The Gregory–Laflamme Instability and Conservation Laws for Linearised Gravity

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To my parents, Stephen and Aileen.

Declaration

This dissertation is based on research done while I was a graduate student at the Department of Pure Mathematics and Mathematical Statistics and the Cambridge Centre for Analysis of the University of Cambridge, in the period between October 2018 and April 2022. This dissertation has not been submitted for any other degree or qualification. The thesis consists of four chapters.

Chapter 1 is my own work and not outcome of work done in collaboration. It is based on the paper:

> Sam C. Collingbourne, "The Gregory–Laflamme instability of the Schwarzschild black string exterior", J. Math. Phys. 62, 032502 (2021).

Chapter 2 contains very little original work; most of chapter 2 can be found in the literature in some form, it is included to provide the necessary prerequisites for the following chapters. Chapters 3 and 4 are my own work and not outcome of work done in collaboration. They are unpublished at the time of writing.

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Abstract

This thesis is concerned with the black hole stability problem in general relativity. In particular, it presents stability and instability results associated to the *linearised* vacuum Einstein equation on black hole backgrounds.

The first chapter of this thesis gives a direct rigorous mathematical proof of the Gregory–Laflamme instability for the 5-dimensional Schwarzschild black string. Under a choice of ansatz for the perturbation and a gauge choice, the linearised vacuum Einstein equation reduces to an ODE problem for a single function. In this work, the ODE is cast into a Schrödinger eigenvalue equation to which an energy functional is assigned. It is then shown by direct variational methods that the lowest eigenfunction gives rise to an exponentially growing mode solution which has admissible behaviour at the future event horizon and spacelike infinity. After the addition of a pure gauge solution, this gives rise to a regular exponentially growing mode solution of the linearised vacuum Einstein equation in harmonic/transverse-traceless gauge.

The remainder of this thesis is concerned with conservation laws associated to the linearised vacuum Einstein equation. For later application, chapter 2 of this thesis contains a review of the double null gauge for the vacuum Einstein equations. In chapter 3, the 'canonical energy' conservation law of Hollands and Wald is studied. This canonical energy conservation law gives an appealing criterion for stability of black holes based upon a conserved current. The method is appealing in its simplicity as it requires one to 'simply' check the sign of the canonical energy $\mathcal{E}$ with $\mathcal{E} > 0$ implying *weak* stability and $\mathcal{E} < 0$ implying instability. However, in practice establishing the sign of $\mathcal{E}$ proves difficult. Indeed, even for the 4-dimensional Schwarzschild black hole exterior the positivity was not previously established. In this thesis, a resolution to this issue for the Schwarzschild black hole is presented by connecting to another weak stability result of Holzegel which exploits the double null gauge. Further weak stability statements for the Schwarzschild black hole (including a proof of mode stability) arising from the canonical energy are also established.

In chapter 4, some preliminary results associated to a novel conserved current associated to the linearised vacuum Einstein equation are presented. This can be viewed as a modification/simplification of the conserved current associated to the canonical energy. In particular, applications of this current to other black hole spacetimes are discussed.

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Introduction

General relativity is a theory that models gravity. Mathematically, the fundamental objects in general relativity are a n-dimensional Lorentzian manifold, M, and its associated metric, g. These describe the gravitational field and dictate how matter moves. The metric is constrained to satisfy the celebrated Einstein equation:

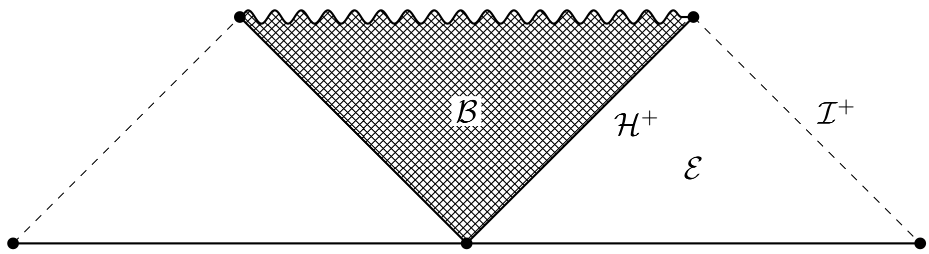
$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g = 8\pi\mathbb{T}. \tag{I.1}$$

This equation relates the geometry of the spacetime, through the combination of the Ricci and scalar curvature on the left-hand side known as the Einstein tensor, to the matter content of the theory, which is modelled with the energy-momentum tensor, $\mathbb{T}$, on the right-hand side. One can write down energy-momentum tensors for many matter models including Yang–Mills fields, scalar fields or fluids. In contrast to the Newtonian predecessor, one can even set $\mathbb{T} = 0$ to yield non-trivial solutions in the absence of matter. In this case, the reduced equation,

$$\mathrm{Ric}(g) = 0, \tag{I.2}$$

is called the *vacuum Einstein equation*.

Famously, the theory of general relativity predicts the existence of black holes, the simplest of which was written down by Schwarzschild [1] mere months after Einstein proposed the theory in 1915 [2]. Informally, these black hole solutions are characterised by the property that they contain a region of spacetime from which not even light can escape. The following 'Penrose diagram' (to which one can attach a precise mathematical meaning) provides a depiction of the geometry of the Schwarzschild black hole:



In this diagram, $\mathcal{E}$ is the exterior of the black hole which is the region in which external observers live and $\mathcal{B}$ is the black hole region which is the region from which nothing can escape to $\mathcal{E}$. The boundary between $\mathcal{B}$ and $\mathcal{E}$, denoted by $\mathcal{H}^+$, is the future event horizon and can be thought of as the point (surface) of no return for observers; once an observer crosses this boundary from $\mathcal{E}$ they cannot return to $\mathcal{E}$. The dashed line denoted $\mathcal{I}^+$ is called 'future null infinity' and is, roughly speaking, where idealised gravitational wave experiments take place. The 'wavey' line is the black hole curvature singularity where infinite tidal forces tear apart anyone who is unfortunate enough to enter the black hole region.

Remark 0.0.1. *For the more astrophysically minded reader, one can construct the Penrose diagram of a black hole arising from the gravitational collapse of a star. This can be modelled with the Oppenheimer–Snyder solution [3] to the Einstein equation which represents a homogeneous, spherically symmetric, collapsing dust star.*[a] *Outside of the star, the spacetime is vacuum and therefore, by Birkoff's theorem [4, 5], is a region of the Schwarzschild spacetime.*

The modern perspective of the vacuum Einstein equation is to view it as a system of 2^{nd}-order quasi-linear partial differential equations for the metric. The equation's type is obscured by the issue of diffeomorphism invariance; one must choose coordinates, (x^α), locally for the spacetime to determine the type of the vacuum Einstein equation. This is often called picking a 'gauge'. For example, one can show that in 'harmonic gauge', which is defined by requiring

$$\Box_g(x^\alpha) \doteq \frac{1}{\sqrt{\det(g)}}\partial_\mu\Big(\sqrt{\det(g)}g^{\mu\nu}\partial_\nu x^\alpha\Big) = 0, \tag{I.3}$$

the vacuum Einstein equation reduces to a system of quasi-linear wave equations

$$\Box_g(g_{\alpha\beta}) = \mathrm{N}_{\alpha\beta}(g, \partial g), \tag{I.4}$$

where $\mathrm{N}_{\alpha\beta}(g, \partial g)$ only depends on g and its first derivatives. Therefore, in harmonic gauge the vacuum Einstein equation is hyperbolic. The hyperbolic nature of the vacuum Einstein equation in harmonic gauge was exploited in the monumental work of Choquet–Bruhat [6] to formulate and prove that the vacuum Einstein equation admits a locally well-posed initial value problem. Global aspects of the initial value problem for the vacuum Einstein equation were subsequently formulated and proven in the work of Choquet–Bruhat–Geroch [7]. In particular, given initial data for the vacuum Einstein equation, the authors prove the existence and uniqueness (up to diffeomorphism) of a 'maximal globally hyperbolic Cauchy development'.[b]

Associated to the initial value formulation is the question of stability of solutions to the Einstein equation: suppose you start with initial data close (in a suitable norm) to initial data which would lead to a known solution, does the solution that results asymptotically approach the

[a] This solution is a weak global solution due to the discontinuity of the dust across the boundary of the star.

[b] The reader can consult [8, 9] for further details on the initial value formulation of the vacuum Einstein equation.

known solution, tend to something else or blow up? Based on many works (for example [10–35]) conducted over the last 60 years starting with the seminal works of Regge–Wheeler [36] in 1957, stability is expected to be true of all 4-dimensional (the 3 spatial and 1 time that we experience) stationary vacuum black holes.

Motivation for Higher-Dimensional Relativity

Since some of the results in this thesis are concerned with higher dimensional relativity, some motivational and expository remarks are in order (for reviews on the topic of black holes in higher dimensions see [37–39]). The convention adopted in this work is that n will denote the *spacetime* dimension.

First, from a purely mathematical perspective, it is of interest to see how general relativity differs in higher dimensions from the 4-dimensional case. This sheds light on how general Lorentzian manifolds obeying the vacuum Einstein equation behave. There are many differences in higher dimensional relativity as apposed to the usual 4-dimensional case. These are effectively due to the increased number of degrees of freedom inherent in the metric, g. In higher dimensions, many results from 4-dimensional general relativity no longer hold. A few examples are the following:

(1) *Event horizon topology does not have to be spherical*. As shown by Hawking, in 4-dimensions the cross-sections of the event horizon of an asymptotically flat stationary black hole spacetime must be homeomorphic to $\mathbb{S}^2$ (under the dominant energy condition) [40]. In $(n \geq 5)$-dimensions, cross-sectional horizon topology does not *have* to be spherical. There exist examples of black holes with spherical horizon topology such as the Myers–Perry black hole [41] (the generalisation of the Kerr solution to arbitrary dimension), which has cross-sectional horizon topology homeomorphic to $\mathbb{S}^{n-2}$. However, it is possible to construct explicit examples of asymptotically flat black hole spacetimes with non-spherical cross-sectional horizon topology. For example, the $5D$ Emparan–Reall and Pomeransky–Sen'kov black ring solutions have horizon topology $\mathbb{S}^2 \times \mathbb{S}^1$ [42]. Hawking's theorem has been generalised to higher dimensions in [43], which shows that the horizon topology of asymptotically flat black hole spacetimes must be of positive scalar curvature. In 5 dimensions a more precise result is known: under the assumptions of stationarity, asymptotic flatness, two commuting axisymmetries and a 'rod structure', the horizon topology is either $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$ or a quotient of $\mathbb{S}^3$ [44].

(2) *Naive black hole uniqueness fails*. In 4-dimensions, under the assumption of either analyticity [40] or axisymmetry [45, 46], the Kerr family is the unique family of stationary vacuum black hole solutions, i.e., a (analytic or axisymmetric) stationary vacuum black hole in 4-dimensions is uniquely specified by its mass, M, and its angular momentum per unit

mass, a. Uniqueness of the Kerr family in the class of *smooth* stationary vacuum black hole solutions is conjectured to hold but a proof seems elusive.[c] This has led to the further conjecture that maximal developments of 'generic' asymptotically flat initial data sets can asymptotically be described by a finite number of Kerr black holes. These uniqueness theorems and, therefore, this 'final state conjecture' cannot generalise immediately to higher dimensions since there exist at least two distinct families of vacuum black hole solutions that can have the same mass and angular momentum: the Myers–Perry black hole and the Emparan–Reall black ring. Moreover, there exist distinct black ring solutions with the same mass and angular momentum [37, 48]. The final state conjecture may need to be modified to include the property of stability. Again, the work [44] provides a more precise result in 5 dimensions.

(3) *Black holes in higher dimensions can be unstable*. As mentioned above, stability is expected to be true of all 4-dimensional stationary vacuum black holes. Indeed, the subextremal Kerr family is conjectured to be asymptotically stable as a solution of the vacuum Einstein equation which, in view of the uniqueness of the Kerr family mentioned in point (2), would constitute all 4-dimensional stationary vacuum black holes (see section IV.1 of [35] for a precise formulation of the Kerr stability conjecture). In stark contrast, many stationary vacuum black hole solutions in higher dimensions are expected to be unstable [37, 38, 49–69]. This is a topic that is addressed in Chapter 1 of this thesis.

Secondly, the physics community is very interested in higher-dimensional gravity from the point of view of producing a 'grand unified theory', i.e., a theory that rectifies the apparent incompatibility of general relativity and quantum field theory [39]. One of the earliest ideas of unification goes back to the 1920s to the works of Kaluza [70] and Klein [71] who produced a classical unified theory of general relativity and electromagnetism in 5 dimensions. Many of the current proposed unifying theories are also formulated in dimensions more than 4. For example, certain types of 'string theory' are formulated with 10 or 11 dimensions [72]. The belief is that we only perceive 4 out of the actual number of dimensions since the rest are sufficiently small and compact that we traverse them imperceptibly fast. For such a theory to be considered 'good', there has to be some limit which produces the previous tried and tested theory, i.e., general relativity. Therefore, understanding how general relativity behaves in higher dimensions is therefore of relevance to the low energy limit of these grand unified theories such as string theory [37, 73].

Summary of the Thesis

The research presented in this PhD thesis focuses on stability problems in general relativity. In particular, it presents a study of the linearised vacuum Einstein equation on two backgrounds:

[c]A partial result in this direction is [47] which shows that if a stationary vacuum black hole solution is 'close' to a Kerr solution then it is isometric to that Kerr solution.

the Schwarzschild black string [49] and the Schwarzschild(–Tangherlini) black hole [74].

The linearised vacuum Einstein equation can be written as an equation for a symmetric 2-tensor h:

$$g^{cd}\nabla_c\nabla_d h_{ab} + \nabla_a\nabla_b(\mathrm{Tr}_g h) - \nabla_a(\mathrm{div}h)_b - \nabla_b(\mathrm{div}h)_a + 2R_a{}^c{}_b{}^d h_{cd} = 0 \qquad \text{(I.5)}$$

where ∇ and R are the Levi-Civita connection and Riemann tensor, respectively, of the background metric g. Studying this equation on flat Minkowski spacetime led Einstein to the prediction of gravitational waves [75], which have now been confirmed by LIGO [76]. It has also been fruitful in establishing stability statements about many black hole spacetimes and is often a precursor to understanding the full 'non-linear' stability of a black hole solution to the Einstein equation. For example, the full non-linear stability of the 4-dimensional Schwarzschild spacetime has only very recently been rigorously established in the monumental work of Dafermos, Holzegel, Rodnianski and Taylor [35] which utilised 'double null gauge' (see also the work of Klainerman and Szeftel under symmetry [31]). However, the road to this result came from a deep understanding of the scalar wave equation

$$\Box_g \Psi = 0, \qquad \text{(I.6)}$$

for $\Psi \in C^\infty(M)$ on the Schwarzschild background (see [19] for a summary) and then the linearised Einstein equation [28].[d]

The first chapter of this thesis is devoted to the 'Gregory–Laflamme instability' of the 5-dimensional Schwarzschild black string. This spacetime is constructed from the Schwarzschild black hole by taking its Cartesian product with $\mathbb{R}$ or a circle of radius R. Strong numerical evidence for the existence of an admissible exponentially growing solution to the linearised vacuum Einstein equation was first given in 1993 by Gregory and Laflamme [49]. Since the original paper, this type of instability has been identified numerically [77–79] and heuristically [55, 57] for other black hole solutions in higher dimensions, such as the exotic 5-dimensional Emparan–Reall [80] and Pomeransky–Sen'kov black ring solutions [81], the $6D$ ultra-spinning Myers–Perry solution [41] and the $5D$ Kerr black string [62]. However, even for the original 5-dimensional Schwarzschild black string, there was no direct mathematical proof of the existence of the Gregory–Laflamme instability until my work [69] (see however the work of Prabu–Wald [66] to be discussed in section 1.1.5 of Chapter 1). In Chapter 1, this direct proof of the existence of the Gregory–Laflamme instability for the Schwarzschild black string is presented. Under a choice of ansatz for the perturbation and a gauge choice, the linearised vacuum Einstein equation reduces to an ODE problem for a single function. In this work, a suitable rescaling and change of variables is applied

[d]Due to its hyperbolic nature, the scalar wave equation can be viewed as the 'poor man's' linearised Einstein equation (I.5).

which casts the ODE into a Schrödinger eigenvalue equation to which an energy functional is assigned. It is then shown by direct variational methods that the lowest eigenfunction gives rise to an exponentially growing mode solution which has admissible behaviour at the future event horizon and spacelike infinity. After the addition of a pure gauge solution, this gives rise to a regular exponentially growing mode solution of the linearised vacuum Einstein equation in harmonic/transverse-traceless gauge.

The second chapter provides a detailed discussion of the double null gauge (following the works [82–84]) that is employed in the subsequent chapters of this thesis. In double null coordinates (u, v, θ^A) the metric takes the form

$$g = -2\Omega^2(du \otimes dv + du \otimes dv) + \not{g}_{AB}(d\theta^A - b^A dv) \otimes (d\theta^A - b^A dv). \tag{I.7}$$

The hypersurfaces of constant u and v are then manifestly null hypersurfaces. Therefore, the double null gauge choice is particularly well adapted to the causal structure of spacetime. Associated to such coordinates is a natural normalised double null frame

$$e_3 = \frac{1}{\Omega}\partial_u, \qquad e_4 = \frac{1}{\Omega}(\partial_v + b^A \partial_{\theta^A}). \tag{I.8}$$

Completing this frame with a (local) basis for the horizontal subspace $\langle e_3, e_4\rangle^\perp$ allows for a double null decomposition of the Ricci coefficients and the Weyl curvature tensor. The content of the vacuum Einstein equation can then be encoded in a system of elliptic and transport equations for metric and Ricci coefficients known as the null structure equations and the Bianchi identities (decomposed with respect to the null frame (I.8)). In keeping with the higher-dimensional theme of the first chapter, these equations are derived in arbitrary dimension. To the best of the author's knowledge, until this work, no complete discussion of this topic in higher dimensions has appeared in the literature.[e] This chapter contains a derivation of the linearised null structure equations around the n-dimensional Schwarzschild–Tangherlini spacetime (the higher-dimensional Schwarzschild solution) for use in Chapters 3 and 4. Further, there is some additional discussion on the failure of decoupling of the famous Teukolsky null Weyl curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ for $n > 4$. It should be stressed that the failure of decoupling for $n > 4$ is a known result and has appeared in previous literature (see works [86–88]) albeit in the slightly different higher-dimensional Geroch–Held–Penrose formalism [89]. The original work in this chapter is the discussion of obtaining a Regge–Wheeler system of equations through a physical space version of the Chandrasekhar transformation.

[e]The reader should note that these equations were schematically derived up to error terms in the work [85] which was sufficient for their purposes.

The third chapter of this thesis is concerned with the ‘canonical energy’ linear stability criterion of Hollands and Wald [65]. Their criterion is based upon a divergence-free current for the linearised vacuum Einstein equation associated with a Killing symmetry $X \in \mathfrak{X}(M)$ of the black hole in question (typically for stationary black holes X is the Killing field T associated to stationarity). Applying the divergence theorem on regions of the black hole spacetime yields a conservation law for a quantity called the canonical energy, $\mathcal{E}^X$. For a class of initial data, Hollands and Wald’s criterion is that a stationary vacuum black hole spacetime is *weakly* linearly stable or unstable if $\mathcal{E}^T$ (evaluated on a Cauchy hypersurface) is positive or negative, respectively. The criterion is appealing conceptually since one can write down a conservation law for *any* stationary vacuum black hole spacetime (or more generally a black hole with a symmetry) and one ‘simply’ needs to check the sign of $\mathcal{E}^T$ (for a particular class of data). However, in practice, it is hard to establish positivity or negativity. Indeed, even for the ‘basic’ case of the stability of the 4-dimensional Schwarzschild black hole, the positivity of $\mathcal{E}^T$ is only rectified in this thesis. To do this, the canonical energy is decomposed in a manner that is favourable for the causal structure of the spacetime: the double null gauge (as mentioned above). The conservation law for $\mathcal{E}^T$ is then understood locally and it is shown that the canonical energy conservation law is equivalent to a conservation law for another energy, $\bar{\mathcal{E}}^T$, inherent in the system of double null decomposed gravitational perturbations on $4D$ Schwarzschild established by Holzegel [90]. Since Holzegel established a weak stability statement (an energy boundedness statement) for the 4-dimensional Schwarzschild solution from the energy $\bar{\mathcal{E}}^T$ (in particular, the positivity of this energy), the same statement then follows from the canonical energy. Moreover, a hierarchy of conservation laws are derived for the system of double null decomposed gravitational perturbations from the canonical energy and the respective energy boundedness statements are studied. This hierarchy of conservation laws is then used to prove mode stability of the 4-dimensional Schwarzschild black hole. Additionally, a novel conservation law for the celebrated decoupled Teukolsky null curvature components α and $\underline{\alpha}$ is derived.

In the final chapter of this thesis, a new divergence-free current associated to the linearised vacuum Einstein equation is defined. This current can be written down on *any* vacuum spacetime with a Killing symmetry. This current can be viewed as modification of the current which gives rise to the canonical energy conservation law and is arguably simpler to compute than the canonical energy current. An application of this conservation law is then given by producing a double null decomposed conservation law on the n-dimensional Schwarzschild–Tangherlini spacetime which should be useful in proving a stability statement for this spacetime.

Conventions

The conventions of this thesis are the following:

- The metric signature convention is $(-,+,+,+,\ldots)$.
- Greek Indices $(\alpha,\beta,\gamma,\ldots)$ will be used for expressions that hold in a particular basis.
- Latin Indices $(a,b,c,\ldots)$ are *abstract indices* which will be used for expressions that hold in any basis (see [91] or [92] for more details).
- The notation $\mathfrak{X}(M)$ denotes the set of vector fields on a manifold M.
- The notation $\Omega^k(M)$ denotes the set of k-forms on a manifold M.
- The notation $\mathrm{sym}(T^\star M \otimes T^\star M)$ and $\mathrm{symtr}(T^\star M \otimes T^\star M)$ denotes the set of symmetric $(0,2)$-tensors and symmetric-traceless $(0,2)$-tensors respectively on a manifold M.
- The convention for the Riemann tensor is

  $$R(X,Y)Z \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

  for $X,Y,Z \in \mathfrak{X}(M)$. The convention for the Riemann tensor indices is

  $$R^a{}_{bcd} X^c Y^d Z^b = (R(X,Y)Z)^a.$$

Chapter 1

The Gregory–Laflamme Instability of the Schwarzschild Black String Exterior

1.1 Introduction

The main topic of this chapter is the study of the stability problem for the Schwarzschild black string solution to the Einstein vacuum equation in 5 dimensions. In 1993, the work of Gregory–Laflamme [49] gave strong numerical evidence for the presence of an exponentially growing mode instability. This phenomenon has since been known as the Gregory–Laflamme instability. This work has been widely invoked in the physics community to infer instability of many higher dimensional spacetimes, for example, black rings, ultraspinning Myers–Perry black holes and black Saturns (see [55, 57, 77–79]). For a review and introduction to instabilities in higher dimensions, the interested reader should consult [37, 39] and references therein, as well as [61] and [65, 66] which give a general approach to stability problems. The purpose of the present chapter is to provide a direct, self-contained and elementary mathematical proof of the Gregory–Laflamme instability of the $5D$ Schwarzschild black string.

1.1.1 Schwarzschild Black Holes, Black Strings and Black Branes

The most basic solution to the vacuum Einstein equation (I.2) giving rise to the black hole phenomena is the Schwarzschild–Tangherlini black hole solution (Schw_n, g_s). It arises dynamically as the maximal Cauchy development of the following initial data: an initial hypersurface $\Sigma_0 = \mathbb{R} \times \mathbb{S}^{n-2}$, a first fundamental form (in isotropic coordinates)

$$h_s = \Big(1 + \frac{M}{2\rho^{n-3}}\Big)^{\frac{4}{n-3}} (d\rho \otimes d\rho + \rho^2 \mathring{\gamma}_{n-2}), \qquad \rho \in (0, \infty) \cong \mathbb{R} \tag{1.1.1}$$

and second fundamental form $K = 0$, where $\mathring{\gamma}_{n-2}$ is the metric on the unit $(n-2)$-sphere $\mathbb{S}^{n-2}$. This spacetime is asymptotically flat and spherically symmetric. The Penrose diagram in Fig. 1.1

represents the causal structure of (Schw_n, g_s) arising from this initial data, restricted to the future of Σ_0.

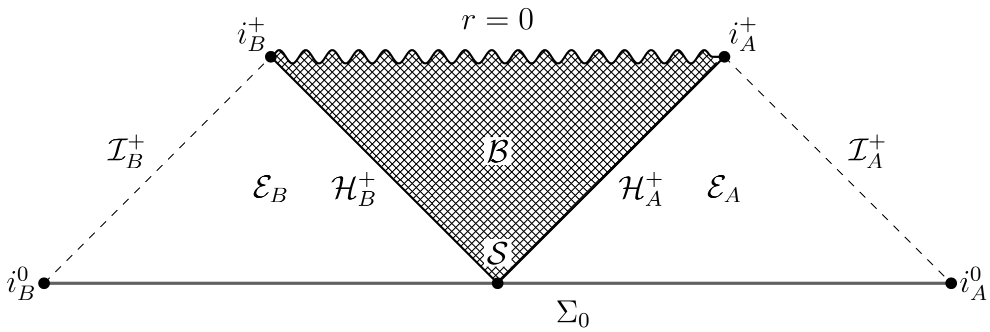


Fig. 1.1 The Penrose diagram of the Schwarzschild–Tangherlini spacetime (Schw_n, g_s).

Here, $\mathcal{I}^+ \doteq \mathcal{I}_A^+ \cup \mathcal{I}_B^+$ is future null infinity, $i^+ \doteq i_A^+ \cup i_B^+$ and $i^0 \doteq i_A^0 \cup i_B^0$ are future timelike infinity and spacelike infinity respectively, $\mathcal{E}_A \doteq J^-(\mathcal{I}_A^+) \cap J^+(\Sigma_0)$ is the distinguished exterior region, $\mathcal{E}_B \doteq J^-(\mathcal{I}_B^+) \cap J^+(\Sigma_0)$ is another exterior region, $\mathcal{B} \doteq \mathrm{Schw}_n \setminus J^-(\mathcal{I}^+)$ is the black hole region, $\mathcal{H}^+ = \mathcal{H}_A^+ \cup \mathcal{H}_B^+ \doteq \mathcal{B} \setminus \mathrm{int}(\mathcal{B})$ is the future event horizon and $\mathcal{S} \doteq \mathcal{H}_A^+ \cap \mathcal{H}_B^+$ is the bifurcation sphere. The wavy line denotes a singular boundary which is not part of the spacetime (Schw_n, g_s) but towards which the Kretchmann curvature invariant diverges. It is in this sense that (Schw_n, g_s) is singular. Note that every point in this diagram is in fact an $(n-2)$-sphere. The metric on the exterior $\mathcal{E}_A$ of the n-dimensional Schwarzschild–Tangherlini black hole in traditional Schwarzschild coordinates $(t, r, \varphi_1, ..., \varphi_{n-2})$ takes the form [1, 74]

$$g_s = -D_n(r) dt \otimes dt + \frac{1}{D_n(r)} dr \otimes dr + r^2 \mathring{\gamma}_{n-2}, \qquad D_n(r) \doteq 1 - \frac{2M}{r^{n-3}}, \tag{1.1.2}$$

where $t \in [0, \infty)$, $r \in ((2M)^{\frac{1}{n-3}}, \infty)$ and $\mathring{\gamma}_{n-2}$ is the metric on the unit $(n-2)$-sphere.

The Lorentzian manifold that is the main topic of this chapter is the Schwarzschild black string spacetime in 5 dimensions which is constructed from the $4D$ Schwarzschild solution (Schw_4, g_s). Before focussing on this spacetime explicitly, it is of interest to discuss more general spacetimes constructed from the n-dimensional Schwarzschild–Tangherlini black hole solution (Schw_n, g_s). Let $\mathbb{S}_R^1$ denote the circle of radius R and let $\mathrm{F}_p \in \{\mathbb{R}^p, \mathbb{R}^{p-1} \times \mathbb{S}_R^1, ..., \mathbb{R} \times \prod_{i=1}^{p-1} \mathbb{S}_{R_i}^1, \prod_{i=1}^{p} \mathbb{S}_{R_i}^1\}$ with its associated p-dimensional Euclidean metric δ_p. If one has the n-dimensional Schwarzschild black hole spacetime (Schw_n, g_s) and takes its Cartesian product with F_p then one realises the $(n+p)$-dimensional Schwarzschild black brane $(\mathrm{Schw}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$. This means that the $(n+p)$-dimensional Schwarzschild black brane $(\mathrm{Schw}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$ is a product manifold made from Ricci-flat manifolds, which is again Ricci-flat and hence satisfies the vacuum Einstein equation (I.2). Note that in contrast to (Schw_n, g_s), the spacetimes $(\mathrm{Schw}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$ are not asymptotically flat but are called 'asymptotically Kaluza–Klein'.

The Schwarzschild black brane spacetimes $(\mathrm{Schw}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$ arise dynamically as the maximal Cauchy development of suitably extended Schwarzschild initial data, i.e., $(\Sigma_0 \times \mathrm{F}_p, h_s \oplus \delta_p, K = 0)$. Hence, the above Penrose diagram in Fig. 1.1 can be reinterpreted as the Penrose diagram for the Schwarzschild black brane, but instead of each point representing a $(n-2)$-sphere, it represents a $\mathbb{S}^{n-2} \times \mathrm{F}_p$. In particular, the notation $\mathcal{E}_A$ will be used henceforth to denote the distinguished exterior region of $(\mathrm{Schw}_n \times \mathrm{F}_p, g_s \oplus \delta_p)$.

Taking $p = 1$ gives rise to the $(n+1)$-dimensional Schwarzschild black string spacetime $\mathrm{Schw}_n \times \mathbb{R}$ or alternatively $\mathrm{Schw}_n \times \mathbb{S}^1_R$. The topic of the present chapter is the $5D$ Schwarzschild black string spacetime $\mathrm{Schw}_4 \times \mathbb{R}$ or alternatively $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. The metric on the exterior $\mathcal{E}_A$ in standard Schwarzschild coordinates is

$$g \doteq -D(r)dt \otimes dt + \frac{1}{D(r)}dr \otimes dr + r^2 \mathring{\gamma}_2 + dz \otimes dz, \qquad D(r) = 1 - \frac{2M}{r}, \tag{1.1.3}$$

where $t \in [0, \infty)$, $r \in (2M, \infty)$ and $z \in \mathbb{R}$ or $\mathbb{R}/2\pi R\mathbb{Z}$.

Finally, to analyse the subsequent problem of linear stability on the exterior region $\mathcal{E}_A$ up to the future event horizon $\mathcal{H}^+_A$, one requires a chart with coordinate functions that are regular up to this hypersurface $\mathcal{H}^+_A \setminus \mathcal{S}$, where $\mathcal{S}$ now denotes the bifurcation surface. A good choice is ingoing Eddington–Finkelstein coordinates defined by

$$v = t + r_\star, \qquad \frac{dr_\star}{dr} = \frac{r^{n-3}}{r^{n-3} - 2M}, \quad \text{with } r_\star(3M) = 3M + 2M\log(M). \tag{1.1.4}$$

The $(n+p)$-dimensional Schwarzschild black brane metric becomes

$$g_s \oplus \delta = -D_n(r)dv \otimes dv + dv \otimes dr + dr \otimes dv + r^2 \mathring{\gamma}_{n-2} + \delta_{ij} dz^i \otimes dz^j, \tag{1.1.5}$$

where $D_n(r)$ is defined in equation (1.1.2) and $v \in \mathbb{R}$ and $r \in (0, \infty)$.

1.1.2 Previous Works

For a good introduction to the Gregory–Laflamme instability and the numerical result of [49] see the book chapter [63]. A detailed survey of the key work [57] related to the present chapter is undertaken in section 1.3. A brief history of the problem is presented here:

(i) In 1988, Gregory–Laflamme examined the Schwarzschild black string spacetime and stated that it is stable [93]. However, an issue in the analysis arose from working in Schwarzschild coordinates which lead to incorrect regularity assumptions for the asymptotic solutions.

(ii) In 1993, Gregory–Laflamme used numerics to give strong evidence for the existence of a low-frequency instability of the Schwarzschild black string and branes in harmonic gauge [49].

(iii) In 1994, Gregory–Laflamme generalised their numerical analysis to show instability of 'magnetically-charged dilatonic' black branes [50] (see [50, 73] for a discussion of these solutions).

(iv) In 2000, Gubser–Mitra discussed the Gregory–Laflamme instability for general black branes. They conjectured that a necessary and sufficient condition for stability of the black brane spacetimes is thermodynamic stability of the corresponding black hole [52, 53].

(v) In 2001, Reall [51], with the aim of addressing the Gubser–Mitra conjecture, explored further the relation between stability of black branes arising from static, spherically symmetric black holes and thermodynamic stability of those black holes. In particular, the work of Reall argues that there is a direct relation between the 'negative mode' of the Euclidean Schwarzschild instanton solution (this mode was initially identified numerically in a paper by Gross, Perry and Yaffe [94]) and the threshold of the Gregory–Laflamme instability. This idea was further explored in a work of Reall et al. [59], which extended the idea that 'negative modes' of the Euclidean extension of a Myers–Perry black hole (the generalisation of the Kerr spacetime to higher dimensions, see [41, 37] for details) correspond to the threshold for the onset of a Gregory–Laflamme instability.

(vi) In 2006, Hovdebo and Myers [57] used a different gauge (which was introduced in [95]) to reproduce the numerics from the original work of Gregory and Laflamme. This gauge choice will be called spherical gauge and will be adopted in the present work. This work discusses the presence of the Gregory–Laflamme instability for the 'boosted' Schwarzschild black string and the Emparan–Reall black ring (for a discussion of this solution see [42, 37, 48]).

(vii) In 2010, Lehner and Pretorius numerically simulated the non-linear evolution of the Gregory–Laflamme instability; see the review [64] and references therein.

(viii) In 2011, Figueras, Murata and Reall [61] put forward the idea that a local Penrose inequality gives a stability criterion. Furthermore, [61] showed numerically that this local Penrose inequality was violated for the Schwarzschild black string for a range of frequency parameters which closely match those found in the original work of Gregory–Laflamme [49].

(ix) In 2012, Hollands and Wald [65] and, later in 2015, Prabu and Wald [66] developed a general method applicable to many linear stability problems which encompasses the problem of linear stability of the Schwarzschild black string exterior $\mathcal{E}_A$. The papers [65] and [66] are explored in detail in section 1.1.5.

A few other works are of relevance to this discussion. The review paper [37] and book chapter [48] discuss the black ring solution [42] in great detail. This relates to the work presented here since the Gregory–Laflamme instability is often heuristically invoked when discussing higher-dimensional black hole solutions. In particular, if the black ring of study has a large radius and is sufficiently thin then it ‘looks like’ a Schwarzschild black string and therefore would be susceptible to the Gregory–Laflamme instability. There has been heuristic and numerical results to give evidence to this claim [57, 67]. Finally, in 2021 Benomio produced the first mathematically rigorous result on the stability problem for the black ring spacetime [68].

1.1.3 Statement of the Main Theorem: Theorem 1.1.2

The purpose of this chapter is to give a direct, self-contained, elementary proof of the Gregory–Laflamme instability for the $5D$ Schwarzschild black string.

For the statement of the main theorem, one should have in mind the Penrose diagram in Fig. 1.2 for the $5D$ Schwarzschild black string spacetime.

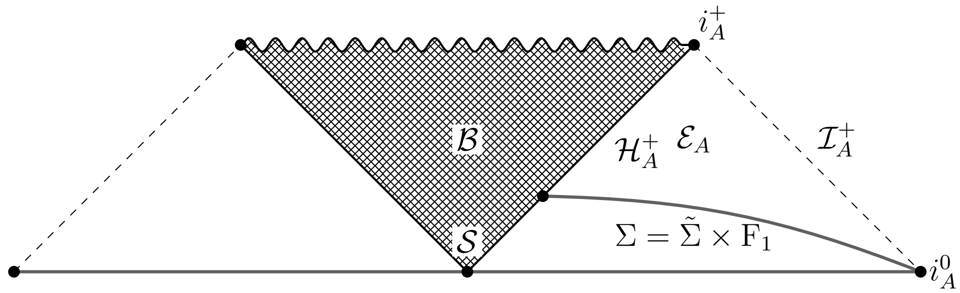


Fig. 1.2 The Penrose diagram for the $5D$ Schwarzschild black string illustrating the set up for the linear instability problem. Every point in this diagram represents a $\mathbb{S}^2 \times \mathrm{F}_1$.

Indicated in figure 1.2 is a spacelike asymptotically flat hypersurface $\tilde{\Sigma}$ which extends from spacelike infinity i_A^0 to intersect the future event horizon $\mathcal{H}_A^+$ to the future of the bifurcation surface $\mathcal{S}$. Further, $\mathrm{F}_1 = \mathbb{R}$ or $\mathbb{S}^1_R$, $\mathcal{B}$ is the black hole region, $\mathcal{E}_A$ is the exterior region, $\mathcal{I}_A^+$ is future null infinity and i_A^+ is future timelike infinity. The hypersurface Σ can be expressed as $\Sigma = \{(t, r_\star, \theta, \varphi, z) : t = f(r_\star)\}$ such that $f = o(r)$ for $r_\star \to \infty$. An explicit example would be a hypersurface of constant $t_\star$ where $t_\star = t + 2M \log(r - 2M)$.

Definition 1.1.1 (Mode Solution)**.** *A solution of the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ of the form*

$$h_{\alpha\beta} = e^{\mu t + ikz} H_{\alpha\beta}(r, \theta) \tag{1.1.6}$$

with $\mu, k \in \mathbb{R}$ and $(t, r, \theta, \varphi, z)$ standard Schwarzschild coordinates will be called a mode solution.

Remark 1.1.1. *The above definition 1.1.1 of a mode solution is not the most general definition one could make. In particular, the above definition restricts h to be axisymmetric and to have $\mu \in \mathbb{R}$; a more general definition of mode solution would allow dependence on φ and $\mu \in \mathbb{C}$.*

A way of establishing the linear instability of an asymptotically flat black hole is exhibiting a mode solution of the linearised Einstein equation (I.5) which is smooth up to and including the future event horizon, decays towards spacelike infinity and such that $\mu > 0$.

Theorem 1.1.2 (Gregory–Laflamme Instability). *For all* $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$*, there exists a non-trivial mode solution* h *of the form (1.1.6) to the linearised vacuum Einstein equation (I.5) on the exterior* $\mathcal{E}_A$ *of the Schwarzschild black string background* $\mathrm{Schw}_4 \times \mathbb{R}$ *with* $\mu > \frac{1}{40\sqrt{10}M} > 0$ *and*

$$H_{\alpha\beta}(r,\theta) = \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & 0 \\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & H_{\theta\theta}(r)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.1.7}$$

The solution h *extends regularly to* $\mathcal{H}_A^+$ *and decays exponentially towards* i_A^0 *and can thus be viewed as arising from regular initial data on a hypersurface* Σ *extending from the future event horizon* $\mathcal{H}_A^+$ *to* i_A^0*. In particular,* $h|_\Sigma$ *and* $\nabla h|_\Sigma$ *are smooth on* Σ*. Moreover, the solution* h *is not pure gauge and can in fact be chosen such that the harmonic/transverse-traceless gauge conditions*

$$\begin{cases} \mathrm{div} h = 0 \\ \mathrm{Tr}_g h = 0 \end{cases} \tag{1.1.8}$$

are satisfied.

Suppose $R > 4M$*, then one can choose* k *such that there exists an integer* $n \in [\frac{3R}{20M}, \frac{8R}{20M}]$ *and therefore* h *induces a smooth solution on the exterior* $\mathcal{E}_A$ *of the Schwarzschild black string* $\mathrm{Schw}_4 \times \mathbb{S}^1_R$*. Moreover, the initial data for such a mode solution on the exterior* $\mathcal{E}_A$ *of* $\mathrm{Schw}_4 \times \mathbb{S}^1_R$ *has finite energy.*

Hence, the exterior $\mathcal{E}_A$ *of the Schwarzschild black string* $\mathrm{Schw}_4 \times \mathbb{R}$ *or* $\mathrm{Schw}_4 \times \mathbb{S}^1_R$ *for* $R > 4M$ *is linearly unstable as a solution of the vacuum Einstein equation (I.5), and the instability can be realised as a mode instability in harmonic/transverse-traceless gauge (1.1.8) which is not pure gauge.*

Remark 1.1.3. *One can construct a gauge invariant quantity, the* $tztz$*-component of the linearised Weyl tensor* $\overset{(1)}{W}$*, which is non-vanishing for a non-trivial mode solution* h *with* $k \neq 0$ *and* $\mu \neq 0$ *and exhibits exponential growth in* t *when* $\mu > 0$*. This allows one to show that the mode solution constructed in Theorem 1.1.2 is not pure gauge. Hence, one expects that the above mode solution persists in any 'good' gauge, not just (1.1.8).*

Remark 1.1.4. *The reader should note that the lower bound on the frequency parameter k should not be interpreted as ruling out the existence of unstable modes with arbitrarily long wavelengths. The lower bound on k in Theorem 1.1.2 results from the use of a test function in the variational argument (see proposition 1.4.5 in section 1.4.3). The numerics of Gregory–Laflamme and Hovdebo–Myers [49, 57] both provide evidence that there are unstable modes for k arbitrarily small.*

1.1.4 Difficulties and Main Ideas of the Proof

It may seem natural to directly consider the problem in harmonic gauge since the equation of study (I.5) reduces to a tensorial wave equation

$$g^{cd}\nabla_c\nabla_d h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} = 0. \tag{1.1.9}$$

The above equation (1.1.9) results from the linearisation of the gauge reduced non-linear vacuum Einstein equation (I.2) which is strongly hyperbolic and therefore well-posed. The equation (1.1.9) reduces to a system of ODEs under the mode solution ansatz (1.1.6) with (1.1.7). This system can be reduced to a single ODE of the form

$$\frac{d^2u}{dr^2} + P_{\mu,k}(r)\frac{du}{dr} + Q_{\mu,k}(r)u = \frac{\mu^2}{D(r)^2}u, \qquad D = 1 - \frac{2M}{r}, \tag{1.1.10}$$

where $u = H_{tt}$, H_{tr}, H_{rr} or $H_{\theta\theta}$ and $P_{\mu,k}(r)$ and $Q_{\mu,k}(r)$ depend on μ, k and r. However, if one insists on this decoupling one introduces a regular singular point in the range $r \in (0, \infty)$. For certain ranges of μ and k, this value occurs on the exterior $\mathcal{E}_A$, i.e., the regular singular point occurs in $r \in (2M, \infty)$. In particular, this regular singularity occurs on the exterior for the numerical values of k and μ for which Gregory–Laflamme identified instability. In the original works of Gregory and Laflamme the decoupled ODE for H_{tr} was studied; see the works [49, 63, 93].

It turns out that, in looking for an instability one can make a different gauge choice called spherical gauge. As shown in section 1.3, the linearised vacuum Einstein equation (I.5) for a mode solution (1.1.6) in spherical gauge can be reduced to a 2^{nd}-order ODE of the form (1.1.10), where, in contrast to harmonic/transverse-traceless gauge, $P_{\mu,k}(r) = P_k(r)$ and $Q_{\mu,k}(r) = Q_k(r)$ depend only on k and r. Hence, existence of solution to the ODE (1.1.10) becomes a simple eigenvalue problem for μ. Spherical gauge was originally introduced in [95] and has another advantage over harmonic/transverse-traceless gauge which is that all $r \in (2M, \infty)$ are ordinary points of the ODE (1.1.10). Hence, the spherical gauge choice also avoids the issues of a regular singularity at some $r \in (2M, \infty)$. However, in contrast to harmonic gauge, for this gauge choice, well-posedness is unclear. If one were trying to prove *stability* then exhibiting a well-posed gauge would be key since well-posedness of the equations is essential for understanding general solutions. For *instability*, it turns out that it is sufficient to exhibit a mode solution of the non-gauge reduced equation (I.5) which is *not pure gauge*. One expects then that such a mode solution will persist in all 'good' gauges, of which harmonic gauge is an example. The discussion of pure gauge mode solutions in spherical gauge in section 1.3.3 provides a proof that if $k \neq 0$ and $\mu \neq 0$ then a mode solution in spherical gauge is *not* pure gauge. This can be shown directly or from the computation of a gauge invariant quantity, namely the $tztz$-component of the linearised Weyl tensor, $\overset{(1)}{W}$. Further, it is shown that if a non-trivial mode solution in spherical gauge grows exponentially in t then $\overset{(1)}{W}_{tztz}$ is non-zero and grows exponentially t.

An issue with spherical gauge is that mode solutions in the spherical gauge do not, in general, extend smoothly to the future event horizon $\mathcal{H}^+_A$, even when they represent physically admissible solutions. However, as shown in section 1.3.4, one can detect what are the admissible boundary conditions at the future event horizon in spherical gauge by adding a pure gauge perturbation to the metric perturbation to try and construct a solution that indeed extends smoothly to $\mathcal{H}^+_A$. In fact, the pure gauge perturbation found is precisely one that transforms the metric perturbation to harmonic/transverse-traceless gauge (1.1.8). Hence, after also identifying the admissible boundary conditions at spacelike infinity i^0_A in section 1.3.4, proving the existence of an unstable mode solution to the linearised vacuum Einstein equation (I.5) that is *not* pure gauge is reduced to showing the existence of a solution to the ODE (1.1.10) with $\mu > 0$ and $k \neq 0$ which satisfies the admissible boundary conditions that are identified in this work.

In this chapter, the ODE problem (1.1.10) is approached from a direct variational point of view in section 1.4. To run a direct variational argument, the solution u of ODE (1.1.10) is rescaled and change of coordinates is applied. It is shown in section 1.4.1 that equation (1.1.10) can be cast into a Schrödinger form

$$-\Delta_{r_\star} u + V_k(r_\star)u = -\mu^2 u, \qquad r_\star = r + 2M\log(r - 2M) \tag{1.1.11}$$

with V_k independent of μ. The ODE (1.1.11) can be interpreted as an eigenvalue problem for $-\mu^2$; finding an eigenfunction, in a suitable space, with a negative eigenvalue will correspond to an instability. As shown in section 1.4.2, this involves assigning the following energy functional to the Schrödinger operator on the left-hand side of (1.1.11):

$$E(u) \doteq \langle \nabla_{r_\star} u, \nabla_{r_\star} u \rangle_{L^2(\mathbb{R})} + \langle V_k u, u \rangle_{L^2(\mathbb{R})}. \tag{1.1.12}$$

Using a suitably chosen test function, one can show that the infimum over functions in $H^1(\mathbb{R})$ of this functional is negative for a range of k. One then needs to argue that this infimum is attained as an eigenvalue, by showing this functional is lower semicontinuous and that the minimizer is non-trivial. The corresponding eigenfunction is then a weak solution in $H^1(\mathbb{R})$ to the ODE (1.1.11) with $\mu > 0$ for a range of $k \in \mathbb{R} \setminus \{0\}$. Elementary one-dimensional elliptic regularity implies the solution is indeed smooth away from the future event horizon, $\mathcal{H}_A^+$, and therefore corresponds to a classical solution of the problem (1.1.11). Finally, the solution can be shown to satisfy the admissible boundary conditions by the condition that the solution lies in $H^1(\mathbb{R})$.

The chapter is organised in the following manner. The remainder of the present section contains additional background on the Gregory–Laflamme instability. In section 1.2, linear perturbation theory is reviewed and the linearised Einstein equation (I.5) is derived. In section 1.3, the analysis in spherical gauge is presented. The decoupled ODE (1.1.10) resulting from the linearised Einstein equation (I.5) is derived and it is established that the problem can be reduced to the existence of a solution to the decoupled ODE with $\mu > 0$ and $k \neq 0$ satisfying admissible boundary conditions. In section 1.4, the proof of the existence of such a solution is presented via the direct variational method.

Appendix A.1 contains a list of the Riemann tensor components and the Christoffel symbols for the Schwarzschild black string spacetime $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. Appendix A.2 collects results on singularities in 2nd order ODE relevant for the discussion of the boundary conditions for the decoupled ODE (1.1.10). Appendix A.3 provides a method of transforming a 2nd order ODE into a Schrödinger equation. Appendix A.4 collects some useful results from analysis that are needed in the proof of theorem 1.1.2. Appendix A.5 compliments theorem 1.1.2 with some stability results.

1.1.5 The Canonical Energy Method

The reader should note that there are two papers [65, 66] concerning a very general class of spacetimes which are of relevance to the stability problem for the Schwarzschild black string. In particular, it follows from [65, 66] that there exists a linear perturbation of the Schwarzschild black string spacetime which is not pure gauge and grows exponentially in the Schwarzschild t-coordinate. The following describes the results of these works.

In 2012, a paper of Hollands and Wald [65] gave a criterion for linear stability of stationary, axisymmetric, vacuum black holes and black branes in $D \geq 4$ spacetime dimensions (see also an extension of this work to a broad class of theories that include matter by Keir [96]). They define a conserved quantity known as the 'canonical energy' $\mathcal{E}$ associated to the space of solutions to the linearised vacuum Einstein equation (I.5) and established a stability criterion based upon it. The canonical energy $\mathcal{E}$ can be expressed as integral over an initial Cauchy surface of an expression quadratic in the perturbation (see definitions 3.2.5 and 3.2.6 in section 3.2.3 of chapter 3) and the associated stability criterion applies to general static or stationary, axisymmetric black hole spacetimes. It can be related to thermodynamic quantities by

$$\mathcal{E} = \delta^2 M - \sum_B \Omega_B \delta^2 J_B - \frac{\kappa}{8\pi}\delta^2 A, \tag{1.1.13}$$

where M and J_B are the ADM mass and ADM angular momenta in the B^{th} plane, and A is the cross-sectional area of the horizon. Note that the right-hand side of (1.1.13) refers to the second variation of thermodynamic quantities. It is remarkable that the combination $\mathcal{E}$ of these second variations is in fact determined by linear perturbations.

The work [65] considers initial data for a perturbation of either a stationary, axisymmetric black hole or black brane with the following properties:

(i) the linearised constraint equations are satisfied.

(ii) the linear change to the ADM charges (momentum, mass and angular momentum) vanish.

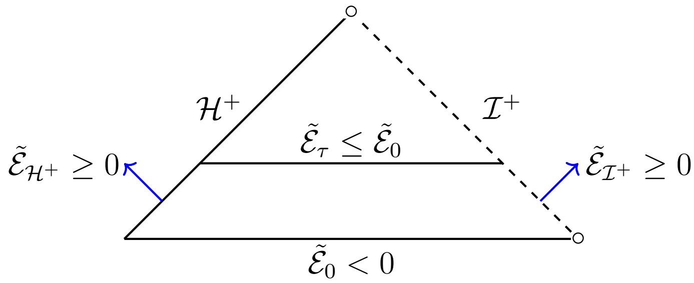
(iii) the perturbation is axisymmetric when the black hole spacetime is stationary, rotating and axisymmetric.

(iv) certain gauge conditions and finiteness/regularity conditions are satisfied at the horizon and infinity.

In what follows, initial data satisfying (i)–(iv) will be referred to as admissible. On this class of data, the canonical energy is gauge invariant and degenerate if and only if the initial data for the perturbation is towards another stationary, axisymmetric black hole. Further, by avoiding superradiance (via point (iii) above), Hollands and Wald establish that the flux of (a modified) canonical energy, $\tilde{\mathcal{E}}$, through (a finite subset) the future event horizon and through (a finite subset) null infinity is positive.

Remark 1.1.5. *The motivation for point (ii) of the admissibility criterion is as follows. For the Schwarzschild black hole, one can take initial data which corresponds simply to a change of the mass parameter $M \mapsto M+\alpha$ and therefore, by equation (1.1.13) and since the cross-sectional area of the horizon is given by $A = 16\pi(M+\alpha)^2$, it follows that $\mathcal{E} < 0$. This is the 'thermodynamic instability' of the Schwarzschild black hole. However, by point (ii), the initial data for a change of mass perturbation is manifestly* <u>*not*</u> *admissible (the family of Schwarzschild black holes is, after all, dynamically stable [28, 35]).*

Hollands and Wald's stability criterion is then formulated as follows. If one can establish positivity of $\mathcal{E}$ for all admissible initial data then this implies the black hole (or black brane) in question is 'weakly' linearly stable. In particular, it rules out growing modes since these would violate the conservation of canonical energy. On the other hand, if there exists admissible initial data for which $\mathcal{E} < 0$ then the positivity of the flux of the (modified) canonical energy, $\tilde{\mathcal{E}}$, through the event horizon and null infinity implies that the canonical energy can only decrease through a foliation that connects the event horizon to null infinity. This is depicted in the following diagram:



This monotonicity property prevents future convergence of the canonical energy to zero when it is negative and therefore implies there exist admissible initial data for a perturbation which cannot approach a stationary perturbation at late times, i.e., one has failure of asymptotic stability.

The canonical energy gives a very clear criterion for stability and instability based on checking the sign of $\mathcal{E}$. However, the complexity of checking that the initial data satisfies the above criteria (i)-(iv) is involved. Further, the expression quoted for the canonical energy in terms of the linearised metric h (see equation (86) of [65]) is complicated and coercivity properties are obscure. Indeed, prior to this work, even the positivity of the canonical energy for the Schwarzschild black hole spacetime (which is known to be linearly stable by [28]) was an open problem. See Chapter 3 for a resolution to this problem.

The work of Hollands and Wald [65] also shows an additional result relevant specifically to the problem of stability of black *strings* (and black branes). Suppose there exist initial data for a perturbation of the ADM parameters of a vacuum black *hole*, (M_{BH}, g), such that $\mathcal{E} < 0$ and let $(M_{\mathrm{BH}} \times \mathbb{S}^1_R, g \oplus \delta)$ be the associated black *string*. The work [65] shows that, starting from such a perturbation of the black *hole*, one can infer the existence of <u>admissible</u> initial data for a perturbation (which is not pure gauge) of the associated black *string* such that again $\mathcal{E} < 0$ as the $\mathbb{S}^1_R$ radius $R \to \infty$. One should note that this argument does not give an explicit bound on R. This is in contrast to the result of theorem 1.1.2 presented in the present chapter which shows that, for $R > 4M$, one can construct an explicit exponentially growing mode solution on the exterior of the Schwarzschild black string. Hollands and Wald's criterion for linear instability of a black string formalised a conjecture by Gubser–Mitra that a necessary and sufficient condition for stability of the black brane spacetimes is thermodynamic stability of the corresponding black hole [52, 53]. Since the change of mass perturbation of Schwarzschild black hole produces $\mathcal{E} < 0$, this argument implies that the Schwarzschild black string fails to be asymptotically stable.

The failure of asymptotic stability does not in itself imply that perturbations grow. However, the results of [65] were strengthened in 2015 by Prabhu and Wald [66]. They showed, using some spectral theory, that if there exist admissible initial data for a perturbation h, which can be written as $h = \mathcal{L}_T \tilde{h}$ for another perturbation $\tilde{h}$, such that $\mathcal{E} < 0$ for a black *brane*, then there exists initially well-behaved perturbations that are not pure gauge and that grow exponentially in time. Having established that there exist admissible initial data for a perturbation such that $\mathcal{E} < 0$ for the Schwarzschild black string in [65], existence of a linear perturbation which is not pure gauge and has exponential growth should follow.

