= x. \end{aligned} \tag{2.2}$$

The transport problem is then, for some initial scalar field $\rho_0(x) : \mathbb{R}^d \to \mathbb{R}$, to find the transported scalar field $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ which is initially equal to $\rho_0(x)$ and parallel transported by the trajectories of $u(x,t)$, that is to satisfy

$$\rho(X(x,t),t) = \rho_0(x). \tag{2.3}$$

One way to solve for $\rho(x,t)$ is to invert the flow map $X(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$. If the inverse $X^{-1}(x,t)$ is continuously differentiable, then for continuously differentiable $\rho_0(x)$ (2.3) is equivalent to solving the PDE (partial differential equation)

$$\begin{aligned} \frac{\partial \rho}{\partial t}(x,t) + \sum_{i=1}^{d} u_i(x,t) \frac{\partial \rho}{\partial x_i}(x,t) = 0, \\ \rho(x,0) = \rho_0(x), \end{aligned} \tag{2.4}$$

for a continuously differentiable $\rho(x,t)$. For brevity, it is standard to abbreviate derivatives in the spatial variables $x \in \mathbb{R}^d$ to the gradient operator $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$. Equation (2.4) is usually referred to as the transport equation.

2.1.2 Classical Solutions

The classical approach to the transport problem is to apply the Cauchy-Lipschitz theorem of ODEs to the flow map (2.2). If $u(x,t)$ is Lipschitz in the spatial variable with Lipschitz constant M, so that for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$

$$\left|u(x,t) - u(x',t)\right| \leq M|x - x'|,$$

and additionally $u(x,t)$ is continuous in the time variable $t \in [0,T]$, then by [56, Section 2 Theorem 1.1] the ODE (2.1) has a unique solution, and solving the ODE (2.1) backwards gives that the flow map $X(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$ is invertible. Additionally one shows the stability estimate for all $x, x' \in \mathbb{R}^d$, for all $t \in [0,T]$

$$\left|\frac{d}{dt}\left|X(x,t) - X(x',t)\right|\right| \leq M\left|X(x,t) - X(x',t)\right|,$$

see for instance [9, Section 2]. The Gronwall Lemma [56, Section 3 Theorem 1.1] then proves the map $X(\cdot,t)$ is Lipschitz with Lipschitz inverse

$$|x - x'|\exp(-Mt) \leq \left|X(x,t) - X(x',t)\right| \leq |x - x'|\exp(Mt). \tag{2.5}$$

Such quantitative estimates will play an essential role in the stability of the transport equation and mixing rates, forming this chapter's main body.

Turning now to the PDE (2.4), we must additionally assume that $u(x,t)$ is continuously differentiable in the spatial variables, $\frac{\partial u_i}{\partial x_j}(x,t)$, so that the flow map (2.2) is continuously differentiable with[1]

$$\frac{d}{dt}\frac{\partial X_i}{\partial x_j}(x,t) = \sum_{k=1}^{d} \frac{\partial u_i}{\partial x_k}(X(x,t),t)\frac{\partial X_k}{\partial x_j}(x,t). \tag{2.6}$$

By the inverse function theorem, $X^{-1}(x,t)$ is also continuously differentiable, and so the chain rule may be applied to (2.3). We conclude that the PDE (2.4) has a unique, continuously differentiable solution given by parallel transport (2.3).

Osgood Moduli of Continuity

If we are content to solve the parallel transport problem (2.3) instead of the PDE (2.4), then the Osgood condition [56, Section 2 Corollary 6.2] on $u(x,t)$ is sufficient for uniqueness and invertibility of the flow map (2.2).

2.1.3 Distributional Solutions

The Osgood condition lowers the regularity on $u(x,t)$ required to solve the transport problem by solving the parallel transport formulation (2.3) instead of the PDE (2.4). We next consider an alternative formulation of the PDE (2.4), as introduced in [41], which similarly lowers the regularity required on $u(x,t)$. The classical transport equation may be written as

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0. \tag{2.7}$$

A related PDE is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (u\rho) = 0, \tag{2.8}$$

which is the linear hyperbolic conservation law [35] when the flux of $\rho(x,t)$ is given by the vector field $u(x,t)$, and governs the evolution of conserved passive scalars. The continuity equation has the advantage that one may consider distributional solutions [58], as long as $u(x,t)\rho(x,t)$ is also a distribution. This vastly lowers the regularity on $u(x,t)$ needed but does not immediately allow us to do the same for the transport equation.

The approach of [41] is to treat the transport equation as a forced continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (u\rho) = (\nabla \cdot u)\rho, \tag{2.9}$$

[1] The convergence of the Picard iterates for (2.6) is a result of the convergence of the Picard iterates for (2.2).

and then, under regularity assumptions on the distribution $\nabla \cdot u$, we may solve a weak version of this forced equation. The assumption of [41] is that the distribution $\nabla \cdot u$ is uniformly bounded, but for this thesis, we will restrict to the simplest case $\nabla \cdot u = 0$ so that the transport equation (2.9) becomes the continuity equation (2.8). This is an entirely separate regime from the Osgood condition; zero or bounded divergence is not enforced by the Osgood condition or vice-versa.

Definition 2.1 (Transport equation - weak solutions [41])**.** Consider a vector field $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ with $\nabla \cdot u = 0$ in the distributional sense. We say $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$ with $u(x,t)\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ is a weak solution to

$$\begin{aligned} \frac{\partial \rho}{\partial t}(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \\ \rho(x,0) = \rho_0(x), \end{aligned} \tag{TE}$$

for initial datum $\rho_0(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$ if, for any $\phi(x,t) \in C^\infty_{\text{c}}(\mathbb{R}^d \times [0,T);\mathbb{R})$,

$$\int_{\mathbb{R}^d \times [0,T)} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right) \, dxdt = -\int_{\mathbb{R}^d} \rho_0(x)\phi_0(x) \, dx,$$

where $\phi_0(x) = \phi(x,0)$. Meanwhile, we say the transport equation is satisfied on an *open* interval $I \subset (0,T)$ if, for any $\phi(x,t) \in C^\infty_{\text{c}}(\mathbb{R}^d \times I;\mathbb{R})$,

$$\int_{\mathbb{R}^d \times I} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right) \, dxdt = 0.$$

Note that in both definitions, we may equally take the test function $\phi(x,t)$ to be Lipschitz in time and space with compact support.

Theorem 2.1 (Existence of weak solutions to (TE))**.** *Let $1 \leq p,q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\rho_0(x) \in L^p_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$, and $u(x,t) \in L^1([0,T];L^q_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ is divergence-free in the distributional sense. If $u(x,t)$ additionally satisfies the growth condition*

$$\frac{u(x,t)}{1+|x|} \in L^1(\mathbb{R}^d \times [0,T];\mathbb{R}^d),$$

then there exists a weak solution $\rho(x,t) \in L^\infty([0,T];L^p(\mathbb{R}^d;\mathbb{R})) = L^\infty_t L^p_x$ to (TE)*, with the bound*

$$\|\rho\|_{L^\infty_t L^p_x} \leq \|\rho_0\|_{L^p_x}.$$

Theorem 2.2 (Weak continuity of transport)**.** *Suppose $\rho(x,t)$ is a weak solution to* (TE)*. Then for any $\phi(x,t) \in C_c^\infty(\mathbb{R}^d \times [0,T];\mathbb{R})$, for a.e. $t \in [0,T]$,*

$$\text{(Trace Formula)} \quad \int_{\mathbb{R}^d} \rho(x,t)\phi(x,t)\, dx$$
$$= \int_{\mathbb{R}^d} \rho_0(x)\phi_0(x)\, dx + \int_{\mathbb{R}^d \times [0,t]} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right) dxdt, \quad (2.10)$$

Suppose further that $\rho(x,t) \in L_t^\infty L_x^p$ for $1 < p \leq \infty$, then (there is a representation of $\rho(x,t)$ with) $\rho(x,t) \in C^0_{\text{weak}-*}([0,T]; L^p(\mathbb{R}^d;\mathbb{R}))$, such that (2.10) *holds for all $t \in [0,T]$. In particular, $\rho(x,0) = \rho_0(x) \in L^p(\mathbb{R}^d;\mathbb{R})$.*

If $\rho(x,t) \in L_t^\infty L_x^1$ with $\{\rho(\cdot,t)\}_{t\in[0,T]} \subset L^1(\mathbb{R}^d;\mathbb{R})$ uniformly integrable, then similarly $\rho(x,t) \in C^0_{\text{weak}}([0,T]; L^1(\mathbb{R}^d;\mathbb{R}))$.

If $\rho_0(x) \in L^p(\mathbb{R}^d;\mathbb{R}) = L_x^p$ for any $1 \leq p \leq \infty$, then the solution constructed in Theorem 2.1 (with p not necessarily the same) satisfies

$$\rho(x,t) \in L_t^\infty L_x^p,$$

with $\{\rho(\cdot,t)\}_{t\in[0,T]} \subset L^1(\mathbb{R}^d;\mathbb{R})$ uniformly integrable if $p = 1$. Moreover, the weakly continuous representative satisfies for all $t \in [0,T]$,

$$\text{(Initial } L^p\text{-Inequality)} \quad \|\rho(\cdot,t)\|_{L^p(\mathbb{R}^d;\mathbb{R})} \leq \|\rho_0\|_{L^p(\mathbb{R}^d;\mathbb{R})}. \quad (2.11)$$

Another standard corollary of Theorem 2.2 is, if any solution of (TE) $\rho(x,t) \in L_t^\infty L_x^1$ and is uniformly integrable, then the weakly continuous representative satisfies

$$\text{(Conservation of Mass)} \quad \int_{\mathbb{R}^d} \rho(x,t)\, dx = \int_{\mathbb{R}^d} \rho_0(x)\, dx. \quad (2.12)$$

The above results are standard and may be found in some form in [41], [9]. See also my work [60, Section 3] for a proof of the precise statement of Theorem 2.2.

While Definition 2.1 successfully generalises (2.9) to much lower regularity vector fields $u(x,t)$, it is highly general and not related to the parallel transport formulation (2.3). To this end, the superposition principle [9, Theorem 12] gives a weak sense in which any positive weak solution $\rho(x,t)$ of (TE) also satisfies parallel transport (2.3) as the pushforward of $\rho_0(x)$ by a measure on the set of trajectories (2.1). The issue, however, is the lack of uniqueness of the flow map. This characteristic of low regularity vector fields was known much earlier. In an attempt to restore uniqueness, the authors of [41] were led to consider weak solutions of (TE), which additionally satisfy the renormalisation property.

Definition 2.2 (Transport equation - renormalised solutions [41])**.** We say $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ is a renormalised solution to (TE) if, for any $\beta(s) \in C^0_{\mathrm{b}}(\mathbb{R};\mathbb{R})$, then $\beta(\rho(x,t)) \in L^\infty(\mathbb{R}^d \times [0,T];\mathbb{R})$ is a weak solution of (TE) for initial datum $\beta(\rho_0(x)) \in L^\infty(\mathbb{R}^d;\mathbb{R})$. It also sufficient to take test functions $\beta(s) \in C^\infty_{\mathrm{c}}(\mathbb{R};\mathbb{R})$.

This is motivated by the parallel transport formulation (2.3), which implies also

$$\beta(\rho(X(x,t),t)) = \beta(\rho_0(x)), \tag{2.13}$$

is parallel transported. The caveat is that if the flow map is not injective, then parallel transport as understood in the superposition principle [9, Theorem 12] fails to satisfy (2.13) as a pushforward of $\beta(\rho_0(x))$, for non-linear $\beta(s)$. Therefore, a renormalised solution to (TE) should correspond to an injective flow map in the superposition principle. Renormalised solutions were used in the seminal paper of DiPerna and Lions [41] to give the first well-posedness class below the Osgood criterion for the transport equation. We state the theorem in the case $\nabla \cdot u = 0$.

Theorem 2.3 (DiPerna-Lions well-posedness [41])**.** *Let* $1 \le p, p', q, q' \le \infty$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *Let*

$$\rho(x,t) \in L^p([0,T];L^q_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R})),$$

be a weak solution to (TE) *with*

$$\nabla u(x,t) \in L^{p'}([0,T];L^{q'}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d \times \mathbb{R}^d)),$$

then $\rho(x,t)$ *is a renormalised weak solution to* (TE)*, and unique if*

$$\frac{u(x,t)}{1+|x|} \in L^1(\mathbb{R}^d \times [0,T];\mathbb{R}^d).$$

The condition on $\frac{u(x,t)}{1+|x|}$ for uniqueness is necessary to prevent trajectories coming from infinity.

There have been many attempts to improve the well-posedness class beyond that of the DiPerna-Lions theory. The most notable improvement is that of Ambrosio [8] to bounded variation vector fields. Again, for simplicity, we present the result for vector fields with zero divergence.

Theorem 2.4 (Ambrosio well-posedness [8])**.** *Let*

$$\rho(x,t) \in L^\infty(\mathbb{R}^d \times [0,T];\mathbb{R}),$$

be a weak solution to (TE) *with*

$$u(x,t) \in L^1([0,T];BV_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)),$$

then $\rho(x,t)$ *is a renormalised solution to* (TE)*, and unique if*

$$\frac{u(x,t)}{1+|x|} \in L^1(\mathbb{R}^d \times [0,T];\mathbb{R}^d).$$

Theorems 2.3 and 2.4 correspond to the selection of an a.e. injective flow map $X(x,t)$ [9, Theorem 16] which preserves the Lebesgue measure, for which we may write the unique solution $\rho(x,t)$ as the parallel transport (2.3) of $\rho_0(x)$.

Definition 2.3 (Regular Lagrangian flow)**.** $X(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is a regular Lagrangian flow along a divergence-free vector field $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ if

1. $X(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$ preserves the Lebesgue measure. Then $u(X(x,t),t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$.

2. For a.e. $x \in \mathbb{R}^d$, that $X(x,\cdot) : [0,T] \to \mathbb{R}^d$ is absolutely continuous, with

$$X(x,t) = x + \int_0^t u(X(x,t),t)\, dt. \tag{2.14}$$

   Alternatively, the distribution $\frac{\partial X}{\partial t}(x,t)$ is equal to $u(X(x,t),t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$.

We say that $X(x,t)$ is an *injective* Lagrangian flow if in addition,

3. For every $t \in [0,T]$, that $X(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$ is injective for a.e. $x \in \mathbb{R}^d$.

Definition 2.4 (Transport equation - Lagrangian solutions)**.** Suppose $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T])$ is a weak solution to (TE). Suppose $X(x,t)$ is an *injective* Lagrangian flow along $u(x,t)$. We say $\rho(x,t)$ is a Lagrangian solution if for every $t \in [0,T]$ we may write

$$\rho(X(x,t),t) = \rho_0(x).$$

Theorem 2.5 (Uniqueness of Lagrangian flows [9])**.** *Let* $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ *be divergence-free in the distributional sense. Under the assumptions of Theorem 2.4, that is*

$$u(x,t) \in L^1([0,T];BV_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)),$$
$$\frac{u(x,t)}{1+|x|} \in L^1(\mathbb{R}^d \times [0,T];\mathbb{R}^d),$$

$u(x,t)$ admits a unique regular Lagrangian flow $X(x,t)$ as in Definition 2.3. One may additionally show that $X(x,t)$ is an injective Lagrangian flow, and the unique renormalised solution of Theorems 2.3 and 2.4 is a Lagrangian solution,

$$\rho(X(x,t),t) = \rho_0(x).$$

This theorem may be proved using standard techniques, such as the lecture series [9].

Moreover, the uniqueness of trajectories (2.1) as integral curves (2.14) holds for a.e. $x \in \mathbb{R}^d$, without the requirement that the flow map preserves the Lebesgue measure, if additionally $u(x,t) \in L^1_t W^{1,p}_x$ with

$$p > d, \tag{2.15}$$

see [29]. In [23] this was extended to give uniqueness of *positive* solutions $\rho(x,t) \in L^\infty_t L^q_x$ along $u(x,t) \in L^1_t W^{1,p}_x$, with

$$\frac{1}{p} + \left(1 - \frac{1}{d}\right)\frac{1}{q} < 1. \tag{2.16}$$

As discussed in the next section, both results are sharp, with only the endpoint case open.

Hamiltonian Flows

Autonomous (time-independent) divergence-free transport may be written as a Hamiltonian system in two space dimensions. For $u(x) : \mathbb{R}^2 \to \mathbb{R}^2$ time-independent, the stream-function, or Hamiltonian, is a scalar function $\psi(x) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$u(x) = \left(-\frac{\partial \psi}{\partial x_2}(x), \frac{\partial \psi}{\partial x_1}(x)\right), \tag{2.17}$$

which we often shorthand to $u(x) = \nabla^\perp \psi(x)$ where $\nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right)$ is the perpendicular-gradient. For $u(x) \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $u(x)$ is divergence-free in the distributional sense, if and only if there exists a Lipschitz stream-function $\psi(x) \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R})$ such that (2.17) holds. Since by design $u(x)$ is perpendicular to the gradient $\nabla\psi(x)$, the trajectories $X(t)$ (2.1) of $u(x)$ should preserve $\psi(x)$,

$$\begin{aligned}\frac{d}{dt}\psi(X(t)) &= \nabla\psi(X(t)) \cdot u(X(t)) \\ &= 0.\end{aligned}$$

Therefore, the trajectories of $u(x)$ lie on the level sets of the stream function $\psi(x)$. For a weak solution $\rho(x,t)$ to (TE), one may take a Lipschitz test function of the form $\beta(\psi(x))\phi(x,t)$

for any $\beta \in C_c^\infty(\mathbb{R};\mathbb{R})$, $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T];\mathbb{R})$, and disintegrate (TE) along the level sets of $\psi(x)$ [4, Lemma 3.7]. This allows one to reduce (TE) to a system of one-dimensional (weak) ODEs, for which one may characterise all solutions [4, Lemma 4.4]. In particular, one obtains [4, Theorem 3.10 and Theorem 4.7] uniqueness of weak solutions to (TE) on $\mathbb{R}^2$ if $u(x)$ is bounded, autonomous, and $u(x) \neq 0$ for a.e. $x \in \mathbb{R}^2$.

There is a little-known generalisation of (2.17) to autonomous vector fields $u(x) \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ in higher dimensions. One requires a multi-dimensional stream-function $\psi(x) \in W^{1,\infty}(\mathbb{R};\mathbb{R}^{d-1})$ such that $u(x)\cdot\nabla\psi(x) = 0$ [4, Section 3]. Given $u(x)$, such a $\psi(x)$ does not, in general, exist in dimensions $d \geq 3$. However, there is a generalisation of (2.17) to produce a divergence-free vector field $u(x)$ from a multi-dimensional stream-function $\psi(x) : \mathbb{R}^d \to \mathbb{R}^{d-1}$. In terms of the wedge product $\wedge$, and Hodge-star operator $*$ on differential forms df, one writes

$$u(x) = *(d\psi_1 \wedge \cdots \wedge d\psi_{d-1}).$$

For $d = 3$ this reads

$$u(x) = \nabla\psi_1(x) \times \nabla\psi_2(x),$$

in terms of the vector cross-product.

Non-zero Divergence

All the results of this chapter have analogues when we instead assume that $\nabla\cdot u$ is uniformly bounded, see for instance [9], and [4, Section 5] for Hamiltonian flows. Often one can relax the condition on divergence to the negative part $[\nabla\cdot u]_- \in L^\infty(\mathbb{R}^d \times [0,T];\mathbb{R})$, where $[\nabla\cdot u]_- = \min\{\nabla\cdot u, 0\}$, see [9].

Lastly, there is an alternative to the weak formulation (2.9) for the transport equation when $u(x,t)$ is not divergence-free. Consider a (weak) solution $\bar{\rho}(x,t)$ to the continuity equation (2.8). Formally, if $\rho(x,t)$ solves the transport equation (2.7) then $\bar{\rho}(x,t)\rho(x,t)$ is another (weak) solution to the continuity equation (2.8). By fixing $\bar{\rho}(x,t)$, one may take this as the definition of a $\bar{\rho}$-weak solution to the transport equation. Essentially, one is using the measure $\bar{\rho}(x,t)dxdt$ so that the transport equation (2.7) is already in divergence form. This has the advantage of not requiring any assumption on $\nabla\cdot u$; however, it may depend on the choice of $\bar{\rho}(x,t)$.

This idea has been studied, and notably, an analogue of the well-posedness Theorem 2.4 is true in this setting for so-called ‘nearly incompressible vector fields’ [16].

2.1.4 Non-uniqueness and Perfect Mixing

We now discuss the non-uniqueness of weak solutions to (TE) that may occur outside the well-posedness classes of the previous section. The most basic example of the non-uniqueness of weak solutions is the checkerboard pattern in [39]. The unique Lagrangian flow along the self-similar vector field $u(x,t) \in L^\infty_t([0,1-2^{-n}];BV_{\mathrm{loc}})$ [39], transforms a periodic checkerboard on $\mathbb{R}^2$ of sidelength 1 to a checkerboard of sidelength 2^{-n}.

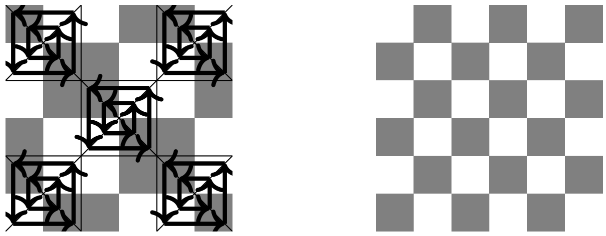


Fig. 2.1 Action of the flow from $t=0$ to $t=1/2$. The shaded region denotes the set $\{\rho = 1\}$. Credit to [36] for this figure.

If the passive scalar $\rho_0(x) \in L^\infty(\mathbb{R}^2;\mathbb{R})$ is $\rho_0(x) = 1$ on the black squares, and $\rho_0(x) = -1$ on the white squares, then $\rho(x,t)$ converges weakly to $\rho(x,1) = 0$, and indeed if the vector field is continued as $u(x,1+t) = 0$ for $t \geq 0$, then the continuation $\rho(x,1+t) = 0$ is the unique weak solution to (TE), by Theorem 2.2 and 2.4.

This illustrates the 'perfect mixing' phenomenon when the passive scalar is mixed from an inhomogeneous initial state to its spatial average. It is intimately related to the non-uniqueness of weak solutions that emerge in low-regularity vector fields. Observing that (TE) is time-reversible, the time-reverse of a perfectly mixed solution begins at zero initial datum $\rho_0(x) = 0$, and 'separates' (or perfectly unmixes) to the non-zero final state. Of course, this is distinct from the expected zero weak solution $\rho(x,t) = 0$. It is natural to enforce physical selection principles to remove these extra (unmixing) solutions and perhaps restore uniqueness. This is an entirely separate topic and will be discussed more deeply in Chapter 4.

The regularity of the vector field considered here is $(1-t)u(x) \in L^\infty([0,1];BV_{\mathrm{loc}})$, which falls just below the well-posedness threshold $u(x) \in L^1_t([0,1];BV_{\mathrm{loc}})$ of Theorem 2.4. Non-uniqueness and perfect mixing can occur even for vector fields with $u(x,t) \in L^\infty([0,1];C^\alpha(\mathbb{R}^2;\mathbb{R}^2))$ [5, Theorem 3.4] for any $\alpha < 1$. We note that it is an open problem to construct such vector fields $u(x,t) \in L^\infty([0,1];C^\alpha(\mathbb{R}^2;\mathbb{R}^2))$ which can perfectly mix all initial data simultaneously.

Surprisingly, perfect mixing is not limited to less than Sobolev regularity. An interesting feature of both the well-posedness Theorems 2.3 and 2.4 is the integrability requirement on $\rho(x,t)$. In [31] (and the pioneering work [86]) it was then shown this is sharp; there exists

$u(x,t) \in L^1_t W^{1,p}_x$ admitting non-unique solutions to (TE) with $\rho(x,t) \in L^\infty_t L^q_x$ for any

$$\frac{1}{p} + \frac{1}{q} > 1.$$

Unlike the example of perfect unmixing in Figure 2.1, these solutions are not related to the non-uniqueness of trajectories (2.1) [2]. This is because the superposition principle [9, Theorem 12] concerns only *positive* weak solutions to (TE). Non-uniqueness of integral curves (2.14) [23], [69], and of positive weak solutions to (TE) [24], then occurs under the slightly weaker integrability condition

$$\frac{1}{p} + \left(1 - \frac{1}{d}\right)\frac{1}{q} > 1,$$

proving that (2.15) and (2.16) are essentially sharp, with only the endpoint cases open.

2.1.5 Mixing and Quantitative Estimates for Sobolev Vector Fields

Mixing of a passive scalar refers to the transfer of mass from large spatial scales/small frequencies to small spatial scales/large frequencies and thus is intimately related to quantitative regularity for either the flow map (2.2) or the passive scalar (2.4). The most basic of these is the exponential Lipschitz estimate (2.5) on the flow map (2.2) along a Lipschitz vector field $u(x,t)$

$$|x - x'| \exp(-Mt) \leq \left|X(x,t) - X(x',t)\right| \leq |x - x'| \exp(Mt). \tag{2.5}$$

This then gives an exponential lower bound on the decay of the length scale or 'mixing scale' of passive scalars advected by Lipschitz vector fields. There are various attempts to define this mixing scale rigorously [5], though they all correspond to some quantitative regularity or stability of the transport equation. To complement the example of perfect mixing in Figure 2.1, there exists Lipschitz vector fields $u(x,t) \in L^\infty_t W^{1,\infty}_x$ which reproduce self-similar copies of the initial datum $\rho_0(x)$ [5], and so saturate (with different constants) any exponential quantitative estimates or mixing bounds on spatial scale such as (2.5).

Below Lipschitz regularity, since the well-posedness classes of Theorems 2.3 and 2.4 (or the Hamiltonian vector fields (2.17)) exclude perfect mixing, they should admit some quantitative estimates on the stability, or spatial scale, of transport. Naively, one obtains from (2.6) a bound on the growth of the Jacobian of the flow map (2.2)

$$\frac{d}{dt} \log |\nabla X(x,t)| \leq |\nabla u(x,t)|, \tag{2.18}$$

as for the Lipschitz estimate (2.5). However, when $u(x,t) \in L^1_t W^{1,p}_x$ for $p < \infty$, then (2.18) fails to generalise to non-smooth vector fields, or even to ensure $X(x,t)$ is differentiable in a suitable sense. On the other hand, when $u(x,t) \in L^1_t W^{1,p}_x$ for $p > 1$, harmonic analysis gives an integrable version of (2.18), see [34] and also [9, Section 6], and in particular we have the bound for all $x, x' \in \mathbb{R}^d$

$$\frac{d}{dt} \log \left| X(x,t) - X(x',t) \right| \leq C \left(M\nabla u(x,t) + M\nabla u(x',t) \right), \tag{2.19}$$

where $M\nabla u(x,t)$ is the maximal function (in $x \in \mathbb{R}^d$) of $\nabla u(x,t) \in L^1_t L^p_x$, with the bound

$$\|M\nabla u\|_{L^1_t L^p_x} \leq C_p \|u\|_{L^1_t W^{1,p}_x},$$

for all $1 < p \leq \infty$, so that (2.19) may be integrated. In particular, one obtains a partial counterpart to (2.5)

$$|x - x'| \left(A(x) + A(x') \right)^{-1} \leq \left| X(x,t) - X(x',t) \right| \leq |x - x'| \left(A(x) + A(x') \right), \tag{2.20}$$

for some function $A(x) : \mathbb{R}^d \to (0, \infty)$ with the bound

$$\|\log A\|_{L^p_x} \leq C_p \|u\|_{L^1_t W^{1,p}_x}. \tag{2.21}$$

As a consequence of these quantitative stability estimates for the flow map, one may also show stability estimates for the passive scalar [97] and growth/decay bounds on specific 'functional mixing scales' [96].

The gap to $p = 1$ remains a significant open problem and the validity of any similar bound to (2.20) was famously conjectured in [22]. For $1 < p < \infty$, there exists a $u(x,t) \in L^\infty_t W^{1,p}_x$ which essentially saturates the bound (2.20) [62, Theorem 1.1, Remark 1.2], and in particular better regularity of the flow map than (2.21) cannot be expected.

There is similarly a vector field $u(x,t) \in L^\infty_t W^{1,p}_x$ for all $p < \infty$, and smooth initial datum $\rho_0(x) \in C^\infty_c(\mathbb{R}^d)$, for which the passive scalar $\rho(x,t)$ immediately leaves every (positive order) Sobolev space [6]. However, using very recent harmonic analysis tools, [95] shows that the passive scalar $\rho(x,t)$ instead propagates a 'logarithm of a derivative' [25]. A drawback of this approach is that such regularity is not necessarily present at the initial time $\rho_0(x)$. The main analysis of this chapter is to give a new approach to quantitative regularity and stability estimates for the passive scalar.

2.2 Quantitative Estimates for Transport along Sobolev Vector Fields

In this chapter, we shall explore how one may use compactness arguments, quantified by particular harmonic techniques, to improve particular decay estimates to be uniform in the required regularity of $\rho(x,t)$ and $u(x,t)$. In Section 2.2.1, we will investigate how this applies to the DiPerna-Lions well-posedness result (Theorem 2.3). Correspondingly, we deduce state-of-the-art mixing and stability estimates in Section 2.2.2 and Section 2.2.3, respectively. Finally, we will approach the infamous Ambrosio well-posedness result (Theorem 2.4) in Section 2.3 to deduce the very first weak stability estimate (and corresponding mixing bound) for transport along a divergence-free vector field $u(x,t) \in L^1_t W^{-1,1}_x$, a notoriously challenging problem.

2.2.1 Quantifying the DiPerna-Lions Theory

To develop the DiPerna-Lions theory, we must first introduce a standard mollifier $\varphi(x) \in L^1(\mathbb{R}^d)$.

Definition 2.5 (Mollifier)**.** We say $\varphi(x) \in L^1(\mathbb{R}^d;\mathbb{R})$ is a standard mollifier if

$$\int_{\mathbb{R}^d} \varphi(x)\, dx = 1 \text{ and define } \varphi_\delta(x) \in L^1(\mathbb{R}^d),\ \varphi_\delta(x) = \delta^{-d}\varphi\left(\frac{x}{\delta}\right),$$

for all $\delta > 0$. Note in the literature that one often asks for additional smoothness such as $\varphi(x) \in C_c^\infty(\mathbb{R}^d)$. We do not restrict ourselves like this unless explicitly stated.

One then defines the DiPerna-Lions commutator as follows.

Definition 2.6 (DiPerna-Lions commutator)**.** Let $u(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, $\rho(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$, with $u(x)\rho(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Let $\varphi(x) \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R})$ with compact support be a standard mollifier. Then, we define the DiPerna-Lions commutator

$$r_\delta(u,\rho)(x) = \int_{\mathbb{R}^d} \rho(y)(u(x)-u(y)) \cdot \nabla\varphi_\delta(x-y)\, dy,$$

where we suppress the dependence on $\varphi(x) \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R})$.

Including time-dependence, $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$, and $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ with $u(x,t)\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$, then we shorthand $r_\delta(u(\cdot,t),\rho(\cdot,t))(x)$ as $r_\delta(u,\rho)(x,t)$.

Note that we may remove the requirement that $\varphi(x)$ is compactly supported if we have L^1_x integrability instead of L^1_{loc} integrability on $u(x,t)$, $\rho(x,t)$, and $u(x,t)\rho(x,t)$.

The motivation behind the DiPerna-Lions commutator is the following. Let $\rho(x,t)$ be a weak solution to (TE), then convolution in the space variable

$$(\rho * \varphi_\delta)(x,t) = \int_{\mathbb{R}^d} \rho(y,t)\varphi_\delta(x-y)\,dy,$$

solves the classical PDE

$$\frac{\partial}{\partial t}(\rho * \varphi_\delta)(x,t) + u(x,t)\cdot\nabla(\rho * \varphi_\delta)(x,t) = r_\delta(u,\rho)(x,t),$$

where one notes that $(\rho * \varphi_\delta)(x,t) \in L^1([0,T];C^n_x)$, and $\frac{\partial}{\partial t}(\rho * \varphi_\delta)(x,t) \in L^1([0,T];C^{n-1}_x)$ when $\varphi(x) \in C^n_x$.

The key theorem of DiPerna-Lions is then that

Theorem 2.6 (DiPerna-Lions commutator decay [41])**.** *Let* $1 \le p,q \le \infty$ *with* $\frac{1}{p}+\frac{1}{q}=1$*. Let* $\nabla u(x) \in L^p(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$ *be divergence-free, and* $\rho(x) \in L^q(\mathbb{R}^d;\mathbb{R})$ *then*

$$r_\delta(u,\rho)(x) \xrightarrow{\delta\to 0} 0 \text{ in } L^1_x.$$

One may equally replace the integrability L_x with local integrability L_{loc} throughout [41]. Theorem 2.3 on well-posedness is a corollary of the above decay. This decay, as shown in [41] is not uniform in $\nabla u(x) \in L^p_x$ or $\rho(x) \in L^q_x$. However, a simple weak compactness argument shows this must be true.

Proposition 2.7. *Let* $1 \le p,q \le \infty$ *with* $\frac{1}{p}+\frac{1}{q}=1$*. Let* $\nabla u(x) \in L^p(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$ *be divergence-free, and* $\rho(x) \in L^q(\mathbb{R}^d;\mathbb{R})$*.*

If $p,q>1$*, then for any* $\varepsilon>0$*, for any* $\bar{\delta}>0$*, there exists* $\delta_\varepsilon>0$ *independent of* $u(x)$ *and* $\rho(x)$ *such that*

$$\inf_{\delta\in[\delta_\varepsilon,\bar{\delta}]} \|r_\delta(u,\rho)\|_{L^1_x} \le \varepsilon \|\nabla u\|_{L^p_x} \|\rho\|_{L^q_x}.$$

Proof. Assume for the sake of contradiction that this is not the case. Then there exists a sequence $\nabla u_n(x) \in L^p_x$, $\rho_n(x) \in L^q_x$ for $n \in \mathbb{N}$ with $n > \bar{\delta}^{-1}$, so that

$$\inf_{\delta\in[\frac{1}{n},\bar{\delta}]} \|r_\delta(u_n,\rho_n)\|_{L^1_x} > \varepsilon \|\nabla u\|_{L^p_x} \|\rho\|_{L^q_x}.$$

By scaling we may take $\|\nabla u_n\|_{L^p_x}, \|\rho_n\|_{L^q_x} = 1$ in the above. Since $p,q>1$, we may take a weakly-* converging subsequence. Without loss of generality, this may be taken to be the original sequence,

$$\nabla u_n(x) \xrightarrow{n\to\infty} \nabla u(x) \in L^p_x,$$
$$\rho_n(x) \xrightarrow{n\to\infty} \rho(x) \in L^q_x,$$

and by the Rellich-Kondrachov theorem [45, Section 5.7], we have the strong convergence

$$u_n(x) \xrightarrow{n\to\infty} u(x) \in L^p_{\mathrm{loc}},$$

and so strong convergence of

$$r_\delta(u_n,\rho_n)(x) \xrightarrow{n\to\infty} r_\delta(u,\rho)(x) \in L^1_{\mathrm{loc}}. \tag{2.22}$$

Since we assumed that

$$\inf_{\delta\in[\frac{1}{n},\bar\delta]} \|r_\delta(u_n,\rho_n)\|_{L^1_x} > \varepsilon,$$

then by (2.22) also

$$\inf_{\delta\in[\frac{1}{n},\bar\delta]} \|r_\delta(u,\rho)\|_{L^1_x} > \varepsilon,$$

but this contradicts the decay of Theorem 2.6. □

As we will see in Section 2.2.3 and 2.2.2, such uniform decay estimates give mixing bounds and weak stability estimates for transport within the DiPerna-Lions setting, i.e. along Sobolev vector fields. Our main goal is to obtain such quantitative uniform decay rates.

We begin by making the argument of Proposition 2.7 quantitative. To this end we introduce the homogeneous Besov space $\dot B^s_{p,q}(\mathbb{R}^d;\mathbb{R}^{d'})$ as the Schwartz distributions $\mathscr{S}'(\mathbb{R}^d;\mathbb{R}^{d'})$ for which the following homogeneous Besov norm is finite,

$$\|f\|_{\dot B^s_{p,q}(\mathbb{R}^d;\mathbb{R}^{d'})} = \left(\sum_{n=-\infty}^{\infty} 2^{ns}\,\|\psi_n * f\|^q_{L^p(\mathbb{R}^d;\mathbb{R}^{d'})}\right)^{\frac{1}{q}},$$

where $\psi_n \in \mathscr{S}(\mathbb{R}^d;\mathbb{R})$ is a Littlewood-Paley decomposition, given by the Fourier transform

$$\hat\psi_n(\xi) = \chi(2^{-n}\xi),$$

for a fixed choice of $\chi \in C^\infty_c(\mathbb{R}^d;\mathbb{R})$ satisfying $\operatorname{supp}\chi = \{\xi : \frac{1}{2} \le |\xi| \le 2\}$, $\chi(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$, and $\sum_{n=-\infty}^{\infty} \chi(2^{-n}\xi) = 1$ for $\xi \neq 0$, as in [15, Chapter 6].

Recall the continuous embedding $L^p_x \hookrightarrow \dot B^0_{p,\max(p,2)}$ for all $1 < p \le \infty$, see [11, Theorem 2.40, 2.41].

We prove the following uniform decay rate of the DiPerna-Lions commutator, where we have removed the time dependence.

Theorem 2.8 (Uniform decay of the DiPerna-Lions commutator)**.** *Let $p, p', q, r \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{r}$.*

Let $\nabla u(x) \in L^p(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)\cap\dot{B}^0_{p,q}(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$ be divergence-free, and $\rho(x)\in L^{p'}(\mathbb{R}^d;\mathbb{R})\cap \dot{B}^0_{p',q}(\mathbb{R}^d;\mathbb{R})$, then

$$\left(\int_0^\infty \|r_\delta(u,\rho)\|^q_{L^r_x}\,\frac{d\delta}{\delta}\right)^{\frac{1}{q}} \leq C\left(\|\nabla u\|_{\dot{B}^0_{p,q}}\|\rho\|_{L^{p'}_x}+\|\nabla u\|_{L^p_x}\|\rho\|_{\dot{B}^0_{p',q}}\right),$$

for some constant $C>0$ depending only on the dimension d, and the following norm of the mollifier $\left\|(1+|x|^2)\nabla\varphi\right\|_{L^1_x}+\left\||x|\nabla^2\varphi\right\|_{L^1_x}$.

Proof. Throughout $\lesssim$ will denote less than or equal to up to a constant depending only on the dimension d, and the norm of the mollifier $\left\|(1+|x|^2)\nabla\varphi\right\|_{L^1_x}+\left\||x|\nabla^2\varphi\right\|_{L^1_x}$, and in particular not on $\delta>0$.

We first rewrite

$$\begin{aligned} r_\delta(u,\rho)(x) &= \int_{\mathbb{R}^d}\rho(y)(u(x)-u(y))\cdot\nabla\varphi_\delta(x-y)\,dy,\\ &= \int_{\mathbb{R}^d}\rho(x-\delta h)\left(\frac{u(x)-u(x-\delta h)}{\delta}\right)\cdot\nabla\varphi(h)\,dh, \end{aligned}$$

and so by testing against $\phi(x)\in L^{r'}_x(\mathbb{R}^d;\mathbb{R})$ for $\frac{1}{r}+\frac{1}{r'}=1$ we see immediately that

$$\|r_\delta(u,\rho)\|_{L^r_x}\lesssim\frac{1}{\delta}\|\rho\|_{L^{p'}_x}\|u\|_{L^p_x}. \tag{2.23}$$

Next, we consider the following approximation

$$\begin{aligned} & r_\delta(u,\rho)(x)-\sum_{i,j=1}^d\frac{\partial u_i}{\partial x_j}(x)\int_{\mathbb{R}^d}\rho(x-\delta h)h_j\frac{\partial\varphi}{\partial h_i}\,dh\\ &= \int_{\mathbb{R}^d}\rho(x-\delta h)\sum_{i=1}^d\left(\frac{u_i(x)-u_i(x-\delta h)}{\delta}-\sum_{j=1}^d h_j\frac{\partial u_i}{\partial x_j}(x)\right)\frac{\partial\varphi}{\partial h_i}\,dh\\ &= \int_0^1\int_{\mathbb{R}^d}\rho(x-\delta h)\sum_{i,j=1}^d\left(h_j\frac{\partial u_i}{\partial x_j}(x-t\delta h)-h_j\frac{\partial u_i}{\partial x_j}(x)\right)\frac{\partial\varphi}{\partial h_i}\,dh\,dt\\ &= \int_0^1\int_0^1\int_{\mathbb{R}^d}\rho(x-\delta h)\sum_{i,j,k=1}^d\left(-t\delta h_jh_k\frac{\partial^2u_i}{\partial x_j\partial x_k}(x-st\delta h)\right)\frac{\partial\varphi}{\partial h_i}\,dh\,dt\,ds, \end{aligned}$$

and so by testing against $\phi(x) \in L_x^{r'}(\mathbb{R}^d;\mathbb{R})$ we have

$$\left\| r_\delta(u,\rho) - \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} \rho(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \right\|_{L_x^r} \lesssim \delta \|\rho\|_{L_x^{p'}} \left\|\nabla^2 u\right\|_{L_x^p}. \tag{2.24}$$

Now notice, since $\nabla \cdot u(x) = 0$,

$$\begin{aligned}
&\sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}(x) \int_{\mathbb{R}^d} \rho(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \\
&\quad = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}(x) \int_{\mathbb{R}^d} \rho(x-\delta h) \frac{\partial (h_j \varphi)}{\partial h_i}\, dh \\
&\quad = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j}(x) \int_{\mathbb{R}^d} \delta \frac{\partial \rho}{\partial x_i}(x-\delta h) h_j \varphi\, dh,
\end{aligned}$$

and so by testing against $\phi(x) \in L_x^{r'}(\mathbb{R}^d;\mathbb{R})$ we have the bound

$$\left\| \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} \rho(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \right\|_{L_x^r} \lesssim \delta \|\nabla u\|_{L_x^p} \|\nabla \rho\|_{L_x^{p'}}. \tag{2.25}$$

Finally, assume $\rho(x) = -\Delta g(x)$ for some $g(x) \in W_x^{1,p'}(\mathbb{R}^d;\mathbb{R})$, then

$$\begin{aligned}
&\sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} \rho(x-\delta h) \frac{\partial (h_j \varphi)}{\partial h_i}\, dh \\
&\quad = \sum_{i,j,k=1}^{d} \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} \frac{\partial^2 g}{\partial x_k \partial x_k}(x-\delta h) \frac{\partial (h_j \varphi)}{\partial h_i}\, dh \\
&\quad = \sum_{i,j,k=1}^{d} \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} -\frac{1}{\delta} \frac{\partial}{\partial h_k} \frac{\partial g}{\partial x_k}(x-\delta h) \frac{\partial (h_j \varphi)}{\partial h_i}\, dh \\
&\quad = \sum_{i,j,k=1}^{d} \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} \frac{1}{\delta} \frac{\partial g}{\partial x_k}(x-\delta h) \frac{\partial^2 (h_j \varphi)}{\partial h_k \partial h_i}\, dh,
\end{aligned}$$

and so

$$\left\| \frac{\partial u_i}{\partial x_j} \int_{\mathbb{R}^d} \rho(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \right\|_{L_x^r} \lesssim \frac{1}{\delta} \|\nabla u\|_{L_x^p} \left\|\nabla(-\Delta)^{-1} \rho\right\|_{L_x^{p'}}. \tag{2.26}$$

Putting equations (2.23) to (2.26) together gives, for any $\bar{u}(x) \in W_x^{2,p}(\mathbb{R}^d;\mathbb{R}^d)$, $\bar{\rho}(x) \in W_x^{1,p'}(\mathbb{R}^d;\mathbb{R})$ such that $(-\Delta)^{-1}(\rho-\bar{\rho})(x) \in W_x^{1,p'}(\mathbb{R}^d;\mathbb{R})$,

$$\begin{aligned}
&\|r_\delta(u,\rho)\|_{L_x^r} \\
&\quad \leq \|r_\delta(u-\bar{u},\rho)\|_{L_x^r} + \left\| r_\delta(\bar{u},\rho) - \sum_{i,j=1}^d \frac{\partial \bar{u}_i}{\partial x_j} \int_{\mathbb{R}^d} \rho(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \right\|_{L_x^r} \\
&\qquad + \left\| \sum_{i,j=1}^d \frac{\partial \bar{u}_i}{\partial x_j} \int_{\mathbb{R}^d} (\rho-\bar{\rho})(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \right\|_{L_x^r} \\
&\qquad + \left\| \sum_{i,j=1}^d \frac{\partial \bar{u}_i}{\partial x_j} \int_{\mathbb{R}^d} \bar{\rho}(x-\delta h) h_j \frac{\partial \varphi}{\partial h_i}\, dh \right\|_{L_x^r} \\
&\quad \lesssim \|\rho\|_{L_x^{p'}} \left(\frac{1}{\delta} \|u-\bar{u}\|_{L_x^p} + \delta \left\|\nabla^2 \bar{u}\right\|_{L_x^p} \right) \qquad (2.27) \\
&\qquad + \|\nabla \bar{u}\|_{L_x^p} \left(\frac{1}{\delta} \left\|\nabla(-\Delta)^{-1}(\rho-\bar{\rho})\right\|_{L_x^{p'}} + \delta \|\nabla \bar{\rho}\|_{L_x^{p'}} \right).
\end{aligned}$$

We now need some standard real interpolation inequalities. For this we recall the definition of $\dot{B}^s_{p,q}$ as the Schwartz distributions $\mathscr{S}'$ such that the following norm is finite,

$$\|\rho\|_{\dot{B}^s_{p,q}} = \left(\sum_{n=-\infty}^{\infty} 2^{ns} \|\psi_n * \rho\|^q_{L_x^p} \right)^{\frac{1}{q}}.$$

By [15, Theorem 6.4.5] the homogeneous Besov spaces $\dot{B}^1_{p,q}$, $\dot{B}^0_{p',q}$ are the real interpolation spaces between the spaces

$$(\dot{B}^0_{p,1}, \dot{B}^2_{p,1})_{\frac{1}{2},q} \text{ and } (\dot{B}^{-1}_{p',1}, \dot{B}^1_{p',1})_{\frac{1}{2},q},$$

respectively. Thus, using the Lions-Peetre K-function method for real interpolation [15, Section 3.1], we have the following characterisation of the norms $\|\cdot\|_{\dot{B}^1_{p,q}}$ and $\|\cdot\|_{\dot{B}^0_{p',q}}$. Defining

$$|u|_{\dot{B}^1_{p,q}} = \left(\int_0^{+\infty} \inf_{\substack{\bar{u} \in \dot{B}^2_{p,1}(\mathbb{R}^d;\mathbb{R}^d) \\ u-\bar{u} \in \dot{B}^0_{p,1}(\mathbb{R}^d;\mathbb{R}^d)}} \left(t^{-\frac{1}{2}} \|u-\bar{u}\|_{\dot{B}^0_{p,1}} + t^{\frac{1}{2}} \|\bar{u}\|_{\dot{B}^2_{p,1}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

$$|\rho|_{\dot{B}^0_{p',q}} = \left(\int_0^{+\infty} \inf_{\substack{\bar{\rho} \in \dot{B}^1_{p',1}(\mathbb{R}^d;\mathbb{R}) \\ \rho-\bar{\rho} \in \dot{B}^{-1}_{p',1}(\mathbb{R}^d;\mathbb{R})}} \left(t^{-\frac{1}{2}} \|\rho-\bar{\rho}\|_{\dot{B}^{-1}_{p',1}} + t^{\frac{1}{2}} \|\bar{\rho}\|_{\dot{B}^1_{p',1}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

there exists a universal constant $C > 0$, independent of $p, p', q \in [1, \infty]$, such that

$$\frac{1}{C}\|u\|_{\dot{B}^1_{p,q}} \le |u|_{\dot{B}^1_{p,q}} \le C\|u\|_{\dot{B}^1_{p,q}},$$
$$\frac{1}{C}\|\rho\|_{\dot{B}^0_{p',q}} \le |\rho|_{\dot{B}^0_{p',q}} \le C\|\rho\|_{\dot{B}^0_{p',q}}.$$

Thus there exists some choice of such $\bar{u}_\delta(x), \bar{\rho}_\delta(x)$ for each $\delta > 0$ so that

$$\left(\int_0^\infty \left(\frac{1}{\delta}\|u-\bar{u}_\delta\|_{\dot{B}^0_{p,1}} + \delta\|\bar{u}_\delta\|_{\dot{B}^2_{p,1}}\right)^q \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \lesssim \|u\|_{\dot{B}^1_{p,q}},$$
$$\left(\int_0^\infty \left(\frac{1}{\delta}\|\rho-\bar{\rho}_\delta\|_{\dot{B}^{-1}_{p',1}} + \delta\|\bar{\rho}_\delta\|_{\dot{B}^1_{p',1}}\right)^q \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \lesssim \|\rho\|_{\dot{B}^0_{p',q}},$$

for all $q \in [1, \infty]$. Moreover, $\bar{u}_\delta(x) \in \dot{B}^2_{p,1}(\mathbb{R}^d; \mathbb{R}^d)$ may always be chosen of the form

$$\bar{u}_\delta(x) = \sum_{n=-\infty}^{K_\delta} (\psi_n * u)(x),$$

for some $K_\delta \in \mathbb{Z}$, see [15, Section 6.4], and so in addition we have the bound $\|\nabla\bar{u}_\delta\|_{L^p_x} \lesssim \|\nabla u\|_{L^p_x}$ independent of $\delta > 0$. In light of this, we may apply (2.27) for each $\delta > 0$ to give the result as follows

$$\begin{aligned}
&\left(\int_0^\infty \|r_\delta(u,\rho)\|^q_{L^r_x} \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \\
&\quad \lesssim \left(\int_0^\infty \left(\|\rho\|_{L^{p'}_x}\left(\frac{1}{\delta}\|u-\bar{u}_\delta\|_{L^p_x} + \delta\left\|\nabla^2\bar{u}_\delta\right\|_{L^p_x}\right)\right)^q \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_0^\infty \left(\|\nabla\bar{u}_\delta\|_{L^p_x}\left(\frac{1}{\delta}\left\|\nabla(-\Delta)^{-1}(\rho-\bar{\rho}_\delta)\right\|_{L^{p'}_x} + \delta\|\nabla\bar{\rho}_\delta\|_{L^{p'}_x}\right)\right)^q \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \\
&\quad \lesssim \|\rho\|_{L^{p'}_x}\left(\int_0^\infty \left(\frac{1}{\delta}\|u-\bar{u}_\delta\|_{\dot{B}^0_{p,1}} + \delta\|\bar{u}_\delta\|_{\dot{B}^2_{p,1}}\right)^q \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \\
&\qquad + \|\nabla u\|_{L^p_x}\left(\int_0^\infty \left(\frac{1}{\delta}\|\rho-\bar{\rho}_\delta\|_{\dot{B}^{-1}_{p',1}} + \delta\|\bar{\rho}_\delta\|_{\dot{B}^1_{p',1}}\right)^q \frac{d\delta}{\delta}\right)^{\frac{1}{q}} \\
&\quad \lesssim \|\rho\|_{L^{p'}_x}\|u\|_{\dot{B}^1_{p,q}} + \|\nabla u\|_{L^p_x}\|\rho\|_{\dot{B}^0_{p',q}},
\end{aligned}$$

where we have used [15, Theorem 6.2.4], that we have the continuous embedding say $\dot{B}^0_{p,1}(\mathbb{R}^d; \mathbb{R}^d) \hookrightarrow L^p_x(\mathbb{R}^d; \mathbb{R}^d)$ and we conclude the proof by $\|u\|_{\dot{B}^1_{p,q}} \lesssim \|\nabla u\|_{\dot{B}^0_{p,q}}$. □

Remark 1. By the continuous embedding $L^p_x \hookrightarrow \dot{B}^0_{p,\max(p,2)}$ for all $1 < p \leq \infty$, Theorem 2.8 provides the quantitative control on $\delta_\varepsilon > 0$ in Proposition 2.7. Namely, we may take

$$\delta_\varepsilon = \bar{\delta} \exp\left(-C_{p,q}\varepsilon^{-\max(p,q)}\right),$$

for some $C_{p,q} > 0$ depending on the parameters $1 < p,q < \infty$ and the dimension $d \geq 2$.

By the continuous embedding $L^p_x \hookrightarrow \dot{B}^0_{p,\max(p,2)}$ for all $1 < p \leq \infty$, and Minkowski's integral inequality we have the following simple corollary.

Corollary 2.9. *Let $\nabla u(x,t) \in L^p_t L^q_x$ be divergence-free. Let $\rho(x,t) \in L^{p'}_t L^{q'}_x$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and $\frac{1}{r} = \frac{1}{q} + \frac{1}{q'} \leq 1$. If $q, q' < \infty$, then*

$$\left(\int_0^\infty \|r_\delta(u,\rho)\|^{\max(q,q')}_{L^1_t L^r_x} \frac{d\delta}{\delta}\right)^{\frac{1}{\max(q,q')}} \leq C \|\nabla u\|_{L^p_t L^q_x} \|\rho\|_{L^{p'}_t L^{q'}_x},$$

for some constant $C > 0$ depending only on p, p', the dimension d, and the following norm of the mollifier $\left\|(1+|x|^2)\nabla\varphi\right\|_{L^1_x} + \left\||x|\nabla^2\varphi\right\|_{L^1_x}$.

Next, we give an example showing that the results of the previous theorem are optimal in the case $p = p' = 2$.

Proposition 2.10 (Sharpness of decay of the DiPerna-Lions commutator)**.** *Let $u(x) \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be the time-independent linear shear*

$$u(x) = (x_2, 0) \in \mathbb{R}^2.$$

For any compactly supported mollifier $\varphi(x) \in C^\infty_c(\mathbb{R}^2;\mathbb{R})$, then both

$$\liminf_{\delta\to 0}\left(\sup_{\substack{\rho(x)\in C^\infty_c(\mathbb{R}^2;\mathbb{R})\\ |\rho(x)|\leq\chi(x)}} \|r_\delta(u,\rho)\|_{L^1_x}\right) > 0, \text{ and } \inf_{q\in[1,2)}\left(\int_0^1 \|r_\delta(u,\bar{\rho})\|^q_{L^1_x}\frac{d\delta}{\delta}\right)^{\frac{1}{q}} = \infty,$$

for any cutoff $\chi(x) \in C^\infty_c(\mathbb{R}^2;\mathbb{R})$ with $\chi(x) = 1$ whenever $|x| \leq 1$, and for some particular choice of $\bar{\rho}(x) : \mathbb{R}^2 \to \mathbb{R}$ which is supported on $|x| \leq 1$, and in L^p_x for all $p \in [1,\infty)$.

Proof. For any $\rho(x) \in L^1_x(\mathbb{R}^2;\mathbb{R})$,

$$\begin{aligned} r_\delta(u,\rho)(x) &= \int_{\mathbb{R}^2} \rho(y)(x_2 - y_2)\frac{\partial \varphi_\delta}{\partial x_1}(x-y)\, dy \\ &= \int_{\mathbb{R}^2} \rho(x-h)\frac{\partial (h_2\varphi_\delta)}{\partial h_1}(h)\, dh, \\ &= (\rho * K_\delta)(x), \end{aligned}$$

where $K_\delta(x) = \delta^{-2}K(\delta^{-1}x)$ and $K(x) = \frac{\partial(x_2\varphi)}{\partial x_1}(x)$.

In particular, the Fourier transforms of $K_\delta(x)$ and $K(x)$ are given by

$$\begin{aligned} \hat{K}_\delta(\xi) &= \hat{K}(\delta\xi), \\ \hat{K}(\xi) &= -\xi_1\frac{\partial\hat{\varphi}}{\partial\xi_2}. \end{aligned}$$

For now consider any $\rho(x) \in \mathscr{S}'(\mathbb{R}^2;\mathbb{R})$ a Schwartz distribution. Then $(\rho * K_\delta)(x)$ will coincide with $r_\delta(u,\rho)(x)$ if in addition $\rho(x) \in L^1_x(\mathbb{R}^2;\mathbb{R})$.

To prove the first statement of the theorem we take the harmonic $\rho(x) = e^{i\delta^{-1}\bar{\xi}\cdot x}$ for some $\bar{\xi} \in \mathbb{R}^2$ such that $|\hat{K}(\bar{\xi})| > 0$, which exists since $\varphi(x) \in L^1_x(\mathbb{R}^2;\mathbb{R})$ with $\int_{\mathbb{R}^2}\varphi(x)\, dx = 1$. Then $(\rho * K_\delta)(x) = e^{i\delta^{-1}\bar{\xi}\cdot x}\hat{K}(\bar{\xi})$.

We now cutoff $\rho(x)$ so that it lies in $L^1_x(\mathbb{R}^d;\mathbb{R})$. Let the mollifier $\varphi(x)$ be supported on $|x| \le R$, then $\varphi_\delta(x)$ is supported on $|x| \le \delta R$. Taking a compactly supported cutoff $\chi(x)$ with $\chi(x) = 1$ on $|x| \le 1$, then $r_\delta(u,\chi\rho)(x) = (\rho * K_\delta)(x)$ whenever $|x| \le 1 - \delta R$. Therefore, for all $\delta \le \frac{1}{2R}$, $\|r_\delta(u,\chi\rho)\|_{L^1_x} \ge |\hat{K}(\bar{\xi})|$ as required.

Note that $(\chi\rho)(x)$ is currently complex-valued, but by taking either the real or imaginary part, a similar non-zero bound must hold.

To prove the second statement of the theorem, pick $\bar{\xi} \in \mathbb{R}^2$ with $|\bar{\xi}| = 1$, and in addition some $0 < c_1 < c_2 < 2c_1$, such that $\inf_{\delta\in[c_1,c_2]} |\hat{K}(\delta\bar{\xi})| = \varepsilon > 0$, noting that $\hat{K}(\xi)$ is continuous if we assume in addition $x_2\varphi(x) \in L^1_x(\mathbb{R}^2;\mathbb{R})$.

Let $\rho(x) \in \mathscr{S}'(\mathbb{R}^2;\mathbb{R})$ be of the form $\rho(x) = \sum_{n=1}^\infty a_n e^{i\delta_n^{-1}\bar{\xi}\cdot x}$ for some $a_n \in [0,1]$ and $\delta_n \in (0,\infty)$.

Now,

$$\begin{aligned}\left|\left(e^{i\delta_n^{-1}\bar{\xi}\cdot x} * K_\delta\right)(x)\right| &= \left|\hat{K}\left(\frac{\delta}{\delta_n}\bar{\xi}\right)\right| \\ &\leq \min\left\{\frac{\delta}{\delta_n}\left\|\frac{\partial\hat{\varphi}}{\partial\xi_2}\right\|_{L_x^\infty}, \frac{\delta_n}{\delta}\left\||\xi|^2\frac{\partial\hat{\varphi}}{\partial\xi_2}\right\|_{L_x^\infty}\right\} \\ &\leq C\min\left\{\frac{\delta}{\delta_n}, \frac{\delta_n}{\delta}\right\}.\end{aligned}$$

Now let $\delta_n = 2^{-n^2}$. Then for $\frac{\delta}{\delta_n} \in [c_1, c_2] \subset [c_1, 2c_1]$,

$$\begin{aligned}|(\rho * K_\delta)(x)| &\geq a_n\left|\hat{K}\left(\frac{\delta}{\delta_n}\bar{\xi}\right)\right| - C\sum_{1\leq m\leq n-1}|a_m|\frac{c_1 2^{-n^2}}{2^{-m^2}} - C\sum_{m=n+1}^{\infty}|a_m|\frac{2^{-m^2}}{c_2 2^{-n^2}} \\ &\geq a_n\left|\hat{K}\left(\frac{\delta}{\delta_n}\bar{\xi}\right)\right| - c_1 C 2^{2-2n} - c_2^{-1}C2^{-2n} \\ &\geq a_n\varepsilon - C(c_1 + c_2^{-1})4^{1-n}.\end{aligned}$$

Therefore, for any $q \in [1,2)$,

$$\int_{\delta_n c_1}^{2\delta_n c_1}\left(\int_{[-\frac{1}{2},\frac{1}{2}]^2}|(\rho * K_\delta)(x)|\,dx\right)^q\frac{d\delta}{\delta} \geq \left(\frac{1}{2}a_n^q\varepsilon^q - (C(c_1+c_2^{-1})4^{1-n})^q\right)\log\left(\frac{c_2}{c_1}\right),$$

and so, for $N \in \mathbb{N}$ such that $2\delta_N c_1 \leq 1$,

$$\int_0^1\left(\int_{[-\frac{1}{2},\frac{1}{2}]^2}|(\rho * K_\delta)(x)|\,dx\right)^q\frac{d\delta}{\delta} \geq \frac{1}{2}\log\left(\frac{c_2}{c_1}\right)\varepsilon^q\sum_{n=N}^{\infty}a_n^q - \frac{C^q(c_1+c_2^{-1})^q}{1-4^{-q}}\log\left(\frac{c_2}{c_1}\right).$$

Now, choose $\{a_n\}_{n=1}^{\infty} \subset (0,\infty)$ so that $\sum_{n=N}^{\infty}a_n^q = \infty$ but $\sum_{n=N}^{\infty}a_n^2 < \infty$. Then $\rho(x) \in \dot{B}^0_{p,2}\left([-\frac{1}{2},\frac{1}{2}]^2;\mathbb{R}\right)$ for all $p \in [1,\infty)$ (noting that $\rho(x)$ is periodic with period 1, and each $e^{i\delta_n^{-1}\bar{\xi}\cdot x}$ lies in a different Littlewood-Paley block). By the embedding $\dot{B}^0_{p,2} \hookrightarrow L^p_x$ whenever $p \in [2,\infty)$ [11, Theorem 2.40], we have that $\rho(x) \in L^p_x\left([-\frac{1}{2},\frac{1}{2}]^2;\mathbb{R}\right)$ for all $p \in [2,\infty)$.

We now cutoff $\rho(x)$ so that it lies in $L^1_x(\mathbb{R}^d;\mathbb{R})$. Let the mollifier $\varphi(x)$ be supported on $|x| \leq R$, then $\varphi_\delta(x)$ is supported on $|x| \leq \delta R$. Taking a compactly supported cutoff $\chi(x)$ with $\chi(x) = 1$ on $|x| \leq \frac{3}{4}$ and $\chi(x) = 0$ on $|x| \geq 1$, then $(\chi\rho)(x) \in L^p_x(\mathbb{R}^2)$ for all $p \in [1,\infty)$, and $r_\delta(u,\chi\rho)(x) = (\rho * K_\delta)(x)$ provided that $|x| \leq \frac{3}{4} - \delta R$.

Therefore,

$$\int_0^1\|r_\delta(u,\rho\chi)\|^q_{L^1_x}\frac{d\delta}{\delta} = \infty.$$

Note that $(\chi\rho)(x)$ above is complex-valued, but the above integral must also be infinite for either the real or imaginary part. □

In particular, the first part of the above result shows that a more standard uniform convergence rate of the form, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ independent of $u(x)$ and $\rho(x)$ such that

$$\sup_{\delta<\delta_\varepsilon} \|r_\delta(u,\rho)\|_{L^1_x} \leq \varepsilon \, \|\nabla u\|_{L^p_x} \|\rho\|_{L^q_x},$$

does not exist.

Despite that the uniform decay of $r_\delta(u,\rho)(x)$ in L^1_x given in Theorem 2.8 is optimal (for $p = p' = 2$), one has significantly faster uniform point-wise decay of $r_\delta(u,\rho)(x)$. This is the content of the following theorem, which relies on recent singular integral estimates in harmonic analysis [95, Theorem 1.1].

Theorem 2.11 (Uniform point-wise decay of the DiPerna-Lions commutator)**.** *Let $1 < p,q \leq \infty$, and $1 \leq r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.*

Let $\nabla u(x) \in L^p_x(\mathbb{R}^d;\mathbb{R}^d \times \mathbb{R}^d)$ be divergence-free, and $\rho(x) \in L^q(\mathbb{R}^d;\mathbb{R})$, then for any $0 < a < b$,

$$\left\| \int_a^b r_\delta(u,\rho)(x) \, \frac{d\delta}{\delta} \right\|_{L^r_x} \leq C \, \|\nabla u\|_{L^p_x} \|\rho\|_{L^q_x},$$

for some constant C depending only on p, q, the dimension d, and some high-regularity norm of the mollifier $\varphi(x)$ (and in particular not on a,b). In particular, this includes any mollifier $\varphi(x)$ as a Schwartz function.

Proof. Throughout, $\lesssim$ will denote less than or equal to up to a constant depending only on p, p', the dimension d, and the mollifier $\varphi(x) \in \mathscr{S}(\mathbb{R}^d;\mathbb{R})$ (and in particular not on a,b).

Now, as before, we write

$$\begin{aligned} r_\delta(u,\rho)(x) &= \int_{\mathbb{R}^d} \rho(x-h)(u(x)-u(x-h)) \cdot \nabla\varphi_\delta(h) \, dy \\ &= \int_{\mathbb{R}^d} \rho(x-h) \sum_{i,j=1}^d \left(\int_0^1 h_j \frac{\partial u_i}{\partial x_j}(x-th) \, dt \right) \frac{\partial \varphi_\delta}{\partial h_i}(h) \, dh. \end{aligned}$$

Recalling that $\varphi_\delta(h) = \delta^{-d}\varphi\left(\frac{h}{\delta}\right)$, for any $0 < a < b$ we have

$$\int_a^b r_\delta(u,\rho)(x) \, \frac{d\delta}{\delta} = \int_{\mathbb{R}^d} \rho(x-h) \sum_{i,j=1}^d \left(\int_0^1 h_j \frac{\partial u_i}{\partial x_j}(x-th) \, dt \right) \left(\int_a^b \frac{\partial \varphi}{\partial h_i}\left(\frac{h}{\delta}\right) \frac{d\delta}{\delta^{d+2}} \right) dh.$$

Using that $\nabla \cdot u(x) = 0$ we may rewrite this as

$$\int_a^b r_\delta(u,\rho)(x)\,\frac{d\delta}{\delta} \tag{2.28}$$

$$= \int_{\mathbb{R}^d} \rho(x-h) \sum_{i,j=1}^d \left(\int_0^1 \frac{\partial u_i}{\partial x_j}(x-th)\,dt \right) \left(\int_a^b \frac{\partial (h_j \varphi)}{\partial h_i}\left(\frac{h}{\delta}\right) \frac{d\delta}{\delta^{d+1}} \right) dh$$

$$= \int_{\mathbb{R}^d} \rho(x-h) \sum_{i,j=1}^d \left(\int_0^1 \frac{\partial u_i}{\partial x_j}(x-th)\,dt \right) K_{i,j}(h)\,dh, \tag{2.29}$$

where

$$K_{i,j}(x) = \int_a^b \frac{\partial (x_j \varphi)}{\partial x_i}\left(\frac{x}{\delta}\right) \frac{d\delta}{\delta^{d+1}}.$$

In particular, we have the following Calderon-Zygmund estimates on $K_{i,j}(x)$

$$\begin{aligned}
|K_{i,j}(x)| &\leq \left\| (1+|x|^{d+1}) \frac{\partial (x_j\varphi)}{\partial x_i} \right\|_{L^\infty_x} \left(\int_{|x|}^\infty \delta^{-d-1} d\delta + \int_0^{|x|} |x|^{-d-1} d\delta \right) \\
&\lesssim |x|^{-d}, \\
\left| \frac{\partial K_{i,j}}{\partial x_k}(x) \right| &= \left| \int_a^b \frac{\partial^2 (x_j \varphi)}{\partial x_k \partial x_i}\left(\frac{x}{\delta}\right) \frac{d\delta}{\delta^{d+2}} \right| \\
&\leq \left\| (1+|x|^{d+2}) \frac{\partial^2 (x_j\varphi)}{\partial x_k \partial x_i} \right\|_{L^\infty_x} \left(\int_{|x|}^\infty \delta^{-d-2} d\delta + \int_0^{|x|} |x|^{-d-2} d\delta \right) \\
&\lesssim |x|^{-d-1}, \\
|\hat{K}_{i,j}(\xi)| &= \left| \int_{\mathbb{R}^d} e^{i\xi\cdot x} \int_a^b \frac{\partial (x_j\varphi)}{\partial x_i}\left(\frac{x}{\delta}\right) \frac{d\delta}{\delta^{d+1}}\,dx \right| \\
&= \left| \int_a^b \left(\int_{\mathbb{R}^d} e^{i\delta\xi\cdot h} \frac{\partial (x_j\varphi)}{\partial x_i}(h)\,dh \right) \frac{d\delta}{\delta} \right| \\
&= \left| \int_a^b -i\delta\xi_i \frac{\partial \hat{\varphi}}{\partial \xi_j}(\delta\xi)\,\frac{d\delta}{\delta} \right| \\
&\leq \int_0^\infty |\xi_i| \left| \frac{\partial \hat{\varphi}}{\partial \xi_j} \right| (\delta\xi)\,d\delta \\
&\leq \left\| (1+|\xi_i|^2) \frac{\partial \hat{\varphi}}{\partial \xi_j} \right\|_{L^\infty_x} \left(\int_0^{|\xi_i|^{-1}} |\xi_i|\,d\delta + \int_{|\xi_i|^{-1}}^\infty |\xi_i|^{-1} \delta^{-2}\,d\delta \right) \\
&\lesssim 1,
\end{aligned}$$

where, in particular, none of the bounds depends on a, b.

