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Bjarne Bergh  ; Robert Salzmann; Nilanjana Datta



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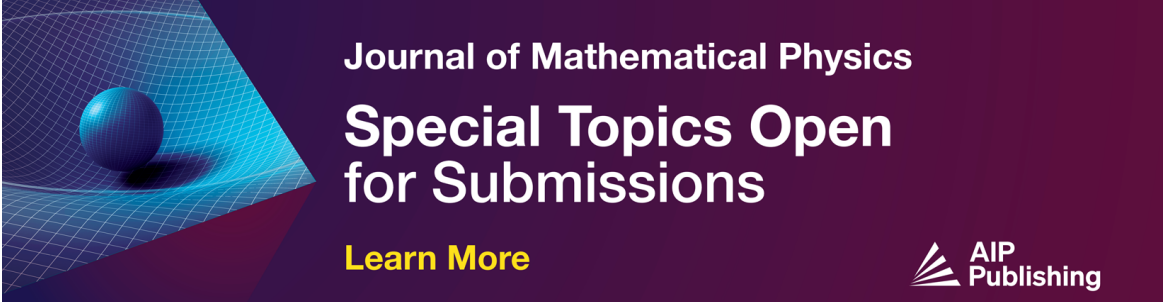

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Bjarne Bergh,^{a)} Robert Salzmann, and Nilanjana Datta

AFFILIATIONS

DAMTP, University of Cambridge, Cambridge, United Kingdom

^{a)} Author to whom correspondence should be addressed: bb536@cam.ac.uk

ABSTRACT

Fawzi and Fawzi [Quantum 5, 387 (2021)] recently defined the sharp Rényi divergence, D_α^\sharp , for $\alpha \in (1, \infty)$, as an additional quantum Rényi divergence with nice mathematical properties and applications in quantum channel discrimination and quantum communication. One of their open questions was the limit $\alpha \rightarrow 1$ of this divergence. By finding a new expression of the sharp divergence in terms of a minimization of the geometric Rényi divergence, we show that this limit is equal to the Belavkin–Staszewski relative entropy. Analogous minimizations of arbitrary generalized divergences lead to a new family of generalized divergences that we call *kringel divergences* for which we prove various properties, including the data-processing inequality.

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I. GEOMETRIC AND SHARP RÉNYI DIVERGENCES

Let \mathcal{H} be a complex finite-dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of linear operators on \mathcal{H} . We write $\mathcal{P}(\mathcal{H})$ for the set of positive semi-definite operators on \mathcal{H} and $\mathcal{P}_+(\mathcal{H})$ for the set of positive definite operators. Let $\mathcal{D}(\mathcal{H})$ denote the set of density matrices, i.e., the set of positive semi-definite operators with trace 1. For $A, B \in \mathcal{P}(\mathcal{H})$, we further write $A \ll B$ if $\text{supp}(A) \subseteq \text{supp}(B)$.

For $A, B \in \mathcal{P}_+(\mathcal{H})$ and $\alpha \in \mathbb{R}$, the *weighted matrix geometric mean* is defined as

$$A\#_\alpha B := A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\alpha A^{\frac{1}{2}}. \quad (1)$$

For $\alpha \geq 0$, this definition can be easily extended to more general $A, B \in \mathcal{P}(\mathcal{H})$, $B \ll A$, by restricting the Hilbert space to the support of A [this also corresponds to directly using pseudo-inverses in (1)]. For $\alpha \in [0, 1]$, the weighted matrix geometric mean satisfies many desirable properties of a matrix mean.²

The geometric Rényi divergence (also called the maximal Rényi divergence), first introduced by Matsumoto,³ can be defined in terms of the weighted matrix geometric mean. For $\rho, \sigma \in \mathcal{P}(\mathcal{H})$, $\rho \ll \sigma$, and $\alpha \in (0, \infty)$, define the *geometric trace function* as

$$\widehat{Q}_\alpha(\rho \parallel \sigma) := \text{Tr}(\sigma\#_\alpha \rho) = \text{Tr} \left(\sigma^{\frac{1}{2}} \left(\sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \sigma^{\frac{1}{2}} \right). \quad (2)$$

This can be extended to general $\rho, \sigma \in \mathcal{P}(\mathcal{H})$ by setting

$$\widehat{Q}_\alpha(\rho \parallel \sigma) := \lim_{\varepsilon \rightarrow 0} \widehat{Q}_\alpha(\rho \parallel \sigma + \varepsilon \mathbb{1}). \quad (3)$$

For $\alpha > 1$, this limit is equal to $+\infty$ if $\rho \not\ll \sigma$, whereas for $\alpha \in (0, 1]$, the limit is always finite and explicit expressions for it can be found in Refs. 3–5. For $\alpha \in (0, 1) \cup (1, \infty)$, one then defines the *geometric Rényi divergence* as^{3,5–7}

$$\widehat{D}_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \widehat{Q}_\alpha(\rho \parallel \sigma). \quad (4)$$