Remark 1.1.6. *The reader should note that the Hollands and Wald paper [65] also showed that a necessary and sufficient condition for stability, with respect to axisymmetric perturbations, is that a 'local Penrose inequality' is satisfied. The idea that a local Penrose inequality gives a stability criterion was originally discussed in the work of Figueras, Murata and Reall [61] which gave strong evidence in favor of sufficiency of this condition for stability. Furthermore [61] showed numerically that this local Penrose inequality was violated for the Schwarzschild black string for a range of frequency parameters which closely match those found in the original work of Gregory–Laflamme [49].*

The present work differs from the above as it gives a direct, self-contained, elementary proof of the Gregory–Laflamme instability following the original formulation of [49, 57, 63, 93] which is completely explicit. In particular, it gives an exponentially growing mode solution with an explicit growth rate, of the form defined by equations (1.1.6) and (1.1.7) in harmonic/transverse-traceless gauge which is not pure gauge.

Remark 1.1.7. *It would also be of interest to see if Theorem 1.1.2 in the form stated could be inferred from the canonical energy method of Hollands, Wald and Prabu [65, 66] in an explicit way bypassing some of the functional calculus applied in the work [66]. In particular, it would be interesting to explore the possible relation between the variational theory applied to $\mathcal{E}$ and that applied here (see section 1.4.2).*

1.1.6 Outlook

This chapter brings together what is known about the Gregory–Laflamme instability as well as providing a direct elementary mathematically rigorous proof of its existence without the use of numerics and with an explicit bound on μ and k. Note that whilst only the $5D$ Schwarzschild black string was considered here, the result of instability readily extends to higher dimensions with the replacement of kz in the exponential factor with $\sum_i k_i z_i$.

Further directions of work could be to study the non-linear problem, the extension to $\mathrm{Kerr}_4 \times \mathbb{S}^1$ or $\mathrm{Kerr}_4 \times \mathbb{R}$, the extension to charged black branes of the work [50], the extension to black rings or ultraspinning Myers–Perry black holes.

1.2 Linear Perturbation Theory

This section provides a derivation and review of the linearised vacuum Einstein equation (I.5) around a general spacetime background metric (M, g) satisfying the vacuum Einstein equation (I.2).

1.2.1 Linearised Vacuum Einstein Equation

Consider a Lorentzian manifold (M, g) with metric satisfying the vacuum Einstein equation (I.2). In this section a 'perturbation' of the spacetime metric will be discussed. This will be represented by a new metric of the form $g + \epsilon h$ with $\epsilon > 0$. Here h is a symmetric bilinear form on the fibres of TM. In the following, a series of results on how various quantities change to $O(\epsilon)$ (the linear level) are derived. This will result in an expression for the Ricci tensor under such a perturbation to linear order.

Remark 1.2.1. *An important point to note that indices are raised and lowered here with respect to g.*

Proposition 1.2.2 (Change to the Levi-Civita Connection)**.** *Consider a Lorentzian manifold (M, g). Suppose the metric $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric. Then the Levi-Civita connection, $\overline{\Gamma}^{\gamma}_{\alpha\beta}$, of $\bar{g}_{ab}$ to $O(\epsilon)$ is*

$$\overline{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \epsilon \overset{(1)}{\Gamma}{}^{\gamma}_{\alpha\beta} \tag{1.2.1}$$

with

$$\overset{(1)}{\Gamma}{}^{a}_{bc} = \frac{1}{2} g^{ad} (\nabla_b h_{cd} + \nabla_c h_{bd} - \nabla_d h_{bc}). \tag{1.2.2}$$

Proof. Proof of this proposition follows from a direct computation of the Christoffel symbols of $g + \epsilon h$ in normal coordinates at some $p \in M$. □

Proposition 1.2.3. *Consider a Lorentzian manifold (M, g). Suppose the metric $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric. Then the Riemann tensor, $\bar{R}^{a}{}_{bcd}$, of $\bar{g}_{ab}$ to $O(\epsilon)$ is*

$$\bar{R}^{a}{}_{bcd} = R^{a}{}_{bcd} + \epsilon \overset{(1)}{R}{}^{a}{}_{bcd} \tag{1.2.3}$$

where

$$\overset{(1)}{R}{}^{a}{}_{bcd} = \nabla_c \overset{(1)}{\Gamma}{}^{a}_{bd} - \nabla_d \overset{(1)}{\Gamma}{}^{a}_{bc}. \tag{1.2.4}$$

Proof. Proof of this proposition follows from a direct computation in normal coordinates at some $p \in M$ and proposition 1.2.2. □

Proposition 1.2.4 (Change in the Ricci Tensor). *Consider a Lorentzian manifold* (M, g). *Suppose the metric* $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab}$ *is a Lorentzian metric. Then the Ricci tensor,* $(\overline{\mathrm{Ric}(g)})_{ab}$, *of* $\bar{g}_{ab}$ *to* $O(\epsilon)$ *is*

$$\overline{\mathrm{Ric}(g)} = \mathrm{Ric}(g) + \epsilon \overset{(1)}{\mathrm{Ric}}, \tag{1.2.5}$$

with

$$\overset{(1)}{\mathrm{Ric}}_{ab} \doteq -\frac{1}{2}\Delta_L h_{ab} \tag{1.2.6}$$

where Δ_L *denotes the Lichnerowicz operator given by*

$$\Delta_L h_{ab} = g^{cd}\nabla_c\nabla_d h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} - 2(\mathrm{Ric}(g))_{c(a}h_{b)}{}^c - 2\nabla_{(a}\nabla^c h_{b)c} + \nabla_a\nabla_b \mathrm{Tr}_g h. \tag{1.2.7}$$

Proof. This proposition follows from a contraction on (a, c) of $\overset{(1)}{R}{}^a{}_{bcd}$ in proposition 1.2.3 and an application of the Ricci identity. □

If one assumes g satisfies the vacuum Einstein equation (I.2) and $g + \epsilon h$ satisfies the vacuum Einstein equation (I.2) to $O(\epsilon)$ then it follows from proposition 1.2.4 that h must satisfy the equation (I.5) to $O(\epsilon)$. This motivates the terminology of 'linearised vacuum Einstein equation' for equation (I.5). This will be the main equation of interest, with g the Schwarzschild black string metric

$$g \doteq -D(r)dt \otimes dt + \frac{1}{D(r)}dr \otimes dr + r^2\Big(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi\Big) + dz \otimes dz, \tag{1.2.8}$$

with $D(r)$ defined in equation (1.1.3).

1.2.2 Pure Gauge Solutions in Linearised Theory

The vacuum Einstein equation (I.2) is a system of second order quasilinear partial differential equations of the pair (M, g) which are invariant under the diffeomorphisms of M. This means that for given initial data, the vacuum Einstein equation (I.2) only determines a spacetime uniquely up to diffeomorphism, i.e., if there exists a diffeomorphism $\Phi : M \to M$ then (M, g) and $(M, \Phi_\star(g))$ are equivalent solutions of the vacuum Einstein equation (I.2). For constructing spacetimes, one often imposes conditions on local coordinates called a gauge choice. Upon linearisation of the theory, this freedom to impose conditions on local coordinates manifests itself as the freedom to impose a form or a condition for linearised metric. A familiar example would be implementing the well known harmonic gauge condition

$$\Box_g(x^\alpha) = \frac{1}{\sqrt{\det(g)}}\partial_\mu\Big(\sqrt{\det(g)}g^{\mu\nu}\partial_\nu x^\alpha\Big) = 0 \Leftrightarrow \Gamma^\alpha_{\mu\nu}g^{\mu\nu} = 0. \tag{1.2.9}$$

Now to linearise this condition, consider two Lorentzian metrics g and $g+\epsilon h$ on a manifold M. Denote the difference of their Levi-Civita connections to $\mathcal{O}(\epsilon)$ as $\overset{(1)}{\Gamma}{}^a_{bc}$ as in proposition 1.2.2. Note that this object is a tensor. Taking some arbitrary point p in (M,g) and using normal coordinates there gives that $\Gamma(p)=0$. Hence, linearising the condition (1.2.9) gives

$$\overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} g^{\mu\nu}\Big|_p = 0. \tag{1.2.10}$$

This is a basis independent result since this is a tensorial expression and the $p\in M$ was arbitrary, so one can promote this condition to

$$\overset{(1)}{\Gamma}{}^c_{ab} g^{ab} = 0 \tag{1.2.11}$$

everywhere. From proposition 1.2.2, $\overset{(1)}{\Gamma}{}^c_{ab}$ can be written in terms of a metric perturbation h as

$$\overset{(1)}{\Gamma}{}^c_{ab} = \frac{1}{2} g^{cd}\Big(\nabla_a h_{bd} + \nabla_b h_{ad} - \nabla_d h_{ab}\Big). \tag{1.2.12}$$

Hence,

$$0 = \overset{(1)}{\Gamma}{}^c_{ab} g^{ab} \implies (\mathrm{div} h)_a - \frac{1}{2}\nabla_a \mathrm{Tr}_g h = 0. \tag{1.2.13}$$

More generally, for linearised theory, gauge choice can be formulated as follows.

Consider a Lorentzian manifold $(M, \bar{g} \doteq g + \epsilon h)$ with $\epsilon > 0$. Let $\{\Phi_\tau\}$ be a 1-parameter family of diffeomorphisms generated by a vector field X and define $\xi \doteq \tau X \in \mathfrak{X}(M)$. Then from the definition of the Lie derivative one has

$$(\Phi_\tau)_\star(\bar{g}) = \bar{g} + \mathcal{L}_\xi g + \mathcal{O}(\epsilon^2) \tag{1.2.14}$$

if one treats $\tau = \mathcal{O}(\epsilon)$. So in the context of linearised theory, one considers two solutions to the linearised vacuum Einstein equation (I.5), h_1 and h_2, as equivalent if

$$h_2 = h_1 + \mathcal{L}_\xi g \Longleftrightarrow (h_2)_{ab} = (h_1)_{ab} + 2\nabla_{(a}\xi_{b)} \tag{1.2.15}$$

for some vector field $\xi \in \mathfrak{X}(M)$.

Definition 1.2.1 (Pure Gauge Solution)**.** *Let (M,g) be a vacuum spacetime. A solution h to the linearised vacuum Einstein equation (I.5) will be called pure gauge if there exists a vector field $\xi \in \mathfrak{X}(M)$ such that*

$$h_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{1.2.16}$$

The notation h_{pg} will be used to denote a pure gauge solution to the linearised vacuum Einstein equation (I.5).

Remark 1.2.5. *One can show by a direct computation that any 2-tensor of the form $2\nabla_{(a}\xi_{b)}$ automatically verifies the linearised vacuum Einstein equation (I.5).*

Showing that a solution h to the linearised vacuum Einstein equation (I.5) is *not* pure gauge is tantamount to showing that h is not equivalent to the trivial solution. It is thus essential that the solution constructed in this chapter *not* be pure gauge. The following propositions establish that the $tztz$-component of linearised Weyl tensor $\overset{(1)}{W}$ is invariant under gauge transformation. This means that if $\overset{(1)}{W}$ is non-zero for a solution h to the linearised vacuum Einstein equation (I.5) then h cannot be pure gauge.

Proposition 1.2.6 (Change to the Weyl Tensor)**.** *Let (M, g) be a vacuum spacetime. Suppose the metric $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric such that h satisfies the linearised vacuum Einstein equation (I.5). Then the Weyl tensor, $\bar{W}_{abcd}$, of $\bar{g}_{ab}$ to $O(\epsilon)$ is*

$$\bar{W}_{abcd} = W_{abcd} + \epsilon \overset{(1)}{W}_{abcd} \tag{1.2.17}$$

where

$$\overset{(1)}{W}_{abcd} = \nabla_c \nabla_{[b} h_{a]d} + \nabla_d \nabla_{[a} h_{b]c} + \frac{1}{2}\Big(R^e{}_{bcd} h_{ae} - R^e{}_{acd} h_{eb}\Big). \tag{1.2.18}$$

Henceforth, $\overset{(1)}{W}$ will be referred to as the linearised Weyl tensor.

Proof. This follows from a direct computation of the linearisation of

$$W_{abcd} = g_{ae} R^e{}_{bcd}, \tag{1.2.19}$$

using proposition 1.2.3. □

Proposition 1.2.7. *For the 5D Schwarszchild black string, $\overset{(1)}{W}_{tztz}$ evaluated on a pure gauge solution vanishes.*

Proof. Let $\overset{(1)}{W}_{\text{pg}}$ denote the linearised Weyl tensor evaluated on a pure gauge solution h_{pg}. Recall that a pure gauge solution h_{pg} can always be written as $h_{\text{pg}} = \mathcal{L}_\xi g$ for some vector field $\xi \in \mathfrak{X}(M)$. Using proposition 1.2.6 one has and that

$$\begin{aligned}(\overset{(1)}{W}_{\text{pg}})_{abcd} =& \nabla_c \nabla_{[b} \nabla_{a]} \xi_d + \nabla_d \nabla_{[a} \nabla_{b]} \xi_c + \frac{1}{2}\Big(R^e{}_{bcd} \nabla_a \xi_e - R^e{}_{acd} \nabla_e \xi_b\Big) \\ &+ \nabla_{[c} \nabla_{|b|} \nabla_{d]} \xi_a + \nabla_{[d} \nabla_{|a|} \nabla_{c]} \xi_b + \frac{1}{2}\Big(R^e{}_{bcd} \nabla_e \xi_a - R^e{}_{acd} \nabla_b \xi_e\Big).\end{aligned} \tag{1.2.20}$$

By repeated use of the Ricci identity with the first and second Bianchi identities one can compute that

$$(\overset{(1)}{W}_{\text{pg}})_{abcd} = 2\nabla_{[a} R_{b]edc}\xi^e + R^e{}_{adc}\nabla_b \xi_e + R^e{}_{bcd}\nabla_a \xi_e + R^e{}_{dab}\nabla_c \xi_e + R^e{}_{cba}\nabla_d \xi_e. \quad (1.2.21)$$

From appendix A.1 one has $R^\mu{}_{\alpha\beta z} = 0$, $R^\mu{}_{\alpha z\beta} = 0$, $R^\mu{}_{z\alpha\beta} = 0$ and $\Gamma^\alpha_{z\beta} = 0$. Further, the black string metric (1.1.3) is independent of t and z. Hence,

$$(\overset{(1)}{W}_{\text{pg}})_{tztz} = 0. \quad (1.2.22)$$

□

1.3 Analysis in Spherical Gauge

In this section a mode solution, h, of the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string spacetime $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$ is considered. One makes the additional assumption that this mode solution preserves the spherical symmetry of Schw_4. So in particular the solution can be expressed in $(t, r, \theta, \varphi, z)$ coordinates as

$$h_{\alpha\beta} = e^{\mu t + ikz} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & H_{tz}(r) \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & H_{rz}(r) \\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & H_{\theta\theta}(r)\sin^2\theta & 0 \\ H_{tz}(r) & H_{rz}(r) & 0 & 0 & H_{zz}(r) \end{pmatrix} \tag{1.3.1}$$

where $\alpha, \beta \in \{t, r, \theta, \varphi, z\}$. Moreover, in search of instability, the most interesting case for the present work is $\mu > 0$.

This section contains the analysis of the ODEs resulting from the linearised Einstein vacuum equation (I.5) for a mode solution of the form (1.3.1) when it is expressed in spherical gauge.

Definition 1.3.1 (Spherical Gauge). *A mode solution h of the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string spacetime $\mathrm{Schw}_4 \times \mathbb{R}$ is said to be in spherical gauge if it is of the form*

$$h_{\mu\nu} = e^{\mu t + ikz} \begin{pmatrix} H_t(r) & \mu H_v(r) & 0 & 0 & 0 \\ \mu H_v(r) & H_r(r) & 0 & 0 & -ikH_v(r) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -ikH_v(r) & 0 & 0 & H_z(r) \end{pmatrix}. \tag{1.3.2}$$

For the Schwarzschild black string spacetime $\mathrm{Schw}_4 \times \mathbb{S}^1_R$ one makes the same definition with the additional assumption that $kR \in \mathbb{Z}$.

Remark 1.3.1. *The terminology 'spherical gauge' is motivated by the fact that a mode solution of this form preserves the area of the spheres of the original spacetime.*

First, it is shown in section 1.3.1 that one can impose the gauge consistently at the level of modes, i.e., if there is a mode solution of the form (1.3.1), with $\mu \neq 0$ and either $k \neq 0$ or $\frac{dH_{tz}}{dr} - H_{rz} = 0$, then there is a mode solution of the form (1.3.2) differing from the original one by a pure gauge solution. In the case where $H_{tz} = 0$, $H_{rz} = 0$ and $H_{zz} = 0$ this consistency condition is already implicit in [95, 57]. In section 1.3.2, the original decoupling of the ODEs resulting from the linearised vacuum Einstein equation (I.5) and the spherical gauge ansatz (1.3.2) is reproduced from [57]. This decoupling results in a single ODE for the component $H_z(r)$ in equation (1.3.2). It is then shown, in section 1.3.3, that if $k \neq 0$ and $\mu \neq 0$, then mode solutions in spherical gauge (1.3.2) are not pure gauge. This is proved by examining the $tztz$-component of the linearised Weyl tensor $\overset{(1)}{W}$ associated to a mode solution in spherical gauge, which is gauge invariant by proposition 1.2.7. In this section it is also proved that if a non-trivial mode solution in spherical gauge has $\mu > 0$ (i.e. it grows exponentially in t) and $k \neq 0$ then $\overset{(1)}{W}_{tztz}$ is non-zero and also grows exponentially. By the gauge invariance of $\overset{(1)}{W}_{tztz}$ this behaviour will persist in all gauges. Next, in section 1.3.4, the admissible boundary conditions for the solution at the future event horizon $\mathcal{H}^+_A$ and finiteness conditions at spacelike infinity i^0_A are identified. Note this issue is subtle since, in general, both 'basis' elements for a mode solution h of the form (1.3.2) are, in fact, singular at the future event horizon $\mathcal{H}^+_A$ in this gauge. By adding a pure gauge perturbation, the admissible boundary conditions for the solution h in the form (1.3.2) can be identified. Moreover, this pure gauge solution can be chosen such that, after adding it, the harmonic/transverse-traceless gauge (1.1.8) conditions are satisfied. Finally, in section 1.3.5, the problem of constructing a linear mode instability of the form (1.3.1) is reduced to showing there exists a solution to the decoupled ODE for $H_z(r)$, with $\mu > 0$ and $k \neq 0$, that satisfies the admissible boundary conditions at the future event horizon $\mathcal{H}^+_A$ and spacelike infinity i^0_A (see proposition 1.3.15).

1.3.1 Consistency

In the paper [57], it is stated that any mode solution of the form in equation (1.3.1) with $H_{tz} = 0$, $H_{rz} = 0$ and $H_{zz} = 0$ can be brought to the spherical gauge form (1.3.2) by the addition of a pure gauge solution. Slightly more generally, one, in fact, has the following:

Proposition 1.3.2 (Consistency of the Spherical Gauge)**.** *Consider a mode solution h to the linearised Einstein vacuum equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string spacetime $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$ of the form (1.3.1) with $\mu \neq 0$. Further suppose that either $k \neq 0$ or $\frac{d}{dr} H_{tz} - \mu H_{rz} = 0$. Then there exists a pure gauge solution h_{pg} such that $h + h_{\mathrm{pg}}$ is of the form (1.3.2). It is in this sense that the spherical gauge (1.3.2) can be consistently imposed on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$.*

Proof. From section 1.2.2, a pure gauge solution is given by $h_{\mathrm{pg}} = 2\nabla_{(a}\xi_{b)}$ for a vector field ξ. So, $\tilde{h}_{ab} = h_{ab} + 2\nabla_{(a}\xi_{b)}$ is the new mode solution. Consider a diffeomorphism generating vector field of the form $\xi = e^{\mu t + ikz}(\zeta_t(r), \zeta_r(r), 0, 0, \zeta_z(r))$.

If $k \neq 0$, one can take

$$\zeta_r(r) = -\frac{H_{\theta\theta}(r)}{2(r-2M)}, \qquad \zeta_z(r) = -\frac{(H_{tz}(r) + ik\zeta_t(r))}{\mu}, \tag{1.3.3}$$

with

$$\zeta_t(r) = \frac{ir(r-2M)}{2Mk}(\partial_r H_{tz}(r) - \mu H_{rz}(r)) + \frac{r(r-2M)}{2M}H_{tr}(r) - \frac{r\mu}{2M}H_{\theta\theta}(r) \tag{1.3.4}$$

and immediately verify that $\tilde{h}$ is of the form (1.3.2).

If $\frac{d}{dr}H_{tz} - \mu H_{rz} = 0$, then one can take

$$\zeta_r(r) = -\frac{H_{\theta\theta}(r)}{2(r-2M)}, \qquad \zeta_z(r) = -\frac{(H_{tz}(r) + ik\zeta_t(r))}{\mu}, \tag{1.3.5}$$

with

$$\zeta_t(r) = \frac{r(r-2M)}{2M}H_{tr}(r) - \frac{r\mu}{2M}H_{\theta\theta}(r) \tag{1.3.6}$$

and immediately verify that $\tilde{h}$ is of the form (1.3.2). □

1.3.2 Reduction to ODE

Under a spherical gauge ansatz (1.3.2) with $\mu \neq 0$ and $k \neq 0$, the linearised vacuum Einstein equation (I.5) reduces to a system of coupled ODEs for the components H_t, H_v, H_r and H_z. This system of ODEs can be decoupled to the single ODE for $\mathfrak{H} \doteq H_z$

$$\frac{d^2\mathfrak{H}}{dr^2}(r) + P_k(r)\frac{d\mathfrak{H}}{dr}(r) + \Big(Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}\Big)\mathfrak{H}(r) = 0, \tag{1.3.7}$$

with

$$P_k(r) \doteq \frac{12M}{r(k^2r^3+2M)} - \frac{5}{r} + \frac{1}{r-2M}, \tag{1.3.8}$$

$$Q_k(r) \doteq \frac{6M}{r^2(r-2M)} - \frac{rk^2}{r-2M} - \frac{12M^2}{r^2(r-2M)(k^2r^3+2M)}. \tag{1.3.9}$$

The following proposition establishes this decoupling of the linearised vacuum Einstein equation (I.5) to the ODE (1.3.7) and the construction of a mode solution h in spherical gauge (1.3.2) from a solution $\mathfrak{H}$ to the ODE (1.3.7).

Proposition 1.3.3. *Given a mode solution h in spherical gauge (1.3.2) with $\mu \neq 0$ and $k \neq 0$ on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$, the ODE (1.3.7) is satisfied by h_{zz}. Conversely, given a $C^2((2M,\infty))$ solution $\mathfrak{H}(r)$ to the ODE (1.3.7) with $k \neq 0$ and $\mu \neq 0$, one can construct a mode solution h in spherical gauge (1.3.2) to the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$. If $kR \in \mathbb{Z}$ then h induces a mode solution on $\mathrm{Schw}_4 \times \mathbb{S}^1_R$.*

Remark 1.3.4. *Since $P_k(r)$ and $Q_k(r)$ are real analytic any $C^2((2M,\infty))$ solution $\mathfrak{H}(r)$ to the ODE (1.3.7) is, in fact, real analytic (see theorem 3.1 in chapter 5 of Olver [97] for more details).*

Proof. Let h be a mode solution in spherical gauge (1.3.2) with $\mu \in \mathbb{R}$ and $k \in \mathbb{R}$ satisfying the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. Equivalently, the following system of ODE has to be satisfied:

$$\mu k H_r = \frac{2M\mu k}{r(r-2M)} H_v, \tag{1.3.10}$$

$$\mu k^2 H_v = \frac{\mu}{2}\frac{dH_z}{dr} - \frac{\mu(r-2M)H_r}{r^2} - \frac{\mu M H_z}{2r(r-2M)}, \tag{1.3.11}$$

$$k\frac{dH_t}{dr} = \frac{kMH_t}{r(r-2M)} - \frac{k(r-2M)(2r-3M)H_r}{r^3} + 2\mu^2 k H_v, \tag{1.3.12}$$

$$\begin{aligned} H_t = {}& \frac{(r-2M)(r(k^2-\mu^2)-2Mk^2)}{M} H_v + \frac{(r-2M)^2(r+M)}{Mr^2} H_r \\ & + \frac{(r-2M)^3}{2Mr}\frac{dH_r}{dr} - \frac{(r-2M)^2}{2M}\frac{dH_z}{dr} + \frac{r(r-2M)}{2M}\frac{dH_t}{dr}, \end{aligned} \tag{1.3.13}$$

$$\frac{d^2H_z}{dr^2} = k^2 H_r + \frac{r^2(\mu^2 H_z - k^2 H_t)}{(r-2M)^2} + \frac{2(r-M)}{r(r-2M)}\Big(2k^2 H_v - \frac{dH_z}{dr}\Big) + 2k^2\frac{dH_v}{dr}, \tag{1.3.14}$$

$$\begin{aligned} \frac{d^2H_z}{dr^2} = {}& \frac{2M(2r-3M)}{r(r-2M)^3} H_t - \frac{(6M^2-(\mu^2+k^2)r^4+2Mr(k^2r^2-2))}{r^3(r-2M)} H_r \\ & - \frac{2M(2Mk^2+r(\mu^2-k^2))}{r(r-2M)^2} H_v - \frac{2\mu^2 r + 4Mk^2 - 2k^2 r}{r-2M}\frac{dH_v}{dr} \\ & + \frac{2r-3M}{r^2}\frac{dH_r}{dr} - \frac{M}{r(r-2M)}\frac{dH_z}{dr} - \frac{M}{(r-2M)^2}\frac{dH_t}{dr} + \frac{r}{r-2M}\frac{d^2H_t}{dr^2}, \end{aligned} \tag{1.3.15}$$

$$\begin{aligned} \frac{d^2H_t}{dr^2} = {}& \frac{k^2r^4 - 2Mk^2r^3 - 2M^2}{r^2(r-2M)^2} H_t - \Big(\mu^2 + \frac{2M^2}{r^4}\Big) H_r - \frac{r\mu^2}{r-2M} H_z \\ & + \frac{4\mu^2 r^2 + 4M^2k^2 - 2Mr(3\mu^2+k^2)}{r^2(r-2M)} H_v - \frac{M(r-2M)}{r^3}\frac{dH_r}{dr} \\ & - \frac{2r-5M}{r(r-2M)}\frac{dH_t}{dr} + 2\mu^2\frac{dH_v}{dr} + \frac{M}{r^2}\frac{dH_z}{dr}. \end{aligned} \tag{1.3.16}$$

Now, if $\mu \neq 0$ and $k \neq 0$, then from equations (1.3.10) and (1.3.11) one can find H_v in terms of H_z and $\frac{dH_z}{dr}$. This can then be used in equation (1.3.12) to give and equation for $\frac{dH_t}{dr}$ in terms of H_t, H_z and $\frac{dH_z}{dr}$. All of these expressions can be used to express H_t in terms of H_z, $\frac{dH_z}{dr}$

and $\frac{d^2 H_z}{dr^2}$ via equation (1.3.13). The resulting equations are

$$H_r(r) = -\frac{M^2 r}{(r-2M)^2(k^2r^2+2M)}H_z(r) + \frac{Mr^2}{(r-2M)(k^2r^2+2M)}\frac{dH_z}{dr}, \tag{1.3.17}$$

$$H_v(r) = -\frac{Mr^2}{(2(r-2M)(k^2r^2+2M)}H_z(r) + \frac{r^3}{2(k^2r^2+2M)}\frac{dH_z}{dr}, \tag{1.3.18}$$

$$\begin{aligned} H_t(r) = {} & \frac{2M^2(r-3M) + Mk^2r^3(2r-5M) - k^4r^6(r-2M)}{r(k^2r^3+2M)^2}H_z \\ & - \frac{2(r-2M)(M(r-4M)+(2r-5M)k^2r^3)}{(k^2r^3+2M)^2}\frac{dH_z}{dr} + \frac{r(r-2M)^2}{k^2r^3+2M}\frac{d^2H_z}{dr^2}. \end{aligned} \tag{1.3.19}$$

Finally, one can use the above expressions to obtain a decoupled ODE for $\mathfrak{H} \doteq H_z$, namely

$$\frac{d^2\mathfrak{H}}{dr^2}(r) + P_k(r)\frac{d\mathfrak{H}}{dr}(r) + \Big(Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}\Big)\mathfrak{H}(r) = 0, \tag{1.3.20}$$

with

$$P_k(r) \doteq \frac{12M}{r(k^2r^3+2M)} - \frac{5}{r} + \frac{1}{r-2M}, \tag{1.3.21}$$

$$Q_k(r) \doteq \frac{6M}{r^2(r-2M)} - \frac{rk^2}{r-2M} - \frac{12M^2}{r^2(r-2M)(k^2r^3+2M)}. \tag{1.3.22}$$

Conversely, given any $C^2((2M,\infty))$ solution $\mathfrak{H}(r)$ to the ODE (1.3.7) with $k \neq 0$ and $\mu \neq 0$ one can define $H_z(r) = \mathfrak{H}(r)$. As noted in remark 1.3.4, $P_k(r)$ and $Q_k(r)$ are real analytic, so any $C^2((2M,\infty))$ solution $\mathfrak{H}(r)$ to the ODE (1.3.7) is, in fact, real analytic. Therefore, since $k \neq 0$, one can use equations (1.3.17)–(1.3.19) to construct $H_t(r)$, $H_r(r)$ and $H_v(r)$. These then define the components of a mode solution h in spherical gauge (1.3.2). Explicitly

$$h = e^{\mu t + ikz}\begin{pmatrix} H_t(r) & \mu H_v(r) & 0 & 0 & 0 \\ \mu H_v(r) & H_r(r) & 0 & 0 & -ikH_v(r) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -ikH_v(r) & 0 & 0 & H_z(r) \end{pmatrix}. \tag{1.3.23}$$

If the ODE (1.3.7) is satisfied and (1.3.17)–(1.3.19) define H_r, H_v and H_t, then equations (1.3.10)–(1.3.16) are also satisfied. Therefore, a mode solution h constructed in this manner solves the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$. If $kR \in \mathbb{Z}$ then this construction also gives a mode solution h which solves the linearised vacuuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. □

Remark 1.3.5. *If $k = 0$ and $\mu \neq 0$, then one can add an additional pure gauge solution h_{pg} to a mode solution h in spherical gauge (1.3.2) such that $h + h_{\text{pg}}$ is also in spherical gauge (1.3.2) with $H_t(r) \equiv 0$. The relevant choice of pure gauge solution is given by $(h_{\text{pg}})_{ab} = 2\nabla_{(a}\xi_{b)}$ with*

$$\xi = e^{\mu t}\Big(-\frac{H_t(r)}{2\mu}, 0, 0, 0, 0\Big). \tag{1.3.24}$$

A mode solution h in spherical gauge with $H_t(r) \equiv 0$ satisfying the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string is then again equivalent to the system of ODE (1.3.10)-(1.3.16) (with $k = 0$ and $H_t \equiv 0$) being satisfied. Equations (1.3.10) and (1.3.12) are automatically satisfied by $k = 0$. The equation (1.3.14) automatically gives the decoupled equation (1.3.7) for H_z. Then, equation (1.3.11) can be solved for H_r in terms of H_z and $\frac{dH_z}{dr}$. This gives the relation in equation (1.3.17) for H_r with $k = 0$. Equation (1.3.13) can be used to solve for H_v in terms of H_z and $\frac{dH_z}{dr}$. At this point, the equations (1.3.15) and (1.3.16) are automatically satisfied. Therefore, again a solution to the ODE (1.3.7) induces a mode solution in spherical gauge with $H_t = 0$.

1.3.3 Excluding Pure Gauge Perturbations

This section contains two proofs that if $k \neq 0$ and $\mu \neq 0$ then a non-trivial mode solution h of the form (1.3.2) cannot be a pure gauge solution. One can prove this directly via the following proposition:

Proposition 1.3.6. *Suppose $k \neq 0$ and $\mu \neq 0$. A non-trivial mode solution h in spherical gauge (1.3.2) of the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$ cannot be pure gauge.*

Proof. If h is pure gauge, it must be possible to write $h_{ab} = 2\nabla_{(a}\xi_{b)}$ for some vector field ξ. Therefore one finds

$$h_{zz} = H_z(r)e^{\mu t+ikz} \implies 2\partial_z\xi_z = H_z(r)e^{\mu t+ikz}, \tag{1.3.25}$$

$$h_{z\theta} = 0 \implies \partial_\theta\xi_z + \partial_z\xi_\theta = 0. \tag{1.3.26}$$

Applying ∂_z to the equation (1.3.26), using that partial derivatives commute and that, from equation (1.3.25), $\partial_z\xi_z$ clearly does not depend on θ gives

$$\partial_z^2\xi_\theta = 0. \tag{1.3.27}$$

Next, $h_{\theta\theta} = 0$ implies

$$\partial_\theta\xi_\theta - \Gamma^r_{\theta\theta}\xi_r = 0. \tag{1.3.28}$$

From appendix A.1, $\Gamma^r_{\theta\theta} = (r-2M)$. Hence, taking two derivatives of (1.3.28) in the z direction and using $\partial_z^2 \xi_\theta = 0$ gives

$$\Gamma^r_{\theta\theta}\partial_z^2\xi_r = (r-2M)\partial_z^2\xi_r = 0. \tag{1.3.29}$$

Therefore, $\partial_z^2\xi_r = 0$ on $\mathcal{E}_A$.

From the h_{rr} component one has,

$$2\partial_r\xi_r - 2\Gamma^r_{rr}\xi_r = 2\partial_r\xi_r + \frac{2M}{r(r-2M)}\xi_r = H_r e^{\mu t + ikz} \tag{1.3.30}$$

where one uses $\Gamma^r_{rr} = -\frac{M}{r(r-2M)}$ from appendix A.1. Taking the second z derivative of equation (1.3.30) and using $\partial_z^2\xi_r = 0$ on $\mathcal{E}_A$ gives $k^2 H_r = 0$ on $\mathcal{E}_A$. Since $k \neq 0$, this implies $H_r \equiv 0$ on the exterior $\mathcal{E}_A$. Since $k \neq 0$ and $\mu \neq 0$, equation (1.3.10) implies that if $H_r = 0$ on $\mathcal{E}_A$, then $H_v \equiv 0$ on $\mathcal{E}_A$. Using the h_{zr} component, one finds

$$\partial_z\xi_r + \partial_r\xi_z = -ikH_v e^{\mu t + ikz} = 0 \implies \partial_r(\partial_z\xi_z) = 0 \implies \frac{dH_z}{dr} = 0 \quad \text{on} \quad \mathcal{E}_A, \tag{1.3.31}$$

where one uses the identity $\partial_z^2\xi_r = 0$ on $\mathcal{E}_A$ in the first implication and that $\partial_z\xi_z = H_z(r)e^{\mu t + ikz}$ in the second implication. The linearised vacuum Einstein equation (I.5) under this ansatz (equation (1.3.11)) then implies $H_z \equiv 0$ on $\mathcal{E}_A$ and therefore, from equations (1.3.12) and (1.3.13), $H_t \equiv 0$ on $\mathcal{E}_A$. Hence, $h \equiv 0$ on $\mathcal{E}_A$. □

Perhaps more satisfactorily one can establish that, if h is a non-trival mode solution in spherical gauge (1.3.2) with $k \neq 0$ and $\mu \neq 0$, then the $tztz$-component of the linearised Weyl tensor $\overset{(1)}{W}$ is non-vanishing. Moreover, if h has $\mu > 0$ then $\overset{(1)}{W}_{tztz}$ grows exponentially. Since $\overset{(1)}{W}_{tztz}$ is gauge invariant this behaviour persists in all gauges. More precisely, one has the following proposition:

Proposition 1.3.7. *Suppose $k \neq 0$, $\mu \neq 0$ and h is a non-trivial mode solution in spherical gauge (1.3.2) of the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of the Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ or $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. Then $\overset{(1)}{W}_{tztz}$ is non-vanishing and h is not pure gauge. Moreover, if $\mu > 0$ then $\overset{(1)}{W}_{tztz}$ also grows exponentially.*

Proof. By proposition 1.2.7, $\overset{(1)}{W}_{tztz}$ is gauge invariant. Hence, if $\overset{(1)}{W}_{tztz}$ is non-zero when evaluated on a non-trivial mode solution h in spherical gauge (1.3.2), h cannot be pure gauge. Using proposition 1.2.6 gives the following expression for $\overset{(1)}{W}_{tztz}$:

$$\overset{(1)}{W}_{tztz} = \frac{e^{\mu t + ikz}}{2}\Big(k^2 H_t(r) - \frac{2Mk^2(r-2M)}{r^3}H_v(r) + \frac{M(r-2M)}{r^3}\frac{dH_z}{dr} - \mu^2 H_z(r)\Big). \tag{1.3.32}$$

If $k \neq 0$ and $\mu \neq 0$, one can use the equations (1.3.17)-(1.3.19) and the ODE (1.3.7) to simplify this to

$$\overset{(1)}{W}_{tztz} = - e^{\mu t + ikz}\Big(\frac{M(r-2M)(k^2r^3(3r-7M)-2M^2)}{r^3(k^2r^3+2M)^2}\frac{dH_z}{dr}(r) + \frac{M(k^4r^3(r-2M)+k^2(\mu^2r^4-Mr+2M^2)+2M\mu^2r)}{r(k^2r^3+2M)^2}H_z(r)\Big). \tag{1.3.33}$$

Suppose $\overset{(1)}{W}_{tztz} \equiv 0$ identically, then

$$\frac{dH_z}{dr}(r) = \frac{r^2(Mr(2k^4r^2+k^2-2\mu^2)-k^2(\mu^2+k^2)r^4-2M^2k^2)}{(r-2M)(3k^2r^4-7Mk^2r^3-2M^2)}H_z(r). \tag{1.3.34}$$

Substituting this into the ODE (1.3.7) gives that either

$$k^4r^3(r-2M)^2 + Mr(4r-9M)\mu^2 + r^5\mu^4 + k^2(r-2M)(2r^4\mu^2-2Mr+5M^2) = 0 \tag{1.3.35}$$

for all $r \in (2M, \infty)$ or $H_z(r) \equiv 0$. If $\mu \neq 0$ and $k \neq 0$ then the polynomial in equation (1.3.35) has at most 5 roots in $r \in (2M, \infty)$. Therefore, if $\overset{(1)}{W}_{tztz} = 0$ then $H_z(r) = 0$ which is a contradiction. Moreover, since $\overset{(1)}{W}_{tztz} \neq 0$, it is clear from equation (1.3.33) that if $\mu > 0$ then $\overset{(1)}{W}_{tztz}$ grows exponentially. □

1.3.4 Admissible Boundary Conditions

One can construct two sets of distinguished solutions to the ODE (1.3.7) associated to the "end points" of the interval $(2M, \infty)$. Note that, by definition A.2.1 from appendix A.2, $r = 2M$ is a regular singularity, as $2M$ is not an ordinary point and

$$(r-2M)P_k(r) \quad \text{and} \quad (r-2M)^2\Big(Q_k(r) - \frac{\mu^2r^2}{(r-2M)^2}\Big) \tag{1.3.36}$$

are analytic near $r = 2M$. By definition A.2.3, the ODE (1.3.7) has an irregular singularity at infinity, since there exist convergent series expansions

$$P_k(r) = \sum_{n=0}^{\infty}\frac{p_n}{r^n} \quad \text{and} \quad Q_k(r) - \frac{\mu^2r^2}{(r-2M)^2} = \sum_{n=0}^{\infty}\frac{q_n}{r^n} \tag{1.3.37}$$

in a neighbourhood of infinity with $p_0 = 0$, $p_1 = -4$, $q_0 = -(k^2+\mu^2)$ and $q_1 = -2M(k^2+2\mu^2)$. The asymptotic analysis of the ODEs around these points is examined in the following two subsections. This analysis of the ODE (1.3.7) near $r = 2M$ and $r = \infty$ will lead to the identification of the admissible boundary conditions for a mode solution h in spherical gauge (1.3.2) of the linearised Einstein vacuum equation (I.5).

The Future Event Horizon $\mathcal{H}^+_A$

The goal of this section is to identify the admissible boundary conditions for a solution $\mathfrak{H}$ to the ODE (1.3.7) near $r = 2M$. This requires one to understand the behaviour near $r = 2M$ of the mode solution h in spherical gauge (1.3.2) of the linearised vacuum Einstein equation (I.5) which results (through the construction in proposition 1.3.3) from $\mathfrak{H}$.

Associated with the future event horizon $\mathcal{H}^+_A$, there exists a basis $\mathfrak{H}^{2M,\pm}$ for solutions to the ODE (1.3.7). From $\mathfrak{H}^{2M,\pm}$ one can examine the behaviour near $r = 2M$ of any mode solution h in spherical gauge (1.3.2) with $\mu \neq 0$ and $k \neq 0$ through proposition 1.3.3. A mode solution h in spherical gauge (1.3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{H}^{2M,-}$ never smoothly extends to the future event horizon. A mode solution h in spherical gauge (1.3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{H}^{2M,+}$ also does not smoothly extend to the future event horizon unless μ satisfies particular conditions. However, if h is a mode solution in spherical gauge (1.3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{H}^{2M,+}$ then, after the addition of a pure gauge solution h_{pg}, it turns out one can smoothly extend $h + h_{\mathrm{pg}}$ to the future event horizon. Moreover, it will be shown that $h + h_{\mathrm{pg}}$ satisfies the harmonic/transverse-traceless gauge (1.1.8) conditions. This will be the content of proposition 1.3.9.

Remark 1.3.8. *Suppose h is a mode solution in spherical gauge (1.3.2) with $\mu > 0$ and $k \neq 0$ constructed from $\mathfrak{H}^{2M,-}$. This work does not claim that there does not exist a pure gauge solution h_{pg} such that $h + h_{\mathrm{pg}}$ extends smoothly to the future event horizon $\mathcal{H}^+_A$. It is simply that, after adding the particular pure gauge solution h_{pg} generated by $\xi \in \mathfrak{X}(M)$ in equation* (1.3.65) *below, $h + h_{\mathrm{pg}}$ does not extend smoothly to the future event horizon if h arises from $\mathfrak{H}^{2M,-}$ (see equations* (1.3.78)-(1.3.81)*). It could be the case that there does not exist a pure gauge solution h_{pg} such that $h + h_{\mathrm{pg}}$ extends smoothly to the future event horizon $\mathcal{H}^+_A$. However, it is unnecessary to establish such a statement since, for the purposes of establishing an instability, only needs to exhibit a mode which extends smoothly to the future event horizon. This is precisely what proposition 1.3.9 guarantees if the mode solution is associated to $\mathfrak{H}^{2M,+}$.*

First, some preliminaries. The coefficients of the ODE (1.3.7) extend meromorphically to $r = 2M$ and behave asymptotically as

$$P_k(r) = \frac{1}{r-2M} + \mathcal{O}(1) \qquad Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2} = -\frac{4M^2\mu^2}{(r-2M)^2} + \mathcal{O}\Big(\frac{1}{r-2M}\Big). \quad (1.3.38)$$

So one may write the ODE (1.3.7) as

$$\frac{d^2\mathfrak{H}}{dr^2} + \Big(\frac{1}{r-2M} + \mathcal{O}(1)\Big)\frac{d\mathfrak{H}}{dr} - \Big(\frac{4M^2\mu^2}{(r-2M)^2} + \mathcal{O}\Big(\frac{1}{r-2M}\Big)\Big)\mathfrak{H} \;=\; 0. \quad (1.3.39)$$

From appendix A.2, the indicial equation associated to the ODE (1.3.39) is

$$I(\alpha) = \alpha^2 - 4M^2\mu^2, \tag{1.3.40}$$

which has roots

$$\alpha_\pm \doteq \pm 2M\mu. \tag{1.3.41}$$

If $\alpha_+ - \alpha_- = 4M\mu \notin \mathbb{Z}$, then one can deduce from theorem A.2.3 the asymptotic basis for solutions near $r = 2M$. If $\alpha_+ - \alpha_- = 4M\mu \in \mathbb{Z}$ then the relevant result for the asymptotic basis of solutions is theorem A.2.5. Combining the results of theorems A.2.3 and A.2.5 one has the following basis for solutions for $\mu > 0$

$$\mathfrak{H}^{2M,+}(r) \doteq (r-2M)^{2M\mu}\sum_{n=0}^{\infty} a_n^+ (r-2M)^n, \tag{1.3.42}$$

$$\mathfrak{H}^{2M,-}(r) \doteq \begin{cases} \sum_{n=0}^{\infty} \frac{a_n^-(r-2M)^n}{(r-2M)^{2M\mu}} + C_N \mathfrak{H}^{2M,+}\ln(r-2M) & \text{if} \quad 4M\mu = N \in \mathbb{Z}_{>0} \\ (r-2M)^{-2M\mu}\sum_{n=0}^{\infty} a_n^-(r-2M)^n & \text{otherwise,} \end{cases} \tag{1.3.43}$$

where the coefficents a_n^+, a_n^- and the anomalous term C_N can be calculated recursively (see theorems A.2.3 and A.2.5). A general solution to the ODE (1.3.7) will be of the form

$$\mathfrak{H}(r) = k_1 \mathfrak{H}^{2M,+}(r) + k_2 \mathfrak{H}^{2M,-}(r) \tag{1.3.44}$$

with $k_1, k_2 \in \mathbb{R}$.

If $4M\mu$ is *not* an integer or $4M\mu$ is an integer and $C_N = 0$, then the asymptotic basis for solutions for $\mu > 0$ reduces to

$$\mathfrak{H}^{2M,+}(r) = (r-2M)^{2M\mu}\sum_{n=0}^{\infty} a_n^+ (r-2M)^n, \tag{1.3.45}$$

$$\mathfrak{H}^{2M,-}(r) = (r-2M)^{-2M\mu}\sum_{n=0}^{\infty} a_n^- (r-2M)^n. \tag{1.3.46}$$

In equations (1.3.45) and (1.3.46), the first order coefficients of the basis can be calculated to be

$$a_1^\pm = \frac{\pm\mu(20M^2k^2 - 1) + 4M(\mu^2 - k^2 + 4M^2\mu^2k^2 + 2M^2k^4)}{(1 \pm 4M\mu)(4M^2k^2+1)}. \tag{1.3.47}$$

The main result of this section is the following:

Proposition 1.3.9. *Suppose $\mu > 0$, $k \neq 0$ and let $\mathfrak{H}$ be a solution to the ODE (1.3.7). Let h be the mode solution on the exterior $\mathcal{E}_A$ of the Schwarzschild black string* $\mathrm{Schw}_4 \times \mathbb{R}$ *constructed* *from $H_z = \mathfrak{H}$ in proposition 1.3.3. Then there exists a pure gauge solution h_{pg} such that $h + h_{\mathrm{pg}}$ extends to a smooth solution of the linearised vacuum Einstein equation (I.5) at the future event horizon $\mathcal{H}_A^+$ if $k_2 = 0$, where k_2 is defined in equation (1.3.44). Moreover, $h + h_{\mathrm{pg}}$ can be chosen to satisfy the harmonic/transverse-traceless gauge (1.1.8) conditions.*

Remark 1.3.10. *To determine admissible boundary conditions of $\mathfrak{H}$ at $r = 2M$ it is essential that one works in coordinates that extend regularly across this hypersurface. A good choice is ingoing Eddington–Finkelstein coordinates $(v, r, \theta, \varphi, z)$ defined by*

$$v = t + r_\star(r), \qquad r_\star(r) = r + 2M \log |r - 2M|. \tag{1.3.48}$$

Also note that for the boundary conditions to be admissible, one needs to consider all components of the mode solution h constructed from $\mathfrak{H}$ via proposition 1.3.3. These remarks will be implemented in the proof of proposition 1.3.9.

Before proving the statement of proposition 1.3.9 it is useful to prove the following lemma:

Lemma 1.3.11. *Let h be a mode solution of the linearised vacuum Einstein equation (I.5) of the form*

$$h_{\alpha\beta} = e^{\mu t + ikz} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & 0 \\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & H_{\theta\theta}(r)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.3.49}$$

Then h satisfies the harmonic/transverse-traceless gauge conditions:

$$\begin{cases} \nabla^a h_{ab} = 0 \\ g^{ab} h_{ab} = 0 \end{cases} \tag{1.3.50}$$

if $k \neq 0$.