The result then follows by applying [95, Theorem 1.1] to (2.29). □

2.2.2 Mixing Estimates for Transport along Sobolev Vector Fields

Next, we give simple mixing bounds that follow directly from the analysis of the previous sections. 'Mixing' of a passive scalar refers to transferring mass to smaller spatial scales/higher frequencies. One approach to measure mixing is through the decay of weak norms such as $\|\rho(\cdot,t)\|_{H_x^{-1}}$. A stronger notion is the geometric mixing scale $\delta \in (0,\infty)$, which is defined by a condition of the form $\|\rho(\cdot,t) * \varphi_\delta\|_{L_x^\infty} \leq \kappa$, see for example [22]. In particular, the DiPerna-Lions commutator controls exactly the change in such geometric mixing scales since

$$\left(\frac{\partial}{\partial t} + u(x,t)\cdot\nabla\right)(\rho * \varphi_\delta)(x,t) = r_\delta(u,\rho)(x,t).$$

Therefore, bounds on the commutator should immediately give bounds on the permitted growth/decay of the mixing scale. Since mixing refers to the decrease of these mixing scales, one tends to prefer lower bounds on the decay of the mixing scale, though upper bounds may be found similarly.

To illustrate this approach, we first give a simple but sub-optimal mixing estimate using the norm-decay of the commutator, Corollary 2.9. Through standard energy estimates

$$\|\rho(\cdot,T) * \varphi_\delta\|_{L_x^1} \geq \|\rho_0 * \varphi_\delta\|_{L_x^1} - \|r_\delta(u,\rho)\|_{L_t^1 L_x^1},$$

for all $\delta > 0$. Integrating over δ, and applying Corollary 2.9 as in the previous section gives the lower bound

$$\left\|\rho(\cdot,T) * \varphi_{\delta_1}\right\|_{L_x^1} \geq \left\|\rho_0 * \varphi_{\delta_2}\right\|_{L_x^1} - \frac{C}{\sqrt{\log\left(\frac{\delta_2}{\delta_1}\right)}} \|\nabla u\|_{L_t^1 L_x^2} \|\rho_0\|_{L_x^2},$$

for all $0 < \delta_1 < \delta_2$, (where we have again supposed for simplicity φ is a Gaussian).

If we then define the geometric mixing scale $\delta > 0$ at time $t = T$ by the mixing condition

$$\|\rho(\cdot,T) * \varphi_\delta\|_{L_x^1} \leq \|\rho_0 * \varphi_\varepsilon\|_{L_x^1} - \kappa,$$

for some $\varepsilon > 0$, $\kappa > 0$, this then gives a Gaussian lower bound on the mixing scale,

$$\delta \geq \varepsilon \exp\left(-\frac{C^2}{\kappa^2} \|\nabla u\|_{L_t^1 L_x^2}^2 \|\rho_0\|_{L_x^2}^2\right),$$

and for this reason, we refer to the decay of Theorem 2.8 in the sharp case $p, p' = 2$ as a 'Gaussian' decay rate on the commutator.

However, the mixing scale is known to be bounded below by an exponential in $\|\nabla u\|_{L_t^1 L_x^p}$ when $1 < p \leq \infty$, see [34], [77], [55]. It is, therefore, quite surprising that the 'Gaussian' norm-decay of the commutator is optimal, Proposition 2.10. This discrepancy can be rectified by considering the commutator's 'exponential' point-wise decay rate, Theorem 2.11 instead.

Theorem 2.12. *Consider a weak solution $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, with $\rho(x,t) \in L_t^\infty L_x^q$, and intial data $\rho_0(x) \in L_x^q$, to the transport equation*

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = 0,$$

along a divergence-free $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, with $\nabla u(x,t) \in L_t^1 L_x^p$, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with $1 < p \leq \infty$, $1 \leq r < \infty$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d;\mathbb{R})$ be the Gaussian mollifier, that is $\varphi(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}x^2}$.

Then for any $\kappa < 1$, there exists some $\varepsilon > 0$ depending on the initial datum $\rho_0(x)$, such that

$$\|\rho(\cdot,T) * \varphi_\delta\|_{L_x^r} \leq (1-\kappa)\,\|\rho_0\|_{L_x^r},$$

then $\delta > 0$ must satisfy

$$\delta \geq \varepsilon \exp\left(-\frac{C_{p,r}}{\kappa} \frac{\|\rho_0\|_{L_x^q}}{\|\rho_0\|_{L_x^r}} \|\nabla u\|_{L_t^1 L_x^p}\right),$$

where the constant $C_{p,r} > 0$ depends only on the parameters $1 < p \leq \infty$, $1 \leq r < \infty$, and the dimension $d \geq 2$.

Essentially, $\varepsilon > 0$ is the scale at which $\rho_0(x)$ is not already mixed, which must satisfy

$$\sup_{\delta' < \varepsilon} \|\rho_0 - \rho_0 * \varphi_{\delta'}\|_{L_x^r} \leq \frac{1}{2}\kappa\,\|\rho_0\|_{L_x^r}.$$

Proof. As in the previous section, for any $0 < \delta_1 < \delta_2$, we have transport of

$$\left(\frac{\partial}{\partial t} + u(x,t)\cdot\nabla\right)\left(\int_{\delta_1}^{\delta_2} (\rho * \varphi_{\delta'})(x,t)\,\frac{d\delta'}{\delta'}\right) = \int_{\delta_1}^{\delta_2} r_{\delta'}(u,\rho)\,\frac{d\delta'}{\delta'}.$$

and so, by standard energy estimates,

$$\left\|\int_{\delta_1}^{\delta_2} \rho(\cdot,T) * \varphi_{\delta'}\,\frac{d\delta'}{\delta'}\right\|_{L_x^r} \geq \left\|\int_{\delta_1}^{\delta_2} \rho_0 * \varphi_{\delta'}\,\frac{d\delta'}{\delta'}\right\|_{L_x^r} - \left\|\int_{\delta_1}^{\delta_2} r_{\delta'}(u,\rho)\,\frac{d\delta'}{\delta'}\right\|_{L_t^1 L_x^r}.$$

Then, by Theorem 2.11, and taking again for simplicity φ to be a Gaussian, we obtain the following lower bound for all $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ with $p>1$, $r<\infty$.

$$\begin{aligned}\|\rho(\cdot,T)*\varphi_{\delta_1}\|_{L^r_x} &\geq \frac{1}{\log\left(\frac{\delta_2}{\delta_1}\right)}\left(\left\|\int_{\delta_1}^{\delta_2}\rho_0\,\frac{d\delta'}{\delta'}\right\|_{L^r_x}-\left\|\int_{\delta_1}^{\delta_2}(\rho_0-\rho_0*\varphi_{\delta'})\,\frac{d\delta'}{\delta'}\right\|_{L^r_x}\right)\\ &\quad -\frac{C_{p,r}}{\log\left(\frac{\delta_2}{\delta_1}\right)}\|\nabla u\|_{L^1_tL^p_x}\|\rho_0\|_{L^q_x}\\ &\geq \left(\|\rho_0\|_{L^r_x}-\sup_{\delta'<\delta_2}\|\rho_0-\rho_0*\varphi_{\delta'}\|_{L^r_x}\right)-\frac{C_{p,r}}{\log\left(\frac{\delta_2}{\delta_1}\right)}\|\nabla u\|_{L^1_tL^p_x}\|\rho_0\|_{L^q_x},\end{aligned}$$

for all $0<\delta_1<\delta_2$.

If we then define the geometric mixing scale $\delta>0$ at time $t=T$ by the mixing condition

$$\|\rho(\cdot,T)*\varphi_\delta\|_{L^r_x}\leq\left(\|\rho_0\|_{L^r_x}-\sup_{\delta'<\varepsilon}\|\rho_0-\rho_0*\varphi_{\delta'}\|_{L^r_x}\right)-\kappa,$$

for some $\varepsilon>0$, $\kappa>0$, we deduce the following exponential lower bound on the mixing scale

$$\delta\geq\varepsilon\left(-\frac{C_{p,r}}{\kappa}\|\nabla u\|_{L^1_tL^p_x}\|\rho_0\|_{L^q_x}\right), \tag{2.30}$$

and the result now follows by relabeling κ. □

As a simple corollary, we also obtain control over the 'functional' mixing scale, defined as a (weak) function norm of the passive scalar, say $W_x^{-1,r}$.

Theorem 2.13. *Consider a weak solution $\rho(x,t):\mathbb{R}^d\times[0,T]\to\mathbb{R}$, with $\rho(x,t)\in L^\infty_tL^q_x$, and intial data $\rho_0(x)\in L^q_x$, to the transport equation*

$$\frac{\partial\rho}{\partial t}(x,t)+\nabla\cdot(u(x,t)\rho(x,t))=0,$$

along a divergence-free $u(x,t):\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$, with $\nabla u(x,t)\in L^1_tL^p_x$, for $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, with $1<p\leq\infty$, $1\leq r<\infty$.

Then there exists some $\varepsilon>0$ depending on the initial datum $\rho_0(x)$, such that,

$$\|\rho(\cdot,T)\|_{W_x^{-1,r}}\geq\varepsilon\|\rho_0\|_{L^r_x}\exp\left(-C_{p,r}\frac{\|\rho_0\|_{L^q_x}}{\|\rho_0\|_{L^r_x}}\|\nabla u\|_{L^1_tL^p_x}\right),$$

where the constant $A > 0$ depends only on the dimension d, and $C_{p,r} > 0$ depends only on the parameters $1 < p \leq \infty$, $1 \leq r < \infty$, and the dimension $d \geq 2$.

The parameter $\varepsilon > 0$ is related to the scale at which $\rho_0(x)$ is not already mixed. Specifically, there is some constant $c > 0$ depending only on dimension, such that ε must satisfy $\varepsilon \leq c$ and

$$\sup_{\delta' < \frac{\varepsilon}{c}} \|\rho_0 - \rho_0 * \varphi_{\delta'}\|_{L^r_x} \leq \frac{1}{4} \|\rho_0\|_{L^r_x}.$$

Proof. As in the statement of Theorem 2.12, we take the mollifier $\varphi(x) \in \mathscr{S}(\mathbb{R}^d;\mathbb{R})$ to be the Gaussian mollifier $\varphi(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}x^2}$. Then we may bound

$$\begin{aligned} \|\rho(\cdot,T) * \varphi_\delta\|_{L^r_x} &\leq \|\rho(\cdot,T)\|_{W^{-1,r}_x} \|\varphi_\delta\|_{W^{1,1}_x} \\ &\leq A\left(1 + \frac{1}{\delta}\right) \|\rho(\cdot,T)\|_{W^{-1,r}_x}, \end{aligned}$$

where $A > 0$ is a constant that depends only on dimension d.

Apply now Theorem 2.13 with $\kappa = \frac{1}{2}$. Then for some $\varepsilon > 0$ depending on the initial datum,

$$\delta = \frac{\varepsilon}{2} \exp\left(-2C_{p,r} \frac{\|\rho_0\|_{L^q_x}}{\|\rho_0\|_{L^r_x}} \|\nabla u\|_{L^1_t L^p_x}\right),$$

we have

$$\|\rho(\cdot,T) * \varphi_\delta\|_{L^r_x} \geq \frac{1}{2} \|\rho_0\|_{L^r_x}.$$

Therefore, assuming also $\varepsilon \leq 1$,

$$\begin{aligned} \|\rho(\cdot,T)\|_{W^{-1,r}_x} &\geq A^{-1}\left(1 + \frac{1}{\delta}\right)^{-1} \|\rho(\cdot,T) * \varphi_\delta\|_{L^r_x} \\ &\geq \frac{1}{2} A^{-1} \left(1 + \frac{2}{\varepsilon}\right)^{-1} \exp\left(-2C_{p,r} \frac{\|\rho_0\|_{L^q_x}}{\|\rho_0\|_{L^r_x}} \|\nabla u\|_{L^1_t L^p_x}\right) \|\rho_0\|_{L^r_x} \\ &\geq \frac{1}{6} A^{-1} \varepsilon \exp\left(-2C_{p,r} \frac{\|\rho_0\|_{L^q_x}}{\|\rho_0\|_{L^r_x}} \|\nabla u\|_{L^1_t L^p_x}\right) \|\rho_0\|_{L^r_x}, \end{aligned}$$

where have the requirement

$$\sup_{\delta' < \varepsilon} \|\rho_0 - \rho_0 * \varphi_{\delta'}\|_{L^r_x} \leq \frac{1}{4} \|\rho_0\|_{L^r_x}.$$

Therefore, by redefining ε, we obtain the result. □

In contrast, the best-known results in the literature require regularity assumptions on the initial datum [96], [61], and so fail to control mixing for more irregular initial data $\rho_0(x) \in L^q_x$. For more regular initial data, one can also show the propagation of 'fractional derivatives' of the passive scalar [77], [55]. We show how our approach can also generalise these results.

Theorem 2.14. *Consider a weak solution $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, with $\rho(x,t) \in L^\infty_t L^q_x$, and initial data $\rho_0(x) \in L^q_x$, to the transport equation*

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = 0,$$

along a divergence-free $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, with $\nabla u(x,t) \in L^1_t L^p_x$, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with $1 < p \leq \infty$, $1 \leq r < \infty$. Then for any convolution kernel $K(x) : \mathbb{R}^d \to \mathbb{R}$,

$$\frac{d}{dt} \|(\rho * K)(\cdot,t)\|_{L^r_x} \leq C \|\rho(\cdot,t)\|_{L^q_x} \|\nabla u(\cdot,t)\|_{L^p_x} \left(\sum_{i,j=1}^{d} \left\| \frac{\partial (x_j K)}{\partial x_i} \right\|_{CZ} \right),$$

for some constant $C > 0$ depending only on p, q, and the dimension $d \geq 2$, and where $\|K\|_{CZ}$ refers to the Calderon-Zygmund norm, defined by

$$\begin{gathered} |K(x)| \leq |x|^{-d} \|K\|_{CZ}, \\ |\nabla K(x)| \leq |x|^{-d-1} \|K\|_{CZ}, \\ \left|\hat{K}(\xi)\right| \leq \|K\|_{CZ}, \end{gathered}$$

for all $x, \xi \in \mathbb{R}^d$, where $\hat{K}(\xi) : \mathbb{R}^d \to \mathbb{R}$ denotes the Fourier transform.

Proof. We have, as for the DiPerna-Lions commutator,

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho * K)(x,t) + u(x,t) \cdot \nabla(\rho * K)(x,t) \\ &\quad = \int_{\mathbb{R}^d} \rho(y)(u(x) - u(y)) \cdot \nabla K(x-y)\, dy \\ &\quad = \int_{\mathbb{R}^d} \rho(x-h) \sum_{i,j=1}^{d} \left(\int_0^1 h_j \frac{\partial u_i}{\partial x_j}(x-th)\, dt \right) \frac{\partial K}{\partial h_i}(h)\, dh \\ &\quad = \int_{\mathbb{R}^d} \rho(x-h) \sum_{i,j=1}^{d} \left(\int_0^1 \frac{\partial u_i}{\partial x_j}(x-th)\, dt \right) \frac{\partial (h_j K)}{\partial h_i}(h)\, dh \\ &\qquad - \int_{\mathbb{R}^d} \rho(x-h) \left(\int_0^1 (\nabla \cdot u)(x-th)\, dt \right) K(h)\, dh. \end{aligned}$$

Then by energy estimates, and since $\nabla \cdot u(x) = 0$, we have

$$\frac{d}{dt}\|(\rho * K)(\cdot,t)\|_{L^r_x} \leq \sum_{i,j=1}^{d} \left\| \int_{\mathbb{R}^d} \rho(x-h) \left(\int_0^1 \frac{\partial u_i}{\partial x_j}(x-th)\, dt \right) \frac{\partial (h_j K)}{\partial h_i}(h)\, dh \right\|_{L^r_x},$$

and so we conclude by the key harmonic analysis estimate [95, Theorem 1.1]. □

Remark 2. We note that one may take the following kernel in the above theorem, which corresponds to the propagation of a logarithm of a derivative of the passive scalar.

$$\hat{K}(\xi) = \log\left(1 + |\xi|^2\right),$$

where $\hat{K}(\xi) : \mathbb{R}^d \to \mathbb{R}$ is the Fourier transform of the desired kernel $K(x) = \log(1-\Delta)$. Then $\frac{\partial (x_j K)}{\partial x_i}$ is a Calderon-Zygmund kernel by the Mikhlin-multiplier theorem [87, Theorem 8.2].

2.2.3 Weak Stability of Transport along Sobolev Vector Fields

We now consider the issue of the stability of transport solutions. Specifically, let $\rho_n(x,t)$ be weak solutions to the forced transport equation

$$\frac{\partial \rho_n}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho_n(x,t)) = \psi_n(x,t),$$

with the same initial datum $\rho_0(x)$, and weak convergence of $\psi_n \xrightharpoonup{n\to\infty} 0$. This includes the case where ρ_n solves the transport equation along $u_n(x,t)$ with convergence of $u_n \xrightarrow{n\to\infty} u$. Then, by the equation's well-posedness (and linearity), one may show that $\rho_n(x,t)$ converges weakly to the unique solution.

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = 0.$$

Based on the above argument, a quantitative version of well-posedness should give a quantitative rate of such convergence. We now use the tools from the previous section to develop such weak stability estimates. We show a few different results, which are significant improvements over the state of the art [97].

To compare to the results of [97] it is helpful in the following to note that expressions of the form $\kappa + Ae^{\frac{T}{\kappa}}$ may be crudely optimised for small $A \leq T^3$, $A \leq 1$ by $\kappa = \frac{3T}{-\log A}$.

Theorem 2.15. *Consider a weak solution $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, with $\rho(x,t) \in L^\infty_t L^q_x$, and initial data $\rho_0(x) \in L^1_x \cap L^q_x$, to the forced transport equation*

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = \psi(x,t),$$

along a divergence-free $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, with $\nabla u(x,t) \in L^1_t L^p_x$, for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 < p < \infty$, and distributional force $\psi \in L^1_t W^{-1,1}_x$. Then for any $\kappa > 0$ we have the quantitative bound

$$\|\rho\|_{L^1_t W^{-1,1}_x} \leq \kappa T + AT \exp\left(\frac{C_p}{\kappa} \|\rho\|_{L^\infty_t L^q_x} \|\nabla u\|_{L^1_t L^p_x}\right) \left(\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x}\right)^{\frac{1}{2}}$$
$$\times \left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right)^{\frac{1}{2}} + \|\psi\|_{W^{-1,1}_{x,t}}.$$

in terms of constants $A > 0$ depending only on dimension d, and $C_p > 0$ depending only on the paramater $1 < p < \infty$ and dimension d.

Proof. Let $\lesssim$ denote less than or equal to up to a constant depending only on the dimension d.

Consider a test function $\theta_T(x) \in C^\infty_c$, and let $\theta(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ be a distributional solution to the transport equation

$$\frac{\partial \theta}{\partial t}(x,t) + u(x,t) \cdot \nabla \theta(x,t) = 0,$$

with final datum $\theta(x,T) = \theta_T(x)$, that is for all test functions $\phi(x,t) \in C^\infty_c(\mathbb{R}^d \times (0,T];\mathbb{R})$,

$$\int_{\mathbb{R}^d \times [0,T]} \theta(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t)\right) dxdt = \int_{\mathbb{R}^d} \theta_T(x)\phi(x,T)\, dx.$$

Such a solution exists with $\theta(x,t) \in L^\infty_t L^1_x \cap L^\infty_t L^\infty_x$, $\frac{\partial \theta}{\partial t}(x,t) \in L^1_t W^{-1,1}_x$ by standard theory [9].

Take a mollifier $\varphi(x) \in C^\infty_c(\mathbb{R}^d;\mathbb{R})$ (Definition 2.5), supported on $|x| \leq 1$, then in terms of the DiPerna-Lions commutator $r_\delta(u,\theta)(x,t)$ (Definition 2.6) for all $t \in (0,T)$,

$$\frac{\partial}{\partial t}(\theta * \varphi_\delta)(x,t) + u(x,t) \cdot \nabla(\theta * \varphi_\delta)(x,t) = r_\delta(u,\theta)(x,t), \tag{2.31}$$

where $(\theta * \varphi_\delta)(x,t) \in L_t^\infty W_x^{1,p}$, $\frac{\partial}{\partial t}(\theta * \varphi_\delta)(x,t) \in L_t^1 L_x^p$, and so (by a suitable approximation argument) can be taken as a test function for

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = \psi(x,t),$$

that is

$$\begin{aligned}\int_{\mathbb{R}^d \times [0,T]} \rho(x,t) r_\delta(u,\theta)(x,t)\, dxdt + \int_{\mathbb{R}^d \times [0,T]} \psi(x,t)(\theta * \varphi_\delta)(x,t)\, dxdt \\ = \int_{\mathbb{R}^d} \rho(x,T)(\theta_T * \varphi_\delta)(x)\, dx - \int_{\mathbb{R}^d} \rho_0(x)(\theta * \varphi_\delta)(x,0)\, dx.\end{aligned}$$

Therefore, for any test function $\theta_T(x) \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho(x,T)\theta_T(x)\, dx \\ &\quad = \int_{\mathbb{R}^d} \rho(x,T)(\theta_T - \theta_T * \varphi_\delta)(x)\, dx \\ &\qquad + \int_{\mathbb{R}^d} \rho_0(x)(\theta * \varphi_\delta)(x,0)\, dx \\ &\qquad + \int_{\mathbb{R}^d \times [0,T]} \rho(x,t) r_\delta(u,\theta)(x,t)\, dxdt \\ &\qquad + \int_{\mathbb{R}^d \times [0,T]} \psi(x,t)(\theta * \varphi_\delta)(x,t)\, dxdt.\end{aligned} \tag{2.32}$$

Term by term, we have, since by standard energy estimates $\|\rho(\cdot,T)\|_{L_x^1} \le \|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}$,

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho(x,T)(\theta_T - \theta_T * \varphi_\delta)(x)\, dx \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x,T)(\theta_T(x) - \theta_T(y))\varphi_\delta(x-y)\, dy\, dx \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 \rho(x,T)(x-y) \cdot \nabla\theta_T(x-(1-s)(x-y))\varphi_\delta(x-y)\, ds\, dy\, dx \\ &\quad \lesssim \delta\, \|\rho(\cdot,T)\|_{L_x^1} \|\nabla\theta_T\|_{L_x^\infty} \\ &\quad \lesssim \delta \left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1} \right) \|\nabla\theta_T\|_{L_x^\infty}.\end{aligned}$$

Since $\|\theta(\cdot,0)\|_{L^\infty_x} \le \|\theta_T\|_{L^\infty_x}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_0(x)(\theta * \varphi_\delta)(x,0)\, dx \\
&\quad \lesssim \|\rho_0\|_{W^{-1,1}_x} \|(\theta * \varphi_\delta)(\cdot,0)\|_{W^{1,\infty}_x} \\
&\quad \lesssim \|\rho_0\|_{W^{-1,1}_x} \left(\|(\theta * \varphi_\delta)(\cdot,0)\|_{L^\infty_x} + \|\nabla(\theta * \varphi_\delta)(\cdot,0)\|_{L^\infty_x} \right) \\
&\quad \lesssim \|\rho_0\|_{W^{-1,1}_x} \left(1 + \frac{1}{\delta}\right) \|\theta(\cdot,0)\|_{L^\infty_x} \\
&\quad \lesssim \|\rho_0\|_{W^{-1,1}_x} \left(1 + \frac{1}{\delta}\right) \|\theta_T\|_{L^\infty_x}.
\end{aligned}$$

By Theorem 2.11, for $0 < \delta_1 < \delta_2$,

$$\begin{aligned}
&\int_{\delta_1}^{\delta_2} \int_{\mathbb{R}^d \times [0,T]} \rho(x,t) r_\delta(u,\theta)(x,t)\, dxdt\, \frac{d\delta}{\delta} \\
&\qquad \le \|\rho\|_{L^\infty_t L^q_x} \left\| \int_{\delta_1}^{\delta_2} r_\delta(u,\theta)(x,t)\, \frac{d\delta}{\delta} \right\|_{L^1_t L^p_x} \\
&\qquad \le C_p \|\rho\|_{L^\infty_t L^q_x} \|\nabla u\|_{L^1_t L^p_x} \|\theta\|_{L^\infty_t L^\infty_x},
\end{aligned}$$

for some $C_p > 0$ depending only on the parameter $1 < p < \infty$, and the dimension $d \ge 2$. Finally,

$$\begin{aligned}
&\int_{\mathbb{R}^d \times [0,T]} \psi(x,t)(\theta * \varphi_\delta)(x,t)\, dxdt \\
&\quad \lesssim \|\psi\|_{L^1_t W^{-1,1}_x} \|\theta * \varphi_\delta\|_{L^\infty_t W^{1,\infty}_x} \\
&\quad \lesssim \|\psi\|_{L^1_t W^{-1,1}_x} \left(\|\theta * \varphi_\delta\|_{L^\infty_t L^\infty_x} + \|\nabla(\theta * \varphi_\delta)\|_{L^\infty_t L^\infty_x} \right) \\
&\quad \lesssim \|\psi\|_{L^1_t W^{-1,1}_x} \left(1 + \frac{1}{\delta}\right) \|\theta\|_{L^\infty_t L^\infty_x} \\
&\quad \lesssim \|\psi\|_{L^1_t W^{-1,1}_x} \left(1 + \frac{1}{\delta}\right) \|\theta_T\|_{L^\infty_x}.
\end{aligned}$$

Integrating (2.32) over $\delta \in [\delta_1, \delta_2]$ with respect to the measure $\frac{d\delta}{\delta}$ then gives, for some constant $A > 0$ depending only on the dimension d,

$$\log\left(\frac{\delta_2}{\delta_1}\right) \int_{\mathbb{R}^d} \rho(x,T)\theta_T(x)\, dx$$

$$\leq A\int_{\delta_1}^{\delta_2} \delta\left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right) \frac{d\delta}{\delta}\,\|\nabla\theta_T\|_{L^\infty_x}$$
$$+A\int_{\delta_1}^{\delta_2}\left(1+\frac{1}{\delta}\right)\left(\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x}\right)\frac{d\delta}{\delta}\,\|\theta_T\|_{L^\infty_x}$$
$$+C_p\,\|\rho\|_{L^\infty_t L^q_x}\|\nabla u\|_{L^1_t L^p_x}\|\theta_T\|_{L^\infty_x},$$

and so

$$\|\rho(\cdot,T)\|_{W^{-1,1}_x} \leq A\delta_2\left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right)$$
$$+A\left(1+\frac{1}{\delta_1}\right)\left(\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x}\right)$$
$$+C_p\left(\log\left(\frac{\delta_2}{\delta_1}\right)\right)^{-1}\|\rho\|_{L^\infty_t L^q_x}\|\nabla u\|_{L^1_t L^p_x}.$$

For $\kappa > 0$, let

$$\delta_2 = \left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right)^{-\frac{1}{2}}\left(\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x}\right)^{\frac{1}{2}}\exp\left(\frac{C_p}{2\kappa}\|\rho\|_{L^\infty_t L^q_x}\|\nabla u\|_{L^1_t L^p_x}\right),$$
$$\delta_1 = \left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right)^{-\frac{1}{2}}\left(\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x}\right)^{\frac{1}{2}}\exp\left(-\frac{C_p}{2\kappa}\|\rho\|_{L^\infty_t L^q_x}\|\nabla u\|_{L^1_t L^p_x}\right),$$

Then, noting also that

$$\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x} \leq \|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x},$$

it follows that

$$\|\rho(\cdot,T)\|_{W^{-1,1}_x} \leq \kappa + 3A\exp\left(\frac{C_p}{2\kappa}\|\rho\|_{L^\infty_t L^q_x}\|\nabla u\|_{L^1_t L^p_x}\right)\left(\|\rho_0\|_{W^{-1,1}_x} + \|\psi\|_{L^1_t W^{-1,1}_x}\right)^{\frac{1}{2}}$$
$$\times\left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right)^{\frac{1}{2}}.$$

We conclude the statement of the theorem by taking an arbitrary $T > 0$. □

We next present a weak stability estimate valid for a distributional force $\psi(x,t)$ which is only controlled in a spatial-temporal weak norm, namely $\psi(x,t) \in W^{-1,1}_{x,t}$.

Theorem 2.16. *Consider a weak solution $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, with $\rho(x,t) \in L^\infty_t L^q_x$, and intial data $\rho_0(x) \in L^1_x \cap L^q_x$, to the forced transport equation*

$$\frac{\partial\rho}{\partial t}(x,t) + \nabla\cdot(u(x,t)\rho(x,t)) = \psi(x,t),$$

along a divergence-free $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$*, with* $\nabla u(x,t) \in L^1_t L^p_x$*, for* $\frac{1}{p} + \frac{1}{q} = 1$*, with* $1 < p < \infty$*, and* $u(x,t) \in L^\infty_t L^1_x$*, with distributional force* $\psi \in W^{-1,1}_{x,t}$*. Then for any* $\kappa > 0$ *we have the quantitative bound*

$$\|\rho\|_{L^\infty_t W^{-1,1}_x} \leq \kappa + A \exp\left(\frac{C_p}{\kappa} \|\rho\|_{L^\infty_t L^q_x} \|\nabla u\|_{L^1_t L^p_x}\right) \left(\|\rho_0\|_{L^1_x} + \|\psi\|_{L^1_t L^1_x}\right)^{\frac{d+1}{d+2}}$$
$$\times \left(\|\rho_0\|_{W^{-1,1}_x} + \left(1 + \|u\|_{L^\infty_t L^1_x}\right) \|\psi\|_{W^{-1,1}_{x,t}}\right)^{\frac{1}{d+2}}.$$

in terms of constants $A > 0$ *depending only on dimension* d*, and* $C_p > 0$ *depending only on the paramater* $1 < p < \infty$ *and dimension* d*.*

Proof. Let $\lesssim$ denote less than or equal to up to a constant depending only on the dimension d.

Consider a test function $f(x) \in C^\infty_c(\mathbb{R}^d \times (0,T); \mathbb{R})$, and let $\theta(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ be a distributional solution to the forced *backwards* transport equation

$$\frac{\partial \theta}{\partial t}(x,t) + u(x,t) \cdot \nabla \theta(x,t) = f(x,t),$$
$$\theta(x,T) = 0.$$

That is, for all test functions $\phi(x,t) \in C^\infty_c(\mathbb{R}^d \times (0,T]; \mathbb{R})$,

$$\int_{\mathbb{R}^d \times [0,T]} \theta(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t)\right) dxdt = -\int_{\mathbb{R}^d \times [0,T]} f(x,t)\phi(x,t)\, dxdt.$$

Such a solution exists with $\theta(x,t) \in L^\infty_t L^1_x \cap L^\infty_t L^\infty_x$, $\frac{\partial \theta}{\partial t}(x,t) \in L^1_t W^{-1,1}_x$ by standard theory [9].

Take a mollifier $\varphi(x) \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$ (Definition 2.5), supported on $|x| \leq 1$ then in terms of the DiPerna-Lions commutator $r_\delta(u,\theta)(x,t)$ (Definition 2.6) for all $t \in (0,T)$,

$$\frac{\partial}{\partial t}(\theta * \varphi_\delta)(x,t) + u(x,t) \cdot \nabla(\theta * \varphi_\delta)(x,t) = r_\delta(u,\theta)(x,t) + (f * \varphi_\delta)(x,t), \tag{2.33}$$

where $(\theta * \varphi_\delta)(x,t) \in L^\infty_t W^{1,p}_x$, $\frac{\partial}{\partial t}(\theta * \varphi_\delta)(x,t) \in L^1_t L^p_x$, and so (by a suitable approximation argument) can be taken as a test function for

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = \psi(x,t),$$

that is

$$\int_{\mathbb{R}^d\times[0,T]} \rho(x,t) r_\delta(u,\theta)(x,t)\,dxdt + \int_{\mathbb{R}^d\times[0,T]} \psi(x,t)(\theta*\varphi_\delta)(x,t)\,dxdt$$
$$= -\int_{\mathbb{R}^d\times[0,T]} \rho(x,t)(f*\varphi_\delta)(x,t)\,dxdt - \int_{\mathbb{R}^d} \rho_0(x)(\theta*\varphi_\delta)(x,0)\,dx.$$

Therefore, for any test function $\theta_T(x)\in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,T]} &\rho(x,t)f(x,t)\,dxdt \\ &= \int_{\mathbb{R}^d\times[0,T]} \rho(x,t)(f-f*\varphi_\delta)(x,t)\,dxdt \\ &\quad -\int_{\mathbb{R}^d} \rho_0(x)(\theta*\varphi_\delta)(x,0)\,dx \\ &\quad -\int_{\mathbb{R}^d\times[0,T]} \rho(x,t)r_\delta(u,\theta)(x,t)\,dxdt \\ &\quad -\int_{\mathbb{R}^d\times[0,T]} \psi(x,t)(\theta*\varphi_\delta)(x,t)\,dxdt.\end{aligned} \tag{2.34}$$

Term by term, we have, since by standard energy estimates $\|\rho\|_{L_t^\infty L_x^1} \le \|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}$,

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,T]} &\rho(x,t)(f-f*\varphi_\delta)(x,t)\,dxdt \\ &\lesssim \delta\,\|\rho\|_{L_t^\infty L_x^1}\|\nabla f\|_{L_t^1 L_x^\infty} \\ &\lesssim \delta T\left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}\right)\|\nabla f\|_{L_t^\infty L_x^\infty}.\end{aligned}$$

Using next, again by standard energy estimates, that $\|\theta\|_{L_t^\infty L_x^\infty} \le T\,\|f\|_{L_t^\infty L_x^\infty}$,

$$\begin{aligned}\int_{\mathbb{R}^d} &\rho_0(x)(\theta*\varphi_\delta)(x,0)\,dx \\ &\lesssim \|\rho_0\|_{W_x^{-1,1}}\|(\theta*\varphi_\delta)(\cdot,0)\|_{W_x^{1,\infty}} \\ &\lesssim \|\rho_0\|_{W_x^{-1,1}}\left(\|(\theta*\varphi_\delta)(\cdot,0)\|_{L_x^\infty} + \|\nabla(\theta*\varphi_\delta)(\cdot,0)\|_{L_x^\infty}\right) \\ &\lesssim \|\rho_0\|_{W_x^{-1,1}}\left(1+\frac{1}{\delta}\right)\|\theta\|_{L_t^\infty L_x^\infty} \\ &\lesssim \|\rho_0\|_{W_x^{-1,1}}\left(1+\frac{1}{\delta}\right)T\,\|f\|_{L_t^\infty L_x^\infty}.\end{aligned}$$

By Theorem 2.11, for $0 < \delta_1 < \delta_2$,

$$\begin{aligned}
&\int_{\delta_1}^{\delta_2} \int_{\mathbb{R}^d \times [0,T]} \rho(x,t) r_\delta(u,\theta)(x,t)\, dxdt\, \frac{d\delta}{\delta} \\
&\quad \leq \|\rho\|_{L_t^\infty L_x^q} \left\| \int_{\delta_1}^{\delta_2} r_\delta(u,\theta)(x,t)\, \frac{d\delta}{\delta} \right\|_{L_t^1 L_x^p} \\
&\quad \leq C_p \|\rho\|_{L_t^\infty L_x^q} \|\nabla u\|_{L_t^1 L_x^p} \|\theta\|_{L_t^\infty L_x^\infty} \\
&\quad \leq T C_p \|\rho\|_{L_t^\infty L_x^q} \|\nabla u\|_{L_t^1 L_x^p} \|f\|_{L_t^\infty L_x^\infty}
\end{aligned}$$

for some $C_p > 0$ depending only on the parameter $1 < p < \infty$, and the dimension $d \geq 2$. Finally, using that $\frac{\partial \theta}{\partial t}(x,t) = -\nabla \cdot (u\theta)(x,t) + f(x,t)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d \times [0,T]} \psi(x,t)(\theta * \varphi_\delta)(x,t)\, dxdt \\
&\quad \lesssim \|\psi\|_{W_{x,t}^{-1,1}} \|\theta * \varphi_\delta\|_{W_{x,t}^{1,\infty}} \\
&\quad \lesssim \|\psi\|_{W_{x,t}^{-1,1}} \left(\|\theta * \varphi_\delta\|_{L_t^\infty L_x^\infty} + \|\nabla(\theta * \varphi_\delta)\|_{L_t^\infty L_x^\infty} + \left\| \frac{\partial \theta}{\partial t} * \varphi_\delta \right\|_{L_t^\infty L_x^\infty} \right) \\
&\quad \lesssim \|\psi\|_{W_{x,t}^{-1,1}} \left(1 + \frac{1}{\delta}\right) \|\theta\|_{L_t^\infty L_x^\infty} + \|\psi\|_{W_{x,t}^{-1,1}} \|\nabla \cdot (u\theta) * \varphi_\delta\|_{L_t^\infty L_x^\infty} + \|\psi\|_{W_{x,t}^{-1,1}} \|f * \varphi_\delta\|_{L_t^\infty L_x^\infty} \\
&\quad \lesssim \|\psi\|_{W_{x,t}^{-1,1}} \left(1 + \frac{1}{\delta}\right) \|\theta\|_{L_t^\infty L_x^\infty} + \frac{1}{\delta^{d+1}} \|u\|_{L_t^\infty L_x^1} \|\theta\|_{L_t^\infty L_x^\infty} + \|\psi\|_{W_{x,t}^{-1,1}} \|f\|_{L_t^\infty L_x^\infty} \\
&\quad \lesssim \|\psi\|_{W_{x,t}^{-1,1}} \left(T \left(1 + \frac{1}{\delta} + \frac{1}{\delta^{d+1}} \|u\|_{L_t^\infty L_x^1} \right) + 1 \right) \|f\|_{L_t^\infty L_x^\infty}.
\end{aligned}$$

Integrating (2.34) over $\delta \in [\delta_1, \delta_2]$ with respect to the measure $\frac{d\delta}{\delta}$ then gives, for some constant $A > 0$ depending only on the dimension d,

$$\begin{aligned}
&\log\left(\frac{\delta_2}{\delta_1}\right) \int_{\mathbb{R}^d \times [0,T]} \rho(x,t) f(x,t)\, dxdt \\
&\quad \leq AT \int_{\delta_1}^{\delta_2} \delta \left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1} \right) \frac{d\delta}{\delta} \|\nabla f\|_{L_t^\infty L_x^\infty} \\
&\qquad + AT \int_{\delta_1}^{\delta_2} \left(1 + \frac{1}{\delta}\right) \left(\|\rho_0\|_{W_x^{-1,1}} + \|\psi\|_{W_{x,t}^{-1,1}} \right) \frac{d\delta}{\delta} \|f\|_{L_t^\infty L_x^\infty} \\
&\qquad + AT \int_{\delta_1}^{\delta_2} \frac{1}{\delta^{d+1}} \|u\|_{L_t^\infty L_x^1} \|\psi\|_{W_{x,t}^{-1,1}} \frac{d\delta}{\delta} \|f\|_{L_t^\infty L_x^\infty} \\
&\qquad + A \log\left(\frac{\delta_2}{\delta_1}\right) \|\psi\|_{W_{x,t}^{-1,1}} \|f\|_{L_t^\infty L_x^\infty} \\
&\qquad + C_p \|\rho\|_{L_t^\infty L_x^q} \|\nabla u\|_{L_t^1 L_x^p} \|\theta_T\|_{L_x^\infty},
\end{aligned}$$

and so, since $f(x,t) \in C_c^\infty(\mathbb{R}^d \times (0,T);\mathbb{R})$ is arbitrary,

$$\begin{aligned}
\|\rho\|_{L_t^1 W_x^{-1,1}} \le AT\delta_2 &\left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}\right) \\
&+ AT\left(1+\frac{1}{\delta_1}\right)\left(\|\rho_0\|_{W_x^{-1,1}} + \|\psi\|_{W_{x,t}^{-1,1}}\right) \\
&+ AT\frac{1}{\delta_1^{d+1}}\|u\|_{L_t^\infty L_x^1}\|\psi\|_{W_{x,t}^{-1,1}} \\
&+ A\|\psi\|_{W_{x,t}^{-1,1}} \\
&+ TC_p\left(\log\left(\frac{\delta_2}{\delta_1}\right)\right)^{-1}\|\rho\|_{L_t^\infty L_x^q}\|\nabla u\|_{L_t^1 L_x^p}
\end{aligned}$$

For $\kappa > 0$, let

$$\begin{aligned}
\delta_2 = &\left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}\right)^{-\frac{1}{d+2}}\left(\|\rho_0\|_{W_x^{-1,1}} + \left(1+\|u\|_{L_t^\infty L_x^1}\right)\|\psi\|_{W_{x,t}^{-1,1}}\right)^{\frac{1}{d+2}} \\
&\times \exp\left(\frac{C_p}{2\kappa}\|\rho\|_{L_t^\infty L_x^q}\|\nabla u\|_{L_t^1 L_x^p}\right), \\
\delta_1 = &\left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}\right)^{-\frac{1}{d+2}}\left(\|\rho_0\|_{W_x^{-1,1}} + \left(1+\|u\|_{L_t^\infty L_x^1}\right)\|\psi\|_{W_{x,t}^{-1,1}}\right)^{\frac{1}{d+2}} \\
&\times \exp\left(-\frac{C_p}{2\kappa}\|\rho\|_{L_t^\infty L_x^q}\|\nabla u\|_{L_t^1 L_x^p}\right),
\end{aligned}$$

Then, noting also that

$$\|\rho_0\|_{W_x^{-1,1}} + \|\psi\|_{W_{x,t}^{-1,1}} \le \|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1},$$

it follows that

$$\begin{aligned}
\|\rho\|_{L_t^1 W_x^{-1,1}} \le \kappa T + 4AT\exp&\left((d+1)\frac{C_p}{2\kappa}\|\rho\|_{L_t^\infty L_x^q}\|\nabla u\|_{L_t^1 L_x^p}\right)\left(\|\rho_0\|_{L_x^1} + \|\psi\|_{L_t^1 L_x^1}\right)^{\frac{d+1}{d+2}} \\
&\times\left(\|\rho_0\|_{W_x^{-1,1}} + \left(1+\|u\|_{L_t^\infty L_x^1}\right)\|\psi\|_{W_{x,t}^{-1,1}}\right)^{\frac{1}{d+2}} + \|\psi\|_{W_{x,t}^{-1,1}}.
\end{aligned}$$

We conclude the statement of the theorem by taking an arbitrary $T > 0$. □

2.3 Quantitative Estimates for Vector Fields in $L_t^1 BV_x$

One of the significant limitations of the previous analysis is the failure of any of the bounds to extend to $\nabla u(x,t) \in L_t^1 L_x^1$. This is a well-known issue in similar previous work due to the failure of particular harmonic analysis estimates in L^1_x [34], [95], and the same issue is faced in the proof of Theorem 2.11. However, our original argument, mainly the bound of Theorem 2.8, does not involve harmonic analysis and instead uses weak compactness of L^p_x to obtain uniform estimates. The issue with the case $p = 1$ is that L^1_x is not weakly compact. The dual of C^0_b is instead the Banach space of signed Radon measure $\mathscr{M}(\mathbb{R}^d)$ which contains L^1_x. To obtain quantitative well-posedness results for $\nabla u(x,t) \in L^1_t L^1_x$ we would therefore first require a well-posedness theory when $\nabla u(x,t) \in L^1([0,T];\mathscr{M}(\mathbb{R}^d))$, i.e. for $u(x,t) \in L^1_t BV_x$, which is precisely the theorem of Ambrosio, Theorem 2.4. This section aims to quantify this and use the weak compactness arguments as before to obtain the first weak stability and mixing estimates for transport along BV_x vector fields.

2.3.1 The Non-quantitative Weak Compactness Argument

We present in this section the analogous argument to Proposition 2.7 in the Ambrosio well-posedness theory. This will be slightly more involved due to the nature of the compactness; in particular, we shall have to employ the Aubin-Lions compactness lemma.

Proposition 2.17. *Let $\rho(x,t) \in L^\infty_t L^\infty_x$ be a weak solution to the transport equation* (TE) *along some divergence-free $\nabla u(x,t) \in L^1_t L^1_x$, and initial datum $\rho_0(x,t) \in L^\infty_x$.*

Then for any $\varepsilon > 0$, there exists $\delta\left(\varepsilon, \|\rho_0\|_{L^\infty_x}, \|\nabla u\|_{L^1_t L^1_x}\right) > 0$ depending only on $\varepsilon > 0$, $\|\rho_0\|_{L^\infty_x}$, $\|\nabla u\|_{L^1_t L^1_x}$, and the dimension $d \geq 2$ such that if

$$\|\rho_0\|_{W^{-1,1}_x} \leq \delta\left(\varepsilon, \|\rho_0\|_{L^\infty_x}, \|\nabla u\|_{L^1_t L^1_x}\right),$$

then also

$$\|\rho(\cdot,T)\|_{W^{-1,1}_x(B;\mathbb{R})} \leq \varepsilon,$$

where $B = \{x \in \mathbb{R}^d : |x| \leq 1\} \subset \mathbb{R}^d$ is a fixed bounded set.

Proof. To simplify the proof, we first show that it is sufficient to assume $\nabla u(x,t) \in L^\infty_t L^1_x$. To see that this is equivalent, consider the continuous, invertible, time rescaling $s(t) : [0,T] \to [0,1]$ for some $\alpha > 0$ given by

$$\frac{ds}{dt} = \frac{\|\nabla u(\cdot,t)\|_{L^1_x} + \frac{\alpha}{T}}{\|\nabla u\|_{L^1_t L^1_x} + \alpha}.$$

Define now

$$\bar{\rho}(x,s(t)) = \rho(x,t),$$
$$\bar{u}(x,s(t)) = \frac{dt}{ds}u(x,t),$$

where we notice $\|\nabla \bar{u}\|_{L^\infty_s L^1_x} \le \|\nabla u\|_{L^1_t L^1_x} + \alpha$.

We claim that $\bar{\rho}(x,s) \in L^\infty_t L^\infty_x$ (with the same norm as before) solves (TE) along $\bar{u}(x,s)$ on the time interval $[0,1]$, with the same initial datum $\rho_0(x) \in L^\infty_x$. To this end, take a test function $\bar{\phi}(x,s) \in C^\infty_c(\mathbb{R}^d \times [0,1);\mathbb{R})$. By chain rule, substitution, and that $\frac{ds}{dt} \in L^1([0,T];\mathbb{R})$, $\frac{dt}{ds} \in L^\infty([0,1];\mathbb{R})$ have the required integrability, we have for $\phi(x,t) = \bar{\phi}(x,s(t))$, with the same trace $\phi(x,0) = \bar{\phi}(x,0)$,

$$\int_{\mathbb{R}^d \times [0,1)} \bar{\rho}(x,s) \left(\frac{\partial \bar{\phi}}{\partial s}(x,s) + \bar{u}(x,s) \cdot \nabla \bar{\phi}(x,s) \right) dxds$$
$$= \int_{\mathbb{R}^d \times [0,T)} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right) dxdt.$$

Now $\phi(x,t)$ is not in $C^\infty_c(\mathbb{R}^d \times [0,T);\mathbb{R})$, but since $\frac{ds}{dt} \in L^1([0,T];\mathbb{R})$ we do have $\phi(x,t) \in W^{1,1}_t C^1_x \subset C^0_t C^1_x$ with trace $\phi(\cdot,T) = 0$. Therefore, we may approximate $\phi(x,t)$ by a sequence of true test functions $\phi_n(x,t) \in C^\infty_c(\mathbb{R}^d \times [0,T);\mathbb{R})$ such that $\nabla \phi_n \xrightarrow{n\to\infty} \nabla \phi$ converges in $L^\infty_t L^\infty_x$, $\frac{\partial \phi_n}{\partial t} \xrightarrow{n\to\infty} \frac{\partial \phi}{\partial t}$ in $L^1_t L^\infty_x$, and $\phi_n(\cdot,0) \xrightarrow{n\to\infty} \phi(\cdot,0)$ in L^∞_x. By testing (TE) with $\phi_n(x,t)$, we then deduce that $\bar{\rho}(x,s)$ solves (TE) along $\bar{u}(x,s)$ on the time interval $s \in [0,1]$, with the same initial datum $\rho_0(x)$.

By this equivalence, it is enough to show the result for a function $\delta\left(\varepsilon, \|\rho_0\|_{L^\infty_x}, \|\nabla u\|_{L^\infty_t L^1_x}\right)$ instead of $\delta\left(\varepsilon, \|\rho_0\|_{L^\infty_x}, \|\nabla u\|_{L^1_t L^1_x}\right)$. That is, assume $\nabla u \in L^\infty_t L^1_x$.

For the sake of contradiction suppose that there exists some $M \ge 0$, $\varepsilon > 0$, and a sequence

$$\left\{\rho_0^{(n)}\right\}_{n\in\mathbb{N}} \subset L^\infty_x,$$
$$\{\nabla u_n\}_{n\in\mathbb{N}} \subset L^\infty_t L^1_x,$$

such that, denoting by $\rho_n(x,t) \in L_t^\infty L_x^\infty$ the unique bounded solution to (TE) along $u_n(x,t)$ with initial datum $\rho_0^{(n)}(x)$, we have

$$\begin{aligned} \left\|\rho_0^{(n)}\right\|_{L_x^\infty} &\leq M \text{ for all } n \in \mathbb{N}, \\ \|\nabla u_n\|_{L_t^\infty L_x^1} &\leq M \text{ for all } n \in \mathbb{N}, \\ \left\|\rho_0^{(n)}\right\|_{W_x^{-1,1}} &\xrightarrow{n\to\infty} 0, \qquad (2.35) \\ \|\rho_n(\cdot,T)\|_{W_x^{-1,1}(B;\mathbb{R})} &> \varepsilon. \qquad (2.36) \end{aligned}$$

Notice that $\nabla u_n(x,t) \in L_t^\infty L_x^1$, and so $u_n(x,t) \in L_t^\infty L_x^{\frac{d}{d-1}}$ are uniformly bounded, and $\rho_n(x,t) \in L_t^\infty L_x^\infty$ are uniformly bounded, so also $\frac{\partial \rho_n}{\partial t}(x,t) \in L_t^\infty W_x^{-1,\frac{d}{d-1}}$ are uniformly bounded by (TE); $\frac{\partial \rho_n}{\partial t}(x,t) = -\nabla \cdot (u_n(x,t)\rho_n(x,t))$. The goal is then to apply the Aubin-Lions compactness lemma [101], to show that $\rho_n(x,t)$ is compact in $L_t^\infty C_x^{-1}$ where C_x^{-1} is the dual of BV_x. However, C_x^{-1} is not strictly speaking contained in $W_x^{-1,\frac{d}{d-1}}$. Instead, for any $R > 0$, consider a cutoff $\chi_R(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ which is 1 on $|x| \leq R$ and zero on $|x| \geq R+1$. Then $(\chi_R\rho_n)(x,t) \in L_t^\infty L_x^\infty$, $\frac{\partial \chi_R\rho_n}{\partial t}(x,t) \in L_t^\infty W_x^{-1,\frac{d}{d-1}}$ are uniformly bounded (with dependence on the fixed choice of $\chi_R(x)$). Working on the compact domain $\{x \in \mathbb{R}^d : |x| \leq R+1\}$, then we have the continuous embeddings

$$L_x^\infty \subset C_x^{-1} \subset W^{-1,\frac{d}{d-1}},$$

where C_x^{-1} is the Banach space dual of BV_x, and in particular the embedding $L_x^\infty \subset C_x^{-1}$ is compact. Then by the Aubin-Lions compactness lemma, $(\chi_R\rho_n)(x,t)$ are compactly embedded in $C_t^0 C_x^{-1}$.

Therefore, we see that $\rho_n(x,t)$ has a subsequence converging strongly locally in $C_t^0 C_x^{-1}$, with the limit also in $L_t^\infty L_x^\infty$. Without loss of generality, let this subsequence be the original sequence $\rho_n(x,t)$. Denote the limit by $\bar{\rho}(x,t) \in C_t^0 C_{\text{loc}}^{-1} \cap L_t^\infty L_x^\infty$. Likewise, taking a further subsequence if necessary, we see that $\nabla u_n(x,t) \in L_t^\infty L_x^1$ converges weakly-$*$ in $L_t^\infty \mathscr{M}_x$ to some limit $\nabla \bar{u}(x,t) \in L_t^\infty \mathscr{M}_x$, where $\mathscr{M}_x$ denotes the space of signed Radon measures. Therefore, $u_n(x,t)$ converges weakly locally in $L_t^\infty BV_x$ to $\bar{u}(x,t) \in L_t^\infty BV_{\text{loc}}$.

Since C_x^{-1} is dual BV_x, it then follows that also $u_n(x,t)\rho_n(x,t) \xrightarrow{n\to\infty} \bar{u}(x,t)\bar{\rho}(x,t)$ converges in distribution (i.e. against test functions with *compact* support) to $\bar{u}(x,t)\bar{\rho}(x,t) \in L_t^\infty L_x^1$. Note also that $\rho_0^{(n)}(x) \xrightarrow{n\to\infty} 0$ in $W_x^{-1,1}$ by (2.35).

Then, by Definition 2.1, for any test function $\phi(x,t) \in C_c^\infty(\mathbb{R}^d \times [0,T);\mathbb{R})$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d\times[0,T)} \bar{\rho}(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t)\cdot \nabla \phi(x,t) \right) dxdt \\
&\quad = \lim_{n\to\infty} \int_{\mathbb{R}^d\times[0,T)} \rho_n(x,t) \frac{\partial \phi}{\partial t}(x,t) + \rho_n(x,t) u_n(x,t) \cdot \nabla \phi(x,t)\, dxdt \\
&\quad = -\lim_{n\to\infty} \int_{\mathbb{R}^d} \rho_0^{(n)}(x)\phi(x,0)\, dx, \\
&\quad = 0,
\end{aligned}$$

and so $\bar{\rho}(x,t) \in L_t^\infty L_x^\infty$ is the unique (by Theorem 2.4) bounded solution to (TE) along $\bar{u}(x,t) \in L_t^\infty BV_x$ on the time interval $t \in [0,1]$ with zero initial datum, that is $\bar{\rho}(x,t) = 0$ and in particular $\bar{\rho}(x,T) = 0$.

However, since $\rho_n(\cdot,T)$ converges to $\bar{\rho}(\cdot,T)$ strongly locally in $W_x^{-1,1}$ (since $\rho_n(x,t)$ converges strongly locally in $C_t^0 C_x^{-1}$), and $\|\rho_n(\cdot,T)\|_{W_x^{-1,1}(B;\mathbb{R})} > \varepsilon > 0$ by (2.36), then also

$$\|\bar{\rho}(\cdot,T)\|_{W_x^{-1,1}(B;\mathbb{R})} \geq \varepsilon > 0,$$

a contradiction as required. □

The goal of the remainder of Section 2.3 is to quantify $\delta\left(\varepsilon, \|\rho_0\|_{L_x^\infty}, \|\nabla u\|_{L_t^\infty L_x^1}\right)$. We aim to give a first such bound. To aid the proof we quantify a mildly weaker result of the same nature, see Theorem 2.22.

2.3.2 Quantifying the Ambrosio Theory

There are various steps to quantifying the proof of Proposition 2.17. The most important of these is to quantify the argument of Ambrosio that allows us to claim that $\bar{\rho}(x,t) = 0$ is the zero solution if the initial datum is the zero initial datum.

As for the DiPerna-Lions theory, the proof of well-posedness along $L_t^1 BV_x$ vector fields proceeds by showing the decay of a similar commutator [8]. However, the key difference is that the appropriate mollifier $\varphi_\delta(x)$ of the passive scalar must depend an-isotropically on the matrix $\frac{\nabla u(x,t)}{|\nabla u(x,t)|}$ for decay to occur. As such, the mollifier must depend on space and time in a particular way. To start, fix a spatial mollifier $\varphi(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, and a space-time mollifier $\bar{\varphi}(x,t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$. Fix a matrix $M \in \mathbb{R}^{d\times d}$ with trace$(M) = 0$, and l^2-norm $|M| \leq 1$. We denote the linear map by $(M\cdot x)_i = \sum_{j=1}^d M_{i,j} x_j$. Following the approach in [9], we define the following an-isotropic space-time dependent mollifier, with an-isotropic parameter $\Lambda > 0$,

$$\varphi^{\Lambda,M}(x) = \frac{1}{\Lambda}\int_0^{\Lambda} \varphi\left(\exp(-\lambda M^{\dagger})\cdot(x-y)\right)\,d\lambda,$$

where $M^{\dagger}$ is the transpose of M,

$$M^{\dagger}_{i,j} = M_{j,i},$$

and $\exp(-\lambda M^{\dagger})$ refers to matrix exponentiation

$$\exp(-\lambda M^{\dagger}) = \sum_{i=0}^{\infty}\frac{(-\lambda)^i}{i!}(M^{\dagger})^i.$$

The key observation is that for any $(\bar{x},\bar{t}) \in \mathbb{R}^d \times [0,T]$ if $M = \frac{\nabla u(\bar{x},\bar{t})}{|\nabla u(\bar{x},\bar{t})|}$, then the usual commutator

$$r\left(u,\rho;\varphi_{\delta}^{\Lambda,M}\right)(x,t) = \int_{\mathbb{R}^d}\rho(y,t)(u(x,t)-u(y,t))\cdot\nabla\varphi_{\delta}^{M,\Lambda}(x-y)\,dy,$$

is small *near* $(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}$ as $\Lambda \to \infty$, largely of $\delta > 0$.

Using this idea, now let $\rho(x,t) \in L^{\infty}_t L^{\infty}_x$ be a solution of (TE) with force $\psi(x,t) \in L^1_t L^1_x$, and with a bound $u(x,t) \in L^1_t BV_x$,

$$\frac{\partial\rho}{\partial t}(x,t) + u(x,t)\cdot\nabla\rho(x,t) = \psi(x,t).$$

Define the unit space-time dependent matrix $M(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times d}$ by

$$M_{i,j}(x,t) = \frac{\frac{\partial u_j}{\partial x_i}(x,t)}{|\nabla u(x,t)|}, \tag{2.37}$$

where $\frac{\nabla u(x,t)}{|\nabla u(x,t)|}$ is the Radon-Nikodym derivative [74, Theorem 4.1]. Let $\bar{M}(\bar{x},\bar{t}) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d\times d}$ be any matrix field, with $\mathrm{trace}(\bar{M}(\bar{x},\bar{t})) = 0$ and l^2-norm $|\bar{M}(\bar{x},\bar{t})| \leq 1$. We show the following quantitative bound on the transport of $\rho * \varphi_{\delta}^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x,t)$. Recall that $\bar{\varphi}(x,t) \in C_c^{\infty}(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$ is a standard space-time mollifier.

The following is a carefully quantified version of the Ambrosio well-posedness proof [8] with special attention paid to the presence of a force term which is small in some weak norm. This is to later complete the weak compactness argument. This requires control by additional mollifiers $\varphi_{\delta_1}(x) \in C_c^{\infty}(\mathbb{R}^d;\mathbb{R})$, and $\varphi'_{\tau}(t) \in C_c^{\infty}(\mathbb{R};\mathbb{R})$ on the passive scalar.

Meanwhile the mollifier $\bar{\varphi}_{\varepsilon}(x,t) \in C_c^{\infty}(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$ ensures the smoothness of the required space-time dependence of the an-isotropic mollifer $\varphi_{\delta}^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x) \in C_c^{\infty}(\mathbb{R}^d;\mathbb{R})$.

Theorem 2.18. *Let $\rho(x,t) \in L_t^\infty L_x^\infty$ solve the forced transport equation*

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = \psi(x,t),$$

for some divergence-free $\nabla u(x,t) \in L_t^\infty L_x^1$, and weak force $\psi(x,t) \in L_t^1 W_x^{-1,1}$. Denote by $\rho_T(x) = \rho(x,T)$, the trace.

Then we have the following bound, for any choice of parameters $\delta, \delta_1, \delta_2, \tau, \tau_2, \varepsilon > 0$, and $\Lambda > 0$. For any choice of mollifiers $\varphi(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, $\varphi'(t) \in C_c^\infty(\mathbb{R};\mathbb{R})$, and a symmetric mollifier $\bar{\varphi}(x,t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$. For any choice of matrix field $\bar{M}(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d\times d}$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(\rho_T * \varphi_{\delta_1})(x)|^2 \, dx \\
&\lesssim \int_{\mathbb{R}^d} |(\rho_0 * \varphi_{\delta_1})(x)|^2 \, dx \\
&\quad + \|\rho\|_{L_t^\infty L_x^\infty} \left(\tau' \|\nabla u\|_{L_t^\infty L_x^1} \|\rho\|_{L_t^\infty L_x^d} + \|\psi\|_{L_t^1 W_x^{-1,1}} \right) \frac{1}{\delta^{d+1}} \left(\delta + e^\Lambda \right) \\
&\quad + e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^2} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho_0 * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right) - \rho_0 * \varphi_{\delta_1} \right| \right\|_{L_x^2} \\
&\quad + e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^2} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho_T * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right) - \rho_T * \varphi_{\delta_1} \right| \right\|_{L_x^2} \\
&\quad + \frac{e^{(d+1)\Lambda}}{\delta^{d+1}} (1+\delta) \|\rho\|_{L_t^\infty L_x^\infty} \|\psi\|_{L_t^1 W_x^{-1,1}} \\
&\quad + \delta_1 e^{(d+2)\Lambda} \left(\frac{1}{\delta\tau} \|\rho\|_{L_t^1 L_x^1} + \frac{1}{\delta} \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1} + \frac{1}{\delta^2} \|\rho\|_{L_t^\infty L_x^d} \|\nabla u\|_{L_t^1 L_x^1} \right) \\
&\quad + \delta_2 e^{(d+1)\Lambda} \frac{1}{\delta} \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1} \\
&\quad + \tau_2 e^{(d+1)\Lambda} \left(\frac{1+\delta_1}{\delta_1^d} \frac{1}{\delta} \left(\|\psi\|_{L_t^1 W_x^{-1,1}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1} \right) \|\nabla u\|_{L_t^\infty L_x^1} + \frac{1}{\delta\tau} \|\rho\|_{L_t^\infty L_x^d} \|\nabla u\|_{L_t^1 L_x^1} \right) \\
&\quad + e^{2\Lambda} \|\rho\|_{L_t^\infty L_x^\infty}^2 \int_{\mathbb{R}^d\times[0,T]} \varepsilon^{-d-1} \int_{\substack{|x-\bar{x}|\le\varepsilon R+\delta e^\Lambda R \\ |t-\bar{t}|\le\varepsilon R+\tau R}} \left| \frac{\nabla u * \varphi_{\delta_2,\tau_2}}{|\nabla u * \varphi_{\delta_2,\tau_2}|}(x,t) - \bar{M}(\bar{x},\bar{t}) \right| d\bar{x} d\bar{t} \, |(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)| dx dt \\
&\quad + \frac{1}{\Lambda} \left(1 + \frac{\delta e^\Lambda}{\varepsilon} \right)^d \left(1 + \frac{\tau}{\varepsilon} \right) \|\rho\|_{L_t^\infty L_x^\infty}^2 \|\nabla u\|_{L_t^1 L_x^1} \\
&\quad + \frac{\tau e^{(d+3)\Lambda}}{\delta^d} \left(1 + \frac{1}{\delta} \right) \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \left(\|\psi\|_{L_t^1 W_x^{-1,1}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1} \right) \\
&\quad + \frac{1}{\varepsilon} \left(e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d},
\end{aligned}$$

where $\tau' = \tau + \tau_2$, and $R > 0$ is a constant depending only on the choice of mollifiers.

In the definition of $(u * \varphi_{\delta_2,\tau_2})(x,t)$ for $t \in [0,T]$, we extended $u(x,t)$ first by zero outside of $t \in [0,T]$.

Proof. We denote throughout by $\lesssim$ less than or equal to up to a constant depending only on the dimension d, and the choice of mollifiers $\varphi(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, $\varphi'(t) \in C_c^\infty(\mathbb{R};\mathbb{R})$, and $\bar{\varphi}(x,t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$, where we additionally require that $\bar{\varphi}(x,t)$ is symmetric, that is

$$\bar{\varphi}(x,t) = \bar{\varphi}(-x,-t). \tag{2.38}$$

Denote by $R > 0$ a constant such that both $\varphi(x)$ is supported on $|x| \le R$, and $\bar{\varphi}(x,t)$ is supported on $|x|,|t| \le R$.

