Accepted: 2 April 2025

DOI: 10.1002/cpa.22256

RESEARCH ARTICLE

Boundary conditions and universal finite-size scaling for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher

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Funding information
NSERC; European Research Council, Grant/Award Number: 851682 SPINRG

Abstract

We analyse and clarify the finite-size scaling of the weakly-coupled hierarchical n-component $|\varphi|^4$ model for all integers $n \geq 1$ in all dimensions $d \geq 4$, for both free and periodic boundary conditions. For $d > 4$, we prove that for a volume of size R^d with periodic boundary conditions the infinite-volume critical point is an effective finite-volume critical point, whereas for free boundary conditions the effective critical point is shifted smaller by an amount of order R^{-2}. For both boundary conditions, the average field has the same non-Gaussian limit within a critical window of width $R^{-d/2}$ around the effective critical point, and in that window we compute the universal scaling profile for the susceptibility. In contrast, and again for both boundary conditions, the average field has a massive Gaussian limit when above the effective critical point by an amount R^{-2}. In particular, at the infinite-volume critical point the susceptibility scales as $R^{d/2}$ for periodic boundary conditions and as R^2 for free boundary conditions. We identify a mass generation mechanism for free boundary conditions that is responsible for this distinction and which we believe has wider validity, in particular to Euclidean (non-hierarchical) models on $\mathbb{Z}^d$ in dimensions $d \geq 4$. For $d = 4$ we prove a similar picture with logarithmic corrections. Our

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analysis is based on the rigorous renormalisation group method of Bauerschmidt, Brydges and Slade, which we improve and extend.

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1 | INTRODUCTION AND MAIN RESULTS

1.1 | Critical behaviour of the $|\varphi|^4$ model

The φ^4 model on the Euclidean lattice $\mathbb{Z}^d$ is a close relative of the Ising model and has been studied for decades as one of the most fundamental spin models in statistical mechanics and Euclidean quantum field theory [36, 42]. Given $n \in \mathbb{N}$, $g > 0$, $\nu \in \mathbb{R}$, a finite set $\Lambda \subset \mathbb{Z}^d$, and a spin field $\varphi : \Lambda \to \mathbb{R}^n$, the finite-volume $|\varphi|^4$ *Hamiltonian* is

$$H(\varphi) = \frac{1}{2}(\varphi, (-\Delta)\varphi) + \sum_{x\in\Lambda} \Big(\frac{1}{4} g|\varphi_x|^4 + \frac{1}{2}\nu|\varphi_x|^2 \Big). \tag{1.1}$$

Our interest here is the theory's critical behaviour, which occurs for ν at and near a critical value ν_c which is negative and corresponds to a specific double-well potential when $n = 1$. The disordered phase is $\nu > \nu_c$ (this would be $\beta < \beta_c$ for the Ising model), while $\nu < \nu_c$ is the ordered phase.

Recent advances include a proof that the spontaneous magnetisation vanishes at the critical point in all dimensions $d \geq 3$ [43], and a proof of the critical theory's Gaussian nature in the upper critical dimension $d = 4$ [3] following a long history for dimensions $d \geq 4$ [2, 5, 39] which in particular established mean-field critical behaviour for $d > 4$. The critical scaling and logarithmic corrections to mean-field behaviour at the upper critical dimension $d = 4$ also have a long history via rigorous renormalisation group (RG) analysis in the case of weak coupling, as we discuss in detail in Section 1.4.1.

Finite-size scaling in dimensions $d \geq 4$ has been widely discussed in the physics literature, both via scaling arguments and numerical simulations (e.g., [13, 29, 56, 57, 68, 69]), and it is desirable to have rigorous results which specify the behaviour definitively. Our purpose here is to analyse

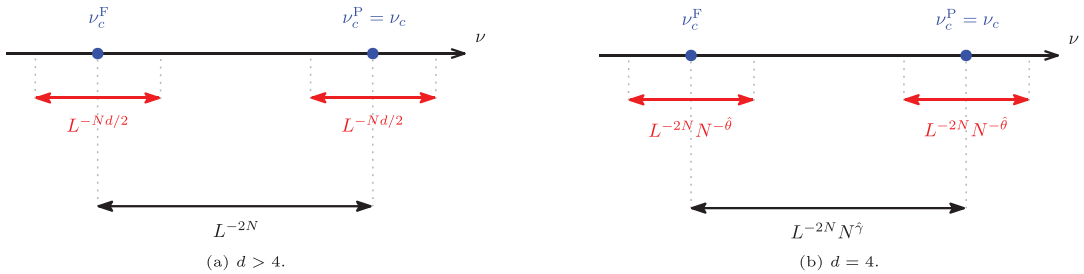

FIGURE 1 Effective critical points ν_c^{F}, ν_c^{P} and scaling windows (red) for FBC and PBC for (a) $d > 4$ and (b) $d = 4$. The scaling windows for FBC and PBC do not overlap. At the infinite-volume critical point ν_c, the average field has a Gaussian limit with FBC and a non-Gaussian limit with PBC.

the critical finite-size scaling of the weakly-coupled n-component $|\varphi|^4$ model in dimensions $d \geq 4$, to indicate the differing effects of free boundary conditions (FBC) versus periodic boundary conditions (PBC), to elucidate the universal scaling profile in the vicinity of the *effective* finite-volume critical point (sometimes called a *pseudocritical* point), and to compute the logarithmic corrections present for $d = 4$.

Following a long tradition in rigorous RG analysis going back to Dyson [31], we work with the d-dimensional *hierarchical* model rather than the Euclidean lattice $\mathbb{Z}^d$, which provides for a simpler analysis yet still exhibits behaviour which we believe to apply exactly also for $\mathbb{Z}^d$. Among the copious previous work on hierarchical models, we mention [1, 9, 14, 18, 19, 40, 44, 66]. Hierarchical models have also recently attracted attention in percolation theory, for example, [46, 47]. Our approach is based on the rigorous RG method for the 4-dimensional hierarchical $|\varphi|^4$ model in [9]. We improve and extend the method of [9] in order to apply it to the finite-size scaling. We also extend the method to apply to dimensions $d > 4$. No additional complication arises in treating n-component fields for general $n \geq 1$ compared to the treatment of $n = 1$. Our RG method is inspired by Wilson's progressive integration over scales, but with no uncontrolled approximations.

Finite-size scaling has a long history in physics, which is natural given that laboratory systems are finite by definition. Numerical simulations of course also involve only finite systems, and finite-size effects are important for the interpretation of simulation data. The early history of finite-size scaling is summarised in [24], an introduction is provided in [23, Section 4.4], and an extensive account is given in [70, Chapter 32]. In brief, the physics picture is as follows:

- The exact critical point for an infinite system is replaced by a critical window whose size scales with the volume of Λ as $|\Lambda|^{-\theta}$ with *window* or *rounding exponent* θ, and with a logarithmic correction at the upper critical dimension.
- Within the critical window, the scaling of the susceptibility and other moments of the average field is governed by universal profiles.
- The location of the critical window is affected by FBC versus PBC.

Our results give a complete description of the above three points for the weakly-coupled hierarchical $|\varphi|^4$ model in a finite volume Λ_N of size L^{dN}, in dimensions $d \geq 4$, with L fixed and N large. A summary is given in Figure 1. We write the average field as $\Phi_N = L^{-dN} \sum_{x \in \Lambda_N} \varphi_x$.

For $d > 4$, we prove that for all $n \geq 1$:

- For PBC, there is a critical window of width $L^{-Nd/2}$ containing the infinite-volume critical point ν_c, within which the rescaled average field $L^{Nd/4}\Phi_N$ converges to a non-Gaussian distribution with density proportional to $e^{-\frac{1}{4}|x|^4-\frac{1}{2}s|x|^2}$. The variable $s \in (-\infty, \infty)$ parametrises the location within the critical window. We explicitly compute the universal profiles of the susceptibility and other moments of the average field within the window. Within the window, the susceptibility scales as $L^{Nd/2}$.
- For FBC, the effective critical point is not ν_c (as it is for PBC) but instead is shifted left to a value ν_c^{F} with $\nu_c - \nu_c^{\mathrm{F}} \asymp L^{-2N}$. Around the shifted effective critical point ν_c^{F} there is again a critical window of width $L^{-Nd/2}$ within which the average field and the susceptibility scale *exactly* as they do for PBC, with the same universal profiles.
- For both FBC and PBC, when ν is above the effective critical point by an amount sL^{-2N} with $s > 0$, the rescaled average field $L^{N(d-2)/2}\Phi_N$ converges to a Gaussian distribution with density proportional to $e^{-\frac{1}{2}s|x|^2}$. The susceptibility scales as L^{2N}. For FBC, the infinite-volume critical point ν_c lies in this Gaussian range and hence the susceptibility at ν_c scales as L^{2N}, unlike for PBC where the scaling is $L^{Nd/2}$.

For $d = 4$, let

$$\hat{\gamma} = \frac{n+2}{n+8}, \qquad \hat{\theta} = \frac{4-n}{2(n+8)} = \frac{1}{2} - \hat{\gamma}. \tag{1.2}$$

We prove that the $d > 4$ picture applies also for $d = 4$ but with logarithmic corrections (polynomial in N) for all $n \geq 1$:

- For PBC, there is a critical window of width $L^{-2N}N^{-\hat{\theta}}$ containing the infinite-volume critical point ν_c, within which the rescaled average field $N^{-1/4}L^N\Phi_N$ converges to a non-Gaussian distribution with density proportional to $e^{-\frac{1}{4}|x|^4-\frac{1}{2}s|x|^2}$. The universal profile of the susceptibility and other moments of the average field within the window are *identical* to those for $d > 4$. Within the window the susceptibility scales as $L^{2N}N^{1/2}$.
- For FBC, the effective critical point is shifted left to a value ν_c^{F} with $\nu_c - \nu_c^{\mathrm{F}} \asymp L^{-2N}N^{\hat{\gamma}}$. Around the shifted effective critical point ν_c^{F} there is a critical window of width $L^{-2N}N^{-\hat{\theta}}$ within which the average field, the susceptibility and other moments of the average field scale *exactly* as they do in the critical window for PBC.
- For both FBC and PBC, above their effective critical points by an amount $sL^{-2N}N^{\hat{\gamma}}$ with $s > 0$, the rescaled average field $L^N\Phi_N$ converges to a Gaussian distribution with density proportional to $e^{-\frac{1}{2}s|x|^2}$. The susceptibility scales as L^{2N}. For FBC, the infinite-volume critical point ν_c lies in this Gaussian range and hence the susceptibility at ν_c scales as L^{2N}, unlike for PBC where the scaling is $L^{2N}N^{1/2}$.

For dimensions $d > 4$ (*above* the upper critical dimension) with PBC, the physics predictions for the universal profile, the window width, and the size of the susceptibility in the window are stated in [70, Section 32.3.1]. For both FBC and PBC in dimensions $d > 4$, [38, Table 1] presents predictions which are consistent with our theorems for the scaling of the susceptibility, the scaling of the average field, and the shift in the effective critical point for FBC (see also [13, p.38]). For $d = 4$, the physics predictions for the logarithmic correction to the susceptibility and the window size are given in [50, (3.6)] and [50, (4.3)].

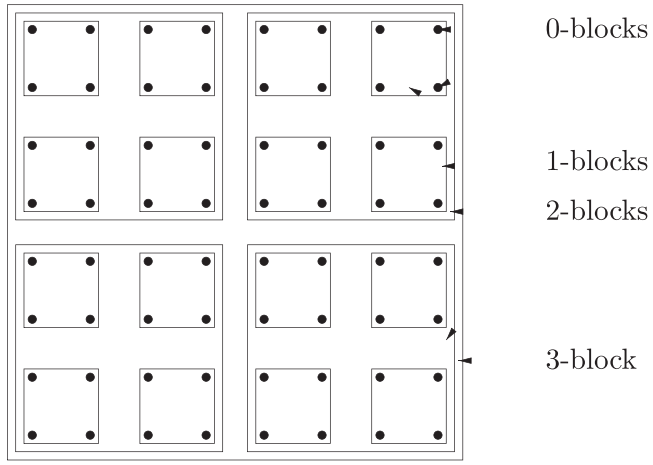

FIGURE 2 The block Λ_3 with its blocks in $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ for the case $d = L = 2$.

Our results provide a rigorous justification of the physics predictions, for the hierarchial lattice in all dimensions $d \geq 4$ and for all $n \geq 1$. Our proof presents a clear mechanism responsible for the shift in the effective critical point for FBC: the free boundary condition generates an effective mass in the Hamiltonian which must be compensated by a shift in the value of ν to attain effective FBC critical behaviour. In addition, although our results are proved only for the hierarchial model, they lead to precise conjectures for the behaviour for Euclidean models on $\mathbb{Z}^d$ for all $d \geq 4$, which we spell out in Section 1.6, not just for the n-component $|\varphi|^4$ model but also for $O(n)$ models (e.g., Ising, XY, Heisenberg models) and self-avoiding walk models ($n = 0$). A summary of further results and conjectures in this direction is given in [53].

Notation. We write $f_N \sim g_N$ to denote $\lim_N f_N/g_N = 1$, and use $f_N \asymp g_N$ to denote the existence of $C > 0$ such that $C^{-1}g_N \leq f_N \leq Cg_N$. We also use the notation $\sim$ and $\asymp$ in limits when f, g depend on a different variable than N.

1.2 | The hierarchical Laplacian

The hierarchial $|\varphi|^4$ model is defined using the hierarchical Laplacian in the Hamiltonian (1.1). We define the hierarchical Laplacian in this section as the generator for a random walk with FBC or PBC (see also [47] for discussion of hierarchical BC). The definitions in this section apply to any dimension $d \geq 1$. Also, we introduce a parameter α (in (1.7)) even though we only use the case $\alpha = 2$, in order to emphasise its role.

1.2.1 | The hierarchical group

Given integers $L > 1$ and $N \geq 1$, let Λ_N denote the cube $\Lambda_N = \{x \in \mathbb{Z}^d : 0 \leq x_i < L^N \text{ for } i = 1, \dots, d\}$, which has *volume* (cardinality) L^{dN}. The thermodynamic limit is the limit $N \to \infty$, for which $\Lambda_\infty = \{x \in \mathbb{Z}^d : 0 \leq x_i \text{ for } i = 1, \dots, d\}$. Given an integer $0 \leq j \leq N$, we can partition Λ_N into $L^{d(N-j)}$ disjoint *blocks* which are each translates of Λ_j and which each contain L^{dj} vertices. We denote the set of such *j-blocks* by $\mathcal{B}_j$. See Figure 2.

An equivalent representation of Λ_∞ is as follows. With $\mathbb{Z}_L = \mathbb{Z}/L\mathbb{Z}$ the cyclic group, let

$$\mathbb{H}_\infty = \{\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ...) : \tilde{x}_i \in \mathbb{Z}_L^d \text{ with only finitely many nonzero } \tilde{x}_i\}. \tag{1.3}$$

This is an Abelian group with coordinate-wise addition mod-L; we denote the group addition as $+$. We define $\mathbb{H}_N$ to be the subgroup of $\mathbb{H}_\infty$ with $\tilde{x}_i = 0$ for $i > N$. The map $\sigma : \mathbb{H}_\infty \to \Lambda_\infty$ defined by

$$\sigma(\tilde{x}) = \sum_{j=1}^{\infty} L^{j-1} x_j \tag{1.4}$$

is a bijection, where x_j is the representative of $\tilde{x}_j$ in $\Lambda_1 \subset \mathbb{Z}^d$. Addition on the right-hand side of (1.4) is in $\mathbb{Z}^d$. Note that σ restricts to a bijection $\sigma_N : \mathbb{H}_N \to \Lambda_N$. The bijection induces an addition and a group structure on Λ_N via $x \oplus y = \sigma_N(\sigma_N^{-1}x + \sigma_N^{-1}y)$ (with $+$ the addition on $\mathbb{H}_N$); this makes σ_N and σ into group isomorphisms.

For $x, y \in \Lambda_N$, we define the *coalescence scale* j_{xy} to be the smallest j such that x and y lie in the same j-block. In terms of $\mathbb{H}_N$, j_{xy} is the largest coordinate i such that $(\sigma_N^{-1}x)_i$ differs from $(\sigma_N^{-1}y)_i$, and from this observation we see that

$$j_{x\oplus z, y\oplus z} = j_{xy} \qquad (x, y, z \in \Lambda_N). \tag{1.5}$$

In particular, with subtraction in Λ_N denoted by $\ominus$,

$$j_{xy} = j_{0, y\ominus x} \qquad (x, y \in \Lambda_N). \tag{1.6}$$

1.2.2 | Hierarchical random walk

Random walk on the infinite hierarchial group. Given $\alpha > 0$, we define a discrete-time random walk on the group Λ_∞ via the transition probabilities $J_{xx} = 0$ $(x \in \Lambda_\infty)$ and

$$J_{xy} = \frac{1}{z} \frac{1}{L^{(d+\alpha)j_{xy}}} \qquad (x, y \in \Lambda_\infty,\ x \neq y), \tag{1.7}$$

with the constant $z = z(d, L, \alpha)$ chosen so that $\sum_{x\neq 0} J_{0x} = 1$. By (1.6), the transition probabilities are translation invariant in the sense that $J_{x,y} = J_{0,y\ominus x}$. Since the number of sites $x \in \Lambda_\infty$ with $j_{0x} = k$ is

$$n_k = L^{dk} - L^{d(k-1)} = L^{dk}(1 - L^{-d}), \tag{1.8}$$

the constant z is given by

$$z = \sum_{x\neq 0} L^{-(d+\alpha)j_{0x}} = \sum_{k=1}^{\infty} n_k L^{-(d+\alpha)k} = \frac{1 - L^{-d}}{L^\alpha - 1}. \tag{1.9}$$

Random walk with FBC. We define a random walk on $\Lambda_N \times \{\dagger\}$ which is killed when it lands on the cemetery state $\dagger$. The killing occurs when a random walk on the infinite hierarchical lattice

Λ_∞ exits Λ_N for the first time. This corresponds to FBC. Explicitly, for $x \in \Lambda_N$,

$$J^{\mathrm{F}}_{x\dagger} = \sum_{y\,:\,j_{xy}>N} J_{xy} = \frac{1}{z}\sum_{k=N+1}^{\infty} n_k L^{-(d+\alpha)k} = L^{-\alpha N}, \tag{1.10}$$

whereas for $x, y \in \Lambda_N$ we maintain the infinite-volume transition probability from (1.7), namely $J^{\mathrm{F}}_{xx} = 0$ and

$$J^{\mathrm{F}}_{xy} = \frac{1}{z} L^{-(d+\alpha)j_{xy}} \qquad (x, y \in \Lambda_N,\ x \neq y). \tag{1.11}$$

This produces a defective (unnormalised) distribution J^{F} on Λ_N for which

$$\sum_{x\in\Lambda_N} J^{\mathrm{F}}_{0x} = 1 - L^{-\alpha N}. \tag{1.12}$$

Random walk with PBC. With N fixed, we define an equivalence relation on Λ_∞ by $x \sim y$ if $\tilde{x}_i = \tilde{y}_i$ for $i = 1, \dots, N$, where $\tilde{x} = \sigma^{-1}x$, $\tilde{y} = \sigma^{-1}y$. The quotient space $\Lambda_\infty/\!\sim$ can be identified with Λ_N. The random walk on Λ_∞ with transition matrix (1.7) projects via the quotient map to a random walk on Λ_N. This projection corresponds to PBC. Explicitly, since for $k > N$ the block Λ_k contains exactly $n_k L^{-dN}$ points with $j_{0x} = k$ that are equivalent to $y \in \Lambda_N$, for the same normalising constant z as in (1.9) the quotient walk has transition probabilities

$$\begin{aligned} J^{\mathrm{P}}_{xy} &= \frac{1}{z} L^{-(d+\alpha)j_{xy}} + L^{-dN} \sum_{k=N+1}^{\infty} n_k \frac{1}{z} L^{-(d+\alpha)k} \\ &= \frac{1}{z} L^{-(d+\alpha)j_{xy}} + L^{-(d+\alpha)N} \qquad (x, y \in \Lambda_N,\ x \neq y) \end{aligned} \tag{1.13}$$

and $J^{\mathrm{P}}_{xx} = L^{-(d+\alpha)N}$.

1.2.3 | The hierarchical Laplacian with boundary conditions

In general, given a transition matrix J_{xy} (defective or not), we define a Laplacian by $-\tilde{\Delta} = 1 - J$ and we let $-\tilde{\Delta}$ act on an n-component spin field $\varphi : \Lambda_N \to \mathbb{R}^n$ component-wise. Here 1 denotes the identity matrix. If J is defective then the Laplacian is massive (i.e., has a spectral gap), and otherwise it is massless. For the $|\varphi|^4$ model, the critical behaviour is not affected by replacing the Laplacian term in the Hamiltonian by a multiple of the Laplacian. We use this flexibility in the following definitions in order to achieve coherence between our Laplacian with PBC and the hierarchial Laplacian defined in [9, Definition 4.1.7]. Since the random walks with FBC and PBC are translation invariant (with respect to $\oplus$) by definition, the Laplacians they define obey

$$\tilde{\Delta}_{xy} = \tilde{\Delta}_{0,y\ominus x} \qquad (x, y \in \Lambda_N). \tag{1.14}$$

In particular, it suffices to specify the matrix elements $\tilde{\Delta}_{0x}$ to define the Laplacians.

For $* = \mathrm{P}, \mathrm{F}$, and with J^* the matrix of transition probabilities with boundary condition $*$, we define

$$-\tilde{\Delta}^* = 1 - J^*. \tag{1.15}$$

For $x, y \in \Lambda_N$ we define $Q_{N;xy} = L^{-dN}$. It follows from the definitions of J^* that

$$1 - J^{\mathrm{F}} = 1 - J^{\mathrm{P}} + L^{-\alpha N} Q_N. \tag{1.16}$$

The matrix $-\tilde{\Delta}^{\mathrm{P}}$ obeys $\sum_{x\in\Lambda_N}(1 - J^{\mathrm{P}})_{0x} = 0$, whereas $-\tilde{\Delta}^{\mathrm{F}}$ obeys $\sum_{x\in\Lambda_N}(1 - J^{\mathrm{F}})_{0x} = L^{-\alpha N}$.

Multiplication of the Laplacian by a scalar amounts to a rescaling of the field. We do the rescaling in the same way for both boundary conditions, by defining

$$-\Delta^* = q(-\tilde{\Delta}^*) = q(1 - J^*) \tag{1.17}$$

with

$$q = \frac{1 - L^{-d}}{1 - L^{-(d+\alpha)}}. \tag{1.18}$$

With this definition, Δ^{P} is exactly the Laplacian in [9, Definition 4.1.7], as we indicate in Section 2.1. By (1.16),

$$-\Delta^{\mathrm{F}} = -\Delta^{\mathrm{P}} + qL^{-\alpha N} Q_N. \tag{1.19}$$

The Green function (inverse Laplacian) on Λ_∞ obeys

$$(-\Delta)^{-1}_{0x} \asymp |x|^{-(d-\alpha)} \qquad (0 < \alpha < d). \tag{1.20}$$

On the right-hand side of (1.20), $|x|$ denotes the Euclidean norm of the point in $\mathbb{Z}^d$ under the identification of Λ_∞ with an orthant of $\mathbb{Z}^d$ (as in Section 1.2.1). The proof of (1.20) given in [9, (4.1.29)] for $\alpha = 2$ applies also for $0 < \alpha < d$. For $\alpha = 2$, the right-hand side of (1.20) is the decay of the Green function for the standard nearest-neighbour Laplacian on $\mathbb{Z}^d$. This indicates a close connection between the hierarchical Laplacian and the Euclidean Laplacian.

1.3 | The hierarchical $|\varphi|^4$ model

Let $-\Delta^*$ represent either $-\Delta^{\mathrm{F}}$ or $-\Delta^{\mathrm{P}}$ on Λ_N, so that either boundary condition can be used in the following definitions. We *always* take $\alpha = 2$ in the definition of the Laplacian. Given $n \in \mathbb{N}$, $g > 0$, $\nu \in \mathbb{R}$, and a spin field $\varphi : \Lambda_N \to \mathbb{R}^n$, we define the hierarchical *Hamiltonian*

$$H^*_{\nu,N}(\varphi) = \frac{1}{2}(\varphi, (-\Delta^*)\varphi) + \sum_{x\in\Lambda_N} \Big(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2}\nu |\varphi_x|^2 \Big), \tag{1.21}$$

the *partition function*

$$Z^*_{\nu,N} = \int_{(\mathbb{R}^n)^{\Lambda_N}} e^{-H^*_{\nu,N}(\varphi)} d\varphi, \tag{1.22}$$

and the associated expectation

$$\langle F\rangle^*_{\nu,N} = \frac{1}{Z^*_{\nu,N}} \int_{(\mathbb{R}^n)^{\Lambda_N}} F(\varphi) e^{-H^*_{\nu,N}(\varphi)} d\varphi. \tag{1.23}$$

In (1.21), and in what follows, we use the notation $(\varphi,\psi) = \sum_{x\in\Lambda_N} \varphi_x \cdot \psi_x$, where the dot product is for $\mathbb{R}^n$. Also, $|\varphi_x| = (\varphi_x \cdot \varphi_x)^{1/2}$ denotes the Euclidean norm of $\varphi_x \in \mathbb{R}^n$. The definitions (1.21)–(1.23) depend on g but since we regard g as fixed (and small) in the following, we do not make this dependence explicit. We will however be interested in varying ν. By definition, the distribution of the field φ under (1.23) is invariant under the map $(\varphi_x)_{x\in\Lambda_N} \mapsto (R\varphi_x)_{x\in\Lambda_N}$ for any orthogonal transformation $R \in O(n)$.

Our main objects of study are the distribution of the *average field*

$$\Phi_N = \frac{1}{L^{dN}} \sum_{x\in\Lambda_N} \varphi_x \in \mathbb{R}^n \tag{1.24}$$

and the expectations $\langle |\Phi_N|^{2p}\rangle_{\nu,N}$ for $p \in \mathbb{N}$. An important special case is $2p = 2$, the finite-volume *susceptibility*:

$$\chi^*_N(\nu) = \sum_{x\in\Lambda_N} \langle \varphi^{(1)}_0 \varphi^{(1)}_x\rangle^*_{\nu,N} = \frac{1}{n} \sum_{x\in\Lambda_N} \langle \varphi_0 \cdot \varphi_x\rangle^*_{\nu,N} = \frac{|\Lambda_N|}{n} \langle |\Phi_N|^2\rangle^*_{\nu,N}, \tag{1.25}$$

where $\varphi^{(j)}_x$ denotes the jth component of $\varphi_x \in \mathbb{R}^n$. The second equality of (1.25) is a consequence of the $O(n)$ invariance. The third follows from the fact that, by (1.14), under the probability measure given by (1.23) the pair (φ_x, φ_y) has the same distribution as $(\varphi_0, \varphi_{y\ominus x})$ for any $x, y \in \Lambda_N$, so (φ_x, Φ_N) and (φ_y, Φ_N) have the same distribution too. The susceptibility is more naturally defined using a truncated expectation in which the square of the infinite-volume spontaneous magnetisation is subtracted from the expectation $\langle \varphi_0 \cdot \varphi_x\rangle_{\nu,N}$. We discuss the distinction between these two definitions in more detail at the end of Section 1.5.1.

1.4 | Main results

Our main results are valid in all dimensions $d \geq 4$ and for all $n \geq 1$. They are of three types:

(i) In Theorem 1.1, we identify the infinite-volume critical point ν_c, and for $\nu = \nu_c + \varepsilon$ with $\varepsilon > 0$ fixed, we prove a Gaussian limit theorem for the average field, and compute moments of the average field. In this limit theorem, the average field is scaled by $L^{Nd/2}$ (Gaussian scaling). The results apply for both FBC and PBC, and there are logarithmic corrections for $d = 4$.

(ii) In Theorem 1.2, we identify effective finite-volume critical points $\nu^*_{c,N}$ for FBC and PBC, with $\nu^{\mathrm{P}}_{c,N} = \nu_c$, and with $\nu_c - \nu^{\mathrm{F}}_{c,N} \asymp L^{-2N}$ for $d > 4$ and $\nu_c - \nu^{\mathrm{F}}_{c,N} \asymp N^{\frac{n+2}{n+8}} L^{-2N}$ for $d = 4$. We prove a non-Gaussian limit for the average field for $\nu = \nu^*_{c,N} + sw_N$ for $s \in (-\infty, \infty)$, where the window scale w_N is of order $L^{-Nd/2}$ for $d > 4$ and of order $N^{\frac{4-n}{2(n+8)}} L^{-2N}$ for $d = 4$. The average field is scaled by $L^{Nd/4}$ for $d > 4$ and by $N^{-1/4}L^N$ for $d = 4$ (non-Gaussian scaling for all $d \geq 4$). Universal profiles for the moments of the average field are identified, and there is a logarithmic correction when $d = 4$. The same non-Gaussian limit applies for both FBC and PBC, and the FBC and PBC windows do not overlap.

(iii) In Theorem 1.3, for FBC and PBC we prove a Gaussian limit for the average field for $\nu = \nu^*_{c,N} + sv_N$ with $s \in (0,\infty)$, where v_N is of order L^{-2N} for $d > 4$ and of order $N^{\frac{n+2}{n+8}} L^{-2N}$ for $d = 4$. The field is scaled by $L^{N(d-2)/2}$ for all $d \geq 4$, which is non-Gaussian scaling. Theorem 1.3 probes the vicinity of the critical point on a finer scale than Theorem 1.1, but on a scale that is less fine than Theorem 1.2.

The proofs of the three theorems are given in the bulk of the paper.

1.4.1 | Gaussian limit and critical behaviour

The following theorem identifies a *critical point* ν_c, for large L and for small $g > 0$, at which the infinite-volume hierarchial susceptibility diverges. Although our proof requires L to be large and g to be small, we believe the theorem remains true for all $L \geq 2$ and all $g > 0$. The theorem also describes the manner of the susceptibility's divergence. For $d = 4$ the theorem is partly proved in [9, Theorem 4.2.1], and we adapt that proof here to include dimensions $d > 4$. For the statement of the theorem, for $s > 0$ we define a Gaussian measure on $\mathbb{R}^n$, together with its moments, by

$$d\mu_{n,s} = \frac{e^{-\frac{1}{2}s|x|^2}dx}{\int_{\mathbb{R}^n} e^{-\frac{1}{2}s|x|^2}dx}, \qquad M_{n,k}(s) = \int_{\mathbb{R}^n} |x|^k d\mu_{n,s}. \tag{1.26}$$

Theorem 1.1 (Massive Gaussian limit). *Let $d \geq 4$, let $n \in \mathbb{N}$, let L be sufficiently large, and let $g > 0$ be sufficiently small (depending on L). There is a critical value $\nu_c \in \mathbb{R}$ (depending on d, n, g, L), constants A_d (depending on d, n, g, L), and a strictly positive continuously differentiable function m_ε^2 defined for all $\varepsilon > 0$ with*

$$m_\varepsilon^2 \sim \begin{cases} A_4^{-1}\,\varepsilon\,(\log \varepsilon^{-1})^{-\hat{\gamma}} & (d=4) \\ A_d^{-1}\,\varepsilon & (d>4) \end{cases} \qquad \text{as } \varepsilon \downarrow 0, \tag{1.27}$$

with $\hat{\gamma}$ given by (1.2), such that the following infinite-volume limits exist for $\varepsilon > 0$ and have the values indicated, independent of the boundary condition $* = \mathrm{F}$ or $* = \mathrm{P}$:

(i) For any $J \in \mathbb{R}^n$,

$$\lim_{N\to\infty} \langle e^{J\cdot\Phi_N L^{Nd/2}} \rangle^*_{\nu_c+\varepsilon,N} = \int_{\mathbb{R}^n} e^{J\cdot x} d\mu_{n,m_\varepsilon^2}. \tag{1.28}$$

(ii) For $p \in \mathbb{N}$,

$$\lim_{N\to\infty} L^{dNp} \langle |\Phi_N|^{2p} \rangle^*_{\nu_c+\varepsilon,N} = M_{n,2p}(m_\varepsilon^2). \tag{1.29}$$

Also, as $g \downarrow 0$, the amplitude in (1.27) and the critical point satisfy

$$A_4 \sim \Big(\frac{(n+8)(1-L^{-d})g}{\log L^2}\Big)^{\hat{\gamma}}, \quad A_d = 1 + O(g) \quad (d>4), \quad \nu_c \sim -(n+2)g(-\Delta)^{-1}_{00}, \tag{1.30}$$

with $(-\Delta)^{-1}_{00}$ *the diagonal matrix element of the Green function of the infinite-volume hierarchical Laplacian.*

Equation (1.28) states that the total field $\sum_{x\in\Lambda_N}\varphi_x$, after a Gaussian rescaling $L^{-Nd/2}$, converges in distribution under $\langle\,\cdot\,\rangle_{\nu_c+\varepsilon,N}$ to a Gaussian random variable with mean zero and variance $1/m_\varepsilon^2$. The moments in (1.29) can be explicitly evaluated, so that for $p\geq 1$, $\varepsilon>0$, and $*=\mathrm{F}$ or $*=\mathrm{P}$,

$$\lim_{N\to\infty} L^{dNp}\langle|\Phi_N|^{2p}\rangle^*_{\nu_c+\varepsilon,N} = \left(\frac{2}{m_\varepsilon^2}\right)^p \frac{\Gamma(\frac{n+2p}{2})}{\Gamma(\frac{n}{2})}. \tag{1.31}$$

By the definition of the susceptibility in (1.25), together with (1.27) and the $p=1$ case of (1.31), the infinite-volume susceptibility $\chi^*_\infty = \lim_{N\to\infty}\chi^*_N(\nu_c+\varepsilon)$ obeys, independent of the boundary condition,

$$\chi_\infty(\nu_c+\varepsilon) = \frac{1}{m_\varepsilon^2} \sim \begin{cases} A_4\,\varepsilon^{-1}\big(\log\varepsilon^{-1}\big)^{\hat\gamma} & (d=4) \\ A_d\,\varepsilon^{-1} & (d>4) \end{cases} \qquad \text{as } \varepsilon\downarrow 0. \tag{1.32}$$

For $d=4$ and PBC, this recovers [9, Theorem 4.2.1]. We emphasise that the equality in (1.32) is not a definition—we compute the infinite-volume limit of the susceptibility defined by (1.25) and find that it is equal to a function governed by the asymptotic formula (1.27).

Several related results have been proved for the $|\varphi|^4$ model on the Euclidean lattice $\mathbb{Z}^4$, which we now summarise. For this, we write c for a g- and n-dependent constant that varies from line to line.

- For $n\geq 1$, the asymptotic formula (1.32) has been proved with $A_4\sim[(n+8)g/(16\pi^2)]^{\hat\gamma}$ as $g\downarrow 0$ [6], as part of a large RG literature for weakly-coupled $|\varphi|^4$ models on $\mathbb{Z}^4$.
- For 1-component spins, it has been proved that the critical two-point function decays as $\langle\varphi_0\varphi_x\rangle\sim c|x|^{-2}$ as $|x|\to\infty$ [35, 41, 65]. The susceptibility and correlation length have logarithmic corrections $\chi\asymp\varepsilon^{-1}|\log\varepsilon|^{1/3}$ and $\xi\asymp\varepsilon^{-1/2}|\log\varepsilon|^{1/6}$ as $\varepsilon\downarrow 0$ [45] (these are also proved for $d>4$ with no logarithmic corrections in [45]). The critical magnetisation vanishes with the magnetic field h as $M\sim ch^{1/3}|\log h|^{1/3}$ [55].
- For general $n\geq 1$, the $|x|^{-2}$ decay of the critical two-point function is proved in [65]. As $\varepsilon\downarrow 0$, the susceptibility obeys $\chi\sim c\varepsilon^{-1}|\log\varepsilon|^{\frac{n+2}{n+8}}$ [6]. The correlation length of order p obeys $\xi_p\sim c_p\varepsilon^{-1/2}|\log\varepsilon|^{\frac{n+2}{2(n+8)}}$ for all $p>0$ [12]. The specific heat obeys $c_H\sim c|\log\varepsilon|^{\frac{4-n}{n+8}}$ $(n=1,2,3)$, $c_H\sim c\log|\log\varepsilon|$ $(n=4)$, and $c_H\sim c$ $(n\geq 5)$ [6].

  In [7, 8, 12], the above asymptotic formulas for the critical two-point function, the susceptibility, and the correlation length of order p are extended to $n=0$. The case $n=0$ corresponds to the continuous-time weakly self-avoiding walk via an exact supersymmetric representation.

Still missing from the above catalogue is a proof that the spontaneous magnetisation obeys $M_+\sim c\varepsilon^{1/2}|\log\varepsilon|^{\frac{3}{n+8}}$ as $\varepsilon=\nu_c-\nu$ (sign change) approaches zero from the ordered phase. The critical behaviour from the ordered phase is beyond the scope of current RG methods.

Apart from [45], we are not aware of rigorous RG results for φ^4 in dimensions $d>4$. On the other hand, lace expansion methods have been used to prove $|x|^{-(d-2)}$ decay of the critical

two-point function in dimensions $d > 4$ for 1-component φ^4 [61], for 1- and 2-component $|\varphi|^4$ [17], and for the (spread-out) Ising model [62]. A different proof of $|x|^{-(d-2)}$ decay for the Ising and φ^4 models for $d > 4$ is given in [30].

Scaling relations among the many logarithmic exponents for $\mathbb{Z}^4$ are presented in [51].

1.4.2 | Non-Gaussian limit inside the critical window

To state our results for the non-Gaussian limit, we need several definitions.

The non-Gaussian limiting measure. Given $n \in \mathbb{N}$ and $s \in \mathbb{R}$ we define a non-Gaussian probability measure on $\mathbb{R}^n$ and its kth moments ($k > -n$) by

$$d\sigma_{n,s} = \frac{e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx}{\int_{\mathbb{R}^n} e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx}, \qquad \Sigma_{n,k}(s) = \int_{\mathbb{R}^n} |x|^k d\sigma_{n,s}. \tag{1.33}$$

The quartic coupling constant. Our RG analysis involves a critical (massless) quartic running coupling constant g_j whose value at scale $j = N$ plays an important role. The sequence g_j is nonnegative and depends on g, L, n and d. For $d = 4$, $g_N \sim (BN)^{-1}$, and for $d > 4$, $\lim_{N\to\infty} g_N = g_\infty$. The positive constants B and g_∞ satisfy

$$B = (n+8)(1 - L^{-d}), \qquad g_\infty = g + O(g^2). \tag{1.34}$$

The window scale. Recall the logarithmic correction exponents $\hat{\gamma} = \frac{n+2}{n+8}$ and $\hat{\theta} = \frac{1}{2} - \hat{\gamma}$ from (1.2), and the amplitudes A_d from Theorem 1.1. We define the *window scale* (or *rounding* scale)

$$w_N = \begin{cases} A_4 (\log L^2)^{\hat{\gamma}} B^{-1/2} N^{-\hat{\theta}} L^{-2N} & (d = 4) \\ A_d\, g_\infty^{1/2} L^{-Nd/2} & (d > 4). \end{cases} \tag{1.35}$$

For $d = 4$, the factors $(\log L^2)^{\hat{\gamma}} B^{-1/2} N^{-\hat{\theta}}$ can equivalently be written as the product of $(\log L^{2N})^{\hat{\gamma}}$, which is a correction logarithmic in the volume, multiplied by $(BN)^{-1/2} \sim g_N^{1/2}$ which is the 4-dimensional counterpart of the $g_\infty^{1/2}$ appearing for $d > 4$. To understand the choice of window scale via a rough computation, for $d > 4$ we start with the ansatz $\chi_N^{\mathrm{P}}(\nu_c) \asymp L^{Nd/2}$ and choose w_N to scale in such a manner that $\chi_\infty(\nu_c + w_N) \asymp \chi_N^{\mathrm{P}}(\nu_c)$. Since $\chi_\infty(\nu_c + w_N) \asymp w_N^{-1}$ by (1.32), we conclude that the window should obey $w_N \asymp L^{-Nd/2}$ for $d > 4$. For $d = 4$, similar reasoning leads to $w_N \asymp L^{-2N} N^{-\hat{\theta}}$ when we assume that $\chi_N^{\mathrm{P}}(\nu_c) \asymp L^{2N} N^{1/2}$ (a prediction of [50] for $\mathbb{Z}^4$, which we prove and extend in Corollary 1.4 for the hierarchical model).

The effective critical points. For PBC we define the *effective critical point* to equal the infinite-volume critical point from Theorem 1.1:

$$\nu_{c,N}^{\mathrm{P}} = \nu_c. \tag{1.36}$$

Although $\nu_{c,N}^{\mathrm{P}}$ is independent of the volume parameter N, we include the subscript to emphasise its role as a finite-volume critical point. For FBC we first define

$$v_N = \begin{cases} A_4 (\log L^2)^{\hat{\gamma}} N^{\hat{\gamma}} L^{-2N} & (d = 4) \\ A_d L^{-2N} & (d > 4). \end{cases} \tag{1.37}$$

By definition, v_N is much larger than w_N, namely,

$$v_N = \begin{cases} (BN)^{1/2} w_N & (d=4) \\ g_\infty^{-1/2} L^{N(d-4)/2} w_N & (d>4). \end{cases} \tag{1.38}$$

For FBC, we specify a volume-dependent effective critical point $\nu_{c,N}^{\mathrm{F}}$, which we compute exactly when $d = 4, 5$ and approximately when $d > 5$, by

$$\nu_{c,N}^{\mathrm{F}} = \begin{cases} \nu_c - q v_N (1 + c^{\mathrm{F}} N^{-\hat{\gamma}}) & (d=4) \\ \nu_c - q v_N & (d=5) \\ \nu_c - q v_N (1 + O(L^{-N})) & (d>5); \end{cases} \tag{1.39}$$

here $q = 1 + O(L^{-d})$ is the constant from (1.18) and $c^{\mathrm{F}} = O(g)$ is determined in the proof. The error term $O(L^{-N} v_N)$ for $d > 5$ is the same order as w_N for $d = 6$ and is larger than w_N for $d > 6$. Although we do not precisely identify $\nu_{c,N}^{\mathrm{F}}$ for $d \geq 6$, we do prove its existence as an anchor for the critical window. See Remark 3.5 for further discussion.

The large-field scale. In terms of the constants B and g_∞ in (1.34), we define the *large-field scale*

$$h_N = \begin{cases} (BN)^{1/4} L^{-N} & (d=4) \\ g_\infty^{-1/4} L^{-Nd/4} & (d>4). \end{cases} \tag{1.40}$$

The prefactors for $L^{-Nd/4}$ in (1.40) are each essentially equal to $g_N^{-1/4}$, which is large. The definition of h_N ensures that if $|\varphi_x| \asymp h_N$, then $\sum_{x \in \Lambda_N} g_N |\varphi_x|^4 \asymp 1$ is positive uniformly in N, L, g, so the exponential decay in the Hamiltonian due to $\exp[-\frac{1}{4} g_N \sum_{x \in \Lambda_N} |\varphi_x|^4]$ becomes significant at the large-field scale.

Recall that the average field Φ_N is defined by (1.24). The following theorem concerns the finite-size near-critical scaling of the weakly-coupled hierarchical $|\varphi|^4$ model in dimensions $d \geq 4$. It is part of the statement of the theorem that there are constants g_∞ and c^{F} as above, and a sequence $\nu_{c,N}^{\mathrm{F}}$ obeying (1.39), such that the conclusions of the theorem hold. The d-dependence has not been made explicit in the notation of the theorem. For part (ii) of the theorem, we introduce two different restrictions on the real sequence s_N:

$$s_N \leq \begin{cases} L^{\sqrt{N}} & (d=4) \\ L^{N(d-3)/2} & (d \geq 5), \end{cases} \qquad |s_N| \leq \begin{cases} (\log N)^{1/4} & (d=4) \\ N^{1/4} & (d \geq 5). \end{cases} \tag{1.41}$$

The definitions of the measure $d\sigma_{n,s}$ and its moments $\Sigma_{n,k}(s)$ are given in (1.33). Note that the integral and the moments on the right-hand sides of (1.42) and (1.43) are *independent* of g and of the dimension $d \geq 4$.

Theorem 1.2 (Non-Gaussian limit in the critical window). *Let $d \geq 4$, let $n \in \mathbb{N}$, let L be sufficiently large, and let $g > 0$ be sufficiently small (depending on L). The following statements hold for the n-component hierarchical model, for boundary conditions $* = \mathrm{F}$ and $* = \mathrm{P}$:*

(i) For any $J \in \mathbb{R}^n$ and any real sequence s_N converging to $s \in (-\infty, \infty)$,

$$\lim_{N\to\infty} \left\langle e^{J\cdot\Phi_N/\hbar_N} \right\rangle^*_{g,\nu^*_{c,N}+s_N w_N,N} = \int_{\mathbb{R}^n} e^{J\cdot x}\, d\sigma_{n,s}. \tag{1.42}$$

(ii) Let $p \geq 1$, let s_N be any real sequence such that either: (a) s_N converges, or (b) $\lim s_N = +\infty$ in such a manner that the first inequality of (1.41) holds, or (c) $\lim s_N = -\infty$ in such a manner that the second inequality of (1.41) holds. Then, as $N \to \infty$,

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N w_N,N} = \hbar_N^{2p} \Sigma_{n,2p}(s_N)\big(1 + o(1)\big). \tag{1.43}$$

Theorem 1.2(i) is a statement of convergence of moment generating functions and implies that $\Phi_N/\hbar_N$ converges in distribution to a random variable on $\mathbb{R}^n$ with distribution $d\sigma_{n,s}$. In other words,

$$(BN)^{-1/4} L^N \Phi_N \Rightarrow d\sigma_{n,s} \quad (d = 4), \tag{1.44}$$

$$g_\infty^{1/4} L^{Nd/4} \Phi_N \Rightarrow d\sigma_{n,s} \quad (d > 4). \tag{1.45}$$

The convergence in (1.42) may be compared with the problem of determining the limiting distribution of the average spin of the Curie–Weiss model (the Ising model on the complete graph $\mathbb{K}_V$ on V vertices). Several authors have considered this problem at the critical point [25, 28, 33, 34, 63], which corresponds to $n = 1$ and the choice $s = 0$, and identify the non-Gaussian distribution $d\sigma_{1,0}$ as the limiting distribution of the rescaled average spin $V^{1/4} S_V$. The scaling window is also treated in [27, 32], and in particular it is shown in [32, Theorem 3.2] that the limiting distribution of $V^{1/4} S_V$ is $d\sigma_{1,s}$ for β of the form $1 + sV^{-1/2}$ (critical window) for all $s \in \mathbb{R}$. Related results for the Ising model on Erdős–Rényi random graphs are obtained in [49]. The scaling we prove for dimensions $d > 4$ in (1.45) similarly involves the fourth root of the volume $L^{Nd/4}$, whereas for $d = 4$ in (1.44) there is an additional logarithmic (in the volume) correction in the factor $N^{-1/4}$.

Theorem 1.2(ii) states that, within a window of ν values of order w_N around the effective critical point $\nu^*_{c,N}$, the expectation of $|\Phi_N|^{2p}$ is given by a factor of $\hbar_N$ for each field, multiplied by a p-dependent but g-independent universal profile $\Sigma_{n,2p}(s)$. This holds even for the crossover out of the critical window, when $s_N \to \pm\infty$ in the manner specified in part (ii). We have not attempted to obtain optimal restrictions on how rapidly $s_N \to \pm\infty$. The case (ii)(a) with $s_N \to 0$ indicates that there can be no "hidden" window smaller than the critical window w_N we have identified.

1.4.3 | Gaussian limit above the critical window

We recall the definitions of $d\mu_{n,s}$ and $M_{n,k}(s)$ from (1.26) and also define ℓ_N as

$$d\mu_{n,s} = \frac{e^{-\frac{1}{2}s|x|^2} dx}{\int_{\mathbb{R}^n} e^{-\frac{1}{2}s|x|^2} dx}, \qquad M_{n,k}(s) = \int_{\mathbb{R}^n} |x|^k d\mu_{n,s}, \qquad \ell_N = L^{-N(d-2)/2}. \tag{1.46}$$

These definitions parallel $d\sigma_{n,s}$, $\Sigma_{n,k}(s)$, $\hbar_N$. The sequences υ_N and w_N are related as in (1.38), so $\upsilon_N \gg w_N$ and the ν values treated in the following theorem are well above the critical window.

We again introduce growth restrictions on the sequence s_N, this time as

$$s_N \le \begin{cases} L^{\sqrt{N}} & (d=4) \\ L^{N/2} & (d \ge 5). \end{cases} \tag{1.47}$$

As in Theorem 1.2, the limiting distribution and the moments in the conclusions of Theorem 1.3 are independent of g and of $d \ge 4$.

Although the limiting distribution in Theorem 1.3 is Gaussian, in its statement the scaling $\ell_N^{-1}\Phi_N = L^{-N(d+2)/2}\sum_{x\in\Lambda_N}\varphi_x$ is non-Gaussian, as opposed to the central limit theorem scaling $L^{-Nd/2}\sum_{x\in\Lambda_N}\varphi_x$ of Theorem 1.1.

Theorem 1.3 (Gaussian limit above the critical window). *Let $d \ge 4$, let $n \in \mathbb{N}$, let L be sufficiently large, and let $g > 0$ be sufficiently small (depending on L). The following statements hold for the n-component hierarchical model, for boundary conditions $* = \mathrm{F}$ and $* = \mathrm{P}$:*

(i) For any $J \in \mathbb{R}^n$ and any real sequence s_N converging to $s > 0$,

$$\lim_{N\to\infty} \left\langle e^{J\cdot\Phi_N/\ell_N} \right\rangle^*_{g,\nu^*_{c,N}+s_N v_N,N} = \int_{\mathbb{R}^n} e^{J\cdot x}\, d\mu_{n,s}. \tag{1.48}$$

(ii) Let $p \ge 1$, let s_N be any real sequence such that $s_N \ge \varepsilon$ for some fixed $\varepsilon > 0$ and s_N obeys (1.47). Then, as $N \to \infty$,

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N v_N,N} = \ell_N^{2p} M_{n,2p}(s_N)\big(1+o(1)\big). \tag{1.49}$$

We have chosen to present Theorem 1.3 as such for comparison with the statement of Theorem 1.2, but since the limiting distribution is Gaussian we can make (1.48)–(1.49) more explicit:

$$\lim_{N\to\infty} \left\langle e^{J\cdot\Phi_N/\ell_N} \right\rangle^*_{g,\nu^*_{c,N}+sv_N,N} = e^{|J|^2/(2s)}, \tag{1.50}$$

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N v_N,N} = \ell_N^{2p} \left(\frac{2}{s_N}\right)^p \frac{\Gamma(\frac{n+2p}{2})}{\Gamma(\frac{n}{2})}\big(1+o(1)\big), \tag{1.51}$$

under the hypotheses of Theorem 1.3.

Theorem 1.3 covers a range of ν values that is outside of the scaling window around the effective critical point $\nu^*_{c,N}$, where we know from Theorem 1.2 that the limiting law is the non-Gaussian measure $d\sigma_{n,s}$ rather than $d\mu_{n,s}$. It follows from (1.39) that there is a solution $\tilde{s}_N$ to the equation $\nu^{\mathrm{F}}_{c,N} + \tilde{s}_N v_N = \nu_c$, which obeys $\tilde{s}_N \sim q$ for all $d \ge 4$. With FBC, Theorem 1.3 therefore gives a Gaussian limit at the infinite-volume critical point ν_c, unlike the non-Gaussian limit established in Theorem 1.2 for the critical window around the effective critical point $\nu^{\mathrm{F}}_{c,N}$, or around ν_c with PBC. In particular, ν_c is outside of the critical window centred at $\nu^{\mathrm{F}}_{c,N}$.

An exact analogue of (1.48) is proved in [6, Theorem 1.3(ii)] for the Euclidean (non-hierarchical) model with PBC in dimension $d = 4$.

1.5 | Discussion of main results

In this section we analyse the susceptibility and its universal profile, compute universal ratios of moments of the average field in the critical window, and discuss the case $n = 0$ of self-avoiding walk. This discussion is useful for the interpretation of our results, but none of it plays a role in the proofs of Theorems 1.1–1.3 which are given in the bulk of the paper.

1.5.1 | The susceptibility

The following corollary of Theorem 1.2 shows that the universal profile $n^{-1}\Sigma_{n,2}(s)$ governs the susceptibility in the critical window around the effective critical point, both for FBC and for PBC, and for all dimensions $d \geq 4$.

Corollary 1.4. *Let $d \geq 4$, let $n \geq 1$, let L be sufficiently large, and let $g > 0$ be sufficiently small (depending on L). Then for boundary conditions $* = \mathrm{F}$ and $* = \mathrm{P}$, and for s_N obeying the hypothesis of Theorem 1.2, as $N \to \infty$ the susceptibility obeys*

$$\chi_N^*(\nu_{c,N}^* + s_N w_N) = \begin{cases} n^{-1}\Sigma_{n,2}(s_N)(BN)^{1/2}L^{2N}[1 + o(1)] & (d = 4) \\ n^{-1}\Sigma_{n,2}(s_N)g_\infty^{-1/2}L^{Nd/2}[1 + o(1)] & (d > 4). \end{cases} \tag{1.52}$$

Proof. By (1.25) and by (1.43) with $p = 1$,

$$\chi_N^*(\nu_{c,N}^* + s_N w_N) = \frac{1}{n}L^{dN}\left\langle |\Phi_N|^2 \right\rangle_{g,\nu_{c,N}^*+s_N w_N,N}^* = L^{dN}\hbar_N^2 \frac{1}{n}\Sigma_{n,2}(s_N)\big(1 + o(1)\big), \tag{1.53}$$

which is the same as (1.52) due to the definition of $\hbar_N$ in (1.40). □

It follows from (1.32) and the definition of w_N that, for $s_N > 0$ with $s_N w_N \to 0$ (with the added restriction $s_N \geq \varepsilon > 0$ for $d = 4$), as $N \to \infty$ the infinite-volume susceptibility obeys

$$\chi_\infty(\nu_c + s_N w_N) \sim \begin{cases} s_N^{-1}(BN)^{1/2}L^{2N} & (d = 4) \\ s_N^{-1}g_\infty^{-1/2}L^{Nd/2} & (d > 4). \end{cases} \tag{1.54}$$

When $d > 4$, the above right-hand side is simply $1/s_N w_N$, whereas for $d = 4$, there is a logarithmic correction. To compare this with Corollary 1.4, we use the elementary fact (see 1.72) that

$$n^{-1}\Sigma_{n,2}(s) \sim \begin{cases} s^{-1} & (s \to +\infty) \\ n^{-1}|s| & (s \to -\infty), \end{cases} \tag{1.55}$$

so for $s_N \to +\infty$ obeying the first inequality of (1.41), it follows from (1.52) that

$$\chi_N^*(\nu_{c,N}^* + s_N w_N) \sim \begin{cases} s_N^{-1}(BN)^{1/2}L^{2N} & (d = 4) \\ s_N^{-1}g_\infty^{-1/2}L^{Nd/2} & (d > 4). \end{cases} \tag{1.56}$$

Comparison of (1.54) and (1.56) shows that, when $s_N \to +\infty$ subject to the first inequality of (1.41), the finite-volume and infinite-volume susceptibilities are asymptotically equivalent:

$$\chi_N^*(\nu_{c,N}^* + s_N w_N) \sim \chi_\infty(\nu_c + s_N w_N). \tag{1.57}$$

This demonstrates a crossover out of the critical window.

On the other hand, if $s_N \to -\infty$ obeys the second inequality of (1.41), then (1.52) and (1.55) give

$$\chi_N^*(\nu_{c,N}^* + s_N w_N) \sim \frac{1}{\Sigma_{n,2}(0)} |s_N| \chi_N^*(\nu_{c,N}^*). \tag{1.58}$$

This should be contrasted with the behaviour of the susceptibility when it is more naturally defined in terms of the truncated two-point function—this truncated susceptibility should have a peak inside the scaling window and decrease, not grow, as s_N becomes increasingly negative (see, e.g., [23, Section 4.4]). For $\nu < \nu_c$, our definition of χ_N does not subtract a term $L^{dN} M_+(\nu)^2$ with M_+ the spontaneous magnetisation which is predicted to obey $M_+ \sim |\varepsilon|^{1/2} |\log \varepsilon|^{\frac{3}{n+8}}$ for $d = 4$ and $M_+ \sim |\varepsilon|^{1/2}$ for $d > 4$ (in infinite volume). The linear growth of $\chi_N^*(\nu_c^* + s w_N)$ as $s \to -\infty$, shown in (1.58), is a manifestation of the growing magnetisation we expect to contribute to χ_N on the low-temperature side of the window, namely (for $d = 4$) $L^{4N}[(|s| w_N)^{1/2} N^{\frac{3}{n+8}}]^2 \asymp |s| L^{2N} N^{-\hat{\theta} + \frac{6}{n+8}} = |s| L^{2N} N^{1/2} \asymp |s| \chi_N(\nu_c)$. There is related discussion in [70, p. 768]. Thus (1.58) provides an intriguing peek into the low-temperature phase, whose full critical behaviour remains an outstanding open problem. For the 4-dimensional Ising model it has been proved that the spontaneous magnetisation obeys $c|\varepsilon|^{1/2} \le M_+ \le c'|\varepsilon|^{1/2} |\log|\varepsilon||^{3/2}$—see [4] and [36, Section 14.4]—but to our knowledge this has not been proved for the $|\varphi|^4$ model, even for $n = 1$.