Proof. First, it is instructive to write out explicit expressions for $\nabla_c h_{ab}$ and $\nabla_c \nabla_d h_{ab}$ in coordinates. These are the following:

$$\nabla_\gamma h_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} - \Gamma^\lambda_{\gamma\alpha} h_{\lambda\beta} - \Gamma^\lambda_{\gamma\beta} h_{\alpha\lambda} \tag{1.3.51}$$

$$\begin{aligned} \nabla_\gamma \nabla_\delta h_{\alpha\beta} = {} & \partial_\gamma(\partial_\delta h_{\alpha\beta} - \Gamma^\lambda_{\delta\alpha} h_{\lambda\beta} - \Gamma^\lambda_{\delta\beta} h_{\alpha\lambda}) - \Gamma^\mu_{\gamma\delta}(\partial_\mu h_{\alpha\beta} - \Gamma^\lambda_{\mu\alpha} h_{\lambda\beta} - \Gamma^\lambda_{\mu\beta} h_{\alpha\lambda}) \\ & - \Gamma^\mu_{\gamma\alpha}(\partial_\delta h_{\mu\beta} - \Gamma^\lambda_{\delta\mu} h_{\lambda\beta} - \Gamma^\lambda_{\delta\beta} h_{\mu\lambda}) - \Gamma^\mu_{\gamma\beta}(\partial_\delta h_{\alpha\mu} - \Gamma^\lambda_{\delta\alpha} h_{\lambda\mu} - \Gamma^\lambda_{\delta\mu} h_{\alpha\lambda}). \end{aligned} \tag{1.3.52}$$

If one takes the ansatz (1.3.49) and $\alpha = z$ in equation (1.3.52), then, since $h_{z\beta} = 0$ for all $\beta \in \{t, r, \theta, \varphi, z\}$ and, from appendix A.1, $\Gamma^{\lambda}_{z\beta} = 0$ for all $\beta, \lambda \in \{t, r, \theta, \varphi, z\}$,

$$\nabla_\gamma \nabla_\delta h_{\alpha\beta} = 0 \qquad (\alpha = z). \tag{1.3.53}$$

Hence,

$$g^{\gamma\delta} \nabla_\gamma \nabla_\delta h_{\alpha\beta} = 0 \qquad (\alpha = z) \tag{1.3.54}$$

$$g^{\delta\beta} \nabla_\gamma \nabla_\delta h_{\alpha\beta} = 0 \qquad (\alpha = z). \tag{1.3.55}$$

Consider the linearised vacuum Einstein equation (I.5) in coordinates

$$g^{\gamma\delta} \nabla_\gamma \nabla_\delta h_{\alpha\beta} + \nabla_\alpha \nabla_\beta \mathrm{Tr}_g h - \nabla_\alpha \nabla^\gamma h_{\beta\gamma} - \nabla_\beta \nabla^\gamma h_{\alpha\gamma} + 2R_\alpha{}^\gamma{}_\beta{}^\delta h_{\gamma\delta} = 0. \tag{1.3.56}$$

Since, from equations (1.3.54)–(1.3.55) and, from appendix A.1, $R_{z\beta\gamma\delta} = 0$, it follows that the linearised vacuum Einstein equation (I.5) in local coordinates with $\alpha = z$ and under the ansatz (1.3.49) reduces to

$$\nabla_z (\nabla_\beta \mathrm{Tr}_g h - \nabla^\gamma h_{\beta\gamma}) = 0. \tag{1.3.57}$$

Further, $\nabla_z = \partial_z$, so using the explicit z-dependence of the ansatz (1.3.49), the equation (1.3.57) reduces to

$$k(\nabla_\beta \mathrm{Tr}_g h - \nabla^\gamma h_{\beta\gamma}) = 0. \tag{1.3.58}$$

Since $k \neq 0$, the harmonic gauge condition

$$\nabla_\beta \mathrm{Tr}_g h - \nabla^\gamma h_{\beta\gamma} = 0 \tag{1.3.59}$$

is satisfied. If $\beta = z$ then, using equation (1.3.51) and $\nabla_z = \partial_z$, equation (1.3.59) reduces to

$$\partial_z \mathrm{Tr}_g h = k \mathrm{Tr}_g h = 0 \implies \mathrm{Tr}_g h = 0 \tag{1.3.60}$$

since $k \neq 0$. Substituting (1.3.60) into equation (1.3.59) gives the transverse condition

$$\nabla^\gamma h_{\beta\gamma} = 0. \tag{1.3.61}$$

□

Proof of Proposition 1.3.9. Consider $H_z^{2M,\pm} \doteq \mathfrak{H}^{2M,\pm}$ where $\mathfrak{H}^{2M,\pm}$ are given by equations (1.3.45) and (1.3.46) with first order coefficients (1.3.47). Taking $k_2 = 0$ is equivalent to examining the basis element $H_z^{2M,+}$. Since $\mu > 0$ and $k \neq 0$, one can use proposition 1.3.3 to construct the components H_t, H_r and H_v associated to $H_z^{2M,\pm}$. Substituting the basis into equations (1.3.17)–(1.3.19), one finds

$$H_r^{2M,\pm} = \frac{(r-2M)^{\pm 2M\mu}}{(r-2M)^2}\Big(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(4M^2(2\mu^2+k^2) \pm 6M\mu - 1)}{2(1+4M^2k^2)}(r-2M) + \mathcal{O}((r-2M)^2)\Big), \tag{1.3.62}$$

$$H_t^{2M,\pm} = (r-2M)^{\pm 2M\mu}\Big(\frac{(1+4M\mu)(4M\mu-1)}{4(1+4M^2k^2)} \tag{1.3.63}$$
$$+ \frac{3+4M^2(8\mu^2-k^2) \pm 2M\mu(8M^2(2\mu^2+k^2)-11)}{8M(1+4M^2k^2)}(r-2M) + \mathcal{O}((r-2M)^2)\Big),$$

$$H_v^{2M,\pm} = (r-2M)^{-1+2M\mu}\Big(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(2M^2(2\mu^2+k^2) - 1 \pm 5M\mu)}{1+4M^2k^2}(r-2M) + \mathcal{O}((r-2M)^2)\Big). \tag{1.3.64}$$

Consider a pure gauge solution $h_{\mathrm{pg}} = 2\nabla_{(a}\xi_{b)}$ generated by the following vector field

$$\xi = e^{\mu t + ikz}\Big(-\frac{\mu H_z(r)}{2k^2}, \frac{2k^2 H_v(r) - \frac{dH_z}{dr}(r)}{2k^2}, 0, 0, \frac{iH_z(r)}{2k}\Big) \tag{1.3.65}$$

where H_v is defined via equation (1.3.18). This gives a new solution to the linearised vacuum Einstein equation (I.5)

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)} = e^{\mu t + ikz}\begin{pmatrix} \tilde{H}_{tt}(r) & \tilde{H}_{tr}(r) & 0 & 0 & 0 \\ \tilde{H}_{tr}(r) & \tilde{H}_{rr}(r) & 0 & 0 & 0 \\ 0 & 0 & \tilde{H}_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & \tilde{H}_{\theta\theta}(r)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.3.66}$$

with the following expressions for the matrix components

$$\tilde{H}_{tt}(r) = c_1(r)H_z(r) + c_2(r)\frac{dH_z}{dr}(r), \tag{1.3.67}$$

$$\tilde{H}_{\theta\theta}(r) = c_3(r)H_z(r) + c_4(r)\frac{dH_z}{dr}(r), \tag{1.3.68}$$

$$\tilde{H}_{rr}(r) = \frac{r^2}{(r-2M)^2}\tilde{H}_{tt}(r) - \frac{2}{r(r-2M)}\tilde{H}_{\theta\theta}(r), \tag{1.3.69}$$

$$\tilde{H}_{tr}(r) = -\frac{2M\mu}{k^2(2M+r^3k^2)}\Big(\frac{dH_z}{dr}(r) - \frac{M}{r(r-2M)}H_z(r)\Big), \tag{1.3.70}$$

where

$$
\begin{aligned}
c_1(r) &\doteq \frac{6M^2(r-2M)}{r(k^2r^3+2M)^2} - \frac{2M(r-2M)}{r(k^2r^3+2M)} + \frac{\mu^2 r^3}{k^2r^3+2M} - \frac{\mu^2}{k^2}, \\
c_2(r) &\doteq \frac{M(r-2M)}{k^2r^3} - \frac{M(r-2M)}{k^2r^3+2M} - \frac{6M(4M^2-4Mr+r^2)}{(k^2r^3+2M)^2}, \\
c_3(r) &\doteq -\frac{Mr^2}{k^2r^3+2M}, \qquad c_4(r) \doteq \frac{r^3(r-2M)}{k^2r^3+2M} - \frac{r-2M}{k^2}.
\end{aligned} \tag{1.3.71}
$$

Note that equations (1.3.7) and (1.3.17)–(1.3.19) have been used to derive equations (1.3.67)–(1.3.70). By lemma 1.3.11, this new mode solution (1.3.66) satisfies the harmonic/transverse-traceless gauge:

$$
\begin{cases} g^{\mu\nu}\tilde{h}_{\mu\nu} = 0 \\ \nabla^{\mu}\tilde{h}_{\mu\nu} = 0. \end{cases} \tag{1.3.72}
$$

As remarked above (see remark 1.3.10), to determine admissible boundary conditions of $\mathfrak{H}$ at $r = 2M$ it is essential that one works in coordinates that extend regularly across this hypersurface. Moreover, to identify the boundary conditions to be admissible, one needs to consider all components of the mode solution h constructed from $\mathfrak{H}$ via proposition 1.3.3. The following formulas give the transformation to ingoing Eddington–Finkelstein coordinates for the components of the mode solution h defined in equation (1.3.66):

$$
\begin{aligned}
\tilde{H}'_{vv} &= \Big(\frac{\partial t}{\partial v}\Big)^2 \tilde{H}_{tt}, \\
\tilde{H}'_{vr} &= \Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial r}\Big)\tilde{H}_{tr} + \Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial t}{\partial r}\Big)\tilde{H}_{tt} \\
&= \tilde{H}_{tr} - \frac{r}{r-2M}\tilde{H}_{tt}, \\
\tilde{H}'_{rr} &= \Big(\frac{\partial t}{\partial r}\Big)^2 \tilde{H}_{tt} + \Big(\frac{\partial t}{\partial r}\Big)\Big(\frac{\partial r}{\partial r}\Big)\tilde{H}_{tr} + \Big(\frac{\partial r}{\partial r}\Big)^2 \tilde{H}_{rr} \\
&= \frac{r^2}{(r-2M)^2}\tilde{H}_{tt} - \frac{r}{r-2M}\tilde{H}_{tr} + \tilde{H}_{rr},
\end{aligned} \tag{1.3.73}
$$

where one uses $t = v - r_\star(r)$ with $r_\star(r) = r + 2M\log|r-2M|$. Explicitly, the equations (1.3.73) can be computed to be

$$
\begin{aligned}
\tilde{H}'_{vv} &= \frac{2M(2M\mu^2 r + k^2(\mu^2 r^4 - Mr + 2M^2) + k^4 r^3(r-2M))}{r(k^3r^3+2Mk)^2} H_z(r) \\
&\quad - \frac{2M(r-2M)(k^2r^3(3r-7M)-2M^2)}{r^3(k^3r^3+2Mk)^2}\frac{dH_z}{dr}(r),
\end{aligned} \tag{1.3.74}
$$

$$\tilde{H}'_{vr} = \Big(\frac{\mu(\mu r^2+M)}{rk^2(r-2M)} - \frac{\mu^2 r^4 + M\mu r^2 - 2M(r-2M)}{(r-2M)(k^2r^3+2M)} - \frac{6M^2}{(k^2r^3+2M)^2}\Big)H_z \quad (1.3.75)$$
$$+\Big(\frac{6Mr(r-2M)}{(k^2r^3+2M)^2} + \frac{r(\mu r^2+M)}{k^2r^3+2M} - \frac{\mu r^2+M}{k^2r^2}\Big)\frac{dH_z}{dr},$$
$$\tilde{H}'_{rr} = \Big(\frac{2r(M\mu r^2+\mu^2r^4-M(r-2M))}{(r-2M)^2(k^2r^3+2M)} + \frac{12M^2 r}{(k^2r^3+2M)^2(r-2M)} - \frac{2\mu(\mu r+M)}{k^2(r-2M)^2}\Big)H_z$$
$$+\Big(\frac{2(\mu r^2+r-M)}{k^2r(r-2M)} - \frac{12Mr^2}{(k^2r^3+2M)^2} - \frac{2r^2(\mu r^2+r-M)}{(r-2M)(k^2r^3+2M)}\Big)H'_z, \quad (1.3.76)$$
$$\tilde{H}'_{\theta\theta} = -\frac{Mr^2}{k^2r^3+2M}H_z(r) - \frac{2M(r-2M)}{k^4r^3+2Mk^2}\frac{dH_z}{dr}(r) \quad (1.3.77)$$

where the ODE (1.3.7) with $\mathfrak{H} = H_z$ has been used. To determine the behaviour of these new metric perturbation components close to the future event horizon $\mathcal{H}^+_A$ one must substitute $H_z^{2M,\pm}(r) \doteq \mathfrak{H}^{2M,\pm}(r)$ from equations (1.3.42)–(1.3.43). Substituting $H_z^{2M,\pm}(r) \doteq \mathfrak{H}^{2M,\pm}(r)$ from equations (1.3.45) and (1.3.46) into these expressions gives leading order behaviour close to the future event horizon $\mathcal{H}^+_A$ determined by the relations

$$\tilde{H}^{2M,\pm}_{vv} = f_{vv}(r)(r-2M)^{\pm 2M\mu}, \quad (1.3.78)$$
$$\tilde{H}^{2M,\pm}_{vr} = \Big(\frac{(\mu\mp\mu)(1+4M\mu)}{2k^2(1+4M^2k^2)}(r-2M)^{-1} + f_{vr}(r)\Big)(r-2M)^{\pm 2M\mu}, \quad (1.3.79)$$
$$\tilde{H}^{2M,\pm}_{rr} = \Big(\frac{-2(1\mp 1)M\mu(1+4M\mu)}{k^2(1+4M^2k^2)(r-2M)^2} + \frac{k_\pm}{(r-2M)} + f_{rr}(r)\Big)(r-2M)^{\pm 2M\mu}, \quad (1.3.80)$$
$$\tilde{H}^{2M,\pm}_{\theta\theta} = f_{\theta\theta}(r)(r-2M)^{\pm 2M\mu} \quad (1.3.81)$$

with f_{vv}, f_{vr}, f_{rr}, $f_{\theta\theta}$ smooth functions of $r \in [2M,\infty)$ which are non-vanishing at $2M$, $k_+ = 0$ and k_- a non-zero constant depending on k, M and μ. Therefore, multiplying $\tilde{H}^{2M,+}_{vv}$, $\tilde{H}^{2M,+}_{vr}$, $\tilde{H}^{2M,+}_{rr}$ and $\tilde{H}^{2M,+}_{\theta\theta}$ by $e^{\mu t} = e^{\mu v}e^{-\mu r}(r-2M)^{-2M\mu}$ gives

$$e^{\mu t+ikz}\tilde{H}^{2M,+}_{vv} = f_{vv}(r)e^{\mu v-\mu r+ikz}, \quad (1.3.82)$$
$$e^{\mu t+ikz}\tilde{H}^{2M,+}_{vr} = f_{vr}(r)e^{\mu v-\mu r+ikz}, \quad (1.3.83)$$
$$e^{\mu t+ikz}\tilde{H}^{2M,+}_{rr} = f_{rr}(r)e^{\mu v-\mu r+ikz}, \quad (1.3.84)$$
$$e^{\mu t+ikz}\tilde{H}^{2M,+}_{\theta\theta} = f_{\theta\theta}(r)e^{\mu v-\mu r+ikz}, \quad (1.3.85)$$

which can indeed be smoothly extended to the future event horizon $\mathcal{H}^+_A$. □

Remark 1.3.12. *The form of the pure gauge solution defined by equation (1.3.65) can be derived as follows. From lemma 1.3.11, a mode solution $\tilde{h}$ of the form (1.3.49) satisfies the harmonic/transverse-traceless (1.1.8) gauge conditions. Take a mode solution h in spherical gauge (1.3.2) add the pure gauge solution $h_{\mathrm{pg}} = 2\nabla_{(a}\xi_{b)}$ for some vector field*

$$\xi = e^{\mu t+ikz}\zeta \quad (1.3.86)$$

where ζ is a vector field which depends only on r. From a direct calculation of $h + h_{\mathrm{pg}}$ one can see that, to obtain a solution $\tilde{h}$ of the form (1.3.49), ζ must be given by equations (1.3.65).

Remark 1.3.13. *To explicitly see the singular behaviour of the mode solution $h^{\pm}$ in spherical gauge (1.3.2) with $\mu > 0$ and $k \neq 0$ associated, via proposition 1.3.3, to either $\mathfrak{H}^{2M,\pm}$, consider directly transforming to ingoing Eddington–Finkelstein coordinates. This transformation gives the following basis elements:*

$$H_{rr}^{2M,\pm'} = \Big(\frac{\partial t}{\partial r}\Big)^2 H_t^{2M,\pm} + 2\Big(\frac{\partial t}{\partial r}\Big)\mu H_v^{2M,\pm} + H_r^{2M,\pm}(r), \tag{1.3.87}$$

$$H_{vv}^{2M,\pm'} = H_t^{2M,\pm}(r), \tag{1.3.88}$$

$$H_{vr}^{2M,\pm'} = \Big(\frac{\partial t}{\partial r}\Big) H_t^{2M,\pm}(r) + \mu H_v^{2M,\pm}(r), \tag{1.3.89}$$

$$H_{zz}^{2M,\pm'} = H_z^{2M,\pm}(r), \tag{1.3.90}$$

where $H_v^{2M,\pm}$, $H_t^{2M,\pm}$ and $H_r^{2M,\pm}$ are the basis for solutions for H_v, H_t and H_r constructed from proposition (1.3.3). These relevant expressions can be found from equations (1.3.17)–(1.3.19).

First, if $4M\mu$ is a positive integer and the coefficent C_N does not vanish then, by equation (1.3.90), the basis element $H_{zz}^{2M,-'}(r) = H_z^{2M,-} = \mathfrak{H}^{2M,-}$ has an essential logarithmic divergence and is therefore always singular at the future event horizon $\mathcal{H}_A^+$.

If $C_N = 0$ or $4M\mu$ is not a positive integer then the basis elements $H_z^{2M,\pm} = \mathfrak{H}^{2M,\pm}$ are given by equations (1.3.45) and (1.3.46) with first order coefficients (1.3.47). Substituting the basis into equations (1.3.17)–(1.3.19) for the other metric perturbation component, one finds

$$H_r^{2M,\pm} = \frac{(r-2M)^{\pm 2M\mu}}{(r-2M)^2}\Big(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(4M^2(2\mu^2+k^2) \pm 6M\mu - 1)}{2(1+4M^2k^2)}(r-2M) + \mathcal{O}\big((r-2M)^2\big)\Big), \tag{1.3.91}$$

$$H_t^{2M,\pm} = (r-2M)^{\pm 2M\mu}\Big(\frac{(1+4M\mu)(4M\mu-1)}{4(1+4M^2k^2)} + \frac{3+4M^2(8\mu^2-k^2) \pm 2M\mu(8M^2(2\mu^2+k^2)-11)}{8M(1+4M^2k^2)}(r-2M) + \mathcal{O}\big((r-2M)^2\big)\Big), \tag{1.3.92}$$

with

$$H_v^{2M,\pm} = \frac{(r-2M)^{\pm 2M\mu}}{(r-2M)}\Big(\frac{M^2(\pm 4M\mu - 1)}{1+4M^2k^2} + \frac{M(2M^2(2\mu^2+k^2) - 1 \pm 5M\mu)}{1+4M^2k^2}(r-2M) + \mathcal{O}\big((r-2M)^2\big)\Big). \tag{1.3.93}$$

Transforming to ingoing Eddington–Finkelstein coordinates gives

$$\begin{aligned}H^{2M,\pm\prime}_{rr} &= (r-2M)^{-2\pm 2M\mu}\Big(\frac{2M^2(1-2M\mu(1\mp 1))(4M\mu-1)}{1+4M^2k^2} \\ &\quad + \frac{2M^2\mu((3\mp 4)+2(9\mp 7)M\mu-(1\mp 1)4M^2(2\mu^2+k^2))}{1+4M^2k^2}(r-2M) \\ &\quad + \mathcal{O}((r-2M)^2)\Big),\end{aligned} \tag{1.3.94}$$

$$H^{2M,\pm\prime}_{vv} = (r-2M)^{\pm 2M\mu}\Big(\frac{(1+4M\mu)(4M\mu-1)}{4(1+4M^2k^2)} + \mathcal{O}(r-2M)\Big), \tag{1.3.95}$$

$$H^{2M,\pm\prime}_{vr} = (r-2M)^{-1\pm 2M\mu}\Big(\frac{M(2M\mu(1\mp 2)-1)(\pm 4M\mu-1)}{2(1+4M^2k^2)} + \mathcal{O}(r-2M)\Big), \tag{1.3.96}$$

$$H^{2M,\pm\prime}_{zz} = (r-2M)^{\pm 2M\mu}(1+\mathcal{O}(r-2M)). \tag{1.3.97}$$

Note that the full mode solution h constructed from proposition 1.3.3 involves a factor of $e^{\mu t} = e^{\mu v}e^{-\mu r}(r-2M)^{-2M\mu}$ so, after multiplication by this exponential factor, one can see that the basis elements $H^{2M,-\prime}_{\mu\nu}$ are always singular, i.e., a solution with $k_2 \neq 0$ is always singular at the future event horizon. The components $e^{\mu t}H^{2M,+\prime}_{vv}$ and $e^{\mu t}H^{2M,+\prime}_{z}$ are unconditionally smooth. However, in general, the components $e^{\mu t}H^{2M,+\prime}_{rr}$ and $e^{\mu t}H^{2M,+\prime}_{vr}$ remain singular at the future event horizon $\mathcal{H}^+_A$ unless $4M\mu = 1$ or $-2+2M\mu \in \mathbb{N}\cup\{0\}$ or $-2+2M\mu > 2$. (In appendix A.5 it is shown that for existence of a solution $\mathfrak{H}$ with $\mu > 0$ which has $k_2 = 0$ and is finite at infinity (see section 1.3.4) then $\mu < \frac{3}{16M}\sqrt{\frac{3}{2}} < \frac{1}{4M}$.) So neither basis perturbation $h^\pm$ in spherical gauge (1.3.2) extends, in general, smoothly across the future event horizon $\mathcal{H}^+_A$.

Spacelike Infinity i^0_A

The goal of this section is to identify the admissible boundary conditions for a solution $\mathfrak{H}$ to the ODE (1.3.7) as $r\to\infty$. This requires one to understand the behaviour as $r\to\infty$ of the mode solution h in spherical gauge (1.3.2) of the linearised vacuum Einstein equation (I.5) which results (through the construction in proposition 1.3.3) from $\mathfrak{H}$.

In this section, a basis for solution $\mathfrak{H}^{\infty,\pm}$ associated to $r\to\infty$ is constructed. This basis $\mathfrak{H}^{\infty,\pm}$ captures the asymptotic behavior of any solution to the ODE (1.3.7) as $r\to\infty$. In particular, as $r\to\infty$, $\mathfrak{H}^{\infty,+}$ grows exponentially and $\mathfrak{H}^{\infty,-}$ decays exponentially. It will be shown that after the addition of the pure gauge solution h_{pg} defined in equations (1.3.65) and (1.3.66), $h+h_{\text{pg}}$ is a mode solution in harmonic/transverse-traceless gauge (1.1.8) to the linearised Einstein vacuum equation which is a linear combination of solutions which grow or decay exponentially as $r\to\infty$. The admissible boundary condition will be that the solution should decay exponentially, from which it will follow that $\mathfrak{H} = a\mathfrak{H}^{\infty,-}$.

One should note that the functions $P_k(r)$ and $Q_k(r) - \frac{\mu^2 r^2}{(r-2M)^2}$ admit convergent series expansions in a neighbourhood of $r = \infty$

$$P_k(r) = \sum_{n=0}^{\infty} \frac{p_n}{r^n} \qquad Q_k(r) = \sum_{n=0}^{\infty} \frac{q_n}{r^n} \tag{1.3.98}$$

with $p_0 = 0$, $p_1 = -4$, $q_0 = -(k^2 + \mu^2)$ and $q_1 = -2M(k^2 + 2\mu^2)$. Therefore, $r = \infty$ is an irregular singular point of the ODE (1.3.7) according to the discussion of appendix A.2. The equations (A.2.19) and (A.2.20) from appendix A.2 give

$$\lambda_\pm = \pm\sqrt{\mu^2 + k^2}, \quad \mu_\pm = 2 \pm \frac{M(2\mu^2 + k^2)}{\sqrt{\mu^2 + k^2}}. \tag{1.3.99}$$

From theorem A.2.7, there exists a unique basis for solutions $\mathfrak{H}^{\infty,\pm}(r)$ to the ODE (1.3.7) satisfying

$$\mathfrak{H}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\Big(e^{\pm\sqrt{\mu^2+k^2}r} r^{1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\Big). \tag{1.3.100}$$

Therefore a general solution will be of the form

$$\mathfrak{H} = c_1 \mathfrak{H}^{\infty,+} + c_2 \mathfrak{H}^{\infty,-} \tag{1.3.101}$$

with $c_1, c_2 \in \mathbb{R}$.

Proposition 1.3.14. *Let $\mathfrak{H}$ be a solution to the ODE (1.3.7). Let h be the mode solution to the linearised vacuum Einstein equation (I.5) in spherical gauge (1.3.2) associated to the solution $\mathfrak{H}$ and let h_{pg} the pure gauge solution defined by equations (1.3.65) and (1.3.66) such that $h + h_{\mathrm{pg}}$ satisfies the harmonic/transverse-traceless gauge (1.1.8) conditions. Then the solution $h + h_{\mathrm{pg}}$ to the ODE (1.3.7) decays exponentially towards spacelike infinity i^0_A if $c_1 = 0$, where c_1 is defined by equation (1.3.101).*

Proof. Defining $H_z^{\infty,\pm}(r) \doteq \mathfrak{H}^{\infty,\pm}(r)$ and using equations (1.3.67)–(1.3.68) one can construct the corresponding basis for solutions as $\tilde{H}_{tt}$, $\tilde{H}_{tr}$, $\tilde{H}_{rr}$ and $\tilde{H}_{\theta\theta}$ from proposition 1.3.3. Note that equations (1.3.67)–(1.3.68) define the components of the mode solution $h + h_{\mathrm{pg}}$ to the linearised vacuum Einstein equation (I.5) which satisfies harmonic/transverse-traceless gauge (1.1.8). Asymptotically $\tilde{H}_{tt}$, $\tilde{H}_{tr}$, $\tilde{H}_{rr}$ and $\tilde{H}_{\theta\theta}$ have the following behavior:

$$H_{tt}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{-1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\Big(e^{\pm\sqrt{\mu^2+k^2}r} r^{-2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\Big), \tag{1.3.102}$$

$$H_{tr}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{-1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\Big(e^{\pm\sqrt{\mu^2+k^2}r} r^{-2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\Big), \tag{1.3.103}$$

$$H_{rr}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{-1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\Big(e^{\pm\sqrt{\mu^2+k^2}r} r^{-2\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\Big) \tag{1.3.104}$$

and

$$H_{\theta\theta}^{\infty,\pm} = e^{\pm\sqrt{\mu^2+k^2}r} r^{1\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}} + \mathcal{O}\Big(e^{\pm\sqrt{\mu^2+k^2}r} r^{\pm\frac{M(2\mu^2+k^2)}{\sqrt{\mu^2+k^2}}}\Big). \tag{1.3.105}$$

It is clear from these expressions that, if $c_1 = 0$, then the mode solution $h + h_{\mathrm{pg}}$ decays exponentially as $r \to \infty$. □

1.3.5 Reduction of Theorem 1.1.2 to Proposition 1.3.16

This section summarises propositions 1.3.3, 1.3.6, 1.3.7, 1.3.9 and 1.3.14 to give a full description of the permissible asymptotic behavior of a mode solution h in spherical gauge (1.3.2) which is not pure gauge. This provides a reduction of theorem 1.1.2 to proving that there exists a solution $\mathfrak{H}$ to the ODE (1.3.7) which has $\mu > 0$, $k \neq 0$ and obeys the admissible boundary conditions: $k_2 = 0$ and $c_1 = 0$.

Proposition 1.3.15. *Let $\mu > 0$ and $k \in \mathbb{R}$ with $k \neq 0$. Let $\mathfrak{H}^{2M,\pm}$ be the basis for the space of solutions to the ODE (1.3.7) as defined in equations (1.3.42) and (1.3.43) and $\mathfrak{H}^{\infty,\pm}$ be the basis for the space of solutions to the ODE (1.3.7) as defined in equation (1.3.100). In particular, to any solution $\mathfrak{H}$ of the ODE (1.3.7) one can ascribe four numbers $k_1, k_2, c_1, c_2 \in \mathbb{R}$ defined by*

$$\mathfrak{H}(r) = k_1\mathfrak{H}^{2M,+}(r) + k_2\mathfrak{H}^{2M,-}(r), \tag{1.3.106}$$

$$\mathfrak{H}(r) = c_1\mathfrak{H}^{\infty,+}(r) + c_2\mathfrak{H}^{\infty,-}(r). \tag{1.3.107}$$

Let h be the mode solution in spherical gauge (1.3.2) to the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{R}$ associated to $\mathfrak{H}$ via proposition 1.3.3. Let h_{pg} be the pure gauge solution as defined in equations (1.3.65) and (1.3.66). Then $h + h_{\mathrm{pg}}$ decays exponentially towards spacelike infinity i^0_A and is smooth at the future event horizon $\mathcal{H}^+_A$ if $k_2 = 0$ and $c_1 = 0$. Moreover, $h + h_{\mathrm{pg}}$ satisfies the harmonic/transverse-traceless gauge conditions (1.1.8) and cannot be a pure gauge solution.

Under the additional assumption that $kR \in \mathbb{Z}$ the mode solution h defined above can be interpreted as a mode solution to the linearised vacuum Einstein equation (I.5) on the exterior $\mathcal{E}_A$ of Schwarzschild black string $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. Hence, if $kR \in \mathbb{Z}$ the above statement applies to the exterior $\mathcal{E}_A$ of $\mathrm{Schw}_4 \times \mathbb{S}^1_R$.

The next section will prove the existence of a solution $\mathfrak{H}$ to the ODE (1.3.7) satisfying the properties of proposition 1.3.15. In particular, for all $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$, a solution $\mathfrak{H}$ to the ODE (1.3.7) with $\mu > \frac{1}{40\sqrt{10}M} > 0$, $k_2 = 0$ and $c_1 = 0$ is constructed. If $R > 4M$, then there exists an integer $n \in [\frac{3R}{20M}, \frac{8R}{20M}]$. Hence, one can choose k such that the constructed $\mathfrak{H}$ gives rise to a mode solution on $\mathrm{Schw}_4 \times \mathbb{S}^1_R$. Moreover, on $\mathrm{Schw}_4 \times \mathbb{S}^1_R$, h will manifestly have finite energy in the sense that $||h|_\Sigma||_{H^1}$ and $||\partial_{t_\star} h|_\Sigma||_{L^2}$ are finite. (Note that, on $\mathrm{Schw}_4 \times \mathbb{R}$, h will not have finite energy due to the periodic behaviour in z on $\mathbb{R}$.) Thus, theorem 1.1.2 follows from proposition 1.3.15 and the following proposition:

Proposition 1.3.16. *For all $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$ there exists a $C^\infty((2M, \infty))$ solution $\mathfrak{H}$ to ODE (1.3.7) with $\mu > 0$, and in the language of proposition 1.3.15, $k_2 = 0$ and $c_1 = 0$.*

1.4 The Variational Argument

By proposition 1.3.15, the proof of theorem 1.1.2 has now been reduced to proposition 1.3.16 which exhibits a solution $\mathfrak{H}$ to (1.3.7) with $\mu > 0$, $k \neq 0$, $k_2 = 0$ and $c_1 = 0$. This section establishes the proposition 1.3.16 thus completing the proof of theorem 1.1.2.

In order to exhibit such a solution $\mathfrak{H}$ to the ODE (1.3.7), it is convenient to rescale the solution and change coordinates in the ODE (1.3.7) so as to recast as a Schrödinger equation for a function u. This transformation is given in section 1.4.1. In section 1.4.2 an energy functional is assigned to the resulting Schrödinger operator. With the use of a test function (constructed in section 1.4.3), a direct variational argument can be run to establish that for $|k| \in [\frac{3}{20M}, \frac{8}{20M}]$, there exists a weak solution $u \in H^1(\mathbb{R})$ with $||u||_{H^1(\mathbb{R})} = 1$ such that $\mu > 0$. The proof of proposition 1.3.16 concludes by showing that the solution u is indeed smooth for $r \in (2M, \infty)$ and satisfies the conditions of proposition 1.3.15, i.e., $k_2 = 0$ and $c_1 = 0$.

1.4.1 Schrödinger Reformulation

To reduce the number of parameters in the ODE (1.3.7), one can eliminate the mass parameter with $x \doteq \frac{r}{2M}$, $\hat{\mu} \doteq 2M\mu$ and $\hat{k} \doteq 2Mk$ to find

$$\frac{d^2\mathfrak{H}}{dx^2}(x) + \mathrm{P}_{\hat{k}}(x)\frac{d\mathfrak{H}}{dx} + \Big(\mathrm{Q}_{\hat{k}}(x) - \frac{\hat{\mu}^2 x^2}{(x-1)^2}\Big)\mathfrak{H}(x) = 0, \tag{1.4.1}$$

with

$$\mathrm{P}_{\hat{k}}(x) \doteq \frac{1}{x-1} - \frac{5}{x} + \frac{6}{x(\hat{k}^2x^3+1)}, \tag{1.4.2}$$

$$\mathrm{Q}_{\hat{k}}(x) \doteq \frac{3}{x^2(x-1)} - \frac{\hat{k}^2 x}{x-1} - \frac{3}{x^2(x-1)(1+\hat{k}^2x^3)}. \tag{1.4.3}$$

Following proposition A.3.1 from appendix A.3 one can now transform the equation (1.4.1) into regularised Schrödinger form by introducing a weight function $\mathfrak{H}(x) = w(x)\tilde{\mathfrak{H}}(x)$ and changing coordinates to $x_\star = \frac{r_\star}{2M} = x + \log|x-1|$. This will produce a Schrödinger operator with a potential which decays to zero at the future event horizon and tends to the constant $\hat{k}^2$ at spatial infinity. From proposition A.3.1 the weight function must satisfy the ODE

$$\frac{dw}{dx} + \frac{(1-2k^2x^3)}{x(1+k^2x^3)}w = 0. \tag{1.4.4}$$

The desired solution for the weight function is

$$w(x) = \frac{(1+\hat{k}^2x^3)}{x}. \tag{1.4.5}$$

The ODE (1.4.1) becomes

$$-\frac{d^2\tilde{\mathfrak{H}}}{dx_\star^2}(x_\star) + V(x_\star)\tilde{\mathfrak{H}}(x_\star) = -\hat{\mu}^2\tilde{\mathfrak{H}}(x_\star), \tag{1.4.6}$$

where $V : \mathbb{R} \to \mathbb{R}$ can be found from equation (A.3.12) to be

$$V(x_\star) = \hat{k}^2\frac{(x-1)}{x} + \frac{(6x-11)(x-1)}{x^4} + \frac{18(x-1)^2}{x^4(1+\hat{k}^2x^3)^2} - \frac{6(4x-5)(x-1)}{x^4(1+\hat{k}^2x^3)}, \tag{1.4.7}$$

where $x \in (1,\infty)$ is understood as an implicit function of $x_\star$.

As a trivial consequence of proposition 1.3.15 in section 1.3.4 on asymptotics of the solution to the ODE (1.3.7), one has the following proposition for the asymptotics of the Schrödinger equation (1.4.6).

Proposition 1.4.1. *Assume $\hat{\mu} > 0$. To any solution $\tilde{\mathfrak{H}}$ to the Schrödinger equation (1.4.6) one can ascribe four numbers $\tilde{k}_1, \tilde{k}_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ defined by*

$$\tilde{\mathfrak{H}}(x_\star) = \tilde{k}_1\tilde{\mathfrak{H}}^{2M,+}(x_\star) + \tilde{k}_2\tilde{\mathfrak{H}}^{2M,-}(x_\star) \quad \text{as} \quad x_\star \to -\infty, \tag{1.4.8}$$

$$\tilde{\mathfrak{H}}(x_\star) = \tilde{c}_1\tilde{\mathfrak{H}}^{\infty,+}(x_\star) + \tilde{c}_2\tilde{\mathfrak{H}}^{\infty,-}(x_\star) \quad \text{as} \quad x_\star \to \infty \tag{1.4.9}$$

with

$$\tilde{\mathfrak{H}}^{2M,\pm} \doteq \frac{\mathfrak{H}^{2M,\pm}}{w}, \tag{1.4.10}$$

$$\tilde{\mathfrak{H}}^{\infty,\pm} \doteq \frac{\mathfrak{H}^{\infty,\pm}}{w}. \tag{1.4.11}$$

The conditions that $\tilde{c}_1 = 0$ and $\tilde{k}_2 = 0$ are equivalent to, in the language of proposition 1.3.15, $c_1 = 0$ and $k_2 = 0$.

Remark 1.4.2. *In the case $4M\mu$ is not a positive integer or $4M\mu$ is a positive integer and $C_N = 0$ the leading order terms of these basis elements are*

$$\tilde{\mathfrak{H}}^{2M,\pm} = (x-1)^{\pm\hat{\mu}}\Big(\frac{1}{1+\hat{k}^2} + \mathcal{O}(x-1)\Big), \tag{1.4.12}$$

$$\tilde{\mathfrak{H}}^{\infty,\pm} = e^{\pm\sqrt{\hat{\mu}^2+\hat{k}^2}x}x^{\pm\frac{(2\hat{\mu}^2+\hat{k}^2)}{2\sqrt{\hat{\mu}^2+\hat{k}^2}}}\Big(\frac{1}{\hat{k}^2} + \mathcal{O}\Big(\frac{1}{x}\Big)\Big). \tag{1.4.13}$$

1.4.2 Direct Variational Argument

This section establishes a variational argument which will be used to infer the existence of a negative eigenvalue to the Schrödinger operator in equation (1.4.6).

Proposition 1.4.3. *Let $W : \mathbb{R} \to \mathbb{R}$ and define*

$$E_0 \doteq \inf_{\substack{v \in H^1(\mathbb{R}) \\ ||v||_{L^2(\mathbb{R})}=1}} \Big\{ E(v) \doteq \langle \nabla v, \nabla v \rangle_{L^2(\mathbb{R})} + \langle Wv, v \rangle_{L^2(\mathbb{R})} \Big\}. \tag{1.4.14}$$

Suppose that $W = p + q$ with $q \in C^0(\mathbb{R})$ such that $\lim_{|x|\to\infty} q(x) = 0$ and $p(x) \in C^0(\mathbb{R})$ bounded and positive. If $E_0 < 0$, then there exists $u \in H^1(\mathbb{R})$ such that $||u||_{L^2(\mathbb{R})} = 1$ and $E(u) = E_0$.

Proof. By the definition of the infimum there exists a minimizing sequence $(u_m)_m \subset H^1(\mathbb{R})$ and $||u_m||_{L^2} = 1$ such that

$$\lim_{n\to\infty} E(u_n) = E_0. \tag{1.4.15}$$

Now, u_n are bounded in $H^1(\mathbb{R})$ by the following argument. There exists an $M \in \mathbb{N}$ such that, for all $m \geq M$,

$$E(u_m) \leq E_0 + 1. \tag{1.4.16}$$

So, for $m \geq M$,

$$\langle \nabla u_m, \nabla u_m \rangle_{L^2(\mathbb{R})} \leq E_0 + 1 + \sup_{x\in\mathbb{R}} |p(x)| + \sup_{x\in\mathbb{R}} |q(x)|. \tag{1.4.17}$$

Hence, $||u_m||_{H^1(\mathbb{R})}$ is controlled. Now using theorem A.4.1 from appendix A.4 there exists a subsequence $(u_{m_n})_n$ such that $u_{m_n} \rightharpoonup u$ in $H^1(\mathbb{R})$.

Consider

$$E(u_m) = \int_{\mathbb{R}} |\nabla u_m|^2 + p(x)|u_m|^2 + q(x)|u_m|^2 dx. \tag{1.4.18}$$

Since the Dirichlet energy is lower semicontinuous, only the latter two terms under the integral (1.4.18) need to be examined more carefully. The middle term in integral (1.4.18) are simply a weighted L^2 integral, so lower semicontinuity is established via

$$||u_n - u||^2_{L^2_p} = \langle u_n - u, u_n - u \rangle_{L^2_p} = ||u_n||^2_{L^2_p} - 2\langle u_n - u, u \rangle_{L^2_p} - ||u||^2_{L^2_p}. \tag{1.4.19}$$

So,

$$||u||^2_{L^2_p} \leq ||u_n||^2_{L^2_p} - 2\langle u, u_n - u \rangle_{L^2_p}. \tag{1.4.20}$$

Hence by weak convergence

$$||u||^2_{L^2_p} \leq \liminf_{n\to\infty} ||u_n||^2_{L^2_p}. \tag{1.4.21}$$

The proposition A.4.2 from appendix A.4 establishes that the multiplication operator $M_q : u \to qu$ is compact from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$. Hence, by the characterisation of compactness through weak convergence (theorem A.4.1 from appendix A.4), $qu_m \to qu$ in $L^2(\mathbb{R})$. Therefore

$$\langle qu, u\rangle_{L^2} = \lim_{m\to\infty}\langle qu_m, u_m\rangle_{L^2} = \liminf_{m\to\infty}\langle qu_m, u_m\rangle_{L^2}. \tag{1.4.22}$$

Hence, the last term under the integral (1.4.18) is also lower semicontinuous. Therefore

$$E(u) \le \liminf_{n\to\infty} E(u_n) = E_0. \tag{1.4.23}$$

Since the infimum is negative the minimiser is non-trivial. One needs to show that there is no loss of mass, i.e., $||u||_{L^2} = 1$. Note $||u||_{L^2} \le \liminf_{n\to\infty} ||u_n||_{L^2} = 1$. So suppose $||u||_{L^2} < 1$ and define $\tilde{u} = \frac{u}{||u||_{L^2}}$ so $||\tilde{u}||_{L^2} = 1$, then

$$E(\tilde{u}) = \frac{E_0}{||u||^2_{L^2(\mathbb{R})}} \le E_0 \tag{1.4.24}$$

since $||u||_{L^2} \le 1$. Hence one would obtain a contradition to the infimum if $||u||_{L^2} < 1$. □

Corollary 1.4.4. *Let $W = V$ with V as defined in equation (1.4.7) then*

$$E(v) \doteq \langle \nabla v, \nabla v\rangle_{L^2(\mathbb{R})} + \langle Vv, v\rangle_{L^2(\mathbb{R})} \qquad E_0 \doteq \inf_{\substack{v\in H^1(\mathbb{R})\\ ||v||_{L^2(\mathbb{R})}=1}} E(v) \tag{1.4.25}$$

satisfies the assumptions of proposition 1.4.6.

Proof. The function $V : \mathbb{R} \to \mathbb{R}$ can be written as $V = p + q$ with p and q as follows. Define

$$p(x_\star) \doteq \hat{k}^2\frac{x-1}{x}, \tag{1.4.26}$$

$$q(x_\star) \doteq \frac{(6x-11)(x-1)}{x^4} + \frac{18(x-1)^2}{x^4(1+\hat{k}^2x^3)^2} - \frac{6(4x-5)(x-1)}{x^4(1+\hat{k}^2x^3)} \tag{1.4.27}$$

where in these expressions x considered as a implicit function of $x_\star$. Since $x \in (1, \infty)$, it follows that $p(x_\star) > 0$ for all $x_\star \in \mathbb{R}$. Moreover, $\sup_{x_\star\in\mathbb{R}} |p(x_\star)| = 1$. Therefore, p is bounded. Note that the function q satisfies $\lim_{|x_\star|\to\infty} q(x_\star) = 0$. So the assumptions of proposition 1.4.3 hold. □

1.4.3 The Test Function and Existence of a Minimiser

The ODE (1.4.6) is now in a form where a direct variational argument can be used to prove that there exists an eigenfunction of the Schrödinger operator associated to the left-hand side of the ODE (1.4.6) with a negative eigenvalue, i.e. $-\hat{\mu}^2 < 0$. The following proposition constructs a suitable test function such that it is in the correct function space, $H^1(\mathbb{R})$, and, for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, implies that the infimum of the energy functional in equation (1.4.25) is negative. (As will be apparent, the negativity is inferred via complicated but purely algebraic calculations.)

Proposition 1.4.5. *Define* $u_T(x_\star) \doteq x(1+|\hat{k}|^2x^3)(x-1)^{\frac{1}{n}}e^{-4|\hat{k}|(x-1)}$ *where* x *is an implicit function of* $x_\star$*,* n *is a finite non-zero natural number,* $\hat{k} \in \mathbb{R}\setminus\{0\}$ *and define* E *and* E_0 *as in equation (1.4.25) of corollary 1.4.4. Then* $u_T \in H^1(\mathbb{R})$ *and, for* $n = 100$ *and* $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$*,* $E_0 \leq \frac{E(u_T)}{||u_T||^2_{L^2(\mathbb{R})}} < -\frac{1}{4000}$.

Proof. Let $k \in \mathbb{N}\cup\{0\}$ and define the following functions

$$f_j(x) \doteq x^{j-1}(x-1)^{\frac{2}{n}-1}e^{-8|\hat{k}|(x-1)}. \tag{1.4.28}$$

The $H^1(\mathbb{R})$-norm of u_T can be expressed as

$$||u_T||^2_{H^1(\mathbb{R})} = \int_1^\infty \Big|\frac{x-1}{x}\frac{du_T}{dx}\Big|^2\frac{x}{x-1}dx + \int_1^\infty |u_T|^2\frac{x}{x-1}dx \tag{1.4.29}$$

where on the right-hand side the change of variables from $x_\star \in \mathbb{R}$ to $x \in (1,\infty)$ has been made. To calculate $||u_T||_{L^2(\mathbb{R})}$ it is useful to write it as a linear combination of the functions f_k in equation (1.4.28). Explicitly, one can show that

$$|u_T|^2\frac{x}{x-1} = f_4(x) + 2|\hat{k}|^2f_7(x) + |\hat{k}|^4f_{10}(x). \tag{1.4.30}$$

Similarly, one can show that

$$\Big|\frac{x-1}{x}\frac{du_T}{dx}\Big|^2\frac{x}{x-1} = \sum_{j=1}^{11}c_jf_{j-1}(x) \tag{1.4.31}$$

with $c_1 = 1$ and

$$\begin{aligned} c_3 &= 1+\frac{1}{n^2}+\frac{2}{n}+16|\hat{k}|+\frac{8|\hat{k}|}{n}+16|\hat{k}|^2, & c_2 &= -2-\frac{2}{n}-8|\hat{k}|,\\ c_5 &= -\frac{10|\hat{k}|^2}{n}-40|\hat{k}|^3, & c_4 &= -8|\hat{k}|-\frac{8|\hat{k}|}{n}-24|\hat{k}|^2,\\ c_7 &= -40|\hat{k}|^3-\frac{16|\hat{k}|^3}{n}-48|\hat{k}|^4, & c_8 &= -\frac{8|\hat{k}|^4}{n}-32|\hat{k}|^5,\\ c_{11} &= 16|\hat{k}|^6, & c_{10} &= -32|\hat{k}|^5-\frac{8|\hat{k}|^5}{n}-32|\hat{k}|^6, \end{aligned} \tag{1.4.32}$$

and

$$c_6 = 8|\hat{k}|^2+\frac{2|\hat{k}|^2}{n^2}+\frac{10|\hat{k}|^2}{n}+80|\hat{k}|^3+\frac{16|\hat{k}|^3}{n}+32|\hat{k}|^4, \tag{1.4.33}$$

$$c_9 = 16|\hat{k}|^4(1+|\hat{k}|^2)+\frac{|\hat{k}|^4}{n^2}+\frac{8|\hat{k}|^4(1+|\hat{k}|)}{n}+64|\hat{k}|^5. \tag{1.4.34}$$

One can express $E(u_T)$ with the change of variables from $x_\star$ to x as

$$E(u_T) = \int_1^\infty \Big(\Big|\frac{x-1}{x}\frac{du_T}{dx}\Big|^2 + V|u_T|^2\Big)\frac{x}{x-1}dx. \tag{1.4.35}$$

The integrand can be written as

$$\Big(\Big|\frac{x-1}{x}\frac{du_T}{dx}\Big|^2 + V|u_T|^2\Big)\frac{x}{x-1} = \sum_{j=1}^{11} a_j f_{j-1}(x), \tag{1.4.36}$$

with $a_1 = 0$

$$\begin{aligned}
a_2 &= -\frac{2+n+8n|\hat{k}|}{n}, & a_3 &= 1 + \frac{1}{n^2} + 16|\hat{k}| + 16|\hat{k}|^2 + \frac{2+8|\hat{k}|}{n},\\
a_4 &= -|\hat{k}|\Big(\frac{8(1+n)}{n} + 33|\hat{k}|\Big), & a_5 &= \frac{(21n-10)|\hat{k}|^2}{n} - 40|\hat{k}|^3,\\
a_8 &= -|\hat{k}|^4\Big(\frac{(8+15n)}{n} + 32|\hat{k}|\Big), & a_7 &= -|\hat{k}|^3\Big(\frac{8(2+5n)}{n} + 39|\hat{k}|\Big),\\
a_{10} &= -|\hat{k}|^5\Big(\frac{8(1+4n)}{n} + 33|\hat{k}|\Big), & a_{11} &= 17|\hat{k}|^6
\end{aligned} \tag{1.4.37}$$

and

$$a_6 = |\hat{k}|^2\Big(\frac{2(1+5n-2n^2)}{n^2} + \frac{16(1+5n)|\hat{k}|}{n} + 32|\hat{k}|^2\Big), \tag{1.4.38}$$

$$a_9 = |\hat{k}|^4\Big(\frac{1+8n+22n^2}{n^2} + \frac{8(1+8n)}{n} + 16|\hat{k}|^2\Big). \tag{1.4.39}$$

Therefore, if one can compute the integrals

$$I_j \doteq \int_1^\infty f_j(x)dx \tag{1.4.40}$$

for $k = 0, ..., 10$ then one can compute $||u_T||_{L^2(\mathbb{R})}$, $||\frac{du_T}{dx_\star}||_{L^2(\mathbb{R})}$ and $E(u_T)$.

Defining a change variables in the integrals (1.4.40) by $t = x - 1$, the integrals (1.4.40) become

$$I_j = \int_0^\infty (t+1)^{j-1} t^{\frac{2}{n}-1} e^{-8|\hat{k}|t}. \tag{1.4.41}$$

Note that the confluent hypergeometric function of the second kind $U(a, b; z)$ can be defined as

$$U(a,b;z) \doteq \frac{1}{\Gamma(a)}\int_0^\infty (t+1)^{b-a-1} t^{a-1} e^{-zt} \tag{1.4.42}$$

for $a, b, z \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(z) > 0$ where $\Gamma(a)$ is the Euler Gamma function, which can be defined through the integral

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt \tag{1.4.43}$$

for $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. For a reference see chapter 9 of [98]. Therefore, setting $a = \frac{2}{n}$, $b = k + \frac{2}{n}$ and $z = 8|\hat{k}|$ gives

$$I_j = \Gamma\Big(\frac{2}{n}\Big) U\Big(\frac{2}{n}, j + \frac{2}{n}; 8|\hat{k}|\Big). \tag{1.4.44}$$

The function $U(a, b; z)$ satisfies the following recurrence properties (see chapter 9 of [98] and chapter 16 of [99]):

$$U(0, b; z) = 1, \tag{1.4.45}$$

$$U(a, b; z) - z^{1-b} U(1 + a - b, 2 - b; z) = 0, \tag{1.4.46}$$

$$U(a, b; z) - aU(a + 1, b; z) - U(a, b - 1; z) = 0, \tag{1.4.47}$$

$$(b - a - 1)U(a, b - 1; z) + (1 - b - z)U(a, b; z) + zU(a, b + 1; z) = 0. \tag{1.4.48}$$

Setting $a = \frac{2}{n}$, $b = 1 + \frac{2}{n}$ and $z = 8|\hat{k}|$ in equation (1.4.46), and using equation (1.4.45) allows one to calculate I_1. Setting $a = \frac{2}{n}$, $b = 2 + \frac{2}{n}$ and $z = 8|\hat{k}|$ in equation (1.4.47), using I_1 and equation (1.4.45) allows one to calculate I_2. Setting $a = \frac{2}{n}$, $b = j + \frac{2}{n}$ and $z = 8|\hat{k}|$ in equation (1.4.48), using $I_{j-1}, ..., I_1$ and equation (1.4.45) allows one to calculate I_j. Finally, one can show that $I_0 < \infty$ by the following argument. One can see from the definition of I_j in equation (1.4.41) that

$$I_0 = \int_1^\infty \frac{1}{x(x-1)} (x-1)^{\frac{2}{n}} e^{-8|\hat{k}|(x-1)} dx. \tag{1.4.49}$$

Now, since $e^{-8|\hat{k}|(x-1)} < 1$ on $x \in (1, \infty)$ and $\frac{(x-1)^{\frac{2}{n}-1}}{x} < \frac{1}{2}(x-1)$ for $n \geq 1$ on $x \in (2, \infty)$,

$$I_0 \leq \int_1^2 \frac{1}{x(x-1)} (x-1)^{\frac{2}{n}} + \frac{1}{2} \int_2^\infty (x-1) e^{-8|\hat{k}|(x-1)} < \infty. \tag{1.4.50}$$

Using the recurrence properties in equations (1.4.45)–(1.4.48) and the estimate (1.4.50) allows one to explicitly show that $||u_T||_{H^1(\mathbb{R})} < \infty$ for $n \geq 1$, $\hat{k} \in \mathbb{R} \setminus \{0\}$, i.e., $u_T \in H^1(\mathbb{R})$. Moreover, one can calculate $\frac{E(u_T)}{||u_T||^2_{L^2(\mathbb{R})}}$. To ease notation let $\hat{\mathrm{E}}(u_T) \doteq \frac{E(u_T)}{||u_T||^2_{L^2(\mathbb{R})}}$. Explicitly, $\hat{\mathrm{E}}(u_T)$ is given by

$$\hat{\mathrm{E}}(u_T) = \frac{|\hat{k}|^2 \sum_{i=1}^{9} p_i(n) |\hat{k}|^{i-1}}{\sum_{j=1}^{10} q_j(n) |\hat{k}|^{i-1}} \tag{1.4.51}$$

with

$$\begin{aligned}
p_1(n) &\doteq 16 + 416n + 5576n^2 + 36176n^3 + 123809n^4 + 234794n^5 + 244459n^6 \\
&\quad + 128034n^7 + 25560n^8, \\
p_2(n) &\doteq 32n(16 + 336n + 3296n^2 + 15572n^3 + 29107n^4 + 21238n^5 \\
&\quad + 4361n^6 - 366n^7), \\
p_3(n) &\doteq 128n^2(56 + 924n + 6130n^2 + 20133n^3 + 11972n^4 - 3365n^5 - 466n^6), \\
p_4(n) &\doteq 1024n^3(56 + 700n + 2750n^2 + 6041n^3 - 1715n^4 - 18n^5), \\
p_5(n) &\doteq 2048n^4(140 + 1260n + 2225n^2 + 3443n^3 - 1758n^4), \\
p_6(n) &\doteq 32768n^5(28 + 168n + 43n^2 + 111n^3), \\
p_7(n) &\doteq 917504n^6(2 + 7n - 3n^2), \\
p_8(n) &\doteq 1048576n^7(2 + 3n), \\
p_9(n) &\doteq 1048576n^8,
\end{aligned} \tag{1.4.52}$$

and

$$\begin{aligned}
q_1(n) &\doteq 16 + 288n + 2184n^2 + 9072n^3 + 22449n^4 + 33642n^5 + 29531n^6 \\
&\quad + 13698n^7 + 2520n^8, \\
q_2(n) &\doteq 4n(144 + 2016n + 12104n^2 + 39120n^3 + 71801n^4 + 73494n^5 \\
&\quad + 38171n^6 + 7590n^7), \\
q_3(n) &\doteq 128n^2(72 + 756n + 3534n^2 + 8535n^3 + 11180n^4 + 7137n^5 + 1642n^6), \\
q_4(n) &\doteq 1536n^3(56 + 420n + 1510n^2 + 2535n^3 + 2351n^4 + 706n^5), \\
q_5(n) &\doteq 2048n^4(252 + 1260n + 3485n^2 + 3495n^3 + 2554n^4), \\
q_6(n) &\doteq 8192n^5(252 + 756n + 1653n^2 + 669n^3 + 512n^4), \\
q_7(n) &\doteq 393216n^6(14 + 21n + 39n^2), \\
q_8(n) &\doteq 524288n^7(18 + 9n + 16n^2), \\
q_9(n) &\doteq 9437184n^8, \\
q_{10}(n) &\doteq 4194304n^9.
\end{aligned} \tag{1.4.53}$$

Taking $n = 100$, one can check, via Sturm's algorithm [100], that the polynomial

$$\mathfrak{p}(n, |\hat{k}|) \doteq \sum_{i=1}^{9} p_i(n)|\hat{k}|^{i-1} \tag{1.4.54}$$

has two distinct real roots in $|\hat{k}| \in (0,1)$. Evaluating $\mathfrak{p}(100, |\hat{k}|)$ at $|\hat{k}| = 0$, $|\hat{k}| = \frac{3}{10}$, $|\hat{k}| = \frac{8}{10}$ and $|\hat{k}| = 1$ yields

$$\mathfrak{p}(100,0) > 0, \quad \mathfrak{p}\Big(100, \frac{3}{10}\Big) < 0, \quad \mathfrak{p}\Big(100, \frac{8}{10}\Big) < 0 \quad \text{and} \quad \mathfrak{p}(100,1) > 0. \tag{1.4.55}$$

So, $\hat{\mathrm{E}}(u_T)$ must be negative for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$. Taking the derivative of $\hat{\mathrm{E}}(u_T)$ with respect to $|\hat{k}|$ yields another rational function of $|\hat{k}|$ with positive denominator. Evaluating at the end points of the interval $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ yields $\frac{d\hat{\mathrm{E}}(u_T)}{d|\hat{k}|} < 0$ at $|\hat{k}| = \frac{3}{10}$ and $\frac{d\hat{\mathrm{E}}(u_T)}{d|\hat{k}|} > 0$ at $|\hat{k}| = \frac{8}{10}$ for $n = 100$. Using Sturm's algorithm once again, one can check that the numerator of $\frac{d\hat{\mathrm{E}}(u_T)}{d|\hat{k}|}$ has one distinct root in $|\hat{k}| \in (\frac{3}{10}, \frac{8}{10})$ for $n = 100$. Hence, $\hat{\mathrm{E}}(u_T)$ with $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ and $n = 100$ attains its maximum in at one of the end points. Further evaluating $\hat{\mathrm{E}}(u_T)$ with $n = 100$ at the end points of the interval $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ one finds

$$\frac{E(u_T)}{||u_T||^2_{L^2(\mathbb{R})}}\bigg|_{n=100} = \hat{\mathrm{E}}(u_T)|_{n=100} < -\frac{1}{4000} \tag{1.4.56}$$

for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$. Hence, $E_0 \leq \frac{E(u_T)}{||u_T||^2_{L^2(\mathbb{R})}}|_{n=100} < -\frac{1}{4000} < 0$ for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$. □

1.4.4 Proof of Proposition 1.3.16

To prove proposition 1.3.16, one can clearly reformulate as follows:

Proposition 1.4.6. *For all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, there exists a $C^\infty(\mathbb{R})$ solution $\tilde{\mathfrak{H}}$ to the Schrödinger equation (1.4.6) with $\hat{\mu} > \frac{1}{20\sqrt{10}} > 0$, and in the language of proposition 1.4.1, $\tilde{k}_2 = 0$ and $\tilde{c}_1 = 0$.*

Proof. By proposition 1.4.3, corollary 1.4.4 and proposition 1.4.5, for all $k \in [\frac{3}{10}, \frac{8}{10}]$, there exists a minimiser $u \in H^1(\mathbb{R})$ with $||u||_{L^2(\mathbb{R})} = 1$ such that

$$E(u) = E_0 \doteq \inf\left\{\langle \nabla v, \nabla v\rangle_{L^2(\mathbb{R})} + \langle Vv, v\rangle_{L^2(\mathbb{R})} : v \in H^1(\mathbb{R}), ||v||_{L^2(\mathbb{R})} = 1\right\} \tag{1.4.57}$$

with V as defined in equation (1.4.7). Moreover, by proposition 1.4.5, $E_0 < -\frac{1}{4000} < 0$.

By standard Euler–Lagrange methods (see theorem 3.21 and example 3.22 in [101]), u will weakly solve the ODE

$$-\frac{d^2u}{dx_\star^2} + V(x_\star)u = -\hat{\mu}^2 u \tag{1.4.58}$$

with $-\hat{\mu}^2 = E_0$. From proposition 1.4.5, $\hat{\mu}^2 = -E_0 > \frac{1}{4000}$. Hence, for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, there exists a weak solution $u \in H^1(\mathbb{R})$ to the Schrödinger equation (1.4.6) with $||u||_{L^2(\mathbb{R})} = 1$ and $\hat{\mu} = \sqrt{-E_0} > \frac{1}{20\sqrt{10}}$.

From the regularity theorem A.4.3, any $u \in H^1(\mathbb{R})$ which weakly solves the Schödinger equation (1.4.6) is in fact smooth. Therefore, for all $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$, there exists a solution $u \in C^\infty(\mathbb{R})$ to the Schrödinger equation (1.4.6) with $\hat{\mu} = \sqrt{-E_0} > \frac{1}{20\sqrt{10}}$.

