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A Continuous/Discontinuous Galerkin Formulation for a Strain Gradient-Dependent Damage Model: 2D Results

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Abstract The numerical solution of strain gradient-dependent continuum problems has been hindered by continuity demands on the basis functions. The presence of terms in constitutive models that involve gradients of the strain field means that the C^0 continuity of standard finite element shape functions is insufficient. Despite a resurgence of research interest in strain gradient continuum models to represent micro-mechanical effects, a sound, effective and simple framework for the numerical solution of strain gradient-dependent problems is lacking. Here, a formulation is presented which allows the use of C^0 finite element shape functions for the solution of a prototype strain gradient-dependent damage model. The formulation is examined in two dimensions for the simulation of crack propagation. Particular attention is paid to the application of non-standard boundary conditions.

Key words: damage, fracture, discontinuous Galerkin method

INTRODUCTION

Strain gradient-dependent continuum models are being developed at a rapid rate in an attempt to model mechanical phenomena which cannot be captured by classical continuum theories, such as size effects and strain localisation. In addition to the classical dependency on the strain field, the stress (or 'microstresses') in a body is dependent on gradients of the strain field, as well as the strain. A small selection of such models can be found in References [1–7]. However, significant difficulties arise in the numerical solution of strain gradient-dependent continuum models. The difficulty lies in the presence of higher-order derivatives (higher than order two) in the governing boundary value problem. For most strain gradient models, the problem is governed by a fourth-order partial differential equation. The situation is further complicated by the fact that for many models, higher-order derivatives are active only in specific regions (typically where inelastic deformations are developing). Numerous techniques have been developed to model strain gradient problems using the finite element method [2, 8, 9], but have been plagued by difficulties.
Here, guidance is taken from recent developments in discontinuous Galerkin methods (see Arnold et al. [10] for an overview) and continuous/discontinuous Galerkin methods [11] to develop a formulation for a strain gradient-dependent damage model which allows the use of C^0 basis functions. The formulation is examined here for a two-dimensional problem involving damage propagation. The Galerkin formulation involves the usual volume integrals, plus integrals over element edges which involve jumps in the strain field. Numerical tests illustrate the performance for different meshes. Special attention is paid to the role and application of non-standard boundary conditions.

CONTINUOUS/DISCONTINUOUS GALERKIN FORMULATION

Consider a body Ω in $\mathbb{R}^n$, with boundary $\Gamma = \partial\Omega$. The outward normal vector to the body is denoted $\boldsymbol{n}$. The strong form of the equilibrium equation for the body Ω, in the absence of body forces, and associated standard boundary conditions, are given by:

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \qquad \text{in } \Omega \tag{1}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{h} \qquad \text{on } \Gamma_h \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_g \tag{3}$$

where ∇ is the gradient operator, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{h}$ is the prescribed traction on Γ_h and $\boldsymbol{g}$ is the prescribed displacement on the boundary Γ_g ($\Gamma_g \cup \Gamma_h = \Gamma$, $\Gamma_g \cap \Gamma_h = \emptyset$). The stress $\boldsymbol{\sigma}$ at a material point $\boldsymbol{x} \in \Omega$ is given by:

$$\boldsymbol{\sigma} = (1-\omega)\mathscr{C} : \nabla^{\mathrm{s}}\boldsymbol{u} \tag{4}$$

where the scalar $\omega \in [0,1]$ is the 'damage', $\mathscr{C}$ is the usual linear, isotropic elasticity tensor, and $\nabla^{\mathrm{s}}(\cdot)$ represents the symmetric gradient of $(\cdot)$. The damage ω is a function of a scalar history parameter κ, which in turn is related to a scalar 'equivalent strain' measure, $\bar{\varepsilon}$. In a classical formulation, $\bar{\varepsilon}$ is simply an invariant of the strain tensor. For a strain gradient-dependent damage model, a gradient dependency of the form proposed by Aifantis [1] is included,

$$\bar{\varepsilon} = \varepsilon_{\mathrm{eq}} + c^2 \Delta \varepsilon_{\mathrm{eq}} \tag{5}$$

where $\varepsilon_{\mathrm{eq}}$ is an invariant of the local strain tensor, and Δ is the Laplacian operator. For dimensional consistency, a length scale c is included. The chosen invariant $\varepsilon_{\mathrm{eq}}$ reflects the mechanical processes which drive damage in a particular material. Importantly, the form of equation (5) is common to a wide range of strain gradient-dependent continuum models. In this sense, the examined model can be considered a prototype for a range of different models.