Recall the definition of $\varphi^{\Lambda,\bar{M}}(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$,

$$\varphi^{\Lambda,\bar{M}}(x) = \frac{1}{\Lambda}\int_0^\Lambda \varphi\left(\exp(-\lambda \bar{M}^\dagger)\cdot x\right)\,d\lambda, \tag{2.39}$$

noting that since $\operatorname{trace}(\bar{M}) = 0$, then $\exp(-\lambda \bar{M}^\dagger)$ has unit determinant and so does its inverse, and so $\varphi^{\Lambda,\bar{M}}(x)$ is a standard mollifier. Since $|\bar{M}| \le 1$, $\varphi^{\Lambda,\bar{M}}(x)$ is supported on $|x| \le e^\Lambda R$. Moreover, we have the uniform bounds

$$\left\|\varphi^{\Lambda,\bar{M}}\right\|_{L_x^1} \lesssim 1, \tag{2.40}$$

$$\left\|\nabla\varphi^{\Lambda,\bar{M}}\right\|_{L_x^1} \lesssim e^\Lambda, \tag{2.41}$$

independent of $\bar{M}$. We will also need pointwise estimates on $\varphi_\delta^{\Lambda,M}(x) = \frac{1}{\delta^d}\varphi^{\Lambda,M}\left(\frac{x}{\delta}\right)$. In particular,

$$\left|\varphi_\delta^{\Lambda,M}(x)\right| \lesssim \delta^{-d} 1_{|x|\le\delta e^\Lambda R}, \tag{2.42}$$

$$\left|\nabla\varphi_\delta^{\Lambda,M}(x)\right| \lesssim \delta^{-d-1} e^\Lambda 1_{|x|\le\delta e^\Lambda R}, \tag{2.43}$$

$$\left|\nabla^2\varphi_\delta^{\Lambda,M}(x)\right| \lesssim \delta^{-d-2} e^{2\Lambda} 1_{|x|\le\delta e^\Lambda R}. \tag{2.44}$$

We start by writing, for each $(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}$,

$$\begin{aligned}\frac{\partial}{\partial t}&\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t) + u(x,t)\cdot\nabla\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t) \\ &= \left(\psi * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t) + r\left(u,\rho;\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t),\end{aligned} \tag{2.45}$$

where the commutator is given by

$$
\begin{aligned}
r(u,\rho;\varphi_\delta)(x,t) &= \int_{\mathbb{R}^d} \rho(y,t)(u(x,t)-u(y,t))\cdot\nabla\varphi_\delta(x-y)\,dy, \\
&= \int_{\mathbb{R}^d} \rho(x-\delta h,t)\left(\frac{u(x,t)-u(x-\delta h,t)}{\delta}\right)\cdot\nabla\varphi(h)\,dh, \\
&= \int_0^1\int_{\mathbb{R}^d} \rho(x-\delta h,t)(h\cdot\nabla u(x-s\delta h,t)\cdot\nabla\varphi(h))\,dh\,ds.
\end{aligned}
\tag{2.46}
$$

Then, for the an-isotropic mollifier given by (2.39), we have the expression

$$
\begin{aligned}
h\cdot\bar{M}(\bar{x},\bar{t})\cdot\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}(h) &= \frac{1}{\Lambda}\int_0^\Lambda -\frac{\partial}{\partial\lambda}\varphi\left(\exp(-\lambda\bar{M}(\bar{x},\bar{t})^\dagger)\cdot h\right)\,d\lambda, \\
&= \frac{1}{\Lambda}\left(\varphi(h)-\varphi\left(\exp(-\Lambda\bar{M}(\bar{x},\bar{t})^\dagger)\cdot h\right)\right).
\end{aligned}
\tag{2.47}
$$

Let $\tau'\ge\tau>0$, then by (2.45) we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,t)\ \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt \\
&\quad= \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\frac{\partial}{\partial t}\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t) \\
&\qquad\quad\times\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt \\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,t)\frac{\partial}{\partial t}\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt \\
&\quad= \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\left(\psi*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t) \\
&\qquad\quad\times\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt \\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t) \\
&\qquad\quad\times\left(r\left(u,\rho;\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)*\varphi'_\tau\right)(x,t)\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt \\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t) \\
&\qquad\quad\times\left(\left(-u\cdot\nabla\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)\right)*\varphi'_\tau\right)(x,t)\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt \\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,t)\frac{\partial}{\partial t}\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt,
\end{aligned}
$$

which gives us the following four terms.

$$\begin{aligned}
&\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,t)\ \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\ d\bar{x}d\bar{t}\ dxdt \qquad (2.48)\\
&\quad=\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\left(\psi*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\\
&\qquad\quad\times\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\ d\bar{x}d\bar{t}\ dxdt\\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\\
&\qquad\quad\times\left(r\left(u,\rho;\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)*\varphi'_\tau\right)(x,t)\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\ d\bar{x}d\bar{t}\ dxdt\\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\left(\left(u\cdot\nabla\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)^2\right)*\varphi'_\tau\right)(x,t)\right.\\
&\qquad\quad\left.-2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\left(\left(u\cdot\nabla\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)\right)*\varphi'_\tau\right)(x,t)\right)\\
&\qquad\quad\times\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\ d\bar{x}d\bar{t}\ dxdt\\
&\qquad+\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,t)\frac{\partial}{\partial t}\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\\
&\qquad\quad+\left(\left(u\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)^2\right)*\varphi'_\tau\right)(x,t)\cdot\nabla\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\ d\bar{x}d\bar{t}\ dxdt.
\end{aligned}$$

Using (2.40) we first bound

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\int_{\mathbb{R}^d\times\mathbb{R}}2\left(\rho*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\left(\psi*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\right. \qquad (2.49)\\
&\qquad\left.\times\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\ d\bar{x}d\bar{t}\ dxdt\right|\\
&\qquad\lesssim\|\rho\|_{L_t^\infty L_x^\infty}\int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]}\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left(\left|\left(\psi*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)(x,t)\right|\right)\ dxdt.
\end{aligned}$$

Now define $M:\mathbb{R}^d\times[0,T]\to\mathbb{R}^{d\times d}$ by

$$M_{i,j}(x,t)=\frac{\frac{\partial u_j}{\partial x_i}(x,t)}{|\nabla u(x,t)|}, \qquad (2.50)$$

Then using (2.46), and (2.47) we bound

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x,t)\left(r\left(u,\rho;\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right) * \varphi'_\tau\right)(x,t)\right.\\
&\qquad \left.\times\, \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x}d\bar{t}\, dxdt \right|\\
&\qquad = \left| \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x,t)\left(\int_0^T \int_0^1 \int_{\mathbb{R}^d}\right.\right.\\
&\qquad\qquad \rho(x-\delta h,t')(h\cdot M(x-s\delta h,t')\cdot\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}(h))|\nabla u|(x-s\delta h,t')\, dh\, ds\\
&\qquad\qquad \left.\left.\times\, \varphi'_\tau(t-t')\, dt'\right)\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x}d\bar{t}\, dxdt\right|\\
&\qquad \lesssim \|\rho\|^2_{L_t^\infty L_x^\infty}\|\bar{\varphi}_\varepsilon\|_{L_t^\infty L_x^\infty}\|\varphi'_\tau\|_{L_t^1}\int_{\mathbb{R}^d\times[0,T]}\int_{\substack{|x-\bar{x}|\le\varepsilon R\\ |t'-\bar{t}|\le\varepsilon R+\tau R}}\int_0^1\int_{\mathbb{R}^d}\\
&\qquad\qquad \left|h\cdot M(x-s\delta h,t')\cdot\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}(h)\right||\nabla u(x-s\delta h,t')|\, dh\, ds\, d\bar{x}d\bar{t}\, dxdt'\\
&\qquad \lesssim \|\rho\|^2_{L_t^\infty L_x^\infty}\varepsilon^{-d-1}\int_{\mathbb{R}^d\times[0,T]}\int_{\substack{|x'-\bar{x}|\le\varepsilon R+\delta e^\Lambda R\\ |t'-\bar{t}|\le\varepsilon R+\tau R}}\int_{\mathbb{R}^d}\left|h\cdot M(x',t')\cdot\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}(h)\right|\\
&\qquad\qquad \times|\nabla u(x',t')|\, dh\, d\bar{x}d\bar{t}\, dx'dt'\\
&\qquad \lesssim \|\rho\|^2_{L_t^\infty L_x^\infty}\varepsilon^{-d-1}\int_{\mathbb{R}^d\times[0,T]}\int_{\substack{|x-\bar{x}|\le\varepsilon R+\delta e^\Lambda R\\ |t-\bar{t}|\le\varepsilon R+\tau R}}\int_{\mathbb{R}^d}\Big(|h|\,|M(x,t)-\bar{M}(\bar{x},\bar{t})|\\
&\qquad\qquad \times\left|\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}(h)\right|+\left|h\cdot\bar{M}(\bar{x},\bar{t})\cdot\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}(h)\right|\Big)\, dh\, d\bar{x}d\bar{t}\,|\nabla u(x,t)|dxdt\\
&\qquad \lesssim \|\rho\|^2_{L_t^\infty L_x^\infty}\varepsilon^{-d-1}\int_{\mathbb{R}^d\times[0,T]}\int_{\substack{|x-\bar{x}|\le\varepsilon R+\delta e^\Lambda R\\ |t-\bar{t}|\le\varepsilon R+\tau R}}\\
&\qquad\qquad \left(e^\Lambda R\,|M(x,t)-\bar{M}(\bar{x},\bar{t})|\left\|\nabla\varphi^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right\|_{L_x^1}+\frac{1}{\Lambda}\right)d\bar{x}d\bar{t}\,|\nabla u(x,t)|dxdt\\
&\qquad \lesssim e^{2\Lambda}\|\rho\|^2_{L_t^\infty L_x^\infty}\int_{\mathbb{R}^d\times[0,T]}\varepsilon^{-d-1}\int_{\substack{|x-\bar{x}|\le\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)\varepsilon R\\ |t-\bar{t}|\le\left(1+\frac{\tau}{\varepsilon}\right)\varepsilon R}}|M(x,t)-\bar{M}(\bar{x},\bar{t})|\, d\bar{x}d\bar{t}\,|\nabla u(x,t)|dxdt\\
&\qquad\quad +\frac{1}{\Lambda}\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)^d\left(1+\frac{\tau}{\varepsilon}\right)\|\rho\|^2_{L_t^\infty L_x^\infty}\|\nabla u\|_{L_t^1L_x^1}.
\end{aligned}$$

We thus have the bound

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} 2\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x',t') \left(r\left(u,\rho;\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right) * \varphi'_\tau\right)(x',t') \right.\\
&\qquad \left. \times\, \bar{\varphi}_\varepsilon(x'-\bar{x},t'-\bar{t})\, d\bar{x}d\bar{t}\, dx'dt' \right| \\
&\qquad \lesssim e^{2\Lambda}\|\rho\|^2_{L^\infty_t L^\infty_x} \int_{\mathbb{R}^d\times[0,T]} \varepsilon^{-d-1} \int_{\substack{|x-\bar{x}|\le\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)\varepsilon R\\ |t-\bar{t}|\le\left(1+\frac{\tau}{\varepsilon}\right)\varepsilon R}} |M(x,t)-\bar{M}(\bar{x},\bar{t})|\; d\bar{x}d\bar{t}\; |\nabla u(x,t)| dxdt \\
&\qquad\quad + \frac{1}{\Lambda}\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)^d \left(1+\frac{\tau}{\varepsilon}\right) \|\rho\|^2_{L^\infty_t L^\infty_x} \|\nabla u\|_{L^1_t L^1_x}.
\end{aligned} \tag{2.51}$$

Using (2.38), (2.40), (2.41), and (2.43), we next bound

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left(\left(\left(u\cdot\nabla\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)^2\right) * \varphi'_\tau\right)(x,t) \right.\right.\\
&\qquad -2\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x,t) \left(\left(u\cdot\nabla\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)\right) * \varphi'_\tau\right)(x,t)\Bigg) \\
&\qquad \left. \times\, \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x}d\bar{t}\, dxdt \right| \\
&\qquad = \left| \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \left(2\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t') \right.\right.\\
&\qquad\qquad -2\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x,t)\Bigg) u(x,t')\cdot\nabla\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t')\varphi'_\tau(t'-t) \\
&\qquad\qquad \left. \times\, \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\; dt'\; d\bar{x}d\bar{t}\; dxdt \right| \\
&\qquad \lesssim \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t') \right.\\
&\qquad\qquad \left. -\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x,t)\right| |u(x,t')| \left|\nabla\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t')\right| \left|\varphi'_\tau(t'-t)\right| \\
&\qquad\qquad \times\, |\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})|\; dt'\; d\bar{x}d\bar{t}\; dxdt \\
&\qquad \lesssim \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \int_0^T \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t') \right.\\
&\qquad\qquad \left. -\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t'')\right| |\varphi'_\tau(t-t'')||u(x,t')| \left|\nabla\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t')\right| \\
&\qquad\qquad \times \left|\varphi'_\tau(t'-t)\right| |\bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})|\; dt''\; dt'\; d\bar{x}d\bar{t}\; dxdt
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \int_0^T \int_0^1 \left| \left(\frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right)(x, t' - s(t'-t'')) \right| \\
&\qquad \times |t-t''| |\varphi'_\tau(t-t'')| |u(x,t')| \left| \nabla \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right)(x,t') \right| \left| \varphi'_\tau(t'-t) \right| \\
&\qquad \times |\bar{\varphi}_\varepsilon(x-\bar{x}, t-\bar{t})| \; ds\, dt''\, dt'\, d\bar{x} d\bar{t}\, dx dt \\
&\lesssim \tau \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \rho * \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^\infty_t L^\infty_x} \|\bar{\varphi}_\varepsilon\|_{L^1_t L^1_x} \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_0^T \int_0^T \int_0^1 \\
&\qquad \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| (x, t' - s(t'-t'')) |\varphi'_\tau(t-t'')| |u(x,t')| \left| \varphi'_\tau(t'-t) \right| \\
&\qquad ds\, dt''\, dt'\, dx dt \\
&\lesssim \tau \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \rho * \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^\infty_t L^\infty_x} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^1_t L^d_x} \\
&\qquad \times \left\| \varphi'_\tau \right\|_{L^1_t} \|u\|_{L^\infty_t L_x^{\frac{d}{d-1}}} \left\| \varphi'_\tau \right\|_{L^1_t} \\
&\lesssim \tau \|\rho\|_{L^\infty_t L^\infty_x} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^1_x} \|\nabla u\|_{L^\infty_t L^1_x} \\
&\qquad \times \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^1_t L^d_x} \\
&\lesssim \frac{\tau e^{(d+1)\Lambda}}{\delta} \|\rho\|_{L^\infty_t L^\infty_x} \|\nabla u\|_{L^\infty_t L^1_x} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^1_t L^d_x}.
\end{aligned}
$$

We thus have the bound

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^d\times[\tau' R, T-\tau' R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left(\left(\left(u \cdot \nabla \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right)^2 \right) * \varphi'_\tau \right)(x',t') \right. \right. \\
&\qquad - 2 \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau \right)(x',t') \left. \left(\left(u \cdot \nabla \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right) \right) * \varphi'_\tau \right)(x',t') \right) \\
&\qquad \left. \times \bar{\varphi}_\varepsilon(x'-\bar{x}, t'-\bar{t})\, d\bar{x} d\bar{t}\, dx' dt' \right| \\
&\qquad \lesssim \frac{\tau e^{(d+1)\Lambda}}{\delta} \|\rho\|_{L^\infty_t L^\infty_x} \|\nabla u\|_{L^\infty_t L^1_x} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right| \right\|_{L^1_t L^d_x}.
\end{aligned} \tag{2.52}
$$

Finally, we bound

$$\begin{aligned}
&\Bigg| \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)^2 (x,t) \frac{\partial}{\partial t} \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t}) \\
&\quad + \left(\left(u \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right)^2 \right) * \varphi'_\tau \right) (x,t) \cdot \nabla \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x} d\bar{t}\, dxdt \Bigg| \\
&\quad \lesssim \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)^2 (x,t) - \left(\rho * \varphi_{\delta_1} * \varphi'_\tau\right)^2 (x,t) \right| \\
&\qquad\qquad \times \left| \frac{\partial \bar{\varphi}_\varepsilon}{\partial t}(x-\bar{x},t-\bar{t}) \right| d\bar{x} d\bar{t}\, dxdt \\
&\qquad + \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left| \left(\left(u \left(\left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right)^2 - \left(\rho * \varphi_{\delta_1}\right)^2 \right) \right) * \varphi'_\tau \right) (x,t) \right| \\
&\qquad\qquad \times \left| \nabla \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t}) \right| d\bar{x} d\bar{t}\, dxdt \\
&\qquad + \Bigg| \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left(\rho * \varphi_{\delta_1} * \varphi'_\tau\right)^2 (x,t) \frac{\partial}{\partial t} \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t}) \\
&\qquad\qquad + \left(\left(u\rho * \varphi_{\delta_1}^2 \right) * \varphi'_\tau \right) (x,t) \cdot \nabla \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x} d\bar{t}\, dxdt \Bigg| \\
&\quad \lesssim \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right) (x,t) - \left(\rho * \varphi_{\delta_1} * \varphi'_\tau\right) (x,t) \right| \\
&\qquad\qquad \times \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right) (x,t) + \left(\rho * \varphi_{\delta_1} * \varphi'_\tau\right) (x,t) \right| \left| \frac{\partial \bar{\varphi}_\varepsilon}{\partial t}(x-\bar{x},t-\bar{t}) \right| d\bar{x} d\bar{t}\, dxdt \\
&\qquad + \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \left| u(x,t') \right| \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)^2 (x,t') - \left(\rho * \varphi_{\delta_1}\right)^2 (x,t') \right| \\
&\qquad\qquad \times \left| \varphi'_\tau(t-t') \right| \left| \nabla \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t}) \right| dt'\, d\bar{x} d\bar{t}\, dxdt \\
&\qquad + \Bigg| \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \left(\rho * \varphi_{\delta_1} * \varphi'_\tau\right)^2 (x,t) \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^d\times\mathbb{R}} \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x} d\bar{t} \right) \\
&\qquad\qquad + \left(\left(u(\rho * \varphi_{\delta_1})^2 \right) * \varphi'_\tau \right) (x,t) \cdot \nabla \left(\int_{\mathbb{R}^d\times\mathbb{R}} \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t})\, d\bar{x} d\bar{t} \right) dxdt \Bigg| \\
&\quad \lesssim \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right) (x,t') - (\rho * \varphi_{\delta_1})(x,t') \right| \left| \varphi'_\tau(t-t') \right| \\
&\qquad\qquad \times \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right) (x,t) + \left(\rho * \varphi_{\delta_1} * \varphi'_\tau\right) (x,t) \right| \left| \frac{\partial \bar{\varphi}_\varepsilon}{\partial t}(x-\bar{x},t-\bar{t}) \right| dt'\, d\bar{x} d\bar{t}\, dxdt \\
&\qquad + \|\rho\|_{L_t^\infty L_x^\infty} \int_{\mathbb{R}^d\times[\tau'R,T-\tau'R]} \int_{\mathbb{R}^d\times\mathbb{R}} \int_0^T \left| u(x,t') \right| \left| \left(\rho * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right) (x,t') - (\rho * \varphi_{\delta_1})(x,t') \right| \\
&\qquad\qquad \times \left| \varphi'_\tau(t-t') \right| \left| \nabla \bar{\varphi}_\varepsilon(x-\bar{x},t-\bar{t}) \right| dt'\, d\bar{x} d\bar{t}\, dxdt
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\| \frac{\partial \bar{\varphi}_\varepsilon}{\partial t} \right\|_{L_t^1 L_x^1} \left\| \varphi_\tau' \right\|_{L_t^1 L_x^1} \left(\left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} * \varphi_\tau' \right| \right\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \left\| \rho * \varphi_{\delta_1} * \varphi_\tau' \right\|_{L_t^\infty L_x^{\frac{d}{d-1}}} \right) \\
&\qquad \times \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d} \\
&\quad + \left\| \nabla \bar{\varphi}_\varepsilon \right\|_{L_t^1 L_x^1} \left\| \rho \right\|_{L_t^\infty L_x^\infty} \left\| u \right\|_{L_t^\infty L_x^{\frac{d}{d-1}}} \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d} \left\| \varphi_\tau' \right\|_{L_t^1 L_x^1} \\
&\lesssim \frac{1}{\varepsilon} \left(e^{d\Lambda} \left\| \rho \right\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \left\| \rho \right\|_{L_t^\infty L_x^\infty} \left\| \nabla u \right\|_{L_t^\infty L_x^1} \right) \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d} .
\end{aligned}
$$

We thus have the bound

$$
\begin{aligned}
&\Bigg| \int_{\mathbb{R}^d \times [\tau' R, T - \tau' R]} \int_{\mathbb{R}^d \times \mathbb{R}} \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} * \varphi_\tau' \right)^2 (x,t) \frac{\partial}{\partial t} \bar{\varphi}_\varepsilon (x - \bar{x}, t - \bar{t}) \qquad (2.53) \\
&\qquad + \left(\left(u \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right)^2 \right) * \varphi_\tau' \right) (x,t) \cdot \nabla \bar{\varphi}_\varepsilon (x - \bar{x}, t - \bar{t}) \, d\bar{x} d\bar{t} \, dx dt \Bigg| \\
&\qquad \lesssim \frac{1}{\varepsilon} \left(e^{d\Lambda} \left\| \rho \right\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \left\| \rho \right\|_{L_t^\infty L_x^\infty} \left\| \nabla u \right\|_{L_t^\infty L_x^1} \right) \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d} .
\end{aligned}
$$

By combining (2.48) with (2.49), (2.51), (2.52), and (2.53)

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^d \times [\tau' R, T - \tau' R]} \frac{\partial}{\partial t} \int_{\mathbb{R}^d \times \mathbb{R}} \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} * \varphi_\tau' \right)^2 (x,t) \, \bar{\varphi}_\varepsilon (x - \bar{x}, t - \bar{t}) \, d\bar{x} d\bar{t} \, dx dt \right| \qquad (2.54) \\
&\lesssim \left\| \rho \right\|_{L_t^\infty L_x^\infty} \int_{\mathbb{R}^d \times [\tau' R, T - \tau' R]} \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left(\left| \left(\psi * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} * \varphi_\tau' \right) (x,t) \right| \right) dx dt \\
&\quad + e^{2\Lambda} \left\| \rho \right\|_{L_t^\infty L_x^\infty}^2 \int_{\mathbb{R}^d \times [0,T]} \varepsilon^{-d-1} \int_{\substack{|x - \bar{x}| \le \left(1 + \frac{\delta e^\Lambda}{\varepsilon}\right) \varepsilon R \\ |t - \bar{t}| \le \left(1 + \frac{\tau}{\varepsilon}\right) \varepsilon R}} \left| M(x,t) - \bar{M}(\bar{x},\bar{t}) \right| \, d\bar{x} d\bar{t} \, \left| \nabla u(x,t) \right| dx dt \\
&\quad + \frac{1}{\Lambda} \left(1 + \frac{\delta e^\Lambda}{\varepsilon} \right)^d \left(1 + \frac{\tau}{\varepsilon} \right) \left\| \rho \right\|_{L_t^\infty L_x^\infty}^2 \left\| \nabla u \right\|_{L_t^1 L_x^1} \\
&\quad + \frac{\tau e^{(d+1)\Lambda}}{\delta} \left\| \rho \right\|_{L_t^\infty L_x^\infty} \left\| \nabla u \right\|_{L_t^\infty L_x^1} \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \frac{\partial \rho}{\partial t} * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right| \right\|_{L_t^1 L_x^d} \\
&\quad + \frac{1}{\varepsilon} \left(e^{d\Lambda} \left\| \rho \right\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \left\| \rho \right\|_{L_t^\infty L_x^\infty} \left\| \nabla u \right\|_{L_t^\infty L_x^1} \right) \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d} .
\end{aligned}
$$

At this point we have redone the proof of Ambrosio [8], as one may now take $\Lambda > 0$ large enough, $\delta, \varepsilon, \tau > 0$ small enough, and appropriate $\bar{M}$, to make the above arbitrarily small.

However, for a uniform bound, we must employ weak compactness. Using (2.41), (2.46), we show the following weak estimate. Let $\rho(x,t)$, divergence-free $u(x,t)$, and $\psi(x,t)$ solve

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = \psi.$$

Extending u by zero outside $t \in [0,T]$, then $\rho * \varphi_{\delta_1}$ and $u * \varphi_{\delta_2,\tau_2}$ solve the following forced equation on the time interval $t \in (0,T)$.

$$\frac{\partial}{\partial t}(\rho * \varphi_{\delta_1}) + (u * \varphi_{\delta_2,\tau_2}) \cdot \nabla(\rho * \varphi_{\delta_1}) = \psi + \psi', \tag{2.55}$$

where

$$\psi' = \frac{\partial}{\partial t}(\rho * \varphi_{\delta_1} - \rho) + \nabla \cdot ((u * \varphi_{\delta_2,\tau_2})(\rho * \varphi_{\delta_1}) - u\rho).$$

In line with (2.54) we wish to bound, for $x \in \mathbb{R}^d$, and $t \in [(\tau + \tau_2)R, T - (\tau + \tau_2)R]$.

$$\begin{aligned}
&\left(\psi' * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)(x,t) \\
&= \int_{\mathbb{R}^d \times [0,T]} (\rho * \varphi_{\delta_1} - \rho)(x',t') \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') \frac{\partial \varphi'_\tau}{\partial t}(t-t')\, dx'dt' \\
&\quad + \int_{\tau_2 R}^{T-\tau_2 R} \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})(x',t')(u * \varphi_{\delta_2,\tau_2} - u * \varphi'_{\tau_2})(x',t') \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') \varphi'_\tau(t-t')\, dx'\, dt' \\
&\quad + \int_{\tau_2 R}^{T-\tau_2 R} \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})(x',t')(u * \varphi'_{\tau_2} - u)(x',t') \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') \varphi'_\tau(t-t')\, dx'\, dt' \\
&\quad + \int_0^T \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1} - \rho)(x',t') u(x',t') \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') \varphi'_\tau(t-t')\, dx'\, dt' \\
&= \int_{\mathbb{R}^d \times [0,T]} \int_{\mathbb{R}^d} \rho(x'',t') \varphi_{\delta_1}(x'-x'') \frac{\partial \varphi'_\tau}{\partial t}(t-t') \left(\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') - \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x'')\right) dx''\, dx'dt' \\
&\quad + \int_{\tau_2 R}^{T-\tau_2 R} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})(x',t')(u * \varphi'_{\tau_2}(x'',t') - u * \varphi'_{\tau_2}(x',t')) \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') \\
&\qquad \times \varphi_{\delta_2}(x'-x'') \varphi'_\tau(t-t')\, dx''\, dx'\, dt' \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^T \left((\rho * \varphi_{\delta_1})(x',t') \varphi'_\tau(t-t') - (\rho * \varphi_{\delta_1})(x',t'') \varphi'_\tau(t-t'')\right) \\
&\qquad \times u(x',t'') \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') \varphi'_{\tau_2}(t'-t'')\, dt''\, dx'\, dt' \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x'',t') \varphi_{\delta_1}(x'-x'') \left(u(x',t') \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x') - u(x'',t') \cdot \nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x'')\right) \\
&\qquad \times \varphi'_\tau(t-t')\, dx''\, dx'\, dt'
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d\times[0,T]}\int_{\mathbb{R}^d} \rho(x'',t')\varphi_{\delta_1}(x'-x'')\frac{\partial\varphi'_\tau}{\partial t}(t-t') \\
&\qquad \int_0^1 \left((x''-x')\cdot\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right)(x-sx'-(1-s)x''))\,ds\,dx''\,dx'dt' \\
&\quad + \int_{\tau_2 R}^{T-\tau_2 R}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 (\rho*\varphi_{\delta_1})(x',t')\left(((x''-x')\cdot\nabla)u*\varphi'_{\tau_2}\right)(x'+s(x''-x'),t') \\
&\qquad \cdot\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}(x-x')\varphi_{\delta_2}(x'-x'')\varphi'_\tau(t-t')\,ds\,dx''\,dx'\,dt' \\
&\quad + \int_0^T\int_{\mathbb{R}^d}\int_0^T\int_0^1 \left((t'-t'')\left(\frac{\partial\rho}{\partial t}*\varphi_{\delta_1}\right)(x',t''+s(t'-t''))\varphi'_\tau(t-st'-(1-s)t'')\right. \\
&\qquad \left. +\left(\rho*\varphi_{\delta_1}\right)(x',t''+s(t'-t''))(t''-t')\frac{\partial\varphi'_\tau}{\partial t}(t-st'-(1-s)t'')\right) \\
&\qquad \times u(x',t'')\varphi'_{\tau_2}(t'-t'')\cdot\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}(x-x')\,ds\,dt''\,dx'\,dt' \\
&\quad + \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 \rho(x'',t')\varphi_{\delta_1}(x'-x'')\varphi'_\tau(t-t') \\
&\qquad \times\left(\left(((x'-x'')\cdot\nabla)u\right)(x''+s(x'-x''),t')\cdot\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}(x-sx'-(1-s)x'')\right. \\
&\qquad \left. +u(x''+s(x'-x''),t')\cdot\left(((x''-x')\cdot\nabla)\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right)(x-sx'-(1-s)x'')\right)\,ds \\
&\qquad dx''\,dx'\,dt'.
\end{aligned}
$$

Therefore, for $\tau'\geq\tau+\tau_2$, by (2.42), (2.43), and (2.44).

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times[\tau' R,T-\tau' R]} \sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}} \left(\left|\left(\psi*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right)(x,t)\right|\right)\,dxdt \\
&\quad\lesssim \delta_1\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L_x^1}\left\|\frac{\partial\varphi'_\tau}{\partial t}\right\|_{L_t^1}\|\varphi_{\delta_1}\|_{L_x^1}\|\rho\|_{L_t^1L_x^1} \\
&\qquad +\delta_2\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L_x^1}\|\varphi'_\tau\|_{L_t^1}\|\rho*\varphi_{\delta_1}\|_{L_t^\infty L_x^\infty}\|\nabla u*\varphi'_{\tau_2}\|_{L_t^1L_x^1}\|\varphi_{\delta_2}\|_{L_x^1} \\
&\qquad +\tau_2\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L_x^1}\|\varphi'_\tau\|_{L_t^1}\left\|\frac{\partial\rho}{\partial t}*\varphi_{\delta_1}\right\|_{L_t^1L_x^d}\|u\|_{L_t^\infty L_x^{\frac{d}{d-1}}}\|\varphi'_{\tau_2}\|_{L_t^1} \\
&\qquad +\tau_2\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L_x^1}\left\|\frac{\partial\varphi'_\tau}{\partial t}\right\|_{L_t^1}\|\rho*\varphi_{\delta_1}\|_{L_t^\infty L_x^d}\|u\|_{L_t^1L_x^{\frac{d}{d-1}}}\|\varphi'_{\tau_2}\|_{L_t^1} \\
&\qquad +\delta_1\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\nabla\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L_x^1}\|\varphi'_\tau\|_{L_t^1}\|\rho\|_{L_t^\infty L_x^\infty}\|\nabla u\|_{L_t^1L_x^1}\|\varphi_{\delta_1}\|_{L_x^1} \\
&\qquad +\delta_1\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\nabla^2\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L_x^1}\|\varphi'_\tau\|_{L_t^1}\|\rho\|_{L_t^\infty L_x^d}\|u\|_{L_t^1L_x^{\frac{d}{d-1}}}\|\varphi_{\delta_1}\|_{L_x^1}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \delta_1 e^{(d+2)\Lambda}\left(\frac{1}{\delta\tau}\|\rho\|_{L^1_tL^1_x}+\frac{1}{\delta}\|\rho\|_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^1_tL^1_x}+\frac{1}{\delta^2}\|\rho\|_{L^\infty_tL^d_x}\|\nabla u\|_{L^1_tL^1_x}\right)\\
&\quad+\delta_2 e^{(d+1)\Lambda}\frac{1}{\delta}\|\rho\|_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^1_tL^1_x}\\
&\quad+\tau_2 e^{(d+1)\Lambda}\left(\frac{1}{\delta}\left\|\frac{\partial\rho}{\partial t}*\varphi_{\delta_1}\right\|_{L^1_tL^d_x}\|\nabla u\|_{L^\infty_tL^1_x}+\frac{1}{\delta\tau}\|\rho\|_{L^\infty_tL^d_x}\|\nabla u\|_{L^1_tL^1_x}\right).
\end{aligned}
$$

Combining with (2.54), we have, for $\tau' \geq \tau+\tau_2$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d\times[\tau' R,T-\tau' R]}\frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right)^2(x,t)\;\bar\varphi_\varepsilon(x-\bar x,t-\bar t)\;d\bar x d\bar t\;dxdt\right|\\
&\quad\lesssim \|\rho\|_{L^\infty_tL^\infty_x}\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\psi*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right|\right\|_{L^1_tL^1_x}\\
&\quad+\delta_1 e^{(d+2)\Lambda}\left(\frac{1}{\delta\tau}\|\rho\|_{L^1_tL^1_x}+\frac{1}{\delta}\|\rho\|_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^1_tL^1_x}+\frac{1}{\delta^2}\|\rho\|_{L^\infty_tL^d_x}\|\nabla u\|_{L^1_tL^1_x}\right)\\
&\quad+\delta_2 e^{(d+1)\Lambda}\frac{1}{\delta}\|\rho\|_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^1_tL^1_x}\\
&\quad+\tau_2 e^{(d+1)\Lambda}\left(\frac{1}{\delta}\left\|\frac{\partial\rho}{\partial t}*\varphi_{\delta_1}\right\|_{L^1_tL^d_x}\|\nabla u\|_{L^\infty_tL^1_x}+\frac{1}{\delta\tau}\|\rho\|_{L^\infty_tL^d_x}\|\nabla u\|_{L^1_tL^1_x}\right)\\
&\quad+e^{2\Lambda}\|\rho\|^2_{L^\infty_tL^\infty_x}\int_{\mathbb{R}^d\times[0,T]}\\
&\qquad\times\varepsilon^{-d-1}\int_{\substack{|x-\bar x|\le\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)\varepsilon R\\ |t-\bar t|\le\left(1+\frac{\tau}{\varepsilon}\right)\varepsilon R}}\left|\frac{\nabla u*\varphi_{\delta_2,\tau_2}}{|\nabla u*\varphi_{\delta_2,\tau_2}|}(x,t)-\bar M(\bar x,\bar t)\right|\,d\bar x d\bar t\;|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)|dxdt\\
&\quad+\frac{1}{\Lambda}\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)^d\left(1+\frac{\tau}{\varepsilon}\right)\|\rho\|^2_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^1_tL^1_x}\\
&\quad+\frac{\tau e^{(d+1)\Lambda}}{\delta}\|\rho\|_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^\infty_tL^1_x}\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\frac{\partial\rho}{\partial t}*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right|\right\|_{L^1_tL^d_x}\\
&\quad+\frac{1}{\varepsilon}\left(e^{d\Lambda}\|\rho\|_{L^\infty_tL_x^{\frac{d}{d-1}}}+\left\|\rho*\varphi_{\delta_1}\right\|_{L^\infty_tL_x^{\frac{d}{d-1}}}+\|\rho\|_{L^\infty_tL^\infty_x}\|\nabla u\|_{L^\infty_tL^1_x}\right)\\
&\qquad\times\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right)-\rho*\varphi_{\delta_1}\right|\right\|_{L^1_tL^d_x},
\end{aligned}
\tag{2.56}
$$

where we recall, in the definition of $(u*\varphi_{\delta_2,\tau_2})(x,t)$ for $t\in[0,T]$, that in (2.55) we extended $u(x,t)$ first by zero outside of $t\in[0,T]$.

We now control the force $\psi(x,t) \in L_t^1 W_x^{-1,1}$, by writing $\psi = \psi_0 + \sum_{i=1}^d \frac{\partial \psi_i}{\partial x_i}$, with $\|\psi\|_{L_t^1 W_x^{-1,1}} = \sum_{i=0}^d \|\psi_i\|_{L_t^1 L_x^1}$. Then by and by (2.42), and (2.43),

$$\begin{aligned}
&\Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\psi * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\Big| \Big\|_{L_t^1 L_x^1} \\
&\lesssim \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\psi_0 * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\Big| \Big\|_{L_t^1 L_x^1} + \sum_{i=1}^d \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\frac{\partial \psi_i}{\partial x_i} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\Big| \Big\|_{L_t^1 L_x^1} \\
&\lesssim \|\psi_0\|_{L_t^1 L_x^1} \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\Big| \Big\|_{L_x^1} \|\varphi'_\tau\|_{L_t^1} + \sum_{i=1}^d \|\psi_i\|_{L_t^1 L_x^1} \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\Big| \Big\|_{L_x^1} \|\varphi'_\tau\|_{L_t^1} \\
&\lesssim \frac{e^{(d+1)\Lambda}}{\delta^{d+1}} (1+\delta) \|\psi\|_{L_t^1 W_x^{-1,1}}. \qquad (2.57)
\end{aligned}$$

Next, using the expression $\frac{\partial \rho}{\partial t} = \psi - \nabla \cdot (u\rho)$, we have

$$\begin{aligned}
\Big\| \frac{\partial \rho}{\partial t} * \varphi_{\delta_1} \Big\|_{L_t^1 L_x^d} &\lesssim \Big\| \Big(\psi_0 + \sum_{i=1}^d \frac{\partial \psi_i}{\partial x_i} - \nabla\cdot(u\rho)\Big) * \varphi_{\delta_1} \Big\|_{L_t^1 L_x^d} \\
&\lesssim \|\psi_0\|_{L_t^1 L_x^1} \|\varphi_{\delta_1}\|_{L_x^d} + \sum_{i=1}^d \|\psi_i\|_{L_t^1 L_x^1} \|\nabla \varphi_{\delta_1}\|_{L_x^d} + \|\rho\|_{L_t^\infty L_x^\infty} \|u\|_{L_t^1 L_x^{\frac{d}{d-1}}} \|\nabla \varphi_{\delta_1}\|_{L_x^{\frac{d}{2}}} \\
&\lesssim \frac{1}{\delta_1^d} (1+\delta_1) \|\psi\|_{L_t^1 W_x^{-1,1}} + \frac{1}{\delta_1^{d-1}} \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1}. \qquad (2.58)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big| \frac{\partial \rho}{\partial t} * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \Big| \Big\|_{L_t^1 L_x^d} \\
&\quad\lesssim \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big| \Big(\psi_0 + \sum_{i=1}^d \frac{\partial \psi_i}{\partial x_i} - \nabla\cdot(u\rho)\Big) * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \Big| \Big\|_{L_t^1 L_x^d} \\
&\quad\lesssim \|\psi_0\|_{L_t^1 L_x^1} \|\varphi_{\delta_1}\|_{L_x^1} \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} |\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}| \Big\|_{L_x^d} + \sum_{i=1}^d \|\psi_i\|_{L_t^1 L_x^1} \|\varphi_{\delta_1}\|_{L_x^1} \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\Big| \Big\|_{L_x^d} \\
&\qquad + \|\rho\|_{L_t^\infty L_x^\infty} \|u\|_{L_t^1 L_x^{\frac{d}{d-1}}} \|\varphi_{\delta_1}\|_{L_x^1} \Big\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \Big|\nabla \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\Big| \Big\|_{L_x^{\frac{d}{2}}} \\
&\quad\lesssim \frac{e^\Lambda}{\delta^{d-1}} \|\psi_0\|_{L_t^1 L_x^1} + \frac{e^\Lambda}{\delta^d} \sum_{i=1}^d \|\psi_i\|_{L_t^1 L_x^1} + \frac{e^{2\Lambda}}{\delta^{d-1}} \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1} \\
&\quad\lesssim \frac{e^\Lambda}{\delta^d} (1+\delta) \|\psi\|_{L_t^1 W_x^{-1,1}} + \frac{e^{2\Lambda}}{\delta^{d-1}} \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^1 L_x^1}. \qquad (2.59)
\end{aligned}$$

Combining (2.56) with (2.57), (2.58), and (2.59), we have finally

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\times[\tau' R,T-\tau' R]} \frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}} \left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right)^2(x,t)\ \bar\varphi_\varepsilon(x-\bar x,t-\bar t)\,d\bar x d\bar t\,dxdt\right|\\
&\lesssim \frac{e^{(d+1)\Lambda}}{\delta^{d+1}}(1+\delta)\,\|\rho\|_{L^\infty_t L^\infty_x}\|\psi\|_{L^1_t W^{-1,1}_x}\\
&\quad+\delta_1 e^{(d+2)\Lambda}\left(\frac{1}{\delta\tau}\|\rho\|_{L^1_t L^1_x}+\frac{1}{\delta}\|\rho\|_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^1_t L^1_x}+\frac{1}{\delta^2}\|\rho\|_{L^\infty_t L^d_x}\|\nabla u\|_{L^1_t L^1_x}\right)\\
&\quad+\delta_2 e^{(d+1)\Lambda}\frac{1}{\delta}\|\rho\|_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^1_t L^1_x}\\
&\quad+\tau_2 e^{(d+1)\Lambda}\left(\frac{1+\delta_1}{\delta_1^d}\frac{1}{\delta}\left(\|\psi\|_{L^1_t W^{-1,1}_x}+\|\rho\|_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^1_t L^1_x}\right)\|\nabla u\|_{L^\infty_t L^1_x}+\frac{1}{\delta\tau}\|\rho\|_{L^\infty_t L^d_x}\|\nabla u\|_{L^1_t L^1_x}\right)\\
&\quad+e^{2\Lambda}\|\rho\|^2_{L^\infty_t L^\infty_x}\int_{\mathbb{R}^d\times[0,T]}\varepsilon^{-d-1}\int_{\substack{|x-\bar x|\le \varepsilon R+\delta e^\Lambda R\\ |t-\bar t|\le\varepsilon R+\tau R}}\left|\frac{\nabla u*\varphi_{\delta_2,\tau_2}}{|\nabla u*\varphi_{\delta_2,\tau_2}|}(x,t)-\bar M(\bar x,\bar t)\right|\,d\bar x d\bar t\,|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)|dxdt\\
&\quad+\frac{1}{\Lambda}\left(1+\frac{\delta e^\Lambda}{\varepsilon}\right)^d\left(1+\frac{\tau}{\varepsilon}\right)\|\rho\|^2_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^1_t L^1_x}\\
&\quad+\frac{\tau e^{(d+3)\Lambda}}{\delta^d}\left(1+\frac{1}{\delta}\right)\|\rho\|_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^\infty_t L^1_x}\left(\|\psi\|_{L^1_t W^{-1,1}_x}+\|\rho\|_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^1_t L^1_x}\right)\\
&\quad+\frac{1}{\varepsilon}\left(e^{d\Lambda}\|\rho\|_{L^\infty_t L^{\frac{d}{d-1}}_x}+\left\|\rho*\varphi_{\delta_1}\right\|_{L^\infty_t L^{\frac{d}{d-1}}_x}+\|\rho\|_{L^\infty_t L^\infty_x}\|\nabla u\|_{L^\infty_t L^1_x}\right)\\
&\qquad\times\left\|\sup_{(\bar x,\bar t)\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}\right)-\rho*\varphi_{\delta_1}\right|\right\|_{L^1_t L^d_x}.
\end{aligned}\tag{2.60}$$

We are left to use the above to control the trace at time $t=T$ in terms of $\rho_0(x)$ and $\psi(x,t)$. To this end consider the following approximation:

$$\begin{aligned}
&\int_{\mathbb{R}^d}|(\rho*\varphi_{\delta_1})(x,T)|^2\,dx\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}}\left(|(\rho*\varphi_{\delta_1})(x,T)|^2-\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right)^2(x,T-\tau' R)\right)\\
&\qquad\times\bar\varphi_\varepsilon(x-\bar x,T-\tau' R-\bar t)\,d\bar x d\bar t\,dx\\
&\quad+\int_{\mathbb{R}^d\times[\tau' R,T-\tau' R]}\frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right)^2(x,t)\ \bar\varphi_\varepsilon(x-\bar x,t-\bar t)\,d\bar x d\bar t\,dxdt\\
&\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar M(\bar x,\bar t)}*\varphi'_\tau\right)^2(x,\tau' R)-|(\rho*\varphi_{\delta_1})(x,0)|^2\right)\bar\varphi_\varepsilon(x-\bar x,\tau' R-\bar t)\,d\bar x d\bar t\,dx\\
&\quad+\int_{\mathbb{R}^d}|(\rho*\varphi_{\delta_1})(x,0)|^2\,dx.
\end{aligned}\tag{2.61}$$

Recall the expression $\frac{\partial \rho}{\partial t} = -\nabla \cdot (u\rho) + \psi$, and so we have the bound

$$\left\|\rho(\cdot,t) - \rho(\cdot,t')\right\|_{W_x^{-1,1}} \leq |t-t'| \, \|u\rho\|_{L_t^\infty L_x^1} + \|\psi\|_{L_t^1 W_x^{-1,1}},$$

then, since $\tau \leq \tau'$, and also (2.42), (2.43)

$$\begin{aligned}
&\left| \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)^2 (x,T) - \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)^2 (x, T-\tau' R)\right| \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \left| \int_0^T \left(\left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,T) - \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,s) \right) \right. \\
&\qquad\qquad \left. \varphi'_\tau(T-\tau' R - s)\, ds \right| \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \left| \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\rho(x'',T) - \rho(x'',s)\right) \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x'-x'')\, dx'' \varphi_{\delta_1}(x-x')\, dx' \right. \\
&\qquad\qquad \left. \varphi'_\tau(T-\tau' R - s)\, ds \right| \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \int_0^T \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \left(\rho(x'',T) - \rho(x'',s)\right) \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x'-x'')\, dx'' \right| \left|\varphi_{\delta_1}(x-x')\right|\, dx' \\
&\qquad\qquad \left|\varphi'_\tau(T-\tau' R - s)\right|\, ds \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \sup_{|T-\tau' R - s| \leq \tau R} \left(\|\rho(\cdot,T) - \rho(\cdot,s)\|_{W_x^{-1,1}} \right) \left\| \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right\|_{W_x^{1,\infty}} \|\varphi_{\delta_1}\|_{L_x^1} \|\varphi'_\tau\|_{L_x^1} \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \left((\tau' R + \tau R) \|u\|_{L_t^\infty L_x^{\frac{d}{d-1}}} \|\rho\|_{L_t^\infty L_x^d} + \|\psi\|_{L_t^1 W_x^{-1,1}} \right) \frac{1}{\delta^{d+1}} \left(\delta + e^\Lambda\right) \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \left(\tau' \|\nabla u\|_{L_t^\infty L_x^1} \|\rho\|_{L_t^\infty L_x^d} + \|\psi\|_{L_t^1 W_x^{-1,1}} \right) \frac{1}{\delta^{d+1}} \left(\delta + e^\Lambda\right).
\end{aligned}$$

Therefore, using the shorthand $\rho_T(x) = \rho(x,T)$, and also (2.42), we may bound the difference

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}} \left(|(\rho * \varphi_{\delta_1})(x,T)|^2 - \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})} * \varphi'_\tau\right)^2 (x, T - \tau' R) \right) \\
&\qquad \times \bar{\varphi}_\varepsilon(x-\bar{x}, T-\tau' R - \bar{t})\, d\bar{x} d\bar{t}\, dx \\
&\quad \lesssim \|\rho\|_{L_t^\infty L_x^\infty} \left(\tau' \|\nabla u\|_{L_t^\infty L_x^1} \|\rho\|_{L_t^\infty L_x^d} + \|\psi\|_{L_t^1 W_x^{-1,1}} \right) \frac{1}{\delta^{d+1}} \left(\delta + e^\Lambda\right) \\
&\qquad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}} \left(|(\rho * \varphi_{\delta_1})(x,T)|^2 - \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)^2 (x,T) \right) \\
&\qquad\quad \times \bar{\varphi}_\varepsilon(x-\bar{x}, T-\tau' R - \bar{t})\, d\bar{x} d\bar{t}\, dx
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\rho\|_{L_t^\infty L_x^\infty}\left(\tau'\|\nabla u\|_{L_t^\infty L_x^1}\|\rho\|_{L_t^\infty L_x^d}+\|\psi\|_{L_t^1 W_x^{-1,1}}\right)\frac{1}{\delta^{d+1}}\left(\delta+e^{\Lambda}\right)\\
&\quad+\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_T*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)+\rho_T*\varphi_{\delta_1}\right|\right\|_{L_x^2}\\
&\qquad\times\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_T*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_T*\varphi_{\delta_1}\right|\right\|_{L_x^2}\|\bar{\varphi}_\varepsilon\|_{L_t^1 L_x^1}\\
&\lesssim \|\rho\|_{L_t^\infty L_x^\infty}\left(\tau'\|\nabla u\|_{L_t^\infty L_x^1}\|\rho\|_{L_t^\infty L_x^d}+\|\psi\|_{L_t^1 W_x^{-1,1}}\right)\frac{1}{\delta^{d+1}}\left(\delta+e^{\Lambda}\right)\\
&\quad+\left(\|\rho\|_{L_t^\infty L_x^2}+\left\|\rho_T*\varphi_{\delta_1}*\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right|\right\|_{L_x^2}\right)\\
&\qquad\times\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_T*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_T*\varphi_{\delta_1}\right|\right\|_{L_x^2},\\
&\lesssim \|\rho\|_{L_t^\infty L_x^\infty}\left(\tau'\|\nabla u\|_{L_t^\infty L_x^1}\|\rho\|_{L_t^\infty L_x^d}+\|\psi\|_{L_t^1 W_x^{-1,1}}\right)\frac{1}{\delta^{d+1}}\left(\delta+e^{\Lambda}\right)\\
&\quad+e^{d\Lambda}\|\rho\|_{L_t^\infty L_x^2}\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_T*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_T*\varphi_{\delta_1}\right|\right\|_{L_x^2},
\end{aligned}
$$

and so we have the bound:

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}}\left(|(\rho*\varphi_{\delta_1})(x,T)|^2-\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,T-\tau'R)\right)\\
&\quad\times\bar{\varphi}_\varepsilon(x-\bar{x},T-\tau'R-\bar{t})\,d\bar{x}d\bar{t}\,dx\\
&\lesssim \|\rho\|_{L_t^\infty L_x^\infty}\left(\tau'\|\nabla u\|_{L_t^\infty L_x^1}\|\rho\|_{L_t^\infty L_x^d}+\|\psi\|_{L_t^1 W_x^{-1,1}}\right)\frac{1}{\delta^{d+1}}\left(\delta+e^{\Lambda}\right)\\
&\quad+e^{d\Lambda}\|\rho\|_{L_t^\infty L_x^2}\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_T*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_T*\varphi_{\delta_1}\right|\right\|_{L_x^2}.
\end{aligned}
\tag{2.62}
$$

Similarly, for $\rho_0(x)$ we also have the bound

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\left(\rho*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}*\varphi'_\tau\right)^2(x,\tau'R)-|(\rho*\varphi_{\delta_1})(x,0)|^2\right)\bar{\varphi}_\varepsilon(x-\bar{x},\tau'R-\bar{t})\,d\bar{x}d\bar{t}\,dx\\
&\lesssim \|\rho\|_{L_t^\infty L_x^\infty}\left(\tau'\|\nabla u\|_{L_t^\infty L_x^1}\|\rho\|_{L_t^\infty L_x^d}+\|\psi\|_{L_t^1 W_x^{-1,1}}\right)\frac{1}{\delta^{d+1}}\left(\delta+e^{\Lambda}\right)\\
&\quad+e^{d\Lambda}\|\rho\|_{L_t^\infty L_x^2}\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_0*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_0*\varphi_{\delta_1}\right|\right\|_{L_x^2}.
\end{aligned}
\tag{2.63}
$$

Combining (2.61) with (2.60), (2.62), and (2.63) then gives the result. □

To complete the quantitative argument, we must now find a suitable weak open cover of $u(x,t) \in L_t^\infty BV_x$ and $\rho(x,t) \in L_t^\infty L_x^1 \cap L_t^\infty L_x^\infty$. The idea here is to consider differences such as $\|\nabla u * \bar{\varphi}_{\varepsilon_1}\|_{L_t^1 L_x^1} - \|\nabla u * \bar{\varphi}_{\varepsilon_2}\|_{L_t^1 L_x^1} > 0$ for $\varepsilon_1 < \varepsilon_2$, which must be small for some values of $\varepsilon_1, \varepsilon_2 > 0$ by the pigeonhole principle. We, therefore, need to bound the remaining non-quantitative terms in (2.60) by such differences. This result is essentially a harmonic analysis decomposition valid for functions in L^1, or even measures $\mathscr{M}$, and is the first of its kind.

Proposition 2.19. *For any $C > 0$, for a suitable choice of $\bar{M}(\bar{x},\bar{t}) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d \times d}$, we have the bound*

$$\int_{\mathbb{R}^d \times [0,T]} \varepsilon^{-d-1} \int_{\substack{|x-\bar{x}| \le \varepsilon C \\ |t-\bar{t}| \le \varepsilon C}} \left| \frac{\nabla u * \varphi_{\delta_2,\tau_2}}{|\nabla u * \varphi_{\delta_2,\tau_2}|}(x,t) - \bar{M}(\bar{x},\bar{t}) \right| d\bar{x} d\bar{t} \, |(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)| dx dt \tag{2.64}$$
$$\lesssim C^{d+1} \|\nabla u\|_{L_t^1 L_x^1}^{\frac{1}{2}} \left(\left\|\nabla u * \varphi_{\delta_2,\tau_2}\right\|_{L_t^1 L_x^1} - \left\|\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\varepsilon C R', \varepsilon C R'}\right\|_{L_t^1 L_x^1} \right)^{\frac{1}{2}},$$

for some $R' > 0$ dependent only on the mollifiers $\varphi(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ and $\varphi'(t) \in C_c^\infty(\mathbb{R};\mathbb{R})$, with the requirement that $\varphi(x)$, $\varphi'(t)$ are non-negative, symmetric, with $\varphi(0), \varphi'(0) > 0$.

In the definition of say $(u * \varphi_{\delta_2,\tau_2})(x,t)$ for $t \in [0,T]$, we extended $\rho(x,t)$ and $u(x,t)$ first by zero outside of $t \in [0,T]$.

Proof. Let $R' > 0$ be such that both $\varphi(x), \varphi'(t) > 0$ for all $|x|, |t| \le \frac{1}{R'}$. Then

$$\int_{\mathbb{R}^d \times [0,T]} \varepsilon^{-d-1} \int_{\substack{|x-\bar{x}| \le \varepsilon C \\ |t-\bar{t}| \le \varepsilon C}} \left| \frac{\nabla u * \varphi_{\delta_2,\tau_2}}{|\nabla u * \varphi_{\delta_2,\tau_2}|}(x,t) - \bar{M}(\bar{x},\bar{t}) \right| d\bar{x} d\bar{t} \, |(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)| dx dt$$
$$\lesssim \int_{\mathbb{R}^d \times [0,T]} C^{d+1} \int_{\mathbb{R}^d \times \mathbb{R}} \left| \frac{\nabla u * \varphi_{\delta_2,\tau_2}}{|\nabla u * \varphi_{\delta_2,\tau_2}|}(x,t) - \bar{M}(\bar{x},\bar{t}) \right| \varphi_{\varepsilon C R', \varepsilon C R'}(x-\bar{x}, t-\bar{t}) \, d\bar{x} d\bar{t}$$
$$\times |(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)| dx dt$$
$$\lesssim C^{d+1} \left(\left\|\nabla u * \varphi_{\delta_2,\tau_2}\right\|_{L_t^1 L_x^1} \left\|\varphi_{\varepsilon C R', \varepsilon C R'}\right\|_{L_t^1 L_x^1} \right)^{\frac{1}{2}}$$
$$\times \left(\int_{\mathbb{R}^d \times \mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}} \left| \frac{\nabla u * \varphi_{\delta_2,\tau_2}}{|\nabla u * \varphi_{\delta_2,\tau_2}|}(x,t) - \bar{M}(\bar{x},\bar{t}) \right|^2 \varphi_{\varepsilon C R', \varepsilon C R'}(x-\bar{x}, t-\bar{t}) \, d\bar{x} d\bar{t} \right.$$
$$\left. \times |(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)| dx dt \right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\lesssim C^{d+1}\,\|\nabla u\|_{L^1_tL^1_x}^{\frac12}\Bigg(\int_{\mathbb{R}^d\times\mathbb{R}}\int_{\mathbb{R}^d\times\mathbb{R}}\Big(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|-2\bar{M}(\bar{x},\bar{t})\cdot(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\\
&\qquad+\bar{M}^2(\bar{x},\bar{t})\,\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|\Big)\varphi_{\varepsilon CR',\varepsilon CR'}(x-\bar{x},t-\bar{t})\,d\bar{x}d\bar{t}\,dxdt\Bigg)^{\frac12}\\
&\lesssim C^{d+1}\,\|\nabla u\|_{L^1_tL^1_x}^{\frac12}\Bigg(\big\|\nabla u*\varphi_{\delta_2,\tau_2}\big\|_{L^1_tL^1_x}-2\int_{\mathbb{R}^d\times\mathbb{R}}\bar{M}(\bar{x},\bar{t})\cdot(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\,d\bar{x}d\bar{t}\\
&\qquad+\int_{\mathbb{R}^d\times\mathbb{R}}\bar{M}^2(\bar{x},\bar{t})\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t})\,d\bar{x}d\bar{t}\Bigg)^{\frac12},
\end{aligned}
$$

where $\delta_2'=\varepsilon CR'$, $\tau_2'=\varepsilon CR'$.

Take now

$$
\bar{M}(\bar{x},\bar{t})=\frac{(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})}{\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t})}.
$$

Then,

$$
\begin{aligned}
&\|\nabla u\|_{L^1_tL^1_x}^{\frac12}\Bigg(\big\|\nabla u*\varphi_{\delta_2,\tau_2}\big\|_{L^1_tL^1_x}-2\int_{\mathbb{R}^d\times\mathbb{R}}\bar{M}(\bar{x},\bar{t})\cdot(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\,d\bar{x}d\bar{t}\\
&\quad+\int_{\mathbb{R}^d\times\mathbb{R}}\bar{M}^2(\bar{x},\bar{t})\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t})\,d\bar{x}d\bar{t}\Bigg)^{\frac12}\\
&\quad\lesssim\|\nabla u\|_{L^1_tL^1_x}^{\frac12}\left(\big\|\nabla u*\varphi_{\delta_2,\tau_2}\big\|_{L^1_tL^1_x}-\int_{\mathbb{R}^d\times\mathbb{R}}\frac{\left|(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right|^2}{\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t})}\,d\bar{x}d\bar{t}\right)^{\frac12}\\
&\quad\lesssim\|\nabla u\|_{L^1_tL^1_x}^{\frac12}\Bigg(\big\|\nabla u*\varphi_{\delta_2,\tau_2}\big\|_{L^1_tL^1_x}-\int_{\mathbb{R}^d\times\mathbb{R}}\left|(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right|\,d\bar{x}d\bar{t}\\
&\qquad+\int_{\mathbb{R}^d\times\mathbb{R}}\bigg(\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t})-\left|(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right|\bigg)\\
&\qquad\qquad\times\frac{\left|(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right|}{\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t})}\,d\bar{x}d\bar{t}\Bigg)^{\frac12},
\end{aligned}
$$

where we now observe that

$$
\left|(\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right|\le\left(\big|(\nabla u*\varphi_{\delta_2,\tau_2})(x,t)\big|*\varphi_{\delta_2',\tau_2'}\right)(\bar{x},\bar{t}),
$$

and so, finally,

$$\|\nabla u\|_{L_t^1 L_x^1}^{\frac{1}{2}} \Bigg(\left\|\nabla u * \varphi_{\delta_2,\tau_2}\right\|_{L_t^1 L_x^1} - \int_{\mathbb{R}^d \times \mathbb{R}} \left|(\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right| \, d\bar{x} d\bar{t}$$
$$+ \int_{\mathbb{R}^d \times \mathbb{R}} \left(\left(\left|(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)\right| * \varphi_{\delta_2',\tau_2'} \right)(\bar{x},\bar{t}) - \left|(\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right| \right)$$
$$\times \frac{\left|(\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right|}{\left(\left|(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)\right| * \varphi_{\delta_2',\tau_2'} \right)(\bar{x},\bar{t})} \, d\bar{x} d\bar{t} \Bigg)^{\frac{1}{2}}$$
$$\lesssim \|\nabla u\|_{L_t^1 L_x^1}^{\frac{1}{2}} \Bigg(\left\|\nabla u * \varphi_{\delta_2,\tau_2}\right\|_{L_t^1 L_x^1} - \int_{\mathbb{R}^d \times \mathbb{R}} \left|(\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right| \, d\bar{x} d\bar{t}$$
$$+ \int_{\mathbb{R}^d \times \mathbb{R}} \left(\left|(\nabla u * \varphi_{\delta_2,\tau_2})(x,t)\right| * \varphi_{\delta_2',\tau_2'} \right)(\bar{x},\bar{t}) - \left|(\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\delta_2',\tau_2'})(\bar{x},\bar{t})\right| \, d\bar{x} d\bar{t} \Bigg)^{\frac{1}{2}}$$
$$\lesssim \|\nabla u\|_{L_t^1 L_x^1}^{\frac{1}{2}} \left(\left\|\nabla u * \varphi_{\delta_2,\tau_2}\right\|_{L_t^1 L_x^1} - \left\|\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\delta_2',\tau_2'}\right\|_{L_t^1 L_x^1} \right)^{\frac{1}{2}}.$$

□

The next necessary bound is then:

Proposition 2.20.

$$\left(e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_{\delta}^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d}$$
$$\lesssim e^{2d\Lambda} \left(\|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \|\rho\|_{L_t^1 L_x^\infty}^{\frac{d-2}{d}} \|\rho\|_{L_t^1 L_x^2}^{\frac{1}{d}}$$
$$\times \left(\left\|\rho * \varphi_{\delta_1}\right\|_{L_t^1 L_x^2} - \left\|\rho * \varphi_{\delta_1} * \varphi_{\delta e^\Lambda R^{-1} R'}\right\|_{L_t^1 L_x^2} \right)^{\frac{1}{d}},$$

for some $R' > 0$ dependent only on the mollifiers $\varphi(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ and $\varphi'(t) \in C_c^\infty(\mathbb{R};\mathbb{R})$, with the requirement that $\varphi(x)$, $\varphi'(t)$ are non-negative, symmetric, with $\varphi(0), \varphi'(0) > 0$. Note that dimension $d \geq 2$.

In the definition of say $(\rho * \varphi_{\delta_1})(x,t)$ for $t \in [0,T]$, we extended $\rho(x,t)$ and $u(x,t)$ first by zero outside of $t \in [0,T]$.

Proof. Firstly, by interpolation between $L_t^1 L_x^\infty$ and $L_t^1 L_x^2$, and by (2.42),

$$\left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_{\delta}^{\Lambda,\bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d}$$

$$
\begin{aligned}
&\lesssim \left(\|\rho\|_{L^1_t L^\infty_x} \|\varphi_{\delta_1}\|_{L^1_x} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left|\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right| \right\|_{L^1_x} + \|\rho\|_{L^1_t L^\infty_x} \|\varphi_{\delta_1}\|_{L^1_x} \right)^{\frac{d-2}{d}} \\
&\quad \times \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right) - \rho * \varphi_{\delta_1} \right| \right\|_{L^1_t L^2_x}^{\frac{2}{d}} \\
&\lesssim \left(1 + e^{(d-2)\Lambda}\right) \|\rho\|_{L^1_t L^\infty_x}^{\frac{d-2}{d}} \left\| \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right) - \rho * \varphi_{\delta_1} \right| \right\|_{L^1_t L^2_x}^{\frac{2}{d}} .
\end{aligned}
$$

As in the previous proof, let $R' > 0$ be such that $\varphi(x) > 0$ for all $|x| \leq \frac{1}{R'}$ (and for the previous theorem, $\varphi'(t) > 0$ for all $|t| \leq \frac{1}{R'}$), so that $\varphi_{\delta e^\Lambda R^{-1}R'}(x) \gtrsim \left(\delta e^\Lambda R^{-1} R'\right)^{-d}$ is bounded below for $|x| \leq \delta e^\Lambda R$. Then, also $(\varphi_{\delta e^\Lambda R^{-1}R'} * \varphi_{\delta e^\Lambda R^{-1}R'})(x) \gtrsim \left(\delta e^\Lambda R^{-1} R'\right)^{-d}$ is bounded below for $|x| \leq \delta e^\Lambda R$. For each $t \in [0,T]$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x,t) - (\rho * \varphi_{\delta_1})(x,t) \right|^2 dx \\
&\quad = \int_{\mathbb{R}^d} \sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}} \left| \int_{\mathbb{R}^d} \left(\left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right) \varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x')\, dx' \right|^2 dx \\
&\quad \lesssim \int_{\mathbb{R}^d} \left| \delta^{-d} \int_{|x-x'|\leq \delta e^\Lambda R} \left| \left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right| dx' \right|^2 dx \\
&\quad \lesssim \delta^{-d} e^{d\Lambda} \int_{\mathbb{R}^d} \int_{|x-x'|\leq \delta e^\Lambda R} \left(\left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right)^2 dx'\, dx \\
&\quad \lesssim \delta^{-d} e^{d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\delta e^\Lambda\right)^d \left(\left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right)^2 \\
&\qquad\qquad \times \left(\varphi_{\delta e^\Lambda R^{-1}R'} * \varphi_{\delta e^\Lambda R^{-1}R'}\right)(x-x')\, dx'\, dx,
\end{aligned}
$$

and then

$$
\begin{aligned}
&e^{2d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right)^2 \left(\varphi_{\delta e^\Lambda R^{-1}R'} * \varphi_{\delta e^\Lambda R^{-1}R'}\right)(x-x')\, dx'\, dx \\
&\quad = e^{2d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\rho * \varphi_{\delta_1}\right)(x',t) \left(\left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right) \\
&\qquad\qquad \times \left(\varphi_{\delta e^\Lambda R^{-1}R'} * \varphi_{\delta e^\Lambda R^{-1}R'}\right)(x-x')\, dx'\, dx \\
&\qquad - e^{2d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\left(\rho * \varphi_{\delta_1}\right)(x',t) - (\rho * \varphi_{\delta_1})(x,t) \right) (\rho * \varphi_{\delta_1})(x,t) \\
&\qquad\qquad \times \left(\varphi_{\delta e^\Lambda R^{-1}R'} * \varphi_{\delta e^\Lambda R^{-1}R'}\right)(x'-x)\, dx'\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= e^{2d\Lambda} \int_{\mathbb{R}^d} \left(\rho * \varphi_{\delta_1}\right)(x',t) \Big(\left(\rho * \varphi_{\delta_1}\right)(x',t) \\
&\qquad - \left(\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'}\right)(x',t) \Big)\, dx' \\
&\quad - e^{2d\Lambda} \int_{\mathbb{R}^d} \Big(\left(\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'}\right)(x,t) \\
&\qquad - (\rho * \varphi_{\delta_1})(x,t) \Big) (\rho * \varphi_{\delta_1})(x,t)\, dx \\
&= 2e^{2d\Lambda} \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})^2(x,t)\, dx \\
&\quad - 2e^{2d\Lambda} \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})(x,t)(\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'})(x,t)\, dx \\
&= 2e^{2d\Lambda} \left(\int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})^2(x,t)\, dx - \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'})^2(x,t)\, dx \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left(e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \\
&\times \left\| \sup_{(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}} \left| \left(\rho * \varphi_{\delta_1} * \varphi_\delta^{\Lambda, \bar{M}(\bar{x},\bar{t})} \right) - \rho * \varphi_{\delta_1} \right| \right\|_{L_t^1 L_x^d} \\
&\quad \lesssim \left(e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \left(1 + e^{(d-2)\Lambda} \right) \|\rho\|_{L_t^1 L_x^\infty}^{\frac{d-2}{d}} \\
&\qquad \times \left(\int_0^T \left(e^{2d\Lambda} \left(\int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})^2(x,t)\, dx - \int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'})^2(x,t)\, dx \right) \right)^{\frac{1}{2}} dt \right)^{\frac{2}{d}} \\
&\quad \lesssim \left(e^{2\Lambda} + e^{d\Lambda} \right) \left(e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \|\rho\|_{L_t^1 L_x^\infty}^{\frac{d-2}{d}} \\
&\qquad \times \left(\int_0^T \left(\left(\int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})^2(x,t)\, dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'})^2(x,t)\, dx \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right. \\
&\qquad \times \left. \left(\left(\int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1})^2(x,t)\, dx \right)^{\frac{1}{2}} - \left(\int_{\mathbb{R}^d} (\rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'})^2(x,t)\, dx \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} dt \right)^{\frac{2}{d}} \\
&\quad \lesssim \left(e^{2\Lambda} + e^{d\Lambda} \right) \left(e^{d\Lambda} \|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}} + \|\rho\|_{L_t^\infty L_x^\infty} \|\nabla u\|_{L_t^\infty L_x^1} \right) \|\rho\|_{L_t^1 L_x^\infty}^{\frac{d-2}{d}} \\
&\qquad\quad \times \left(\left\| \rho * \varphi_{\delta_1} \right\|_{L_t^1 L_x^2} + \left\| \rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L_t^1 L_x^2} \right)^{\frac{1}{d}} \\
&\qquad\quad \times \left(\left\| \rho * \varphi_{\delta_1} \right\|_{L_t^1 L_x^2} - \left\| \rho * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L_t^1 L_x^2} \right)^{\frac{1}{d}}
\end{aligned}
$$

$$\lesssim \left(e^{2\Lambda}+e^{d\Lambda}\right)\left(e^{d\Lambda}\|\rho\|_{L_t^\infty L_x^{\frac{d}{d-1}}}+\|\rho\|_{L_t^\infty L_x^\infty}\|\nabla u\|_{L_t^\infty L_x^1}\right)\|\rho\|_{L_t^1 L_x^\infty}^{\frac{d-2}{d}}\|\rho\|_{L_t^1 L_x^2}^{\frac{1}{d}}$$
$$\times\left(\left\|\rho*\varphi_{\delta_1}\right\|_{L_t^1 L_x^2}-\left\|\rho*\varphi_{\delta_1}*\varphi_{\delta e^\Lambda R^{-1}R'}\right\|_{L_t^1 L_x^2}\right)^{\frac{1}{d}}.$$

□

And the final necessary bounds are the following:

Proposition 2.21.

$$e^{d\Lambda}\|\rho\|_{L_t^\infty L_x^2}\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_0*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_0*\varphi_{\delta_1}\right|\right\|_{L_x^2}$$
$$\lesssim e^{2d\Lambda}\|\rho\|_{L_t^\infty L_x^2}^{\frac{3}{2}}\left(\left\|\rho_0*\varphi_{\delta_1}\right\|_{L_x^2}-\left\|\rho_0*\varphi_{\delta_1}*\varphi_{\delta e^\Lambda R^{-1}R'}\right\|_{L_x^2}\right)^{\frac{1}{2}},$$
$$e^{d\Lambda}\|\rho\|_{L_t^\infty L_x^2}\left\|\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_T*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)-\rho_T*\varphi_{\delta_1}\right|\right\|_{L_x^2}$$
$$\lesssim e^{2d\Lambda}\|\rho\|_{L_t^\infty L_x^2}^{\frac{3}{2}}\left(\left\|\rho_T*\varphi_{\delta_1}\right\|_{L_x^2}-\left\|\rho_T*\varphi_{\delta_1}*\varphi_{\delta e^\Lambda R^{-1}R'}\right\|_{L_x^2}\right)^{\frac{1}{2}},$$

for some $R'>0$ dependent only on the mollifiers $\varphi(x)\in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ and $\varphi'(t)\in C_c^\infty(\mathbb{R};\mathbb{R})$, with the requirement that $\varphi(x)$, $\varphi'(t)$ are non-negative, symmetric, with $\varphi(0),\varphi'(0)>0$. Note that dimension $d\geq 2$.