To verify the boundary conditions of u, recall by proposition 1.4.1 the solution u can be expressed, in the bases associated to $r = 2M$ and $r \to \infty$, as

$$u = \tilde{k}_1 \tilde{\mathfrak{H}}^{2M,+} + \tilde{k}_2 \tilde{\mathfrak{H}}^{2M,-}, \qquad u = \tilde{c}_1 \tilde{\mathfrak{H}}^{\infty,+} + \tilde{c}_2 \tilde{\mathfrak{H}}^{\infty,-}, \tag{1.4.59}$$

with $\tilde{k}_1, \tilde{k}_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$. Note that

$$\int_{-\infty}^{0} |\tilde{\mathfrak{H}}^{2M,-}|^2 dx_\star = \int_1^{\frac{3}{2}} |\tilde{\mathfrak{H}}^{2M,-}|^2 \frac{x}{x-1} dx = \infty, \tag{1.4.60}$$

whilst

$$\int_{-\infty}^{0} |\tilde{\mathfrak{H}}^{2M,+}|^2 + |\nabla_{x_\star} \mathfrak{H}^{2M,+}|^2 dx_\star = \int_1^{\frac{3}{2}} \left(|\tilde{\mathfrak{H}}^{2M,+}|^2 + \left| \frac{x-1}{x} \nabla_x \mathfrak{H}^{2M,+} \right|^2 \right) \frac{x}{x-1} dx \tag{1.4.61}$$

which is finite. Similarly, for $X_\star > 0$ sufficently large

$$\int_{X_\star}^{\infty} |\tilde{\mathfrak{H}}^{\infty,+}|^2 dx_\star = \int_{x(X_\star)}^{\infty} |\tilde{\mathfrak{H}}^{\infty,+}|^2 \frac{x}{x-1} dx = \infty, \tag{1.4.62}$$

whilst

$$\int_{X_\star}^{\infty} |\tilde{\mathfrak{H}}^{\infty,-}|^2 + |\nabla_{x_\star} \mathfrak{H}^{\infty,-}|^2 dx_\star = \int_{x(X_\star)}^{\infty} \left(|\tilde{\mathfrak{H}}^{\infty,-}|^2 + \left| \frac{x-1}{x} \nabla_x \mathfrak{H}^{\infty,-} \right|^2 \right) \frac{x}{x-1} dx < \infty. \tag{1.4.63}$$

Therefore, since $u \in H^1(\mathbb{R})$, the solution u, in the language of proposition 1.4.1, must have $\tilde{k}_2 = 0$ and $\tilde{c}_1 = 0$.

Therefore, taking $\tilde{\mathfrak{H}} = u$ and $|\hat{k}| \in [\frac{3}{10}, \frac{8}{10}]$ gives a $C^\infty(\mathbb{R})$ solution to the Schrödinger equation (1.4.6) with $\hat{\mu} > \frac{1}{20\sqrt{10}} > 0$, $\tilde{k}_2 = 0$ and $\tilde{c}_1 = 0$. □

Chapter 2

The Einstein Equation in Double Null Gauge

In keeping with the higher-dimensional theme of the first chapter, this chapter gives an introduction to the double null decomposition of an n-dimensional spacetime following [84] (see also the lecture notes [82] for an alternative introduction in the $4D$ case). To the best of the author's knowledge the results of this chapter have not been previously derived completely. However, one should note that the paper [85] provides a schematic derivation of the results in this section up to error terms that was sufficient for their purposes.

This chapter starts in section 2.1 with the introduction of the double null foliation of spacetime and canonical coordinates. It proceeds with a double null decomposition of the Ricci coefficients and Weyl tensor in sections 2.2 and 2.3. Some useful operations and calculations are introduced in section 2.4. The null structure equations and Bianchi equations are introduced in sections 2.6 and 2.7. Sections 2.8 and 2.9 provide double null foliations of the Schwarzschild(–Tangherlini) and Kerr spacetimes. Section 2.10 discusses linearisation of the vacuum Einstein equation in the double null gauge. In particular, it focuses on linearisation in double null gauge around the Schwarzschild(–Tangherlini) spacetime. It provides a detailed derivation of the Teukolsky and Regge–Wheeler (systems of) equations on Schw_n associated to the double null gauge and discusses the relation to the traditional Teukolsky equation [13] in $n = 4$ as derived with the Newman–Penrose formalism [102].

2.1 Double Null Foliation and Canonical Coordinates

A double null gauge is a set of coordinates (u, v, θ^A), with $A = 1, .., n-2$, such that the metric takes the form

$$g = -2\Omega^2(du \otimes dv + dv \otimes du) + \not{g}_{AB}(d\theta^A - b^A dv) \otimes (d\theta^B - b^B dv). \tag{2.1.1}$$

Remark 2.1.1. *Any Lorentzian metric can be locally put into the form (2.1.1).*

One can construct a double null gauge associated to a double null foliation in a region of n-dimensional spacetime (M, g) in the following manner. One starts by picking an embedded codimension-2 closed submanifold $\mathcal{S}_{0,0}$ of M. Let C_0 and $\underline{C}_0$ the null hypersurfaces spanned by outgoing and ingoing null geodesics emanating orthogonally from $\mathcal{S}_{0,0}$ as depicted in the following diagram:

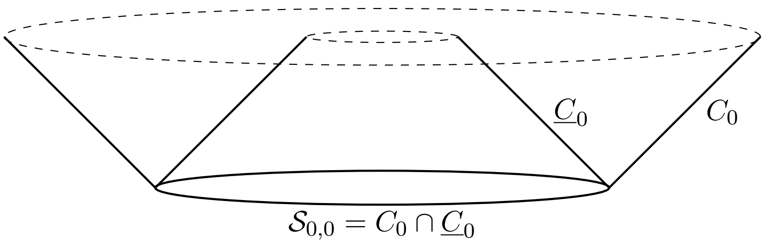


Fig. 2.1 Illustration of starting point for the double null foliation.

One now considers two optical functions u and v that satisfy the Eikonal equation

$$|\nabla u|_g^2 = 0, \qquad |\nabla v|_g^2 = 0, \tag{2.1.2}$$

with initial data $u = 0$ on C_0 where the normal derivative of u is chosen so that $|\nabla u|_g^2 = 0$ on C_0 and ∇u is chosen to be transversal to C_0 and similarly for v with $\underline{C}_0$. Denote the level sets of u as C_u and C_v (so that $\underline{C}_0 = C_{v=0}$ and $C_0 = C_{u=0}$). Since (u, v) satisfy the Eikonal equation C_u and C_v are null hypersurfaces. Note that the foliation being described here will only exist in general up to some C_{v_f} and C_{u_f} for v_f, u_f small. Denote this region D_{u_f,v_f}. One additionally assumes that there are no cut or conjugate points along C_u and C_v in D_{u_f,v_f}.

Define the two null vectors

$$L' \doteq -2(du)^\sharp, \qquad \underline{L}' \doteq -2(dv)^\sharp \tag{2.1.3}$$

and the *null lapse function* as

$$\Omega^2 \doteq -\frac{2}{g(L', \underline{L}')}. \tag{2.1.4}$$

Note that by virtue of the Eikonal equation, $(L', \underline{L}')$ are geodesic. Additionally, define the normalised null frame as

$$e_3 \doteq \Omega \underline{L}' \qquad e_4 \doteq \Omega L', \tag{2.1.5}$$

so that $g(e_3, e_4) = -2$. Finally, the sets at fixed values of (u, v) are homeomorphic to the $(n-2)$-dimensional surface $\mathcal{S}_{0,0}$. In what follows these sets will be denoted $\mathcal{S}_{u,v}$.

Suppose one is given a double null foliation of $D_{u_f,v_f} \subset M$ and let $p \in D_{u_f,v_f}$. One can assign double null canonical coordinates to p as follows:

(i) On $\mathcal{S}_{0,0}$ let θ^A be local coordinates for some patch $U \subset \mathcal{S}_{0,0}$.

(ii) Use v on C_0 as a parameter so that on C_0 one can take the coordinate system (v, θ^A) by parallelly propagating the coordinates θ^A, i.e., take $L'(\theta^A) = 0$ on C_0.

(iii) Suppose $p \in C_{v_0}$ where C_{v_0} is the null hypersurface of ingoing null geodesics emanating from $\mathcal{S}_{0,v_0}$ in C_0. Parallel propagate θ^A along the null geodesic in $\underline{C}_{v_0}$ that intersects p, i.e. impose $\underline{L}'(\theta^A) = 0$ on C_{v_0}. Note that doing this for all v_0 imposes $\underline{L}'(\theta^A) = 0$ everywhere.

(iv) Suppose additionally that $p \in C_{u_0}$ where C_{u_0} is the null hypersurface of out null geodesics emanating from $\underline{\mathcal{S}}_{u_0,0}$ in C_0. Therefore, one assigns the coordinates (u_0, v_0, θ^A) to p.

The following figure illustrates this construction:

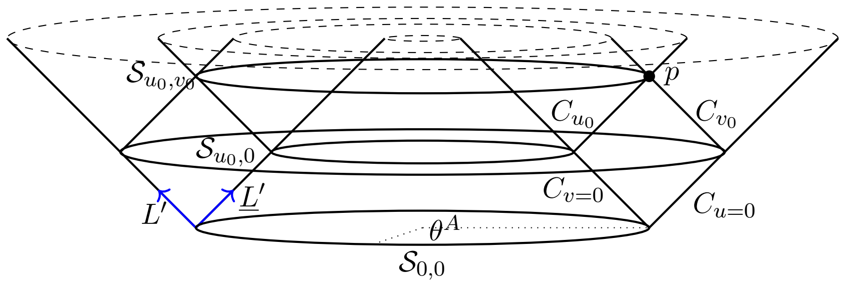


Fig. 2.2 Illustration of the double null foliation and canonical coordinates.

In canonical coordinates one has

$$e_3 = \frac{1}{\Omega}\partial_u, \qquad e_4 = \frac{1}{\Omega}\Big(\partial_v + b^A e_A\Big), \tag{2.1.6}$$

for some b^A such that $b^A|_{C_{u=0}} = 0$ for all $A = 1, ..., n-2$ and where $e_A \doteq \frac{\partial}{\partial \theta^A} \in \mathfrak{X}(\mathcal{S}_{u,v})$ for all $A = 1, ..., n-2$ and u, v. The metric in canonical coordinates for the double null foliation is then in form (2.1.1), where $\not{g}$ is the induced metric on $\mathcal{S}_{u,v} = C_u \cap C_v$.

The inverse metric in the normalised null frame is

$$g^{-1} = -\frac{1}{2}(e_3 \otimes e_4 + e_4 \otimes e_3) + \not{g}^{AB}\partial_{\theta^A} \otimes \partial_{\theta^B}. \tag{2.1.7}$$

The metric in the dual normalised null frame is given by

$$g = -2\Big(f^3 \otimes f^4 + f^4 \otimes f^3\Big) + \not{g}_{AB} f^A \otimes f^B, \tag{2.1.8}$$

where (f^3, f^4, f^A) is the dual basis to (e_3, e_4, ∂_A) which can be explicitly computed to be

$$f^3 = \Omega du, \qquad f^4 = \Omega dv, \qquad f^A = d\theta^A - b^A dv. \tag{2.1.9}$$

Associated to the embedded submanifolds $\mathcal{S}_{u,v}$ are the notion of $\mathcal{S}_{u,v}$-tensors. These are defined as follows:

Definition 2.1.1 ($\mathcal{S}_{u,v}$-tensors). *A vector field $X \in \mathfrak{X}(M)$ is an $\mathcal{S}_{u,v}$-vector field if*

$$g(X, e_4) = 0 = g(X, e_3). \tag{2.1.10}$$

Further, a $X \in \mathfrak{X}(\mathcal{S}_{u,v})$ can be viewed as a vector field in $\mathfrak{X}(M)$ satisfying (2.1.10). A one-form $\omega \in \Omega^1(M)$ is an $\mathcal{S}_{u,v}$-one-form if

$$\omega(e_3) = 0 = \omega(e_4). \tag{2.1.11}$$

Similarly, one can view $\omega \in \Omega^1(\mathcal{S}_{u,v})$ as $\omega \in \Omega^1(M)$ satisfying (2.1.11). One extends these definitions naturally to arbitrary tensors.

Remark 2.1.2. *Note that in double null coordinates this definition means $\omega_u = 0$ and $\omega_v = -b^A \omega_A$.*

Remark 2.1.3. *Note that $\not{g}$ induces a musical isomorphism between $T\mathcal{S}_{u,v}$ and $T^\star\mathcal{S}_{u,v}$.*

2.2 Null Decomposition of Ricci Coefficients

It is particularly convenient to decompose the Ricci coefficients in the normalised null frame. One makes the following definition:

Definition 2.2.1 (Connection coefficients). *Define the following $\mathcal{S}_{u,v}$-tensor fields:*

$$\begin{aligned} \chi_{AB} &\doteq g(\nabla_A e_4, e_B), & \underline{\chi}_{AB} &\doteq g(\nabla_A e_3, e_B), \\ \eta_A &\doteq \frac{1}{2} g(\nabla_3 e_4, e_A), & \underline{\eta}_A &\doteq \frac{1}{2} g(\nabla_4 e_3, e_A), \\ \hat{\omega} &\doteq -\frac{1}{2} g(\nabla_4 e_4, e_3), & \underline{\hat{\omega}} &\doteq -\frac{1}{2} g(\nabla_3 e_3, e_4) \end{aligned} \tag{2.2.1}$$

and

$$\zeta_A \doteq \frac{1}{2} g(\nabla_A e_4, e_3). \tag{2.2.2}$$

Extend these to tensor fields on M by zero on e_3 and e_4. Additionally, it is useful to define

$$\omega \doteq \Omega\hat{\omega}, \qquad \underline{\omega} \doteq \Omega\hat{\underline{\omega}}. \tag{2.2.3}$$

Remark 2.2.1. *Note that since $e_3 = \Omega L'$ and $e_4 = \Omega \underline{L}'$ and $(L', \underline{L}')$ satisfy the (affinely parameterised) geodesic equation that*

$$\hat{\omega} = \frac{e_4(\Omega)}{\Omega}, \qquad \hat{\underline{\omega}} = \frac{e_3(\Omega)}{\Omega}. \tag{2.2.4}$$

Since χ and $\underline{\chi}$ are defined with respect to the double null canonical coordinates one has the following proposition:

Proposition 2.2.2. *The $\mathcal{S}_{u,v}$-tensors χ and $\underline{\chi}$ are symmetric.*

Proof. Let $X, Y \in \mathfrak{X}(\mathcal{S}_{u,v})$. Then, using the properties of the Levi-Civita connection, one has

$$\chi(X, Y) = \chi(Y, X) - g(e_4, [X, Y]) \tag{2.2.5}$$

and similarly for $\underline{\chi}$. Noting that $X, Y \in \mathfrak{X}(\mathcal{S}_{u,v})$ can be decomposed as $X = X^A\partial_A$ and $Y = Y^A\partial_A$ gives that $[X, Y] = [X, Y]^A\partial_A$ and therefore $g(e_3, [X, Y]) = 0 = g(e_4, [X, Y])$. □

The following relations are particularly useful for computations.

Proposition 2.2.3. *The connection coefficients of definition 2.2.1 satisfy the following relations:*

$$\nabla_A e_B = \nabla\!\!\!/_A e_B + \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\underline{\chi}_{AB} e_4 \tag{2.2.6}$$

and

$$\begin{aligned}
\nabla_3 e_A &= \nabla\!\!\!/_3 e_A + \eta_A e_3, & \nabla_4 e_A &= \nabla\!\!\!/_4 e_A + \underline{\eta}_A e_4,\\
\nabla_A e_3 &= \underline{\chi}_A^B e_B + \zeta_A e_3, & \nabla_A e_4 &= \chi_A^B e_B - \zeta_A e_4,\\
\nabla_3 e_4 &= 2\eta^A e_A - \hat{\underline{\omega}} e_4, & \nabla_4 e_3 &= 2\underline{\eta}^A e_A - \hat{\omega} e_3,\\
\nabla_3 e_3 &= \hat{\underline{\omega}} e_3, & \nabla_4 e_4 &= \hat{\omega} e_4,
\end{aligned} \tag{2.2.7}$$

where $\chi_A^B = \chi_{AC}\not{g}^{CB}$ one makes the following definitions:

$$\nabla\!\!\!/_3 X = \nabla_3 X + \frac{1}{2} g(\nabla_3 X, e_3) e_4 + \frac{1}{2} g(\nabla_3 X, e_4) e_3, \tag{2.2.8}$$

$$\nabla\!\!\!/_4 X = \nabla_4 X + \frac{1}{2} g(\nabla_4 X, e_3) e_4 + \frac{1}{2} g(\nabla_4 X, e_4) e_3, \tag{2.2.9}$$

$$\nabla\!\!\!/_X Y = \nabla_X Y + \frac{1}{2} g(\nabla_X Y, e_3) e_4 + \frac{1}{2} g(\nabla_X Y, e_4) e_3, \tag{2.2.10}$$

for all X, Y $\mathcal{S}_{u,v}$-vector fields.

Proof. This follows from a direct computation and using definition 2.2.1. □

Remark 2.2.4. *Let $\not{d}$ be the exterior derivative on $\mathcal{S}_{u,v}$. One has the following relations for the torsion and the null-lapse*

$$\zeta = \frac{1}{2}(\eta - \underline{\eta}), \qquad \not{d}\log(\Omega) = \frac{1}{2}(\eta + \underline{\eta}). \tag{2.2.11}$$

Now one can show

$$[\Omega e_4, \Omega e_3] = -4\Omega^2\zeta^\sharp \implies \partial_u b^A = 4\Omega^2\zeta^A. \tag{2.2.12}$$

It is conventional to decompose χ and $\underline{\chi}$ in the following manner:

Definition 2.2.2 (Shear,Expansion)**.** *The traceless part of χ is called the shear $\hat{\chi}$ and the trace of χ is called the expansion. Therefore,*

$$\chi = \hat{\chi} + \frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\not{g}. \tag{2.2.13}$$

2.3 Null Decomposition of the Weyl Tensor

One can also decompose the Riemann tensor

$$R_{\alpha\beta\gamma\delta} \doteq g(e_\alpha, R(e_\gamma, e_\delta)e_\beta), \tag{2.3.1}$$

with respect to the normalised null frame as follows.

Definition 2.3.1 (Curvature Components)**.** *One defines the following $\mathcal{S}_{u,v}$ tensors:*

$$\begin{aligned}
\nu_{ABC} &\doteq R_{ABC4}, & \underline{\nu}_{ABC} &\doteq R_{ABC3}, \\
\tau_{AB} &\doteq \frac{1}{2}(R_{3A4B} + R_{3B4A}), & \varsigma_{AB} &\doteq \frac{1}{2}(R_{3A4B} - R_{3B4A}), \\
\alpha_{AB} &\doteq R_{A4B4}, & \underline{\alpha}_{AB} &\doteq R_{A3B3}, \\
\beta_A &\doteq \frac{1}{2}R_{A434}, & \underline{\beta}_A &\doteq \frac{1}{2}R_{A334}
\end{aligned} \tag{2.3.2}$$

and

$$\rho \doteq \frac{1}{4}R_{3434}. \tag{2.3.3}$$

For $n = 4$, one defines the scalar σ as

$$\sigma \doteq \frac{1}{4}(\star R)_{3434} \tag{2.3.4}$$

where $(\star R)_{abcd} \doteq \frac{1}{2}\varepsilon_{efab}R^{ef}{}_{cd}$ and ε is the volume form for (M, g).

Remark 2.3.1. *For $n = 4$, note that $\varepsilon(e_4, e_3, e_2, e_1) = 2\sqrt{|\det(\not{g})|}$, hence $\varepsilon_{AB34} = 2\not{\epsilon}_{AB}$ where $\not{\epsilon}$ is the volume form of $(\mathcal{S}_{u,v}, \not{g})$. So,*

$$(\star R)_{3434} = \frac{1}{2}\varepsilon_{AB34}R^{AB}{}_{34} = \not{\epsilon}^{AB}R_{AB34}. \tag{2.3.5}$$

Therefore, $\sigma = \frac{1}{4}\not{\epsilon}^{AB}R_{AB34}$.

Proposition 2.3.2. *The curvature components of definition 2.3.1 satisfy*

$$\begin{aligned} \mathrm{Tr}_{\not{g}}\alpha &= \mathrm{Ric}_{44}, & \mathrm{Tr}_{\not{g}}\underline{\alpha} &= \mathrm{Ric}_{33} \\ \beta_A &= \nu_{AB}{}^{B} + \mathrm{Ric}_{A4}, & \underline{\beta}_A &= -\underline{\nu}_{AB}{}^{B} - \mathrm{Ric}_{A3}, \\ \mathrm{Tr}_{\not{g}}\tau &= -2\rho + \mathrm{Ric}_{34}, & \tau_{AB} &= \not{g}^{CD}R_{CADB} - \mathrm{Ric}_{AB}, \\ \nu_{[ABC]} &= 0, & \nu_{(AB)C} &= 0, \\ \nu_{A[BC]} &= \frac{1}{2}\nu_{CBA}, & \nu_{ABC} &= \frac{4}{3}\nu_{A(BC)} + \frac{2}{3}\nu_{C(BA)}. \end{aligned} \tag{2.3.6}$$

and identical for $\underline{\nu}$.

Proof. Observe that $\mathrm{Ric}(e_4, e_4) = R^3{}_{434} + R^A{}_{4A4} = g^{34}R_{4434} + \mathrm{Tr}_{\not{g}}\alpha = \mathrm{Tr}_{\not{g}}\alpha$. The relation for $\mathrm{Tr}_{\not{g}}\underline{\alpha}$ is completely analogous.

Now,

$$(\mathrm{Ric}(g))_{A4} = -\frac{1}{2}R_{4A34} + \not{g}^{BC}R_{BAC4}, \tag{2.3.7}$$

which gives the relation between ν and β. Additionally,

$$\mathrm{Ric}_{AB} = -\frac{1}{2}R_{4A3B} - \frac{1}{2}R_{3A4B} + \not{g}^{CD}R_{CADB}, \tag{2.3.8}$$

which gives the relation for τ. Further,

$$(\mathrm{Ric}(g))_{34} = -\frac{1}{2}R_{4334} + \not{g}^{AB}R_{A3B4} \tag{2.3.9}$$

which gives the relation between τ and ρ.

For the ν identities one notes

$$R_{(AB)C4} = 0 \implies \nu_{(AB)C} = 0 \tag{2.3.10}$$

and

$$R_{[ABC]4} = 0 \implies \nu_{[ABC]} = 0, \tag{2.3.11}$$

which can be written as

$$\nu_{ABC} + \nu_{BCA} + \nu_{CAB} = 0. \tag{2.3.12}$$

Combining with equation (2.3.10) gives

$$\nu_{A[BC]} = \frac{1}{2}\nu_{CBA}. \tag{2.3.13}$$

One can now derive the last identity from this

$$\nu_{ABC} = \nu_{A(BC)} + \nu_{A[BC]} = \nu_{A(BC)} + \frac{1}{2}\nu_{CBA} = \nu_{A(BC)} + \frac{1}{2}\nu_{C(BA)} + \frac{1}{2}\nu_{C[BA]} \tag{2.3.14}$$

$$= \nu_{A(BC)} + \frac{1}{2}\nu_{C(BA)} + \frac{1}{4}\nu_{ABC}, \tag{2.3.15}$$

which gives the last identity for ν. □

The following proposition details the reduction to $4D$:

Proposition 2.3.3. *Suppose* (M, g) *is a* $4D$ *spacetime satisfying* $\mathrm{Ric}(g) = 0$ *then the following relations are satisfied*

$$\begin{aligned} \underline{\nu}_{BCA} &= \not{g}_{AB}\underline{\beta}_C - \not{g}_{AC}\underline{\beta}_B, & \varsigma &= \sigma\not{\epsilon}, \\ \nu_{BCA} &= \not{g}_{AC}\beta_B - \not{g}_{AB}\beta_C, & R_{ABCD} &= \rho(\not{g}_{AD}\not{g}_{BC} - \not{g}_{AC}\not{g}_{BD}), \end{aligned} \tag{2.3.16}$$

and $\hat{\tau} = 0$.

Proof. One starts by noting that

$$0 = (\mathrm{Ric}(g))_{4A} = R^{\mu}{}_{4\mu A} = \beta_A + R^{C}{}_{4CA}. \tag{2.3.17}$$

Further, by symmetry

$$R_{C4AB} = \xi_C \not{\epsilon}_{AB}. \tag{2.3.18}$$

So,

$$-\beta_B = \not{g}^{CA} R_{C4AB} = -(\star\xi)_B \implies \xi = -(\star\beta). \tag{2.3.19}$$

If one then notes that

$$\not{\epsilon}_{AB}\not{\epsilon}_{CD} = \not{g}_{AC}\not{g}_{BD} - \not{g}_{BC}\not{g}_{AD}, \tag{2.3.20}$$

then one has the relations in the first column of the statement.

To show the relations in the second column, start by noting that

$$(\mathrm{Ric}(g))_{AB} = 0 \implies R^3{}_{A3B} + R^4{}_{A4B} + \not{g}^{CD} R_{CADB} = 0. \tag{2.3.21}$$

Now, by symmetry, $R_{CADB} = f \not{\epsilon}_{CA} \not{\epsilon}_{DB}$, so $\not{g}^{CD} R_{CADB} = f \not{g}_{AB}$. Therefore,

$$-\frac{1}{2} R_{4A3B} - \frac{1}{2} R_{3A4B} + f \not{g}_{AB} = 0. \tag{2.3.22}$$

Tracing gives

$$2f = \mathrm{Ric}_{34} + \frac{1}{2} R_{4334} = -2\rho. \tag{2.3.23}$$

Hence, $f = -\rho$ and $\hat{\tau} = 0$. Using the first Bianchi identity

$$R_{A3B4} - R_{B3A4} = R_{AB34} = h \not{\epsilon}_{AB}. \tag{2.3.24}$$

One has $\not{\epsilon}_{AB} \not{\epsilon}^{AB} = 2$ so $h = 2\sigma$ and $\varsigma = \sigma \not{\epsilon}$. □

2.4 Algebra Calculus of $\mathcal{S}_{u,v}$-Tensor Fields

In this section some useful operations on $\mathcal{S}_{u,v}$ tensors are defined.

Definition 2.4.1 (Operations). *Let Θ, Φ be $(0,2)$ $\mathcal{S}_{u,v}$-tensor fields and ξ_i be $\mathcal{S}_{u,v}$ one-forms*

$$\hat{\Phi}_{AB} \doteq \frac{1}{2}\Big(\Phi_{AB} + \Phi_{BA} - \frac{2}{n-2}(\mathrm{Tr}_{\not{g}}\Phi)\not{g}_{AB}\Big), \tag{2.4.1}$$

$$(\Theta \times \Phi)_{AB} \doteq \not{g}^{CD}\Theta_{AC}\Phi_{BD}, \tag{2.4.2}$$

$$(\Theta \wedge \Phi) \doteq \begin{cases} \frac{1}{2}(\Theta \times \Phi - \Phi \times \Theta), & n > 4 \\ \frac{1}{2}\not{\epsilon}^{AB}(\Theta \times \Phi - \Phi \times \Theta)_{AB}, & n = 4 \end{cases} \tag{2.4.3}$$

$$\langle \Theta, \Phi \rangle \doteq \Theta_{AB}\Phi^{AB}, \tag{2.4.4}$$

$$\xi_1 \hat{\otimes} \xi_2 \doteq \xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1 - \frac{2}{n-2}\langle \xi_1, \xi_2 \rangle \not{g}. \tag{2.4.5}$$

Additionally, in $n = 4$ one defines the left and right Hodge dual as

$$\begin{aligned} (\star\xi)_A &\doteq \not{\epsilon}_{AB}\xi^B, & (\xi\star)_A &\doteq \xi^B \not{\epsilon}_{BA}, \\ (\star\Theta)_{AB} &\doteq \not{\epsilon}_{AC}\Theta^C{}_B, & (\Theta\star)_{AB} &\doteq \Theta_A{}^C \not{\epsilon}_{CB}. \end{aligned} \tag{2.4.6}$$

Definition 2.4.2 (Projected Covariant and Lie Derivatives). *The projected covariant derivative $\nabla\!\!\!/_3$, $\nabla\!\!\!/_4$ and $\nabla\!\!\!/_A$ on a rank-$(0,p)$ $\mathcal{S}_{u,v}$-tensor field T is defined as*

$$(\nabla\!\!\!/_3 T)(X_1, ..., X_p) \doteq (\nabla_3 T)(X_1, ..., X_p), \tag{2.4.7}$$
$$(\nabla\!\!\!/_4 T)(X_1, ..., X_p) \doteq (\nabla_4 T)(X_1, ..., X_p), \tag{2.4.8}$$
$$(\nabla\!\!\!/_A T)(X_1, ..., X_p) \doteq (\nabla_A T)(X_1, ..., X_p), \tag{2.4.9}$$

for all $X_i \in \mathfrak{X}(\mathcal{S}_{u,v})$. Further one defines the projected Lie derivatives as

$$(\mathcal{L}\!\!\!/_4 T)(X_1, ..., X_p) \doteq (\mathcal{L}_4 T)(X_1, ..., X_p), \tag{2.4.10}$$
$$(\mathcal{L}\!\!\!/_3 T)(X_1, ..., X_p) \doteq (\mathcal{L}_3 T)(X_1, ..., X_p), \tag{2.4.11}$$

for all $X_i \in \mathfrak{X}(\mathcal{S}_{u,v})$.

One defines $(p-1)$-covariant tensor field $\mathrm{d}\!\!\!/\mathrm{iv} T$ as

$$(\mathrm{d}\!\!\!/\mathrm{iv} T)_{A_1 \ldots A_{p-1}} \doteq g\!\!\!/^{BC} (\nabla\!\!\!/_B T)_{C A_1 \ldots A_{p-1}}. \tag{2.4.12}$$

For $n = 4$, one additionally defines

$$(\mathrm{cu\!\!\!/rl} T)_{A_1 \ldots A_{p-1}} \doteq \epsilon\!\!\!/^{BC} (\nabla\!\!\!/_B T)_{C A_1 \ldots A_{p-1}}, \tag{2.4.13}$$

where $\epsilon\!\!\!/$ is the induced volume form on $\mathcal{S}_{u,v}$.

Remark 2.4.1. *By the Leibniz rule for ∇ one has the Leibniz rule for $\nabla\!\!\!/$, i.e.,*

$$(\nabla\!\!\!/_\alpha T)(X_1, ..., X_p) = e_\alpha(T(X_1, ..., X_p)) - T(\nabla\!\!\!/_\alpha X_1, ..., X_p) - ... - T(X_1, ..., \nabla\!\!\!/_\alpha X_p) \tag{2.4.14}$$

for $\alpha = 3, 4, A$ and for all $X_i \in \mathfrak{X}(\mathcal{S}_{u,v})$.

Definition 2.4.3 (Symmetrised Derivative). *One defines the operator $\nabla\!\!\!/ \hat{\otimes}$ on $\mathcal{S}_{u,v}$-one-forms as*

$$(\nabla\!\!\!/ \hat{\otimes} \xi)_{AB} = (\nabla\!\!\!/_A \xi)_B + (\nabla\!\!\!/_B \xi)_A - \frac{2}{n-2} (\mathrm{d}\!\!\!/\mathrm{iv}\, \xi) g\!\!\!/_{AB}. \tag{2.4.15}$$

Definition 2.4.4 (Formal Adjoint Operators). *Let ξ be an arbitrary $\mathcal{S}_{u,v}$ one-form and Θ an arbitrary symmetric traceless 2-tensor on $\mathcal{S}_{u,v}$. Define $\mathcal{D}\!\!\!/_2 : \mathrm{symtr}(T^\star \mathcal{S}_{u,v} \otimes T^\star \mathcal{S}_{u,v}) \to \Omega^1(\mathcal{S}_{u,v})$ by*

$$\mathcal{D}\!\!\!/_2 \Theta \doteq \mathrm{d}\!\!\!/\mathrm{iv} \Theta \tag{2.4.16}$$

and $\not{\mathcal{D}}_2^\star : \Omega^1(\mathcal{S}_{u,v}) \to \mathrm{symtr}(T^\star\mathcal{S}_{u,v} \otimes T^\star\mathcal{S}_{u,v})$ *by*

$$\not{\mathcal{D}}_2^\star \xi \doteq -\frac{1}{2}\not{\nabla}\hat{\otimes}\xi. \tag{2.4.17}$$

For $n = 4$, *define the following operators* $\not{\mathcal{D}}_1 : \Omega^1(\mathcal{S}_{u,v}) \to C^\infty(M) \times C^\infty(M)$ *by*

$$\xi \mapsto (\not{\mathrm{d}}\mathrm{iv}(\xi), \mathrm{c}\not{\mathrm{u}}\mathrm{rl}(\xi)) \tag{2.4.18}$$

and $\not{\mathcal{D}}_1^\star : C^\infty(M) \times C^\infty(M) \to \Omega^1(\mathcal{S}_{u,v})$ *by*

$$[\not{\mathcal{D}}_1^\star(f_1, f_2)]_A = -\not{\nabla}_A f_1 + \not{\in}_{AB}\not{\nabla}^B f_2 \qquad \forall f_1, f_2 \in C^\infty(M). \tag{2.4.19}$$

Proposition 2.4.2. *The operators* $\not{\mathcal{D}}_1^\star$ *and* $\not{\mathcal{D}}_2^\star$ *are the formal* L^2 *adjoints of* $\not{\mathcal{D}}_1$ *and* $\not{\mathcal{D}}_2$ *respectively.*

This section concludes with a collection of useful results for $n = 4$.

Lemma 2.4.3 (Useful Identities in $4D$)**.** *Let* $n = 4$, $X \in \mathfrak{X}(\mathcal{S}_{u,v})$, $\Phi, \Theta \in \mathrm{symtr}(T^\star\mathcal{S}_{u,v} \otimes T^\star\mathcal{S}_{u,v})$ *and* $\xi \in \Omega^1(\mathcal{S}_{u,v})$. *Then*

$$(\Phi \times \Theta)_{(AB)} = \frac{1}{2}\langle \Phi, \Theta\rangle \not{g}_{AB} \tag{2.4.20}$$

and

$$\star\Theta = \widehat{(\star\Theta)}, \qquad \star\Theta = -(\Theta\star). \tag{2.4.21}$$

Additionally, one has the following identity relating $\not{\mathrm{d}}\mathrm{iv}$ *and* $\mathrm{c}\not{\mathrm{u}}\mathrm{rl}$,

$$\mathrm{c}\not{\mathrm{u}}\mathrm{rl}\Theta = \star(\not{\mathrm{d}}\mathrm{iv}\Theta) \tag{2.4.22}$$

and the following identities for the projected Lie derivative

$$(\not{\mathcal{L}}_X\xi)_A = (\not{\nabla}_X\xi)_A - (\not{\mathcal{D}}_2^\star\xi)_{AB}\xi^B + \frac{1}{2}(\not{\mathrm{d}}\mathrm{iv}X)\xi_A + \frac{1}{2}(\mathrm{c}\not{\mathrm{u}}\mathrm{rl}X)(\star\xi)_A, \tag{2.4.23}$$

$$(\not{\mathcal{L}}_X\Theta) = (\not{\nabla}_X\Theta) - \langle\not{\mathcal{D}}_2^\star X, \Theta\rangle\not{g} + (\not{\mathrm{d}}\mathrm{iv}X)\Theta + (\mathrm{c}\not{\mathrm{u}}\mathrm{rl}X)(\star\Theta). \tag{2.4.24}$$

Further, the following integrated identities hold:

$$\int_{\mathcal{S}_{u,v}} |\not{\nabla}\Theta|^2 \not{\in} = \int_{\mathcal{S}_{u,v}} \Big(2|\not{\mathrm{d}}\mathrm{iv}\Theta|^2 - \mathrm{Scal}(\not{g})|\Theta|^2\Big)\not{\in}, \tag{2.4.25}$$

$$\int_{\mathcal{S}_{u,v}} |\not{\nabla}\xi|^2 \not{\in} = \int_{\mathcal{S}_{u,v}} \Big(|\mathrm{c}\not{\mathrm{u}}\mathrm{rl}\xi|^2 + |\not{\mathrm{d}}\mathrm{iv}\xi|^2 - \frac{1}{2}\mathrm{Scal}(\not{g})|\xi|^2\Big)\not{\in}. \tag{2.4.26}$$

Proof. The first two results (equations (2.4.20) and (2.4.21)) in this lemma can be proved by computing explicitly the components of each object.

The identity in equation (2.4.22) relating $\operatorname{d\!\!/iv}$ and $\operatorname{cu\!\!/rl}$ is proved by noting by metric compatibility

$$\star(\operatorname{d\!\!/iv}\Theta) = -\operatorname{d\!\!/iv}(\Theta\star) \tag{2.4.27}$$

and hence, by the second result in equation (2.4.21),

$$\star(\operatorname{d\!\!/iv}\Theta) = \operatorname{d\!\!/iv}(\star\Theta) = \operatorname{cu\!\!/rl}\Theta. \tag{2.4.28}$$

The results (equations (2.4.23) and (2.4.24)) on the Lie derivative in the lemma follow from writing

$$(\not{\mathcal{L}}_X\xi)_A = (\nabla\!\!\!/_X\xi)_A + \xi^B(\nabla\!\!\!/_A X)_B \tag{2.4.29}$$

and similarly for Θ. Writing

$$(\nabla\!\!\!/_A X)_B = -(\not{\mathcal{D}}_2^\star X)_{AB} + \frac{1}{2}\operatorname{d\!\!/iv}X\not{g} + \frac{1}{2}\operatorname{cu\!\!/rl}X\not{\epsilon}, \tag{2.4.30}$$

then gives the result (in the Θ case one uses (2.4.20) to conclude).

Turning to the integrated identities in equations (2.4.25) and (2.4.26). Start by noting a standard identity for the Levi-Civita symbol $\not{\epsilon}$

$$\not{\epsilon}^{AB}\not{\epsilon}^{CD} = \not{g}^{AC}\not{g}^{BD} - \not{g}^{AD}\not{g}^{BC}. \tag{2.4.31}$$

Further, for a $2D$ surface

$$\not{R}^A{}_{BCD} = \frac{1}{2}\mathrm{Scal}(\not{g})(\not{\delta}^A_C\not{g}_{BD} - \not{\delta}^A_D\not{g}_{BC}). \tag{2.4.32}$$

This allows one to establish, by considering $|\operatorname{cu\!\!/rl}\Theta|^2$ and using the Ricci identity, that

$$|\nabla\!\!\!/\theta|^2 = |\operatorname{d\!\!/iv}\theta|^2 + |\operatorname{cu\!\!/rl}\theta|^2 - \mathrm{Scal}(\not{g})|\theta|^2 + \operatorname{d\!\!/iv}(X), \tag{2.4.33}$$

where $X_A = \theta^{BC}(\nabla\!\!\!/_B\theta)_{AC} - \theta_{AC}(\operatorname{d\!\!/iv}\theta)^C$. The result then follows from noting that

$$|\operatorname{cu\!\!/rl}\Theta|^2 = |\star\operatorname{d\!\!/iv}\Theta|^2 = |\mathrm{div}\Theta|^2. \tag{2.4.34}$$

Turning to the result for the 1-form, one compute directly using the Ricci identity that

$$|\nabla\!\!\!/\xi|^2 = |\mathrm{c}\!\!/\mathrm{url}\xi|^2 + |\mathrm{d}\!\!/\mathrm{iv}\xi|^2 - \frac{1}{2}\mathrm{Scal}(g\!\!\!/)|\xi|^2 + \mathrm{d}\!\!/\mathrm{iv}(Y), \tag{2.4.35}$$

with $Y = \nabla\!\!\!/_\xi \xi - (\mathrm{d}\!\!/\mathrm{iv}\xi)\xi$. □

2.5 Computing in Double Null Coordinates

It is often the case that one wants to compute explicitly objects such as $\nabla\!\!\!/_3 X$ for X an $\mathcal{S}_{u,v}$-vector field or $(\nabla\!\!\!/_A T)_{B_1\ldots B_s}$. This section elaborates on how to compute such things.

Proposition 2.5.1. *In double null coordinates* (u, v, θ^A) *one has*

$$\nabla\!\!\!/_3 e_B = \underline{\chi}_A^B e_B, \tag{2.5.1}$$

$$\nabla\!\!\!/_4 e_B = (\chi_A^B - \frac{1}{\Omega}\partial_A b^B) e_B, \tag{2.5.2}$$

$$\nabla\!\!\!/_A e_B = \Gamma\!\!\!/^C_{AB} e_C. \tag{2.5.3}$$

Proof. The first two follow from the torsion-free condition on ∇ and proposition 2.2.3.

The last result follows from writing,

$$\nabla_A e_B = \Gamma^u_{AB}\partial_u + \Gamma^v_{AB}\partial_v + \Gamma^C_{AB}\partial_C \tag{2.5.4}$$

and noting that

$$\Gamma^C_{AB} = \Gamma\!\!\!/^C_{AB} + b^C \Gamma^v_{AB}. \tag{2.5.5}$$

Projecting onto the horizontal subspace gives the result. □

Proposition 2.5.2 (Projected Derivatives of p-covariant $\mathcal{S}$-Tensor Fields)**.** *Let T be a p-covariant $\mathcal{S}_{u,v}$-tensor field. Then, in double null coordinates* (u, v, θ^A)*,*

$$(\nabla\!\!\!/_3 T)_{A_1\ldots A_p} = \frac{1}{\Omega}\Big(\partial_u(T_{A_1\ldots A_p}) - \frac{p}{n-2}(\Omega \mathrm{Tr}_{g\!\!\!/}\underline{\chi})T_{A_1\ldots A_p} - \Omega\sum_{i=1}^{p} \hat{\underline{\chi}}_{A_i}{}^{B} T_{A_1\ldots \hat{A}_i B\ldots A_p}\Big), \tag{2.5.6}$$

$$(\nabla\!\!\!/_4 T)_{A_1\ldots A_p} = \frac{1}{\Omega}\Big((\partial_v + b^A\partial_A)(T_{A_1\ldots A_p}) - \frac{p}{n-2}(\Omega \mathrm{Tr}_{g\!\!\!/}\chi)T_{A_1\ldots A_p} - \sum_{i=1}^{p}\Big(\Omega\hat{\chi}_{A_i}{}^{B} - (\partial_{A_i} b)^B\Big) T_{A_1\ldots \hat{A}_i B\ldots A_p}\Big), \tag{2.5.7}$$

$$(\nabla\!\!\!/_A T)_{B_1\ldots B_p} = \partial_A(T_{B_1\ldots B_p}) - \sum_{i=1}^{p} \Gamma\!\!\!/^C_{AB_i} T_{B_1\ldots \hat{B}_i C\ldots B_p}. \tag{2.5.8}$$

where $\hat{A}_i$ denotes removing the i^{th} index and replacing it by B.

Proof. By the remark 2.4.1 following definition 2.4.2 above one has

$$(\nabla\!\!\!/_{\alpha}T)_{B_1...B_p} = e_\alpha(T_{B_1...B_p}) - T(\nabla\!\!\!/_{\alpha}e_{B_1},...,e_{B_p}) - ... - T(e_{B_1},...,\nabla\!\!\!/_{\alpha}e_{B_p}), \tag{2.5.9}$$

for $\alpha = 3, 4, A$. Using proposition 2.5.1, one gets the result. □

Remark 2.5.3. *This shows that induced metric on $\mathcal{S}_{u,v}$ satisfies*

$$(\nabla\!\!\!/_{3}g\!\!\!/)_{AB} = 0 = (\nabla\!\!\!/_{4}g\!\!\!/)_{AB}. \tag{2.5.10}$$

2.6 Null Structure Equations

In this section a series of results are stated about the geometry of the double null foliation. Note that the vacuum Einstein equations are not assumed. For completeness, a proof of these statements can be found in appendix B.1.

Proposition 2.6.1 (First Variation Formulas). *The metric coefficients Ω, b and $g\!\!\!/$ satisfy*

$$\begin{aligned} \mathcal{L}\!\!\!/_{4}g\!\!\!/ &= 2\chi, & \mathcal{L}\!\!\!/_{3}g\!\!\!/ &= 2\underline{\chi}, \\ e_3(\Omega) &= \underline{\omega}, & e_4(\Omega) &= \omega, \\ \partial_u b^A &= 2\Omega^2(\eta - \underline{\eta})^A, & \nabla\!\!\!/\Omega &= \frac{\Omega}{2}(\eta + \underline{\eta}). \end{aligned} \tag{2.6.1}$$

Proposition 2.6.2 (Raychaudhuri/Shear Equations). *The expansions $\mathrm{Tr}_{g\!\!\!/}\chi$, $\mathrm{Tr}_{g\!\!\!/}\underline{\chi}$ and the shears $\hat{\chi}$, $\hat{\underline{\chi}}$ satisfy*

$$\nabla\!\!\!/_{4}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi) = -\Omega\mathrm{Ric}_{44} - \Omega|\hat{\chi}|^2 - \frac{1}{(n-2)\Omega}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2 + 2\omega\mathrm{Tr}_{g\!\!\!/}\chi, \tag{2.6.2}$$

$$\nabla\!\!\!/_{3}(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}) = -\Omega\mathrm{Ric}_{33} - \Omega|\hat{\underline{\chi}}|^2 - \frac{1}{(n-2)\Omega}(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})^2 + 2\underline{\omega}\mathrm{Tr}_{g\!\!\!/}\underline{\chi}, \tag{2.6.3}$$

$$\nabla\!\!\!/_{4}\hat{\chi} = \hat{\omega}\hat{\chi} - \frac{2}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\chi} - \widehat{\hat{\chi}\times\hat{\chi}} - \hat{\alpha}, \tag{2.6.4}$$

$$\nabla\!\!\!/_{3}\hat{\underline{\chi}} = \hat{\underline{\omega}}\hat{\underline{\chi}} - \frac{2}{n-2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\hat{\underline{\chi}} - \widehat{\hat{\underline{\chi}}\times\hat{\underline{\chi}}} - \hat{\underline{\alpha}}. \tag{2.6.5}$$

where $\widehat{\hat{\chi}\times\hat{\underline{\chi}}} = 0$ if $n = 4$. If (M, g) satisfies the vacuum Einstein equation (I.2), $\hat{\underline{\alpha}} = \underline{\alpha}$ and $\hat{\alpha} = \alpha$.

Proposition 2.6.3 (Torsion Propagation Equations). *The torsions η and $\underline{\eta}$ satisfy*

$$(\nabla\!\!\!/_4\eta)_A = -\beta_A - \hat{\chi}_{AB}(\eta-\underline{\eta})^B - \frac{1}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi(\eta-\underline{\eta})_A, \tag{2.6.6}$$

$$(\nabla\!\!\!/_3\underline{\eta})_A = \underline{\beta}_A + \hat{\underline{\chi}}_{AB}(\eta-\underline{\eta})^B + \frac{1}{n-2}\mathrm{Tr}_{g\!\!\!/}\underline{\chi}(\eta-\underline{\eta})_A. \tag{2.6.7}$$

Proposition 2.6.4 (Torsion Constraints). *The torsions η and $\underline{\eta}$ satisfy*

$$d\!\!\!/\underline{\eta} = \frac{1}{2}\chi\wedge\underline{\chi} - \varsigma, \tag{2.6.8}$$

$$d\!\!\!/\eta = -\frac{1}{2}\chi\wedge\underline{\chi} + \varsigma. \tag{2.6.9}$$

For $n=4$, the torsions η and $\underline{\eta}$ satisfy

$$\mathrm{c\!\!\!/url}\underline{\eta} = \frac{1}{2}\chi\wedge\underline{\chi} - \sigma, \tag{2.6.10}$$

$$\mathrm{c\!\!\!/url}\eta = -\frac{1}{2}\chi\wedge\underline{\chi} + \sigma. \tag{2.6.11}$$

Proposition 2.6.5 (Propagation Equations for $\hat{\omega}, \hat{\underline{\omega}}$). *The functions $\hat{\omega}, \hat{\underline{\omega}}$ satisfy*

$$\nabla\!\!\!/_4\underline{\omega} = \Omega(2\langle\eta,\underline{\eta}\rangle - |\eta|^2 - \rho), \tag{2.6.12}$$

$$\nabla\!\!\!/_3\omega = \Omega(2\langle\eta,\underline{\eta}\rangle - |\underline{\eta}|^2 - \rho). \tag{2.6.13}$$

Proposition 2.6.6 (Transversal Propagation Equations for χ and $\underline{\chi}$). *The expansions $\mathrm{Tr}_{g\!\!\!/}\chi$, $\mathrm{Tr}_{g\!\!\!/}\underline{\chi}$ and the shears $\hat{\chi}$, $\hat{\underline{\chi}}$ satisfy*

$$\nabla\!\!\!/_4(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}) = 2\Omega\mathrm{d\!\!\!/iv}\underline{\eta} + 2\Omega|\underline{\eta}|^2 - \Omega\langle\hat{\chi},\hat{\underline{\chi}}\rangle - \frac{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}{(n-2)\Omega} + 2\Omega\rho - \Omega\mathrm{Ric}_{34}, \tag{2.6.14}$$

$$\nabla\!\!\!/_3(\Omega\mathrm{Tr}_{g\!\!\!/}\chi) = 2\Omega\mathrm{d\!\!\!/iv}\eta + 2\Omega|\eta|^2 - \Omega\langle\hat{\chi},\hat{\underline{\chi}}\rangle - \frac{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}{(n-2)\Omega} + 2\Omega\rho - \Omega\mathrm{Ric}_{34}, \tag{2.6.15}$$

$$\nabla\!\!\!/_4\hat{\underline{\chi}} = \underline{\eta}\hat{\otimes}\underline{\eta} - \hat{\tau} - 2\mathcal{D}\!\!\!/_2^\star\underline{\eta} - \frac{1}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\underline{\chi}} - \frac{1}{n-2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\hat{\chi} - \hat{\omega}\hat{\underline{\chi}} - \widehat{\hat{\chi}\times\hat{\underline{\chi}}}, \tag{2.6.16}$$

$$\nabla\!\!\!/_3\hat{\chi} = \eta\hat{\otimes}\eta - \hat{\tau} - 2\mathcal{D}\!\!\!/_2^\star\eta - \frac{1}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\underline{\chi}} - \frac{1}{n-2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\hat{\chi} - \hat{\underline{\omega}}\hat{\chi} - \widehat{\hat{\chi}\times\hat{\underline{\chi}}}. \tag{2.6.17}$$

where $\widehat{\hat{\chi}\times\hat{\underline{\chi}}} = 0$ if $n=4$. Additionally, if $n=4$ and (M,g) satisfies the vacuum Einstein equation (I.2), $\hat{\tau}=0$.

Proposition 2.6.7 (Gauss Constraint Equation(s))**.** *Let* $\not{\mathrm{S}}\mathrm{cal}$, $\not{\mathrm{R}}\mathrm{ic}$ *and* $\not{R}$ *be the Ricci scalar, Ricci curvature and Riemann tensor associated to* $\mathcal{S}_{u,v}$. *Then,*

$$\not{R}_{ABCD} = R_{ABCD} + \frac{1}{2}\Big[\chi_{AD}\underline{\chi}_{BC} + \chi_{BC}\underline{\chi}_{AD} - \chi_{AC}\underline{\chi}_{BD} - \underline{\chi}_{AC}\chi_{BD}\Big], \tag{2.6.18}$$

$$\widehat{\not{\mathrm{R}}\mathrm{ic}}_{AB} = \hat{\tau}_{AB} + \widehat{\mathrm{Ric}}_{AB} + (\widehat{\hat{\chi} \times \hat{\underline{\chi}}})_{AB} - \frac{n-4}{2(n-2)}\Big(\mathrm{Tr}_{\not{g}}\chi\hat{\underline{\chi}}_{AB} + \mathrm{Tr}_{\not{g}}\underline{\chi}\hat{\chi}_{AB}\Big), \tag{2.6.19}$$

$$\not{\mathrm{S}}\mathrm{cal} = \mathrm{Scal} + 2\mathrm{Ric}_{34} - 2\rho + \frac{3-n}{n-2}\mathrm{Tr}_{\not{g}}\chi\mathrm{Tr}_{\not{g}}\underline{\chi} + \langle\hat{\chi}, \hat{\underline{\chi}}\rangle. \tag{2.6.20}$$

For $n = 4$, *let* K *be the Gauss curvature of* $\mathcal{S}_{u,v}$. *Then,*

$$\not{R}_{ABCD} = \mathrm{K}(\not{g}_{AC}\not{g}_{BD} - \not{g}_{BC}\not{g}_{AD}) \tag{2.6.21}$$

and

$$\mathrm{K} = \frac{\mathrm{Scal}}{2} + \mathrm{Ric}_{34} - \rho - \frac{1}{4}\mathrm{Tr}_{\not{g}}\chi\mathrm{Tr}_{\not{g}}\underline{\chi} + \frac{1}{2}\langle\hat{\chi}, \hat{\underline{\chi}}\rangle. \tag{2.6.22}$$

Proposition 2.6.8 (Codazzi Constraint Equations)**.** *The shears* $\hat{\chi}$, $\hat{\underline{\chi}}$ *satisfy*

$$\not{\nabla}_{[A}\hat{\chi}_{B]C} = \frac{1}{n-2}\not{g}_{C[A}\not{\nabla}_{B]}(\Omega\mathrm{Tr}_{\not{g}}\chi) + \frac{1}{2}\nu_{ABC} + \hat{\chi}_{C[A}\zeta_{B]} - \frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\not{g}_{C[A}\underline{\eta}_{B]}, \tag{2.6.23}$$

$$\not{\nabla}_{[A}\hat{\underline{\chi}}_{B]C} = \frac{1}{n-2}\not{g}_{C[A}\not{\nabla}_{B]}(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi}) + \frac{1}{2}\underline{\nu}_{ABC} - \hat{\underline{\chi}}_{C[A}\zeta_{B]} - \frac{\mathrm{Tr}_{\not{g}}\underline{\chi}}{n-2}\not{g}_{C[A}\eta_{B]}, \tag{2.6.24}$$

$$(\not{\mathrm{d}}\mathrm{iv}\hat{\chi})_A = \frac{n-3}{(n-2)\Omega}\not{\nabla}_A(\Omega\mathrm{Tr}_{\not{g}}\chi) - \frac{1}{2}\hat{\chi}_{AB}(\eta - \underline{\eta})^B - \frac{n-3}{n-2}\mathrm{Tr}_{\not{g}}\chi\underline{\eta}_A - \beta_A + \mathrm{Ric}_{4A}, \tag{2.6.25}$$

$$(\not{\mathrm{d}}\mathrm{iv}\hat{\underline{\chi}})_A = \frac{n-3}{(n-2)\Omega}\not{\nabla}_A(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi}) + \frac{1}{2}\hat{\underline{\chi}}_{AB}(\eta - \underline{\eta})^B - \frac{n-3}{n-2}\mathrm{Tr}_{\not{g}}\underline{\chi}\eta_A + \underline{\beta}_A + \mathrm{Ric}_{3A}. \tag{2.6.26}$$

If $n = 4$, *then*

$$\not{\nabla}_{[A}\hat{\chi}_{B]C} = \frac{1}{2}(\star\not{\mathrm{d}}\mathrm{iv}\hat{\chi})_C\not{\epsilon}_{AB}, \tag{2.6.27}$$

$$\not{\nabla}_{[A}\hat{\underline{\chi}}_{B]C} = \frac{1}{2}(\star\not{\mathrm{d}}\mathrm{iv}\hat{\underline{\chi}})_C\not{\epsilon}_{AB}. \tag{2.6.28}$$

Remark 2.6.9. *If* (M, g) *satisfies the vacuum Einstein equation (I.2), i.e., one sets* $\mathrm{Ric}(g) = 0 = \mathrm{Scal}$, *then one has encoded the vacuum Einstein equation in the null structure equations of propositions 2.6.1-2.6.8.*

2.7 The Bianchi Identities in Double Null Gauge

In this subsection a collection of results are stated for the Weyl tensor. These are derived in appendix B.2. Note that the vacuum Einstein equations <u>are</u> assumed.