The history parameter κ is equal to the largest (positive) value of $\bar{\varepsilon}$ reached at a material point during loading. It is akin to the equivalent plastic strain, and its evolution obeys the Kuhn-Tucker conditions,

$$\dot{\kappa} \geq 0, \qquad f \leq 0, \qquad \dot{\kappa} f = 0 \tag{6}$$

where f is a loading function, $f = \bar{\varepsilon} - \kappa$. Both the invariant of the strain tensor $\varepsilon_{\mathrm{eq}}$, and the dependency of the damage ω on κ reflect properties of the material being modelled. The dependency of ω on κ is typically complex.

Upon insertion of the constitutive model, the fundamental problem is locally fourth-order (in regions where damage is developing). This requires higher-order boundary conditions on these regions. If these boundaries are internal to Ω, boundary conditions do not need to be explicitly supplied as they are implied by continuity (at least weak continuity) between the damaging and undamaged (or unloading) regions. Special attention is however required when the boundary of a damaging region coincides with the boundary of Ω. At this point, extra boundary conditions must be supplied on the boundary Γ_κ, which is defined as:

$$\Gamma_\kappa = \{\boldsymbol{x} \in \Gamma \mid \dot{\kappa} > 0\}. \tag{7}$$

A commonly accepted boundary condition is

$$\nabla \varepsilon_{\mathrm{eq}} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_\kappa. \tag{8}$$

The governing equation is fourth-order only in regions where damage is developing – elsewhere the problem is governed by a second-order PDE. This is a particular point of difficulty for the considered model. Initially, a computation is typically elastic, and at some stage damage will reach the boundary Γ. At this point, extra boundary conditions are required. However, imposition of the boundary conditions in equation (7) will

result in a ‘jump’ in the conditions at the boundary as in the general case, $\nabla \varepsilon_{\text{eq}} \cdot \boldsymbol{n} \neq 0$ prior to the onset of damage. Given that the gradient terms in the considered models are not physically motivated, guidance as to appropriate values for the non-standard boundary conditions is lacking. Gradient-dependent models for which gradient effects are active in the elastic regime may have a decided advantage in this case.
Before proceeding, it is necessary to consider a partition of the domain Ω into finite elements Ω_e such that

$$\bar{\Omega} = \bigcup_{e=1}^{n_{el}} \bar{\Omega}_e. \tag{9}$$

where $\bar{\Omega}_e$ is a closed set (i.e., it includes the boundary of the element). A domain $\tilde{\Omega}$ is also defined

$$\tilde{\Omega} = \bigcup_{e=1}^{n_{el}} \Omega_e \tag{10}$$

where $\tilde{\Omega}$ does not include element boundaries. It is also useful to define the ‘interior’ boundary $\tilde{\Gamma}$,

$$\tilde{\Gamma} = \bigcup_{i=1}^{n_b} \Gamma_i \tag{11}$$

where Γ_i is the ith interior element boundary and n_b is the number of internal inter-element boundaries. The function spaces $\mathscr{S}^h$, $\mathscr{V}^h$ and $\mathscr{W}^h$ are introduced:

$$\mathscr{S}^h = \left\{ u_i^h \in H_0^1(\Omega) \;\middle|\; u_i^h|_{\Omega_e} \in P_{k_1}(\Omega_e) \forall e,\; u_i = g_i \text{ on } \Gamma_g \right\} \tag{12}$$

$$\mathscr{V}^h = \left\{ w_i^h \in H_0^1(\Omega) \;\middle|\; w_i^h|_{\Omega_e} \in P_{k_1}(\Omega_e) \forall e,\; w_i = 0 \text{ on } \Gamma_g \right\} \tag{13}$$

$$\mathscr{W}^h = \left\{ q^h \in L^2(\Omega) \;\middle|\; q^h|_{\Omega_e} \in P_{k_2}(\Omega_e) \forall e \right\} \tag{14}$$

where P_k represents the space of polynomial finite element shape functions (of polynomial order k). The spaces $\mathscr{S}^h$ and $\mathscr{V}^h$ represent usual C^0 continuous finite element shape functions. The space $\mathscr{W}^h$ can contain discontinuous functions.
The difficulty with many strain gradient-dependent continuum model is that the nonlinear nature of the governing equation and the complex dependency on the higher-order derivatives prevents the straightforward application of integration by parts to derive a weak form which involves second-order derivatives, and two natural boundary conditions. Hence, here both equations (1) and (5) are addressed, leading to a coupled set of equations. A Galerkin formulation which allows the solution of the problem using the previously defined function spaces is of the form [12]: find $\boldsymbol{u}^h \in \mathscr{S}^h$ and $\bar{\varepsilon}^h \in \mathscr{W}^h$ such that