In the definition of say $(\rho*\varphi_{\delta_1})(x,t)$ for $t\in[0,T]$, we extended $\rho(x,t)$ and $u(x,t)$ first by zero outside of $t\in[0,T]$.

Proof. Since both bounds are essentially the same, we prove only the former, involving $\rho_0(x)$. As in the previous proof, let $R'>0$ be such that $\varphi(x)>0$ for all $|x|\leq\frac{1}{R'}$ (and for the previous theorem, $\varphi'(t)>0$ for all $|t|\leq\frac{1}{R'}$), so that $\varphi_{\delta e^\Lambda R^{-1}R'}(x)\gtrsim\left(\delta e^\Lambda R^{-1}R'\right)^{-d}$ is bounded below for $|x|\leq\delta e^\Lambda R$. Then, also $(\varphi_{\delta e^\Lambda R^{-1}R'}*\varphi_{\delta e^\Lambda R^{-1}R'})(x)\gtrsim\left(\delta e^\Lambda R^{-1}R'\right)^{-d}$ is bounded below for $|x|\leq\delta e^\Lambda R$. For each $t\in[0,T]$,

$$\int_{\mathbb{R}^d}\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\left(\rho_0*\varphi_{\delta_1}*\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}\right)(x)-(\rho_0*\varphi_{\delta_1})(x)\right|^2\,dx$$
$$=\int_{\mathbb{R}^d}\sup_{(\bar{x},\bar{t})\in\mathbb{R}^d\times\mathbb{R}}\left|\int_{\mathbb{R}^d}\left(\left(\rho_0*\varphi_{\delta_1}\right)(x')-(\rho_0*\varphi_{\delta_1})(x)\right)\varphi_\delta^{\Lambda,\bar{M}(\bar{x},\bar{t})}(x-x')\,dx'\right|^2\,dx$$
$$\lesssim\int_{\mathbb{R}^d}\left|\delta^{-d}\int_{|x-x'|\leq\delta e^\Lambda R}\left|\left(\rho_0*\varphi_{\delta_1}\right)(x')-(\rho_0*\varphi_{\delta_1})(x)\right|\,dx'\right|^2\,dx$$

$$
\begin{aligned}
&\lesssim \delta^{-d} e^{d\Lambda} \int_{\mathbb{R}^d} \int_{|x-x'|\le \delta e^{\Lambda} R} \left(\left(\rho_0 * \varphi_{\delta_1}\right)(x') - (\rho_0 * \varphi_{\delta_1})(x) \right)^2 \, dx' \, dx \\
&\lesssim \delta^{-d} e^{d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\delta e^{\Lambda}\right)^d \left(\left(\rho_0 * \varphi_{\delta_1}\right)(x') - (\rho_0 * \varphi_{\delta_1})(x) \right)^2 \\
&\qquad \times \left(\varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right)(x - x') \, dx' \, dx,
\end{aligned}
$$

and then

$$
\begin{aligned}
& e^{2d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\left(\rho_0 * \varphi_{\delta_1}\right)(x',t) - (\rho_0 * \varphi_{\delta_1})(x,t) \right)^2 \left(\varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right)(x - x') \, dx' \, dx \\
&\quad = e^{2d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\rho_0 * \varphi_{\delta_1}\right)(x',t) \left(\left(\rho_0 * \varphi_{\delta_1}\right)(x',t) - (\rho_0 * \varphi_{\delta_1})(x,t) \right) \\
&\qquad\qquad \times \left(\varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right)(x - x') \, dx' \, dx \\
&\qquad - e^{2d\Lambda} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\left(\rho_0 * \varphi_{\delta_1}\right)(x',t) - (\rho_0 * \varphi_{\delta_1})(x,t) \right) (\rho_0 * \varphi_{\delta_1})(x,t) \\
&\qquad\qquad \times \left(\varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right)(x' - x) \, dx' \, dx \\
&\quad = e^{2d\Lambda} \int_{\mathbb{R}^d} \left(\rho_0 * \varphi_{\delta_1}\right)(x',t) \Big(\left(\rho_0 * \varphi_{\delta_1}\right)(x',t) \\
&\qquad\qquad - \left(\rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'}\right)(x',t) \Big) \, dx' \\
&\qquad - e^{2d\Lambda} \int_{\mathbb{R}^d} \Big(\left(\rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'}\right)(x,t) \\
&\qquad\qquad - (\rho_0 * \varphi_{\delta_1})(x,t) \Big) (\rho_0 * \varphi_{\delta_1})(x,t) \, dx \\
&\quad = 2e^{2d\Lambda} \int_{\mathbb{R}^d} (\rho_0 * \varphi_{\delta_1})^2(x,t) \, dx \\
&\qquad - 2e^{2d\Lambda} \int_{\mathbb{R}^d} (\rho_0 * \varphi_{\delta_1})(x,t) (\rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} * \varphi_{\delta e^{\Lambda} R^{-1} R'})(x,t) \, dx \\
&\quad = 2e^{2d\Lambda} \left(\int_{\mathbb{R}^d} (\rho_0 * \varphi_{\delta_1})^2(x,t) \, dx - \int_{\mathbb{R}^d} (\rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'})^2(x,t) \, dx \right) \\
&\quad = 2e^{2d\Lambda} \left(\left\| \rho_0 * \varphi_{\delta_1} \right\|_{L_x^2}^2 - \left\| \rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L_x^2}^2 \right) \\
&\quad = 2e^{2d\Lambda} \left(\left\| \rho_0 * \varphi_{\delta_1} \right\|_{L_x^2} + \left\| \rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L_x^2} \right) \left(\left\| \rho_0 * \varphi_{\delta_1} \right\|_{L_x^2} - \left\| \rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L_x^2} \right) \\
&\quad \lesssim 2e^{2d\Lambda} \left\| \rho \right\|_{L_t^\infty L_x^2} \left(\left\| \rho_0 * \varphi_{\delta_1} \right\|_{L_x^2} - \left\| \rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L_x^2} \right).
\end{aligned}
$$

Putting this together then gives the result. □

We now ask for parameters such that the expression in Theorem 2.18 is bounded by some $\kappa > 0$.

2.3.3 The Tetration Bound

For the following result, note that as in the proof of Proposition 2.17, one may always rescale time for any $\nabla u(x,t) \in L_t^1 L_x^1$ so that $\|\nabla u\|_{L_t^\infty L_x^1} \leq 1$ and the final time is given by $T = \|\nabla u\|_{L_t^1 L_x^1}$.

The following is a weak stability estimate, but the contrapositive implies an analogous lower bound on $\|\rho_0\|_{H_x^{-1}}$, namely a (backwards) mixing bound of the form

$$\|\rho_0\|_{H_x^{-1}} \geq \left(\exp^{\lceil A\exp(\frac{B}{\kappa}T)\rceil}(1)\right)^{-1},$$

with κ depending on the final datum $\rho(\cdot,T)$. This is an exponential-tetration bound in the time T.

Theorem 2.22. *Consider a solution to the forced transport equation*

$$\frac{\partial\rho}{\partial t}(x,t) + u(x,t)\cdot\nabla\rho(x,t) = \psi(x,t),$$

with $u(x,t)$ divergence-free.

Assume $\|\rho\|_{L_t^\infty L_x^\infty}, \|\rho\|_{L_t^\infty L_x^1}, \|\nabla u\|_{L_t^\infty L_x^1} \leq 1$, and $T \geq 1$. Then constants $A,B > 1$ exist depending only on the dimension $d \geq 2$.

For any $0 < \kappa < 1$, if

$$\|\rho_0\|_{H_x^{-1}}, \|\psi\|_{L_t^\infty W_x^{-1,1}} \leq \left(\exp^{\lceil A\exp(\frac{B}{\kappa}T)\rceil}(1)\right)^{-1},$$

where $\exp^n(x)$ refers to repeated exponentiation (tetration), $\exp(\exp(\ldots(\exp(x))))$, then

$$\|\rho(\cdot,T)\|_{H_x^{-1}} \leq \kappa.$$

Proof. We ask for parameters such that each of the terms in Theorem 2.18 is bounded by κ. That is, using that $\|\rho\|_{L_t^\infty L_x^\infty}, \|\rho\|_{L_t^\infty L_x^1}, \|\nabla u\|_{L_t^\infty L_x^1} \leq 1$, and Proposition 2.19, Proposition 2.20, and Proposition 2.21,

$$\int_{\mathbb{R}^d} |(\rho_0 * \varphi_{\delta_1})(x)|^2\, dx \lesssim \kappa, \tag{2.65}$$

$$\left(\tau + \tau_2 + T\,\|\psi\|_{L_t^\infty W_x^{-1,1}}\right)\frac{1}{\delta^{d+1}}\left(\delta + e^{\Lambda}\right) \lesssim \kappa, \tag{2.66}$$

$$e^{2d\Lambda}\left(\|\rho_0 * \varphi_{\delta_1}\|_{L_x^2} - \|\rho_0 * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda}R^{-1}R'}\|_{L_x^2}\right)^{\frac{1}{2}} \lesssim \kappa, \tag{2.67}$$

$$e^{2d\Lambda}\left(\|\rho_T * \varphi_{\delta_1}\|_{L_x^2} - \|\rho_T * \varphi_{\delta_1} * \varphi_{\delta e^{\Lambda}R^{-1}R'}\|_{L_x^2}\right)^{\frac{1}{2}} \lesssim \kappa, \tag{2.68}$$

$$\frac{e^{(d+1)\Lambda}}{\delta^{d+1}}(1+\delta)T\|\psi\|_{L_t^\infty W_x^{-1,1}} \lesssim \kappa, \tag{2.69}$$

$$\delta_1 e^{(d+2)\Lambda}\left(\frac{T}{\delta\tau}+\frac{T}{\delta^2}(1+\delta)\right) \lesssim \kappa, \tag{2.70}$$

$$\delta_2 e^{(d+1)\Lambda}\frac{T}{\delta} \lesssim \kappa, \tag{2.71}$$

$$\tau_2 e^{(d+1)\Lambda}\left(\frac{1+\delta_1}{\delta_1^d}\frac{1}{\delta}\left(T\|\psi\|_{L_t^\infty W_x^{-1,1}}+T\right)+\frac{T}{\delta\tau}\right) \lesssim \kappa, \tag{2.72}$$

$$e^{2\Lambda}C^{d+1}T^{\frac{1}{2}}\left(\left\|\nabla u*\varphi_{\delta_2,\tau_2}\right\|_{L_t^1L_x^1}-\left\|\nabla u*\varphi_{\delta_2,\tau_2}*\varphi_{\varepsilon CR',\varepsilon CR'}\right\|_{L_t^1L_x^1}\right)^{\frac{1}{2}} \lesssim \kappa, \tag{2.73}$$

$$\left(1+\frac{\delta e^{\Lambda}}{\varepsilon}\right) \le C, \tag{2.74}$$

$$\left(1+\frac{\tau}{\varepsilon}\right) \le C, \tag{2.75}$$

$$\frac{1}{\Lambda}\left(1+\frac{\delta e^{\Lambda}}{\varepsilon}\right)^d\left(1+\frac{\tau}{\varepsilon}\right)T \lesssim \kappa, \tag{2.76}$$

$$\frac{\tau e^{(d+3)\Lambda}}{\delta^{d+1}}(1+\delta)\left(T\|\psi\|_{L_t^\infty W_x^{-1,1}}+T\right) \lesssim \kappa, \tag{2.77}$$

$$\frac{1}{\varepsilon}e^{2d\Lambda}T^{\frac{d-1}{d}}\left(\left\|\rho*\varphi_{\delta_1}\right\|_{L_t^1L_x^2}-\left\|\rho*\varphi_{\delta_1}*\varphi_{\delta e^{\Lambda}R^{-1}R'}\right\|_{L_t^1L_x^2}\right)^{\frac{1}{d}} \lesssim \kappa, \tag{2.78}$$

Note that (2.65) requires smallness of $\left\|\rho_0*\varphi_{\delta_1}\right\|_{L_x^2}$, while both (2.66), and (2.69) require smallness of $\|\psi\|_{L_t^\infty W_x^{-1,1}}$. We therefore deal with this at the end, and after we have chosen δ,Λ, it will become a quantitative requirement on $\rho_0(x),\psi(x,t)$.

We will also, for later use, require

$$\tau \lesssim \kappa T^{-1}e^{-(d+1)\Lambda}\delta^d, \tag{2.79}$$

$$\tau_2 \lesssim \kappa T^{-1}e^{-(d+1)\Lambda}\delta^d, \tag{2.80}$$

$$\delta_1 \lesssim \kappa^2 T^{-2}, \tag{2.81}$$

$$\tau \lesssim \kappa\delta^{d+1}e^{-\Lambda}, \tag{2.82}$$

$$\tau_2 \lesssim \kappa\delta^{d+1}e^{-\Lambda} \tag{2.83}$$

We begin with (2.76),

$$\Lambda \gtrsim \frac{1}{\kappa}\left(1+\frac{\delta e^{\Lambda}}{\varepsilon}\right)^d\left(1+\frac{\tau}{\varepsilon}\right)T,$$

which we instead enforce via

$$\Lambda = \frac{T}{\kappa}, \tag{2.84}$$

$$\tau \lesssim \varepsilon, \tag{2.85}$$

$$\delta \lesssim \varepsilon e^{-\Lambda}. \tag{2.86}$$

Now we deal with (2.77), (2.79), (2.82) and (2.85) by defining $\tau > 0$ in terms of $\delta > 0$,

$$\tau(\delta) = \kappa \delta^{d+1} e^{-(d+3)\Lambda} T^{-1}, \tag{2.87}$$

$$\delta \lesssim 1, \tag{2.88}$$

$$\|\psi\|_{L_t^\infty W_x^{-1,1}} \lesssim \kappa T^{-1}, \tag{2.89}$$

$$\delta^{d+1} \lesssim \kappa^{-1} \varepsilon T. \tag{2.90}$$

Note that in fact (2.89) is enforced by (2.69) and (2.88). The next aim is to choose $\delta > 0$ to make (2.67), (2.68), (2.78) small. First pick $\delta_1 > 0$ to satisfy (2.70) and (2.81). We exploit (2.87), (2.88) and (2.89).

$$\delta_1(\delta, \varepsilon) = \kappa^2 \delta^{d+2} e^{-2(d+2)\Lambda} T^{-2}. \tag{2.91}$$

To exploit the pigeonhole principle argument for weak compactness, define the function $\Phi_1 : (0,\infty) \to (0,\infty)$ by

$$\Phi_1(x) = \delta_1\left(x e^{-\Lambda} R \left(R'\right)^{-1}, \varepsilon\right), \tag{2.92}$$

where if

$$x^d \lesssim \kappa^{-2} T^2, \tag{2.93}$$

then,

$$\Phi_1(x) \lesssim x^2. \tag{2.94}$$

Fix some $N_1 \in \mathbb{N}$, and $\delta_0 > 0$, then define

$$\rho^{(0)}(x,t) = \left(\rho * \varphi_{\Phi_1^{N_1}(\delta_0)}\right)(x,t),$$

$$\rho_0^{(0)}(x) = \left(\rho_0 * \varphi_{\Phi_1^{N_1}(\delta_0)}\right)(x),$$

$$\rho_T^{(0)}(x) = \left(\rho_T * \varphi_{\Phi_1^{N_1}(\delta_0)}\right)(x),$$

and for $n = 1, \dots, N_1$,

$$\rho^{(n)}(x,t) = \left(\rho^{(n-1)} * \varphi_{\Phi_1^{N_1-n}(\delta_0)}\right)(x,t), \tag{2.95}$$

$$\rho_0^{(n)}(x) = \left(\rho_0^{(n-1)} * \varphi_{\Phi_1^{N_1-n}(\delta_0)}\right)(x), \tag{2.96}$$

$$\rho_T^{(n)}(x) = \left(\rho_T^{(n-1)} * \varphi_{\Phi_1^{N_1-n}(\delta_0)}\right)(x). \tag{2.97}$$

where $\Phi_1^n(x)$ for $n \in \mathbb{N}$ refers to repeated application of $\Phi_1(\Phi_1(\dots(\Phi_1(x))))$, and $\Phi_1^0(x) = x$ is the identity map.

We now define our pigeonholes as follows,

$$\begin{aligned}
&\sum_{n=1}^{N_1} \left(\left\|\rho^{(n-1)}\right\|_{L_t^1 L_x^2} - \left\|\rho^{(n)}\right\|_{L_t^1 L_x^2} + T\left\|\rho_0^{(n-1)}\right\|_{L_x^2} - T\left\|\rho_0^{(n)}\right\|_{L_x^2} + T\left\|\rho_T^{(n-1)}\right\|_{L_x^2} - T\left\|\rho_T^{(n)}\right\|_{L_x^2} \right) \\
&\quad = \left\|\rho^{(0)}\right\|_{L_t^1 L_x^2} - \left\|\rho^{(N_1)}\right\|_{L_t^1 L_x^2} + T\left\|\rho_0^{(0)}\right\|_{L_x^2} - T\left\|\rho_0^{(N_1)}\right\|_{L_x^2} + T\left\|\rho_T^{(0)}\right\|_{L_x^2} - T\left\|\rho_T^{(N_1)}\right\|_{L_x^2} \\
&\quad \lesssim T,
\end{aligned}$$

where each

$$\begin{aligned}
\left\|\rho^{(n)}\right\|_{L_t^1 L_x^2} &\le \left\|\rho^{(n-1)}\right\|_{L_t^1 L_x^2}, \\
\left\|\rho_0^{(n)}\right\|_{L_x^2} &\le \left\|\rho_0^{(n-1)}\right\|_{L_x^2}, \\
\left\|\rho_T^{(n)}\right\|_{L_x^2} &\le \left\|\rho_T^{(n-1)}\right\|_{L_x^2}.
\end{aligned}$$

Therefore, there exists some $n \in \{1, \dots, N_1\}$ such that

$$\left\|\rho^{(n-1)}\right\|_{L_t^1 L_x^2} - \left\|\rho^{(n-1)} * \varphi_{\Phi_1^{N_1-n}(\delta_0)}\right\|_{L_t^1 L_x^2} \lesssim \frac{T}{N_1}, \tag{2.98}$$

$$\left\|\rho_0^{(n-1)}\right\|_{L_x^2} - \left\|\rho_0^{(n-1)} * \varphi_{\Phi_1^{N_1-n}(\delta_0)}\right\|_{L_x^2} \lesssim \frac{1}{N_1}, \tag{2.99}$$

$$\left\|\rho_T^{(n-1)}\right\|_{L_x^2} - \left\|\rho_T^{(n-1)} * \varphi_{\Phi_1^{N_1-n}(\delta_0)}\right\|_{L_x^2} \lesssim \frac{1}{N_1}. \tag{2.100}$$

In order for (2.98) to bound (2.78), and similarly for (2.99) with (2.67), and (2.100) with (2.68), we need to control the differences

$$\left\|\rho^{(n-1)} - \rho * \varphi_{\Phi_1^{N_1-(n-1)}(\delta_0)}\right\|_{L_t^1 L_x^2},$$
$$\left\|\rho_0^{(n-1)} - \rho_0 * \varphi_{\Phi_1^{N_1-(n-1)}(\delta_0)}\right\|_{L_x^2},$$
$$\left\|\rho_T^{(n-1)} - \rho_T * \varphi_{\Phi_1^{N_1-(n-1)}(\delta_0)}\right\|_{L_x^2}.$$

We do this for the first case, with the others following similarly. By the expression (2.95), we have for each $m \in \{1, \dots, N_1\}$,

$$\begin{aligned}\left\|\rho^{(m)} - \rho * \varphi_{\Phi_1^{N_1-m}(\delta_0)}\right\|_{L_t^1 L_x^2} \leq{}& \left\|\rho^{(m-1)} - \rho * \varphi_{\Phi_1^{N_1-(m-1)}(\delta_0)}\right\|_{L_t^1 L_x^2} \left\|\varphi_{\Phi_1^{N_1-m}(\delta_0)}\right\|_{L_x^1} \\ &+ \left\|\rho * \varphi_{\Phi_1^{N_1-(m-1)}(\delta_0)} * \varphi_{\Phi_1^{N_1-m}(\delta_0)} - \rho * \varphi_{\Phi_1^{N_1-m}(\delta_0)}\right\|_{L_t^1 L_x^2}.\end{aligned}$$

Since $\varphi(x)$ is non-negative and a standard mollifier, so $\|\varphi\|_{L_x^1} = 1$, we have by induction, and (2.93) and (2.94), that if

$$\delta_0 \lesssim 1, \tag{2.101}$$
$$\delta_0^d \lesssim \kappa^{-2} T^2, \tag{2.102}$$

then $\Phi_1^i(\delta_0) \leq \delta_0$ for all $i \in \{1, \dots, N_1\}$, and

$$\begin{aligned}\left\|\rho^{(m)} - \rho * \varphi_{\Phi_1^{N_1-m}(\delta_0)}\right\|_{L_t^1 L_x^2} &\leq \sum_{i=1}^{m} \left\|\rho * \varphi_{\Phi_1^{N_1-(i-1)}(\delta_0)} * \varphi_{\Phi_1^{N_1-i}(\delta_0)} - \rho * \varphi_{\Phi_1^{N_1-i}(\delta_0)}\right\|_{L_t^1 L_x^2} \\ &\lesssim \sum_{i=1}^{m} \Phi_1^{N_1-(i-1)}(\delta_0) \left\|\rho * \nabla\varphi_{\Phi_1^{N_1-i}(\delta_0)}\right\|_{L_t^1 L_x^2} \\ &\lesssim \sum_{i=1}^{m} T \frac{\Phi_1^{N_1-(i-1)}(\delta_0)}{\Phi_1^{N_1-i}(\delta_0)} \\ &\lesssim \sum_{i=1}^{m} T \Phi_1^{N_1-i}(\delta_0) \\ &\lesssim T N_1 \delta_0.\end{aligned}$$

and similarly

$$\left\| \rho_0^{(m)} - \rho_0 * \varphi_{\Phi_1^{N_1-m}(\delta_0)} \right\|_{L^2_x} \lesssim N_1 \delta_0,$$
$$\left\| \rho_T^{(m)} - \rho_T * \varphi_{\Phi_1^{N_1-m}(\delta_0)} \right\|_{L^2_x} \lesssim N_1 \delta_0.$$

So by (2.98), (2.99), (2.100), for

$$\delta(\varepsilon) = \Phi_1^{N_1-n}(\delta_0) e^{-\Lambda} R (R')^{-1} \lesssim \delta_0, \tag{2.103}$$

then, by (2.92),

$$\left\| \rho * \varphi_{\delta_1(\delta,\varepsilon)} \right\|_{L^1_t L^2_x} - \left\| \rho * \varphi_{\delta_1(\delta,\varepsilon)} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L^1_t L^2_x} \lesssim \frac{T}{N_1} + T N_1 \delta_0, \tag{2.104}$$
$$\left\| \rho_0 * \varphi_{\delta_1(\delta,\varepsilon)} \right\|_{L^2_x} - \left\| \rho_0 * \varphi_{\delta_1(\delta,\varepsilon)} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L^2_x} \lesssim \frac{1}{N_1} + N_1 \delta_0, \tag{2.105}$$
$$\left\| \rho_T * \varphi_{\delta_1(\delta,\varepsilon)} \right\|_{L^2_x} - \left\| \rho_T * \varphi_{\delta_1(\delta,\varepsilon)} * \varphi_{\delta e^{\Lambda} R^{-1} R'} \right\|_{L^2_x} \lesssim \frac{1}{N_1} + N_1 \delta_0, \tag{2.106}$$

Therefore, to satisfy (2.88), (2.90), (2.101), (2.102), and most importantly, (2.67), (2.68), (2.78), we must choose $N_1 \in \mathbb{N}$, and $\delta_0 > 0$, such that

$$\delta_0 \lesssim 1,$$
$$\delta_0^{d+1} \lesssim \kappa^{-1} \varepsilon T,$$
$$\delta_0^d \lesssim \kappa^{-1} T,$$
$$\delta_0^d \lesssim \kappa^{-2} T^2,$$
$$\frac{T}{N_1} + T \delta_0 N_1 \lesssim \kappa^d \varepsilon^d \exp\left(-2d^2 \frac{T}{\kappa} \right) T^{1-d},$$
$$\frac{1}{N_1} + \delta_0 N_1 \lesssim \kappa^2 \exp\left(-4d \frac{T}{\kappa} \right).$$

Recalling that $T \geq 1$, this is possible for some $C_d > 1$ depending only on dimension and the mollifiers, where we choose N_1, δ_0 by

$$\frac{1}{\varepsilon^d} \exp\left(C_d \left(1 + \frac{T}{\kappa} \right) \right) \leq N_1 \leq \frac{1}{\varepsilon^d} \exp\left(C_d \left(1 + \frac{T}{\kappa} \right) \right) + 1,$$
$$\frac{1}{\delta_0} = \frac{1}{\varepsilon^{2d}} \exp\left(2C_d \left(1 + \frac{T}{\kappa} \right) \right),$$
$$\varepsilon \lesssim 1.$$

Then, by the expressions (2.91) and (2.92), i.e. $\Phi_1(x) = \frac{\kappa^2}{T^2} e^{-(d+2)\Lambda} x^{d+2} (R')^{d+2}$, then for some $C'_d > 1$,

$$\log(\Phi_1(x)) \geq C'_d \left(\log(x) - 1 - \frac{T}{\kappa} \right),$$

so

$$\log(\Phi_1^n(x)) \geq (C'_d)^n \log(x) - n (C'_d)^n \left(1 + \frac{T}{\kappa}\right).$$

Therefore, by (2.103)

$$\begin{aligned} \frac{1}{\delta(\varepsilon)} &\lesssim e^{\frac{T}{\kappa}} \left(\frac{1}{\delta_0}\right)^{(C'_d)^{N_1}} \exp\left(N_1 (C'_d)^{N_1} \left(1 + \frac{T}{\kappa}\right)\right) \\ &\lesssim \exp\left(\exp\left(\exp\left(C''_d \left(1 + \frac{1}{\varepsilon} + \frac{T}{\kappa}\right)\right)\right)\right), \end{aligned}$$

for some $C''_d > 1$. To summarise, under the assumptions (recalling (2.69)),

$$\|\psi\|_{L_t^\infty W_x^{-1,1}} \lesssim \kappa T^{-1} \delta^{d+1} e^{-(d+1)\Lambda}, \tag{2.107}$$

$$\varepsilon \lesssim 1, \tag{2.108}$$

we have chosen

$$\Lambda = \frac{T}{\kappa},$$

and $\delta(\varepsilon), \delta_1(\varepsilon), \tau(\varepsilon)$ with the bounds (recall (2.87), (2.91))

$$\frac{1}{\varepsilon} \lesssim \frac{1}{\delta(\varepsilon)}, \frac{1}{\delta_1(\varepsilon)}, \frac{1}{\tau(\varepsilon)} \lesssim \exp^3\left(C'''_d \left(1 + \frac{1}{\varepsilon} + \frac{T}{\kappa}\right)\right), \tag{2.109}$$

for some $C'''_d > 1$, so that (2.69), (2.70), (2.76), (2.77), and (2.67), (2.68), (2.78) are satisfied. That is we are left to choose $\varepsilon, \delta_2, \tau_2$ such that (recall also (2.85), and (2.86))

$$\delta_2 e^{(d+1)\Lambda} \frac{T}{\delta} \lesssim \kappa, \tag{2.110}$$

$$\tau_2 e^{(d+1)\Lambda} \left(\frac{1}{\delta_1^d} \frac{T}{\delta} + \frac{T}{\delta \tau} \right) \lesssim \kappa, \tag{2.111}$$

$$\tau_2 \lesssim \kappa T^{-1} e^{-(d+1)\Lambda} \delta^d, \tag{2.112}$$

$$\tau_2 \lesssim \kappa \delta^{d+1} e^{-\Lambda}, \tag{2.113}$$

$$e^{2\Lambda} T^{\frac{1}{2}} \left(\left\|\nabla u * \varphi_{\delta_2,\tau_2}\right\|_{L_t^1 L_x^1} - \left\|\nabla u * \varphi_{\delta_2,\tau_2} * \varphi_{\varepsilon CR',\varepsilon CR'}\right\|_{L_t^1 L_x^1} \right)^{\frac{1}{2}} \lesssim \kappa. \tag{2.114}$$

To satisfy (2.110), (2.111), and (2.112), (2.113), we use (2.109) and define

$$\frac{1}{\delta_2(\varepsilon)}, \frac{1}{\tau_2(\varepsilon)} = \exp^3\left(C_d''''\left(1+\frac{1}{\varepsilon}+\frac{T}{\kappa}\right)\right), \tag{2.115}$$

for some $C_d'''' > 1$. It remains now the choose $\varepsilon \lesssim 1$ to satisfy (2.114). To exploit the pigeonhole principle argument for weak compactness once more, define the function $\Phi_2 : (0,\infty) \to (0,\infty)$ by

$$\Phi_2(x) = \left(\exp^3\left(C_d''''\left(1+\frac{CR'}{x}+\frac{T}{\kappa}\right)\right)\right)^{-1}, \tag{2.116}$$

where if

$$x \lesssim 1,$$

then,

$$\Phi_2(x) \lesssim x^2.$$

Then, as for (2.104), for any $\varepsilon_0 > 0$, $N_2 \in \mathbb{N}$, for some $n \in \{1,\dots,N_2\}$, and for

$$\varepsilon = \Phi^{N_2-n}(\varepsilon_0)C^{-1}(R')^{-1} \lesssim \varepsilon_0, \tag{2.117}$$

then, by (2.115) and (2.116),

$$\left\|\nabla u * \varphi_{\delta_2(\varepsilon),\tau_2(\varepsilon)}\right\|_{L^1_t L^1_x} - \left\|\nabla u * \varphi_{\delta_2(\varepsilon),\tau_2(\varepsilon)} * \varphi_{\varepsilon CR',\varepsilon CR'}\right\|_{L^1_t L^1_x} \lesssim \frac{T}{N_2} + TN_2\varepsilon_0.$$

So, we satisfy (2.114) by

$$\frac{T^2}{\kappa^2}e^{4\Lambda} \le N_2 \le \frac{T^2}{\kappa^2}e^{4\Lambda}+1,$$
$$\varepsilon_0 \lesssim \frac{\kappa^4}{T^4}e^{-8\Lambda}.$$

We then observe that

$$\Phi_2^n(x) \ge \left(\exp^{C_d'''''(1+n)}\left(1+\frac{1}{x}+\frac{T}{\kappa}\right)\right)^{-1},$$

for some $C_d''''' > 1$ independent of $n \in \mathbb{N}$, and $x \le C_d''''''$.

Putting everything together, there exists constants $A_d, B_d > 1$ depending only on the dimension d and the mollifiers, so that (by choosing a sufficiently small $\varepsilon_0 > 0$ in (2.117)), there exists $\varepsilon, \delta, \delta_1, \delta_2, \tau, \tau_2 > 0$ with

$$\exp^{\left\lceil \frac{A_d}{2} \exp\left(B_d \frac{T}{\kappa}\right) \right\rceil}(1) \leq \frac{1}{\varepsilon}, \frac{1}{\delta}, \frac{1}{\delta_1}, \frac{1}{\delta_2}, \frac{1}{\tau}, \frac{1}{\tau_2} \leq \exp^{\left\lceil A_d \exp\left(B_d \frac{T}{\kappa}\right) \right\rceil}(1). \tag{2.118}$$

where $\exp^n(x)$ refers to repeated exponentiation (tetration), such that if also (2.65), (2.66), (2.69) are satisfied, then

$$\int_{\mathbb{R}^d} |(\rho_T * \varphi_{\delta_1})(x)|^2 \, dx \lesssim \kappa. \tag{2.119}$$

Now, by (2.82), (2.83), (2.88), we see that (2.66) reduces to

$$T \|\psi\|_{L_t^\infty W_x^{-1,1}} \frac{1}{\delta^{d+1}} e^\Lambda \lesssim \kappa.$$

Meanwhile, (2.65) can be reduced to (since also $\delta \lesssim 1$, (2.88))

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(\rho_0 * \varphi_{\delta_1})(x)|^2 \, dx \\
&\qquad \lesssim \left\|\rho_0 * \varphi_{\delta_1}\right\|_{H_x^{-1}} \left\|\rho_0 * \varphi_{\delta_1}\right\|_{H_x^1} \\
&\qquad \lesssim \|\rho_0\|_{H_x^{-1}} \left(1 + \frac{1}{\delta}\right) \|\rho_0\|_{L_x^2} \\
&\qquad \lesssim \frac{1}{\delta} \|\rho_0\|_{H_x^{-1}}.
\end{aligned}$$

Therefore, (2.65), (2.66), (2.69) are satisfied if

$$\|\rho_0\|_{H_x^{-1}}, \|\psi\|_{L_t^\infty W_x^{-1,1}} \leq \left(\exp^{\left\lceil A'_d \exp\left(B'_d \frac{T}{\kappa}\right) \right\rceil}(1)\right)^{-1}, \tag{2.120}$$

for some constants $A'_d, B'_d > 0$ depending only on the dimension d and the mollifiers. Note we have used that $T \geq 1$.

Then we have bounded all the terms in Theorem 2.22 by κ, and so

$$\int_{\mathbb{R}^d} |(\rho_T * \varphi_{\delta_1})(x)|^2 \, dx \lesssim \kappa.$$

Now

$$\|\rho(\cdot, T)\|_{H_x^{-1}} = \sup_{\|\phi\|_{H_x^1} \leq 1} \int_{\mathbb{R}^d} \rho_T(x) \phi(x) \, dx,$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_T(x)\phi(x)\,dx &= \int_{\mathbb{R}^d} \rho_T(x)\left(\phi - \phi * \varphi_{\delta_1}\right)(x)\,dx + \int_{\mathbb{R}^d} (\rho_T * \varphi_{\delta_1})(x)\phi(x)\,dx \\ &\leq \|\rho_T\|_{L^2_x} \left\|\phi - \phi * \varphi_{\delta_1}\right\|_{L^2_x} + \left\|\rho_T * \varphi_{\delta_1}\right\|_{L^2_x} \|\phi\|_{L^2_x} \\ &\lesssim (\delta_1 + \kappa)\, \|\phi\|_{H^1_x} \\ &\lesssim \kappa \|\phi\|_{H^1_x},\end{aligned}$$

where we have used (2.118), and that $T \geq 1$, to bound $\delta_1 \lesssim \kappa$.

So, under the assumption (2.120),

$$\|\rho(\cdot,T)\|_{H^{-1}_x} \lesssim \kappa,$$

which completes the proof, by redefining $\kappa > 0$ and the constants $A'_d, B'_d > 0$ in (2.120). □

2.4 Concluding Remarks

This chapter has demonstrated the use of weak compactness arguments in deriving uniform estimates for the transport equation. Our novel approach has yielded two completely novel results: uniform decay of the DiPerna-Lions commutator (Theorem 2.8) and uniform weak stability of transport along BV_x vector fields (Theorem 2.22). These findings not only advance our understanding of the transport equation but also illustrate the broad applicability of this technique in the analysis of partial differential equations.

Applying, in particular, the uniform decay of the DiPerna-Lions commutator, we give substantial improvements to classical mixing estimates for the transport equation. We establish exponential lower bounds on the mixing scale for all initial data $\rho_0(x) \in L^q_x \cap L^1_x$ (Section 2.2.2, Theorems 2.12 and 2.13). Notably, an exponential lower bound of some form is known to be sharp, as evidenced by examples of self-similar mixing [5]. An intriguing aspect of our results, and those in the literature, is that the mixing rate depends on the choice of the mixing parameter $\kappa > 0$ and the initial datum. Our results improve upon existing work by showing that the mixing rate depends on the initial datum only through the ratio $\frac{\|\rho_0\|_{L^q_x}}{\|\rho_0\|_{L^r_x}}$. Whether dependence on the initial datum or the parameter $\kappa > 0$ is necessary remains an open question. We note that when $\nabla u(x,t) \in L^1_t L^\infty_x$, this dependence vanishes due to the uniform exponential bound on trajectory separation, (2.5). In this setting, our result allows us to take $q = r$, making the ratio $\frac{\|\rho_0\|_{L^q_x}}{\|\rho_0\|_{L^r_x}}$ equal to 1, thus independent of the initial datum.

A key contribution of this chapter is the state-of-the-art weak stability estimates presented in Section 2.2.3, Theorems 2.15 and 2.16. These results are likely sub-optimal, as one might expect a bound of the form:

$$\|\rho\|_{L_t^\infty W_x^{-1,1}} \leq \kappa + \exp\left(\frac{C_p}{\kappa}\|\rho\|_{L_t^\infty L_x^q}\|\nabla u\|_{L_t^1 L_x^p}\right)\left(\|\rho_0\|_{W_x^{-1,1}} + \|\psi\|_{L_t^1 W_x^{-1,1}}\right), \qquad (2.121)$$

where there is no fractional power on the weak norms of the initial datum $\rho_0(x)$ or force $\psi(x,t)$, and the constant preceding the exponential is 1, ensuring the estimate's utility for short time scales. However, for small $A > 0$, expressions of the form $\kappa + A\exp\left(\frac{T}{\kappa}\right)$ are crudely optimised by $\kappa = \frac{2T}{-\log A}$. Consequently, the bound (2.121) does not improve upon the implied logarithmic decay

$$\|\rho\|_{L_t^\infty W_x^{-1,1}} \lesssim \left(-\log\left(\|\rho\|_{W_x^{-1,1}} + \|\psi\|_{L_t^1 W_x^{-1,1}}\right)\right)^{-1},$$

implied by Theorem 2.15, and similarly for Theorem 2.16. Whether such logarithmic stability is optimal for transport along Sobolev vector fields remains an open question. In contrast, when $\nabla u \in L_t^1 L_x^\infty$, it is straightforward to demonstrate the estimate

$$\|\rho\|_{L_t^\infty W_x^{-1,1}} \leq e^{\|\nabla u\|_{L_t^1 L_x^\infty}}\left(\|\rho_0\|_{W_x^{-1,1}} + \|\psi\|_{L_t^1 W_x^{-1,1}}\right),$$

which is linear in $\|\rho_0\|_{W_x^{-1,1}} + \|\psi\|_{L_t^1 W_x^{-1,1}}$.

The remaining challenge is determining the optimal values of the constants C_p appearing in these results, mainly their behaviour as $p \to 1$. We have provided the first weak stability estimate in the regime $p = 1$ by combining weak compactness arguments with a specific harmonic analysis technique (Proposition 2.19) to decompose measures uniformly.

While representing a significant advance, this result falls short of an exponential bound of the form (2.121). Whether any such exponential mixing bounds or logarithmic stability hold in the regime $p = 1$ is widely regarded as the most critical open problem in the study of the linear transport equation, first conjectured in some form by Bressan [22]. The current estimate, the exponential-tetration weak stability in Theorem 2.22, might be improved to the rate

$$\|\rho_0\|_{W_x^{-1,1}}, \|\psi\|_{L_t^1 W_x^{-1,1}} \leq \left(\exp^{\left\lceil \frac{A}{\kappa}\|\nabla u\|_{L_t^1 BV_x}\right\rceil}(1)\right)^{-1}, \qquad (2.122)$$

where the argument in the tetration is linear in $\|\nabla u\|_{L_t^1 BV_x}$ rather than exponential. This value essentially represents the number of 'pigeon-holes' of the form $\|\nabla u * \varphi_\delta\|_{L_t^1 L_x^1} - \|\nabla u * \varphi_\delta * \varphi_\varepsilon\|_{L_t^1 L_x^1}$ required in the weak compactness argument in Theorem 2.22. Improving the regularity of the anisotropic mollifier in Theorem 2.18 from exponential to linear in the parameter Λ would then correspondingly reduce the number of pigeon-holes. We refer to Ambrosio's original proof [8], which employs Alberti's rank-one theorem [3] to construct a different anisotropic mollifier with significantly better regularity. Finally, Proposition 2.19 should be replaced with an estimate that is linear in the differences $\|\nabla u * \varphi_\delta\|_{L^1_t L^1_x} - \|\nabla u * \varphi_\delta * \varphi_\varepsilon\|_{L^1_t L^1_x}$, perhaps analogous to the improved pointwise decay of the DiPerna-Lions commutator (Theorem 2.11) compared to norm decay (Corollary 2.9).

Improvement beyond a tetration rate such as (2.122) is not achievable using the methods developed in this chapter. To this end, it is perhaps interesting to note that these weak compactness arguments yield the optimal rate for the DiPerna-Lions commutator (Theorem 2.10). The implied possibility that the tetration rate (2.122) might be optimal has not been previously considered and would be extremely surprising.

In conclusion, the results presented in this chapter provide significant new insights into the properties of solutions transported by Sobolev vector fields below Lipschitz regularity, particularly regarding the evolution of specific length scales of the solution. This has led to state-of-the-art mixing bounds and weak stability estimates, including in the notoriously challenging regime where $u(x,t) \in L^1_t BV_x$.

Chapter 3

Improved Regularity and Well-posedness of the Transport-Diffusion Equation

3.1 Introduction

This chapter examines the transport of passive scalars in the context of less regular divergence-free vector fields, specifically those belonging to the space $L_t^p L_x^q$ for $1 \leq p, q \leq \infty$. As previously discussed in Section 2.1.4, these conditions can lead to non-unique solutions in passive scalar transport, including the counterintuitive "perfect unmixing" phenomenon where non-zero solutions emerge from zero initial data. We investigate how introducing isotropic diffusion can restore well-posedness to this model.

We focus on the transport-diffusion equation, also known as the advection-diffusion equation. Including a diffusion parameter $\kappa > 0$ allows for more robust energy estimates and leads to weak solutions' uniqueness and regularity, largely independent of the vector field $u(x,t)$. While such estimates are standard, they often fail to capture additional regularity in more general function spaces or uniqueness for weak solutions lacking a-priori regularity for energy estimates. Such challenges have been extensively studied for related problems like the heat and Navier-Stokes equations. Still, the literature on the regularity of the transport-diffusion equation remains comparatively sparse.

This chapter aims to address the gap in the literature regarding the regularity of the transport-diffusion equation. We introduce:

- The definition of mild solutions (Section 3.1.3)
- Weak convolution estimates (Section 3.1.4)
- Maximal regularity of the heat equation (Section 3.1.5)

- Mixed derivative estimates (Section 3.1.5)

As a result of these techniques, we present new regularity estimates for the transport-diffusion equation in equations (3.18) and (3.19). Section 3.2 provides state-of-the-art well-posedness and regularity estimates for the transport-diffusion equation, comparable to known results for other parabolic equations. We present three key results:

1. Theorem 3.10: A condition for weak solutions to lie in the classical energy class
2. Theorem 3.11: Improved uniqueness
3. Theorem 3.13: Enhanced regularity of weak solutions matching that of the heat equation

A key focus is on the Ladyzhenskaya-Prodi-Serrin condition for the vector field, which captures the interplay between diffusion and transport. This condition, originally introduced in the context of the Navier-Stokes equations [91], [99], [70], captures the interplay between diffusion and transport through the integrability of vector field. When $u(x,t) \in L_t^p L_x^q$ it may be expressed by a condition on the Hölder exponents:

$$\frac{2}{p} + \frac{d}{q} \leq 1,$$

in terms of the dimension d.

The chapter concludes with Section 3.3, where we discuss various extensions and further applications of our findings.

3.1.1 The Transport-Diffusion Equation

The passive transport of a scalar $\rho(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ along a (divergence-free) vector field $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is given by

$$\begin{gathered} \frac{\partial \rho}{\partial t}(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \\ \rho(x,0) = \rho_0(x). \end{gathered} \tag{3.1}$$

Including a diffusivity constant $\kappa > 0$, the equation instead becomes

$$\begin{gathered} \frac{\partial \rho}{\partial t}(x,t) + u(x,t) \cdot \nabla \rho(x,t) - \kappa \Delta \rho(x,t) = 0, \\ \rho(x,0) = \rho_0(x), \end{gathered} \tag{3.2}$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the spatial coordinates $x \in \mathbb{R}^d$. Similar to the transport equation, classical solutions to parabolic equations of this type are well-posed when the coefficients are continuously differentiable, $u(x,t) \in C^1(\mathbb{R}^d \times [0,T])$, but also permit additional regularity and decay due to the smoothing effect of diffusion [49]. Like the parallel transport formulation (2.3) of passive scalar transport (3.1), one may equally treat the transport-diffusion problem (3.2) as a parallel transport along stochastic trajectories solving the diffused/stochastic differential equation

$$\begin{aligned} dX(x,t,\omega) &= u(X(x,t,\omega),t)dt + \sqrt{2\kappa}dW(t,\omega), \\ X(x,0,\omega) &= x, \end{aligned} \tag{3.3}$$

for a Brownian motion $W(t,\omega)$ on a filtered probability space $\omega \in (\Omega, \mathscr{F}, \mathscr{F}_t, P)$ [64, Section 5]. The Feynman-Kac representation [64, Section 5 Theorem 7.6], then gives an explicit representation of classical solutions $\rho(x,t)$ of (3.2) in terms of the stochastic flow $X(x,t,\omega)$. If we parameterise the paths so that $X'(x,s,\omega) = x$, namely:

$$\begin{aligned} dX'(x,t,\omega) &= u(X'(x,t,\omega),t)dt + \sqrt{2\kappa}dW(t,\omega), \\ X'(x,s,\omega) &= x, \end{aligned}$$

then $\rho(x,s)$ is parallel transported by the average over $\omega \in (\Omega, \mathscr{F}, P)$

$$\rho(x,s) = \int_{\Omega} \rho(X'(x,t,\omega),t)d\omega. \tag{3.4}$$

We will not work further with classical solutions to (3.2). Instead, we define weak solutions to (3.2) analogously to Definition 2.1 for the transport equation.

Definition 3.1 (Transport-diffusion equation - weak solutions)**.** Consider a vector field $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ with $\nabla \cdot u(x,t) = 0$ in the distributional sense, and some positive constant diffusion $\kappa > 0$ (also called viscosity).

We say $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T]; \mathbb{R})$ with $u(x,t)\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ is a weak solution to

$$\begin{aligned} \frac{\partial \rho}{\partial t}(x,t) + u(x,t) \cdot \nabla \rho(x,t) - \kappa \Delta \rho(x,t) &= 0, \\ \rho(x,0) &= \rho_0(x), \end{aligned} \tag{κ-TDE}$$

for initial datum $\rho_0(x) \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R})$ if, for any $\phi(x,t) \in C_c^\infty(\mathbb{R}^d \times [0,T); \mathbb{R})$,

$$\int_{\mathbb{R}^d \times [0,T)} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) + \kappa \Delta \phi(x,t) \right) dxdt = -\int_{\mathbb{R}^d} \rho_0(x) \phi_0(x)\, dx,$$

where $\phi_0(x) = \phi(x,0)$. Meanwhile, we say the transport-diffusion equation (κ-TDE) is satisfied on an *open* interval $I \subset (0,T)$ if, for any $\phi(x,t) \in C_c^\infty(\mathbb{R}^d \times I;\mathbb{R})$,

$$\int_{\mathbb{R}^d \times I} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) + \kappa \Delta \phi(x,t) \right) dxdt = 0.$$

As for the transport equation, we have a basic weak continuity result; see, for instance, [60, Section 3].

Theorem 3.1 (Weak continuity of (κ-TDE))**.** *Suppose $\rho(x,t)$ is a weak solution to* (κ-TDE)*. Then for any $\phi(x,t) \in C_c^\infty(\mathbb{R}^d \times [0,T];\mathbb{R})$, for a.e. $t \in [0,T]$,*

$$\text{(Trace Formula)} \quad \int_{\mathbb{R}^d} \rho(x,t)\phi(x,t)\, dx$$
$$= \int_{\mathbb{R}^d} \rho_0(x)\phi_0(x)\, dx + \int_{\mathbb{R}^d \times [0,t]} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) + \kappa \Delta \phi(x,t) \right) dxdt, \tag{3.5}$$

Suppose further that $\rho(x,t) \in L_t^\infty L_x^p$ for $1 < p \leq \infty$, then (there is a representation of $\rho(x,t)$ with) $\rho(x,t) \in C^0_{\text{weak}-*}([0,T];L^p(\mathbb{R}^d;\mathbb{R}))$, such that (2.10) *holds for all $t \in [0,T]$. In particular, $\rho(x,0) = \rho_0(x) \in L_x^p$.*

If $\rho(x,t) \in L_t^\infty L_x^1$ with $\{\rho(\cdot,t)\}_{t\in[0,T]} \subset L_x^1$ uniformly integrable, then similarly $\rho(x,t) \in C^0_{\text{weak}}([0,T];L^1(\mathbb{R}^d;\mathbb{R}))$.

We mention briefly that, as for the transport equation, the transport-diffusion equations also permit a superposition principle for weak solutions [48, Section 2]. This allows one to rewrite *positive* weak solutions to the PDE (κ-TDE) in terms of martingale solutions to the SDE and vice-versa, via the Fokker-Planck formula (3.4).

In contrast to weak solutions to the transport equation, Theorem 4.2, the parabolic nature of the PDE (κ-TDE) should give rise to solutions with additional smoothness. Indeed, one may show that strong solutions satisfy a range of additional regularity estimates (3.6) below [45, Section 7.1], and (3.7) below [30, Theorem 1.1]. One may combining these results with the standard regularisation method, see [41]; one then constructs weak solutions with the following additional regularity.

Theorem 3.2 (Existence of weak solutions to (κ-TDE))**.** *Let $1 \leq p,q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\rho_0(x) \in L_x^p$, and $u(x,t) \in L_t^1 L_x^q$ is divergence-free in the distributional sense. Then a weak solution exists $\rho(x,t) \in L_t^\infty L_x^p$ to* (κ-TDE).

If in addition $\rho_0(x) \in L^2_x$, *then for a.e.* $t \in [0,T]$,

$$\text{(Energy Inequality)} \quad \int_{\mathbb{R}^d} |\rho(x,t)|^2 \, dx + 2\kappa \int_0^t \int_{\mathbb{R}^d} |\nabla \rho(x,s)|^2 \, dxds \le \int_{\mathbb{R}^d} |\rho_0(x)|^2 \, dx. \tag{3.6}$$

If in addition $\rho_0(x) \in L^r_x$ *for any* $1 \le r \le \infty$, *then for all* $r \le q' \le \infty$, *for a.e.* $t \in [0,T]$

$$\|\rho(\cdot,t)\|_{L^{q'}_x} \le t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q'}\right)} C_{r,q'} \|\rho_0\|_{L^r_x}, \tag{3.7}$$

for some constant $C_{r,q'} > 0$, *with in particular* $C_{r,r} = 1$, *depending only on the parameters* r, q' *and the dimension* $d \ge 1$.

This existence may be taken as a linear map in the initial data $\rho_0(x)$. Then, by applying suitable interpolation theorems, we will show in Section 3.2, Lemma 3.12, that the bound (3.7) may be improved to

$$\left(\int_0^T \|\rho(\cdot,t)\|^{p'}_{L^{q'}_x} \, dt \right)^{\frac{1}{p'}} \le C_{r,q'} \|\rho_0\|_{L^r_x}, \tag{3.8}$$

where

$$\frac{2}{p'} + \frac{d}{q'} = \frac{d}{r},$$

for $0 < p' \le \infty$, $1 < r \le q' < \infty$, and some constant $C_{r,q'} > 0$ with in particular $C_{r,r} = 1$, depending only on the parameters r, q' and the dimension $d \ge 1$. The case $q' = \infty$ perhaps also holds, though not by these methods. We discuss this more in the concluding remarks, Section 3.3.

3.1.2 Well-posedness in the Energy Class

Parabolic equations such as the transport-diffusion equation (κ-TDE) are also well-posed under far less regularity on the coefficients of the equation (on $u(x,t)$). The standard theory proceeds via energy estimates, requiring the solution $\rho(x,t)$ to have enough regularity to be taken as a test function $\phi(x,t)$. To this end, we introduce the notion of an energy class of solutions, also sometimes called parabolic solutions [20, Definition 2.3], to the transport-diffusion equation.

Definition 3.2 (Transport-diffusion equation - energy class)**.** We say a weak solution to the transport-diffusion equation (κ-TDE) with $\rho_0(x) \in L^2_x$ is in the energy class if $\rho(x,t) \in L^\infty_t L^2_x$, and $\nabla \rho(x,t) \in L^2_t L^2_x$.

Unlike for the inviscid transport equation (TE), one should expect the above regularity even for weak solutions. Indeed from the energy inequality (3.6) in Theorem 3.2, we have existence of weak solutions with $\rho(x,t) \in L_t^\infty L_x^2$ and $\nabla\rho(x,t) \in L_t^2 L_x^2$ for any vector field $u(x,t) \in L_t^1 L_x^2$ if $\rho_0(x) \in L_x^2$. This may even be improved to $u(x,t) \in L_t^1 L_x^1$ if additionally $\rho_0(x) \in L_x^2 \cap L_x^\infty$.

This is perhaps reminiscent of the non-linear Navier-Stokes equations, for which existence of 'Leray-Hopf' weak solutions in $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ can be guaranteed by a similar energy inequality to (3.6) [78]. To this end, it may make sense to call weak solutions to (κ-TDE), which are not just in the energy class but also satisfy (3.6), 'Leray-Hopf'-like solutions. However, we shall see in the following theorem that the introduction of such solutions is not particularly necessary, as one can improve the energy inequality (3.6) to equality under the mild integrability $u(x,t) \in L_t^2 L_x^2$.

By standard methods for renormalised solutions and energy estimates, see [20, Theorem 2.7], [45, Chapter 7.1], we have the following theorem.

Theorem 3.3. *Suppose $\rho(x,t) \in L_t^\infty L_x^2$ with $\nabla\rho(x,t) \in L_t^2 L_x^2$ and $\rho_0(x) \in L_x^2$ is a weak solution to* (κ-TDE).

If $u(x,t) \in L_t^2 L_x^2$, then $\rho(x,t)$ additionally satisfies the estimate (for a.e. $t \in [0,T]$)

$$\text{(Energy Identity)} \quad \int_{\mathbb{R}^d} |\rho(x,t)|^2 \, dx + 2\kappa \int_0^t \int_{\mathbb{R}^d} |\nabla\rho(x,s)|^2 \, dx\, ds = \int_{\mathbb{R}^d} |\rho_0(x)|^2 \, dx. \quad (3.9)$$

We note in particular that the energy identity (3.9) with $\rho_0(x) = 0$ implies the uniqueness of weak solutions in the energy class under the mild integrability assumption $u(x,t) \in L_t^2 L_x^2$.

This is far more general than for the well-posedness result for the transport equation, Theorem 2.3, which requires Sobolev regularity on the vector field. On the other hand, non-uniqueness of weak solutions in the energy class, $\rho(x,t) \in L_t^\infty L_x^2$ and $\nabla\rho(x,t) \in L_t^2 L_x^2$, with $u(x,t) \notin L_t^2 L_x^2$ also holds, see [85, Thoerem 1.4]. One may also show non-uniqueness of weak solutions outside the energy class when $u(x,t) \in L_t^2 L_x^2$ [85, Thoerem 1.3]. However, the sharpness of the condition $L_t^2 L_x^2$ in both cases remains an open problem.

We will later show in Section 3.2, Theorem 3.10, that when $u(x,t)\rho(x,t) \in L_t^2 L_x^2$ then any weak solution is in the energy class, $\rho(x,t) \in L_t^\infty L_x^2$ and $\nabla\rho(x,t) \in L_t^2 L_x^2$, and so also the above well-posedness result applies. This may be applied quite generally under integrability assumptions on $\rho(x,t)$ and $u(x,t)$. To this end, we note that the bound (3.8) is quite useful, or that one may interpolate and apply Sobolev embedding between the energy regularity $\rho(x,t) \in L_t^\infty L_x^2$ and $\nabla\rho(x,t) \in L_t^2 L_x^2$ [1]. As a simple corollary, we see the importance of another condition on $u(x,t)$. This is the Ladyzhenskaya-Prodi-Serrin condition [71, Chapter

3]; $u(x,t) \in L_t^p L_x^q$ where

$$\frac{2}{p} + \frac{d}{q} = 1. \tag{3.10}$$

Then any solution given by say Theorem 3.2 also satisfies $u(x,t)\rho(x,t) \in L_t^2 L_x^2$.

The original motivation for studying the Ladyzhenskaya-Prodi-Serrin condition (3.10) comes from the non-linear Navier-Stokes equations [78], for which this class of vector fields are scaling critical, and various uniqueness and smoothing properties have been proved [91], [99], [70]. It is perhaps somewhat surprising then that for the linear transport-diffusion equation, one also has well-posedness under the weaker (naively super-critical) condition $u(x,t) \in L_t^2 L_x^2$, Theorem 3.3. This is essentially explained by the maximum principle for the the transport-diffusion equation.

We will see later that the Ladyzhenskaya-Prodi-Serrin condition (3.10) is related to further regularity and uniqueness of weak solutions to the transport-diffusion equation. For example, it is known that, under the sub-critical Ladyzhenskaya-Prodi-Serrin condition $u(x,t) \in L_t^p L_x^q$,

$$\frac{2}{p} + \frac{d}{q} < 1, \tag{3.11}$$

then the stochastic formulation of transport-diffusion, the SDE (3.3), is well-posed [68]. The unique path measure even admits an explicit Radon-Nikodym derivative in terms of the Weiner measure [68, Lemma 3.3]. By the superposition principle [48, Lemma 2.3], one may then write any non-negative weak solution to (κ-TDE) in this form.

3.1.3 Mild Solutions

So far, our analysis has focused on energy estimates for solutions to the transport-diffusion equation, which is the most standard approach. However, stronger regularity and well-posedness may be shown by carefully analysing the heat kernel's regularity against the Ladyzhenskaya-Prodi-Serrin condition (3.10). To this end, we must first rewrite a weak solution to (κ-TDE) as an integral solution to the heat equation with force. Such solutions are usually referred to as mild solutions [79], [46, Section 2]. For the transport-diffusion equation, this reads as follows.

Definition 3.3 (Transport-diffusion equation - mild solutions)**.** Consider a vector field $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ with $\nabla \cdot u(x,t) = 0$ in the distributional sense, and some positive constant diffusion $\kappa > 0$ (also called viscosity).

We say $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$ with $u(x,t)\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ is a mild solution to (κ-TDE) for initial datum $\rho_0(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$ if additionally

$$u(x,t)\rho(x,t) \in \mathscr{S}'(\mathbb{R}^d \times [0,T];\mathbb{R}),$$
$$\rho_0(x) \in \mathscr{S}'(\mathbb{R}^d;\mathbb{R}),$$

are Schwartz distributions [58, Section 7], and in the sense of distribution

$$\rho(x,t) = e^{\kappa t\Delta}\rho_0(x) - \nabla \cdot \left(\int_0^t e^{\kappa(t-s)\Delta} \left(u(x,s)\rho(x,s) \right) \, ds \right), \tag{3.12}$$

where

$$e^{\kappa t\Delta}\psi(x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \exp\left(-\frac{|x-y|^2}{4\kappa t} \right) \psi(y) \, dy,$$

denotes convolution in space with the heat kernel. That is, for any test function $\psi(x,t) \in C_c^\infty(\mathbb{R}^d \times [0,T],\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}^d \times [0,T]} \rho(x,t)\psi(x,t) \, dxdt = & \int_{\mathbb{R}^d} \rho_0(x) \left(\int_0^T e^{\kappa t\Delta}\psi(x,t) \, dt \right) \, dx \\ & + \int_{\mathbb{R}^d \times [0,T]} \rho(x,s)u(x,s) \cdot \nabla \left(\int_s^T e^{\kappa(t-s)\Delta}\psi(x,t) \, dt \right) \, dxds. \end{aligned}$$

Remark 3. The expression for mild solutions, (3.12), may equally be written in terms of convolution $*$ over $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ as

$$\rho(x,t)\mathbb{1}_{t\in[0,T]} = e^{\kappa t\Delta}\rho_0(x)\mathbb{1}_{t\in[0,T]} - \nabla \cdot \left(K(x,t) * \left(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]} \right) \right) \mathbb{1}_{t\in[0,T]}, \tag{3.13}$$

where $K(x,t)$ denotes the heat kernel with diffusivity $\kappa > 0$ on $\mathbb{R}^d \times \mathbb{R}$, given by

$$K(x,t) = \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4\kappa t} \right) \mathbb{1}_{t>0}, \tag{3.14}$$

and $\mathbb{1}_{t\in I}$ denotes the indicator function of the set $(x,t) \in \mathbb{R}^d \times I$. The heat kernel $K(x,t) \in C^\infty(\mathbb{R}^d \times \mathbb{R} \setminus (0,0))$ is smooth except at the origin.

Under the assumption that $\rho_0 \in \mathscr{S}'(\mathbb{R}^d;\mathbb{R})$, and $u(x,t)\rho(x,t) \in \mathscr{S}'(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ are Schwartz distributions, then weak and mild solutions to (κ-TDE) are equivalent by rewriting the test functions $\phi(x,t)$ in Definition 3.1, and $\psi(x,t)$ in Definition 3.3, as

$$\phi(x,s) = \int_s^T e^{\kappa(t-s)\Delta}\psi(x,t)\,dt,$$
$$\psi(x,t) = \frac{\partial\phi}{\partial t}(x,t) + \kappa\Delta\phi(x,t).$$

Proposition 3.4. *Consider a vector field $u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$ with $\nabla \cdot u(x,t) = 0$ in the distributional sense.*

Let $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$ with $u(x,t)\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$, and $\rho_0(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$. Suppose, in addition, that

$$\rho_0(x) \in \mathscr{S}'(\mathbb{R}^d;\mathbb{R}),$$
$$u(x,t)\rho(x,t) \in \mathscr{S}'(\mathbb{R}^d \times [0,T];\mathbb{R}).$$

Then $\rho(x,t)$ is a weak solution to (κ-TDE)*, Definition 3.1, if and only if it is a mild solution to* (κ-TDE)*, Definition 3.3.*

One of the main advantages of rewriting the equation in this way is that we may study the regularity of weak solutions to (κ-TDE) via the regularity of the heat kernel. This strategy is very successful for non-linear equations [51], [46].

3.1.4 Convolution Estimates for the Heat Kernel

The immediate advantage of working with mild solutions, is that instead of having to use energy estimates, one may apply standard convolution theorems to the expression (3.13),

$$\rho(x,t)\mathbb{1}_{t\in[0,T]} = e^{\kappa t\Delta}\rho_0(x)\mathbb{1}_{t\in[0,T]} - \nabla\cdot\left(K(x,t) * \left(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]}\right)\right)\mathbb{1}_{t\in[0,T]},$$

without assuming any regularity on the mild solution $\rho(x,t)$. Indeed, one may even control inhomogeneous $L^p_t L^q_x$ spatial-temporal norms by an inhomogeneous version of Young's convolution inequality [18, Theorem 3.9.4].

To this end, estimating the regularity of the heat kernel $K(x,t)$ and its derivatives becomes necessary. This is easily done by using the self-similar scaling $K(x,t) = K(xt^{-\frac{1}{2}},1)$ for $t > 0$, and the regularity $K(x,1) \in \mathscr{S}(\mathbb{R}^d;\mathbb{R})$ a Schwartz function. Similar to [99], we have

Theorem 3.5 (Heat regularity)**.** *For the heat kernel*

$$K(x,t) = \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4\kappa t}\right) \mathbb{1}_{t>0},$$

we have

$$\left\| t^{\frac{1}{p}} \frac{\partial^r}{\partial t^r} \nabla^s K(x,t) \right\|_{L_t^\infty L_x^q} < \infty, \tag{3.15}$$

where

$$\frac{2}{p} + \frac{d}{q} = d + 2r + s,$$

for any $1 \leq p,q \leq \infty$*,* $r,s \in \mathbb{N} \cup \{0\}$.

A suitable version for fractional derivatives may be shown similarly.

The expression (3.15) is unwieldy. Instead, it is convenient to work with the weak Lebesgue space $L_t^{p,\infty}$, obtained by real interpolation of regular Lebesgue spaces [15, Theorem 5.2.1]. Theorem 3.5 now states that

$$\frac{\partial^r}{\partial t^r} \nabla^s K(x,t) \in L_t^{p,\infty} L_x^q, \tag{3.16}$$

where

$$\frac{2}{p} + \frac{d}{q} = d + 2r + s.$$

This is useful when combined with the following non-standard convolution theorem. The following is a consequence of the Banach-valued Marcinkiewicz interpolation theorem [15, Theorem 5.3.1], applied to the inhomogeneous convolution theorem [18, Theorem 3.9.4], and is essentially an inhomogeneous version of the standard weak convolution theorem [54, Theorem 1.4.25].

Theorem 3.6 (Young's inhomogeneous convolution inequality for weak spaces)**.** *Let* $1 < p, p_1, p_2 < \infty$*, and* $1 \leq q, q_1, q_2 \leq \infty$ *where*

$$1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$
$$1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

then

$$\|f * g\|_{L_t^p L_x^q} \leq C_{p,p_1,p_2} \|f\|_{L_t^{p_1} L_x^{q_1}} \|g\|_{L_t^{p_2,\infty} L_x^{q_2}},$$

for some constant $C_{p,p_1,p_2} > 0$ *depending only on the parameters* $1 < p, p_1, p_2 < \infty$ *and the dimension* $d \geq 1$.

We remark that one may similarly prove an analogue of Theorem 3.6 for general Lorentz spaces, akin to [89, Theorem 2.6], which even applies in the endpoint case $p = \infty$ if we replace $L_t^{p_1}$ with the Lorentz space $L_t^{p_1,1}$, see for instance [89, Theorem 3.6].

Applying the convolution Theorem 3.6 to the heat kernel regularity $\frac{\partial^r}{\partial t^r}\nabla^s K(x,t) \in L_t^{p,\infty} L_x^q$ (3.16), one has the following important estimate, similar to the analysis in [51].

Theorem 3.7. *Let* $F \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R})$ *be a Schwartz distribution, and let* $f(x,t) = (K * F)(x,t) \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R})$ *a distributional solution to the heat equation*

$$\frac{\partial f}{\partial t}(x,t) - \kappa \Delta f(x,t) = F(x,t).$$

Then, for some constant $C_{p,p'} > 0$ *depending only on the parameters* $1 < p < p' < \infty$, *the diffusivity* $\kappa > 0$, *and the dimension* $d \geq 1$, *we have the following bound.*

$$\left\| \frac{\partial^r}{\partial t^r} \nabla^s f \right\|_{L_t^{p'} L_x^{q'}} \leq C_{p,p'} \|F\|_{L_t^p L_x^q},$$

where

$$\frac{2}{p'} + \frac{d}{q'} = \frac{2}{p} + \frac{d}{q} + 2r + s - 2,$$

with $1 < p < p' < \infty$, $1 \leq q \leq q' \leq \infty$, $r, s \in \mathbb{N} \cup \{0\}$.