Proposition 2.7.1. *Suppose* $\mathrm{Ric}(g) = 0$*. The null Weyl curvature components of definition 2.3.1 satisfy the following alterations of the usual* $n = 4$ *(null-decomposed) Bianchi identities:*

$$\nabla\!\!\!/_4\rho = \langle 2\underline{\eta} + \zeta, \beta\rangle - \Big(\frac{n-1}{n-2}\Big)\rho\mathrm{Tr}_{g\!\!\!/}\chi + \mathrm{d}i\!\!\!/\mathrm{v}\beta + \frac{1}{2}\langle\hat{\tau}, \hat{\chi}\rangle - \frac{1}{2}\langle\alpha, \hat{\underline{\chi}}\rangle, \tag{2.7.1}$$

$$\nabla\!\!\!/_3\rho = -\langle 2\eta - \zeta, \underline{\beta}\rangle - \Big(\frac{n-1}{n-2}\Big)\rho\mathrm{Tr}_{g\!\!\!/}\underline{\chi} - \mathrm{d}i\!\!\!/\mathrm{v}\underline{\beta} + \frac{1}{2}\langle\hat{\tau}, \hat{\underline{\chi}}\rangle - \frac{1}{2}\langle\underline{\alpha}, \hat{\chi}\rangle, \tag{2.7.2}$$

$$(\nabla\!\!\!/_4\beta)_A = (\underline{\eta} + 2\zeta)^B\alpha_{AB} + \hat{\omega}\beta_A + (\mathrm{d}i\!\!\!/\mathrm{v}\alpha)_A - \frac{n}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\beta_A - \hat{\chi}^C_A\beta_C + \hat{\chi}^{CD}\nu_{DAC}, \tag{2.7.3}$$

$$(\nabla\!\!\!/_3\underline{\beta})_A = (2\zeta - \eta)^B\underline{\alpha}_{AB} + \hat{\underline{\omega}}\underline{\beta}_A - (\mathrm{d}i\!\!\!/\mathrm{v}\underline{\alpha})_A - \hat{\underline{\chi}}^B_A\underline{\beta}_B - \frac{n}{n-2}\mathrm{Tr}_{g\!\!\!/}\underline{\chi}\underline{\beta}_A - \hat{\underline{\chi}}^{DB}\underline{\nu}_{DAB}, \tag{2.7.4}$$

$$\begin{aligned}(\nabla\!\!\!/_4\underline{\beta})_A = {} & 3\varsigma_{AB}\underline{\eta}^B + \hat{\tau}_{AB}\underline{\eta}^B - \frac{2(n-1)}{n-2}\rho\underline{\eta}_A - \hat{\omega}\underline{\beta}_A + 2\hat{\underline{\chi}}_{AB}\beta^B - \frac{n-4}{n-2}\mathrm{Tr}_{g\!\!\!/}\underline{\chi}\beta_A \\ & - \frac{2(n-3)}{n-2}\nabla\!\!\!/_A\rho - \mathrm{d}i\!\!\!/\mathrm{v}(\hat{\tau} + \varsigma)_A - \hat{\chi}^B_A\underline{\beta}_B + \hat{\chi}^{BD}\underline{\nu}_{ABD} - \frac{2}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi\underline{\beta}_A,\end{aligned} \tag{2.7.5}$$

$$\begin{aligned}(\nabla\!\!\!/_3\beta)_A = {} & 3\varsigma_{AB}\eta^B - \hat{\tau}_{AB}\eta^B + \frac{2(n-1)}{n-2}\rho\eta_A - \hat{\underline{\omega}}\beta_A + 2\hat{\chi}_{AB}\underline{\beta}^B - \frac{n-4}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi\underline{\beta}_A \\ & + \frac{2(n-3)}{n-2}\nabla\!\!\!/_A\rho + \mathrm{d}i\!\!\!/\mathrm{v}(\hat{\tau} - \varsigma)_A - \hat{\underline{\chi}}^B_A\beta_B - \hat{\underline{\chi}}^{BD}\nu_{ABD} - \frac{2}{n-2}\mathrm{Tr}_{g\!\!\!/}\underline{\chi}\beta_A,\end{aligned} \tag{2.7.6}$$

$$\begin{aligned}(\nabla\!\!\!/_4\varsigma)_{AB} = {} & \underline{\eta}^C\nu_{ABC} + (\beta\wedge(\underline{\eta} + \zeta))_{AB} - (d\!\!\!/\beta)_{AB} + (\hat{\tau}\wedge\hat{\chi})_{AB} - (\alpha\wedge\hat{\underline{\chi}})_{AB} \\ & - 3(\varsigma\wedge\hat{\chi})_{AB} - \frac{3}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\varsigma_{AB},\end{aligned} \tag{2.7.7}$$

$$\begin{aligned}(\nabla\!\!\!/_3\varsigma)_{AB} = {} & -\eta^C\underline{\nu}_{ABC} + (\underline{\beta}\wedge(\eta - \zeta))_{AB} - (d\!\!\!/\underline{\beta})_{AB} - (\hat{\tau}\wedge\hat{\underline{\chi}})_{AB} + (\underline{\alpha}\wedge\hat{\chi})_{AB} \\ & - 3(\varsigma\wedge\hat{\underline{\chi}})_{AB} - \frac{3}{n-2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\varsigma_{AB},\end{aligned} \tag{2.7.8}$$

$$\begin{aligned}(\nabla\!\!\!/_3\alpha)_{AB} = {} & \Big(\Big(2\eta + \frac{1}{2}\zeta\Big)\hat{\otimes}\beta\Big)_{AB} - 2\hat{\underline{\omega}}\alpha_{AB} - (\mathcal{D}\!\!\!/^\star_2\beta)_{AB} + \widehat{(\hat{\tau}\times\hat{\chi})}_{AB} - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\tau}_{AB} \\ & - \rho\frac{n}{(n-2)}\hat{\chi}_{AB} - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\alpha_{AB} + \hat{\chi}^{CE}R_{CABE} - 3\widehat{(\varsigma\times\hat{\chi})}_{AB} - \widehat{(\mathrm{d}i\!\!\!/\mathrm{v}\nu)}_{AB} \\ & - (\zeta + 4\eta)^C\Big(\nu_{C(AB)} - \frac{1}{n-2}\beta_C g\!\!\!/_{AB}\Big) + \frac{1}{n-2}\langle\hat{\tau}, \hat{\chi}\rangle g\!\!\!/_{AB},\end{aligned} \tag{2.7.9}$$

$$\begin{aligned}(\nabla\!\!\!/_4\underline{\alpha})_{AB} = {} & \Big(\Big(\frac{1}{2}\zeta - 2\underline{\eta}\Big)\hat{\otimes}\underline{\beta}\Big)_{AB} - 2\hat{\omega}\underline{\alpha}_{AB} + (\mathcal{D}\!\!\!/^\star_2\underline{\beta})_{AB} + \widehat{(\hat{\tau}\times\hat{\underline{\chi}})}_{AB} - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\hat{\tau}_{AB} \\ & - \rho\frac{n}{(n-2)}\hat{\underline{\chi}}_{AB} - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\chi)\underline{\alpha}_{AB} + \hat{\underline{\chi}}^{CE}R_{CABE} + 3\widehat{(\varsigma\times\hat{\underline{\chi}})}_{AB} - \widehat{(\mathrm{d}i\!\!\!/\mathrm{v}\underline{\nu})}_{AB} \\ & + (\zeta - 4\underline{\eta})^C\Big(\underline{\nu}_{C(AB)} + \frac{1}{n-2}\underline{\beta}_C g\!\!\!/_{AB}\Big) + \frac{1}{n-2}\langle\hat{\tau}, \hat{\underline{\chi}}\rangle g\!\!\!/_{AB},\end{aligned} \tag{2.7.10}$$

and the additional (null-decomposed) Bianchi identities for $n > 4$ (which are automatically satisfied in $n = 4$ due to the above)

$$
\begin{aligned}
(\nabla\!\!\!/_4\hat{\tau})_{AB} = &-(\underline{\eta}\hat{\otimes}\beta)_{AB} + (\mathcal{D}\!\!\!/^\star_2\beta)_{AB} + \widehat{(\alpha\times\underline{\hat{\chi}})}_{AB} - \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{g\!\!\!/}\underline{\chi}\alpha_{AB} + \hat{\chi}_{AB}\rho \\
&- \frac{1}{2}(\beta\hat{\otimes}\zeta)_{AB} - \widehat{(\mathrm{d}\!\!\!/\mathrm{iv}\nu)}_{AB} - \frac{n}{2(n-2)}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\tau}_{AB} + \frac{1}{n-2}\langle\hat{\tau},\hat{\chi}\rangle g\!\!\!/_{AB} \\
&- (\zeta + 2\underline{\eta})^C\Big(\nu_{C(AB)} - \frac{1}{n-2}\beta_C g\!\!\!/_{AB}\Big) + \hat{\chi}^{CE}R_{CABE},
\end{aligned}
\tag{2.7.11}
$$

$$
\begin{aligned}
(\nabla\!\!\!/_3\hat{\tau})_{AB} = &(\eta\hat{\otimes}\underline{\beta})_{AB} - (\mathcal{D}\!\!\!/^\star_2\underline{\beta})_{AB} + \widehat{(\underline{\alpha}\times\hat{\chi})}_{AB} - \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{g\!\!\!/}\chi\underline{\alpha}_{AB} + \underline{\hat{\chi}}_{AB}\rho \\
&- \frac{1}{2}(\underline{\beta}\hat{\otimes}\zeta)_{AB} - \widehat{(\mathrm{d}\!\!\!/\mathrm{iv}\underline{\nu})}_{AB} - \frac{n}{2(n-2)}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\hat{\tau}_{AB} + \frac{1}{n-2}\langle\hat{\tau},\underline{\hat{\chi}}\rangle g\!\!\!/_{AB} \\
&+ (\zeta - 2\eta)^C\Big(\underline{\nu}_{C(AB)} + \frac{1}{n-2}\underline{\beta}_C g\!\!\!/_{AB}\Big) + \underline{\hat{\chi}}^{CE}R_{CABE},
\end{aligned}
\tag{2.7.12}
$$

$$
\begin{aligned}
(\nabla\!\!\!/_3\underline{\nu})_{ABC} = &2\Big(\nabla\!\!\!/_{[B}\underline{\alpha}_{A]C} + \underline{\chi}_{C[B}\underline{\beta}_{A]} - \underline{\chi}^D_{[B}\underline{\nu}_{|CD|A]} - \underline{\chi}^D_{[B}\underline{\nu}_{A]DC} + \underline{\eta}_{[B}\underline{\alpha}_{A]C}\Big) \\
&+ \underline{\hat{\omega}}\underline{\nu}_{ABC},
\end{aligned}
\tag{2.7.13}
$$

$$
\begin{aligned}
(\nabla\!\!\!/_4\nu)_{ABC} = &2\Big(\nabla\!\!\!/_{[B}\alpha_{A]C} - \chi_{C[B}\beta_{A]} - \chi^D_{[B}\nu_{|CD|A]} - \chi^D_{[B}\nu_{A]DC} + \eta_{[B}\alpha_{A]C}\Big) \\
&+ \hat{\omega}\nu_{ABC},
\end{aligned}
\tag{2.7.14}
$$

$$
\begin{aligned}
(\nabla\!\!\!/_4\underline{\nu})_{ABC} = &2\underline{\eta}_C\varsigma_{BA} + 2(\tau+\varsigma)_{C[A}\underline{\eta}_{B]} + 2\underline{\eta}^D R_{ABCD} - \hat{\omega}\underline{\nu}_{ABC} - 2\underline{\chi}_{C[A}\beta_{B]} \\
&- 2(\nabla\!\!\!/_{[A}(\tau-\varsigma))_{B]C} + 2\chi^D_{[A}\underline{\nu}_{B]DC} + 2\nu_{CD[B}\underline{\chi}^D_{A]},
\end{aligned}
\tag{2.7.15}
$$

$$
\begin{aligned}
(\nabla\!\!\!/_3\nu)_{ABC} = &2\eta_C\varsigma_{AB} + 2(\tau-\varsigma)_{C[A}\eta_{B]} + 2\eta^D R_{ABCD} - \underline{\hat{\omega}}\nu_{ABC} + 2\chi_{C[A}\underline{\beta}_{B]} \\
&- 2(\nabla\!\!\!/_{[A}(\varsigma+\tau))_{B]C} + 2\underline{\chi}^D_{[A}\nu_{B]DC} + 2\underline{\nu}_{CD[B}\chi^D_{A]},
\end{aligned}
\tag{2.7.16}
$$

$$
\begin{aligned}
\nabla\!\!\!/_4 R_{ABCD} = &-2(\zeta+\underline{\eta})_{[C}\nu_{|AB|D]} - 2\underline{\eta}_{[A}\nu_{|CD|B]} - 2\nabla\!\!\!/_{[C}\nu_{|AB|D]} + 2\chi^E_{[C}R_{|AB|D]E} \\
&- \underline{\chi}_{A[C}\alpha_{D]B} + \underline{\chi}_{B[C}\alpha_{D]A} - \chi_{A[C}(\tau_{D]B} - \varsigma_{D]B}) + \chi_{B[C}(\tau_{D]A} - \varsigma_{D]A}),
\end{aligned}
\tag{2.7.17}
$$

$$
\begin{aligned}
\nabla\!\!\!/_3 R_{ABCD} = &2(\zeta-\eta)_{[C}\underline{\nu}_{|AB|D]} - 2\eta_{[A}\underline{\nu}_{|CD|B]} - 2\nabla\!\!\!/_{[C}\underline{\nu}_{|AB|D]} + 2\underline{\chi}^E_{[C}R_{|AB|D]E} \\
&- \chi_{A[C}\underline{\alpha}_{D]B} + \chi_{B[C}\underline{\alpha}_{D]A} - \underline{\chi}_{A[C}(\tau_{D]B} + \varsigma_{D]B}) + \underline{\chi}_{B[C}(\tau_{D]A} + \varsigma_{D]A}).
\end{aligned}
\tag{2.7.18}
$$

Proposition 2.7.2 (Additional Constraint Equations)**.** *Suppose* $\mathrm{Ric}(g) = 0$*. The* $\mathcal{S}_{u,v}$*-tensors* ν *and* $\underline{\nu}$ *satisfy the constraints*

$$(\mathrm{d\!\!/iv}\underline{\nu})_{[AB]} = \underline{\chi}^C_{[A}(\varsigma + \tau)_{B]C} - \frac{1}{2}(\mathrm{d\!\!/}\underline{\beta})_{AB} - \underline{\beta}_{[A}\zeta_{B]} + \frac{1}{2}(\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi})\varsigma_{AB} + \zeta^C\underline{\nu}_{C[AB]}, \tag{2.7.19}$$

$$(\mathrm{d\!\!/iv}\nu)_{[AB]} = \chi^C_{[A}(\tau - \varsigma)_{B]C} + \frac{1}{2}(\mathrm{d\!\!/}\beta)_{AB} - \beta_{[A}\zeta_{B]} - \frac{1}{2}(\mathrm{Tr}_{\mathrm{g\!\!/}}\chi)\varsigma_{AB} - \zeta^C\nu_{C[AB]}. \tag{2.7.20}$$

The $\mathcal{S}_{u,v}$*-tensor* $\hat{\tau}$ *satisfies the constraint*

$$(\mathrm{d\!\!/iv}\hat{\tau})_A = -\frac{n-4}{n-2}\nabla\!\!\!\!/_A\rho + \frac{1}{2}\nu_{ABC}\underline{\chi}^{BC} - \frac{1}{2}\underline{\nu}_{ABC}\chi^{BC} - \frac{1}{2}\chi^B_A\underline{\beta}_B + \frac{1}{2}\underline{\chi}^B_A\beta_B \tag{2.7.21}$$
$$- \frac{1}{2}\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi}\beta_A + \frac{1}{2}\mathrm{Tr}_{\mathrm{g\!\!/}}\chi\underline{\beta}_A.$$

2.7.1 The Bianchi Identities in Double Null Gauge in $4D$

In this subsection a collection of results are stated about the Weyl tensor for $n = 4$. The proof of these statements follow from reducing the above equation for general n with proposition 2.3.3 and lemma 2.4.3.

Proposition 2.7.3 (Bianchi Identities)**.** *Suppose* $\mathrm{Ric}(g) = 0$*. Then, for* $n = 4$*, the double null decomposed Weyl tensor satisfies the following relations:*

$$\nabla\!\!\!\!/_3\alpha = -2\hat{\underline{\omega}}\alpha + (4\eta + \zeta)\hat{\otimes}\beta - 3\rho\hat{\chi} - 3\sigma(\star\hat{\chi}) - 2(\mathcal{D}\!\!\!\!/^\star_2\beta) - \frac{1}{2}(\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi})\alpha, \tag{2.7.22}$$

$$\nabla\!\!\!\!/_4\rho = \mathrm{d\!\!/iv}\beta - \frac{3}{2}\rho\mathrm{Tr}_{\mathrm{g\!\!/}}\chi - \frac{1}{2}\langle\alpha, \hat{\underline{\chi}}\rangle + \langle 2\underline{\eta} + \zeta, \beta\rangle, \tag{2.7.23}$$

$$\nabla\!\!\!\!/_4\sigma = -\mathrm{cu\!\!/rl}\beta - \frac{3}{2}\sigma\mathrm{Tr}_{\mathrm{g\!\!/}}(\chi) + \frac{1}{2}\hat{\underline{\chi}} \wedge \alpha + \beta \wedge (\zeta + 2\underline{\eta}), \tag{2.7.24}$$

$$\nabla\!\!\!\!/_4\beta = \mathrm{d\!\!/iv}\alpha - 2(\mathrm{Tr}_{\mathrm{g\!\!/}}\chi)\beta + i_{\eta^\sharp + 2\zeta^\sharp}\alpha + \hat{\omega}\beta, \tag{2.7.25}$$

$$\nabla\!\!\!\!/_3\beta = 3\rho\eta + 3\sigma(\star\eta) - \hat{\underline{\omega}}\beta + \mathcal{D}\!\!\!\!/^\star_1(-\rho, \sigma) + 2i_{\underline{\beta}^\sharp}\hat{\chi} - (\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi})\beta, \tag{2.7.26}$$

$$\nabla\!\!\!\!/_4\underline{\alpha} = -2\hat{\omega}\underline{\alpha} - (4\underline{\eta} - \zeta)\hat{\otimes}\underline{\beta} - 3\rho\hat{\underline{\chi}} + 3\sigma(\star\hat{\underline{\chi}}) + 2(\mathcal{D}\!\!\!\!/^\star_2\underline{\beta}) - \frac{1}{2}(\mathrm{Tr}_{\mathrm{g\!\!/}}\chi)\underline{\alpha}, \tag{2.7.27}$$

$$\nabla\!\!\!\!/_3\rho = -\mathrm{d\!\!/iv}\underline{\beta} - \frac{3}{2}\rho\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi} - \frac{1}{2}\langle\underline{\alpha}, \hat{\chi}\rangle - \langle 2\eta - \zeta, \underline{\beta}\rangle, \tag{2.7.28}$$

$$\nabla\!\!\!\!/_3\sigma = -\mathrm{cu\!\!/rl}\underline{\beta} - \frac{3}{2}\sigma\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi} - \frac{1}{2}\hat{\chi} \wedge \underline{\alpha} + \underline{\beta} \wedge (2\eta - \zeta), \tag{2.7.29}$$

$$\nabla\!\!\!\!/_3\underline{\beta} = -\mathrm{d\!\!/iv}\underline{\alpha} - 2(\mathrm{Tr}_{\mathrm{g\!\!/}}\underline{\chi})\underline{\beta} - i_{\underline{\eta}^\sharp - 2\zeta^\sharp}\underline{\alpha} + \hat{\underline{\omega}}\underline{\beta}, \tag{2.7.30}$$

$$\nabla\!\!\!\!/_4\underline{\beta} = -3\rho\underline{\eta} + 3\sigma(\star\underline{\eta}) - \hat{\omega}\underline{\beta} + \mathcal{D}\!\!\!\!/^\star_1(\rho, \sigma) + 2i_{\beta^\sharp}\hat{\underline{\chi}} - (\mathrm{Tr}_{\mathrm{g\!\!/}}\chi)\underline{\beta}. \tag{2.7.31}$$

2.8 The Double Null Foliation of the Schw_n Exterior

The n-dimensional Schwarzschild–Tangherlini black hole solution (Schw_n, g_s) has been discussed in detail in section 1.1.1 in chapter 1. Here it is simply recalled that the metric on the exterior $\mathcal{E}_A$

in traditional Schwarzschild coordinates $(t, r, \varphi^1, ..., \varphi^{n-2})$ takes the form

$$g_s = -D_n(r)dt \otimes dt + \frac{1}{D_n(r)}dr \otimes dr + r^2\mathring{\gamma}_{n-2}, \qquad D_n(r) = 1 - \frac{2M}{r^{n-3}}, \tag{2.8.1}$$

where $t \in [0, \infty)$, $r \in \left((2M)^{\frac{1}{n-3}}, \infty\right)$ and $\mathring{\gamma}_{n-2}$ is the metric on the unit $(n-2)$-sphere.

Remarkably, the maximally extended Schwarzschild–Tangherlini spacetime can be globally covered by a double null coordinate system known as the Kruskal coordinate system (see the works [103–105]). The distinguished exterior region $\mathcal{E}_A$ can be covered by a convenient set of double null coordinates known as the double null Eddington–Finkelstein coordinates $(u, v, \varphi^1, ..., \varphi^{n-2})$ which can be introduced as follows. Let $u = \frac{1}{2}(t - r_\star)$ and $v = \frac{1}{2}(t + r_\star)$ where $r_\star : ((2M)^{\frac{1}{n-3}}, \infty) \to \mathbb{R}$ is defined in the usual way by the ODE

$$\frac{dr_\star}{dr} = \frac{1}{D_n(r)}, \tag{2.8.2}$$

with the initial condition $r_\star((4M)^{\frac{1}{n-3}}) = (4M)^{\frac{1}{n-3}}$ and $D_n(r)$ is given in equation (2.8.1). The metric in $(u, v, \varphi^1, ..., \varphi^{n-2})$ coordinates becomes

$$g = -2D_n(r(u,v))(du \otimes dv + dv \otimes du) + r(u,v)^2\mathring{\gamma}_{n-2}, \tag{2.8.3}$$

where $\mathring{\gamma}_{n-2}$ is the metric on the unit $(n-2)$-sphere and $(u, v) \in \mathbb{R}^2$. So one has

$$\Omega(u,v)^2 = D_n(r(u,v)), \qquad b^A \equiv 0, \qquad \not{g} = r(u,v)^2\mathring{\gamma}_{n-2}. \tag{2.8.4}$$

and coordinate r is now viewed as a function $r : \mathbb{R}_u \times \mathbb{R}_v \to ((2M)^{\frac{1}{n-3}}, \infty)$ defined implicitly as a function of (u, v). The exterior is covered by $(u, v, \varphi^1, ..., \varphi^{n-2})$. In particular, $u, v \in \mathbb{R}$ so that $\mathcal{E}_A = \mathbb{R}_u \times \mathbb{R}_v \times \mathbb{S}^{n-2}_{r(u,v)}$. Strictly speaking the coordinates $(u, v, \varphi^1, ..., \varphi^{n-2})$ do not cover the future event horizon $\mathcal{H}^+_A$ or future null infinity $\mathcal{I}^+_A$. However, formally one can parameterise the future event horizon as $(\infty, v, \varphi^1, ..., \varphi^{n-2})$ and future null infinity as $(u, \infty, \varphi^1, ..., \varphi^{n-2})$.

The normalised null frame is simply

$$e_3 = \frac{1}{\Omega}\partial_u, \qquad e_4 = \frac{1}{\Omega}\partial_v, \qquad e_A = \partial_{\varphi^A}. \tag{2.8.5}$$

Remark 2.8.1. *One should note that the frame (e_3, e_4, e_A) does not extend regularly to the future event horizon $\mathcal{H}^+$. However, one can check that, by transforming to Kruskal coordinates, the re-scaled frame $(\frac{1}{\Omega}e_3, \Omega e_4, e_A)$ does extend regularly to a non-vanishing null frame on $\mathcal{H}^+$.*

One can calculate all Ricci coefficients and curvature components explicitly in terms of r. The only non-vanishing Ricci coefficients are

$$(\Omega \mathrm{Tr}_{\not g}\chi) = -(\Omega \mathrm{Tr}_{\not g}\underline{\chi}) = \frac{(n-2)\Omega^2}{r}, \quad \omega = -\underline{\omega} = \frac{(n-3)M}{r^{n-2}} \tag{2.8.6}$$

and the only non-vanishing double null curvature component is

$$\rho = -\frac{(n-2)(n-3)M}{r^{n-1}}. \tag{2.8.7}$$

One has that the Riemann and Ricci curvature of $\mathbb{S}^{n-2}_{u,v}$ are

$$\not R_{ABCD} = \frac{1}{r^2}(\not g_{AC}\not g_{BD} - \not g_{AD}\not g_{BC}), \qquad \not{\mathrm{Ric}} = \frac{(n-3)}{r^2}\not g \tag{2.8.8}$$

and scalar curvature of $\not g$ is

$$\not{\mathrm{Scal}} = \frac{(n-2)(n-3)}{r^2}. \tag{2.8.9}$$

Finally, one can compute that

$$R_{ABCD} = -\frac{2\rho}{(n-2)(n-3)}(\not g_{AC}\not g_{BD} - \not g_{AD}\not g_{BC}). \tag{2.8.10}$$

Recall that in $n = 4$ one has $\not{\mathrm{Scal}} = 2\mathrm{K}$ where K is the Gauss curvature of $\mathbb{S}^2_{u,v}$. So, $\mathrm{K} = \frac{1}{r^2}$.

Further proposition 2.2.3 simplifies to

Proposition 2.8.2. *The connection coefficients of definition 2.2.3 for* Schw_n *in double null Eddington–Finkelstein coordinates satisfy the following relations:*

$$\nabla_A e_B = \not\Gamma^C_{AB} e_C + \frac{1}{2(n-2)}\mathrm{Tr}_{\not g}\chi(e_3 - e_4)\not g_{AB} \tag{2.8.11}$$

and

$$\begin{aligned} \nabla_3 e_A &= -\frac{\mathrm{Tr}_{\not g}\chi}{n-2}e_A, & \nabla_4 e_A &= \frac{\mathrm{Tr}_{\not g}\chi}{n-2}e_A, \\ \nabla_A e_3 &= -\frac{\mathrm{Tr}_{\not g}\chi}{n-2}e_A, & \nabla_A e_4 &= \frac{\mathrm{Tr}_{\not g}\chi}{n-2}e_A, \\ \nabla_3 e_4 &= \hat\omega e_4, & \nabla_4 e_3 &= -\hat\omega e_3, \\ \nabla_3 e_3 &= -\hat\omega e_3, & \nabla_4 e_4 &= \hat\omega e_4. \end{aligned} \tag{2.8.12}$$

Also, proposition 2.5.2 becomes

Proposition 2.8.3 (Projected Derivatives of p-covariant $\mathbb{S}^{n-2}_{u,v}$-Tensor Fields)**.** *Let T be a p-covariant $\mathbb{S}^{n-2}_{u,v}$-tensor field. Then in double null Eddington–Finkelstein coordinates on the*

n-dimensional Schwarzschild–Tangherlini exterior one has

$$(\nabla\!\!\!/_3 T)_{A_1\dots A_p} = \frac{1}{\Omega}\Big(\partial_u(T_{A_1\dots A_p}) + \frac{p}{n-2}(\Omega \mathrm{Tr}_{g\!\!/}\chi)T_{A_1\dots A_p}\Big), \tag{2.8.13}$$

$$(\nabla\!\!\!/_4 T)_{A_1\dots A_p} = \frac{1}{\Omega}\Big(\partial_v(T_{A_1\dots A_p}) - \frac{p}{n-2}(\Omega \mathrm{Tr}_{g\!\!/}\chi)T_{A_1\dots A_p}\Big), \tag{2.8.14}$$

$$(\nabla\!\!\!/_A T)_{B_1\dots B_p} = \partial_A(T_{B_1\dots B_p}) - \sum_{i=1}^{p} \Gamma\!\!\!/^{C}_{AB_i} T_{B_1\dots \hat{B}_i C\dots B_p}. \tag{2.8.15}$$

where $\hat{B}_i$ denotes removing the i^{th} index and replacing it by C.

Finally this subsection concludes with the following commutation lemma

Lemma 2.8.4 (Commutation Lemma). *Let T be a p-covariant $\mathbb{S}^{n-2}_{u,v}$-tensor field. Then in double null Eddington–Finkelstein coordinates on the n-dimensional Schwarzschild–Tangherlini exterior one has*

$$(\nabla\!\!\!/_3\nabla\!\!\!/_B T - \nabla\!\!\!/_B\nabla\!\!\!/_3 T)_{A_1\dots A_p} = \frac{\mathrm{Tr}_{g\!\!/}\chi}{n-2}\nabla\!\!\!/_B T_{A_1\dots A_p}, \tag{2.8.16}$$

$$(\nabla\!\!\!/_4\nabla\!\!\!/_B T - \nabla\!\!\!/_B\nabla\!\!\!/_4 T)_{A_1\dots A_p} = -\frac{\mathrm{Tr}_{g\!\!/}\chi}{n-2}\nabla\!\!\!/_B T_{A_1\dots A_p}, \tag{2.8.17}$$

$$(\nabla\!\!\!/_3\nabla\!\!\!/_4 T - \nabla\!\!\!/_4\nabla\!\!\!/_3 T)_{A_1\dots A_p} = \hat{\omega}(\nabla\!\!\!/_3 T + \nabla\!\!\!/_4 T)_{A_1\dots A_p}, \tag{2.8.18}$$

or equivalently,

$$[\nabla\!\!\!/_3, r\nabla\!\!\!/_B]T = 0, \qquad [\nabla\!\!\!/_4, r\nabla\!\!\!/_B]T = 0, \qquad [\Omega\nabla\!\!\!/_3, \Omega\nabla\!\!\!/_4]T = 0. \tag{2.8.19}$$

2.9 The Kerr Exterior in Double Null Canonical Coordinates

The Schwarzschild–Tangherlini solution sits as a 1-parameter subfamily of a $(\lfloor\frac{n-1}{2}\rfloor+1)$-parameter family of rotating black holes known as the Myers–Perry solutions [41] which was first written down in 1986. These generalised, to arbitrary dimension, the 4-dimensional Kerr black hole spacetimes (Kerr_4, g_K) written down in 1963 by Roy Kerr [106]. So for $n=4$, the Schwarzschild black hole family sits as a 1-parameter subfamily of a 2-parameter family of rotating black holes. The Kerr family verifies the vacuum Einstein equation (I.2) and also gives rise to the black hole phenomena (as does the Myers–Perry family). It arises dynamically as the maximal Cauchy development of suitable initial data $(\Sigma_0, h_K, \mathrm{K}_K)$. This spacetime is stationary, asymptotically flat and axisymmetric. The following Penrose diagram is for the submanifold of the spacetime corresponding to the axis of symmetry ($\theta = 0$ or $\theta = \pi$) arising from initial data, restricted to the future of Σ_0.

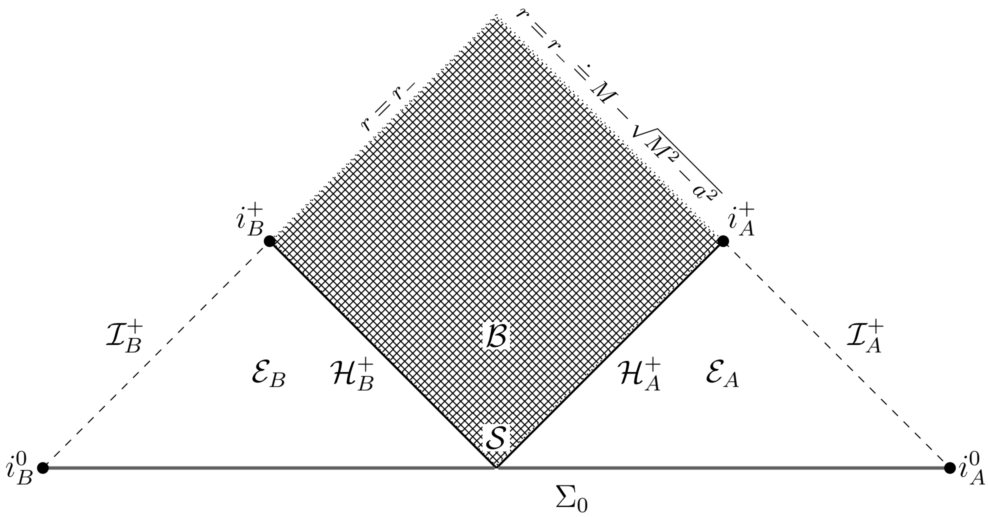


Fig. 2.3 The Penrose diagram for the submanifold corresponding to the axis of symmetry of the Kerr spacetime. The notation here is completely analogous to the Schwarzschild case described in section 1.1.1.

The metric on its exterior $\mathcal{E}_A$ in traditional Boyer-Lindquist coordinates [107] (t, r, θ, φ) is

$$g_{\mathrm{K}} = -\Big(1 - \frac{2Mr}{\Sigma(r,\theta)}\Big)dt \otimes dt + \frac{\Sigma(r,\theta)}{\Delta(r)}dr \otimes dr + R(r,\theta)^2 \sin^2\theta d\varphi \otimes d\varphi \qquad (2.9.1)$$
$$+ \Sigma(r,\theta)d\theta \otimes d\theta - \frac{2Mar\sin^2\theta}{\Sigma(r,\theta)}(dt \otimes d\varphi + d\varphi \otimes dt),$$

where

$$\Sigma(r,\theta) \doteq r^2 + a^2\cos^2\theta, \qquad R(r,\theta)^2 \doteq r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma(r,\theta)} \qquad (2.9.2)$$

and

$$\Delta(r) \doteq r^2 - 2Mr + a^2. \qquad (2.9.3)$$

The coordinate ranges are $t \in [0, \infty)$, $r \in (r_+ \doteq M + \sqrt{M^2 - a^2}, \infty)$, $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$.

Even more remarkably (than the Schwarzschild case), one can construct (see [108] or appendix A of [109] or section 9.1 of [110]) a set of double null coordinates $(u, v, \vartheta^1, \vartheta^2)$ for the whole exterior of the Kerr spacetime Kerr_4. The construction goes through defining a tortoise coordinate $r_\star(r,\theta)$ and the coordinate $\vartheta^1(r,\theta)$ such that hypersurfaces given by the level sets of

$$u \doteq \frac{1}{2}(t - r_\star) \qquad v = \frac{1}{2}(t + r_\star) \qquad (2.9.4)$$

are characteristic. The Boyer-Lindquist coordinates (r, θ) are now interpreted as functions of (u, v, ϑ^1). In particular,

$$r = r(r_\star = u - v, \vartheta^1), \qquad \theta = \theta(r_\star = u - v, \vartheta^1). \tag{2.9.5}$$

The coordinate ϑ^2 is constructed from φ by defining

$$\vartheta^2 \doteq \varphi - h(v - u, \vartheta^1), \tag{2.9.6}$$

such that h satisfies

$$\partial_u h = \frac{2Mar}{\Sigma(r,\theta)R(r,\theta)^2} = -\partial_v h. \tag{2.9.7}$$

In other words, h is defined by solving the ODE (for every fixed $\vartheta^1 \in (0, \pi)$)

$$\frac{dh}{dr_\star}(r_\star, \vartheta^1) = -\frac{2Mar}{\Sigma(r,\theta)R(r,\theta)^2}, \tag{2.9.8}$$

with initial data $h(r_\star = 0, \vartheta^1) = 0$.

The Kerr metric in double null form is the following

$$g = -2\Omega_{\mathrm{K}}^2(du \otimes dv + dv \otimes du) + (\not{g}_{\mathrm{K}})_{AB}(d\vartheta^A - b_{\mathrm{K}}^A dv) \otimes (d\vartheta^B - b_{\mathrm{K}}^B dv) \tag{2.9.9}$$

with

$$\Omega_{\mathrm{K}}^2 = \frac{\Delta(r)}{R(r,\theta)^2}, \quad b_{\mathrm{K}}^1 = 0, \quad b_{\mathrm{K}}^2 = \frac{4Mar}{\Sigma(r,\theta)R(r,\theta)^2}, \tag{2.9.10}$$

$$\not{g}_{\mathrm{K}} \doteq \Big[\frac{L(r,\theta)^2}{R(r,\theta)^2} + \Big(\frac{\partial h}{\partial \vartheta^1}\Big)^2 R(r,\theta)^2 \sin^2\theta\Big] d\vartheta^1 \otimes d\vartheta^1 + R(r,\theta)^2 \sin^2\theta d\vartheta^2 \otimes d\vartheta^2 \tag{2.9.11}$$
$$+ \Big(\frac{\partial h}{\partial \vartheta^1}\Big) R^2 \sin^2\theta (d\vartheta^1 \otimes d\vartheta^2 + d\vartheta^2 \otimes d\vartheta^2),$$

and

$$L \doteq -aG(r,\theta)\sqrt{\big(\sin^2\vartheta^1 - \sin^2\theta\big)\big((r^2+a^2)^2 - a^2\Delta(r)\sin^2\vartheta^1\big)}, \tag{2.9.12}$$

$$G(r,\theta) \doteq \frac{\partial}{\partial \vartheta^1}\Big|_{(r,\theta)\text{ fixed}} F(\vartheta^1; r, \theta), \tag{2.9.13}$$

$$F(\vartheta^1; r, \theta) \doteq \int_r^\infty \frac{dr'}{\sqrt{((r')^2+a^2)^2 - a^2\Delta(r')\sin^2\vartheta^1}} + \int_{\vartheta^1}^\theta \frac{d\theta'}{a\sqrt{\sin^2\vartheta^1 - \sin^2\theta'}}. \tag{2.9.14}$$

At this point one can indeed compute (implicitly) the Ricci coefficients and curvature components associated to the double null foliation. One finds that none vanish. However, in these coordinates the stationarity and axisymmetry of the spacetime is encoded now in the fact that the metric coefficients only depend on $r_\star = u - v$ and ϑ^1, not $t = u + v$ or ϑ^2. Therefore, one can derive a set of relations between Ricci coefficients and between curvature components due to these symmetries.

2.9.1 The Algebraically Special Frame

Instead of choosing a double null foliation of the Kerr exterior one could choose the 'algebraically special frame' of Kerr. This frame arises naturally in the Petrov classification of 4-dimensional black holes which states that there are 6 types of algebraic symmetry associated to Weyl tensor of a 4-dimensional spacetime [111]. The Kerr spacetime is Petrov type D. This frame can be defined (up to rescaling) with respect to Boyer–Lindquist coordinates as

$$e_3 \doteq \frac{r^2+a^2}{\sqrt{\Delta(r)\Sigma(r,\theta)}}\partial_t - \sqrt{\frac{\Delta(r)}{\Sigma(r,\theta)}}\partial_r + \frac{a}{\sqrt{\Delta(r)\Sigma(r,\theta)}}\partial_\varphi, \tag{2.9.15}$$

$$e_4 \doteq \frac{r^2+a^2}{\sqrt{\Delta(r)\Sigma(r,\theta)}}\partial_t + \sqrt{\frac{\Delta(r)}{\Sigma(r,\theta)}}\partial_r + \frac{a}{\sqrt{\Delta(r)\Sigma(r,\theta)}}\partial_\varphi, \tag{2.9.16}$$

for the null vectors and

$$e_1 \doteq \frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta, \qquad e_2 \doteq \frac{a\sin\theta}{\sqrt{\Sigma(r,\theta)}}\partial_t + \frac{1}{\sin\theta\sqrt{\Sigma(r,\theta)}}\partial_\varphi, \tag{2.9.17}$$

for the horizontal subspace. This frame arises by defining coordinates u and v, called the Eddington–Kerr or simply Kerr coordinates which can be introduced by defining the coordinate one-forms

$$\begin{aligned} d\mathrm{u} &\doteq dt - \frac{r^2+a^2}{\Delta(r)}dr, \qquad & d\Xi_{\mathrm{u}} &\doteq d\varphi - \frac{a}{\Delta(r)}dr, \\ d\mathrm{v} &\doteq dt + \frac{r^2+a^2}{\Delta(r)}dr, \qquad & d\Xi_{\mathrm{v}} &\doteq d\varphi + \frac{a}{\Delta(r)}dr, \end{aligned} \tag{2.9.18}$$

It is important to note that a hypersurfaces of constant u or v are spacelike if $a \neq 0$ and <u>not</u> null.[a] Moreover, one can check that

$$[e_1, e_2] = \frac{a\cos\theta\sqrt{\Delta(r)}}{\Sigma(r,\theta)^{\frac{3}{2}}}(e_3+e_4) - \frac{(r^2+a^2)\cot\theta}{\Sigma(r,\theta)^{\frac{3}{2}}}e_2. \tag{2.9.19}$$

Therefore, the frame is non-integrable and not tangent to the (Boyer–Lindquist) spheres. One can generalise the definition 2.2.1 to deal with this case.[b]

Definition 2.9.1 (Connection coefficients II)**.** *Suppose that* (e_3, e_4) *are a null pair normalised such that* $g(e_3,e_4) = -2$ *and that* $(e_A)_{A=1,2}$ *is a basis for the horizontal subspace,* $\mathrm{span}(e_3,e_4)^{\perp_g}$.

[a] One can make a consistent choice of Ξ to provide a coordinate system (u, v, θ, Ξ); see section 4.6.2 in [112].

[b] The formalism described here is a relabelling of the usual Geroch–Held–Penrose formalism [113]; see section 2 of [114].

Then one can define the following horizontal tensor fields:

$$\begin{aligned} \chi_{AB} &\doteq g(\nabla_A e_4, e_B), & \underline{\chi}_{AB} &\doteq g(\nabla_A e_3, e_B), \\ \eta_A &\doteq \frac{1}{2} g(\nabla_3 e_4, e_A), & \underline{\eta}_A &\doteq \frac{1}{2} g(\nabla_4 e_3, e_A), \\ \xi_A &\doteq \frac{1}{2} g(\nabla_4 e_4, e_A), & \underline{\xi}_A &\doteq \frac{1}{2} g(\nabla_3 e_3, e_A), \\ \hat{\omega} &\doteq -\frac{1}{2} g(\nabla_4 e_4, e_3), & \underline{\hat{\omega}} &\doteq -\frac{1}{2} g(\nabla_3 e_3, e_4) \end{aligned} \tag{2.9.20}$$

and

$$\zeta_A \doteq \frac{1}{2} g(\nabla_A e_4, e_3). \tag{2.9.21}$$

If one uses a non-integrable frame in this definition (such as the algebraically special frame of Kerr) instead of the normalised null frame associated to the double null gauge, one has two 'extra' horizontal one-forms $(\xi, \underline{\xi})$, which previously vanished, and that χ and $\underline{\chi}$ are no longer symmetric (in contrast to proposition 2.2.2).

If one now calculates the Ricci coefficients and curvature components of Kerr with respect to the algebraically special frame using definitions 2.9.1 and 2.3.1 respectively, one finds that the only non-vanishing Ricci-coefficients are

$$\chi_{[AB]} = \underline{\chi}_{[AB]}, \quad \mathrm{Tr}\chi = -\mathrm{Tr}\underline{\chi}, \quad \eta, \quad \underline{\eta}, \quad \hat{\omega} = -\underline{\hat{\omega}} \tag{2.9.22}$$

and the only non-vanishing null curvature components are the scalars ρ and σ.

Remark 2.9.1. *The reader should note that in the Schwarzschild case, the algebraically special frame coincides with the normalised null frame of the double null Eddington–Finkelstein foliation (if one takes an orthonormal basis on the sphere) as one can see by taking $a = 0$ in the above equations.*

2.10 Linearisation in Double Null Gauge

To linearise the null structure equations and Bianchi equations of sections 2.6 and 2.7, consider a one-parameter family of metrics $g(\epsilon)$ in double null canonical coordinates of the form

$$\begin{aligned} g(\epsilon) = &-2\Omega^2(\epsilon)(du \otimes dv + dv \otimes du) + b^A \not{g}_{AB}(\epsilon)(d\theta^B \otimes dv + dv \otimes d\theta^B) \\ &+ b^A(\epsilon) b^B(\epsilon) \not{g}_{AB}(\epsilon) dv \otimes dv + \not{g}_{AB}(\epsilon) d\theta^A \otimes d\theta^B \end{aligned} \tag{2.10.1}$$

where $\Omega(0)$, $\slashed{g}(0)$ and $b(0)$ are the background values for the spacetime one wants to linearise around. Therefore, one takes

$$\Omega(\epsilon) = \Omega(0) + \epsilon\overset{(1)}{\Omega}, \qquad \slashed{g}(\epsilon) = \slashed{g}(0) + \epsilon\slashed{h}, \qquad b(\epsilon) = b(0) + \epsilon\overset{(1)}{b} \tag{2.10.2}$$

where the quantities with a superscript '(1)' denote linear perturbations. In general, the metric to linear order becomes

$$g(\epsilon) = g(0) + \epsilon h + \mathcal{O}(\epsilon^2),$$

where $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ given by

$$\begin{aligned} h \doteq &-4\Omega\overset{(1)}{\Omega}(du \otimes dv + dv \otimes du) + (2b^A\overset{(1)}{b}{}^B\slashed{g}_{AB} + b^A b^B \slashed{h}_{AB})dv \otimes dv \\ &- (b^A\slashed{h}_{AB} + \overset{(1)}{b}{}^A\slashed{g}_{AB})(d\theta^B \otimes dv + dv \otimes d\theta^B) + \slashed{h}_{AB}d\theta^A \otimes d\theta^B. \end{aligned} \tag{2.10.3}$$

where one uses the abuse of notation that $\Omega = \Omega(0)$, $b = b(0)$ and $\slashed{g} = \slashed{g}(0)$. This leads to the following definition of what it means for a linearised metric h to be in double null gauge:

Definition 2.10.1 (Double Null Gauge for h)**.** *A solution $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ to the linearised vacuum Einstein equation (I.5) is said to be in double null gauge if there exists a function $\overset{(1)}{\Omega} : M \to \mathbb{R}$, a vector $\overset{(1)}{b}{}^A \in \mathfrak{X}(\mathcal{S}_{u,v})$ and a symmetric 2-tensor $\slashed{h} \in \mathrm{sym}(T^\star\mathcal{S}_{u,v} \otimes T^\star\mathcal{S}_{u,v})$ such that*

$$h = -\frac{4\overset{(1)}{\Omega}}{\Omega}(f^3 \otimes f^4 + f^4 \otimes f^3) - \frac{\overset{(1)}{b}_A}{\Omega}(f^4 \otimes f^A + f^A \otimes f^4) + \slashed{h}_{AB}f^A \otimes f^B, \tag{2.10.4}$$

in the dual basis to (e_3, e_4, e_A) for the background metric $g(0)$.

Remark 2.10.1. *The paper [28] uses the notation $\overset{(1)}{\slashed{g}}_{AB}$ for $\slashed{h}_{AB}$.*

One can work out the linear perturbation to the inverse metric in the following manner. Recall that

$$(g^{-1}(\epsilon))^{ab}g_{bc}(\epsilon) = \delta^a_c, \tag{2.10.5}$$

where, in particular the right-hand side is independent of ϵ. Now, to $\mathcal{O}(\epsilon)$

$$(g^{ab} + \epsilon(h^{-1})^{ab})(g_{bc} + \epsilon h_{bc}) = \delta^a_c, \implies (h^{-1})^{ab} = -g^{ac}g^{bd}h_{cd}. \tag{2.10.6}$$

Similarly, the inverse of $\slashed{h}$ is

$$(\slashed{h}^{-1})^{AB} = -\slashed{g}^{AC}\slashed{g}^{BD}\slashed{h}_{CD}. \tag{2.10.7}$$

Recall that the normalised null frame is given by

$$e_3 = \frac{1}{\Omega(\epsilon)}\partial_u, \qquad e_4 = \frac{1}{\Omega(\epsilon)}(\partial_v + b^A(\epsilon)e_A), \qquad e_A(\epsilon) = \partial_{\theta^A}. \tag{2.10.8}$$

So, to linear order, the normalised frame is

$$e_3(\epsilon) = e_3 - \epsilon\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)e_3 + \mathcal{O}(\epsilon^2), \quad e_4(\epsilon) = e_4 + \epsilon\Big(\frac{\overset{(1)}{b}{}^A}{\Omega}e_A - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)e_4\Big) + \mathcal{O}(\epsilon^2). \tag{2.10.9}$$

Remark 2.10.2. *One should note that the covariant derivative associated to $g(0)$ is* <u>*not*</u> *the same as the covariant derivative associated to $g(\epsilon)$, i.e., $\overset{(\epsilon)}{\nabla}_a \neq \nabla_a$.*

2.10.1 The Linearised Null Structure Equations Around Schw_n

In this section the formal linearisation of the null structure equations in propositions 2.6.1-2.10.17 around the Schwarzschild–Tangherlini is performed. Recall that the non-vanishing background metric quantities are

$$\Omega^2 = D_n(r), \qquad \not{g} = r^2\mathring{\not{\gamma}}_{n-2}, \tag{2.10.10}$$

the non-vanishing Ricci coefficients are

$$(\Omega\mathrm{Tr}_{\not{g}}\chi) = \frac{(n-2)D_n(r)}{r} = -(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi}), \qquad \omega = \frac{(n-3)M}{r^{n-2}} = -\underline{\omega}, \tag{2.10.11}$$

and the non-vanishing curvature components are

$$\begin{aligned} \rho &= -\frac{(n-2)(n-3)M}{r^{n-1}}, \qquad & \not{\mathrm{R}}\mathrm{ic}(\not{g}) &= \frac{(n-3)}{r^2}\not{g}, \\ \mathrm{S}\not{\mathrm{c}}\mathrm{al}(\not{g}) &= \frac{(n-2)(n-3)}{r^2}, & \not{R}_{ABCD} &= \frac{1}{r^2}(\not{g}_{AC}\not{g}_{BD} - \not{g}_{AD}\not{g}_{BC}) \end{aligned} \tag{2.10.12}$$

and

$$R_{ABCD} = -\frac{2\rho}{(n-2)(n-3)}(\not{g}_{AC}\not{g}_{BD} - \not{g}_{AD}\not{g}_{BC}). \tag{2.10.13}$$

In the following, it will be assumed that h in double null gauge satisfies the linearised vacuum Einstein equation (I.5) on Schw_n. Additionally, recall that $\mathrm{Ric} = 0$ for the Schwarzschild–

Tangherlini metric. Therefore, to formally linearise, one takes

$$
\begin{aligned}
\Omega(\epsilon) &= \Omega + \epsilon\overset{(1)}{\Omega}, & \not{g}(\epsilon) &= \not{g} + \epsilon\not{h}, \\
b(\epsilon) &= 0 + \epsilon\overset{(1)}{b}, & \underline{\omega}(\epsilon) &= \underline{\omega} + \epsilon\overset{(1)}{\underline{\omega}}, \\
\omega(\epsilon) &= \omega + \epsilon\overset{(1)}{\omega}, & (\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})(\epsilon) &= (\Omega\mathrm{Tr}_{\not{g}}\underline{\chi}) + \epsilon\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}, \\
(\Omega\mathrm{Tr}_{\not{g}}\chi)(\epsilon) &= (\Omega\mathrm{Tr}_{\not{g}}\chi) + \epsilon\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}, & \underline{\eta}(\epsilon) &= 0 + \epsilon\overset{(1)}{\underline{\eta}}, \\
\eta(\epsilon) &= 0 + \epsilon\overset{(1)}{\eta}, & \hat{\underline{\chi}}(\epsilon) &= 0 + \epsilon\overset{(1)}{\hat{\underline{\chi}}}, \\
\hat{\chi}(\epsilon) &= 0 + \epsilon\overset{(1)}{\hat{\chi}}, & \rho(\epsilon) &= \rho + \epsilon\overset{(1)}{\rho} \\
\hat{\tau}(\epsilon) &= 0 + \overset{(1)}{\hat{\tau}}, & \varsigma(\epsilon) &= 0 + \epsilon\overset{(1)}{\varsigma} \\
\beta(\epsilon) &= 0 + \epsilon\overset{(1)}{\beta}, & \underline{\beta}(\epsilon) &= 0 + \epsilon\overset{(1)}{\underline{\beta}}, \\
\alpha(\epsilon) &= 0 + \epsilon\overset{(1)}{\alpha}, & \underline{\alpha}(\epsilon) &= 0 + \epsilon\overset{(1)}{\underline{\alpha}}, \\
\nu(\epsilon) &= 0 + \overset{(1)}{\nu}, & \underline{\nu}(\epsilon) &= 0 + \overset{(1)}{\underline{\nu}}, \\
\mathrm{S}\not{c}\mathrm{al}(\epsilon) &= \mathrm{S}\not{c}\mathrm{al}(\not{g}) + \epsilon\overset{(1)}{\mathrm{S}\not{c}\mathrm{al}}, & \widehat{\mathrm{R}\not{i}\mathrm{c}}(\epsilon) &= 0 + \epsilon\overset{(1)}{\widehat{\mathrm{R}\not{i}\mathrm{c}}}, \\
R_{ABCD}(\epsilon) &= R_{ABCD} + \epsilon\overset{(1)}{R}_{ABCD}, & \not{R}_{ABCD}(\epsilon) &= \not{R}_{ABCD} + \epsilon\overset{(1)}{\not{R}}_{ABCD},
\end{aligned}
\tag{2.10.14}
$$

where linearised quantities are denoted with '(1)'.

Remark 2.10.3. *If h in double null gauge solves the linearised vacuum Einstein equation (I.5), then $\overset{(1)}{\mathrm{Ric}} = 0$ and $\overset{(1)}{\mathrm{Scal}} = 0$ and the linearised null structure equations (and linearised Bianchi equations) in the following propositions 2.10.7-2.10.20 are satisfied. Conversely, if one has a solution to the linearised null structure equations and linearised Bianchi equations in propositions 2.10.7-2.10.20 then, in particular, one has a $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ in double null gauge. The linearised null structure equations of propositions 2.10.7-2.10.17 imply that this h solves the linearised vacuum Einstein equation (I.5). In the rest of this thesis, the terminology that $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ solves the linearised vacuum Einstein equation (I.5) in double null gauge will be used synonymously with the terminology that $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ satisfies definition 2.10.1 and solves the linearised null structure equations of propositions 2.10.7-2.10.17 and the linearised Bianchi equations of proposition 2.10.20.*

Remark 2.10.4. *Rather than linearising the propositions 2.6.1-2.6.8 directly, there is an alternative (but equivalent) route to obtain propositions 2.10.7-2.10.17 below from the linearised vacuum Einstein equation (I.5). First, one should note the perturbations to the normalised double null basis in equation (2.10.9). Then from directly linearising the connection coefficients via definition 2.2.1 with the basis independent formula for the linearised Christoffel symbols given in proposition 1.2.2 one will arrive at the proposition 2.10.7. Additionally, from substituting h in double null gauge into the linearised vacuum Einstein equation (I.5) and from directly linearising the curvature* *components via definition 2.3.1 with the basis independent formula for the linearised Riemann tensor given in proposition 1.2.3, one will arrive at the rest of the linearised null structure equations in propositions 2.10.8-2.10.17.*

Direct linearisation of the null structure equations in propositions 2.6.1-2.10.17 around the Schwarzschild–Tangherlini is fairly trivial since so many of the background quantities vanish. However, to be clear the general procedure for derivatives is the following. If one has a p-covariant tensor T which does not necessarily vanish on the background then to linearise $\nabla\!\!\!/_3 T$ one expands $(\Omega\nabla\!\!\!/_3 T)$ using proposition 2.5.2 as

$$(\Omega\nabla\!\!\!/_3 T)_{A_1\dots A_p} = \partial_u(T_{A_1\dots A_p}) - \frac{p}{n-2}(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})T_{A_1\dots A_p} - \sum_{i=1}^{p}\Omega\hat{\underline{\chi}}_{A_i}^{B}T_{A_1\dots\hat{A}_iB\dots A_p} \tag{2.10.15}$$

where all quantities here are associated to $g(\epsilon)$ of equation (2.10.1). Since $\hat{\underline{\chi}}_{A_iB}$ vanishes on the background the last term picks up no linear contribution from Ω, $g\!\!\!/^{-1}$ or T. Hence (using the abuse of notation where T now denotes the background quantity),

$$\begin{aligned}(\overset{(1)}{\nabla\!\!\!/_3 T})_{A_1\dots A_p} = (\nabla\!\!\!/_3\overset{(1)}{T})_{A_1\dots A_p} - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\nabla\!\!\!/_3 T - \sum_{i=1}^{p}\overset{(1)}{\hat{\underline{\chi}}}{}_{A_i}^{B}T_{A_1\dots\hat{A}_iB\dots A_p}\\ - \frac{p}{(n-2)\Omega}(\Omega\overset{(1)}{\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})T_{A_1\dots A_p}.\end{aligned} \tag{2.10.16}$$

For $\nabla\!\!\!/_4$ one similarly has

$$\begin{aligned}(\overset{(1)}{\nabla\!\!\!/_4 T})_{A_1\dots A_p} = (\nabla\!\!\!/_4\overset{(1)}{T})_{A_1\dots A_p} - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\nabla\!\!\!/_4 T - \sum_{i=1}^{p}\Big(\overset{(1)}{\hat{\chi}}{}_{A_i}^{B} - \frac{1}{\Omega}\nabla\!\!\!/_{A_i}\overset{(1)}{b}{}^{B}\Big)T_{A_1\dots\hat{A}_iB\dots A_p}\\ - \frac{p}{(n-2)\Omega}(\Omega\overset{(1)}{\mathrm{Tr}_{g\!\!\!/}\chi})T_{A_1\dots A_p}\end{aligned} \tag{2.10.17}$$

where $\nabla\!\!\!/_{A_i}$ is the background covariant derivative associated to $g\!\!\!/$. Finally, if T vanishes for the background (which will usually be the case of interest for linearising around Schw_n) one simply has

$$(\overset{(1)}{\nabla\!\!\!/_A T})_{A_1\dots A_p} = (\nabla\!\!\!/_A\overset{(1)}{T})_{A_1\dots A_p}. \tag{2.10.18}$$

Note that for Schw_n one does not have to be very careful with linearising contractions, i.e., one usually does not need to account for linear perturbations to the inverse metric, since most $\mathbb{S}^{n-2}_{u,v}$-tensors for the background vanish.