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{w}^h : \left(1 - \omega\left(\bar{\varepsilon}^h\right)\right) \mathscr{C} : \nabla^{\text{s}} \boldsymbol{u}^h \, d\Omega - \int_{\Gamma_\kappa} \alpha_2 \nabla w_{\text{eq}}^h \cdot \boldsymbol{n} E c^2 \nabla \varepsilon_{\text{eq}}^h \cdot \boldsymbol{n} \, d\Gamma \\ = \int_{\Gamma_h} \boldsymbol{w}^h \cdot \boldsymbol{h} \, d\Gamma - \int_{\Gamma_\kappa} \alpha_2 \nabla w_{\text{eq}}^h \cdot \boldsymbol{n} E c^2 h_{\nabla \varepsilon} \, d\Gamma \quad \forall \boldsymbol{w}^h \in \mathscr{V}^h \end{aligned} \tag{15}$$

$$\begin{aligned} \int_\Omega q^h \bar{\varepsilon}^h \, d\Omega - \int_\Omega q^h \varepsilon_{\text{eq}}^h \, d\Omega + \int_{\tilde{\Omega}} \nabla q^h \cdot c^2 \nabla \varepsilon_{\text{eq}}^h \, d\Omega - \int_\Gamma q^h \nabla \varepsilon_{\text{eq}}^h \cdot \boldsymbol{n} \, d\Gamma - \int_{\tilde{\Gamma}} \left[\!\left[q^h \right]\!\right] \cdot c^2 \left\langle \nabla \varepsilon_{\text{eq}}^h \right\rangle d\Gamma \\ - \int_{\tilde{\Gamma}} \left\langle \nabla q^h \right\rangle \cdot c^2 \left[\!\left[\varepsilon_{\text{eq}}^h \right]\!\right] d\Gamma + \int_{\tilde{\Gamma}} \frac{\alpha_1 c^2}{h_e} \left[\!\left[q^h \right]\!\right] \cdot \left[\!\left[\varepsilon_{\text{eq}}^h \right]\!\right] d\Gamma = 0 \quad \forall q^h \in \mathscr{W}^h \end{aligned} \tag{16}$$

where α_1 is a penalty-like parameter related to the stabilising term, α_2 is a penalty term for weakly enforcing a non-standard boundary condition and h_e is a measure of element size.
For the numerical results presented herein, the case in which $\bar{\varepsilon}^h \in C^0$ is considered. Hence the space $\mathscr{W}^h$ is defined by:

$$\mathscr{W}^h = \left\{ q^h \in H_0^1(\Omega) \;\middle|\; q^h|_{\Omega_e} \in P_{k_2}(\Omega_e) \forall e \right\}. \tag{17}$$

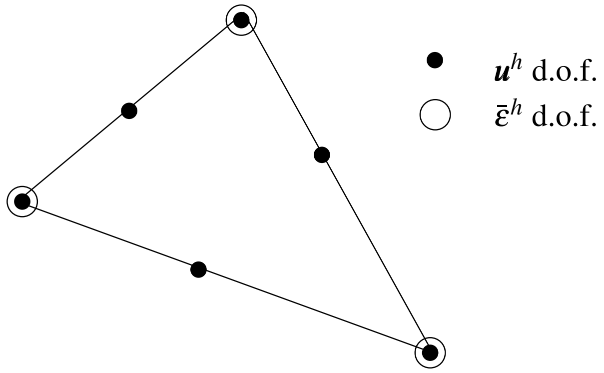

Figure 1: Triangular element and associated degrees of freedom (d.o.f).

This choice leads to a significant simplification of the Galerkin problem as all terms involving $[\![q^h]\!]$ vanish. The Galerkin problem for $\bar{\varepsilon}^h$ is then of the form: find $\bar{\varepsilon}^h \in \mathscr{W}^h$ such that

$$\int_{\Omega} q^h \bar{\varepsilon}^h \, d\Omega - \int_{\Omega} q^h \varepsilon^h_{\text{eq}} \, d\Omega + \int_{\tilde{\Omega}} \nabla q^h \cdot c^2 \nabla \varepsilon^h_{\text{eq}} \, d\Omega - \int_{\Gamma} q^h \nabla \varepsilon^h_{\text{eq}} \cdot \boldsymbol{n} \, d\Gamma \\ - \int_{\tilde{\Gamma}} \left\langle \nabla q^h \right\rangle \cdot c^2 \left[\!\left[\varepsilon^h_{\text{eq}} \right]\!\right] \, d\Gamma = 0 \quad \forall q^h \in \mathscr{W}^h. \quad (18)$$

A particularly attractive feature of this formulation is the penalty-like term in the more general term is not present. Furthermore, this formulation does not require 'double' degrees of freedom for $\bar{\varepsilon}^h$ at element interfaces.