A suitable version for fractional derivatives may be shown similarly. Note also that the sub-critical case

$$\frac{2}{p'} + \frac{d}{q'} < \frac{2}{p} + \frac{d}{q} + 2r + s - 2, \tag{3.17}$$

now in the full range $1 \leq p \leq p' \leq \infty$, may be shown more straightforwardly using the standard inhomogeneous convolution estimate [18, Theorem 3.9.4], with the regularity of the heat kernel (3.15) stated in Theorem 3.5. However, in the sub-critical case, the constant $C_{p,p'}$ will also depend on the size of the time interval $T < \infty$.

As a simple application, applying Theorem 3.7 with $s = 1$ and $F(x,t) = u(x,t)\rho(x,t)$ to the expression for mild solutions (3.13), we obtain the following bound for any mild solution to (κ-TDE).

$$\|\rho\|_{L_t^{p'} L_x^{q'}} \leq \left\| e^{t\kappa\Delta} \rho_0(x) \right\|_{L_t^{p'} L_x^{q'}} + C_{p,p'} \|u\rho\|_{L_t^p L_x^q}, \tag{3.18}$$

where

$$\frac{2}{p'} + \frac{d}{q'} = \frac{2}{p} + \frac{d}{q} - 1,$$

with $1 < p < p' < \infty$, $1 \leq q \leq q' \leq \infty$.

The critical observation is that we have not required that $\rho(x,t)$ is in the energy class, Definition 3.2. We remark also that the regularity of the heat solution $e^{t\kappa\Delta}\rho_0(x)$ can be found from the explicit formula for the heat kernel.

We will later show how such estimates may be used to prove the uniqueness of mild solutions outside the energy class, as long as $u(x,t)$ satisfies the Ladyzhenskaya-Prodi-Serrin condition (3.10), see Section 3.2.

3.1.5 Maximal Regularity for the Heat Equation

We notice that we cannot take $p = p'$ in Theorem 3.7, as this would require taking the $L_t^{1,\infty}$ endpoint in the weak convolution estimate, Theorem 3.6. This is a severe obstruction to proving higher regularity of mild solutions to the transport-diffusion equation as it prevents us from taking $r = 1$ or $s = 2$ in Theorem 3.7. However, it turns out that convolution with the heat kernel is similar to a Calderon-Zymund operator [54, Section 5]. So, one may still obtain bounds via Fourier-multiplier theorems [52]. This is commonly referred to as the maximal regularity of the heat equation since it takes the following form.

Theorem 3.8 (Maximal $L_t^pL_x^q$-regularity [57]). *Let $F \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$ be a Schwartz distribution, and $f(x,t) = (K * F)(x,t) \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R};\mathbb{R})$, a solution to the heat equation*

$$\frac{\partial f}{\partial t}(x,t) - \kappa\Delta f(x,t) = F(x,t).$$

Then for some constant $C_{p,q} > 0$ depending only on the parameters $1 < p,q < \infty$, the diffusivity κ, and the dimension $d \geq 1$,

$$\left\| \frac{\partial f}{\partial t} \right\|_{L_t^pL_x^q}, \|\Delta f\|_{L_t^pL_x^q} \leq C_{p,q} \|F\|_{L_t^pL_x^q},$$

with $1 < p,q < \infty$.

So at the cost of the endpoints $q = 1,\infty$, we may extend Theorem 3.7 to give control over $\frac{\partial f}{\partial t}(x,t)$, and $\Delta f(x,t)$. The analogue of the bound (3.18) for mild solutions $\rho(x,t)$ to (κ-TDE) is now

$$\begin{aligned} \left\| \frac{\partial \rho}{\partial t} \right\|_{L_t^pW_x^{-1,q}} &\leq \left\| \frac{\partial}{\partial t} e^{t\kappa\Delta}\rho_0(x) \right\|_{L_t^pW_x^{-1,q}} + C_{p,q} \|u\rho\|_{L_t^pL_x^q}, \\ \|\nabla\rho\|_{L_t^pL_x^q} &\leq \left\| \nabla e^{t\kappa\Delta}\rho_0(x) \right\|_{L_t^pL_x^q} + C_{p,q} \|u\rho\|_{L_t^pL_x^q}, \end{aligned} \tag{3.19}$$

with $1 < p,q < \infty$. We remark that the regularity of the heat solution $e^{t\kappa\Delta}\rho_0(x)$ can be found from the explicit formula for the heat kernel.

We will later show how these estimates may be used to prove the more general existence and regularity of weak solutions to the transport-diffusion equation, in contrast with the more classical result of Theorem 3.2.

Mixed Derivative Estimates

Although we will not need it, we mention for completion that it is often helpful to interpolate maximal regularity, Theorem 3.8, between $\frac{\partial f}{\partial t}(x,t)$ and $\Delta f(x,t)$. However, such a result is not covered by standard interpolation theory. The first such result was given in [102], and such estimates are now known as mixed derivative estimates.

For any Banach space X, we define the homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R}^n;X)$ as in [15, Section 6.3], as the Schwartz distributions $\mathscr{S}'(\mathbb{R}^n;X)$ for which the following homogeneous Sobolev norm is finite,

$$\|f\|_{\dot{H}^{s,p}(\mathbb{R}^n;X)} = \left\|\mathscr{F}^{-1}\left(|\xi|^s\mathscr{F}(f)\right)\right\|_{L^p(\mathbb{R}^n;X)},$$

where $\mathscr{F}$ denotes the Fourier transform in the coordinates $\mathbb{R}^n$, and we denote the Fourier variable by $\xi \in \mathbb{R}^n$. We note that for $1 < p < \infty$ and $s = n \in \mathbb{Z}$ this is equivalent to the usual integer homogeneous Sobolev norm $\dot{W}^{n,p}(\mathbb{R}^n;X)$.

We state the theorem similar to [105, Proposition 4.3]. We use the shorthand $\dot{H}^{r,p}_t\dot{H}^{s,q}_x = \dot{H}^{r,p}(\mathbb{R};\dot{H}^{s,q}(\mathbb{R}^d;\mathbb{R}))$.

Theorem 3.9 (The mixed derivative theorem [102])**.** *Let $f(x,t) \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R})$ be a Schwartz distribution, then*

$$\|f\|_{\dot{H}^{r,p}_t\dot{H}^{s,q}_x} \leq C\,\|f\|^{\theta}_{\dot{H}^{r_1,p}_t\dot{H}^{s_1,q}_x}\,\|f\|^{1-\theta}_{\dot{H}^{r_2,p}_t\dot{H}^{s_2,q}_x},$$

where $1 < p,q < \infty$, $0 < \theta < 1$, and $r_1,r_2,s_1,s_2 \in \mathbb{R}$ with

$$r = \theta r_1 + (1-\theta)r_2,$$
$$s = \theta s_1 + (1-\theta)s_2,$$

and the constant $C > 0$ crudely depends on all the parameters including the dimension $d \geq 1$.

For more general mixed derivative estimates see [84, Theorem 3.1], or [92, Corollary 4.5.10].

3.2 Uniqueness and Regularity beyond the Energy Class

In general, the tools presented in Sections 3.1.4 and 3.1.5 may be used to prove an ocean of different regularity results for weak solutions to the transport-diffusion equation, depending on the integrability or regularity of the vector field $u(x,t)$. Despite being widely used to analyse more general non-linear PDEs, including the Navier-Stokes equations [46], these techniques have not yet readily been applied to the transport-diffusion equation. This section presents a handful of significant applications beyond the current literature on the transport-diffusion equation.

3.2.1 Energy Class Criterion for Weak Solutions

Firstly, we give a general criterion for a weak solution in the energy class, Definition 3.2, namely $\rho(x,t) \in L^\infty_t L^2_x$ and $\nabla\rho(x,t) \in L^2_t L^2_x$. Combined with Theorem 3.3, the following is an improvement over a similar result using commutator estimates [20, Theorem 3.3].

Theorem 3.10. *Let $\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$ be a weak solution to (κ-TDE), so that $\rho_0(x) \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R})$ and $u(x,t)\rho(x,t) \in L^1_{\text{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R}^d)$.*

Suppose both $\rho_0(x) \in L^2_x$, and $u(x,t)\rho(x,t) \in L^2_t L^2_x$.

Then $\rho(x,t) \in L^\infty_t L^2_x$ and $\nabla\rho(x,t) \in L^2_t L^2_x$, with the bounds

$$\|\nabla\rho\|_{L^2_t L^2_x} \leq \frac{1}{\sqrt{2\kappa}} \|\rho_0\|_{L^2_x} + C\|u\rho\|_{L^2_t L^2_x},$$

$$\left\|\frac{\partial\rho}{\partial t}\right\|_{L^2_t \dot{H}^{-1}_x} \leq \sqrt{\frac{\kappa}{2}} \|\rho_0\|_{L^2_x} + C\|u\rho\|_{L^2_t L^2_x},$$

$$\|\rho\|_{L^\infty_t L^2_x} \leq \sqrt{2}\|\rho_0\|_{L^2_x} + C\|u\rho\|_{L^2_x},$$

for some constant $C > 0$ depending only on the diffusivity $\kappa > 0$ and the dimension $d \geq 1$.

Proof. Firstly, the assumed regularity $\rho_0(x) \in L^2_x$ and $u(x,t)\rho(x,t) \in L^2_t L^2_x$ is sufficient for $\rho(x,t)$ to be a mild solution to (κ-TDE), Proposition 3.4. Therefore, we have the expression (3.13),

$$\rho(x,t)\mathbb{1}_{t\in[0,T]} = e^{\kappa t\Delta}\rho_0(x)\mathbb{1}_{t\in[0,T]} - \nabla\cdot\left(K(x,t) * \left(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]}\right)\right)\mathbb{1}_{t\in[0,T]}.$$

Applying Theorem 3.8 with $F(x,t) = u(x,t)\rho(x,t)$ gives

$$\|\nabla\rho\|_{L^2_t L^2_x} \leq \left\|\nabla e^{t\kappa\Delta}\rho_0(x)\right\|_{L^2_t L^2_x} + C\|u\rho\|_{L^2_t L^2_x},$$

for some constant $C > 0$ depending on the diffusivity $\kappa > 0$ and the dimension $d \geq 1$. Note that for the heat solution $e^{t\kappa\Delta}\rho_0(x)$, we immediately have the energy identity (3.9), and so

$$\left\|\nabla e^{t\kappa\Delta}\rho_0(x)\right\|_{L_t^2 L_x^2} \leq \frac{1}{\sqrt{2\kappa}} \|\rho_0\|_{L_x^2},$$

giving the bound

$$\|\nabla\rho\|_{L_t^2 L_x^2} \leq \frac{1}{\sqrt{2\kappa}} \|\rho_0\|_{L_x^2} + C\|u\rho\|_{L_t^2 L_x^2}.$$

Now consider the expression (κ-TDE),

$$\frac{\partial\rho}{\partial t}(x,t) = \kappa\Delta\rho - \nabla\cdot(u(x,t)\rho(x,t)),$$

and so

$$\begin{aligned}\left\|\frac{\partial\rho}{\partial t}\right\|_{L_t^2 \dot{H}_x^{-1}} &\leq \kappa\|\nabla\rho\|_{L_t^2 L_x^2} + \|u\rho\|_{L_t^2 L_x^2} \\ &\leq \sqrt{\frac{\kappa}{2}} \|\rho_0\|_{L_x^2} + (1+\kappa C)\|u\rho\|_{L_t^2 L_x^2}.\end{aligned}$$

It is then left to deduce that $\rho(x,t) \in L_t^\infty L_x^2$. To this end, we wish to use the Lions–Magenes lemma [103, Lemma 1.2]. Since we have $\nabla\rho(x,t) \in L_t^2 L_x^2$ and $\frac{\partial\rho}{\partial t}(x,t) \in L_t^2 \dot{H}_x^{-1}$, it is left to prove that $\rho(x,t) \in L_t^2 L_x^2$ to satisfy the requirements on the Lions-Magenes lemma. We postpone this for now. Then $\rho(\cdot,t) : [0,T] \to L_x^2$ is a continuous function within the sense of the distribution

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} |\rho(x,t)|^2\, dx = 2\int_{\mathbb{R}^d} \frac{\partial\rho}{\partial t}(x,t)\rho(x,t)\, dx,$$

and so by weak continuity, Theorem 3.1, we then see that for $t \in [0,T]$,

$$\int_{\mathbb{R}^d} |\rho(x,t)|^2\, dx = \int_{\mathbb{R}^d} |\rho_0(x,t)|^2 + 2\int_0^t \int_{\mathbb{R}^d} \frac{\partial\rho}{\partial t}(x,t)\rho(x,t)\, dx\, dt,$$

which finally gives the bound

$$\begin{aligned}\|\rho\|_{L_t^\infty L_x^2}^2 &\leq \|\rho_0\|_{L_x^2}^2 + 2\left\|\frac{\partial\rho}{\partial t}\right\|_{L_t^2 \dot{H}_x^{-1}} \|\nabla\rho\|_{L_t^2 L_x^2} \\ &\leq \|\rho_0\|_{L_x^2}^2 + 2\left(\frac{1}{\sqrt{2\kappa}}\|\rho_0\|_{L_x^2} + C\|u\rho\|_{L_t^2 L_x^2}\right)\left(\sqrt{\frac{\kappa}{2}}\|\rho_0\|_{L_x^2} + (1+\kappa C)\|u\rho\|_{L_x^2}\right) \\ &\leq \left(\sqrt{2}\|\rho_0\|_{L_x^2} + C'\|u\rho\|_{L_t^2 L_x^2}\right)^2,\end{aligned}$$

for some constant $C' > 0$ depending only the diffusivity $\kappa > 0$ and the dimension $d \geq 1$.

To prove that indeed $\rho(x,t) \in L_t^2 L_x^2$, we could repeat the argument above with a cutoff on $\rho(x,t)$, but for brevity, we present the following approach instead. We appeal once more to the expression for mild solutions, (3.13),

$$\rho(x,t)\mathbb{1}_{t\in[0,T]} = e^{\kappa t\Delta}\rho_0(x)\mathbb{1}_{t\in[0,T]} - \nabla\cdot\left(K(x,t)*\left(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]}\right)\right)\mathbb{1}_{t\in[0,T]},$$

and this time apply Theorem 3.7 with $F(x,t) = u(x,t)\rho(x,t)$ and $s = 1$ to give

$$\|\rho\|_{L_t^{p'}L_x^2} \leq \left\|e^{t\kappa\Delta}\rho_0(x)\right\|_{L_t^{p'}L_x^2} + C_{p,p'}\|u\rho\|_{L_t^p L_x^2},$$

where

$$\frac{2}{p'} = \frac{2}{p} - 1$$

with $1 < p < p' < \infty$, for some constant $C_{p,p'} > 0$ depending only on the parameters $1 < p < p' < \infty$, the diffusivity $\kappa > 0$, and the dimension $d \geq 1$. Taking say $p = \frac{3}{2}$ and $p' = 3$, and noting that $T < \infty$, then $u(x,t)\rho(x,t) \in L_t^2 L_x^2 \subset L_t^{\frac{3}{2}} L_x^2$, and $e^{t\kappa\Delta}\rho_0(x) \in L_t^\infty L_x^2 \subset L_t^3 L_x^2$. Therefore, $\rho(x,t) \in L_t^3 L_x^2 \subset L_t^2 L_x^2$ as required. □

3.2.2 Well-posedness and Regularity under Ladyzhenskaya-Prodi-Serrin Conditions

This section will study improvements to well-posedness outside the energy class, Definition 3.2, namely $\rho(x,t) \in L_t^\infty L_x^2$ and $\nabla\rho(x,t) \in L_t^2 L_x^2$. We shall see that for a vector field $u(x,t) \in L_t^p L_x^q$ where $1 \leq p,q \leq \infty$ satisfy the Ladyzhenskaya-Prodi-Serrin condition (3.10)

$$\frac{2}{p} + \frac{d}{q} = 1,$$

stronger uniqueness and regularity of weak solutions to the transport-diffusion equation holds. We mention that certain analogues of these results are known for the related Navier-Stokes equations [46].

Our first result is a uniqueness result valid for all weak solutions, and in particular not requiring the regularity $\rho(x,t) \in L_t^\infty L_x^2$ and $\nabla\rho(x,t) \in L_t^2 L_x^2$ of the standard well-posedness result, Theorem 3.3.

Theorem 3.11. *Suppose* $u(x,t) \in L_t^p L_x^q$ *where* $1 \leq p,q \leq \infty$ *satisfy the Ladyzhenskaya-Prodi-Serrin condition* (3.10)

$$\frac{2}{p} + \frac{d}{q} = 1,$$

with $p < \infty$.

Suppose $\rho(x,t)$ *is a weak solution to* (κ-TDE) *with* $\rho_0(x) = 0$, *and* $\rho(x,t)u(x,t) \in L_t^{p'} L_x^{q'}$ *for any* $1 < p' \leq \infty$ *and* $1 \leq q' \leq \infty$. *Then also* $\rho(x,t) = 0$.

Proof. Fixing a finite time interval $T < \infty$, and noting that $p \geq 2$, we may assume that $1 < p' < p$.

Notice first that $\rho_0(x) = 0$ and $u(x,t)\rho(x,t) \in L_t^{p'} L_x^{q'}$ are Schwartz distributions, and so by Proposition 3.4, $\rho(x,t)$ is also mild solution, Definition 3.3. In particular we have the expression (3.13),

$$\rho(x,t)\mathbb{1}_{t\in[0,T]} = e^{\kappa t\Delta}\rho_0(x)\mathbb{1}_{t\in[0,T]} - \nabla\cdot\left(K(x,t)*\left(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]}\right)\right)\mathbb{1}_{t\in[0,T]},$$

with $\rho_0(x) = 0$.

We now apply Theorem 3.7 with $F(x,t) = u(x,t)\rho(x,t) \in L_t^{p'} L_x^1$, $r = 0$, and $s = 1$ to bound

$$\begin{aligned} \|\rho\|_{L_t^{p''} L_x^{q''}} &\leq \left\|\nabla\cdot\left(K(x,t)*\left(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]}\right)\right)\right\|_{L_t^{p''} L_x^{q''}}, \\ &\leq C_{p',p''}\|u\rho\|_{L_t^{p'} L_x^{q'}}, \end{aligned} \tag{3.20}$$

where

$$\frac{2}{p''} + \frac{d}{q''} = \frac{2}{p'} + \frac{d}{q'} - 1, \tag{3.21}$$

with $1 < p' < p'' < \infty$, $1 \leq q' \leq q'' \leq \infty$, and the constant $C_{p',p''} > 0$ depends only on the parameters $1 < p' < p'' < \infty$, the diffusivity $\kappa > 0$, and the dimension $d \geq 2$.

Let

$$\begin{aligned} \frac{1}{p''} &= \frac{1}{p'} - \frac{1}{p}, \\ \frac{1}{q''} &= \frac{1}{q'} - \frac{1}{q}, \end{aligned}$$

where we note that $1 < p' < p < \infty$ and $1 \leq q' \leq q \leq \infty$ so that $1 < p' < p'' < \infty$ and $1 \leq q' \leq q'' \leq \infty$. Then (3.21) is satisfied due to the Ladyzhenskaya-Prodi-Serrin condition (3.10)

$$\frac{2}{p} + \frac{d}{q} = 1,$$

and so

$$\|u\rho\|_{L_t^{p'}L_x^{q'}} \leq \|u\|_{L_t^pL_x^q} \|\rho\|_{L_t^{p''}L_x^{q''}},$$

where in particular $\|\rho\|_{L_t^{p''}L_x^{q''}} < \infty$ by (3.20). Moreover, we then have the inequality

$$\|\rho\|_{L_t^{p''}L_x^{q''}} \leq C_{p',p''} \|u\|_{L_t^pL_x^q} \|\rho\|_{L_t^{p''}L_x^{q''}}. \tag{3.22}$$

Therefore, if $\|u\|_{L_t^pL_x^q} < \frac{1}{C_{p',p''}}$, then $\rho(x,t) = 0$. We must first take the time interval $T < \infty$ sufficiently small to apply this. Noting that we assume $p < \infty$, we may find a sequence of increasing time intervals $[0,T] = \bigcup_{i=0}^{N}[t_i, t_{i+1}]$, with $t_i < t_{i+1}$, such that

$$\left\|u\mathbb{1}_{t\in[t_i,t_{i+1}]}\right\|_{L_t^pL_x^q} < \frac{1}{C_{p',p''}}. \tag{3.23}$$

Firstly, since

$$\rho(x,t)\mathbb{1}_{t\in[0,t_1]} : \mathbb{R}^d \times [0,t_1] \to \mathbb{R},$$

is a mild solution to (κ-TDE) with zero initial data, then by (3.22) and (3.23),

$$\begin{aligned}\left\|\rho\mathbb{1}_{t\in[0,t_1]}\right\|_{L_t^{p''}L_x^{q''}} &\leq C_{p',p''} \left\|u\mathbb{1}_{t\in[0,t_1]}\right\|_{L_t^pL_x^q} \left\|\rho\mathbb{1}_{t\in[0,t_1]}\right\|_{L_t^{p''}L_x^{q''}} \\ &< \left\|\rho\mathbb{1}_{t\in[0,t_1]}\right\|_{L_t^{p''}L_x^{q''}}\end{aligned}$$

with $\left\|\rho\mathbb{1}_{t\in[0,t_1]}\right\|_{L_t^{p''}L_x^{q''}} < \infty$ by (3.20), and so $\rho(x,t)\mathbb{1}_{t\in[0,t_1]}(x,t) = 0$.

Then, by induction, if $\rho(x,t)\mathbb{1}_{t\in[t_i,t_{i+1}]} = 0$, then also

$$\rho(x,t+t_{i+1})\mathbb{1}_{t+t_{i+1}\in[t_{i+1},t_{i+2}]} : \mathbb{R}^d \times [0, t_{i+2} - t_{i+1}] \to \mathbb{R},$$

is a mild solution to (κ-TDE) with zero initial data, and so by (3.22) and (3.23),

$$\begin{aligned}\left\|\rho\mathbb{1}_{t\in[t_{i+1},t_{i+2}]}\right\|_{L_t^{p''}L_x^{q''}} &\leq C_{p',p''} \left\|u\mathbb{1}_{t\in[t_{i+1},t_{i+2}]}\right\|_{L_t^pL_x^q} \left\|\rho\mathbb{1}_{t\in[t_{i+1},t_{i+2}]}\right\|_{L_t^{p''}L_x^{q''}} \\ &< \left\|\rho\mathbb{1}_{t\in[t_{i+1},t_{i+2}]}\right\|_{L_t^{p''}L_x^{q''}},\end{aligned}$$

with $\left\|\rho\mathbb{1}_{t\in[t_{i+1},t_{i+2}]}\right\|_{L_t^{p''}L_x^{q''}} < \infty$ by (3.20), and so also $\rho(x,t)\mathbb{1}_{t\in[t_{i+1},t_{i+2}]} = 0$.

From the induction, we then conclude that $\rho(x,t) = 0$ is required. □

Remark 4. In the case $p = \infty$ we may recover uniqueness with instead a smallness assumption on the norm $\|u\|_{L_t^\infty L_x^d} \leq \varepsilon_\kappa$, depending in particular on the diffusivity $\kappa > 0$.

Remark 5. If we assume $u(x,t) \in L_t^p L_x^q$ where $1 \le p, q \le \infty$ satisfy the sub-critical Ladyzhenskaya-Prodi-Serrin condition (3.11),

$$\frac{2}{p} + \frac{d}{q} < 1,$$

then by applying the more straightforward sub-critical version of Theorem 3.7 valid when $1 \le p \le p' \le \infty$, we also obtain uniqueness when $u(x,t)\rho(x,t) \in L_t^1 L_x^{q'}$, i.e. for $p' = 1$.

We next provide the following primary interpolation result, which shall be used in the proof of the following theorem, and also finish the proof of expression (3.8) in the introduction.

Lemma 3.12. *Suppose, for some $\alpha > 0$ and $\beta \ge 0$, that the linear map*

$$T : C_c^\infty(\mathbb{R}^d; \mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R}^d \times [0,T]; \mathbb{R}),$$

satisfies the bounds

$$\|T(\rho_0)(\cdot,t)\|_{L_x^q} \le t^{-\frac{1}{\alpha}\left(\frac{1}{r} - \frac{1}{q} + \beta\right)} C_{q,r} \|\rho_0\|_{L_x^r},$$

for any $1 \le r \le q \le \infty$, and some constant $C_{q,r} > 0$, depending only on the parameters q, r and the dimension $d \ge 1$. Then also

$$\left(\int_0^T \|\rho(\cdot,t)\|_{L_x^q}^p \, dt \right)^{\frac{1}{p}} \le C'_{q,r} \|\rho_0\|_{L_x^r},$$

where

$$\frac{\alpha}{p} = \frac{1}{r} - \frac{1}{q} + \beta$$

for $0 < p \le \infty$, $1 < r \le q < \infty$, and some constant $C'_{q,r} > 0$, depending only on the parameters q, r and the dimension $d \ge 1$.

The case $q = \infty$ also holds if $\alpha \ge 1$, $\beta = 0$.

Proof. We first interpolate in the parameter $r \in (1,q)$. Notice that we have, for

$$\frac{\alpha}{p} = \frac{1}{r} - \frac{1}{q} + \beta,$$

with $0 < p \le \infty$, that

$$\|T(\rho_0)\|_{L_t^{p,\infty} L_x^q} \le C_{q,r} \|\rho_0\|_{L_x^r},$$

including, in particular, quasi-norms $p \le 1$.

By the Banach-valued Marcinkiewicz interpolation theorem for Hölder exponents in the full range $(0,\infty]$ [15, Theorem 5.3.1], one then has

$$\|T(\rho_0)\|_{L_t^p L_x^q} \le C'_{q,r} \|\rho_0\|_{L_x^{r,p}},$$

where

$$\frac{\alpha}{p} = \frac{1}{r} - \frac{1}{q} + \beta, \tag{3.24}$$

for $0 < p \le \infty$, $1 < r < q \le \infty$ (with the $r = q$ case already known), and some constant $C'_{q,r} > 0$, depending only on the parameters q, r and the dimension $d \ge 1$.

To remove the dependence on the Lorentz norm $L_x^{r,p}$ of the initial data, we would require $r \le p$ (which is implied when $\alpha \ge 1$, $\beta = 0$). However, in general we now fix the parameter $\frac{\alpha}{1+\beta} < p \le \infty$, and interpolate between $1 < r \le \left(\frac{\alpha}{p} - \beta\right)^{-1}$, corresponding to $\left(1 + \beta - \frac{\alpha}{p}\right)^{-1} < q \le \infty$. As a consequence of, say [106, Section 1.18.4] and [15, Theorem 3.11.6], we then have

$$\|T(\rho_0)\|_{L_t^p L_x^q} \le C''_{q,r} \|\rho_0\|_{L_x^{r,q}},$$

where

$$\frac{\alpha}{p} = \frac{1}{r} - \frac{1}{q} + \beta,$$

for $\frac{\alpha}{1+\beta} < p \le \infty$, $1 < r \le q < \infty$, and some constant $C''_{p,r} > 0$, depending only on the parameters p, r and the dimension $d \ge 1$.

The dependence on the Lorentz norm $L_x^{r,q}$ of the initial data can now be controlled by the L_x^r norm since $r \le q$ to deduce the result. □

Our final result is an existence result with further regularity. In particular, in the Ladyzhenskaya-Prodi-Serrin regime $u(x,t) \in L_t^{p'} L_x^{q'}$, we obtain integrability of $\frac{\partial \rho}{\partial t}(x,t)$ and $\nabla \rho(x,t)$ analogous to solutions of the heat equation.

Theorem 3.13. *Suppose $u(x,t) \in L_t^{p'} L_x^{q'}$ where $1 \le p', q' \le \infty$ satisfy the Ladyzhenskaya-Prodi-Serrin condition* (3.10)

$$\frac{2}{p'} + \frac{d}{q'} = 1.$$

Suppose $\rho_0(x) \in L_x^{r'}$ for some $1 \le r' \le \infty$, then there exists a weak (even a mild) solution to (κ-TDE) *such that for any $1 < p, q < \infty$, $1 < r \le q$, with $p \le p'$, $q < q'$, and*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{r} + 1,$$

then

$$\left\| \frac{\partial \rho}{\partial t} \right\|_{L_t^p \dot{W}_x^{-1,q}}, \|\nabla \rho\|_{L_t^p L_x^q} \leq C \left(1 + \|u\|_{L_t^{p'} L_x^{q'}}\right) \|\rho_0\|_{L_x^r}, \tag{3.25}$$

for some constant $C > 0$ *depending on the parameters* $1 < p, q < \infty$*, the diffusivity* $\kappa > 0$*, and the dimension* $d \geq 1$.

Proof. To aid the proof, we shall show the additional bound for any $0 < p \leq \infty$, $1 < r \leq q < \infty$, with

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{r},$$

then

$$\left(\int_0^T \|\rho(\cdot,t)\|_{L_x^q}^p \, dt\right)^{\frac{1}{p}} \leq C'_{q,r} \|\rho_0\|_{L_x^r}, \tag{3.26}$$

for some constant $C'_{q,r} > 0$ with in particular $C'_{r,r} = 1$, depending only on the parameters q, r and the dimension $d \geq 1$.

The solution $\rho(x,t)$ will be given by Theorem 3.2 with initial data $\rho_0(x)$. Note that, since we have $u(x,t) \in L_t^{p'} L_x^{q'}$, the statement of Theorem 3.2 strictly speaking requires the initial datum to be in $L_x^{\frac{q'}{q'-1}}$. To circumvent this issue, we take an approximating sequence of initial data and note that the existence map given by Theorem 3.2 may be taken to be linear in the initial data. By proving the bounds (3.25) and (3.26) for the approximating sequence, it follows that the sequences converge with the limit also satisfying the bounds (3.25) and (3.26). That the limit shall still be a weak (even a mild) solution to (κ-TDE) shall follow from convergence also of $u(x,t)\rho(x,t)$.

The proof, therefore, is divided into three steps. First, we show the bound (3.26) for a solution given by Theorem 3.2 (with suitably regular initial datum). Second, we show that $u(x,t)\rho(x,t)$ is bounded in some appropriate space by $\|\rho_0\|_{L_x^{r'}}$ (and so converges). Third, we show that the bound (3.25) follows from the bound (3.26) (for a solution with suitably regular initial datum).

We first do step 1.

Fix $0 < p \leq \infty$, $1 \leq r \leq q \leq \infty$, with

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{r},$$

then by (3.7), we have

$$\|\rho(\cdot,t)\|_{L_x^q} \leq t^{-\frac{1}{p}} C_{q,r} \|\rho_0\|_{L_x^r}, \tag{3.27}$$

for some constant $C_{q,r} > 0$ with in particular $C_{r,r} = 1$, depending only on the parameters q, r and the dimension $d \geq 1$.

Consider the linear map $T(\rho_0)(x,t) = \rho(x,t)$ given by Theorem 3.2. By (3.27), this map satisfies the assumptions of Lemma 3.12 with $\alpha = \frac{2}{d}$, $\beta = 0$.

Therefore, if $1 < r < q < \infty$, then (3.26) follows from Lemma 3.12,

$$\left(\int_0^T \|\rho(\cdot,t)\|_{L_x^q}^p \, dt\right)^{\frac{1}{p}} \leq C'_{q,r} \|\rho_0\|_{L_x^r}, \tag{3.28}$$

for some constant $C'_{q,r} > 0$ depending only on the parameters q, r and the dimension $d \geq 1$. The case $q = r$ in (3.26) follows from the case $q = r$ in (3.27).

We now do step 2.

We split into two cases. First, consider the case

$$\frac{1}{q} = \frac{1}{r'} + \frac{1}{q'} \leq 1,$$

then (3.27) with $\rho_0(x) \in L_x^{r'}$ gives us immediately

$$\|\rho\|_{L_t^\infty L_x^{r'}} \leq \|\rho_0\|_{L_x^{r'}},$$

and so

$$\|u\rho\|_{L_t^{p'} L_x^q} \leq \|u\|_{L_t^{p'} L_x^{q'}} \|\rho_0\|_{L_x^{r'}},$$

as required.

The second case is when

$$\frac{1}{r'} + \frac{1}{q'} > 1,$$

and so, in particular, $1 < r', q' \leq \infty$. Then let $\frac{1}{q} = 1 - \frac{1}{q'} < \frac{1}{r'}$.

Since then $1 < r' < q < \infty$, (3.28) gives

$$\|\rho(\cdot,t)\|_{L_t^p L_x^q} \leq C'_{q,r'} \|\rho_0\|_{L_x^{r'}},$$

where

$$\frac{1}{p} = \frac{d}{2}\left(\frac{1}{r'} - \frac{1}{q}\right) > 0,$$

and so $\frac{2r'}{d} \leq p < \infty$.

Since $\frac{1}{q} = 1 - \frac{1}{q'}$, and $\frac{2}{p'} + \frac{d}{q'} = 1$, we also have

$$\frac{1}{p} = \frac{d}{2r'} - \frac{d}{2} + \frac{1}{2} - \frac{1}{p'} < \frac{1}{2} - \frac{1}{p'},$$

and so if we define
$$\frac{1}{p''}=\frac{1}{p}+\frac{1}{p'}<\frac{1}{2},$$
then $2<p''<\infty$, and
$$\begin{aligned}\|u\rho\|_{L_t^{p''}L_x^1} &\le \|u\|_{L_t^{p'}L_x^{q'}}\|\rho\|_{L_t^pL_x^q},\\ &\le C'_{q,r'}\|u\|_{L_t^{p'}L_x^{q'}}\|\rho_0\|_{L_x^{r'}},\end{aligned}$$
as required.

We now do step 3.

Fix $1\le p\le p'$, $1\le q\le q'$, $1\le r\le\infty$ with
$$\frac{2}{p}+\frac{d}{q}=\frac{d}{r}+1,$$
and $\rho_0(x)\in L_x^r$.

Define $p\le p''\le\infty$, $q\le q''\le\infty$ by
$$\begin{aligned}\frac{1}{p''}&=\frac{1}{p}-\frac{1}{p'},\\ \frac{1}{q''}&=\frac{1}{q}-\frac{1}{q'}.\end{aligned}$$

Assume now that $1<p,q<\infty$, and $1<r\le q<q'$ so that $1<r\le q\le q''<\infty$.

We take p'',q'',r in (3.26), noting that $\frac{2}{p''}+\frac{d}{q''}=\frac{d}{r}$, so that
$$\|\rho\|_{L_t^{p''}L_x^{q''}}\le C_1\|\rho_0\|_{L_x^r}. \tag{3.29}$$
for some constant $C_1>0$.

Next, one has the bound
$$\|u(x,t)\rho(x,t)\|_{L_t^pL_x^q}\le\|u\|_{L_t^{p'}L_x^{q'}}\|\rho\|_{L_t^{p''}L_x^{q''}}. \tag{3.30}$$

We consider the expression for mild solutions (3.13),
$$\rho(x,t)\mathbb{1}_{t\in[0,T]}=e^{\kappa t\Delta}\rho_0(x)\mathbb{1}_{t\in[0,T]}-\nabla\cdot\big(K(x,t)*\big(u(x,t)\rho(x,t)\mathbb{1}_{t\in[0,T]}\big)\big)\mathbb{1}_{t\in[0,T]}.$$

By maximal regularity, Theorem 3.8 with $F(x,t)=u(x,t)\rho(x,t)\in L_t^pL_x^q$ and $s=1$, we see that
$$\|\nabla\rho\|_{L_t^pL_x^q}\le\left\|\nabla e^{t\kappa\Delta}\rho_0(x)\right\|_{L_t^pL_x^q}+C_2\|u\rho\|_{L_t^pL_x^q}, \tag{3.31}$$

for some constant $C_2 > 0$.

Consider the linear map $T(\rho_0)(x,t) = \nabla e^{t\kappa\Delta}\rho_0(x)$. By the heat kernel regularity (3.15) with $r = 0$, $s = 1$, we have by Young's convolution theorem the bound in Theorem 3.12 for $\alpha = \frac{2}{d}$, $\beta = \frac{1}{2}$. Using the expression

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{r} + 1,$$

we may take p, q, r in Theorem 3.12, since $1 < r \leq q < \infty$, which gives

$$\left\| \nabla e^{t\kappa\Delta}\rho_0(x) \right\|_{L^p_t L^q_x} \leq C_3 \left\| \rho_0 \right\|_{L^r_x}, \tag{3.32}$$

for some constant $C_3 > 0$.

Combining the equations (3.30), (3.29), (3.31), (3.32), we see that

$$\left\| \nabla\rho \right\|_{L^p_t L^q_x} \leq C_4 \left(1 + \left\| u \right\|_{L^{p'}_t L^{q'}_x} \right) \left\| \rho_0 \right\|_{L^r_x}, \tag{3.33}$$

for some constant $C_4 > 0$.

Now consider the expression (κ-TDE),

$$\frac{\partial\rho}{\partial t}(x,t) = -\nabla \cdot (u(x,t)\rho(x,t)) + \kappa\Delta\rho.$$

Therefore, we see

$$\left\| \frac{\partial\rho}{\partial t} \right\|_{L^p_t \dot{W}^{-1,q}_x} \leq \left\| u\rho \right\|_{L^p_t L^q_x} + \kappa \left\| \nabla\rho \right\|_{L^p_t L^q_x},$$

and so by equations (3.30), (3.29), (3.33), we see that

$$\left\| \frac{\partial\rho}{\partial t} \right\|_{L^p_t \dot{W}^{-1,q}_x} \leq C_5 \left(1 + \left\| u \right\|_{L^{p'}_t L^{q'}_x} \right) \left\| \rho_0 \right\|_{L^r_x},$$

for some constant $C_5 > 0$. This finishes the proof of (3.25). □

Remark 6. Of course, we may also require that the solution requires the estimates in Theorem 3.2, namely (3.6), (3.7) and (2.11). However this do not specifically require the Ladyzhenskaya-Prodi-Serrin regularity $u(x,t) \in L^{p'}_t L^{q'}_x$ (3.10).

Remark 7. If $u(x,t)$ or $\rho_0(x)$ is more regular, then we may iteratively apply maximal regularity Theorem 3.8 with $F(x,t) = u(x,t)\rho(x,t)$, i.e. the bound (3.19), to improve the regularity of (3.25).

Remark 8. The case $q = q'$ in (3.25) would follow from the case $q = \infty$ in (3.26), and so perhaps also holds. We will discuss this further in the concluding remarks, Section 3.3.

3.3 Concluding Remarks

This chapter has explored the impact of incorporating diffusion ($\kappa > 0$) into the transport problem, demonstrating significant improvements in both the well-posed class of vector fields and the regularity of solutions. A striking example is the case of a divergence-free vector field satisfying the Ladyzhenskaya-Prodi-Serring condition (3.10), where the transport-diffusion equation (κ-TDE) exhibits markedly enhanced regularity compared to its transport equation counterpart (TE).

Drawing inspiration from analogous analyses of the heat equation [57] and other non-linear parabolic equations [51], including the Navier-Stokes equations [46], we identified and addressed a gap in the literature regarding the transport-diffusion equation. Sections 3.1.4 and 3.1.5 introduced advanced techniques that surpass standard energy estimates for parabolic equations. These methods were then applied in Section 3.2 to derive novel regularity and well-posedness results, extending the current state of the art.

Our first significant result, Theorem 3.10, establishes a general condition for weak solutions to belong to the classical energy class (Definition 3.2). This finding is particularly relevant in light of the more conventional well-posedness result for weak solutions in this class, as stated in Theorem 3.3. Our second and third significant contributions, Theorems 3.11 and 3.13, address enhanced uniqueness and regularity of weak solutions when the vector field $u(x,t) \in L^p_t L^q_x$ satisfies the Ladyzhenskaya-Prodi-Serrin condition (3.10):

$$\frac{2}{p} + \frac{d}{q} = 1.$$

It is worth noting that classical parabolic estimates guarantee the existence of weak solutions exhibiting the decay properties described in (3.7) and (3.8). The latter, in particular, proves especially useful. However, an intriguing open question remains regarding the validity of the decay estimate (3.8) for $r > 1$ when $q' = \infty$, i.e. $\frac{2}{p} = \frac{d}{r}$. We observe that when $u(x,t) \in L^1_t L^\infty_x$, the problem can be essentially reduced to that of the heat equation by applying the upper bound for the fundamental solution of the transport-diffusion equation, as established in [30, Theorem 3]. For the heat equation, one might demonstrate this endpoint case through interpolation techniques similar to those used in Lemma 3.12, but employing higher regularity norms than L^∞_x. A successful proof of the case $q' = \infty$ would immediately yield the endpoint case $q = q'$ in (3.25) of Theorem 3.13.

The uniqueness result in Theorem 3.11 currently includes two potentially superfluous conditions. The first is the constraint $p < \infty$ in the Ladyzhenskaya-Prodi-Serrin condition (3.10), which could be removed by imposing a smallness condition on $\|u\|_{L_t^\infty L_x^d}$. The second condition requires $\rho(x,t)u(x,t) \in L_t^r L_x^q$ with $r > 1$. This constraint might be eliminated by working in the sub-critical Ladyzhenskaya-Prodi-Serrin regime (3.11), as this would only necessitate the more straightforward sub-critical case (3.17) of Theorem 3.7. Furthermore, it may be feasible to extend the uniqueness result to more general cases where $\rho(x,t)u(x,t) \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times [0,T];\mathbb{R})$ by employing weighted spatial norms and relevant convolution estimates.

Regarding the regularity of the passive scalar when the vector field $u(x,t)$ belongs to a higher regularity Sobolev space, one can exploit the relationship between the regularity of $u(x,t)\rho(x,t)$ and $\rho(x,t)$ in (κ-TDE). This relationship is comprehensively described by a fractional version of the weak convolution estimate, Theorem 3.7. Alternatively, one could combine maximal regularity (Theorem 3.8) with the mixed derivative estimate (Theorem 3.9) and Sobolev embedding. The goal would be to identify a fixed point in this relationship between $u(x,t)\rho(x,t)$ and $\rho(x,t)$, taking into account the regularity of $u(x,t)$ and the heat solution $e^{t\kappa\Delta}\rho_0(x)$. This approach motivated the development of Theorem 3.13.

In conclusion, the transport-diffusion equation exhibits markedly different behaviour compared to the inviscid transport problem analysed in Chapter 2, demonstrating well-posedness under significantly lower regularity conditions on the vector field $u(x,t)$. This chapter presents the standard well-posedness theory via energy estimates and extends these results to more general weak solutions and regularity conditions. We have filled a notable gap in the existing literature by applying sophisticated techniques developed for other parabolic equations to the transport-diffusion equation.

Chapter 4

Vanishing Diffusion Limit and Solution Selection

4.1 Introduction

This final chapter investigates the problem of solution selection for the passive scalar transport equation:

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = 0. \tag{TE}$$

As demonstrated in Chapter 2, this equation admits non-unique weak solutions when the vector field does not lie in the Sobolev class $L^1_t W^{1,1}_x$, often leading to physically inadmissible behaviour such as "perfect unmixing".

To address this issue, we examine the inclusion of molecular diffusivity/viscosity $\kappa > 0$, resulting in the transport-diffusion equation:

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) - \kappa \Delta \rho(x,t) = 0. \tag{κ-TDE}$$

Since this parabolic equation is well-posed under lower regularity conditions (as shown in Chapter 3), our focus is on understanding whether the vanishing viscosity limit ($\kappa \to 0$) can effectively rule out physically inadmissible solutions. Indeed, one immediately sees that for a constant initial datum, the vanishing viscosity limit always selects the constant solution uniquely, avoiding any solutions which perfectly unmix.

Contrary to the prevailing belief that vanishing viscosity is a benchmark for physical admissibility in fluid models, we present examples where this limit fails to restore uniqueness or physical admissibility to the inviscid system. Our main contributions are:

1. Theorem 4.10: We demonstrate the existence of a bounded, divergence-free vector field and two vanishing viscosity subsequences that converge to different solutions of the inviscid problem for every non-constant initial datum.

2. Theorem 4.13: We prove the existence of a bounded, divergence-free vector field for which the vanishing viscosity limit converges uniquely to a solution that exhibits perfect mixing followed by perfect unmixing, thus losing any arrow of time enforced by diffusion.

These results challenge the conventional understanding of vanishing viscosity limits and their role in selecting physically admissible solutions.

For convenience, this chapter considers passive scalar transport on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ rather than on $\mathbb{R}^d$. Section 4.2.1 restates all necessary definitions and theorems for transport on $\mathbb{T}^d$.

The results in this section have been developed in collaboration with Edriss Titi and reported in [60].

4.1.1 Hyperbolic Conservation Laws and Entropy

The non-uniqueness of weak solutions (or generalised solutions) to PDEs is well-known beyond the literature on the transport equation. Hyperbolic conservation laws were among the first PDEs for which this was studied in depth. These equations take the form:

$$\frac{\partial f}{\partial t} + \nabla \cdot A(x,t,f(x,t)) = 0, \tag{4.1}$$

where $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d'}$ is a vector field driven by a non-linear, often smooth, flux $A(x,t,f(x)) : \mathbb{R}^d \times [0,T] \times \mathbb{R}^{d'} \to \mathbb{R}^{d \times d'}$ [35, Chapter 3]. Equation (4.1) is in divergence form, and so, like the continuity equation $A(x,t,\rho) = u(x,t)\rho$, (2.8), naturally admits a weak formulation. It has long been known that classical solutions to (4.1) generally have a finite maximal time of existence [35, Section 4.2 and Theorem 6.1.1], particularly for the infamous $d = 1$ Burgers equation, $A(x,t,f) = \frac{1}{2}f^2$ [28]. However, weak solutions can be extended beyond the formation of such shocks and, in fact, non-uniquely [35, Section 4.4]. The need for admissibility criteria to then restore uniqueness thus stems from the failure of a global existence theory for classical solutions to (4.1).

Lax [76] introduced inequalities to restore uniqueness on the specific discontinuous profiles, 'shocks', that were observed to emerge (see also [35, Section 8]). However, it was Kruzkov who successfully provided a well-posedness class of weak solutions to (4.1) with smooth flux in the case $d' = 1$ (scalar hyperbolic conservation laws) by imposing abstract entropy inequalities on all weak solutions [67]. Moreover, the vanishing viscosity scheme is proved to converge to the unique entropy solution [35, Section 4.6], in line with the heuristic that in physical systems, these entropy inequalities express the second law of thermodynamics [35, Section 3.3], see also [75].

Turning to the topic of this thesis, (TE) when written in conservation form is a (linear) scalar hyperbolic conservation law, and so any scalar function $\eta(\rho(x,t))$ (for $\eta(s):\mathbb{R}\to\mathbb{R}$) is an appropriate entropy as in [67], [35, Section 6]. As in [35], we argue that the second law of thermodynamics applied to weak solutions to the fluid model should ensure that physically relevant weak solutions $\rho(x,t)$ of (TE) satisfy the following:

Definition 4.1 (Transport equation - entropy solutions [35])**.** We say a solution $\rho(x,t):\mathbb{R}^d\times[0,T]\to\mathbb{R}$ to (TE) is an entropy solution if any convex function (entropy) $\beta(s)\in C^1_{\mathrm{b}}(\mathbb{R};\mathbb{R})$ is *sub*-transported, satisfying the distributional inequality

$$\frac{\partial}{\partial t}\beta(\rho(x,t))+\nabla\cdot(u(x,t)\beta(\rho(x,t)))\leq 0, \tag{4.2}$$

i.e. for any *positive* test function $\phi(x,t)\in C_c^\infty(\mathbb{R}^d\times[0,T);\mathbb{R})$,

$$\int_{\mathbb{R}^d\times[0,T]}\beta(\rho(x,t))\left(\frac{\partial\phi}{\partial t}(x,t)+u(x,t)\cdot\nabla\phi(x,t)\right)\,dxdt+\int_{\mathbb{R}^d}\beta(\rho_0(x))\phi(x,0)\,dx\geq 0.$$

This is equality for a renormalised weak solution, Definition 2.2, so the solution preserves entropy.

The well-posedness theory of Kruzkov [67] does not apply unless $u(x,t)$ is sufficiently smooth, for which we already have a classical well-posedness theory, see Chapter 2 Section 2.1.2.

Nonetheless, the only entropy solution satisfying (4.2) with $\rho_0(x)=0$ is the zero solution, and the inequality in (4.2) therefore successfully rules out the phenomenon of perfect unmixing, Section 2.1.4. As for the second law of thermodynamics, an admissible solution forward in time is not necessarily admissible when time is reversed. That this does not in general hold for the vanishing viscosity limit, Theorem 4.13, is one of the main results of this chapter and should be seen as an indication of the failure of the vanishing viscosity limit as a physically appropriate admissibility criterion.

Moreover, the construction in Theorem 4.13 also violates entropy-admissibility since, for example, the 'bulk energy' $\int_{\mathbb{T}^d}|\rho(x,t)|^2\,dx$ is strictly increasing at $t=58$ for any (non-constant) initial datum. And indeed, despite its physical significance, the existence of entropy-admissible solutions (4.2) to the transport equation (TE) is an open problem for vector fields with low regularity.

1D Hyperbolic Conservation Laws

Outside of the theory of scalar hyperbolic conservation laws, much less is known regarding physical selection mechanisms. One may have blow up (rather than shock formation) of, say, the compressible Euler equations [109], [82, 83], for which the conditions of Lax do not have a clear analogue. Even the existence of entropy-admissible weak solutions is an open problem in this setting [81].

Much more is known when one restricts to the case of one-dimensional hyperbolic conservation laws, (4.1) with $d = 1$, see [35]. In particular, the existence of entropy solutions [53], uniqueness of entropy solutions [21], and convergence of the vanishing viscosity scheme [17] were successfully shown with small initial data and strict hyperbolicity of the flux.

4.1.2 Incompressible Fluids and Turbulence

Turning instead to incompressible fluids, the search for a well-posedness theory of solutions to the incompressible inviscid Euler or viscous Navier-Stokes equations has a separate history [72], [103]. The blow-up problem for classical solutions remains one of the most challenging open problems [73]. In two spatial dimensions for suitable initial data, there is no need for an admissible theory of weak solutions due to the strong well-posedness class of Yudovich [63], and similarly in three dimensions under axial symmetry with no swirl [107].

In general, one may construct Leray-Hopf admissible weak solutions to the Navier-Stokes equations [78]. However, their uniqueness remains open. These 'Leray-Hopf' admissible solutions are the analogue of energy solutions, Definition 3.2, to (κ-TDE), which additionally satisfy the so-called energy inequality (or local-energy inequality) in the sense of the distribution

$$\frac{\partial}{\partial t}\rho^2(x,t) + \nabla\cdot(u(x,t)\rho^2(x,t)) - \kappa\Delta\rho^2(x,t) \leq -2\kappa|\nabla\rho|^2(x,t), \tag{4.3}$$

i.e. for any *positive* test function $\phi(x,t) \in C_c^\infty(\mathbb{R}^d \times [0,T) : \mathbb{R})$,

$$\int_{\mathbb{R}^d\times[0,T]} \rho^2(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right) dxdt \\ - 2\kappa\int_{\mathbb{R}^d\times[0,T]} |\nabla\rho|^2(x,t)\phi(x,t)\, dxdt + \int_{\mathbb{R}^d} \rho_0^2(x)\phi_0(x)\, dx \geq 0,$$

where $\phi_0(x) = \phi(x,0)$.

This is the viscous ($\kappa > 0$) analogue of the entropy admissibility condition (4.2) with $\beta(\rho) = \rho^2$. Indeed, for a symmetric system of hyperbolic conservation laws, such as the compressible Euler equations, the energy $|u(x,t)|^2$ is the unique entropy [35, Section 3.2]. As such, the local energy inequality (4.3) and its inviscid analogue ($\kappa = 0$) are often used as admissibility criteria for weak solutions in incompressible fluids.

In the viscous case $\kappa > 0$ (κ-TDE) under very mild assumptions on the vector field $u(x,t)$ such as the Ladyzhenskaya-Prodi-Serrin condition (3.10), the local energy inequality (4.3) becomes an equality, Section 3.1.2. While this picture similarly holds in the non-linear case of the Navier-Stokes equations, and even for the lower regularity $u(x,t) \in L_t^4 L_x^4$ [91], it is an open problem whether the $L_t^4 L_x^4$ or Ladyzhenskaya-Prodi-Serrin condition hold for any Leray-Hopf admissible weak solution $u(x,t)$ of the three dimensional Navier-Stokes equations.

As for the phenomenon of perfect unmixing in passive scalar transport, Section 2.1.4, the local energy inequality (4.3) successfully rules out the 'wild' non-uniqueness of weak solutions with zero initial data, say for the inviscid incompressible Euler equations [94] and [100].

Meanwhile, the vanishing viscosity limit $\kappa \to 0$ is an a-priori separate selection mechanism to rule out such 'wild' solutions [13], [12], [88], selecting in these examples a unique solution to the inviscid model, deemed physical.

If the convergence is strong in $L_t^2 L_x^2$, then the local energy equality (4.3) will be preserved in the limit $\kappa \to 0$. Under weak convergence, this might fail in general, and for non-linear equations such as the incompressible Euler equations, one can show only that the limit is a sub-solution. Such solutions have been vital to studying weak solutions' non-uniqueness and low regularity to incompressible Euler, passive scalar transport, and many other equations [37].

Anomalous Dissipation

The vanishing viscosity limit for the local energy inequality (4.3) is particularly interesting to the mechanisms behind hydrodynamic turbulence. Consider the negative right-hand side of the local energy inequality (4.3). If the solution $\rho(x,t)$ fails to preserve suitable regularity in the limit $\kappa \to 0$, then there is no reason why this term should vanish; that is

$$\limsup_{\kappa \to 0} \left(2\kappa \int_{\mathbb{R}^d \times [0,T]} |\nabla \rho(x,t)|^2 \, dxdt \right) > 0. \tag{4.4}$$

This anomalous dissipation is the fundamental assumption to derive the energy cascade of hydrodynamic turbulence [65]. This is motivated by the energy balance (for classical solutions $u(x,t)$) to the Navier-Stokes equations with viscosity $\nu > 0$,

$$\int_{\mathbb{R}^d} |u(x,T)|^2 \, dx = \int_{\mathbb{R}^d} |u(x,0)|^2 \, dx - 2\nu \int_{\mathbb{R}^d \times [0,T]} |\nabla u(x,t)|^2 \, dxdt,$$

and so, in particular, if there is anomalous dissipation in the sense of Kolmogorov, the inviscid limit cannot conserve energy.

In the construction of this chapter, the limiting solution in Theorem 4.13 exhibits anomalously, i.e. without viscosity, a dissipation of energy (and later a reverse of that dissipation). However, this is not the same as anomalous dissipation in the sense of above. Such anomalous dissipation (4.4), like the inviscid local energy inequality (4.3) with $\kappa = 0$, gives an arrow of time on solutions of the transport equation and energy lost by (4.4) cannot be recovered. In the construction of this chapter all the energy is recovered, since the anomalous loss is associated with weak convergence of the limit, rather than a lower bound of the form (4.4).

That additional energy will dissipate if the inviscid limit fails to converge strongly is one of the key observations made in this chapter. None of the examples in the current literature exhibiting anomalous dissipation for passive scalar transport [42], [32], [10], [43] show that strong convergence of the inviscid limit fails, and whether further dissipation occurred or is even possible was unknown. Theorem 4.13 then answers these questions, proving also that mixing does not imply anomalous dissipation in the sense of Kolmogorov.

Correspondingly, whether the dissipation of energy in the infinite Reynolds number limit of full hydrodynamic turbulence is due solely to anomalous dissipation, or whether there is additional dissipation through weak convergence as above, and correspondingly whether the arrow of time given by a local energy inequality for the incompressible Euler equations should hold in the vanishing viscosity limit of Navier-Stokes, is an important open question. In particular, wild solutions (those violating the local energy inequality) to the Euler equations by now have a long history of study [94], [100], [33], [38], [26], [27]. We highlight that it remains an open question whether such solutions may arise in the vanishing viscosity limit of strong solutions to the Navier-Stokes equations. Our result shows that the answer to the analogue of this question for the passive scalar transport equation is positive.

We mention that some of these recent works on anomalous dissipation for passive scalars also, as a corollary, show non-uniqueness of the vanishing viscosity limit of the transport-diffusion equation, paralleling Theorem 4.10 [32], [10].

The authors of [32] demonstrate the non-uniqueness of weak limit points of vanishing viscosity for a single initial datum, with one viscosity subsequence converging weakly to an energy-dissipating solution and another to an energy-preserving solution. Meanwhile, [10] establishes non-uniqueness of weak limit points for all initial data in H^1_x, with the required subsequences depending on the H^1_x-norm of the initial datum.

Our non-uniqueness result presented in Theorem 4.10 is stronger since it is valid for all initial datum in L^∞_x, with strong convergence of the vanishing viscosity limit, and with no dependence of the required vanishing viscosity subsequences on the initial datum.

4.2 Main Results

We first develop a simple framework (different to homogenisation [10], or stochastic calculus [32]) to give control of the vanishing viscosity limit, see Section 4.2.2 and specifically Theorem 4.6. Despite diffusion not acting directly on the background velocity field $u(x,t)$, diffusion of $\rho(x,t)$ gives an indirect emergent smoothing of $u(x,t)$. Therefore, we can approximate a weak solution $\rho(x,t)$ to (κ-TDE) by inviscid transport (TE) along the vector field $\bar{u}$ which smooths spatial scales of $u(x,t)$ below a dissipation length.

The construction of Theorem 4.10 relies on an arrangement of alternating shear flows obeying a specific commutation property, Proposition 4.8. By playing with a second length scale independent of this commutation property, well-adapted viscosity activates only some of these shear flows. By design, the global effect switches back and forth between null and a large-scale shear as a sequence of viscosities is chosen to vanish vanishes. We mention also that alternating shear flows have been the basis of numerous recent constructions for perfect mixing and enhanced and anomalous dissipation [5], [42], [32], [10], [44], [43].

The construction of Theorem 4.13 relies on moving slabs to mix the passive scalar for any initial datum, reminiscent of work on non-uniqueness of trajectories [2]. To ensure unique convergence of the vanishing viscosity limit in-between the well-adapted viscosity subsequence given by the following analysis (and in particular Theorem 4.6), one introduces a third intermittency length scale to control the error to the nearest neighbor in the well-adapted viscosity subsequence. This technique is perhaps reminiscent of the intermittency of small-scale structures in hydrodynamic turbulence.

4.2.1 Definitions and Preliminary Results

In this section, we recap the relevant definitions and results from Chapter 2 and Chapter 3; in particular, we rephrase them on the d-dimension unit torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Definition 4.2 (Weak solution to the transport equation (TE))**.** Consider a vector field $u(x,t) \in L^1([0,T];L^1(\mathbb{T}^d;\mathbb{R}^d))$ with $\nabla \cdot u(x,t) = 0$ in the distributional sense.

We say $\rho(x,t) \in L^1([0,T];L^1(\mathbb{T}^d;\mathbb{R}))$ with $u(x,t)\rho(x,t) \in L^1_t L^1_x$ is a weak solution to the transport equation along $u(x,t)$

$$\frac{\partial \rho}{\partial t}(x,t) + \nabla \cdot (u(x,t)\rho(x,t)) = 0, \tag{TE}$$

with initial data $\rho_0(x) \in L^1_x$ if, for any $\phi(x,t) \in C_c^\infty(\mathbb{T}^d \times [0,T);\mathbb{R})$,

$$\int_{\mathbb{T}^d \times [0,T)} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right) dxdt = -\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\, dx,$$

where $\phi_0(x) = \phi(x,0)$.

Meanwhile, we say the transport equation is satisfied on an *open* interval $I \subset (0,T)$ if, for any $\phi(x,t) \in C_c^\infty(\mathbb{T}^d \times I;\mathbb{R})$,

$$\int_{\mathbb{T}^d \times I} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right) dxdt = 0.$$

Remark 9. Note that in the above definition, we may equivalently take the test function $\phi(x,t)$ to be Lipschitz in time and space with compact support in $\mathbb{T}^d \times [0,T)$. This is done by finding a sequence of smooth functions $\phi_n(x,t) \in C_c^\infty(\mathbb{T}^d \times [0,T);\mathbb{R})$ such that $\phi_n(x,t)$ are bounded in $C^1_{x,t}$ by the Lipschitz norm of $\phi(x,t)$, and $\frac{\partial \phi_n}{\partial t}(x,t) \xrightarrow{n\to\infty} \frac{\partial \phi}{\partial t}(x,t)$, $\nabla\phi_n(x,t) \xrightarrow{n\to\infty} \nabla\phi(x,t)$ pointwise almost everywhere.

Definition 4.3 (Renormalised weak solutions to (TE))**.** Following the definition introduced in [41], suppose $\rho(x,t)$ is a weak solution to (TE) along $u(x,t)$ with initial data $\rho_0(x)$.

If, for any $\beta(s) \in C^0_b(\mathbb{R};\mathbb{R})$, $\beta(\rho(x,t))$ is a weak solution to (TE) along $u(x,t)$ with initial data $\beta(\rho_0(x))$, then we say $\rho(x,t)$ is a renormalised weak solution of (TE).

Remark 10. This definition is well motivated by the expression $\left(\frac{\partial}{\partial t} + u(x,t)\cdot\nabla\right)\beta(\rho(x,t)) = \beta'(\rho(x,t))\left(\frac{\partial}{\partial t} + u(x,t)\cdot\nabla\right)\rho(x,t)$ when $\beta(s)$ is a differentiable function. Indeed some authors require $\beta(s) \in C^1(\mathbb{R};\mathbb{R})$ as in [9], or even with decay at infinity as in [41]. In our case ($\nabla \cdot u(x,t) = 0$), it is straightforward to show that these give equivalent definitions.

Definition 4.4 (Lagrangian flows and Lagrangian solutions to (TE))**.** For a divergence-free vector field $u(x,t) \in L^1_t L^1_x$, we say a family ($t \in [0,T]$) of Lebesgue-measure preserving bijections $y_t(x) : \mathbb{T}^d \to \mathbb{T}^d$ is a Lagrangian flow along $u(x,t)$ if for a.e. $x \in \mathbb{T}^d$ the map $t \mapsto y_t(x)$ is absolutely continuous and the derivative satisfies $\frac{dy_t(x)}{dt} = u(y_t(x),t)$ as a function class in $L^1_t L^1_x$.

We say that a function $\rho(x,t) \in L^1_t L^1_x$ is a Lagrangian solution to (TE) along $u(x,t)$ (with initial data $\rho_0(y_0^{-1}(x))$) if $\rho(x,t) = \rho_0(y_t^{-1}(x))$ for some $\rho_0(x) \in L^1_x$, and $\{y_t\}_{t\in[0,T]}$

a Lagrangian flow along $u(x,t)$. It is often convenient to take $y_0(x) = x$ without loss of generality.

Remark 11. One may show that, if additionally $u(x,t)\rho(x,t) \in L_t^1 L_x^1$, then a Lagrangian solution to (TE) along $u(x,t)$ with initial data $\rho_0(x)$ is also a weak solution in the sense of Definition 4.2. We reduce the problem to bounded $\rho(x,t)$ and $\rho_0(x)$ by the point-wise approximation with $\rho(x,t)\mathbb{1}_{|\rho(x,t)|\leq k}$, so that $u(x,t)\rho(x,t)\mathbb{1}_{|\rho(x,t)|\leq k} \xrightarrow{k\to\infty} u(x,t)\rho(x,t)$ in $L_t^1 L_x^1$ by dominated convergence. Observe then that $\rho(x,t)\mathbb{1}_{|\rho(x,t)|\leq k}$ are already Lagrangian solutions for the initial data $\rho_0(x)\mathbb{1}_{|\rho_0(x)|\leq k}$, since $\rho_0(y_t^{-1}(x))\mathbb{1}_{|\rho_0(y_t^{-1}(x))|\leq k} = \left((\rho_0\mathbb{1}_{|\rho_0|\leq k})\circ y_t^{-1}\right)(x)$. The result for bounded $\rho(x,t)$ and $\rho_0(x)$ follows from changing variables in the integral

$$\int_{\mathbb{T}^d\times[0,T)} \rho_0(y_t^{-1}(x)) \left(\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t)\right) dxdt,$$

and using that $\left(\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t)\right)(y_t(x)) = \frac{\partial}{\partial t}(\phi(y_t(x),t))$ by chain rule for absolutely continuous functions. Moreover, since for any $\beta(s) \in C_b^0(\mathbb{R};\mathbb{R})$ we can rewrite $\beta(\rho_0(y_t^{-1}(x))) = \left((\beta\circ\rho_0)\circ y_t^{-1}\right)(x)$, these solutions are then also renormalised weak solutions in the sense of Definition 4.3.

Definition 4.5 (Weak solution to the transport-diffusion equation (κ-TDE))**.** Consider a vector field $u(x,t) \in L_t^1 L_x^1$ with $\nabla\cdot u(x,t) = 0$ in the distributional sense, and some positive constant viscosity $\kappa > 0$ (also called diffusivity).

We say $\rho(x,t) \in L_t^1 L_x^1$ with $u(x,t)\rho(x,t) \in L_t^1 L_x^1$ is a weak solution to the transport-diffusion equation along $u(x,t)$

$$\frac{\partial\rho}{\partial t}(x,t) + \nabla\cdot(u(x,t)\rho(x,t)) - \kappa\Delta\rho(x,t) = 0, \tag{κ-TDE}$$

with initial data $\rho_0(x) \in L_x^1$, if for any $\phi(x,t) \in C_c^\infty(\mathbb{T}^d\times[0,T);\mathbb{R})$,

$$\int_{\mathbb{T}^d\times[0,T]} \rho(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right) dxdt = -\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\,dx,$$

where $\phi_0(x) = \phi(x,0)$.

Meanwhile, we say the transport-diffusion equation (κ-TDE) is satisfied on an *open* interval $I \subset (0,T)$ if, for any $\phi(x,t) \in C_c^\infty(\mathbb{T}^d\times I;\mathbb{R})$,

$$\int_{\mathbb{T}^d\times I} \rho(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right) dxdt = 0.$$

Definition 4.6 (Vanishing viscosity solution to (TE))**.** Consider a vector field $u(x,t) \in L_t^1 L_x^1$ with $\nabla \cdot u(x,t) = 0$ in the distributional sense.

We say a weak solution $\rho(x,t)$ to (TE) along $u(x,t)$ with initial data $\rho_0(x) \in L_x^1$ is a vanishing viscosity solution, if there exists a positive sequence $\kappa_n \to 0$ and corresponding weak solutions $\rho^{(n)}(x,t)$ to (κ-TDE) with viscosity κ_n along $u(x,t)$ with initial data $\rho_0(x)$, such that

$$\rho^{(n)}(x,t) \xrightarrow{n\to\infty} \rho(x,t),$$

converges as distributions in $\mathscr{D}'(\mathbb{T}^d \times [0,T);\mathbb{R})$.

Remark 12. In the literature $\rho^{(n)}(x,t)$ is often taken to have, in addition, non-constant initial data $\rho_0^{(n)}(x)$, where $\rho_0^{(n)}(x)$ converges to $\rho_0(x)$ in a suitable topology. We do not consider such a more general definition. Still, we note that under strong convergence of $\rho_0^{(n)}(x) \to \rho_0(x)$, the appropriate analogue of our main results, Theorem 4.10 and Theorem 4.13, follow by comparing between the viscous solutions with initial data $\rho_0^{(n)}(x,t)$ and $\rho_0(x)$. This can be done by equality (4.11) in Theorem 4.3.

The following is by no means a complete exposition of standard existence, uniqueness, and regularity theory for (TE) and (κ-TDE), but contains some of the more salient points, and in particular those relevant to this chapter.

Theorem 4.1 (Weak continuity of transport)**.** *Suppose $\rho(x,t)$ is a weak solution to* (κ-TDE) *or* (TE) *(i.e. $\kappa = 0$) along $u(x,t) \in L_t^1 L_x^1$ with initial data $\rho_0(x)$.*

Then for any $\phi(x,t) \in C^\infty(\mathbb{T}^d \times [0,T];\mathbb{R})$, for a.e. $t \in [0,T]$,

$$\textit{(Trace Formula)} \quad \int_{\mathbb{T}^d} \rho(x,t)\phi(x,t)\, dx$$
$$= \int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\, dx + \int_{\mathbb{T}^d \times [0,t]} \rho(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t) \cdot \nabla\phi(x,t) + \kappa\Delta\phi(x,t) \right) dxdt, \tag{4.5}$$

Suppose further that $\rho(x,t) \in L_t^\infty L_x^p$ for $p \in (1,\infty]$, then (there is a representation of $\rho(x,t)$ with) $\rho(x,t) \in C^0_{\text{weak}-*} L_x^p$, such that (4.5) *holds for all $t \in [0,T]$.*

In particular $\rho(x,0) = \rho_0(x)$ in L_x^q.

