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A Decentralized General Equilibrium Model for Distributed Social Welfare in Mesoeconomics via Input Output Multiplex Networks

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Abstract

We study a multicommodity trade and production model across multiple subeconomies, focusing on social welfare. Trade flows between subeconomies are represented by a multiplex network. This framework allows us to explore how production shocks in one subeconomy can impact others by examining the interconnected trade dimensions through a Leontief input-output model. The approach aligns with mesoeconomics, assuming cooperation among subeconomies to enhance welfare. We formulate the social welfare problem as a distributed optimization problem using the multiplex network Laplacian matrix and solve it with a primal-dual algorithm. The model functions as a decentralized general equilibrium system, encompassing the production process.

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1 Introduction

The study of networks in economic theory is quite extensive [Goyal \(2012\)](#); [Jackson \(2010\)](#); [Bramoullé et al. \(2014\)](#) especially in the cases of trade [Choi et al. \(2017\)](#); [Bhattacharya et al. \(2008\)](#); [Rauch \(2001\)](#) and production [Carvalho and Tahbaz-Salehi \(2019\)](#); [Elliott and Jackson \(2024\)](#). In our study we focus on the so called mesoeconomic level [Mann \(2011\)](#), i.e., we are interested for the population welfare of various neighboring regions, e.g., the municipalities of a country, the member states of a union like the E.U., etc. We will call these regions subeconomies and each of them may have their unique geographical and socio-economic characteristics that are being reflected in their respective economic sectors. We will study this setting from the scope of an agent based modeling, i.e., the interests of each subeconomy are represented by an agent, the respective policy maker. All the agents of the regions under consideration act in a cooperative way in order to achieve the optimum levels of welfare for their respective populations via trade.

We will assume that we have a finite number of subeconomies trading a finite number of goods that can be used both as intermediate and final goods. Each subeconomy acts both as a producer and as a consumer of the goods and has a utility function for each good that consumes and a cost function for each good that produces. The welfare function of each subeconomy, for each good is formed as the summation of the consumption welfare and the production welfare respectively for the corresponding good. The total welfare of each subeconomy is formed as the sum of the welfare functions for each good that produces and consumes and the goal of each subeconomy is to optimize the total welfare function given its trade interactions with the other subeconomies. In this economic setting the trade network actually describes the trade flows of goods among different subeconomies but since the trade takes place among many goods and not just one we will not use the conventional notion of network but we will use instead the notion of multiplex network which can be viewed as a multidimensional network [Sole-Ribalta et al. \(2013\)](#); [Tomasini \(2015\)](#); [Gomez et al. \(2013\)](#). In multiplex networks each dimension is an actual network which in our setting describes the trade flows of the respective good. The edges of each dimension of the multiplex trade network describe the trade flows of a single good among the different subeconomies and we will assume that the respective edge weights are given from standard domestic trade models like [Yotov \(2022\)](#).

The trade process among the subeconomies is decentralized since there are no centralized coordinators but instead we have the ISOs (Independent System Operators) that act as Walrasian auctioneers and what they do is actually a tâtonnement process in order to bring supply into consistency with demand (price adjustment mechanism). Depending on which market belongs the good under consideration the ISOs could be commodity exchanges, market operators, warehouse receipt systems, etc., that operate on the geographical region under consideration. The idea of tâtonnement was proposed by Walras himself in [Walras \(1900\)](#) and after it was put in the right orbit of study by Hicks in [Hicks \(1939\)](#), it took its modern form from the works [Samuelson \(1941\)](#); [Metzler \(1945\)](#); [Arrow et al. \(1959\)](#); [McKenzie \(1960\)](#). Since our model is decentralized, it is natural that the tâtonnement process of our model to be decentralized as well. We have multiple ISOs that observe and correct non overlapping partitions of

the supply and demand in the subset of the economy under consideration where these supply and demand partitions are defined by the trade network structure of the problem. Relevant literature regarding decentralized tâtonnement processes is provided in [Cheng and Wellman \(1998\)](#); [Mandel and Gintis \(2016\)](#); [Gintis \(2007\)](#); [Malamud and Rostek \(2017\)](#). The notion of tâtonnement is closely related with standard general equilibrium theory [Arrow and Debreu \(1954\)](#); [Arrow \(1974\)](#); [Debreu \(1974\)](#) and our model can be viewed as a decentralized general equilibrium model given that we can calculate both the supply/demand quantities and the prices given the welfare of the subeconomies and their respective trade interconnections.

The goods of each subeconomy can be either independent of each other or their respective supply and demand can be linked. In the case that the trade processes of the goods are independent of each other then the problem is fully separable and the trade process of each good can be studied separately. A more realistic scenario is the existence of interconnections among the different goods of the same subeconomy via their production process. A suitable model to describe this case would be a Leontief input output model [Leontief \(1986\)](#) for the goods of each subeconomy. The Leontief input output model has a network interpretation since the Leontief matrix can be viewed as the adjacency matrix of the network that is formed by the supply and demand variables needed in the production process of the goods of each subeconomy. There are numerous works in the literature in the modelling of production networks as we can see from works like [Acemoglu and Azar \(2020\)](#); [Carvalho and Voigtländer \(2014\)](#); [Carvalho \(2008\)](#); [Acemoglu et al. \(2012\)](#). The case is the same in modelling the interregional trade of the economy with the use of a Leontief input output model [Isard \(1951\)](#); [Serrano and Boguná \(2003\)](#). By incorporating the Leontief network structures of each subeconomy in the existing trade network model we have a unified network expression of the production and trade relationships of each subeconomy. These additional input output edges create links among the trade processes of different goods and they provide more insights in potential variations of the supply and demand of the goods and their possible effects on the production and trade of other goods of both the same and other subeconomies as well. These new channels of information due to the input output Leontief edges among seemingly unrelated goods and subeconomies create a new augmented interactions model among subeconomies. This novel information structure can provide more accurate conclusions regarding the production and trade process of the goods and potential shocks within these processes. It is important to note that the decentralization of the model will not vanish completely since it is not expected in the general case the existence of input-output Leontief relationships among all goods of a subeconomy nor trade relationships among all subeconomies for each good. Given all the above, the study of multicommodity trade among different subeconomies without taking into account the Leontief input output relations among the goods of each subeconomy could lead to false results and conclusions with potentially huge error. The proposed model could be seen as an integration of the trade and production processes of different subeconomies in one unified model.