If one assumes h satisfies the linearised vacuum Einstein equation (I.5) this implies $\overset{(1)}{\mathrm{Ric}} = 0$ (see equation (1.2.6)) and, therefore, combining this fact with proposition 2.3.2 gives the following linearised identities:

Proposition 2.10.5. *Suppose h is a solution to the linearised vacuum Einstein equation (I.5) in double null gauge on Schw_n. Then one has the following linearised identities:*

$$
\begin{aligned}
&\mathrm{Tr}_{\slashed{g}}\overset{(1)}{\alpha} = 0, && \mathrm{Tr}_{\slashed{g}}\overset{(1)}{\underline{\alpha}} = 0,\\
&\overset{(1)}{\beta}_A = \overset{(1)}{\nu}_{AB}{}^{B}, && \overset{(1)}{\underline{\beta}}_A = -\overset{(1)}{\underline{\nu}}_{AB}{}^{B},\\
&\overset{(1)}{\nu}_{[ABC]} = 0, && \overset{(1)}{\nu}_{(AB)C} = 0,\\
&\overset{(1)}{\nu}_{A[BC]} = \frac{1}{2}\nu_{CBA}, && \overset{(1)}{\nu}_{ABC} = \frac{4}{3}\overset{(1)}{\nu}_{A(BC)} + \frac{2}{3}\overset{(1)}{\nu}_{C(BA)}
\end{aligned}
\tag{2.10.19}
$$

and identically for $\overset{(1)}{\underline{\nu}}$.

The following proposition details the reduction to $4D$ as in proposition 2.3.3:

Proposition 2.10.6. *Suppose h is a solution to the linearised vacuum Einstein equation (I.5) in double null gauge on Schw_4. Then the following relations are satisfied by the linear perturbations of $\underline{\nu}$, ν, $\underline{\beta}$, β and σ:*

$$
\overset{(1)}{\underline{\nu}}_{BCA} = \slashed{\epsilon}_{AB}\overset{(1)}{\underline{\beta}}_C - \slashed{\epsilon}_{AC}\overset{(1)}{\underline{\beta}}_B, \qquad \overset{(1)}{\nu}_{BCA} = \slashed{\epsilon}_{AC}\overset{(1)}{\beta}_B - \slashed{\epsilon}_{AB}\overset{(1)}{\beta}_C,
\tag{2.10.20}
$$

along with $\overset{(1)}{\hat{\tau}} = 0$ and $\overset{(1)}{\varsigma} = \overset{(1)}{\sigma}\slashed{\epsilon}$.

The linearisation of propositions 2.6.1-2.6.8 around (Schw_n, g_s) are now stated below. It should be stressed that the linearised vacuum Einstein equation (I.5) is assumed.

Proposition 2.10.7 (Linearised First Variation Formulas)**.** *The linearised metric coefficients $\overset{(1)}{\Omega}$, $\overset{(1)}{b}$ and $\slashed{h}$ satisfy:*

$$
\begin{aligned}
&\slashed{\nabla}_3(\mathrm{Tr}_{\slashed{g}}\slashed{h}) = \frac{2}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}), && \slashed{\nabla}_4(\mathrm{Tr}_{\slashed{g}}\slashed{h}) = \frac{2}{\Omega}\Big((\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi}) - \slashed{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big)\\
&\slashed{\nabla}_3\hat{\slashed{h}}_{AB} = 2\overset{(1)}{\hat{\underline{\chi}}}_{AB}, && \slashed{\nabla}_4\hat{\slashed{h}}_{AB} = 2\overset{(1)}{\hat{\chi}}_{AB} + \frac{2}{\Omega}(\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{b})_{AB},\\
&e_3\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \frac{1}{\Omega}\overset{(1)}{\underline{\omega}}, && e_4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \frac{1}{\Omega}\overset{(1)}{\omega},\\
&\partial_u\overset{(1)}{b}{}^A = 2\Omega^2(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})^A, && \slashed{\nabla}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \frac{1}{2}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}).
\end{aligned}
\tag{2.10.21}
$$

Proposition 2.10.8 (Linearised Transversal Propagation Equations for Expansions)**.** *The linearised expansions* $(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)$ *and* $(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})$ *satisfy*

$$\slashed{\nabla}_4(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) = 2\Omega\Big[\slashed{\mathrm{div}}\,\overset{(1)}{\underline{\eta}} + \Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)\Big] - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\Big), \tag{2.10.22}$$

$$\slashed{\nabla}_3(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) = 2\Omega\Big[\slashed{\mathrm{div}}\,\overset{(1)}{\eta} + \Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)\Big] - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\Big). \tag{2.10.23}$$

Proposition 2.10.9 (Linearised Raychauduri Equations)**.** *The linearised expansions* $(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)$ *and* $(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})$ *satisfy*

$$\slashed{\nabla}_4(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) = -\frac{2\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + 2\overset{(1)}{\omega}\mathrm{Tr}_{\slashed{g}}\chi + \frac{2}{\Omega}\omega(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi), \tag{2.10.24}$$

$$\slashed{\nabla}_3(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) = \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) - 2\overset{(1)}{\underline{\omega}}\mathrm{Tr}_{\slashed{g}}\chi - \frac{2}{\Omega}\omega(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}). \tag{2.10.25}$$

Proposition 2.10.10 (Linearised Equations for the Shears)**.** *The linearised shears* $\overset{(1)}{\hat{\chi}}$ *and* $\overset{(1)}{\underline{\hat{\chi}}}$ *satisfy*

$$\slashed{\nabla}_4\overset{(1)}{\hat{\chi}} = \Big(\hat{\omega} - \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{n-2}\Big)\overset{(1)}{\hat{\chi}} - \overset{(1)}{\alpha}, \tag{2.10.26}$$

$$\slashed{\nabla}_3\overset{(1)}{\underline{\hat{\chi}}} = -\Big(\hat{\omega} - \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{n-2}\Big)\overset{(1)}{\underline{\hat{\chi}}} - \overset{(1)}{\underline{\alpha}}, \tag{2.10.27}$$

$$\slashed{\nabla}_4\overset{(1)}{\underline{\hat{\chi}}} = -\overset{(1)}{\hat{\tau}} - 2\slashed{\mathcal{D}}^{\star}_2\overset{(1)}{\underline{\eta}} + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{n-2}(\overset{(1)}{\hat{\chi}} - \overset{(1)}{\underline{\hat{\chi}}}) - \hat{\omega}\overset{(1)}{\underline{\hat{\chi}}}, \tag{2.10.28}$$

$$\slashed{\nabla}_3\overset{(1)}{\hat{\chi}} = -\overset{(1)}{\hat{\tau}} - 2\slashed{\mathcal{D}}^{\star}_2\overset{(1)}{\eta} + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{n-2}(\overset{(1)}{\hat{\chi}} - \overset{(1)}{\underline{\hat{\chi}}}) + \hat{\omega}\overset{(1)}{\hat{\chi}}, \tag{2.10.29}$$

where $\overset{(1)}{\hat{\tau}} = 0$ *if* $n = 4$.

Proposition 2.10.11 (Linearised Torsion Propagation Equations)**.** *The linearised torsions* $\overset{(1)}{\eta}$ *and* $\overset{(1)}{\underline{\eta}}$ *satisfy*

$$\begin{aligned} \slashed{\nabla}_4\overset{(1)}{\eta} &= -\overset{(1)}{\beta} - \frac{1}{n-2}\mathrm{Tr}_{\slashed{g}}\chi(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}), & \slashed{\nabla}_3\overset{(1)}{\underline{\eta}} &= \overset{(1)}{\underline{\beta}} - \frac{1}{n-2}\mathrm{Tr}_{\slashed{g}}\chi(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}), \\ \slashed{\nabla}_3\overset{(1)}{\eta} &= \frac{2}{\Omega}\slashed{\nabla}\overset{(1)}{\underline{\omega}} + \frac{2}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\eta} - \overset{(1)}{\underline{\beta}}, & \slashed{\nabla}_4\overset{(1)}{\underline{\eta}} &= \frac{2}{\Omega}\slashed{\nabla}\overset{(1)}{\omega} - \frac{2}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\underline{\eta}} + \overset{(1)}{\beta}. \end{aligned} \tag{2.10.30}$$

Remark 2.10.12. *The reader may notice that one has two extra linearised equations here for the torsions* $\overset{(1)}{\eta}$ *and* $\overset{(1)}{\underline{\eta}}$*. This has resulted from considering*

$$\slashed{\nabla}_3(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}) = 2\slashed{\nabla}_3\slashed{\nabla}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \tag{2.10.31}$$

and applying the commutation lemma 2.8.4 and propositions 2.10.7 and the first two equations of 2.10.11.

Proposition 2.10.13. *The functions* $\overset{(1)}{\underline{\omega}}$ *and* $\overset{(1)}{\omega}$ *satisfy*

$$\nabla\!\!\!/_4\overset{(1)}{\underline{\omega}} = -\Omega\Big(2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho + \overset{(1)}{\rho}\Big), \qquad \nabla\!\!\!/_3\overset{(1)}{\omega} = -\Omega\Big(2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho + \overset{(1)}{\rho}\Big). \tag{2.10.32}$$

Proposition 2.10.14 (Linearised Torsion Constraints)**.** *For $n > 4$, the linearised torsions* $\overset{(1)}{\eta}$ *and* $\overset{(1)}{\underline{\eta}}$ *satisfy*

$$d\!\!\!/\overset{(1)}{\underline{\eta}} = -\overset{(1)}{\varsigma}, \qquad d\!\!\!/\overset{(1)}{\eta} = \overset{(1)}{\varsigma}. \tag{2.10.33}$$

For $n = 4$, the linearised torsions $\overset{(1)}{\eta}$ *and* $\overset{(1)}{\underline{\eta}}$ *satisfy*

$$\mathrm{cu\!\!\!/rl}\overset{(1)}{\underline{\eta}} = -\overset{(1)}{\sigma}, \qquad \mathrm{cu\!\!\!/rl}\overset{(1)}{\eta} = \overset{(1)}{\sigma}. \tag{2.10.34}$$

Proposition 2.10.15 (Linearised Gauss Equations)**.** *The linearised scalar curvature satisfies*

$$\overset{(1)}{\mathrm{Sc\!\!\!/al}} = -2\overset{(1)}{\rho} - \frac{n-3}{(n-2)\Omega}\mathrm{Tr}_{g\!\!\!/}\chi\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})} - \overset{(1)}{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)}\Big] - \frac{2(n-3)}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \tag{2.10.35}$$

and the linearised Ricci curvature satisfies

$$\widehat{\overset{(1)}{\mathrm{Ri\!\!\!/c}}} = \overset{(1)}{\hat{\tau}} - \frac{n-4}{2(n-2)}\mathrm{Tr}_{g\!\!\!/}\chi\Big(\overset{(1)}{\hat{\underline{\chi}}} - \overset{(1)}{\hat{\chi}}\Big), \tag{2.10.36}$$

where $\overset{(1)}{\hat{\tau}} = 0$ *if* $n = 4$.

Corollary 2.10.16. *The linearised metric coefficent* $\hat{h\!\!\!/}$ *satisfies*

$$\Delta\!\!\!\!/\,\hat{h\!\!\!/} = \frac{2\mathrm{Sc\!\!\!/al}(g\!\!\!/)}{(n-2)(n-3)}\hat{h\!\!\!/} - 2\overset{(1)}{\hat{\tau}} + \frac{n-4}{n-2}\Big[\mathrm{Tr}_{g\!\!\!/}\chi\Big(\overset{(1)}{\hat{\underline{\chi}}} - \overset{(1)}{\hat{\chi}}\Big) + (\mathcal{D\!\!\!/}_2^\star\nabla\!\!\!/\mathrm{Tr}_{g\!\!\!/}h\!\!\!/)\Big] - 2(\mathcal{D\!\!\!/}_2^\star\mathrm{di\!\!\!/v}\hat{h\!\!\!/}) \tag{2.10.37}$$

and

$$\begin{aligned}\mathrm{di\!\!\!/v}\,\mathrm{di\!\!\!/v}\hat{h\!\!\!/} = {}& -2\overset{(1)}{\rho} + \frac{n-3}{(n-2)\Omega}\mathrm{Tr}_{g\!\!\!/}\chi\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)} - \overset{(1)}{(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}\Big] - \frac{2(n-3)}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \\ & + \frac{n-3}{n-2}\Delta\!\!\!\!/\,\mathrm{Tr}_{g\!\!\!/}h\!\!\!/ + \frac{1}{n-2}\mathrm{Sc\!\!\!/al}(g\!\!\!/)\mathrm{Tr}_{g\!\!\!/}h\!\!\!/,\end{aligned} \tag{2.10.38}$$

where $\overset{(1)}{\hat{\tau}} = 0$ *if* $n = 4$.

Proof. Recall the formula for the linearised Ricci curvature in proposition 1.2.4. Using the background values for $\not{R}$ and $\not{\mathrm{Ric}}$ for Schw_n gives

$$\overset{(1)}{\not{\mathrm{Ric}}}_{AB} = -\frac{1}{2}\Big((\not{\Delta}\not{h})_{AB} - 2\not{\nabla}_{(A}(\not{\mathrm{div}}\not{h})_{B)} + \not{\nabla}_A\not{\nabla}_B\mathrm{Tr}_{\not{g}}\not{h}\Big) + \frac{\not{\mathrm{Scal}}}{n-3}\hat{\not{h}}_{AB}. \tag{2.10.39}$$

Decomposing h into its trace and trace-free parts gives

$$\begin{aligned}-2\overset{(1)}{\not{\mathrm{Ric}}}_{AB} = (\not{\Delta}\hat{\not{h}})_{AB} + \frac{1}{n-2}(\not{\Delta}\mathrm{Tr}_{\not{g}}\not{h})\not{g}_{AB} - 2\not{\nabla}_{(A}(\not{\mathrm{div}}\hat{\not{h}})_{B)} + \frac{n-4}{n-2}\not{\nabla}_A\not{\nabla}_B\mathrm{Tr}_{\not{g}}\not{h} \\ - \frac{2\not{\mathrm{Scal}}}{n-3}\hat{\not{h}}_{AB}.\end{aligned} \tag{2.10.40}$$

Noting that

$$\overset{(1)}{\not{\mathrm{Ric}}} = \widehat{\overset{(1)}{\not{\mathrm{Ric}}}} + \frac{1}{n-2}\overset{(1)}{\not{\mathrm{Scal}}}\not{g} + \frac{1}{n-2}\not{\mathrm{Scal}}\not{h}, \tag{2.10.41}$$

gives that

$$\widehat{\overset{(1)}{\not{\mathrm{Ric}}}} = \widehat{\overset{(1)}{\not{\mathrm{Ric}}}} + \frac{\not{\mathrm{Scal}}}{n-2}\hat{\not{h}}, \qquad \mathrm{Tr}_{\not{g}}(\overset{(1)}{\not{\mathrm{Ric}}}) = \overset{(1)}{\not{\mathrm{Scal}}} + \frac{1}{n-2}\not{\mathrm{Scal}}\mathrm{Tr}_{\not{g}}\not{h}. \tag{2.10.42}$$

Using equation (2.10.41) and proposition 2.10.15 gives the results. □

Proposition 2.10.17 (Linearised Codazzi Constraints)**.** *The linearised shears* $\overset{(1)}{\hat{\chi}}$ *and* $\overset{(1)}{\hat{\underline{\chi}}}$ *satisfy*

$$\not{\mathrm{div}}\overset{(1)}{\hat{\chi}} = \frac{n-3}{(n-2)\Omega}\not{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) - \frac{n-3}{n-2}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{\underline{\eta}} - \overset{(1)}{\beta}, \tag{2.10.43}$$

$$\not{\mathrm{div}}\overset{(1)}{\hat{\underline{\chi}}} = \frac{n-3}{(n-2)\Omega}\not{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}) + \frac{n-3}{n-2}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{\eta} + \overset{(1)}{\underline{\beta}}. \tag{2.10.44}$$

Additionally, one has that that $\overset{(1)}{\hat{\chi}}$ *and* $\overset{(1)}{\hat{\underline{\chi}}}$ *satisfy*

$$\not{\nabla}_{[A}\overset{(1)}{\hat{\chi}}_{B]C} = \frac{1}{(n-2)\Omega}\not{g}_{C[A}\not{\nabla}_{B]}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) + \frac{1}{2}\overset{(1)}{\nu}_{ABC} - \frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\not{g}_{C[A}\overset{(1)}{\underline{\eta}}_{B]}, \tag{2.10.45}$$

$$\not{\nabla}_{[A}\overset{(1)}{\hat{\underline{\chi}}}_{B]C} = \frac{1}{(n-2)\Omega}\not{g}_{C[A}\not{\nabla}_{B]}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}) + \frac{1}{2}\overset{(1)}{\underline{\nu}}_{ABC} + \frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\not{g}_{C[A}\overset{(1)}{\eta}_{B]}. \tag{2.10.46}$$

Corollary 2.10.18. *The linearised shears* $\overset{(1)}{\hat{\chi}}$ *and* $\overset{(1)}{\hat{\underline{\chi}}}$ *satisfy*

$$\not{\Delta}\overset{(1)}{\hat{\chi}} = \widehat{(\not{\mathrm{div}}\overset{(1)}{\nu})} + \not{\mathcal{D}}_2^\star\overset{(1)}{\beta} + \mathrm{Tr}_{\not{g}}\chi\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}} - \frac{1}{\Omega}\not{\mathcal{D}}_2^\star\not{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) + \frac{\not{\mathrm{Scal}}}{n-3}\overset{(1)}{\hat{\chi}}, \tag{2.10.47}$$

$$\not{\Delta}\overset{(1)}{\hat{\underline{\chi}}} = \widehat{(\not{\mathrm{div}}\overset{(1)}{\underline{\nu}})} - \not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}} - \mathrm{Tr}_{\not{g}}\chi\not{\mathcal{D}}_2^\star\overset{(1)}{\eta} - \frac{1}{\Omega}\not{\mathcal{D}}_2^\star\not{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}) + \frac{\not{\mathrm{Scal}}}{n-3}\overset{(1)}{\hat{\underline{\chi}}}. \tag{2.10.48}$$

Proof. To prove this statement apply $\slashed{\nabla}^A$ to the latter two equations of proposition 2.10.17. At this point one can use the Ricci identity for $\slashed{R}$ and substitute in the first two equations of proposition 2.10.17. The proof then concludes by taking the symmetric traceless part of the resulting equation. □

Remark 2.10.19. *One can also linearise the Gauss constraint for $\slashed{R}_{ABCD}$ but this will not be required in this work.*

2.10.2 The Linearised Bianchi Identities

Proposition 2.10.20. *Suppose h in double null gauge satisfies the linearised vacuum Einstein equation (I.5). Then, the linearised null Weyl curvature components satisfy the following alterations of the usual $n = 4$ (null-decomposed) linearised Bianchi identities on Schw_n:*

$$\slashed{\nabla}_4 \overset{(1)}{\rho} = -\Big(\frac{n-1}{n-2}\Big)\Big(\overset{(1)}{\rho}\mathrm{Tr}_{\slashed{g}}\chi + \frac{1}{\Omega}\rho(\Omega \overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\Big) + \slashed{\mathrm{div}}\overset{(1)}{\beta}, \tag{2.10.49}$$

$$\slashed{\nabla}_3 \overset{(1)}{\rho} = \Big(\frac{n-1}{n-2}\Big)\Big(\overset{(1)}{\rho}\mathrm{Tr}_{\slashed{g}}\chi - \frac{1}{\Omega}\rho(\Omega \overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\Big) - \slashed{\mathrm{div}}\overset{(1)}{\underline{\beta}}, \tag{2.10.50}$$

$$\slashed{\nabla}_4 \overset{(1)}{\beta} = \hat{\omega}\overset{(1)}{\beta} + \slashed{\mathrm{div}}\overset{(1)}{\alpha} - \frac{n}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\beta}, \tag{2.10.51}$$

$$\slashed{\nabla}_3 \overset{(1)}{\underline{\beta}} = \underline{\hat{\omega}}\overset{(1)}{\underline{\beta}} - \slashed{\mathrm{div}}\overset{(1)}{\underline{\alpha}} - \frac{n}{n-2}\mathrm{Tr}_{\slashed{g}}\underline{\chi}\overset{(1)}{\underline{\beta}}, \tag{2.10.52}$$

$$\slashed{\nabla}_4 \overset{(1)}{\underline{\beta}} = \frac{n-4}{n-2}\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\beta} - \frac{2(n-1)}{n-2}\rho\overset{(1)}{\underline{\eta}} - \frac{2(n-3)}{n-2}\slashed{\nabla}\overset{(1)}{\rho} - \slashed{\mathrm{div}}(\overset{(1)}{\varsigma} + \overset{(1)}{\hat{\tau}}) - \Big(\frac{2\mathrm{Tr}_{\slashed{g}}\chi}{n-2} + \hat{\omega}\Big)\overset{(1)}{\underline{\beta}}, \tag{2.10.53}$$

$$\slashed{\nabla}_3 \overset{(1)}{\beta} = \frac{2(n-1)}{n-2}\rho\overset{(1)}{\eta} - \frac{n-4}{n-2}\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\underline{\beta}} + \frac{2(n-3)}{n-2}\slashed{\nabla}\overset{(1)}{\rho} - \slashed{\mathrm{div}}(\overset{(1)}{\varsigma} - \overset{(1)}{\hat{\tau}}) + \Big(\frac{2\mathrm{Tr}_{\slashed{g}}\chi}{n-2} + \hat{\omega}\Big)\overset{(1)}{\beta}, \tag{2.10.54}$$

$$\slashed{\nabla}_3 \overset{(1)}{\alpha} = \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\overset{(1)}{\alpha} - \slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\beta} - \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\hat{\tau}} - \frac{n-1}{n-3}\rho\overset{(1)}{\hat{\chi}} - \widehat{(\slashed{\mathrm{div}}\overset{(1)}{\nu})}, \tag{2.10.55}$$

$$\slashed{\nabla}_4 \overset{(1)}{\underline{\alpha}} = -\Big(2\hat{\omega} + \frac{1}{2}\mathrm{Tr}_{\slashed{g}}\chi\Big)\overset{(1)}{\underline{\alpha}} + \slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\underline{\beta}} - \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\underline{\chi})\overset{(1)}{\hat{\tau}} - \frac{n-1}{(n-3)}\rho\overset{(1)}{\underline{\hat{\chi}}} - \widehat{(\slashed{\mathrm{div}}\overset{(1)}{\underline{\nu}})}, \tag{2.10.56}$$

$$\slashed{\nabla}_4 \overset{(1)}{\varsigma} = -\slashed{d}\overset{(1)}{\beta} - \frac{3}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\varsigma}, \tag{2.10.57}$$

$$\slashed{\nabla}_3 \overset{(1)}{\varsigma} = -\slashed{d}\overset{(1)}{\underline{\beta}} + \frac{3}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\varsigma} \tag{2.10.58}$$

where, if $n=4$, then $\overset{(1)}{\hat{\tau}}=0$, $\overset{(1)}{\varsigma}=\overset{(1)}{\sigma}\not{\epsilon}$, $\not{d}\overset{(1)}{\beta}=\mathrm{c\not{u}rl}\overset{(1)}{\beta}\not{\epsilon}$, $\not{d}\overset{(1)}{\underline{\beta}}=\mathrm{c\not{u}rl}\overset{(1)}{\underline{\beta}}\not{\epsilon}$ and

$$\widehat{(\mathrm{d\not{i}v}\overset{(1)}{\underline{\nu}})}=-\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}},\qquad \widehat{(\mathrm{d\not{i}v}\overset{(1)}{\nu})}=\not{\mathcal{D}}_2^\star\overset{(1)}{\beta},\qquad \mathrm{d\not{i}v}\overset{(1)}{\varsigma}=-\not{\mathcal{D}}_1^\star(\pm\overset{(1)}{\rho},\overset{(1)}{\sigma})\mp\not{\nabla}\overset{(1)}{\rho}. \tag{2.10.59}$$

Further, if $n>4$, one has the following additional (null-decomposed) linearised Bianchi identities for the null decomposed linearised Weyl curvature components,

$$\not{\nabla}_4\overset{(1)}{\hat{\tau}}=\not{\mathcal{D}}_2^\star\overset{(1)}{\beta}-\frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\not{g}}\underline{\chi}\overset{(1)}{\alpha}+\frac{(n-4)(n-1)}{(n-2)(n-3)}\overset{(1)}{\hat{\chi}}\rho-\widehat{(\mathrm{d\not{i}v}\overset{(1)}{\nu})}-\frac{n(\mathrm{Tr}_{\not{g}}\chi)}{2(n-2)}\overset{(1)}{\hat{\tau}}, \tag{2.10.60}$$

$$\not{\nabla}_3\overset{(1)}{\hat{\tau}}=-\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}}-\frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{\underline{\alpha}}+\frac{(n-4)(n-1)}{(n-2)(n-3)}\overset{(1)}{\underline{\hat{\chi}}}\rho-\widehat{(\mathrm{d\not{i}v}\overset{(1)}{\underline{\nu}})}+\frac{n(\mathrm{Tr}_{\not{g}}\chi)}{2(n-2)}\overset{(1)}{\hat{\tau}}, \tag{2.10.61}$$

$$(\not{\nabla}_3\overset{(1)}{\underline{\nu}})_{ABC}=2\not{\nabla}_{[B}\overset{(1)}{\underline{\alpha}}_{A]C}+\frac{2}{n-2}\mathrm{Tr}_{\not{g}}\underline{\chi}\not{g}_{C[B}\overset{(1)}{\underline{\beta}}_{A]}-\frac{3}{n-2}\mathrm{Tr}_{\not{g}}\underline{\chi}\overset{(1)}{\underline{\nu}}_{ABC}+\underline{\hat{\omega}}\overset{(1)}{\underline{\nu}}_{ABC}, \tag{2.10.62}$$

$$(\not{\nabla}_4\overset{(1)}{\nu})_{ABC}=2\not{\nabla}_{[B}\overset{(1)}{\alpha}_{A]C}-\frac{2}{n-2}\mathrm{Tr}_{\not{g}}\chi\not{g}_{C[B}\overset{(1)}{\beta}_{A]}-\frac{3}{n-2}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{\nu}_{ABC}+\hat{\omega}\overset{(1)}{\nu}_{ABC}, \tag{2.10.63}$$

$$\begin{aligned}(\not{\nabla}_3\overset{(1)}{\nu})_{ABC}&=\Big(\hat{\omega}+\frac{2}{n-2}\mathrm{Tr}_{\not{g}}\chi\Big)\overset{(1)}{\nu}_{ABC}-\frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\overset{(1)}{\underline{\nu}}_{ABC}+\frac{4\rho(n-1)}{(n-2)(n-3)}\overset{(1)}{\eta}_{[A}\not{g}_{B]C}\\&\quad-\frac{2\mathrm{Tr}_{\not{g}}\chi}{n-2}\overset{(1)}{\underline{\beta}}_{[A}\not{g}_{B]C}+\frac{4}{n-2}\not{\nabla}_{[A}\overset{(1)}{\rho}\not{g}_{B]C}-2\not{\nabla}_{[A}(\overset{(1)}{\varsigma}+\overset{(1)}{\hat{\tau}})_{B]C},\end{aligned} \tag{2.10.64}$$

$$\begin{aligned}(\not{\nabla}_4\overset{(1)}{\underline{\nu}})_{ABC}&=-\Big(\hat{\omega}+\frac{2}{n-2}\mathrm{Tr}_{\not{g}}\chi\Big)\overset{(1)}{\underline{\nu}}_{ABC}+\frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\overset{(1)}{\nu}_{ABC}+\frac{4\rho(n-1)}{(n-2)(n-3)}\overset{(1)}{\underline{\eta}}_{[A}\not{g}_{B]C}\\&\quad-\frac{2\mathrm{Tr}_{\not{g}}\chi}{n-2}\overset{(1)}{\beta}_{[A}\not{g}_{B]C}+\frac{4}{n-2}\not{\nabla}_{[A}\overset{(1)}{\rho}\not{g}_{B]C}-2\not{\nabla}_{[A}(\overset{(1)}{\hat{\tau}}-\overset{(1)}{\varsigma})_{B]C},\end{aligned} \tag{2.10.65}$$

which are automatically satisfied when $n=4$ due to the above linearised Bianchi identities.

Remark 2.10.21. *One can additionally derive linearised equations for $\not{\nabla}_3\overset{(1)}{R}$ and $\not{\nabla}_4\overset{(1)}{R}$.*

One can also derive the following linearised constraint for curvature which is trivially satisfied when $n=4$:

Proposition 2.10.22. *Suppose h in double null gauge satisfies the linearised vacuum Einstein equation (I.5). Then the curvature component $\hat{\tau}$ is constrained to satisfy*

$$\mathrm{d\not{i}v}\overset{(1)}{\hat{\tau}}-\frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi(\overset{(1)}{\beta}+\overset{(1)}{\underline{\beta}})+\frac{n-4}{(n-2)}\not{\nabla}\overset{(1)}{\rho}=0, \tag{2.10.66}$$

where $\overset{(1)}{\hat{\tau}}=0$ if $n=4$.

2.10.3 Residual Gauge Freedom in Double Null Gauge

Restricting the coordinates to ensure the metric is of double null form (2.10.1) is not sufficient to uniquely determine coordinates on an abstract Lorentzian manifold. In linearised theory, this non-uniqueness manifests itself with the existence of residual pure gauge solutions that preserves double null form for a solution h to the linearised vacuum Einstein equation (I.5) in double null gauge. In the following the residual gauge freedom is identified.

Proposition 2.10.23 (Residual Pure Gauge Freedom)**.** *Suppose h is a solution in double null gauge to the linearised vacuum Einstein equation (I.5) on Schw_n. Further, let h_{pg} be a pure gauge solution to the linearised vacuum Einstein equation (I.5) such that $h + h_{\mathrm{pg}}$ is a solution in double null gauge. Then, in double null Eddington–Finkelstein coordinates $(u, v, \underline{\varphi})$ on the Schw_n exterior, the one-form ξ generating h_{pg} can be written as*

$$\xi^u = f^3(u, \underline{\varphi}), \tag{2.10.67}$$

$$\xi^v = f^4(v, \underline{\varphi}), \tag{2.10.68}$$

$$\xi^A = f^A(v, \underline{\varphi}) + 2r\slashed{g}^{AB}\partial_B\big(f^4(v, \underline{\varphi})\big), \tag{2.10.69}$$

for some arbitrary smooth functions (f^3, f^4, f^A) on the exterior of Schw_n such that $(\Omega^2 f^3, f^4, f^A)$ extend smoothly to $\mathcal{H}^+$. Moreover, the pure gauge solution has a double null decomposition as

$$\Big(\frac{\overset{(1)}{\Omega}_{\mathrm{pg}}}{\Omega}\Big) = \frac{1}{2\Omega^2}\Big(\partial_u(\Omega^2 f^3) + \partial_v(\Omega^2 f^4)\Big), \tag{2.10.70}$$

$$(\overset{(1)}{b}_{\mathrm{pg}})_A = 2\Omega^2\partial_A(f^3) - \slashed{g}_{AB}\partial_v(\xi^B), \tag{2.10.71}$$

$$(\slashed{h}_{\mathrm{pg}})_{AB} = \slashed{\nabla}_A\xi_B + \slashed{\nabla}_B\xi_A + (\Omega\mathrm{Tr}_{\slashed{g}}\chi)(f^4 - f^3)\slashed{g}_{AB}, \tag{2.10.72}$$

$$(\mathrm{Tr}_{\slashed{g}}\slashed{h})_{\mathrm{pg}} = 2\slashed{\mathrm{div}}\slashed{\xi} + 2(\Omega\mathrm{Tr}_{\slashed{g}}\chi)(f^4 - f^3), \tag{2.10.73}$$

$$(\hat{\slashed{h}}_{\mathrm{pg}})_{AB} = -2(\slashed{\mathcal{D}}_2^\star\slashed{\xi})_{AB}, \tag{2.10.74}$$

where the notation $\slashed{\xi} \doteq (\xi_1, ...\xi_{n-2})$ has been introduced. Henceforth, (f^3, f^4, f^A) will be referred to as residual pure gauge functions, and any pure gauge solution arising from (f^3, f^4, f^A) will be referred to as a residual pure gauge solution.

Proof. Recall that a linearised metric is in double null gauge if it is of the form:

$$h = -4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(f^3 \otimes f^4 + f^4 \otimes f^3) - \frac{\overset{(1)}{b}_A}{\Omega}(f^4 \otimes f^A + f^A \otimes f^4) + \slashed{h}_{AB}f^A \otimes f^B. \tag{2.10.75}$$

In preserving this form one must have

$$\nabla_3\xi_3 = 0, \qquad \nabla_4\xi_4 = 0. \tag{2.10.76}$$

These conditions give

$$\partial_u\Big(\frac{1}{\Omega}\xi_3\Big) = 0 \implies \xi_3 = -2\Omega f^4(v,\underline{\varphi}), \tag{2.10.77}$$

$$\partial_v\Big(\frac{1}{\Omega}\xi_4\Big) = 0 \implies \xi_4 = -2\Omega f^3(u,\underline{\varphi}). \tag{2.10.78}$$

Recalling that $\xi^v = \frac{1}{-2\Omega^2}\xi_u$ and that $\xi_3 = \frac{1}{\Omega}\xi_u$ and similarly for ξ^u one finds

$$\xi^u = f^3(u,\underline{\varphi}), \qquad \xi^v = f^4(v,\underline{\varphi}). \tag{2.10.79}$$

Further, using the proposition 2.8.2, one must also have

$$\nabla_3\xi_A + \nabla_A\xi_3 = 0 \implies e_3(\xi_A) + e_A(\xi_3) + \frac{2}{n-2}\mathrm{Tr}_{\not{g}}\chi\xi_A = 0. \tag{2.10.80}$$

One can write that $r^2 e_3\Big(\frac{1}{r^2}\Big) = \frac{2}{n-2}\mathrm{Tr}_{\not{g}}\chi$ gives

$$\partial_u\Big(\frac{\xi_A}{r^2}\Big) = -\frac{\Omega}{r^2}e_A(\xi_3) = \frac{2\Omega^2}{r^2}e_A(f^4(v,\underline{\varphi})). \tag{2.10.81}$$

One can check

$$\xi_A(u,v,\underline{\varphi}) = \not{g}_{AB}f^B(v,\underline{\varphi}) + 2re_A(f^4(v,\underline{\varphi})) \tag{2.10.82}$$

is the solution. Raising the index gives

$$\xi^A = f^A(v,\underline{\varphi}) + 2r\not{g}^{AB}\partial_B(f^4(v,\underline{\varphi})). \tag{2.10.83}$$

Now, using the relations in proposition 2.2.3, for a pure gauge solution that preserves double null form, one has that

$$\Big(\frac{\overset{(1)}{\Omega}_{\mathrm{pg}}}{\Omega}\Big) = -\frac{1}{4}\Big(\nabla_3\xi_4 + \nabla_4\xi_3\Big) = -\frac{1}{4}\Big(e_3(\xi_4) + e_4(\xi_3) + \hat{\omega}(\xi_3 - \xi_4)\Big), \tag{2.10.84}$$

$$\frac{(\overset{(1)}{b}_{\mathrm{pg}})_A}{\Omega} = -(\nabla_A\xi_4 + \nabla_4\xi_A) = -e_A(\xi_4) - e_4(\xi_A) + \frac{2}{n-2}\mathrm{Tr}_{\not{g}}\chi\xi_A, \tag{2.10.85}$$

$$(\not{h}_{\mathrm{pg}})_{AB} = \not{\nabla}_A\xi_B + \not{\nabla}_B\xi_A - \frac{1}{n-2}\mathrm{Tr}_{\not{g}}\chi(\xi_3 - \xi_4)\not{g}_{AB}, \tag{2.10.86}$$

which simplifies to the relations stated. □

Remark 2.10.24. *The requirement of $(\Omega^2 f^3, f^4, f^A)$ to extend smoothly to $\mathcal{H}^+$ comes from observing that $(\frac{1}{\Omega}e_3, \Omega e_4, e_A)$ extends regularly to the future event horizon. Hence,*

$$\xi(\frac{1}{\Omega}e_3) = \frac{1}{\Omega}\xi_3 \propto f^4 \tag{2.10.87}$$
$$\xi(\Omega e_4) = \Omega\xi_4 \propto \Omega^2 f^3 \tag{2.10.88}$$

shows that $\Omega^2 f^3$ and f^4 have to be a regularly extendible gauge functions.

The following two subsets of residual pure gauge solutions will be useful in chapter 3 (see section 3.4.4). The residual pure gauge solution arising from $(f^3, f^4, f^A) = (0, f(v, \underline{\varphi}), \underline{0})$ gives the following solution to the linearised vacuum Einstein equation (I.5) in double null gauge:

Lemma 2.10.25. *Let $f = f(v, \underline{\varphi})$ be smooth and $(f^3, f^4, f^A) = (0, f(v, \underline{\varphi}), \underline{0})$ then the residual pure gauge solution arising from these residual gauge functions is*

$$\begin{aligned}
\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)_{\text{pg}} &= \frac{1}{2\Omega^2}\partial_v(\Omega^2 f), & (\mathrm{Tr}_{\not g}\not h)_{\text{pg}} &= 4r\not\Delta f + 2(\Omega\mathrm{Tr}_{\not g}\chi)f,\\
& & \hat{\not h}_{\text{pg}} &= -4r(\not{\mathcal{D}}_2^\star\not\nabla f),\\
(\overset{(1)}{b}_{\text{pg}})_\flat &= -2r^2\not d\Big(\partial_v\Big(\frac{f}{r}\Big)\Big), & (\Omega\mathrm{Tr}_{\not g}\overset{(1)}{\underline{\chi}})_{\text{pg}} &= 2\Omega^2\Big(\frac{(\mathrm{Tr}_{\not g}\chi)^2}{n-2} + 2\rho\Big)f + 2\Omega^2\not\Delta f,\\
(\Omega\mathrm{Tr}_{\not g}\overset{(1)}{\chi})_{\text{pg}} &= \partial_v(\Omega\mathrm{Tr}_{\not g}\chi f), & \overset{(1)}{\underline{\eta}}_{\text{pg}} &= \frac{r}{\Omega^2}\not d\Big[\partial_v\Big(\frac{\Omega^2}{r}f\Big)\Big],\\
(\overset{(1)}{\eta}_{\text{pg}}) &= \frac{\Omega^2}{r}\not d f, & \overset{(1)}{\hat{\underline{\chi}}}_{\text{pg}} &= -2\Omega(\not{\mathcal{D}}_2^\star\not\nabla f),\\
\overset{(1)}{\rho}_{\text{pg}} &= -\frac{3}{2}\rho(\Omega\mathrm{Tr}_{\not g}\chi)f, & \overset{(1)}{\underline{\beta}}_{\text{pg}} &= -\frac{2(n-1)\Omega}{n-2}\rho\not d f,
\end{aligned} \tag{2.10.89}$$

and

$$\overset{(1)}{\mathrm{S\not cal}}_{\text{pg}} = \frac{(\Omega\mathrm{Tr}_{\not g}\chi)}{n-2}\Big[\Big(n\rho - \frac{2(n-3)}{(n-2)}(\mathrm{Tr}_{\not g}\chi)^2\Big)f - 2(n-3)\not\Delta f\Big], \tag{2.10.90}$$

with $\overset{(1)}{\hat{\chi}}_{\text{pg}} = 0$, $\overset{(1)}{\beta}_{\text{pg}} = 0$, $\overset{(1)}{\varsigma}_{\text{pg}} = 0$, $\overset{(1)}{\hat{\tau}}_{\text{pg}} = 0$, $\overset{(1)}{\alpha}_{\text{pg}} = 0$ and $\overset{(1)}{\underline{\alpha}}_{\text{pg}} = 0$.

Proof. The explicit computation of all linearised Ricci coefficients can be performed with proposition 2.10.23 and 2.10.7. After obtaining the Ricci coefficients, the curvature components can be computed from propositions 2.10.8-2.10.17. □

The residual pure gauge solution arising from $(f^3, f^4, f^A) = (f(u, \underline{\varphi}), 0, \underline{0})$ gives the following solution to the linearised vacuum Einstein equation (I.5) in double null gauge:

Lemma 2.10.26. *Let $f = f(u, \underline{\varphi})$ be smooth such that $\Omega^2 f$ extends regularly to the future event horizon $\mathcal{H}^+$ and $(f^3, f^4, f^A) = (f(u, \underline{\varphi}), 0, \underline{0})$ then the residual pure gauge solution arising* *from these residual gauge functions is*

$$\begin{aligned}
\Big(\frac{\overset{(1)}{\Omega}_{\rm pg}}{\Omega}\Big) &= \frac{1}{2\Omega^2}\partial_u(\Omega^2 f), & (\mathrm{Tr}_{\not g}\not h)_{\rm pg} &= -2(\Omega\mathrm{Tr}_{\not g}\chi)f,\\
(\overset{(1)}{b}_{\rm pg})_\flat &= 2\Omega^2 \not d f, & \overset{(1)}{(\Omega\mathrm{Tr}_{\not g}\chi)}_{\rm pg} &= 2\Omega^2\Big(\frac{(\mathrm{Tr}_{\not g}\chi)^2}{n-2}+2\rho\Big)f + 2\Omega^2 \not\Delta f,\\
\overset{(1)}{(\Omega\mathrm{Tr}_{\not g}\underline{\chi})}_{\rm pg} &= -\partial_u\Big((\Omega\mathrm{Tr}_{\not g}\chi)f\Big), & \overset{(1)}{\hat\chi}_{\rm pg} &= -2\Omega(\not{\mathcal{D}}_2^\star \not\nabla f),\\
\overset{(1)}{\underline{\eta}}_{\rm pg} &= -\frac{\Omega^2}{r}\not d f, & \overset{(1)}{\eta}_{\rm pg} &= \frac{r}{\Omega^2}\not d\Big[\partial_u\Big(\frac{\Omega^2}{r}f\Big)\Big],\\
\overset{(1)}{\rho}_{\rm pg} &= \frac{3}{2}\rho(\Omega\mathrm{Tr}_{\not g}\chi)f, & \overset{(1)}{\beta}_{\rm pg} &= \frac{2(n-1)\Omega}{n-2}\rho\not d f,
\end{aligned} \tag{2.10.91}$$

and

$$\overset{(1)}{\mathrm{S\not cal}}_{\rm pg} = \frac{(\Omega\mathrm{Tr}_{\not g}\chi)}{n-2}\Big[\Big(\frac{2(n-3)}{(n-2)}(\mathrm{Tr}_{\not g}\chi)^2 - n\rho\Big)f + 2(n-3)\not\Delta f\Big], \tag{2.10.92}$$

with $\hat{\not h}_{\rm pg}=0$, $\overset{(1)}{\hat{\underline{\chi}}}_{\rm pg}=0$, $\overset{(1)}{\underline{\beta}}_{\rm pg}=0$, $\overset{(1)}{\varsigma}_{\rm pg}=0$, $\overset{(1)}{\hat\tau}_{\rm pg}=0$, $\overset{(1)}{\alpha}_{\rm pg}=0$ *and* $\overset{(1)}{\underline{\alpha}}_{\rm pg}=0$.

Proof. The explicit computation of all linearised Ricci coefficients can be performed with proposition 2.10.23 and 2.10.7. After obtaining the Ricci coefficients, the curvature components can be computed from propositions 2.10.8-2.10.17. □

2.10.4 The Teukolsky and Regge–Wheeler Equations on Schw_n

This section studies the partial decoupling (or full decoupling for $n=4$) of linearised curvature components on the Schwarzschild–Tangherlini spacetime. In particular, the Teukolsky system of equations is derived for the Schw_n. This for all intents and purpose reproduces the results of [86, 87] (or [12, 13, 28] for $n=4$) in the double null gauge and associated notation. Note that the relation between the 'WAND frame' (l, n, m_i) used for the higher-dimensional Geroch–Held–Penrose formalism (introduced in [89]) of [86] and the double null frame is

$$l = -\frac{1}{\Omega}e_3, \qquad n = \frac{\Omega}{2}e_4, \tag{2.10.93}$$

and that m_i are the *unit* vectors on $\mathbb{S}^{n-2}_r$. As [87], the obstruction to decoupling for $n > 4$ is found.[c] This section concludes with a discussion of the physical space 'Chandrasekhar transformation' introduced for $n=4$ in [28] in arbitrary dimension.

[c] In the work [87], the authors prove that for a Teukolsky type equation to decouple the background spacetime has to be of Kundt type (see chapter 31 of [115] and [116]). Unfortunately, black hole spacetimes are not Kundt. However, if the black hole is extreme, then its 'near-horizon geometry' is Kundt. This was exploited in [117] to study instability.

The Teukolsky (System of) Equation(s) on Schw_n

For $n > 4$ decoupling of the null curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ fails. One can derive a coupled system of equations for $\overset{(1)}{\hat{\tau}}$, $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$. The main result of this section is the following proposition:

Proposition 2.10.27 (The Teukolsky (System of) Equation(s)). *The null curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy*

$$\begin{aligned}\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} &= \slashed{\Delta}\overset{(1)}{\alpha} + \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\slashed{\nabla}_4\overset{(1)}{\alpha} - \Big(\frac{6+n}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi - \hat{\omega}\Big)\slashed{\nabla}_3\overset{(1)}{\alpha} \\ &\quad + \Big[\frac{2(n-4)^2}{(n-3)(n-2)}\rho - 4\hat{\omega}^2 + \frac{2(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)} - \frac{\mathrm{S\slashed{c}al}(\slashed{g})}{(n-3)}\Big]\overset{(1)}{\alpha} - \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)}\overset{(1)}{\hat{\tau}},\end{aligned} \tag{2.10.94}$$

$$\begin{aligned}\slashed{\nabla}_3\slashed{\nabla}_4\overset{(1)}{\underline{\alpha}} &= \slashed{\Delta}\overset{(1)}{\underline{\alpha}} - \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\slashed{\nabla}_3\overset{(1)}{\underline{\alpha}} + \Big(\frac{6+n}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi - \hat{\omega}\Big)\slashed{\nabla}_4\overset{(1)}{\underline{\alpha}} \\ &\quad + \Big[\frac{2(n-4)^2}{(n-3)(n-2)}\rho - 4\hat{\omega}^2 + \frac{2(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)} - \frac{\mathrm{S\slashed{c}al}(\slashed{g})}{(n-3)}\Big]\overset{(1)}{\underline{\alpha}} - \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)}\overset{(1)}{\hat{\tau}}.\end{aligned} \tag{2.10.95}$$

Additionally the linearised curvature component $\overset{(1)}{\hat{\tau}}$ satisfies

$$\begin{aligned}\slashed{\nabla}_3\slashed{\nabla}_4\overset{(1)}{\hat{\tau}} &= \slashed{\Delta}\overset{(1)}{\hat{\tau}} + \Big(\frac{n+2}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi + \hat{\omega}\Big)\slashed{\nabla}_4\overset{(1)}{\hat{\tau}} - \frac{n+2}{2(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{\nabla}_3\overset{(1)}{\hat{\tau}} \\ &\quad + \Big(\frac{n(\mathrm{Tr}_{\slashed{g}}\chi)}{(n-2)} + \frac{2(n-4)(n-1)}{(n-3)}\hat{\omega}\Big)\frac{\mathrm{Tr}_{\slashed{g}}\chi}{n-2}\overset{(1)}{\hat{\tau}} - \frac{(n-4)(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)^2}(\overset{(1)}{\alpha} + \overset{(1)}{\underline{\alpha}})\end{aligned} \tag{2.10.96}$$

and an angular commuted version

$$\begin{aligned}\slashed{\nabla}_3\slashed{\nabla}_4\slashed{\Delta}\overset{(1)}{\hat{\tau}} &= \slashed{\Delta}\slashed{\Delta}\overset{(1)}{\hat{\tau}} + \Big(\frac{n+6}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi + \hat{\omega}\Big)\slashed{\nabla}_4\slashed{\Delta}\overset{(1)}{\hat{\tau}} - \frac{n+6}{2(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{\nabla}_3\slashed{\Delta}\overset{(1)}{\hat{\tau}} \\ &\quad + \Big(\frac{3(n+2)(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)^2} - \frac{2(n^2-3n-2)}{(n-3)(n-2)}\rho\Big)\slashed{\Delta}\overset{(1)}{\hat{\tau}} - \frac{(n-4)(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)^2}\slashed{\Delta}(\overset{(1)}{\alpha} + \overset{(1)}{\underline{\alpha}}).\end{aligned} \tag{2.10.97}$$

Proof. Taking $\slashed{\nabla}_4$ of $\slashed{\nabla}_3\overset{(1)}{\alpha}$ in proposition 2.10.20 gives

$$\begin{aligned}\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} &= e_4\Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\overset{(1)}{\alpha} + \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)(\slashed{\nabla}_4\overset{(1)}{\alpha}) - \frac{n-1}{n-3}e_4(\rho)\overset{(1)}{\hat{\chi}} \\ &\quad - \frac{1}{2}e_4(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\hat{\tau}} + \frac{1}{n-2}\mathrm{Tr}_{\slashed{g}}\chi\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\beta} + \frac{1}{n-2}\mathrm{Tr}_{\slashed{g}}\chi\widehat{(\slashed{\mathrm{d}}\mathrm{iv}\overset{(1)}{\nu})} \\ &\quad - (\slashed{\mathcal{D}}_2^{\star}\slashed{\nabla}_4\overset{(1)}{\beta}) - \frac{n-1}{n-3}\rho(\slashed{\nabla}_4\overset{(1)}{\hat{\chi}}) - \widehat{(\slashed{\mathrm{d}}\mathrm{iv}\slashed{\nabla}_4\overset{(1)}{\nu})} - \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{\nabla}_4\overset{(1)}{\hat{\tau}}\end{aligned} \tag{2.10.98}$$

where one uses the commutation lemma 2.8.4. One can substitute the other linearised Bianchi equations and linearised null structure equations in propositions 2.10.20 and 2.10.7-2.10.17

respectively to give

$$
\begin{aligned}
(\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha})_{AB} = {}& \Big[\frac{n-1}{n-3}\rho + e_4\Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big) - \frac{(n-4)}{4(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)^2\Big]\overset{(1)}{\alpha} \\
&+ \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)(\slashed{\nabla}_4\overset{(1)}{\alpha})_{AB} + \slashed{\Delta}\overset{(1)}{\alpha} - \frac{n-2}{r^2}\overset{(1)}{\alpha} \\
&+ \Big(\frac{6+n}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi - \hat{\omega}\Big)\Big[\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\beta} + \widehat{(\slashed{\mathrm{div}}\overset{(1)}{\nu})} + \frac{n-1}{n-3}\rho\overset{(1)}{\hat{\chi}}\Big] \\
&+ \Big[\frac{n}{4(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)^2 - \frac{1}{2}e_4(\mathrm{Tr}_{\slashed{g}}\chi)\Big]\overset{(1)}{\hat{\tau}},
\end{aligned}
\tag{2.10.99}
$$

where one uses that,

$$
\slashed{\nabla}^D\slashed{\nabla}_{(A}\overset{(1)}{\alpha}_{B)D} = -(\slashed{\mathcal{D}}_2^{\star}\slashed{\mathrm{div}}\overset{(1)}{\alpha})_{AB} + \frac{1}{n-2}\slashed{\mathrm{div}}\slashed{\mathrm{div}}\overset{(1)}{\alpha}\slashed{g}_{AB} + \frac{n-2}{r^2}\overset{(1)}{\alpha}_{AB},
\tag{2.10.100}
$$

by the Ricci identity for $\slashed{R}$. The linearised Bianchi equations in proposition 2.10.20 give

$$
\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\beta} + \frac{n-1}{n-3}\rho\overset{(1)}{\hat{\chi}} + \widehat{(\slashed{\mathrm{div}}\overset{(1)}{\nu})} = -\slashed{\nabla}_3\overset{(1)}{\alpha} + \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\overset{(1)}{\alpha} - \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\hat{\tau}}.
\tag{2.10.101}
$$

So,

$$
\begin{aligned}
\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} = {}& \slashed{\Delta}\overset{(1)}{\alpha} + \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\slashed{\nabla}_4\overset{(1)}{\alpha} - \Big(\frac{6+n}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi - \hat{\omega}\Big)\slashed{\nabla}_3\overset{(1)}{\alpha} \\
&+ \Big(\frac{6+n}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi - \hat{\omega}\Big)\Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\overset{(1)}{\alpha} - \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)}\overset{(1)}{\hat{\tau}} \\
&+ \Big[\frac{n-1}{n-3}\rho + e_4\Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big) - \frac{(n-4)}{4(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)^2 - \frac{n-2}{r^2}\Big]\overset{(1)}{\alpha}.
\end{aligned}
\tag{2.10.102}
$$

Using that

$$
e_4\Big(2\hat{\omega} + \frac{1}{2}\mathrm{Tr}_{\slashed{g}}\chi\Big) = 2\rho - 2\hat{\omega}^2 + \frac{1}{2}\hat{\omega}\mathrm{Tr}_{\slashed{g}}\chi - \frac{1}{2(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)^2,
\tag{2.10.103}
$$

simplifies equation (2.10.102) to the desired result.