It reduces to the classical α -Rényi divergence for commuting states and satisfies the data-processing inequality for $\alpha \in (0, 1) \cup (1, 2]$.³ Furthermore, it is known^{3,5,8} that for $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{P}(\mathcal{H})$,

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho \| \sigma) = \widehat{D}(\rho \| \sigma) := \text{Tr}(\rho \log(\rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}})), \quad (5)$$

the Belavkin–Staszewski relative entropy.⁹

In Ref. 1, Fawzi and Fazwi defined the sharp trace function for $\alpha \in (1, \infty)$,

$$Q_\alpha^\#(\rho \| \sigma) = \min_{A \geq 0} \left\{ \text{Tr} A \mid \rho \leq \sigma \#_{\frac{1}{\alpha}} A \right\}. \quad (6)$$

They defined the sharp Rényi divergence of order α in terms of it as follows:

$$D_\alpha^\#(\rho \| \sigma) := \frac{1}{\alpha - 1} \log(Q_\alpha^\#(\rho \| \sigma)) \quad (7)$$

and proved that it satisfies the data-processing inequality and that it also reduces to the classical α -Rényi divergence for commuting states. Furthermore, they showed that this divergence has several desirable computational properties, such as an efficient semidefinite programming representation for states and quantum channels (i.e., linear, completely positive trace-preserving maps), and a crucial chain rule property that can be exploited to obtain important information-theoretic results concerning quantum channel discrimination and quantum channel capacities.

A natural question to ask is as follows: *What is $\lim_{\alpha \rightarrow 1} D_\alpha^\#(\rho \| \sigma)$?*

This was left as an open question in Ref. 1. In this paper, we answer this question by proving that this limit is given by the Belavkin–Staszewski relative entropy (see Theorem 7).

To address the limit of $\alpha \rightarrow 1$, we want to make use of the following alternative characterization of the sharp trace function and sharp Rényi divergence:

Proposition 1. For $\alpha \in (1, \infty)$ and $\rho, \sigma \in \mathcal{P}(\mathcal{H})$,

$$Q_\alpha^\#(\rho \| \sigma) = \min_{A \geq \rho} \widehat{Q}_\alpha(A \| \sigma), \quad (8)$$

$$D_\alpha^\#(\rho \| \sigma) = \min_{A \geq \rho} \widehat{D}_\alpha(A \| \sigma). \quad (9)$$

Remark 2. Note that the A in the above expressions are, in general, unnormalized states and we use the definitions of the geometric trace function and the geometric Rényi divergence as in (2) and (4) without additional normalization factors.

For what is going to follow, we prove a slightly stronger version of the above statement, given by the following lemma:

Lemma 3. For $\alpha \in (1, a]$, with $a \in (1, \infty)$, and $\rho, \sigma \in \mathcal{P}(\mathcal{H})$,

$$Q_\alpha^\#(\rho \| \sigma) = \min_{A \geq \rho} \widehat{Q}_\alpha(A \| \sigma), \quad (10)$$

and the minimization can be restricted to $A \in K$, where K is a compact subset of $\{A \in \mathcal{P}(\mathcal{H}) \mid A \ll \sigma\}$ depending only on ρ, σ, a but not on α .

Proof. Recall the definition of $Q_\alpha^\#(\rho \| \sigma)$ for $\alpha > 1$,¹

$$Q_\alpha^\#(\rho \| \sigma) = \min_{A \geq 0} \left\{ \text{Tr} A \mid \rho \leq \sigma^{\frac{1}{2}} \left(\sigma^{-\frac{1}{2}} A \sigma^{-\frac{1}{2}} \right)^{\frac{1}{\alpha}} \sigma^{\frac{1}{2}} \right\}. \quad (11)$$

Fawzi and Fawzi¹ showed that the minimization can be further restricted to $0 \leq A \leq c^{\alpha-1} \text{Tr}(\rho) \Pi_\sigma$, where $c = \left\| \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right\|$ is the spectral norm and Π_σ is the projection onto the support of σ . It is easy to see that if $\rho \not\ll \sigma$, the minimization is infeasible, and so $Q_\alpha^\#(\rho \| \sigma) = +\infty$. Moreover, if $\rho \ll \sigma$ and $A \geq \rho$, then also $A \ll \sigma$, and hence, $\min_{A \geq \rho} \widehat{Q}_\alpha(A \| \sigma) = +\infty$ and the statement holds. Thus, we can assume $\rho \ll \sigma$. Taking $C := \text{Tr}(\rho) \sup_{\alpha \in [1, a]} c^{\alpha-1}$, which is always finite, we get