1.5.2 | Universal profile for the susceptibility

Next, we establish basic properties of the universal profile for the susceptibility in (1.52), which we denote here by

$$f_n(s) = \frac{1}{n} \Sigma_{n,2}(s) = \frac{\int_{\mathbb{R}^n} |x|^2 e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx}{n \int_{\mathbb{R}^n} e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx}. \tag{1.59}$$

The fact that $f_n(s)$ is strictly decreasing, both as a function of s and as a function of n, is proved in Lemma A.1. Plots of f_n are also given in Appendix A.

For $k \in (-1, \infty)$ we define the integral

$$I_k(s) = \int_0^\infty x^k e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2} dx, \tag{1.60}$$

and rewrite the integrals defining $\Sigma_{n,k}(s)$ and $f_n(s)$ in polar form to obtain

$$\Sigma_{n,k}(s) = \frac{I_{k+n-1}(s)}{I_{n-1}(s)}, \qquad f_n(s) = \frac{I_{n+1}(s)}{n I_{n-1}(s)}. \tag{1.61}$$

Although our main concern is for positive integers n, the formula (1.61) for $f_n(s)$ remains meaningful for real values $n \in (0, \infty)$. Furthermore, integration by parts in the denominator of $f_n(s)$

gives

$$nI_{n-1}(s) = \int_0^\infty (x^{n+3} + sx^{n+1}) e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2} dx = I_{n+3}(s) + sI_{n+1}(s), \tag{1.62}$$

and therefore, for all $n > 0$,

$$f_n(s) = \frac{I_{n+1}(s)}{nI_{n-1}(s)} = \frac{I_{n+1}(s)}{I_{n+3}(s) + sI_{n+1}(s)}. \tag{1.63}$$

The right-hand side of (1.63) extends the definition of f_n from $n \in (0, \infty)$ to $n \in (-2, \infty)$.

In particular, for $n = 0$ the denominator becomes

$$I_3(s) + sI_1(s) = \int_0^\infty (x^3 + sx) e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2} dx = -\int_0^\infty d\big(e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2}\big) = 1, \tag{1.64}$$

so (1.63) gives

$$f_0(s) = I_1(s) = \int_0^\infty x e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2} dx. \tag{1.65}$$

As we discuss in Section 1.5.4, the case $n = 0$, although not included in our main results, has significance for the self-avoiding walk. In particular, f_0 arises as the profile for both strictly and weakly self-avoiding walk on the complete graph.

For $n > -2$, (1.63) provides the recursion relation

$$f_n(s) = \frac{1}{(n+2) f_{n+2}(s) + s}. \tag{1.66}$$

It follows from (1.66) that $\lim_{n \to -2} f_n(s) = s^{-1}$ for $s > 0$, and that this limiting function f_{-2} is maximal in the sense that $f_n(s) < s^{-1}$ for all $s > 0$ and $n > -2$. This profile $f_{-2}(s) = s^{-1}$ is consistent with the value $n = -2$ corresponding to a Gaussian model ($n = -2$ corresponds to a two-component fermion field for which the fourth power vanishes), since the susceptibility of the Gaussian model is simply $\chi(\nu) = \nu^{-1}$. Values of n in the interval $[-2, 0)$ are not completely understood. The model with $n = -2$ is identified as loop-erased random walk in [67] (which is also Gaussian for $d \geq 4$), and there is speculation about walk interpretations for $n \in (-2, 0)$ in [67, Section 2].

By the change of variables $t = x^4/4$, the integral $I_k(s)$ can be rewritten in terms of the *Faxén integral* (see [59, p. 332])

$$\mathrm{Fi}(\alpha, \beta; y) = \int_0^\infty e^{-t + yt^\alpha} t^{\beta - 1} dt \qquad (0 \leq \alpha < 1, \ \ \beta > 0) \tag{1.67}$$

as

$$I_k(s) = 2^{\frac{k-3}{2}} \, \mathrm{Fi}\big(\tfrac{1}{2}, \tfrac{k+1}{4}; -s\big). \tag{1.68}$$

Since $I_k(0) = 2^{\frac{k-3}{2}}\,\mathrm{Fi}(\frac{1}{2}, \frac{k+1}{4}; 0) = 2^{\frac{k-3}{2}}\,\Gamma(\frac{k+1}{4})$ for $k > -1$, $f_n(0)$ is given explicitly by

$$f_n(0) = \frac{\Gamma(\frac{n+2}{4})}{2\Gamma(\frac{n+4}{4})} \qquad (n > -2), \tag{1.69}$$

where we used (1.66) for $n \in (-2, 0]$. The known asymptotic behaviour of the Faxén integral (see [59, Ex. 7.3, p. 84]) gives, for $k > -1$,

$$I_k(s) \sim \begin{cases} 2^{\frac{k-1}{2}}\Gamma(\frac{k+1}{2})s^{-(k+1)/2} & (s \to \infty) \\ \sqrt{\pi}|s|^{(k-1)/2}e^{s^2/4} & (s \to -\infty), \end{cases} \tag{1.70}$$

and hence, by (1.61),

$$\Sigma_{n,k}(s) \sim \begin{cases} \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n}{2})}\left(\frac{2}{s}\right)^{k/2} & (s \to +\infty) \\ |s|^{k/2} & (s \to -\infty). \end{cases} \tag{1.71}$$

In particular, by setting $k = 2$ and by the recursion relation for the Gamma function,

$$f_n(s) \sim \frac{1}{s} \qquad (n > -2,\ s \to +\infty), \qquad\qquad f_n(s) \sim \frac{|s|}{n} \qquad (n > 0,\ s \to -\infty), \tag{1.72}$$

where again we used (1.66) for $n \in (-2, 0]$ and $s \to +\infty$.

1.5.3 | Universal ratios

This section applies for all dimensions $d \geq 4$. All statements in this section are valid for either choice $* = \mathrm{P}$ or $* = \mathrm{F}$ for the boundary condition, but to declutter the notation we do not place asterisks on the expectations or on $\nu_{c,N}$, which should always be interpreted as $\langle\,\cdot\,\rangle^*$ and $\nu^*_{c,N}$.

Theorem 1.2 allows for explicit computation of the universal ratios

$$\mathcal{R}^{(2p)}_n(s) = \lim_{N\to\infty} \frac{\left\langle |\Phi_N|^{2p}\right\rangle_{g,\nu_{c,N}+sw_N,N}}{\left\langle |\Phi_N|^2\right\rangle^p_{g,\nu_{c,N}+sw_N,N}} = \frac{\Sigma_{n,2p}(s)}{[\Sigma_{n,2}(s)]^p} \tag{1.73}$$

for $n \geq 1$, $p \geq 1$, and $s \in \mathbb{R}$. For $s = 0$, by (1.61) and with the value of $I_k(0)$ stated above (1.69), this gives

$$\mathcal{R}^{(2p)}_n(0) = \frac{\Gamma(\frac{n+2p}{4})[\Gamma(\frac{n}{4})]^{p-1}}{[\Gamma(\frac{n+2}{4})]^p}, \tag{1.74}$$

which is the same as the value stated in [70, (32.36)] for $\mathbb{Z}^d$ with PBC in dimensions $d > 4$.

In particular, by (1.74), $\mathcal{R}_n^{(4)}(0)$ (known in statistics as the *kurtosis*) takes the universal value

$$\mathcal{R}_n^{(4)}(0) = \frac{\Gamma(\frac{n+4}{4})\Gamma(\frac{n}{4})}{[\Gamma(\frac{n+2}{4})]^2} = \frac{4}{n}\left(\frac{\Gamma(\frac{n+4}{4})}{\Gamma(\frac{n+2}{4})}\right)^2 = \frac{1}{nf_n(0)^2}. \tag{1.75}$$

For $n = 1$ the reciprocal, $Q = 1/\mathcal{R}_1^{(4)}(0)$ is $Q = 4[\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})]^2 = 0.456\,947\ldots$. Related to $\mathcal{R}_n^{(4)}(0)$ is the *Binder cumulant* $U_n(s) = 1 - \frac{1}{3}\mathcal{R}_n^{(4)}(s)$ (in statistics $-3U_n(s)$ is the *excess kurtosis*).

A fundamental quantity is the *renormalised coupling constant*, which in finite volume and for general $n \geq 1$ is defined by

$$\lambda_{n,N}(s) = -\frac{\langle|\Phi_N|^4\rangle_{g,\nu_{c,N}+sw_N,N} - \frac{n+2}{2}\langle|\Phi_N|^2\rangle^2_{g,\nu_{c,N}+sw_N,N}}{\langle|\Phi_N|^2\rangle^2_{g,\nu_{c,N}+sw_N,N}}. \tag{1.76}$$

For a Gaussian distribution, $\lambda_N = 0$. In the limit as $N \to \infty$, (1.73) gives

$$\lambda_n(s) = \lim_{N\to\infty}\lambda_{n,N}(s) = \frac{n+2}{2} - \mathcal{R}_n^{(4)}(s) = \frac{n+2}{2} - \frac{\int_{\mathbb{R}^n}|x|^4 d\sigma_{n,s}}{(\int_{\mathbb{R}^n}|x|^2 d\sigma_{n,s})^2}. \tag{1.77}$$

In particular, for $n = 1$,

$$\lambda_1(0) = 3 - 4\left(\frac{\Gamma(5/4)}{\Gamma(3/4)}\right)^2 = 0.81156\ldots \qquad (d \geq 4). \tag{1.78}$$

Also, the asymptotic formulas (1.71) imply that

$$\lim_{s\to\infty}\lambda_n(s) = \frac{n+2}{n} - \frac{n+2}{n} = 0, \tag{1.79}$$

$$\lim_{s\to-\infty}\lambda_n(s) = \frac{n+2}{n} - 1 = \frac{2}{n}. \tag{1.80}$$

By the proof of Lemma A.1 (put $n = 1$ in A.2), the right-hand side of (1.77) is strictly monotone decreasing, so the renormalised coupling constant varies monotonically over the range $[0, 2/n]$ as the window is traversed. For $n = 1$, the Lebowitz and Griffiths inequalities imply that that $\lambda(s) \in [0, 2]$, so as s decreases through the window we see a value of the renormalised coupling constant that increases from the Gaussian value of zero to the maximum possible non-Gaussian value 2.

1.5.4 | Self-avoiding walk and $n = 0$

The value $n = 0$ is not part of our main results, which are restricted to $n \geq 1$ components. However, as we discuss in this section, there is compelling evidence of the universal nature of the profile f_0 for self-avoiding walk in dimensions $d \geq 4$. In particular, we confirm that f_0 is the profile for: (i) self-avoiding walk on the complete graph, and (ii) the continuous-time weakly self-avoiding walk (a.k.a. discrete Edwards model) on the complete graph.

Self-avoiding walk on the complete graph. The number of n-step self-avoiding walks on the complete graph $\mathbb{K}_N$ on N vertices, starting from a fixed origin, is simply $\prod_{j=1}^{n}(N-j)$ for $n \le N$. The susceptibility is the generating function for this sequence, that is,

$$\chi_N^{\mathbb{K}}(z) = \sum_{n=0}^{N-1} z^n \prod_{j=1}^{n}(N-j) \qquad (z \in \mathbb{R}). \tag{1.81}$$

The effective critical point is N^{-1}. The following proposition, which is proved in Appendix B, shows that the profile $f_0(s)$ applies in this setting, with window and susceptibility scaling as square root of the volume N (this corresponds to $L^{Nd/2}$ in (1.52)). The minus sign on the left-hand side of (1.82) compensates for the different monotonicity of z compared to $e^{-\nu}$.

Proposition 1.5. *For self-avoiding walk on the complete graph, as $N \to \infty$,*

$$\chi_N^{\mathbb{K}}(N^{-1}(1 - s(2N)^{-1/2})) \sim (2N)^{1/2} f_0(s) \qquad (s \in \mathbb{R}). \tag{1.82}$$

Continuous-time weakly self-avoiding walk on the complete graph. Let X denote the continuous-time random walk on the complete graph $\mathbb{K}_N$. The walk takes steps at the events of a rate-$(1 - N^{-1})$ Poisson process, with steps taken to a vertex chosen uniformly at random from all other vertices (the choice being independent of the Poisson process). The *local time* of X at a vertex $v \in \mathbb{K}_N$, up to time T, is the random variable

$$L_{T,v} = \int_0^T \mathbb{1}_{X(t)=v}\, dt. \tag{1.83}$$

For $x, y \in \mathbb{K}_N$, for fixed $g > 0$, and for $\nu \in \mathbb{R}$, the *two-point function* is

$$G_{xy}(\nu) = \int_0^\infty E_x\Big(e^{-g\sum_{v\in\mathbb{K}_N} L_{T,v}^2} \mathbb{1}_{X(T)=y}\Big) e^{-\nu T}\, dT. \tag{1.84}$$

By symmetry, the *susceptibility* is

$$\chi_N^{\mathrm{WSAW}}(\nu) = \sum_{y\in\mathbb{K}_N} G_{xy}(\nu) = G_{00}(\nu) + (N-1)G_{01}(\nu), \tag{1.85}$$

where 0,1 denote two distinct vertices of $\mathbb{K}_N$.

As we discuss in more detail in Appendix B, this model has a critical point $\nu_c = \nu_c(g)$. The following proposition shows that the universal profile $f_0(s)$ arises for the susceptibility in this model in its critical window. Note that the profile is independent of the value of $g > 0$, as also seen for $n \ge 1$ in Corollary 1.4. The proof of Proposition 1.6, which applies results of [11], is given in Appendix B.

Proposition 1.6. *Let $g > 0$. There are positive constants λ_1, λ_2 (explicit and g-dependent) such that*

$$\lim_{N\to\infty} N^{-1/2} \chi_N^{\mathrm{WSAW}}(\nu_c + sN^{-1/2}) = \lambda_1 f_0(\lambda_2 s) \qquad (s \in \mathbb{R}). \tag{1.86}$$

1.6 | Open problems

Our results suggest several open problems.

1.6.1 | Intermediate scales and the plateau

We have studied the average field Φ_N, which is averaged over the entire finite volume. It would be of interest to analyse the average field Φ_M on Λ_M for $1 \ll M \ll N$, that is, for intermediate scales.

Problem 1. Compute the scaling limit of the average field on intermediate scales.

Intermediate scales have been recently studied at the infinite-volume critical point for the 1-component φ^4 model on $\mathbb{Z}^4$ with FBC [3], where a generalised Gaussian limit is obtained. For the hierarchical model in dimensions $d \geq 4$ and for all $n \geq 1$, we expect that an extension of our method could be used to analyse the average field on intermediate scales at the effective critical point $\nu^*_{c,N}$ (for either FBC or PBC), and to prove that there is a crossover from Gaussian to non-Gaussian behaviour as M increases.

Problem 2. Prove the "plateau" phenomenon for the two-point function for all dimensions $d \geq 4$ and for all $n \geq 1$.

More precisely, prove that within the critical windows, for both FBC and PBC and for all $d \geq 4$ and $n \geq 1$, the finite-volume two-point function obeys (with s-dependent constants)

$$\langle \varphi_0 \cdot \varphi_x \rangle_{\nu^*_{c,N}+s_N w_N,N} \asymp |x|^{-(d-2)} + \begin{cases} N^{1/2}L^{-2N} & (d=4) \\ L^{-Nd/2} & (d>4). \end{cases} \tag{1.87}$$

The constant term $N^{1/2}L^{-2N}$ or $L^{-Nd/2}$ is the "plateau" term. Note that in summation of the right-hand side of (1.87) over $x \in \Lambda_N$, the constant term dominates the sum, and also the resulting sum has the same scaling as the susceptibility in (1.52). A related problem is to prove that, for FBC and general n, at the infinite-volume critical point (which is outside the FBC window) the constant plateau term is absent. (For $n = 1, 2$, the FBC two-point function is dominated by the infinite volume one, by Griffith's inequality, so the absence follows in this case.) For the hierarchical model with $n \geq 1$, we expect that an extension of our method could be used to prove (1.87).

Note added before publication: Problem 2 has now been solved by an extension of our method [60]; the universal profile f_n also gives the amplitude for the plateau term.

The effect of boundary conditions on the decay of the two-point function of statistical mechanical models in finite-volume above the upper critical dimension has been widely discussed, for example, in [57, 68, 69], and there has been some debate about the plateau. Early numerical evidence for a plateau with PBC can be seen in [56, fig. 4]. For the Ising model in dimensions $d > 4$ at the infinite-volume critical point, with FBC the absence of the plateau has been proved in [22], and with PBC a plateau lower bound is proved in [52] (for PBC a matching upper bound remains unproved). For (spread-out) percolation in dimensions $d > 6$ with PBC, an analogous plateau is proved to exist throughout the critical window in [48], whereas with FBC the absence of the plateau at the infinite-volume critical point is proved in [26]. For (spread-out) lattice trees and lattice animals in dimensions $d > 8$, the existence of a plateau is established in [54]. A general theory of the plateau is presented in [53].

1.6.2 | Extension to Euclidean models

Problem 3. Prove the results of Theorems 1.2 and 1.3 for the $|\varphi|^4$ model on $\mathbb{Z}^d$ for dimensions $d \geq 4$ and for $n \geq 1$. Analyse intermediate scales and prove the "plateau" phenomenon for the two-point function in this Euclidean setting. Do the same for the Ising, XY, and Heisenberg models, and more generally for all N-vector models.

We expect that the conclusions of Theorems 1.2 and 1.3 hold *verbatim* in the Euclidean setting for all these models, with the only change being a need to modify constant prefactors in the large-field scale $\hbar_N$, the window width w_N, and the FBC shift v_N of the effective critical point. However, to extend our methods to the Euclidean setting in order to prove this would require new ideas, despite the fact that a large part of our analysis has already been extended [20, 21]. One challenge in the Euclidean setting would be to improve the large-field regulator used in [20, 21], which bounds the non-perturbative RG coordinate by an exponentially *growing* factor (see [20, 1.38]) rather than the exponentially *decaying* factor that we exploit (see 5.25). Also, the covariance decomposition we use for the hierarchial model has a constant covariance at the final scale, whereas the decomposition used in [20, 21] does not have this useful feature. This issue could likely be overcome using the decomposition in [10, Section 3] which does have a constant covariance at scale N and which obeys similar estimates to the decomposition used in [20, 21] (see [10, Corollary 4.1] for $d = 2$ and [10, Proposition 3.4] for the general case).

For the Ising model at the infinite-volume critical point in dimensions $d > 4$, for PBC a lower bound on the susceptibility of order $L^{Nd/2}$ is proved in [52] (a matching upper bound has not yet been proved), and for FBC the L^{2N} behaviour of the susceptibility is proved in [22].

1.6.3 | Self-avoiding walk

Problem 4. Prove the statements of both Theorems 1.2 and 1.3 with $n = 0$ for the weakly self-avoiding walk in dimension $d \geq 4$, both on the hierarchical lattice and on $\mathbb{Z}^d$. Much more ambitiously, prove this for the strictly self-avoiding walk on $\mathbb{Z}^d$ in dimensions $d \geq 4$.

For $n = 0$, the values of the logarithmic correction exponents in (1.2) are $\hat{\gamma} = \frac{1}{4}$ and $\hat{\theta} = \frac{1}{4}$, which indicates window scaling $L^{-2N}N^{-1/4}$ for $d = 4$ and $L^{-Nd/2}$ for $d > 4$. The scaling of $\hbar_N$ remains as it is for $n \geq 1$, namely $L^{-2N}N^{1/4}$ for $d = 4$ and $L^{-Nd/4}$ for $d > 4$.

It is a consequence of [58, Theorem 1.4] that for $d > 4$ with PBC, the susceptibility of weakly self-avoiding walk is at least of order $L^{Nd/2}$ at the $\mathbb{Z}^d$ critical point. A matching upper bound has not been proved. For FBC, numerical results are presented in [69]. A related conjecture for the universality of the profile for the expected length of self-avoiding walks in dimensions $d > 4$ is investigated numerically in [29].

1.7 | Structure of proof and guide to the paper

The proof of our main results in Theorems 1.1–1.3 is based on a rigorous renormalisation group analysis. The RG analysis is a multiscale analysis based on a finite-range decomposition of the resolvent of the hierarchial Laplacian, which is used to perform expectations via progressive

integration over scales. The finite-range decomposition is described in Section 2, along with the form of the integration at the final scale. The difference between FBC and PBC appears only at this final scale, where there is a mass generation effect for FBC which is responsible for shifting $\nu_{c,N}^{\mathrm{F}}$ smaller than $\nu_{c,N}^{\mathrm{P}}$. The mass is produced due to the fact that the random walk with the FBC Laplacian is transient on Λ_N, with a defective distribution as discussed in Section 1.2.3. The random walk with PBC is recurrent and its distribution is not defective. The mass generation with FBC compared to PBC is apparent in Lemma 2.1. At the final scale, which is scale $\hat{N}$ and follows scale N, the remaining field to integrate is a constant $\mathbb{R}^n$-valued field.

In Section 3, we introduce the scale-N effective potential $V_N(\varphi) = |\Lambda_N|(\frac{1}{4}g_N|\varphi|^4 + \frac{1}{2}\nu_N|\varphi|^2)$ and its non-perturbative counterpart K_N. In Theorems 3.1 and 3.3 we summarise properties of g_N, ν_N, K_N that are proved in later sections. These theorems and further results in Section 3 also reveal properties of a renormalised (squared) mass which can in some cases be negative—negative mass is what permits negative values of s in the critical window in Theorem 1.2.

In Section 4, we prove our main results Theorems 1.1–1.3 assuming the deep results of Section 3. Because this requires only the integration over the final scale, which is an integral over $\mathbb{R}^n$, the analysis in Section 4 involves only calculus.

The principal part of our effort, which occupies approximately half the paper in Sections 5–9, is to establish the properties listed in Section 3 for the renormalised mass and for V_N and K_N. This entails the consideration of effective potentials V_j and non-perturbative coordinates K_j for each scale $j = 1, \dots, N$, and of the RG map that advances (V_j, K_j) to (V_{j+1}, K_{j+1}). Our RG analysis in Sections 5–9 follows the general approach laid out by Bauerschmidt, Brydges and Slade in [9]. Sections 5–9 depend in important ways on the results of [9], and we frequently refer to and rely on [9].

We develop important improvements to the method of [9]. Specific improvements include: (i) the introduction of a large-field regulator which captures the decay of K_j for large fields, (ii) the control of mass derivatives of V_j and K_j, and (iii) the inclusion of dimensions $d > 4$ and not only $d = 4$.

The latter point may appear to be minor, since for $d > 4$ the monomial $|\varphi|^4$ is irrelevant (in the RG sense) and there are no logarithmic corrections to deal with. However, $|\varphi|^4$ is a so-called *dangerous irrelevant monomial*, and its inclusion in the analysis requires care and innovation. The concept of a dangerous irrelevant variable was introduced in [37]. In our context, this concept is reflected by the fact that, although irrelevant, g gets renormalised even when $d > 4$: the constant g_∞ of (1.34) accumulates contributions of all orders. This g_∞ occurs in the window scale w_N in (1.35) and in the large-field scale $\hbar_N$ in (1.40). It is therefore centrally important in Theorem 1.2. Also, the irrelevance of $|\varphi|^4$ for $d > 4$ means that its contribution to the Hamiltonian scales down exponentially in the RG flow. Since the quartic term in the Hamiltonian is what gives stability to integrals, and since we are working in a regime where the quadratic term can have a bad sign (recall that $\nu_c < 0$), care is required in order to exploit the quartic term despite its irrelevance. We therefore must track the RG flow of the irrelevant quartic term carefully when $d > 4$. This makes the extension of the RG method from $d = 4$ to $d > 4$ nontrivial. For Euclidean models it would be worse: $d > 4$ also involves dangerously irrelevant terms that involve gradients of φ. These gradient terms do not arise in the hierarchical model.

In Section 5, we set up the necessary spaces and norms used to formulate the RG map, and we state estimates which control the RG map. The main bounds on the RG map are stated in Theorems 5.6–5.7.

In Section 6, the main bounds on the RG map are used to prove three propositions, Propositions 6.2–6.4, which themselves are used to prove Theorems 1.1–1.3. The proofs of Propositions 6.2–6.4 are given in Section 6, subject to Proposition 6.6 which captures properties of derivatives of the RG flow with respect to the initial value of ν and with respect to the mass. It then remains to prove the main RG results Theorems 5.6–5.7, as well as Proposition 6.6.

Section 7 is devoted to the proof of Proposition 6.6. The proof relies entirely on Theorems 5.6–5.7, and, although lengthy, is essentially computational.

Section 8 proves the first of the main RG theorems, Theorem 5.6. This theorem controls the perturbative part of the RG map, by estimating the corrections to the perturbative flow as the effective potential V_j is advanced to V_{j+1}. The proof involves the estimation of polynomials measured in the norms defined in Section 5. Related estimates that are needed to control the advance of the non-perturbative coordinate from K_j to K_{j+1} are also performed in Section 8.

Section 9 concludes the proof of Theorem 5.7 and thereby concludes the proof of our main results Theorems 1.1–1.3. It is in Section 9 that our analysis becomes the most sophisticated. This is not surprising, since it requires the control of an essentially infinite-dimensional problem in a complicated space whose norm tracks derivatives of K_{j+1} with respect to V_j, K_j, the mass, and the field φ. It is here that we rely most heavily on results from [9], although as mentioned previously we include new features: the large-field regulator, mass derivatives and dimensions $d > 4$.

The paper ends with two appendices. Appendix A proves a monotonicity theorem for the universal profile f_n. Appendix B proves the two Propositions 1.5 and 1.6 for self-avoiding walk which demonstrate its connection to the profile f_0.

2 | INTEGRATION TO THE FINAL SCALE

2.1 | Covariance decomposition and the hierarchical field

The progressive integration underlying the multi-scale analysis of the RG method has its basis in a decomposition of the resolvent of the hierarchical Laplacian, which we discuss next. Recall the definitions and notations of the hierarchical Laplacian from Section 1.2, which we specialise here to $\alpha = 2$. The following discussion applies for all dimensions $d \geq 1$.

Given $x \in \Lambda_N$ and given j between 0 and N, let $B_j(x)$ denote the unique block in $\mathcal{B}_j$ that contains x. We define the matrices of symmetric operators Q_j and P_j on $\ell^2(\Lambda_N)$ by

$$Q_{j;xy} = \begin{cases} L^{-dj} & B_j(x) = B_j(y) \\ 0 & B_j(x) \neq B_j(y) \end{cases} \quad (j = 0, 1, \dots, N), \tag{2.1}$$

$$P_j = Q_{j-1} - Q_j \quad (j = 1, \dots, N). \tag{2.2}$$

As is shown in [9, Lemma 4.1.5], the operators $P_1, \dots, P_N, Q_N$ are orthogonal projections whose ranges are disjoint and provide a direct sum decomposition of $\ell^2(\Lambda)$.

The PBC Laplacian defined in (1.17) can be seen after some arithmetic to have matrix elements

$$-\Delta^{\mathrm{P}}_{N;0x} = \begin{cases} \frac{1-L^{-d}}{1-L^{-(d+2)}}(1 - L^{-(d+2)N}) & (x = 0) \\ -\frac{L^2-1}{1-L^{-d}} L^{-(d+2)j_{0x}} - \frac{1-L^{-d}}{1-L^{-(d+2)}} L^{-(d+2)N} & (x \in \Lambda_N \setminus \{0\}). \end{cases} \tag{2.3}$$

The above is identical to the representation of the hierarchical Laplacian in [9, ((4.1.8)]. Thus [9, (4.1.7)] provides a representation of the Laplacian as a sum over scales, namely

$$-\Delta_N^{\mathrm{P}} = \sum_{j=1}^{N} L^{-2(j-1)} P_j. \tag{2.4}$$

By (1.19), the FBC Laplacian can therefore be written as

$$-\Delta_N^{\mathrm{F}} = \sum_{j=1}^{N} L^{-2(j-1)} P_j + qL^{-2N} Q_N. \tag{2.5}$$

Let

$$\gamma_j(a) = \frac{L^{2(j-1)}}{1 + aL^{2(j-1)}}. \tag{2.6}$$

It follows exactly as in [9, Proposition 4.1.9] that the resolvents of $-\Delta_N^{\mathrm{P}}$ and $-\Delta_N^{\mathrm{F}}$ have the decompositions

$$(-\Delta_N^{\mathrm{P}} + a)^{-1} = \sum_{j=1}^{N} \gamma_j(a) P_j + a^{-1} Q_N, \tag{2.7}$$

$$(-\Delta_N^{\mathrm{F}} + a)^{-1} = \sum_{j=1}^{N} \gamma_j(a) P_j + (a + qL^{-2N})^{-1} Q_N. \tag{2.8}$$

In order to ensure that $\gamma_j(a) > 0$ for all $j \in \{1, 2, \dots, N\}$, we assume that $a > -L^{-2(N-1)}$. We also assume for now that $a \neq 0$ and $a \neq -qL^{-2N}$ for the coefficients of Q_N terms, respectively, though we will find a way to relax this in Section 2.2. We use the variable a, rather than m^2 as in [9, Proposition 4.1.9], because unlike in [9] we allow both positive and negative values of the "mass" parameter a. The qL^{-2N} term in the coefficient of Q_N for FBC, which is absent for PBC, is ultimately the source of the shift in the effective critical point for FBC.

The matrices defined by

$$C_j(a) = \gamma_j(a) P_j, \qquad C_{\leq N}(a) = \sum_{j=1}^{N} C_j(a), \tag{2.9}$$

are symmetric, real and positive semi-definite. At the last scale we define

$$C_{\hat{N}}^*(a) = Q_N \times \begin{cases} a^{-1} & (* = \mathrm{P}) \\ (a + qL^{-\alpha N})^{-1} & (* = \mathrm{F}), \end{cases} \tag{2.10}$$

so that

$$(-\Delta_N^* + a)^{-1} = C_{\leq N} + C_{\hat{N}}^*. \tag{2.11}$$

The matrix $C_{\hat{N}}^{\mathrm{P}}$ is positive semi-definite for $a > 0$ and negative semi-definite for $a < 0$, whereas $C_{\hat{N}}^{\mathrm{F}}$ is positive semi-definite for $a > -qL^{-2N}$ and negative semi-definite for $a < -qL^{-2N}$. Note that

$-L^{-2(N-1)} < -qL^{-2N}$ when L is large (then q is close to 1), so $a > -qL^{-2N}$ is a greater restriction than the requirement $a > -L^{-2(N-1)}$ imposed in the previous paragraph.

By definition, $P_j \mathbb{1} = 0$ where $\mathbb{1}$ is the constant field $\mathbb{1}(x) = 1$ for all $x \in \Lambda_N$, and therefore also $C_j \mathbb{1} = 0$ for $j = 1, \dots, N$. The non-interacting ($g = 0$) massive susceptibility $\chi_N^{*,0}(a) = (-\Delta^* + a)^{-1}\mathbb{1}$ is therefore equal to $C_{\hat{N}}^* \mathbb{1}$, and hence

$$\chi_N^{\mathrm{P},0}(a) = \frac{1}{a}, \qquad \chi_N^{\mathrm{F},0}(a) = \frac{q^{-1}L^{2N}}{1 + aq^{-1}L^{2N}}. \tag{2.12}$$

The infinite-volume critical value for $g = 0$ is $a = 0$, and

$$\chi_N^{\mathrm{P},0}(0) = \infty, \qquad \chi_N^{\mathrm{F},0}(0) = q^{-1}L^{2N}, \qquad \chi_N^{\mathrm{F},0}(-qL^{-2N}) = \infty. \tag{2.13}$$

Thus we see that the non-interacting PBC critical value agrees with the infinite-volume value, whereas the FBC critical value is shifted by $-qL^{-2N}$. This is a harbinger of the shift in (1.39) for the interacting model, and it is ultimately responsible for that shift in the effective FBC critical value and its window.

Now an insignificant adaption of [9, Proposition 4.1.9] from $m^2 > 0$ to $a > -L^{-2(N-1)}$ shows that, for independent Gaussian fields ζ_j with covariance C_j, the field $\varphi = \zeta_1 + \cdots + \zeta_N$ is a *hierarchical field* in the sense that:

(i) on any two blocks $B, B' \in \mathcal{B}_j$ that are not identical, $\zeta_j|_B$ and $\zeta_j|_{B'}$ are independent for all $j = 1, \dots, N$, and
(ii) on any block $b \in \mathcal{B}_{j-1}$ the field ζ_j is constant, that is, $\zeta_{j,x} = \zeta_{j,y}$ for all $x, y \in b$, for all $j = 1, \dots, N$.

In addition, by [9, Exercise 4.1.6], the field ζ_j vanishes with probability 1 when summed over a block at scale j, namely:

$$\sum_{b \in \mathcal{B}_{j-1}(B)} \zeta_j(b) = 0 \quad \text{a.s. for all } B \in \mathcal{B}_j, \tag{2.14}$$

where $\mathcal{B}_{j-1}(B)$ is the set of $(j-1)$-blocks inside B. Degenerate Gaussian fields (with non-invertible covariance) are discussed in [9, Section 2.1]. In the above decomposition of the field φ we exclude a last field corresponding to $C_{\hat{N}}^*$, which is dealt with in Section 2.2.

2.2 | Integration at the final scale

Throughout this section we consider both FBC and PBC but to lighten the notation we indicate the boundary condition only sometimes for emphasis. Also, we often do not make the dependence on a or g explicit in the notation. To keep the covariance matrices $C_j(a)$ and $C_{\hat{N}}^*(a)$ positive semi-definite, we restrict a initially to $a > 0$ for PBC, and to $a + qL^{-2N} > 0$ for FBC, though later we will show that these restrictions can be relaxed.

Our goal is to study expectations (with either FBC or PBC)

$$\langle F \rangle_{\nu,N} = \frac{1}{Z_{\nu,N}} \int_{(\mathbb{R}^n)^{\Lambda_N}} F(\varphi) e^{-H_{\nu,N}(\varphi)} d\varphi, \tag{2.15}$$

where the Hamiltonian is

$$H_{\nu,N}(\varphi) = \frac{1}{2}(\varphi, (-\Delta_N)\varphi) + V_0(\varphi), \tag{2.16}$$

with

$$V_0(\varphi) = \sum_{x\in\Lambda_N} \Big(\frac{1}{4}g|\varphi_x|^4 + \frac{1}{2}\nu|\varphi_x|^2\Big). \tag{2.17}$$

The hierarchical Laplacian in (2.16) acts component-wise on the field φ. Let

$$Z_0(\varphi) = \exp(-V_0(\varphi)). \tag{2.18}$$

With $\mathbb{E}_C$ denoting expectation with respect to the Gaussian measure with covariance $C = (-\Delta_N + a)^{-1}$, the definitions lead to

$$\langle F\rangle_{\nu+a,N} = \frac{\mathbb{E}_C F Z_0}{\mathbb{E}_C Z_0}. \tag{2.19}$$

The Gaussian expectation $\mathbb{E}_C$ can be computed in two steps as

$$\mathbb{E}_C F = \mathbb{E}_{C_{\hat{N}}}\mathbb{E}_{C_{\leq N}} F(\varphi + \hat{\varphi}), \tag{2.20}$$

where $\mathbb{E}_{C_{\leq N}}$ involves integration over φ, and $\mathbb{E}_{C_{\hat{N}}}$ involves integration over $\hat{\varphi}$. In particular,

$$\mathbb{E}_C Z_0(\psi) e^{(J\mathbb{1},\psi)} = \mathbb{E}_{C_{\hat{N}}}\mathbb{E}_{C_{\leq N}} Z_0(\varphi + \hat{\varphi}) e^{(J\mathbb{1},\varphi+\hat{\varphi})}, \tag{2.21}$$

where on the left-hand side ψ is a dummy integration variable, and on the right-hand side $\mathbb{1}$ is (again) the constant field with value 1. By completing the square in the integral $\mathbb{E}_{C_{\leq N}}$ (as in [9, Exercise 2.1.10]), and since $C_{\leq N}\mathbb{1} = 0$, we see that

$$\begin{aligned}\mathbb{E}_{C_{\leq N}} Z_0(\varphi + \hat{\varphi}) e^{(J\mathbb{1},\varphi+\hat{\varphi})} &= e^{(J\mathbb{1},\hat{\varphi})} e^{\frac{1}{2}(J\mathbb{1}, C_{\leq N} J\mathbb{1})} \mathbb{E}_{C_{\leq N}} Z_0(\varphi + \hat{\varphi} + C_{\leq N} J\mathbb{1}) \\ &= e^{(J\mathbb{1},\hat{\varphi})} \mathbb{E}_{C_{\leq N}} Z_0(\varphi + \hat{\varphi}).\end{aligned} \tag{2.22}$$

With the definition

$$Z_N(\hat{\varphi}; a) = Z_N(\hat{\varphi}) = \mathbb{E}_{C_{\leq N}} Z_0(\varphi + \hat{\varphi}), \tag{2.23}$$

this leads to the identity

$$\mathbb{E}_C Z_0(\psi) e^{(J\mathbb{1},\psi)} = \mathbb{E}_{C_{\hat{N}}} e^{(J\mathbb{1},\hat{\varphi})} Z_N(\hat{\varphi}). \tag{2.24}$$

For $a > 0$ with PBC, and for $a > -qL^{-2N}$ with FBC, the matrix $C^*_{\hat{N}}(a)$ has rank 1 so the expectation $\mathbb{E}_{C_{\hat{N}}}$ is supported on constant fields (the *zero mode*) and therefore reduces to an integral over $\mathbb{R}^n$. This is reflected by our notation in Lemma 2.1 (and also later) where for simplicity inside integrals we evaluate Z_N on $y \in \mathbb{R}^n$ rather than on a constant field $y \in (\mathbb{R}^n)^{\Lambda_N}$. The lemma relaxes the restrictions on a to the interval $a \in (-L^{-2(N-1)}, \infty)$ on which the covariance $C_{\leq N}(a)$ is positive semi-definite. We occasionally write

$$\Omega_N = L^{dN} \tag{2.25}$$

for the volume of Λ_N.

Lemma 2.1. *Let $d \geq 1$, $g > 0$, $\nu \in \mathbb{R}$, $a \in (-L^{-2(N-1)}, \infty)$, and $J \in \mathbb{R}^n$. Then*

$$\left\langle e^{J\cdot\Phi_N}\right\rangle^{\mathrm{P}}_{\nu+a,N} = \frac{\int_{\mathbb{R}^n} e^{J\cdot y}\, Z_N(y) e^{-\frac{1}{2}a\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}a\Omega_N|y|^2} dy}, \tag{2.26}$$

$$\left\langle e^{J\cdot\Phi_N}\right\rangle^{\mathrm{F}}_{\nu+a,N} = \frac{\int_{\mathbb{R}^n} e^{J\cdot y}\, Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2 dy}}, \tag{2.27}$$

where Z_N is defined in (2.23) using the covariance $C_{\leq N}(a)$ and with ν in (2.17).

Proof. We give the proof for PBC; the proof for FBC is completely analogous. Consider first the case $a > 0$ (or $a + qL^{-2N} > 0$ for FBC), and set $\sigma^2 = a^{-1}\Omega_N^{-1}$. Our starting point is (2.19) and (2.24), from which we see that

$$\left\langle e^{J\cdot\Phi_N}\right\rangle_{g,\nu+a,N} = \frac{\mathbb{E}_{C_{\hat{N}}} e^{(J\mathbb{1},\hat{\varphi})/\Omega_N} Z_N(\hat{\varphi})}{\mathbb{E}_{C_{\hat{N}}} Z_N(\hat{\varphi})}. \tag{2.28}$$

Now we use the fact that under $C_{\hat{N}}$ the field $\hat{\varphi}$ has the same distribution as the constant field $y\mathbb{1}$ where $y \in \mathbb{R}^n$ has independent components each with a normal distribution with mean zero and variance σ^2. We write this Gaussian measure on $\mathbb{R}^n$ as $d\mu_{\sigma^2}(y)$. The exponent in the right-hand side of (2.28) simplifies to $J \cdot y$, so

$$\mathbb{E}_{C_{\hat{N}}} e^{(J\mathbb{1},\hat{\varphi})} Z_N(\hat{\varphi}) = \int_{\mathbb{R}^n} e^{J\cdot y} Z_N(y) d\mu_{\sigma^2}(y). \tag{2.29}$$

For $J = 0$ this is simply

$$\mathbb{E}_C Z_0(\varphi) = \int_{\mathbb{R}^n} Z_N(y) d\mu_{\sigma^2}(y), \tag{2.30}$$

and the above three equations give the desired result when $a > 0$.

To extend the result to $a \in (-L^{-2(N-1)}, 0]$, we use an analytic continuation argument, as follows. By its definition in (2.15), the function $f : a \mapsto \langle F\rangle_{\nu+a,N}$ is meromorphic in $a \in \mathbb{C}$, since both numerator and denominator in (2.15) are entire functions of ν. It is sufficient to show that the function of a appearing in the right-hand side of (2.26), which we denote by g, is meromorphic on the domain $\mathcal{H}_N = \{z \in \mathbb{C},\ \mathrm{Re}(z) > -L^{-2(N-1)}\}$, since this implies that the difference $f - g$ is meromorphic on $\mathcal{H}_N$ and equal to zero on $[0, \infty)$, so identically zero on $\mathcal{H}_N \supset (-L^{-2(N-1)}, \infty]$.

To see that g is meromorphic, we first extend the definition of $Z_N(y)$ from a function of real a to a function of complex a by noting that the measure

$$\exp\Big(-\frac{1}{2}(\varphi, C_{\leq N}(a)^{-1}\varphi)\Big)\, d\varphi \tag{2.31}$$

is integrable over $\mathbb{R}^n$ provided that $a \in \mathcal{H}_N$, since this ensures that $\mathrm{Re}(\gamma_j(a)) > 0$ for all $j \leq N$. This measure is supported on the orthogonal complement of the kernel of $C_{\leq N}(a)$ as in [9, (2.1.3)], which is the same as $\mathrm{Im}(Q_N)^\perp$ by the observation below (2.2). It then follows that $Z_N(y; a)$ is holomorphic on $\mathcal{H}_N$ by application of Morera's (and Fubini's) theorem together with the facts that $a \mapsto \exp(-\frac{1}{2}(\varphi, C_{\leq N}(a)^{-1}\varphi))$ is holomorphic on $\mathcal{H}_N$ and that $\exp(-\frac{1}{4}g\sum_x |\varphi_x|^4 - \frac{1}{2}\nu\sum_x |\varphi_x|^2)$ is

bounded and integrable with respect to the measure (2.31). We now wish to conclude that the two functions

$$g_1(a) = \int_{\mathbb{R}^n} e^{J\cdot y} Z_N(y;a) e^{-\frac{1}{2}a\Omega_N y^2} dy, \qquad g_2(a) = \int_{\mathbb{R}^n} Z_N(y;a) e^{-\frac{1}{2}a\Omega_N y^2} dy \tag{2.32}$$

are holomorphic in $a \in \mathcal{H}_N$, since then their ratio $g = g_1/g_2$ is meromorphic on $\mathcal{H}_N$, as desired. To see this, by another application of Morera's theorem it is sufficient to prove that

$$\int_{\mathbb{R}^n} F(y) Z_N(y;a) e^{-\frac{1}{2}a\Omega_N y^2} dy \tag{2.33}$$

is integrable for any function F growing at most exponentially. This last fact follows from

$$\begin{aligned} &\Big| \int_{\mathbb{R}^n \times \mathrm{Im}(Q_N)^\perp} F(y) Z_0(\varphi + y\mathbb{1}) e^{-\frac{1}{2}(\varphi, C_{\le N}(a)^{-1}\varphi) - \frac{1}{2}a\Omega_N y^2} d(y \otimes \varphi) \Big| \\ &\quad \le \int_{\mathbb{R}^n \times \mathrm{Im}(Q_N)^\perp} |F(y)| Z_0(\varphi + y\mathbb{1}) e^{\frac{1}{2}|a|\Omega_N y^2} d(y \otimes \varphi), \end{aligned} \tag{2.34}$$

by noting that $\mathbb{R}^n \times \mathrm{Im}(Q_N)^\perp \cong \mathbb{R}^{n\Lambda_N}$ (under $(y,\varphi) \mapsto y\mathbb{1} + \varphi$) and that the integrand is integrable over $\mathbb{R}^{n\Lambda_N}$ since $g > 0$ and $\sum_x |\varphi_x + y|^4 \ge \sum_x (|\varphi_x|^4 + |y|^4)$ (the dot product $\varphi \cdot y\mathbb{1}$ vanishes). This completes the proof. □

We use multi-index notation. For $p \ge 1$, let A_p denote the set of p-tuples $\alpha = (\alpha_1, \dots, \alpha_p)$ with each $\alpha_i \in \{1, \dots, n\}$. Given $\varphi : \Lambda_N \to \mathbb{R}^n$ and $\alpha \in A_p$, we write

$$\Phi_N^\alpha = \frac{1}{\Omega_N^p} \sum_{x_1,\dots,x_p \in \Lambda_N} \varphi_{x_1}^{(\alpha_1)} \cdots \varphi_{x_p}^{(\alpha_p)}, \tag{2.35}$$

where $\varphi_{x_i}^{(\alpha_i)}$ denotes the α_i^{th} component of $\varphi_{x_i} \in \mathbb{R}^n$. Similarly, for $y \in \mathbb{R}^n$ we write $y^\alpha = y^{(\alpha_1)} \cdots y^{(\alpha_p)}$.

Corollary 2.2. *Let $d \ge 1$, $g > 0$, $\nu \in \mathbb{R}$, $p \ge 1$, $\alpha \in A_p$, and $a \in (-L^{-2(N-1)}, \infty)$. Then*

$$\left\langle \Phi_N^\alpha \right\rangle_{\nu+a,N}^{\mathrm{P}} = \frac{\int_{\mathbb{R}^n} y^\alpha Z_N(y) e^{-\frac{1}{2}a\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}a\Omega_N|y|^2} dy}, \tag{2.36}$$

$$\left\langle \Phi_N^\alpha \right\rangle_{\nu+a,N}^{\mathrm{F}} = \frac{\int_{\mathbb{R}^n} y^\alpha Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2 dy}}, \tag{2.37}$$

where Z_N is defined in (2.23) using the covariance $C_{\le N}(a)$ and with ν in (2.17).

Proof. The identities (2.36)–(2.37) follow simply by differentiation with respect to J in Lemma 2.1. □

3 | THE EFFECTIVE POTENTIAL AND THE RENORMALISED MASS

The proof of our main results is given in Section 4, based on Theorems 3.1, 3.2 and 3.3 which are proved by extending the RG analysis of the 4-dimensional hierarchical n-component $|\varphi|^4$ model from [9]. The proofs of Theorems 3.1–3.3 are given in Sections 5–9.

3.1 | The effective potential at the final scale

To apply Lemma 2.1 and its corollary, we need a good understanding of Z_N. Since the choice of BC enters only in the last integral in Lemma 2.1, Z_N itself is independent of the boundary condition. Theorems 3.1–3.3, concern the behaviour of Z_N, and are thus free of any mention to BC. We express Z_N in terms of a scale-N effective potential V_N, together with a non-perturbative coordinate $K_N : \mathbb{R}^n \to \mathbb{R}$.

Theorem 3.1 produces a critical value $\nu_c(a)$, for each choice of $a \geq 0$, such that Z_N can be controlled for all scales N when Z_N is defined with the initial ν-value in V_0 tuned to equal $\nu_c(a)$. It extends the results from [9] by the inclusion of an additional large-field decay of the form $e^{-\kappa g_N \Omega_N |\varphi|^4}$ in the bounds on K_N in (3.8) and (3.12), by the addition of the case $d > 4$ to the statement, and finally by the control of the mass dependence of the critical point via its derivative. Theorem 3.2 proves the existence and provides the asymptotic behaviour of a renormalised mass $m_\varepsilon^2 \geq 0$ such that $\nu_c(0) + \varepsilon = m_\varepsilon^2 + \nu_c(m_\varepsilon^2)$. Theorem 3.3 is analogous to Theorem 3.1 but now determines the behaviour of Z_N also for negative and small N-dependent mass a. The counterpart of $\nu_c(a)$ in Theorem 3.3 is $\nu_{0,N}(a)$ which is also well-understood by estimates on its derivative. An adequate notion of renormalised mass, that is, a version of Theorem 3.2 for $\nu_{0,N}(a)$ rather than $\nu_c(a)$, is developed later, in Section 3.3. In all three theorems, logarithmic corrections occur for $d = 4$ but not in higher dimensions.

Recall that $\Omega_N = L^{dN}$ denotes the volume. Given $g_N, \nu_N \in \mathbb{R}$, we define

$$V_N(\varphi) = \Omega_N\big(\tfrac{1}{4} g_N |\varphi|^4 + \tfrac{1}{2}\nu_N |\varphi|^2\big) \qquad (\varphi \in \mathbb{R}^n). \tag{3.1}$$

We also define the volume-dependent mass interval

$$\mathbb{I}_{N-1}(0) = \big(-\tfrac{1}{2}L^{-2(N-1)}, \tfrac{1}{2}L^{-2(N-1)}\big) \subset \mathbb{R}. \tag{3.2}$$

To state estimates on the non-perturbative coordinate K_N, we define

$$e_N = \begin{cases} N^{-3/4} & (d = 4) \\ g^{3/4} L^{-3(d-4)N/4} & (4 < d \leq 12) \\ g^{3/4} L^{-Nd/2} & (d > 12). \end{cases} \tag{3.3}$$

We believe that the second option actually applies for all $d > 4$, and that for $d > 12$ our estimates on K_N are not sharp (they are, however, sufficient for our needs).

For the statement of part (i) of the following theorem, given squared mass $a \in \mathbb{R}$, we define the *mass scale* $j_a \in \mathbb{Z}$ by

$$j_a = \max\{j \in \mathbb{Z} : L^{2j} a \leq 1\}, \tag{3.4}$$

with the degenerate case $j_a = \infty$ if $a \le 0$. By definition, $j_a < 0$ when $a > 1$. We do not state estimates on the *vacuum energy* u_N appearing in (3.5), as it cancels in numerator and denominator when calculating expectations as in (2.36)–(2.37). A detailed computation of u_N is given in [60, Theorem 2.4].

Theorem 3.1 (Final scale for nonnegative mass). *Let $d \ge 4$ and $n \ge 1$. Fix L sufficiently large, $g > 0$ sufficiently small. There exists a continuous strictly increasing function $\nu_c(a)$ of $a \ge 0$ which is continuously differentiable for $a > 0$ if $d = 4$ and for $a \ge 0$ if $d > 4$, and there exist (u_N, g_N, ν_N, K_N) (all depending on a), such that Z_N defined by (2.23) with the choice a for the squared mass in the covariance $C_{\le N}$ and with the choice $\nu = \nu_c(a)$ for the ν in Z_0, satisfies*

$$Z_N(\varphi) = e^{-u_N|\Lambda_N|}\big(e^{-V_N(\varphi)} + K_N(\varphi)\big) \tag{3.5}$$

for all $\varphi \in \mathbb{R}^n$ with, uniformly in $a \ge 0$ and for some $\kappa = \kappa(n)$:

(i) For $d = 4$, with e_N defined in (3.3), there exists $A_4 > 0$ such that

$$g_N = \begin{cases} (BN)^{-1}(1 + O(N^{-1}\log N)) & (a = 0) \\ B^{-1}(N \wedge j_a)^{-1}(1 + O((N \wedge j_a)^{-1}\log(N \wedge j_a))) & (j_a \in [1, \infty)) \\ g + O(g^2) & (j_a \le 0), \end{cases} \tag{3.6}$$

$$|\nu_N| \le O(g_N L^{-2N}), \tag{3.7}$$

$$|K_N(\varphi)| \le O(e_N) e^{-\kappa g_N \Omega_N |\varphi|^4}, \tag{3.8}$$

$$\frac{d}{da}\nu_c(a) = A_4(\log a^{-1})^{\hat{\gamma}}(1 + o(1)) \qquad \text{as } a \downarrow 0. \tag{3.9}$$

(ii) For $d > 4$, with e_N defined in (3.3), there exist $A_d > 1$ and a continuous function g_∞ of $a \ge 0$ satisfying $g_\infty(a) = g + O(g^2)$, such that

$$g_N = g_\infty + O(g^2 L^{-(d-4)N}), \tag{3.10}$$

$$|\nu_N| \le O(gL^{-(d-2)N}), \tag{3.11}$$

$$|K_N(\varphi)| \le O(e_N) e^{-\kappa g_N \Omega_N |\varphi|^4}, \tag{3.12}$$

$$\frac{d}{da}\nu_c(a) = (A_d - 1)(1 + o(1)) \qquad \text{as } a \downarrow 0. \tag{3.13}$$

Also, for $a \in [0, 1]$, the following bound (useful when $a = o(L^{-N})$) relates the massive g_N and massless g_∞:

$$g_N(a) = g_\infty(0) + O(g^2 L^{-(d-4)N}) + O(g^2 a L^N). \tag{3.14}$$

Finally, the critical value $\nu_c(0)$ and constants A_d (for $d \ge 4$) satisfy the asymptotic formulas (1.30) as $g \downarrow 0$.

Theorem 3.2. *In the setting of Theorem 3.1, there exists a strictly increasing function m_ε^2 of $\varepsilon \geq 0$, with $m_0^2 = 0$, such that*

$$\nu_c(0) + \varepsilon = \nu_c(m_\varepsilon^2) + m_\varepsilon^2, \tag{3.15}$$

with

$$m_\varepsilon^2 \sim \begin{cases} A_4^{-1}\,\varepsilon\,(\log \varepsilon^{-1})^{-\hat{\gamma}} & (d=4) \\ A_d^{-1}\,\varepsilon & (d>4) \end{cases} \qquad (\text{as } \varepsilon \downarrow 0). \tag{3.16}$$

Theorem 3.3 (Final scale for mass $a \in \mathbb{I}_{N-1}(0)$). *Let $d \geq 4$ and $n \geq 1$. Fix L sufficiently large, $g > 0$ sufficiently small. There exists a continuously differentiable strictly increasing function $\nu_{0,N}(a)$ of $a \in \mathbb{I}_{N-1}(0)$ with $\nu_{0,N}(0) = \nu_c(0)$, and there exist (u_N, g_N, ν_N, K_N) (all depending on a), such that Z_N defined by (2.23) with the choice a for the mass in the covariance $C_{\leq N}$ and with the choice $\nu = \nu_{0,N}(a)$ for the ν in Z_0, satisfies*

$$Z_N(\varphi) = e^{-u_N|\Lambda_N|}\big(e^{-V_N(\varphi)} + K_N(\varphi)\big) \tag{3.17}$$

for all $\varphi \in \mathbb{R}^n$ with, uniformly in a and for some $\kappa = \kappa(n)$:

(i) For $d = 4$, with e_N defined in (3.3) and with $c^{\mathrm{F}} = O(g)$,

$$g_N = \frac{1}{BN} + O(N^{-2}\log N), \tag{3.18}$$

$$|\nu_N| \leq O(g_N L^{-2N}), \tag{3.19}$$

$$|K_N(\varphi)| \leq O(e_N) e^{-\kappa g_N \Omega_N |\varphi|^4}, \tag{3.20}$$

$$\frac{d}{da}\nu_{0,N}(a) = A_4(\log L^2)^{\hat{\gamma}} N^{\hat{\gamma}} + c^{\mathrm{F}} - 1 + O(N^{\hat{\gamma}-1}). \tag{3.21}$$

(ii) For $d > 4$, with e_N defined in (3.3),

$$g_N(a) = g_\infty(0) + O(g^2 L^{-N}), \tag{3.22}$$

$$|\nu_N| \leq O(gL^{-(d-2)N}), \tag{3.23}$$

$$|K_N(\varphi)| \leq O(e_N) e^{-\kappa g_N \Omega_N |\varphi|^4}, \tag{3.24}$$

$$\frac{d}{da}\nu_{0,N}(a) = (A_d - 1)(1 + O(L^{-N})). \tag{3.25}$$

We connect the two critical values ν_c and $\nu_{0,N}$ by defining

$$\nu_{1,N}(a) = \begin{cases} \nu_c(a) & (a \geq 0) \\ \nu_{0,N}(a) & (a \in (-\frac{1}{2}L^{-2(N-1)}, 0)). \end{cases} \tag{3.26}$$

By Theorems 3.1 and 3.3, the function $\nu_{1,N} : (-\frac{1}{2}L^{-2(N-1)}, \infty) \to \mathbb{R}$ is continuous, and it is also differentiable except possibly at 0.

3.2 | Effective critical points

We define the effective critical points for PBC and FBC by

$$\nu_{c,N}^{\mathrm{P}} = \nu_c(0), \qquad \nu_{c,N}^{\mathrm{F}} = \nu_{0,N}(-qL^{-2N}) - qL^{-2N}. \tag{3.27}$$

The choice of $\nu_c(0)$ as the effective critical point for PBC is natural. For FBC, we first observe that $q = 1 + O(L^{-d})$ by (1.18), so $-qL^{-2N} \in \mathbb{I}_{N-1}(0)$ and thus $\nu_{c,N}^{\mathrm{F}}$ is well-defined.