Turning to the $\overset{(1)}{\hat{\tau}}$ equation and computing directly from proposition 2.10.20 (using the commutation lemma 2.8.4) gives

$$
\begin{aligned}
\slashed{\nabla}_3\slashed{\nabla}_4\overset{(1)}{\hat{\tau}} = {}& (\slashed{\mathcal{D}}_2^{\star}\slashed{\nabla}_3\overset{(1)}{\beta}) + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{n-2}(\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\beta}) + \frac{(n-4)}{2(n-2)}e_3(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\alpha} + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi\slashed{\nabla}_3\overset{(1)}{\alpha} \\
&+ \frac{(n-4)(n-1)}{(n-2)(n-3)}e_3(\rho)\overset{(1)}{\hat{\chi}} + \frac{(n-4)(n-1)}{(n-2)(n-3)}\rho\slashed{\nabla}_3\overset{(1)}{\hat{\chi}} - \widehat{(\slashed{\mathrm{div}}\slashed{\nabla}_3\overset{(1)}{\nu})} \\
&- \frac{\mathrm{Tr}_{\slashed{g}}\chi}{n-2}\widehat{(\slashed{\mathrm{div}}\overset{(1)}{\nu})} - \frac{n}{2(n-2)}(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{\nabla}_3\overset{(1)}{\hat{\tau}} - \frac{n}{2(n-2)}e_3(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\hat{\tau}}.
\end{aligned}
\tag{2.10.104}
$$

From the linearised Bianchi equation for $\nabla\!\!\!/_3 \overset{(1)}{\nu}$ in proposition 2.10.20, one can then compute that

$$\widehat{\mathrm{d}\!\!/\mathrm{iv}(\nabla\!\!\!/_3 \overset{(1)}{\nu})} = \Big(\hat{\omega} + \frac{2}{n-2}\mathrm{Tr}_{g\!\!/}\chi\Big)\widehat{\mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\nu}} - \frac{\mathrm{Tr}_{g\!\!/}\chi}{n-2}\widehat{\mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\underline{\nu}}} + \frac{2\rho(n-1)}{(n-2)(n-3)}\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\eta} \\ - \frac{\mathrm{Tr}_{g\!\!/}\chi}{n-2}\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\underline{\beta}} + \frac{2}{n-2}\mathcal{D}\!\!\!/_2^{\star}\nabla\!\!\!/\overset{(1)}{\rho} - \Delta\!\!\!/\overset{(1)}{\hat{\tau}} - \mathcal{D}\!\!\!/_2^{\star}(\mathrm{d}\!\!/\mathrm{iv}(\overset{(1)}{\varsigma} + \overset{(1)}{\hat{\tau}})) + \frac{\mathrm{Scal}(g\!\!/)}{n-3}\overset{(1)}{\hat{\tau}}. \tag{2.10.105}$$

So substituting this relation along with the linearised Bianchi identities 2.10.20 and then the linearised constraint of proposition 2.10.22 gives

$$\nabla\!\!\!/_3\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}} = \Delta\!\!\!/\overset{(1)}{\hat{\tau}} + \Big(\frac{n+2}{2(n-2)}\mathrm{Tr}_{g\!\!/}\chi + \hat{\omega}\Big)\Big(\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\beta} - \widehat{\mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\nu}} + \frac{(n-4)(n-1)}{(n-2)(n-3)}\rho\overset{(1)}{\hat{\chi}}\Big) \\ + \frac{\mathrm{Tr}_{g\!\!/}\chi}{n-2}\Big(\widehat{\mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\underline{\nu}}} + \mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\underline{\beta}} - \frac{(n-4)(n-1)}{(n-2)(n-3)}\rho\overset{(1)}{\underline{\hat{\chi}}}\Big) - \frac{n}{2(n-2)}(\mathrm{Tr}_{g\!\!/}\chi)\nabla\!\!\!/_3\overset{(1)}{\hat{\tau}} \\ - \Big(\frac{(n-4)(n-1)}{(n-2)(n-3)}\rho + \frac{n}{2(n-2)}e_3(\mathrm{Tr}_{g\!\!/}\chi) + \frac{(n-4)}{4(n-2)}(\mathrm{Tr}_{g\!\!/}\chi)^2 + \frac{\mathrm{Scal}(g\!\!/)}{n-3}\Big)\overset{(1)}{\hat{\tau}} \\ + \frac{(n-4)}{2(n-2)}\Big[e_3(\mathrm{Tr}_{g\!\!/}\chi) + \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{g\!\!/}\chi)\Big)\mathrm{Tr}_{g\!\!/}\chi\Big]\overset{(1)}{\alpha}, \tag{2.10.106}$$

One can now substitute the expressions for $\nabla\!\!\!/_3\overset{(1)}{\hat{\tau}}$ and $\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}}$ from linearised Bianchi (proposition 2.10.20), i.e.,

$$(\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\beta}) + \frac{(n-4)(n-1)}{(n-2)(n-3)}\overset{(1)}{\hat{\chi}}\rho - (\widehat{\mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\nu}}) = (\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}}) + (\mathrm{Tr}_{g\!\!/}\chi)\Big(\frac{n}{2(n-2)}\overset{(1)}{\hat{\tau}} - \frac{(n-4)}{2(n-2)}\overset{(1)}{\alpha}\Big), \tag{2.10.107}$$

$$\frac{(n-4)(n-1)}{(n-2)(n-3)}\overset{(1)}{\underline{\hat{\chi}}}\rho - (\widehat{\mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\underline{\nu}}}) - (\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\underline{\beta}}) = (\nabla\!\!\!/_3\overset{(1)}{\hat{\tau}}) + (\mathrm{Tr}_{g\!\!/}\chi)\Big(\frac{(n-4)}{2(n-2)}\overset{(1)}{\underline{\alpha}} - \frac{n}{2(n-2)}\overset{(1)}{\hat{\tau}}\Big), \tag{2.10.108}$$

to gives the first result.

To obtain the angular commuted version, note that two applications of the commutation lemma gives

$$\nabla\!\!\!/_4\Delta\!\!\!/\overset{(1)}{\hat{\tau}} = \Delta\!\!\!/\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}} - \frac{2\mathrm{Tr}_{g\!\!/}\chi}{n-2}\Delta\!\!\!/\overset{(1)}{\hat{\tau}} \tag{2.10.109}$$

and therefore

$$\nabla\!\!\!/_3\nabla\!\!\!/_4\Delta\!\!\!/\overset{(1)}{\hat{\tau}} = \Delta\!\!\!/\nabla\!\!\!/_3\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}} + \frac{2\mathrm{Tr}_{g\!\!/}\chi}{n-2}(\Delta\!\!\!/\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}} - \Delta\!\!\!/\nabla\!\!\!/_3\overset{(1)}{\hat{\tau}}) - \frac{2}{n-2}\rho\Delta\!\!\!/\overset{(1)}{\hat{\tau}} - \frac{6(\mathrm{Tr}_{g\!\!/}\chi)^2}{(n-2)^2}\Delta\!\!\!/\overset{(1)}{\hat{\tau}}. \tag{2.10.110}$$

Hence,

$$\nabla\!\!\!/_3\nabla\!\!\!/_4\Delta\!\!\!/\overset{(1)}{\hat{\tau}} = \Delta\!\!\!/\Delta\!\!\!/\overset{(1)}{\hat{\tau}} + \Big(\frac{n+6}{2(n-2)}\mathrm{Tr}_{g\!\!\!/}\chi + \hat{\omega}\Big)\Delta\!\!\!/\nabla\!\!\!/_4\overset{(1)}{\hat{\tau}} - \frac{n+6}{2(n-2)}(\mathrm{Tr}_{g\!\!\!/}\chi)\Delta\!\!\!/\nabla\!\!\!/_3\overset{(1)}{\hat{\tau}} \\ + \Big(\frac{(n-6)(\mathrm{Tr}_{g\!\!\!/}\chi)^2}{(n-2)^2} - \frac{2(n^2-4n+1)}{(n-3)(n-2)}\rho\Big)\Delta\!\!\!/\overset{(1)}{\hat{\tau}} - \frac{(n-4)(\mathrm{Tr}_{g\!\!\!/}\chi)^2}{2(n-2)^2}\Delta\!\!\!/(\overset{(1)}{\alpha} + \overset{(1)}{\underline{\alpha}}). \tag{2.10.111}$$

Commuting once again so that its an equation for $\Delta\!\!\!/\overset{(1)}{\hat{\tau}}$ gives the result. □

A Brief Note on the Regge–Wheeler Equation on Schw_n

It turns out that, even in the $n = 4$ case, the Teukolsky equation is difficult to analyze directly due to problematic first order t-derivative of $\overset{(1)}{\alpha}$ or $\overset{(1)}{\underline{\alpha}}$ appearing in the second and third terms on the right hand of proposition 2.10.4. In $n = 4$, the work [28] introduced so-called 'Regge–Wheeler' unknowns $(P, \underline{P})$ via a physical space interpretation of the 'Chandrasekhar transformation' (see [15] for the mode decomposed version). The unknowns $(P, \underline{P})$ satisfy the Regge-Wheeler equation (initially found for metric perturbations [36]) that can be treated with methods employed for the wave equation. This section turns to the possibility of a generalisation of $(P, \underline{P})$ to higher-dimensions. The unknowns $(P, \underline{P})$ should allow one to recover control on $\overset{(1)}{\alpha}$, $\overset{(1)}{\underline{\alpha}}$ and $\overset{(1)}{\hat{\tau}}$ from control on Regge–Wheeler variable. In $4D$, $(P, \underline{P})$ transform solutions $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ of the Teukolsky equations of proposition 2.10.4 to solutions of the (tensorial) Regge–Wheeler equation. In particular, the transformation is

$$\begin{aligned} P &\doteq \frac{1}{\Omega r^3}\nabla\!\!\!/_3(\Omega r^3\psi), \qquad & \psi &\doteq -\frac{1}{2r\Omega^2}\nabla\!\!\!/_3(r\Omega^2\overset{(1)}{\alpha}), \\ \underline{P} &\doteq -\frac{1}{\Omega r^3}\nabla\!\!\!/_4(\Omega r^3\underline{\psi}), \qquad & \underline{\psi} &\doteq \frac{1}{2r\Omega^2}\nabla\!\!\!/_4(r\Omega^2\overset{(1)}{\underline{\alpha}}). \end{aligned} \tag{2.10.112}$$

The unknown P satisfies an equation of the form

$$\nabla\!\!\!/_3\nabla\!\!\!/_4 P + \nabla\!\!\!/_4\nabla\!\!\!/_3 P - 2\Delta\!\!\!/ P + f(\nabla\!\!\!/_3 - \nabla\!\!\!/_4)P + gP = R[\overset{(1)}{\alpha}] \tag{2.10.113}$$

where $R[\overset{(1)}{\alpha}]$ is some error term depending on $\overset{(1)}{\alpha}$ and the unknown $\underline{P}$ satisfies an analogous equation with $R[\overset{(1)}{\underline{\alpha}}]$ on the right-hand side. Whilst not immediately obvious, it is the second order operator acting on $\overset{(1)}{\alpha}$ or $\overset{(1)}{\underline{\alpha}}$ in the transformation that gets rid of the problematic t-derivative of $\overset{(1)}{\alpha}$ or $\overset{(1)}{\underline{\alpha}}$ appearing in the second and third terms on the right hand of proposition 2.10.4. Remarkably, in $n = 4$, P and $\underline{P}$ satisfy completely decoupled equations, i.e. $R[\overset{(1)}{\alpha}] \equiv 0$! This particular property seems to be peculiar to 4-dimensional Schwarzschild. Indeed, in the case of Kerr [29] or Reissner–Nordström [118] such complete decoupling does not occur.

A naive generalisation to higher dimensions would be to modify the r-weights in the definition of P and $\underline{P}$. In particular, the one could take the ansatz for the generalisation of P and $\underline{P}$ to be

$$\mathcal{P} \doteq \frac{1}{\Omega r^q}\nabla_3(\Omega r^q \Psi) = -\Big(\frac{q}{n-2}\mathrm{Tr}_{\not g}\chi + \hat{\omega}\Big)\Psi + \nabla_3\Psi, \tag{2.10.114}$$

$$\underline{\mathcal{P}} \doteq -\frac{1}{\Omega r^q}\nabla_4(\Omega r^q \underline{\Psi}) = -\Big(\frac{q}{n-2}\mathrm{Tr}_{\not g}\chi + \hat{\omega}\Big)\underline{\Psi} - \nabla_4\underline{\Psi}, \tag{2.10.115}$$

with

$$\Psi \doteq -\frac{1}{2r^p\Omega^2}\nabla_3(r^p\Omega^2\overset{(1)}{\alpha}) = \Big(\frac{p}{2(n-2)}\mathrm{Tr}_{\not g}\chi + \hat{\omega}\Big)\overset{(1)}{\alpha} - \frac{1}{2}\nabla_3\overset{(1)}{\alpha}, \tag{2.10.116}$$

$$\underline{\Psi} \doteq \frac{1}{2r^p\Omega^2}\nabla_4(r^p\Omega^2\overset{(1)}{\underline{\alpha}}) = \Big(\frac{p}{2(n-2)}\mathrm{Tr}_{\not g}\chi + \hat{\omega}\Big)\overset{(1)}{\underline{\alpha}} + \frac{1}{2}\nabla_4\overset{(1)}{\underline{\alpha}}. \tag{2.10.117}$$

Hence,

$$\mathcal{P} = -\Big(\frac{(q-1)p}{2(n-2)^2}(\mathrm{Tr}_{\not g}\chi)^2 + \frac{n-2-(p+q)}{n-2}\rho\Big)\overset{(1)}{\alpha} + \Big(\frac{p+q}{2(n-2)}\mathrm{Tr}_{\not g}\chi + \frac{3}{2}\hat{\omega}\Big)\nabla_3\overset{(1)}{\alpha} - \frac{1}{2}\nabla_3\nabla_3\overset{(1)}{\alpha}. \tag{2.10.118}$$

Motivated by the form of equation (2.10.113) in the $n=4$ case with $p=1$ and $q=3$, this section is dedicated to looking for a $\mathcal{P}$ in $n>4$ which satisfies an equation of the form

$$\nabla_3\nabla_4\mathcal{P} + \nabla_4\nabla_3\mathcal{P} - 2\Delta\mathcal{P} + f(\nabla_3 - \nabla_4)\mathcal{P} + g\mathcal{P} = \mathcal{R} \tag{2.10.119}$$

where $\mathcal{R}$ is an error term independent of $\mathcal{P}$ which ideally vanishes. In particular, one has the following proposition:

Proposition 2.10.28. *Let $\mathcal{P}$ be defined as*

$$\mathcal{P} \doteq \mathfrak{c}(r)\overset{(1)}{\alpha} + \mathfrak{d}(r)\nabla_3\overset{(1)}{\alpha} - \frac{1}{2}\nabla_3\nabla_3\overset{(1)}{\alpha}. \tag{2.10.120}$$

Then $\mathcal{P}$ satisfies

$$\nabla_3\nabla_4\mathcal{P} + \nabla_4\nabla_3\mathcal{P} - 2\Delta\mathcal{P} + \mathfrak{f}(\nabla_3\mathcal{P} - \nabla_4\mathcal{P}) + \mathfrak{g}\mathcal{P} = 2\mathcal{R}[\overset{(1)}{\alpha}, \overset{(1)}{\hat{\tau}}], \tag{2.10.121}$$

where

$$\mathfrak{f} \doteq \frac{n+6}{n-2}\mathrm{Tr}_{\not g}\chi + \hat{\omega}, \tag{2.10.122}$$

$$\mathfrak{g} \doteq 2\nabla_4\mathfrak{d} + 2\hat{\omega}\mathfrak{d} - \hat{\omega}^2 - \frac{(n^2+8n+44)(\mathrm{Tr}_{\not g}\chi)^2}{4(n-2)^2} + \frac{(n^2+3n-22)\rho}{(n-3)(n-2)} \tag{2.10.123}$$

and the error term $\mathcal{R}[\overset{(1)}{\alpha},\overset{(1)}{\hat{\tau}}]$ *is of the form*

$$\mathcal{R}[\overset{(1)}{\alpha},\overset{(1)}{\hat{\tau}}] \doteq \not{\mathfrak{g}}(r)\not{\Delta}\overset{(1)}{\alpha} + \frac{(\mathrm{Tr}_{\not{g}}\chi)^2}{2(n-2)}\not{\nabla}_3\not{\nabla}_3\overset{(1)}{\hat{\tau}} + \underline{\mathfrak{q}}_{1,\alpha}(r)\not{\nabla}_3\overset{(1)}{\alpha} + \mathfrak{q}_{1,\alpha}(r)\not{\nabla}_4\overset{(1)}{\alpha} + \mathfrak{q}_{0,\alpha}(r)\overset{(1)}{\alpha} + \mathfrak{q}_{1,\tau}(r)\not{\nabla}_3\overset{(1)}{\hat{\tau}} + \mathfrak{q}_{0,\tau}(r)\overset{(1)}{\hat{\tau}}, \tag{2.10.124}$$

where $\not{\mathfrak{g}}$, $\mathfrak{q}_{1,\alpha}$ *are given by*

$$\not{\mathfrak{g}} \doteq \not{\nabla}_3\mathfrak{d} - \frac{2\mathrm{Tr}_{\not{g}}\chi}{(n-2)}\mathfrak{d} + \hat{\omega}\mathfrak{d} + \frac{n(\mathrm{Tr}_{\not{g}}\chi)^2}{2(n-2)^2} + \frac{3(n-4)\rho}{2(n-2)} - 3\hat{\omega}^2, \tag{2.10.125}$$

$$\mathfrak{q}_{1,\alpha} \doteq \frac{\Omega^2}{r^4}\not{\nabla}_3\Big(\frac{r^4}{\Omega^2}\Big[\mathfrak{c} - \frac{1}{2}\Big(6\hat{\omega}^2 + \frac{n(\mathrm{Tr}_{\not{g}}\chi)^2}{4(n-2)} - \frac{7\rho}{2}\Big) + \mathfrak{d}\Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\not{g}}\chi)\Big)\Big]\Big) \tag{2.10.126}$$

and $\underline{\mathfrak{q}}_{1,\alpha}$, $\mathfrak{q}_{0,\alpha}$, $\mathfrak{q}_{1,\tau}$ *and* $\mathfrak{q}_{0,\tau}$ *can written in terms of* $\mathfrak{d}$, $\mathfrak{c}$ *and background Ricci coefficients (see equations (2.10.146-2.10.149)). Additionally, a completely analogous proposition holds for* $\underline{\mathcal{P}}$ *with the relevant quantities 'barred' and* $e_3 \mapsto e_4$*. In particular, if* $\mathcal{P}$ *is given by equation (2.10.118) with* $p = \frac{n-2}{2}$ *and* $q = \frac{n+2}{2}$ *then* $\not{\mathfrak{g}} = 0$, $\mathfrak{q}_{1,\alpha} = 0$ *and*

$$\mathfrak{g} = \frac{2(n^2-n-8)}{(n-2)(n-3)}\rho - \frac{3n+4}{(n-2)^2}(\mathrm{Tr}_{\not{g}}\chi)^2, \tag{2.10.127}$$

$$\underline{\mathfrak{q}}_{1,\alpha} = \frac{n-4}{2(n-2)}\rho\mathrm{Tr}_{\not{g}}\chi, \tag{2.10.128}$$

$$\mathfrak{q}_{0,\alpha} = \frac{n-4}{n-2}\rho\Big(\rho - \frac{n(\mathrm{Tr}_{\not{g}}\chi)^2}{2(n-2)}\Big), \tag{2.10.129}$$

$$\mathfrak{q}_{1,\tau} = \frac{\mathrm{Tr}_{\not{g}}\chi}{2(n-2)}\Big(7\rho - \frac{n(\mathrm{Tr}_{\not{g}}\chi)^2}{(n-2)}\Big), \tag{2.10.130}$$

$$\mathfrak{q}_{0,\tau} = \frac{4\rho^2}{n-2} + \frac{n(\mathrm{Tr}_{\not{g}}\chi)^4}{8(n-2)^2} - \frac{4\rho(\mathrm{Tr}_{\not{g}}\chi)^2}{(n-2)^2}. \tag{2.10.131}$$

If $n = 4$*, then* $\overset{(1)}{\hat{\tau}} = 0$ *and, therefore,* $\mathcal{R}[\overset{(1)}{\alpha},\overset{(1)}{\hat{\tau}}] \equiv 0$*.*

Proof. Computing naively gives

$$\not{\nabla}_4\mathcal{P} = (\not{\nabla}_4\mathfrak{c})\overset{(1)}{\alpha} + \mathfrak{c}\not{\nabla}_4\overset{(1)}{\alpha} + (\not{\nabla}_4\mathfrak{d})\not{\nabla}_3\overset{(1)}{\alpha} + \mathfrak{d}\not{\nabla}_4\not{\nabla}_3\overset{(1)}{\alpha} - \frac{1}{2}\not{\nabla}_4\not{\nabla}_3\not{\nabla}_3\overset{(1)}{\alpha} \tag{2.10.132}$$

Now by the commutation relations in lemma 2.8.4

$$\not{\nabla}_4\not{\nabla}_3\not{\nabla}_3\overset{(1)}{\alpha} = \not{\nabla}_3\not{\nabla}_4\not{\nabla}_3\overset{(1)}{\alpha} - \hat{\omega}(\not{\nabla}_3\not{\nabla}_3\overset{(1)}{\alpha} + \not{\nabla}_4\not{\nabla}_3\overset{(1)}{\alpha}). \tag{2.10.133}$$

So,

$$\begin{aligned}\slashed{\nabla}_4\mathcal{P} = &(\slashed{\nabla}_4\mathfrak{c})\overset{(1)}{\alpha} + \mathfrak{c}\slashed{\nabla}_4\overset{(1)}{\alpha} + (\slashed{\nabla}_4\mathfrak{d})\slashed{\nabla}_3\overset{(1)}{\alpha} + (\mathfrak{d} + \frac{1}{2}\hat{\omega})\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} \\ &+ \frac{1}{2}\hat{\omega}\slashed{\nabla}_3\slashed{\nabla}_3\overset{(1)}{\alpha} - \frac{1}{2}\slashed{\nabla}_3\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha}.\end{aligned} \tag{2.10.134}$$

From the Teukolsky equation in proposition (2.10.4) and the commutation lemma 2.8.4 one can calculate

$$\begin{aligned}\slashed{\nabla}_3\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} = &\slashed{\Delta}\slashed{\nabla}_3\overset{(1)}{\alpha} + \frac{2(\mathrm{Tr}_{\slashed{g}}\chi)}{n-2}\slashed{\Delta}\overset{(1)}{\alpha} - \Big(\frac{6+n}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\chi - \hat{\omega}\Big)\slashed{\nabla}_3\slashed{\nabla}_3\overset{(1)}{\alpha} \\ &+ \Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} + \Big(4\hat{\omega}^2 + \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)} - 2\rho\Big)\slashed{\nabla}_4\overset{(1)}{\alpha} \\ &+ \Big[\frac{(n-10)(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)^2} - \frac{8n-28}{(n-3)(n-2)}\rho - \hat{\omega}^2\Big]\slashed{\nabla}_3\overset{(1)}{\alpha} - \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)}\slashed{\nabla}_3\overset{(1)}{\hat{\tau}} \\ &+ \Big[\frac{2(n^3-7n^2+16n-8)}{(n-3)(n-2)^2}\rho\mathrm{Tr}_{\slashed{g}}\chi - 8\hat{\omega}^3 + 8\hat{\omega}\rho + \frac{2(\mathrm{Tr}_{\slashed{g}}\chi)^3}{(n-2)^2}\Big]\overset{(1)}{\alpha} \\ &- \frac{2(\mathrm{Tr}_{\slashed{g}}\chi)}{(n-2)}\Big[\frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{n-2} + \rho\Big]\overset{(1)}{\hat{\tau}}.\end{aligned} \tag{2.10.135}$$

Hence,

$$\begin{aligned}\slashed{\nabla}_4\mathcal{P} = &-\frac{1}{2}\slashed{\Delta}\slashed{\nabla}_3\overset{(1)}{\alpha} + \Big[\mathfrak{d} - \frac{1}{2}\Big(\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big)\Big]\slashed{\nabla}_4\slashed{\nabla}_3\overset{(1)}{\alpha} - \frac{(\mathrm{Tr}_{\slashed{g}}\chi)}{n-2}\slashed{\Delta}\overset{(1)}{\alpha} \\ &+ \Big[\frac{6+n}{4(n-2)}\mathrm{Tr}_{\slashed{g}}\chi\Big]\slashed{\nabla}_3\slashed{\nabla}_3\overset{(1)}{\alpha} + \Big[\mathfrak{c} - \frac{1}{2}\Big(4\hat{\omega}^2 + \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)} - 2\rho\Big)\Big]\slashed{\nabla}_4\overset{(1)}{\alpha} \\ &+ \Big[(\slashed{\nabla}_4\mathfrak{d}) - \frac{1}{2}\Big(\frac{(n-10)(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)^2} - \frac{8n-28}{(n-3)(n-2)}\rho - \hat{\omega}^2\Big)\Big]\slashed{\nabla}_3\overset{(1)}{\alpha} + \frac{1}{2}\frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{(n-2)}\slashed{\nabla}_3\overset{(1)}{\hat{\tau}} \\ &+ \Big[(\slashed{\nabla}_4\mathfrak{c}) + -\frac{1}{2}\Big(\frac{2(n^3-7n^2+16n-8)}{(n-3)(n-2)^2}\rho\mathrm{Tr}_{\slashed{g}}\chi - 8\hat{\omega}^3 + 8\hat{\omega}\rho + \frac{2(\mathrm{Tr}_{\slashed{g}}\chi)^3}{(n-2)^2}\Big)\Big]\overset{(1)}{\alpha} \\ &+ \frac{(\mathrm{Tr}_{\slashed{g}}\chi)}{(n-2)}\Big(\frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{n-2} + \rho\Big)\overset{(1)}{\hat{\tau}}.\end{aligned} \tag{2.10.136}$$

One can use the Teukolsky equation in proposition 2.10.4 and

$$\slashed{\nabla}_3\slashed{\nabla}_3\overset{(1)}{\alpha} = 2\mathfrak{c}\overset{(1)}{\alpha} + 2\mathfrak{d}\slashed{\nabla}_3\overset{(1)}{\alpha} - 2\mathcal{P}, \tag{2.10.137}$$

to produce

$$\begin{aligned}\slashed{\nabla}_4\mathcal{P} = &-\frac{1}{2}\slashed{\Delta}\slashed{\nabla}_3\overset{(1)}{\alpha} + \Big[\mathfrak{d} - \frac{1}{2}\Big(\hat{\omega} + \frac{(n+2)\mathrm{Tr}_{\slashed{g}}\chi}{2(n-2)}\Big)\Big]\slashed{\Delta}\overset{(1)}{\alpha} - \frac{(n+6)\mathrm{Tr}_{\slashed{g}}\chi}{2(n-2)}\mathcal{P} \\ &+ \mathfrak{f}_{1,\alpha}\slashed{\nabla}_4\overset{(1)}{\alpha} + \underline{\mathfrak{f}}_{1,\alpha}\slashed{\nabla}_3\overset{(1)}{\alpha} + \mathfrak{f}_{0,\alpha}\overset{(1)}{\alpha} + \frac{(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2(n-2)}\slashed{\nabla}_3\overset{(1)}{\hat{\tau}} + \mathfrak{f}_{0,\tau}\overset{(1)}{\hat{\tau}},\end{aligned} \tag{2.10.138}$$

where

$$\mathfrak{f}_{1,\alpha} \doteq \mathfrak{c} - \frac{1}{2}\Big(6\hat{\omega}^2 + \frac{n(\mathrm{Tr}_{\not g}\chi)^2}{4(n-2)} - \frac{7\rho}{2}\Big) + \mathfrak{d}\Big(2\hat{\omega} + \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)\Big), \tag{2.10.139}$$

$$\underline{\mathfrak{f}}_{1,\alpha} \doteq \not\nabla_4\mathfrak{d} + \hat{\omega}\mathfrak{d} - \frac{1}{2}\Big(\hat{\omega}^2 + \frac{2(10-n)(\mathrm{Tr}_{\not g}\chi)^2}{4(n-2)^2} + \frac{4\rho}{(n-3)(n-2)}\Big), \tag{2.10.140}$$

$$\begin{aligned}\mathfrak{f}_{0,\alpha} \doteq{}& \not\nabla_4\mathfrak{c} + \Big[\frac{2(n-4)^2}{(n-3)(n-2)}\rho - 4\hat{\omega}^2 + \frac{2(\mathrm{Tr}_{\not g}\chi)^2}{(n-2)} - \frac{\mathrm{S\not c al}(\not g)}{(n-3)}\Big]\mathfrak{d} + \frac{(n+6)\mathrm{Tr}_{\not g}\chi}{2(n-2)}\mathfrak{c} \\ &+ 6\hat{\omega}^3 - \frac{3n^3 - 24n^2 + 65n - 50}{2(n-2)^2(n-3)}\rho\mathrm{Tr}_{\not g}\chi - \frac{6n^2 - 32n + 44}{(n-2)(n-3)}\hat{\omega}\rho - \frac{(n+2)(\mathrm{Tr}_{\not g}\chi)^3}{4(n-2)^2},\end{aligned} \tag{2.10.141}$$

$$\mathfrak{f}_{0,\tau} \doteq \frac{(\mathrm{Tr}_{\not g}\chi)^2}{2(n-2)}\Big(\frac{(n+2)(\mathrm{Tr}_{\not g}\chi)}{2(n-2)} - \hat{\omega}\Big) - \mathfrak{d}\frac{(\mathrm{Tr}_{\not g}\chi)^2}{(n-2)}. \tag{2.10.142}$$

One can now compute using the commutation lemma 2.8.4,

$$-\frac{1}{2}\not\Delta\not\nabla_3\not\nabla_3\overset{(1)}{\alpha} = -\mathfrak{c}\not\Delta\overset{(1)}{\alpha} - \mathfrak{d}\not\Delta\not\nabla_3\overset{(1)}{\alpha} + \not\Delta\mathcal{P}, \tag{2.10.143}$$

$$-\frac{1}{2}\not\nabla_3\not\nabla_3\overset{(1)}{\alpha} = -\mathfrak{c}\overset{(1)}{\alpha} - \mathfrak{d}\not\nabla_3\overset{(1)}{\alpha} + \mathcal{P}, \tag{2.10.144}$$

and that $\not\Delta\not\nabla_3\overset{(1)}{\alpha}$ can be expressed in terms of $\not\nabla_4\mathcal{P}$ to show

$$\begin{aligned}\not\nabla_3\not\nabla_4\mathcal{P} ={}& \not\Delta\mathcal{P} + \hat{\omega}\not\nabla_4\mathcal{P} - \frac{(n+6)\mathrm{Tr}_{\not g}\chi}{2(n-2)}(\not\nabla_3\mathcal{P} - \not\nabla_4\mathcal{P}) - \mathfrak{g}\mathcal{P} \\ &+ \not\mathfrak{g}\not\Delta\overset{(1)}{\alpha} + \frac{(\mathrm{Tr}_{\not g}\chi)^2}{2(n-2)}\not\nabla_3\not\nabla_3\overset{(1)}{\hat{\tau}} + \underline{\mathfrak{q}}_{1,\alpha}\not\nabla_3\overset{(1)}{\alpha} + \mathfrak{q}_{1,\alpha}\not\nabla_4\overset{(1)}{\alpha} + \mathfrak{q}_{0,\alpha}\overset{(1)}{\alpha} \\ &+ \mathfrak{q}_{1,\tau}\not\nabla_3\overset{(1)}{\hat{\tau}} + \mathfrak{q}_{0,\tau}\overset{(1)}{\hat{\tau}},\end{aligned} \tag{2.10.145}$$

where

$$\underline{\mathfrak{q}}_{1,\alpha} \doteq (\not\nabla_3\underline{\mathfrak{f}}_{1,\alpha}) + \mathfrak{f}_{0,\alpha} + 2\mathfrak{d}\underline{\mathfrak{f}}_{1,\alpha} - \frac{(n+6)\mathrm{Tr}_{\not g}\chi}{2(n-2)}(\underline{\mathfrak{f}}_{1,\alpha} + \mathfrak{f}_{1,\alpha}) + 2\hat{\omega}(\mathfrak{f}_{1,\alpha} - \underline{\mathfrak{f}}_{1,\alpha}), \tag{2.10.146}$$

$$\begin{aligned}\mathfrak{q}_{0,\alpha} \doteq{}& \not\nabla_3\mathfrak{f}_{0,\alpha} + 2\mathfrak{c}\underline{\mathfrak{f}}_{1,\alpha} - \Big(\hat{\omega} + \frac{(n+6)\mathrm{Tr}_{\not g}\chi}{2(n-2)}\Big)\mathfrak{f}_{0,\alpha} \\ &+ \mathfrak{f}_{1,\alpha}\Big[\frac{2(n-4)^2}{(n-3)(n-2)}\rho - 4\hat{\omega}^2 + \frac{2(\mathrm{Tr}_{\not g}\chi)^2}{(n-2)} - \frac{\mathrm{S\not c al}(\not g)}{(n-3)}\Big],\end{aligned} \tag{2.10.147}$$

$$\mathfrak{q}_{1,\tau} \doteq \mathfrak{f}_{0,\tau} + \frac{1}{2}e_3\Big(\frac{(\mathrm{Tr}_{\not g}\chi)^2}{(n-2)}\Big) - \frac{1}{2}\Big(\hat{\omega} + \frac{(n+6)\mathrm{Tr}_{\not g}\chi}{2(n-2)}\Big)\frac{(\mathrm{Tr}_{\not g}\chi)^2}{(n-2)}, \tag{2.10.148}$$

$$\mathfrak{q}_{0,\tau} \doteq (\not\nabla_3\mathfrak{f}_{0,\tau}) - \mathfrak{f}_{0,\tau}\Big(\hat{\omega} + \frac{(n+6)\mathrm{Tr}_{\not g}\chi}{2(n-2)}\Big) - \mathfrak{f}_{1,\alpha}\frac{(\mathrm{Tr}_{\not g}\chi)^2}{(n-2)}. \tag{2.10.149}$$

If $\mathcal{P}$ is given by equation (2.10.118) then one can compute directly from equation (2.10.146)–(2.10.149) the concluding result in the proposition. □

2.10.5 Recovering the Newman–Penrose Teukolsky Equation

For later use and the reader familiar with the Newman–Penrose formalism [102] the relation between the Teukolsky equation (arising in the double null gauge) derived above in proposition 2.10.4 and the traditional Teukolsky equation of [13] is given here. To be clear, this section only applies to 4-dimensional Schwarzschild spacetime.

The traditional Teukolsky equation (or Bardeen–Press equation) [13, 12] on Schw_4 is

$$\Box_g \alpha^{[s]} + \frac{2s}{r^2}(r-M)\partial_r \alpha^{[s]} + i\frac{2s\cos\theta}{r^2\sin^2\theta}\partial_\varphi \alpha^{[s]} + \frac{2s}{r^2}\Big(\frac{M}{D(r)} - r\Big)\partial_t \alpha^{[s]} + \frac{1}{r^2}\Big(s - s^2\cot^2\theta\Big)\alpha^{[s]} = 0, \tag{2.10.150}$$

where $\alpha^{[s]}$ is a smooth complex-valued spin s-weighted function on the exterior $\mathcal{E}$ of the Schwarzschild spacetime (see section 2.2.1 of [29] or section 2.2 of [33] for a precise definition and discussion of smooth complex-valued spin s-weighted functions). One can consider this equation for arbitrary $s \in \frac{1}{2}\mathbb{Z}$. For $s = 0$, equation (2.10.150) reduces to the wave equation on Schw_4. For $s = \pm 1$ one can show that equation (2.10.150) governs the extreme components of the Maxwell equations on Schw_4. For $s = \pm 2$, the equation (2.10.150) governs the extremal curvature components of the metric in the Newman–Penrose formalism [102]. There is a precise relation between equation (2.10.150) for $s = \pm 2$ and the Teukolsky equation(s) written down in proposition 2.10.4 for $n = 4$ as shall now be elaborated on.

In the Newman–Penrose (NP) formalism one takes a arbitrary null pair (l, n) normalised such that

$$g(l, n) = -1, \tag{2.10.151}$$

and two orthonormal vectors (m_1, m_2) for the space $\langle l, n\rangle^\perp$. One then constructs a (complex) null tetrad by complexifying the space $\langle l, n\rangle^\perp$ by taking $m \doteq \frac{1}{\sqrt{2}}(m_1 + i m_2)$.[d] The metric is then given by

$$g_{ab} = -l_{(a}n_{b)} + m_{(a}\bar{m}_{b)}, \tag{2.10.152}$$

where $\bar{m}$ is the complex conjugate of m. One then defines the following complex Weyl NP scalars

$$\Psi_0 \doteq R_{abcd}l^a m^b l^c m^d, \qquad \Psi_1 \doteq R_{abcd}l^a n^b l^c m^d, \qquad \Psi_2 \doteq R_{abcd}l^a m^b \bar{m}^c n^d, \qquad \Psi_3 \doteq R_{abcd}l^a n^b \bar{m}^c n^d, \tag{2.10.153}$$

[d] The original convention of Newman and Penrose was to use signature $(+,-,-,-)$ and therefore, the convention $g(l, n) = 1$ [102].

and

$$\Psi_4 \doteq R_{abcd} n^a \bar{m}^b n^c \bar{m}^d. \tag{2.10.154}$$

Using the conventions of [13], the complex null tetrad on Schw_4 is

$$l \doteq \frac{1}{\Omega} e_4 = \frac{1}{\Omega^2}\partial_v \qquad n \doteq \Omega e_3 = \frac{1}{2}\partial_u \qquad m \doteq \frac{1}{r\sqrt{2}}\Big(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\Big). \tag{2.10.155}$$

The frame written here in equation (2.10.155) is not regular at $\mathcal{H}^+$. Denoting the linearised versions of Ψ_A as $\overset{(1)}{\Psi}_A$ $(A = 1, ..., 4)$, the papers [12] and [13] showed, working in the frame (2.10.155), that $\upalpha^{[+2]} = \overset{(1)}{\Psi}_0$ and $\upalpha^{[-2]} = r^4 \overset{(1)}{\Psi}_4$ satisfiy the equation (2.10.150).

With the frame (2.10.155) relations, the relation of Ψ_0 and Ψ_4 to the $\mathbb{S}^2_{u,v}$-tensors α and $\underline{\alpha}$ is now straightforward

$$\Psi_0 = \frac{1}{r^2\Omega^2}\Big(\alpha_{\theta\theta} + \frac{i}{\sin\theta}\alpha_{\theta\varphi}\Big), \qquad \Psi_4 = \frac{\Omega^2}{r^2}\Big(\underline{\alpha}_{\theta\theta} - \frac{i}{\sin\theta}\underline{\alpha}_{\theta\varphi}\Big), \tag{2.10.156}$$

where the trace-free property of α and $\underline{\alpha}$ has been used. Therefore, under linearisation around the Schw_4 background one has

$$\overset{(1)}{\Psi}_0 = \frac{1}{r^2\Omega^2}\Big(\overset{(1)}{\alpha}_{\theta\theta} + \frac{i}{\sin\theta}\overset{(1)}{\alpha}_{\theta\varphi}\Big), \qquad \overset{(1)}{\Psi}_4 = \frac{\Omega^2}{r^2}\Big(\overset{(1)}{\underline{\alpha}}_{\theta\theta} - \frac{i}{\sin\theta}\overset{(1)}{\underline{\alpha}}_{\theta\varphi}\Big). \tag{2.10.157}$$

Using (2.10.157), the relation between equation (2.10.150) and proposition 2.10.4 for $n = 4$ can be stated. One can show using proposition 2.10.4 that

$$\upalpha^{[+2]} = \frac{1}{r^2\Omega^2}\Big(\overset{(1)}{\alpha}_{\theta\theta} + \frac{i}{\sin\theta}\overset{(1)}{\alpha}_{\theta\varphi}\Big), \tag{2.10.158}$$

$$\upalpha^{[-2]} = \Omega^2 r^2\Big(\overset{(1)}{\underline{\alpha}}_{\theta\theta} - \frac{i}{\sin\theta}\overset{(1)}{\underline{\alpha}}_{\theta\varphi}\Big), \tag{2.10.159}$$

satisfy the equation (2.10.150). Conversely, if $\upalpha^{[\pm 2]}$ satisfy the equation (2.10.150) for $s = \pm 2$ respectively then one can show that

$$\overset{(1)}{\alpha} = \Omega^2 r^2\Big(\Re(\upalpha^{[+2]})(d\theta\otimes d\theta - \sin^2\theta d\varphi\otimes d\varphi) + \sin\theta\,\Im(\upalpha^{[+2]})(d\theta\otimes d\varphi + d\varphi\otimes d\theta)\Big), \tag{2.10.160}$$

$$\overset{(1)}{\underline{\alpha}} = \frac{1}{\Omega^2 r^2}\Big(\Re(\upalpha^{[-2]})(d\theta\otimes d\theta - \sin^2\theta d\varphi\otimes d\varphi) - \sin\theta\,\Im(\upalpha^{[-2]})(d\theta\otimes d\varphi + d\varphi\otimes d\theta)\Big), \tag{2.10.161}$$

satisfies proposition 2.10.4 for $n = 4$.

Chapter 3

Weak Stability of Schwarzschild from Canonical Energy

3.1 Introduction

The main topic of this chapter is the study of the linear stability problem for the exterior of the 4-dimensional Schwarzschild black hole spacetime [1]. The aim of this chapter is to establish the 'weak' linear stability of the $4D$ Schwarzschild black hole exterior spacetime using the canonical energy conservation law of Hollands and Wald [65]. The main step in achieving this aim is to establish a explicit connection between the conservation laws of Holzegel [90] and the conservation law for the canonical energy. This work also acts as a blueprint for exploring the use of the canonical energy to prove weak stability results on other spacetimes; for example the Kerr black hole spacetime [106].

3.1.1 Previous Works and Context

The definitive result on the *linear* stability of the Schwarzschild spacetime was published in 2019 in the monumental work of Dafermos, Holzegel and Rodnianski [28] (the reader should also note the definitive work of Dafermos, Holzegel, Rodnianski and Taylor [35] on the *non-linear* problem). Their result can be stated roughly as follows

Theorem 3.1.1 (Linear Stability of the Schwarzschild Solution [28])**.** *All solutions to the linearised vacuum Einstein equation ($I.5$) (in double null gauge) around Schwarzschild arising from regular asymptotically flat initial data remain uniformly bounded on the exterior and (after adding a pure gauge solution which can be estimated by the size of the data) decay inverse polynomially (through a suitable foliation) to a linearised Kerr solution.*

Remark 3.1.2. *This statement is the best one could expect for the linearised vacuum Einstein equation ($I.5$) on the Schwarzschild exterior; in view of the existence of the Kerr solution, the* *best one can expect of general solutions to the linearised vacuum Einstein equation (I.5) on the Schwarzschild exterior is that the decay to a linear combination of a pure gauge solution and a linearised Kerr solution.*

This result came approximately 60 years after the seminal results of Regge and Wheeler [36] on the mode stability of the Schwarzschild black hole spacetime in 1957. In the successive decades there were numerous works studying linear perturbations of the Schwarzschild and Kerr spacetimes (see for example [10–18]) which led to the conjecture that the Schwarzschild spacetime and Kerr spacetime are (linearly) stable. From this early literature, the proof of theorem 3.1.1 exploits the celebrated (tensorial) Teukolsky equations for the (residually) gauge invariant quantities $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ which decouple from the full system of linear equations [13]. The work [28] finds a transformation from $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ to solutions P and $\underline{P}$ of the (tensorial) Regge–Wheeler equation by a physical space interpretation of the Chandrasekhar transformation [15] (for a derivation of the Teukolsky equations and Regge–Wheeler equations see the discussion in section 2.10.4). Additionally, the proof of theorem 3.1.1 relies on more recent work which advanced the understanding of ways to produce robust boundedness and decay statements for the scalar wave equation on black hole exteriors [19–28]. These recent advances for the scalar wave equation can be applied in the proof of theorem 3.1.1 to produce robust decay estimates for P and $\underline{P}$. Through a hierarchical structure in the linearised system, that the authors identify, the rest of the linearised system can be estimated through transport equations.

Despite the above theorem 3.1.1, it is still of interest to study alternative methods to approach the problem of linear stability of the Schwarzschild spacetime. Amongst possible others, there are two important reasons that are of relevance in this chapter:

(1) It is useful to have a method to investigate stability of a black hole spacetime which avoids the physical space Chandrasekhar transformation theory and, ideally, the use of the decoupled Teukolsky equations. This is because decoupling of the linearised system on other black hole backgrounds is not always possible, let alone a transformation to an equation which can be treated with the methods available for the wave equation. As illustrated by section 2.10.4 on the Teukolsky equation on n-dimensional Schwarzschild–Tangherlini, decoupling often fails in higher dimensions, *even in highly symmetric spacetimes*.

(2) The proof of [28] requires initial boundedness of (up to) second derivatives of curvature to obtain control of some linearised Ricci coefficients (the linearised shear) on the future event horizon, it is therefore of interest to investigate methods which require less control on initial data to produce such estimates.

One such method was suggested by Holzegel [90] which relies upon a conservation law inherent in the system of gravitational perturbations on Schwarzschild in double null gauge. The conservation law holds on a characteristic rectangle on the exterior of the $4D$ Schwarzschild spacetime as depicted in blue in the following Penrose diagram:

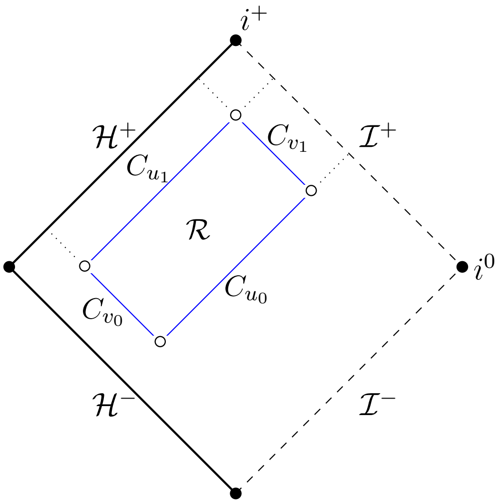


Fig. 3.1 A region $\mathcal{R}$ bounded by a characteristic rectangle on the Schwarzschild exterior.

Here C_u and C_v are the null hypersurfaces given by the level sets of the double null Eddington–Finkelstein coordinates, i.e., $\{u = \mathrm{const.}\}$ and $\{v = \mathrm{const.}\}$ respectively. Define the following 'modified T-canonical energies' (in terminology that will become apparent in the body of the work) on C_u and C_v respectively

$$\overline{\mathcal{E}}^T_u[h](v_0, v_1) \doteq \int_{v_0}^{v_1} \Big[|\Omega\overset{(1)}{\hat{\chi}}|^2 + 2|\Omega\overset{(1)}{\underline{\eta}}|^2 - 2\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) - \frac{1}{2}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})^2 + 4\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\Big]dv\slashed{\epsilon}, \tag{3.1.1}$$

$$\overline{\mathcal{E}}^T_v[h](u_0, u_1) \doteq \int_{u_0}^{u_1} \Big[|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2|\Omega\overset{(1)}{\eta}|^2 - 2\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi}) - \frac{1}{2}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})^2 - 4\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\Big]du\slashed{\epsilon}, \tag{3.1.2}$$

where $\slashed{\epsilon}$ is the volume form on $\mathbb{S}^2_{u,v}$. The work [90] then shows directly from the linearised null structure equations (see section 2.10.1) that one has a conservation law for these fluxes (3.1.1) and (3.1.2). Using this conservation law together with an understanding of the pure gauge solutions Holzegel proves the following statement (stated here roughly):

Theorem 3.1.3 (Holzegel)**.** *Let h be a smooth solution of the linearised vacuum Einstein equation (I.5) in double null gauge. Then the modified canonical energies in equations (3.1.1) and (3.1.2) satisfy the following conservation law:*

$$\overline{\mathcal{E}}^T_{v_0}[h](u_0, u_1) + \overline{\mathcal{E}}^T_{u_0}[h](v_0, v_1) = \overline{\mathcal{E}}^T_{v_1}[h](u_0, u_1) + \overline{\mathcal{E}}^T_{u_1}[h](v_0, v_1). \tag{3.1.3}$$

Further, suppose h arises from suitably normalised asymptotically flat (see section 3.4.3) initial data. Then there exists a constant $K_M > 0$ independent of u such that along any outgoing cone C_u with $u \geq u_0$

$$\int_{v_0}^{\infty} \int_{\mathbb{S}^2_{u,v}} |\Omega \overset{(1)}{\hat{\chi}}|^2(u,v) dv \epsilon\!\!/ \leq K_M \mathrm{E}_{\mathrm{data}} \tag{3.1.4}$$

for some suitable initial data energy $\mathrm{E}_{\mathrm{data}}$. Moreover, the total energy fluxes along the future horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$ are bounded by initial data, i.e.,

$$\int_{\mathcal{H}^+} |\Omega \overset{(1)}{\hat{\chi}}|^2 dv \epsilon\!\!/ + \int_{\mathcal{I}^+} |\overset{(1)}{\underline{\hat{\chi}}}|^2 du \epsilon\!\!/ \leq \mathcal{E}_{\mathrm{data}} \tag{3.1.5}$$

for some suitable initial data energy $\mathcal{E}_{\mathrm{data}}[h]$.

Remark 3.1.4. *The initial data energy $\mathcal{E}_{\mathrm{data}}[h]$ as identified in Holzegel's work is <u>not</u> manifestly positive but the bound (3.1.5) proves positivity of the energy assigned to the initial data a posteriori.*

The boundedness of these fluxes in theorem 3.1.3 can be viewed as a *weak* stability statement. In particular, the interpretation of the estimate (3.1.5) is that the sum of the energy leaving the exterior region through the horizon $\mathcal{H}^+$ and the energy radiated to null infinity $\mathcal{I}^+$ is bounded by initial data.

Remark 3.1.5. *It is evident that a growing mode ansatz for the shear would contradict the uniform bound (3.1.4). What is non-trivial to show (and will be shown in section 3.5.3 of this chapter) is that, by commuting the bound in the estimate (3.1.5) with the T Killing field, one can rule out solutions of the linearised vacuum Einstein equation* (I.5) *of the form*

$$h = e^{-i\omega t} e^{im\varphi} H_{\alpha\beta}(r,\theta) \tag{3.1.6}$$

with $\mathrm{Im}(\omega) \geq 0$ for the metric.

The method of Holzegel [90] addresses both points (1) and (2) above, i.e., it does not rely on the decoupled Teukolsky equations for $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ or the associated Chandrasekhar transformation theory and it only requires initial boundedness of Ricci coefficients to produce such estimates. However, [90] gives no systematic method to derive the conservation law in equation (3.1.3). Incredibly, he spots them by eye in the linearised null structure equations! Hence, it is unclear what their generalisation to other spacetimes is, for example to the Kerr spacetime or Schwarzschild–Tangherlini spacetimes.

Recall from section 1.1.5, Hollands and Wald [65] gave another alternative approach to *general* linear stability problems for stationary, axisymmetric vacuum black holes with their criterion associated to the ‘canonical energy’. One may wonder if the canonical energy can be used to prove a weak stability statement for the 4-dimensional Schwarzschild black hole. As mentioned in section 1.1.5 of chapter 1, this has remained an open problem until now. The present chapter rectifies this issue by showing that, after considering the canonical energy *locally* and exploiting a double null decomposition, the canonical energy conservation law allows one to derive Holzegel's conservation laws and, therefore, one can infer the positivity of the (modified) canonical energy. Due to the generality of the canonical energy method, it is clear how to apply it to other spacetimes (see section 3.1.3 for more discussion). Hence, it becomes clear from the present work how to derive useful conservation laws from the canonical energy in double null gauge.

Remark 3.1.6. *There has been progress towards establishing the positivity of the canonical energy on Schwarzschild by Prabu and Wald [119] who prove that the canonical energy of a metric perturbation of Schwarzschild that is generated by a 'Hertz potential' is positive. They conjecture (but do not prove) that any real, smooth metric perturbation of Schwarzschild can be obtained as the real part of a metric perturbation generated by a smooth Hertz potential. Initial data giving rise to a Hertz potential is unconstrained and thus gets around point (i) of Hollands and Wald's admissibility criterion. Moreover, they relate the energy quantity associated to the Regge–Wheeler equation arising in the linear stability proof of Dafermos, Holzegel and Rodnianski [28] to the canonical energy of an associated metric perturbation generated by a Hertz potential. This conservation law occurs at the level of (up to) three derivatives of curvature. In contrast, Holzegel's weak stability result relies upon a conservation law that occurs at a much lower regularity. In particular, the conservation law in question is at the level of Ricci coefficients.*

3.1.2 Overview and Main Results: Theorems 3.1.7-3.1.13

This section contains a brief overview and outline of the main results of the chapter. The main body of the chapter starts with section 3.2 which discusses the canonical energy in detail. However, the exposition in this chapter will be slightly different to the original work of Hollands and Wald [65]; rather than viewing the canonical energy as a constrained variational principle evaluated on Cauchy hypersurfaces, the canonical energy is simply viewed as a quantity for the linearised metric h arising from a 'vector field current' which can be evaluated *locally*. In particular, for a vector field X the X-canonical energy will be associated to a current $\mathcal{J}[h]^X$ which is divergence free if X is Killing and h satisfies the linearised vacuum Einstein equation (I.5). The X-canonical energy on some hypersurface Σ with unit normal n_Σ will then be

$$\mathcal{E}^X_\Sigma[h] = \int_\Sigma n_\Sigma(\mathcal{J}[h]^X)\mathrm{dvol}_\Sigma. \tag{3.1.7}$$

Now, since $\mathcal{J}[h]^X$ if divergence free is X is Killing, one can construct conservation laws for $\mathcal{E}^X[h]$ associated to the boundary of some spacetime region and the Killing symmetries of the spacetime.

Section 3.3 contains the main body of the author's original work. By evaluating the canonical energy associated to the Killing field $T \doteq \partial_t$ for the Schwarzschild spacetime on the characteristic rectangle depicted in figure 3.1 and choosing the metric perturbation h to be in double null gauge (see definition 2.10.1), a double null decomposition of the canonical energy is achieved. Since T is a Killing vector field one obtains a conservation law for the T-canonical energy

$$\mathcal{E}^T_{u_1}[h](v_0,v_1) + \mathcal{E}^T_{v_1}[h](u_0,u_1) = \mathcal{E}^T_{u_0}[h](v_0,v_1) + \mathcal{E}^T_{v_0}[h](u_0,u_1), \tag{3.1.8}$$

where, to ease notation, one denotes

$$\mathcal{E}^T_u[h](v_0,v_1) \doteq \mathcal{E}^T_{C_u\cap\{v_0\le v\le v_1\}}[h], \quad \mathcal{E}^T_v[h](u_0,u_1) \doteq \mathcal{E}^T_{C_v\cap\{u_0\le u\le u_1\}}[h]. \tag{3.1.9}$$

A natural question to ask is: is this conservation law for the T-canonical energy related to Holzegel's conservation law of equation (3.1.3)? One of the main results of section 3.3 is to prove the following precise relation between these conservation laws.