$$Q_\alpha^\#(\rho\|\sigma) = \min_{A \geq 0} \left\{ \text{Tr} A \mid \rho \leq \sigma^{\frac{1}{2}} \left(\sigma^{-\frac{1}{2}} A \sigma^{-\frac{1}{2}} \right)^{\frac{1}{\alpha}} \sigma^{\frac{1}{2}}, A \leq C \Pi_\sigma \right\} \quad (12)$$

$$= \min_{A \geq 0} \left\{ \text{Tr} \left(\sigma^{\frac{1}{2}} A \sigma^{\frac{1}{2}} \right) \mid \rho \leq \sigma^{\frac{1}{2}} A^{\frac{1}{\alpha}} \sigma^{\frac{1}{2}}, A \leq C \sigma^{-\frac{1}{2}} \Pi_\sigma \sigma^{-\frac{1}{2}} \leq C \|\sigma^{-1}\| \Pi_\sigma \right\} \quad (13)$$

$$= \min_{A \geq 0} \left\{ \text{Tr} \left(\sigma^{\frac{1}{2}} A^\alpha \sigma^{\frac{1}{2}} \right) \mid \rho \leq \sigma^{\frac{1}{2}} A \sigma^{\frac{1}{2}}, A^\alpha \leq C \|\sigma^{-1}\| \Pi_\sigma \right\}, \quad (14)$$

where we redefined A multiple times and understand $\|\sigma^{-1}\|$ as the operator norm of the pseudo-inverse. While for general matrices $A, B \geq 0$, the statements $A^\alpha \leq B$ and $A \leq B^{\frac{1}{\alpha}}$ are not equivalent, they are equivalent in the case in which B is a constant times a projector onto a subspace that includes the support of A . This is because in this case, B is diagonal in the same basis in which A (and hence also A^α) is diagonal. Hence, the operator inequality $A^\alpha \leq B$ turns into a condition only on the eigenvalues of A , which is then equivalent to $A \leq B^{\frac{1}{\alpha}}$. Thus, we finally get

$$Q_\alpha^\#(\rho\|\sigma) = \min_{A \geq 0} \left\{ \text{Tr} \left(\sigma^{\frac{1}{2}} A^\alpha \sigma^{\frac{1}{2}} \right) \mid \rho \leq \sigma^{\frac{1}{2}} A \sigma^{\frac{1}{2}}, A \leq (C \|\sigma^{-1}\|)^{\frac{1}{\alpha}} \Pi_\sigma \right\} \quad (15)$$

$$= \min_{A \geq 0} \left\{ \widehat{Q}_\alpha(A\|\sigma) \mid \rho \leq A, A \leq (C \|\sigma^{-1}\|)^{\frac{1}{\alpha}} \sigma^{\frac{1}{2}} \Pi_\sigma \sigma^{\frac{1}{2}} \leq (C \|\sigma^{-1}\|)^{\frac{1}{\alpha}} \|\sigma\| \Pi_\sigma \right\}. \quad (16)$$

With $\tilde{C} := \sup_{\alpha \in [1, \alpha]} (C \|\sigma^{-1}\|)^{\frac{1}{\alpha}} \|\sigma\|$, which exists, and $K := \{A \in \mathcal{P}(\mathcal{H}) \mid \rho \leq A \leq \tilde{C} \Pi_\sigma\}$, which is compact, we have for all $\alpha \in (1, b]$

$$Q_\alpha^\#(\rho\|\sigma) = \min_{A \in K} \widehat{Q}_\alpha(A\|\sigma). \quad (17)$$

□

Since the logarithm on \mathbb{R} is monotone, Lemma 3 implies that also $D_\alpha^\#(\rho\|\sigma) = \min_{A \geq \rho} \widehat{D}_\alpha(A\|\sigma)$ for all $\alpha > 1$. This completes the Proof of Proposition 1. Note that for $\alpha > 2$, the geometric divergence does not satisfy the data-processing inequality, while the sharp divergence does.

II. THE $\alpha \rightarrow 1$ LIMIT

We show in Theorem 7 that the limit $\alpha \rightarrow 1$ of the sharp divergence is the Belavkin–Staszewski relative entropy. The key step in the proof is to show that $\frac{d}{d\alpha} Q_\alpha^\#(\rho\|\sigma)|_{\alpha=1}$ exists and is equal to $\frac{d}{d\alpha} \widehat{Q}_\alpha(\rho\|\sigma)|_{\alpha=1}$. One way to establish this is to use the following theorem [Ref. 10, Theorem 2.2.1]:

Theorem 4 (Ref. 10). *Let $U \subset \mathbb{R}^n$ be open, $K \subset \mathbb{R}^m$ be compact, and $f : U \times K \rightarrow \mathbb{R}$ be continuous and also $\nabla_u f(u, k)$ be (jointly) continuous. Then, the function $g(u) = \min_{k \in K} f(u, k)$ has for every $u \in U$ a one-sided directional derivative along every $v \in \mathbb{R}^n$, which can be computed as*

$$\lim_{t \searrow 0} \frac{g(u + vt) - g(u)}{t} = \min_{k \in R(u)} \langle \nabla_u f(u, k), v \rangle, \quad (18)$$

where $R(u) = \{k \in K \mid f(u, k) = g(u)\}$.¹⁸

The following lemmas establish the properties of \widehat{Q}_α , which are necessary in order to apply Theorem 4:

Lemma 5. *Let Ω be a subset of \mathbb{R}^n and $h : \Omega \times [0, \infty) \rightarrow [0, \infty)$ be a jointly continuous function. Then, the corresponding matrix function $h : \Omega \times \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H})$ is jointly continuous on $\Omega \times \mathcal{P}(\mathcal{H})$.*

Proof. For a fixed $\alpha \in \Omega$, the continuity of the matrix function $h(\alpha, \rho)$ in ρ follows from the continuity of $h(\alpha, x)$ in x , for $x \in [0, \infty)$, by Ref. 11 (Theorem 6.2.37). To see joint continuity, let $\rho_n \rightarrow \rho$ be a converging sequence in $\mathcal{P}(\mathcal{H})$ and $\alpha_n \rightarrow \alpha$ be a converging sequence in Ω , as $n \rightarrow \infty$. Then, taking operator norms,

$$\|h(\alpha_n, \rho_n) - h(\alpha, \rho)\| \leq \|h(\alpha_n, \rho_n) - h(\alpha, \rho_n)\| + \|h(\alpha, \rho_n) - h(\alpha, \rho)\|. \quad (19)$$

The second term on the right-hand side goes to zero as $n \rightarrow \infty$ by the continuity of $h(\alpha, \rho)$ in ρ . For the first term, note that $h(\alpha_n, \rho_n)$ and $h(\alpha, \rho_n)$ commute and the operator norm can be evaluated as

$$\max_{\lambda \in \text{spec}(\rho_n)} |h(\alpha_n, \lambda) - h(\alpha, \lambda)|. \quad (20)$$

Since the limit $\alpha_n \rightarrow \alpha$ exists, α_n is eventually bounded, so there exists a compact subset K of Ω such that $\alpha_n \in K$ for all large enough n . Analogously, the limit $\rho_n \rightarrow \rho$ exists, and so the spectrum of ρ_n is eventually bounded, and hence, there exists a compact subset K' of $[0, \infty)$

such that $\text{spec}(\rho_n) \subset K'$ for all large enough n . Since $K \times K'$ is compact, h is, in fact, uniformly continuous on $K \times K'$. Hence, the expression in (20) goes to zero as $n \rightarrow \infty$ uniformly in all elements of the maximum and also uniformly in ρ_n . This shows that

$$\lim_{n \rightarrow \infty} \|h(\alpha_n, \rho_n) - h(\alpha, \rho)\| = 0, \tag{21}$$

which completes the proof. \square

Lemma 6. For $\sigma \in \mathcal{P}(\mathcal{H})$ fixed, the functions $f_\rho(\alpha) = f(\alpha, \rho) = \widehat{Q}_\alpha(\rho\|\sigma)$ and $f'_\rho(\alpha) = f'(\alpha, \rho) = \frac{d}{d\alpha}\widehat{Q}_\alpha(\rho\|\sigma)$ are jointly continuous on $(0, \infty) \times \{A \in \mathcal{P}(\mathcal{H}) | A \ll \sigma\}$.

Proof. We have

$$\widehat{Q}_\alpha(\rho\|\sigma) = \text{Tr}\left(\sigma^{\frac{1}{2}}\left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)^\alpha\sigma^{\frac{1}{2}}\right), \tag{22}$$

$$\frac{d}{d\alpha}\widehat{Q}_\alpha(\rho\|\sigma) = \text{Tr}\left(\sigma^{\frac{1}{2}}\log\left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)\left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)^\alpha\sigma^{\frac{1}{2}}\right). \tag{23}$$

It is easy to check that the real functions $(\alpha, x) \mapsto x^\alpha$ and $(\alpha, x) \mapsto x^\alpha \log(x)$ are jointly continuous on $(0, \infty) \times [0, \infty)$. In addition, for $A \in \{A \in \mathcal{P}(\mathcal{H}) | A \ll \sigma\}$, the expression $\sigma^{-\frac{1}{2}}A\sigma^{-\frac{1}{2}}$ is continuous in A (remember that the inverses are all pseudo-inverses). Hence, the continuity follows from Lemma 5, the continuity of the matrix product, and the continuity of the trace. \square

Theorem 7. For $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{P}(\mathcal{H})$,

$$\lim_{\alpha \rightarrow 1} D_\alpha^\#(\rho\|\sigma) = \widehat{D}(\rho\|\sigma), \tag{24}$$

the Belavkin–Staszewski relative entropy.