Applying integration by parts to equations (15) and (16) for the case $\alpha_2 = 0$, and identifying the Euler-Lagrange equations, it can be shown that the weak form implies that equations (1)–(3) and equation (5) are satisfied. However, satisfying these equations yields only one boundary condition on $\boldsymbol{x} \in \Gamma$, while as discussed previously, an extra boundary condition is required on Γ_κ. A boundary condition on $\boldsymbol{u}$ can be applied by requiring that functions in the the space $\mathscr{S}^h$ (equation (12)) satisfy the boundary condition, or through the addition of appropriate terms to the weak form. It appears that, through a fortuitous choice of basis functions, the necessary extra boundary conditions can be applied. If $\mathscr{S}^h$ contains piecewise quadratic functions, naturally, the third derivative of the displacement field (and hence $\Delta\varepsilon_{\text{eq}}$) is equal to zero. This is the case for the problems examined in the following section. It is emphasised however that this is not a general solution to the problem of higher-order boundary conditions, rather a fortunate coincidence. It is an issue which requires significant further examination.

NUMERICAL EXAMPLES

The proposed formulation has previously been examined extensively in one dimension for various elements for cases in which damage does not reach the boundary [12–14]. The simplest, most reliable element in one dimension utilised a piecewise quadratic interpolation for $\boldsymbol{u}^h$, and a continuous, piecewise linear interpolation for $\bar{\varepsilon}^h$. Based on this experience, a triangular element with quadratic shape functions for $\boldsymbol{u}^h$ and linear shape functions for $\bar{\varepsilon}^h$ is adopted for the analysis of a beam subjected to three-point bending. The adopted element is illustrated in Figure 1. As discussed in the previous section, the choice of a quadratic basis for $\boldsymbol{u}^h$ implies that $\Delta\varepsilon_{\text{eq}} = 0$ on Γ, hence no boundary conditions on $\nabla\varepsilon_{\text{eq}} \cdot \boldsymbol{n}$ are supplied.

The geometry and boundary conditions for the three-point bending specimen to be tested are shown in Figure 2. For this case, the scalar equivalent strain ε_{eq} is defined as the trace of the strain tensor,

$$\varepsilon_{\text{eq}} = \text{tr}\,(\boldsymbol{\varepsilon})\,. \quad (19)$$

This is not particularly realistic, but for testing purposes a simple invariant dramatically simplifies the lin-

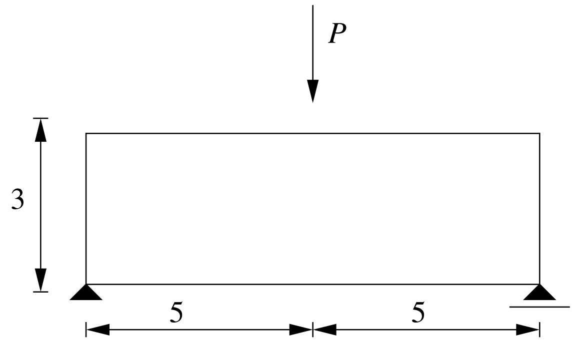

Figure 2: Three-point bending specimen.