We will formulate the problem as an optimization problem where the objective function would be the summation of the subeconomies welfare functions while the

trade constraints among the subeconomies would be described by the respective Laplacian matrix of the multiplex network. The summation of the welfare functions in the objective denotes the cooperation among the subeconomies. Before we proceed with the proposed solution method for the problem we will provide an insight of the motivation for selecting this specific method. We notice that in the trade process among the different subeconomies the variable that has to do with the amount of a specific product appears in both the cost function of the producer and in the utility function of the consumer. This variable can be viewed as a coupling variable and the case is the same for the respective variables of all the goods. As a result, we want to optimize functions with coupling variables and this class of problems is studied in distributed optimization [Yang et al. \(2019\)](#) which is a class of methods for optimizing the objectives of multiagent interconnected systems in a decentralized way. The coupling variables of the welfare functions are captured by the Laplacian matrix which is something natural if we think that these couplings actually create the network effects in our model. This methodology has been used extensively in the case of a single good (electricity) and one market (power market). Discussions regarding the economics of power markets are provided in [Biggar and Hesamzadeh \(2014\)](#); [Joskow \(2008\)](#); [De Vany and Walls \(1999\)](#). In [Yamamoto and Tsumura \(2012\)](#) the authors studied the social welfare problem for multiple subeconomies from the viewpoint of distributed optimization for the case of smart grids and for just one commodity (electricity).

A commonly used method of solution for a distributed optimization problem is the primal dual algorithm. The primal variables would describe the supply and demand variables of the system while the dual variables would describe the prices of the respective goods [Kelly et al. \(1998\)](#). While we will require the positivity of the primal variables something that would be trivially satisfied from the structure of the welfare functions we will not restrict the dual variables, i.e., we will allow them to take negative values and this has the economic interpretation of subsidization policies from the ISOs in order to make supply meet demand, a phenomenon that is quite common at the inter-regional level within a country or a union of countries. In the case of any active inequality constraints, the primal dual dynamics take the form of a projected primal dual dynamics and a solution approach is provided in [Feijer and Paganini \(2010\)](#). We prove that the equilibrium point of the primal dual dynamical system solves the distributed optimization problem and we prove its convergence with non linear control theoretic techniques.

The overall significance of this new information structure is that it enables us to examine how production shocks in each subeconomy can impact the social welfare of the subeconomies being analyzed by conducting a comparative static analysis of the optimal solutions to the corresponding distributed social welfare optimization problems, which are based on two distinct Leontief input-output production models: the original one and the altered version after the production shock. A formal comparative static analysis of the model is part of ongoing work.

The paper is organized as follows: section 2 presents the mathematical tools that will be used from convex analysis and optimization [Boyd et al. \(2004\)](#), control theory [Khalil \(2002\)](#) and network theory [Vitaly \(2009\)](#). Section 3 introduces the information structure of the trade process among the different subeconomies and the different

goods while in section 4 we present the distributed social welfare optimization problem and its respective primal dual algorithm. Finally, in section 5 we present the social welfare problem with Leontief input output interconnections. Numerical examples are provided alongside with the evolution of the theory.

2 Preliminaries

2.1 Convex Analysis-Optimization Theory

Definition 1. A differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be convex if the following inequality holds for all $x, y \in \mathbb{R}^n$,

$$f(x) - f(y) \leq \nabla f(x)^T(x - y). \quad (1)$$

The function is said to be strictly convex if the inequality (1) strictly holds whenever $x \neq y$. We say that a function f is concave if $-f$ is convex. A function f is strictly concave if $-f$ is strictly convex.

Lemma 1. Consider a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Then, its gradient $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is monotonically increasing, i.e., the following inequality holds for any $x, y \in \mathbb{R}^n$,

$$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0. \quad (2)$$

If f is strictly convex, the inequality strictly holds as long as $x \neq y$. In that case, ∇f is strictly monotonically increasing. In the case of a concave and a strictly concave function the inequality (2) changes direction respectively and ∇f is a (strictly) monotonically decreasing function.

For the following optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } Ax = b \end{aligned} \quad (3)$$

x is the decision variable, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function, $A \in \mathbb{R}^{r \times n}$ and $b \in \mathbb{R}^r$ are the constant matrix and vector for equality constraints respectively. We define the Lagrangian of (3) as $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\mathcal{L}(x, v) = f(x) + v^T(Ax - b)$$

where $x \in \mathbb{R}^n$ is the primal variable and $v \in \mathbb{R}^n$ is the dual variable. The primal-dual dynamics as a solution to (3) are given by:

$$\dot{x} = \nabla f(x) + A^T v$$

$$\dot{v} = -(Ax - b)$$

and the set of the optimality solutions is defined as

$$X^* := \{(x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n\},$$

where x^* and v^* satisfy the following equations

$$\nabla f(x^*) + A^T v^* = 0, \quad Ax^* - b = 0.$$

Throughout this paper, we consider that for each optimization problem the constraint set $\{x \in \mathbb{R}^n | Ax - b = 0\}$ is nonempty, and the Slater's condition holds so that the duality gap is zero.

2.2 Non Linear Control Theory

We will study a continuous, autonomous nonlinear system

$$\dot{x}(t) = f(x(t)) \tag{4}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies $\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$ is called radially unbounded. The function $\dot{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ is called the Lie derivative of V and is defined as:

$$\dot{V}(x(t)) = \nabla V(x(t))^T \cdot \dot{x}(t) = \nabla V(x(t))^T \cdot f(x(t)).$$

Theorem 2. *Let x^* be an equilibrium point of (4). If $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is radially unbounded and $\dot{V}(x) < 0, \forall x \neq x^*$ then x^* is globally asymptotically stable and V is a valid Lyapunov function of (4).*

Definition 2. *A domain $\mathcal{D} \subseteq \mathbb{R}^n$ is called invariant for the system $\dot{x} = f(x)$ if*

$$\forall x(t_0) \in \mathcal{D} \Rightarrow x(t) \in \mathcal{D}, \forall t \in \mathbb{R}.$$

Definition 3. *A domain $\mathcal{D} \subseteq \mathbb{R}^n$ is called positively invariant for the system $\dot{x} = f(x)$, if*

$$\forall x(t_0) \in \mathcal{D} \Rightarrow x(t) \in \mathcal{D}, \forall t \geq t_0.$$

Theorem 3 (LaSalle's Invariance Principle). *Let $\Omega \subset \mathcal{D}$ be a compact positively invariant set with respect to $\dot{x} = f(x)$ and $V : \mathcal{D} \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω . Let $\mathcal{X}$ be the set of all points in Ω where $\dot{V}(x) = 0$ and M be the largest invariant set in $\mathcal{X}$, then every solution starting in Ω approaches M as $t \rightarrow \infty$.*