Proof. As the sharp Rényi divergence, $D_\alpha^\#$, is only defined for $\alpha > 1$, all there is to prove is a limit from above. Let us fix $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{P}(\mathcal{H})$. We can restrict our proof to the case $\rho \ll \sigma$, since otherwise $D_\alpha^\#(\rho\|\sigma) = \widehat{D}_\alpha(\rho\|\sigma) = \infty$ for all $\alpha > 1$ and the statement clearly holds. By Lemma 3, there exists a compact set $K \subseteq \{A \in \mathcal{P}(\mathcal{H}) | A \ll \sigma\}$ ¹⁹ such that for $\alpha \in (1, 2]$,

$$Q_\alpha^\#(\rho\|\sigma) = \min_{A \in K} \widehat{Q}_\alpha(A\|\sigma). \tag{25}$$

For $\alpha \in (0, \infty)$,²⁰ we define

$$f(\alpha, A) := \widehat{Q}_\alpha(A\|\sigma), \tag{26}$$

$$g(\alpha) := \min_{A \in K} f(\alpha, A) = \min_{A \in K} \widehat{Q}_\alpha(A\|\sigma). \tag{27}$$

By Lemma 6, $f(\alpha, A)$ is jointly continuous on $(0, \infty) \times K$ and has a jointly continuous derivative in α . Hence, by Theorem 4, $g(\alpha)$ has the following one-sided derivatives:

$$\lim_{\alpha \searrow 1} \frac{g(\alpha) - g(1)}{\alpha - 1} = \min_{A \in R(\alpha)} \frac{d}{d\alpha} f(\alpha, A), \tag{28}$$

$$\lim_{\alpha \nearrow 1} \frac{g(\alpha) - g(1)}{\alpha - 1} = - \min_{A \in R(\alpha)} \left(- \frac{d}{d\alpha} f(\alpha, A) \right), \tag{29}$$

where $R(\alpha) := \{A \in K | f(\alpha, A) = \min_{B \in K} f(\alpha, B)\}$. Recall that the sharp Rényi divergence is only defined for $\alpha > 1$, and so all we really need here is the limit from above (28). However, establishing that this is also equal to the limit from below makes things slightly simpler, as we are then able to directly use the chain rule later in (32).

For $\alpha = 1$, the set $R(\alpha)$ only contains ρ , since

$$\min_{A \geq \rho} \widehat{Q}_1(A\|\sigma) = \min_{A \geq \rho} \text{Tr}(A) = \text{Tr}(\rho), \tag{30}$$

and $A \geq \rho$, together with $\text{Tr}(A) = \text{Tr}(\rho)$, imply that $A = \rho$. Hence, the two one-sided derivatives are equal and

$$\left. \frac{d}{d\alpha} g(\alpha) \right|_{\alpha=1} = \left. \frac{d}{d\alpha} f(\alpha, \rho) \right|_{\alpha=1} = \left. \frac{d}{d\alpha} \widehat{Q}_\alpha(\rho\|\sigma) \right|_{\alpha=1}. \tag{31}$$

We further have $g(\alpha) = Q_\alpha^\#(\rho\|\sigma)$ for $\alpha \in (1, 2]$ and $g(1) = \text{Tr}(\rho) = 1$, so

$$\begin{aligned} \lim_{\alpha \searrow 1} D_\alpha^\#(\rho\|\sigma) &= \lim_{\alpha \searrow 1} \frac{\log Q_\alpha^\#(\rho\|\sigma)}{\alpha - 1} = \lim_{\alpha \searrow 1} \frac{\log g(\alpha) - \log g(1)}{\alpha - 1} \\ &= \left. \frac{d}{d\alpha} \log g(\alpha) \right|_{\alpha=1} = \left. \frac{d}{d\alpha} g(\alpha) \right|_{\alpha=1} = \left. \frac{d}{d\alpha} g(\alpha) \right|_{\alpha=1}. \end{aligned} \tag{32}$$

As the same argument also gives

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho\|\sigma) = \left. \frac{d}{d\alpha} \widehat{Q}_\alpha(\rho\|\sigma) \right|_{\alpha=1}, \tag{33}$$

by using (31), we get

$$\lim_{\alpha \rightarrow 1} D_\alpha^\#(\rho\|\sigma) = \lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho\|\sigma) = \widehat{D}(\rho\|\sigma). \tag{34}$$

□

III. KRINGEL DIVERGENCES AND THEIR PROPERTIES

A key ingredient of our main result, Theorem 7, was Proposition 1, which allowed us to express the sharp divergence as a minimization of the geometric Rényi divergence. Analogous minimizations of arbitrary generalized divergences lead to an interesting new family of generalized divergences, which we call *kringel divergences*.²¹ We introduce them in this section and prove some of their properties, including the data-processing inequality.