To explain our choice of $\nu_{c,N}^{\mathrm{F}}$, we recall from (2.27) that

$$\left\langle e^{J\cdot\Phi_N} \right\rangle_{\nu+a,N}^{\mathrm{F}} = \frac{\int_{\mathbb{R}^n} e^{J\cdot y}\, Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2 dy}}. \tag{3.28}$$

The massless choice in (3.28) is $a = -qL^{-2N}$, since it causes the exponential factor to become simply $e^0 = 1$. With this choice, to apply Theorem 3.3 we must tune ν to the correct value $\nu = \nu_{0,N}(-qL^{-2N})$ in order to represent Z_N in terms of e^{-V_N} and K_N with good estimates. In particular $K_N \to 0$. With this tuning, we obtain

$$\nu + a = \nu_{0,N}(-qL^{-2N}) - qL^{-2N}, \tag{3.29}$$

which is our definition of the FBC effective critical point $\nu_{c,N}^{\mathrm{F}}$.

The following corollary and remark justify the formulas for $\nu_{c,N}^{\mathrm{F}}$ in (1.39), where we recall from (1.37) the definition of v_N.

Corollary 3.4. *The effective critical point for FBC satisfies*

$$\nu_{c,N}^{\mathrm{F}} = \begin{cases} \nu_c - qv_N(1 + c^{\mathrm{F}}N^{-\hat{\gamma}} + O(N^{-1})) & (d = 4) \\ \nu_c - qv_N(1 + O(L^{-N})) & (d > 4). \end{cases} \tag{3.30}$$

Proof. By definition and by the Fundamental Theorem of Calculus,

$$\begin{aligned} \nu_{c,N}^{\mathrm{F}} &= \nu_c(0) + [\nu_{0,N}(-qL^{-2N}) - \nu_{0,N}(0)] - qL^{-2N} \\ &= \nu_c(0) - \int_{-qL^{-2N}}^{0} \Big(\frac{d}{da}\nu_{0,N}(s) + 1\Big) ds. \end{aligned} \tag{3.31}$$

The desired result follows by substituting (3.21) and (3.25) in the right-hand side, together with (1.37), the definition of v_N. □

Remark 3.5. In (1.39), for $d = 4$ we defined

$$\nu_{c,N}^{\mathrm{F}} = \nu_c - qv_N(1 + c^{\mathrm{F}}N^{-\hat{\gamma}}) \tag{3.32}$$

without the $O(N^{-1})$ error present in (3.30). In the rest of the paper, we use the exact definition of $\nu_{c,N}^{\mathrm{F}}$ in (3.27) as it enables simplifications later on. This does not affect the validity of our claims in Section 1.4 since the neglected error term is $O(N^{\hat{\gamma}-1}L^{-2N}) = O(N^{-1/2}w_N)$ which is smaller than the window scale w_N. Similar reasoning applies for $d = 5$, for which the relation between v_N and w_N in (1.38) gives $L^{-N}v_N \asymp L^{-N/2}w_N$. However, for $d > 5$, by (1.38) $O(L^{-N}v_N)$ is of the same order as w_N for $d = 6$ and is larger than w_N for $d > 6$. This issue is due to the fact that our error bound is not sharp for $d > 5$. We therefore retain the error term $O(L^{-N})$ in (1.39) when $d > 5$.

3.3 | The renormalised masses

The effective critical points $\nu_{c,N}^{\mathrm{P}}$ and $\nu_{c,N}^{\mathrm{F}}$ are defined in Section 3.2. For FBC, $\nu_{c,N}^{\mathrm{F}} = \nu_{0,N}(-qL^{-2N}) - qL^{-2N}$ is defined in terms of a *renormalised mass* $-qL^{-2N}$, with $\nu_{0,N}$ evaluated at this renormalised mass. In this section, for both FBC and PBC we show how to define the renormalised mass corresponding to a value of ν which need not be the effective critical point, and we state asymptotic properties of the renormalised mass in Proposition 3.9. The renormalised mass and its asymptotic properties play an important role in Section 4 where we prove our main results.

The need for the renormalised mass goes back to the formula (2.19). After making explicit the dependence on the Hamiltonian's quadratic coefficient ν' and on the mass a, (2.19) expresses $\langle F\rangle_{\nu'+a,N}$ in terms of an expectation $\mathbb{E}_{C(a)}FZ_0(\nu')$. We want to use this formula for $\langle F\rangle_{\nu,N}$ with a given value ν, which must therefore be expressed as a sum $\nu = \nu' + a$ in a useful way. The term "useful" means that given a we require that ν' be the critical value $\nu_c(a)$ from Theorem 3.1 (or $\nu_{0,N}(a)$ from Theorem 3.3), as needed in order to apply the theorem. Values of ν that we need to consider are of the form $\nu_{c,N}^* + sw_N$ or $\nu_{c,N}^* + sv_N$. The renormalised mass achieves the useful decomposition. As an instance of this, for PBC with $s > 0$ the renormalised mass $a_N^{\mathrm{P}}(s)$ will achieve the desired decomposition $\nu_c + sw_N = \nu_c(a_N^{\mathrm{P}}(s)) + a_N^{\mathrm{P}}(s)$.

Recall the function $\nu_{1,N} : (-\frac{1}{2}L^{-2(N-1)}, \infty) \to \mathbb{R}$ from (3.26), which is continuous, strictly increasing, and also differentiable on $(-\frac{1}{2}L^{-2(N-1)}, 0) \cup (0, \infty)$ for all $d \geq 4$. The same applies to the function $a \mapsto \nu_{1,N}(a) + a$, which must therefore be a bijection onto its range, which is the interval $R_N = (\nu_{0,N}(-\frac{1}{2}L^{-2(N-1)}) - \frac{1}{2}L^{-2(N-1)}, \infty)$. For $*\in \{\mathrm{F}, \mathrm{P}\}$, we define intervals in $\mathbb{R}$ by

$$\mathcal{W}_N^* = \{s \in \mathbb{R} : \nu_{c,N}^* + sw_N \in R_N\}, \qquad \tilde{\mathcal{W}}_N^* = \{s \in \mathbb{R} : \nu_{c,N}^* + sv_N \in R_N\}, \tag{3.33}$$

and define the renormalised masses as the inverse maps to these bijections, as in the next definition.

Definition 3.6. The *renormalised masses* are the functions $a_N^* : \mathcal{W}_N^* \to (-\frac{1}{2}L^{-2(N-1)}, \infty)$ and $\tilde{a}_N^* : \tilde{\mathcal{W}}_N^* \to (-\frac{1}{2}L^{-2(N-1)}, \infty)$ given by the unique solutions to

$$\nu_{c,N}^* + sw_N = \nu_{1,N}(a_N^*(s)) + a_N^*(s) \text{ and } \qquad \nu_{c,N}^* + sv_N = \nu_{1,N}(\tilde{a}_N^*(s)) + \tilde{a}_N^*(s), \tag{3.34}$$

respectively.

Since they are inverses of increasing bijections, the two renormalised masses are also increasing bijections. In particular, since $\nu_{c,N}^{\mathrm{F}} < \nu_c = \nu_{c,N}^{\mathrm{P}}$ by (3.30), we see that $a_N^{\mathrm{F}}(s) < a_N^{\mathrm{P}}(s)$ and $\tilde{a}_N^{\mathrm{F}}(s) < \tilde{a}_N^{\mathrm{P}}(s)$.

In Definition 3.6, there is a point of non-differentiability when the mass is zero due to the non-differentiability of $\nu_{1,N}(a)$ at $a = 0$. This non-differentiability is due to the somewhat discretionary definition of $\nu_{1,N}$ as the concatenation of $\nu_{0,N}$ and ν_c at $a = 0$. We deal with the two regimes separately.

Since $\nu^{\mathrm{P}}_{c,N} = \nu_c(0)$, it follows from the monotonicity in s that $a^{\mathrm{P}}_N(s) \le 0$ if and only if $s \le 0$. Similarly, $\tilde{a}^{\mathrm{P}}_N(s) \le 0$ if and only if $s \le 0$. Thus $s = 0$ is the point of regime splitting for PBC. For FBC, we also define threshold values s^{F}_N and $\tilde{s}^{\mathrm{F}}_N$, at which a^{F}_N and $\tilde{a}^{\mathrm{F}}_N$ change signs. Again, by monotonicity in s, these are defined by

$$\nu^{\mathrm{F}}_{c,N} + s^{\mathrm{F}}_N w_N = \nu_c(0), \qquad \nu^{\mathrm{F}}_{c,N} + \tilde{s}^{\mathrm{F}}_N \upsilon_N = \nu_c(0). \tag{3.35}$$

We summarise the above considerations with the following lemma.

Lemma 3.7. *Let $d \ge 4$, and recall $\mathcal{W}^*_N$ and $\tilde{\mathcal{W}}^*_N$ from (3.33).*

(i) For $s \in \mathcal{W}^*_N$, the mass $a^*_N(s)$ is strictly increasing in s and satisfies $a^{\mathrm{P}}_N(s) \le 0$ if and only if $s \le 0$, and $a^{\mathrm{F}}_N(s) \le 0$ if and only if $s \le s^{\mathrm{F}}_N$.
(ii) For $s \in \tilde{\mathcal{W}}^*_N$, the mass $\tilde{a}^*_N(s)$ is strictly increasing in s and satisfies $\tilde{a}^{\mathrm{P}}_N(s) \le 0$ if and only if $s \le 0$, and $\tilde{a}^{\mathrm{F}}_N(s) \le 0$ if and only if $s \le \tilde{s}^{\mathrm{F}}_N$.

Asymptotic formulas can also be obtained for s^{F}_N and $\tilde{s}^{\mathrm{F}}_N$. Indeed, by directly solving the defining relations (3.35), and by using the formula for $\nu^{\mathrm{F}}_{c,N}$ in (3.30) and the definitions of w_N and υ_N in (1.35) and (1.37), we find that, as $N \to \infty$,

$$s^{\mathrm{F}}_N \sim qL^{(d-2)N}\hbar^2_N = \begin{cases} q(BN)^{1/2} & (d=4) \\ qg_\infty^{-\frac{1}{2}} L^{N(d-4)/2} & (d>4), \end{cases} \tag{3.36}$$

$$\tilde{s}^{\mathrm{F}}_N \sim qL^{(d-2)N}\ell^2_N = q \qquad (d \ge 4).$$

Here, as in (1.40) and (1.46),

$$\hbar_N = \begin{cases} (BN)^{1/4}L^{-N} & (d=4) \\ g_\infty^{-1/4}L^{-Nd/4} & (d>4), \end{cases} \qquad \ell_N = L^{-N(d-2)/2}. \tag{3.37}$$

The next lemma is used in the proof of Proposition 3.9. Its hypothesis that $s \ge -\frac{1}{4}L^2\hbar^2_N L^{N(d-2)}$ can be written more explicitly as $s \ge -\frac{1}{4}L^2(BN)^{1/2}$ for $d = 4$, and as $s \ge -\frac{1}{4}L^2 g_\infty^{-1/2} L^{N(d-4)/2}$ for $d > 4$, so for all $d \ge 4$ the bound diverges to $-\infty$ as $N \to \infty$. Lemma 3.8 provides a range of s values for which the renormalised masses lie in the interval $\mathbb{I}_{N-1}(0)$ for which Theorem 3.3 applies. In particular, the interval $[-\frac{1}{4}L^2\hbar^2_N L^{Nd/2}, \infty)$ is a subset of $\mathcal{W}^{\mathrm{F}}_N \cap \mathcal{W}^{\mathrm{P}}_N$, and the interval $[-\frac{1}{4}L^2, \infty)$ is a subset of $\tilde{\mathcal{W}}^{\mathrm{F}}_N \cap \tilde{\mathcal{W}}^{\mathrm{P}}_N$. This shows that the masses $a^*_N(s)$ and $\tilde{a}^*_N(s)$ are well-defined for s in these intervals, respectively.

Lemma 3.8. *Let $d \ge 4$ and $* \in \{\mathrm{F}, \mathrm{P}\}$. If we have $s \in [-\frac{1}{4}L^2\hbar^2_N L^{N(d-2)}, 0]$ for PBC, or if we have $s \in [-\frac{1}{4}L^2\hbar^2_N L^{N(d-2)}, s^{\mathrm{F}}_N]$ for FBC, then $a^*_N(s) \in \mathbb{I}_{N-1}(0)$. Also, if $s \in [-\frac{1}{4}L^2, 0]$ for PBC, or if $s \in [-\frac{1}{4}L^2, s^{\mathrm{F}}_N]$ for FBC, then $\tilde{a}^*_N(s) \in \mathbb{I}_{N-1}(0)$.*

Proof. By Lemma 3.7, our assumptions on s imply that the renormalised masses are not positive, so (by monotonicity of the mass in s) we only need to verify the conclusion for the lower limits $\underline{\sigma} = -\frac{1}{4}L^2 h_N^2 L^{N(d-2)}$ (for a_N^*) and $\underline{\sigma} = -\frac{1}{4}L^2$ (for $\tilde{a}_N^*$). Since the renormalised mass for PBC is larger than the mass for FBC, it suffices to consider only FBC. We give the details only for a_N^{F}, as similar but simpler calculations apply for $\tilde{a}_N^{\mathrm{F}}$.

Let $\sigma_{\inf} = \inf\{s \in \mathbb{R} : a_N^{\mathrm{F}}(s) \in \mathbb{I}_{N-1}(0)\}$. By continuity of a_N^* and by Lemma 3.7, $\sigma_{\inf} < s_N^{\mathrm{F}}$. Our goal now is to show that $\underline{\sigma} > \sigma_{\inf}$.

Suppose first that $d = 4$. Given $s \in (\sigma_{\inf}, s_N^{\mathrm{F}})$, the Fundamental Theorem of Calculus and (3.21) give

$$\nu_{0,N}(a_N^{\mathrm{F}}(s)) - \nu_{0,N}(0) = a_N^{\mathrm{F}}(s) A_4 (\log L^2)^{\hat{\gamma}} N^{\hat{\gamma}} (1 + o(1)). \tag{3.38}$$

Since $\nu_{0,N}(a_N^{\mathrm{F}}(s)) + a_N^{\mathrm{F}}(s) = \nu_{c,N}^{\mathrm{F}} + s w_N$ and $\nu_{0,N}(0) = \nu_c$ by definition, this gives

$$\begin{aligned} \nu_{c,N}^{\mathrm{F}} + s w_N &= \nu_c + a_N^{\mathrm{F}}(s) A_4 (\log L^2)^{\hat{\gamma}} N^{\hat{\gamma}} (1 + o(1)) \\ &= \nu_c + a_N^{\mathrm{F}}(s) (BN)^{1/2} L^{2N} w_N (1 + o(1)), \end{aligned} \tag{3.39}$$

where we used the definition of w_N in (1.35) for the second equality. By the formula $\nu_{c,N}^{\mathrm{F}} = \nu_c - q v_N (1 + o(1))$ in (3.30), together with the relation $v_N = (BN)^{1/2} w_N$ from (1.38), we can rewrite the above equation as

$$s = (BN)^{1/2} \Big[q(1 + o(1)) + a_N^{\mathrm{F}}(s) L^{2N} (1 + o(1)) \Big]. \tag{3.40}$$

Now we choose s_0 so that $a_N^{\mathrm{F}}(s_0) = -\frac{3}{8} L^{-2(N-1)}$. This $a_N^{\mathrm{F}}(s_0)$ is in $\mathbb{I}_{N-1}(0)$ by definition, so $s_0 \geq \sigma_{\inf}$ and it suffices to prove that $\underline{\sigma} \geq s_0$. But, since $q = 1 + O(L^{-d})$, s_0 obeys

$$s_0 = -(BN)^{1/2} \Big[\frac{3}{8} L^2 - q \Big] (1 + o(1)) < -\frac{1}{4} (BN)^{1/2} L^2 = \underline{\sigma} \tag{3.41}$$

if L is large enough. This completes the proof for $d = 4$.

For $d > 4$ the proof is similar. Now the Fundamental Theorem of Calculus and (3.25) lead, as above, to

$$\nu_{c,N}^{\mathrm{F}} + s w_N - a_N^{\mathrm{F}}(s) = \nu_c + a_N^{\mathrm{F}}(s)(A_d - 1)(1 + o(1)). \tag{3.42}$$

In turn, again by (3.30), this leads to $s w_N \sim q v_N + a_N^{\mathrm{F}}(s) A_d$ as $N \to \infty$. With s_0 chosen such that $a_N^{\mathrm{F}}(s_0) = -\frac{3}{8} L^2 g_\infty^{-1/2} L^{N(d-4)/2}$, we find that

$$s_0 = -g_\infty^{-1/2} L^{N(d-4)/2} \Big[\frac{3}{8} L^2 A_d - q \Big], \tag{3.43}$$

and the proof is completed for $d > 4$ in the same manner as for $d = 4$. □

In addition to the assumption in Lemma 3.8 that s does not become too negative, our analysis of the renormalised mass in the next proposition requires that $a_N^*(s) \to 0$ and $\tilde{a}_N^*(s) \to 0$. By (3.34), these requirements are satisfied respectively if $s w_N \to 0$ or $s v_N \to 0$. In fact, for $d = 4$, we assume

more. This leads us to define the s-intervals

$$\Sigma_N = \begin{cases} (-\frac{1}{4}L^2\hbar_N^2 L^{N(d-2)}, L^{\sqrt{N}}) & (d=4) \\ (-\frac{1}{4}L^2\hbar_N^2 L^{N(d-2)}, o(w_N^{-1})) & (d>4), \end{cases}$$
$$\tilde{\Sigma}_N = \begin{cases} (-\frac{1}{4}L^2, L^{\sqrt{N}}) & (d=4) \\ (-\frac{1}{4}L^2, o(v_N^{-1})) & (d>4). \end{cases} \tag{3.44}$$

The upper bound $L^{\sqrt{N}}$ for $d=4$ above has not been optimised but is sufficient for our needs. For the $d>4$ upper bounds, it is sufficient to fix any sequence which is $o(1)$ as $N\to\infty$, and take the upper bounds to be $o(1)w_N^{-1}$ and $o(1)v_N^{-1}$. In Proposition 3.9, the appearance of $L^{-\sqrt{N}}$ as the boundary between (3.46) and (3.47) is a convenient but non-canonical choice and we invite the reader to focus on (3.45) and (3.46).

Proposition 3.9. *Let $d\geq 4$ and $s\in\Sigma_N$. Then $a_N^*(s)$ obeys the following estimates. For FBC,*

$$a_N^{\mathrm{F}}(s) = s\hbar_N^{-2}L^{-dN}(1+o(1)) - qL^{-2N}. \tag{3.45}$$

For PBC, if $d\geq 4$ and $s\notin(0,L^{-\sqrt{N}})$,

$$a_N^{\mathrm{P}}(s) = s\hbar_N^{-2}L^{-dN}(1+o(1)). \tag{3.46}$$

For $d=4$ with $s\in(0,L^{-\sqrt{N}})$, (3.46) is supplemented by the inequality

$$0\leq a_N^{\mathrm{P}}(s)\leq s\hbar_N^{-2}L^{-dN}(1+o(1)) \qquad (d=4). \tag{3.47}$$

In addition, (3.45)–(3.47) all hold with the replacement of $a_N^*(s)$, Σ_N, $\hbar_N$ by $\tilde{a}_N^*(s)$, $\tilde{\Sigma}_N$, ℓ_N, respectively.

Proof. Let y_N denote either w_N or v_N. In the case $y_N=v_N$, $a_N(s)$ should be replaced by $\tilde{a}_N(s)$ and s_N^{F} by $\tilde{s}_N^{\mathrm{F}}$. We separate the analysis into two parts depending on the sign of $a_N^*(s)$, which by Lemma 3.7 corresponds to studying $a_N^{\mathrm{P}}(s)$ above or below 0, or $a_N^{\mathrm{F}}(s)$ above or below s_N^{F}. Note that $a_N^*(s)\to 0$ since $sy_N\to 0$ by the definition of y_N and by our assumption on s.

Case $s\leq s_N^{\mathrm{F}}$ if $*=\mathrm{F}$ and $s\leq 0$ if $*=\mathrm{P}$. In this case $a_N^*(s)\leq 0$ by Lemma 3.7 and $a_N^*(s)$ is the unique value of a satisfying

$$\nu_{0,N}(a) + a = \nu_{c,N}^* + sy_N. \tag{3.48}$$

By the Implicit Function Theorem, $a_N^*(s)$ is differentiable in $s<0$ $(*=\mathrm{P})$ or in $s<s_N^{\mathrm{F}}$ $(*=\mathrm{P})$, and by the chain rule, its derivative satisfies

$$\frac{d}{ds}a_N^*(s)\Big(\frac{d}{da}\nu_{0,N}(a)+1\Big) = y_N \tag{3.49}$$

for this range of s-values. By Lemma 3.8, $a_N^*(s) \in \mathbb{I}_{N-1}(0)$. Thus, by inserting the derivative estimates for $\frac{d}{da}\nu_{0,N}(a)$ from (3.21) and (3.25) (valid since $a_N^*(s) \to 0$) into (3.49), we see that

$$\frac{d}{ds}a_N^*(s) = \begin{cases} A_4^{-1} y_N (\log L^2)^{-\hat{\gamma}} N^{-\hat{\gamma}} (1 + o(1)) & (d = 4) \\ A_d^{-1} y_N (1 + o(1)) & (d > 4). \end{cases} \tag{3.50}$$

We consider first the case $y_N = w_N$. For all $d \geq 4$, it follows from the definitions of w_N and $\hbar_N$ that, for $y_N = w_N$, (3.50) simplifies to

$$\frac{d}{ds}a_N^*(s) = \hbar_N^{-2} L^{-dN}(1 + o(1)). \tag{3.51}$$

For PBC, we integrate between 0 and $s < 0$ and deduce from (3.50) and the Fundamental Theorem of Calculus that

$$a_N^{\mathrm{P}}(s) - a_N^{\mathrm{P}}(0) = -\int_s^0 \frac{d}{du}a_N^{\mathrm{P}}(u)du = s\hbar_N^{-2} L^{-dN}(1 + o(1)). \tag{3.52}$$

This gives the desired result for PBC because $a_N^{\mathrm{P}}(0) = 0$. For FBC and $s < s_N^{\mathrm{F}}$, since $a_N^{\mathrm{F}}(0) = -qL^{-2N}$, we can again integrate on the interval $[0, s]$ if $s \geq 0$ or on $[s, 0]$ if $s < 0$ to obtain

$$a_N^{\mathrm{F}}(s) + qL^{-2N} = s\hbar_N^{-2} L^{-dN}(1 + o(1)). \tag{3.53}$$

This proves (3.45)–(3.46) for w_N in this case. The computations are analogous for $\tilde{a}_N(s)$ and $s \in \tilde{\Sigma}_N$, with ℓ_N instead of $\hbar_N$. In particular, the definitions of υ_N and ℓ_N imply that, for all $d \geq 4$, (3.51) is replaced by

$$\frac{d}{ds}\tilde{a}_N^*(s) = \ell_N^{-2} L^{-dN}(1 + o(1)), \tag{3.54}$$

and also, by (3.34), $\tilde{a}_N^{\mathrm{P}}(0) = 0$ and $\tilde{a}_N^{\mathrm{F}}(0) = -qL^{-2N}$. This completes the proof of (3.45)–(3.46) in this case.

Case $s > s_N^{\mathrm{F}}$ if $* = \mathrm{F}$ and $s > 0$ if $* = \mathrm{P}$. In this case $a_N^*(s) > 0$ by Lemma 3.7, so $a_N^*(s)$ is the unique value of a satisfying

$$\nu_c(a) + a = \nu_{c,N}^* + sy_N. \tag{3.55}$$

As before, we use the Implicit Function Theorem and the derivative estimates for $\frac{d}{da}\nu_c(a)$ in (3.9) and (3.13) to obtain

$$\frac{d}{ds}a_N^*(s) = \begin{cases} A_4^{-1} y_N (-\log a_N^*(s))^{-\hat{\gamma}} (1 + o(1)) & (d = 4) \\ A_d^{-1} y_N (1 + o(1)) & (d > 4). \end{cases} \tag{3.56}$$

When $d > 4$, we have the same formula as in (3.50), and hence obtain the same conclusion. For the remaining case of $d = 4$, we define $\psi : [0, 1] \mapsto \mathbb{R}$ by

$$\psi(x) = x(-\log x)^{-\hat{\gamma}}. \tag{3.57}$$

For PBC we integrate over $[0, s]$ and obtain (see [9, p. 261] for an elementary proof)

$$a_N^{\mathrm{P}}(s) = \psi(A_4^{-1} s y_N)(1 + o(1)). \tag{3.58}$$

The logarithm in ψ is

$$(-\log(A_4^{-1}sy_N))^{-\hat{\gamma}} = (-\log y_N)^{-\gamma}\Big(1+\frac{\log s}{\log y_N}\Big)^{-\gamma}(1+o(1)). \tag{3.59}$$

It is the $\log s$ that will create the distinction between smaller and larger positive s in the statement of the proposition for PBC. By hypothesis our hypothesis that $s \in \Sigma_N$, we have $s \le L^{\sqrt{N}}$. For $s \in [L^{-\sqrt{N}}, L^{\sqrt{N}}]$

$$\Big|\frac{\log s}{\log y_N}\Big| \le O(N^{-1}\times\sqrt{N}) = o(1). \tag{3.60}$$

We first consider the case $y_N = w_N$. By definition,

$$(-\log w_N)^{-\gamma} = (\log L^2)^{-\hat{\gamma}}N^{\hat{\gamma}}(1+o(1)), \tag{3.61}$$

so with the above restrictions on s,

$$\psi(A_4^{-1}sw_N) = A_4^{-1}sw_N(\log L^2)^{-\hat{\gamma}}N^{-\hat{\gamma}}(1+o(1)) = s\hbar_N^{-2}L^{-4N}(1+o(1)). \tag{3.62}$$

This proves (3.46) for $d=4$. For $s \in (0, L^{-\sqrt{N}})$ the left-hand side of (3.60) can become arbitarily large and we obtain instead the inequality

$$0 \le a_N^{\mathrm{P}}(s) \le s\hbar_N^{-2}L^{-4N}(1+o(1)). \tag{3.63}$$

This proves (3.47) for PBC and $y_N = w_N$, and the same proof applies with minor adjustments for $y_N = \upsilon_N$.

For FBC with $y_N = w_N$ and $s - s_N^{\mathrm{F}} > L^{-\sqrt{N}}$, we instead integrate over $[s_N^{\mathrm{F}}, s]$ and similarly obtain

$$a_N^{\mathrm{F}}(s) = \psi(A_4^{-1}(s-s_N^{\mathrm{F}})y_N)(1+o(1)). \tag{3.64}$$

The calculations are then identical with s replaced by $s - s_N^{\mathrm{F}}$, and give

$$a_N^{\mathrm{F}}(s) = (s-s_N^{\mathrm{F}})\hbar_N^{-2}L^{-4N}(1+o(1)). \tag{3.65}$$

By (3.36),

$$s_N^{\mathrm{F}}\hbar_N^{-2}L^{-dN} = qL^{-2N}(1+o(1)), \tag{3.66}$$

and hence when $s > s_N^{\mathrm{F}}$ the error $o(qL^{-2N})$ can be absorbed by $o(s\hbar_N^{-2}L^{-4N})$. This gives

$$a_N^{\mathrm{F}}(s) = -qL^{-2N} + s\hbar_N^{-2}L^{-4N}(1+o(1)). \tag{3.67}$$

For $s \in (s_N^{\mathrm{F}}, s_N^{\mathrm{F}} + L^{-\sqrt{N}})$ we have instead the bound

$$0 \le a_N^{\mathrm{F}}(s) \le -qL^{-2N} + s\hbar_N^{-2}L^{-4N}(1+o(1)), \tag{3.68}$$

which we can rewrite as

$$qL^{-2N} \le a_N^{\mathrm{F}}(s) + qL^{-2N} \le s\hbar_N^{-2}L^{-4N}(1+o(1)). \tag{3.69}$$

But by (3.66) and our assumption that s is close to s_N^{F}, we can rewrite the lower bound as

$$qL^{-2N} = \frac{s_N^{\mathrm{F}}}{s} s\hbar_N^{-2} L^{-4N}(1 + o(1)) = s\hbar_N^{-2} L^{-4N}(1 + o(1)). \tag{3.70}$$

The lower and upper bounds in (3.69) therefore match, and this completes the proof of (3.45) for FBC, and for $y_N = w_N$. The proof is analogous for $y_N = v_N$. This completes the proof. □

3.4 | The critical mass domain

In order to accommodate sequences $s_N \to +\infty$ in Theorems 1.2 and 1.3, we need to work on a larger mass interval than the interval $\mathbb{I}_{N-1}(0) = (-\frac{1}{2}L^{-2(N-1)}, \frac{1}{2}L^{-2(N-1)})$ of Theorem 3.3. The larger interval we use is

$$\mathbb{I}_{\mathrm{crit}} = \begin{cases} (-\frac{1}{2}L^{-2(N-1)}, L^{-2N}L^{o(N)}) & (d = 4) \\ (-\frac{1}{2}L^{-2(N-1)}, 2L^{-3N/2}) & (d > 4). \end{cases} \tag{3.71}$$

The $L^{o(N)}$ for $d = 4$ is present in order to make the interval larger than if the upper bound were just L^{-2N}. For concreteness, it will appear in the proof of Lemma 3.11 that the choice $L^{\sqrt{N}}$ for $L^{o(N)}$ is permitted and adequate. By definition, $\mathbb{I}_{N-1}(0) \subset \mathbb{I}_{\mathrm{crit}}$, with the inclusion strict due to the larger positive values permitted in $\mathbb{I}_{\mathrm{crit}}$ compared to $\mathbb{I}_{N-1}(0)$. These larger values are restricted so that the bounds in the next lemma hold, and those bounds will play a role in Section 4.1. Lemma 3.10 shows that for a mass $a \in \mathbb{I}_{\mathrm{crit}}$, the coupling constant $g_N(a)$ is close to its massless counterpart. It is for this reason that the interval bears a subscript indicating criticality.

Lemma 3.10. *For $a \in \mathbb{I}_{\mathrm{crit}}$, let $\nu = \nu_{1,N}(a)$, and let g_N be as in Theorem 3.1 if $a > 0$ and as in Theorem 3.3 if $a \le 0$. Then*

$$g_N(a) = \begin{cases} (BN)^{-1}(1 + O(N^{-1/2})) & (d = 4) \\ g_\infty(0)(1 + O(gL^{-N/2})) & (d > 4). \end{cases} \tag{3.72}$$

Proof. Let $a \in \mathbb{I}_{\mathrm{crit}}$. Suppose first that $a \le 0$. For $d = 4$, (3.72) is an immediate consequence of (3.18). For $d > 4$, (3.72) is a consequence of (3.22) since the error term in (3.22) is better than in (3.72).

Suppose next that $a > 0$. For $d > 4$, (3.72) follows from (3.14) applied to a mass a that is the upper bound $2L^{-3N/2}$ of $\mathbb{I}_{\mathrm{crit}}$. Finally, for $d = 4$, when $a \in (0, L^{-2N+N^{1/2}})$ the mass scale j_a (defined in 3.4) obeys $N \wedge j_a = N + O(N^{1/2})$, so (3.72) follows from (3.6). This completes the proof. □

Recall the intervals Σ_N and $\tilde{\Sigma}_N$ defined in (3.44). We now define intervals $S_N \subset \Sigma_N$ and $\tilde{S}_N \subset \tilde{\Sigma}_N$ by

$$S_N = \begin{cases} (-\frac{1}{4}L^2\hbar_N^2 L^{N(d-2)}, L^{\sqrt{N}}) & (d = 4) \\ (-\frac{1}{4}L^2\hbar_N^2 L^{N(d-2)}, L^{N(d-3)/2}) & (d > 4), \end{cases} \tag{3.73}$$

$$\tilde{S}_N = \begin{cases} (-\frac{1}{4}L^2, L^{\sqrt{N}}) & (d=4) \\ (-\frac{1}{4}L^2, L^{N/2}) & (d>4). \end{cases} \tag{3.74}$$

The intervals S_N and $\tilde{S}_N$ are balanced to serve two purposes: (i) they are small enough to be included in Σ_N and $\tilde{\Sigma}_N$ respectively, so Proposition 3.9 can be applied, and (ii) they are large enough to allow $s_N \to +\infty$, and in such a way that $a_N(s_N)$ and $\tilde{a}_N(s_N)$ remain inside $\mathbb{I}_{\text{crit}}$. This second point requires the S domains to be strict subsets of the larger Σ domains. The upper bounds for S_N and $\tilde{S}_N$ are somewhat arbitrary and have not been optimised; they match the upper bounds in the restrictions on s_N in (1.41) and (1.47) that occur in our main results. The lower bounds for S_N have N dependence of order $-N^{1/2}$ $(d=4)$ and $-L^{N(d-4)/2}$ $(d>4)$ so are considerably more generous than (1.41); they will be further restricted in the proofs of Theorems 1.2 and 1.3, which appear in Section 4.2.

Lemma 3.11. *Let $d \geq 4$. For both choices of BC, if $s \in S_N$ then $a_N^*(s) \in \mathbb{I}_{\text{crit}}$, and if $s \in \tilde{S}_N$ then $\tilde{a}_N^*(s) \in \mathbb{I}_{\text{crit}}$.*

Proof. By Definition 3.6, $a_N^{\text{F}}(s)$ and $a_N^{\text{P}}(s)$ map into $(-\frac{1}{2}L^{-2(N-1)}, \infty)$. We therefore only need to verify that their range do not exceed the upper limits of the interval $\mathbb{I}_{\text{crit}}$. We write $\mathbb{I}_{\text{crit}}$ as $(\underline{\alpha}, \overline{\alpha})$, where according to (3.71) and the choice $L^{o(N)} = L^{\sqrt{N}}$ indicated below (3.71),

$$\overline{\alpha} = \begin{cases} L^{-2N(1-N^{-1/2})} & (d=4) \\ L^{-3N/2} & (d>4). \end{cases} \tag{3.75}$$

We write y_N to represent w_N or v_N. Let $\overline{\sigma}$ denote the upper limit of S_N or $\tilde{S}_N$ (as context dictates), that is,

$$\overline{\sigma} = L^{\sqrt{N}} \quad (d=4), \qquad \overline{\sigma} = \begin{cases} L^{N(d-3)/2} & (d>4,\ y_N = w_N) \\ L^{N/2} & (d>4,\ y_N = v_N). \end{cases} \tag{3.76}$$

Since $a_N^{\text{F}}(s) < a_N^{\text{P}}(s)$, it suffices to prove that $a_N^{\text{P}}(\overline{\sigma}) < \overline{\alpha}$ and $\tilde{a}_N^{\text{P}}(\overline{\sigma}) < \overline{\alpha}$.

For $d > 4$, we see from Proposition 3.9 that $a_N^{\text{P}}(\overline{\sigma}) \sim \overline{\sigma} g_\infty^{1/2} L^{-dN/2}$ for $y_N = w_N$, and $\tilde{a}_N^{\text{P}}(\overline{\sigma}) \sim \overline{\sigma} L^{-2N}$ for $y_N = v_N$. Therefore,

$$\begin{aligned} a_N^{\text{P}}(\overline{\sigma}) &\sim g_\infty^{1/2} L^{-3N/2} && (y_N = w_N) \\ \tilde{a}_N^{\text{P}}(\overline{\sigma}) &\sim L^{-3N/2} && (y_N = v_N). \end{aligned} \tag{3.77}$$

Since g_∞ is small, this completes the proof for $d > 4$. For $d = 4$, we have instead

$$\begin{aligned} a_N^{\text{P}}(\overline{\sigma}) &\sim L^{\sqrt{N}} (BN)^{-1/2} L^{-2N} && (y_N = w_N) \\ \tilde{a}_N^{\text{P}}(\overline{\sigma}) &\sim L^{\sqrt{N}} L^{-2N} && (y_N = v_N), \end{aligned} \tag{3.78}$$

which is less than $\overline{\alpha}$ for $d = 4$. This completes the proof. □

4 | PROOF OF MAIN RESULTS

The common point of departure for the proofs of Theorems 1.1, 1.2 and 1.3 is Lemma 2.1, which states that for $n \geq 1$, $g > 0$, $\nu \in \mathbb{R}$, and $a \in (-L^{-2(N-1)}, \infty)$,

$$\left\langle e^{J\cdot\Phi_N} \right\rangle^{\mathrm{P}}_{\nu+a,N} = \frac{\int_{\mathbb{R}^n} e^{J\cdot y}\, Z_N(y) e^{-\frac{1}{2}a\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}a\Omega_N|y|^2} dy}, \tag{4.1}$$

$$\left\langle e^{J\cdot\Phi_N} \right\rangle^{\mathrm{F}}_{\nu+a,N} = \frac{\int_{\mathbb{R}^n} e^{J\cdot y}\, Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2} dy}{\int_{\mathbb{R}^n} Z_N(y) e^{-\frac{1}{2}(a+qL^{-2N})\Omega_N|y|^2 dy}}. \tag{4.2}$$

Here Z_N is defined in (2.23) using the covariance $C_{\leq N}(a)$ and with Z_0 given by (2.17)–(2.18), and its form is given by (3.5) and (3.17) with ν and a chosen appropriately. The common factor $e^{-u_N|\Lambda_N|}$ cancels in numerator and denominator.

Since we are also interested in different test functions in Theorems 1.1–1.3, and with different scalings, we consider integrals with the field rescaled by a parameter $\mathfrak{b}_N$, of the form

$$\int_{\mathbb{R}^n} f(x/\mathfrak{b}_N) e^{-V_N(x)-\frac{1}{2}a\Omega_N|x|^2} dx \quad \text{and} \quad \int_{\mathbb{R}^n} f(x/\mathfrak{b}_N) K_N(x) e^{-\frac{1}{2}a\Omega_N|x|^2} dx, \tag{4.3}$$

where ν and a are properly chosen depending on the regimes we want to study. We use three different field scalings $\mathfrak{b}_N$, corresponding to the three scalings in Theorems 1.1–1.3. We will show that the integral containing K_N is relatively negligible compared to the integral containing e^{-V_N}, and we will control V_N precisely using Theorems 3.1 and 3.3. In Section 4.1, we establish the asymptotic behaviour of the integrals (4.3), and in Section 4.2, we use the results of Section 4.1 to prove Theorems 1.1–1.3.

4.1 | Scaling of integrals

In this section, we prove estimates on the integrals (4.3) under different BCs, different choices of ν and a, and different scalings. To specify the choice of ν and a, recall from (3.34) that (for $* = \mathrm{P}$ or $* = \mathrm{F}$)

$$\nu^*_{c,N} + s w_N = \nu_{1,N}(a^*_N(s)) + a^*_N(s), \qquad \nu^*_{c,N} + s v_N = \nu_{1,N}(\tilde{a}^*_N(s)) + \tilde{a}^*_N(s), \tag{4.4}$$

where, as in (3.26),

$$\nu_{1,N}(a) = \begin{cases} \nu_c(a) & (a \geq 0) \\ \nu_{0,N}(a) & (a < 0). \end{cases} \tag{4.5}$$

For notational convenience we focus on the case w_N (the case v_N can be obtained by adding tildes). Then our choice of ν and a is to take $\nu = \nu_{1,N}(a^*_N(s))$ and $a = a^*_N(s)$.

The different regimes studied in Theorems 1.1, 1.2 and 1.3 correspond respectively to the three choices $\mathfrak{b}_N = p_N, \hbar_N, \ell_N$ in (4.3), where

$$p_N = L^{-Nd/2}, \qquad \hbar_N = \begin{cases} (BN)^{1/4}L^{-N} & (d=4) \\ g_\infty^{-1/4}L^{-Nd/4} & (d>4), \end{cases} \qquad \ell_N = L^{-N(d-2)/2}. \tag{4.6}$$

The sequences $\hbar_N$ and ℓ_N are as defined in (1.40) and (1.46), and p_N is introduced here for the first time. For s in the domain S_N defined in (3.73), the asymptotic behaviour of a_N^* is given in Proposition 3.9 as (with a proviso for PBC when $s \to 0$ that we do not discuss here)

$$a_N^{\mathrm{P}}(s) = s\hbar_N^{-2}L^{-dN}(1+o(1)), \qquad a_N^{\mathrm{F}}(s) + qL^{-2N} = s\hbar_N^{-2}L^{-dN}(1+o(1)). \tag{4.7}$$

After inserting this into (4.1)–(4.2), we see that the $|y|^2$ terms in the exponent become identical for FBC and PBC. The same occurs for $\tilde{a}_N^*(s)$, with ℓ_N in place of $\hbar_N$.

We focus on the first integral in (4.3). The change of variable $y = x/\mathfrak{b}_N$, and the choices m^2, a_N^{P} and a_N^{F} for a, lead us to consider the three integrals

$$\int_{\mathbb{R}^n} f(y) e^{-V_N(\mathfrak{b}_N y)} e^{-\frac{1}{2}m^2\mathfrak{b}_N^2\Omega_N|y|^2}\,dy, \tag{4.8}$$

$$\int_{\mathbb{R}^n} f(y) e^{-V_N(\mathfrak{b}_N y)} e^{-\frac{1}{2}a_N^{\mathrm{P}}\mathfrak{b}_N^2\Omega_N|y|^2}\,dy, \tag{4.9}$$

$$\int_{\mathbb{R}^n} f(y) e^{-V_N(\mathfrak{b}_N y)} e^{-\frac{1}{2}(a_N^{\mathrm{F}}+qL^{-2N})\mathfrak{b}_N^2\Omega_N|y|^2}\,dy. \tag{4.10}$$

For $\mathfrak{b}_N$ equal to p_N, $\hbar_N$ and ℓ_N, and since $p_N^2\Omega_N = 1$, the above reduce, respectively, to

$$\int_{\mathbb{R}^n} f(y) e^{-V_N(p_N y)} e^{-\frac{1}{2}m^2|y|^2}\,dy, \tag{4.11}$$

$$\int_{\mathbb{R}^n} f(y) e^{-V_N(\hbar_N y)} e^{-\frac{1}{2}s(1+o(1))|y|^2}\,dy, \tag{4.12}$$

$$\int_{\mathbb{R}^n} f(y) e^{-V_N(\ell_N y)} e^{-\frac{1}{2}s(1+o(1))|y|^2}\,dy. \tag{4.13}$$

For all $d \geq 4$, the coupling constants g_N and ν_N in $V_N(y) = L^{dN}(\frac{1}{4}g_N|y|^4 + \frac{1}{2}\nu_N|y|^2)$ are prescribed by Theorems 3.1 and 3.3. In particular, $|\nu_N| \leq O(g_N L^{-(d-2)N})$. Thus,

$$V_N(p_N y) = \frac{1}{4}g_N L^{-dN}|y|^4 + O(g_N L^{-(d-2)N})|y|^2, \tag{4.14}$$

$$V_N(\hbar_N y) = \frac{1}{4}g_N \hbar_N^4 L^{dN}|y|^4 + O(g_N^{1/2}L^{-(d-4)N/2})|y|^2, \tag{4.15}$$

$$V_N(\ell_N y) = \frac{1}{4}g_N L^{-(d-4)N}|y|^4 + O(g_N L^{-(d-4)N})|y|^2. \tag{4.16}$$

For $V_N(p_N y)$, the coefficients of $|y|^4$ and $|y|^2$ both vanish as $N \to \infty$, and it is $e^{-\frac{1}{2}m^2|y|^2}$ that survives in the limit. Similarly, the coefficients of $V_N(\ell_N y)$ again go to zero, and it is $e^{-\frac{1}{2}s|y|^2}$ that survives in the limit. On the other hand, for $V_N(\hbar_N y)$ the quadratic term does vanish but

now $g_N \hbar_N^4 L^{dN} \to 1$ in the limit precisely due to our choice of $\hbar_N$, and the non-Gaussian limit $e^{-\frac{1}{4}|y|^4 - \frac{1}{2}s|y|^2}$ emerges.

We make the above considerations more precise in the following lemma. For its statement, we introduce $V_N^{(i)}(y)$ and $K_N^{(i)}$ $(i = c, 1)$, to distinguish each of two mass regimes we investigate: $a = m^2 \geq 0$ fixed and $a \in \mathbb{I}_{\text{crit}}$, as follows.

- $V_N^{(c)}, K_N^{(c)}$ are equal to the functions V_N and K_N obtained from (3.5), with parameters $\nu = \nu_c(a)$ and mass $a \geq 0$. The estimates of Theorem 3.1 then apply uniformly in $a \geq 0$. We do not make explicit the dependence of $V_N^{(c)}, K_N^{(c)}$ on a.
- $V_N^{(1)}, K_N^{(1)}$ are equal to the functions V_N and K_N with parameters $\nu = \nu_{1,N}(a)$ and mass $a \in \mathbb{I}_{\text{crit}}$ (defined in 3.71), obtained from (3.5) if $a \geq 0$, or from (3.17) if $a \leq 0$. The estimates of Theorem 3.1 and Theorem 3.3 then apply uniformly in $a \in \mathbb{I}_{\text{crit}}$. We do not make explicit the dependence of $V_N^{(1)}, K_N^{(1)}$ on a. In this case, Lemma 3.10 applies and gives

$$g_N = \begin{cases} (BN)^{-1}(1 + O(N^{-1}\log N)) & (d = 4) \\ g_\infty(0)(1 + O(gL^{-N/2})) & (d > 4) \end{cases} \tag{4.17}$$

uniformly in $a \in \mathbb{I}_{\text{crit}}$.

Finally, we define function domains for the next lemma:

$$\tilde{\mathcal{F}} = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \geq 0,\ |f(x)| \leq Me^{u|x|^\alpha} \text{ for some } u \in \mathbb{R},\ \alpha \in [0, 2)\}, \tag{4.18}$$

$$\mathcal{F} = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \geq 0,\ |f(x)| \leq Me^{u|x|^4} \text{ for some } u < \tfrac{1}{2}\min(\kappa, \tfrac{1}{4})\}, \tag{4.19}$$

where κ is the parameter occurring in the exponential decay rate of K_N in Theorems 3.1 and 3.3.

Lemma 4.1. *Let $d \geq 4$ and let λ_N be a real sequence. For $V^{(c)}$ and $K^{(c)}$ we use $a \geq 0$ and $\nu = \nu_c(a)$. For $V^{(1)}$ and $K^{(1)}$ we use $a \in \mathbb{I}_{\text{crit}}$ and $\nu = \nu_{1,N}(a)$. The following asymptotic formulas hold as $N \to \infty$, uniformly for these parameters a and ν, and with c_N, $\tilde{c}_N$ defined by*

$$c_N = \begin{cases} N^{1/4} & (d = 4) \\ L^{N/4} & (d > 4), \end{cases} \qquad \tilde{c}_N = \begin{cases} N^{1/4} & (d = 4) \\ L^{N(d-4)/4} & (d > 4). \end{cases} \tag{4.20}$$

(i) For any $f \in \tilde{\mathcal{F}}$, if $\lambda_N \geq \varepsilon$ for some $\varepsilon > 0$, then

$$\int_{\mathbb{R}^n} f(y) e^{-V_N^{(c)}(p_N y)} e^{-\frac{1}{2}\lambda_N |y|^2} dy = (1 + o(1)) \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}\lambda_N |x|^2} dx + O(e^{-L^N \lambda_N}), \tag{4.21}$$

$$\int_{\mathbb{R}^n} f(y) e^{-V_N^{(1)}(\hbar_N y)} e^{-\frac{1}{2}\lambda_N |y|^2} dy = (1 + o(1)) \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}\lambda_N |x|^2} dx + O(e^{-\tilde{c}_N \lambda_N}). \tag{4.22}$$

For any $f \in \mathcal{F}$ and any sequence λ_N whose negative part $\lambda_N^- = |\lambda_N \wedge 0|$ obeys $\lambda_N^- = o(c_N)$,

$$\int_{\mathbb{R}^n} f(y) e^{-V_N^{(1)}(\hbar_N y)} e^{-\frac{1}{2}\lambda_N |y|^2} dy = (1 + o(1)) \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{4}|x|^4 - \frac{1}{2}\lambda_N |x|^2} dx + O(e^{-c_N(c_N + \lambda_N)}). \tag{4.23}$$

(ii) Recall the definition of e_N in (3.3). For any $f \in \tilde{\mathcal{F}}$, if $\lambda_N \geq \varepsilon$ for some $\varepsilon > 0$, then

$$\Big| \int_{\mathbb{R}^n} f(y) K_N^{(c)}(p_N y) e^{-\frac{1}{2}\lambda_N |y|^2} dy \Big| \leq O(e_N) \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}\lambda_N |x|^2} dx, \tag{4.24}$$

$$\Big| \int_{\mathbb{R}^n} f(y) K_N^{(1)}(\ell_N y) e^{-\frac{1}{2}\lambda_N |y|^2} dy \Big| \leq O(e_N) \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}\lambda_N |x|^2} dx. \tag{4.25}$$

For any $f \in \mathcal{F}$ and for any sequence λ_N,

$$\Big| \int_{\mathbb{R}^n} f(y) K_N^{(1)}(\hbar_N y) e^{-\frac{1}{2}\lambda_N |y|^2} dy \Big| \leq O(e_N) \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}\varkappa |x|^4 - \frac{1}{2}\lambda_N |x|^2} dx. \tag{4.26}$$

Proof.

(i) We first prove (4.21). By (4.14), the exponent (up to sign) becomes

$$\frac{1}{2}\lambda_N |y|^2 + \eta_N(y) \quad \text{with} \quad \eta_N(y) = \frac{1}{4} g_N L^{-dN} |y|^4 + O(g_N L^{-(d-2)N}) |y|^2, \tag{4.27}$$

and we rewrite the integral in (4.21) as

$$\int_{\mathbb{R}^n} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy + I_N(f) \quad \text{with} \quad |I_N(f)| \leq \int_{\mathbb{R}^n} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} \big| e^{-\eta_N(y)} - 1 \big| dy. \tag{4.28}$$

We now make three observations. By our hypothesis that $f \in \tilde{\mathcal{F}}$, there is a $C_1 > 0$ depending on f and $C_2, C_3 > 0$, such that

$$\int_{|y|^2 \geq c} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy \leq C_1 e^{-\frac{1}{4} c \lambda_N}, \tag{4.29}$$

$$\sup_{y \in \mathbb{R}^n} \big| e^{-\eta_N(y)} - 1 \big| \leq C_2, \tag{4.30}$$

$$\sup_{|y|^2 \leq c} \big| e^{-\eta_N(y)} - 1 \big| \leq C_3 c^2 g_N L^{-(d-2)N}, \tag{4.31}$$

where $c > 0$ is arbitrary for (4.29), and $1 \leq c \leq 4L^N$ for (4.31) (the upper bound is convenient but not optimal). The first point is clear. The second point follows from the fact that the minimal value of η_N is (by elementary calculus) at worst of order $-(g_N L^{-(d-2)N})^2 / (g_N L^{-dN}) = -g_N L^{-(d-4)N}$ (it occurs at $|y|^2 \asymp L^{2N}$ but we do not use this fact). The third point follows from the bound $|e^w - 1| \leq |w| e^{|w|}$.

We now estimate $I_N(f)$ by splitting the integral at $|y|^2 = 4L^N$. This gives

$$\begin{aligned} |I_N(f)| &\leq 16 C_3 g_N L^{-(d-4)N} \int_{|y|^2 \leq 4L^N} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy \\ &\quad + C_2 g_N L^{-(d-4)N} \int_{|y|^2 \geq 4L^N} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy \\ &\leq 16 C_3 g_N L^{-(d-4)N} \int_{\mathbb{R}^n} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy + C_1 C_2 e^{-L^N \lambda_N}. \end{aligned} \tag{4.32}$$

This proves the desired estimate (4.21).

The proof of (4.22) follows similarly using (4.16), now with

$$\eta_N(y) = g_N L^{-(d-4)N}\Big(\frac{1}{4}|y|^4 + O(1)|y|^2\Big). \tag{4.33}$$

The inequalities (4.29)–(4.30) remain valid, and (4.31) is replaced (with the same proof) by

$$\sup_{|y|^2\le c} |e^{-\eta(y)} - 1| \le O(1)c^2 g_N L^{-(d-4)N} \tag{4.34}$$

with $1 \le c \le g_N^{-1/2} L^{N(d-4)/2}$. With a similar definition for $I_N(f)$, by splitting the integral at $4\tilde{c}_N$ (defined in 4.20), and by using the asymptotic behaviour of g_N in (4.17), we see that

$$|I_N(f)| \le O(g_N^{1/2})L^{-N(d-4)/2} \int_{\mathbb{R}^n} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy + O(1) e^{-\lambda_N g_N^{-1/4} L^{N(d-4)/4}}, \tag{4.35}$$

from which (4.22) follows.

For (4.23), the proof varies a little since the $|y|^4$ term survives in the limit. Also, we now permit $\lambda_N \to -\infty$. It follows from (4.15), the definition of $\hbar_N$, and (4.17), that in the exponent we now have

$$\frac{1}{4}|y|^4 + \frac{1}{2}\lambda_N |y|^2 + \eta_N(y) \quad \text{with} \quad \eta_N(y) = \zeta_N |y|^4 + \xi_N |y|^2, \tag{4.36}$$

with

$$\zeta_N = \begin{cases} O(N^{-1/2}) & (d=4) \\ O(gL^{-N/2}) & (d>4), \end{cases} \qquad \xi_N = O(g_N^{1/2} L^{-N(d-4)/2}). \tag{4.37}$$

By taking the worst decay of ζ_N and ξ_N, we deduce that

$$\eta_N(y) = \rho_N(|y|^4 + |y|^2) \quad \text{with} \quad \rho_N = \begin{cases} O(N^{-1/2}) & (d=4) \\ O(L^{-N/2}) & (d>4). \end{cases} \tag{4.38}$$

We estimate

$$\int_{\mathbb{R}^n} f(y) e^{-\frac{1}{4}|y|^4 - \frac{1}{2}\lambda_N |y|^2} \big|e^{-\eta_N(y)} - 1\big| dy \tag{4.39}$$

by splitting it at $|y|^2 = 4c_N$ (defined in 4.20, the 4 simplifies the final result). Note that $|e^{-\eta_N(y)} - 1|$ is no longer bounded, but by splitting the integral we obtain

$$|I_N(f)| \le o(1) \int_{\mathbb{R}^n} f(y) e^{-\frac{1}{4}|y|^4 - \frac{1}{2}\lambda_N |y|^2} dy + I_{N,c_N}(f), \tag{4.40}$$

where $I_{N,c_N}(f)$ is an upper bound on the integral $I_N(f)$ restricted to $|y|^2 \ge 4c_N$. To estimate $I_{N,c_N}(f)$ we bound $|e^{-\eta_N(y)} - 1|$ by $1 + e^{|\rho_N|(|y|^4+|y|^2)}$ to obtain

$$I_{N,c_N} \le O(1) \int_{|y|^2\ge 4c_N} f(y) e^{-\frac{1}{8}|y|^4} e^{-(\frac{1}{8}|y|^4 + \frac{1}{2}\lambda_N |y|^2)} (1 + e^{|\rho_N|(|y|^4+|y|^2)}) dy \tag{4.41}$$

$$\le O(1) e^{-4c_N(\frac{1}{2}c_N + \frac{1}{2}\lambda_N - |\rho_N|(1+c_N))} \int_{|y|^2\ge c_N} f(y) e^{-\frac{1}{8}|y|^4} dy \tag{4.42}$$

$$\leq O(1)e^{-c_N(c_N+\lambda_N)}, \tag{4.43}$$

where in the second line we employed our assumption on the growth of λ_N, and in the last line we used our hypothesis that $f \in \mathcal{F}$. This completes the proof of (4.23).

(ii) We now prove the bounds on the K_N integrals in (4.24)–(4.26). By (3.8) and (3.12) (we do not need the large-field decay of K_N here),

$$\Big| \int_{\mathbb{R}^n} f(y) K_N^{(1)}(yp_N) e^{-\frac{1}{2}\lambda_N |y|^2} dy \Big| \leq O(e_N) \int_{\mathbb{R}^n} f(y) e^{-\frac{1}{2}\lambda_N |y|^2} dy, \tag{4.44}$$

which is (4.24). The inequality (4.25) is obtained in the same way using (3.20) and (3.24), again without making use of the large-field decay in those bounds on K_N.