2.3 Network Theory

A directed network is a pair $G = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the node set and $\mathcal{E}$ is the edge set. The edge set is a set of ordered pairs of elements in V , e.g., $e_{u,v} \in \mathcal{E}$ with $u, v \in V$ where the element u is called the tail of the edge and is denoted as $T(e_{u,v})$ while v is called the head of the edge and is denoted as $H(e_{u,v})$. The total number of nodes in network G is called the order of the network and is denoted by $|\mathcal{V}|$ while the total number of edges in G is called the size of the network and is denoted by $|\mathcal{E}|$. The

incidence matrix of a directed network $\mathcal{G}$ is a $|\mathcal{V}| \times |\mathcal{E}|$ matrix $[e_{ij}]$ defined as follows

$$E = e_{ij} = \begin{cases} -1, & v_i \in T(e_{ij}) \\ 1, & v_j \in H(e_{ij}) \\ 0, & \text{otherwise.} \end{cases}$$

The Laplacian matrix L of a directed network G is given by the formula,

$$L = EE^T.$$

A multiplex directed network is a triple $G = (\mathcal{V}, \mathcal{E}, D)$ where $\mathcal{V}$ and $\mathcal{E}$ denote the node set and the oriented edge set respectively while D defines the different dimensions of the network, with each dimension to be a different type of relationships among the node set $\mathcal{V}$. We can think of the node set to be repeated $|D|$ times and be expressed as $\mathcal{V} = \mathcal{V}^1 \cup \dots \cup \mathcal{V}^D$. The edge set $\mathcal{E}$ can be partitioned into two types of edge sets, $\mathcal{E}^I$ and $\mathcal{E}^X$ where $\mathcal{E} = \mathcal{E}^I \cup \mathcal{E}^X$. The set $\mathcal{E}^I$ consists of triplets $e_{u,v}^d$ with $u, v \in \mathcal{V}$ and $e_{u,v} \in \mathcal{E}^d, 1 \leq d \leq |D|$ depicting the directed edges belonging to the edge set of the d th dimension of the multiplex network. As a result, $\mathcal{E}^I = \mathcal{E}^1 \cup \dots \cup \mathcal{E}^D$. The set $\mathcal{E}^X$ consists of directed edges connecting the nodes of different dimensions and is of the form $\mathcal{E}^X = \mathcal{E}^{1,2} \cup \dots \cup \mathcal{E}^{1,D} \cup \dots \cup \mathcal{E}^{D,1} \cup \dots \cup \mathcal{E}^{D,D-1}$ where $\mathcal{E}^{i,j}$ describes the connections of the dimensions $i, j, \forall i \neq j$ with $\mathcal{E}^{i,i}$ not being permitted $\forall i$. The incidence matrix of the multiplex network G has dimensions $(|\mathcal{V}| \times |D|) \times |\mathcal{E}|$ and is given by the following form

$$E = e_{ij} = \begin{cases} -1, & v_i \in T(e_{ij}) \\ 1, & v_j \in H(e_{ij}) \\ 0, & \text{otherwise} \end{cases}$$

$\forall i, j \in \mathcal{V}$ and $e_{ij} \in \mathcal{E}$. The respective $(|\mathcal{V}| \times |D|) \times (|\mathcal{V}| \times |D|)$ Laplacian matrix L is given by $L = EE^T$. In the case where there are no edges among different dimensions, i.e., $\mathcal{E}^X = \emptyset$, the incidence matrix can be expressed in the following form: $E = \text{diag}\{E^1, \dots, E^{|D|}\}$ where

$$E^d = e_{ij}^d = \begin{cases} -1, & v_i \in T(e_{ij}^d) \\ 1, & v_j \in H(e_{ij}^d) \\ 0, & \text{otherwise} \end{cases} \quad \forall i, j \in \mathcal{V}^d, e_{ij}^d \in \mathcal{E}^d.$$

The Laplacian matrix is given by $L = \text{diag}\{L^1, \dots, L^{|D|}\}$ where $L^d = E^d(E^d)^T, 1 \leq d \leq |D|$. If G is a weighted multiplex network with edge weights represented by the diagonal matrix $W = \text{diag}(w_1, \dots, w_m) \in \mathbb{R}^{m \times m}$ where m represents the cardinality of all the edges in the multiplex network then the multiplex network Laplacian is given by $L = EWE^T$ where E is the incidence matrix.

3 Trade Information Structure with Independent Production Processes

We assume that there exist N subeconomies (municipalities of a country or states in a union of countries) which trade M different goods with independent production processes among the goods of each subeconomy. For each good there are $m^k, 1 \leq k \leq M$ trade lines among the subeconomies that act both as consumers and producers. This trade process is described by the multiplex trade network $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}, M\}$ where $\mathcal{V} = \{\mathcal{V}_1, \dots, \mathcal{V}_M\}$ is the node set of the network and for each dimension corresponding to the k th good $\forall 1 \leq k \leq M$ we have $\mathcal{V}_k = \{d_1^k, s_1^k, \dots, d_N^k, s_N^k\}$ with d_i^k and s_i^k representing the demand and supply of i th subeconomy and k th good respectively, $1 \leq i \leq N$. For the visualization of the multiplex network structure we define a block that contains two nodes. The two nodes represent the supply and demand of the subeconomy for the k th good respectively. As a result, the node set that describes the trade for the k th good has order $|\mathcal{V}_k| = 2N$ and consequently the node set of the entire multiplex network has order $|\mathcal{V}| = 2N \cdot M$. The edge set $\mathcal{E}$ similarly with the node set can be partitioned in the following way $\mathcal{E} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_M$ where $\mathcal{E}_k = \{e_{ij}^k\}_{i \sim j}, 1 \leq k \leq M$ describes the trade lines for the k th good among the i th and j th subeconomy. The size of the multiplex network is the summation of the sizes of the different dimensions that describe the trade lines of each good, i.e., $|\mathcal{E}| = |\mathcal{E}_1| + \dots + |\mathcal{E}_M|$. The edge weight set $\mathcal{W} = \{w_{ij}^k\}_{i \sim j} \forall 1 \leq k \leq M$ describes the weights of the k th good and how the trade portion of total supply of i th subeconomy is allocated to j th subeconomy. The m_k edges of $\mathcal{G}$ for the k th good are of the form $e_{ij}^k = \{s_i^k, d_j^k\}$ with their respective weights $w_{ij}^k, \forall 1 \leq k \leq m$ to satisfy the relationship

$$d_j^k = w_{ij}^k s_i^k \quad (5)$$

that means the demand of j th subeconomy for the k th good equals w_{ij}^k times the supply of i th subeconomy. The weights of the edges are considered known and as we have mentioned in the introduction they can be assigned with the use of domestic trade models. We assume that the total supply of a subeconomy is not used solely for the trade process but rather a portion of it which is distributed to the locally interconnected consumers. The edge weights $w_{ij}^k, \forall 1 \leq k \leq M$ describe how the trade portion of total supply of a subeconomy is allocated to the local consumers. Given our assumptions we have for the edge weights, $0 \leq w_{ij}^k < 1$ and $\sum_{j \sim i} w_{ij}^k < 1$. The remaining