For a function $\mathbf{D} : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, we define for $\rho, \sigma \in \mathcal{P}(\mathcal{H})$

$$\mathbf{D}^\circ(\rho\|\sigma) = \inf_{A \geq \rho} \mathbf{D}(A\|\sigma). \tag{35}$$

Moreover, we say $\mathbf{D} : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is a *generalized divergence* if it satisfies the data-processing inequality, i.e., for any $\rho, \sigma \in \mathcal{P}(\mathcal{H})$ and any quantum channel \mathcal{N} , we have

$$\mathbf{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \leq \mathbf{D}(\rho\|\sigma). \tag{36}$$

Remark 8. Note that the above definition of the generalized divergence is an extension of the standard definition (see, e.g., in Ref. 12). In the latter, the generalized divergence is considered as a function $\mathbf{D} : \mathcal{D}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{\infty\}$, where $\mathcal{D}(\mathcal{H})$ denotes the set of density matrices (quantum states) on \mathcal{H} .

The following lemma shows that if \mathbf{D} is a generalized divergence then so is \mathbf{D}° . In this case, we call \mathbf{D}° the *kringel divergence* of \mathbf{D} :

Lemma 9 (Data-processing inequality). *Let \mathbf{D} be a generalized divergence, $\rho, \sigma \in \mathcal{P}(\mathcal{H})$, and \mathcal{N} be a quantum channel. Then,*

$$\mathbf{D}^\circ(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \leq \mathbf{D}^\circ(\rho\|\sigma). \tag{37}$$

Proof. We have

$$\begin{aligned} \mathbf{D}^\circ(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) &= \inf_{A \geq \mathcal{N}(\rho)} \mathbf{D}(A\|\mathcal{N}(\sigma)) \leq \inf_{\mathcal{N}(A) \geq \mathcal{N}(\rho)} \mathbf{D}(\mathcal{N}(A)\|\mathcal{N}(\sigma)) \\ &\leq \inf_{\mathcal{N}(A) \geq \mathcal{N}(\rho)} \mathbf{D}(A\|\sigma) \leq \inf_{A \geq \rho} \mathbf{D}(A\|\sigma) = \mathbf{D}^\circ(\rho\|\sigma). \end{aligned} \tag{38}$$

Here, for the first inequality, we have used the fact that the minimum increases when we only optimize over operators of the form $\mathcal{N}(A)$, and for the second inequality, we have used the data-processing inequality for \mathbf{D} . For the third inequality, we have used the fact that $A \geq \rho$ implies $\mathcal{N}(A) \geq \mathcal{N}(\rho)$, which follows by positivity and linearity of \mathcal{N} . □

Remark 10. By Proposition 1, we know that for $\alpha \in (1, \infty)$, we have

$$\widehat{D}_\alpha^\circ = D_\alpha^\#, \tag{39}$$

where the left-hand side denotes the kringel divergence corresponding to the geometric Rényi divergence (4). Note that even though the geometric Rényi divergence satisfies the data-processing inequality only for $\alpha \in (0, 1) \cup (1, 2]$, by [Ref. 1, Proposition 3.2], its corresponding kringel divergence actually satisfies the data-processing inequality for all $\alpha \in (0, 1) \cup (1, \infty)$.

Lemma 11. If \mathbf{D} is subadditive, then so is \mathbf{D}° , i.e., for all $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathcal{P}(\mathcal{H})$, we have

$$\mathbf{D}^\circ(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) \leq \mathbf{D}^\circ(\rho_1 \| \sigma_1) + \mathbf{D}^\circ(\rho_2 \| \sigma_2). \tag{40}$$

Proof. We have

$$\begin{aligned} \mathbf{D}^\circ(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) &= \inf_{A \geq \rho_1 \otimes \rho_2} \mathbf{D}(A \| \sigma_1 \otimes \sigma_2) \leq \inf_{A_1 \otimes A_2 \geq \rho_1 \otimes \rho_2} \mathbf{D}(A_1 \otimes A_2 \| \sigma_1 \otimes \sigma_2) \\ &\leq \inf_{A_1 \otimes A_2 \geq \rho_1 \otimes \rho_2} (\mathbf{D}(A_1 \| \sigma_1) + \mathbf{D}(A_2 \| \sigma_2)) \\ &= \inf_{A_1 \geq \rho_1} \mathbf{D}(A_1 \| \sigma_1) + \inf_{A_2 \geq \rho_2} \mathbf{D}(A_2 \| \sigma_2) \\ &= \mathbf{D}^\circ(\rho_1 \| \sigma_1) + \mathbf{D}^\circ(\rho_2 \| \sigma_2). \end{aligned} \tag{41}$$