Finally, (4.26) also follows from (3.20) and (3.24), but now we do use the large-field decay, namely

$$|K_N(y\hbar_N)| \leq O(e_N) e^{-\kappa g_N L^{dN} \hbar_N^4 |y|^4}. \tag{4.45}$$

According to the behaviour of g_N in (4.17) and the definition of $\hbar_N$ in (1.40), the exponent in the above right-hand side is $-\kappa[1 + o(1)]|y|^4$. This completes the proof. □

4.2 | Proof of Theorems 1.1, 1.2 and 1.3

In this section we prove our three main results, Theorems 1.1, 1.2 and 1.3, subject to the results stated in Section 3. The proofs are given in parallel. We start with a joint proof of part (i) of the three theorems, concerning the limit of the Laplace transform of the rescaled average field; the proof follows directly from the results of the previous section. We then turn to the proof of part (ii) of the three theorems, concerning the asymptotic behaviour of moments of the average field; this requires additional effort to accommodate infinite limits for the sequence s_N. Recall that the infinite-volume critical point is $\nu_c = \nu_c(0)$, and that the effective critical points for PBC and FBC are

$$\nu_{c,N}^{\mathrm{P}} = \nu_c(0) \qquad \nu_{c,N}^{\mathrm{F}} = \nu_{0,N}(a_N^{\mathrm{F}}(0)) + a_N^{\mathrm{F}}(0). \tag{4.46}$$

Proof of Theorems 1.1(i), 1.2(i) and 1.3(i). We prove (1.28), (1.42) and (1.48). Our starting point is (2.26)–(2.27).

We first prove (1.28) for FBC. Given $\varepsilon \geq 0$, we apply Theorem 3.2 to obtain $m^2 = m_\varepsilon^2$, satisfying the asymptotic formula (1.27), such that

$$\nu_c(0) + \varepsilon = \nu_c(m^2) + m^2. \tag{4.47}$$

By (2.27), with $\nu = \nu_c(m^2)$ and $a = m^2 \geq 0$, and since $p_N^2 \Omega_N = 1$,

$$\big\langle e^{J\cdot\Phi_N/p_N} \big\rangle^{\mathrm{F}}_{\nu_c(m^2)+m^2,N} = \frac{\int_{\mathbb{R}^n} e^{J\cdot y}(e^{-V_N(p_N y)} + K_N(p_N y)) e^{-\frac{1}{2}(m^2+qL^{-2N})|y|^2} dy}{\int_{\mathbb{R}^n} (e^{-V_N(p_N y)} + K_N(p_N y)) e^{-\frac{1}{2}(m^2+qL^{-2N})|y|^2 dy}}. \tag{4.48}$$

For the e^{-V_N} terms we apply (4.21) and for the K_N terms we apply (4.24), with $f(y) = e^{J\cdot y}$ for the numerator and $f = 1$ for the denominator, and with $\lambda_N^{\mathrm{F}} = m^2 + qL^{-2N}$. This gives

$$\langle e^{J\cdot\Phi_N L^{dN/2}}\rangle^{\mathrm{F}}_{\nu_c(0)+\varepsilon,N} = (1+o(1))\int_{\mathbb{R}^n} e^{J\cdot x} d\mu_{\lambda_N^{\mathrm{F}}}(x). \tag{4.49}$$

Since $\lambda_N^{\mathrm{F}} \to m_\varepsilon^2$ as $N\to\infty$, this proves (1.28). For PBC, the proof instead uses $\lambda_N^{\mathrm{P}} = m^2$ but otherwise is identical.

Next, we prove (1.42). Let s_N be a real sequence with $s_N \to s \in \mathbb{R}$. Then s_N is eventually in the interval S_N defined in (3.73). As discussed in the paragraph above Lemma 3.8, this implies that $S_N \subset \mathcal{W}_N^*$. According to Definition 3.6, we can therefore define $a_N^*(s_N)$ as in (3.34), so that

$$\nu_{c,N}^* + s_N w_N = \nu_{1,N}(a_N^*(s_N)) + a_N^*(s_N). \tag{4.50}$$

Then we use Lemma 2.1 with $\nu = \nu_{1,N}(a_N^*(s_N))$ and $a = a_N^*(s_N)$. We define a sequence λ_N^* by

$$\lambda_N^* \hbar_N^{-2} = \Omega_N a_N^*(s_N) + \begin{cases} 0 & (* = \mathrm{P}) \\ \Omega_N q L^{-2N} & (* = \mathrm{F}). \end{cases} \tag{4.51}$$

Since s_N converges to s, it follows from Proposition 3.9 that

$$\begin{aligned} &\lambda_N^* = s(1+o(1)) && (* = \mathrm{F},\ \text{or}\ * = \mathrm{P}\ \text{and}\ s \neq 0), \\ &|\lambda_N^*| \le |s_N|(1+o(1)) && (* = \mathrm{P},\ s = 0). \end{aligned} \tag{4.52}$$

Note that for FBC there is a cancellation of the qL^{-2N} term due to (3.45). The different cases in (4.52) are only needed for $d = 4$ but are also true for $d > 4$ by Proposition 3.9 again. In any case, we see from (4.52) that λ_N^* and s_N both converge to s, and therefore the above dichotomy becomes inconsequential. As in the proof of (1.28), we now apply (4.23) and (4.26) to obtain

$$\langle e^{J\cdot\Phi_N \hbar_N^{-1}}\rangle^*_{\nu_{c,N}^*+s_N w_N,N} = (1+o(1))\int_{\mathbb{R}^n} e^{J\cdot x} d\sigma_{\lambda_N^*}(x), \tag{4.53}$$

and (1.42) then follows since $\lambda_N^* \to s$.

The proof of (1.48) uses instead the tilde version of Proposition 3.9, (4.22) and (4.25). Since it involves nothing new, we omit the details. □

Proof of Theorems 1.1(ii), 1.2(ii) and 1.3(ii). The asymptotic formulas (1.30) are part of the statement of Theorem 3.1, so we are left to prove (1.29), (1.43), (1.49).

We first prove (1.29), which is equivalent to the statement that for $p \ge 1$ we have

$$\langle |\Phi_N|^{2p}\rangle^*_{\nu_c(0)+\varepsilon,N} = (1+o(1))p_N^{2p}\int_{\mathbb{R}^n} |x|^{2p} d\mu_{m_\varepsilon^2}. \tag{4.54}$$

We apply the identities (2.36)–(2.37) (using linearity to obtain their counterparts with $|\Phi_N|^{2p}$ and $|y|^{2p}$), with $\nu = \nu_c(m^2)$ and $a = m^2$, and define $\lambda_N^{\mathrm{P}} = m^2$ and $\lambda_N^{\mathrm{F}} = m^2 + qL^{-2N}$. Exactly as in the proof of (1.28), we apply (4.21) and (4.24), now with with $f(y) = |y|^{2p}$ in the numerator. This leads to (4.54).

Next we prove (1.43). Recall that S_N is defined in (3.73) by

$$S_N = \begin{cases} (-\frac{1}{4}L^2(BN)^{1/2}, L^{\sqrt{N}}) & (d=4) \\ (-\frac{1}{4}L^2 g_\infty^{-1/2} L^{N(d-4)/2}, L^{N(d-3)/2}) & (d \geq 5). \end{cases} \tag{4.55}$$

We are assuming that either: (a) s_N converges, or (b) $s_N \to +\infty$ in such a manner that $s_N \leq L^{(d-3)N/2}$ for $d \geq 5$ or $s_N \leq L^{\sqrt{N}}$ for $d=4$, or (c) $s_N \to -\infty$ in such a manner that $|s_N| \leq (\log N)^{1/4}$ for $d=4$ or $|s_N| \leq N^{1/4}$ for $d>4$. In all these cases, s_N remains inside S_N for large enough N. Therefore, by Lemma 3.11, the solution (recall 3.34) to

$$\nu^*_{c,N} + s_N w_N = \nu_{1,N}(a^*_N(s_N)) + a^*_N(s_N). \tag{4.56}$$

obeys $a^*_N(s_N) \in \mathbb{I}_{\text{crit}}$.

We use (2.36)–(2.37) with $\nu = \nu_{1,N}(a^*_N(s_N))$ and $a = a^*_N(s_N)$, and apply (4.23) and (4.26) with $f(y) = |y|^{2p}$ in the numerator and $f(y)=1$ in the denominator, and with λ^*_N defined as in (4.51). By assumption, s_N converges to $s \in \mathbb{R}$ or diverges to $\pm\infty$ while staying in the domain S_N. Thus we can apply Proposition 3.9 as in (4.52) to see that

$$\begin{aligned} &\lambda^*_N = s_N(1+o(1)) && (* = \text{F, or } * = \text{P and } s_N \nrightarrow 0) \\ &\lambda^*_N = o(1) && (* = \text{P, } s_N \to 0). \end{aligned} \tag{4.57}$$

The dichotomy in (4.57) is only needed for $d=4$, since for $d>4$ we always have $\lambda^*_N = s_N(1+o(1))$ when $s_N \in S_N$. Our assumption that $|s_N|$ (and thus $|\lambda^*_N|$) diverges at worst like $o(c_N)$ when $s_N \to -\infty$ ensures that we can apply Lemma 4.1. Overall, this gives (as in 1.61 for the integrals $I_k(s)$)

$$\begin{aligned} \left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N w_N,N} &= \hbar_N^{2p} \frac{(1+o(1))\int_{\mathbb{R}^n} |y|^{2p} e^{-\frac{1}{4}|x|^4 - \frac{1}{2}\lambda^*_N |x|^2} dx + \eta_{N,2p}}{(1+o(1))\int_{\mathbb{R}^n} e^{-\frac{1}{4}|x|^4 - \frac{1}{2}\lambda^*_N |x|^2} dx + \eta_{N,0}} \\ &= \hbar_N^{2p} \frac{(1+o(1))I_{2p+n-1}(\lambda^*_N) + \eta_{N,2p}}{(1+o(1))I_{n-1}(\lambda^*_N) + \eta_{N,0}}, \end{aligned} \tag{4.58}$$

where $\eta_{N,2p}$ comprises error terms from both (4.23) and (4.26). Explicitly,

$$\eta_{N,2p} = O(e^{-c_N(c_N+\lambda^*_N)}) + O(e_N) I_{2p+n-1}(\lambda^*_N/\sqrt{2\kappa}), \tag{4.59}$$

with c_N given in (4.20) and, as defined in (3.3),

$$e_N = \begin{cases} N^{-3/4} & (d=4) \\ g^{3/4} L^{-3(d-4)N/4} & (4 < d \leq 12) \\ g^{3/4} L^{-Nd/2} & (d > 12). \end{cases} \tag{4.60}$$

For the case $s_N \to s \in \mathbb{R}$, so that also $\lambda^*_N \to s$, with (1.61) we immediately conclude that

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N w_N,N} = \hbar_N^{2p}(1+o(1))\Sigma_{n,2p}(s), \tag{4.61}$$

which proves (1.43) in this case. It remains to consider infinite limits. The asymptotic behaviour of the integral on the right-hand side of (4.59) is given in (1.70). Using this, we verify that the first term in (4.59) is always much smaller than the second one and hence

$$\eta_{N,2p} = O(e_N) \times \begin{cases} (\lambda_N^*)^{-(2p+n)/2} & (\lambda_N^* \to +\infty) \\ e^{\frac{1}{8\kappa}(\lambda_N^*)^2} |\lambda_N^*|^{\frac{2p+n-2}{2}} & (\lambda_N^* \to -\infty). \end{cases} \tag{4.62}$$

To obtain (1.43) it is enough to show that $\eta_{N,2p}$ is always much smaller than the integral $I_{2p+n-1}(\lambda_N^*)$. This in turn follows by our assumptions on the growth of s_N and from (1.70) again. Indeed when $s_N \to +\infty$ this is immediate by the extra $O(e_N)$ decay. If $s_N \to -\infty$ then we use our assumption that $|s_N| \le (\log N)^{1/4}$ if $d = 4$, or $|s_N| \le N^{1/4}$ if $d > 4$, to see that for some unimportant constants $c_\kappa, c_L > 0$,

$$\eta_{N,2p} = O(e_N) I_{2p+n-2}(\lambda_N^*) e^{c_\kappa (\lambda_N^*)^2} = O(1) I_{2p+n-2}(\lambda_N^*) \times \begin{cases} e^{-\frac{3}{4}(1-o(1))\log N} & (d=4) \\ e^{-c_L(1-o(1))N} & (d>4). \end{cases} \tag{4.63}$$

Finally, from (1.71), (4.57) and the above considerations, in the case of either of the two infinite limits we obtain

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N w_N,N} = \hbar_N^{2p}(1+o(1))\Sigma_{n,2p}(\lambda_N^*) = \hbar_N^{2p}(1+o(1))\Sigma_{n,2p}(s_N), \tag{4.64}$$

which completes the proof of (1.43).

It remains to prove (1.49). The structure of the proof is essentially the same as for (1.43), but it is simpler since we do not consider $s_N \to -\infty$. As before, we check that $s_N \in \tilde{S}_N$ first. Recall that

$$\tilde{S}_N = \begin{cases} (-\frac{1}{4}L^2, L^{\sqrt{N}}) & (d=4) \\ (-\frac{1}{4}L^2, L^{N/2}) & (d \ge 5). \end{cases} \tag{4.65}$$

Our hypothesis on s_N in (1.47) ensures that $s_N \in \tilde{S}_N$ and we assume that $s_N \ge \varepsilon > 0$. We then let $\tilde{a}^* = \tilde{a}_N^*(s_N)$ as in (3.34), so that

$$\nu^*_{c,N} + s_N \upsilon_N = \nu_{0,N}(\tilde{a}_N^*(s_N)) + \tilde{a}_N^*(s_N). \tag{4.66}$$

Again, we use (2.36)–(2.37) with $\nu = \nu_{0,N}(\tilde{a}_N^*(s_N))$ and $a = \tilde{a}_N^*(s_N)$, and we apply (4.22) and (4.25) with $f(y) = |y|^{2p}$ and $f(y) = 1$, and with

$$\lambda_N^* = s_N(1+o(1)) \tag{4.67}$$

which follows from Proposition 3.9 and the fact that $s_N \in \tilde{S}_N$. We obtain this time the similar but neater equation

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N\upsilon_N,N} = \hbar_N^{2p} \frac{(1+o(1))\int_{\mathbb{R}^n} |y|^{2p} e^{-\frac{1}{2}\lambda_N^*|x|^2} dx + O(e^{-\tilde{c}_N\lambda_N^*})}{(1+o(1))\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_N^*|x|^2} dx + O(e^{-\tilde{c}_N\lambda_N^*})}. \tag{4.68}$$

As before, $O(e^{-\tilde{c}_N \lambda_N^*})$ is always much smaller than the integrals appearing in the numerator and denominator above (whether λ_N^* is bounded or $\lambda_N^* \to +\infty$), so we conclude that

$$\left\langle |\Phi_N|^{2p} \right\rangle^*_{g,\nu^*_{c,N}+s_N v_N,N} = \ell_N^{2p}(1+o(1))M_{n,2p}(\lambda_N^*) = \ell_N^{2p}(1+o(1))M_{n,2p}(s_N). \tag{4.69}$$

This completes the proof of (1.49). □

5 | THE RG MAP

To complete the proof of our main results, it remains to prove Theorems 3.1–3.3. In this section, we formulate our main RG theorems, Theorems 5.6–5.7. These are used to prove Theorems 3.1–3.3 in Section 6.

The following are the main improvements in our RG analysis compared to [9]:

(i) We include dimensions $d > 4$ as well as the upper critical dimension $d = 4$. Since $|\varphi|^4$ is a so-called *dangerous irrelevant monomial* in dimensions $d > 4$, the inclusion of higher dimensions in the analysis requires care and innovation.
(ii) The non-perturbative coordinate $K_N(\varphi)$ has the additional large-field decay $\exp[-\kappa g_N L^{dN}|\varphi|^4]$, which is essential for the control of the integration over the last scale in Lemma 4.1. This major improvement is accomplished via the introduction of a new large-field regulator.
(iii) We permit *negative* mass in Theorem 3.3, which allows for the analysis of the critical window for values of ν both above and below the effective critical point, and in particular below the infinite-volume critical point ν_c. This is feasible because an initial condition involving a small N-dependent deviation from the infinite-volume critical point will still allow the RG map to be applied until scale N. The proof requires precise control of derivatives of the RG flow with respect to the initial condition ν_0 and the mass a. This control of mass derivatives is another new feature of our work.

The notation and definitions used throughout Sections 5–9 largely follow those of [9]. In particular, we assume familiarity with [9, Chapter 4] where the hierarchical $|\varphi|^4$ model is defined and discussed, and on the definition of the RG map and the perturbative flow discussed in [9, Chapter 5]. Our proof requires improvements and extensions to the arguments in [9, Chapters 8–10], and those three chapters play an important role in Sections 8 and 9.

5.1 | The renormalisation group map

We now examine all scales j and not only the final scale N. Thus we write the scale-$(j+1)$ partition function Z_{j+1} in terms of the scale-j partition function Z_j as

$$Z_{j+1}(\varphi) = \mathbb{E}_{C_{j+1}} Z_j(\varphi + \zeta), \qquad j \in \{0, \dots, N-1\}, \tag{5.1}$$

where the Gaussian integral on the right-hand side integrates over ζ with φ held fixed, and the covariance C_{j+1} arises from the decomposition (2.9). For notational simplicity, we write (5.1) more

compactly as

$$Z_+(\varphi) = \mathbb{E}_{C_+} Z(\varphi + \zeta) = \mathbb{E}_+ Z(\varphi + \zeta). \tag{5.2}$$

This is consistent with the usage in [9], namely that when two generic scales j and $j+1$ are under discussion, we drop the subscript j and replace $j+1$ simply by $+$. As another example, we write $\mathcal{B}$ for the set of j-blocks (defined at the beginning of Section 2.1) and $\mathcal{B}_+$ for the set of $(j+1)$-blocks. Such notational reduction will be used throughout the rest of the paper.

We introduce several definitions needed to define the RG map. Further details can be found in [9].

Definition 5.1 [9, Definition 5.1.5]. A *polymer activity* at scale j is a smooth function $F : \mathcal{B} \times (\mathbb{R}^n)^{\mathcal{B}} \to \mathbb{R}$ such that $F(b, \varphi)$ only depends on $\varphi(b)$ (but not otherwise on $b \in \mathcal{B}$) and $F(b, \varphi) = F(b, M\varphi)$ for any $M \in O(n)$. The space of polymer activities is denoted $\mathcal{F}$.

A polymer activity F can simply be identified with the function $\tilde{F} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{F}(\varphi(b)) = F(b, \varphi)$, so we will often interpret F as $\tilde{F}$, with $\varphi \in \mathbb{R}^n$ rather than in $(\mathbb{R}^n)^{\mathcal{B}}$, when the meaning is clear from the context. Given a constant field φ on a block $b \in \mathcal{B}$, and given real parameters g, ν, u, important examples of polymer activities are polynomials in the field of the form

$$V(b, \varphi) = L^{jd}\Big(\frac{1}{4} g|\varphi|^4 + \frac{1}{2}\nu|\varphi|^2\Big), \tag{5.3}$$

$$U(b, \varphi) = L^{jd} u + V(b, \varphi). \tag{5.4}$$

We denote the vector spaces of such polynomials V and U as $\mathcal{V}$ and $\mathcal{U}$ respectively. By definition, $\mathcal{V} \simeq \mathbb{R}^2$ and $\mathcal{U} \simeq \mathbb{R}^3$, and $\mathcal{V}$ can be identified with the subspace of $\mathcal{U}$ which has u equal to 0. The spaces $\mathcal{V}$ and $\mathcal{U}$ describe the freedom to choose the parameters in (5.3)–(5.4).

The following defines the *localisation* operator. It extracts from a functional of the field its *relevant* and *marginal* parts; see [9, Section 5.2.2] for further discussion. Although $|\varphi|^4$ is irrelevant in dimensions $d > 4$, we nevertheless extract it with Loc. This allows a unified treatment of dimensions $d > 4$ and $d = 4$ via second-order perturbation theory. It is also important because the scaling we use for Theorem 1.2 makes $|\varphi|^4$ appear to be marginal even in dimensions $d > 4$. More generally, as discussed in Section 1.7, we need the $|\varphi|^4$ term to stabilise integrals, and by extracting it we can track it for this purpose. This is connected with the fact that $|\varphi|^4$ is a dangerous irrelevant monomial in dimensions $d > 4$.

Definition 5.2 [9, Definition 5.2.2]. For a C^4-function $F : \mathbb{R}^n \to \mathbb{R}$, the *localisation* of F is its fourth order Taylor expansion at 0:

$$\mathrm{Loc}\, F(\varphi) = \sum_{p=0}^{4} \frac{1}{p!} F^{(p)}(0; \varphi, \dots, \varphi), \tag{5.5}$$

where $\varphi \in \mathbb{R}^n$ and $F^{(p)}(\varphi; \dot{\varphi}_1, \dots, \dot{\varphi}_p) = \partial_{t_1}|_{t_1=0} \cdots \partial_{t_p}|_{t_p=0} F(\varphi + \sum_{i=1}^{p} t_i \dot{\varphi}_i)$.

Definition 5.3 [9, Definition 2.1.5]. Given $\zeta : \mathcal{B} \to \mathbb{R}^n$ and $F : (\mathbb{R}^n)^{\mathcal{B}} \to \mathbb{R}$, we define $\theta F : \mathbb{R}^n \to \mathbb{R}$ by $(\theta F)(\varphi) = F(\varphi + \zeta)$, where we interpret φ in the sum on the right-hand side as a constant

field in $(\mathbb{R}^n)^{\mathcal{B}_+}$. We do not indicate ζ explicitly in the notation θF, but it will always occur inside a convolution integral of the form

$$(\mathbb{E}_+\theta F)(\varphi) = \mathbb{E}_+ F(\varphi + \zeta), \tag{5.6}$$

where $\varphi \in \mathbb{R}^n$ is fixed and the integration on the right-hand side is over ζ in $(\mathbb{R}^n)^{\mathcal{B}}$.

Definition 5.4 [9, Definition 5.2.5]. For $B \in \mathcal{B}_+$, we define $\Phi_{\mathrm{pt}} : \mathcal{U} \to \mathcal{U}_+$ by

$$\Phi_{\mathrm{pt}}(U;B) = \mathbb{E}_+\theta U(B) - \frac{1}{2}\operatorname{Loc}\operatorname{Cov}_+[\theta U(B), \theta U(B)], \tag{5.7}$$

where $\operatorname{Cov}_+[X,Y] = \mathbb{E}_+(XY) - (\mathbb{E}_+X)(\mathbb{E}_+Y)$. The subscript "pt" refers to "perturbation theory."

Explicitly, for $V \in \mathcal{V}$ with *coupling constants* g, ν, it is shown in [9, Proposition 5.3.5] that $\Phi_{\mathrm{pt}}(V)$ has coupling constants given in terms of explicit j-dependent coefficients β_j, η_j, ξ_j by

$$g_{\mathrm{pt}} = g - \beta_j g^2 \tag{5.8}$$

$$\nu_{\mathrm{pt}} = (1 - \hat{\gamma}\beta_j g)\nu + \eta_j g_j - \xi_j g^2. \tag{5.9}$$

Here $\hat{\gamma} = \frac{n+2}{n+8}$, and, as in [9, (5.3.7)] (we drop the primes from [9, (5.3.7)]),

$$\beta_j = (n+8)(1-L^{-d})(1+aL^{2j})^{-2}L^{-(d-4)j}, \tag{5.10}$$

$$\eta_j = (n+2)(1-L^{-d})(1+aL^{2j})^{-1}L^{-(d-2)j}, \tag{5.11}$$

$$\xi_j = \xi_0^0(1+aL^{2j})^{-3}L^{-(2d-6)j}, \tag{5.12}$$

with an (unimportant) n- and L-dependent constant ξ_0^0. There is also the coupling constant u_{pt} (vacuum energy) but since it cancels in ratios we require only limited information about it, which we defer to Section 8.5 where it begins to play a role.

In the following definition, V represents a polynomial in $\mathcal{V}$, and $U_+ = (u_+, V_+)$ represents a polynomial in $\mathcal{U}$. The non-perturbative coordinate K is an element of the space $\mathcal{F}$ defined with blocks $b \in \mathcal{B}$ on scale j, while K_+ lies in the space $\mathcal{F}_+$ defined with blocks $B \in \mathcal{B}_+$ on the next scale $j+1$.

Definition 5.5 [9, Definition 5.2.8]. For $a > -L^{-2j}$, the *renormalisation group map* (RG map)

$$\Phi_+ : (V,K) \mapsto (U_+, K_+) = (u_+, V_+, K_+) \tag{5.13}$$

is defined by

$$U_+ = \Phi_{\mathrm{pt}}(V - \operatorname{Loc}(e^V K)), \tag{5.14}$$

$$K_+(B) = e^{u_+|B|}\mathbb{E}_+\theta\Big(\prod_{b\in\mathcal{B}(B)}(e^{-V(b)} + K(b))\Big) - e^{-V_+(B)}, \tag{5.15}$$

where $B \in \mathcal{B}_+$ and $\mathcal{B}(B) = \{b \in \mathcal{B} : b \subset B\}$. The domain of Φ_+ consists of those $(V, K) \in \mathcal{V} \times \mathcal{F}$ such that the expectation in (5.15) is convergent. We write the components of Φ_+ as

$$\Phi_+ = (\Phi_+^U, \Phi_+^K) = (\Phi_+^u, \Phi_+^V, \Phi_+^K). \tag{5.16}$$

The notation $\Phi_+ = (\Phi_+^U, \Phi_+^K)$ is used to distinguish the RG flow of the coupling constants (u_+, g_+, ν_+) and the non-perturbative coordinate K_+. The map Φ_+^U, which gives the flow of coupling constants, is the extension of the explicit map $\Phi_{\mathrm{pt}}(V)$ by a remainder term given by

$$R_+^U(V, K) = (r_{g,j}, r_{\nu,j}, r_{u,j}) = \Phi_+^U(V, K) - \Phi_{\mathrm{pt}}(V). \tag{5.17}$$

With the above definition of the RG map, it follows as in [9, Section 5.2.1] from basic algebra that $Z_+ = \mathbb{E}_+ \theta Z$ (recall 5.1) is given by

$$Z_+ = e^{-(u+u_+)|\Lambda|} \prod_{B \in \mathcal{B}_+} (e^{-V_+(B)} + K_+(B)) = e^{-u|\Lambda|} \mathbb{E}_+ \theta \prod_{b \in \mathcal{B}} (e^{-V(b)} + K(b)). \tag{5.18}$$

(This slightly differs from [9, (5.2.6)] because in (5.18) u_+ is the increment in the constant term produced by the RG map whereas in [9, (5.2.6)] it represents the accumulation of increments. Our usage agrees with [9, Definition 5.2.8] and (5.13).)

5.2 | Regulator, norms and domains

In this section, we formulate several basic elements of our analysis.

Fix $\tilde{a} \geq 0$. As in [9, Section 8.3], we define a sequence $\tilde{g}_j$ by

$$\tilde{g}_{j+1} = \tilde{g}_j - \beta_j(\tilde{a}) \tilde{g}_j^2 \quad \text{with initial condition } \tilde{g}_0 = g. \tag{5.19}$$

Thus the sequence $\tilde{g}_j$ is defined via the recursion for g_{pt} of (5.8), with the mass in β_j set equal to $\tilde{a}$. For $g > 0$ sufficiently small, it follows from (5.10) that this sequence obeys the condition $\tilde{g}_{j+1} \in [\frac{1}{2}\tilde{g}_j, \tilde{g}_j]$.

We define the *large-field* and *fluctuation-field scales*

$$h_j = k_0 \tilde{g}_j^{-1/4} L^{-dj/4}, \qquad \ell_j = \ell_0 L^{-(d-2)j/2}, \tag{5.20}$$

with $\ell_0 = L^{1+d/2}$, and with k_0 a constant which is chosen sufficiently small depending on n (see Remark 6.8 and Lemma 8.12). According to the definition of our Gaussian scaling factor ℓ_N in (1.46), $\ell_N = \ell_0 \ell_N$. Also, when the mass obeys $a \in \mathbb{I}_{\mathrm{crit}}$, it follows from Lemma 3.10 and the definition of our non-Gaussian scaling factor $\hbar_N$ in (1.40) that h_N is essentially a multiple of $\hbar_N$. We require that g be sufficient small depending on n and L so that $\ell_j < h_j$ for all j.

We need to control various derivatives of Φ_+. For a function $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$, its Fréchet derivative at $x \in X$ (when it exists) is the unique linear operator $Df(x) : X \to Y$ satisfying

$$\lim_{\|h\|_X \to 0} \frac{\|f(x+h) - f(x) - Df(x;h)\|_Y}{\|h\|_X} = 0. \tag{5.21}$$

Given a normed space $(\mathcal{Z}, |\cdot|_{\mathcal{Z}})$ and an infinitely Fréchet-differentiable function $F : \mathcal{Z} \to \mathbb{R}$, we define

$$\|F\|_{T_z} = \sum_{p=0}^{\infty} \frac{1}{p!} \|F^{(p)}(z)\|_{\mathcal{Z}} \quad \text{with} \quad \|F^{(p)}(z)\|_{\mathcal{Z}} = \sup\{F^{(p)}(z; \dot{z}_1, \dots, \dot{z}_p) : |\dot{z}_i|_{\mathcal{Z}} \le 1\}, \tag{5.22}$$

where $F^{(p)}(z; \dot{z}_1, \dots, \dot{z}_p)$ is the pth Fréchet derivative at z in directions $\dot{z}_1, \dots, \dot{z}_p$. We use properties of this norm which are developed in [9, Chapter 7].

Although a more elaborate choice of the space $\mathcal{Z}$ will be used in Section 9, throughout Section 5 we take $\mathcal{Z} = \mathbb{R}^n$ with $|\varphi|_{\mathcal{Z}} = \|\varphi\|_2/\mathfrak{h}$ where $\mathfrak{h}$ is either ℓ or h. In this case we write the T_z norm as $\|\cdot\|_{T_\varphi(\mathfrak{h})}$. For example, for integers $p \ge 0$ the $T_0(\mathfrak{h})$-norm of a monomial on a block $b \in \mathcal{B}_j$ is $\||\varphi|^{2p}\|_{T_0(\mathfrak{h})} = \mathfrak{h}^{2p}$. Note that here in $T_\varphi(\mathfrak{h})$ the subscript is $\varphi = 0$ so derivatives are evaluated at 0. In particular, if we keep track only of powers of L^j (permitting constants to depend on ℓ_0 and $\tilde{g}_j$) we see that

$$L^{dj}\||\varphi|^{2p}\|_{T_0(\ell)} \asymp \begin{cases} L^{dj} & (2p=0) \\ L^{2j} & (2p=2) \\ L^{-(d-4)j} & (2p=4) \\ L^{-(2d-6)j} & (2p=6), \end{cases} \qquad L^{dj}\||\varphi|^{2p}\|_{T_0(h)} \asymp \begin{cases} L^{dj} & (2p=0) \\ L^{dj/2} & (2p=2) \\ L^{0} & (2p=4) \\ L^{-dj/2} & (2p=6). \end{cases} \tag{5.23}$$

Thus, monomials with power $2p = 0, 2$ are relevant, the monomial with $2p = 4$ is marginal for $d = 4$ and irrelevant for $d > 4$, and the monomial with power $2p = 6$ is irrelevant in dimensions $d \ge 4$. The L^0 for the case $2p = 4$ of the $T_0(h)$-norm is a sign that this monomial will play an important role, in particular for ensuring stability of integrals, and we need to track it carefully even when $d > 4$; it is a so-called *dangerous irrelevant variable* [37] for $d > 4$. The monomial $|\varphi|^6$ plays a role in the determination of our upper bounds on the non-perturbative coordinate K, which we believe are not sharp for $d \ge 6$ due to the fact that our estimates do not take into account the decay in coefficients of $|\varphi|^6$ arising in perturbation theory. Although not sharp for $d \ge 6$, our estimates on K are sufficient to prove our main results.

We define the *regulator* at scale j by

$$G_j(b, \varphi) = e^{-\kappa|\varphi/h_j|^4} \qquad (b \in \mathcal{B}_j, \ \ \varphi \in \mathbb{R}^n), \tag{5.24}$$

where the dependence on b merely indicates that φ is constant on b. The constant $\kappa > 0$ is chosen to satisfy smallness conditions (depending only on n), see Lemma 8.11 and the proofs of Propositions 9.2 and 9.5. Some of the conditions on κ are technical ones, but the crucial restriction is that κ is smaller than the constant c^{st} in Lemma 8.11. The latter restriction ensures that the exponent of G_j remains smaller than the natural decay rate of $\|e^{-V_j(b)}\|_{T_\varphi(h_j)}$, namely $L^{dj}\|g_j|\varphi|^4\|_{T_\varphi(h_j)} \asymp |\varphi/h_j|^4$. The regulator is used to define the norm

$$\|K\|_{T_j^G(h_j)} = \sup_{\varphi \in \mathbb{R}^n} \frac{\|K(b)\|_{T_\varphi(h_j)}}{G_j(b, \varphi)}. \tag{5.25}$$

Often regulators control the divergence of K_j as $|\varphi| \to \infty$ (e.g., [16, 20]), but it is the opposite here—the regulator ensures a decay rate on K_j for large fields.

We define

$$\rho_j = L^{-(d-4)j}. \tag{5.26}$$

Our bounds on K_j involve ρ_j raised to the d-dependent exponents

$$\mathfrak{a}(d) = \begin{cases} 3 & (4 \le d < 6) \\ \frac{2d-6}{d-4}(1-\varepsilon(d)) & (d \ge 6), \end{cases} \qquad \mathfrak{p}(d) = \begin{cases} \frac{3}{4} & (4 \le d < 12) \\ \frac{d}{2(d-4)}(1-\varepsilon(d)) & (d \ge 12), \end{cases} \tag{5.27}$$

and $\mathfrak{b}(d) = \mathfrak{a}(d) - \mathfrak{p}(d) > 0$, where $\varepsilon(d) > 0$ is such that $\mathfrak{a}(d) > 2$, $\mathfrak{p}(d) > \frac{1}{2}$. These two conditions are satisfied if and only if $\varepsilon(d) \in (0, \frac{1}{d-3})$.

Using these parameters, we define

$$\|K\|_{\mathcal{W}_j^\kappa} = \|K\|_{T_0(\ell_j)} + \tilde{g}_j^{9/4} \rho_j^{\mathfrak{b}} \|K\|_{T_j^G(h_j)}. \tag{5.28}$$

We write $\mathcal{W}_j^\kappa$ for the space of $K \in \mathcal{F}_j$ such that $\|K\|_{\mathcal{W}_j^\kappa} < \infty$. The weight $\tilde{g}_j^{9/4} \rho_j^{\mathfrak{b}}$ serves to equalise our bounds on the two terms on the right-hand side, since we will eventually establish estimates of the form

$$\|K\|_{T_0(\ell_j)} \le O(g_j^3 \rho_j^{\mathfrak{a}}), \qquad \|K\|_{T_j^G(h_j)} \le O(g_j^{3/4} \rho_j^{\mathfrak{p}}). \tag{5.29}$$

We invite the reader to focus on the cases $d = 4, 5$, for which $\mathfrak{a} = 3$, $\mathfrak{b} = 9/4$, $\mathfrak{p} = 3/4$, and the above estimates become

$$\|K\|_{T_0(\ell_j)} \le O(g_j^3 \rho_j^3), \qquad \|K\|_{T_j^G(h_j)} \le O(g_j^{3/4} \rho_j^{3/4}) \qquad (d = 4, 5). \tag{5.30}$$

For $d = 4$, the $\mathcal{W}_j^\kappa$-norm of (5.28) is designed to replace the norm

$$\|K\|_{\mathcal{W}_j} = \|K(b)\|_{T_0(\ell_j)} + \tilde{g}_j^{9/4} \sup_{\varphi \in \mathbb{R}^n} \|K(b)\|_{T_\varphi(h_j)} \tag{5.31}$$

of [9, (8.2.9)], both by taking into account dimensions $d > 4$, and by imposing a specific exponential decay on K for large fields. By definition,

$$\|K\|_{T_j^G(h_j)} \ge \sup_{\varphi \in \mathbb{R}^n} \|K(b)\|_{T_\varphi(h_j)} \qquad \text{and} \qquad \|K\|_{\mathcal{W}_j^\kappa} \ge \|K\|_{\mathcal{W}_j}. \tag{5.32}$$

The $\mathcal{W}^\kappa$-norm dominates the $\mathcal{W}$-norm (which has no regulator), so upper bounds on the former imply upper bounds on the latter.

Given $\tilde{a} \in \mathbb{R}$, recall the mass scale $j_{\tilde{a}}$ given by (3.4), that is, the greatest integer j such that $L^{2j}\tilde{a} \le 1$, with the degenerate case $j_{\tilde{a}} = \infty$ if $\tilde{a} \le 0$. We define

$$\tilde{\vartheta}_j = \begin{cases} 2^{-(j-j_{\tilde{a}})_+} & (\tilde{a} < 1) \\ \tilde{a}^{-1} 2^{-j} & (\tilde{a} \ge 1), \end{cases} \tag{5.33}$$

which we use to keep track of decay that arises when j is above the mass scale. As in [9, (6.1.19)], we define a domain $\mathbb{I}_j$ for the mass a by

$$\mathbb{I}_j(\tilde{a}) = \begin{cases} (-\frac{1}{2}L^{-2j}, \frac{1}{2}L^{-2j}) & (\tilde{a} = 0) \\ (\frac{1}{2}\tilde{a}, 2\tilde{a}) & (\tilde{a} > 0). \end{cases} \tag{5.34}$$

Note that the interval $\mathbb{I}_j$ actually has no j-dependence unless $\tilde{a} = 0$. We emphasise that $\mathbb{I}_j(0)$ permits negative a; as mentioned previously this is a major novelty of our analysis which is absent in [9] and related RG papers. Our choice of the domain $\mathbb{I}_{N-1}(0)$ ensures that the potentially ill-behaved factors $(1 + aL^{-2j})^{-k}$ appearing in (5.10)–(5.12) at scale $j < N$ are bounded by absolute constants for $a \in \mathbb{I}_{N-1}(0)$. If we define ϑ_j using $a \in \mathbb{I}_j(\tilde{a})$ instead of $\tilde{a}$, then by definition $\vartheta_j \in [\frac{1}{2}\tilde{\vartheta}_j, 2\tilde{\vartheta}_j]$ so ϑ_j and $\tilde{\vartheta}_j$ can be used interchangeably in inequalities up to a usually unimportant factor 2 or $1/2$.

At scale j, derivatives with respect to a create factors L^{2j}. This can be inferred, for example, from the γ_j terms in the covariance matrix in (2.6). It is therefore natural to place these scaling factors inside a norm on $\mathbb{I}_j$, and we equip $\mathbb{I}_j(\tilde{a})$ with the norm where $x \in \mathbb{R}$ has norm $|x|L^dL^{2j}$. The decision to include also the factor L^d has been made for a reason which arises in the proof of Lemma 8.1, where mass derivatives are investigated in detail. With this definition, the norm of the Fréchet derivative of $f : \mathbb{R} \to \mathbb{R}$, with the above norm on the domain of f (but not the range), is

$$\|Df(a)\| = \sup_{|x|L^dL^{2j} \le 1} \Big| \frac{d}{dt}\Big|_{t=0} f(a + tx) \Big| = \frac{1}{L^dL^{2j}} \Big| \frac{df(a)}{da} \Big|, \tag{5.35}$$

and the prefactor on the right-hand side cancels the growth typical of mass derivatives.

As in [9, (8.2.1)], with the sequence $\tilde{g}_j$ given by (5.19), with k_0 as in (5.20), and given $\tilde{a} \ge 0$, we define a domain for $\mathcal{V}$ by

$$\mathcal{D}_j = \big\{ (g, \nu) \in \mathbb{R}^2 : 2k_0\tilde{g}_j < g < \tfrac{1}{2k_0}\tilde{g}_j, \ \ |\nu| < \tfrac{1}{2k_0}\tilde{g}_j\rho_j L^{-2j} \big\}. \tag{5.36}$$

Elements $V \in \mathcal{D}_j$ are equipped with the norm $\|V(b)\|_{T_0(\ell)}$ for the space $\mathcal{V}_j$, and we denote the normed space by $\mathcal{V}(\ell)$. The space $\mathcal{U}(\ell) \simeq \mathbb{R}^3$ is defined similarly with the norm $\|U(b)\|_{T_0(\ell)} = \|V(b)\|_{T_0(\ell)} + |u||b|$ which at scale j obeys (by 5.23)

$$\|U(b)\|_{T_0(\ell)} \asymp |g|\rho_j + |\nu|L^{2j} + |u|L^{dj}. \tag{5.37}$$

Given $C_{\mathrm{RG}} > 0$ (we make a specific choice in Theorem 5.7), we define a domain in $\mathcal{F}_j$ by

$$\mathcal{K}_j = \{K \in \mathcal{F}_j : \|K\|_{\mathcal{W}^\kappa} < C_{\mathrm{RG}}\tilde{\vartheta}_j^3\tilde{g}_j^3\rho_j^{\mathfrak{a}}\}. \tag{5.38}$$

Finally, we restrict the domain of Φ_+ of Definition 5.5, as in [9, (8.2.12)], to be the set of (V, K, a) in

$$\mathbb{D}_j = \mathcal{D}_j \times \mathcal{K}_j \times \mathbb{I}_j. \tag{5.39}$$

By definition, $\mathbb{D}$ depends on the choice of $\tilde{a}$ and on the initial condition g for the recursion (5.19). For $d = 4$, $\mathcal{D}_j \times \mathcal{K}_j$ is smaller than the domain that is defined in [9, (8.2.12)] using the $\mathcal{W}$ norm instead of the $\mathcal{W}^\kappa$ norm, due to (5.32), and also our $\mathbb{D}_j$ in (5.39) has an extra coordinate for a. It

should be noted that the dependence of the RG map Φ_+ on the coordinate a arises solely from the covariance C_+ of the Gaussian expectation $\mathbb{E}_+$.

5.3 | Main theorems for the RG map

To iterate and control the RG map, we need certain estimates involving Fréchet derivatives with respect to V and K. We consider maps $F : \mathcal{V}(\ell) \times \mathcal{W}^\varkappa \times \mathbb{I} \to \mathcal{X}$ taking values in a normed space $\mathcal{X}$, with $\mathcal{X}$ either $\mathcal{U}_+(\ell_+)$ or $\mathcal{W}^\varkappa_+$. For $(V, K, a) \in \mathcal{V}(\ell) \times \mathcal{W}^\varkappa \times \mathbb{I}$, the Fréchet derivative with respect to V or K or a is denoted D_V or D_K or D_a, respectively, so that $D_V^p D_K^q D_a^r F(V, K, a)$ is a multilinear map $\mathcal{V}(\ell)^p \times \mathcal{F}^q \times \mathbb{I}^r \to \mathcal{X}$. We write

$$\|D_V^p D_K^q D_a^r F(V, K, a)\|_{\mathbb{D}\to\mathcal{X}} = \sup_{\dot V^p, \dot K^q, \dot a^r} \|D_V^p D_K^q D_a^r F(V, K, a; \dot V^p, \dot K^q, \dot a^r)\|_{\mathcal{X}} \tag{5.40}$$

for the norm of this multilinear map and the supremum is taken over all $\dot V^p$, $\dot K^q$, $\dot a^r$ in $\mathcal{V}(\ell)^p$, $\mathcal{F}^q$ and $\mathbb{I}^r$ of unit norm in their respective spaces. The notation $\mathbb{D}$ for the operator norm on the left hand-side of (5.40) keeps track of the domain where (V, K, a) (*not* the directions $\dot V, \dot K, \dot a$) lives. In bounds that are uniform in (V, K, a), we omit the arguments appearing in (5.40).

In Theorems 5.6–5.7, we use the following hypothesis.

($\mathbf{A}_\Phi$) Let $\tilde a \geq 0$, let L be sufficiently large, and let $\varkappa$ be sufficiently small (the latter depending only on n). At scale j, let $\tilde g = \tilde g_j$ be sufficiently small depending on L, and let $(V, K) \in \mathbb{D} = \mathbb{D}_j$.

The following two theorems lie at the heart of the proof of Theorems 3.1, 3.2 and 3.3. Recall from (5.27) that

$$\mathfrak{a}(d) = \begin{cases} 3 & (4 \leq d < 6) \\ \frac{2d-6}{d-4}(1 - \varepsilon(d)) & (d \geq 6), \end{cases} \qquad \mathfrak{b}(d) = \begin{cases} \mathfrak{a}(d) - \frac{3}{4} & (4 \leq d < 12) \\ \frac{3}{2}(1 - \varepsilon(d)) & (d \geq 12), \end{cases} \tag{5.41}$$

with $\varepsilon(d) \in (0, \frac{1}{d-3})$. In the two theorems, the limitation p_a on the number of a-derivatives is arbitrary but fixed and finite. For their application to prove Theorems 1.1–1.3, we need p_a to be at least 2.

Theorem 5.6. *Let $d \geq 4$. At scale $j \in \{0, \dots, N-1\}$, assume ($\mathbf{A}_\Phi$) and let $r \in \{0, \dots, p_a\}$. There exist positive constants $\tilde M_{p,q}$ (for integers $p, q \geq 0$), which depend on C_{RG}, such that the map $R_+^U : \mathbb{D} \to \mathcal{U}_+$ satisfies*

$$\|D_V^p D_K^q D_a^r R_+^U\|_{\mathbb{D}\to\mathcal{U}_+(\ell_+)} \leq \begin{cases} M_{p,0} \tilde\vartheta_+^3 \tilde g_+^3 \rho_+^{\mathfrak{a}} & (p \geq 0,\ q = 0) \\ M_{p,q} & (p \geq 0,\ q \geq 1) \\ 0 & (p \geq 0,\ q \geq 3). \end{cases} \tag{5.42}$$

By (5.37), the bound (5.42) on R_+^U implies bounds on its coupling constants (r_g, r_ν):

$$\|D^\alpha r_g\|_{\mathbb{D}\to\mathbb{R}} \le O(\rho_+^{-1})\|D^\alpha R_+^U\|_{\mathbb{D}\to\mathcal{U}_+(\ell_+)}, \qquad \|D^\alpha r_\nu\|_{\mathbb{D}\to\mathbb{R}} \le O(L^{-2(j+1)})\|D^\alpha R_+^U\|_{\mathbb{D}\to\mathcal{U}_+(\ell_+)}, \tag{5.43}$$

for any mixed derivative D^α with respect to (V, K, a).

Theorem 5.7. *Let $d \ge 4$. At scale $j \in \{0, \dots, N-1\}$, assume $(\mathbf{A}_\Phi)$ and let $r \in \{0, \dots, p_a\}$. There exists a positive constant C_{RG} such that Φ_+^K is well-defined as a map $\mathbb{D} \to \mathcal{W}^\varkappa$ (i.e., the integral (5.15) converges), and there exist positive $M_{p,q}$ (for integers $p, q \ge 0$) and L-independent $M'_{0,1}$ such that*

$$\|D_V^p D_K^q D_a^r \Phi_+^K\|_{\mathbb{D}\to\mathcal{W}_+^\varkappa} \le \begin{cases} C_{\mathrm{RG}}\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{a}} & (p=0,\ q=0) \\ M_{p,0}\tilde{\vartheta}_+^3\tilde{g}_+^{3-p}\rho_+^{\mathfrak{a}-p} & (p>0,\ q=0) \\ M'_{0,1}L^{-\max\{2,(d-4)\mathfrak{a}\}} & (p=0,\ q=1) \\ M_{p,q}\tilde{g}_+^{-p-\frac{9}{4}(q-1)}\rho_+^{-p-\mathfrak{b}(q-1)} & (p\ge 0,\ q\ge 1). \end{cases} \tag{5.44}$$

The case $(p, q) = (0, 1)$ provides a bound on the K-derivative of Φ_+^K which we refer to as the *crucial contraction*. It is this latter estimate that requires most of our work. The case $(p, q) = (0, 0)$ (which turns out to follow from the crucial contraction) implies that under $(\mathbf{A}_\Phi)$, the non-perturbative coordinate at the next scale, $K_+ = \Phi_+^K$, satisfies $K_+ \in \mathcal{K}_+$. This allows for the RG map to be iterated over scales to create an RG flow which we formalise and study in detail in the next section.

6 | THE RG FLOW: PROOF OF THEOREMS 3.1, 3.2 AND 3.3

In this section, we prove Theorems 3.1, 3.2, 3.3, subject to Theorems 5.6 and 5.7. An ingredient in the proof is Proposition 6.6, and its proof is deferred to Section 7. Thus, after Section 6, what remains is to prove Theorems 5.6–5.7 and Proposition 6.6.

6.1 | Reduction of proofs of Theorems 3.1, 3.2 and 3.3

In this section, we state Propositions 6.2–6.4 and Lemma 6.5, and use them to prove Theorems 3.1–3.3. The following definition formalises the notion of an RG flow.

Definition 6.1. Let $k \in \mathbb{N}$. Given $\tilde{a} \ge 0$, initial coupling constants $g_0 > 0$ and $\nu_0 \in \mathbb{R}$ defining V_0, and $K_0 = 0$, we say that $(V_j, K_j)_{j\le k}$ is an *RG flow to scale k with initial condition (g_0, ν_0) and mass a*, if it obeys

$$(V_{j+1}, K_{j+1}) = \big(\Phi_{j+1}^V(V_j, K_j, a), \Phi_{j+1}^K(V_j, K_j, a)\big) \qquad (0 \le j < k) \tag{6.1}$$

with

$$(V_j, K_j, a) \in \mathbb{D}(\tilde{a}) \qquad (0 \le j < k). \tag{6.2}$$

For fixed g_0, we write $\mathcal{X}_k$ for the set of (ν_0, a) consisting of such initial ν_0 and mass a. When an RG flow exists to scale k for all $k \in \mathbb{N}$ then we refer to it as a *global RG flow*, and write $\mathcal{X}_\infty = \cap_{k=1}^\infty \mathcal{X}_k$.

An important observation is that, by (6.2), the bounds of Theorems 5.6–5.7 hold for all scales $j \le k$ for an RG flow to scale k. Also, an RG flow to scale $k+1$ with a given initial condition and mass simply extends the RG flow to scale k with the same initial condition and mass. In other words, lengthening an RG flow does not alter its initial portion. This follows from the definition of the RG map, in which (V_j, K_j) is determined by scales prior to scale j and is not affected by future scales.

Concerning the relative roles of a vs $\tilde{a}$, given any $a > 0$ we can always choose (many) $\tilde{a} > 0$ such that $a \in \mathbb{I}(\tilde{a}) = (\frac{1}{2}\tilde{a}, 2\tilde{a})$. On the other hand, if $\tilde{a} = 0$ then a requirement that $a \in \mathbb{I}_j(0) = (-\frac{1}{2}L^{-2j}, \frac{1}{2}L^{-2j})$ is restrictive, and if the requirement is to hold for all scales j then we must have $a = 0$.

Each RG map Φ_{j+1} depends on a through the covariance C_{j+1}. Although these maps do not depend on ν_0, an RG flow to scale k with initial condition (g, ν_0) does depend on both a and ν_0, as does any function of (V_j, K_j) in an RG flow with initial condition (g, ν_0). Partial derivatives of a function $f(\nu_0, a)$ with respect to ν_0 and a are denoted as

$$f' = \frac{\partial f}{\partial \nu_0}, \qquad \dot{f} = \frac{\partial f}{\partial a}. \tag{6.3}$$

We use this notation heavily. Moreover, in order to construct an RG flow it is necessary also to choose the initial value ν_0 to depend on a.

We equip the subsets $\mathcal{X}_k$ and $\mathcal{X}_\infty$ of $\mathbb{R}^2$ with the subset topology. Thus, when we say a function $f : \mathcal{X}_k \to \mathbb{R}$ is continuous, it means that f is continuous with respect to this topology. Given a mass $a \ge 0$, we will determine a critical value $\nu_c(a)$ from which we can construct a global RG flow, and in this case the set $\mathcal{X}_\infty$ consists of elements of the form $(\nu_c(a), a)$, and continuity reduces to continuity in a.

6.1.1 | The main propositions

Proposition 6.2 (Existence of critical ν_0). *Let $d \ge 4$. Let $g > 0$ and $\kappa > 0$ be sufficiently small and let $a \ge 0$. There exists a continuous function $\nu_c : [0, \infty) \to \mathbb{R}$ (depending also on g) such that (V_j, K_j) is a global RG flow with initial condition $(g, \nu_c(a))$ and mass a, with $\mathbb{D}_j$ defined by any $\tilde{a}$ such that $a \in \mathbb{I}_j(\tilde{a})$ for all $j \ge 0$. The critical value $\nu_c(a)$ satisfies $|\nu_c(a)| \le O(\vartheta_0 g)$, and $\nu_c(0) \sim -(n+2)g(-\Delta_H)^{-1}_{00}$ as $g \downarrow 0$.*

The occurrence of ϑ_0 in the last sentence of Proposition 6.2 has significance only for large mass, since by definition $\vartheta_0 = 1$ for $a < 1$ and $\vartheta_0 = a^{-1}$ for $a \ge 1$. For $d > 4$, it is shown in the proof of Theorem 3.1(ii) in Section 6.1.2 that

$$A_d = 1 + \dot{\nu}_c(0) \qquad (d > 4). \tag{6.4}$$

The constant A_4 in Theorem 1.1 appears in the next proposition.

Proposition 6.3 (Derivative of the critical value). *For $d > 4$, the critical value $\nu_c(\cdot)$ is a strictly increasing continuously differentiable function of $a \geq 0$, with $\dot{\nu}_c(a) = O(g)$. For $d = 4$, $\nu_c(\cdot)$ is strictly increasing and is continuously differentiable for $a > 0$, and there is a constant $A_4 > 0$ such that*

$$\dot{\nu}_c(a) \sim A_4(\log a^{-1})^{\hat{\gamma}} \qquad \text{as } a \downarrow 0, \tag{6.5}$$

with $A_4 \sim (\frac{Bg}{\log L^2})^{\hat{\gamma}}$ as $g \downarrow 0$.

The next proposition permits small negative mass a.

Proposition 6.4. *Let $d \geq 4$. Let $g > 0$ and $\kappa > 0$ be sufficiently small. For $N \in \mathbb{N}$, there exists a continuously differentiable function $\nu_{0,N} : \mathbb{I}_{N-1}(0) \to \mathbb{R}$, with $\nu_{0,N}(0) = \nu_c(0)$, such that (V_j, K_j) is an RG flow to scale N with initial condition $(g, \nu_{0,N}(a))$ and mass $a \in \mathbb{I}_{N-1}(0)$. There exists a constant $c^{\mathrm{F}} = O(g)$ such that the derivative of $\nu_{0,N}$ satisfies*

$$\dot{\nu}_{0,N}(a) = \begin{cases} A_4(\log L^2)^{\hat{\gamma}}\big(N^{\hat{\gamma}} - 1 + c^{\mathrm{F}}\big)\big(1 + O(N^{-1})\big) & (d = 4) \\ \dot{\nu}_c(0) + O(g_N \rho_N^{\mathfrak{a}-2}) & (d > 4). \end{cases} \tag{6.6}$$

Under the hypotheses of Proposition 6.2, the estimates of Theorems 5.6 and 5.7 all hold for all scales j. In particular, by (5.43) the correction $r_{g,j}$ to the perturbative flow of g_j is $O(\tilde{\vartheta}_j^3 g_j^3 \rho_j^{\mathfrak{a}-1})$. From this we can determine the asymptotic behaviour of g_j, as in the following lemma. Recall that the mass scale j_a is defined in (3.4). The constant g_∞ which appears in the definition of the window scale w_N in (1.35) and in the formula for $\hbar_N$ in (1.40) is equal to $g_\infty = g_\infty(0)$. We postpone the proof of Lemma 6.5 to Section 6.2 where we prepare the derivative estimates (6.8); the rest of Lemma 6.5 follows easily from the recursion relation for g_j. Derivatives of g_j are also needed for $d = 4$, but not yet, and their analysis is postponed to Section 7.

Lemma 6.5. *Let $d \geq 4$. Let $\tilde{a} \geq 0$. Let $g > 0$ and $\kappa > 0$ be sufficiently small. The following statements also hold when g_j is replaced by $\tilde{g}_j$.*

(i) Let $k \in \mathbb{N}$ and suppose that (V_j, K_j) is an RG flow to scale k with initial condition (g, ν_0) (for some ν_0) and mass a. Then, for each $j \leq k$,

$$g_j \begin{cases} = (Bj)^{-1} + O(j^{-2}\log j) & (d = 4,\ \tilde{a} = 0) \\ \asymp (j \wedge j_a)^{-1} + O((j \wedge j_a)^{-2}\log(j \wedge j_a)) & (d = 4,\ j_{\tilde{a}} \geq 1) \\ = g + O(g^2) & (d = 4,\ j_{\tilde{a}} \leq 0 \text{ or } d > 4), \end{cases} \tag{6.7}$$

and, for $d > 4$, g_j is a differentiable function of $(\nu_0, a) \in \mathcal{X}_k$ with

$$|g_j'|, \ \ |\dot{g}_j| \leq O(L^{2j})\tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2}. \tag{6.8}$$

(ii) For $d > 4$, or for $d = 4$ with $j_{\tilde{a}} < \infty$, suppose that (V_j, K_j) is a global RG flow with initial condition (g, ν_0) (for some ν_0) and mass a. Then there exists a continuous function $g_\infty(\nu_0, a) = g + O(g^2)$ of $(\nu_0, a) \in \mathcal{X}_\infty$ such that

$$g_j = \begin{cases} g_\infty + O(\vartheta_j^2 g^2) & (d = 4,\ j \geq j_{\tilde{a}}) \\ g_\infty + O(g^2 \rho_j) & (d > 4). \end{cases} \tag{6.9}$$

6.1.2 | Proofs of Theorems 3.1, 3.2 and 3.3

Proof of Theorem 3.1. For $a \geq 0$, we take $\nu_0 = \nu_c(a)$ to be the critical value from Proposition 6.2. The asymptotic formulas (1.30) for $\nu_c(0)$ and A_d as $g \downarrow 0$, which are claimed in Theorem 3.1, are part of the statements of Proposition 6.2 and Proposition 6.3. We proceed to prove the estimates for $d = 4$ in (i) and for $d > 4$ in (ii).