portion of the total supply of a subeconomy that is not used for trade purposes could be used for internal consumption reasons. The edge weights for the k th good can be described more compactly by the diagonal $m_k \times m_k$ matrix $W_k = \text{diag}(w_1^k, \dots, w_{m_k}^k)$. A compact expression of the edges for all the goods is described by the block diagonal matrix $W = \text{diag}(W_1, \dots, W_M)$. We denote the $2N \times m_k$ incidence matrix of the trade network $\mathcal{G}$ for k th good by E_k . The Laplacian matrix that describes the trade among the subeconomies of the k th good is given by the formula $L_k = E_k W_k E_k^T$ while the Laplacian matrix for the trade of all the goods is given by the block diagonal matrix $L = \text{diag}(L_1, \dots, L_M)$ since the production processes of the goods for each subeconomy are independent. All the lines and columns of the Laplacian matrix correspond to the

trade interconnections of the respective production and consumption sectors of the subeconomies and the summation of the elements of each line and column to zero has the economic interpretation of supply meeting demand. In order for the trade network market to clear the following relationship that utilizes the Laplacian matrix has to hold

$$L \cdot \begin{bmatrix} d_1^1 \\ \vdots \\ s_N^1 \\ \vdots \\ d_1^M \\ \vdots \\ s_N^M \end{bmatrix} = 0. \quad (6)$$

Finally, in the model we also have the ISOs (independent system operators) that do not pursue the profit and act as Walrasian auctioneers and what they do is actually a tâtonnement process i.e. they observe the local total supply and demand gap among the interconnected subeconomies and they adjust the prices accordingly. We have multiple ISOs that observe and correct non overlapping partitions of the supply and demand in the global economy where these supply and demand partitions are defined by the trade network $\mathcal{G}$. The total number of ISOs in the global economy coincides with the number of paths in the undirected version of the network $\mathcal{G}$ that results from not taking into account the orientation of the edges. Since our model is decentralized due to its network structure, it is natural that the tâtonnement processes of our model to be decentralized as well. In case of total absence of decentralization, i.e, there is a path among all nodes of the network then we would have a centralized problem and the need for just one ISO. Below we present a graphical example of our setting.

Example 1. We assume the existence of four subeconomies and two networked markets for two independent goods respectively. The graphical representations of the markets are given in Figure 1 and Figure 2 respectively. The node set for both goods is similar and is given by $\mathcal{V} = \{d_1^i, s_1^i, d_2^i, s_2^i, d_3^i, s_3^i, d_4^i, s_4^i\}$ where d_j^i and s_j^i corresponds to demand and supply of j th subeconomy and i th good respectively, $1 \leq i \leq 2, 1 \leq j \leq 4$. The edge set that represents the trade lines of the first good is $\mathcal{E}_1 = \{e_{12}^1, e_{14}^1, e_{23}^1, e_{43}^1, e_{31}^1\}$ with the respective weights to be $w_{12}^1 = 0.3, w_{14}^1 = 0.5, w_{23}^1 = 0.4, w_{43}^1 = 0.9$ and $w_{31}^1 = 0.7$. The edge $e_{31}^1 = \{s_3^1, d_1^1\}$ represents the constraint $d_1^1 = w_{31}^1 \cdot s_3^1$ which means that demand of the first subeconomy equals the 70% of total supply of the third subeconomy. Similarly, $e_{12}^1 = \{s_1^1, d_2^1\}, e_{14}^1 = \{s_1^1, d_4^1\}$ means that 30% and 50% of the total supply of the first subeconomy are allocated to the second and fourth subeconomy respectively. Finally, we have $e_{23}^1 = \{s_2^1, d_3^1\}, e_{43}^1 = \{s_4^1, d_3^1\}$ which means that 40% of the total supply of the second subeconomy and 70% of the total supply of the fourth subeconomy are allocated to the third subeconomy. Without taking into account the edge orientation of the trade network $\mathcal{G}$ we have three paths, $\{e_{31}^1\}, \{e_{12}^1, e_{14}^1\}, \{e_{23}^1, e_{43}^1\}$ and as a result, three ISOs. The incidence matrix of the

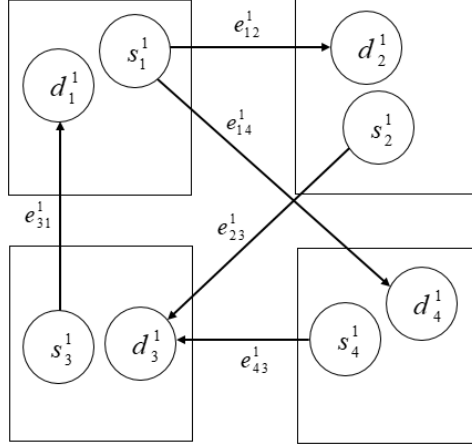


Fig. 1 Market of 1st good

trade network market for this good is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and since the weight matrix is

$$\mathcal{W}_1 = \begin{pmatrix} 0,3 & 0 & 0 & 0 & 0 \\ 0 & 0,5 & 0 & 0 & 0 \\ 0 & 0 & 0,4 & 0 & 0 \\ 0 & 0 & 0 & 0,9 & 0 \\ 0 & 0 & 0 & 0 & 0,7 \end{pmatrix}.$$

the respective Laplacian matrix is

$$L_1 = \begin{pmatrix} \frac{3}{10} & 0 & 0 & 0 & 0 & \frac{-3}{10} & 0 & 0 \\ 0 & \frac{9}{10} & \frac{-1}{2} & 0 & 0 & 0 & \frac{-2}{5} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{10} & \frac{-9}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-9}{10} & \frac{9}{5} & 0 & 0 & \frac{-7}{10} \\ \frac{-3}{10} & 0 & 0 & 0 & 0 & \frac{3}{10} & 0 & 0 \\ 0 & \frac{-2}{5} & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{-7}{10} & 0 & 0 & \frac{7}{10} \end{pmatrix}.$$

For the second good in Figure 2 we have the edge set to be $\mathcal{E}_2 = \{e_{13}^2, e_{32}^2, e_{21}^2, e_{41}^2, e_{24}^2\}$.