Remark 13. We remark without proof that the analogous result holds when $p = 1$ if we additionally assume $\rho(x,t) : [0,T] \to L_x^1$ is uniformly integrable in the indexing variable t (for a.e. $t \in [0,T]$).

Proof. This is similar to Theorem 2.2 in Chapter 2, and Theorem 3.1 in Chapter 3.

For any weak solution of (TE) or (κ-TDE) we have $\rho(x,t) \in L_t^1 L_x^1$, $u(x,t)\rho(x,t) \in L_t^1 L_x^1$.

Fix some $\phi(x,t) \in C^\infty\left(\mathbb{T}^d \times [0,T];\mathbb{R}\right)$ and consider the following $L^1([0,T];\mathbb{R})$ function of $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \rho(x,t)\phi(x,t)\,dx. \tag{4.6}$$

Then for all $\psi(t) \in C_c^\infty([0,T);\mathbb{R})$, by Definitions 4.2, 4.5,

$$\begin{aligned}
&\int_{\mathbb{T}^d\times[0,T)} \rho(x,s)\phi(x,s)\frac{d\psi}{ds}(s)\,dxds \\
&\quad= \int_{\mathbb{T}^d\times[0,T)} \rho(x,s)\phi(x,s)\frac{d\psi}{ds}(s)\,dxds - \left(\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\psi_0(x)\,dx\right. \\
&\qquad\qquad \left.+ \int_{\mathbb{T}^d\times[0,T)} \rho(x,s)\left(\frac{\partial}{\partial s} + u(x,s)\cdot\nabla + \kappa\Delta\right)(\phi(x,t)\psi(s))\,dxds\right) \\
&\quad= -\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\psi_0\,dx \\
&\qquad -\int_{\mathbb{T}^d\times[0,T)} \rho(x,s)\left(\frac{\partial\phi}{\partial s}(x,s) + u(x,s)\cdot\nabla\phi(x,s) + \kappa\Delta\phi(x,s)\right)\psi(s)\,dxds,
\end{aligned}$$

where $\psi_0 = \psi(0)$ and $\phi_0(x) = \phi(x,0)$.

And so the function defined in (4.6) is an absolutely continuous function of $t \in [0,T]$, with derivative $\int_{\mathbb{T}^d} \rho(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right)\,dx$, and the initial value at $t=0$ is $\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\,dx$, see Chapter 3, Lemma 1.1 in [103]. That is (4.5) holds for a.e. $t \in [0,T]$.

Suppose now, for some $p \in (1,\infty]$, that $\rho(x,t) \in L_t^\infty L_x^p$. Define for each $t \in [0,T]$ the distribution $F_t(x) \in \mathscr{D}'(\mathbb{T}^d;\mathbb{R})$ acting on test functions $\chi(x) \in C^\infty(\mathbb{T}^d;\mathbb{R})$, given by

$$\langle F_t,\chi\rangle = \int_{\mathbb{T}^d} \rho_0(x)\chi(x)\,dx + \int_{\mathbb{T}^d\times[0,t]} \rho(x,t)\left(u(x,t)\cdot\nabla\chi(x,t) + \kappa\Delta\chi(x,t)\right)dxdt.$$

Then, thanks to (4.5), for a.e. $t \in [0,T]$, $F_t(x) = \rho(x,t)$ as distributions in $\mathscr{D}'(\mathbb{T}^d;\mathbb{R})$, which is assumed uniformly bounded in L_x^q. We therefore have for a.e. $t \in [0,T]$, for all $\chi(x) \in C^\infty(\mathbb{T}^d;\mathbb{R})$, the bound $|\langle F_t,\chi\rangle| \le \|\rho\|_{L_t^\infty L_x^p}\|\chi\|_{L_x^q}$ where $\frac{1}{p}+\frac{1}{q}=1$. By the absolute continuity of $t \mapsto \langle F_t,\chi\rangle$ the bound, in fact, holds for all $t \in [0,T]$. Since $p \in (1,\infty]$ this implies further that $F_t(x)$ can be extended to a function $\bar\rho(x,t) \in L_x^q$ for all $t \in [0,T]$, and the absolute continuity of $\langle F_t,\chi\rangle$ implies $\bar\rho(x,t) \in C^0_{\text{weak}-*}L_x^p$ with (4.5) holding for the representative $\bar\rho(x,t)$ now for all $t \in [0,T]$, as required. □

Theorem 4.2 (Existence for (TE) and (κ-TDE))**.** *Suppose $\rho_0(x) \in L_x^\infty$, and $u(x,t) \in L_t^1 L_x^1$ is divergence-free in the distributional sense, then there exists a weak solution $\rho(x,t) \in C^0_{\text{weak}-*}L_x^\infty$ to* (TE) *or* (κ-TDE) *along $u(x,t)$ with initial data $\rho_0(x)$. Moreover, for all*

$p \in [1, \infty]$, $t \in [0, T]$,

$$\text{(Initial } L^p_x\text{-Inequality)} \quad \|\rho(\cdot, t)\|_{L^p_x} \leq \|\rho_0\|_{L^p_x}, \tag{4.7}$$

$$\text{(Conservation of Mass)} \quad \int_{\mathbb{T}^d} \rho(x,t)\, dx = \int_{\mathbb{T}^d} \rho_0\, dx. \tag{4.8}$$

Proof. This is similar to Theorem 2.1 in Chapter 2, and Theorem 3.2 in Chapter 3.

More generally, for $\rho_0(x) \in L^q_x$, $u(x,t) \in L^1_t L^r_x$ with $\frac{1}{q} + \frac{1}{r} = 1$, existence of a weak solution $\rho(x,t)$ with $\|\rho\|_{L^\infty_t L^p_x} \leq \|\rho_0\|_{L^p_x}$ (for all $p \in [1, \infty]$, permitting infinite values of the norms) follows from a standard approximation scheme (regularisation of $u(x,t)$ and $\rho_0(x)$), see [41] for details;. However, the proof considers only $\kappa = 0$ (i.e. (TE)), and the spatial domain $\mathbb{R}^d$ instead of $\mathbb{T}^d$, an identical argument goes through here.

By assuming $u(x,t) \in L^1_t L^1_x$, we may apply the above result with $q = \infty$.

By Theorem 4.1 the solution is in $C^0_{\text{weak}-*} L^\infty_x$, and so the bound $\|\rho\|_{L^\infty_t L^p_x} \leq \|\rho_0\|_{L^p_x}$ implies the Initial L^p_x-Inequality (4.7).

Conservation of Mass (4.8) follows from the Trace Formula (4.5) in Theorem 4.1 with $\phi(x,t) = 1$ on $\mathbb{T}^d \times [0, T]$. □

Remark 14. Conservation of Mass (4.8) will hold for a.e. $t \in [0, T]$ for every weak solution on the torus, and not only those constructed in Theorem 4.2. This does not hold in $\mathbb{R}^d$.

Next, we give the following well-posedness and regularity result for (κ-TDE). To illustrate the stark contrast between (TE) and its regularisation (κ-TDE), we show well-posedness for any $\rho_0(x) \in L^1_x$. We obtain further regularity if $\rho_0(x)$ is more integrable. Therefore, we shall later assume $\rho_0(x) \in L^\infty_x$.

Theorem 4.3 (Well-posedness of (κ-TDE))**.** *Suppose $u(x,t) \in L^\infty_t L^\infty_x$, then for any initial data $\rho_0(x) \in L^1_x$ any weak solution (in the class $L^1_t L^1_x$) to* (κ-TDE) *along $u(x,t)$ with initial data $\rho_0(x)$ is unique. Moreover this solution exists, $\rho(x,t) \in C^0_t L^1_x$ with $\|\rho\|_{C^0_t L^1_x} \leq \|\rho_0\|_{L^1_x}$ and becomes immediately bounded, $\rho(x,t) \in C^0([\varepsilon, T]; C^0_x)$ for all $\varepsilon \in (0, T]$.*

Furthermore, when $\rho_0(x) \in L^\infty_x$ we have the following additional regularity (for all $p \in [1,\infty)$)

$$\rho(x,t) \in C^0_t L^p_x \cap L^2_t H^1_x, \tag{4.9}$$

$$\rho(x,t) \in C^0_{\text{weak}-*} L^\infty_x, \tag{4.10}$$

$$\text{(}L^p_x\text{-Inequality)} \quad 0 \le s \le t \implies \|\rho(\cdot,t)\|_{L^p_x} \le \|\rho(\cdot,s)\|_{L^p_x}, \tag{4.11}$$

$$\text{(Energy Identity)} \quad \int_{\mathbb{T}^d} |\rho(x,t)|^2 \, dx + 2\kappa \int_{\mathbb{T}^d \times [0,t]} |\nabla \rho(x,s)|^2 \, dxds = \int_{\mathbb{T}^d} |\rho_0(x)|^2 \, dx, \tag{4.12}$$

$$\text{(Equicontinuity)} \quad \left\| \frac{\partial \rho}{\partial t} \right\|_{L^2_t H^{-1}_x} \le \left(\|u\|_{L^2_t L^\infty_x} + \sqrt{\frac{\kappa}{2}} \right) \|\rho_0\|_{L^2_x}. \tag{4.13}$$

Proof. The standard well-posedness result is for $\rho_0(x) \in L^2_x$ and is done via energy estimates. However, if we wish to obtain uniqueness in the class $L^1_t L^1_x$, we must be more careful. The following is similar to the proof of the results given in Chapter 3, Section 3.2.

Given by the spatial Fourier transform on $\mathbb{R}^d$, $\mathscr{F}(K_\kappa(\cdot,t))(\xi) = e^{-\kappa \xi^2 t}$, denote by $K_\kappa(x,t) \in C^\infty(\mathbb{R}^d \times (0,\infty);\mathbb{R}) \cap L^\infty_t L^1_x$ the heat kernel for the heat equation with diffusivity κ, that is for any $\phi(x,t) \in C^\infty_c(\mathbb{T}^d \times (-\infty,\infty);\mathbb{R})$ we have

$$\left(\frac{\partial}{\partial t} - \kappa \Delta \right) (\phi * (K_\kappa \mathbb{1}_{t>0}))(x,t) = \phi(x,t),$$

with convolution in space and time. Denoting by $\bar{K}_\kappa(x,t) \in C^\infty(\mathbb{R}^d \times (-\infty,0);\mathbb{R})$ the backwards heat kernel $\bar{K}_\kappa(x,t) = K_\kappa(-x,-t)$, then we have for the backwards heat equation, for any $\phi(x,t) \in C^\infty_c(\mathbb{T}^d \times (-\infty,\infty);\mathbb{R})$

$$\left(\frac{\partial}{\partial t} + \kappa \Delta \right) (\phi *_{x,t} (\bar{K}_\kappa \mathbb{1}_{t<0}))(x,t) = -\phi(x,t).$$

For $\phi(x,t) \in C_c^\infty(\mathbb{T}^d \times [0,T];\mathbb{R})$ we may take $(\phi * (\bar{K}_\kappa \mathbb{1}_{t<0}))(x,t)$ as a test function in (κ-TDE), which (by expanding out all convolutions) can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{T}^d\times[0,T]} \rho(x,t)\phi(x,t)\,dxdt \\
&\quad = \int_{\mathbb{T}^d} \rho_0(x)\,(\phi * (\bar{K}_\kappa \mathbb{1}_{t<0}))\,(x,0)\,dx \\
&\quad\quad + \int_{\mathbb{T}^d\times[0,T]} \rho(x,t)u(x,t)\cdot(\phi * (\nabla \bar{K}_\kappa \mathbb{1}_{t<0}))\,(x,t)\,dxdt \\
&\quad = \int_{\mathbb{T}^d} \phi(x,t)\left((\rho_0 * (K_\kappa(\cdot,t)))\,(x) - \left((\rho u \mathbb{1}_{t\in[0,T]}) * (\nabla K_\kappa \mathbb{1}_{t>0})\right)(x,t)\right)\,dxdt,
\end{aligned}$$

where $(\rho_0 * (K_\kappa(\cdot,t)))\,(x)$ is convolution over the spatial variable $x \in \mathbb{T}^d$ only.

We have shown, indeed for any weak solution $\rho(x,t)$ to (κ-TDE) along $u(x,t)$ with initial data $\rho_0(x) \in L_x^1$,

$$\rho(x,t) = (\rho_0 * K_\kappa(\cdot,t)(x,t) - \left((\rho u \mathbb{1}_{t\in[0,T]}) *_{x,t} (\nabla K_\kappa \mathbb{1}_{t>0})\right)(x,t). \tag{4.14}$$

When $\rho_0(x) = 0$ we have by Young's convolution inequality,

$$\left\|\rho\,\mathbb{1}_{t\in[0,\varepsilon]}\right\|_{L_t^1L_x^1} \leq \left\|\rho\,\mathbb{1}_{t\in[0,\varepsilon]}\right\|_{L_t^1L_x^1} \|u\|_{L_t^\infty L_x^\infty} \left\|\nabla K_\kappa \mathbb{1}_{t\in(0,\varepsilon]}\right\|_{L_t^1L_x^1}.$$

It is straightforward to check that $\nabla K_\kappa(x,t)\mathbb{1}_{t\in(0,T]} \in L_t^1L_x^1$, and so for ε small enough (depending only on $\|u\|_{L_t^\infty L_x^\infty}$ and κ) the above implies that $\rho(x,t)\mathbb{1}_{t\in[0,\varepsilon]} = 0$. Repeating the argument then shows that for all $n \in \mathbb{N}$, $\rho(x,t)\mathbb{1}_{t\in[0,n\varepsilon]} = 0$, and so indeed $\rho(x,t) = 0$, proving uniqueness.

Existence of a weak solution $\rho(x,t) \in L_t^\infty L_x^1$ with $\|\rho\|_{L_t^\infty L_x^1} \leq \|\rho_0\|_{L_x^1}$ follows by solving the equation for mollified $u(x,t)$ and $\rho_0(x)$ as in Theorem 4.2. It can be checked that this produces a sequence of smooth functions $\rho_n(x,t)$ converging to the solution $\rho(x,t)$ weakly, and by the argument in [30], with an a priori bound on $\|\rho_n\|_{L^\infty([\varepsilon,T];L_x^\infty)}$ depending only on $\|\rho_0\|_{L_x^1}$, $\|u\|_{L_t^1L_x^\infty}$, $\varepsilon > 0$, and κ. We therefore have $\rho(x,t) \in L_t^\infty L_x^1$ and $\rho(x,t) \in L^\infty([\varepsilon,T];L_x^\infty)$ for any $\varepsilon > 0$. Though we do not make use of it in the remainder of the chapter, the improved regularity $\rho(x,t) \in C_t^0L_x^1$, and $\rho(x,t) \in C^0([\varepsilon,T];C_x^0)$ for all $\varepsilon \in (0,T]$, follow from the formula (4.14) and the regularity of $K_\kappa(x,t)$.

When in addition $\rho_0(x) \in L_x^2$ the regularised sequence $\rho_n(x,t)$ can further be shown to converge in $C_t^0L_x^2 \cap L_t^2H_x^1$. Statements (4.11) and (4.12) then follow from their counterparts for smooth $\rho_0(x)$ and $u(x,t)$. If $\rho_0(x) \in L_x^\infty$ we have, say by (4.11), $\rho(x,t) \in L_t^\infty L_x^\infty$, and hence by Theorem 4.1 we prove (4.10). The continuity $\rho(x,t) \in C_t^0L_x^p$ then follows from $\rho(x,t) \in C_t^0L_x^2$, for $p \in [1,2)$ by compactness of $\mathbb{T}^d$, and for $p \in (2,\infty)$ by interpolation with $\rho(x,t) \in L_t^\infty L_x^\infty$. It remains to show (4.13). From (4.12) we have the bounds $\|\rho\|_{L_t^\infty L_x^2}, \sqrt{2\kappa}\|\nabla\rho\|_{L_t^2 L_x^2} \leq \|\rho_0\|_{L_x^2}$, and hence for any $\phi(x,t) \in C_c^\infty(\mathbb{T}^d \times (0,T);\mathbb{R})$, by the equation ($\kappa$-TDE),

$$\begin{aligned}\left|\int_{\mathbb{T}^d\times(0,T)} \rho(x,t)\frac{\partial\phi}{\partial t}(x,t)\,dxdt\right| &= \left|\int_{\mathbb{T}^d\times(0,T)} \rho(x,t)\left(u(x,t)\cdot\nabla\phi(x,t)+\kappa\Delta\phi(x,t)\right)\,dxdt\right| \\ &\leq \|\rho\|_{L_t^\infty L_x^2}\|u\|_{L_t^2 L_x^\infty}\|\nabla\phi\|_{L_t^2 L_x^2} + \kappa\|\nabla\rho\|_{L_t^2 L_x^2}\|\nabla\phi\|_{L_t^2 L_x^2} \\ &\leq \|\rho_0\|_{L_x^2}\|u\|_{L_t^2 L_x^\infty}\|\nabla\phi\|_{L_t^2 L_x^2} + \sqrt{\frac{\kappa}{2}}\|\rho_0\|_{L_x^2}\|\nabla\phi\|_{L_t^2 L_x^2},\end{aligned}$$

which proves (4.13). □

4.2.2 Control of the Vanishing Viscosity Limit

The main purpose of this section is to prove Theorem 4.6, which allows us to construct bounded divergence-free vector fields in a way that permits control of the corresponding vanishing viscosity limit of (κ-TDE). This relies on two Propositions.

The first, Proposition 4.4 below, gives a general criterion for which the vanishing viscosity limit of (κ-TDE) converges strongly to (TE). That is, for a suitable divergence-free vector field $u(x,t) : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$, and small viscosity $\kappa > 0$, that solutions of (κ-TDE) along $u(x,t)$ are well-approximated by a weak solution of (TE) along $u(x,t)$. This result generalises the Selection Theorem in [19].

The second, Proposition 4.5 below, uses a similar argument to show, for fixed viscosity $\kappa > 0$, how solutions of (κ-TDE) depend little on the small spatial scales of the vector field $u(x,t)$. We quantify these scales through the weak-$*$ topology of vector fields in $L_t^\infty L_x^\infty$. The intuition is that the viscosity 'blurs' these small spatial scales.

The key idea of Theorem 4.6 is then that solving (κ-TDE) with reduced viscosity $\kappa > 0$ is akin to adding small spatial scales while solving (TE).

The advantage of solving (TE) is then that Lagrangian solutions (Definition 4.4) can be designed rather explicitly.

Proposition 4.4. *Consider a vector field $u(x,t) \in L_t^1 L_x^1$ with $\nabla \cdot u(x,t) = 0$ in the distributional sense. Fix some initial data $\rho_0(x) \in L_x^\infty$.*

Suppose that there is a unique weak solution $\rho(x,t)$ (in the class $L_t^\infty L_x^\infty$) to (TE) *along $u(x,t)$ with initial data $\rho_0(x)$ and that additionally $\rho(x,t)$ is a renormalised weak solution (Definition 4.3).*

For each $\kappa > 0$ denote by $\rho^\kappa(x,t)$ any weak solution to (κ-TDE) *along $u(x,t)$ with initial data $\rho_0(x)$. Suppose in addition that $\rho^\kappa(x,t) \in C^0_{\text{weak}-*} L_x^\infty$ and satisfies the Initial*

L^p_x*-Inequality* (4.7) *for all* $p \in [1,\infty]$. *(This would be the case if, say,* $u(x,t) \in L^\infty_t L^\infty_x$*, see Theorem 4.3.)*

Then, for each $p \in [1,\infty)$, $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ *converges strongly in* $L^p_t L^p_x$ *and also in weak-*$*$ $L^\infty_t L^\infty_x$.

If additionally $\rho(x,t) \in C^0_t L^1_x$*, then for each* $p \in [1,\infty)$, $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ *also converges strongly in* $L^\infty_t L^p_x$.

Proof. Suppose that $\rho^\kappa(x,t)$ does not converge in weak-$*$ $L^\infty_t L^\infty_x$ to $\rho(x,t)$ as $\kappa \to 0$. Then there exists some $g(x,t) \in L^1_t L^1_x$, and a sequence $\kappa_i \xrightarrow{i\to\infty} 0$ and $c > 0$ such that for all $i \in \mathbb{N}$

$$\left| \int_{\mathbb{T}^d\times[0,T]} (\rho^{\kappa_i}(x,t) - \rho(x,t)) g(x,t)\, dxdt \right| \geq c. \tag{4.15}$$

By the Initial L^p_x-Inequality (4.7) $\rho^{\kappa_i}(x,t)$ is uniformly bounded for all $i \in \mathbb{N}$ in $L^\infty_t L^\infty_x$, and so by taking a subsequence if necessary, we may assume $\rho^{\kappa_i}(x,t) \xrightarrow{i\to\infty} \bar{\rho}(x,t)$ converges in weak-$*$ $L^\infty_t L^\infty_x$ to some $\bar{\rho}(x,t) \in L^\infty_t L^\infty_x$. Then, for any $\phi(x,t) \in C^\infty_c(\mathbb{T}^d \times [0,T);\mathbb{R})$,

$$\begin{aligned}
&\int_{\mathbb{T}^d\times[0,T)} \bar{\rho}(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) \right) dxdt \\
&= \lim_{i\to\infty} \int_{\mathbb{T}^d\times[0,T)} \rho^{\kappa_i}(x,t) \left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) + \kappa_i \Delta\phi(x,t) \right) dxdt \\
&= -\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\, dx,
\end{aligned}$$

and so the limit $\bar{\rho}(x,t)$ is a weak solution to (TE) along $u(x,t)$ with initial data $\rho_0(x)$. Moreover, it is in $L^\infty_t L^\infty_x$, so by assumption, it must be the unique weak solution $\rho(x,t)$, contradicting (4.15). Therefore $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ converges in weak-$*$ $L^\infty_t L^\infty_x$.

Recall by Theorem 4.1 that $\rho(x,t), \rho^\kappa(x,t) \in C^0_{\text{weak}-*} L^\infty_x$.

Since by assumption $\rho^\kappa(x,t)$ satisfies the Initial L^p_x-Inequality (4.7) for all $p \in [1,\infty]$, we may bound

$$\|\rho^\kappa\|_{L^p_t L^p_x} \leq T^{\frac{1}{p}} \|\rho_0\|_{L^p_x}. \tag{4.16}$$

If $p \in (1,\infty]$, weak-$*$ convergence in $L^\infty_t L^\infty_x$ implies weak-$*$ convergence in $L^p_t L^p_x$. Whenever $p \in (1,\infty)$, weak-$*$ convergence in $L^p_t L^p_x$ is also strong in $L^p_t L^p_x$ if and only if $\limsup_{\kappa\to 0} \|\rho^\kappa\|_{L^p_t L^p_x} \leq \|\rho\|_{L^p_t L^p_x}$. This is a standard result from uniform convexity of L^q_x for $p \in (1,\infty)$.

To show that this is satisfied, we use that $\rho(x,t) \in C^0_{\text{weak}-*} L^\infty_x$ is a renormalised weak solution to (TE) (Definition 4.3). Denoting by $a \wedge b = \min\{a,b\}$, let $M \in \mathbb{N}$ and $\beta(x) = M \wedge |x|^p$, then $\beta(\rho(x,t))$ is a weak solution to (TE) along $u(x,t)$ with initial data $\beta(\rho_0(x)) \in L^\infty_x$. Taking $\phi(x,t) = 1$ in the Trace Formula (4.5) shows that there exists a subset $E_M \subset [0,T]$

with zero Lebesgue-measure, such that for all $t \in [0,T] \setminus E_M$ we have,

$$\int_{\mathbb{T}^d} M \wedge |\rho(x,t)|^p \, dx = \int_{\mathbb{T}^d} M \wedge |\rho_0(x)|^p \, dx.$$

In particular, the above holds for all $t \in [0,T] \setminus \bigcup_{M\in\mathbb{N}} E_M$. By the Lebesgue monotone convergence Theorem, taking $M \to \infty$ shows that $\|\rho(\cdot,t)\|_{L^p_x} = \|\rho_0\|_{L^p_x}$ for all $t \in [0,T] \setminus \bigcup_{M\in\mathbb{N}} E_M$, which implies $\|\rho\|_{L^p_t L^p_x} = T^{\frac{1}{p}} \|\rho_0\|_{L^p_x}$. Combined with (4.16) this implies that $\limsup_{\kappa\to 0} \|\rho^\kappa\|_{L^p_t L^p_x} \leq \|\rho\|_{L^p_t L^p_x}$, and hence $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ in $L^p_t L^p_x$ as required.

Convergence of $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ in $L^1_t L^1_x$ follows from convergence in $L^p_t L^p_x$ for any $p \in (1,\infty)$ and the compactness of the domain $\mathbb{T}^d \times [0,T]$.

We now assume that $\rho(x,t) \in C^0_t L^1_x$ and wish to upgrade to convergence in $L^\infty_t L^p_x$ for each $p \in [1,\infty)$. The idea is to use the Trace Formula (4.5) to show for each $t \in [0,T]$ that $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ converges in weak-$*$ L^∞_x, and then upgrade to (uniform in time) strong convergence in L^q_x by convergence of the norm $\|\rho^\kappa(\cdot,t)\|_{L^p_x}$ for $p \in (1,\infty)$.

Since the L^2_x-inner product makes life easier, we will only prove convergence in $L^\infty_t L^2_x$, and notice that convergence in $L^\infty_t L^p_x$ follows for $p \in [1,2)$ by compactness of $\mathbb{T}^d$, and for $p \in (2,\infty)$ by interpolation with the existing uniform bound in $L^\infty_t L^\infty_x$. Though we do not elaborate on it, convergence in $L^\infty_t L^p_x$ for $p \in (1,\infty)$ can also be shown directly if, say, $\rho_0(x) \notin L^2_x$.

We have already shown that $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ in $L^1_t L^1_x$, with uniform bound in $L^\infty_t L^\infty_x$. Suppose that $u(x,t)\rho^\kappa(x,t)$ does not converge strongly in $L^1_t L^1_x$ to $u(x,t)\rho(x,t)$ as $\kappa \to 0$. Then there exists a sequence $\kappa_i \xrightarrow{i\to\infty} 0$ and $c > 0$ such that for all $i \in \mathbb{N}$

$$\|u\rho^{\kappa_i} - u\rho\|_{L^1_t L^1_x} \geq c. \tag{4.17}$$

Now $\rho^{\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho(x,t)$ strongly in $L^1_t L^1_x$, and so by taking a further subsequence if necessary, we may assume that $\rho^{\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho(x,t)$ point-wise a.e. in $\mathbb{T}^d \times [0,T]$. But then $u(x,t)\rho^{\kappa_i}(x,t) \xrightarrow{i\to\infty} u(x,t)\rho(x,t)$ converge point-wise a.e. in $\mathbb{T}^d \times [0,T]$, and are also uniformly bounded for all $i \in \mathbb{N}$ in $L^1_t L^1_x$ by $\|u\|_{L^1_t L^1_x} \|\rho_0\|_{L^\infty_x}$, and so the dominated convergence Theorem yields a contradiction to (4.17). Therefore the product $u(x,t)\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} u(x,t)\rho(x,t)$ converges strongly in $L^1_t L^1_x$.

We now use that $\rho(x,t) \in C^0_t L^1_x$, and therefore $\rho(x,t) \in C^0_t L^2_x$ (by interpolation with the existing bound $\rho(x,t) \in L^\infty_t L^\infty_x$), to take a smooth approximation $\phi_\varepsilon(x,t) \in C^\infty(\mathbb{T}^d \times [0,T];\mathbb{R})$, such that $\|\phi_\varepsilon - \rho\|_{L^\infty_t L^2_x} \leq \varepsilon$. Then by the Trace Formula (4.5), and that $\|\rho^\kappa(\cdot,t)\|_{L^2_x} \leq$

$\|\rho_0\|_{L^2_x} = \|\rho(\cdot,t)\|_{L^2_x}$ for a.e. $t \in [0,T]$, we may write for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\|\rho^\kappa(\cdot,t) - \rho(\cdot,t)\|^2_{L^2_x} \\
&\quad = \|\rho^\kappa(\cdot,t)\|^2_{L^2_x} + \|\rho(\cdot,t)\|^2_{L^2_x} - 2\int_{\mathbb{T}^d} \rho^\kappa(x,t)\rho(x,t)\,dx \\
&\quad \le 2\|\rho(\cdot,t)\|^2_{L^2_x} + 2\varepsilon\|\rho_0\|_{L^2_x} - 2\int_{\mathbb{T}^d} \rho^\kappa(x,t)\phi_\varepsilon(x,t)\,dx \\
&\quad \le 2\|\rho(\cdot,t)\|^2_{L^2_x} + 2\varepsilon\|\rho_0\|_{L^2_x} - 2\int_{\mathbb{T}^d} \rho_0(x)\phi_\varepsilon(x,0)\,dx \\
&\qquad - 2\int_{\mathbb{T}^d\times[0,t]} \rho^\kappa(x,t)\left(\frac{\partial\phi_\varepsilon}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi_\varepsilon(x,t) + \kappa\Delta\phi_\varepsilon(x,t)\right)\,dxdt \\
&\quad \le 2\|\rho(\cdot,t)\|^2_{L^2_x} + 2\varepsilon\|\rho_0\|_{L^2_x} - 2\int_{\mathbb{T}^d} \rho_0(x)\phi_\varepsilon(x,0)\,dx \\
&\qquad - 2\int_{\mathbb{T}^d\times[0,t]} \rho(x,t)\left(\frac{\partial\phi_\varepsilon}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi_\varepsilon(x,t)\right)\,dxdt \\
&\qquad + 2\|\rho^\kappa - \rho\|_{L^1_tL^1_x}\left\|\frac{\partial\phi_\varepsilon}{\partial t}\right\|_{L^\infty_tL^\infty_x} + 2\kappa\|\rho^\kappa\|_{L^1_tL^1_x}\|\Delta\phi_\varepsilon\|_{L^\infty_tL^\infty_x} \\
&\qquad + 2\|u\rho^\kappa - u\rho\|_{L^1_tL^1_x}\|\nabla\phi_\varepsilon\|_{L^\infty_tL^\infty_x} \\
&\quad = 2\|\rho(\cdot,t)\|^2_{L^2_x} + 2\varepsilon\|\rho_0\|_{L^2_x} - 2\int_{\mathbb{T}^d} \rho(x,t)\phi_\varepsilon(x,t)\,dx \\
&\qquad + 2\|\rho^\kappa - \rho\|_{L^1_tL^1_x}\left\|\frac{\partial\phi_\varepsilon}{\partial t}\right\|_{L^\infty_tL^\infty_x} + 2\kappa\|\rho^\kappa\|_{L^1_tL^1_x}\|\Delta\phi_\varepsilon\|_{L^\infty_tL^\infty_x} \\
&\qquad + 2\|u\rho^\kappa - u\rho\|_{L^1_tL^1_x}\|\nabla\phi_\varepsilon\|_{L^\infty_tL^\infty_x} \\
&\quad \le 4\varepsilon\|\rho_0\|_{L^2_x} + C_\varepsilon\left(\|\rho^\kappa - \rho\|_{L^1_tL^1_x} + \kappa\|\rho^\kappa\|_{L^1_tL^1_x} + \|u\rho^\kappa - u\rho\|_{L^1_tL^1_x}\right),
\end{aligned}$$

with $\varepsilon > 0$ arbitrary, and C_ε a constant that depends only on ε and $\rho(x,t)$, and not on $t \in [0,T]$. Then since $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$, and $u(x,t)\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} u(x,t)\rho(x,t)$ in $L^1_tL^1_x$ we see that $\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t)$ in $L^\infty_tL^2_x$ as required. □

Proposition 4.5. *Consider a sequence of uniformly bounded, divergence-free vector fields $u_n(x,t) \in L^\infty_tL^\infty_x$, such that $u_n(x,t) \xrightarrow{n\to\infty} u(x,t)$ converges in weak-$*$ $L^\infty_tL^\infty_x$ to some $u(x,t) \in L^\infty_tL^\infty_x$. Fix some initial data $\rho_0(x) \in L^\infty_x$.*

For each $n \in \mathbb{N}$, and $\kappa > 0$ denote by $\rho^{n,\kappa}(x,t)$ the unique (by Theorem 4.3) weak solution to (κ-TDE) *along $u_n(x,t)$ with initial data $\rho_0(x)$. Similarly denote by $\rho^\kappa(x,t)$ the unique weak solution to* (κ-TDE) *along $u(x,t)$ with initial data $\rho_0(x)$. Then for each $0 < a \le b$, $p \in [1,\infty)$*

$$\sup_{\kappa\in[a,b]} \|\rho^{n,\kappa} - \rho^\kappa\|_{L^2_tH^1_x \cap L^\infty_tL^p_x} \xrightarrow{n\to\infty} 0.$$

Proof. The case $p \in (2,\infty)$ follows from the same result for $p=2$ by interpolation with the existing uniform bound on $\rho^{n,\kappa}(x,t)$, $\rho^{\kappa}(x,t)$ in $L^\infty_t L^\infty_x$ from the L^p_x-Inequality (4.11), while for the case $p \in (1,2)$ it follows from the case for $p=2$ by compactness of $\mathbb{T}^d$. Therefore, we may restrict to $p=2$.

Assume to the contrary that for some $0 < a \le b$ that there exists $c > 0$ and sequences $\{n_i\}_{i\in\mathbb{N}}$, $\{\kappa_i\}_{i\in\mathbb{N}}$ with $n_i \in \mathbb{N}$ increasing, $a \le \kappa_i \le b$, such that for all $i \in \mathbb{N}$

$$\|\rho^{n_i,\kappa_i} - \rho^{\kappa_i}\|_{L^2_t H^1_x} + \|\rho^{n_i,\kappa_i} - \rho^{\kappa_i}\|_{L^\infty_t L^2_x} \ge c. \tag{4.18}$$

By taking a subsequence if necessary we may assume $\kappa_i \xrightarrow{i\to\infty} \kappa$ for some $\kappa \in [a,b]$.

Since κ_i are bounded above and below, by Energy Identity (4.12) $\rho^{n_i,\kappa_i}(x,t)$, $\rho^{\kappa_i}(x,t)$ are uniformly bounded for all $i \in \mathbb{N}$ in $L^2_t H^1_x$, and by Equicontinuity (4.13) $\frac{\partial}{\partial t}\rho^{n_i,\kappa_i}(x,t)$, $\frac{\partial}{\partial t}\rho^{\kappa_i}(x,t)$ are uniformly bounded for all $i \in \mathbb{N}$ in $L^2_t H^{-1}_x$.

Since $H^1_x \Subset L^2_x \subset H^{-1}_x$ we may apply the Aubin-Lions compactness Lemma (see Chapter 3, Theorem 2.1 in [103]) to deduce that the set $\{\rho(x,t) \in L^2_t H^1_x : \left\|\frac{\partial \rho}{\partial t}\right\|_{L^2_t H^{-1}_x} \le C\}$ is compactly embedded into $L^2_t L^2_x$.

Hence, again taking a subsequence if necessary, we may assume that both $\rho^{n_i,\kappa_i}(x,t)$, $\rho^{\kappa_i}(x,t)$ converge, to some limits, strongly in $L^2_t L^2_x$ (and weakly in $L^2_t H^1_x$) as $i \to \infty$.

We next show that both $\rho^{n_i,\kappa_i}(x,t)$, and $\rho^{\kappa_i}(x,t)$ converge to $\rho^{\kappa}(x,t)$ strongly in $L^2_t H^1_x$ and $L^\infty_t L^2_x$ as $i \to \infty$, contradicting the assumption (4.18).

We will only show the required convergence for $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^{\kappa}(x,t)$, since the same result for $\rho^{\kappa_i}(x,t)$ follows the same proof by additionally assuming that $u_{n_i}(x,t)$ are a constant sequence $u_{n_i}(x,t) = u(x,t)$ for all $i \in \mathbb{N}$.

Since $u_{n_i}(x,t) \xrightarrow{i\to\infty} u(x,t)$ are uniformly bounded and converge in weak-$*$ $L^\infty_t L^\infty_x$, and as we have already shown that $\rho^{n_i,\kappa_i}(x,t)$ converges to some $\bar{\rho}$ strongly in $L^2_t L^2_x$ as $i \to \infty$, then the product $\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t)$ converges to $\bar{\rho}(x,t)u(x,t)$ weakly in $L^2_t L^2_x$ as $i \to \infty$.

Then by the weak formulation of (κ-TDE), for any $\phi(x,t) \in C^\infty_c(\mathbb{T}^d \times [0,T);\mathbb{R})$, one has

$$\begin{aligned}
&\int_{\mathbb{T}^d\times[0,T]} \bar{\rho}(x,t)\left(\frac{\partial \phi}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right)\,dxdt \\
&\qquad = \lim_{i\to\infty}\int_{\mathbb{T}^d\times[0,T]} \rho^{n_i,\kappa_i}(x,t)\left(\frac{\partial \phi}{\partial t}(x,t) + u_{n_i}(x,t)\cdot\nabla\phi(x,t) + \kappa_i\Delta\phi(x,t)\right)\,dxdt \\
&\qquad = -\int_{\mathbb{T}^d} \rho_0(x)\phi_0(x)\,dx.
\end{aligned}$$

Consequently, we see that $\bar{\rho}(x,t)$ is a weak solution to (κ-TDE) along $u(x,t)$ with initial data $\rho_0(x)$. By Theorem 4.3, this solution is unique and must be $\rho^{\kappa}(x,t)$.

We are left to upgrade the convergence $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \bar{\rho}(x,t)$ from strong in $L_t^2L_x^2$ and weak in $L_t^2H_x^1$, to strong in $L_t^2H_x^1$ and strong in $L_t^\infty L_x^2$.

In light of the Trace Formula (4.5), and the uniform bound on $\rho^{n_i,\kappa_i}(x,t)$ for all $i \in \mathbb{N}$ in $C_t^0L_x^2$ (by Theorem 4.3), strong convergence $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^\kappa(x,t)$ in $L_t^2L_x^2$ implies $\rho^{n_i,\kappa_i}(x,T) \xrightarrow{i\to\infty} \rho^\kappa(x,T)$ converges weakly in L_x^2 (note that $\rho^{n_i,\kappa_i}(x,t)$, $\rho^\kappa(x,t)$ are uniformly bounded in $C_t^0L_x^2$ by Theorem 4.3). In particular

$$\liminf_{i\to\infty} \int_{\mathbb{T}^d} |\rho^{n_i,\kappa_i}(x,T)|^2 \, dx \geq \int_{\mathbb{T}^d} |\rho^\kappa(x,T)|^2 \, dx.$$

In light of the Energy Identity (4.12), this implies

$$\limsup_{i\to\infty} \int_{\mathbb{T}^d\times[0,T]} |\nabla\rho^{n_i,\kappa_i}(x,t)|^2 \, dxdt \leq \int_{\mathbb{T}^d\times[0,T]} |\nabla\rho^\kappa(x,t)|^2 \, dxdt.$$

Since also $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^\kappa(x,t)$ weakly in $L_t^2H_x^1$ we have

$$\begin{aligned}\liminf_{i\to\infty} &\int_{\mathbb{T}^d\times[0,T]} \left(|\rho^{n_i,\kappa_i}(x,t)|^2 + |\nabla\rho^{n_i,\kappa_i}(x,t)|^2\right) dxdt \\ &\geq \int_{\mathbb{T}^d\times[0,T]} \left(|\rho^\kappa(x,t)|^2 + |\nabla\rho^\kappa(x,t)|^2\right) dxdt.\end{aligned}$$

However, since $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^\kappa(x,t)$ strongly in $L_t^2L_x^2$, from the above we must have convergence of the norms $\|\rho^{n_i,\kappa_i}\|_{L_t^2H_x^1}^2 \xrightarrow{i\to\infty} \|\rho^\kappa\|_{L_t^2H_x^1}^2$. Thus the weak convergence $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^\kappa(x,t)$ in $L_t^2H_x^1$ is in fact strong, as required.

To extend to strong convergence in $L_t^\infty L_x^2$, we use the fact that $\rho^\kappa(x,t) \in C_t^0L_x^2$ to take a smooth approximation $\phi_\varepsilon(x,t) \in C^\infty(\mathbb{T}^d \times [0,T];\mathbb{R})$, such that $\|\phi_\varepsilon - \rho^\kappa\|_{L_t^\infty L_x^2} \leq \varepsilon$. Then by the Trace Formula (4.5), and the Energy Identity (4.12), for all $t \in [0,T]$ we see that

$$\begin{aligned}&\|\rho^{n_i,\kappa_i}(\cdot,t) - \rho^\kappa(\cdot,t)\|_{L_x^2}^2 \\ &\quad= \|\rho^{n_i,\kappa_i}(\cdot,t)\|_{L_x^2}^2 + \|\rho^\kappa(\cdot,t)\|_{L_x^2}^2 - 2\int_{\mathbb{T}^d} \rho^{n_i,\kappa_i}(x,t)\rho^\kappa(x,t)\, dx \\ &\quad\leq \|\rho^{n_i,\kappa_i}(\cdot,t)\|_{L_x^2}^2 + \|\rho^\kappa(\cdot,t)\|_{L_x^2}^2 + 2\varepsilon\|\rho_0\|_{L_x^2} - 2\int_{\mathbb{T}^d} \rho^{n_i,\kappa_i}(x,t)\phi_\varepsilon(x,t)\, dx \\ &\quad= \|\rho^{n_i,\kappa_i}(\cdot,t)\|_{L_x^2}^2 + \|\rho^\kappa(\cdot,t)\|_{L_x^2}^2 + 2\varepsilon\|\rho_0\|_{L_x^2} - 2\int_{\mathbb{T}^d} \rho_0(x)\phi_\varepsilon(x,0)\, dx \\ &\qquad -2\int_{\mathbb{T}^d\times[0,t]} \rho^{n_i,\kappa_i}(x,t)\left(\frac{\partial\phi_\varepsilon}{\partial t}(x,t) + u_{n_i}(x,t)\cdot\nabla\phi_\varepsilon(x,t) + \kappa_i\Delta\phi_\varepsilon(x,t)\right) dxdt\end{aligned}$$

$$
\begin{aligned}
&\leq \|\rho^{n_i,\kappa_i}(\cdot,t)\|_{L_x^2}^2 + \|\rho^{\kappa}(\cdot,t)\|_{L_x^2}^2 + 2\varepsilon \|\rho_0\|_{L_x^2} - 2\int_{\mathbb{T}^d} \rho_0(x)\phi_\varepsilon(x,0)\,dx \\
&\quad - 2\int_{\mathbb{T}^d\times[0,t]} \rho^{\kappa}(x,t)\left(\frac{\partial \phi_\varepsilon}{\partial t}(x,t) + u(x,t)\cdot\nabla\phi_\varepsilon(x,t) + \kappa\Delta\phi_\varepsilon(x,t)\right)\,dxdt \\
&\quad + 2\|\rho^{n_i,\kappa_i} - \rho^{\kappa}\|_{L_t^2L_x^2}\left\|\frac{\partial\phi_\varepsilon}{\partial t}\right\|_{L_t^2L_x^2} + 2|\kappa_i - \kappa|\,\|\rho^{n_i,\kappa_i}\|_{L_t^2L_x^2}\|\Delta\phi_\varepsilon\|_{L_t^2L_x^2} \\
&\quad + 2\left|\int_{\mathbb{T}^d\times[0,t]} \left(\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \rho^{\kappa}(x,t)u(x,t)\right)\cdot\nabla\phi_\varepsilon(x,t)\,dxdt\right| \\
&= \|\rho^{n_i,\kappa_i}(\cdot,t)\|_{L_x^2}^2 + \|\rho^{\kappa}(\cdot,t)\|_{L_x^2}^2 + 2\varepsilon\|\rho_0\|_{L_x^2} - 2\int_{\mathbb{T}^d}\rho^{\kappa}(x,t)\phi_\varepsilon(x,t)\,dx \\
&\quad + 2\|\rho^{n_i,\kappa_i} - \rho^{\kappa}\|_{L_t^2L_x^2}\left\|\frac{\partial\phi_\varepsilon}{\partial t}\right\|_{L_t^2L_x^2} + 2|\kappa_i - \kappa|\,\|\rho^{n_i,\kappa_i}\|_{L_t^2L_x^2}\|\Delta\phi_\varepsilon\|_{L_t^2L_x^2} \\
&\quad + 2\left|\int_{\mathbb{T}^d\times[0,t]} \left(\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \rho^{\kappa}(x,t)u(x,t)\right)\cdot\nabla\phi_\varepsilon(x,t)\,dxdt\right| \\
&\leq \|\rho^{n_i,\kappa_i}(\cdot,t)\|_{L_x^2}^2 - \|\rho^{\kappa}(\cdot,t)\|_{L_x^2}^2 + 4\varepsilon\|\rho_0\|_{L_x^2} \\
&\quad + 2\|\rho^{n_i,\kappa_i} - \rho^{\kappa}\|_{L_t^2L_x^2}\left\|\frac{\partial\phi_\varepsilon}{\partial t}\right\|_{L_t^2L_x^2} + 2|\kappa_i - \kappa|\,\|\rho^{n_i,\kappa_i}\|_{L_t^2L_x^2}\|\Delta\phi_\varepsilon\|_{L_t^2L_x^2} \\
&\quad + 2\left|\int_{\mathbb{T}^d\times[0,t]} \left(\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \rho^{\kappa}(x,t)u(x,t)\right)\cdot\nabla\phi_\varepsilon(x,t)\,dxdt\right| \\
&= 4\varepsilon\|\rho_0\|_{L_x^2} - \int_{\mathbb{T}^d\times[0,t]}\left(2\kappa_i|\nabla\rho^{n_i,\kappa_i}(x,t)|^2 - 2\kappa|\nabla\rho^{\kappa}(x,t)|^2\right)\,dxdt \\
&\quad + 2\|\rho^{n_i,\kappa_i} - \rho^{\kappa}\|_{L_t^2L_x^2}\left\|\frac{\partial\phi_\varepsilon}{\partial t}\right\|_{L_t^2L_x^2} + 2|\kappa_i - \kappa|\,\|\rho^{n_i,\kappa_i}\|_{L_t^2L_x^2}\|\Delta\phi_\varepsilon\|_{L_t^2L_x^2} \\
&\quad + 2\left|\int_{\mathbb{T}^d\times[0,t]} \left(\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \rho^{\kappa}(x,t)u(x,t)\right)\cdot\nabla\phi_\varepsilon(x,t)\,dxdt\right| \\
&\leq 4\varepsilon\|\rho_0\|_{L_x^2} + 2\left\|\kappa_i^{\frac{1}{2}}\rho^{n_i,\kappa_i} - \kappa^{\frac{1}{2}}\rho^{\kappa}\right\|_{L_t^2H_x^1}\left(\kappa_i^{\frac{1}{2}}\|\rho^{n_i,\kappa_i}\|_{L_t^2H_x^1} + \kappa^{\frac{1}{2}}\|\rho^{\kappa}\|_{L_t^2H_x^1}\right) \\
&\quad + C_\varepsilon\left(\|\rho^{n_i,\kappa_i} - \rho^{\kappa}\|_{L_t^2L_x^2} + |\kappa_i - \kappa|\,\|\rho^{n_i,\kappa_i}\|_{L_t^2L_x^2}\right) \\
&\quad + 2\left|\int_{\mathbb{T}^d\times[0,t]} \left(\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \rho^{\kappa}(x,t)u(x,t)\right)\cdot\nabla\phi_\varepsilon(x,t)\,dxdt\right|,
\end{aligned}
$$

with $\varepsilon > 0$ arbitrary, and C_ε a constant that depends only on $\phi_\varepsilon(x,t)$, and not on $t \in [0,T]$.

Therefore by the convergence $\kappa_i \xrightarrow{i\to\infty} \kappa$, and $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^{\kappa}(x,t)$ strongly in $L_t^2H_x^1$, to show strong convergence of $\rho^{n_i,\kappa_i}(x,t) \xrightarrow{i\to\infty} \rho^{\kappa}(x,t)$ in $L_t^\infty L_x^2$ we need only show, for fixed $\phi_\varepsilon(x,t)$

$$
\sup_{t\in[0,T]}\left|\int_{\mathbb{T}^d\times[0,t]} \left(\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \rho^{\kappa}(x,t)u(x,t)\right)\cdot\nabla\phi_\varepsilon(x,t)\,dxdt\right| \xrightarrow{i\to\infty} 0.
$$

Suppose not, then there exists a sequence $t_i \in [0,T]$ for $i \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{T}^d \times [0,t_i]} (\rho^{n_i,\kappa_i}(x,t) u_{n_i}(x,t) - \rho^{\kappa}(x,t) u(x,t)) \cdot \nabla \phi_\varepsilon(x,t)\, dxdt \right| \geq c, \tag{4.19}$$

for some $c > 0$.

By compactness of $[0,T]$, and taking a subsequence if necessary, we may further assume that $t_i \xrightarrow{i\to\infty} \bar{t}$ for some $\bar{t} \in [0,T]$. However, there is strong convergence of both $\mathbb{1}_{t\in[0,t_i]}\rho^{n_i,\kappa_i}(x,t)$, $\mathbb{1}_{t\in[0,t_i]}\rho^{\kappa}(x,t) \xrightarrow{i\to\infty} \mathbb{1}_{t\in[0,\bar{t}]}\rho^{\kappa}(x,t)$ in $L^2_t L^2_x$, and boundedness of the sequence $u_{n_i}(x,t)$ in $L^\infty_t L^\infty_x$, and weak-$*$ convergence $u_{n_i}(x,t) \xrightarrow{i\to\infty} u(x,t)$ in $L^\infty_t L^\infty_x$. Therefore, the products $\mathbb{1}_{t\in[0,t_i]}\rho^{n_i,\kappa_i}(x,t)u_{n_i}(x,t) - \mathbb{1}_{t\in[0,t_i]}\rho^{\kappa}(x,t)u(x,t) \xrightarrow{i\to\infty} 0$ converge weakly in $L^2_t L^2_x$, contradicting (4.19), and hence proving the claim. □

We now give the following corollary of the previous two propositions. By first fixing a viscosity $\kappa_i > 0$, and applying Proposition 4.5, we find that transport-diffusion is not sensitive to suitably small scale changes to the vector field $u(x,t)$. However, inviscid transport is, so we adjust the inviscid solution to suit our requirements without much change to transport-diffusion with viscosity κ. Having done so, we then apply Proposition 4.4 to find a smaller $\kappa_{i+1} > 0$ for which transport-diffusion is close to the adjusted inviscid transport. Repeating, we may make finer and finer adjustments to our vector field, for which a well-prepared vanishing sequence of viscosity $\{\kappa_i\}_{i\in\mathbb{N}}$ permits a good approximation of transport-diffusion to suitable inviscid transport.

Theorem 4.6. *Fix some $M > 0$, and a metric d_* inducing the weak-$*$ topology on*

$$X = \left\{ u(x,t) \in L^\infty_t L^\infty_x : \|u\|_{L^\infty_t L^\infty_x} \leq M \right\}.$$

Let $Y \subset X$ be the set of all divergence-free vector fields admitting a unique renormalised weak solution (unique in the class of all $L^\infty_t L^\infty_x$ weak solutions) to (TE) *for any initial data $\rho_0(x) \in L^\infty_x$.*

Let $\{u_i(x,t)\}_{i\in\mathbb{N}} \subset Y$, $\rho_0(x) \in L^\infty_x$. For each $n \in \mathbb{N}$, and $\kappa > 0$, denote by $\rho^{n,\kappa}(x,t)$, respectively $\rho^n(x,t)$, the unique weak solution to (κ-TDE)*, respectively* (TE)*, along $u_n(x,t)$ with initial data $\rho_0(x)$. Then,*

- ***S.1*** *For all $n \in \mathbb{N}$, there exists $\kappa_n > 0$, $\varepsilon_n > 0$ depending only on $\{u_i(x,t)\}_{i=1}^n$ (and in particular not on $\rho_0(x)$), with $\kappa_n \xrightarrow{n\to\infty} 0$ monotonically, such that the following hold true:*

- ***S.2*** *For all $p \in [1,\infty)$,*

$$\sup_{0<\kappa\leq\kappa_n} \|\rho^{n,\kappa} - \rho^n\|_{L^\infty_t L^p_x} \xrightarrow{n\to\infty} 0.$$

S.3 *If $d_*(u_{n+1},u_n) \le \varepsilon_n$ for all $n \in \mathbb{N}$, then $u_n(x,t) \xrightarrow{n\to\infty} u_\infty(x,t)$ converges in weak-$*$ $L_t^\infty L_x^\infty$ to some divergence-free vector field $u_\infty(x,t) \in L_t^\infty L_x^\infty$,*

S.4 *and if we denote by $\rho^{\infty,\kappa}(x,t)$ the unique weak solution to* (κ-TDE) *along $u_\infty(x,t)$ with initial data $\rho_0(x)$, then for all $p \in [1,\infty)$,*

$$\sup_{\kappa_n \le \kappa \le \kappa_1} \|\rho^{n,\kappa} - \rho^{\infty,\kappa}\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0.$$

S.5 *In particular, for all $p \in [1,\infty)$,*

$$\|\rho^{\infty,\kappa_n} - \rho^n\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0.$$

Proof. By separability of L_x^1, take a countable dense sequence $\{\rho_0^m\}_{m\in\mathbb{N}}$ in L_x^1, satisfying $\rho_0^m(x) \in L_x^\infty$.

For all $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $\kappa > 0$, denote by $\rho^{m;n,\kappa}(x,t)$, respectively $\rho^{m;n}(x,t)$, the unique weak solution to (κ-TDE), respectively (TE), along $u_n(x,t)$ with initial data $\rho_0^m(x)$. If in addition $u_n(x,t) \xrightarrow{n\to\infty} u_\infty$ in weak-$*$ $L_t^\infty L_x^\infty$, denote by $\rho^{m;\infty,\kappa}(x,t)$ the unique weak solution to (κ-TDE) along $u_\infty(x,t)$ with initial data $\rho_0^m(x)$.

For the given $\rho_0(x) \in L_x^\infty$, for each $n \in \mathbb{N}$, and $\kappa > 0$, denote by $\rho^{n,\kappa}(x,t)$, respectively $\rho^n(x,t)$, the unique weak solution to (κ-TDE), respectively (TE), along $u_n(x,t)$ with initial data $\rho_0(x)$. If in addition $u_n(x,t) \xrightarrow{n\to\infty} u_\infty$ in weak-$*$ $L_t^\infty L_x^\infty$, denote by $\rho^{\infty,\kappa}(x,t)$ the unique weak solution to (κ-TDE) along $u_\infty(x,t)$ with initial data $\rho_0(x)$.

Notice, by linearity of (κ-TDE), (TE), and the Initial L_x^p-Inequality (4.7), that for each $m \in \mathbb{N}$, $n \in \mathbb{N}$, $\kappa > 0$, we have

$$\|\rho^{n,\kappa} - \rho^{m;n,\kappa}\|_{L_t^\infty L_x^1} \le \|\rho_0 - \rho_0^m\|_{L_x^1}, \tag{4.20}$$

$$\|\rho^{n} - \rho^{m;n}\|_{L_t^\infty L_x^1} \le \|\rho_0 - \rho_0^m\|_{L_x^1}, \tag{4.21}$$

$$\|\rho^{\infty,\kappa} - \rho^{m;\infty,\kappa}\|_{L_t^\infty L_x^1} \le \|\rho_0 - \rho_0^m\|_{L_x^1}. \tag{4.22}$$

We will use this and L_x^1-density to only consider the countable set of initial data $\{\rho_0^m(x)\}_{m\in\mathbb{N}}$.

Given $n \in \mathbb{N}$ and $\{u_i(x,t)\}_{i=1}^n \subset Y$ we need to find $\kappa_n > 0$, $\varepsilon_n > 0$ such that (S.1)-(S.5) hold. For the purpose of the proof we additionally find $\delta_n > 0$ such that, for all $m \in \{1,\ldots,n\}$, $\kappa \in [\kappa_n, \kappa_1]$, and $w(x,t) \in Y$ with $d_*(w,u_n) \le \delta_n$,

$$\|\rho^{m;n,\kappa} - g^{m;\kappa}\|_{L_t^\infty L_x^1} \le \frac{1}{n}, \tag{4.23}$$

where $g^{m,\kappa}(x,t)$ is the unique weak solution to (κ-TDE) along $w(x,t)$ with initial data $\rho_0^m(x)$.

We do so by induction. Given $\{u_i\}_{i=1}^n$ as in the statement of the theorem assume $\{(\kappa_i,\varepsilon_i,\delta_i)\}_{i=1}^{n-1}$ are already chosen (an empty set if $n=1$).

By applying Proposition 4.4 for each initial data $\{\rho_0^m(x)\}_{m=1}^n$ there exists some $\kappa_n>0$ such that for all $m\in\{1,...,n\}$, $\kappa\in(0,\kappa_n]$,

$$\|\rho^{m;n,\kappa}-\rho^{m;n}\|_{L_t^\infty L_x^1}\leq\frac{1}{n}. \tag{4.24}$$

One may choose κ_n to be smaller, so that $\kappa_n\leq\frac{1}{n}$, and if $n\neq 1$, $\kappa_n<\kappa_{n-1}$.

We next find $\delta_n>0$ satisfying (4.23). Assume to the contrary that no such δ_n exists. There is a sequence of divergence-free vector fields $\{w_k(x,t)\}_{k\in\mathbb{N}}\subset Y$ with $w_k(x,t)\xrightharpoonup{k\to\infty}u_n(x,t)$ converging in weak-$*$ $L_t^\infty L_x^\infty$, which violate Proposition 4.5 for at least one of the initial data $\{\rho_0^m(x)\}_{m=1}^n$, where we have substituted $a=\kappa_n$, $b=\kappa_1$ into Proposition 4.5.

One may choose δ_n smaller so that $\delta_n\leq\frac{1}{n}$. Finally, take

$$\varepsilon_n=\min_{k\in\{0,...,n-1\}}\{\delta_{n-k}2^{-k-1}\}. \tag{4.25}$$

(S.1) follows immediately from our choice of κ_n. Meanwhile (S.2) with $p=1$ follows from (4.20), (4.21), (4.24). Interpolation with the existing uniform bound on $\rho^{n,\kappa}(x,t)$, $\rho^n(x,t)$ in $L_t^\infty L_x^\infty$ then gives (S.2) for $p\in(1,\infty)$.

We are now left to show (S.3)-(S.5) are satisfied when in addition $d_*(u_{n+1},u_n)\leq\varepsilon_n$ for all $n\in\mathbb{N}$.

By (4.25), $\sum_{i=n}^\infty\varepsilon_i\leq\delta_n$, and also $\delta_n\to 0$, so $\{u_i(x,t)\}_{i\in\mathbb{N}}$ is a d_*-Cauchy sequence, uniformly bounded in $L_t^\infty L_x^\infty$ by M, and so converges in weak-$*$ $L_t^\infty L_x^\infty$ to some limit $u_\infty(x,t)$, proving (S.3).

Moreover, for all $n\in\mathbb{N}$, $d_*(u_\infty,u_n)\leq\delta_n$, so taking in (4.23) $w(x,t)=u_\infty(x,t)$, $g^{m;\kappa}(x,t)=\rho^{m;\infty,\kappa}(x,t)$ then (S.4) follows from (4.20), (4.22), and interpolation with the existing uniform bound on $\rho^{n,\kappa}(x,t)$, $\rho^{\infty,\kappa}(x,t)$ in $L_t^\infty L_x^\infty$.

Finally (S.5) is a simple corollary of (S.2) and (S.4). □

4.2.3 Non-uniqueness of the Vanishing Viscosity Limit

For $x\in\mathbb{R}^2$ we set $x=(x_1,x_2)$, and define the corresponding unit vectors $e_1=(1,0)$, $e_2=(0,1)$. For $i\in\{1,2\}$ an index, we define by $\hat{i}$ the other index, that is $\hat{i}\in\{1,2\}\setminus\{i\}$.

When working on the 2-torus, $[x]\in\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ shall instead always be written in terms of a representative $x\in\mathbb{R}^2$, usually $x\in[0,1)^2$. For notational convenience, functions on the torus $\mathbb{T}^2$ will often be considered periodic functions on $\mathbb{R}^2$, and vice versa. Integrals over $\mathbb{T}^2$ are strictly speaking over $[0,1)^2$, such as when taking norms.

First, we define the shear flows that form the building block of our construction. We shall later apply these shears in a carefully designed order reminiscent of a cantor subset of $[0,T]$. The arising self-cancellation of these shear flows will produce non-uniqueness of renormalised weak solutions (Definition 4.3) for any initial data to (TE). Moreover, the small spatial scale of the shears, see (4.26) below, will allow us to approximate the vanishing viscosity limit of (κ-TDE).

Definition 4.7 (Lagrangian shear flow)**.** For $L \in \mathbb{N}$, we divide $\mathbb{R}$ into disjoint intervals $\bigcup_{m\in\mathbb{Z}} \left[\frac{m}{2L}, \frac{m+1}{2L}\right)$. We define for each $i \in \{1,2\}$, $L \in \mathbb{N}$, the periodic vector field $u^{(i;L)}(x) : \mathbb{T}^2 \to \mathbb{R}^2$,

$$u^{(i;L)}(x) = \begin{cases} e_i & \text{if } x_{\hat{i}} \in \left[\frac{m}{2L}, \frac{m+1}{2L}\right) \text{ for even } m, \\ -e_i & \text{if } x_{\hat{i}} \in \left[\frac{m}{2L}, \frac{m+1}{2L}\right) \text{ for odd } m, \end{cases}$$

where, since $2L$ is even, the definition is a periodic function of $x \in \mathbb{R}^2$, and so is well-defined on $\mathbb{T}^2$. $u^{(i;L)}(x)$ is bounded and divergence-free in the distributional sense since it is of the form $u^{(1;L)}(x) = (g(x_2),0)$, or $u^{(2;L)}(x) = (0,g(x_1))$ for some $g(x) \in L^\infty_x$. We refer to this vector field as the $(i;L)$-Lagrangian shear.

We shall denote by $\left\{y_t^{(i;L)}(x)\right\}_{t\in(-\infty,\infty)}$ the following Lagrangian flow (Definition 4.4) along $u^{(i;L)}(x)$,

$$\begin{aligned} y_t^{(i;L)}(x) : \ \mathbb{T}^2 &\to \mathbb{T}^2 \\ x &\mapsto \begin{cases} x + te_i \mod \mathbb{Z}^2 & \text{if } x_{\hat{i}} \in \left[\frac{m}{2L}, \frac{m+1}{2L}\right) \text{ for even } m, \\ x - te_i \mod \mathbb{Z}^2 & \text{if } x_{\hat{i}} \in \left[\frac{m}{2L}, \frac{m+1}{2L}\right) \text{ for odd } m. \end{cases} \end{aligned}$$

As per Definition 4.4 of Lagrangian flows, this preserves the Lebesgue-measure on $\mathbb{T}^2$, e.g. for $i = 1$ we can decompose a Lebesgue-measurable subset $A \subset \mathbb{T}^2$ into $A_m = A \cap \left(\mathbb{T} \times \left[\frac{m}{2L}, \frac{m+1}{2L}\right)\right)$, for which each $\left(y_t^{(1;L)}\right)^{-1}(A_m)$ is a translation. Moreover, it is invertible with inverse given by $\left(y_t^{(i;L)}\right)^{-1}(x) = y_{-t}^{(i;L)}(x)$, and is absolutely continuous since with respect to t since it is differentiable with derivative $\frac{\partial}{\partial t} y_t^{(i;L)}(x) = u^{(i;L)}(x) = u^{(i;L)}(y_t^{(i;L)}(x))$. Therefore, $\left\{y_t^{(i;L)}\right\}_{t\in(-\infty,\infty)}$ is a Lagrangian flow along $u^{(i;L)}(x)$.

Finally, for each $i \in \{1,2\}$, we show

$$u^{(i;L)}(x) \xrightharpoonup{L\to\infty} 0, \tag{4.26}$$

with convergence in weak-$*$ L^∞_x.

The proof of (4.26) is similar to that of the Riemann-Lebesgue lemma. Notice that $u^{(i;L)}\left(x + \frac{1}{2L} e_{\hat{i}}\right) = -u^{(i;L)}(x)$ and so for any test function $\phi(x) \in L^1_x$, by changing variables we have that

$$\int_{\mathbb{T}^2} \phi(x) u^{(i;L)}(x)\, dx = \int_{\mathbb{T}^2} \frac{\phi(x) - \phi\left(x + \frac{1}{2L} e_{\hat{i}}\right)}{2} u^{(i;L)}(x)\, dx.$$

Now $\phi\left(x + \frac{1}{2L} e_{\hat{i}}\right) \xrightarrow{L\to\infty} \phi(x)$ converges strongly in L^1_x. Together with the uniform bound $\left\| u^{(i;L)} \right\|_{L^\infty_x} \leq 1$ this implies the above integral converges to zero as $L \to \infty$, as required.

Next, we show the well-posedness of (TE) along these vector fields. A weaker version of the following result, valid when $\rho(x,t) \in L^\infty_t L^\infty_x$, follows from the breakthrough well-posedness theory of Ambrosio for vector fields of bounded variation in space [8]. At its heart, this is more involved than required for the simple case of shear flows. Therefore, we prefer to give a direct elementary proof of the uniqueness for all $\rho(x,t) \in L^1_t L^1_x$.

Proposition 4.7 (Shear flow uniqueness)**.** *We follow the notation introduced in Definition 4.7.*

Suppose $\rho(x,t) \in L^1_t L^1_x$ is a weak solution to (TE) *along $u^{(i;L)}(x)$ on an open interval $I \subset (0,T)$. Then there exists some $g(x) \in L^1_x$ such that $\rho(x,t)$ is (a.e. in $\mathbb{T}^2 \times I$) equal to the following Lagrangian solution (Definition 4.4) associated with $g(x)$,*

$$\rho(\cdot,t) = g \circ \left(y_t^{(i;L)}\right)^{-1}.$$

In particular, $\rho(x,t)$ is a renormalised weak solution to (TE) *(Definition 4.3). Moreover, $g\left(\left(y_t^{(i;L)}\right)^{-1}(x)\right) \in C^0((-\infty,\infty); L^1_x)$.*

Proof. Without loss of generality suppose $i = 1$, and use the shorthand $y_t = y_t^{(1;L)}$ for each $t \in (-\infty,\infty)$, that is

$$y_t(x) = \begin{cases} x + te_1 & \text{if } x_2 \in \left[\frac{m}{2L}, \frac{m+1}{2L}\right) \text{ for even } m, \\ x - te_1 & \text{if } x_2 \in \left[\frac{m}{2L}, \frac{m+1}{2L}\right) \text{ for odd } m. \end{cases}$$

Since the e_2-component of $y_t(x)$ is unchanged, that is $[y_t(x)]_2 = x_2$, we see

$$u^{(1;L)}(y_t(x),t) = u^{(1;L)}(x,t).$$

Fix $m \in \{0,1,...2L-1\}$. Take a test function $\phi(x,t) \in C^\infty_c\left(\left(\mathbb{T} \times \left(\frac{m}{2L}, \frac{m+1}{2L}\right)\right) \times I\right)$, and let $\psi(x,t) = \phi(y_t^{-1}(x),t)$. This is well-defined since $y_t^{-1}(x)$ preserves the set $\mathbb{T} \times \left(\frac{m}{2L}, \frac{m+1}{2L}\right)$.