(i) For $d = 4$, the three bounds on g_N in (3.6) are stated in (6.7), and the bounds on ν_N and K_N in (3.7)–(3.8) follow from the fact that $(V_N, K_N) \in \mathbb{D}_N$ by Proposition 6.2. In fact, the bound on ν_N is immediate from the definition of $\mathbb{D}_N$, while for K_N, for all $d \geq 4$ we have

$$|K_N(\varphi)| \leq C_{\mathrm{RG}} \tilde{\vartheta}_N^3 (g_N \rho_N)^{3/4} e^{-\kappa \tilde{g}_N L^{dN} |\varphi|^4}. \tag{6.10}$$

For $d = 4$, $\rho_N = 1$. Also, $\tilde{\vartheta}_N^3 g_N^{3/4} \leq O(N^{-3/4})$ due to the bounds on g_N in (3.6) and the fact that $\tilde{\vartheta}_N$ decays exponentially starting from the mass scale. Since $e_N = N^{-3/4}$ by definition, the bound (3.8) on K_N follows.

The fact that ν_c is a differentiable function of the mass is part of Proposition 6.3, and the bound (3.9) is given by (6.5). This completes the proof of (i).

(ii) For $d \geq 5$, the existence of $g_\infty(a)$ and the bound (3.10) on g_N follow from (6.9). The bound (3.8) on ν_N follows from the fact that $(V_N, K_N) \in \mathbb{D}_N$ by Proposition 6.2, and similarly the bound (3.12) on K_N follows from (6.10) (in fact (6.10) gives $g^{3/4} L^{-3(d-4)N/4}$ for *all* $d > 4$ which is better than e_N for $d > 12$).

The fact that ν_c is a strictly increasing differentiable function of the mass is a part of Proposition 6.3, whose statement includes the continuity of $\dot{\nu}_c$ on $[0, \infty)$ with $\dot{\nu}_c(a) = O(g)$. Thus (3.13) holds with $A_d = 1 + \dot{\nu}_c(0)$ as in (6.4).

To verify (3.14), we first observe from (3.10) that it suffices to prove that, when $d \geq 5$, $|g_N(a) - g_N(0)| \leq O(g^2 a L^N)$. To do so, we temporarily make the dependence of g_j on ν explicit, and use (6.8), the Mean Value Theorem, and Proposition 6.4 to see that

$$\begin{aligned} |g_N(\nu_c(a), a) - g_N(\nu_c(0), 0)| &\leq \sup_{\overline{a} \in [0,a]} |g'_N(\nu_c(\overline{a}), \overline{a})| \, |\nu_c(a) - \nu_c(0)| + \sup_{\overline{a} \in [0,a]} |\dot{g}_N(\nu_c(\overline{a}), \overline{a})| \, a \\ &\leq O\big(g^2 L^{2N} \rho_N^{\mathfrak{a}-2} a\big). \end{aligned} \tag{6.11}$$

It therefore suffices to prove that $L^{2N} \rho_N^{\mathfrak{a}-2} \leq L^N$, that is, that $2 - (d-4)(\mathfrak{a}-2) \leq 1$ with $\mathfrak{a}$ given by (5.27) and $\varepsilon(d) \in (0, \frac{1}{d-3})$ small enough when $d \geq 5$. For $d = 5$, since $\mathfrak{a} = 3$ we obtain $2 - (d-4)(\mathfrak{a}-2) = 1$, which is what we want. For $d \geq 6$, we get $2 - (d-4)(\mathfrak{a}-2) = (2d-6)\varepsilon(d)$ and this is sufficient because we can make the choice $\varepsilon(d) = \frac{1}{2d-6}$. □

Proof of Theorem 3.2. Consider the function $f(a) = \nu_c(a) + a - \nu_c(0)$. Our goal is to compute the inverse function of f, as $m_\varepsilon^2 = f^{-1}(\varepsilon)$. By Proposition 6.3, $f(a)$ is strictly increasing with $\dot{f}(a) \geq 1 - O(g)$, so $f : [0, \infty) \to [0, \infty)$ is a bijection and its inverse does exist.

For $d = 4$, it follows from Proposition 6.3 that

$$f(a) \sim A_4 \, a (\log a^{-1})^{\hat{\gamma}} \qquad \text{as } a \downarrow 0. \tag{6.12}$$

From this, we conclude that

$$f^{-1}(\varepsilon) \sim A_4^{-1}\varepsilon(\log \varepsilon^{-1})^{-\hat{\gamma}} \qquad \text{as } \varepsilon \downarrow 0, \tag{6.13}$$

which is the $d = 4$ case of (3.16).

For $d > 4$, Proposition 6.3 says that f is continuously differentiable with $f'(a) = 1 + \dot{\nu}_c(a)$ near 0. By the Inverse Function Theorem, $m_\varepsilon^2 = f^{-1}(\varepsilon)$ grows approximately linearly for small ε with slope $A_d^{-1} = (1 + \dot{\nu}_c(0))^{-1}$, and this proves the $d > 4$ case of (3.16). □

Proof of Theorem 3.3. Except for (3.22), the estimates of Theorem 3.3 follow from Proposition 6.4 and the estimates inherent to an RG flow due to the definition of the domain, together with Lemma 6.5 for g_N. The strategy to prove (3.22) is as in (6.11), again with $\varepsilon(d) = \frac{1}{2d-6}$. Indeed, for $a \in \mathbb{I}_{N-1}(0)$, and since (again by Proposition 6.4) (6.8) holds for this range of a,

$$g_N(\nu_c(a), a) = g_N(\nu_c(0), 0) + O(g^2 L^N |a|). \tag{6.14}$$

Finally, since $|a| \le O(L^{-2N})$, the error term is at most $O(g^2 L^{-N})$, which proves (3.22) and completes the proof. □

6.2 | Proofs of Propositions 6.2, 6.3, 6.4 and Lemma 6.5

6.2.1 | Derivatives of ν_j and proof of Lemma 6.5

We start with Proposition 6.6, which concerns bounds on derivatives with respect to the mass a and the initial value ν_0. Its proof is lengthy but is essentially only calculus, and the proof is deferred to Section 7. As mentioned previously, derivatives of g_j are also needed for $d = 4$, but we postpone those to Section 7, alongside derivatives of K_j for $d \ge 4$, as part of the proof of Proposition 6.6. In Proposition 6.6, note that all statements for $d = 4$ agree with those for $d > 4$ if we set $\hat{\gamma} = 0$ and $\rho_j = 1$. Recall that we denote the derivatives of a function $f(\nu_0, a)$ as

$$f' = \frac{\partial f}{\partial \nu_0}, \qquad \dot{f} = \frac{\partial f}{\partial a}. \tag{6.15}$$

Proposition 6.6. *Let $d \ge 4$ and let g_0 be sufficiently small. Let $N \in \mathbb{N}$ and $(\nu_0, a) \in \mathbb{R} \times \mathbb{I}_{N-1}(\tilde{a})$. Suppose that (V_j, K_j) is an RG flow to scale N with initial condition (g_0, ν_0) and mass a. Then, for $j \le N$, (g_j, ν_j, K_j) are twice differentiable functions of $(\nu_0, a) \in \mathcal{X}_N$ and satisfy the following bounds.*

(i) For $d = 4$, there are g_0-dependent constants $c_{0,N}(\nu_0, a) = 1 + O(\tilde{\mathfrak{d}}_0 g_0)$ and $c_{1,N}(\nu_0, a) = 1 + O(\tilde{\mathfrak{d}}_0 g_0)$ such that

$$\nu_j' = \Big(\frac{g_j}{g_0}\Big)^{\hat{\gamma}} \big(c_{0,N} + O(\tilde{\mathfrak{d}}_j \tilde{g}_j)\big), \qquad \dot{\nu}_j = -1 + \Big(\frac{g_j}{g_0}\Big)^{\hat{\gamma}} c_{1,N} + O(\tilde{\mathfrak{d}}_j \tilde{g}_j), \tag{6.16}$$

$$|\nu_j''| \le O\Big(L^{2j} \tilde{\mathfrak{d}}_j^3 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}\Big), \qquad |\dot{\nu}_j'| \le O\Big(L^{2j} \tilde{\mathfrak{d}}_j^3 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g}\Big)^{\hat{\gamma}}\Big), \qquad |\ddot{\nu}_j| \le O(L^{2j} \tilde{\mathfrak{d}}_j^3 \tilde{g}_j) \tag{6.17}$$

Also, there is a constant $C > 0$ such that $\dot{\nu}_j \le -C\tilde{\vartheta}_0^2 \tilde{g}_j$. For a global RG flow, the limits $c_{i,\infty} = \lim_{N\to\infty} c_{i,N}$ exist for $i = 0, 1$. Each $c_{i,N}$ is continuous in $(\nu_0, a) \in \mathcal{X}_N$ and each $c_{i,\infty}$ is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$.

(ii) For $d > 4$, there are g_0-dependent constants constants $c_{2,N}(\nu_0, a) = 1 + O(\tilde{\vartheta}_0 g_0)$ and positive $c_{3,N}(\nu, a) = O(\tilde{\vartheta}_0 g_0)$ such that

$$\nu_j' = c_{2,N} + O(\tilde{\vartheta}_j \tilde{g}_j \rho_j), \qquad \dot{\nu}_j = -c_{3,N} + O(\tilde{\vartheta}_j \tilde{g}_j \rho_j), \tag{6.18}$$

$$|\nu_j''|,\ |\dot{\nu}_j'|,\ |\ddot{\nu}_j| \le O(L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j \rho_j^{\mathfrak{a}-2}). \tag{6.19}$$

Also, there are constants $C_1, C_2 > 0$ such that $\dot{\nu}_j \le -C_1 \tilde{\vartheta}_0^2 \tilde{g}_j$ for sufficiently large j and

$$|g_j'| \le C_2 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2}, \qquad |\dot{g}_j| \le C_2 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2}. \tag{6.20}$$

For a global RG flow, the limits $c_{i,\infty} = \lim_{N\to\infty} c_{i,N}$ exist for $i = 2, 3$. Each $c_{i,N}$ is continuous in $(\nu_0, a) \in \mathcal{X}_N$ and each $c_{i,\infty}$ is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$.

Proof of Lemma 6.5. Let $\tilde{a} \ge 0$ and fix $a \in \mathbb{I}_{k-1}(\tilde{a})$. By (5.17), (5.8), and since the bound of Theorem 5.6 applies, for $j < k$ we have

$$g_{j+1} = g_j - \beta_j g_j^2 + O(\tilde{\vartheta}_j^3 g_j^3 \rho_j^{\mathfrak{a}}), \tag{6.21}$$

with $\beta_j(a) = (n+8)L^{-(d-4)j}(1 + aL^{2j})^{-2}(1 - L^{-d})$ by (5.10).

Suppose first that $d = 4$. In this case, the asymptotic behaviour of solutions to this recursion are given by [9, Proposition 6.1.3]. This implies (6.7) for $j_{\tilde{a}} \ge 1$ (i.e., $\tilde{a} \in (0, 1)$), and also for $\tilde{a} = a = 0$ once we observe that the constant B of (1.34) is identical to β_0^0 (see [9, Lemma 5.3.4]). The extension to the case $a \in \mathbb{I}_{k-1}(0)$ follows as in the proof of [9, (6.1.20)] and we omit the details, other than to mention that our restriction on a guarantees that the factor $(1 + aL^{2j})^{-2}$ remains uniformly bounded. For $\tilde{a} > 1$, the third line of (6.7) and (6.9) follow by summing $g_{j+1} - g_j = O(\beta_j g_j^2) = O(g^2 L^{-4j})$. The fact that the function $(\nu_0, a) \mapsto g_\infty(\nu_0, a)$ is continuous on $\mathcal{X}_\infty$ follows from the Dominated Convergence Theorem.

For $d > 4$, for both (6.7) and (6.9), we use $g_{j+1} - g_j = O(g_j^2 \rho_j) = O(g_j^2 L^{-(d-4)j})$ instead, for which the exponentially decaying error term simplifies matters. Continuity of $(\nu_0, a) \mapsto g_\infty(\nu_0, a)$ follows from the Dominated Convergence Theorem and the fact that each g_j is continuous in $(\nu_0, a) \in \mathcal{X}_j$.

Finally, (6.8) is the same as (6.20). □

6.2.2 | The Bleher–Sinai argument

The construction of the critical point is based on Lemma 6.7. Lemma 6.7 is analogous to the Bleher–Sinai argument of [9, Proposition 8.3.1], which itself originated in [15]. Unlike [9, Proposition 8.3.1], Lemma 6.7 applies for all dimensions $d \ge 4$. The lemma produces a sequence of decreasing intervals of ν-values $I_j(a; \tilde{a})$ for which the RG flow with mass a exists up to scale j when started with initial condition $\nu \in I_j(a; \tilde{a})$. These intervals shrink to a point which uniquely identifies the critical point $\nu_c(a)$. The lemma also gives the rate at which $|I_j(a; \tilde{a})| \to 0$ when

$j \to \infty$. This turns out to be essential in the proof of Lemma 6.10, from which we prove our estimates on the derivative $\dot{\nu}_c(a)$ in Proposition 6.3.

For the statement of the lemma, given $\tilde{a} \geq 0$, we define the intervals

$$H_j = \big(\tfrac{1}{2}\tilde{g}_j(\tilde{a}), 2\tilde{g}_j(\tilde{a})\big), \qquad J_j = (-6(n+2)\tilde{\vartheta}_j\tilde{g}_j(\tilde{a})\rho_j L^{-2j}, 6(n+2)\tilde{\vartheta}_j\tilde{g}_j(\tilde{a})\rho_j L^{-2j}). \tag{6.22}$$

By definition of the domain $\mathcal{D}_j$ in (5.36), if we take any $k_0 \leq \frac{1}{24(n+2)}$, then $H_j \times J_j \times \mathbb{I}_j(\tilde{a}) \subset \mathcal{D}_j$. Recall from (5.38) that $\mathcal{K}_j$ consists of those $K \in \mathcal{F}_j$ for which $\|K\|_{\mathcal{W}^\kappa} < C_{\mathrm{RG}}\tilde{\vartheta}_j^3\tilde{g}_j^3\rho_j^{\mathfrak{a}}$. We use (5.34) to define $\mathbb{I}_j(\tilde{a})$ for $j \geq -1$.

Lemma 6.7. *Let L be sufficiently large, g_0 be sufficiently small, and $\tilde{a} \geq 0$. For $0 \leq j \leq N$ and $a \in \mathbb{I}_{j-1}$, there are open intervals $I_j(a;\tilde{a}) \subset \mathbb{R}$, nested in the sense that $I_{j+1} \subset I_j$, which satisfy the following for an RG flow to scale j with initial condition (g_0, ν_0), mass a, and $\nu_0 \in I_j$:*

- *For all $i \leq j$, we have $(g_i, \nu_i, K_i) \in H_i \times J_i \times \mathcal{K}_i$.*
- *For each fixed $a \in \mathbb{I}_{j-1}(\tilde{a})$, the set $\{\nu_j : \nu_0 \in I_j(a;\tilde{a})\}$ of ν_j values with initial condition $\nu_0 \in I_j(a;\tilde{a})$ is exactly J_j.*
- *For each fixed $a \in \mathbb{I}_{j-1}(\tilde{a})$, the interval $I_j(a;\tilde{a})$ is connected, has length bounded by*

$$|I_j(a;\tilde{a})| \leq \begin{cases} O\big(\big(\frac{\tilde{g}_j}{g_0}\big)^{-\hat{\gamma}}|J_j|\big) & (d=4) \\ O(|J_j|) & (d>4), \end{cases} \tag{6.23}$$

and $\bar{I}_{j+1}(a;\tilde{a}) \subset I_j(a;\tilde{a})$ (here $\bar{I}$ is the closure of I).

Proof. Fix $\tilde{a} \geq 0$. We write $\mathbb{I}_{j-1} = \mathbb{I}_{j-1}(\tilde{a})$, $\tilde{g}_j = \tilde{g}_j(\tilde{a})$, and set $\mu_j = L^{2j}\nu_j$. Let $a \in \mathbb{I}_{k-1}$. Fix $k \leq j$. We make the induction hypothesis that there exists $I_k(a;\tilde{a})$ such that the statement of the lemma holds when j is replaced by k. For $k = 0$, the statement does hold with $I_0 = J_0$.

To advance the induction, consider $\nu_0 \in I_k(a)$. By Lemma 6.5, $g_{k+1} \in H_{k+1}$. By Theorem 5.7, $K_{k+1} \in \mathcal{K}_{k+1}$. By the perturbative recursion for ν_k in (5.9) and by Theorems 5.6,

$$\mu_{k+1} = L^2(\mu_k + e_k), \tag{6.24}$$

with

$$e_k = -\hat{\gamma}\beta_k\mu_k g_k + L^{2k}\eta_k g_k - L^{2k}\xi_k g_k^2 + O(\tilde{\vartheta}_k^3\tilde{g}_k^3\rho_k^{\mathfrak{a}}). \tag{6.25}$$

Since $|\hat{\gamma}\beta_k\mu_k g_k| \leq O(\tilde{\vartheta}_k^3\tilde{g}_k^2\rho_k^2)$, $L^{2k}|\eta_k| \leq 2(n+2)\tilde{\vartheta}_k\rho_k$ and $L^{2k}\xi_k \leq O(\tilde{\vartheta}_j^3\rho_k^2)$, e_k satisfies

$$|e_k| \leq |\eta_k|g_k + O(\tilde{\vartheta}_k^3\tilde{g}_k^2\rho_k^{\mathfrak{a}}) \leq \frac{5}{2}(n+2)\tilde{\vartheta}_k g_k\rho_k \leq 5(n+2)\tilde{\vartheta}_k\tilde{g}_k\rho_k, \tag{6.26}$$

whenever $g_k \leq 2\tilde{g}_k \leq 2g_0$ is sufficiently small. By the assumption that the range of μ_k is exactly $L^{2k}J_k$, and by (6.24) and (6.25),

$$\begin{aligned} \sup_{\nu\in I_k(a)} \mu_{k+1} &\geq \sup_{\nu\in I_k(a)} L^2(\mu_k - 5(n+2)\tilde{\vartheta}_k\tilde{g}_k\rho_k) \geq (n+2)L^2\tilde{\vartheta}_k\tilde{g}_k\rho_k, \\ \inf_{\nu\in I_k(a)} \mu_{k+1} &\leq \inf_{\nu\in I_k(a)} L^2(\mu_k + 5(n+2)\tilde{\vartheta}_k\tilde{g}_k\rho_k) \leq -(n+2)L^2\tilde{\vartheta}_k\tilde{g}_k\rho_k. \end{aligned} \tag{6.27}$$

For large L, the bounds in (6.27) exceed the limit $6(n+2)\tilde{\vartheta}_{k+1}\tilde{g}_{k+1}\rho_{k+1}$ of the interval $L^{2(k+1)}J_{k+1}$, since the ratio of $\tilde{\vartheta}_{k+1}\tilde{g}_{k+1}$ to $\tilde{\vartheta}_k\tilde{g}_k$ is bounded and since $\rho_{k+1} \le \rho_k$. Therefore, by the continuity of the function $\psi : \nu_0 \mapsto \mu_{k+1}$, the set $\psi(I_k)$ contains the interval $L^{2(k+1)}J_{k+1}$. Thus we can define the preimage $\psi^{-1}(L^{2(k+1)}J_{k+1})$ (inside $I_k(a;\tilde{a})$) to be $I_{k+1}(a;\tilde{a})$.

This I_{k+1} is an open set because it is the preimage of an open set under the continuous function ψ. By Proposition 6.6, μ_{k+1} is a monotone function of $\nu \in I_k$, so since I_k is a connected interval, I_{k+1} is also a connected interval. Since $\psi(I_{k+1}) = L^{2(k+1)}J_{k+1}$, it follows from the estimates on $\mu'_{k+1} = L^{2(k+1)}\nu'_{k+1}$ from (6.16) and (6.18) that

$$|I_{k+1}| \le \frac{1}{\inf_{\nu_0 \in I_k} \psi'(\nu_0)} L^{2(k+1)}|J_{k+1}|, \tag{6.28}$$

from which we conclude that (6.23) holds. Finally, since μ_{k+1} is a continuous function of ν_0, the image of $\bar{I}_{k+1}(a;\tilde{a})$ is $L^{2(k+1)}\bar{J}_{k+1}$, which is contained in the image of $I_k(a;\tilde{a})$, so we have $\bar{I}_{k+1}(a;\tilde{a}) \subset I_k(a;\tilde{a})$. □

6.2.3 | Proofs of Propositions 6.2, 6.3 and 6.4

Proof of Proposition 6.2. Let $a \ge 0$ and set $\tilde{a} = a$ in Lemma 6.7. Then there exists a nested sequence of intervals $I_j(a;a)$ satisfying the properties of Lemma 6.7, and we can define the infinite-volume critical point as the unique point in their intersection:

$$\{\nu_c(a)\} = \cap_{j\ge 0} I_j(a;a). \tag{6.29}$$

The intersection is non-empty because $\bar{I}_{j+1} \subset I_j$ for each j, and it contains a unique point since $|I_j| \to 0$ as $j \to \infty$. This holds for every $j \ge 0$, so since $H_j \times J_j \times \mathbb{I}_j(\tilde{a}) \subset \mathcal{D}_j$ there is a global RG flow with initial condition $(g, \nu_c(a))$ and mass a. The properties of the flow (g_j, ν_j, K_j) with initial condition $\nu_0 = \nu_c(a)$ then follow from Lemma 6.7. The proof that $\nu_c(a)$ is continuous in a is as in [9, Proposition 8.3.3].

It remains to prove that $|\nu_c(a)| \le O(\vartheta_0 g)$, and $\nu_c(0) \sim -(n+2)g(-\Delta_H)^{-1}_{00}$ as $g \downarrow 0$. The upper bound follows from that $\nu_c(a) \in I_0(a) = J_0(a)$. For $d = 4$, the asymptotic formula for $\nu_c(0)$ follows from [9, (4.2.7)]. The proof is similar for $d > 4$, as follows. By solving the recursion relation for ν_j backwards as in [9, (6.1.31)], we obtain

$$\nu_0 = \Pi^{-1}_{0,j}\nu_{j+1} + \sum_{k=0}^{j}(-\eta_k g_k + \xi_k g_k^2 - r_{\nu,k})\Pi^{-1}_{0,k-1} \tag{6.30}$$

with

$$\Pi_{i,j} = \prod_{k=i}^{j}(1 - \hat{\gamma}\beta_k g_k). \tag{6.31}$$

With the choice $(\nu_0, a) = (\nu_c(a), a)$, the infinite product converges since β_j decays exponentially, so $\Pi^{-1}_{0,j}\nu_{j+1} \to 0$ as $j \to \infty$. By (5.11) and [9, (5.3.5)], $(n+2)^{-1}\eta_k$ is the 00 component of C_{k+1}, so

$$\sum_{k=0}^{\infty}\eta_k = (n+2)(-\Delta)^{-1}_{00}. \tag{6.32}$$

It follows from the asymptotic behaviour of g_k in Lemma 6.5 that $\Pi_{0,k-1} = 1 + O(g_0)$, and, with the exponential decay of ξ_k and $r_{\nu,k}$, that

$$\nu_c(0) = \sum_{k=0}^{\infty} (-\eta_k g_k + \xi_k g_k^2 - r_{\nu,k}) \Pi_{0,k-1}^{-1} = -(n+2)(-\Delta)_{00}^{-1} g_0 (1 + O(g_0)). \tag{6.33}$$

This completes the proof. □

Remark 6.8. In the above proof, we made the specific choice $a = \tilde{a}$. Although each $I_j(a;\tilde{a})$ depends on $\tilde{a}$, as long as $k_0 \le \frac{1}{24(n+2)}$ and $a \in (\frac{1}{2}\tilde{a}, 2\tilde{a})$, the intersection $\cap_{j\ge 0} I_j(a;\tilde{a})$ does not depend on $\tilde{a}$ by the uniqueness argument of [9, Proposition 8.3.1], that is,

$$\{\nu_c(a)\} = \cap_{j\ge 0} I_j(a;\tilde{a}) \tag{6.34}$$

whenever $a \in \cap_{j\ge 0} \mathbb{I}_j(\tilde{a})$ (the set $\cap_{j\ge 0}\mathbb{I}_j(\tilde{a})$ is $(\frac{1}{2}\tilde{a}, 2\tilde{a})$ if $\tilde{a} > 0$ and is $\{0\}$ if $\tilde{a} = 0$).

The value of ν_N depends on the initial value $\nu_0 = \nu$ and on the mass a (also on g but we suppress this dependence since we fix g). We write $\nu_N|_{(\nu_0,a)}$ when we need to display the dependence on (ν_0, a). The following lemma defines the function $\nu_{0,N}(a;\tilde{a})$. The interval I_N in Lemma 6.9 is the one determined by Lemma 6.7. We also need the subset of $\mathbb{R}^2$ defined by $\mathbb{A}_N(\tilde{a}) = \{(\nu, a) : a \in \mathbb{I}_{N-1}(\tilde{a}),\ \nu \in I_N(a;\tilde{a})\}$. By construction of each $I_N(a;\tilde{a})$ in Lemma 6.7, $\mathbb{A}_N(\tilde{a})$ is a preimage of an open set under the continuous map $(\nu_0, a) \mapsto \mu_N$, thus is open.

Lemma 6.9. *For $\tilde{a} \ge 0$ and $a \in \mathbb{I}_{N-1}(\tilde{a})$, there is a unique $\nu_{0,N}(a;\tilde{a}) \in I_N(a;\tilde{a})$ such that*

$$\nu_N|_{(\nu_{0,N}(a;\tilde{a}),a)} = \nu_N|_{(\nu_c(\tilde{a}),\tilde{a})}. \tag{6.35}$$

This $\nu_{0,N}$ satisfies

$$\nu_c(a) = \lim_{N\to\infty} \nu_{0,N}(a,\tilde{a}). \tag{6.36}$$

The function $\nu_{0,N}(\cdot;\tilde{a}) : \mathbb{I}_{N-1}(\tilde{a}) \to \mathbb{R}$ is twice continuously differentiable, obeys $\nu_{0,N}(\tilde{a};\tilde{a}) = \nu_c(\tilde{a})$, and, for all $a \in \mathbb{I}_{N-1}(\tilde{a})$ its derivative satisfies

$$\dot{\nu}_{0,N}(a;\tilde{a}) = -\frac{\dot{\nu}_N}{\nu_N'}\Big|_{(\nu_{0,N}(a;\tilde{a}),a)}. \tag{6.37}$$

Proof. We first verify the existence and uniqueness of $\nu_{0,N}(a;\tilde{a})$ satisfying (6.35). Let $\tilde{a} \ge 0$ and $a \in \mathbb{I}_{N-1}(\tilde{a})$. By Lemma 6.7 together with the identification of $\nu_c(\tilde{a})$ in (6.29), we have $(\nu_c(\tilde{a}), \tilde{a}) \in \mathbb{A}_N(\tilde{a})$, and hence in particular $\nu_N|_{(\nu_c(\tilde{a}),\tilde{a})} \in J_N(\tilde{a})$. We have seen in the proof of Lemma 6.7 that the map $\nu \mapsto \nu_N|_{(\nu,a)}$ from $I_N(a;\tilde{a})$ to $J_N(\tilde{a})$ is surjective. It is also strictly increasing since $\nu_N' > 0$ by Proposition 6.6, so the existence and uniqueness of $\nu_{0,N}(a)$ obeying (6.35) follows. Then by Lemma 6.7, $(g_j, \nu_j, K_j, a) \in \mathbb{D}_j$ for each $j \le N$ when $\nu = \nu_{0,N}(a)$.

The limit (6.36) follows from Remark 6.8 and the fact that $\nu_{0,N}(a;\tilde{a}) \in I_N(a;\tilde{a})$ for each N.

It remains to establish the claims about differentiability of $\nu_{0,N}(a;\tilde{a})$. Given $d \ge 4$, we fix $\tilde{a} \ge 0$ and consider $\nu_{0,N}(a_0;\tilde{a}) \in I_N(a_0;\tilde{a})$ for $a_0 \in \mathbb{I}_{N-1}(\tilde{a})$. By Proposition 6.6, the map $(\nu_0, a) \mapsto \nu_N$ on $\mathbb{A}_N$ is twice continuously differentiable on a neighbourhood of $(\nu_{0,N}(a_0;\tilde{a}), a_0)$, with non-vanishing derivative ν_N'. By the Implicit Function Theorem, there exists an open neighbourhood

$U \subset \mathbb{I}_{N-1}(\tilde{a})$ of a_0 and $f \in C^2(U;\mathbb{R})$ such that

$$\nu_N|_{(f(a),a)} = \nu_N|_{(\nu_{0,N}(a_0;\tilde{a}),a_0)}. \tag{6.38}$$

We choose U smaller if necessary to ensure that $(f(a),a) \in \mathbb{A}_N(\tilde{a})$ for each $a \in U$, and then it follows from the uniqueness of $\nu_{0,N}(\cdot;\tilde{a}) \in I_N(a;\tilde{a})$ satisfying (6.35) that

$$f(a) = \nu_{0,N}(a;\tilde{a}) \qquad (a \in U). \tag{6.39}$$

The formula (6.37) for the derivative also follows from the Implicit Function Theorem. □

In terms of the functions $c_{i,\infty}$ of Proposition 6.6, we define

$$C_4^{(1)}(a) = \frac{1}{c_{0,\infty}}\Big|_{(\nu_c(a),a)}, \qquad C_4^{(2)}(a) = -\frac{c_{1,\infty}}{c_{0,\infty}}\Big|_{(\nu_c(a),a)}, \qquad C_d(a) = \frac{c_{3,\infty}}{c_{2,\infty}}\Big|_{(\nu_c(a),a)} \ (d>4), \tag{6.40}$$

which are continuous in $a \geq 0$ by Propositions 6.6 and 6.2. (Proposition 6.6 asserts continuity in (ν_0,a) with the subset topology on $\mathcal{X}_\infty$, so continuity in a also uses the continuity of $a \mapsto \nu_c(a)$). The constants in (6.40) are all defined in terms of the infinite-volume critical point $\nu_c(a)$, and the next lemma shows that the leading behaviour of the mass derivative of $\nu_{0,N}$ is given in terms of these quantities. The statement is slightly stronger for $a \in \mathbb{I}_{N-1}(0)$, since the massless critical point $\nu_c(0)$ appears instead of $\nu_c(a)$.

Lemma 6.10. *The mass derivative of $\nu_{0,N}(\cdot;\tilde{a})$ has the following asymptotic behaviour:*

(i) Let $\tilde{a} > 0$ and $a \in \mathbb{I}_{N-1}(\tilde{a}) = (\frac{1}{2}\tilde{a}, 2\tilde{a})$. Then

$$\dot{\nu}_{0,N}(a;\tilde{a}) = \begin{cases} \big(\big(\frac{g_N}{g}\big)^{-\hat{\gamma}}\big|_{(\nu_c(a),a)} C_4^{(1)}(a) + C_4^{(2)}(a)\big)(1+O(\tilde{\vartheta}_N\tilde{g}_N)) & (d=4) \\ C_d(a)(1+O(\tilde{\vartheta}_N\tilde{g}_N\rho_N^{a-1})) & (d>4), \end{cases} \tag{6.41}$$

and also $\liminf_{N\to\infty} \dot{\nu}_{0,N}(a) > 0$.

(ii) Let $\tilde{a} = 0$ and $a \in \mathbb{I}_{N-1}(0) = (-\frac{1}{2}L^{-2(N-1)}, \frac{1}{2}L^{-2(N-1)})$. Then

$$\dot{\nu}_{0,N}(a;0) = \begin{cases} \big(\big(\frac{g_N}{g}\big)^{-\hat{\gamma}}\big|_{(\nu_c(0),0)} C_4^{(1)}(0) + C_4^{(2)}(0)\big)(1+O(\tilde{g}_N)) & (d=4) \\ C_d(0)(1+O(\tilde{g}_N\rho_N^{\mathfrak{a}-2})) & (d>4). \end{cases} \tag{6.42}$$

Proof. Since $\dot{\nu}_{0,N}(a) = -\frac{\dot{\nu}_N}{\nu_N'}|_{(\nu_{0,N}(a),a)}$ by (6.37), our strategy is to approximate $\dot{\nu}_N$ and ν_N' by quantities that do not have strong N-dependence. In the proof, we often omit $\tilde{a}$ for brevity.

(i) Let $\tilde{a} \geq 0$ and $a \in \mathbb{I}_{N-1}(\tilde{a})$. We prove the statement of part (i) including $\tilde{a} = 0$ as we will use this in the proof of the stronger part (ii). To lighten the notation, we leave implicit the dependence of $\nu_{0,N}$ and I_N on $\tilde{a}$.

We have $\nu_{0,N}(a) \in I_N(a)$ by construction, and $\nu_c(a) \in I_N(a)$ by Remark 6.8. By the bound on $|I_N|$ in (6.23), by the second-derivative bounds of (6.17) and (6.19), and by the Mean Value

Theorem,

$$\begin{aligned}
\left|\nu_N'|_{(\nu_c(a),a)} - \nu_N'|_{(\nu_{0,N}(a),a)}\right| &\le \begin{cases} O(\tilde{\vartheta}_N^4 \tilde{g}_N^2 \left(\frac{\tilde{g}_N}{g}\right)^{\hat{\gamma}}) & (d=4) \\ O(\tilde{\vartheta}_N^4 \tilde{g}_N^2 \rho_N^{\mathfrak{a}-1}) & (d>4), \end{cases} \\
\left|\dot{\nu}_N|_{(\nu_c(a),a)} - \dot{\nu}_N|_{(\nu_{0,N}(a),a)}\right| &\le \begin{cases} O(\tilde{\vartheta}_N^4 \tilde{g}_N^2) & (d=4) \\ O(\tilde{\vartheta}_N^4 \tilde{g}_N^2 \rho_N^{\mathfrak{a}-1}) & (d>4). \end{cases}
\end{aligned} \tag{6.43}$$

Therefore, we can approximate $\dot{\nu}_N$ and ν_N' by their counterparts at the critical point $\nu_c(a)$. But by Proposition 6.2, when the initial value is tuned to $\nu_c(a)$ we can take the limit as $N \to \infty$ in the formulas for ν_j' and $\dot{\nu}_j$ in (6.16) and (6.18) to obtain

$$\nu_N'|_{(\nu_c(a),a)} = \begin{cases} \left(\frac{g_N}{g}\right)^{\hat{\gamma}} (c_{0,\infty} + O(\tilde{\vartheta}_N \tilde{g}_N))|_{(\nu_c(a),a)} & (d=4) \\ c_{2,\infty}|_{(\nu_c(a),a)} + O(\tilde{\vartheta}_N \tilde{g}_N \rho_N) & (d>4), \end{cases} \tag{6.44}$$

$$\dot{\nu}_N|_{(\nu_c(a),a)} = \begin{cases} -1 + c_{1,\infty} \left(\frac{g_N}{g}\right)^{\hat{\gamma}}|_{(\nu_c(a),a)} + O(\tilde{\vartheta}_N \tilde{g}_N) & (d=4) \\ -c_{3,\infty}|_{(\nu_c(a),a)} + O(\tilde{\vartheta}_N \tilde{g}_N \rho_N) & (d>4). \end{cases} \tag{6.45}$$

After division, we find the desired formula for $\dot{\nu}_{0,N}(a)$.

For the case $\tilde{a} > 0$, the uniform positivity of $\dot{\nu}_{0,N}(a)$ then follows from the facts that $\dot{\nu}_j(a) \le -C\vartheta_0^2 \tilde{g}_j$ and $c_{0,\infty}, c_{2,\infty} = 1 + O(\vartheta_0 g)$ by Proposition 6.6. Indeed,

$$\dot{\nu}_{0,N}(a) = -\frac{\dot{\nu}_N}{\nu_N'}\Big|_{(\nu_{0,N}(a),a)} \ge \begin{cases} C\vartheta_0^2 \tilde{g}_N^{1-\hat{\gamma}} g^{\hat{\gamma}} (1 + O(\vartheta_0 g)) & (d=4) \\ C\vartheta_0^2 \tilde{g}_N (1 + O(\vartheta_0 g)) & (d>4) \end{cases} \tag{6.46}$$

and Lemma 6.5 gives a strictly positive lower bound on $\tilde{g}_N$ as $N \to \infty$ when $\tilde{a} > 0$.

(ii) Let $\tilde{a} = 0$. The case $a = 0$ has already been established in the proof of part (i) and we use it as a reference point. For $a \in \mathbb{I}_{N-1}(0) \backslash \{0\}$, we compute the second derivative of $\dot{\nu}_{0,N}(a)$ using (6.37) and the chain rule, and obtain

$$\ddot{\nu}_{0,N} = \Big(-\frac{\dot{\nu}_N'}{\nu_N'} + \frac{\dot{\nu}_N \nu_N''}{(\nu_N')^2} \Big) \dot{\nu}_{0,N} + \Big(-\frac{\ddot{\nu}_N}{\nu_N'} + \frac{\dot{\nu}_N \dot{\nu}_N'}{(\nu_N')^2} \Big)\Big|_{(\nu_{0,N}(a),a)}. \tag{6.47}$$

Let $M = \sup_{a \in \mathbb{I}_{N-1}(0)} |\dot{\nu}_{0,N}(a) - \dot{\nu}_{0,N}(0)|$. By the Mean Value Theorem,

$$M \le \frac{1}{2} L^{-2(N-1)} \big(A(M + |\dot{\nu}_{0,N}(0)|) + B \big), \tag{6.48}$$

where

$$A = \sup_{a \in \mathbb{I}_{N-1}(0)} \Big| -\frac{\dot{\nu}_N'}{\nu_N'} + \frac{\dot{\nu}_N \nu_N''}{(\nu_N')^2} \Big|, \qquad B = \sup_{a \in \mathbb{I}_{N-1}(0)} \Big| -\frac{\ddot{\nu}_N}{\nu_N'} + \frac{\dot{\nu}_N \dot{\nu}_N'}{(\nu_N')^2} \Big|. \tag{6.49}$$

Using (6.17) and (6.19), we see that

$$|A| \le \begin{cases} O(L^{2N}\tilde{g}_N) & (d=4) \\ O(L^{2N}\tilde{g}_N\rho_N^{\mathfrak{a}-2}) & (d>4), \end{cases} \qquad |B| \le \begin{cases} O((\frac{g_N}{g})^{-\hat{\gamma}}L^{2N}\tilde{g}_N) & (d=4) \\ O(L^{2N}\tilde{g}_N\rho_N^{\mathfrak{a}-2}) & (d>4). \end{cases} \tag{6.50}$$

Thus (6.48) reduces to

$$M \le O(L^{-2N})\big(A|\dot{\nu}_{0,N}(0)| + B\big) \le \begin{cases} O((\frac{g_N}{g})^{-\hat{\gamma}}\tilde{g}_N) & (d=4) \\ O(\tilde{g}_N\rho_N^{\mathfrak{a}-2}) & (d>4). \end{cases} \tag{6.51}$$

This proves that $\dot{\nu}_{0,N}(a)$ is within a small enough error of $\dot{\nu}_{0,N}(0)$ to give the desired asymptotic formula for part (ii), once we recall that we have already established it for $\dot{\nu}_{0,N}(0)$ in part (i). □

Proof of Proposition 6.3. We first consider $\tilde{a} > 0$ and $a \in (\frac{1}{2}\tilde{a}, 2\tilde{a}) = \cap_{j\ge 0}\mathbb{I}_j(\tilde{a})$. By (6.36), $\nu_c(a)$ is the limit of $\nu_{0,N}(a;\tilde{a})$. Let $a_1, a_2 \in (\frac{1}{2}\tilde{a}, 2\tilde{a})$. By Lemma 6.10(i) and the Dominated Convergence Theorem,

$$\nu_c(a_1) - \nu_c(a_2) = \lim_{N\to\infty}\int_{a_2}^{a_1}\dot{\nu}_{0,N}(x;\tilde{a})dx = \begin{cases} \int_{a_2}^{a_1}[(\frac{g_\infty}{g})^{-\hat{\gamma}}\big|_{(\nu_c(x),x)}C_4^{(1)}(x) + C_4^{(2)}(x)]dx & (d=4) \\ \int_{a_2}^{a_1} C_d(x)dx & (d>4). \end{cases} \tag{6.52}$$

Since this holds for any $a_1, a_2 \in (\frac{1}{2}\tilde{a}, 2\tilde{a})$ and $\tilde{a} > 0$, it in fact holds for any $a_1, a_2 > 0$. This establishes the continuous differentiability at strictly positive a. The positivity of $\dot{\nu}_c(a)$ follows from the statement of uniform positivity for $\dot{\nu}_{0,N}$ in Lemma 6.10(i).

For $d > 4$, since $\nu_c(a)$ and $C_d(a)$ are continuous also at $a = 0^+$, taking the limit $a_2 \downarrow 0$ gives

$$\nu_c(a_1) - \nu_c(0) = \int_0^{a_1} C_d(x)dx. \tag{6.53}$$

This extends the continuous differentiability to $a = 0^+$. Since $c_{3,\infty} = O(g)$ and $c_{2,\infty} = 1 + O(g)$, we also have $C_d(x) = \frac{c_{3,\infty}}{c_{2,\infty}} = O(g)$ and hence $\dot{\nu}_c(a) = O(g)$.

For $d = 4$, ν_c is not differentiable at 0. But since $g_\infty(a) \sim (Bj_a)^{-1}$ as $a \downarrow 0$, we see that

$$\dot{\nu}_c(a) \sim C_4^{(1)}(a)g^{\hat{\gamma}}(Bj_a)^{\hat{\gamma}} \sim C_4^{(1)}(0)(Bg)^{\hat{\gamma}}(\log_{L^2} a^{-1})^{\hat{\gamma}} \sim A_4(\log a^{-1})^{\hat{\gamma}} \qquad \text{as } a \downarrow 0, \tag{6.54}$$

with

$$A_4 = \frac{(Bg)^{\hat{\gamma}}C_4^{(1)}(0)}{(\log L^2)^{\hat{\gamma}}}. \tag{6.55}$$

The asymptotic formula $A_4 \sim (Bg/\log L^2)^{\hat{\gamma}}$ as $g \downarrow 0$ follows from the fact that $C_4^{(1)}(0) = 1 + O(g)$. □

Proof of Proposition 6.4. Take $\tilde{a} = 0$ and let $a \in \mathbb{I}_{N-1}(0)$. The formula for $\dot{\nu}_{0,N}(a;0)$ in Lemma 6.10(ii) and the asymptotic formula for g_N in Lemma 6.5 imply that

$$\dot{\nu}_{0,N}(a) = \dot{\nu}_{0,N}(a;0) = \begin{cases} \big(A_4(\log L^2)^{\hat{\gamma}} N^{\hat{\gamma}} - 1 + c^{\mathrm{F}}\big)(1 + O(N^{-1})) & (d=4) \\ \dot{\nu}_c(0) + O(g\rho_N^{\mathfrak{a}-2}) & (d>4), \end{cases} \tag{6.56}$$

where we set $c^{\mathrm{F}} = 1 + C_4^{(2)}(0) = O(g)$. This completes the proof. □

7 | DERIVATIVES OF THE RG FLOW: PROOF OF PROPOSITION 6.6

In this section, we prove Proposition 6.6, which entails computation of derivatives of all the RG coordinates. The proof requires extension of the ideas in [9, Section 8.4] to include mass derivatives, derivatives of second order, and dimensions $d > 4$. We divide the proof into two parts: $d = 4$ in Section 7.2 and $d > 4$ in Section 7.3. Section 7 is independent of the rest of the paper, apart from proving Proposition 6.6, and it could be omitted in a first reading. We always assume Theorems 5.6 and 5.7 in this section.

7.1 | Leading behaviour

The recursion

$$\nu_{j+1} = (1 - \hat{\gamma}\beta_j g_j)\nu_j + \eta_j g_j - \xi_j g_j^2 + r_{\nu,j} \tag{7.1}$$

follows from the perturbative recursion (5.9) together with its nonperturbative remainder $r_{\nu,j}$. This recursion has derivatives

$$\nu'_{j+1} = (1 - \hat{\gamma}\beta_j g_j)\nu'_j - \hat{\gamma}\beta_j g'_j \nu_j + \eta_j g'_j - 2\xi_j g_j g'_j + r'_{\nu,j}, \quad \nu'_0 = 1, \tag{7.2}$$

$$\dot{\nu}_{j+1} = (1 - \hat{\gamma}\beta_j g_j)\dot{\nu}_j - \hat{\gamma}\beta_j g_j + \eta_j \dot{g}_j - \hat{\gamma}(\dot{\beta}_j g_j + \beta_j \dot{g}_j)\nu_j - \dot{\xi}_j g_j^2 - 2\xi_j g_j \dot{g}_j + \dot{r}_{\nu,j}, \quad \dot{\nu}_0 = 0, \tag{7.3}$$

where in (7.3) we have used (5.10)–(5.11) to replace $\dot{\eta}_j g_j$ by its equivalent $-\hat{\gamma}\beta_j g_j$.

A guiding principle throughout Section 7 is that the recursions for ν'_j and $\dot{\nu}_j$ have their leading behaviours governed by the solutions q'_j and $\dot{q}_j$ to the recursions

$$q'_{j+1} = (1 - \hat{\gamma}\beta_j g_j)q'_j, \quad q'_0 = 1, \tag{7.4}$$

$$\dot{q}_{j+1} = (1 - \hat{\gamma}\beta_j g_j)\dot{q}_j - \hat{\gamma}\beta_j g_j, \quad \dot{q}_0 = 0. \tag{7.5}$$

The solution to (7.4) is

$$q'_j = \Pi_{0,j-1}, \tag{7.6}$$

where, as in (6.31),

$$\Pi_{k,j} = \prod_{i=k}^{j} (1 - \hat{\gamma}\beta_i g_i). \tag{7.7}$$

Also, it can be verified that the recursion (7.5) for $\dot{q}_j$ is solved by

$$\dot{q}_j = -1 + \Pi_{0,j-1}. \tag{7.8}$$

Our estimates for ν'_j and $\dot{\nu}_j$ are stated in terms of the sequences

$$\Sigma_j = \frac{\nu'_j - \Pi_{0,j-1}}{\Pi_{0,j-1}}, \qquad F_j = \frac{\dot{\nu}_j + 1 - \Pi_{0,j-1}}{\Pi_{0,j-1}}, \tag{7.9}$$

which respectively compare ν'_j and $\dot{\nu}_j$ with the sequences q'_j and $\dot{q}_j$ from (7.6) and (7.8). The denominators in (7.9) are included for convenience. By definition, $\Sigma_1 = \Sigma_0 = F_0 = 0$ and

$$F_1 - F_0 = F_1 = \Pi_{0,0}^{-1}\big(-\hat{\gamma}\dot{\beta}_0 g_0 \nu_0 - \dot{\xi}_0 g_0^2 \big). \tag{7.10}$$

7.2 | Proof of Proposition 6.6 for $d = 4$

For $d = 4$, the important product $\Pi_{k,j}$ is well understood, according to the following lemma.

Lemma 7.1. *Let $d = 4$. Suppose that $(V_j, K_j)_{j\le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$, with $g_0 > 0$ sufficiently small. Then for $t \in (0, 2]$ and $i \le j < k$,*

$$\prod_{m=i}^{j}(1 - t\beta_m g_m) = \Big(\frac{g_{j+1}}{g_i}\Big)^t \bar{c}_{i,j}[t], \tag{7.11}$$

where $\bar{c}_{i,j}[t] = 1 + O_t(\vartheta_i g_i)$. For a global RG flow the limit $\bar{c}_i[t] = \lim_{j\to\infty} \bar{c}_{i,j}[t]$ exists, is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$, and obeys $\bar{c}_i[t] = 1 + O_t(\vartheta_i g_i)$.

Proof. The arithmetic used in the proof of [9, Lemma 6.1.6] shows that the formula (7.11) holds with $\bar{c}_{i,j}[t]$ given by

$$\bar{c}_{i,j}[t] = \prod_{m=i}^{j}(1 + \upsilon_m), \qquad \upsilon_m = -1 + \Big(1 - \frac{r_{g,m}}{g_{m+1}}\Big)^t \frac{1 - t\beta_m g_m}{(1 - \beta_m g_m)^t}. \tag{7.12}$$

A small calculation, using Theorem 5.6 to bound $r_{g,m}$, shows that $|\upsilon_m| \le O(\tilde{\vartheta}_m \tilde{g}_m^2)$. Since this last upper bound sums to $O(\tilde{\vartheta}_m \tilde{g}_m)$ as in [9, (6.1.17)], we conclude that $\bar{c}_{i,j}[t] = 1 + O_t(\vartheta_i g_i)$. For a global RG flow the infinite product converges and also obeys $\bar{c}_i[t] = 1 + O_t(\vartheta_i g_i)$.

To see the continuity of $\bar{c}_i[t]$, we first observe that Theorems 5.6 and 5.7 imply that each RG map $\Phi_{j+1} : (V_j, K_j, a) \mapsto (V_{j+1}, K_{j+1})$ is continuous, so (V_j, K_j) is continuous in $(\nu_0, a) \in \mathcal{X}_k$. Thus each $\bar{c}_{i,j}[t]$ given by (7.12) is continuous in $(\nu_0, a) \in \mathcal{X}_k$, and the continuity of $\bar{c}_i$ in $(\nu_0, a) \in \mathcal{X}_\infty$ follows by the Dominated Convergence Theorem. □

Proposition 7.2 is the main ingredient in the proof of Proposition 6.6 for $d = 4$. It includes derivatives of all the RG coordinates because we must bound them all together.

Recall the definition of an RG flow from Definition 6.1, and the definition $\hat{\gamma} = \frac{n+2}{n+8}$ from (1.2).

Proposition 7.2 (Case $d = 4$). *Suppose that L is sufficiently large and g_0 is sufficiently small. Suppose that $(V_j, K_j)_{j\le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$. There exist positive constants C_0, C_1, C_2, C (independent of k) such that for all $j \le k$, (g_j, ν_j, K_j) are continuously differentiable functions of $(\nu_0, a) \in \mathcal{X}_k$ and satisfy the following bounds.*

(i) The ν_0-derivatives satisfy

$$|\Sigma_j - \Sigma_{j-1}| \le C_0 \tilde{\vartheta}_j^3 \tilde{g}_j^2, \qquad |g_j'| \le C_1 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2 \Pi_{0,j-1}, \qquad \|K_j'\|_{\mathcal{W}_j^\kappa} \le C_2 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2 \Pi_{0,j-1}. \tag{7.13}$$

(ii) The a-derivatives satisfy

$$|F_j - F_{j-1}| \le C_0 \tilde{\vartheta}_j^3 \tilde{g}_j^2 \Pi_{0,j-1}^{-1}, \qquad |\dot{g}_j| \le C_1 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2, \qquad \|\dot{K}_j\|_{\mathcal{W}_j^\kappa} \le C_2 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2, \tag{7.14}$$

and $\dot{\nu}_j \le -C \tilde{\vartheta}_0^2 \tilde{g}_j$.

(iii) The second derivatives satisfy

$$|\nu_j''| \le O\Big(L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}\Big), \qquad |\dot{\nu}_j'| \le O\Big(L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{\hat{\gamma}}\Big), \qquad |\ddot{\nu}_j| \le O(L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j). \tag{7.15}$$

Remark 7.3. It is important that C_0, C_1 and C_2 are independent of j and k. The constants will eventually be chosen to satisfy

$$M = \max_{0\le p,q\le 2} M_{p,q} \ll C_2 \ll C_1 \ll C_0 \tag{7.16}$$

where $M_{p,q}$ are the L-dependent constants in Theorems 5.6 and 5.7 (including $M_{0,0} = C_{\mathrm{RG}}$) and (7.16) defines M.

Since it will be important to keep track of the constants C_i in the following, we often write $O(C_1)$, $O(M_{0,1}C_2)$, etc. In particular, with this notation, $O(1)$ is independent of L, C_1, C_2, C_3, and of course of the scale. When a constant does depend on L, but not on the others, we instead write $O_L(1)$. In addition, we will take $(g_0)^{-1}$ to be large compared to these constants. The same holds for the constants C_0', C_1', C_2' of Lemma 7.5, and for the corresponding constants in Section 7.3 with the additional assumption that C_0', C_1', C_2' are themselves much larger than C_0, C_1, C_2.

Proof of Proposition 6.6(i). We assume that (V_j, K_j) is an RG flow to scale N with initial condition (g_0, ν_0) and mass a.

The negative upper bound on $\dot{\nu}_j$ is a part of the statement of Proposition 7.2(ii), the required bounds on the second derivatives are restated in (7.15), and (6.20) is restated in (7.13) and (7.14). It therefore suffices to prove the bounds

$$\nu_j' = \Big(\frac{g_j}{g_0}\Big)^{\hat{\gamma}} (c_{0,N} + O(\tilde{\vartheta}_j \tilde{g}_j)), \qquad \dot{\nu}_j + 1 = \Big(\frac{g_j}{g_0}\Big)^{\hat{\gamma}} c_{1,N} + O(\tilde{\vartheta}_j \tilde{g}_j) \tag{7.17}$$

on the first derivatives, as well as facts that the constants $c_{0,N}$ and $c_{1,N}$ are continuous in $(\nu_0, a) \in \mathcal{X}_N$ and that their limits $c_{i,\infty}$ are continuous in $(\nu_0, a) \in \mathcal{X}_\infty$.

By (7.13), which holds by assumption for all $j \le N$,

$$\Sigma_j = \Sigma_N + O\Big(\sum_{k=j+1}^{N} \tilde{\vartheta}_k^3 \tilde{g}_k^2\Big) = \Sigma_N + O(\tilde{\vartheta}_j^3 \tilde{g}_j), \tag{7.18}$$

where the sum has been bounded as in [9, Exercise 6.1.4]. Since $\nu'_j = \Pi_{0,j-1}(1+\Sigma_j)$ by definition, it follows from the formula for $\Pi_{0,j-1}$ in Lemma 7.1 that

$$\nu'_j = \Big(\frac{g_j}{g}\Big)^{\hat\gamma}\big(1+\Sigma_N + O(\tilde\vartheta_j^3\tilde g_j)\big)\big(\bar c_{0,N}[\hat\gamma] + O(\vartheta_j g_j)\big). \tag{7.19}$$

This gives the formula (7.17) for ν'_j with $c_{0,N} = (1+\Sigma_N)\bar c_{0,N}[\hat\gamma] = 1 + O(\tilde\vartheta_0 g)$. The continuity of $\bar c_{0,N}$ and $\bar c_{0,\infty}$ is provided by Lemma 7.1. By definition, $\Sigma_0 = 0$. The limit $\Sigma_\infty = \lim_{N\to\infty}\sum_{i=0}^{N-1}(\Sigma_{i+1}-\Sigma_i)$ is continuous in (ν, a) by the Dominated Convergence Theorem, and therefore $c_{0,\infty}$ is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$ as claimed.

For $\dot\nu_j$, we start with $\dot\nu_j + 1 = \Pi_{0,j-1}(1+F_j)$ and proceed similarly. With Lemma 7.1, we obtain

$$F_j = F_N + O\Big(g_0^{\hat\gamma}\sum_{k=j+1}^{N}\tilde\vartheta_k^3\tilde g_k^{2-\hat\gamma}\Big) = F_N + O(g_0^{\hat\gamma}\tilde\vartheta_j^3\tilde g_j^{1-\hat\gamma}), \tag{7.20}$$

and hence

$$\dot\nu_j + 1 = \Pi_{0,j-1}(1+F_N) + O(\tilde\vartheta_j^3\tilde g_j). \tag{7.21}$$

This gives (7.17) for $\dot\nu_j$ with $c_{1,N} = (1+F_N)\bar c_{0,N}[\hat\gamma] = 1 + O(\tilde\vartheta_0 g)$. Again the limit $c_{1,\infty}$ exists and is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$. □

We now turn to the proof of Proposition 7.2, which is by induction on j. Part (i) is essentially as in [9, Theorem 8.4.1], while parts (ii) and (iii) require new effort. We begin by applying Theorems 5.6 and 5.7 to see that when the bounds of Proposition 7.2 hold, we can inductively obtain bounds on the derivatives of K_j and R_j^U up to and including scale $k+1$. Note that we write $\dot R_{j+1}^U = \frac{d}{da}(R_{j+1}^U(V_j, K_j))$, so $\dot R_{j+1}^U$ is not the same as $D_a R_{j+1}^U$ which includes the scaling factor present in (5.35). Similarly, $(R_{j+1}^U)' = \frac{d}{d\nu_0}(R_{j+1}^U(V_j, K_j))$.