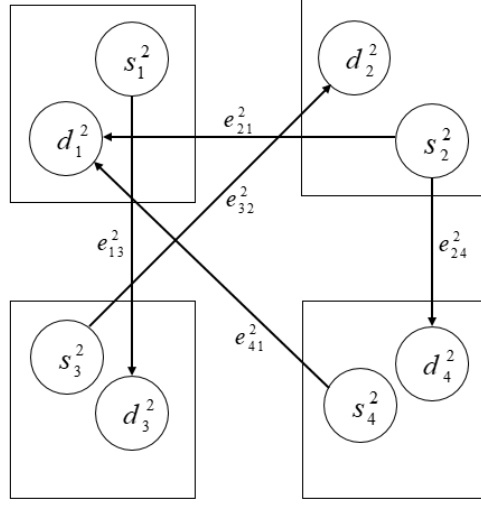


Fig. 2 Market of 2nd good

Without taking into account the edge orientation of the trade network $\mathcal{G}$ we have three paths, $\{e_{13}^2\}$, $\{e_{32}^2\}$, $\{e_{21}^2, e_{24}^2, e_{41}^2\}$ and as a result, three ISOs. The incidence matrix of the trade network market for this good is

$$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and given that the respective weights are given by $w_{13}^2 = 0.7, w_{32}^2 = 0.4, w_{21}^2 = 0.3, w_{24}^2 = 0.2, w_{41}^2 = 0.5$ or in matrix form

$$\mathcal{W}_2 = \begin{pmatrix} 0,7 & 0 & 0 & 0 & 0 \\ 0 & 0,4 & 0 & 0 & 0 \\ 0 & 0 & 0,3 & 0 & 0 \\ 0 & 0 & 0 & 0,2 & 0 \\ 0 & 0 & 0 & 0 & 0,5 \end{pmatrix}$$

the respective Laplacian matrix is

$$L_2 = \begin{pmatrix} \frac{4}{5} & 0 & 0 & \frac{-3}{10} & 0 & 0 & 0 & \frac{-1}{2} \\ 0 & \frac{7}{10} & 0 & 0 & \frac{-7}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 & \frac{-2}{5} & 0 & 0 \\ \frac{-3}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{-1}{5} & 0 \\ 0 & \frac{-7}{10} & 0 & 0 & \frac{7}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{-2}{5} & 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{5} & 0 & 0 & \frac{1}{5} & 0 \\ \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

4 Social Welfare Distributed Optimization

The demand and supply levels for the k th good in i th subeconomy are defined by $d_k^i \in \mathbb{R}_{++}$ and $s_k^i \in \mathbb{R}_{++}$ with their prices to be $p_k^{di} \in \mathbb{R}$ and $p_k^{si} \in \mathbb{R}$ respectively. This practically means that the supply and demand of a subeconomy have different prices for the consumption and production of the same good. The consumption utility function and the production cost function for the k th good in i th subeconomy are given by $u_k^i(\cdot) : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}, \in \mathcal{C}^2$ and $c_k^i(\cdot) : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}, \in \mathcal{C}^2$ respectively and we also assume that utility functions $u_k^i(\cdot)$ are strictly concave while cost functions $c_k^i(\cdot)$ are strictly convex. We have that the subeconomies via the actions of their respective policy makers obtain the following two types of welfare:

The consumption welfare for the k th good of i th subeconomy is given by buying quantity d_k^i with price p_k^{di} per unit of good, i.e.,

$$CW_k^i = u_k^i(d_k^i) - p_k^{di} d_k^i, \quad (7)$$

The production welfare for the k th good of i th subeconomy is given by selling quantity s_k^i with price p_k^{si} per unit of good, i.e.,

$$PW_k^i = p_k^{si} s_k^i - c_k^i(s_k^i), \quad 1 \leq i \leq N. \quad (8)$$

Remark 1. We can compare utilities and costs with prices since market prices are not provided [Elberfeld \(2004\)](#) and we will calculate them from the primal dual algorithm in (12a)-(12b).

The social welfare for the k th good of i th subeconomy results by adding the consumption and production welfare, i.e., $SW_k^i = CW_k^i + PW_k^i$ and the social welfare for all the subeconomies regarding the k th good is given by $SW_k = \sum_{i=1}^N [CW_k^i + PW_k^i]$

while the social welfare for all the goods is given by $SW = \sum_{k=1}^M SW_k$. By taking into

account the trade network $\mathcal{G}$ and the relationship (6) we have that

$$SW = \sum_{k=1}^M \sum_{i=1}^N u_k^i(d_k^i) - c_k^i(s_k^i) \quad (9)$$

since

$$\sum_{k=1}^M \sum_{i=1}^N -p_k^{di} d_k^i + p_k^{si} s_k^i = - [p_1^{d1} \dots p_1^{sN} \dots p_M^{d1} \dots p_M^{sN}]^T L \begin{bmatrix} d_1^1 \\ \vdots \\ s_N^1 \\ \vdots \\ d_1^M \\ \vdots \\ s_N^M \end{bmatrix} = 0$$

from (6).

By including the supply and demand variables of all subeconomies into a vector $x^T = (x_1 \dots x_M) \in \mathbb{R}_{++}^{2N \cdot M}$ where $x_k^T = (d_k^1 \ s_k^1 \dots d_k^N \ s_k^N) \in \mathbb{R}_{++}^{2N} \ \forall 1 \leq k \leq M$ are the subvectors with the demand and supply of the respective subeconomies for the k th good, then the clearing of the market can be described by the relationship

$$Lx = 0.$$

By setting $f_k(x_k) = SW_k = \sum_{i=1}^N [u_k^i(d_k^i) - c_k^i(s_k^i)]$, $\forall 1 \leq k \leq M$ the social welfare problem under the trade network constraints can be expressed as the following distributed optimization problem

$$\begin{aligned} \max_{x=[x_1, \dots, x_M]^T} & \sum_{k=1}^M f_k(x_k) \\ \text{s.t.} & Lx = 0 \end{aligned} \quad (10)$$

with x being the primal variable of the optimization problem (10). We form the Lagrangian of (10) as

$$\mathcal{L}(x, p) = f(x) + p^T Lx \quad (11)$$

where $f(x) = \sum_{k=1}^M f_k(x_k)$ and $p \in \mathbb{R}^{2N \cdot M}$ is the dual variable with the respective Lagrange multipliers. The primal dual algorithm of problem (10) is given below:

$$\dot{x}(t) = \nabla \mathcal{L}_x = \nabla f(x(t)) + Lp(t) \quad (12a)$$

$$\dot{p}(t) = -\nabla \mathcal{L}_p = -Lx(t). \quad (12b)$$

Dual variable p can be interpreted as prices for the goods. Each subeconomy operates in order to maximize its social welfare function for the given prices. The ISOs via the

primal dual algorithm act as local price adjustment mechanisms in order to bring the supply into consistency with the demand at each path of $\mathcal{G}$ (decentralized tâtonnement process). The optimization problem (10) can be interpreted as a general equilibrium problem and its solution via the primal dual algorithm will provide the values of supply, demand and prices in a decentralized way based on the structure of the multiplex network that describes the various markets of goods.