Then $\psi(x,t) \in C^\infty_c\left(\left(\mathbb{T} \times \left(\frac{m}{2L}, \frac{m+1}{2L}\right)\right) \times I\right)$ is bounded, and supported and smooth on $\mathbb{T} \times \left(\frac{m}{2L}, \frac{m+1}{2L}\right) \times I$ since $y_t^{-1}(x)$ is smooth there. By the chain rule, the following point-wise equality holds

$$\frac{\partial \psi}{\partial t}(x,t) + u^{(1;L)}(x) \cdot \nabla \psi(x,t) = \frac{\partial \phi}{\partial t}(y_t^{-1}(x),t).$$

Since $\rho(x,t) \in L_t^1 L_x^1$ is assumed to be a weak solution to (TE) along $u^{(1;L)}(x)$ on the open interval $I \subset (0,T)$,

$$\int_{\mathbb{T}\times\left(\frac{m}{2L},\frac{m+1}{2L}\right)\times I} \rho(x,t) \left(\frac{\partial \psi}{\partial t}(x,t) + u^{(1;L)}(x) \cdot \nabla \psi(x,t)\right) \, dxdt = 0,$$

which implies

$$\int_{\mathbb{T}\times\left(\frac{m}{2L},\frac{m+1}{2L}\right)\times I} \rho(x,t) \frac{\partial \phi}{\partial t}(y_t^{-1}(x),t) \, dxdt = 0,$$

and after changing variables,

$$\int_{\mathbb{T}\times\left(\frac{m}{2L},\frac{m+1}{2L}\right)\times I} \rho(y_t(x),t) \, \frac{\partial \phi}{\partial t}(x,t) \, dxdt = 0.$$

This holds for all $\phi(x,t) \in C_c^\infty\left(\left(\mathbb{T}\times\left(\frac{m}{2L},\frac{m+1}{2L}\right)\right)\times I\right)$, and so as a distribution $\rho(y_t(x),t) \in L_t^1 L_x^1$ is independent of $t \in I$, see for example Theorem 3.1.4' in [58]. Therefore, there must exists some $g_m(x) \in L^1\left(\mathbb{T}\times\left(\frac{m}{2L},\frac{m+1}{2L}\right);\mathbb{R}\right)$ such that (for a.e. $t \in I$)

$$\rho(y_t(x),t) = g_m(x) \text{ a.e. on } \mathbb{T}\times\left(\frac{m}{2L},\frac{m+1}{2L}\right).$$

Repeating for each $m \in \{0,1,...2L-1\}$ gives some $g \in L_x^1$ such that,

$$\rho(y_t(x),t) = g(x) \text{ a.e. on } \mathbb{T}^2.$$

And so $\rho(x,t) = g(y_t^{-1}(x))$ is a Lagrangian solution to (TE) (Definition 4.4), and hence also a renormalised weak solution (Definition 4.3, Remark 11).

To prove that $g(y_t^{-1}(x)) \in C_t^0 L_x^1$ we shall use the fact that $y_t(x)$ is bijective, measure preserving, and 1-Lipschitz in time, i.e. for all $x \in \mathbb{T}^2$, $t,s \in \mathbb{R}$

$$|y_t(x) - y_s(x)| \leq |t-s|,$$

and so, by replacing x with $y_s^{-1}(x)$,

$$\left|(y_t \circ y_s^{-1})(x) - x\right| \leq |t-s|. \tag{4.27}$$

The following follows a standard argument. We fix $t \in I$ and write

$$g \circ y_s^{-1} = \left(g \circ y_t^{-1}\right) \circ y_t \circ y_s^{-1}.$$

Now mollify $(g \circ y_t^{-1})(x)$ in L^1_x, that is for each $\varepsilon > 0$ take some $\phi_\varepsilon(x) \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that

$$\left\| \phi_\varepsilon - g \circ y_t^{-1} \right\|_{L^1_x} \leq \varepsilon.$$

But also, since $y_t(x)$, $y_s^{-1}(x)$ are measure preserving,

$$\left\| \phi_\varepsilon \circ y_t \circ y_s^{-1} - g \circ y_s^{-1} \right\|_{L^1_x} \leq \varepsilon.$$

Then since $\phi_\varepsilon(x)$ is uniformly continuous on $\mathbb{T}^2$, by (4.27) there exists $\delta \in (0,1)$ such that if $|t-s| < \delta$ then $\|\phi_\varepsilon \circ y_t \circ y_s^{-1} - \phi_\varepsilon\|_{L^1_x} \leq \varepsilon$, and so

$$\|g \circ y_t^{-1} - g \circ y_s^{-1}\|_{L^1_x} \leq 3\varepsilon,$$

which completes the proof. □

Next, we demonstrate how these shear flows self-cancel. It is this property that gives rise to non-uniqueness in our later construction. This behaviour is quite different to recent works on anomalous dissipation [42], [43], and even [32], [10], which also achieve some non-uniqueness of the vanishing viscosity limit. Contrary to our approach, these constructions aim to exploit some mixing effect of shear flows, and non-uniqueness occurs by exploiting different mixing along different viscosity subsequences. In contrast, the commutator relation below ensures that shear flows in our construction precisely reverse mixing that occurs previously. Non-uniqueness instead occurs by quantitatively different transport along different viscosity subsequences rather than by quantitatively different mixing along different viscosity subsequences.

Proposition 4.8 (Cancellation). *Let $L_1, L_2 \in \mathbb{N}$, and $i_1, i_2 \in \{1,2\}$ such that $i_1 \neq i_2$. Suppose that $2\tau_2 = \frac{1}{2L_1}$, and that $2L_2\tau_1$ is an odd integer.*

Then the composition

$$\left(y_{\tau_2}^{(i_2;L_2)}\right)^2 \circ y_{\tau_1}^{(i_1;L_1)} \circ \left(y_{\tau_2}^{(i_2;L_2)}\right)^2 \circ y_{\tau_1}^{(i_1;L_1)} = \mathrm{Id},$$

where we have denoted by $\phi^2 := \phi \circ \phi$.

Proof. Without loss of generality, suppose $i_1 = 1, i_2 = 2$.

Since $\left(y_{\tau_2}^{(2;L_2)}\right)^2 = y_{2\tau_2}^{(2;L_2)}$ we instead prove the more general result

$$y_{\tau_2}^{(2;L_2)} \circ y_{\tau_1}^{(1;L_1)} \circ y_{\tau_2}^{(2;L_2)} \circ y_{\tau_1}^{(1;L_1)} = \mathrm{Id},$$

whenever both $2L_1\tau_2$ and $2L_2\tau_1$ are odd integers.

To this end, we divide $\mathbb{T}^2$ into tiles. Given $x \in [0,1)^2$ we find the unique integers $m_1, m_2 \in \mathbb{Z}$ such that

$$x \in \left[\frac{m_1}{2L_2}, \frac{m_1+1}{2L_2}\right) \times \left[\frac{m_2}{2L_1}, \frac{m_2+1}{2L_1}\right). \tag{4.28}$$

We are concerned with the parity of m_1 and m_2 (even or odd); we have four 'colours' on our tiling, two for each coordinate.

This defines two equivalence relations $[\cdot]_{m_1}, [\cdot]_{m_2}$ on $\mathbb{T}^2$, each with two equivalence classes corresponding to the parity of either m_1 or m_2 in (4.28).

Since $2L_1\tau_2$, $2L_2\tau_1$ are odd integers, the action of $y_{\tau_1}^{(1;L_1)}(x)$ changes the parity of m_1 and not m_2, while the action of $y_{\tau_2}^{(2;L_2)}(x)$ changes the parity of m_2 and not m_1.

Introduce the shorthand

$$\begin{aligned}
x^{(0)} &= x, \\
x^{(1)} &= y_{\tau_1}^{(1;L_1)}(x^{(0)}), \\
x^{(2)} &= y_{\tau_2}^{(2;L_2)}(x^{(1)}), \\
x^{(3)} &= y_{\tau_1}^{(1;L_1)}(x^{(2)}), \\
x^{(4)} &= y_{\tau_2}^{(2;L_2)}(x^{(3)}).
\end{aligned}$$

Then, by the above discussion,

$$\begin{aligned}
\left[x^{(2)}\right]_{m_2} &\neq \left[x^{(0)}\right]_{m_2}, \\
\left[x^{(3)}\right]_{m_1} &\neq \left[x^{(1)}\right]_{m_1}.
\end{aligned}$$

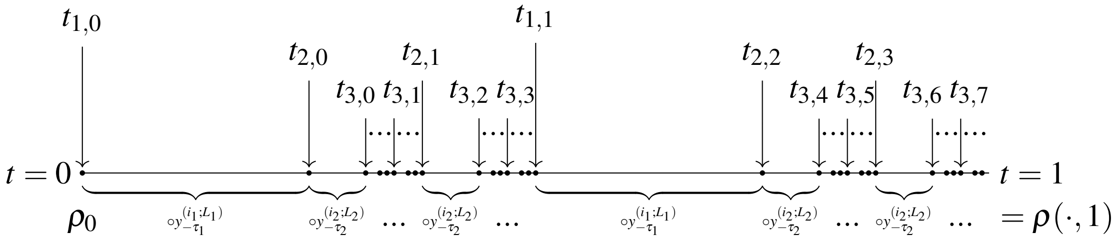
From the definitions of $y_{\tau_1}^{(1;L_1)}, y_{\tau_2}^{(2;L_2)}$ this implies

$$\begin{aligned}
x^{(3)} - x^{(2)} &= -\left(x^{(1)} - x^{(0)}\right), \\
x^{(4)} - x^{(3)} &= -\left(x^{(2)} - x^{(1)}\right),
\end{aligned}$$

and hence $x^{(4)} = x^{(0)}$ as required. □

Now for some sequences $\{L_k\}_{k\in\mathbb{N}}$, $\{i_k\}_{k\in\mathbb{N}}$, $\{\tau_k\}_{k\in\mathbb{N}}$ to be specified later, we first choose the order in which to apply these shear flows $y_{\tau_k}^{(i_k;L_k)}(x)$. We use a double index $(k,m) \in \mathscr{D} \subset \mathbb{Z}^2$, where k shall index the parameters of the Lagrangian shear $y_{\tau_k}^{(i_k;L_k)}(x)$, while m denotes the m^{th} occurrence of that particular shear.

To exploit Proposition 4.8, we shall ensure that there are exactly two occurrences of a suitable shear $y_{\tau_{k+1}}^{(i_{k+1};L_{k+1})}(x)$ between each occurrence of $y_{\tau_k}^{(i_k;L_k)}(x)$. If we denote by $t_{k,m} \in [0,1]$ the times at which the shears are first applied, we may illustrate this construction in the following diagram:



If we include only the shears $y_{\tau_k}^{(i_k;L_k)}(x)$ for $k \le K$, it can now be seen how Proposition 4.8 may create two different behaviours of the trace $\rho(x,1)$ as $2K \to \infty$, and $2K+1 \to \infty$.

The frequencies of the shears, $L_k \in \mathbb{N}$, will later be chosen to grow sufficiently quickly so that transport-diffusion (κ-TDE) along a viscosity subsequence $\kappa_k > 0$ will only include the effect of the first k shears, as per Theorem 4.6.

To this end, we first define a total order $<_{\text{time}}$ on the indexing set $\mathscr{D}$, and then define $t_{k,m}$ so that they respect this ordering.

Definition 4.8 (Lexicographic dyadic ordering)**.** We define the following set of ‘dyadic’ pairs

$$\mathscr{D} = \{(k,m) : k \in \mathbb{N}, m \in \mathbb{Z}, 0 \le m < 2^k\},$$

and define a total order $<_{\text{time}}$ on $\mathscr{D}$ via

$$(k_1,m_1) <_{\text{time}} (k_2,m_2) \text{ if and only if } \begin{cases} m_1 2^{-k_1} < m_2 2^{-k_2}, \text{ or} \\ m_1 2^{-k_1} = m_2 2^{-k_2} \text{ and } k_1 < k_2. \end{cases} \tag{4.29}$$

Define also for each $K \in \mathbb{N}$ the finite subset

$$\mathscr{D}_K = \{(k,m) \in \mathscr{D} : k \le K\},$$

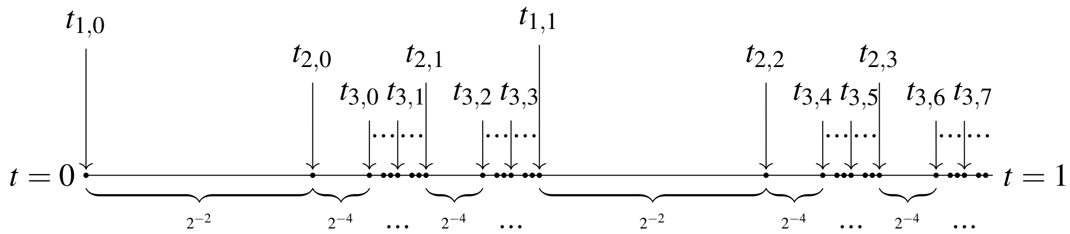
which inherits the total order $(\mathscr{D}, <_{\text{time}})$, which is now a finite order.

We define for each $k \in \mathbb{N}$, and $m \in \mathbb{Z}$ with $0 \leq m < 2^k$,

$$t_{k,m} = \sum_{(k',m') <_{\text{time}} (k,m)} 2^{-2k'} < 1, \tag{4.30}$$

where the bound on $t_{k,m}$ follows from direct calculation of $\sum_{k \in \mathbb{N}} 2^k 2^{-2k} = 1$, and an empty sum is zero.

We illustrate this arrangement in the following diagram:



(4.31)

Next, we construct the ‘fractal’ vector fields exploiting the above.

Definition 4.9 (Fractal shear flow)**.** Consider a sequence of tuples $\{(i_k, L_k, \tau_k)\}_{k \in \mathbb{N}}$ with $i_k \in \{1,2\}$, $L_k \in \mathbb{N}$, and $\tau_k > 0$.

(i) We say $\{(i_k, L_k, \tau_k)\}_{k \in \mathbb{N}}$ satisfies the Finiteness Condition if for all $k \in \mathbb{N}$ we have

$$\text{(Finiteness Condition)} \quad \tau_k < 2^{-2k}. \tag{4.32}$$

(ii) We say $\{(i_k, L_k, \tau_k)\}_{k \in \mathbb{N}}$ satisfies the Cancellation Conditions if for all $k \in \mathbb{N}$ we have

$$\text{(Cancellation Conditions)} \qquad \begin{gathered} i_{k+1} \neq i_k, \\ 2\tau_{k+1} = \frac{1}{2L_k}, \\ 2L_{k+1}\tau_k \text{ an odd integer.} \end{gathered} \tag{4.33}$$

Following the notation of Definition 4.8, in particular (4.30), when $\{(i_k, L_k, \tau_k)\}_{k \in \mathbb{N}}$ satisfy the Finiteness Condition (4.32), we may define for each $K \in \mathbb{N}$ the following fractal shear flows on $\mathbb{T}^2 \times [0,1] \to \mathbb{R}^2$,

$$u_K^{\{(i_k,L_k,\tau_k)\}_{k=1}^K}(x,t) = \begin{cases} u^{(i_k;L_k)}(x) & \text{if } t \in [t_{k,m}, t_{k,m}+\tau_k] \text{ for some } (k,m) \in \mathscr{D}_K, \\ 0 & \text{otherwise,} \end{cases}$$
$$u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t) = \begin{cases} u^{(i_k;L_k)}(x) & \text{if } t \in [t_{k,m}, t_{k,m}+\tau_k] \text{ for some } (k,m) \in \mathscr{D}, \\ 0 & \text{otherwise.} \end{cases}$$

By (4.30), this is well-defined since the Finiteness Condition (4.32) implies that the intervals $[t_{k,m}, t_{k,m}+\tau_k]$ are disjoint subsets of $[0,1]$.

Next, we show how this construction exploits Proposition 4.8 to create non-uniqueness of renormalised solutions to (TE).

Proposition 4.9 (Fractal behaviour)**.** *We follow the notation introduced in Definitions 4.8, 4.9.*

Let $\rho_0(x) \in L_x^\infty$, and suppose we have an infinite sequence of tuples $\{(i_k, L_k, \tau_k)\}_{k\in\mathbb{N}}$ with $i_k \in \{1,2\}$, $L_k \in \mathbb{N}$, and $\tau_k > 0$, satisfying the Finiteness Condition (4.32)*. Then for each $K \in \mathbb{N}$ there exists a unique weak solution $\rho^K(x,t)$ to* (TE) *along $u_K^{\{(i_k,L_k,\tau_k)\}_{k=1}^K}(x,t)$ with initial data $\rho_0(x)$. Moreover, $\rho^K(x,t) \in C_t^0 L_x^1 \cap L_t^\infty L_x^\infty$, is a Lagrangian solution, and hence also a renormalised weak solution (Definitions 4.3, 4.4).*

If in addition $\{(i_k, L_k, \tau_k)\}_{k\in\mathbb{N}}$ satisfy the Cancellation Conditions (4.33) *then*

$$\rho^{2K}(x,t) \xrightarrow{K\to\infty} \rho^{\text{even}}(x,t),$$
$$\rho^{2K+1}(x,t) \xrightarrow{K\to\infty} \rho^{\text{odd}}(x,t),$$

with the above convergence in weak-$*$ $L_t^\infty L_x^\infty$, and strong in $L_t^p L_x^\infty$ for any $p \in [1,\infty)$. Moreover, the sequences $\rho^{2K}(x,t) \in C_t^0 L_x^1$ and $\rho^{2K+1}(x,t) \in C_t^0 L_x^1$ are eventually constant as $K \to \infty$ when restricted to the spatio-temporal domain $\mathbb{T}^2 \times [t_{k,m}, t_{k,m}+2^{-2k}]$ for any $(k,m) \in \mathscr{D}$, or to the spatio-temporal domain $\mathbb{T}^2 \times \{1\}$. Furthermore, the limit functions $\rho^{\text{even}}(x,t), \rho^{\text{odd}}(x,t)$ are renormalised weak solutions to (TE) *along $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$ with initial data $\rho_0(x)$.*

If $\rho_0(x)$ is not constant, then $\rho^{\text{even}}(x,t) \neq \rho^{\text{odd}}(x,t)$, and in particular,

$$\begin{aligned} \rho^{\text{even}}(x,1) &= \rho_0(x), \\ \rho^{\text{odd}}(x,1) &= \rho_0\big(y_{-2\tau_1}^{(i_1;L_1)}(x)\big). \end{aligned} \tag{4.34}$$

Proof. Recall the language and notation introduced in Definitions 4.8, 4.9, as well as Definitions 4.3, 4.4 of renormalised and Lagrangian solutions to (TE).

By Theorems 4.1, 4.2, there exists some weak solution $\rho^K(x,t) \in C^0_{\mathrm{weak}-*}L^\infty_x$ to (TE) along $u_K^{\{(i_k,L_k,\tau_k)\}_{k=1}^K}(x,t)$ with initial data $\rho_0(x)$.

We aim to show $\rho(x,t) \in C^0_t L^1_x$ with

$$\rho^K(\cdot,t) = \rho_0 \circ \left(\tilde{y}^K_t\right)^{-1}, \tag{4.35}$$

for $\left\{\tilde{y}^K_t(x)\right\}_{t\in[0,1]}$ a Lagrangian flow along $u_K^{\{(i_k,L_k,\tau_k)\}_{k=1}^K}(x,t)$ which does not depend on $\rho_0(x)$, thus proving uniqueness.

From the definition of $u_K^{\{(i_k,L_k,\tau_k)\}_{k=1}^K}(x,t)$, finiteness of the set $\mathscr{D}_K$, and disjointness of the intervals $[t_{k,m}, t_{k,m}+\tau_k]$ for all $(k,m) \in \mathscr{D}_K$, we may piecewise apply Proposition 4.7 to $\rho^K(x,t)$. We use that $\rho^K(x,t) \in C^0_{\mathrm{weak}-*}L^\infty_x$ to glue together the pieces, and that any weak solution to (TE) along $u(x,t) = 0$ on an open time interval I is a constant function of $t \in I$, see for example Theorem 3.1.4' in [58]. This constructs for each $K \in \mathbb{N}$ a Lagrangian flow $\tilde{y}^K_t(x)$ satisfying (4.35), and moreover gives an expression for $\tilde{y}^K_t(x)$ in terms of the Lagrangian shear flows $y_t^{(i_k;L_k)}(x)$, see (4.37) below.

For any $(k,m) \in \mathscr{D}_K$, define

$$t_{\mathrm{suc}_K(k,m)} = \begin{cases} 1 & \text{if } (k,m) \text{ is maximal in } (\mathscr{D}_K, <_{\mathrm{time}}), \text{ else} \\ t_{\mathscr{S}} & \text{for } \mathscr{S} \in \mathscr{D}_K \text{ the successor of } (k,m) \text{ in } (\mathscr{D}_K, <_{\mathrm{time}}). \end{cases}$$

Note from (4.30), and the Finiteness Condition (4.32), that

$$\tau_k < 2^{-2k} \le t_{\mathrm{suc}_K(k,m)} - t_{k,m}. \tag{4.36}$$

Therefore, for each $t \in [t_{k,m}, t_{\mathrm{suc}_K(k,m)}]$,

$$\tilde{y}^K_t \circ \left(\tilde{y}^K_{t_{k,m}}\right)^{-1} = \begin{cases} y^{(i_k;L_k)}_{t-t_{k,m}} & \text{if } t \in [t_{k,m}, t_{k,m}+\tau_k], \\ y^{(i_k;L_k)}_{\tau_k} & \text{if } t \in [t_{k,m}+\tau_k, t_{\mathrm{suc}_K(k,m)}]. \end{cases} \tag{4.37}$$

This expression defines $\tilde{y}^K_t(x)$ for all $t \in [0,1]$, since $t_{1,0} = 0$ and $\tilde{y}^K_0(x) = x$. This completes the proof of uniqueness.

Next, by considering (4.29), (4.30), we may show that (see the illustration (4.31))

$$t_{\mathrm{suc}_K(k,m)} = \begin{cases} t_{k,m} + 2^{-2k} & \text{if } k < K, \\ t_{k,m} + 2^{1-2k} & \text{if } k = K. \end{cases} \tag{4.38}$$

Moreover, for all $(k,m) \in \mathscr{D}$,

$$\begin{aligned} t_{k,m} + 2^{-2k} &= t_{k+1,2m}, \\ t_{k,m} + 2^{1-2k} &= t_{k,m+1} \quad \text{when } m \text{ even.} \end{aligned} \tag{4.39}$$

Assume that additionally $\{(i_k, L_k, \tau_k)\}_{k\in\mathbb{N}}$ satisfy the Cancellation Conditions (4.33).

Fix some $K \in \mathbb{N}$. We claim that, for each $(k,m) \in \mathscr{D}_K$, and for all $t \in [t_{k,m}, t_{k,m} + 2^{-2k}]$, one has

$$\tilde{y}_t^{K+2}(x) = \tilde{y}_t^K(x). \tag{4.40}$$

By (4.36), (4.37), it is sufficient to show, for all $(k,m) \in \mathscr{D}_K$, that as maps $\mathbb{T}^2 \to \mathbb{T}^2$,

$$\tilde{y}^{K+2}_{t_{\mathrm{suc}_K(k,m)}} \circ \left(\tilde{y}^{K+2}_{t_{\mathrm{suc}_{K+2}(k,m)}}\right)^{-1} = \tilde{y}^{K}_{t_{\mathrm{suc}_K(k,m)}} \circ \left(\tilde{y}^{K}_{t_{\mathrm{suc}_{K+2}(k,m)}}\right)^{-1}.$$

By (4.38), this is immediate for all $k < K$. Meanwhile, by (4.38), (4.39), for $k = K$, we may rewrite this, and now need to prove that for all $m \in \mathbb{Z}$ with $0 \le m < 2^K$,

$$\tilde{y}^{K+2}_{t_{K+1,2m}+2^{-2K}} \circ \left(\tilde{y}^{K+2}_{t_{K+1,2m}}\right)^{-1} = \tilde{y}^{K}_{t_{K+1,2m}+2^{-2K}} \circ \left(\tilde{y}^{K}_{t_{K+1,2m}}\right)^{-1}.$$

In particular, it is certainly sufficient to show, for any $(k,m) \in \mathscr{D}_{K+1}$ with m even, that as maps $\mathbb{T}^2 \to \mathbb{T}^2$,

$$\begin{aligned} \tilde{y}^{K}_{t_{k,m}+2^{2-2k}} \circ \left(\tilde{y}^{K}_{t_{k,m}}\right)^{-1} &= \begin{cases} y^{(i_k;L_k)}_{2\tau_k} & \text{if } k = K \pmod 2, \\ \mathrm{Id} & \text{otherwise.} \end{cases} \\ &= Y^K_{k,m}, \end{aligned} \tag{4.41}$$

where we have defined the shorthand $Y^K_{k,m}(x) : \mathbb{T}^2 \to \mathbb{T}^2$.

Fixing an even $m \in \mathbb{Z}$ with $0 \le m < 2^{K+1}$, we prove this by induction. For $k = K+1$, by (4.36), (4.37) we see that as maps $\mathbb{T}^2 \to \mathbb{T}^2$,

$$\tilde{y}^{K}_{t_{\mathrm{suc}_K(k,m)}} \circ \left(\tilde{y}^{K}_{t_{\mathrm{suc}_{K+2}(k,m)}}\right)^{-1} = \mathrm{Id}.$$

After rewriting this using (4.38), (4.39), this proves (4.41) for $k = K+1$.

Now let $k \leq K$. By (4.39), and since by assumption m is even, we may rewrite (it may be helpful to consider the illustration (4.31))

$$\begin{aligned} Y^K_{k,m} = & \left(\tilde{y}^K_{t_{k+1,2m+2}+2^{-2k}} \circ \left(\tilde{y}^K_{t_{k+1,2m+2}}\right)^{-1}\right) \\ & \circ \left(\tilde{y}^K_{t_{k,m+1}+2^{-2k}} \circ \left(\tilde{y}^K_{t_{k,m+1}}\right)^{-1}\right) \\ & \circ \left(\tilde{y}^K_{t_{k+1,2m}+2^{-2k}} \circ \left(\tilde{y}^K_{t_{k+1,2m}}\right)^{-1}\right) \\ & \circ \left(\tilde{y}^K_{t_{k,m}+2^{-2k}} \circ \left(\tilde{y}^K_{t_{k,m}}\right)^{-1}\right). \end{aligned}$$

Then by the definition of $Y^K_{k+1,m'}(x)$, (4.36), and (4.37), the right hand side may be rewritten again as

$$Y^K_{k,m} = Y^K_{k+1,2m+2} \circ y^{(i_k;L_k)}_{\tau_k} \circ Y^K_{k+1,2m} \circ y^{(i_k;L_k)}_{\tau_k}.$$

By Proposition 4.8, the result (4.41) for k now follows from the same result for $k+1$. This completes the induction, thus the proof of (4.41) and the proof of (4.40).

Next, note by (4.30), that

$$\begin{aligned} E = & \left(\bigcup_{(k,m)\in\mathscr{D}} [t_{k,m}, t_{k,m} + 2^{-2k}]\right) \cup \left\{t_{k,m} + 2^{2-2k} : (k,m) \in \mathscr{D} \text{ with } m \text{ even}\right\} \\ & \subset [0,1], \end{aligned}$$

has Lebesgue-measure 1. Moreover, by (4.40), (4.41), the sequences $\rho^{2K}(x,t)$, $\rho^{2K+1}(x,t)$ are eventually constant on $\mathbb{T}^2 \times [t_{k,m}, t_{k,m} + 2^{-2k}]$, and each $\mathbb{T}^2 \times \{t_{k,m} + 2^{2-2k}\}$ with m even.

Therefore, for all $t \in E$, and so for a.e. $t \in [0,1]$, we see that

$$\begin{aligned} \rho^{2K}(\cdot,t) &\xrightarrow{K\to\infty} \rho_0 \circ \left(\tilde{y}^{\text{even}}_t\right)^{-1}, \\ \rho^{2K+1}(\cdot,t) &\xrightarrow{K\to\infty} \rho_0 \circ \left(\tilde{y}^{\text{odd}}_t\right)^{-1}, \end{aligned} \tag{4.42}$$

for some Lebesgue-measure preserving $\{\tilde{y}^{\text{even}}_t(x)\}_{t\in E}$, $\{\tilde{y}^{\text{odd}}_t(x)\}_{t\in E}$ independent of $\rho_0(x)$, with the convergence strong in L^∞_x.

Moreover, by the dominated convergence Theorem, we see that (4.42) converges strongly in $L^p(E; L^\infty_x)$ for all $p \in [1,\infty)$, and hence also in weak-$*$ $L^\infty(E; L^\infty_x)$.

By Definition 4.9, for all $t \in [0,1]$ (in particular $t \in E$), $u_K^{\{(i_k,L_k,\tau_k)\}_{k=1}^K}(x,t)$ is bounded by 1 in L^∞_x, and as $K \to \infty$ the sequence is eventually equal to $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$. Therefore it too converges strongly in $L^p(E; L^\infty_x)$ for all $p \in [1,\infty)$.

Fix $t \in E$, and apply the Trace Formula (4.5) to $\rho^{2K}(x,t)$, $\rho^{2K+1}(x,t)$. Taking now the limit $K \to \infty$ then gives that for all $t \in E$, for any $\phi(x,t) \in C^\infty(\mathbb{T}^d \times [0,T];\mathbb{R})$,

$$\begin{aligned}
&\int_{\mathbb{T}^2} \rho_0\big(\big(\tilde{y}_t^{\text{even}}\big)^{-1}(x)\big)\phi(x,t)\,dx \\
&\quad = \int_{\mathbb{T}^2} \rho_0(x)\phi_0(x)\,dx + \int_{\mathbb{T}^2\times(E\cap[0,t])} \rho_0\big(\big(\tilde{y}_t^{\text{even}}\big)^{-1}(x)\big) \\
&\qquad\qquad \times \left(\frac{\partial\phi}{\partial t}(x,t) + u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right) dxdt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^2} \rho_0\big(\big(\tilde{y}_t^{\text{odd}}\big)^{-1}(x)\big)\phi(x,t)\,dx \\
&\quad = \int_{\mathbb{T}^2} \rho_0(x)\phi_0(x)\,dx + \int_{\mathbb{T}^2\times(E\cap[0,t])} \rho_0\big(\big(\tilde{y}_t^{\text{odd}}\big)^{-1}(x)\big) \\
&\qquad\qquad \times \left(\frac{\partial\phi}{\partial t}(x,t) + u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)\cdot\nabla\phi(x,t) + \kappa\Delta\phi(x,t)\right) dxdt,
\end{aligned}$$

where $\phi_0(x) = \phi(x,0)$.

This implies that $\rho_0\big(\big(\tilde{y}_t^{\text{even}}\big)^{-1}(x)\big), \rho_0\big(\big(\tilde{y}_t^{\text{odd}}\big)^{-1}(x)\big) \in C^0_{\text{weak}-*}(E;L^\infty_x)$ and so may be extended to $\rho^{\text{even}}(x,t), \rho^{\text{odd}}(x,t) \in C^0_{\text{weak}-*}([0,1];L^\infty_x)$, as argued in Theorem 4.1.

For any $\phi(x,t) \in C_c^\infty(\mathbb{T}^2 \times [0,1);\mathbb{R})$ let $t = 1$ in the above (noting that $1 = t_{1,0} + 2^{2-2} \in E$). This proves that $\rho^{\text{even}}(x,t)$, $\rho^{\text{odd}}(x,t)$ are both weak solutions to (TE) along $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$ with initial data $\rho_0(x)$.

Moreover, for any $\beta(s) \in C_b^0(\mathbb{R};\mathbb{R})$, we may rewrite $\beta\big(\rho_0\big(\big(\tilde{y}_t^{\text{even}}\big)^{-1}(x)\big)\big) = \big((\beta\circ\rho_0)\circ\big(\tilde{y}_t^{\text{even}}\big)^{-1}\big)(x)$. By repeating the above arguments we have also that $\big((\beta\circ\rho_0)\circ\big(\tilde{y}_t^{\text{even}}\big)^{-1}\big)(x)$ (can be extended to) a weak solution to (TE) along $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$ with initial data $\beta(\rho_0(x))$. Therefore, we see that $\rho^{\text{even}}(x,t)$ is a renormalised weak solution to (TE) along $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$ with initial data $\rho_0(x)$. Similarly for $\rho^{\text{odd}}(x,t)$.

Next, observe that by (4.41), (4.42), for all $(k,m) \in \mathscr{D}$ with m even, that

$$\begin{aligned}
\rho^{\text{even}}(\cdot,t_{k,m}+2^{2-2k}) &= \rho^{\text{even}}(\cdot,t_{k,m})\circ\begin{cases}\big(y_{2\tau_k}^{(i_k;L_k)}\big)^{-1} & \text{for even } k,\\ \text{Id} & \text{for odd } k.\end{cases}\\
\rho^{\text{odd}}(\cdot,t_{k,m}+2^{2-2k}) &= \rho^{\text{odd}}(\cdot,t_{k,m})\circ\begin{cases}\text{Id} & \text{for even } k,\\ \big(y_{2\tau_k}^{(i_k;L_k)}\big)^{-1} & \text{for odd } k.\end{cases}
\end{aligned} \tag{4.43}$$

In particular, since $t_{1,0} = 0$, we have proved (4.34).

It remains to show, when $\rho_0(x)$ is not constant, that from (4.43) we deduce $\rho^{\text{even}}(x,t) \neq \rho^{\text{odd}}(x,t)$. Assume to the contrary that they are equal and call this solution $\rho(x,t)$. Then by (4.43) (noting that by (4.29), (4.30), $1-2^{2-2k} = t_{k,2^k-2}$, see for example the illustration (4.31)), we have for all $k \in \mathbb{N}$, that

$$\rho(x,1) = \rho(x, 1-2^{2-2k}),$$
$$\rho(x,1) = \rho\big(\big(y_{2\tau_k}^{(i_k;L_k)}\big)^{-1}(x), 1-2^{2-2k}\big).$$

In particular, taking $k=1$, we see $\rho(x,1) = \rho_0(x)$. Substituting this back in gives $\rho_0(x) = \rho(x, 1-2^{2-2k})$ for all $k \in \mathbb{N}$. Again, substituting this back in gives that for all $k \in \mathbb{N}$,

$$\rho_0(x) = \rho_0\big(y_{2\tau_k}^{(i_k;L_k)}(x)\big).$$

In terms of the unit vectors e_1, e_2, we have that for a.e. $x \in \mathbb{T}^2$, $\rho_0(x) = \rho_0(x + 2\tau_k e_{i_k})$. Convolving with a smooth function $\varphi(x) \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$ then gives that for all $x \in \mathbb{T}^2$,

$$(\varphi * \rho_0)(x) = (\varphi * \rho_0)(x + 2\tau_k e_{i_k}). \tag{4.44}$$

By the Finiteness and Cancellation Conditions (4.32), (4.33), we see that $\tau_k \xrightarrow{k\to\infty} 0$, and $i_{k+1} \neq i_k$ (so equal to both 1 and 2 infinitely often). Now $(\varphi * \rho_0)(x) \in C^\infty(\mathbb{T}^2;\mathbb{R})$, and so by (4.44) we must have that $(\varphi * \rho_0)(x)$ is a constant. This holds for all $\varphi(x) \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$, and therefore implies also that $\rho_0(x)$ is a constant, reaching the required contradiction. □

Finally, for a suitably fast growing sequence $\{L_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, we apply Theorem 4.6 to the sequence of vector fields $u_n^{\{(i_k,L_k,\tau_k)\}_{k=1}^n}(x,t)$. This allows us to control the vanishing viscosity limit along the vector field $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$.

Theorem 4.10 (Non-unique renormalised vanishing viscosity limit)**.** *There exists a divergence-free vector field $u(x,t) \in L_t^\infty L_x^\infty$, and a sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ with $\kappa_n > 0$ and $\kappa_n \xrightarrow{n\to\infty} 0$, such that for any initial data $\rho_0(x) \in L_x^\infty$, and for $\rho^\kappa(x,t)$ the unique solution to* (κ-TDE) *along $u(x,t)$ with initial data $\rho_0(x)$, one has*

$$\rho^{\kappa_{2n}}(x,t) \xrightarrow{n\to\infty} \rho^{\text{even}}(x,t),$$
$$\rho^{\kappa_{2n+1}}(x,t) \xrightarrow{n\to\infty} \rho^{\text{odd}}(x,t),$$

with the above convergence in weak-$*$ $L_t^\infty L_x^\infty$, and strong in $L_t^p L_x^p$ for all $p \in [1,\infty)$. Furthermore, the limit functions $\rho^{\text{even}}(x,t), \rho^{\text{odd}}(x,t)$ are renormalised weak solutions to (TE) *along $u(x,t)$ with initial data $\rho_0(x)$.*

If $\rho_0(x)$ is not constant, then $\rho^{\text{even}}(x,t) \neq \rho^{\text{odd}}(x,t)$, and moreover the set of weak-$*$ limit points of $\rho^\kappa(x,t) \in L_t^\infty L_x^\infty$ as $\kappa \to 0$ is uncountable.

Proof. Recall the language and notation introduced in Definitions 4.8, 4.9, as well as Definitions 4.3, 4.4 of renormalised and Lagrangian solutions to (TE).

Consider any infinite sequence of tuples $\{(i_k, L_k, \tau_k)\}_{k\in\mathbb{N}}$ with $i_k \in \{1,2\}$, $L_k \in \mathbb{N}$, $\tau_k > 0$, and satisfying the Finiteness Condition (4.32).

By Proposition 4.9 there exists a Lagrangian solution $\rho^n(x,t) \in C_t^0 L_x^1$, unique in the class of all weak solutions, to (TE) along $u_n^{\{(i_k,L_k,\tau_k)\}_{k=1}^n}(x,t)$ for any initial data $\rho_0(x) \in L_x^\infty$. Moreover, $u_n^{\{(i_k,L_k,\tau_k)\}_{k=1}^n}(x,t)$ is bounded by 1 in $L_t^\infty L_x^\infty$.

Let d_* be a metric inducing the weak-$*$ topology on

$$X = \left\{ u(x,t) \in L_t^\infty L_x^\infty : \|u\|_{L_t^\infty L_x^\infty} \leq 1 \right\}.$$

Let $\rho_0(x) \in L_x^\infty$, and denote for each $n \in \mathbb{N}$, $\kappa > 0$, by $\rho^{n,\kappa}(x,t)$, respectively $\rho^n(x,t)$, the unique weak solution to (κ-TDE), respectively (TE), along $u_n(x,t)$ with initial data $\rho_0(x)$. Moreover denote by $\rho^{\infty,\kappa}(x,t)$ the unique weak solution to (κ-TDE) along $u_\infty^{\{(i_k,L_k,\tau_k)\}_{k\in\mathbb{N}}}(x,t)$ with initial data $\rho_0(x)$.

Then by Theorem 4.6,

S.1 For all $n \in \mathbb{N}$, there exists $\kappa_n > 0$, $\varepsilon_n > 0$ depending only on $\{(i_k, L_k, \tau_k)\}_{k=1}^n$ (and in particular not on $\rho_0(x)$), with $\kappa_n \xrightarrow{n\to\infty} 0$ monotonically, such that the following holds true:

S.5 If $d_*(u_{n+1}, u_n) \leq \varepsilon_n$ for all $n \in \mathbb{N}$, then for all $p \in [1,\infty)$

$$\|\rho^{\infty,\kappa_n} - \rho^n\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0.$$

We now construct such a sequence $\{(i_k, L_k, \tau_k)\}_{k\in\mathbb{N}}$. Let $(i_1, L_1, \tau_1) = (1, 2^2, 2^{-2})$. We proceed by induction on $n \in \mathbb{N}$. Assume $\{(i_k, L_k, \tau_k)\}_{k=1}^n$ are given, and satisfy for $k \in \{1, ..., n\}$.

$$i_k \in \{1,2\},$$
$$L_k \in \mathbb{N},$$
$$L_k \geq 2^{2k},$$
$$0 < \tau_k \leq 2^{-2k},$$

It is straightforward to check that this is satisfied for the base case $(i_1, L_1, \tau_1) = (1, 2^2, 2^{-2})$. If $n \geq 2$, we assume in the inductive hypothesis that also, for $k \in \{1, ..., n-1\}$,

$$\begin{gathered} i_{k+1} \neq i_k, \\ 2\tau_{k+1} = \frac{1}{2L_k}, \\ 2L_{k+1}\tau_k \text{ is an odd integer}, \\ d_*\left(u_{k+1}^{\{(i_l,L_l,\tau_l)\}_{l=1}^{k+1}}, u_k^{\{(i_l,L_l,\tau_l)\}_{l=1}^{k}}\right) \leq \varepsilon_k. \end{gathered}$$

We then choose $(i_{n+1}, L_{n+1}, \tau_{n+1})$ as follows. Let $i_{n+1} \in \{1,2\} \setminus \{i_n\}$. Let $\tau_{n+1} = \frac{1}{4L_n}$ which by the inductive hypothesis satisfies $\tau_{n+1} \leq 2^{-2k-2}$. Let

$$L_{n+1} = 2L_{n-1}(2M+1), \tag{4.45}$$

for some large $M \in \mathbb{N}$ to be chosen, where when $n = 1$ we take $L_0 = 1$. This will ensure $2L_{n+1}\tau_n$ is an odd integer.

By (4.26) and Definition 4.9, we see for $n \in \mathbb{N}$ fixed, that

$$d_*\left(u_{n+1}^{\{(i_k,L_k,\tau_k)\}_{k=1}^{n+1}}, u_n^{\{(i_k,L_k,\tau_k)\}_{k=1}^{n}}\right) \xrightarrow{(L_{n+1})\to\infty} 0.$$

Therefore, by taking $M \in \mathbb{N}$ large enough in (4.45), we have that both $L_{n+1} \geq 2^{2n+2}$, and

$$d_*\left(u_{n+1}^{\{(i_k,L_k,\tau_k)\}_{k=1}^{n+1}}, u_n^{\{(i_k,L_k,\tau_k)\}_{k=1}^{n}}\right) \leq \varepsilon_n,$$

completing the induction.

We have constructed $\{(i_k, L_k, \tau_k)\}_{k\in\mathbb{N}}$ satisfying the Finiteness and Cancellation Conditions (4.32), (4.33), and also (S.1), (S.5). The main statement of the theorem is now a straightforward corollary of Proposition 4.9.

Finally, we show that if $\rho_0(x)$ is not constant, then the set of weak-$*$ limit points of $\rho^\kappa(x,t) \in L_t^\infty L_x^\infty$ as $\kappa \to 0$ is uncountable. Working in the weak-$*$ topology of $L_t^\infty L_x^\infty$, the set of vanishing viscosity limit points is bounded, and so is a metric space. Moreover, the map $\kappa \to \rho^\kappa(x)$ is continuous, which implies that the set of limit points is connected. If it is a connected metric space, then it is either a singleton or uncountable. Therefore, we conclude by observing that $\rho^{\text{even}}(x,t) \neq \rho^{\text{odd}}$ are at least two limit points. □

4.2.4 Inadmissibility of the Vanishing Viscosity Limit

We continue with the same notation for the 2-torus introduced in Section 4.2.3. However, we no longer make use of the notation in Definitions 4.7, 4.8, 4.9. In addition, we define the binary expansion of some $x = (x_1, x_2) \in \mathbb{T}^2$ as follows. For the representative $(x_1, x_2) \in [0,1)^2$ denote for $i \in \{1,2\}$, $k \in \mathbb{N}$, by $x_{i,k}$ the k^{th} binary digit of the i^{th} coordinate of x. That is, for each $i \in \{1,2\}$,

$$x_i = \sum_{k=1}^{\infty} x_{i,k} 2^{-k}, \tag{4.46}$$

where $x_{i,k} \in \{0,1\}$ and $x_{i,k} \overset{k\to\infty}{\not\longrightarrow} 1$ (as is standard to ensure uniqueness of the binary expansion).

First, we define vector fields and corresponding Lagrangian flows, which swap points in $\mathbb{T}^2$ according to their binary expansion. These 'binary swaps' form the building block of our construction. Subsequently, we shall give a divergence-free vector field in $L^\infty([0,100]; L^\infty_x)$ with $L^\infty_t L^\infty_x$-norm equal to 1, which perfectly mixes the transported scalar to its spatial average, and subsequently unmixes, any initial data to (TE). We aim to show that this behaviour is the unique limit point of vanishing viscosity of the associated solution to (κ-TDE). To ensure the uniqueness of the limit points, it is necessary to 'gradually' perform these binary swaps; that is, they must be restricted to gradually smaller regions of space, see (4.49) below.

Definition 4.10 (Lagrangian binary swap)**.** Suppose $i \in \{1,2\}$, $k \in \mathbb{N}$, $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$, $L \in \mathbb{N}$, with $L \geq k+1$.

The proof below will define a time-dependent, divergence-free vector field called the $(i,k,n;L)$-binary swap

$$u^{(i,k,n;L)}(x,t) : \mathbb{T}^2 \times [0, 3\cdot 2^{-k}] \to \mathbb{R}^2,$$

and a corresponding Lagrangian flow map (Definition 4.4) $\{y_t^{(i,k,n;L)}(x)\}_{t\in[0,3\cdot 2^{-k}]}$ with the properties (4.47)-(4.51) below.

Define $J_{k,n} = \left[(n-1)2^{-\lfloor k/2 \rfloor}, n2^{-\lfloor k/2 \rfloor}\right] \subset \mathbb{T}$. Then at time $t = 3\cdot 2^{-k}$, for a.e. $x = (x_1, x_2) \in \mathbb{T}^2$, $y^{(i,k,n;L)}_{3\cdot 2^{-k}}(x)$ will swap the k^{th} and $(k+1)^{\text{th}}$ binary digits of x_i if $x_i \in J_{k,n}$.

That is $y_0^{(i,k,n;L)}(x) = x$. For a.e. $x = (x_1,x_2) \in \mathbb{T}^2$ denote by $(x_1', x_2') = y^{(i,k,n;L)}_{3\cdot 2^{-k}}(x)$. Following the notation for binary expansions in (4.46), for $j \in \{1,2\}$ the coordinate, and for

$l \in \mathbb{N}$ the binary digit,

$$
\begin{aligned}
x'_j &= x_j \text{ if } j \neq i, \\
x'_{i,l} &= x_{i,l} \text{ for } l \notin \{k, k+1\}, \\
x'_{i,k+1} &= \begin{cases} x_{i,k} & \text{if } x_i \in J_{k,n}, \\ x_{i,k+1} & \text{otherwise,} \end{cases} \\
x'_{i,k} &= \begin{cases} x_{i,k+1} & \text{if } x_i \in J_{k,n}, \\ x_{i,k} & \text{otherwise.} \end{cases}
\end{aligned}
\tag{4.47}
$$

Additionally, the vector field $u^{(i,k,n;L)}(x,t)$ will satisfy

$$\left\| u^{(i,k,n;L)} \right\|_{L_t^\infty L_x^\infty} \leq 1, \tag{4.48}$$

and for all $t \in [0, 3 \cdot 2^{-k}]$, and all $x = (x_1, x_2) \in \mathbb{T}^2$ with $x_i \notin J_{k,n}$,

$$u^{(i,k,n;L)}(x,t) = 0. \tag{4.49}$$

Moreover, for $i \in \{1,2\}$, $k \in \mathbb{N}$ fixed, as $L \to \infty$,

$$u^{(i,k,n;L)}(x,t) \xrightarrow{L \to \infty} 0, \text{ in weak-}* \ L_t^\infty L_x^\infty. \tag{4.50}$$

Finally, for any $r, r' \in \{1, ..., 2^{k-1}\}$, for the spatial intervals $J = [(r-1)2^{1-k}, r2^{1-k}]$, and $J' = [(r'-1)2^{1-k}, r'2^{1-k}]$, the Lagrangian-flow $y_t^{(i,k,n;L)}$ preserves the squares

$$y_t^{(i,k,n;L)} : J \times J' \leftrightarrow J \times J'. \tag{4.51}$$

Proof. We now construct the above vector field and Lagrangian flow. We shall give the construction for $i = 1$, then define its coordinate reflection for $i = 2$. That is, for each $(x_1, x_2) \in \mathbb{T}^2$, $t \in [0, 3 \cdot 2^{-k}]$

$$u^{(2,k,n;L)}((x_1, x_2), t) = u^{(1,k,n;L)}((x_2, x_1), t), \tag{4.52}$$

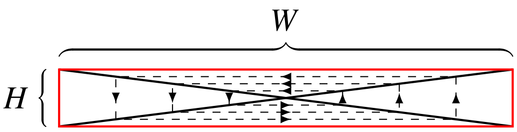
$$y_t^{(2,k,n;L)}((x_1, x_2)) = y_t^{(1,k,n;L)}((x_2, x_1)). \tag{4.53}$$

so that (4.48), (4.50), (4.47) with $i = 2$ follow from the same result for $i = 1$.

We shall achieve the required binary-swaps (4.47) by piecing together particular rotating vector fields which perform half rotations of rectangular regions of $\mathbb{T}^2$. Suppose $W, H > 0$, and define the following 1-Lipschitz stream-function

$$\psi : \left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)\subset\mathbb{R}^2\to\mathbb{R},$$
$$\psi((x_1,x_2)) = \min\{W,H\}\cdot\max\left\{\left(\frac{x_1}{W}\right)^2,\left(\frac{x_2}{H}\right)^2\right\}.$$

This defines a 1-bounded, time-independent, divergence-free, vector field $\nabla^{\perp}\psi(x) = \left(-\frac{\partial\psi}{\partial x_2}(x),\frac{\partial\psi}{\partial x_1}(x)\right)$ on the open rectangle $\left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)$. $\nabla^{\perp}\psi(x)$ has rectangular flow lines, as illustrated in the following diagram:



(4.54)

Moreover, on each triangular segment, $\nabla^{\perp}\psi(x)$ is a linear function of space, and each flow line within each segment has a time period equal to $\max\{W,H\}$.

Therefore, $\nabla^{\perp}\psi(x)$ admits a Lagrangian flow $\{y_t(x)\}_{t\in(-\infty,\infty)}$ on $\left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)$. That is for all $t\in(-\infty,\infty)$, $y_t(x) : \left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)\leftrightarrow\left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)$ is a Lebesgue-measure preserving bijection, $y_0(x) = x$, and for all $x\in\left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)$, and all $t\in(-\infty,\infty)$, $\frac{dy_t}{dt}(x) = \nabla^{\perp}\psi(y_t(x))$.

Moreover, we have $y_0(x) = x$, while $y_{2\max\{W,H\}}(x)$ is exactly a half rotation of $\left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)$ around its centre. Finally, both $y_t(x)$ and $y_t^{-1}(x)$ are Lipschitz in both x and t. That is, for all $x,x'\in\left(-\frac{W}{2},\frac{W}{2}\right)\times\left(-\frac{H}{2},\frac{H}{2}\right)$, and all $t,s\in(-\infty,\infty)$,

$$\begin{aligned}\left|y_t(x)-y_s(x')\right| &\le C(|x-x'|+|t-s|),\\ \left|y_t^{-1}(x)-y_s^{-1}(x')\right| &\le C(|x-x'|+|t-s|),\end{aligned}\tag{4.55}$$

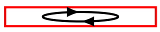
for some constant $C>0$.

Consider now any open rectangle $Q\subset\mathbb{R}^2$ with width W and height H. By translating (4.54) to Q, we have the same vector field on Q, which, to simplify the following presentation, we symbolically notate by

 $(x) : Q\to\mathbb{R}^2.$

We iterate that this is a 1-bounded, time-independent, divergence-free vector field, admitting a Lipschitz Lagrangian flow $\{y_t(x)\}_{t\in(-\infty,\infty)}$, as in (4.55). Moreover, $y_0(x) = x$, while $y_{2\max\{W,H\}}(x)$ is exactly a half rotation of Q around its centre.

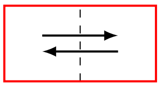
We also denote the same vector field multiplied by -1 by the same diagram with the arrows reversed,

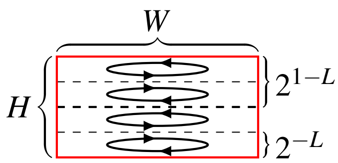
 $(x) = -$  (x).

This admits the Lagrangian flow $\{y_t^{-1}(x)\}_{t\in(-\infty,\infty)}$, and so enjoys the same properties as above.

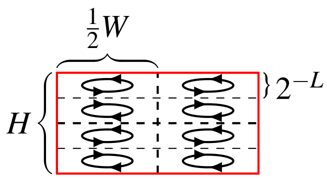
Next, we piece these half-rotations together to perform orientation-preserving swaps. This building block is required to perform the binary swaps (4.47). Let $Q \subset \mathbb{R}^2$ be again an open rectangle with width $W > 0$ and height $H > 0$. Let $L \in \mathbb{N}$ and suppose in addition that H is an integer multiple of 2^{1-L}, and $W \geq 2^{1-L}$.

Subsequently, we define a time-dependent vector field, symbolically notated by  $(x,t): Q \times [0,3W] \to \mathbb{R}^2$, by

 $(\cdot,t) =$

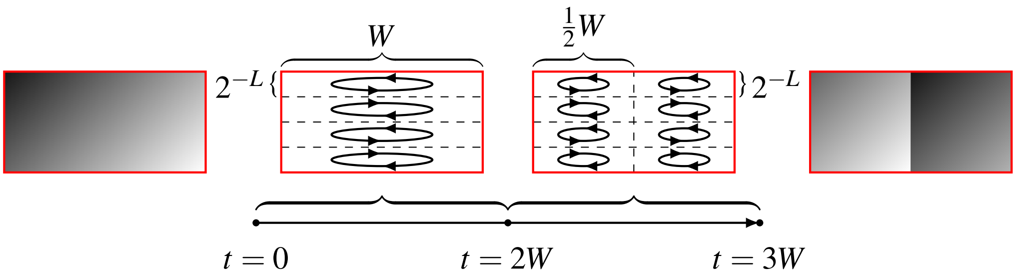


if $t \in [0,2W)$,

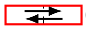


if $t \in [2W,3W]$. (4.56)

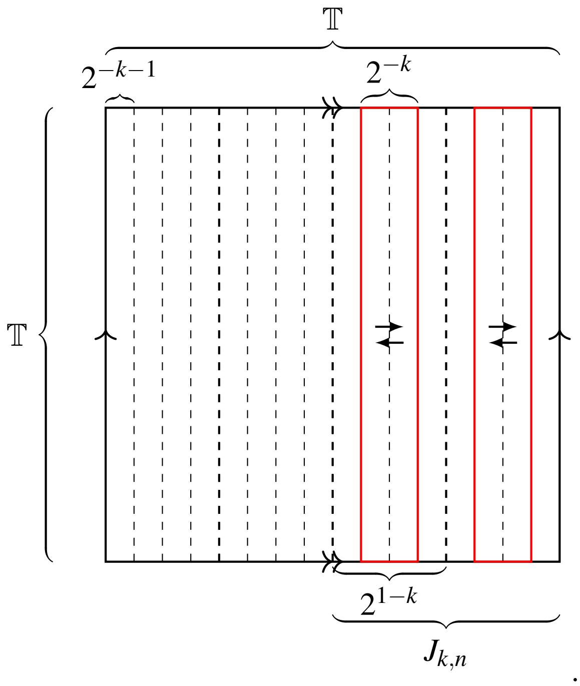
This has the property that at time $t = 3W$, the left and right halves are swapped in an orientation-preserving manner. This is illustrated below:



Finally, suppose $k \in \mathbb{N}$, $n \in \{1,\ldots,2^{\lfloor k/2 \rfloor}\}$, and $L \in \mathbb{N}$, with $L \geq k+1$.
Let $J_{k,n} = \left[(n-1)2^{-\lfloor k/2 \rfloor}, n2^{-\lfloor k/2 \rfloor}\right] \subset \mathbb{T}$.

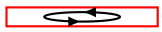
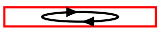
For $W = 2^{-k}$, $H = 1$, and $L \geq k+1$ we may use (x,t) to define the time-dependent vector field

$$u^{(1,k,n;L)}(x,t) : \mathbb{T}^2 \times [0, 3 \cdot 2^{-k}] \to \mathbb{R}^2,$$



Note that $u^{(1,k,n;L)}(x,t)$ is piecewise given by (4.54), and so is 1-bounded and divergence-free.

By construction (4.47), (4.49) are satisfied, while (4.51) follows from the illustration (4.56) after noting the bound $2^{-L} \leq 2^{1-k}$.

Finally, to prove (4.50), notice in (4.56) we alternate between (x) and (x), and so for all $x \in \mathbb{T}^2$, $t \in [0, 3 \cdot 2^{-k}]$

$$u^{(1,k,n;L)}(x,t) = -u^{(1,k,n;L)}(x + 2^{-L} e_2, t).$$

The proof of (4.50) now follows the same argument as the proof of (4.26). □

Next, we show the well-posedness of (TE) along these vector fields. As for Proposition 4.7, a weaker version of the following result, when $\rho(x,t) \in L_t^\infty L_x^\infty$, follows directly from the well-posedness theory of Ambrosio [8]. However, for completeness, we again prefer to give a direct elementary proof of uniqueness, valid for all $\rho(x,t) \in L_t^1 L_x^1$.

Proposition 4.11 (Binary-swap uniqueness)**.** *Suppose* $i \in \{1,2\}$*,* $k \in \mathbb{N}$*,* $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$*,* $L \in \mathbb{N}$*, with* $L \geq k+1$*.*

Suppose $\rho(x,t) \in L^1_t L^1_x$ *is a weak solution to* (TE) *along* $u^{(i,k,n;L)}(x,t)$ *on the open time interval* $(0, 3 \cdot 2^{-k})$*.*

Then there exists some $\rho_0(x) \in L^1_x$ *such that* $\rho(x,t)$ *is (a.e. in* $\mathbb{T}^2 \times (0, 3 \cdot 2^{-k})$*) equal to the Lagrangian solution (Definition 4.4)*

$$\rho(\cdot,t) = \rho_0 \circ \left(y_t^{(i,k,n;L)}\right)^{-1}. \tag{4.57}$$

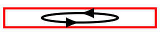
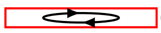
In particular, $\rho(x,t)$ *is a renormalised weak solution to* (TE) *(Definition 4.3) along* $u^{(i,k,n;L)}(x,t)$ *with initial data* $\rho_0(x)$*. Moreover* $\rho_0\big(\left(y_t^{(i,k,n;L)}\right)^{-1}(x)\big) \in C^0_t L^1_x$*.*

Proof. We shall give only a brief proof since this follows the proof of Proposition 4.7. Alternatively, one may prove the result via the $L^1_t BV_x$-theory of transport developed by Ambrosio in [8].

The case $i = 2$ follows the same result for $i = 1$ and the definitions (4.52), (4.53). Hence, we may assume $i = 1$.

Notice, from the construction (4.56), that $u^{(1,k,n;L)}(x,t)$ is stationary/time-independent on the time intervals $I_1 = [0, 2 \cdot 2^{-k}]$, and $I_2 = [2 \cdot 2^{-k}, 3 \cdot 2^{-k}]$.

We claim it is sufficient to prove (4.57) for some $\rho_0(x) \in L^1_x$ for a.e. $t \in I_1$, and for some $\rho_0'(x) \in L^1_x$ for a.e. $t \in I_2$. This is because, after we prove the continuity $\rho_0\big(\left(y_t^{(1,k,n;L)}\right)^{-1}(x)\big) \in C^0_t L^1_x$, we apply Theorem 4.1 to deduce $\rho_0(x) = \rho_0'(x)$.

Restrict now to one of the time intervals $t \in I_1$ or $t \in I_2$. Then we may divide $\mathbb{T}^2$ into open rectangles $Q \subset \mathbb{T}^2$ (and a set of Lebesgue-measure zero between rectangles) on which $u^{(1,k,n;L)}(x,t)$ is given either by zero, (x), or (x).

As in the proof of Proposition 4.7 we take a test function $\phi(x) \in C^\infty_c(Q \times I_1)$, supported on $Q \times I_1$ (or $Q \times I_2$ respectively), and let $\psi(x,t) = \phi\big(\left(y_t^{(1,k,n;L)}\right)^{-1}(x), t\big)$. From the Lipschitz bound (4.55), $\psi \in L^\infty(Q \times I_1; \mathbb{R})$ is then Lipschitz (so may be taken as a test function in (TE)), supported on Q, and by chain rule the following point-wise equality holds

$$\frac{\partial \psi}{\partial t}(x,t) + u^{(1,k,n;L)}(x,t) \cdot \nabla \psi(x,t) = \frac{\partial \phi}{\partial t}\left(\left(y_t^{(1,k,n;L)}\right)^{-1}(x), t\right).$$

Following the argument as in Proposition 4.7 then proves (4.57) holds a.e. on each $Q \times I_1$, for some $\rho_0(x)$. By gluing together the $\rho_0(x)$ for each Q, we show that (4.57) holds a.e. on $\mathbb{T}^2 \times I_1$ (or $\mathbb{T}^2 \times I_2$ respectively).

We are left to show $\rho_0\big(\circ \left(y_t^{(1,k,n;L)}\right)^{-1}(x)\big) \in C^0_t L^1_x$. As in the proof of Proposition 4.7, this will follow from the global Lipschitz in time bound. That is we must show, for all $x \in \mathbb{T}^2$, and $t,s \in [0,3\cdot 2^{-k}]$,

$$\left| y_t^{(1,k,n;L)}\left(\left(y_s^{(1,k,n;L)}\right)^{-1}(x)\right) - x\right| \le C|t-s|.$$

This follows by taking $x = x' = \left(y_s^{(1,k,n;L)}\right)^{-1}(z)$ and applying the local Lipschitz bound (4.55).

As discussed, by Theorem 4.1 we may glue together $\mathbb{T}^2 \times I_1$ and $\mathbb{T}^2 \times I_2$, proving (4.57). $\rho(x,t)$ is now a Lagrangian solution to (TE) along $u^{(1,k,n;L)}(x,t)$ with initial data $\rho_0(x)$, and so a renormalised weak solution by Remark 11. □

Next, we define the times at which to apply a binary swap $y_{3\cdot 2^{-k}}^{(i,k,n;L)}(x)$. We use a quadruple index $(k,m,i,n) \in \mathscr{N} \subset \mathbb{N}^4$ where k,i,n shall determine which binary swap $y_{3\cdot 2^{-k}}^{(i,k,n;L)}(x)$ to perform, with $L \in \mathbb{N}$, $L \ge k+1$ chosen later, while m denotes the m^{th} occurrence of that binary swap. To define the times $t_{(k,m,i,n)} \in [0,1]$ at which the binary swaps will be applied, we first define an ordering $<_{\text{time}}$ on the indexing set $\mathscr{N}$, and then define $t_{(k,m,i,n)}$ so that they respect this ordering. The resulting time ordering is illustrated in the diagram (4.61) below, with the ordering designed to iteratively make 4^m copies of the initial data on a square lattice with widths 2^{-m}, thus mixing the initial data to its spatial average as $m \to \infty$.

We additionally define a well-order $<_{\text{lex}}$ on the indexing set $\mathscr{N}$. This will inform which binary swaps $y_{3\cdot 2^{-k}}^{(i,k,n;L)}(x)$ will be included when approximating the vanishing viscosity limit of (κ-TDE), as in Proposition 4.6.

Definition 4.11 (Total-orders $<_{\text{lex}}$, $<_{\text{time}}$)**.** For any $p \in \mathbb{N}$, we define the lexicographic well-order $<_{\text{lex}}$ on $\mathbb{N}^p$ via

$$a <_{\text{lex}} b \iff (\exists i \in \{1,...,p\}), \begin{cases} (\forall j < i),\ a_j = b_j, \\ a_i < b_i. \end{cases}$$

Let

$$\mathscr{N} = \left\{(k,m,i,n) \in \mathbb{N}^4 :\ m \le k, i \in \{1,2\},\ n \le 2^{\lfloor k/2 \rfloor}\right\},$$

then note that the well-order $(\mathscr{N}, <_{\text{lex}})$ is order isomorphic to $\mathbb{N}$ with the usual order.

We also define another total-order $<_{\text{time}}$ on $\mathscr{N}$, via

$$(k_1,m_1,i_1,n_1) <_{\text{time}} (k_2,m_2,i_2,n_2) \text{ if } (m_1,k_2,i_2,n_2) <_{\text{lex}} (m_2,k_1,i_1,n_1), \tag{4.58}$$

which is not a well-order.

Define also for each $\mathscr{K} \in \mathscr{N}$ the finite subset

$$\mathscr{N}_{\mathscr{K}} = \{(k,m,i,n) \in \mathscr{N} : (k,m,i,n) \leq_{\text{lex}} \mathscr{K}\},$$

which inherits the finite (and therefore well-) orders $(\mathscr{N}_{\mathscr{K}}, <_{\text{lex}})$, $(\mathscr{N}_{\mathscr{K}}, <_{\text{time}})$.

For each $(k,m,i,n) \in \mathscr{N}$ we define

$$T_{(k,m,i,n)} = \sum_{(k',m',i',n') <_{\text{time}} (k,m,i,n)} 3 \cdot 2^{-k'} < 42, \tag{4.59}$$

where the bound on $T_{(k,m,i,n)}$ follows from direct calculation of

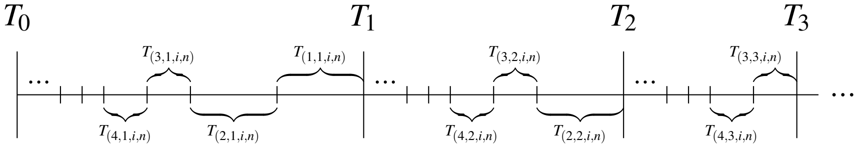
$$\sum_{k \in \mathbb{N}} 2k \cdot 2^{\lfloor k/2 \rfloor} \cdot 3 \cdot 2^{-k} = 42,$$

and an empty sum is zero.

Moreover, for each $m \in \mathbb{N}$, let

$$T_m = \sum_{\substack{m' \leq m \\ (k',m',i',n') \in \mathscr{N}}} 3 \cdot 2^{-k'} < 42. \tag{4.60}$$

Writing also $T_0 = 0$, we illustrate the time ordering $<_{\text{time}}$ in the following diagram,



(4.61)

where for fixed $k, m \in \mathbb{N}$ with $m \leq k$, the bracket $\overbrace{\quad}^{T_{(k,m,i,n)}}$ contains for all $i \in \{1,2\}$, $n \in \mathbb{N}$ with $n \leq 2^{\lfloor k/2 \rfloor}$, precisely the time intervals

$$\left[T_{(k,m,i,n)}, T_{(k,m,i,n)} + 3 \cdot 2^{-k}\right).$$

Next, we apply the divergence-free vector fields $u^{(i,k,n;L)}(x,t)$ (for some $L \in \mathbb{N}$ to be chosen) in each time interval $\left[T_{(k,m,i,n)}, T_{(k,m,i,n)} + 3 \cdot 2^{-k}\right)$, with $m \in \{1,...,k\}$ an index denoting the m^{th} occurrence of this swap. The order is chosen so that the solution to (TE) at time $t = T_m$ creates 4^m identical copies of the initial data on a square lattice with widths 2^{-m}.

To see how this can be achieved, $i \in \{1,2\}$ denotes which coordinate the binary swap (4.47) acts on, while $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$ is an integer denoting which region of $\mathbb{T}^2$ the binary swap is completed on.

For fixed $k \in \mathbb{N}$, $m \in \{1,...,k\}$, these binary swap commute, and together the bracket $\underbrace{}_{}^{T_{(k,m,i,n)}}$ swaps the k^{th} and $(k+1)^{\text{th}}$ binary digit of both coordinates of every $x = (x_1, x_2) \in \mathbb{T}^2$.

The order $\leq_{\text{lex}}$ is chosen such that between T_{m-1} and T_m, an 'undefined' binary digit passes from infinitely far up the binary expansion, down to the m^{th} position. Since its values are undefined, this creates four copies (2 in each coordinate) of the $t = T_{m-1}$ data at $t = T_m$. See Proposition 4.12 below for the details.

But first, we define this vector field and its finite approximations, which by Theorem 4.6 will control the vanishing viscosity limit of (κ-TDE).

Definition 4.12 (Gradual perfect mixing)**.** We follow the notation introduced in Definition 4.11.

Notice that a particular $\mathscr{P} = (k,m,i,n) \in \mathscr{N}$ fixes $k \in \mathbb{N}$, $i \in \{1,2\}$, $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$.