The third line above follows from the fact that $A_1 \geq \rho_1$ and $A_2 \geq \rho_2$ imply that $A_1 \otimes A_2 \geq \rho_1 \otimes \rho_2$ because $A_1 \otimes A_2 - \rho_1 \otimes \rho_2 = (A_1 - \rho_1) \otimes A_2 + \rho_1 \otimes (A_2 - \rho_2)$. \square

A. Kringel divergences for α -Rényi divergences

We say $\mathbf{D}_\alpha : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is a quantum generalization of the α -Rényi divergence if it reduces to the corresponding classical Rényi divergence if both entries commute. That is, if $\rho, \sigma \in \mathcal{P}(\mathcal{H})$ are commuting operators, then

$$\mathbf{D}_\alpha(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \left(\sum_i p_i^\alpha q_i^{1-\alpha} \right), \tag{42}$$

where the $\{p_i\}_i$ and $\{q_i\}_i$ are eigenvalues of ρ and σ , respectively, with respect to a simultaneous eigenbasis. The following lemma states that, for $\alpha > 1$, if \mathbf{D}_α is a quantum generalization of the Rényi relative entropy, then \mathbf{D}_α° too reduces to the classical Rényi divergence in the commuting case and hence is itself a quantum generalization of the Rényi relative entropy:

Lemma 12. Let $\alpha > 1$ and \mathbf{D}_α be a quantum generalization of the α -Rényi divergence satisfying the data-processing inequality (36). Then, also \mathbf{D}_α° is a quantum generalization of the α -Rényi divergence.

Proof. Let $\rho, \sigma \in \mathcal{P}(\mathcal{H})$ such that $[\rho, \sigma] = 0$. Then, there exists a simultaneous eigenbasis $\{|i\rangle\}_i$ of ρ and σ such that

$$\rho = \sum_i p_i |i\rangle\langle i|, \quad \sigma = \sum_i q_i |i\rangle\langle i|. \tag{43}$$

Clearly,

$$\mathbf{D}_\alpha^\circ(\rho \| \sigma) \leq \mathbf{D}_\alpha(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \left(\sum_i p_i^\alpha q_i^{1-\alpha} \right). \tag{44}$$

For the reversed inequality, let \mathcal{P} be the pinching map defined as $\mathcal{P}(\cdot) := \sum_i |i\rangle\langle i| \cdot |i\rangle\langle i|$. Hence, denoting for $A \geq \rho$ the diagonal entries by $a_i = \langle i|A|i\rangle \geq p_i$, we see

$$\begin{aligned} \mathbf{D}_\alpha^\circ(\rho \| \sigma) &= \inf_{A \geq \rho} \mathbf{D}_\alpha(A \| \sigma) \geq \inf_{A \geq \rho} \mathbf{D}_\alpha(\mathcal{P}(A) \| \mathcal{P}(\sigma)) \\ &= \inf_{A \geq \rho} \frac{1}{\alpha - 1} \log \left(\sum_i a_i^\alpha q_i^{1-\alpha} \right) = \frac{1}{\alpha - 1} \log \left(\sum_i p_i^\alpha q_i^{1-\alpha} \right), \end{aligned} \tag{45}$$

where we have used the data-processing inequality for \mathbf{D}_α in the first inequality. \square

Remark 13. Note that for $\alpha \in (0, 1)$ and \mathbf{D}_α being a quantum generalization of the α -Rényi divergence, one easily sees that

$$\mathbf{D}_\alpha^\circ(\rho \| \sigma) = -\infty \tag{46}$$

for all $\rho, \sigma \in \mathcal{P}(\mathcal{H})$ with $\sigma \neq 0$, which is why we excluded this range of α in Lemma 12 and Proposition 14.

Important quantum generalizations of the α -Rényi divergence include the geometric- and sharp Rényi divergences, as well as the Petz Rényi divergence (D_α)¹³ and the sandwiched Rényi divergence (\tilde{D}_α).^{14,15} The latter two are defined, respectively, as follows: for $\rho, \sigma \in \mathcal{P}(\mathcal{H})$,

$$\begin{aligned}
 D_\alpha(\rho\|\sigma) &:= \frac{1}{\alpha-1} \log \text{Tr}(\rho^\alpha \sigma^{1-\alpha}), \\
 \tilde{D}_\alpha(\rho\|\sigma) &:= \frac{1}{\alpha-1} \log \text{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha
 \end{aligned}
 \tag{47}$$

if $\alpha \in (0, 1)$ or $\alpha \in (1, \infty)$ and $\rho \ll \sigma$. If $\alpha \in (1, \infty)$ and $\rho \not\ll \sigma$, then $D_\alpha(\rho\|\sigma) = \tilde{D}_\alpha(\rho\|\sigma) = \infty$. Both the Petz- and the sandwiched Rényi divergences are additive. Moreover, D_α for $\alpha \in (0, 1) \cup (1, 2]$ and \tilde{D}_α for $\alpha \in [1/2, 1) \cup (1, \infty)$ satisfy the data-processing inequality (36), as was shown in Refs. 13, 16, and 17, respectively. Therefore, both of these divergences (in the corresponding ranges of α) can be used as \mathbf{D}_α in the following proposition:

Proposition 14. Let $\alpha > 1$ and \mathbf{D}_α be a quantum generalization of the α -Rényi divergence, which is subadditive and satisfies the data-processing inequality (36). Then,

$$\tilde{D}_\alpha \leq \mathbf{D}_\alpha^\circ \leq D_\alpha^\#
 \tag{48}$$

In particular, this gives for any $\rho, \sigma \in \mathcal{P}(\mathcal{H})$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{D}_\alpha^\circ(\rho^{\otimes n} \|\sigma^{\otimes n}) = \tilde{D}_\alpha(\rho\|\sigma).
 \tag{49}$$

Moreover, in the case of the sandwiched divergence,

$$\tilde{D}_\alpha^\circ = \tilde{D}_\alpha.
 \tag{50}$$

Proof. As \mathbf{D}_α satisfies the data-processing inequality, we know that $\mathbf{D}_\alpha \leq \tilde{D}_\alpha$; this follows from the argument [Ref. 8, Sec. 4.2.3] (also see Ref. 3 where the argument originally appeared).²² This gives

$$\mathbf{D}_\alpha^\circ(\rho\|\sigma) = \min_{A \geq \rho} \mathbf{D}_\alpha(A\|\sigma) \leq \min_{A \geq \rho} \tilde{D}_\alpha(A\|\rho) = \tilde{D}_\alpha^\circ(\rho\|\sigma) = D_\alpha^\#(\rho\|\sigma).
 \tag{51}$$

Moreover, as $\alpha > 1$ and additionally \mathbf{D}_α is subadditive, Lemmas 9, 11, and 12 give that also \mathbf{D}_α° is a subadditive quantum generalization of the α -Rényi divergence satisfying the data-processing inequality. Therefore, by the argument in [Ref. 8, Sec. 4.2.2] (note that actually only subadditivity instead of additivity is used there), we have

$$\tilde{D}_\alpha \leq \mathbf{D}_\alpha^\circ,
 \tag{52}$$

which gives (48). Since, trivially, $\tilde{D}_\alpha^\circ \leq \tilde{D}_\alpha$, (50) follows immediately.

Finally, using $\alpha > 1$ again and Ref. 1, Proposition 3.4 gives for any $\rho, \sigma \in \mathcal{P}(\mathcal{H})$

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_\alpha^\#(\rho^{\otimes n} \|\sigma^{\otimes n}) = \tilde{D}_\alpha(\rho\|\sigma),
 \tag{53}$$

which together with the additivity of the sandwiched Rényi divergence and (48) gives (49). □

IV. SUMMARY AND OPEN QUESTIONS

By finding a new expression of the sharp divergence in terms of a minimization of the geometric α -Rényi divergence, we have shown that its $\alpha \rightarrow 1$ limit is equal to the Belavkin–Staszewski relative entropy. Applying this minimization procedure to arbitrary generalized α -Rényi divergences yields a new family of divergences that we call *kringel divergences*. We derive various properties of these divergences, including the data-processing inequality, and show that similar to the sharp divergences, their regularization is also given by the sandwiched Rényi divergence. It should be worthwhile to investigate whether the new kringel divergences can find similar mathematical use in the analysis of quantum information-theoretic problems, such as the sharp divergence.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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- ¹⁸In Ref. 10, the theorem is phrased with a maximum instead of a minimum, but it is easy to see that this is equivalent upon setting $f \mapsto -f$.
- ¹⁹ K depends on ρ and σ , but since they are fixed in the entire proof, we do not make this dependence explicit.
- ²⁰To apply Theorem 4, we require an open interval in α that includes 1.
- ²¹Since the symbols D , \tilde{D}_α , \widehat{D}_α , D_α^b , and D_α^f all refer to existing divergences and generalized divergences are usually denoted as \mathbf{D} , we use the symbol \mathbf{D}° to denote this new family, hence the name *kringel divergences* (kringel means circle in German).
- ²²Note that in Refs. 3 and 8, the argument is presented for the case in which ρ and σ are states. However, it can be easily seen that, by a slight modification, the argument also works for the case $\rho, \sigma \in \mathcal{P}(\mathcal{H})$.