Remark 2. *We have not included positivity and budget type constraints of supply and demand variables in (10) for reasons of simplicity, since our main focus is on the information/network structure of the model. We will satisfy the positivity constraints of supply and demand throughout this work via an appropriate formulation of the objective functions, e.g., logarithmic utility functions. Since we do not have budget type inequality constraints in (10) the dual variables are not restricted in $\mathbb{R}_+^{2N}$ and they can take negative values as well. This can be viewed as a form of subsidization policies in order for supply to meet demand. In the case of inequality constraints of any nature (positivity, budget, etc.), the primal dual dynamics take the form of a projected primal dual dynamical system and an appropriate solution approach is provided in [Feijer and Paganini \(2010\)](#).*

Theorem 4. *Let (x^*, p^*) be an equilibrium point of the dynamical system (12a)-(12b) then (x^*, p^*) satisfies the optimality conditions of the optimization problem (10).*

Proof. We find the equilibrium point of the dynamical system from $\dot{x} = 0 \Rightarrow \nabla f(x^*) = -Lp^*$ and $\dot{p}(t) = 0 \Rightarrow Lx^* = 0$. We notice that the equilibrium point satisfies the optimality conditions as they were presented in the introduction and as a result, the equilibrium point of the dynamical system (12a)-(12b) solves the optimization problem (10). □

Theorem 5. *The equilibrium point of the dynamical system (12a)-(12b) is globally asymptotically stable.*

Proof. The Lyapunov function of the dynamical system (12a)-(12b) system is constructed as the sum of the respective storage functions. We choose as candidate Lyapunov function $V(t) : \mathbb{R}^{2N \cdot M} \rightarrow \mathbb{R}$ where $V(t) = \frac{1}{2}\|x(t) - x^*\|_2^2 + \frac{1}{2}\|p(t) - p^*\|_2^2$. The respective Lie derivative is

$$\begin{aligned}
\dot{V}(t) &= \dot{x}^T(t)(x(t) - x^*) + \dot{p}^T(t)(p(t) - p^*) \\
&= (\nabla f(x(t)) + Lp(t))^T(x(t) - x^*) \\
&\quad + (-Lx(t))^T(p(t) - p^*) \\
&= \nabla f(x(t))^T(x(t) - x^*) + p^T(t)L(x(t) - x^*) \\
&\quad - x^T(t)L(p(t) - p^*) \\
&= \nabla f(x(t))^T(x(t) - x^*) + p^T(t)Lx(t) - p^T(t)Lx^* \\
&\quad - x^T(t)Lp(t) + x^T(t)Lp^*
\end{aligned}$$

$$\begin{aligned}
&= \nabla f(x(t))^T (x(t) - x^*) - p^T(t) Lx^* + x^T(t) Lp^* \\
&= \nabla f(x(t))^T (x(t) - x^*) + \nabla f(x^*)^T x(t) - \nabla f(x^*)^T x^* \\
&= (\nabla f(x(t)) - \nabla f(x^*))^T (x(t) - x^*) < 0, \quad \forall x \neq x^*
\end{aligned}$$

since $\nabla f(x)$ is strictly monotonically decreasing. From LaSalle's invariance principle we have convergence to the largest invariant set for which $\dot{V} = 0$, i.e., $x = x^*$. This set includes only the equilibrium point (x^*, p^*) . As a result, the dynamical system (12a)-(12b) converges asymptotically to the solution (x^*, p^*) . In the fifth line of the proof we have used that $p^T(t)Qx(t) = x^T(t)Qp(t)$ while in the sixth line of the proof we have used that $p^T(t)Qx^* = 0$, $x^T(t)Qp^* = \nabla f(x^*)^T x(t)$ and $\nabla f(x^*)^T x^* = 0$ from the equilibrium properties. $\square$

5 Trade Information Structure with Production Input Output Interconnections

We assume that all N subeconomies follow a Leontief input output production model for the M goods where the production process of i th subeconomy for all its goods can be described by the following system of equations

$$\begin{aligned}
s_i^1 &= a_i^{11} s_i^1 + \dots + a_i^{1M} s_i^M + d_i^1 \\
&\vdots \\
s_i^M &= a_i^{M1} s_i^1 + \dots + a_i^{MM} s_i^M + d_i^M
\end{aligned} \tag{14}$$

where $s_i^k, d_i^k \forall 1 \leq i \leq N, 1 \leq k \leq M$ are the supply and demand quantities respectively of the k th good that are needed in the production process of the M goods of i th subeconomy. The system of equations (14) can also be written in matrix form as

$$s_i = A_i s_i + d_i \tag{15}$$

where $s_i, d_i \in \mathbb{R}_+^M$ represent the supply and demand quantities of all M goods that are needed for their production processes in i th subeconomy. Matrix $A_i \in \mathbb{R}^{M \times M} \forall 1 \leq i \leq N$ is the matrix of interproduction coefficients of i th subeconomy. From system (15) we can express the production supply in terms of its respective production demand in the following way:

$$s_i = (I - A_i)^{-1} d_i$$

where matrix $(I - A_i)^{-1}$ is called the Leontief inverse matrix. We define by A_i^* the normalized $(I - A_i)^{-1}$ so that each row of the matrix sums to one. Since s_i is the supply and d_i the demand of the M goods respectively for the production process of i th subeconomy we can express the supply of each good in terms of the demand of all other goods of the same subeconomy. Practically this means that in the multiplex network we connect for each subeconomy all the supply and demand variables of all the goods that the subeconomy produces in terms of the normalized Leontief matrix

coefficients. The different dimensions of the multiplex network stop to be independent and they are connected in terms of the normalized Leontief matrix. By interpreting the normalized Leontief matrix A^* as the adjacency matrix of a weighted directed network and by including this network in our previous model of the independent production processes of the goods, the multiplex network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}, M)$ remains the same except from the edge set $\mathcal{E}$ which now has two types of edges. The edge set is now of the form $\mathcal{E} = \mathcal{E}^I \cup \mathcal{E}^X$ where $\mathcal{E}^I = \{\mathcal{E}_1^I, \dots, \mathcal{E}_M^I\}$ represents the trade lines of the M goods among the different subeconomies and the edge set $\mathcal{E}^X = \{\mathcal{A}^1 \cup \dots \cup \mathcal{A}^N\}$ that represents the Leontief input output networks for the goods of all subeconomies. The respective Laplacian of the multiplex network will be of the form $L^* = EW^*E^T$ where E will be the incidence matrix of this new augmented multiplex network and W^* will be the new diagonal weight matrix which would be of the form

$$W^* = \text{diag}\{W_1, \dots, W_M, \text{diag}A_1^*, \dots, \text{diag}A_N^*\}$$