For $\mathscr{K} \in \mathscr{N}$, and a finite sequence $\{L_{\mathscr{P}}\}_{\mathscr{P} \in \mathscr{N}_{\mathscr{K}}} \subset \mathbb{N}$ with $L_{(k,m,i,n)} \geq k+1$ for all $(k,m,i,n) \in \mathscr{N}_{\mathscr{K}}$, we define the following time-dependent vector field.

$$u_{\mathscr{K}}^{\{L_{\mathscr{P}}\}_{\mathscr{P} \in \mathscr{N}_{\mathscr{K}}}}(x,t) : \mathbb{T}^2 \times [0,50] \to \mathbb{R}^2,$$

$$u_{\mathscr{K}}^{\{L_{\mathscr{P}}\}_{\mathscr{P} \in \mathscr{N}_{\mathscr{K}}}}(x,t) = \begin{cases} u^{(i,k,n;L_{\mathscr{P}})}(x, t - T_{\mathscr{P}}) & \text{for } \begin{cases} \mathscr{P} = (k,m,i,n) \in \mathscr{N}_{\mathscr{K}}, \text{ and} \\ t \in \left[T_{\mathscr{P}}, T_{\mathscr{P}} + 3 \cdot 2^{-k}\right), \end{cases} \\ 0 & \text{otherwise, in particular for } t \geq 42. \end{cases}$$

Meanwhile, for an infinite sequence $\{L_{\mathscr{P}}\}_{\mathscr{P} \in \mathscr{N}} \subset \mathbb{N}$ with $L_{(k,m,i,n)} \geq k+1$ for all $(k,m,i,n) \in \mathscr{N}$, we define the following time-dependent vector field.

$$u_{\infty}^{\{L_{\mathscr{P}}\}_{\mathscr{P} \in \mathscr{N}}}(x,t) : \mathbb{T}^2 \times [0,50] \to \mathbb{R}^2,$$

$$u_{\infty}^{\{L_{\mathscr{P}}\}_{\mathscr{P} \in \mathscr{N}}}(x,t) = \begin{cases} u^{(i,k,n;L_{\mathscr{P}})}(x, t - T_{\mathscr{P}}) & \text{for } \begin{cases} \mathscr{P} = (k,m,i,n) \in \mathscr{N}, \text{ and} \\ t \in \left[T_{\mathscr{P}}, T_{\mathscr{P}} + 3 \cdot 2^{-k}\right), \end{cases} \\ 0 & \text{otherwise, in particular for } t \geq 42. \end{cases}$$

These vector fields are well-defined since by (4.59), the time intervals

$$\left[T_{(k,m,i,n)}, T_{(k,m,i,n)} + 3 \cdot 2^{-k}\right),$$

are disjoint subsets of $[0,42]$.

Proposition 4.12 (Mixing). *We follow the notation introduced in Definitions 4.11, 4.12.*

Let $\rho_0(x) \in L^\infty_x$, and suppose we have an infinite sequence $\{L_\mathscr{P}\}_{\mathscr{P}\in\mathscr{N}} \subset \mathbb{N}$ with $L_{(k,m,i,n)} \geq k+1$ for all $(k,m,i,n) \in \mathscr{N}$. Then for each $\mathscr{K} \in \mathscr{N}$ there exists a unique weak solution $\rho^\mathscr{K}(x,t)$ to (TE) *along $u_\mathscr{K}^{\{L_\mathscr{P}\}_{\mathscr{P}\in\mathscr{N}_\mathscr{K}}}(x,t)$ with initial data $\rho_0(x)$. Moreover, $\rho^\mathscr{K}(x,t) \in C^0_t L^1_x \cap L^\infty_t L^\infty_x$, is a Lagrangian solution, and hence also a renormalised weak solution (Definitions 4.3, 4.4).*

By the well-order $(\mathscr{N}, <_{\text{lex}})$ (which is order isomorphic to $(\mathbb{N}, <)$), as $\mathscr{K} \xrightarrow{\text{lex}} \infty$,

$$\rho^\mathscr{K}(x,t) \xrightarrow{\mathscr{K}\to\infty} \rho^\infty(x,t),$$

with the above convergence in weak-$*$ $L^\infty_t L^\infty_x$, strong in $L^p([0,42];L^p_x)$ and $C^0([0,42-\varepsilon];L^p_x)$ for all $p \in [1,\infty)$, and all $\varepsilon > 0$. The limit function $\rho^\infty(x,t) \in C^0_{\text{weak}-*}L^\infty_x$ is a weak solution to (TE) *along $u_\infty^{\{L_\mathscr{P}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$ with initial data $\rho_0(x)$.*

Moreover, for all $t \in [42,50]$ and all $x \in \mathbb{T}^2$,

$$\rho^\infty(x,t) = \int_{\mathbb{T}^2} \rho_0(y)\, dy, \tag{4.62}$$

i.e., $\rho^\infty(x,t)$ is perfectly mixed after $t = 42$.

Proof. Recall the language and notation introduced in Definitions 4.10, 4.11, 4.12, as well as Definitions 4.3, 4.4 of renormalised and Lagrangian solutions to (TE).

By Theorems 4.1, 4.2, there exists some weak solution $\rho^\mathscr{K}(x,t) \in C^0_{\text{weak}-*}L^\infty_x$ to (TE) along $u_\mathscr{K}^{\{L_\mathscr{P}\}_{\mathscr{P}\in\mathscr{N}_\mathscr{K}}}(x,t)$ with initial data $\rho_0(x)$.

We aim to show $\rho(x,t) \in C^0_t L^1_x$ with

$$\rho^\mathscr{K}(\cdot,t) = \rho_0 \circ \left(\tilde{y}_t^\mathscr{K}\right)^{-1}, \tag{4.63}$$

for $\left\{\tilde{y}_t^\mathscr{K}(x)\right\}_{t\in[0,50]}$ a Lagrangian flow along $u_\mathscr{K}^{\{L_\mathscr{P}\}_{\mathscr{P}\in\mathscr{N}_\mathscr{K}}}(x,t)$ which does not depend on $\rho_0(x)$, thus proving uniqueness.

From the definition of $u_\mathscr{K}^{\{L_\mathscr{P}\}_{\mathscr{P}\in\mathscr{N}_\mathscr{K}}}(x,t)$, finiteness of the set $\mathscr{N}_\mathscr{K}$, and disjointness of the intervals $\left[T_{(k,m,i,n)}, T_{(k,m,i,n)} + 3\cdot 2^{-k}\right)$ for all $(k,m,i,n) \in \mathscr{N}_\mathscr{K}$, we may piecewise apply Proposition 4.11 to $\rho^\mathscr{K}(x,t)$. We use that $\rho^\mathscr{K}(x,t) \in C^0_{\text{weak}-*}L^\infty_x$ to glue together the pieces, and that any weak solution to (TE) along $u(x,t) = 0$ on an open time interval I is a constant function of $t \in I$, see for example Theorem 3.1.4' in [58]. This constructs for each $\mathscr{K} \in \mathscr{N}$ a Lagrangian flow $\tilde{y}_t^\mathscr{K}(x)$ satisfying (4.63), and moreover gives an expression for $\tilde{y}_t^\mathscr{K}(x)$ in terms of the binary swaps $y_t^{(i,k,n;L)}(x)$, see (4.65) below.

For each $\mathscr{P} = (k,m,i,n) \in \mathscr{N}_{\mathscr{K}}$, define

$$T_{\mathrm{suc}(\mathscr{P})} = \begin{cases} 50 & \text{if } \mathscr{P} \text{ is maximal in } (\mathscr{N}_{\mathscr{K}}, <_{\mathrm{time}}), \text{ else} \\ T_{\mathscr{P}'} & \text{for } \mathscr{P}' \text{ the successor of } \mathscr{P} \text{ in } (\mathscr{N}_{\mathscr{K}}, <_{\mathrm{time}}). \end{cases}$$

With each $\mathscr{P} = (k,m,i,n) \in \mathscr{N}_{\mathscr{K}}$ fixed, and therefore fixed $k \in \mathbb{N}$, by (4.59) we have that

$$T_{\mathscr{P}} + 3 \cdot 2^{-k} \leq T_{\mathrm{suc}(\mathscr{P})}. \tag{4.64}$$

With each $\mathscr{P} = (k,m,i,n) \in \mathscr{N}_{\mathscr{K}}$ fixed, and therefore fixed $k \in \mathbb{N}$, $i \in \{1,2\}$, $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$, we have that for all $t \in [T_{\mathscr{P}}, T_{\mathrm{suc}(\mathscr{P})}]$

$$\tilde{y}_t^{\mathscr{K}} \circ \left(\tilde{y}_{T_{\mathscr{P}}}^{\mathscr{K}}\right)^{-1} = \begin{cases} y_{t-T_{\mathscr{P}}}^{(i,k,n;L_{\mathscr{P}})} & \text{if } t \in [T_{\mathscr{P}}, T_{\mathscr{P}} + 3 \cdot 2^{-k}], \\ y_{3\cdot 2^{-k}}^{(i,k,n;L_{\mathscr{P}})} & \text{if } t \in [T_{\mathscr{P}} + 3 \cdot 2^{-k}, T_{\mathrm{suc}(\mathscr{P})}]. \end{cases} \tag{4.65}$$

Meanwhile, if $\mathscr{P}_{\min}$ is minimal in $(\mathscr{N}_{\mathscr{K}}, <_{\mathrm{time}})$, we have that for all $t \in [0, T_{\mathscr{P}_{\min}}]$, $\tilde{y}_t^{\mathscr{K}}(x) = x$. This completes the proof of uniqueness.

Next, we wish to show, for any $m \in \mathbb{N}$, that the sequence $\rho^{\mathscr{K}}(x,t) \in C_t^0 L_x^1$ is Cauchy in $C^0([T_{m-1}, T_m]; L_x^1)$ as $\mathscr{K} \xrightarrow{\mathrm{lex}} \infty$, as follows.

We proceed by induction on $m \in \mathbb{N}$. By the inductive hypothesis we have that $\rho^{\mathscr{K}}(x,t)(\cdot, T_{m-1}) \in L_x^1$ is Cauchy in L_x^1 as $\mathscr{K} \xrightarrow{\mathrm{lex}} \infty$ (for the base case $m = 1$ this follows by $\rho^{\mathscr{K}}(x, T_0) = \rho_0(x)$). From this we wish to deduce that $\rho^{\mathscr{K}}(x,t) \in C_t^0 L_x^1$ is Cauchy in $C^0([T_{m-1}, T_m]; L_x^1)$ as $\mathscr{K} \xrightarrow{\mathrm{lex}} \infty$. Before we proceed with the induction, we will need some properties of the Lagrangian flow $\tilde{y}_t^{\mathscr{K}}(x)$ on the interval $t \in [T_{m-1}, T_m]$.

Fix any $(k,m,i,n) \in \mathscr{N}$, which fixes $k \in \mathbb{N}$, $m \in \{1,...,k\}$, $i \in \{1,2\}$, $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$.

Recall the definition (4.60) of T_m and T_{m-1}, where we set $T_0 = 0$ if necessary. By (4.58), we have (it may be helpful to consider the illustration (4.61))

$$\begin{gathered} T_{m-1} < T_{(k,m,i,n)}, \\ T_{(k,m,i,n)} + 3 \cdot 2^{-k} \leq T_m, \end{gathered}$$

and also for each $\mathscr{P} = (k', m', i', n') >_{\mathrm{lex}} (k,m,i,n)$, which fixes $k' \in \mathbb{N}$, $m' \in \{1,...,k'\}$,

$$\begin{gathered} T_{\mathscr{P}} + 3 \cdot 2^{-k'} \leq T_{(k,m,i,n)} \quad \text{if } m' \leq m, \\ T_m < T_{\mathscr{P}} \quad \text{if } m < m'. \end{gathered}$$

Therefore, by the piecewise structure (4.65), for each $t \in [T_{(k,m,i,n)}, T_m]$, and for each $\mathscr{K}' >_{\text{lex}} \mathscr{K} \geq_{\text{lex}} (k,m,i,n)$, we have that

$$\tilde{y}_t^{\mathscr{K}'} \circ \left(\tilde{y}_{T_{(k,m,i,n)}}^{\mathscr{K}'}\right)^{-1} = \tilde{y}_t^{\mathscr{K}} \circ \left(\tilde{y}_{T_{(k,m,i,n)}}^{\mathscr{K}}\right)^{-1},$$

is unchanged. We shall need the inverse of these Lagrangian maps,

$$\tilde{y}_{T_{(k,m,i,n)}}^{\mathscr{K}'} \circ \left(\tilde{y}_t^{\mathscr{K}'}\right)^{-1} = \tilde{y}_{T_{(k,m,i,n)}}^{\mathscr{K}} \circ \left(\tilde{y}_t^{\mathscr{K}}\right)^{-1}. \tag{4.66}$$

Next, for each $\mathscr{P} = (k', m', i', n') \in \mathscr{N}$ with $k' < k$, which fixes $k' \in \mathbb{N}$, $m' \in \{1, ..., k'\}$, we have that

$$\begin{aligned} T_{\mathscr{P}} + 3 \cdot 2^{-k'} &\leq T_{m-1} \quad \text{if } m' < m, \\ T_{(k,m,i,n)} &< T_{\mathscr{P}} \quad \text{if } m \leq m'. \end{aligned}$$

Therefore, we may apply (4.51) and the piecewise structure (4.65) to show the following.

For any $r, r' \in \{1, ..., 2^{k-1}\}$, denote the intervals $J = [(r-1)2^{1-k}, r2^{1-k}]$, $J' = [(r'-1)2^{1-k}, r'2^{1-k}]$.

For all $\mathscr{K} \in \mathscr{N}$, for all $t \in [T_{m-1}, T_{(k,m,i,n)}]$, we see that the Lagrangian-flow $\tilde{y}_t^{\mathscr{K}} \circ \left(\tilde{y}_{T_{m-1}}^{\mathscr{K}}\right)^{-1}$ preserves the square lattice

$$\tilde{y}_t^{\mathscr{K}}\left(\left(\tilde{y}_{T_{m-1}}^{\mathscr{K}}\right)^{-1}(x)\right) : J \times J' \leftrightarrow J \times J'. \tag{4.67}$$

We are now ready to proceed with the induction. Recall we have assumed that for some $m \in \mathbb{N}$, $\rho^{\mathscr{K}}(x,t)(\cdot, T_{m-1}) \in L^1_x$ is Cauchy in L^1_x as $\mathscr{K} \xrightarrow{\text{lex}} \infty$.

We wish to deduce $\rho^{\mathscr{K}}(x,t) \in C^0_t L^1_x$ is Cauchy in $C^0([T_{m-1}, T_m]; L^1_x)$ as $\mathscr{K} \xrightarrow{\text{lex}} \infty$.

Denote by $\rho_{m-1}(x) \in L^1_x$ the limit $\rho^{\mathscr{K}}(x, T_{m-1}) \xrightarrow{\mathscr{K} \to \infty} \rho_{m-1}(x)$ in L^1_x.

Note, by (4.63), for all $\mathscr{K} \in \mathscr{N}$, and for all $t \in [T_{m-1}, T_m]$, we may write

$$\rho^{\mathscr{K}}(\cdot, t) = \rho^{\mathscr{K}}(\cdot, T_{m-1}) \circ \tilde{y}_{T_{m-1}}^{\mathscr{K}} \circ \left(\tilde{y}_t^{\mathscr{K}}\right)^{-1}. \tag{4.68}$$

Fix some $\varepsilon > 0$. Let $\phi_\varepsilon(x) \in C^\infty(\mathbb{T}^2; \mathbb{R})$ be a smooth approximation of $\rho_{m-1}(x)$ in L^1_x so that $\|\phi_\varepsilon - f_{m-1}\|_{L^1_x} \leq \varepsilon$.

Fixing also some $k \in \mathbb{N}$, let $\mathscr{K} \in \mathscr{N}$ be large enough that we have $\mathscr{K} \geq_{\text{lex}} (k, m, 1, 1)$, and that for all $\mathscr{K}' \geq_{\text{lex}} \mathscr{K}$,

$$\left\|\rho^{\mathscr{K}'}(\cdot, T_{m-1}) - \rho_{m-1}\right\|_{L^1_x} \leq \varepsilon.$$

Then, it also follows that

$$\left\|\rho^{\mathscr{K}'}(\cdot,T_{m-1})-\phi_\varepsilon\right\|_{L^1_x}\leq 2\varepsilon.$$

Therefore, by (4.68), and since Lagrangian flows are Lebesgue-measure preserving, for any $\mathscr{K}'\geq_{\text{lex}}\mathscr{K}$, and for all $t\in[T_{m-1},T_m]$, we have that

$$\left\|\rho^{\mathscr{K}'}(\cdot,t)-\rho^{\mathscr{K}}(\cdot,t)\right\|_{L^1_x}\leq 4\varepsilon+\left\|\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}'}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}'}_t\right)^{-1}-\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}}_t\right)^{-1}\right\|_{L^1_x}.$$

We now split into two cases, $t\in[T_{m-1},T_{(k,m,1,1)}]$, and $t\in[T_{(k,m,1,1)},T_m]$. We first consider the former.

For the square lattice of widths 2^{1-k}, $J\times J'\subset\mathbb{T}^2$ defined in (4.67), we have for all $x,y\in J\times J'$,

$$\begin{aligned}|\phi_\varepsilon(x)-\phi_\varepsilon(y)|&\leq\|\nabla\phi_\varepsilon\|_{L^\infty_x}|x-y|\\&\leq\sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x}2^{1-k}.\end{aligned}$$

So by (4.67), for all $t\in[T_{m-1},T_{(k,m,1,1)}]$, and for all $\mathscr{K}'\geq_{\text{lex}}\mathscr{K}$, we have that

$$\left\|\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}'}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}'}_t\right)^{-1}-\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}}_t\right)^{-1}\right\|_{L^1_x}\leq\sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x}2^{1-k}.\tag{4.69}$$

Next, by (4.66), for all $t\in[T_{(k,m,1,1)},T_m]$, we have that

$$\begin{aligned}&\left\|\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}'}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}'}_t\right)^{-1}-\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}}_t\right)^{-1}\right\|_{L^1_x}\\&\quad=\left\|\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}'}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}'}_{T_{(k,m,1,1)}}\right)^{-1}\circ\tilde{y}^{\mathscr{K}}_{T_{(k,m,1,1)}}\circ\left(\tilde{y}^{\mathscr{K}}_t\right)^{-1}-\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}}_t\right)^{-1}\right\|_{L^1_x}\\&\quad=\left\|\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}'}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}'}_{T_{(k,m,1,1)}}\right)^{-1}-\phi_\varepsilon\circ\tilde{y}^{\mathscr{K}}_{T_{m-1}}\circ\left(\tilde{y}^{\mathscr{K}}_{T_{(k,m,1,1)}}\right)^{-1}\right\|_{L^1_x},\end{aligned}$$

where the last line is already bounded in (4.69), by $\sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x}2^{1-k}$. Putting everything together, we see that

$$\left\|\rho^{\mathscr{K}'}-\rho^{\mathscr{K}}\right\|_{L^\infty([T_{m-1},T_m];L^1_x)}\leq 4\varepsilon+\sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x}2^{1-k}.$$

$\|\nabla\phi_\varepsilon\|_{L^\infty_x}$ depends only on ε and $\rho_{m-1}(x)$, and in particular not on k. Thus for k sufficiently large, i.e. for $\mathscr{K}',\mathscr{K}\in(\mathscr{N},<_{\text{lex}})$ sufficiently large, we can make the right-hand side arbitrarily small. Therefore $\rho^{\mathscr{K}}(x,t)$ is Cauchy in $L^\infty([T_{m-1},T_m];L^1_x)$ as $\mathscr{K} \xrightarrow{\text{lex}} \infty$, as required.

This completes the induction. Observe that $T_m \xrightarrow{m\to\infty} 42$, and so we have proven that $\rho^{\mathscr{K}}(x,t)$ converges in $C^0([0,42-\varepsilon];L^1_x)$ as $\mathscr{K} \xrightarrow{\text{lex}} \infty$, for any $\varepsilon > 0$.

$\rho^{\mathscr{K}}(x,t)$ are Lagrangian (and weak) solutions to (TE) along $u_{\mathscr{K}}^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$ with initial data $\rho_0(x)$. Therefore, since $\rho_0(x) \in L^\infty_x$, $\rho^{\mathscr{K}}(x,t)$ are uniformly bounded in $L^\infty_t L^\infty_x$.

Moreover, by Definition 4.12, $u_{\mathscr{K}}^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}}}}(x,t)$ is eventually constant, and equal to $u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$ as $\mathscr{K} \xrightarrow{\text{lex}} \infty$ for each $t \in [0,50]$, and are uniformly bounded in $L^\infty_t L^\infty_x$.

Denote by $\bar\rho(x,t)$ a weak-$*$ limit point of $\{\rho^{\mathscr{K}}(x,t)\}_{\mathscr{K}\in\mathscr{N}}$ in $L^\infty([0,50];L^\infty_x)$, that is there exists a subsequence $\mathscr{K}_n \xrightarrow{n\to\infty} \infty$ with $\rho^{\mathscr{K}_n}(x,t) \xrightarrow{n\to\infty} \bar\rho(x,t)$ in weak-$*$ $L^\infty_t L^\infty_x$. Then for all $\phi(x,t) \in C^\infty_c(\mathbb{T}^2 \times [0,50);\mathbb{R})$, we see that

$$\begin{aligned}
&\int_{\mathbb{T}^2\times[0,50]} \bar\rho(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)\cdot\nabla\phi(x,t)\right)dxdt\\
&= \lim_{n\to\infty}\int_{\mathbb{T}^2\times[0,50]} \rho^{\mathscr{K}_n}(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)\cdot\nabla\phi(x,t)\right)dxdt\\
&= \lim_{n\to\infty}\int_{\mathbb{T}^2\times[0,50]} \rho^{\mathscr{K}_n}(x,t)\left(\frac{\partial\phi}{\partial t}(x,t) + u_{\mathscr{K}_n}^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}_n}}}(x,t)\cdot\nabla\phi(x,t)\right)dxdt\\
&= -\int_{\mathbb{T}^2} \rho_0(x)\phi_0(x)\,dx.
\end{aligned}$$

Therefore, $\bar\rho(x,t)$ is a weak solution to (TE) along $u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$ with initial data $\rho_0(x)$. Moreover, it is bounded in $L^\infty_t L^\infty_x$ and so by Theorem 4.1 the limit is in $C^0_{\text{weak}-*}([0,50];L^\infty_x)$. However, $u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t) = 0$ for each $t \in [42,50]$ and $x \in \mathbb{T}^2$, and so (say by (4.5)), for all $t \in [42,50]$, $\bar\rho(x,t)$ is determined by $\bar\rho(x,s)$ for $s \in [0,42)$.

However, we have already shown $\lim_{\mathscr{K}\to\infty} \rho^{\mathscr{K}}(x,t)$ converges in $C^0([0,42-\varepsilon];L^1_x)$ for any $\varepsilon > 0$. Therefore, the limit $\bar\rho(x,t)$ is unique. Assume then that $\lim_{\mathscr{K}\to\infty}\rho^{\mathscr{K}}(x,t)$ does not converge in weak-$*$ $L^\infty([0,50];L^\infty_x)$. Then, by the uniform bound in $L^\infty([0,50];L^\infty_x)$, at least two limit points exist, contradicting uniqueness.

Denote now the limit by $\rho^\infty(x,t)$, that is $\rho^{\mathscr{K}}(x,t) \xrightarrow{\mathscr{K}\to\infty} \rho^\infty(x,t)$ in weak-$*$ $L^\infty([0,50];L^\infty_x)$ and strongly in $C^0([0,42-\varepsilon];L^p_x)$ for all $\varepsilon > 0$.

We are left to show convergence in $L^p([0,42];L^p_x)$ and $C^0([0,42-\varepsilon];L^p_x)$ for all $p \in [1,\infty)$, and all $\varepsilon > 0$. The latter follows by interpolation between convergence in $C^0([0,42-\varepsilon];L^1_x)$ and the existing uniform bound in $L^\infty_t L^\infty_x$. When combined with the uniform bound in $L^2([0,42];L^2_x)$ this further implies convergence of the norm $\|\rho^{\mathscr{K}}\|_{L^2([0,42];L^2_x)}$. Convergence in $L^2([0,42];L^2_x)$ then follows from the already proved weak convergence in $L^2([0,42];L^2_x)$. The analogous result for $p \in [1,2)$ follows from compactness of the domain $\mathbb{T}^2 \times [0,42]$.

Moreover, the convergence for $p \in (2,\infty)$ follows by interpolation with the existing uniform bound in $L^\infty([0,42];L^\infty_x)$.

Finally, we show the mixing formula (4.62).

For $K \in \mathbb{N}$ denote by $\rho^K(x,t) = \rho^{(K,K,2,2^{\lfloor K/2 \rfloor})}(x,t)$, $\mathcal{N}_K = \mathcal{N}_{(K,K,2,2^{\lfloor K/2 \rfloor})}$, and by $\tilde{y}^K_t(x) = \tilde{y}_t^{(K,K,2,2^{\lfloor K/2 \rfloor})}(x)$.

By (4.58), observe that $(k,m,i,n) \in \mathcal{N}_K$ if and only if $k \le K$, and moreover $(K,K,2,2^{\lfloor K/2 \rfloor}) \xrightarrow{\text{lex}} \infty$ as $K \to \infty$.

Fix $k,m \in \mathbb{N}$ with $k \le K$, $m \in \{1,...,k\}$. For all $i \in \{1,2\}$, for all $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$, we see that $(k,m,i,n) \in \mathcal{N}_K$, and so consider the maps

$$\tilde{y}^K_{T_{(k,m,i,n)}+3\cdot 2^{-k}} \circ \left(\tilde{y}^K_{T_{(k,m,i,n)}}\right)^{-1}.$$

By (4.47), (4.64), (4.65), these maps commute for all $i \in \{1,2\}$, for all $n \in \{1,...,2^{\lfloor k/2 \rfloor}\}$. Moreover, by (4.58), (4.59) (illustrated in (4.61)), their composition is equal to

$$\tilde{y}^K_{T_{(k,m,1,1)}+3\cdot 2^{-k}} \circ \left(\tilde{y}^K_{T_{(k,m,2,2^{\lfloor k/2 \rfloor})}}\right)^{-1}.$$

Furthermore, we have the following expression for this map regarding binary expansions.

For $x = (x_1,x_2) \in \mathbb{T}^2$ denote by $(x'_1,x'_2) = \tilde{y}^K_{T_{(k,m,1,1)}+3\cdot 2^{-k}}\left(\left(\tilde{y}^K_{T_{(k,m,2,2^{\lfloor k/2 \rfloor})}}\right)^{-1}(x)\right)$.

For a.e. $x \in \mathbb{T}^2$, for all $j \in \{1,2\}$, x'_j has swapping the k^{th} and $(k+1)^{\text{th}}$ binary digits of x_j, illustrated by the map $Y_k(x)$ below. We use colour to denote swaps in the binary expansion for the reader's convenience. Following the notation for binary expansions introduced in (4.46), we have that for $j \in \{1,2\}$ the coordinate, and for $l \in \mathbb{N}$ the binary digit,

$$\left(\tilde{y}^K_{T_{(k,m,1,1)}+3\cdot 2^{-k}}\left(\left(\tilde{y}^K_{T_{(k,m,2,2^{\lfloor k/2 \rfloor})}}\right)^{-1}(x)\right)\right)_j = \left(0\,.\,x_{j,1}\ \ \dots\ \ \overset{Y_k(x)}{\overbrace{x_{j,k}\ \ x_{j,k+1}}}\ \ \dots\right),$$

$$\left(\tilde{y}^K_{T_{(k,m,1,1)}+3\cdot 2^{-k}}\left(\left(\tilde{y}^K_{T_{(k,m,2,2^{\lfloor k/2 \rfloor})}}\right)^{-1}(x)\right)\right)_{j,l} = \begin{cases} x_{j,k+1} & \text{if } l = k, \\ x_{j,k} & \text{if } l = k+1, \\ x_{j,l} & \text{otherwise.} \end{cases}$$

Next, we fix $m \in \{1,...,K\}$, and use (4.58), (4.59), (4.60), (illustrated in (4.61)), to piece together $\tilde{y}^K_{T_{(k,m,1,1)}+3\cdot 2^{-k}}\left(\left(\tilde{y}^K_{T_{(k,m,2,2^{\lfloor k/2 \rfloor})}}\right)^{-1}(x)\right)$ for $k \in \{m,m+1,...\}$.

From this, we deduce that for a.e. $x \in \mathbb{T}^2$, for all $m \in \{1,\dots,K\}$, and for $j \in \{1,2\}$ the coordinate, and $l \in \mathbb{N}$ the binary digit, we have that

$$\left(\tilde{y}^K_{T_m} \circ \left(\tilde{y}^K_{T_{m-1}}\right)^{-1}(x)\right)_j = \left(0\,.\,x_{j,1} \;\; \dots \;\; \overset{Y_m}{x_{j,m}} \;\; \overset{\cdots}{\dots} \;\; \overset{Y_{K-1}}{} \; \overset{Y_K}{x_{j,K}} \;\; x_{j,K+1} \;\; \dots \right),$$

$$\left(\tilde{y}^K_{T_m} \circ \left(\tilde{y}^K_{T_{m-1}}\right)^{-1}(x)\right)_{j,l} = \begin{cases} x_{j,K+1} & \text{if } l = m, \\ x_{j,l-1} & \text{if } m < l \leq K+1, \\ x_{j,l} & \text{otherwise.} \end{cases}$$

Therefore, since $\tilde{y}^K_{T_0}(x) = x$, we see that for a.e. $x \in \mathbb{T}^2$, for all $m \in \{1,\dots,K\}$, and for $j \in \{1,2\}$ the coordinate, and $l \in \mathbb{N}$ the binary digit,

$$\left(\left(\tilde{y}^K_{T_m}\right)^{-1}(x)\right)_j = \left(0\,.\,x_{j,m+1} \;\; x_{j,m+2} \;\; \dots \;\; x_{j,K+1} \;\; x_{j,m} \;\; x_{j,m-1} \;\; \dots \;\; x_{j,1} \;\; x_{j,K+2} \;\; x_{j,K+3} \;\; \dots \right),$$

$$\left(\left(\tilde{y}^K_{T_m}\right)^{-1}(x)\right)_{j,l} = \begin{cases} x_{j,m+l} & \text{if } l \leq K+1-m, \\ x_{j,K+2-l} & \text{if } K+2-m \leq l \leq K+1, \\ x_{j,l} & \text{if } l \geq K+2. \end{cases} \tag{4.70}$$

For each $m \in \mathbb{N}$ define now the map $z_m(x) : \mathbb{T}^2 \to \mathbb{T}^2$ by, for all $x \in \mathbb{T}^2$, for $j \in \{1,2\}$ to coordinate, and $l \in \mathbb{N}$ the binary digit,

$$(z_m(x))_j = \left(0\,.\,x_{j,m+1} \;\; x_{j,m+2} \;\; x_{j,m+3} \;\; \dots \right),$$

$$(z_m(x))_{j,l} = \left\{ x_{j,m+l} \quad \text{for all } l \in \mathbb{N}. \right.$$

This map is an approximation of (4.70). Notice that for the initial data $\rho_0(x) \in L^\infty_x$, $\rho_0(z_m(x))$ contains 4^m scaled copies of $\rho_0(x)$, one on each tile in the square lattice with widths 2^{-m}.

For any $r, r' \in \{1,\dots,2^{k-1}\}$, define the intervals $J = [(r-1)2^{1-k}, r2^{1-k}]$, $J' = [(r'-1)2^{1-k}, r'2^{1-k}]$. Then $z_m(x)$ is a bijection from this tile $J \times J'$ to $\mathbb{T}^2$, that is

$$z_m(x) : J \times J' \leftrightarrow \mathbb{T}^2.$$

Moreover, for $d\mu(x,t)$ the Lebesgue-measure on $\mathbb{T}^2$, $z_m|_{J\times J'} \circ d\mu = 4^{-m} d\mu(x,t)$. In particular $z_m(x) : \mathbb{T}^2 \to \mathbb{T}^2$ is measure preserving, and

$$\int_{J\times J'} \rho_0(z_m(x))\, dx = \frac{1}{4^m}\int_{\mathbb{T}^2} \rho_0(x)\, dx.$$

Therefore, we deduce that

$$\rho_0(z_m(x)) \xrightarrow{m\to\infty} \int_{\mathbb{T}^2} \rho_0(y)\, dy, \tag{4.71}$$

with the above convergence in weak-$*$ L^∞_x.

By (4.63), for all $K \in \mathbb{N}$, $m \in \{1,...,L\}$, we may write $\rho^K(x,T_m) = \rho_0\big(\big(\tilde{y}^K_{T_m}\big)^{-1}(x)\big)$. We now wish to approximate $\rho^K(x,T_m)$ by $\rho_0(z_m(x))$ to prove the mixing formula (4.62).

Fix some $\varepsilon > 0$. Take a smooth approximation $\phi_\varepsilon(x) \in C^\infty(\mathbb{T}^2;\mathbb{R})$ of $\rho_0(x)$ in L^1_x, so that $\|\phi_\varepsilon - \rho_0\|_{L^1_x} \le \varepsilon$. Since $\big(\tilde{y}^K_{T_m}\big)^{-1}(x)$, $z_m(x)$ are measure preserving, we see that also

$$\begin{gathered}\Big\|\phi_\varepsilon \circ \big(\tilde{y}^K_{T_m}\big)^{-1} - \rho_0 \circ \big(\tilde{y}^K_{T_m}\big)^{-1}\Big\|_{L^1_x} \le \varepsilon,\\ \|\phi_\varepsilon \circ z_m - \rho_0 \circ z_m\|_{L^1_x} \le \varepsilon.\end{gathered} \tag{4.72}$$

Next, consider the square lattice with widths 2^{m-K-1}, that is for $r, r' \in \{1,...,2^{1+K-m}\}$ define $J = [(r-1)2^{m-K-1}, r2^{m-K-1}]$, and $J' = [(r-1)2^{m-K-1}, r2^{m-K-1}]$.

Then, by (4.70), we see that for a.e. $x \in \mathbb{T}^2$, $\big(\tilde{y}^K_{T_m}\big)^{-1}(x) \in J\times J'$ if and only if $z_m(x) \in J\times J'$. But then by the Lipschitz bound on $\phi_\varepsilon(x)$,

$$\Big|\phi_\varepsilon\big(\big(\tilde{y}^K_{T_m}\big)^{-1}(x)\big) - \phi_\varepsilon(z_m(x))\Big| \le \sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x} 2^{m-K-1}.$$

Therefore,

$$\Big\|\phi_\varepsilon \circ \big(\tilde{y}^K_{T_m}\big)^{-1} - \phi_\varepsilon \circ z_m\Big\|_{L^\infty_x} \le \sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x} 2^{m-K-1}.$$

In light of (4.72), we deduce that

$$\big\|\rho^K(x,T_m) - \rho_0(z_m(x))\big\|_{L^1_x} \le 2\varepsilon + \sqrt{2}\,\|\nabla\phi_\varepsilon\|_{L^\infty_x} 2^{m-K-1}. \tag{4.73}$$

Recall that $\rho^K(x,t) \xrightarrow{K\to\infty} \rho^\infty(x,t)$ strongly in $C^0([0,42-\varepsilon];L^1_x)$ for all $\varepsilon > 0$. Therefore, for $m \in \mathbb{N}$ fixed, $\rho^K(x,T_m) \xrightarrow{K\to\infty} \rho^\infty(x,T_m)$ strongly in L^1_x.

So, by (4.73), for all $m \in \mathbb{N}$, we see that $\rho^\infty(x,T_m) = \rho_0(z_m(x))$. That is $\rho^\infty(x,T_m)$ contains 4^m scaled copies of $\rho_0(x)$, one on each tile in the square lattice with widths 2^{-m}.

Hence, by (4.71), $\rho^\infty(x,T_m) \xrightarrow{m\to\infty} \int_{\mathbb{T}^2} \rho_0(y)\, dy$ converges in weak-$*$ L^∞_x.

Since $\rho^\infty(x,t) \in C^0_{\text{weak}-*}L^\infty_x$, with $\rho^\infty(x,t)$ independent of $t \in [42,50]$, and $T_m \xrightarrow{m\to\infty} 42$, we have proved the mixing formula (4.62). $\square$

Finally, it remains to find a suitably fast-growing sequence $\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}} \subset \mathbb{N}$ in Definition 4.12. Then to apply Theorem 4.6 to the sequence of vector fields $u_{\mathscr{K}}^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}}}}(x,t)$, indexed by $\mathscr{K} \in \mathscr{N}$. This will allow us to approximate the vanishing viscosity limit to (κ-TDE) along $u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$. Additionally, by subsequently time-reversing the vector field $u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$ on the time interval $[50,100]$, we will obtain the inadmissible behaviour below.

Theorem 4.13 (Inadmissible vanishing viscosity limit)**.** *There exists a divergence-free vector field $u(x,t) \in L^\infty([0,100];L^\infty_x)$, such that for any initial data $\rho_0(x) \in L^\infty_x$, and for $\rho^\kappa(x,t)$ the unique solution to* (κ-TDE) *along $u(x,t)$ with initial data $\rho_0(x)$, one has*

$$\rho^\kappa(x,t) \xrightarrow{\kappa\to 0} \rho(x,t),$$

with the above convergence in weak-$*$ $L^\infty([0,100];L^\infty_x)$, strong in $L^p([0,42];L^p_x)$, $C^0([0,42-\varepsilon];L^p_x)$, $L^p([58,100];L^p_x)$, and $C^0([58+\varepsilon,100];L^p_x)$ for all $p \in [1,\infty)$, and all $\varepsilon > 0$. The limit function $\rho(x,t) \in C^0_{\text{weak}-*}([0,100];L^\infty_x)$ is a weak solution to (TE) *along $u(x,t)$ with initial data $\rho_0(x)$.*

Moreover, for all $t \in [42,58]$ and $x \in \mathbb{T}^2$,

$$\rho(x,t) = \int_{\mathbb{T}^2} \rho_0(y)\, dy,$$

is perfectly mixed to its spatial average.

Furthermore, for all $t \in [0,100]$, $\rho(x,t) = \rho(x,100-t)$ and in particular,

$$\rho(x,100) = \rho_0(x),$$

is perfectly unmixed. In particular, if $\rho_0(x)$ is not constant, any L^p_x norms of $\rho(x,t)$ (for $p \in (1,\infty]$) increase after $t = 58$, contrary to the entropy-admissibility criterion in [35].

Proof. Recall the language and notation introduced in Definitions 4.11, 4.12, as well as Definitions 4.3, 4.4 of renormalised and Lagrangian solutions to (TE).

For each $n \in \mathbb{N}$ we denote by $\mathscr{K}_n \in \mathscr{N}$ the isomorphism between the well orders $(\mathbb{N},<)$ and $(\mathscr{N},<_{\text{lex}})$. That is $n \mapsto \mathscr{K}_n$ is a bijection from $\mathbb{N}$ to $\mathscr{N}$, and for all $n_1,n_2 \in \mathbb{N}$, we have that $n_1 < n_2$ if and only if $\mathscr{K}_{n_1} <_{\text{lex}} \mathscr{K}_{n_2}$.

For an infinite sequence $\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}} \subset \mathbb{N}$ with $L_{(k,m,i,q)} \geq k+1$ for all $(k,m,i,q) \in \mathscr{N}$, we define $u_n(x,t) : \mathbb{T}^2 \times [0,100] \to \mathbb{R}^2$ by

$$u_n(x,t) = \begin{cases} u_{\mathscr{K}_n}^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}_n}}}(x,t) & \text{if } t \in [0,50], \\ -u_{\mathscr{K}_n}^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}_n}}}(x,100-t) & \text{if } t \in [50,100]. \end{cases} \tag{4.74}$$

$u_n(x,t) \in L_t^\infty L_x^\infty$ is then bounded by 1.

Let $\rho_0(x) \in L_x^\infty$, then for $\rho^{\mathscr{K}_n}(x,t)$ given by Proposition 4.12, we have the following Lagrangian solution $\rho^n(x,t) \in C_t^0 L_x^1$ to (TE) along $u_n(x,t)$ with initial data $\rho_0(x)$,

$$\rho^n(x,t) = \begin{cases} \rho^{\mathscr{K}_n}(x,t) & \text{if } t \in [0,50], \\ \rho^{\mathscr{K}_n}(x,100-t) & \text{if } t \in [50,100]. \end{cases} \tag{4.75}$$

We now apply Theorem 4.6. Let d_* be a metric inducing the weak-$*$ topology on

$$\left\{u(x,t) \in L^\infty([0,100];L_x^\infty) : \|u\|_{L_t^\infty L_x^\infty} \leq 1\right\}.$$

Let $\rho_0(x) \in L_x^\infty$, and denote for each $n \in \mathbb{N}$, $\kappa > 0$, by $\rho^{n,\kappa}(x,t)$, respectively $\rho^n(x,t)$, the unique weak solution to (κ-TDE), respectively (TE), along $u_n(x,t)$ with initial data $\rho_0(x)$. Moreover denote by $\rho^{\infty,\kappa}(x,t)$ the unique weak solution to (κ-TDE) along $u_\infty^{\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}}(x,t)$ with initial data $\rho_0(x)$.

Then by Theorem 4.6,

S.1 For all $n \in \mathbb{N}$ there exists $\kappa_n > 0$, $\varepsilon_n > 0$ depending only on $\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}_n}}$ (and in particular not on $\rho_0(x)$), with $\kappa_n \xrightarrow{n\to\infty} 0$ monotonically, such that the following hold true:

S.2 For all $p \in [1,\infty)$

$$\sup_{\kappa\leq\kappa_n} \|\rho^{n,\kappa} - \rho^n\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0,$$

S.4 If $d_*(u_{n+1},u_n) \leq \varepsilon_n$ for all $n \in \mathbb{N}$, then for all $p \in [1,\infty)$

$$\sup_{\kappa_n\leq\kappa\leq\kappa_1} \|\rho^{n,\kappa} - \rho^{\infty,\kappa}\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0.$$

We now choose $\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}}$. Proceeding by induction on $N \in \mathbb{N}$, assume there exists a sequence $\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}_{\mathscr{K}_N}} \subset \mathbb{N}$, so that with $\{u_n(x,t)\}_{n=1}^N \subset L^\infty([0,100];L_x^\infty)$ given by (4.74), and $\{\varepsilon_n\}_{n=1}^N \subset (0,\infty)$ given by (S.1), we have that for all $n \in \{1,...,N-1\}$,

$$d_*(u_{n+1},u_n) \le \varepsilon_n.$$

We next pick $L_{\mathscr{K}_{N+1}} \in \mathbb{N}$. Note that for any $L_{\mathscr{K}_{N+1}} \in \mathbb{N}$ we obtain by (4.74) a vector field $u_{N+1}(x,t)$.

By (4.50) and Definition 4.12, we see that as $L_{\mathscr{K}_{N+1}} \to \infty$,

$$d_*(u_{N+1},u_N) \xrightarrow{(L_{\mathscr{K}_{N+1}})\to\infty} 0,$$

and so we may pick $L_{\mathscr{K}_{N+1}}$ large enough that $d_*(u_{N+1},u_n) \le \varepsilon_N$. This completes the inductive step.

That is, there exists a sequence $\{L_{\mathscr{P}}\}_{\mathscr{P}\in\mathscr{N}} \subset \mathbb{N}$ such that for all $n \in \mathbb{N}$, with $u_n(x,t)$ given by (4.74), we have that $d_*(u_{n+1},u_n) \le \varepsilon_n$. Therefore, (S.2) and (S.4) are satisfied.

Next, for all $n \in \mathbb{N}$, and for all $p \in [1,\infty)$, we see that

$$\begin{aligned}\sup_{\kappa_{n+1}\le\kappa\le\kappa_n} \|\rho^{\infty,\kappa}-\rho^n\|_{L_t^\infty L_x^p} \le &\sup_{\kappa_{n+1}\le\kappa\le\kappa_n} \left\|\rho^{\infty,\kappa}-\rho^{n+1,\kappa}\right\|_{L_t^\infty L_x^p}\\ &+\sup_{\kappa_{n+1}\le\kappa\le\kappa_n} \left\|\rho^{n+1,\kappa}-\rho^{n,\kappa}\right\|_{L_t^\infty L_x^p}\\ &+\sup_{\kappa_{n+1}\le\kappa\le\kappa_n} \|\rho^{n,\kappa}-\rho^n\|_{L_t^\infty L_x^p},\end{aligned}$$

Therefore, by (S.2), (S.4), if for all $p \in [1,\infty)$,

$$\sup_{\kappa_{n+1}\le\kappa\le\kappa_n} \left\|\rho^{n+1,\kappa}-\rho^{n,\kappa}\right\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0, \tag{4.76}$$

then also for all $p \in [1,\infty)$,

$$\sup_{\kappa_{n+1}\le\kappa\le\kappa_n} \|\rho^{\infty,\kappa}-\rho^n\|_{L_t^\infty L_x^p} \xrightarrow{n\to\infty} 0.$$

Therefore, the statement of Theorem 4.13 follows from Proposition 4.12 applied to (4.75).

Notice that for all $p \in (1,\infty)$, (4.76) follows from the case $p=1$ by interpolation with the existing uniform bound in $L_t^\infty L_x^\infty$. We, therefore, only prove (4.76) for $p=1$.

Let $n \in \mathbb{N}$. Express $\mathscr{K}_{n+1} \in \mathscr{N}$ as $\mathscr{K}_{n+1} = (k,m,i,q)$ with $k \in \mathbb{N}$, $m \in \{1,...,k\}$, $i \in \{1,2\}$, and $q \in \{1,...,2^{\lfloor k/2\rfloor}\}$.

Then, by the expression (4.74), and Definitions 4.11, 4.12, we have that for all $t \notin [T_{\mathscr{K}_{n+1}}, T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k}] \cup [100 - T_{\mathscr{K}_{n+1}} - 3\cdot 2^{-k}, 100 - T_{\mathscr{K}_{n+1}}]$, that $u_{n+1}(x,t) = u_n(x,t)$.

Therefore, for all $\kappa > 0$, $\rho^{n+1,\kappa}(x,t) - \rho^{n,\kappa}(x,t) \in C_t^0 L_x^1$ is a solution to (κ-TDE) on the time interval $[0, T_{\mathscr{K}_{n+1}}]$ along the same $u_n(x,t)$ with initial data $0 \in L_x^\infty$.

Similarly, on the time interval $[T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k}, 100 - T_{\mathscr{K}_{n+1}} - 3\cdot 2^{-k}]$ with initial data $(\rho^{n+1,\kappa} - \rho^{n,\kappa})(x, T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k}) \in L_x^\infty$.

Similarly, on the time interval $[100 - T_{\mathscr{K}_{n+1}}, 100]$ with initial data $(\rho^{n+1,\kappa} - \rho^{n,\kappa})(x, 100 - T_{\mathscr{K}_{n+1}}) \in L_x^\infty$.

Applying the L_x^p-Inequality (4.11) to these three cases shows that

$$\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L^\infty([0,T_{\mathscr{K}_{n+1}}];L_x^1)} = 0, \tag{4.77}$$

$$\begin{aligned}\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|&_{L^\infty([T_{\mathscr{K}_{n+1}}+3\cdot 2^{-k},100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k}];L_x^1)} \\ &\leq \left\|\left(\rho^{n+1,\kappa} - \rho^{n,\kappa}\right)(\cdot, T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k})\right\|_{L_x^1},\end{aligned} \tag{4.78}$$

$$\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L^\infty([100-T_{\mathscr{K}_{n+1}},100];L_x^1)} \leq \left\|\left(\rho^{n+1,\kappa} - \rho^{n,\kappa}\right)(\cdot, 100 - T_{\mathscr{K}_{n+1}})\right\|_{L_x^1}.$$

Therefore, using the continuity $\rho^{n+1,\kappa}(x,t) - \rho^{n,\kappa}(x,t) \in C_t^0 L_x^1$, we have the bound

$$\begin{aligned}&\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L_t^\infty L_x^1} \\ &\quad \leq \left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L^\infty([T_{\mathscr{K}_{n+1}},T_{\mathscr{K}_{n+1}}+3\cdot 2^{-k}]\cap[100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k},100-T_{\mathscr{K}_{n+1}}];L_x^1)}.\end{aligned} \tag{4.79}$$

Next, for $k \in \mathbb{N}$ given in terms of $n \in \mathbb{N}$ by the expression $\mathscr{K}_{n+1} = (k,m,i,q)$, we aim to prove the bound, for all $n \in \mathbb{N}$, and for all $\kappa > 0$,

$$\begin{aligned}&\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L^\infty([T_{\mathscr{K}_{n+1}},T_{\mathscr{K}_{n+1}}+3\cdot 2^{-k}];L_x^1)} \\ &\leq 2\left\|\rho_0\right\|_{L_x^\infty} 2^{-\lfloor k/2\rfloor} + C\left\|\rho_0\right\|_{L_x^\infty}\sqrt{\kappa 2^{-k}},\end{aligned} \tag{4.80}$$

with $C > 0$ independent of $n \in \mathbb{N}$ and $\kappa > 0$.

In the expression $\mathscr{K}_{n+1} = (k,m,i,q)$, we assume, without loss of generality, that $i = 1$. The case $i = 2$ then follows the same argument with the coordinates reversed.

Let $J = [(q-1)2^{-\lfloor k/2\rfloor}, q2^{-\lfloor k/2\rfloor}] \subset \mathbb{T}$.

Then, by (4.49), for all $t \in [0, 3\cdot 2^{-k}]$, and for all $x \notin J \times \mathbb{T}$, we see that the binary swap vector field $u^{(i,k,q,L_{\mathscr{K}_{n+1}})}(x,t) = 0$.

Therefore, for all $t \in [T_{\mathscr{K}_{n+1}}, T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k}] \cap [100 - T_{\mathscr{K}_{n+1}} - 3\cdot 2^{-k}, 100 - T_{\mathscr{K}_{n+1}}]$, and for all $x \notin J \times \mathbb{T}$, we have that $u_{n+1}(x,t), u_n(x,t) = 0$.

We first tackle the case $t \in [T_{\mathscr{K}_{n+1}}, T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k}]$.

Let $\Omega = [0, 1-2^{-\lfloor k/2 \rfloor}] \times \mathbb{T} \subset \mathbb{T}^2$, a periodic strip.

Then, on the spatio-temporal domain $((x_1,x_2),t) \in \Omega \times [0, 3\cdot 2^{-k}]$, we see that $(\rho^{n+1,\kappa} - \rho^{n,\kappa})((x_1 + q2^{-\lfloor k/2 \rfloor}, x_2), t + T_{\mathscr{K}_{n+1}})$ is a solution to the heat equation, and by (4.77) has the initial data 0. However, its boundary data is unknown. We will construct a second solution to the heat equation on the same domain, with initial data 0, which is an upper bound on the boundary, and then apply the maximum principle.

We have the a priori bound $\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L^\infty_t L^\infty_x} \leq 2\left\|\rho_0\right\|_{L^\infty_x}$.

Therefore, we may apply hypo-ellipticity for the heat equation (see, for example, Section 4.4 in [58]) to deduce that $(\rho^{n+1,\kappa} - \rho^{n,\kappa})((x_1 + q2^{-\lfloor k/2 \rfloor}, x_2), t + T_{\mathscr{K}_{n+1}})$ is smooth on the interior of the domain $\Omega \times [0, 3\cdot 2^{-k}]$.

We introduce

$$\begin{aligned}
\operatorname{erf}(x) &= \int_{-\infty}^{x} e^{-y^2}\, dy, \\
C_0 &= \operatorname{erf}(0), \\
a &= 1 - 2^{-\lfloor k/2 \rfloor}.
\end{aligned}$$

We define the following solution to the heat equation.

$$\begin{aligned}
&\theta : \Omega \times [0, 3\cdot 2^{-k}] \to \mathbb{R}, \\
&\theta((x_1,x_2),t) = 2\left\|\rho_0\right\|_{L^\infty_x} C_0^{-1}\left(\operatorname{erf}\left(\frac{-x_1}{\sqrt{4\kappa t}}\right) + \operatorname{erf}\left(\frac{x_1 - a}{\sqrt{4\kappa t}}\right)\right).
\end{aligned}$$

Observe that θ has initial data 0, and its value on the boundary $\partial\Omega = \{0, a\} \times \mathbb{T}$ is greater than or equal to $2\left\|\rho_0\right\|_{L^\infty_x}$. Also for all $(x_1,x_2) \in \Omega$, we have that $\theta((x_1,x_2),t)$ is an increasing function of $t \in [0, 3\cdot 2^{-k}]$.

Therefore, by maximum principle, we have the following point-wise bound on the interior of the domain, for all $(x_1,x_2) \in (0, 1-2^{-\lfloor k/2 \rfloor}) \times \mathbb{T}$, and for all $t \in (0, 3\cdot 2^{-k})$,

$$\left|(\rho^{n+1,\kappa} - \rho^{n,\kappa})((x_1 + q2^{-\lfloor k/2 \rfloor}, x_2), t + T_{\mathscr{K}_{n+1}})\right| \leq \theta((x_1,x_2), 3\cdot 2^{-k}).$$

Hence,

$$\begin{aligned}
&\left\|\rho^{n+1,\kappa} - \rho^{n,\kappa}\right\|_{L^\infty([T_{\mathscr{K}_{n+1}}, T_{\mathscr{K}_{n+1}} + 3\cdot 2^{-k}]; L^1((\mathbb{T}\backslash J)\times\mathbb{T};\mathbb{R}))} \\
&\leq 4\left\|\rho_0\right\|_{L^\infty_x} C_0^{-1} \int_0^a \operatorname{erf}\left(\frac{-x}{\sqrt{4\kappa \cdot 3\cdot 2^{-k}}}\right)\, dx.
\end{aligned}$$

After changing variables, for $C = 8\sqrt{3}C_0^{-1}\int_0^\infty \operatorname{erf}(-x)\,dx$, we have that

$$\left\|\rho^{n+1,\kappa}-\rho^{n,\kappa}\right\|_{L^\infty([T_{\mathscr{K}_{n+1}},T_{\mathscr{K}_{n+1}}+3\cdot 2^{-k}];L^1((\mathbb{T}\setminus J)\times\mathbb{T};\mathbb{R}))} \\ \leq C\left\|\rho_0\right\|_{L^\infty_x}\sqrt{\kappa 2^{-k}}.$$

Since additionally $J\times\mathbb{T}$ has Lebesgue-measure $2^{-\lfloor k/2\rfloor}$, and we have the a priori bound $\left\|\rho^{n+1,\kappa}-\rho^{n,\kappa}\right\|_{L^\infty_t L^\infty_x}\leq 2\left\|\rho_0\right\|_{L^\infty_x}$, we deduce (4.80).

Next, since k in determined by n through the map $n\mapsto\mathscr{K}_{n+1}=(k,i,m,q)$, denote this now by $k_n\in\mathbb{N}$.

Since $n\mapsto\mathscr{K}_{n+1}$ respects the well-orders $(\mathbb{N},<)$, $(\mathscr{N},<_{\text{lex}})$, we see that $k_n\xrightarrow{n\to\infty}\infty$. Moreover, by (S.1), we have that $\kappa_n\xrightarrow{n\to\infty}0$.

Therefore, by (4.80), we see that

$$\sup_{\kappa_{n+1}\leq\kappa\leq\kappa_n}\left\|\rho^{n+1,\kappa}-\rho^{n,\kappa}\right\|_{L^\infty([T_{\mathscr{K}_{n+1}},T_{\mathscr{K}_{n+1}}+3\cdot 2^{-k_n}];L^1_x)}\xrightarrow{n\to\infty}0. \tag{4.81}$$

By (4.79), (4.81), the proof of (4.76) will then be complete if also

$$\sup_{\kappa_{n+1}\leq\kappa\leq\kappa_n}\left\|\rho^{n+1,\kappa}-\rho^{n,\kappa}\right\|_{L^\infty([100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}];L^1_x)}\xrightarrow{n\to\infty}0. \tag{4.82}$$

To this end denote by $g^{n,\kappa}(x,t)\in C^0([100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}],L^1_x)$ the solution to (κ-TDE) along $u_n(x,t)$ on the time interval $[100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}]$ with initial data $\rho^{n+1,\kappa}(x,100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n})\in L^\infty_x$.

Then, arguing as in (4.80), we have the same bound. For all $n\in\mathbb{N}$, for all $\kappa>0$,

$$\left\|\rho^{n+1,\kappa}-g^{n,\kappa}\right\|_{L^\infty([100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}];L^1_x)} \\ \leq 2\left\|\rho_0\right\|_{L^\infty_x}2^{-\lfloor k_n/2\rfloor}+C\left\|\rho_0\right\|_{L^\infty_x}\sqrt{\kappa 2^{-k_n}}.$$

Therefore, as before, we deduce that

$$\sup_{\kappa_{n+1}\leq\kappa\leq\kappa_n}\left\|\rho^{n+1,\kappa}-g^{n,\kappa}\right\|_{L^\infty([100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}];L^1_x)}\xrightarrow{n\to\infty}0. \tag{4.83}$$

By the L^p_x-Inequality (4.11), and (4.78), we have that, for all $n\in\mathbb{N}$, and for all $\kappa>0$,

$$\|g^{n,\kappa}-\rho^{n,\kappa}\|_{L^\infty([100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}];L^1_x)} \\ \leq\left\|\left(\rho^{n+1,\kappa}-\rho^{n,\kappa}\right)(\cdot,T_{\mathscr{K}_{n+1}}+3\cdot 2^{-k_n})\right\|_{L^1_x}.$$

So, by (4.81),

$$\sup_{\kappa_{n+1}\leq\kappa\leq\kappa_n} \|g^{n,\kappa}-\rho^{n,\kappa}\|_{L^\infty([100-T_{\mathscr{K}_{n+1}}-3\cdot 2^{-k_n},100-T_{\mathscr{K}_{n+1}}];L^1_x)} \xrightarrow{n\to\infty} 0.$$

Therefore, by also (4.83), we deduce (4.82). □

4.3 Concluding Remarks

This chapter has explored the selection of solutions to the inviscid transport equation (TE) under conditions where uniqueness fails. Our findings reveal that the vanishing diffusion limit not only fails to restore uniqueness (Theorem 4.10) but may even select a single solution that violates the expected arrow of time (Theorem 4.13). These results challenge the intuitive notion that diffusion enforces an arrow of time in its vanishing limit.

It may be possible to develop quantitative versions of the vanishing viscosity control in Section 4.2.2 by adapting techniques from Chapters 2, 3. Previous work on such quantitative methods is done by stochastic calculus [32] or homogenisation [10]. These lead to partial non-uniqueness results under higher regularity of the vector field, particularly for $u(x,t)\in L^1_t C^\alpha_x$ with $\alpha<1$. However, to extend Theorems 4.10 and 4.13 to this regime, it is not the vanishing viscosity control but the construction of the vector field which presents a significant challenge. In particular, there is no construction in the literature of a C^α_x vector field which perfectly mixes every initial datum.

A second point of interest is the potential applicability of our analysis to more general regularisation schemes for the passive scalar transport equation, such as mollification of the velocity field or hyperviscosity on the passive scalar. Indeed, the non-quantitative analysis in Section 4.2.2 relies solely on norm convergence of the vanishing viscosity limit. Consequently, similar methods might hold for more general passive scalar transport equation regularisations.

This presents a serious philosophical challenge, as such methods are the only tools available in the literature for producing solutions to the initial value problem, suggesting the troubling possibility that these tools cannot always construct 'physical' solutions. This problem is not unique to passive scalar transport along rough vector fields. It is also a serious research topic if a weak solution exists to the inviscid Euler equations for any initial datum that satisfies the local energy inequality [108].

This phenomenon can be studied at the level of particle trajectories of the passive scalar, satisfying the (not so) stochastic differential equation (2.2),

$$dX_t = u(X_t, t). \tag{4.84}$$

As discussed in Section 2.1.3, one can associate a stochastic solution (i.e. a path measure) to the above for any positive weak solution to (TE) via Ambrosio's superposition principle [9, Theorem 12]. The failure of a weak solution to be physical then corresponds to the failure of the corresponding solution to (4.84) to satisfy the (strong) Markov property.

This indicates the need for a deeper understanding of the stochastic differential equation for particle trajectories. In particular, the existence and uniqueness of Markov stochastic solutions preserving the Lebesgue measure remain critical questions. Resolving these issues or demonstrating their failure (as counterintuitive as that would be) is essential to further understanding the physical selection problem.

In conclusion, this chapter highlights the limitations of the vanishing viscosity approach in selecting physically admissible solutions for the passive scalar transport model in ill-posed regimes. Our explicit counterexamples underscore the need for novel approaches to tackle this issue, focusing on establishing physical solutions in such challenging regimes.

Chapter 5

Conclusion

5.1 Summary of Main Results

This thesis has explored the passive scalar transport model in incompressible fluid mechanics, focusing on the dynamics of the equation when the vector field exhibits lower regularity. Our investigation has spanned three main areas: quantitative estimates for transport along Sobolev vector fields, improved regularity and well-posedness of the transport-diffusion equation, and the vanishing viscosity limit for solution selection. These studies have provided new insights into fluid flow behaviour in irregular regimes and have important implications for our understanding of turbulent dynamics.

Chapter 2: Quantitative Estimates for Transport along Sobolev Vector Fields

In Chapter 2, we introduced a novel weak compactness technique that yields improved quantitative bounds in the DiPerna-Lions and Ambrosio well-posedness theories for the transport equation with the Sobolev vector field. Key results include:

- Uniform decay of the DiPerna-Lions commutator (Theorems 2.8 and 2.11)
- Improved mixing estimates with exponential lower bounds on the mixing scale for all initial data (Theorems 2.12 and 2.13)
- State-of-the-art weak stability estimates (Theorems 2.15 and 2.16)
- The first quantitative stability estimate for transport along vector fields with bounded variation (Theorem 2.22)

These findings advance our understanding of mixing and stability in sub-Lipschitz regimes, providing new tools for analysing fluid dynamics in irregular vector fields such as those found in hydrodynamic turbulence. The improved mixing estimates with exponential lower bounds on the mixing scale offer deeper insights into the behaviour of passive scalars in turbulent flows. The state-of-the-art weak stability estimates provide a more robust framework for analysing the stability of solutions in Sobolev spaces, which is essential for numerical simulations and theoretical studies of fluid dynamics. Finally, the novel weak compactness technique introduced in this chapter and the uniform weak stability of transport along BV_x vector fields provide powerful new tools for studying fluid flows in less regular regimes, including progress towards resolving the Bressan mixing conjecture.

Chapter 3: Improved Regularity and Well-posedness of the Transport-Diffusion Equation

Chapter 3 examined the transport-diffusion equation, incorporating isotropic diffusion to ensure uniqueness and regularity of weak solutions under much lower regularity of the advecting vector field. In addition to introducing standard energy estimates, we employed more recent techniques borrowed from other parabolic equations, including weak convolution estimates for mild solutions and maximal regularity for evolution equations. Our main contributions include:

- A general condition for weak solutions to belong to the classical energy class (Theorem 3.10)
- Enhanced uniqueness of weak solutions when the vector field satisfies the Ladyzhenskaya-Prodi-Serrin condition (Theorem 3.11)
- Improved regularity of weak solutions matching that of the heat equation (Theorem 3.13)

These findings extend standard energy estimates and address a gap in the existing literature regarding regularity analysis in more general function spaces for the transport-diffusion equation. The general condition for weak solutions to belong to the classical energy class provides a unified framework for analysing solutions under various regularity conditions. Meanwhile, the enhanced uniqueness and regularity results when the vector field satisfies the Ladyzhenskaya-Prodi-Serrin condition offer new insights into the interplay between diffusion and transport.

Chapter 4: Vanishing Diffusion Limit and Solution Selection

In Chapter 4, we explored the vanishing viscosity/diffusion limit to select physically admissible solutions to the transport equation. Historically, this approach has been used to restore physical meaning to weak solutions in regimes where well-posedness fails. Our key results challenge this conventional understanding of vanishing viscosity limits:

- Demonstration of non-unique renormalised vanishing viscosity limits (Theorem 4.10)
- Proof of an inadmissible vanishing viscosity limit that selects a single solution violating the expected arrow of time (Theorem 4.13)

These findings challenge conventional wisdom about the role of vanishing viscosity in selecting physically admissible solutions. The demonstration of non-unique renormalised vanishing viscosity limits and the proof of an inadmissible vanishing viscosity limit that violates the expected arrow of time has profound implications for our understanding of fluid dynamics and the role of regularisation, such as diffusion, in stabilising numerical models. They also highlight the need for new approaches to identifying physically admissible solutions in ill-posed equations. In particular, the existence of physically meaningful solutions in these regimes becomes unclear.

5.2 Significance and Further Work

The main goal of this thesis was to shed light on the phenomenological behaviour of fluid flow permitted by lower regularity velocity fields, as measured in specific function classes. This aspect of incompressible fluid dynamics, both mysterious and relevant, relates to many important open questions regarding turbulent dynamics. These include the mixing properties of incompressible fluid flow and its analytic behaviour in a high Reynolds number limit, such as vanishing viscosity.

To develop insightful mathematical analysis, we isolated the fluid flow from the system, captured by the passive scalar transport model. This approach provided a playground to test the analytical properties of fluid flow in different function classes, informing corresponding hydrodynamic models.

Our primary advancement in the analysis of mixing and stability lies in developing a new functional approach to quantify these phenomena. Mixing and stability lie at the heart of the complex dynamics of turbulent fluid flows. Despite their importance, the mathematical analysis of these phenomena is still in its infancy. This includes the lack of a satisfactory analytic approach to control the qualitative phenomenon we call ‘mixing’. We developed an approach corresponding to averaging tracers across specific length scales, known in mathematics as mollification.

In contrast to previous approaches using direct Fourier analysis or weak functional norms, we captured more general control across a range of integrable norms of the passive tracer and vector field. Whether such a functional approach to the control of ‘mixing’ truly captures the physics observed in incompressible fluids remains to be seen. However, we highlight that the array of possible bounds, in terms of different length scales and functional norms, allows some flexibility in capturing the dynamics of any specific fluid phenomenon. We emphasise the need for further precise control of the mixing constants, which we have only abstractly produced. In particular, their behaviour in the regime $\nabla u(x,t) \in L^1_t L^1_x$ requires further investigation, for which we contribute only the initial analytical techniques for a complete theory.

Our work has provided new insights into the quantitative effects of diffusion and viscosity in fluid dynamics. The quantitative impact of diffusion, or viscosity in active fluid models, on fluid dynamics is mainly understood as an "ultra-violet" cutoff on the frequency spectrum. This manifests qualitatively in the dynamics only by rescaling the equation—the Reynolds number. Beyond this, very little analysis specific to the viscosity parameter, or its vanishing limit, is well-understood. The toolbox of regularity theory is the first step to such quantitative control. Specifically, the dependence on the viscosity/diffusivity parameter is paramount in rescaling these estimates. The phenomenological behaviour of viscosity to simply cut off any dynamics below a particular length scale is an oversimplified picture. Instead, quantitative control—for which we provide an extensive framework—is needed to capture this phenomenon.

A key finding of our research is the demonstration of the limitations of current models in capturing thermodynamical principles in fluid dynamics. The mathematical manifestation of thermodynamical principles, such as arrows of time, is the central theme of an entire discipline of mathematical analysis—kinetic theory—and one of the significant problems of mathematics. Naive energy dissipation in viscous models is one such example. However, our work has demonstrated failure of this phenomenon in the vanishing viscosity limit, highlighting the oversimplification of viscosity to capture such thermodynamical laws accurately. The same shortcoming of any similar approach and the corresponding lack of an existence theory for Markovian solutions to these problems represents a profound failure of our current mathematical analysis or perhaps of our physical models. Addressing these issues is paramount to ensure our models do not deviate from the physics they represent.

In conclusion, the contributions of this thesis provide meaningful progress and insight into significant and profound problems in the search for rigorous mathematical analysis of the physical dynamics of incompressible, turbulent fluid flows. Our novel analytical approach to mixing, insights into diffusion and viscosity effects, and demonstration of model limitations in capturing thermodynamical principles have significantly advanced this field. This thesis on the interplay between regularity and qualitative dynamics of turbulent flow paves the way for a comprehensive mathematical approach to developing new understanding of physics in hydrodynamical systems.

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