Lemma 7.4. *Let $d = 4$. Suppose that $(V_j, K_j)_{j\le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$. Suppose further that the bounds of Proposition 7.2(i,ii) hold for all $i \le j$. Then*

$$\|K'_{j+1}\|_{\mathcal{W}^\kappa_{j+1}} \le C_2 L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2\Big(\frac{\tilde g_{j+1}}{g_0}\Big)^{\hat\gamma}, \tag{7.22}$$

$$\|(R_{j+1}^U)'\|_{\mathcal{V}_{j+1}(\ell_{j+1})} \le O(MC_2)L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2\Big(\frac{\tilde g_{j+1}}{g_0}\Big)^{\hat\gamma}, \tag{7.23}$$

$$\|\dot K_{j+1}\|_{\mathcal{W}^\kappa_{j+1}} \le C_2 L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2, \tag{7.24}$$

$$\|\dot R_{j+1}^U\|_{\mathcal{V}_{j+1}(\ell_{j+1})} \le O(MC_2)L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2. \tag{7.25}$$

Proof. We fix $j \le k$ and make the indicated assumption for $i \le j$. Proposition 6.6(i) then applies for such i, in addition to the assumed bounds of Proposition 7.2(i,ii) for $i \le j$. We apply induction and assume that the bounds of Lemma 7.4 hold when $j+1$ is replaced by i for all $i \le j$. They

hold trivially for $i = 0$ since K_0 and R_0 are both zero. By (7.17) and Lemma 7.1, and with the L-dependence due to $\ell_0 = L^{1+d/2}$ in ℓ_j of (5.20),

$$\begin{aligned}\|V'_j\|_{\mathcal{V}_j(\ell_j)} &\le O_L(1)\big(|g'_j| + |\nu'_j|L^{2j}\big)\\ &\le O_L(1)L^{2j}\big(\frac{\tilde g_j}{\tilde g_0}\big)^{\hat\gamma}\big(O(C_1\tilde g_j^2) + \frac{1}{2}c_{0,N} + O(\tilde\vartheta_j\tilde g_j)\big) \le O_L(1)L^{2j}\big(\frac{\tilde g_j}{\tilde g_0}\big)^{\hat\gamma},\end{aligned} \tag{7.26}$$

for g_0 small enough. Also, by (7.13) and Lemma 7.1, $\|K'_j\|_{\mathcal{W}_j^\kappa} \le 2C_2L^{2j}\tilde\vartheta_j^3\tilde g_j^2\big(\frac{\tilde g_j}{\tilde g_0}\big)^{\hat\gamma}$ for small g_0. We can use these bounds with the chain rule, namely

$$\frac{d}{d\nu_0}\Phi_{j+1}(V_j, K_j) = D_K\Phi_{j+1}(V_j, K_j; K'_j) + D_V\Phi_{j+1}(V_j, K_j; V'_j), \tag{7.27}$$

to bound both K'_{j+1} and R'_{j+1}. Indeed, by Theorem 5.7, and for L large enough to overcome the L-independent constant $M'_{0,1}$,

$$\begin{aligned}\|K'_{j+1}\|_{\mathcal{W}_{j+1}^\kappa} &\le M'_{0,1}L^{-2}2C_2L^{2j}\tilde\vartheta_j^3\tilde g_j^2\Big(\frac{\tilde g_j}{\tilde g_0}\Big)^{\hat\gamma} + M_{1,0}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2O_L(1)L^{2j}\Big(\frac{\tilde g_j}{\tilde g_0}\Big)^{\hat\gamma}\\ &\le C_2L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2\Big(\frac{\tilde g_{j+1}}{g_0}\Big)^{\hat\gamma},\end{aligned} \tag{7.28}$$

where we used (7.16) in the last step. Similarly, by Theorem 5.6, for C_2 large enough we have

$$\begin{aligned}\|(R^U_{j+1})'\|_{\mathcal{V}_{j+1}(\ell_{j+1})} &\le M_{0,1}C_2L^{2j}\tilde\vartheta_j^3\tilde g_j^2\Big(\frac{\tilde g_j}{\tilde g_0}\Big)^{\hat\gamma} + M_{1,0}\tilde\vartheta_j^3\tilde g_j^3O_L(1)L^{2j}\Big(\frac{\tilde g_j}{\tilde g_0}\Big)^{\hat\gamma}\\ &\le 2MC_2L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2\Big(\frac{\tilde g_{j+1}}{g_0}\Big)^{\hat\gamma}.\end{aligned} \tag{7.29}$$

For the mass derivative, by the chain rule, $\frac{d}{da}\Phi_{j+1}(V_j, K_j) = \dot\Phi_{j+1} + D_K\Phi_{j+1}(\dot K_j) + D_V\Phi_{j+1}(\dot V_j)$. The first term is controlled by Theorem 5.7 since our norm contains mass derivatives. Also (7.14) and (7.17) imply that, as in (7.26), $\|\dot V_j\|_{\mathcal{V}(\ell_j)} \le O_L(1)L^{2j}$. Therefore,

$$\begin{aligned}\|\dot K_{j+1}\|_{\mathcal{W}_{i+1}^\kappa} &\le L^{2(j+1)}C_{\mathrm{RG}}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^3 + M'_{0,1}L^{-2}C_2L^{2j}\tilde\vartheta_j^3\tilde g_j^2 + M_{1,0}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2O_L(1)L^{2j}\\ &\le C_2L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2.\end{aligned} \tag{7.30}$$

The bound on $\dot R^U_{j+1}$ is obtained similarly, again using Theorem 5.6. □

We can now prove Proposition 7.2(i).

Proof of Proposition 7.2(i). Fix $j \le k$ and assume the bounds of Proposition 7.2(i) for all scales $i \le j$. This assumption is satisfied for $j = 0$ since $\Sigma_1 - \Sigma_0 = 0 - 0 = 0$, and also g'_0 and K'_0 are zero. Lemma 7.4 gives the desired bound on K'_{j+1} at scale $j+1$. For the bound on g'_{j+1}, we first use Lemma 7.4 and (5.43) to obtain

$$|r'_{g,j+1}| \le O(1)\|(R^U_{j+1})'\|_{\mathcal{V}(\ell_{j+1})} \le O(MC_2)L^{2(j+1)}\tilde\vartheta_{j+1}^3\tilde g_{j+1}^2\big(\frac{\tilde g_{j+1}}{g_0}\big)^{\hat\gamma}. \tag{7.31}$$

Therefore, by Lemma 7.1 and the induction hypothesis,

$$|g'_{j+1}| \le (1 - 2\beta_j g_j)|g'_j| + |r'_{g,j+1}| \le O(C_1 L^{-2} + MC_2) L^{2(j+1)} \tilde{\vartheta}^3_{j+1} \tilde{g}^2_{j+1} \Pi_{0,j}, \tag{7.32}$$

where we used $1 - 2\beta_j g_j \le 1$. With L large, and assuming $C_1 \gg C_2$, this gives the desired bound on g'_{j+1}.

Finally, for the bound on Σ, we first use (7.9) and (7.2) to obtain

$$\begin{aligned} \Sigma_{j+1} - \Sigma_j &= \Pi_{0,j}^{-1}\big(\nu'_{j+1} - (1 - \hat{\gamma}\beta_j g_j)\nu'_j\big) \\ &= \Pi_{0,j}^{-1}\big(- \hat{\gamma}\beta_j g'_j \nu_j + \eta_j g'_j - 2\xi_j g_j g'_j + r'_{\nu,j}\big). \end{aligned} \tag{7.33}$$

Then we use the bound on $r'_{\nu,j}$ from Lemma 7.4, on β_j, η_j, ξ_j from (5.10)–(5.12), on g'_j from the induction hypothesis, and on $\Pi_{0,j}$ from Lemma 7.1, and we obtain the desired bound

$$|\Sigma_{j+1} - \Sigma_j| \le C_0 \tilde{\vartheta}^3_{j+1} \tilde{g}^2_{j+1}, \tag{7.34}$$

assuming $C_0 \gg C_1$. □

Proof of Proposition 7.2(ii). As induction hypotheses, we assume that the bounds of (7.14) hold for scales $i \le j$. To start the induction, $\dot{g}_0$ and $\dot{K}_0$ are both equal to zero, and, by (7.10) and the fact that $|\nu_0| \le O(g_0)$, $F_1 - F_0$ is bounded above by a multiple of $\tilde{\vartheta}^3_1 \tilde{g}^2_1 \Pi^{-1}_{0,0}$ as required.

We first prove that

$$|F_{j+1} - F_j| \le C_0 \tilde{\vartheta}^3_j \tilde{g}^2_j \Pi^{-1}_{0,j}. \tag{7.35}$$

By definition, $F_j = \Pi^{-1}_{0,j-1}(\dot{\nu}_j - \dot{q}_j)$, where $\dot{q}_j$ satisfies the recursion (7.8). From this, together with $\dot{\eta}_j = -\hat{\gamma}\beta_j$ and some arithmetic, we arrive at

$$F_{j+1} - F_j = E_j \Pi^{-1}_{0,j} \tag{7.36}$$

with

$$E_k = -\hat{\gamma}\frac{d}{da}(\beta_k g_k)\nu_k + \eta_k \dot{g}_k + \frac{d}{da}(-\xi_k g_k^2 + r_{\nu,k}). \tag{7.37}$$

By (7.14) and Lemma 7.4, we can bound each constituent of E_j to obtain

$$|E_j| \le O(M(C_1 + C_2))\tilde{\vartheta}^3_j \tilde{g}^2_j. \tag{7.38}$$

This proves (7.35).

The desired bound on $\dot{K}_{j+1}$ follows from Lemma 7.4. For the bound on $\dot{g}_{j+1}$, we apply Lemma 7.1 and (5.43) to see that

$$\begin{aligned} |\dot{g}_{j+1}| &\le (1 - 2\beta_j g_j)|\dot{g}_j| + |\dot{\beta}_j g_j^2| + |\dot{r}_{g,j}| \\ &\le O(L^{-2}C_1 + 1 + MC_2) L^{2(j+1)} \tilde{\vartheta}^3_{j+1} \tilde{g}^2_{j+1} \\ &\le C_1 L^{2(j+1)} \tilde{\vartheta}^3_{j+1} \tilde{g}^2_{j+1}, \end{aligned} \tag{7.39}$$

for sufficiently large L and C_1.

It remains to prove that $\dot\nu_{j+1} \le -C\tilde\vartheta_0^2 \tilde g_{j+1}$. By definition,

$$\tilde\vartheta_j = \begin{cases} 2^{-(j-j_{\tilde a})_+} & (\tilde a < 1) \\ \tilde a^{-1} 2^{-j} & (\tilde a \ge 1), \end{cases} \tag{7.40}$$

so, in particular, $\tilde\vartheta_0 = 1 \vee \tilde a^{-1}$. The proof is in two steps: we use induction for $j \le j_a + 1$ and then argue directly for $j > j_a + 1$.

For the induction, given $j \ge 1$, we assume that $\dot\nu_j \le -C\tilde\vartheta_0^2 \tilde g_j$, with $C = \frac{1}{8}\hat\gamma B$ where $B = (n+8)(1 - L^{-d})$ is the constant defined in (1.34). Of course $\dot\nu_0 = 0$, so we start the induction by showing that $\dot\nu_1 \le -C\tilde\vartheta_0^2 \tilde g_1$. Recall that $\dot\eta_j = -\hat\gamma\beta_j$. By the recursion (7.2) for ν' (for the first equality), and by the induction hypothesis and (7.38) (for the second equality),

$$\begin{aligned} \dot\nu_{j+1} &= (1 - \hat\gamma\beta_j g_j)\dot\nu_j + \dot\eta_j g_j + E_j \\ &= (1 - \hat\gamma\beta_j g_j)\dot\nu_j - \hat\gamma\beta_j g_j + O(\tilde\vartheta_j^3 \tilde g_j^2). \end{aligned} \tag{7.41}$$

In particular, for $j = 0$ the first term on the right-hand side is zero, the second is $-\hat\gamma B(1+a)^{-2} g_0 \le -\frac{1}{4}\hat\gamma B \tilde\vartheta_0^2 g_0$, and the third term is relatively small, so this proves that $\dot\nu_1 \le -\frac{1}{8}\hat\gamma B\tilde\vartheta_0^2 g_0$ which is sufficient since $g_0 > g_1$. To advance the induction, we first consider $j \le j_a \wedge (N-1)$ (so $a \le 1$ and $\tilde\vartheta_0 = 1$). In this case, $\beta_j \ge \beta_{j_a} \ge \frac{1}{4}B$, and the term E_j is relatively small since it is of order g_j^2. We use the induction hypothesis to see that $\dot\nu_j \le 0$. Together, these facts show that $\dot\nu_{j+1} \le -\frac{1}{8}\hat\gamma B g_j \le -\frac{1}{8}\hat\gamma B g_{j+1}$, which advances the induction to $j+1 \le (j_a+1) \wedge N$ with $C = \frac{1}{8}\hat\gamma B$.

Suppose that $N > j_a + 1$. We apply a different argument for $j > j_a$ since β_j decays as j increases beyond the mass scale. We again consider the recursion (7.5) for $\dot q$, but now only for $j \ge i \ge j_a + 1$ with fixed i and with initial condition $\dot\nu_i$. With the initial i indicated by an additional subscript, the solution to this recursion is

$$\dot q_{i,j} = \dot\nu_i \Pi_{i,j-1} - 1 + \Pi_{i,j-1}. \tag{7.42}$$

Then we consider

$$F_{i,j} = \frac{\dot\nu_j - \dot q_{i,j}}{\Pi_{i,j-1}} \qquad (j \ge i). \tag{7.43}$$

In particular, $\dot q_{0,j} = \dot q_j$ and $F_j = F_{0,j}$. The proof of (7.35) similarly gives

$$|F_{i,j} - F_{i,j-1}| \le C_0 \tilde\vartheta_j^3 \tilde g_j^2 \Pi_{i,j-1}^{-1}. \tag{7.44}$$

By definition of $F_{i,j}$,

$$\dot\nu_j = \dot\nu_i \Pi_{i,j-1} - 1 + \Pi_{i,j-1}(1 + F_{i,j}) \qquad (j \ge i). \tag{7.45}$$

Now we choose $i = 1 + (j_a \vee 0)$ and consider $j > i$. Since ϑ_j^3 decays exponentially in $j - j_a$, there are positive c, c' such that $1 - c\tilde\vartheta_i^2 \tilde g_i \le \Pi_{i,j-1} \le 1 - c'\tilde\vartheta_i^2 \tilde g_i$. Also, (7.44) gives

$$|F_{i,j}| \le \sum_{k=i}^{j-1} |F_{i,k} - F_{i,k+1}| \le O(C_0)\tilde\vartheta_i^3 \tilde g_i^2, \tag{7.46}$$

so for sufficiently small $\tilde{g}_i$,

$$-1 + \Pi_{i,j-1}(1 + F_{i,j}) \le 0. \tag{7.47}$$

If we insert this bound in (7.47), use $\Pi_{i,j-1} \ge 1/2$ for small $\tilde{g}_i$ and take $i = j_a + 1$, then we deduce from the upper bound on $\dot{\nu}_i$ that

$$\dot{\nu}_j \le \dot{\nu}_i \Pi_{i,j-1} \le -\frac{1}{8}\hat{\gamma} B \tilde{\mathfrak{d}}_0^2 \tilde{g}_i \times \frac{1}{2} \le -\frac{1}{16}\hat{\gamma} B \tilde{\mathfrak{d}}_0^2 \tilde{g}_j, \tag{7.48}$$

and the proof is complete. □

For the proof of Proposition 7.2(iii), we first prove a counterpart to Lemma 7.4 for these second derivatives. (The bound on ν_j'' in (7.49) is included in the bounds of Proposition 7.2, but we include it explicitly in (7.49) for easy reference.) We provide details of the proof of Proposition 7.2(iii) only for the second derivatives with respect to ν_0. The bounds on $\dot{\nu}_j'$ and $\ddot{\nu}_j$ follow in the same way, once we observe that the bounds on the first derivative of $(\dot{\nu}_j, \dot{g}_j, \dot{K}_j)$ (as in part (ii)) simply have one less factor of $(\tilde{g}_j/g_0)^{\hat{\gamma}}$ compared to (ν_j', g_j', K_j').

Lemma 7.5. *Let $d = 4$. Suppose that $(V_j, K_j)_{j \le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$. Suppose that the bounds of Proposition 7.2 hold for all $i \le j$, and that, in addition,*

$$|\nu_j''| \le C_0' L^{2j} \tilde{\mathfrak{d}}_j^3 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}, \qquad |g_j''| \le C_1' L^{4j} \tilde{\mathfrak{d}}_j^2 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}, \qquad \|K_j''\|_{\mathcal{W}_j^\kappa} \le C_2' L^{4j} \tilde{\mathfrak{d}}_j^3 \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}. \tag{7.49}$$

Then for sufficiently large C_2',

$$\|K_{j+1}''\|_{\mathcal{W}_{j+1}^\kappa} \le C_2' L^{4(j+1)} \tilde{\mathfrak{d}}_{j+1}^3 \tilde{g}_{j+1} \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}, \tag{7.50}$$

$$\|(R_{j+1}^U)''\|_{\mathcal{V}_{j+1}(\ell_{j+1})} \le O(MC_2') L^{4j} \tilde{\mathfrak{d}}_{j+1}^3 \tilde{g}_{j+1} \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}. \tag{7.51}$$

Proof. We use the fact that the second derivative of a function $f(V, K)$ is

$$\begin{aligned} f''(V,K) &= D_V f(V,K;V'') + D_K f(V,K;K'') \\ &\quad + D_V^2 f(V,K;V',V') + 2D_V D_K f(V,K;V',K') + D_K^2 f(V,K;K',K'). \end{aligned} \tag{7.52}$$

We are free to apply Proposition 7.2(i,ii), which have been proved already.

By (7.49) with $C_0' \gg C_1'$, we have

$$\|V_j''\|_{\mathcal{V}_j(\ell_j)} \le O(C_0') L^{4j} \tilde{\mathfrak{d}}_j \tilde{g}_j \Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}. \tag{7.53}$$

With our convention that we omit j and write $+$ for $j+1$, and writing $\mathcal{W}_+^\kappa$ instead of $\mathbb{D} \to \mathcal{W}_+^\kappa$ for the operator norms,

$$\begin{aligned} \|K_+''\|_{\mathcal{W}_+^\kappa} &\le \|D_V \Phi_+^K\|_{\mathcal{W}_+^\kappa} \|V''\|_{\mathcal{V}} + \|D_K \Phi_+^K\|_{\mathcal{W}_+^\kappa} \|K''\|_{\mathcal{W}^\kappa} \\ &\quad + \|D_V^2 \Phi_+^K\|_{\mathcal{W}_+^\kappa} \|V'\|_{\mathcal{V}}^2 + 2\|D_V D_K \Phi_+^K\|_{\mathcal{W}_+^\kappa} \|V'\|_{\mathcal{V}} \|K'\|_{\mathcal{W}^\kappa} + \|D_K^2 \Phi_+^K(V,K)\|_{\mathcal{W}_+^\kappa} \|K'\|_{\mathcal{W}^\kappa}^2 \end{aligned}$$

$$\leq O(L^{4j}\tilde{\mathfrak{g}}_+^3)\big(O_L(1)MC_0'\tilde{g}_+^3 + C_2'L^{-2}\tilde{g} + O_L(1)MC_2\tilde{g}_+ + O_L(1)M\tilde{g}_+ + MC_2^2\tilde{g}_+^{-\frac{9}{4}+4}\big)\Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}}$$

$$\leq C_2'L^{4(j+1)}\tilde{\mathfrak{g}}_+^3\tilde{g}_+\Big(\frac{\tilde{g}_+}{g_0}\Big)^{2\hat{\gamma}}, \tag{7.54}$$

for sufficiently large C_2' and sufficiently small g (to absorb C_0'). Here we used the induction hypotheses to bound the second derivatives, the previously noted bound $\|V_j'\|_{\mathcal{V}(\ell_j)} \leq O_L(1)L^{2j}(\frac{\tilde{g}_j}{\tilde{g}_0})^{\hat{\gamma}}$ and (7.4) to bound the first derivatives, and M is the L-dependent constant which bounds the constants from the main RG Theorems 5.6–5.7. Similarly,

$$\begin{aligned}\|(R_+^U)''\|_{\mathcal{U}_+} &\leq \|D_V R_+^U\|_{\mathcal{U}_+}\|V''\|_{\mathcal{V}} + \|D_K R_+^U(V,K)\|_{\mathcal{U}_+}\|K''\|_{\mathcal{W}^\kappa} + \|D_V^2 R_+^U(V,K)\|_{\mathcal{U}_+}\|V'\|_{\mathcal{V}}^2 \\ &\quad + 2\|D_V D_K R_+^U\|_{\mathcal{U}_+}\|V'\|_{\mathcal{V}}\|K'\|_{\mathcal{W}^\kappa} + \|D_K^2 R_+^U(V,K)\|_{\mathcal{U}_+}\|K'\|_{\mathcal{W}^\kappa}^2 \\ &\leq O(M)L^{4j}\tilde{\mathfrak{g}}_+^3\big(C_0'\tilde{g}_+^4 + C_2'\tilde{g}_+ + \tilde{g}_+^3\tilde{g}_+^{2\hat{\gamma}} + \tilde{g}_+^2 + \tilde{g}_+^4\big)\Big(\frac{\tilde{g}_j}{g_0}\Big)^{2\hat{\gamma}} \\ &\leq O(MC_2')L^{4j}\tilde{\mathfrak{g}}_+^3\tilde{g}_+\Big(\frac{\tilde{g}}{g_0}\Big)^{2\hat{\gamma}}, \end{aligned} \tag{7.55}$$

and the proof is complete. □

Proof of Proposition 7.2(iii). Suppose that $(V_j, K_j)_{j\leq k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \geq 0$. Given $j \leq k$, we assume as induction hypothesis that the bounds (7.49) hold (they hold trivially for $j = 0$ as all are zero). Then the conclusion of Lemma 7.5 also holds, and by the recursion for g_j'' and (5.43),

$$\begin{aligned}|g_+''| &\leq (1-\beta g)|g''| + 2\beta|g'|^2 + O(1)\|(R_+^U)''\|_{\mathcal{U}_+} \\ &\leq C_1'L^{4(j+1)}\tilde{\mathfrak{g}}_+^2\tilde{g}_+\Big(\frac{\tilde{g}_+}{g_0}\Big)^{2\hat{\gamma}}, \end{aligned} \tag{7.56}$$

for sufficiently large C_1'. For the bound on ν_{j+1}'', we begin with

$$|\nu_+''| \leq (1-\hat{\gamma}\beta g)|\nu''| + \eta|g'| + \hat{\gamma}\beta|g'\nu'| + \hat{\gamma}\beta|g''\nu| + 2\xi|gg''| + 2\xi|g'|^2 + |r_\nu''|. \tag{7.57}$$

We already have bounds for all terms on the right-hand side. The constant $C_0'L^{2j}$ enters in our bound on the first term on the right-hand side, via (7.49), and this is smaller by L^{-2} than our target $C_0'L^{2(j+1)}$. This is the only occurrence of C_0' in our bounds on the right-hand side. All the other terms are either higher order in $\tilde{g}$ (e.g., $g'\nu'$) or are of the order desired for our upper bound but contain a constant that can be overcome by a large choice of C_0' (e.g., r_ν''). After bookkeeping, we arrive as desired at

$$|\nu_+''| \leq C_0'L^{2(j+1)}\tilde{\mathfrak{g}}_+^3\tilde{g}_+\Big(\frac{g_+}{g_0}\Big)^{2\hat{\gamma}}, \tag{7.58}$$

for sufficiently large C_0'. This completes the proof. □

7.3 | Proof of Proposition 6.6 for $d > 4$

The proof for $d > 4$ has the same structure as the proof in Section 7.2 for $d = 4$. For $d > 4$, the important product $\Pi_{k,j}$ is much simpler since now β_j contains the exponentially decaying factor $\rho_j = L^{-(d-4)j}$, and this simplicity leads to the absence of logarithmic corrections. On the other hand, we must keep track of powers of ρ_j instead.

Lemma 7.6. *Let $d > 4$. Suppose that $(V_j, K_j)_{j \le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$, with $g_0 > 0$ sufficiently small. Then for $t \in (0, 2]$ and $i \le j < k$,*

$$\Pi_{i,j}[t] = \prod_{m=i}^{j} (1 - t\beta_m g_m) = 1 + O_t(\vartheta_i^2 g_i \rho_i). \tag{7.59}$$

For a global RG flow, the limit $\Pi_i[t] = \lim_{j\to\infty} \Pi_{i,j}[t]$ exists, is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$, and obeys $\Pi_i[t] = 1 + O_t(\vartheta_i g_i)$.

Proof. This follows from the facts that for $m \ge i$ we have $g_m \le O(g_i)$ and $\beta_m \le O(\vartheta_m^2 \rho_m)$, so $\beta_m g_m$ is summable and the sum is dominated by the first term. The continuity statement follows as in the proof of Lemma 7.1. □

Proposition 7.7 is the main ingredient in the proof of Proposition 6.6 for $d > 4$. Recall the definitions of Σ_j and F_j from (7.9), and that $\mathfrak{a}$ is given by (5.27). In particular, $\mathfrak{a} = 3$ for $d = 5$ and $\mathfrak{a} > 2$ for all $d > 4$.

Proposition 7.7 (Case $d > 4$). *Suppose that L is sufficiently large and g_0 is sufficiently small. Suppose that $(V_j, K_j)_{j \le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$. Then for all $j \le k$, (g_j, ν_j, K_j) are continuously differentiable functions of $(\nu_0, a) \in \mathcal{X}_k$ and satisfy the following bounds.*

(i) The ν_0-derivatives satisfy

$$|\Sigma_j - \Sigma_{j-1}| \le C_0 \tilde{\vartheta}_j \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}, \qquad |g_j'| \le C_1 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2}, \qquad \|K_j'\|_{\mathcal{W}_j^\kappa} \le C_2 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}. \tag{7.60}$$

(ii) The a-derivatives satisfy

$$|F_j - F_{j-1}| \le C_0 \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}, \qquad |\dot{g}_j| \le C_1 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2}, \qquad \|\dot{K}_j\|_{\mathcal{W}_j^\kappa} \le C_2 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}. \tag{7.61}$$

(iii) The second derivatives satisfy

$$|\nu_j''|, \ |\dot{\nu}_j'|, \ |\ddot{\nu}_j| \le O(1) L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j \rho_j^{\mathfrak{a}-2}. \tag{7.62}$$

Proof of Proposition 6.6(ii). The bounds on the second derivatives are repeated in (7.62), and the bounds on g_j' and $\dot{g}_j$ are repeated in (7.60) and (7.61). Here we only prove that

$$\nu_j' = c_{2,N} + O(\tilde{\vartheta}_j \tilde{g}_j \rho_j), \qquad \dot{\nu}_j = -c_{3,N} + O(\tilde{\vartheta}_j^2 \tilde{g}_j \rho_j) \tag{7.63}$$

with $c_{2,N}(\nu_0, a) = 1 + O(\tilde{\vartheta}_0 g_0)$ and positive $c_{3,N}(\nu, a) = O(\tilde{\vartheta}_0 g_0)$, obtain the negative upper bound on $\dot{\nu}_j$ for large j, identify the limiting constants $c_{2,\infty}, c_{3,\infty}$, and show that each $c_{i,N}$ is continuous in $(\nu_0, a) \in \mathcal{X}_N$ and each $c_{i,\infty}$ is continuous in $(\nu_0, a) \in \mathcal{X}_\infty$.

It follows from (7.60) that

$$\Sigma_j = \Sigma_N + O\Big(\sum_{k=j+1}^{N} \tilde{\vartheta}_k \tilde{g}_k^2 \rho_k^{\mathfrak{a}-1} \Big) = \Sigma_N + O(\tilde{\vartheta}_j \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}). \tag{7.64}$$

Since $\nu_j' = \Pi_{0,j-1}(1 + \Sigma_j)$ by definition, it follows from Lemma 7.6 that

$$\nu_j' = \Pi_{0,N-1}\big(1 + O(\vartheta_j^2 \tilde{g}_j \rho_j)\big)\big(1 + \Sigma_N + O(\tilde{\vartheta}_j \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1})\big). \tag{7.65}$$

This proves the first item of (7.63) with $c_{2,N} = \Pi_{0,N-1}(1 + \Sigma_N) = 1 + O(\tilde{\vartheta}_0 g_0)$, a continuous function on $\mathcal{X}_N$. For a global RG flow, the limit $\Sigma_\infty = \lim_{N\to\infty} \sum_{i=0}^{N-1} (\Sigma_{i+1} - \Sigma_i)$ is continuous in (ν_0, a) by the Dominated Convergence Theorem, and $\Pi_{0,\infty}$ is continuous by Lemma 7.6. Therefore, $c_{2,\infty} = \lim_{N\to\infty} c_{2,N} = \Pi_{0,\infty}(1 + \Sigma_\infty)$ is a continuous function of $(\nu_0, a) \in \mathcal{X}_\infty$.

The estimate on $\dot{\nu}_j$ follows similarly, since, $\dot{\nu}_j = -1 + (1 + F_j)\Pi_{0,j-1}$ with $F_j = F_N + O(\tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1})$ and

$$\dot{\nu}_j = -1 + \big(1 + F_N + O(\tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1})\big)\Pi_{0,N-1}\big(1 + O(\vartheta_j^2 \tilde{g}_j \rho_j)\big). \tag{7.66}$$

This proves the second item of (7.63) with $-c_{3,N} = -1 + (1 + F_N)\Pi_{0,N-1} = O(\tilde{\vartheta}_0 g)$, a continuous function on $\mathcal{X}_N$. Also, since $F_N = O(\tilde{\vartheta}_0^2 g^2)$ and $\Pi_{0,N-1} \le 1 - \hat{\gamma}\beta_0 g_0$, we see that $c_{3,N} \ge C' \tilde{\vartheta}_0^2 g$ and hence is strictly positive. Thus $\dot{\nu}_j \le -C\tilde{\vartheta}_0^2 g$ for sufficiently large j. For the infinite RG flow, the continuity of $c_{3,\infty} = \lim_{N\to\infty} c_{3,N} = 1 - (1 + F_\infty)\Pi_{0,\infty}$ in $(\nu_0, a) \in \mathcal{X}_\infty$ follows because F_∞ is continuous by the Dominated Convergence Theorem and $\lim_{N\to\infty} \Pi_{0,N-1}$ is continuous by Lemma 7.6. □

Lemma 7.8. *Let $d > 4$, and let M be the constant of (7.16). Suppose that $(V_j, K_j)_{j\le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass $a \ge 0$. Suppose further that the bounds of Proposition 7.7(i,ii) hold for all $i \le j$. Then, for either $\partial_x = \frac{d}{d\nu_0}$ or $\frac{d}{da}$,*

$$\|\partial_x \Phi_{j+1}^K\|_{\mathcal{W}_{j+1}^\kappa} \le C_2 L^{2(j+1)} \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1}^2 \rho_{j+1}^{\mathfrak{a}-1}, \tag{7.67}$$

$$\|\partial_x R_{j+1}^U\|_{\mathcal{U}_{j+1}(\ell_{j+1})} \le O(MC_2) L^{2j} \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1}^2 \rho_j^{\mathfrak{a}-1}. \tag{7.68}$$

Proof. For $\tilde{g}_j$ sufficiently small, (7.60), (7.61) and (7.63) imply that $\|\partial_x V_j\|_{\mathcal{V}_j(\ell_j)} \le O(L^{2j})$ (the cited bounds are implied by our hypothesis). We give the proof only for $\frac{d}{da}$, since $\frac{d}{d\nu_0}$ is treated similarly. By the chain rule and Theorems 5.6–5.7, for sufficiently large C_2 and L we have

$$\begin{aligned}
\Big\| \frac{d}{da} \Phi_{j+1}^K \Big\|_{\mathcal{W}_{j+1}^\kappa} &\le \|\dot{\Phi}_{j+1}^K\|_{\mathcal{W}_{j+1}^\kappa} + \|D_V \Phi_{j+1}^K\|_{\mathcal{W}_{j+1}^\kappa} \|\dot{V}_j\|_{\mathcal{V}_j(\ell_j)} + \|D_K \Phi_{j+1}^K\|_{\mathcal{W}_{j+1}^\kappa} \|\dot{K}_j\|_{\mathcal{W}_j^\kappa} \\
&\le \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1}^2 \Big(M L^{2(j+1)} \tilde{g}_{j+1} \rho_{j+1}^{\mathfrak{a}} + M \rho_{j+1}^{\mathfrak{a}-1} O_L(L^{2j}) + O(L^{-(d-4)\mathfrak{a}}) C_2 L^{2j} \rho_j^{\mathfrak{a}-1} \Big) \\
&\le C_2 L^{2(j+1)} \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1}^2 \rho_{j+1}^{\mathfrak{a}-1},
\end{aligned} \tag{7.69}$$

where in the second line we chose the $L^{-(d-4)\mathfrak{a}}$ alternative for the bound on $D_K\Phi^K_{j+1}$, in the third line we used $L^{-(d-4)\mathfrak{a}}$ to (more than) advance $\rho_j^{\mathfrak{a}-1}$ to $\rho_{j+1}^{\mathfrak{a}-1}$, and we used $O(L^{2j}) \le \frac{1}{2}L^{2(j+1)}$ for large enough L. Similarly,

$$\begin{aligned}\Big\|\frac{d}{da}R^U_{j+1}\Big\|_{\mathcal{V}_{j+1}(\ell_{j+1})} &\le \|\dot R^U_{j+1}\|_{\mathcal{V}_{j+1}(\ell_{j+1})} + \|D_K R^U_{j+1}\|_{\mathcal{V}_{j+1}(\ell_{j+1})}\|\dot K_j\|_{\mathcal{W}_j^\kappa} + \|D_V R^U_{j+1}\|_{\mathcal{V}_{j+1}(\ell_{j+1})}\|\dot V_j\|_{\mathcal{V}_j(\ell_j)}\\ &\le M\tilde\vartheta^3_{j+1}\tilde g^2_{j+1}\big(L^{2(j+1)}\tilde g_{j+1}\rho^{\mathfrak{a}}_{j+1} + C_2L^{2j}\rho_j^{\mathfrak{a}-1} + \tilde g_{j+1}\rho^{\mathfrak{a}}_{j+1}O_L(L^{2j})\big)\\ &\le ML^{2j}\tilde\vartheta^3_{j+1}\tilde g^2_{j+1}(L^2\tilde g_{j+1}\rho^{\mathfrak{a}}_{j+1} + 2C_2\rho_j^{\mathfrak{a}-1}).\end{aligned} \tag{7.70}$$

This completes the proof. □

Recall the definition of $\mathfrak{a}$ from (5.27). Our choice of $\mathfrak{a}$ implies that

$$L^{2j}\rho_j^{\mathfrak{a}-2} = L^{-\varepsilon(2d-6)}L^{2(j+1)}\rho_{j+1}^{\mathfrak{a}-2}. \tag{7.71}$$

This inequality will be important for the advancement of bounds from one scale to the next, with its helpful factor $L^{-\varepsilon(2d-6)}$ which is small for large L.

Proof of Proposition 7.7(i). We wish to prove that

$$|\Sigma_j - \Sigma_{j-1}| \le C_0\tilde\vartheta_j\tilde g_j^2\rho_j^{\mathfrak{a}-1}, \qquad |g_j'| \le C_1L^{2j}\tilde\vartheta_j^2\tilde g_j^2\rho_j^{\mathfrak{a}-2}, \qquad \|K_j'\|_{\mathcal{W}_j^\kappa} \le C_2L^{2j}\tilde\vartheta_j^3\tilde g_j^2\rho_j^{\mathfrak{a}-1}. \tag{7.72}$$

We assume these bounds and proceed via induction. This assumption is satisfied for $j = 0$ since $\Sigma_1 - \Sigma_0 = 0$, and also g_0' and K_0' are zero. The induction for K_j' is already advanced by Lemma 7.8.

For g', from the recursion $g'_{j+1} = (1 - 2\beta_j g_j)g_j' + r'_{g,j}$, the fact that $0 \le 1 - 2\beta_j g_j \le 1$, the induction hypothesis, and (7.71), we obtain

$$\begin{aligned}|g'_{j+1}| &\le C_1L^{2j}\tilde\vartheta_j^2\tilde g_j^2\rho_j^{\mathfrak{a}-2} + |r'_{g,j}|\\ &\le \frac{1}{2}C_1L^{2(j+1)}\tilde\vartheta^2_{j+1}\tilde g^2_{j+1}\rho^{\mathfrak{a}-2}_{j+1} + |r'_{g,j}|.\end{aligned} \tag{7.73}$$

For the remainder term, we apply (5.43) and Lemma 7.8, and obtain

$$|r'_{g,j}| \le O(\rho_{j+1}^{-1})\|(R^U_{j+1})'\|_{\mathcal{V}_{j+1}(\ell_{j+1})} \le O\Big(MC_2\frac{\rho_j}{\rho_{j+1}}\Big)L^{2j}\tilde\vartheta^3_{j+1}\tilde g^2_{j+1}\rho_j^{\mathfrak{a}-2}. \tag{7.74}$$

For C_1 large compared to MC_2, this permits the induction to be advanced.

Finally, we recall (7.33), and use Lemma 7.6 to bound $\Pi_{0,j}$ to see that

$$|\Sigma_{j+1} - \Sigma_j| \le O(1)\Big[\big(\beta_j|\nu_j| + \eta_j + 2\xi_j g_j\big)|g_j'| + |r'_{\nu,j}|\Big]. \tag{7.75}$$

By (5.10)–(5.12) and our bound on ν_j, the factor multiplying $|g_j'|$ is of order $\tilde\vartheta_j^2\rho_j g_j L^{-2j}\rho_j + \tilde\vartheta_j L^{-2j}\rho_j + \tilde\vartheta_j^3 L^{-2j}\rho_j^2 g_j$. The main term here is due to η_j. Also, it follows from Lemma 7.8 that $|r'_{\nu,j}| \le L^{-2(j+1)}O(MC_2)L^{2j}\tilde\vartheta^3_{j+1}\tilde g^2_{j+1}\rho_j^{\mathfrak{a}-1}$. With the induction hypothesis for g_j', this gives

$$|\Sigma_{j+1} - \Sigma_j| \le C_0\tilde\vartheta_{j+1}\tilde g^2_{j+1}\rho^{\mathfrak{a}-1}_{j+1} \tag{7.76}$$

once we choose C_0 sufficiently large compared to $C_1 + C_2$. □

Proof of Proposition 7.7(ii). We wish to prove that

$$|F_j - F_{j-1}| \le C_0 \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}, \qquad |\dot{g}_j| \le C_1 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2}, \qquad \|\dot{K}_j\|_{\mathcal{W}_j^\kappa} \le C_2 L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-1}. \tag{7.77}$$

We assume these bounds and proceed via induction. To start the induction, $\dot{g}_0$ and $\dot{K}_0$ are both equal to zero, and, by (7.10), $F_1 - F_0$ is bounded above by an L-dependent multiple of $\tilde{\vartheta}_1^3 \tilde{g}_1^2 \rho_1^{\mathfrak{a}-1}$ as required. The induction for $\dot{K}_j$ is already advanced by Lemma 7.8.

Since $\dot{g}_{j+1} = (1 - 2\beta_j g_j)\dot{g}_j + \dot{\beta}_j g_j^2 + \dot{r}_{g,j}$ and $\dot{\beta}_j = O(1) L^{2j} \vartheta_j^3 \rho_j$,

$$\begin{aligned} |\dot{g}_{j+1}| &\le |\dot{g}_j| + O(1) L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j^2 \rho_j + \rho_{j+1}^{-1} O(MC_2) L^{2j} \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1}^2 \rho_j^{\mathfrak{a}-1} \\ &\le C_1 L^{2(j+1)} \tilde{g}_{j+1}^2 \tilde{\vartheta}_{j+1}^2 \rho_{j+1}^{\mathfrak{a}-2}, \end{aligned} \tag{7.78}$$

where we argue as in the previous proof to obtain a constant $\frac{1}{2} C_1$ (due to change of scale) from the inductive assumption on $\dot{g}_j$, and we choose C_1 large compared to MC_2 to absorb the other terms.

Finally, by (7.36),

$$|F_{j+1} - F_j| \le O(1) \Big[\Big| \frac{d}{da} (\beta_j g_j) \nu_j \Big| + |\eta_j \dot{g}_j| + \Big| \frac{d}{da} (-\xi_j g_j^2) \Big| + |\dot{r}_{\nu,j})| \Big]. \tag{7.79}$$

The main term is $\eta_j \dot{g}_j$, for which $\eta_j = O(L^{-2j} \rho_j)$. Also, by our hypothesis on $\dot{g}_j$, and since $\mathfrak{a} > 2$,

$$\Big| \frac{d}{da} (\beta_j g_j) \nu_j \Big| = |\dot{\beta}_j g_j \nu_j + \beta_j \dot{g}_j \nu_j| \le O(C_1) \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^2, \tag{7.80}$$

$$\Big| \frac{d}{da} (\xi_j g_j^2) \Big| = |\dot{\xi}_j g_j^2 + 2 \xi_j g_j \dot{g}_j| \le O(C_1) \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^2, \tag{7.81}$$

while by (5.43) and Lemma 7.8,

$$|\dot{r}_{\nu,j}| \le O(L^{-2(j+1)}) \|\dot{R}_{j+1}^U\|_{\mathcal{U}_{j+1}(\ell_{j+1})} \le O(MC_2) \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1}^2 \rho_{j+1}^{\mathfrak{a}-1}. \tag{7.82}$$

Altogether, since $\mathfrak{a} \le 3$, this proves that

$$|F_{j+1} - F_j| \le C_0 \tilde{\vartheta}_{j+1}^2 \tilde{g}_{j+1}^2 \rho_{j+1}^{\mathfrak{a}-1}, \tag{7.83}$$

once we choose C_0 sufficiently large compared to $C_1 + C_2$. □

Lemma 7.9. *Let $d > 4$. Suppose that $(V_j, K_j)_{j \le k}$ is an RG flow to scale k with initial condition (g_0, ν_0) and mass a. Suppose that the bounds of Proposition 7.7 hold for all $i \le j$, and that, in addition,*

$$|\partial_{xy} \nu_j| \le C_0' L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j \rho_j^{\mathfrak{a}-2}, \quad |\partial_{xy} g_j| \le C_1' L^{4j} \tilde{\vartheta}_j^2 \tilde{g}_j \rho_j^{\mathfrak{a}-3}, \quad \|\partial_{xy} K_j\|_{\mathcal{W}_j^\kappa} \le C_2' L^{4j} \tilde{\vartheta}_j^3 \tilde{g}_j \rho_j^{\mathfrak{a}-2}, \tag{7.84}$$

where ∂_{xy} denotes any second derivative with respect to ν_0 and/or a. Then, for sufficiently large C_2',

$$\|\partial_{xy} K_{j+1}\|_{\mathcal{W}_{j+1}^\kappa} \le C_2' L^{4j} \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1} \rho_{j+1}^{\mathfrak{a}-2}, \tag{7.85}$$

$$\|\partial_{xy} R_{j+1}^U\|_{\mathcal{U}_{j+1}(\ell_{j+1})} \le O(MC_2') L^{4j} \tilde{\vartheta}_{j+1}^3 \tilde{g}_{j+1} \rho_j^{\mathfrak{a}-2}. \tag{7.86}$$

Proof. We only consider $\partial_{xy} = \partial_{v_0}^2$ as the other cases are similar. By our hypothesis (7.84), and since $C_1' \ll C_0'$ (recall the paragraph after Remark 7.3),

$$\|V_j''\|_{\mathcal{V}_j} = O(C_0')L^{4j}\tilde{\mathfrak{d}}_j^2\tilde{g}_j\rho_j^{\mathfrak{a}-2}. \tag{7.87}$$

Also, $\|\partial_x V_j\|_{\mathcal{V}_j} \le O_L(L^{2j})$. We apply the chain rule as in (7.52), with the induction hypotheses, to obtain

$$\begin{aligned}\|\partial_{v_0}^2\Phi_+^K\|_{\mathcal{W}_+^\kappa} \le O(L^{4j}\tilde{\mathfrak{d}}_{j+1}^3)\Big(&M\tilde{g}_{j+1}^2\rho_{j+1}^{\mathfrak{a}-1}C_0'\tilde{g}_j\rho_j^{\mathfrak{a}-2} + M_{0,1}'L^{-(d-4)\mathfrak{a}}C_2'\tilde{g}_j\rho_j^{\mathfrak{a}-2}\\ &+ M\tilde{g}_{j+1}\rho_{j+1}^{\mathfrak{a}-2}C_0^2 + M\tilde{g}_{j+1}^{-1}\rho_{j+1}^{-1}C_2\tilde{g}_j^2\rho_j^{\mathfrak{a}-1} + M\tilde{g}_{j+1}^{-9/4}\rho_{j+1}^{-\mathfrak{b}}(C_2\tilde{g}_j^2\rho_j^{\mathfrak{a}-1})^2\Big).\end{aligned} \tag{7.88}$$

The second term on the right-hand side contains C_2', but this is multiplied by an inverse power of L from which we see that the contribution of this term is at most half as big as what is needed to advance the induction. For the last term, we use

$$-\mathfrak{b} + 2(\mathfrak{a}-1) - (\mathfrak{a}-2) = -(\mathfrak{a}-\mathfrak{p}) + 2(\mathfrak{a}-1) - (\mathfrak{a}-2) = \mathfrak{p} > 0 \tag{7.89}$$

to replace the product $\rho_{j+1}^{-\mathfrak{b}}\rho_j^{2\mathfrak{a}-2}$ by $\rho_{j+1}^{\mathfrak{a}-2}$. The other terms are handled by taking C_2' sufficiently large. It follows that, as required,

$$\|\partial_{v_0}^2\Phi_{j+1}^K\|_{\mathcal{W}_{j+1}^\kappa} \le C_2'L^{4j}\tilde{\mathfrak{d}}_{j+1}^3\tilde{g}_{j+1}\rho_{j+1}^{\mathfrak{a}-2}. \tag{7.90}$$

Similarly,

$$\begin{aligned}\|\partial_{v_0}^2 R_{j+1}^U\|_{\mathcal{U}_{j+1}(\ell_{j+1})} &\le O(M)L^{4j}\tilde{\mathfrak{d}}_{j+1}^3\Big(\tilde{g}_{j+1}^3\rho_{j+1}^{\mathfrak{a}}C_0'\tilde{g}_j\rho_j^{\mathfrak{a}-2} + C_2'\tilde{g}_j\rho_j^{\mathfrak{a}-2}\\ &\qquad + \tilde{g}_{j+1}^3\rho_{j+1}^{\mathfrak{a}}C_0^2 + C_2\tilde{g}_j^2\rho_j^{\mathfrak{a}-1} + (C_2\tilde{g}_j^2\rho_j^{\mathfrak{a}-1})^2\Big)\\ &\le O(MC_2')L^{4j}\tilde{\mathfrak{d}}_{j+1}^3\tilde{g}_{j+1}\rho_{j+1}^{\mathfrak{a}-2},\end{aligned} \tag{7.91}$$

and the proof is complete. □

Proof of Proposition 7.7(iii). Our goal is to prove the bounds (7.84), which assert that

$$|\partial_{xy}\nu_j| \le O(1)L^{2j}\tilde{\mathfrak{d}}_j^3\tilde{g}_j\rho_j^{\mathfrak{a}-2}. \tag{7.92}$$

To do so, we require also bounds on second derivatives of g_j, K_j, R_j. We proceed by induction, with the induction hypothesis that the bounds (7.84) hold. The induction holds at scale zero since all the second derivatives are zero. By Lemma 7.9, the induction hypothesis imply the desired bound on $\partial_{xy}K_{j+1}$, as well as a bound on $\partial_{xy}R_{j+1}^U$, so we are only left to bound the derivatives of g_{j+1} and ν_{j+1}.

Differentiation of the recursion relation for g_j gives

$$\begin{aligned}\partial_{xy}g_{j+1} = (1-2\beta_j g_j)\partial_{xy}g_j - g_j^2\partial_{xy}\beta_j - 2g_j\partial_x g_j\partial_y\beta_j - 2g_j\partial_y g_j\partial_x\beta_j\\ - 2\beta_j\partial_x g_j\partial_y g_j + \partial_{xy}r_{g,j}.\end{aligned} \tag{7.93}$$

The coefficient β_j is independent of ν_0, and we take the worst case of a-derivatives of β_j in the following. The terms are bounded by

$$|\partial_{xy} g_j| \le C_1' L^{4j} \tilde{\vartheta}_j^2 \tilde{g}_j \rho_j^{\mathfrak{a}-3}, \tag{7.94}$$

$$|g_j^2 \partial_{xy} \beta_j| \le \tilde{g}_j^2 O(1) L^{4j} \tilde{\vartheta}_j^4 \rho_j, \tag{7.95}$$

$$|g_j \partial_x g_j \partial_y \beta_j| \le \tilde{g}_j (C_1 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2})(O(L^{2j}) \tilde{\vartheta}_j^3 \rho_j) = O(C_1) L^{4j} \tilde{\vartheta}_j^5 \tilde{g}_j^3 \rho_j^{a-1}, \tag{7.96}$$

$$|\beta_j \partial_x g_j \partial_y g_j| \le (O(1) \tilde{\vartheta}_j^2 \rho_j)(C_1 L^{2j} \tilde{\vartheta}_j^2 \tilde{g}_j^2 \rho_j^{\mathfrak{a}-2})^2 = O(C_1^2) L^{4j} \tilde{\vartheta}_j^6 \tilde{g}_j^4 \rho_j^{2a-3}, \tag{7.97}$$

$$|\partial_{xy} r_{g,j}| \le \rho_{j+1}^{-1} \|\partial_{xy} R_{j+1}^U\|_{\mathcal{U}_{j+1}(\ell_{j+1})} \le O(MC_2') L^{4j} \tilde{\vartheta}_j^3 \tilde{g}_j \rho_j^{a-2} \rho_{j+1}^{-1}, \tag{7.98}$$

where we used Proposition 7.7(i,ii) to bound $\partial_x g_j$ and Lemma 7.9 to bound $\partial_{xy} r_{g,j}$. Thus we have the desired bound on $\partial_{xy} g_{j+1}$ for C_1' sufficiently large compared to C_2'.

Each term of

$$\partial_{xy} \nu_{j+1} = \partial_{xy} \big((1 - \hat{\gamma} \beta_j g_j) \nu_j + \eta_j g_j - \xi_j g_j^2 + r_{\nu,j} \big) \tag{7.99}$$

is bounded similarly, with the important difference that

$$|\partial_{xy} r_{\nu,j}| \le O(L^{-2(j+1)}) \|\partial_{xy} R_{j+1}^U\|_{\mathcal{U}_{j+1}(\ell_{j+1})} \le O(MC_2') L^{-2} L^{2j} \tilde{\vartheta}_j^3 \tilde{g}_j \rho_j^{a-2} \tag{7.100}$$

instead of (7.98); the first inequality is by (5.43) and the second by Lemma 7.9. This proves the desired bound on $\partial_{xy} \nu_{j+1}$ for C_0' sufficiently large compared to C_1' and C_2'. □

8 | GAUSSIAN INTEGRALS, THE REGULATOR, AND PROOF OF THEOREM 5.6

In this section, we prove Theorem 5.6 and prepare for the proof of Theorem 5.7. In Section 8.1, we define an extended norm which we use to mechanise estimates on derivatives with respect to the effective potential V, the non-perturbative coordinate K, and the mass a. In Section 8.2, we use the extended norm to obtain estimates on mass derivatives of Gaussian expectations. In Section 8.3, we obtain properties of the regulator $G(b, \varphi)$ of (5.24), which are needed in order to obtain the large-field decay in our estimates for K. The results of Sections 8.2–8.3 are of a general nature and are potentially applicable to any RG analysis of hierarchical spin models. In Section 8.4, we prove Theorem 5.6. Finally, in Section 8.5 we obtain estimates on the building blocks of the RG map Φ_+^K. These estimates will be assembled in Section 9 where the proof of Theorem 5.7 is concluded.

As usual, we often omit labels for the scale j and write $+$ for scale $j+1$, when only two scales are at play.

8.1 | The extended norm

To facilitate the control of Φ_+^K and Φ_+^U together with its derivatives with respect to φ, V, K, a, as in [9, Chapter 10] we use an extended version of the semi-norm employed in Section 5. This extended

norm is again defined by (5.22), but with the space $\mathcal{Z}$ now given in terms of a new space $\mathcal{Y}(\vec{\lambda})$ by $\mathcal{Z} = \mathbb{R}^n \times \mathcal{Y}(\vec{\lambda})$. The space $\mathcal{Y}(\vec{\lambda})$ is the subset of $\mathcal{V} \times \mathcal{F} \times \mathbb{R}$ consisting of $y = (V, K, a)$ such that $\|y\|_{\mathcal{Y}(\vec{\lambda})} < \infty$, where

$$\|(V, K, a)\|_{\mathcal{Y}(\vec{\lambda})} = \max\left\{ \frac{\|V(b)\|_{\mathcal{V}(\ell)}}{\lambda_V}, \frac{\|K\|_{\mathcal{W}^\kappa}}{\lambda_K}, \frac{|a|}{\lambda_a} \right\}. \tag{8.1}$$

Here $\vec{\lambda} = (\lambda_V, \lambda_K, \lambda_a)$ is a triple of nonnegative real numbers, and the choice $\lambda_V = 0$ or $\lambda_K = 0$ or $\lambda_a = 0$ simply means the derivative in the corresponding coordinate is not considered. For F a smooth function of φ, V, K, a, we use the extended seminorm $\|F\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})}$ given by (5.22) with $\mathcal{Z} = \mathbb{R}^n \times \mathcal{Y}(\vec{\lambda})$ and $|(\varphi, y)|_{\mathcal{Z}} = \max\{\frac{\|\varphi\|_2}{\mathfrak{h}}, \|y\|_{\mathcal{Y}}\}$. The norm contains derivatives of all orders with respect to φ, V, K, but we restrict to an arbitrary but fixed finite number p_a of a-derivatives in (5.22). For $y = (V, K, a) \in \mathcal{D} \times \mathcal{F} \times \mathbb{I}$, we extend the definition of the norm in (5.28) by setting

$$\|F\|_{T^G_y(h,\vec{\lambda})} = \sup_{\varphi \in \mathbb{R}^n} G(b, \varphi)^{-1} \|F(b)\|_{T_{\varphi,y}(h,\vec{\lambda})}, \tag{8.2}$$

$$\|F\|_{\mathcal{W}^\kappa_y(\vec{\lambda})} = \|F\|_{T_{0,y}(\ell,\vec{\lambda})} + \tilde{g}^{9/4} \rho^{\mathfrak{b}} \|F\|_{T^G_y(h,\vec{\lambda})}, \tag{8.3}$$

with the exponent $\mathfrak{b}$ defined in (5.41).

8.2 | Mass derivative of the expectation

In this section, we discuss how to control the mass derivative of Gaussian expectations whose covariance depends on a. We have already observed around (5.35) that derivatives with respect to a lead to factors of order L^{2j}, and we have compensated for this by equipping the mass interval $\mathbb{I}_j$ with norm $|x|L^d L^{2j}$. This is equivalent to taking

$$\lambda_{a,j} = k_{a,j}^{-1}, \qquad k_{a,j} = L^d L^{2j} \tag{8.4}$$

in (8.1), and we often drop the scale j from the notation as usual. Throughout this section, when we are concerned with the expectation $\mathbb{E}_+ = \mathbb{E}_{j+1}$ with covariance $C_+ = C_{j+1} = \gamma_+ P_+$ (recall 2.9), we always assume that

$$a \geq -\frac{1}{2} L^{-2j} \tag{8.5}$$

so that $\gamma_+ \leq \frac{1}{2} L^{2j}$. We assume (8.5) explicitly in the statement of Lemma 8.1 for emphasis, but afterwards we assume it tacitly.

In the following, as in Definition 5.3 we write $\theta F(\varphi) = F(\varphi + \zeta)$, where $\varphi \in \mathbb{R}^n$ and ζ is a $(\mathbb{R}^n)^{\mathcal{B}}$-valued Gaussian random variable with covariance C_+. The next lemma indicates how the extended norm, which in particular includes derivatives with respect to the mass in the covariance in $\mathbb{E}_+$, can be moved under the expectation in an upper bound.