where $W_1, \dots, W_M$ are the weight matrices that correspond to the trade lines $\mathcal{E}_1^I, \dots, \mathcal{E}_M^I$ among the subeconomies while $\text{diag}A_1^*, \dots, \text{diag}A_N^*$ are diagonalizations of the elements of matrices $A_1^*, \dots, A_N^*$ that correspond to the Leontief coefficients of each subeconomy. The ordering of the elements of diagonal matrix W is not fixed but it has to correspond to the right weights of the respective edges. With the inclusion of the Leontief coefficients in the edge set of the trade multiplex network we have created interdependencies of the different markets of goods (network dimensions) in terms of the Leontief input output production model for all subeconomies and this results to a unified information structure that describes both the trade and production process for the mesoeconomic setting under consideration. The social welfare problem of the subeconomies for this unified information structure is given by

$$\begin{aligned} \max_{x=[x_1, \dots, x_M]^T} \quad & \sum_{k=1}^M f_k(x_k) \\ \text{s.t.} \quad & L^*x = 0 \end{aligned} \tag{16}$$

and for its solution we can use the primal dual dynamics in (12a)-(12b). The only difference of this optimization problem with its equivalent for the case of independent production processes is the Laplacian matrix which now captures information regarding the production process as well but this does not affect the equilibrium and convergence proofs which remain the same. The importance of this novel information structure lies on the fact that it allows us to study how any production shocks in each of the subeconomies can affect the social welfare of the subeconomies under consideration by contacting a comparative static analysis to the optimum solutions of the respective distributed optimization problems formed by the two different Leontief input output production models, the initial one and the one after the production shock. Below we present a numerical example for the solution of distributed optimization problem.

Example 2. We assume that we have two subeconomies with two independent goods as it is shown in Figure 3. The objective functions of the two subeconomies are:

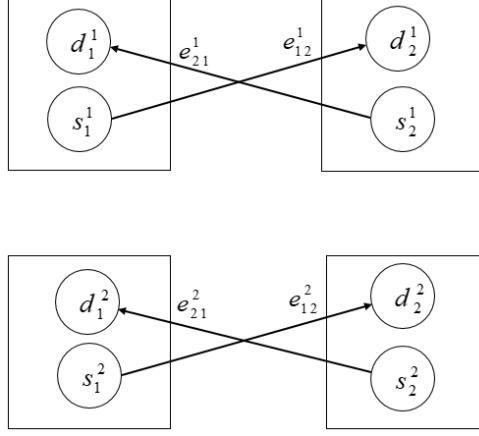


Fig. 3 Market without Input Output Interconnections

$$f_1 = 0.2\log(d_1^1 + 1) + 0.1\log(d_1^2 + 2) - 0.5(s_1^1)^2 - 0.6(s_1^2)^2$$

$$f_2 = 0.3\log(d_2^1 + 3) + 0.5\log(d_2^2 + 4) - 0.15(s_2^1)^2 - 0.2(s_2^2)^2$$

The incidence matrices for both markets and the incidence matrix of the global market are given below:

$$E_1 = E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Assuming that the weights of the trade interconnections are given by $e_{12}^1 = 0.8, e_{21}^1 = 0.9, e_{12}^2 = 0.9$ and $e_{21}^2 = 0.8$ then the diagonal weight matrix is

$$W = \begin{pmatrix} 0,8 & 0 & 0 & 0 \\ 0 & 0,9 & 0 & 0 \\ 0 & 0 & 0,9 & 0 \\ 0 & 0 & 0 & 0,8 \end{pmatrix}$$

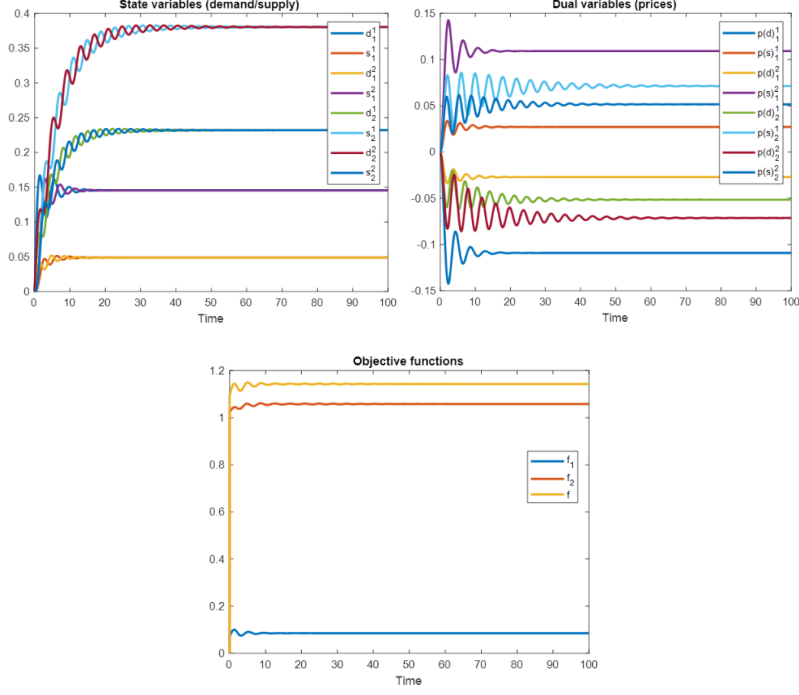


Fig. 4 (a) Supply& Demand Variables (b) Price Variables (c) Welfare Functions

and the resulting Laplacian matrix for the global multiplex networked market is

$$L = EWE^T = \begin{pmatrix} \frac{4}{5} & 0 & 0 & \frac{-4}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{9}{10} & \frac{-9}{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-9}{10} & \frac{9}{10} & 0 & 0 & 0 & 0 & 0 \\ \frac{-4}{5} & 0 & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{9}{10} & 0 & 0 & \frac{-9}{10} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{5} & \frac{-4}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-4}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{-9}{10} & 0 & 0 & \frac{9}{10} \end{pmatrix}.$$

The distributed social welfare problem for this case is given below:

$$\max_{x_1, x_2} f_1(x_1) + f_2(x_2)$$

$$s.t. \quad Lx = 0.$$

where $x = [x_1, x_2]$, $x_1 = (d_1^1, s_1^1, d_2^1, s_2^1)$ and $x_2 = (d_1^2, s_1^2, d_2^2, s_2^2)$. The convergence process of the supply and demand variables (primal variables), the prices (dual variables) and the welfare functions are presented in Figure (4) with their respective asymptotic

values to be,

$$x^* = (0.1455, 0.0488, 0.0488, 0.1455, 0.2321, 0.3805, 0.3805, 0.2321)$$

$$p^* = (-0.1091, 0.0271, -0.0271, 0.1091, -0.0516, 0.0713, -0.0713, 0.0516)$$

$$f_1 = 0.0850, f_2 = 1.0580, f = 1.1430.$$

If we assume that both subeconomies follow a production input output Leontief model with common input output matrix

$$A = \begin{pmatrix} 0.2 & 0.7 \\ 0.1 & 0.6 \end{pmatrix}$$

then the resulting Leontief coefficient matrix is given by

$$(I - A)^{-1} = \begin{pmatrix} 1.6 & 2.8 \\ 0.4 & 3.2 \end{pmatrix}.$$

and its normalized version is:

$$A^* = \begin{pmatrix} 0.36 & 0.64 \\ 0.11 & 0.89 \end{pmatrix}.$$

A graphical expression of the two markets with input output coefficients is given in Figure 5 and the incidence matrix of the global market for this case is