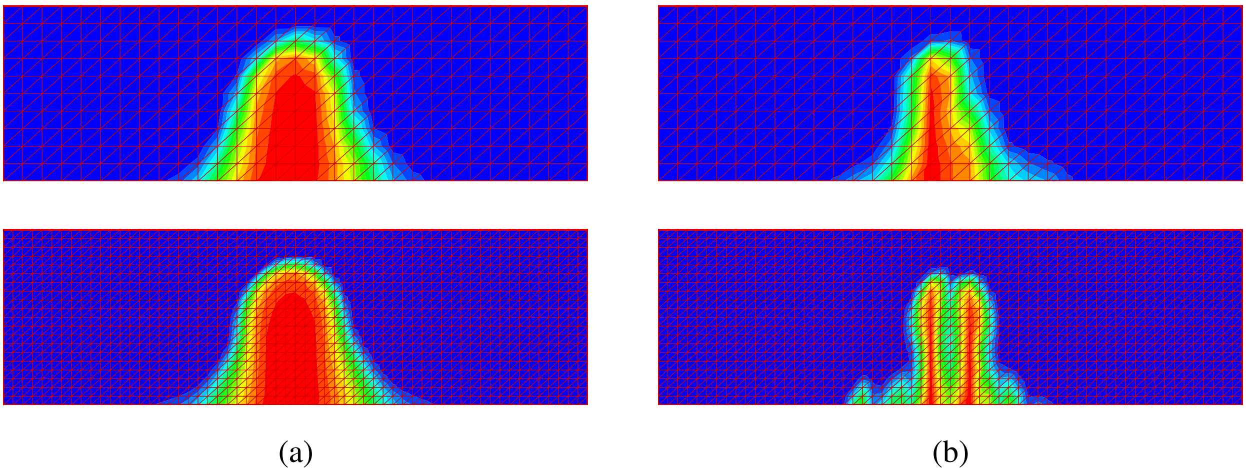

Figure 3: Damage contours for two meshes (a) with gradient effects ($c \neq 0$) and (b) without gradient effects ($c = 0$).

earisation of the equations. Damage evolution is governed by:

$$\omega = \begin{cases} 0 & \text{if } \kappa \leq \kappa_0 \\ 1 - \dfrac{\kappa_0 (\kappa_c - \kappa)}{\kappa (\kappa_c - \kappa_0)} & \text{if } \kappa_0 < \kappa < \kappa_c \\ 1 & \text{if } \kappa \geq \kappa_c \end{cases} \tag{20}$$

where κ_0 is the value of the history parameter at which damage begins to develop and κ_c is the value at which $\omega = 1$. This dependency yields a linear softening response for a one-dimensional test in the absence of strain gradient effects.

For the bending test, the following material properties are adopted: Young's modulus $E = 20 \times 10^4$ MPa, Poisson's ratio $\nu = 0$, $\kappa_0 = 1 \times 10^{-4}$, and $\kappa_c = 1.25 \times 10^{-2}$. For gradient-dependent simulations $c = 8 \times 10^{-2}$ mm. The formulation is tested for two meshes with $c \neq 0$ and $c = 0$. Computations are stopped when damage reaches unity at any point in the mesh. The meshes, with the computed damage contours, are shown in Figure 3. From the damage contours, it is clear that the computed result is similar for the two meshes with $c \neq 0$. In the absence of regularising effects ($c = 0$), the result is clearly affected by the discretisation. The load–displacement responses for the various cases are shown in Figure 4. For the gradient-dependent case, the responses for the two meshes are similar. For the case $c = 0$, the responses are also similar, which is somewhat in contrast to what is normally expected for a strain softening problem. The responses are similar in this case due to the development of two cracks for the finer mesh (in contrast to the single main crack for the coarse mesh). This is evident from the damage contours in Figure 3.

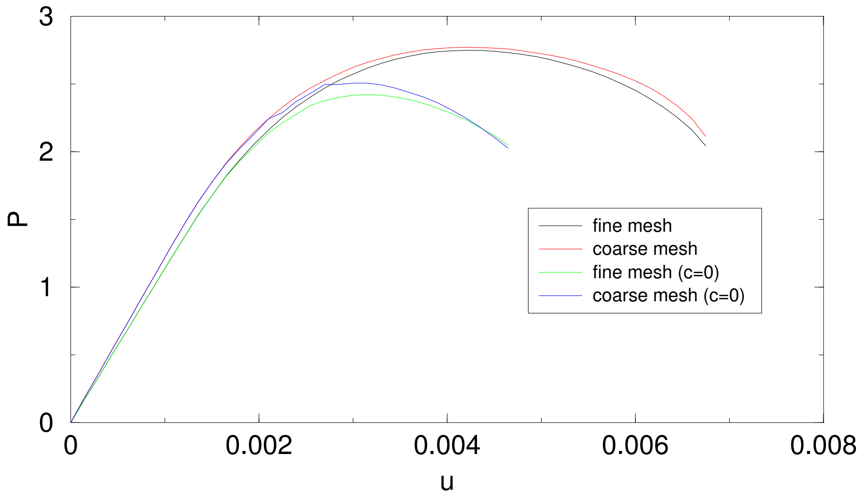

Figure 4: Load–displacement responses for various meshes.

CONCLUSIONS

A continuous/discontinuous Galerkin formulation has been examined for the simulation of damage propagation in two dimensions using a strain gradient dependent model. The approach allows the use of standard C^0 finite element shape functions by taking into account jumps in the strain field across element boundaries in the weak form. While the formulation appears to be effective, a number of issues arise regarding the application of non-standard boundary conditions. This is partly due to the deficiency of the examined model in the sense that the non-standard boundary conditions have no clear physical relevance, and partly due to the complex structure of the governing equations which prevents the standard application of integration by parts to yield two natural boundary conditions. It is likely that this issue will be at least partly resolved by addressing strain gradient-dependent models which are based on physical motivations and have a more natural variational structure.

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