Lemma 8.1. *Let $B \in \mathcal{B}_+$ and let $F^B = \prod_{b \in B(B)} F(b)$ be a product of scale-j polymer activities which may depend on $(V, K) \in \mathcal{V} \times \mathcal{F}$ but not on a. There is a constant C (depending on p_a and n but not*

on L or a) such that, with $\lambda_a = k_a^{-1}$, and for all $\mathfrak{h} > 0$ and all $a \geq -\frac{1}{2}L^{-2j}$,

$$\|\mathbb{E}_+ \theta F^B\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \leq C \left[\mathbb{E}_+ \prod_{b\in\mathcal{B}(B)} \|F(b)\|^2_{T_{\varphi+\zeta_b,y}(\mathfrak{h},\vec{\lambda})}\right]^{1/2}. \tag{8.6}$$

Proof. Let $\vec{\lambda}_0 = (\lambda_V, \lambda_K, 0)$. To simplify the notation, we write $\tilde{F} = \prod_{b\in\mathcal{B}(B)} \|F(b)\|_{T_{\varphi+\zeta_b,y}(\mathfrak{h},\vec{\lambda}_0)}$. By definition of the $T_{\varphi,y}(h,\vec{\lambda})$-seminorm, and by pushing the norm inside the expectation (justified by [9, (7.3.3)]), we have

$$\|\mathbb{E}_+ \theta F^B\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} = \sum_{r=0}^{p_a} \frac{1}{r!} \left\|D_a^r \mathbb{E}_+ \theta F^B\right\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda}_0)} \leq \sum_{r=0}^{p_a} \frac{1}{r!} \frac{1}{k_a^r} \frac{d^r}{da^r} \mathbb{E}_+ \tilde{F}. \tag{8.7}$$

To proceed, we require good control of mass derivatives of $\mathbb{E}_+\tilde{F}$. The proof does not use the specific form of $\tilde{F}$ until the final step in (8.14), and until then $\tilde{F}$ can be an arbitrary function of ζ (with dependence on φ, V, K permitted).

The derivative with respect to a of the Gaussian expectation $\mathbb{E}_+$ acts on $\exp(-\frac{1}{2}(\zeta, C_+^{-1}\zeta))$. Recall the covariance decomposition (2.7). By (2.6), $C_+^{-1} = \gamma_+^{-1} P_+$ depends on a only through the prefactor $\gamma_+^{-1} = L^{-2j} + a \geq \frac{1}{2}L^{-2j}$. It follows that

$$\frac{d}{da}\mathbb{E}_+\tilde{F} = -\mathbb{E}_+[X_+\tilde{F}], \tag{8.8}$$

where

$$X_+(a) = Y_+ - \mathbb{E}_+ Y_+ \qquad \text{with} \qquad Y_+ = \frac{1}{2}(\zeta, P_+\zeta). \tag{8.9}$$

The derivative of X_+ is

$$\frac{d}{da}X_+(a) = -\mathbb{E}_+ X_+ Y_+. \tag{8.10}$$

It can then be seen from induction that there are constants c_p such that

$$\frac{d^r}{da^r}\mathbb{E}_+\tilde{F} = \sum_p c_p \mathbb{E}_+[Y_+^{p_1}\tilde{F}] \prod_{i=2}^{m_p} \mathbb{E}_+[Y_+^{p_i}], \tag{8.11}$$

where the sum ranges over $p = (p_i)_{i=1}^{m_p}$ with nonnegative p_i, for some $m_p \geq 1$, such that $\sum_{i=1}^{m_p} p_i = r$.

By definition of Y_+, and with the integrals over $(\mathbb{R}^n)^{L^d-1}$, the expectation in the product over i in (8.11) is given by

$$\gamma_+^{-p_i}\mathbb{E}_+\left[Y_+^{p_i}\right] = (-1)^{p_i}\frac{d^{p_i}}{ds^{p_i}}\Big|_{s=1} \frac{\int e^{-\frac{1}{2}s(\zeta,C_+^{-1}\zeta)}d\zeta}{\int e^{-\frac{1}{2}(\zeta,C_+^{-1}\zeta)}d\zeta} = (-1)^{p_i}\frac{d^{p_i}}{ds^{p_i}}\Big|_{s=1} s^{-\frac{1}{2}n(L^d-1)}. \tag{8.12}$$

The right-hand side is bounded above by $(\frac{1}{2}nL^d + p_i)^{p_i}$, so the product over i in (8.11) is bounded above by $O_{n,r}(L^d\gamma_+)^{r-p_1}$. Therefore,

$$\Big|\frac{d^r}{da^r}\mathbb{E}_+\tilde{F}\Big| \le O_{r,n}(1)(L^d\gamma_+)^r \max_{0\le p_1\le r}(L^d\gamma_+)^{-p_1}\mathbb{E}_+\big[|Y_+|^{p_1}\tilde{F}\big] \qquad (r\ge 0). \tag{8.13}$$

Since $k_a^{-r}(L^d\gamma_+)^r \le (2k_a^{-1}L^dL^{2j})^r = 2^r$, we can now bound the right-hand side of (8.7) to obtain

$$\|\mathbb{E}_+\theta F\|_{T_{\varphi,y}(h,\vec{\lambda})} \le O_{r,n}(1)\max_{0\le p_1\le p_a}(L^d\gamma_+)^{-p_1}\mathbb{E}_+[|Y_+|^{p_1}\tilde{F}]. \tag{8.14}$$

By the Cauchy–Schwarz inequality and the bound obtained previously for the expectation of powers of Y_+, the quantity under the maximum is bounded above by an L-independent multiple of $\big(\mathbb{E}_+\tilde{F}^2\big)^{1/2}$. Since F has been assumed to be independent of a, the norm with $\vec{\lambda}_0$ in $\tilde{F}$ is equal to the norm with $\vec{\lambda}$, so the proof is complete. □

We need good estimates on polynomials. The following lemma shows that the $T_{\varphi,y}$-norm of an a-independent polynomial can be bounded in terms of the $T_{0,y}$-norm. The lemma combines [9, Exercise 7.5.2] and [9, Lemma 7.2.2] into one. For its statement, given $\mathfrak{h} > 0$ we define

$$P_{\mathfrak{h}}(\varphi) = 1 + |\varphi|/\mathfrak{h}. \tag{8.15}$$

Lemma 8.2. *With $z = (\varphi, y)$, suppose that $F(z)$ is an a-independent polynomial in φ of degree k, with coefficients that are functions of y. Then*

$$\|F\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le \|F\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})}P_{\mathfrak{h}}^k(\varphi). \tag{8.16}$$

Also, for $r \le k$, the norm of the derivative of F obeys

$$\|D_\varphi^r F\|_{\mathbb{R}^n\to T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le 2^k\frac{r!}{\mathfrak{h}^r}\|F\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})}. \tag{8.17}$$

The next lemma is analogous to what appears in [9, Section 7.6], but it needs adaptation since our norm includes the mass derivative. We exclude $p = 0$ in its statement because (8.19)–(8.20) hold trivially in this case. We write

$$\mathfrak{c}_+^2 = C_{+;00} = \frac{L^{-(d-2)j}}{1+aL^{2j}}(1-L^{-d}) \tag{8.18}$$

for the diagonal element of the covariance C_+.

Lemma 8.3. *Let V_1, V_2 be even a-independent polynomials on $B \in \mathcal{B}_+$, of degrees at most $2p$ for some $p \ge 1$. Let $\mathfrak{h} \ge \mathfrak{c}_+$. There is an L-independent constant C such that*

$$\|\mathbb{E}_+(\theta V_1 - V_1)(B)\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le C\Big(\frac{\mathfrak{c}_+}{\mathfrak{h}}\Big)^2\|V_1(B)\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})}P_{\mathfrak{h}}^{2p-2}(\varphi), \tag{8.19}$$

$$\|\operatorname{Cov}_+[\theta V_1(B), \theta V_2(B)]\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le C\Big(\frac{\mathfrak{c}_+}{\mathfrak{h}}\Big)^4\|V_1(B)\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})}\|V_2(B)\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})}P_{\mathfrak{h}}^{2p}(\varphi). \tag{8.20}$$

Proof. For the norm which does not measure a-derivatives, (8.19) is proved in [9, Exercise 7.6.2], and we adapt that proof to our norm which does measure a-derivatives. We use the notation $\alpha = (\alpha_1, \dots, \alpha_n)$ for a multi-index, and we say that α is even if each α_i is even. By Taylor's theorem,

$$\mathbb{E}_+(\theta V_1 - V_1)(B) = \sum_{x\in B} \sum_{2\le|\alpha|\le 2p} \frac{1}{\alpha!} V_{1,x}^{(\alpha)}(\varphi)\mathbb{E}_+\zeta_x^\alpha, \tag{8.21}$$

with the sum restricted to even multi-indices α since otherwise $\mathbb{E}_+\zeta_x^\alpha = 0$. We use Lemma 8.1 (with $F = \zeta_x^\alpha$ and $\varphi = 0$, so F is independent of φ, K, V, a) to move the norm inside the expectation, apply the triangle inequality and the product property of the $T_{\varphi,y}$-seminorm, and finally use Lemma 8.2 to conclude that there is an L-independent constant C such that

$$\|\mathbb{E}_+(\theta V_1 - V_1)(B)\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le C \sum_{x\in B} \sum_{2\le|\alpha|\le 2p} \mathfrak{h}^{-|\alpha|} \|V_{1,x}\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})} P_{\mathfrak{h}}^{2p-|\alpha|}(\varphi) (\mathbb{E}_+ \|\zeta_x^\alpha\|^2_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})})^{1/2}. \tag{8.22}$$

The norm $\|\zeta_x^\alpha\|^2_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})}$ is simply $\zeta_x^{2\alpha}$ for even α, and $\mathbb{E}_+\zeta_x^{2\alpha}$ is bounded by a constant times $\mathfrak{c}_+^{2|\alpha|}$ by [9, Exercise 2.1.7]. With the worst power of $P_{\mathfrak{h}}$, this gives

$$\|\mathbb{E}_+(\theta V_1 - V_1)(B)\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le C\, P_{\mathfrak{h}}^{2p-2}(\varphi) \|V_1(B)\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})} \sum_{2\le|\alpha|\le 2p} \Big(\frac{\mathfrak{c}_+}{\mathfrak{h}}\Big)^{|\alpha|}, \tag{8.23}$$

and then by taking the worst power of $\mathfrak{c}_+/\mathfrak{h}$ (which is at most 1) gives (8.19).

The proof of (8.20) is a similar adaptation of the proof of [9, Lemma 7.6.1], and we omit the details. □

8.3 | Estimates for the regulator

Decay of the nonperturbative coordinate K_j for large fields is controlled by the regulator $G_j(\varphi) = e^{-\kappa|\varphi/h_j|^4}$ defined in (5.24), where $h_j = k_0 \tilde{g}_j^{-1/4} L^{-dj/4}$ as in (5.20). As has been emphasised by Brydges, a good regulator should have a "supermartingale" property (see, e.g., [16, Section 6.5]) which permits the propagation of estimates as the scale is advanced. In our context, the supermartingale property is the inequality (8.25) in the next lemma. We choose to implement quartic exponential decay in the regulator since this is easy to do for the hierarchical model. Quadratic decay would also be sufficient, and may be best possible in a Euclidean setting.

Lemma 8.4. *Fix any $\overline{\kappa} \ge 0$. For any $\varphi \in \mathbb{R}^n$ and $B \in \mathcal{B}_+$,*

$$\prod_{b\in\mathcal{B}(B)} e^{-\overline{\kappa}|\varphi+\zeta_b|^4/h^4} \le e^{-\overline{\kappa}|\varphi/h_+|^4} \quad \text{a.s. } [\mathbb{E}_+]. \tag{8.24}$$

In particular, for any $t \ge 0$ and with $\overline{\kappa} = t\kappa$, this gives

$$\mathbb{E}_+ \prod_{b\in\mathcal{B}(B)} G^t(b, \varphi + \zeta_b) \le G_+^t(B, \varphi). \tag{8.25}$$

Proof. It follows by expansion of $|\varphi+\zeta_b|^4 = [|\varphi|^2 + (2\varphi\cdot\zeta_b + |\zeta_b|^2)]^2$ that

$$|\varphi+\zeta_b|^4 \geq |\varphi|^4 + 4|\varphi|^2\varphi\cdot\zeta_b. \tag{8.26}$$

Since $\sum_{b\in\mathcal{B}(B)}\zeta_b = 0$ with probability 1 by (2.14), this implies that

$$\prod_{b\in\mathcal{B}(B)} e^{-\overline{\kappa}|\varphi+\zeta_b|^4/h^4} \leq e^{-L^d\overline{\kappa}|\varphi/h|^4} \quad \text{a.s.} \tag{8.27}$$

Now we use the definition $h_j = k_0\tilde{g}_j^{-1/4}L^{-dj/4}$, together with the fact that $\tilde{g}_+ \leq \tilde{g}$, to see that

$$L^d h^{-4} = h_+^{-4}\tilde{g}/\tilde{g}_+ \geq h_+^{-4}, \tag{8.28}$$

which proves (8.24). The supermartingale property (8.25) then follows by taking the expectation. □

For $\mathfrak{h} > 0$, we recall the definition

$$P_{\mathfrak{h}}(\varphi) = 1 + |\varphi|/\mathfrak{h} \tag{8.29}$$

from (8.15). One usage of the regulator is to control polynomial factors of the form $P_{\mathfrak{h}}^k(\varphi)$. This is done via the observation that for any $\mathfrak{h}, k, \overline{\kappa} > 0$ and uniformly in $\varphi\in\mathbb{R}^n$,

$$P_{\mathfrak{h}}^k(\varphi)e^{-\overline{\kappa}|\varphi/\mathfrak{h}|^4} \leq O_{\overline{\kappa},k}(1), \tag{8.30}$$

since $1+t$ is bounded above by a multiple of $e^{k^{-1}\overline{\kappa}t^4}$ for $t\geq 0$. A related domination of polynomial factors is provided by the next lemma.

Lemma 8.5. *Let $B\in\mathcal{B}_+$, $b\in\mathcal{B}(B)$, $p\geq 1$, and $t>0$. If $\tilde{g}$ is sufficiently small (depending on L) then there exists $c_{p,t,\kappa}>0$ such that*

$$\mathbb{E}_+\Big(G^t(b,\varphi+\zeta_b)P_{h_+}^p(\varphi+\zeta_b)\prod_{b'\in\mathcal{B}(B),\,b'\neq b} G^{2t}(b',\varphi+\zeta_{b'})\Big) \leq c_{p,t,\kappa}\,G_+^t(B,\varphi). \tag{8.31}$$

Proof. We write c for a (p,t,κ)-dependent constant whose value may change from line to line. Let k be the smallest integer such that $4k\geq p$. We first observe that $P_{h_+}^p(\varphi+\zeta_b) \leq c(1+|\varphi/h_+|^p + |\zeta_b/h_+|^p)$. We use the fact that $h_+^{-4}\leq 2L^dh^{-4}$ together with the fact that φ is constant in B (in the second inequality of 8.32) to obtain

$$\begin{aligned}
1+\Big|\frac{\varphi}{h_+}\Big|^p &\leq c\Big(1+\Big|\frac{\varphi}{h_+}\Big|^4\Big)^k \leq c\Big(1+\sum_{b'\neq b}\Big|\frac{\varphi}{h}\Big|^4\Big)^k \\
&\leq c\Big(1+\sum_{b'\neq b}\Big|\frac{\varphi+\zeta_{b'}}{h}\Big|^4 + \sum_{b'\neq b}\Big|\frac{\zeta_{b'}}{h}\Big|^4\Big)^k \\
&\leq c\Big(e^{\kappa t\sum_{b'\neq b}|(\varphi+\zeta_{b'})/h|^4} + \sum_{b'\neq b}\Big|\frac{\zeta_{b'}}{h}\Big|^{4k}\Big).
\end{aligned} \tag{8.32}$$

From this, we see that

$$P^p_{h_+}(\varphi+\zeta_b) \le c\Big(\prod_{b'\neq b} G(b',\varphi+\zeta_{b'})^{-t} + \sum_{b'\neq b}\Big|\frac{\zeta_{b'}}{h}\Big|^{4k} + \Big|\frac{\zeta_b}{h_+}\Big|^p\Big). \tag{8.33}$$

When multiplied by the regulators on the left-hand side of (8.31), the $G(b')^{-t}$ cancels with $G(b')^{2t}$ to produce $G(b')^t$, and we can then apply Lemma 8.4 to get the desired upper bound for this term. In preparation for the other two terms on the right-hand side of (8.33), let $b'' \in \mathcal{B}(B)$, let q be a nonnegative integer, and let $\mathfrak{h} > 0$. By the Cauchy–Schwarz inequality, Lemma 8.4, and [9, Lemma 10.3.1] (to bound the last factor in the second line),

$$\begin{aligned}
&\mathbb{E}_+\Big[\Big(\prod_{b'\neq b} G(b',\varphi+\zeta_{b'})^{2t}\Big)G(b,\varphi+\zeta_b)^t\Big|\frac{\zeta_{b''}}{\mathfrak{h}}\Big|^q\Big] \\
&\quad\le \Big(\frac{\ell_+}{\mathfrak{h}}\Big)^q\Big[\mathbb{E}_+\Big(\prod_{b'\neq b} G(b',\varphi+\zeta_{b'})^{4t}\Big)G(b,\varphi+\zeta_b)^{2t}\Big]^{1/2}\Big[\mathbb{E}_+\Big(\frac{\zeta_{b''}}{\ell_+}\Big)^{2q}\Big]^{1/2} \\
&\quad\le c\Big(\frac{\ell_+}{\mathfrak{h}}\Big)^q G^t_+(B,\varphi).
\end{aligned} \tag{8.34}$$

For the term $|\zeta_b/h_+|^p$ (so $q=p$ and $\mathfrak{h}=h_+$), this gives the desired bound immediately since $\ell_+ \le h_+$ for sufficiently small $\tilde{g}$ (since $d \ge 4$). Finally, for the term $\sum_{b'\neq b}|\frac{\zeta_{b'}}{h}|^{4k}$ (so $q=4k$ and $\mathfrak{h}=h$), there are L^d-1 terms but we have h not h_+, so for sufficiently small $\tilde{g}$ we gain $(\ell_+/h)^{4k} \le L^{-2(d-2)k}L^{-(d-4)jk} \le L^{-2(d-2)}$. The product of this beneficial factor with the dangerous factor L^d-1 is bounded for $d \ge 4$, and the proof is complete. □

The next lemma gathers and extends estimates from [9] on how norms interact with the expectation. It uses a general inequality from [9, (10.4.5)], namely

$$\|F(b)\|_{T_{\varphi,y}(\ell_+,\vec{\lambda})} \le P^{10}_{\ell_+}(\varphi)\|F(b)\|_{\mathcal{W}^0_{y,+}(\vec{\lambda})}, \tag{8.35}$$

which permits the $T_{\varphi,y}(\ell_+,\vec{\lambda})$ norm to be bounded in terms of the $T_{0,y}(\ell_+,\vec{\lambda})$ and $T_{\varphi,y}(h_+,\vec{\lambda})$ norms.

As usual, the fluctuation field $\zeta \sim \mathcal{N}(0,C_+)$ is constant on blocks $b \in \mathcal{B}(B)$ and φ is constant on B, and we assume $a \ge -\frac{1}{2}L^{-2j}$ as in (8.5). Note that on the right-hand side of (8.37) we have the scale-j regulator G which is defined with h, but that the T_φ norm used to define the $T^G_y(h_+,\vec{\lambda})$ norm instead uses h_+.

Lemma 8.6. *Let $B \in \mathcal{B}_+$, let F be a polymer activity at scale j, and let $F^B = \prod_{b\in\mathcal{B}(B)} F(b)$. There is an L-independent constant C and an L-dependent constant C_L such that for any $\lambda_V, \lambda_K \ge 0$,*

$$\|\mathbb{E}_+\theta F^B\|_{T_{0,y}(\ell_+,\vec{\lambda})} \le C_L\prod_{b\in\mathcal{B}(B)}\|F(b)\|_{\mathcal{W}^0_{y,+}(\vec{\lambda})}, \tag{8.36}$$

$$\|\mathbb{E}_+\theta F^B\|_{T_{\varphi,y}(h_+,\vec{\lambda})} \le CG_+(B,\varphi)\prod_{b\in\mathcal{B}(B)}\|F(b)\|_{T^G_y(h_+,\vec{\lambda})}. \tag{8.37}$$

Proof. For (8.36), we combine (8.35) with (8.6). With $\mathfrak{h} = \ell_+$ and $\varphi = 0$, this gives

$$\|\mathbb{E}_+ \theta F^B\|_{T_{0,y}(\ell_+,\vec{\lambda})} \le C \prod_{b\in\mathcal{B}(B)} \|F(b)\|_{\mathcal{W}^0_{y,+}(\vec{\lambda})} \Big(\mathbb{E}_+ \Big[\prod_{b\in\mathcal{B}(B)} P^{20}_{\ell_+}(\zeta_b) \Big] \Big)^{1/2}. \tag{8.38}$$

The expectation is bounded by an L-dependent constant, as in [9, (10.4.10)], and the proof of (8.36) is complete. Finally, for (8.37), we use (8.6) and (8.25) to obtain

$$\begin{aligned} \|\mathbb{E}_+ \theta F^B\|_{T_{\varphi,y}(h_+,\vec{\lambda})} &\le C \Big(\mathbb{E}_+ \Big[\prod_{b\in\mathcal{B}(B)} \|F(b)\|^2_{T^G_y(h_+,\vec{\lambda})} G^2(b, \varphi + \zeta_b) \Big] \Big)^{1/2} \\ &\le C G_+(B,\varphi) \prod_{b\in\mathcal{B}(B)} \|F(b)\|_{T^G_y(h_+,\vec{\lambda})}. \end{aligned} \tag{8.39}$$

This completes the proof. □

8.4 | The perturbative RG map: Proof of Theorem 5.6

We now prove Theorem 5.6, which contains the main estimates for the perturbative RG map. According to its definition in (5.17), the remainder for the flow of coupling constants is given by $R^U_+(V,K) = \Phi^U_+(V,K) - \Phi_{\mathrm{pt}}(V)$. After some algebra (see [9, (9.1.5)]), this can be written as

$$R^U_+(B) = -\mathbb{E}_+ \theta Q(B) + \mathrm{Cov}_+ \Big[\theta(V(B) - \frac{1}{2} Q(B)), \theta Q(B) \Big] \tag{8.40}$$

where

$$Q(b) = \mathrm{Loc}(e^V K)(b), \qquad Q(B) = \sum_{b\in\mathcal{B}(B)} Q(b). \tag{8.41}$$

In Lemma 8.7, we bound Q, which does not depend on a. The a-dependence of R^U_+ arises after application of $\mathbb{E}_+$ or Cov_+, and is handled by our extended norm in Lemma 8.8.

Lemma 8.7. *For $(V,K) \in \mathbb{D}$,*

$$\|Q(b)\|_{T_{0,y}(\ell,\vec{\lambda})} \le e^{\|V(b)\|_{T_0(\ell)} + \lambda_V} \big(\|K\|_{T_0(\ell)} + \lambda_K \big). \tag{8.42}$$

If $\mathfrak{h} \ge \ell$, $\lambda_V \le 1$, and $\lambda_K \le \tilde{g}\rho$, then for an L-independent constant C,

$$\|Q(b)\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})} \le C \Big(\frac{\mathfrak{h}}{h} \Big)^4. \tag{8.43}$$

Proof. The first bound is [9, (9.3.20)]. For (8.43), we first apply [9, (9.3.24)] and then use (8.42) to obtain

$$\|Q(b)\|_{T_{0,y}(\mathfrak{h},\vec{\lambda})} \le \Big(\frac{\mathfrak{h}}{\ell} \Big)^4 \|Q(b)\|_{T_{0,y}(\ell,\vec{\lambda})} \le \Big(\frac{\mathfrak{h}}{\ell} \Big)^4 e^{\|V(b)\|_{T_0(\ell)} + \lambda_V} \big(\|K\|_{T_0(\ell)} + \lambda_K \big). \tag{8.44}$$

The assumptions on V and λ_V imply that $e^{\|V(b)\|_{T_0(\ell)}+\lambda_V} \le O(1)$. Also, by the assumptions on K and λ_K, and since $\mathfrak{a} > 2$,

$$\|K\|_{T_0(\ell)} + \lambda_K \le C_{\mathrm{RG}}\tilde{g}^3\rho^{\mathfrak{a}} + \tilde{g}\rho \le 2\tilde{g}\rho \tag{8.45}$$

for $\tilde{g}$ small enough compared to C_{RG} (which for now is an arbitrary but fixed L-dependent constant, to be specified in (9.13). Since $\tilde{g}\rho\ell^{-4} \le O(1)h^{-4}$, this completes the proof. □

Lemma 8.8. *Let $(V,K) \in \mathbb{D}$, $r_1 = \|V(b)\|_{T_0(\ell)} + \lambda_V$, and $r_2 = \|K(b)\|_{T_0(\ell)} + \lambda_K$ with $\lambda_V \le 1$. There is an L-dependent $O_L(1)$ such that, for $a \ge -\frac{1}{2}L^{-2j}$ and $\lambda_a = k_a^{-1}$,*

$$\|R_+^U(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} = O_L(1)(1 + r_1 + r_2)r_2. \tag{8.46}$$

Proof. We bound the two terms on the right-hand side of (8.40) separately. For the first term, we use Lemma 8.7, the assumptions on V and λ_V, and $\ell_+ \le \ell$ (with monotonicity of the norm in ℓ), to obtain

$$\|Q(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \le O_L(1)r_2. \tag{8.47}$$

Therefore, by Lemma 8.3,

$$\begin{aligned}\|\mathbb{E}_+\theta Q(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} &\le \|Q(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} + \|\mathbb{E}_+(\theta Q - Q)(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \\ &\le O(1)\|Q(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \le O_L(1)r_2.\end{aligned} \tag{8.48}$$

Also, again by Lemma 8.3,

$$\begin{aligned}\Big\| \mathrm{Cov}_+\left[\theta(V(B) - \frac{1}{2}Q(B)), \theta Q(B)\right]\Big\|_{T_{0,y}(\ell_+,\vec{\lambda})} &\le O_L(1)\|V(b) - \frac{1}{2}Q(b)\|_{T_{0,y}(\ell_+,\vec{\lambda})}\|Q(b)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \\ &\le O_L(1)(r_1 + r_2)r_2.\end{aligned} \tag{8.49}$$

Together, these bounds give the desired estimate and complete the proof. □

Proof of Theorem 5.6. Let $d \ge 4$. We wish to prove that under the hypotheses of Theorem 5.6, the RG remainder obeys the bounds

$$\|D_V^p D_K^q D_a^r R_+^U\|_{\mathbb{D}\to\mathcal{U}_+(\ell_+)} \le \begin{cases} M_{p,0}\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{a}} & (p \ge 0,\ q = 0) \\ M_{p,q} & (p \ge 0,\ q \ge 1) \\ 0 & (p \ge 0,\ q \ge 3). \end{cases} \tag{8.50}$$

For $q \ge 3$, the derivatives vanish because by definition R_+^U is only quadratic in K. For the other cases, we fix $r \le p_a$ and control derivatives using the extended norm as in [9, (9.2.7)], which gives

$$\|D_V^p D_K^q D_a^r R_+^U\|_{\mathbb{D}\to\mathcal{U}_+(\ell_+)} \le \frac{p!q!r!}{\lambda_V^p\lambda_K^q}\|R_+^U\|_{T_{0,y}(\ell_+,\vec{\lambda})}, \tag{8.51}$$

(there is no λ_a^{-r} for the a-derivative because this factor is included inside D_a^{-r}). We apply Lemma 8.8 to bound the right-hand side by $(1 + r_1 + r_2)r_2$, with $r_1 = \|V(b)\|_{T_0(\ell)} + \lambda_V$ and $r_2 = \|K(b)\|_{T_0(\ell)} + \lambda_K$.

For the case $q = 0$, we apply Lemma 8.8 with $(\lambda_V, \lambda_K) = (1, 0)$. The fact that (V, K) is in the RG domain implies that $r_2 = \|K(b)\|_{T_0(\ell)} \le C_{\mathrm{RG}} \tilde{\vartheta}^3 \tilde{g}^3 \rho^{\mathfrak{a}}$, and also that $r_1 = \|V\|_{T_0(\ell)} + \lambda_K$ is bounded. The conversion of $\tilde{\vartheta}^3 \tilde{g}^3 \rho^{\mathfrak{a}}$ to $\tilde{\vartheta}_+^3 \tilde{g}_+^3 \rho_+^{\mathfrak{a}}$ costs an unimportant multiple of $\rho/\rho_+ = L^{d-4}$, so this gives the desired estimate for $q = 0$, with an L-dependent constant $M_{p,0}$.

For $q \ge 1$ and arbitrary $p \ge 0$, we instead use $(\lambda_V, \lambda_K) = (1, 1)$. Then $r_1 = \|V\|_{T_0(\ell)} + \lambda_K$ is again bounded, but now $r_2 = \|K\|_{T_0(\ell)} + \lambda_K$ is $O(1)$, so we obtain an $O(1)$ bound on the norm on the right-hand side of (8.51). This completes the proof. □

8.5 | Preliminary bounds for the non-perturbative RG map

Now we turn our attention to the building blocks of Φ_+^K. The point of departure is to rewrite Φ_+^K as a sum of two terms. To do so, we first recall the definition of Q in (8.41) and define

$$\hat{V} = V - Q, \qquad \hat{K} = K - (e^{-\hat{V}} - e^{-V}), \tag{8.52}$$

and, for $X \subset \mathcal{B}(B)$ with $B \in \mathcal{B}_+$,

$$\hat{V}(B\backslash X) = \sum_{b \in \mathcal{B}(B\backslash X)} \hat{V}(b), \qquad \hat{K}^X = \prod_{b \in X} \hat{K}(b). \tag{8.53}$$

Then, exactly as in [9, (10.1.6)], basic algebra leads to

$$K_+ = \Phi_+^K(V, K, a) = S_0 + S_1 \tag{8.54}$$

where

$$S_0 = e^{u_+|B|} \mathbb{E}_+ \Big(e^{-\theta \hat{V}(B)} - e^{-U_+(B)} \Big), \tag{8.55}$$

$$S_1 = e^{u_+|B|} \sum_{X \subset \mathcal{B}(B), |X| \ge 1} \mathbb{E}_+ \theta \Big(e^{-\hat{V}(B\backslash X)} \hat{K}^X \Big). \tag{8.56}$$

Here $|X|$ denotes the number of blocks in X. In this section, we provide bounds on the constituents of S_0 and S_1.

For situations in which V and K are themselves measured using the norm $\|\cdot\|_{T_{\varphi,y}(h,\vec{\lambda})}$, to avoid confusion we use the notation V^* and K^* for the functions

$$V^* : (V, K) \mapsto V, \qquad K^* : (V, K) \mapsto K. \tag{8.57}$$

Bounds on the norms of these functions are given in [9, (9.3.9) ff.]. We write C_L for a generic L-dependent constant whose value is unimportant and may change from line to line. The real numbers u_{pt} and u_+ are respectively the vacuum energy components (constant terms) of the perturbative map Φ_{pt} and the RG map Φ_+ defined in Definitions 5.4–5.5.

Lemma 8.9. *Let $V \in \mathcal{D}$, $\lambda_V \le \tilde{g}\rho$, $b \in \mathcal{B}$, and let L be sufficiently large. Then*

$$\|V^*(b)\|_{T_{0,y}(h,\vec{\lambda})} \le \frac{11}{8} k_0^3. \tag{8.58}$$

Proof. The proof is a small modification of the proof of [9, Lemma 9.3.2] to include dimensions $d > 4$. Let $\tau = \frac{1}{2}|\varphi|^2$, so that $V(b) = g\tau^2(b) + \nu\tau(b)$. It suffices to prove that

$$\|(g\tau^2)^*(b)\|_{T_{0,y}(h,\vec{\lambda})} \le \frac{3}{8}\frac{g}{\tilde{g}} k_0^4, \qquad \|(\nu\tau)^*(b)\|_{T_{0,y}(h,\vec{\lambda})} \le k_0^4. \tag{8.59}$$

As in the bound on [9, (9.3.16)],

$$\sup_{\dot{V}\in\mathcal{V},\dot{\nu}=0} \frac{\|\dot{V}(b)\|_{T_0(h)}}{\|\dot{V}(b)\|_{T_0(\ell)}} = h^4\ell^{-4} = \ell_0^{-4} k_0^4 (\tilde{g}\rho)^{-1}, \tag{8.60}$$

so that [9, (9.3.17)] gets replaced by

$$\|(g\tau^2)^*(b)\|_{T_{0,y}(h,\vec{\lambda})} \le \Big(\frac{g}{4\tilde{g}} + \frac{\rho^{-1}\lambda_V}{\tilde{g}\ell_0^4}\Big) k_0^4 \le \frac{3g}{8\tilde{g}} k_0^4, \tag{8.61}$$

using $\lambda_V \le \tilde{g}\rho$ and large L. The bound on $\|(\nu\tau)^*(b)\|_{T_{0,y}(h,\vec{\lambda})}$ follows similarly. □

In the following, when we make reference to results in [9] proved for $d = 4$, we in fact use the result generalised to $d \ge 4$ using Lemma 8.9 instead of [9, Lemma 9.3.2]. This generalisation is possible because of the stronger assumptions $\lambda_V, \lambda_K \le \tilde{g}\rho$ (instead of $\lambda_V, \lambda_K \le \tilde{g}$ in [9]).

Remark 8.10. Henceforth we assume that the constant k_0 of (5.20) is sufficiently small depending only on n (as in [9, Proposition 10.2.1]). In particular, $k_0 \le [24(n+2)]^{-1}$. In the important constant c^{st} of (8.64), the label "st" stands for stability; the c^{st} in the exponential decay is what enables us to include the regulator in our norm. This c^{st} depends only on k_0. We choose κ small compared to c^{st}, consistent with our assumption on κ in ($\mathbf{A}_\Phi$) and the discussion below (5.24).

Lemma 8.11. *Assume ($\mathbf{A}_\Phi$) and $\lambda_V, \lambda_K \le \tilde{g}\rho$. There is a constant C and an L-dependent constant C_L such that the following hold for all $\varphi \in \mathbb{R}^n$, $b \in \mathcal{B}$, and $B \in \mathcal{B}_+$:*

(i) For $\mathfrak{h} \in \{\ell, h\}$,

$$\|V(b)\|_{T_\varphi(\mathfrak{h})} \le \begin{cases} C_L \tilde{g}\rho P_\ell^4(\varphi) & (\mathfrak{h} = \ell) \\ C P_h^4(\varphi) & (\mathfrak{h} = h), \end{cases} \tag{8.62}$$

$$\|\hat{V}(B)\|_{T_0(\mathfrak{h}_+,\vec{\lambda})} \le \begin{cases} C_L \tilde{g}_+ \rho & (\mathfrak{h} = \ell) \\ 1 & (\mathfrak{h} = h). \end{cases} \tag{8.63}$$

(ii) Let k_0 be sufficiently small. Let $t \ge 0$ and $s \in [0,1]$, There exists a constant $c^{\mathrm{st}} > 0$ (depending on k_0 only) such that

$$\|e^{-t(V^*-sQ)(b)}\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le 2^{t/4} e^{-4tc^{\mathrm{st}}|\varphi/h|^4} (\mathfrak{h} \le h), \tag{8.64}$$

$$\|e^{-t(V^*-sQ)(b)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \le (2^{t/4} e^{-4tc^{\mathrm{st}}|\varphi/h_+|^4})^{L^{-d}} \quad (\mathfrak{h}_+ \le h_+). \tag{8.65}$$

(iii) For all $\kappa \le 2c^{\mathrm{st}}$,

$$\|\hat{K}(b)\|_{T_{0,y}(\ell,\vec{\lambda})} \le C\big(\|K(b)\|_{\mathcal{W}^\kappa} + \lambda_K\big), \tag{8.66}$$

$$\|\hat{K}(b)\|_{T_{\varphi,y}(h,\vec{\lambda})} \le C\big(\|K(b)\|_{\mathcal{W}^\kappa} + \lambda_K\big)(\tilde{g}^{9/4}\rho^{\mathfrak{b}})^{-1} G(b,\varphi). \tag{8.67}$$

Proof.

(i) The inequality (8.62) follows from combining [9, Lemma 9.3.1] and (8.16), and (8.63) is proved as in [9, Lemma 9.3.4].

(ii) We first prove (8.64). For $s \in [0,1]$, it follows from the product property of the norm that

$$\|e^{-t(V^*-sQ)(b)}\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \le \|e^{-tV^*(b)}\|_{T_{\varphi,y}} e^{t\|Q(b)\|_{T_{\varphi,y}}}. \tag{8.68}$$

The first factor is bounded by the generalisation to $d \ge 4$ of [9, Proposition 10.2.1] which does hold for $\mathfrak{h} \le h$. For the second factor, we follow [9, Lemma 10.2.2] which is stated for $\mathfrak{h} \le h_+$, so we verify that the proof actually permits $\mathfrak{h} \le h$. Indeed, the proof relies on the estimate [9, (9.3.21)] which is our Lemma 8.7, and requires $\mathfrak{h} \ge \ell$ (both here and in [9]). We therefore obtain the desired bound for $\mathfrak{h} \in [\ell, h]$, but since $\|\cdot\|_{T_\varphi(\mathfrak{h})}$ is monotone in $\mathfrak{h}$, it holds also for $\mathfrak{h} \in [0,h]$.

To prove (8.65), we recall from [9, Lemma 10.2.3] that, for $\mathfrak{h}_+ \le h_+$, $t' \ge 0$, and $s \in [0,1]$,

$$\|e^{-t'(V^*-sQ)(B)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \le 2^{t'/4} e^{-4t'c^{\mathrm{st}}|\varphi/h_+|^4}. \tag{8.69}$$

We rewrite $t(V^* - sQ)(b)$ as $L^{-d}t(V^* - sQ)(B)$, and then apply (8.69) with $t' = tL^{-d}$. This gives (8.65).

(iii) The first inequality is contained in [9, (10.4.18), (10.4.19)].

For the second inequality, which incorporates our regulator, we first recall from [9, (10.4.16), (10.4.17)] that

$$\|\hat{K}\|_{T_{\varphi,y}(h,\vec{\lambda})} \le \|K^*\|_{T_{\varphi,y}(h,\vec{\lambda})} + 2P_h^4(\varphi)\|K^*\|_{T_{0,y}(h,\vec{\lambda})} \sup_{s\in[0,1]} \|e^{-(V^*-sQ)(b)}\|_{T_{\varphi,y}(h,\vec{\lambda})}. \tag{8.70}$$

We insert (8.64) into this bound, choose $\kappa \le 2c^{\mathrm{st}}$, and recall from (8.30) that $P_h^4(\varphi)G(b,\varphi) \le O(1)$. This gives

$$\begin{aligned}\|\hat{K}\|_{T_{\varphi,y}(h,\vec{\lambda})} &\le \|K^*\|_{T_{\varphi,y}(h,\vec{\lambda})} + 4P_h^4(\varphi)\|K^*\|_{T_{0,y}(h,\vec{\lambda})} G(b,\varphi)^2 \\ &\le \|K^*\|_{T_{\varphi,y}(h,\vec{\lambda})} + CG(b,\varphi)\|K^*\|_{T_{0,y}(h,\vec{\lambda})}.\end{aligned} \tag{8.71}$$

It follows as in [9, (9.3.10), (9.3.13)], together with the definition of the extended norm in (8.1), that

$$\|K^*\|_{T_{\varphi,y}(h,\vec{\lambda})} \le \|K\|_{T_\varphi(h)} + \lambda_K(\tilde{g}^{9/4}\rho^{\mathfrak{b}})^{-1} G(b,\varphi), \tag{8.72}$$

which gives

$$\|\hat{K}\|_{T_{\varphi,y}(h,\vec{\lambda})} \le \|K\|_{T_\varphi(h)} + C\|K\|_{T_0(h)} G(b,\varphi) + C\lambda_K (\tilde{g}^{9/4}\rho^{\mathfrak{b}})^{-1} G(b,\varphi). \tag{8.73}$$

By definition of the norm in (5.28), $\|K\|_{T_\varphi(h)} \le (\tilde{g}^{9/4}\rho^{\mathfrak{b}})^{-1} G(b,\varphi)\|K\|_{\mathcal{W}^\kappa}$. Since $G(b,0) = 1$, this gives the desired estimate (8.67) and completes the proof. □

The following lemma is an extension of [9, Lemma 10.2.4] which includes mass derivatives in the norm.

Lemma 8.12. *Let k_0 be sufficiently small. Assume (A$_\Phi$), $\lambda_V, \lambda_K \le \tilde{g}\rho$, and $\lambda_a = k_a^{-1}$. There exists an L-dependent constant C_L, such that for all $B \in \mathcal{B}_+$, $\mathfrak{h}_+ \le h_+$, $t \ge 0$, and $r \le p_a$,*

$$\left\| e^{-tU_+(B)} \right\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \le 2^{t/2} e^{-2tc^{\mathrm{st}}|\varphi/h_+|^4}, \tag{8.74}$$

$$\exp\left(\|u_+|B|\|_{T_{\varphi,y}(\mathfrak{h},\vec{\lambda})} \right) \le 2^{1/2}, \tag{8.75}$$

$$\left| D_a^r (u_+ - u_{\mathrm{pt}}) \right| |B| \le C_L \|K\|_{\mathcal{W}^\kappa}. \tag{8.76}$$

Proof. For (8.74), by monotonicity of the norm in $\mathfrak{h}_+$, it is sufficient to set $\mathfrak{h}_+ = h_+$. We make the decomposition $e^{-tU_+} = e^{-t(U_+ - V)} e^{-tV}$, and use [9, Proposition 10.2.1] (which assumes that k_0 is sufficiently small) to bound the norm of e^{-tV^*}. This gives

$$\begin{aligned} \|e^{-tU_+}\|_{T_{\varphi,y}(h_+,\vec{\lambda})} &\le \|e^{-t(U_+ - V^*)}\|_{T_{\varphi,y}(h_+,\vec{\lambda})} \|e^{-tV^*}\|_{T_\varphi(h_+,\vec{\lambda})} \\ &\le \exp\left[t\|U_+ - V^*\|_{T_{\varphi,y}(h_+,\vec{\lambda})} \right] 2^{t/8} e^{-8tc^{\mathrm{st}}|\varphi/h_+|^4}. \end{aligned} \tag{8.77}$$

According to the definition of U_+ in (5.14), $U_+ = \Phi_{\mathrm{pt}}(\hat{V})$ with Φ_{pt} given by (5.7). We write $U_+ - V = \Phi_{\mathrm{pt}}(\hat{V}) - \hat{V} + Q$ and obtain

$$(U_+ - \hat{V})(B) = \mathbb{E}_+[\theta\hat{V}(B) - \hat{V}(B)] - \frac{1}{2}\operatorname{Loc}\operatorname{Var}_+(\theta\hat{V}(B)). \tag{8.78}$$

By Lemma 8.3, by (8.63), and by the fact that Loc is a bounded operator $\|\cdot\|_{T_\varphi(\mathfrak{h})} \to \|\cdot\|_{T_\varphi(\mathfrak{h})}$ (see [9, (7.5.3)] with $k = 4$), we find that

$$\|(U_+ - \hat{V})(B)\|_{T_{0,y}(h_+,\vec{\lambda})} \le C\left(\frac{\mathfrak{c}_+}{h_+}\right)^2. \tag{8.79}$$

With the bound on Q from (8.43) (for which we assume that $\tilde{g}_+$ is small depending on L to accommodate the hypothesis $h_+ \ge \ell$), and by (8.16), this leads to

$$\|U_+ - V^*\|_{T_{\varphi,y}(h_+,\vec{\lambda})} \le \|U_+ - V^*\|_{T_{0,y}(h_+,\vec{\lambda})} P_{h_+}^4(\varphi) \le O(1)\big(L^{-d} + \tilde{g}^{1/2} L^{-(d-4)/2}\big) P_{h_+}^4(\varphi). \tag{8.80}$$

This produces a small multiple of $P_{h_+}^4(\varphi)$ in the exponent, and serves to reduce the 8 in (8.77) to (say) 2. This completes the proof of (8.74).

To obtain (8.75), we simply use

$$\|u_+\|_{T_{\varphi,y}(h_+,\vec{\lambda})} \le \|U_+ - V^*\|_{T_{0,y}(h_+,\vec{\lambda})}. \tag{8.81}$$

Finally, for (8.76), since $R_+^U(V,K) = \Phi_+^U(V,K) - \Phi_+^U(V,0)$, it follows from the Mean Value Theorem and the case $(p,q) = (0,1)$ of Theorem 5.6 (with its constant $M_{0,1}$) that

$$|D_a^r(u_+ - u_{\mathrm{pt}})||B| \le \|D_a^r R_+^U(V,K)\|_{\mathcal{U}_+} \le M_{0,1}\|K\|_{\mathcal{W}^\kappa}. \tag{8.82}$$

This completes the proof. □

9 | THE NON-PERTURBATIVE RG MAP: PROOF OF THEOREM 5.7

In this section, we prove Theorem 5.7, which we restate here as Theorem 9.1 for easy reference, along with its assumption ($\mathbf{A}_\Phi$):

($\mathbf{A}_\Phi$) Let $\tilde{a} \ge 0$, let L be sufficiently large, and let κ be sufficiently small (the latter depending only on n). At scale j, let $\tilde{g} = \tilde{g}_j$ be sufficiently small depending on L, and let $(V,K) \in \mathbb{D} = \mathbb{D}_j$.

Theorem 9.1. *Let $d \ge 4$. At scale $j \in \{0,\dots,N-1\}$, assume ($\mathbf{A}_\Phi$) and let $r \in \{0,\dots,p_a\}$. There exists a positive constant C_{RG} such that Φ_+^K is well-defined as a map $\mathbb{D} \to \mathcal{W}^\kappa$ (i.e., the integral (5.15) converges), and there exist positive $M_{p,q}$ (for integers $p,q \ge 0$) and L-independent $M'_{0,1}$ such that*

$$\|D_V^p D_K^q D_a^r \Phi_+^K\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} \le \begin{cases} C_{\mathrm{RG}}\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{a}} & (p=0,\ q=0) \\ M_{p,0}\tilde{\vartheta}_+^3\tilde{g}_+^{3-p}\rho_+^{\mathfrak{a}-p} & (p>0,\ q=0) \\ M'_{0,1}L^{-\max\{2,(d-4)\mathfrak{a}\}} & (p=0,\ q=1) \\ M_{p,q}\tilde{g}_+^{-p-\frac{9}{4}(q-1)}\rho_+^{-p-\mathfrak{b}(q-1)} & (p\ge 0,\ q\ge 1,\ (p,q)\ne(0,1)). \end{cases} \tag{9.1}$$

We also repeat from (5.41) the definitions:

$$\mathfrak{a}(d) = \begin{cases} 3 & (4 \le d < 6) \\ \frac{2d-6}{d-4}(1-\varepsilon(d)) & (d \ge 6), \end{cases} \qquad \mathfrak{b}(d) = \begin{cases} \mathfrak{a}(d) - \frac{3}{4} & (4 \le d < 12) \\ \frac{3}{2}(1-\varepsilon(d)) & (d \ge 12), \end{cases} \tag{9.2}$$

with $\varepsilon(d) \in (0, \frac{1}{d-3})$. These obey

$$\mathfrak{a}(d) \in (2,3] \qquad \mathfrak{a}(d) - \mathfrak{b}(d) \in (1/2, 3/4], \tag{9.3}$$

and for $d \ge 5$,

$$\mathfrak{a}(d) < \frac{2d-6}{d-4}, \qquad \mathfrak{a}(d) - \mathfrak{b}(d) < \frac{d}{2(d-4)}. \tag{9.4}$$

9.1 | Structure of proof

Recall the decomposition (8.54), which states that

$$K_+ = \Phi_+^K(V, K, a) = S_0 + S_1, \tag{9.5}$$

where

$$S_0 = e^{u_+|B|}\mathbb{E}_+\Big(e^{-\theta\hat{V}(B)} - e^{-U_+(B)}\Big), \tag{9.6}$$

$$S_1 = e^{u_+|B|}\sum_{X\subset\mathcal{B}(B),|X|\geq 1}\mathbb{E}_+\theta\Big(e^{-\hat{V}(B\setminus X)}\hat{K}^X\Big), \tag{9.7}$$

with $|X|$ the number of blocks in X.

We follow the strategy of [9, Section 10.1], and prove Theorem 9.1 using the following three propositions. These propositions improve [9, Lemma 10.1.4], [9, Lemma 10.1.5], and the crucial contraction (the $(p,q) = (0,1)$ case of [9, (10.1.2)]), by replacing the $\mathcal{W}$-norm by the $\mathcal{W}^\varkappa$-norm and including dimensions $d > 4$. We repeatedly use the fact that $(\mathbf{A}_\Phi)$ implies

$$\|K\|_{\mathcal{W}^\varkappa} \leq C_{\mathrm{RG}}\tilde{\mathfrak{d}}^3\tilde{g}^3\rho^a \tag{9.8}$$

where the important constant C_{RG} is specified below in (9.13). In the extended norm (8.1), we always take $\lambda_a = k_a^{-1} = (L^dL^{2j})^{-1}$ as in (8.4) and fix a finite number $p_a \geq 2$ of mass derivatives, but we will vary our choices of λ_V and λ_K.

Proposition 9.2. *Assume $(\mathbf{A}_\Phi)$ and suppose that $\lambda_V, \lambda_K \leq \tilde{g}\rho$. Then*

$$\|S_0\|_{T_{0,y}(\ell_+,\vec{\lambda})} \leq C_L\tilde{\mathfrak{d}}_+^3(\tilde{g}_+\rho_+)^3, \tag{9.9}$$

$$\|S_0\|_{T_y^{G_+}(h_+,\vec{\lambda})} \leq C_L\tilde{\mathfrak{d}}_+^3(\tilde{g}_+\rho_+)^{3/4}. \tag{9.10}$$

Remark 9.3. The constant C_{RG} occurs in the definition of the domain $\mathbb{D}$ that is part of $(\mathbf{A}_\Phi)$, so C_L in Proposition 9.2 potentially depends on C_{RG}. We therefore cannot use C_L to define C_{RG}. We avoid any such circularity as follows. When we specialise Proposition 9.2 by setting $K = 0$ on the left-hand sides of its two bounds, the constant C_L that occurs for this restricted case cannot depend on C_{RG} since there is no K-dependence. We write this L-dependent constant as C_{pt}, which satisfies

$$\|S_0(V, K=0)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \leq C_{\mathrm{pt}}\tilde{\mathfrak{d}}_+^3(\tilde{g}_+\rho_+)^3, \tag{9.11}$$

$$\|S_0(V, K=0)\|_{T_y^{G_+}(h_+,\vec{\lambda})} \leq C_{\mathrm{pt}}\tilde{\mathfrak{d}}_+^3(\tilde{g}_+\rho_+)^{3/4}. \tag{9.12}$$

Then we define

$$C_{\mathrm{RG}} = 4p_a!C_{\mathrm{pt}}, \tag{9.13}$$

where p_a is the number of mass derivatives included in our norm.

Proposition 9.4. *Assume* ($\mathbf{A}_\Phi$) *and suppose that* $\lambda_V \le \tilde{g}\rho$ *and* $\lambda_K \le \tilde{g}^{9/4}\rho^{\mathfrak{b}}$. *Then*

$$\|S_1\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} \le C_L \tilde{\vartheta}_+^3 (\tilde{g}_+^3 \rho^{\mathfrak{a}} + \lambda_K). \tag{9.14}$$

As we show below, Propositions 9.2 and 9.4 imply Theorem 9.1 except for (p,q) equal to (0,0) or (0,1). The case $(p,q) = (0,1)$ provides the crucial contraction of the RG map, and the case $(p,q) = (0,0)$ is proved using this contraction. The crucial contraction requires special attention and we isolate it now as the following proposition, which, as we will see in the proof of Theorem 9.1, is slightly stronger than its counterpart in Theorem 9.1. It does not use the extended norm.

Proposition 9.5. *Assume* ($\mathbf{A}_\Phi$). *There is an* L*-independent* $M'_{0,1}$ *such that, uniformly in* $(V,K,a) \in \mathbb{D}$ *and for* $0 \le r \le p_a$,

$$\|D_a^r D_K \Phi_+^K\|_{\mathbb{D}\to\mathcal{W}^{\kappa}_+} \le M'_{0,1} \max\{L^{-(2d-6)}, L^{-\frac{d}{2}-(d-4)\mathfrak{b}}\}. \tag{9.15}$$

Our restrictions on the powers $\mathfrak{a}$ and $\mathfrak{b}$ are used in the following proof. As usual, throughout this section C_L denotes a generic L-dependent constant whose value may change from one line to another.

Proof of Theorem 9.1. Consider first the cases $(p,q) \notin \{(0,0),(0,1)\}$. By [9, 9.2.8)],

$$\|D_V^p D_K^q D_a^r \Phi_+^K\|_{\mathbb{D}\to\mathcal{W}^{\kappa}_+} \le \frac{p!q!r!}{\lambda_V^p \lambda_K^q} \|\Phi_+^K\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} \tag{9.16}$$

(the absence of a factor λ_a^r is as in 8.51). We bound $\Phi_+^K = S_0 + S_1$ by bounding S_0 and S_1 separately using Propositions 9.2 and 9.4.

For $q = 0$ and $p \ge 1$, we apply (9.16) with $(\lambda_V, \lambda_K) = (\tilde{g}\rho, 0)$, together with

$$\begin{aligned}
\|\Phi_+^K\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} &\le \|S_0\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} + \|S_1\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} \\
&\le C_L \tilde{\vartheta}_+^3 \Big[(\tilde{g}_+\rho_+)^3 + \tilde{g}_+^{9/4} \rho_+^{\mathfrak{b}} (\tilde{g}_+\rho_+)^{3/4} + \tilde{g}_+^3 \rho_+^{\mathfrak{a}} \Big] \\
&\le C_L \tilde{\vartheta}_+^3 \tilde{g}_+^3 \rho_+^{\mathfrak{a}}.
\end{aligned} \tag{9.17}$$

The above holds also for $p = 0$, but it is not sufficient for this case of (9.1) because it does not produce the constant C_{RG}. For $q > 0$, we apply (9.16) with $(\lambda_V, \lambda_K) = (\tilde{g}\rho, \tilde{g}^{9/4}\rho^{\mathfrak{b}})$, together with

$$\begin{aligned}
\|\Phi_+^K\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} &\le \|S_0\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} + \|S_1\|_{\mathcal{W}^{\kappa}_{y,+}(\vec{\lambda})} \\
&\le C_L \tilde{\vartheta}_+^3 \Big[(\tilde{g}_+\rho_+)^3 + \tilde{g}_+^{9/4} \rho_+^{\mathfrak{b}} (\tilde{g}_+\rho_+)^{3/4} + \tilde{g}_+^3 \rho_+^{\mathfrak{a}} + \lambda_K \Big] \\
&\le C_L \tilde{\vartheta}_+^3 \tilde{g}_+^{9/4} \rho_+^{\mathfrak{b}}.
\end{aligned} \tag{9.18}$$

For $(p,q) = (0,1)$, by Proposition 9.5 it suffices to observe that

$$\max\{L^{-(2d-6)}, L^{-\frac{d}{2}-(d-4)\mathfrak{b}}\} \le L^{-\max\{2,(d-4)\mathfrak{a}\}}. \tag{9.19}$$

Indeed, this is an equality for $d = 4$, while for $d > 4$ our assumptions (9.4) on $\mathfrak{a}, \mathfrak{b}$ imply that $\min\{2d-6, \frac{d}{2} + (d-4)\mathfrak{b}\} > (d-4)\mathfrak{a}$. Thus Proposition 9.5 is slightly stronger than what is required for $d > 4$.

Finally, we consider the remaining case $(p, q) = (0, 0)$, whose proof uses the crucial contraction. We make the decomposition

$$\|D_a^r \Phi_+^K(V,K)\|_{\mathcal{W}_+^\kappa} \le \|D_a^r \Phi_+^K(V,0)\|_{\mathcal{W}_+^\kappa} + \|D_a^r(\Phi_+^K(V,K) - \Phi_+^K(V,0))\|_{\mathcal{W}_+^\kappa}, \tag{9.20}$$

and bound each term separately. For the first term in (9.20), by definition $S_1(V,0) = 0$, so it follows from Proposition 9.2, Remark 9.3 and the definition of the norm in (8.3) that

$$\begin{aligned}\|D_a^r \Phi_+^K(V,0)\|_{\mathcal{W}_+^\kappa} &\le r! C_{\mathrm{pt}}[\tilde{\vartheta}_+^3(\tilde{g}_+\rho_+)^3 + \tilde{g}_+^{9/4}\rho_+^{\mathfrak{b}}\tilde{\vartheta}_+^3(\tilde{g}_+\rho_+)^{3/4}] \\ &\le \frac{1}{2} C_{\mathrm{RG}}[\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^3 + \tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{b}+3/4}] \\ &\le \frac{1}{2} C_{\mathrm{RG}}\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{a}},\end{aligned} \tag{9.21}$$

where in the last line we used $\mathfrak{a} \le \min\{3, \mathfrak{b} + 3/4\}$. For the second term in (9.20), by the Fundamental Theorem of Calculus, the bound on $D_K\Phi_+^K$ in Proposition 9.5, and the bound on K from (9.8),

$$\begin{aligned}\|D_a^r(\Phi_+^K(V,K) - \Phi_+^K(V,0))\|_{\mathcal{W}_+^\kappa} &\le \|K\|_{\mathcal{W}^\kappa} \sup_{t\in[0,1]} \|D_a^r D_K \Phi_+^K(V,tK)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} \\ &\le C_{\mathrm{RG}}\tilde{\vartheta}^3\tilde{g}^3\rho^{\mathfrak{a}} M'_{0,1} \max\{L^{-(2d-6)}, L^{-\frac{d}{2}-(d-4)\mathfrak{b}}\}.\end{aligned} \tag{9.22}$$

It suffices now to verify that the right-hand side of (9.22) is bounded above by $\frac{1}{2}C_{\mathrm{RG}}\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{a}}$. This requires some bookkeeping in order to advance the scale. For $\tilde{\vartheta}$ and $\tilde{g}$, the advancement to $\tilde{\vartheta}_+$ and $\tilde{g}_+$ is at the cost of an L-independent constant. For $d = 4$, in which case $\rho = 1$, this cost is overcome by the L^{-2} on the right-hand side of (9.22) with L taken sufficiently large (recall that $M'_{0,1}$ is independent of L) to reduce the right-hand side of (9.22) to $\frac{1}{2}C_{\mathrm{RG}}\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^{\mathfrak{a}}$. For $d > 4$, we need the powers of L on the right-hand side of (9.22) also to overcome the ratio $(\rho/\rho_+)^{\mathfrak{a}} = L^{(d-4)\mathfrak{a}}$. For this, it is sufficient if we have both

$$2d - 6 > (d-4)\mathfrak{a} \qquad \text{and} \qquad \frac{d}{2} + (d-4)\mathfrak{b} > (d-4)\mathfrak{a}. \tag{9.23}$$

Both of these are guaranteed by (9.4), and the proof is complete. □

In the remainder of Section 9, we present the proofs of Propositions 9.2, 9.4, 9.5. We start in Section 9.2 with the short proof of the bound on S_0, Proposition 9.2. The bound on S_1, Proposition 9.4, is obtained in Section 9.3. In Section 9.4, we obtain the crucial contraction estimate, Proposition 9.5, which relies on a careful decomposition of S_1 and the computation of its K-derivative. Throughout, we rely on the estimates of Section 8.5.