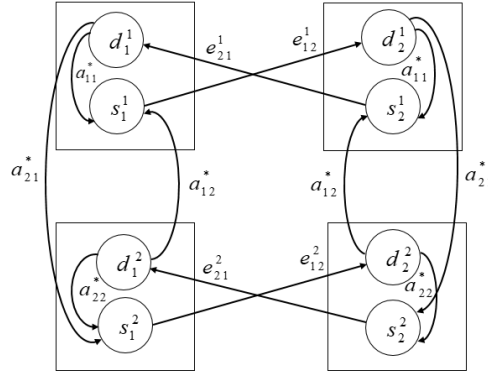


Fig. 5 Market with Input Output Interconnections

$$E = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Considering the same edge weights for the trade process as previously with the independent production processes of the goods and the Leontief normalized weights, the diagonal weight matrix is

$$W = \text{diag}\{0.8, 0.9, 0.36, 0.11, 0.64, 0.89, 0.36, 0.11, 0.64, 0.89, 0.9, 0.8\}.$$

The Laplacian matrix for the global input output multiplex networked market is of the form

$$L^* = \begin{pmatrix} \frac{127}{100} & \frac{-9}{25} & 0 & \frac{-4}{5} & 0 & \frac{-11}{100} & 0 & 0 \\ \frac{-9}{25} & \frac{19}{10} & \frac{-9}{10} & 0 & \frac{-16}{25} & 0 & 0 & 0 \\ 0 & \frac{-9}{10} & \frac{137}{100} & \frac{-9}{25} & 0 & 0 & 0 & \frac{-11}{100} \\ \frac{-4}{5} & 0 & \frac{-9}{25} & \frac{9}{5} & 0 & 0 & \frac{-16}{25} & 0 \\ 0 & \frac{-16}{25} & 0 & 0 & \frac{233}{100} & \frac{-89}{100} & 0 & \frac{-4}{5} \\ \frac{-11}{100} & 0 & 0 & 0 & \frac{-89}{100} & \frac{19}{10} & \frac{-9}{10} & 0 \\ 0 & 0 & 0 & \frac{-16}{25} & 0 & \frac{-9}{10} & \frac{243}{100} & \frac{-89}{100} \\ 0 & 0 & \frac{-11}{100} & 0 & \frac{-4}{5} & 0 & \frac{10}{100} & \frac{9}{5} \end{pmatrix}$$

The respective social welfare problem for the input output case is:

$$\max_{x=[x_1, x_2]^T} f_1 + f_2$$

$$s.t. \quad L^* x = 0.$$

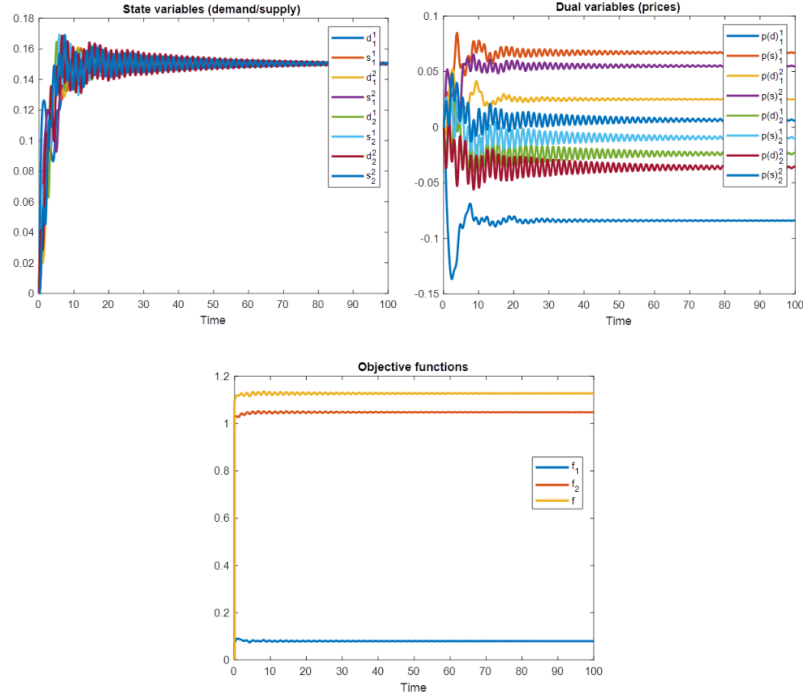


Fig. 6 (a) Supply& Demand Variables (b) Price Variables (c) Welfare Functions

The convergence process of the supply and demand variables (primal variables), the prices (dual variables) and the welfare functions are presented in Figure (6) with their respective asymptotic values to be,

$$\begin{aligned}x^* &= (0.1504, 0.1504, 0.1504, 0.1504, 0.1504, 0.1504, 0.1504, 0.1504) \\p^* &= (-0.0840, 0.0671, 0.0251, 0.0550, -0.0239, -0.0095, -0.0362, 0.0064) \\f_1 &= 0.0797, f_2 = 1.0479, f = 1.1276.\end{aligned}$$

We notice that in the case of independent production processes we have four different convergence values of supply and demand while in the case of input-output interconnections we have convergence in just one value. From an algorithmic point of view this happens because there is a path from each node to any other node in the multiplex network and has the economic interpretation of a holistic balance across all dimensions of the network, driven by the feedback mechanisms created by the input-output relationships. These mechanisms synchronize the flows between supply and demand, where each agent's consumption and production decisions in one dimension influence their behavior in the other dimensions.

Remark 3. It is important to note that in the above example we have assumed the same Leontief coefficients for both subeconomies. It could be the case for each subeconomy to have different Leontief coefficients.

6 Acknowledgements

The authors would like to thank professor Vasco Carvalho from the Faculty of Economics, University of Cambridge for his valuable comments on this work.

7 Conclusion

This study presents a novel approach to optimizing social welfare across multiple subeconomies connected by a multiplex trade network. By incorporating both independent and interconnected production processes through a Leontief input-output model, we provide deeper insights into the dynamics of trade and production, especially in the face of production shocks. The decentralized optimization framework, solved using a primal-dual algorithm, ensures that welfare is optimized while respecting the decentralized nature of the system. The model's integration of trade and production networks offers a more comprehensive understanding of the interdependencies among goods and subeconomies, laying the groundwork for future research into comparative statics and policy implications at the mesoeconomic level.

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