9.2 | Bound on S_0: Proof of Proposition 9.2

Proof of Proposition 9.2. Assume ($\mathbf{A}_\Phi$) and suppose that $\lambda_V, \lambda_K \le \tilde{g}\rho$. We proceed as in [9, Section 10.3.2] with the important improvement that our norm includes the regulator, which is absent in [9].

We start with the formula [9, (10.3.10)], which states that

$$S_0 = e^{-U_+(B)+u_+|B|}\Big(\frac{1}{8}\big(\operatorname{Var}_+ \theta\hat{V}(B)\big)^2 + \mathbb{E}_+ A_3(B)\Big), \tag{9.24}$$

where $A_3(B) = -\frac{1}{2}(\delta\hat{V}(B))^3 \int_0^1 e^{-t\delta\hat{V}(B)}(1-t)^2 dt$ with $\delta\hat{V} = \theta\hat{V}(B) - U_+(B)$. It suffices to prove that

$$\|e^{-U_+(B)+u_+|B|}\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \le 2e^{-2c^{\mathrm{st}}|\varphi/h_+|^4}, \tag{9.25}$$

$$\|\operatorname{Var}_+ \theta\hat{V}(B)\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \le \begin{cases} C_L\tilde{\vartheta}_+^2\tilde{g}_+^2\rho_+^2 P_{\ell_+}^4(\varphi) & (\mathfrak{h}_+ = \ell_+) \\ C_L\tilde{\vartheta}_+^2\tilde{g}_+\rho_+ P_{h_+}^4(\varphi) & (\mathfrak{h}_+ = h_+), \end{cases} \tag{9.26}$$

$$\|e^{-U_+(B)+u_+|B|}\mathbb{E}_+ A_3\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \le \begin{cases} C_L\tilde{\vartheta}_+^3\tilde{g}_+^3\rho_+^3 P_{\ell_+}^{12}(\varphi) & (\mathfrak{h}_+ = \ell_+) \\ C_L\tilde{\vartheta}_+^3\tilde{g}_+^{3/4}\rho_+^{3/4} G_+(B,\varphi) & (\mathfrak{h}_+ = h_+). \end{cases} \tag{9.27}$$

Indeed, for $\mathfrak{h}_+ = h_+$ the product of the exponential factor in (9.25) and the polynomial factor P_{h_+} in (9.26) combine to produce a regulator, since, as in (8.30),

$$P_{h_+}^4(\varphi)e^{-2c^{\mathrm{st}}|\varphi/h_+|^4} \le CG_+(B,\varphi). \tag{9.28}$$

For $\mathfrak{h}_+ = \ell_+$, to prove (9.9) we only need to consider $\varphi = 0$ and no role is played by $P_{\ell_+}(0) = G_+(B,0) = 1$.

The bound (9.25) follows from Lemma 8.12. Also, by (8.63) and Lemma 8.3, the bound (9.26) holds for $\varphi = 0$, and hence it also holds in general by Lemma 8.2 because $\operatorname{Var}_+ \theta\hat{V}(B)$ is a polynomial of degree four.

For (9.27), we observe that

$$\begin{aligned}
&\|e^{-U_+(B)+u_+|B|}\mathbb{E}_+ A_3\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \\
&\quad\le \frac{1}{6}\sup_{t\in[0,1]}\Big\|\mathbb{E}_+\Big(\delta\hat{V}(B)^3 e^{-(U_+-u_+)(B)-t\delta\hat{V}(B)}\Big)\Big\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \\
&\quad\le C_L\sup_{t\in[0,1]}\sup_{b_1,b_2,b_3\in B}\Big\|\mathbb{E}_+\delta V(b_1)\delta V(b_2)\delta V(b_3)\prod_{b\in B} e^{-(U_+-u_+)(b)-t\delta\hat{V}(b)}\Big\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})} \\
&\quad\le C_L\sup_{t\in[0,1]}\sup_{b'\in B}\Big(\mathbb{E}_+\|\delta\hat{V}(b')\|^6_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})}\prod_{b\in B}\|e^{-(U_+-u_+)(b)-t\delta\hat{V}(b)}\|^2_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})}\Big)^{1/2},
\end{aligned} \tag{9.29}$$

where we used $\delta\hat{V}(B) = \sum_{b\in B}\delta\hat{V}(b)$ for the second inequality and Lemma 8.1 for the third inequality. The exponent in the product over b on the right-hand side of (9.29) can be rewritten as

$$-(U_+ - u_+)(b) - t\delta\hat{V}(b) = u_+(b) + (1-t)U_+(b) - t\theta\hat{V}(b). \tag{9.30}$$

The term $u_+(b)$ is insignificant, even after taking the product over blocks b, due to (8.75). For the other terms, we use (8.74) with $t' = (1-t)L^{-d}$, and also (8.65), to see that

$$\begin{aligned}&\|e^{-(1-t)U_+(b)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})}\|e^{-t\theta\hat{V}(b)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})}\\&\quad\le\big(2^{(1-t)L^{-d}/2}e^{-2(1-t)L^{-d}c^{\mathrm{st}}|\varphi/h_+|^4}\big)\big(2^{tL^{-d}/4}e^{-4tc^{\mathrm{st}}L^{-d}|(\varphi+\zeta_b)/h_+|^4}\big).\end{aligned}\tag{9.31}$$

The product over b of the above expression is bounded above by a multiple of

$$e^{-2(1-t)c^{\mathrm{st}}|\varphi/h_+|^4}e^{-4tc^{\mathrm{st}}L^{-d}\sum_b|(\varphi+\zeta_b)/h_+|^4}.\tag{9.32}$$

Since $L^{-d}h_+^{-4}\ge\frac{1}{2}h^{-4}$, it follows from the almost sure bound of Lemma 8.4 that the above two factors combine in an upper bound to produce the regulator $G_+^4(B,\varphi)$ (for κ small compared to c^{st}). This can be factored out of the expectation and after the square root it gives $G_+^2(B,\varphi)$. What remains is the square root of

$$\mathbb{E}_+\|\delta\hat{V}(b')\|^6_{T_{\varphi,y}(\mathfrak{h}_+,\vec{\lambda})}\le C\big(\tfrac{\mathfrak{c}_+}{\mathfrak{h}_+}\big)^6\|V(b')\|^6_{T_{0,y}(\mathfrak{h}_+)}P^{24}_{\mathfrak{h}_+}(\varphi)\,\mathbb{E}_+P^{24}_{\ell_+}(\zeta_b),\tag{9.33}$$

where we used an adaptation of [9, Proposition 7.6.4] for the inequality. The factor $\mathbb{E}_+P^{24}_{\ell_+}(\zeta_b)$ is bounded by a constant by [9, Lemma 10.3.1]. By (8.62) and the definitions of $\mathfrak{c}_+$ (recall 8.18) and h_+, the combination $(\frac{\mathfrak{c}_+}{\mathfrak{h}_+})^6\|V(b)\|^6_{T_{0,y}(\mathfrak{h}_+)}$ gives $(\tilde{\mathfrak{d}}_+\tilde{\mathfrak{g}}_+\rho_+)^6$ for $\mathfrak{h}_+=\ell_+$ and gives $(\tilde{\mathfrak{d}}_+\tilde{\mathfrak{g}}_+^{1/4}\rho_+^{1/4})^6$ for $\mathfrak{h}_+=h_+$ which, after taking the square root, is what appears in (9.27). Finally, if $\mathfrak{h}_+=h_+$ then we use one of our two regulators to control $P^{12}_{h_+}(\varphi)G_+(B,\varphi)\le C$, whereas if $\mathfrak{h}_+=\ell_+$ then we neglect the regulators and simply conserve the polynomial $P^{12}_{\ell_+}(\varphi)$. This completes the proof. □

9.3 | Bound on S_1: Proof of Proposition 9.4

Proof of Proposition 9.4. For $|X|\ge 1$, we define

$$F(B,X)=e^{u_+|B|}\mathbb{E}_+\theta\big(e^{-\hat{V}(B\setminus X)}\hat{K}^X\big)\tag{9.34}$$

so that $S_1=\sum_{X\subset B,|X|\ge1}F(B,X)$. We first prove the estimates

$$\|F(B,X)\|_{T_{0,y}(\ell_+,\vec{\lambda})}\le C_L\big(\tilde{\mathfrak{d}}_+^3\tilde{\mathfrak{g}}_+^3\rho^{\mathfrak{a}}+\lambda_K\big)^{|X|},\tag{9.35}$$

$$\|F(B,X)\|_{T_{\varphi,y}(h_+,\vec{\lambda})}\le C_LG_+(B,\varphi)\big(\tilde{\mathfrak{d}}^3\tilde{\mathfrak{g}}_+^{3/4}\rho^{\mathfrak{a}-\mathfrak{b}}+\lambda_K(\tilde{\mathfrak{g}}^{9/4}\rho^{\mathfrak{b}})^{-1}\big)^{|X|}.\tag{9.36}$$

By (8.75) the norm of $\exp(u_+|B|)$ is bounded by a constant so we only have to bound the norm of $\mathbb{E}_+\theta(e^{-\hat{V}(B\setminus X)}\hat{K}^X)$.

We start with (9.36). Since $\hat{V}=V^*-Q$, and since $L^{-d}h_+^{-4}\ge\frac{1}{2}h^{-4}$, (8.64) gives

$$\|e^{-\hat{V}(b)}\|_{T_{\varphi,y}(h_+,\vec{\lambda})}\le CG(b,\varphi)\qquad(b\in\mathcal{B}).\tag{9.37}$$

This bound, together with (8.37), (8.67), and our hypothesis that $\|K\|_{\mathcal{W}^\kappa} \le C_{\mathrm{RG}} \tilde{\mathfrak{d}}^3 \tilde{g}^3 \rho^{\mathfrak{a}}$, imply

$$\begin{aligned}\|\mathbb{E}_+ \theta e^{-\hat{V}(B\setminus X)} \hat{K}^X\|_{T_{\varphi,y}(h_+,\vec{\lambda})} &\le C_L G_+(B,\varphi) \prod_{b\in B\setminus X} \|e^{-\hat{V}(b)}\|_{T_y^G(h_+,\vec{\lambda})} \prod_{b\in X} \|\hat{K}(b)\|_{T_y^G(h_+,\vec{\lambda})} \\ &\le C_L G_+(B,\varphi)(\tilde{\mathfrak{d}}^3 \tilde{g}^{3/4} \rho^{\mathfrak{a}-\mathfrak{b}} + \lambda_K (\tilde{g}^{9/4}\rho^{\mathfrak{b}})^{-1})^{|X|}. \end{aligned} \tag{9.38}$$

Thus we have (9.36). Similarly, use of (8.36) instead of (8.37) gives

$$\begin{aligned}\|\mathbb{E}_+ \theta e^{-\hat{V}(B\setminus X)} \hat{K}^X\|_{T_{0,y}(\ell_+,\vec{\lambda})} &\le C_L \prod_{b\in B\setminus X} \|e^{-V(b)}\|_{\mathcal{W}^0_{y,+}} \prod_{b\in X} \|\hat{K}(b)\|_{\mathcal{W}^0_{y,+}} \\ &\le C_L (\tilde{\mathfrak{d}}^3 \tilde{g}^3 \rho^{\mathfrak{a}} + \lambda_K)^{|X|}, \end{aligned} \tag{9.39}$$

where we used (8.65) and (8.66) in the second inequality. This gives (9.35).

Next, we bound the sum $S_1(B) = \sum_{X\in\mathcal{B}(B),|X|\ge 1} F(B,X)$. For this, we give a name to factors occurring on the right-hand sides of (9.35)–(9.36):

$$f(\mathfrak{h}_+) = \begin{cases} \tilde{\mathfrak{d}}^3 \tilde{g}^3 \rho^{\mathfrak{a}} + \lambda_K & (\mathfrak{h}_+ = \ell_+) \\ \tilde{\mathfrak{d}}^3 \tilde{g}^{3/4} \rho^{\mathfrak{a}-\mathfrak{b}} + \lambda_K (\tilde{g}^{9/4}\rho^{\mathfrak{b}})^{-1} & (\mathfrak{h}_+ = h_+). \end{cases} \tag{9.40}$$

Then

$$\|S_1(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \le C_L \sum_{X\subset\mathcal{B}(B),\,|X|\ge 1} f(\ell_+)^{|X|} = C_L\big((1+f(\ell_+))^{|\mathcal{B}(B)|} - 1\big). \tag{9.41}$$

Since $\lambda_K \le \tilde{g}^{9/4}\rho^{\mathfrak{b}} \le 1$, we have $f(\ell_+) \le 2$, so there is a constant C_L such that

$$(1+f(\ell_+))^{|\mathcal{B}(B)|} - 1 \le e^{L^d f(\ell_+)} - 1 \le C_L f(\ell_+), \tag{9.42}$$

and hence

$$\|S_1(B)\|_{T_{0,y}(\ell_+,\vec{\lambda})} \le C_L f(\ell_+). \tag{9.43}$$

A similar computation applies with $\|\cdot\|_{T_{0,y}(\ell_+,\vec{\lambda})}$ replaced by $\|\cdot\|_{T_{\varphi,y}(h_+,\vec{\lambda})}$, but with a factor $G_+(B,\varphi)$, so that

$$\|S_1(B)\|_{T_{\varphi,y}(h_+,\vec{\lambda})} \le C_L G_+(B,\varphi) f(h_+). \tag{9.44}$$

Finally, we bundle the bounds (9.41) and (9.44) into a bound on the $\mathcal{W}^\kappa_{y,+}(\vec{\lambda})$ norm (defined by 8.3). The result is the desired conclusion (9.14). □

9.4 | Contraction of Φ^K_+: Proof of Proposition 9.5

We now prove Proposition 9.5, which is the contraction estimate that is crucial for the repeated iteration of the RG map when started at or near the critical point. Our goal is to prove that the K-derivative of $\Phi^K_+(V,K)$ is bounded above by an L-independent multiple of the maximum of

$L^{-(2d-6)}$ and $L^{-\frac{d}{2}-(d-4)\mathfrak{b}}$ (each is L^{-2} when $d=4$). We again use the decomposition

$$K_+ = \Phi_+^K(V,K,a) = S_0 + S_1 \tag{9.45}$$

of (8.54). In the following lemma we see that the K-derivative of S_0 is negligible in the sense that it can be made smaller than any L-dependent constant (upon choosing $\tilde{g}$ small enough).

Lemma 9.6. *Let ε_L be any small L-dependent constant and assume ($\mathbf{A}_\Phi$). Then for $0 \le r \le p_a$,*

$$\|D_a^r D_K S_0\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} \le \varepsilon_L. \tag{9.46}$$

Proof. By Proposition 9.2 with the choice $\lambda_K = \tilde{g}\rho$, and by using the norm to bound the K-derivative as in (9.16),

$$\|D_a^r D_K S_0\|_{\mathcal{W}_+^\kappa} \le C_L \lambda_K^{-1} \tilde{\vartheta}_+^3 \tilde{g}_+^3 \rho_+^3 = C_L \tilde{\vartheta}_+^3 \tilde{g}_+^2 \rho_+^2. \tag{9.47}$$

Since we assume that $\tilde{g}$ is small (depending on L), and since $\tilde{g}_+ \le \tilde{g}$ by the recursion and Theorem 5.6, we have the desired conclusion. □

9.4.1 | Extraction of linear term

It remains to bound the K-derivative of S_1. For this, we extract from S_1 its linear term in K, as in [9, (10.5.9)–(10.5.10)], namely

$$S_1 = e^{u_{\mathrm{pt}}|B|}\mathbb{E}_+\theta\mathcal{L}K + \mathcal{E}, \tag{9.48}$$

with

$$\mathcal{L}K(B) = \sum_{b\in\mathcal{B}(B)} \Big(e^{-V(B)}(1-\mathrm{Loc})(e^{V(b)}K(b)) \Big) \tag{9.49}$$

and with $\mathcal{E}$ then implicitly defined. By definition, S_1 is zero when K is zero, as is $\mathcal{E}$.

While the optimal bound on $D_K\mathbb{E}_+\theta\mathcal{L}K$ with correct L dependence requires careful treatment, the contraction of $\mathcal{E}$ is relatively easy because $\mathcal{E}$ comprises terms that are at least second order in K. The bounds we need are summarised in the following lemmas, whose proofs are given respectively in Sections 9.4.2 and 9.4.3. As we show below the statement of the lemmas, they quickly imply Proposition 9.5.

Lemma 9.7. *Assume ($\mathbf{A}_\Phi$) and $\|\dot{K}\|_{\mathcal{W}_j^\kappa} < \infty$. There is an L-independent constant $C > 0$ such that, for $0 \le r \le p_a$,*

$$\|D_a^r\mathbb{E}_+\theta\mathcal{L}\dot{K}(B)\|_{\mathcal{W}_+^\kappa} \le C\max\{L^{-(2d-6)}, L^{-\frac{d}{2}-(d-4)\mathfrak{b}}\}\|\dot{K}\|_{\mathcal{W}^\kappa}. \tag{9.50}$$

Lemma 9.8. *Assume ($\mathbf{A}_\Phi$). For $0 \le r \le p_a$, the Fréchet derivative of $D_a^r\mathcal{E}(V,\cdot) : \mathcal{W}^\kappa \to \mathcal{W}_+^\kappa$ vanishes at $K = 0$.*

Proof of Proposition 9.5. We seek a bound on the norm of $D_K\Phi_+^K$. By the Fundamental Theorem of Calculus,

$$\begin{aligned}\|D_a^r D_K \Phi_+^K(V,K)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} \\ \le \|D_a^r D_K \Phi_+^K(V,0)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} + \sup_{t\in[0,1]} \|D_a^r D_K^2 \Phi_+^K(V,tK)\|_{\mathbb{D}^2\to\mathcal{W}_+^\kappa} \|K\|_{\mathcal{W}^\kappa}.\end{aligned} \tag{9.51}$$

For the second term we apply Theorem 9.1 with $(p,q)=(0,2)$ (recall that this follows from Propositions 9.2 and 9.4, so our argument is not circular here), and obtain

$$\|D_a^r D_K^2 \Phi_+^K(V,tK)\|_{\mathbb{D}^2\to\mathcal{W}_+^\kappa} \|K\|_{\mathcal{W}^\kappa} \le M_{0,2}\tilde{g}_+^{-9/4}\rho_+^{-\mathfrak{b}} C_{\mathrm{RG}}\tilde{g}^3\rho^{\mathfrak{a}} \le O_L\big(\tilde{g}^{3/4}\rho^{\mathfrak{a}-\mathfrak{b}}\big). \tag{9.52}$$

The right-hand side is smaller than any small fixed L-dependent constant, for $\tilde{g}$ sufficiently small. For the first term in (9.51) we use $\Phi_+ = S_0 + S_1$, Lemma 9.6, and Lemma 9.8, to see that

$$\begin{aligned}\|D_a^r D_K \Phi_+^K(V,0)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} &\le \|D_a^r D_K S_0(V,0)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} + \|D_a^r D_K S_1(V,0)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} \\ &\le \varepsilon_L + \|D_a^r D_K e^{u_{\mathrm{pt}}|B|}\mathbb{E}_+\theta\mathcal{L}\dot{K}(B)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa},\end{aligned} \tag{9.53}$$

where ε_L is as small as desired. Since $\mathcal{L}K$ is linear in K, the Fréchet K-derivative of $\mathbb{E}_+\theta\mathcal{L}K$ in direction $\dot{K}$ is simply $\mathbb{E}_+\theta\mathcal{L}\dot{K}$. Then, since u_{pt} does not depend on K, and using the estimates (8.75)–(8.76) to bound $e^{u_{\mathrm{pt}}|B|}$, together with Lemma 9.7, we obtain finally that

$$\|D_a^r D_K \Phi_+^K(V,K)\|_{\mathbb{D}\to\mathcal{W}_+^\kappa} \le C\max\{L^{-(2d-6)}, L^{-\frac{d}{2}-(d-4)\mathfrak{b}}\}, \tag{9.54}$$

once we take $\varepsilon_L \le \max\{L^{-(2d-6)}, L^{-\frac{d}{2}-(d-4)\mathfrak{b}}\}$. This completes the proof. □

9.4.2 | Contraction of $\mathcal{L}$: Proof of Lemma 9.7

The following lemma from [9] is at the heart of the proof of contraction. It is used to implement the idea that $(1-\mathrm{Loc})F$ contains only irrelevant contributions which contract under change of scale. Note that this good operator $1-\mathrm{Loc}$ appears explicitly in (9.49).

Lemma 9.9 [9, (10.5.11)]. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be $O(n)$-invariant and $\mathfrak{h} \ge \mathfrak{h}' > 0$. Then*

$$\|(1-\mathrm{Loc})F\|_{T_\varphi(\mathfrak{h}')} \le 2\Big(\frac{\mathfrak{h}'}{\mathfrak{h}}\Big)^6 P_{\mathfrak{h}'}(\varphi)^6 \sup_{0\le t\le 1}\|F\|_{T_{t\varphi}(\mathfrak{h})}. \tag{9.55}$$

We decompose the operator $\mathcal{L}$ from (9.49) as

$$\mathcal{L}\dot{K} = \sum_{q=0}^{2}\frac{1}{q!}\mathcal{L}_1(V^q\dot{K}) + \mathcal{L}_2\dot{K}, \tag{9.56}$$

where

$$\mathcal{L}_1(V^q\dot{K})(B) = \sum_{b\in\mathcal{B}(B)}\Big(e^{-V(B)}(1-\mathrm{Loc})(V^q\dot{K})(b)\Big), \tag{9.57}$$

$$\mathcal{L}_2\dot{K}(B) = \sum_{b\in\mathcal{B}(B)} \Big(e^{-V(B)}(1-\mathrm{Loc})\Big[\big(e^V - 1 - V - \frac{1}{2}V^2\big)\dot{K}\Big](b)\Big). \tag{9.58}$$

(This decomposition is different from that in [9, (10.5.16), (10.5.17)].) We prove contraction estimates for $\mathcal{L}_1(V^q\dot{K})$ and $\mathcal{L}_2\dot{K}$ in Lemmas 9.10 and 9.11, respectively. These estimates then imply Lemma 9.7, as we show at the end of this section. The next two lemmas, Lemmas 9.10–9.11, involve extensions of and small corrections to [9, Section 10.5]. The extensions are improvements due to the inclusion of our regulator in the norm.

Lemma 9.10. *Assume* ($\mathbf{A}_\Phi$) *and* $\|\dot{K}\|_{\mathcal{W}^\varkappa} < \infty$. *There is an* L*-independent* C *such that for* $0 \le r \le p_a$ *and for bounded* $q \ge 0$,

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1(V^q\dot{K})(B)\|_{T_0(\ell_+)} \le CL^{-(2d-6)}\|\dot{K}\|_{\mathcal{W}^\varkappa}, \tag{9.59}$$

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1(V^q\dot{K})(B)\|_{T^{G_+}(h_+)} \le CL^{-d/2}\|\dot{K}\|_{T^G(h)}. \tag{9.60}$$

Proof. By the triangle inequality and the definition of $\mathcal{L}_1$,

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1(V^q\dot{K})(B)\|_{T_\varphi(\mathfrak{h}_+)} \le L^d \sup_b \|D_a^r \mathbb{E}_+ \theta e^{-V(B)}(1-\mathrm{Loc})(V^q\dot{K})(b)\|_{T_\varphi(\mathfrak{h}_+)}. \tag{9.61}$$

We write the integrand on the right-hand side as a product over blocks in B, and apply Lemma 8.1. This eliminates the mass derivative and yields

$$\begin{aligned}&\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1(V^q\dot{K})(B)\|_{T_\varphi(\mathfrak{h}_+)} \\ &\quad\le O(L^d)\sup_b \Big[\mathbb{E}_+\Big(\prod_{b'} \|e^{-V(b')}\|^2_{T_{\varphi+\zeta_{b'}}(\mathfrak{h}_+)} \|(1-\mathrm{Loc})(V^q\dot{K})(b)\|^2_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)}\Big)\Big]^{1/2}.\end{aligned} \tag{9.62}$$

The potentially dangerous entropic factor L^d is more than cancelled by good powers of L arising from $1-\mathrm{Loc}$. To see this, we first note that it follows from (9.55) that

$$\|(1-\mathrm{Loc})(V^q\dot{K})(b)\|_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)} \le O\Big(\frac{\mathfrak{h}_+}{\mathfrak{h}}\Big)^6 P^6_{\mathfrak{h}_+}(\varphi+\zeta_b) \sup_{t\in[0,1]} \|\dot{K}(b)\|_{T_{t(\varphi+\zeta_b)}(\mathfrak{h})}, \tag{9.63}$$

where we used Lemma 8.11(i) to bound the norm of V^q by an L-independent constant. Also, by (8.65) (with $\varkappa \le 2c^{\mathrm{st}}$)

$$\|e^{-V(b')}\|_{T_{\varphi+\zeta_{b'}}(\mathfrak{h}_+)} \le 2^{L^{-d}} G^2(b', \varphi+\zeta_{b'}). \tag{9.64}$$

Since $(\frac{\ell_+}{\ell})^6 = O(L^{-3(d-2)})$ and $(\frac{h_+}{h})^6 = O(L^{-3d/2})$, this leads to

$$\begin{aligned}&\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1 \dot{K}(B)\|_{T_\varphi(\mathfrak{h}_+)} \\ &\quad\le O(L^{-p(\mathfrak{h},\mathfrak{h}_+)}) \sup_b \Big[\mathbb{E}_+\Big(\prod_{b'} G(b', \varphi+\zeta_{b'})^4 P^{12}_{\mathfrak{h}_+}(\varphi+\zeta_b) \sup_{t\in[0,1]} \|\dot{K}(b)\|^2_{T_{t(\varphi+\zeta_b)}(\mathfrak{h})}\Big)\Big]^{1/2},\end{aligned} \tag{9.65}$$

where $p(\ell, \ell_+) = 2d-6$ and $p(h, h_+) = d/2$.

For the case $\mathfrak{h} = h$, since $\|\dot{K}\|_{T_{t(\varphi+\zeta_b)}(h)} \le \|\dot{K}\|_{T^G(h)}$ by definition of the norm, it follows that

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1 \dot{K}(B)\|_{T_\varphi(h_+)} \le O(L^{-d/2}) \|\dot{K}\|_{T^G(h)} \sup_b \Big[\mathbb{E}_+\Big(\prod_{b'} G(b', \varphi+\zeta_{b'})^4 P_{h_+}^{12}(\varphi+\zeta_b)\Big)\Big]^{1/2}$$
$$\le O(L^{-d/2}) \|\dot{K}\|_{T^G(h)} G_+(B,\varphi), \tag{9.66}$$

where the second inequality follows from Lemma 8.5. This proves (9.60).

Finally, for the case $\mathfrak{h} = \ell$ with $\varphi = 0$, we bound all regulators by 1. Also, by (8.35) (at scale j),

$$\sup_{t\in[0,1]} \|\dot{K}\|_{T_{t\zeta_b}(\ell)} \le P_\ell^{10}(\zeta_b) \|\dot{K}\|_{\mathcal{W}^0} \le P_{\ell_+}^{10}(\zeta_b) \|\dot{K}\|_{\mathcal{W}^\kappa}. \tag{9.67}$$

Then (9.65) becomes

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_1 \dot{K}(B)\|_{T_0(\ell_+)} \le O(L^{-(2d-6)}) \|\dot{K}\|_{\mathcal{W}^\kappa} \Big[\mathbb{E}_+ P_{\ell_+}^{32}(\zeta_b)\Big]^{1/2}. \tag{9.68}$$

Since the expectation of $P_{\ell_+}^{32}(\zeta_b)$ is bounded by [9, Lemma 10.3.1], we obtain (9.59) and the proof is complete. □

Lemma 9.11. *Assume* (**A**$_\Phi$) *and* $\|\dot{K}\|_{\mathcal{W}^\kappa} < \infty$. *There is an* L*-independent* C *such that for* $0 \le r \le p_a$,

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_2 \dot{K}(B)\|_{T_0(\ell_+)} \le C L^{-(2d-6)} \|\dot{K}\|_{\mathcal{W}^\kappa}, \tag{9.69}$$

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_2 \dot{K}(B)\|_{T^{G_+}(h_+)} \le C L^{-d/2} \|\dot{K}\|_{T^G(h)}. \tag{9.70}$$

Proof. Let $F_V = (e^V - 1 - V - V^2/2)$. Since the Taylor expansion of F_V in φ vanishes up to degree five, $1 - \mathrm{Loc}$ acts as the identity on $F_V \dot{K}$ and $e^{-V} F_V$, and thus

$$\mathcal{L}_2 \dot{K}(B) = \sum_{b\in\mathcal{B}(B)} e^{-V(B)} F_V(b) \dot{K}(b)$$
$$= \sum_{b\in\mathcal{B}(B)} e^{-V(B\setminus b)} \dot{K}(b) (1-\mathrm{Loc})\big(e^{-V(b)} F_V(b)\big). \tag{9.71}$$

By the triangle inequality and Lemma 8.1,

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_2 \dot{K}(B)\|_{T_\varphi(\mathfrak{h}_+)}$$
$$\le O(L^d) \sup_b \Big[\mathbb{E}_+\Big(\|\dot{K}(b)\|^2_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)} \|(1-\mathrm{Loc}) e^{-V(b)} F_V(b)\|^2_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)} \prod_{b'\neq b} \|e^{-V(b')}\|^2_{T_{\varphi+\zeta_{b'}}(\mathfrak{h}_+)}\Big)\Big]^{1/2}. \tag{9.72}$$

By Lemma 9.9 and Lemma 8.11(i,ii),

$$\|(1-\mathrm{Loc}) e^{-V} F_V\|_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)} \le 2\Big(\frac{\mathfrak{h}_+}{\mathfrak{h}}\Big)^6 P_{\mathfrak{h}_+}^6(\varphi+\zeta_b) \sup_{t\in[0,1]} \|e^{-V} F_V\|_{T_{t(\varphi+\zeta_b)}(\mathfrak{h})}$$
$$\le O(L^{-d-p(\mathfrak{h},\mathfrak{h}_+)}) P_{\mathfrak{h}_+}^6(\varphi+\zeta_b), \tag{9.73}$$

for either $(\mathfrak{h},\mathfrak{h}') = (\ell,\ell_+)$ or (h,h_+), where again $p(\ell,\ell_+) = 2d-6$ and $p(h,h_+) = d/2$. With (9.64), this gives

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_2 \dot{K}(B)\|_{T_\varphi(\mathfrak{h}_+)}$$
$$\le O(L^{-p(\mathfrak{h},\mathfrak{h}_+)}) \sup_b \Big[\mathbb{E}_+\Big(\|\dot{K}\|^2_{T_{\varphi+\zeta_b}(\mathfrak{h}_+)} P^{12}_{\mathfrak{h}_+}(\varphi+\zeta_b) \prod_{b'\neq b} G^4(b',\varphi+\zeta_{b'})\Big)\Big]^{1/2}. \tag{9.74}$$

For $\mathfrak{h} = h$, it follows from $h_+ \le h$ and the monotonicity of the $T_\varphi(\mathfrak{h})$-norm in $\mathfrak{h}$ that $\|\dot{K}\|_{T_{\varphi+\zeta_b}(h_+)} \le \|\dot{K}\|_{T_{\varphi+\zeta_b}(h)} \le \|\dot{K}\|_{T^G(h)} G(b,\varphi+\zeta_b)$, so

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L}_2 \dot{K}(B)\|_{T_\varphi(h_+)}$$
$$\le O(L^{-p(h,h_+)}) \|\dot{K}\|_{T^G(h)} \sup_b \Big[\mathbb{E}_+\Big(G(b,\varphi+\zeta_b)^2 P^{12}_{h_+}(\varphi+\zeta_b) \prod_{b'\neq b} G(b',\varphi+\zeta_{b'})^4\Big)\Big]^{1/2}. \tag{9.75}$$

The supremum on the right-hand side is at most $G_+(B,\varphi)$ by Lemma 8.5, so this proves (9.70).

For $\mathfrak{h} = \ell$ and $\varphi = 0$, we apply (8.35) to see that

$$\|\dot{K}\|_{T_{\zeta_b}(\ell_+)} \le \|\dot{K}\|_{T_{\zeta_b}(\ell)} \le P^{10}_\ell(\zeta_b)\|\dot{K}\|_{\mathcal{W}^0} \le P^{10}_\ell(\zeta_b)\|\dot{K}\|_{\mathcal{W}^\kappa}. \tag{9.76}$$

After substitution of this into in (9.74), we bound all regulators by 1, and then the expectation is bounded. This proves (9.69). □

Proof of Lemma 9.7. By definition of the $\mathcal{W}^\kappa$ norm in (8.3),

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L} \dot{K}(B)\|_{\mathcal{W}^\kappa_+} = \|D_a^r \mathbb{E}_+ \theta \mathcal{L} \dot{K}(B)\|_{T_0(\ell_+)} + \tilde{g}_+^{9/4} \rho_+^{\mathfrak{b}} \|D_a^r \mathbb{E}_+ \theta \mathcal{L} \dot{K}(B)\|_{T^G_+(h_+)}. \tag{9.77}$$

With the decomposition (9.56), Lemmas 9.10 and Lemma 9.11 imply that there is an L-independent C such that

$$\|D_a^r \mathbb{E}_+ \theta \mathcal{L} \dot{K}(B)\|_{\mathcal{W}^\kappa_+} \le C\big(L^{-(2d-6)} + L^{-d/2}(\tilde{g}_+/\tilde{g})^{9/4}(\rho_+/\rho)^{\mathfrak{b}}\big)\|\dot{K}\|_{\mathcal{W}^\kappa}. \tag{9.78}$$

Since $\tilde{g}_+/\tilde{g} \le 1$ and $\rho_+/\rho = L^{-(d-4)}$, this gives the desired result with its maximum of $L^{-(2d-6)}$ and $L^{-\frac{d}{2}-(d-4)\mathfrak{b}}$. □

9.4.3 | Vanishing derivative of $\mathcal{E}$: Proof of Lemma 9.8

Let

$$A(b) = e^{-V(b)}\big(1 + Q(b) - e^{Q(b)}\big), \qquad Q(b) = \mathrm{Loc}(e^{V(b)}K(b)). \tag{9.79}$$

It follows from the definition of $\mathcal{E}$ in (9.48) and basic algebra, exactly as in [9, (10.5.10)], that

$$\mathcal{E} = \Delta_1 + \Delta_2 + \Delta_3, \tag{9.80}$$

with

$$\Delta_1 = (e^{u_+|B|} - e^{u_{\mathrm{pt}}|B|})\mathbb{E}_+\theta\mathcal{L}K, \tag{9.81}$$

$$\Delta_2 = e^{u_+|B|}\sum_{b\in\mathcal{B}(B)}\mathbb{E}_+\theta\big(e^{-\hat{V}(B\setminus b)}A(b)\big), \tag{9.82}$$

$$\Delta_3 = e^{u_+|B|}\sum_{X\subset\mathcal{B}(B),\,|X|\geq 2}\mathbb{E}_+\theta\big(e^{-\hat{V}(B\setminus X)}\hat{K}^X\big). \tag{9.83}$$

Although the formula for $\mathcal{E}$ appears complicated, only crude estimates are needed.

Proof of Lemma 9.8. Since $\mathcal{E} = 0$ when $K = 0$, it suffices to prove that

$$\|D_a^r\mathcal{E}(V,K)\|_{\mathcal{W}_+^\kappa} \leq C\|K\|_{\mathcal{W}^\kappa}^2 \tag{9.84}$$

for $\|K\|_{\mathcal{W}^\kappa}$ sufficiently small. Since we are letting K tend to zero with L and $\tilde{g}$ held fixed, we can and do permit the above constant C to depend (badly) on L and $\tilde{g}$, and we use C throughout the proof as a generic such constant. For (9.84), it suffices to prove that, for $i \in \{1,2,3\}$,

$$\|D_a^r\Delta_i(V,K)\|_{T_0(\ell_+)} \leq C\|K\|_{\mathcal{W}^\kappa}^2, \tag{9.85}$$

$$\|D_a^r\Delta_i(V,K)\|_{T^{G_+}(h_+)} \leq C\|K\|_{\mathcal{W}^\kappa}\|K\|_{T^G(h)}. \tag{9.86}$$

We proceed as in the proof of [9, (10.5.29)–(10.5.30)], with adaptations in particular to accommodate our regulator.

For Δ_1, we see from (9.59) that $\|D_a^r\mathbb{E}_+\theta\mathcal{L}K\|_{T_0(\ell_+)} \leq C\|K\|_{\mathcal{W}^\kappa}$ and $\|D_a^r\mathbb{E}_+\theta\mathcal{L}K\|_{T^{G_+}(h_+)} \leq C\|K\|_{T^G(h)}$. Also, (8.75)–(8.76) give $|e^{u_+|B|} - e^{u_{\mathrm{pt}}|B|}| \leq C\|K\|_{\mathcal{W}^\kappa}$. This proves the Δ_1 case of (9.85)–(9.86).

For Δ_2, we first apply Lemma 8.6 and then use (8.65) to see that

$$\|D_a^r\Delta_2(V,K)\|_{T_0(\ell_+)} \leq C\sup_b\|A(b)\|_{\mathcal{W}_+^0}\prod_{b'\neq b}\|e^{-\hat{V}(b')}\|_{\mathcal{W}_+^0} \leq C\sup_b\|A(b)\|_{\mathcal{W}_+^0}, \tag{9.87}$$

$$\|D_a^r\Delta_2(V,K)\|_{T^{G_+}(h_+)} \leq C\sup_b\|A(b)\|_{T^G(h_+)}\prod_{b'\neq b}\|e^{-\hat{V}(b')}\|_{T^G(h_+)} \leq C\sup_b\|A(b)\|_{T^G(h_+)}, \tag{9.88}$$

for sufficiently small $\tilde{g}$. Note that the regulator is G in (9.88), not G_+. By Taylor's Theorem,

$$A(b) = -e^{-V(b)}Q(b)^2\int_0^1(1-s)e^{sQ(b)}ds, \tag{9.89}$$

so

$$\|A(b)\|_{T_\varphi(\mathfrak{h}_+)} \leq \|A(b)\|_{T_\varphi(\mathfrak{h})} \leq \|Q(b)\|_{T_\varphi(\mathfrak{h})}^2\sup_{s\in[0,1]}\|e^{-(V-sQ)(b)}\|_{T_\varphi(\mathfrak{h})}. \tag{9.90}$$

By [9, (9.3.19)], $\|Q(b)\|_{T_\varphi(\mathfrak{h})} \leq C\|K(b)\|_{T_0(\mathfrak{h})}P_{\mathfrak{h}}^4(\varphi)$. With (8.64) for the exponential factor, this gives

$$\|A(b)\|_{T_{\varphi(\mathfrak{h}_+)}} \leq C\|K(b)\|_{T_0(\mathfrak{h})}^2P_{\mathfrak{h}}^8(\varphi)e^{-4c^{\mathrm{st}}|\varphi/h|^4} \leq C\|K(b)\|_{T_0(\mathfrak{h})}^2e^{-2c^{\mathrm{st}}|\varphi/h|^4}. \tag{9.91}$$

Thus for $\varkappa \le 2c^{\mathrm{st}}$,

$$\|A(b)\|_{T_0(\ell_+)} \le C\|K\|^2_{T_0(\ell)}, \qquad \|A(b)\|_{T^G(h_+)} \le C\|K\|^2_{T_0(h)}. \tag{9.92}$$

This proves the required bounds on Δ_2, since we are not concerned with the factor $(\tilde{g}\rho^{\mathfrak{b}})^{-1}$ required to bound $\|K\|_{T_0(h)}$ by $\|K\|_{\mathcal{W}^\varkappa}$.

Finally, Δ_3 is similar to Δ_2, but with $A(b)$ on the right-hand sides of (9.87)–(9.88) now replaced by $\hat{K}^X$ with $|X| \ge 2$. Since $\|K\|_{\mathcal{W}^0_+} \le \|K\|_{\mathcal{W}^0} \le \|K\|_{\mathcal{W}^\varkappa}$, the analogue of (9.87) gives the $i = 3$ version of (9.85). Since $\|K\|_{T^G(h_+)} \le \|K\|_{T^G(h)}$, the analogue of (9.88) gives the $i = 3$ version of (9.86). The sum over X has an L-dependent number of terms, but this does not matter, and the sum is dominated by $|X| = 2$ as $K \to 0$. Also, the factor $(\tilde{g}\rho^{\mathfrak{b}})^{-1}$ needed to convert the $T^G(h)$ norm to the $\mathcal{W}^\varkappa$ norm does not matter. This proves the Δ_3 case of (9.86) and completes the proof. □

ACKNOWLEDGEMENTS

We thank Tom Hutchcroft for a preliminary version of [47] and for discussions about boundary conditions for the hierarchical lattice. We are grateful to Roland Bauerschmidt for discussions, to Yucheng Liu for comments concerning Appendix B, and to Tim Garoni for comments on a preliminary draft. We are grateful to the late Ralph Kenna for comments on a preliminary draft and for advice concerning the physics literature. The work of E.M. and G.S. was supported in part by NSERC of Canada. J.P. was partially supported by the European Research Council under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 851682 SPINRG), and gratefully acknowledges the support and hospitality of the University of British Columbia and of the Pacific Institute for the Mathematical Sciences in Vancouver during part of this work.

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APPENDIX A: MONOTONICITY OF THE UNIVERSAL PROFILE f_n

The universal profile

$$f_n(s) = \frac{\int_{\mathbb{R}^n} |x|^2 e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx}{n \int_{\mathbb{R}^n} e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx} = \frac{I_{n+1}(s)}{n I_{n-1}(s)} \qquad (n > 0,\ s \in \mathbb{R}) \tag{A.1}$$

is introduced in Section 1.5.2. It extends to $n \in [-2, \infty)$ using the recursion (1.66). Plots of f_n are given in Figure A.1.

Lemma A.1 proves elementary monotonicity properties of the universal profile f_n. For its statement we define $s_n^* = -\infty$ if $n \geq 0$, and $s_n^* = \sup\{s : (n+2) f_{n+2}(s) + s = 0\}$ for $n \in [-2, 0)$. The divergence of f_n at s_n^* for $n \in [-2, 0)$, visible in Figure A.1, follows from (1.66).

Lemma A.1. *For $n \in [-2, \infty)$ and $s > s_n^*$, the universal profile $f_n(s)$ is strictly decreasing both as a function of s and as a function of n.*

Proof. We first prove monotonicity in s. For $n = 0, -2$, the functions $f_0(s) = I_1(s)$ and $f_{-2}(s) = s^{-1}$ are monotone in s by definition. For $k > 0$, we define the probability measure $d\mu_{k,s}(x) \propto x^{k-1} e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2} dx$ on $(0, \infty)$. Then for $n > 0$ we have $n f_n(s) = \int_0^\infty x^2 d\mu_{n,s}$, so

$$2n \frac{d}{ds} f_n(s) = -\int x^4 d\mu_{n,s} + \left(\int x^2 d\mu_{n,s} \right)^2. \tag{A.2}$$

Notice that, for any strictly increasing functions f and g,

$$\mu_{k,s}(fg) - \mu_{k,s}(f)\mu_{k,s}(g) = \frac{1}{2} \iint d\mu_{k,s}(x) d\mu_{k,s}(y) \big(f(x) - f(y)\big)\big(g(x) - g(y)\big) > 0, \tag{A.3}$$

(this is the FKG inequality with strict inequality) so we obtain $\frac{d}{ds} f_n(s) < 0$. For $n \in (-2, 0)$, by (1.66) it is sufficient to prove that

$$\frac{d}{ds}(n+2) f_{n+2}(s) > -1 \quad \text{if } s > s_n^*. \tag{A.4}$$

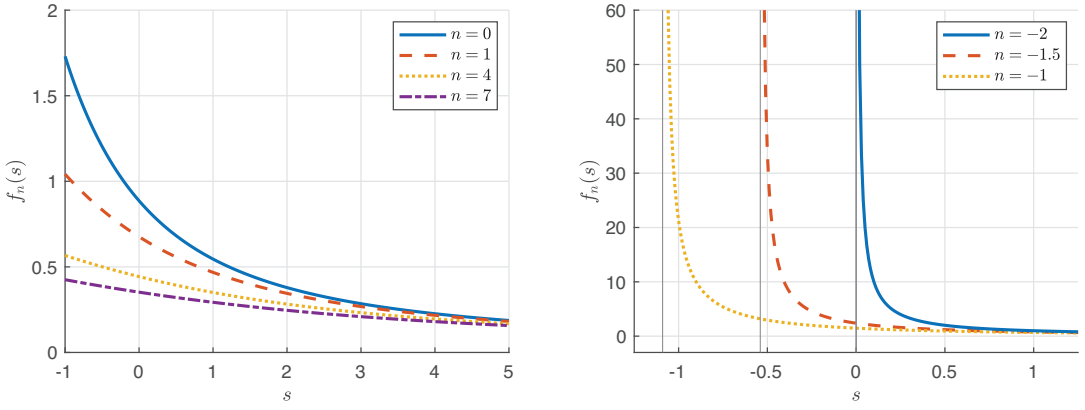

FIGURE A.1 Plots of the universal profile f_n.

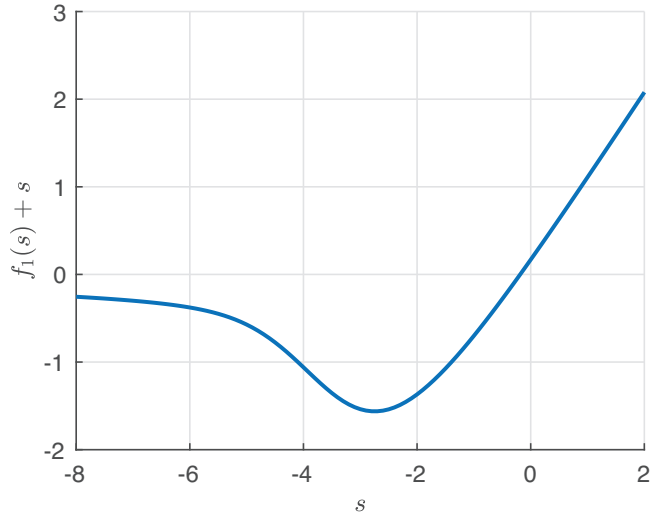

FIGURE A.2 Plot of $f_1(s)+s$ illustrating lack of convexity of $f_1(s)+s$ and hence of $f_1(s)$.

By (A.2) and (1.62), we have

$$2(n+2)\frac{d}{ds}f_{n+2}(s) = -\frac{I_{n+5}(s)}{I_{n+1}(s)} + \left(\frac{I_{n+3}(s)}{I_{n+1}(s)}\right)^2$$
$$= -(n+2) + s(n+2)f_{n+2}(s) + \big((n+2)f_{n+2}(s)\big)^2. \tag{A.5}$$

Since $s+(n+2)f_{n+2}(s) > 0$ for $s > s_n^*$, we see that indeed $(n+2)\frac{d}{ds}f_{n+2}(s) > -\frac{n+2}{2} > -1$.

Finally, for the monotonicity in n, for $k > -1$ and $q > 0$ we have

$$\frac{d}{dk}\frac{I_{q+k}(s)}{I_k(s)} = \int x^q \log x \, d\mu_{k,s} - \left(\int x^q d\mu_{k,s}\right)\left(\int \log x \, d\mu_{k,s}\right) > 0 \tag{A.6}$$

where the 1-dimensional integral $I_k(s)$ is defined in (1.60), and the final inequality follows from (A.3). (Note that this proves that the moments $\Sigma_{n,k}(s)$ in (1.33) are monotone *increasing* in $n > 0$.) Therefore, by (1.63) and by (A.6), for all $n > -2$ we have

$$\frac{d}{dn}\frac{1}{f_n(s)} = \frac{d}{dn}\left(\frac{I_{n+3}(s)}{I_{n+1}(s)} + s\right) > 0. \tag{A.7}$$

This proves the desired monotonicity in n, and the proof is complete. □

A cursory examination of Figure A.1 may suggest that f_n is convex but this is misleading: Figure A.2 shows that $s \mapsto f_1(s)+s$ is not convex and hence neither is f_1.

APPENDIX B: SELF-AVOIDING WALK: PROOF OF PROPOSITIONS 1.5–1.6

In this section we prove Propositions 1.5–1.6, which we restate here as Propositions B.1–B.2.

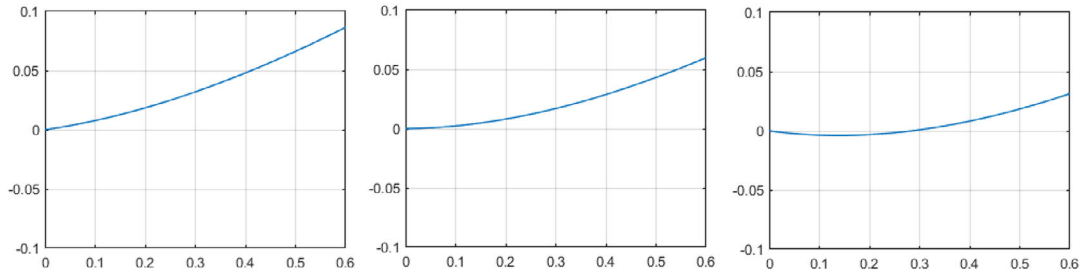

FIGURE B.1 Effective potential V versus t for $g = 1$ and $\nu = -1.1, -1.2, -1.3$. The critical value is close to -1.2.

Proposition B.1. *For self-avoiding walk on the complete graph, as $N \to \infty$,*

$$\chi_N^{\mathbb{K}}(N^{-1}(1 - s(2N)^{-1/2})) \sim (2N)^{1/2} f_0(s) \qquad (s \in \mathbb{R}). \tag{B.1}$$

Proof. It is proved in [64, Theorem 1.1] on the basis of an exact rewriting of the susceptibility in terms of the incomplete Gamma function, that, as $N \to \infty$,

$$\chi_N^{\mathbb{K}}(N^{-1}(1 - s(2N)^{-1/2})) \sim N^{1/2} \sqrt{\frac{\pi}{2}} e^{s^2/4} \operatorname{erfc}(s/2), \tag{B.2}$$

where the complementary error function is defined for $x \in \mathbb{R}$ by $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$. With the change of variable $y = x^2/2$ in the definition of I_1 in (1.60), we see that

$$f_0(s) = I_1(s) = e^{\frac{1}{4}s^2} \int_0^\infty e^{-(y+s/2)^2} dy = \frac{\sqrt{\pi}}{2} e^{s^2/4} \operatorname{erfc}(s/2). \tag{B.3}$$

This proves (B.1). □

A detailed analysis of self-interacting walk on the complete graph is given in [11]. Although the emphasis in [11] is on elucidating the phase diagram for a model with tricritical behaviour, much of [11] applies more generally and in particular to the continuous-time weakly self-avoiding walk. The critical point $\nu_c = \nu_c(g)$ is characterised in [11, Definition 1.2] as follows. First, the effective potential $V : [0, \infty) \to \mathbb{R}$ is defined by

$$V(t) = t - \log(1 + v(t)), \qquad v(t) = \int_0^\infty e^{-gs^2 - \nu s} e^{-s} \sqrt{\frac{t}{s}} \tilde{I}_1(2\sqrt{st})\, ds. \tag{B.4}$$

The critical point is then the choice of ν_c for which $V_c'(0) = 0$, $V_c''(0) > 0$ (the subscript c denotes evaluation at ν_c). The function $\tilde{I}_1$ is the modified Bessel function of the first kind (the tilde is used to avoid confusion with I_1 of 1.60), and $V(0) = 0$ since $\tilde{I}_1(0) = 0$. See Figure B.1 for an illustration.

In the following theorem, we apply the results of [11] to prove that the profile $f_0(s)$ arises for the susceptibility in this model in its critical window. Note that the profile is independent of the value of $g > 0$, as we have also seen for $n \geq 1$ in Corollary 1.4.

Proposition B.2. *There are positive constants λ_1, λ_2 such that*

$$\lim_{N\to\infty} N^{-1/2}\chi_N^{\mathrm{WSAW}}(\nu_c + sN^{-1/2}) = \lambda_1 f_0(\lambda_2 s) \qquad (s \in \mathbb{R}). \tag{B.5}$$

Proof. At the critical point ν_c, it is proved in [11, Theorem 1.3] that $G_{00}(\nu_c) \to 1$ as $N \to \infty$, whereas $G_{01}(\nu_c)$ is of order $N^{-1/2}$. Thus, at the critical point, G_{00} does not contribute to the leading order of χ, which is $\chi(\nu_c) \sim NG_{01}$. We now extend this to the scaling window $\nu(s) = \nu_c + sN^{-1/2}$.

The combination of [11, (3-5)] and [11, (3-17)] gives the formula

$$G_{01}(\nu) = \int_0^\infty e^{-NV(t)}(NV'(t)F(t,t) - \partial_1 F(t,t))\,dt, \tag{B.6}$$

with $F(t_1,t_2) = (1 - V'(t_1))^2 t_2$. Integration by parts then gives

$$\begin{aligned} G_{01}(\nu) &= \int_0^\infty e^{-NV(t)}\Big(\frac{d}{dt}F(t,t) - \partial_1 F(t,t)\Big)\,dt \\ &= \int_0^\infty e^{-NV(t)}\partial_2 F(t,t)\,dt \\ &= \int_0^\infty e^{-NV(t)}(1 - V'(t))^2 dt. \end{aligned} \tag{B.7}$$

(This integration by parts was not done in [11]; its use would simplify some computations in [11].) We use prime and dot to denote derivatives with respect to t and ν, respectively. By Taylor expansion in the variable s, as $N \to \infty$ we have

$$V_{\nu(s)}(t) = V_c(t) + \dot{V}_c(t)sN^{-1/2} + O(s^2N^{-1}). \tag{B.8}$$

The integral on the right-hand side of (B.7) is dominated by its behaviour near $t = 0$ because V grows at least linearly as $t \to \infty$ by [11, Proposition 1.1], so we may restrict the integral to a finite interval and use (B.8) and [59, Ex. 4.1, p.333] to evaluate the asymptotic form of G_{01} in the window. In the notation of [59, Ex. 4.1, p.333], and with (B.8), we have

$$\begin{aligned} &p(t) \sim Pt^2, \quad s(t) = St, \quad r(x,t) = O(1), \quad q(N,t) = (1 - V_c'(t))^2 \sim 1, \\ &P = \frac{1}{2}V_c''(0), \quad S = -\dot{V}_c'(0)s, \quad \mu = 2, \quad \sigma = 1, \quad \lambda = 1. \end{aligned} \tag{B.9}$$

With Fi the Faxén integral (1.67), we conclude that

$$\chi_N^{\mathrm{WSAW}}(\nu_c + sN^{-1/2}) \sim NG_{01}(\nu_c + sN^{-1/2}) \sim N\,\mathrm{Fi}(\tfrac{1}{2},\tfrac{1}{2};SP^{-1/2})\frac{1}{(PN)^{1/2}}. \tag{B.10}$$

According to (1.68), $\mathrm{Fi}(\frac{1}{2},\frac{1}{2};u) = 2I_1(-u) = 2f_0(-u)$, and therefore we arrive at

$$\chi_N^{\mathrm{WSAW}}(\nu_c + sN^{-1/2}) \sim N^{1/2}\frac{2}{(\frac{1}{2}V_c''(0))^{1/2}} f_0\Big(s\frac{\dot{V}_c'(0)}{(\frac{1}{2}V_c''(0))^{1/2}}\Big), \tag{B.11}$$

which identifies the constants λ_1, λ_2 and completes the proof. □