Theorem 3.1.7. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior. Then the T-canonical energy of h on the null cones $C_u\cap\{v_0\le v\le v_1\}$ and $C_v\cap\{u_0\le u\le u_1\}$ is given by*

$$\mathcal{E}^T_u[h](v_0,v_1) = 2\overline{\mathcal{E}}^T_u[h](u_0,u_1) - 2\int_{\mathbb{S}^2_{u,v}} \mathcal{A}[h](u,v,\theta,\varphi)\not{\epsilon}\Big|_{v_0}^{v_1}, \tag{3.1.10}$$

$$\mathcal{E}^T_v[h](u_0,u_1) = 2\overline{\mathcal{E}}^T_v[h](v_0,v_1) + 2\int_{\mathbb{S}^2_{u,v}} \mathcal{A}[h](u,v,\theta,\varphi)\not{\epsilon}\Big|_{u_0}^{u_1}. \tag{3.1.11}$$

where $\overline{\mathcal{E}}^T_u[h]$ and $\overline{\mathcal{E}}^T_v[h]$ are defined in equations (3.1.1) and (3.1.2) and with

$$\begin{aligned}\mathcal{A}[h] \doteq& \frac{1}{4}(\overset{(1)}{\underline{\omega}} - \overset{(1)}{\omega})\mathrm{Tr}_{\not{g}}\not{h} - \frac{1}{4}(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})\overset{(1)}{(b)} + \frac{1}{8}\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} - \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\Big]\mathrm{Tr}_{\not{g}}\not{h} - \frac{\Omega}{4}\langle\overset{(1)}{\underline{\hat{\chi}}} - \overset{(1)}{\hat{\chi}}, \hat{\not{h}}\rangle \\ &+ \frac{3}{2}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\Big] + \frac{1}{2}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(\mathrm{Tr}_{\not{g}}\not{h} - 4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big).\end{aligned} \tag{3.1.12}$$

Moreover, the modified T-canonical energies satisfy

$$\overline{\mathcal{E}}^T_{u_1}[h](v_0,v_1) + \overline{\mathcal{E}}^T_{v_1}[h](u_0,u_1) = \overline{\mathcal{E}}^T_{u_0}[h](v_0,v_1) + \overline{\mathcal{E}}^T_{v_0}[h](u_0,u_1). \tag{3.1.13}$$

Remark 3.1.8. *The last part of this theorem (equation (3.1.13)) follows the from the equations (3.1.10) and (3.1.11) in conjunction with the conservation law for the canonical energy (3.1.8). Therefore, one can view this theorem as a proof that the canonical energy conservation law implies Holzegel's conservation law in equation (3.1.3).*

The proof of theorem 3.1.7 (which can be found in section 3.3.3) relies upon using the linearised null structure equations in propositions 2.10.7-2.10.17 (but not the linearised Bianchi equations in proposition 2.10.20) to integrate by parts and produce the boundary term $\mathcal{A}$. Moreover, in proving this result the *linearised Gauss and Codazzi constraint equations are key* (see proposition 2.10.15 and see proposition 2.10.17). This directly relates to point (i) of Hollands and Wald's admissible data criterion (see section 1.1.5). However, one advantage of using the double null decomposition is that one can readily use these linearised constraint equations.

Section 3.3.4 constructs a 'higher order' T-canonical energy conservation law associated to $\mathcal{L}_{\Omega_i} h$[a] where $\{\Omega_i\}_{i=1}^3$ are the Killing vector fields associated to the $SO(3)$ symmetry of the Schwarzschild spacetime (see equations (3.3.11)-(3.3.13) for explicit expressions for $\{\Omega_i\}_{i=1}^3$). For convenience, define the following 'modified higher order T-canonical energies':

$$\overline{\not{\mathcal{E}}}^T_u[h](v_0,v_1) \doteq \int_{v_0}^{v_1} \Big(\frac{\Omega^2 r^2}{2}|\overset{(1)}{\beta}|^2 + \frac{3\Omega^2 r^2\rho}{2}|\overset{(1)}{\underline{\eta}}|^2 + \frac{\Omega^2 r^2}{2}(|\overset{(1)}{\sigma}|^2 + |\overset{(1)}{\rho}|^2) - \frac{3r^2\rho}{2}\overset{(1)}{\omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\underline{\chi})} \tag{3.1.14}$$
$$- \frac{3r^2\rho}{2}\Big[\frac{1}{2}(\Omega\mathrm{Tr}_{\not{g}}\chi) - 2\omega\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\Big)\, dv\not{\epsilon},$$

$$\overline{\not{\mathcal{E}}}^T_v[h](u_0,u_1) \doteq \int_{u_0}^{u_1} \Big(\frac{\Omega^2 r^2}{2}|\overset{(1)}{\underline{\beta}}|^2 + \frac{3\Omega^2 r^2\rho}{2}|\overset{(1)}{\eta}|^2 + \frac{\Omega^2 r^2}{2}(|\overset{(1)}{\sigma}|^2 + |\overset{(1)}{\rho}|^2) - \frac{3r^2\rho}{2}\overset{(1)}{\underline{\omega}}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)} \tag{3.1.15}$$
$$+ \frac{3r^2\rho}{2}\Big[\frac{1}{2}(\Omega\mathrm{Tr}_{\not{g}}\chi) - 2\omega\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\Big)\, du\not{\epsilon},$$

With these definitions in hand, the following theorem is proved in section 3.3.

Theorem 3.1.9. *Suppose h is a smooth solution of the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior. The T-canonical energy of $\mathcal{L}_{\Omega_k} h$ satisfies*

$$\sum_k \mathcal{E}^T_u[\mathcal{L}_{\Omega_k}h](v_0,v_1) = 8\overline{\not{\mathcal{E}}}^T_u[h](v_0,v_1) + 4\overline{\mathcal{E}}^T_u[h](v_0,v_1) - 2\int_{\mathbb{S}^2_{u,v}} \mathcal{B}(u,v,\theta,\varphi)\not{\epsilon}\Big|_{v_0}^{v_1}, \tag{3.1.16}$$
$$\sum_k \mathcal{E}^T_v[\mathcal{L}_{\Omega_k}h](u_0,u_1) = 8\overline{\not{\mathcal{E}}}^T_v[h](u_0,u_1) + 4\overline{\mathcal{E}}^T_v[h](u_0,u_1) + 2\int_{\mathbb{S}^2_{u,v}} \mathcal{B}(u,v,\theta,\varphi)\not{\epsilon}\Big|_{u_0}^{u_1}. \tag{3.1.17}$$

with

$$\mathcal{B}[h] \doteq r^2\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\,\not{\mathrm{div}}\overset{(1)}{\eta} - \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\,\not{\mathrm{div}}\overset{(1)}{\underline{\eta}} - \overset{(1)}{\rho}\big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\big) + 2\Omega\mathrm{Tr}_{\not{g}}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho} \tag{3.1.18}$$
$$+ (\Omega\mathrm{Tr}_{\not{g}}\chi)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle + \Big[\frac{\omega}{\Omega^2} - \frac{\mathrm{Tr}_{\not{g}}\chi}{2\Omega}\Big]\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\Big) + \sum_k \mathcal{A}[\mathcal{L}_{\Omega_k}h],$$

[a]Note that, by proposition C.1.4 in appendix C.1, if k is a Killing vector field and h solves the linearised vacuum Einstein equation (I.5) then $\mathcal{L}_k h$ also solves the linearised vacuum Einstein equation (I.5).

where $\mathcal{A}[\mathcal{L}_{\Omega_k}h]$ results from expressing $\mathcal{A}[h]$ in equation (3.1.12) in terms of h and replacing it with $\mathcal{L}_{\Omega_k}h$. Moreover, the modified higher order T-canonical energies satisfy

$$\overline{\not{\mathcal{E}}}^{T}_{u_1}[h](v_0,v_1)+\overline{\not{\mathcal{E}}}^{T}_{v_1}[h](u_0,u_1)=\overline{\not{\mathcal{E}}}^{T}_{u_0}[h](v_0,v_1)+\overline{\not{\mathcal{E}}}^{T}_{v_0}[h](u_0,u_1). \tag{3.1.19}$$

The reader should note that Holzegel also identified the conservation law (3.1.19) in [90] by eye (see equations (82) and (83) and proposition 8.1 in [90]).

Section 3.3.5 constructs an additional 'higher order' T-canonical energy conservation law associated to $\mathcal{L}_T h$. For convenience define the following 'modified higher order T-canonical energies':

$$\begin{aligned}\dot{\overline{\mathcal{E}}}^{T}_{u}[h](v_0,v_1)\doteq\int_{v_0}^{v_1}\Big(&\frac{\Omega^4}{4}|\overset{(1)}{\alpha}|^2+\frac{3}{2}\Omega^4\big(|\overset{(1)}{\rho}|^2+|\overset{(1)}{\sigma}|^2+|\overset{(1)}{\beta}|^2\big)+\frac{\Omega^4}{2}|\overset{(1)}{\underline{\beta}}|^2+f_2|\overset{(1)}{\hat{\underline{\chi}}}|^2+f_1|\overset{(1)}{\hat{\chi}}|^2\\&+f_3|\overset{(1)}{\underline{\eta}}|^2-\frac{1}{\Omega^2}f_3\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})+\frac{2}{\Omega^2}\Big(\omega f_3+2\Omega\mathrm{Tr}_{\not{g}}\chi f_2\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)\\&-\frac{f_1}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)^2-\frac{f_2}{2\Omega^2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})+2(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big]^2\Big)dv\not{\epsilon},\end{aligned}\tag{3.1.20}$$

$$\begin{aligned}\dot{\overline{\mathcal{E}}}^{T}_{v}[h](u_0,u_1)\doteq\int_{u_0}^{u_1}\Big(&\frac{\Omega^4}{4}|\overset{(1)}{\underline{\alpha}}|^2+\frac{3}{2}\Omega^4\big(|\overset{(1)}{\rho}|^2+|\overset{(1)}{\sigma}|^2+|\overset{(1)}{\underline{\beta}}|^2\big)+\frac{\Omega^4}{2}|\overset{(1)}{\beta}|^2+f_1|\overset{(1)}{\hat{\underline{\chi}}}|^2+f_2|\overset{(1)}{\hat{\chi}}|^2\\&+f_3|\overset{(1)}{\eta}|^2-\frac{f_3}{\Omega^2}\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)-\frac{2}{\Omega^2}\Big(\omega f_3+2\Omega\mathrm{Tr}_{\not{g}}\chi f_2\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})\\&-\frac{f_1}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})^2-\frac{f_2}{2\Omega^2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)-2(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big]^2\Big)du\not{\epsilon},\end{aligned}\tag{3.1.21}$$

with

$$f_1\doteq-\Omega^2\Big(\omega^2+\frac{5}{4}\Omega^2\rho\Big),\quad f_2\doteq-\frac{3}{4}\Omega^4\rho,\quad f_3\doteq2\Omega^2(\Omega^2\rho-\omega^2). \tag{3.1.22}$$

Additionally, the following theorem is proved in section 3.3.5:

Theorem 3.1.10. *Suppose h is a smooth solution of the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior. Then the modified higher order T-canonical energies satisfy*

$$\dot{\overline{\mathcal{E}}}^{T}_{u_1}[h](v_0,v_1)+\dot{\overline{\mathcal{E}}}^{T}_{v_1}[h](u_0,u_1)=\dot{\overline{\mathcal{E}}}^{T}_{u_0}[h](v_0,v_1)+\dot{\overline{\mathcal{E}}}^{T}_{v_0}[h](u_0,u_1). \tag{3.1.23}$$

Remark 3.1.11. *To the best of the authors knowledge, no such local conservation law for $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ has been derived.*

This chapter then concludes with a discussion of the energy boundedness statements that can be derived from these conservation laws as well as a proof of mode stability. For this purpose, the restrictions on initial data, along with the gauge freedom are discussed in section 3.4. For completeness section 3.5.2 reproves the boundedness statement in [90] which was stated roughly in theorem 3.1.3. Additionally, two other energy boundedness statements are proved. These can be stated roughly as the following:

Theorem 3.1.12. *Let h be a smooth solution of the linearised vacuum Einstein equation (I.5) in double null gauge which is suitably normalised and asymptotically flat. Then there exists a constant $K_M > 0$ independent of u such that along any outgoing cone C_u with $u \geq u_0$*

$$\int_{v_0}^{\infty} \int_{\mathbb{S}^2_{u,v}} |\Omega \overset{(1)}{\beta}|^2(u,v)\, dv\not{\epsilon} \leq K_M \not{\mathbb{E}}_{\text{data}} \tag{3.1.24}$$

for some suitable initial data energy $\not{\mathbb{E}}_{\text{data}}$ (independent of u). Moreover, the following fluxes along the future horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$ are bounded by initial data:

$$\int_{\mathcal{H}^+} |\Omega \overset{(1)}{\beta}|^2 dv\not{\epsilon} + \int_{\mathcal{I}^+} |\overset{(1)}{\underline{\beta}}|^2 du\not{\epsilon} \leq \not{\mathbb{C}}_{\text{data}} \tag{3.1.25}$$

for some suitable initial data energy $\not{\mathbb{C}}_{\text{data}}$.

With the success of the conservation laws in theorems 3.1.7 and 3.1.9 producing L^2-boundedness statements for the shears $(\overset{(1)}{\hat{\chi}}, \overset{(1)}{\underline{\hat{\chi}}})$ and $(\overset{(1)}{\beta}, \overset{(1)}{\underline{\beta}})$, the reader may be wondering about if one can use the local conservation law (3.1.23) in theorem 3.1.10 for $(\overset{(1)}{\alpha}, \overset{(1)}{\underline{\alpha}})$ to produce the analogue of theorems 3.1.3 and 3.1.12? In particular, can the conservation law arising in theorem 3.1.10 produce a boundedness statement for $|\overset{(1)}{\alpha}|^2$ on any outgoing cone C_u and $|\overset{(1)}{\underline{\alpha}}|^2$ at null infinity $\mathcal{I}^+$. However, at the time of writing, any attempt to produce such an estimate has failed. However, one does have a commuted estimate arising from theorem 3.1.3:

Theorem 3.1.13. *Let h be a smooth solution of the linearised vacuum Einstein equation (I.5) in double null gauge which is suitably normalised and asymptotically flat. Then the following fluxes along the future horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$ are bounded by initial data:*

$$\int_{\mathcal{H}^+} |\Omega^2 \overset{(1)}{\alpha}|^2 dv\not{\epsilon} + \int_{\mathcal{I}^+} |\overset{(1)}{\underline{\alpha}}|^2 du\not{\epsilon} \leq \dot{\mathcal{E}}_{\text{data}} \tag{3.1.26}$$

for some suitable initial data energy $\dot{\mathcal{E}}_{\text{data}}$.

As mentioned in remark 3.1.5, section 3.5.3 will establish mode stability for solutions h to the linearised vacuum Einstein equation (I.5) on the Schwarzschild black hole exterior. It is, in fact, this last energy boundedness theorem for $\Omega^2 \overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ in conjunction with some asymptotic ODE analysis for the Teukolsky equation (2.10.150) that allows one to prove the following mode stability statement:

Corollary 3.1.14. *Let h be a smooth solution to the linearised vacuum Einstein equation (I.5) on the Schwarzschild black hole exterior of the form,*

$$h_{\alpha\beta} = e^{-i\omega t} e^{im\varphi} H_{\alpha\beta}(r, \theta), \tag{3.1.27}$$

with $\mathrm{Im}(\omega) \geq 0$ and which obeys suitable boundary conditions at the future event horizon and future null infinity (in particular, boundary conditions consistent with finite energy on a suitable hypersurface). Then it is the sum of a pure gauge and linearised Kerr solution.

3.1.3 Outlook

In this subsection, a few ideas for extensions to the work presented here are outlined.

The Extension to Kerr

It is conjectured that the subextremal Kerr black hole spacetime is asymptotically stable as a solution to the vacuum Einstein equation (see section IV of the introduction of [35] for a precise formulation of this conjecture). In view of the works [29] and [33] on quantitative boundedness and decay for the Teukolsky equation on subextremal Kerr, the full linear stability of the subextremal Kerr spacetime is within reach in analogy with [28]. Alternatively, one could follow the weak stability path in analogy with the work in this chapter and attempt a energy boundedness statement for the subextremal Kerr spacetime as shall be elaborated on in this section.

As was discussed in section 2.9 of chapter 2, the exterior of the Kerr spacetime can be covered globally by double null coordinates. These coordinates were originally introduced in [108]. Naturally, one can consider the T-canonical energy on the exterior of Kerr to give a local conservation law on a characteristic rectangle. Due to the reduced symmetry, most of the background quantities are non-zero (and only implicitly defined). This is in stark contrast to the Schwarzschild case where all but a few Ricci coefficients and curvature components are zero for the normalised null frame associated with the double null coordinate chart. Therefore, the expressions resulting for the T-canonical energy on the exterior of Kerr are far more complicated but, of course, can be derived. The reader should note that here is where the canonical energy has a significant advantage over the direct method of Holzegel [90]; since one starts with a conservation law that can be manipulated rather than having to spot the conservation law in the relevant linearised null structure equations.

Once the conservation law on the exterior of Kerr is derived, it is then very probable that the method of obtaining 'weak' stability results (similar to theorem 3.1.3) presented in this chapter would naturally extend to the Kerr spacetime albeit with some caveats and an increase in technical complexity. Indeed, the main caveat is that one cannot expect to exploit the limits and the residual pure gauge freedom to produce a positive quantity at the horizon for *arbitrary linear perturbations* to the Kerr spacetime due to the issue of superradiance. The expectation in the Kerr case is that one could prove a restricted weak stability result for *axisymmetric perturbations*. This is the topic of work in progress of the author with Holzegel. One should also note the work of Moncrief and Gudapati [120] in this direction.

Remark 3.1.15. *Instead of choosing a double null foliation of the Kerr exterior, one could choose the 'algebraically special null frame' of Kerr to compute the local conservation law (see section 2.9). The algebraically special frame is advantageous since more of the Ricci coefficients and curvature components vanish identically and one has explicit expressions for them in terms of* (r, θ). *The complexity of the problem is pushed into the null structure equations since the space* $\mathrm{span}(e_3, e_4)^{\perp}$ *is not integrable (see, for example, [114]). This complexity manifests itself as additional equations for the quantities such as the twist: the anti-symmetric part of the null second fundamental form* χ. *Additionally the non-integrability of the frame means that the ability to integrate by parts is obscure and ultimately results in one having to understand the relation between the double null foliation and the algebraically special frame. There are none of these issues for the Schwarzschild case since the algebraically special frame coincides with the normalised null frame of the double null foliation.*

The Extension to Reissner–Nordström

The canonical energy arises naturally from the Einstein–Hilbert action for the Einstein vacuum equation by considering antisymmetrised variations of the action. The notion of canonical energy extends naturally to many theories with a Lagrangian formulation (see Keir [96]). In particular, the Einstein–Maxwell system is a natural candidate for investigating the canonical energy outside vacuum.

A natural starting place for investigating the canonical energy for Einstein–Maxwell would be the Reissner–Nordström spacetime [121, 122]. Due to its spherical symmetry, the computations should be reasonably tractable and allow one to investigate how the canonical energy behaves when gravitational and electromagnetic perturbations are coupled. One should note that the full linear stability in analogy with [28] has been established in [123].

3.2 Canonical Energy

When trying to prove boundedness and decay for hyperbolic equations on black hole backgrounds energy estimates have proved invaluable [19–28, 124]. In many cases, one can view such estimates as arising from applications of the divergence theorem to energy currents. If one has an energy–momentum tensor, $\mathbb{T}$, associated to a theory, it provides a natural way to construct such energy currents. In particular, if one has an energy–momentum tensor one can define an X-energy current associated to a vector field X by

$$J_a^X \doteq \mathbb{T}_{ab}X^b \tag{3.2.1}$$

and, from this current, an 'X-energy' on a hypersurface Σ

$$E_\Sigma^X \doteq \int_\Sigma n_\Sigma(J^X)\mathrm{dvol}_\Sigma, \tag{3.2.2}$$

where n_Σ is the (future-directed) unit normal to Σ and dvol_Σ is the induced volume form associated to Σ. One can readily show that

$$\mathrm{div}(J^X) = \mathbb{T}^{ab}\Pi^X_{ab}, \tag{3.2.3}$$

where $\Pi^X \doteq \frac{1}{2}\mathcal{L}_X g$ is called the deformation tensor. Applying the divergence theorem on a region $\mathcal{R}$ bounded by two homologous hypersurfaces, Σ_1 and Σ_2, gives

$$E^X_{\Sigma_1} = \int_{\mathcal{R}} \mathbb{T}^{ab}\Pi^X_{ab}\mathrm{dvol} + E^X_{\Sigma_2}, \tag{3.2.4}$$

where n is the future-directed unit normal to the relevant hypersurface. Note that

$$\mathrm{div}(J^X) = 0, \tag{3.2.5}$$

when X is Killing. Therefore, applying the divergence theorem gives a conservation law.

Famous examples of theories with energy-momentum tensors are the scalar wave equation and Maxwell's equations. Sadly, in the case of the linearised vacuum Einstein equation (I.5), no such energy momentum tensor exists.[b] Nevertheless, there is an alternative way to construct currents associated to some vector field X based on symplectic structure of the space of solutions to the linearised vacuum Einstein equation (I.5). When one has a stationary spacetime and uses the Killing field associated to stationarity T, this method of constructing currents gives rise to the current which generates the 'canonical energy' of Hollands and Wald [65]. In this case, there

[b]For linearised theory around the Minkowski spacetime there are various notions of energy–momentum pseudo-tensors (see for example sections 20.3 and 20.4 in [125] or sections 6.3 and 7.5 in [92] for more discussion on this topic).

is no bulk term and one obtains a conservation law.

In this section a pedestrian approach to the canonical energy is presented. In sections 3.2.1 and 3.2.2, the idea of a canonical energy based on symplectic structure is motivated using the wave equation and Maxwell's equations. The canonical energy for the linearised vacuum Einstein equation (I.5) is defined in section 3.2.3. Note that the exposition in this section gives a different viewpoint on the canonical energy from that of the original work of Hollands and Wald. In particular, the canonical energy is simply viewed as a quantity for the linearised metric h arising from a 'vector field current' which can be evaluated *locally*. This section does not discuss the nice connection of the canonical energy to black hole thermodynamics or the stability criterion associated to the canonical energy when evaluated on Cauchy hypersurfaces. The interested reader should consult [65] and the introductory section 3.1.1 of chapter 1.

3.2.1 Canonical Energy for the Wave Equation

A classic toy model for the Einstein equation (in harmonic coordinates) is the the wave equation

$$\Box_g \Psi = 0, \tag{3.2.6}$$

for $\Psi \in C^\infty(M)$ on a spacetime (M, g). The energy-momentum tensor for this theory is

$$\mathbb{T}[\Psi]_{ab} = \nabla_a \Psi \nabla_b \Psi - \frac{1}{2} g_{ab} |\nabla \Psi|_g^2. \tag{3.2.7}$$

For a vector field $X \in \mathfrak{X}(M)$ one has the associated current

$$(J^X[\Psi])_a = \nabla_X \Psi \nabla_a \Psi - \frac{1}{2} X_a |\nabla \Psi|_g^2. \tag{3.2.8}$$

However, one can also define a current based on symplectic structure:

Definition 3.2.1 (Symplectic Current Associated to the Wave Equation)**.** *Let $\Phi, \Psi \in C^\infty(M)$. Then the symplectic current associated to Φ and Ψ is defined as*

$$\mathfrak{w}[\Phi, \Psi]^a \doteq g^{ab}(\Phi \nabla_b \Psi - \Psi \nabla_b \Phi). \tag{3.2.9}$$

Proposition 3.2.1. *Suppose $\Phi, \Psi \in C^\infty(M)$ satisfy the wave equation (3.2.6) then $\mathfrak{w}[\Phi, \Psi]$ is divergence free.*

Proof. The result of this proposition follows from a direct computation of $\mathrm{div}(\mathfrak{w}[\Phi, \Psi])$ and substitution of the wave equation (3.2.6). □

This leads to a definition of a 'canonical energy' for the wave equation:

Definition 3.2.2 (Canonical Energy for the Wave Equation). *Let $\Psi \in C^\infty(M)$ satisfy the wave equation (3.2.6). Further, let X be a vector field and Σ be a hypersurface in (M, g) with (unit) normal n. One defines the X-canonical energy current for the wave equation as*

$$\mathcal{J}^X[\Psi] \doteq \mathfrak{w}[\Psi, \mathcal{L}_X \Psi]. \tag{3.2.10}$$

The X-canonical energy for the wave equation on Σ is defined as

$$\mathcal{E}^X_\Sigma[\Psi] \doteq \int_\Sigma n(\mathcal{J}^X[\Psi]) \mathrm{dvol}_\Sigma. \tag{3.2.11}$$

Proposition 3.2.2. *Suppose $\Psi \in C^\infty(M)$ satisfies the wave equation (3.2.6) and X is a vector field. Then,*

$$\mathrm{div}(\mathfrak{w}[\Psi, \mathcal{L}_X\Psi]) = -\Psi g^{ab} \mathrm{K}^X_{abc} \nabla^c \Psi - 2\Psi \Pi^X_{ab} \nabla^a \nabla^b \Psi \tag{3.2.12}$$

where

$$\mathrm{K}^X_{abc} \doteq \nabla_a \Pi^X_{bc} + \nabla_b \Pi^X_{ac} - \nabla_c \Pi^X_{ab}, \qquad \Pi^X_{ab} \doteq \frac{1}{2}(\mathcal{L}_X g)_{ab} \tag{3.2.13}$$

Proof. One can calculate directly that

$$\mathrm{div}(\mathfrak{w}[\Psi, \mathcal{L}_X\Psi]) = -\Psi \Box_g (\mathcal{L}_X \Psi). \tag{3.2.14}$$

Using proposition C.1.2 in appendix C.1 one sees $\nabla_a \mathcal{L}_X \Psi = \mathcal{L}_X(\nabla\Psi)_a$ and therefore,

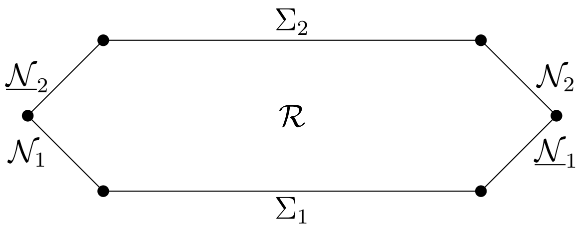
$$\nabla_a \nabla_b \mathcal{L}_X \Psi = \mathcal{L}_X(\nabla\nabla\Psi)_{ab} + \mathrm{K}^X_{abc}\nabla^c\Psi. \tag{3.2.15}$$

One has that $(\mathcal{L}_X g^{-1})_{ab} = -2\Pi^X_{ab}$ and, therefore, using that Ψ satisfies the wave equation (3.2.6), one has

$$\Box_g \mathcal{L}_X \Psi = g^{ab}\mathrm{K}^X_{abc}\nabla^c\Psi + 2\Pi^X_{ab}\nabla^a\nabla^b\Psi. \tag{3.2.16}$$

□

Suppose one applies the divergence theorem to $\mathfrak{w}[\Psi, \mathcal{L}_X\Psi]$ in some region of a vacuum spacetime depicted in the following diagram:



If X is *not* a Killing vector field, then the right-hand side of equation (3.2.12) appears as the bulk term. If one wanted to produce a Morawetz-type [124] spacetime estimate such a bulk term seems undesirable since the right-hand side of equation (3.2.12) is not a quadratic expression in derivatives of Ψ. On the other hand, if X is Killing then $\mathcal{L}_X\Psi$ is also a solution to the wave equation (3.2.6) and therefore the bulk vanishes to yield the conservation law

$$\mathcal{E}^X_{\Sigma_1}[\Psi] + \mathcal{E}^X_{\mathcal{N}_1}[\Psi] + \mathcal{E}^X_{\underline{\mathcal{N}}_1}[\Psi] = \mathcal{E}^X_{\Sigma_2}[\Psi] + \mathcal{E}^X_{\mathcal{N}_2}[\Psi] + \mathcal{E}^X_{\underline{\mathcal{N}}_2}[\Psi], \tag{3.2.17}$$

associated to the boundary of the region $\mathcal{R}$ depicted above.

For a general vector field X, it seems reasonable to expect that the X-canonical energy for a spacetime is related to the standard X-energy constructed from the energy momentum tensor. The following proposition confirms this expectation.

Proposition 3.2.3. *Suppose X is a Killing field for a vacuum spacetime (M,g) and $\Psi \in C^\infty(M)$ solves the wave equation (3.2.6). Then the X-canonical energy current satisfies*

$$(\mathcal{J}^X[\Psi])_a = 2(J^X[\Psi])_a + (j^X[\Psi])_a \tag{3.2.18}$$

where $(J^X[\Psi])_a$ is defined in equation (3.2.8) as the standard X-current associated to the energy-momentum tensor and

$$(j^X[\Psi])_b \doteq \nabla^a A_{ab}, \qquad A_{ab} \doteq X_{[a}\nabla_{b]}\Psi^2, \tag{3.2.19}$$

i.e., $\mathcal{J}^X[\Psi]$ and $J^X[\Psi]$ are related by a divergence. Moreover, $j^X[\Psi]$ is divergence free.

Proof. By the identity in proposition C.1.1 in appendix C.1 one has

$$(\mathcal{J}^X[\Psi])_a = \mathcal{L}_X\Psi\nabla_a\Psi - \Psi\mathcal{L}_X(\nabla\Psi)_a \tag{3.2.20}$$

which can be expressed as

$$(\mathcal{J}^X[\Psi])_a = 2\mathcal{L}_X\Psi\nabla_a\Psi - (\mathcal{L}_X\omega)_a \tag{3.2.21}$$

for $\omega_a \doteq \Psi\nabla_a\Psi$. Now, by the Killing property of X,

$$(\mathcal{L}_X\omega)_a = (\mathrm{div}Q)_a + \omega_b\nabla_a X^b \tag{3.2.22}$$

$$= (\nabla^b Q)_{ba} + (\mathrm{div}\omega)X_a - (\nabla^b Q)_{ab} \tag{3.2.23}$$

for $Q_{ab} \doteq \omega_b X_a$. By the wave equation (3.2.6), $\mathrm{div}\omega = |\nabla\Psi|^2_g$ and hence the result. □

Remark 3.2.4. *The fact that j^X is a total divergence will be important in proposition 3.3.1 below which states the analogous result of theorem 3.1.7 for the wave equation (3.2.6) on the Schwarzschild black hole.*

Remark 3.2.5. *As will be discussed in chapter 4 (see proposition 4.3.1), the above proposition has an interesting generalisation to the linearised vacuum Einstein equation (I.5) despite the fact that there is no energy-momentum tensor for the theory.*

3.2.2 Canonical Energy for Maxwell's Equations

Another well studied example of a field theory is Maxwell's equations in the absence of sources for a vector potential $A \in \Omega^1(M)$ on a spacetime background. Maxwell's equations can be written neatly as

$$\star(d\star F) = 0, \qquad dF = 0, \tag{3.2.24}$$

where $F \doteq dA$ is the Maxwell tensor. One has the following energy momentum tensor for the potential $A \in \Omega^1(M)$

$$\mathbb{T}[A]_{ab} \doteq F_{ac}F_b{}^c - \frac{1}{4}g_{ab}|F|_g^2. \tag{3.2.25}$$

For identifying the symplectic current associated to Maxwell's equations it turns out to be useful to rewrite them in terms of the one form A_a as

$$P^a{}_b{}^{cd}\nabla_a\nabla_c A_d = 0, \tag{3.2.26}$$

with

$$P^{abcd} \doteq g^{ac}g^{bd} - g^{ad}g^{bc}. \tag{3.2.27}$$

Definition 3.2.3 (Symplectic Current Associated to the Maxwell's Equations)**.** *Let $A_1, A_2 \in \Omega^1(M)$. Then the symplectic current associated to A_1 and A_2 is defined as*

$$\mathfrak{w}[A_1, A_2]^a \doteq P^{abcd}((A_1)_b\nabla_c(A_2)_d - (A_2)_b\nabla_c(A_1)_d), \tag{3.2.28}$$

with P^{abcd} defined in equation (3.2.27).

Proposition 3.2.6. *Suppose $A_1, A_2 \in \Omega^1(M)$ satisfy the Maxwell's equations (3.2.26) then $\mathfrak{w}[A_1, A_2]$ is divergence free.*

Proof. This result follows from a direct computation of $\mathrm{div}(\mathfrak{w}[A_1, A_2])$ with the Maxwell equation (3.2.26). □

Once again this leads to the definition of 'canonical energy' for Maxwell's equations:

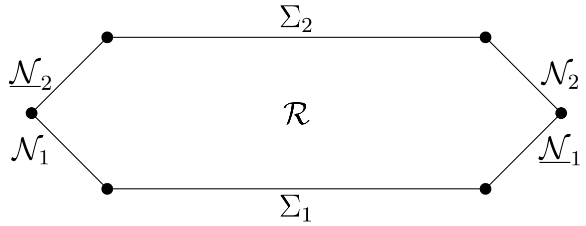
Definition 3.2.4 (Canonical Energy for the Maxwell's Equations)**.** *Suppose $A \in \Omega^1(M)$ satisfies Maxwell's equations (3.2.26). Further, let X be a vector field and Σ be a hypersurface in (M, g) with (unit) normal n. One defines the X-canonical energy current for Maxwell's equations as*

$$\mathcal{J}^X[A] \doteq \mathfrak{w}[A, \mathcal{L}_X A]. \tag{3.2.29}$$

The X-canonical energy for Maxwell's equations on Σ is defined as

$$\mathcal{E}^X_\Sigma[\Psi] \doteq \int_\Sigma n(\mathcal{J}^X[A])\mathrm{dvol}_\Sigma. \tag{3.2.30}$$

One again, one can apply the divergence theorem to $\mathcal{J}^X[A]$ in some region of a vacuum spacetime depicted in the following diagram:



As with solutions to the the wave equation (3.2.6), if X is Killing then $\mathcal{L}_X A$ is also a solution to the Maxwell's equation (3.2.26) and therefore, by proposition 3.2.6, one obtains a conservation law

$$\mathcal{E}^X_{\Sigma_1}[A] + \mathcal{E}^X_{\mathcal{N}_1}[A] + \mathcal{E}^X_{\underline{\mathcal{N}}_1}[A] = \mathcal{E}^X_{\Sigma_2}[A] + \mathcal{E}^X_{\mathcal{N}_2}[A] + \mathcal{E}^X_{\underline{\mathcal{N}}_2}[A], \tag{3.2.31}$$

associated to the boundary of the region $\mathcal{R}$ depicted above.

3.2.3 Canonical Energy for the Linearised Einstein Vacuum Equation

It turns out to be convenient to rewrite the linearised vacuum Einstein equation (I.5) (plus its trace) in the following form:

Proposition 3.2.7. *Suppose (M, g) satisfies the vacuum Einstein equation (I.2) and h satisfies the linearised vacuum Einstein equation (I.5). Then*

$$P^{a}{}_{(bc)}{}^{def}\nabla_a\nabla_d h_{ef} = 0, \tag{3.2.32}$$

where P is defined as

$$P^{abcdef} \doteq g^{ae}g^{bf}g^{cd} - \frac{1}{2}g^{ad}g^{be}g^{cf} - \frac{1}{2}g^{ab}g^{ef}g^{cd} - \frac{1}{2}g^{ae}g^{df}g^{bc} + \frac{1}{2}g^{ad}g^{ef}g^{bc}. \tag{3.2.33}$$

Proof. Computing directly, using equation (3.2.33), gives

$$P^{a}{}_{(bc)}{}^{def}\nabla_a\nabla_d h_{ef} = \nabla^a\nabla_{(c}h_{b)a} - \frac{1}{2}\Box_g h_{bc} - \frac{1}{2}\nabla_b\nabla_c \mathrm{Tr}_g h - \frac{1}{2}\mathrm{div}(\mathrm{div}h)g_{bc} + \frac{1}{2}(\Box_g \mathrm{Tr}_g h)g_{bc}. \tag{3.2.34}$$

The last two terms are the trace of the linearised vacuum Einstein equation (I.5) and therefore cancel. Moreover, one can apply the Ricci identity to the the first term to give

$$P^{a}{}_{(bc)}{}^{def}\nabla_a\nabla_d h_{ef} = \nabla_{(c}\mathrm{div}h_{b)} - R_{(b}{}^{a}{}_{c)}{}^{d}h_{ad} - \frac{1}{2}\Box_g h_{bc} - \frac{1}{2}\nabla_b\nabla_c \mathrm{Tr}_g h, \tag{3.2.35}$$

where one uses $\mathrm{Ric}(g) = 0$. Therefore, using the symmetries of the Riemann tensor and the linearised vacuum Einstein equation (I.5) one has the result. □

One can define the following symplectic current on $\mathrm{sym}(T^\star M \otimes T^\star M)$:

Definition 3.2.5 (Symplectic Current)**.** *Let $h_1, h_2 \in \mathrm{sym}(T^\star M \otimes T^\star M)$. Then the symplectic current $\mathfrak{w}$ associated to h_1 and h_2 is defined by*

$$\mathfrak{w}[h_1,h_2]^a \doteq P^{abcdef}\Big((h_2)_{bc}\nabla_d(h_1)_{ef} - (h_1)_{bc}\nabla_d(h_2)_{ef}\Big), \tag{3.2.36}$$

where P is defined in equation (3.2.33) of proposition 3.2.7.

Proposition 3.2.8. *Suppose (M,g) satisfies the vacuum Einstein equation and h_1 and h_2 satisfy the linearised vacuum Einstein equation (I.5). Then the symplectic current $\mathfrak{w}[h_1,h_2]$ is divergence free.*

Proof. Computing directly once again gives

$$\mathrm{div}(\mathfrak{w}[h_1,h_2]) = P^{abcdef}\nabla_a(h_2)_{bc}\nabla_d(h_1)_{ef} - P^{abcdef}\nabla_a(h_1)_{bc}\nabla_d(h_2)_{ef} \tag{3.2.37}$$

$$+ P^{abcdef}(h_2)_{bc}\nabla_a\nabla_d(h_1)_{ef} - P^{abcdef}(h_1)_{bc}\nabla_a\nabla_d(h_2)_{ef}. \tag{3.2.38}$$

Noting that P has the symmetry

$$P^{abcdef} = P^{dfeacb}, \tag{3.2.39}$$

shows the first two terms in (3.2.38) cancel and proposition 3.2.7 shows that the last two vanish. □

Remark 3.2.9. *As Hollands and Wald illustrate in [65], the current $\mathfrak{w}[h_1,h_2]$ arises naturally by considering antisymmeterised second variations of the Einstein–Hilbert action for a vacuum spacetime (M,g) which has Lagrangian density*

$$L = \frac{1}{16\pi}\mathrm{Scal}(g)\varepsilon, \tag{3.2.40}$$

where $\mathrm{Scal}(g)$ *is the Ricci scalar of* g *and* ε *is the volume form associated to* g*.*

Definition 3.2.6 (Canonical Energy)**.** *Let* (M, g) *be a spacetime satisfying the vacuum Einstein equation (I.2) and* h *be a solution to the linearised vacuum Einstein equation (I.5). Let* X *be a vector field and* Σ *be a hypersurface in* (M, g) *with (unit) normal* n*. The* X*-canonical energy on* Σ *is defined as*

$$\mathcal{E}^X_\Sigma[h] \doteq \int_\Sigma n(\mathcal{J}^X[h])\mathrm{dvol}_\Sigma, \tag{3.2.41}$$

where $\mathcal{J}^X[h] \doteq \mathfrak{w}[h, \mathcal{L}_X h]$*.*

Remark 3.2.10. *This definition extends the definition of the canonical energy given by Hollands and Wald in the sense that they consider the case only where* (M, g) *is stationary with stationary Killing field* T *and then take* $X = T$*.*

Proposition 3.2.11. *Suppose* (M, g) *satisfies the vacuum Einstein equation and* h *satisfies the linearised vacuum Einstein equation (I.5). Then the symplectic current* $\mathcal{J}^X[h]$ *satisfies*

$$\begin{aligned}\mathrm{div}(\mathcal{J}^X[h]) = &- h_{bc}P^{abcdef}\left(\mathrm{K}^X_{adp}\nabla^p h_{ef} + \mathrm{K}^X_{aep}\nabla_d h^p{}_f + \mathrm{K}^X_{afp}\nabla_d h^p{}_e\right) \\ &- h_{bc}P^{abcdef}\left(\nabla_a\mathrm{K}^X_{dep}h^p{}_f + \mathrm{K}^X_{dep}\nabla_a h^p{}_f + \nabla_a\mathrm{K}^X_{dfp}h^p{}_e + \mathrm{K}^X_{dfp}\nabla_a h^p{}_e\right) \\ &+ (\mathcal{L}_X P)^{abcdef}h_{bc}\nabla_a\nabla_d h_{ef},\end{aligned} \tag{3.2.42}$$

where

$$\mathrm{K}^X_{abc} \doteq \nabla_a\Pi^X_{bc} + \nabla_b\Pi^X_{ac} - \nabla_c\Pi^X_{ab}, \qquad \Pi^X_{ab} \doteq \frac{1}{2}(\mathcal{L}_X g)_{ab} \tag{3.2.43}$$

and $(\mathcal{L}_X P)^{abcdef}$ *can be expressed in terms of* Π^X_{ab} *and the inverse metric.*

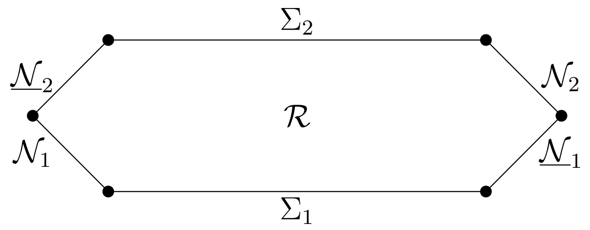
Proof. This follows from a direct computation with the results in appendix C.1 and noting that

$$(\mathcal{L}_X P)^{abcdef}\nabla_a\nabla_d h_{ef} = -P^{abcdef}\mathcal{L}_X(\nabla\nabla h)_{adef}, \tag{3.2.44}$$

by proposition 3.2.7. □

Suppose one applies the divergence theorem to $\mathcal{J}^X[h]$ in some region of a vacuum spacetime. One gets the following proposition:

Proposition 3.2.12. *Let* h *be a smooth solution to the linearised vacuum Einstein equation (I.5),* X *a Killing vector field and* $\mathcal{R}$ *be a region of the vacuum spacetime* (M, g) *with boundary depicted in the following diagram,*



where Σ_i are spacelike and $\mathcal{N}_i$ and $\underline{\mathcal{N}}_i$ are null. Then the X-canonical energy satisfies

$$\mathcal{E}^X_{\mathcal{N}_1}[h] + \mathcal{E}^X_{\underline{\mathcal{N}}_1}[h] + \mathcal{E}^X_{\Sigma_1}[h] = \mathcal{E}_{\mathcal{N}_2}[h] + \mathcal{E}^X_{\underline{\mathcal{N}}_2}[h] + \mathcal{E}^X_{\Sigma_2}[h]. \tag{3.2.45}$$

Proof. This result follows from proposition 3.2.8 and proposition C.1.4 in appendix C.1. □

If X is *not* a Killing vector field then the above proposition 3.2.12 can be modified to include the expression in equation (3.2.42) appearing as a spacetime bulk term. If one wanted to produce a Morawetz-type [124] spacetime estimate such a bulk term seems undesirable since the right-hand side of equation (3.2.42) is not a quadratic expression in derivatives of h.

3.2.4 Higher Order Canonical Energies

Suppose the stationary spacetime (M, g) has a Killing field k (not necessarily the stationary Killing vector field T). By proposition C.1.4, if h is a solution to the linearised vacuum Einstein equation (I.5) then $\mathcal{L}_k h$ is also a solution i.e. one can commute as many Lie derivatives $\mathcal{L}_k$ through the linearised vacuum Einstein operator as one likes. Hence, one can consider a 'higher order' X-canonical energy $\mathcal{E}^X_\Sigma[\mathcal{L}^m_k h]$ resulting from

$$(\mathcal{J}^X[\mathcal{L}^m_k h])^a = P^{abcdef}\Big((\mathcal{L}_X\mathcal{L}^m_k h)_{bc}\nabla_d(\mathcal{L}^m_k h)_{ef} - (\mathcal{L}^m_k h)_{bc}\nabla_d(\mathcal{L}_X\mathcal{L}^m_k h)_{ef}\Big). \tag{3.2.46}$$

For the 4-dimensional Schwarzschild solution one has the Killing fields associated to spherical symmetry Ω_i $i = 1, 2, 3$. In sections 3.3.4 and 3.3.5, the T-canonical energies of $\mathcal{L}_{\Omega_i} h$ and $\mathcal{L}_T h$ will be evaluated.

Remark 3.2.13. *Recall, more generally, that any two solutions of the linearised vacuum Einstein equation (I.5) define a conserved quantity resulting from $\mathfrak{w} \in \mathfrak{X}(M)$ of the form*

$$\mathfrak{w}[h_1, h_2]^a = P^{abcdef}\Big((h_1)_{bc}\nabla_d(h_2)_{ef} - (h_2)_{bc}\nabla_d(h_1)_{ef}\Big). \tag{3.2.47}$$

Hence, one could conceivably get conservation laws resulting from arbitrary combinations of Lie derivatives with respect to Killing fields.

3.3 Canonical Energy in Double Null Gauge

In this section, the proofs of theorems 3.1.7, 3.1.9 and 3.1.10 are given. In section 3.3.1 the computation of the canonical energy in double null gauge is setup with a motivational example of the main computation with the wave equation (3.2.6). Section 3.3.2 collects some preliminary computations which will be useful in the proof of the theorems 3.1.7-3.1.10. The intensive parts of the computations for the canonical energy in double null gauge are then given in sections 3.3.3, 3.3.4 and 3.3.5 as the proofs of theorems 3.1.7, 3.1.9 and 3.1.10.

3.3.1 The Setup

Consider the exterior of Schw_4 in double null Eddington–Finkelstein coordinates as defined in equation (2.8.3). Consider a characteristic rectangle bounded by surfaces of constant u and v on the exterior with vertices (u_0, v_0), (u_1, v_0), (u_0, v_1) and (u_1, v_1) as depicted in blue in the following diagram:

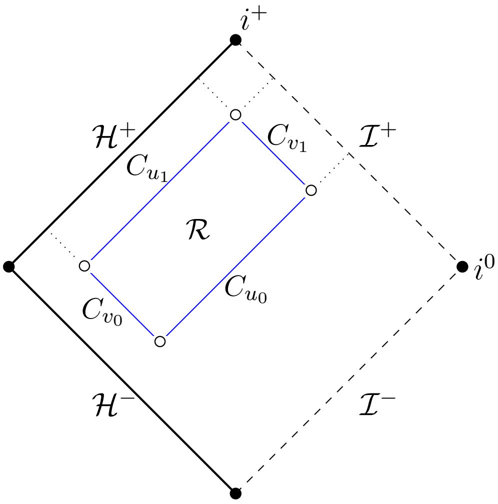


Fig. 3.2 The Penrose diagram depicting the setup up for the computation of the canonical energy on the exterior of the Schw_4 spacetime.

In the dual basis to $e_3 = \frac{1}{\Omega}\partial_u$, $e_4 = \frac{1}{\Omega}\partial_v$ $e_1 = \partial_\theta$ and $e_2 = \partial_\varphi$ the metric is

$$g = -2(f^3 \otimes f^4 + f^4 \otimes f^3) + \not{g} \tag{3.3.1}$$

$$= -2\Omega^2(du \otimes dv + dv \otimes du) + \not{g}. \tag{3.3.2}$$

where $\not{g} = r^2 \mathring{\not{\gamma}}_2$. The volume form for the Schwarzschild black hole exterior is

$$\varepsilon = 2\Omega^2 du \wedge dv \wedge \not{\epsilon} \tag{3.3.3}$$

where $\not\epsilon$ is the induced volume form on $\mathbb{S}^2_{u,v}$. The induced volume forms on the surfaces of constant u and v are given by

$$\mathrm{dvol}_{C_u} = 2\Omega^2 dv \wedge \not\epsilon, \qquad \mathrm{dvol}_{C_v} = 2\Omega^2 du \wedge \not\epsilon, \tag{3.3.4}$$

respectively. To simplify notation, let

$$\mathcal{E}^T_u[h](v_0, v_1) \doteq \mathcal{E}^T_{C_u \cap \{v_0 \le v \le v_1\}}[h], \qquad \mathcal{E}^T_v[h](u_0, u_1) \doteq \mathcal{E}^T_{C_v \cap \{u_0 \le u \le u_1\}}[h]. \tag{3.3.5}$$

By proposition 3.2.12 one has the following canonical energy conservation law for a smooth solution h of the linearised vacuum Einstein equation (I.5):

$$\mathcal{E}^T_{u_0}[h](v_0, v_1) + \mathcal{E}^T_{v_0}[h](u_0, u_1) = \mathcal{E}^T_{u_1}[h](v_0, v_1) + \mathcal{E}^T_{v_1}[h](u_0, u_1). \tag{3.3.6}$$

One can write the terms in this conservation law (3.3.6) explicitly as

$$\mathcal{E}^T_u[h](v_0, v_1) = 2\int_{v_0}^{v_1} \int_{\mathbb{S}^2_{u,v}} du(\mathcal{J}^T[h])\Omega^2 dv \not\epsilon = 2\int_{v_0}^{v_1} \int_{\mathbb{S}^2_{u,v}} (\mathcal{J}^T[h])^3 \Omega dv \not\epsilon, \tag{3.3.7}$$

$$\mathcal{E}^T_v[h](u_0, u_1) = 2\int_{u_0}^{u_1} \int_{\mathbb{S}^2_{u,v}} dv(\mathcal{J}^T[h])\Omega^2 du \not\epsilon = 2\int_{u_0}^{u_1} \int_{\mathbb{S}^2_{u,v}} (\mathcal{J}^T[h])^4 \Omega du \not\epsilon, \tag{3.3.8}$$

where $\mathcal{J}^T[h]$ is the vector defined in definition 3.2.5. For clarity, recall that

$$(\mathcal{J}^T[h])^a = P^{abcdef} \cdot \Big[(\mathcal{L}_T h)_{bc} \nabla_d h_{ef} - h_{bc} \nabla_d (\mathcal{L}_T h)_{ef}\Big], \tag{3.3.9}$$

with P is defined as

$$P^{abcdef} \doteq g^{ae} g^{bf} g^{cd} - \frac{1}{2} g^{ad} g^{be} g^{cf} - \frac{1}{2} g^{ab} g^{ef} g^{cd} - \frac{1}{2} g^{ae} g^{df} g^{bc} + \frac{1}{2} g^{ad} g^{ef} g^{bc}. \tag{3.3.10}$$

Recall that the Schwarzschild black hole spacetime Schw_4 has three additional Killing fields associated to the spherical symmetry of the spacetime. Let Ω_k be the Killing fields on the sphere $\mathbb{S}^2$, i.e.

$$\Omega_1 = \partial_\varphi, \tag{3.3.11}$$

$$\Omega_2 = \sin\varphi \partial_\theta + \cot\theta \cos\varphi \partial_\varphi, \tag{3.3.12}$$

$$\Omega_3 = \cos\varphi \partial_\theta - \cot\theta \sin\varphi \partial_\varphi. \tag{3.3.13}$$

As discussed in section 3.2.4, one has a canonical energy conservation law for $\mathcal{L}_{\Omega_k} h$ for each $k = 1, 2, 3$. In fact, the more appropriate conservation law is the sum of $\mathcal{E}^T[\mathcal{L}_{\Omega_k} h]$. Denote

$$\not\mathcal{E}^T_u[h](v_0, v_1) \doteq \sum_{i=1}^{3} \mathcal{E}_u[\mathcal{L}_{\Omega_i} h](v_0, v_1), \qquad \not\mathcal{E}^T_u[h](v_0, v_1) \doteq \sum_{i=1}^{3} \mathcal{E}_v[\mathcal{L}_{\Omega_i} h](u_0, u_1). \tag{3.3.14}$$

Then $\not{\mathcal{E}}^T$ satisfies

$$\not{\mathcal{E}}^T_{u_0}[h](v_0,v_1)+\not{\mathcal{E}}^T_{v_0}[h](u_0,u_1)=\not{\mathcal{E}}^T_{u_1}[h](v_0,v_1)+\not{\mathcal{E}}^T_{v_1}[h](u_0,u_1). \tag{3.3.15}$$

One can write the terms in this conservation law (3.3.6) explicitly as

$$\not{\mathcal{E}}^T_u[h](v_0,v_1)=2\sum_{i=1}^{3}\int_{v_0}^{v_1}\int_{\mathbb{S}^2}(\mathcal{J}[\mathcal{L}_{\Omega_i}h])^3\Omega dv\not{\epsilon}, \tag{3.3.16}$$

$$\not{\mathcal{E}}^T_v[h](u_0,u_1)=2\sum_{i=1}^{3}\int_{u_0}^{u_1}\int_{\mathbb{S}^2}(\mathcal{J}[\mathcal{L}_{\Omega_i}h])^4\Omega du\not{\epsilon}. \tag{3.3.17}$$

In the following sections the currents $\mathcal{J}^T[h]$, $\sum_k\mathcal{J}^T[\mathcal{L}_{\Omega_k}h]$ and $\mathcal{J}^T[\mathcal{L}_T h]$ are computed in explicitly. The reader should note that the proof of the statements in theorems 3.1.7-3.1.10 are extremely computationally involved. As a instructive first step, the following argument for the wave equation illustrates the key ideas of the proofs.

In double null Eddington–Finkelstein coordinates on the exterior the wave equation (3.2.6) for $\Psi\in C^\infty(\mathrm{Schw}_4)$ reduces to

$$-\frac{1}{\Omega^2}\partial_u\partial_v\Psi+\frac{1}{r}(\partial_v-\partial_u)\Psi+\not{\Delta}\Psi=0 \tag{3.3.18}$$

where $\not{\Delta}$ is the Laplacian on $\mathbb{S}^2_{u,v}$. Define the following energies on C_u and C_v respectively (which are the usual T-energies arising from the current in equation (3.2.8) defined using the energy-momentum tensor):

$$E^T_u[\Psi](v_0,v_1)\doteq\frac{1}{2}\int_{v_0}^{v_1}\int_{\mathbb{S}^2}\Big(|\not{\nabla}_4\Psi|^2+|\not{\nabla}\Psi|^2\Big)\Omega^2 dv\not{\epsilon}, \tag{3.3.19}$$

$$E^T_v[\Psi](u_0,u_1)\doteq\frac{1}{2}\int_{u_0}^{u_1}\int_{\mathbb{S}^2}\Big(|\not{\nabla}_3\Psi|^2+|\not{\nabla}\Psi|^2\Big)\Omega^2 dv\not{\epsilon}. \tag{3.3.20}$$

For brevity, introduce the following notation for the T-canonical energy for the wave equation

$$\mathcal{E}^T_u[\Psi](v_0,v_1)\doteq\mathcal{E}^T_{C_u\cap\{v_0\le v\le v_1\}}[\Psi],\qquad \mathcal{E}^T_v[\Psi](u_0,u_1)\doteq\mathcal{E}^T_{C_v\cap\{u_0\le u\le u_1\}}[\Psi]. \tag{3.3.21}$$

Since T is Killing, $\mathcal{J}^T[\Psi]$ is divergence-free (see proposition 3.2.1). Therefore, the T-canonical energy for the wave equation satisfies:

$$\mathcal{E}^T_{u_0}[\Psi](v_0,v_1)+\mathcal{E}^T_{v_0}[\Psi](u_0,u_1)=\mathcal{E}^T_{u_1}[\Psi](v_0,v_1)+\mathcal{E}^T_{v_1}[\Psi](u_0,u_1). \tag{3.3.22}$$

The analogous result to theorem 3.1.7 for the wave equation is then the following:

Proposition 3.3.1. *Let Ψ be a smooth solution to the wave equation (3.2.6) on the Schwarzschild black hole exterior. Then the T-canonical energy for the wave equation of Ψ on the null cones $C_u \cap \{v_0 \le v \le v_1\}$ and $C_v \cap \{u_0 \le u \le u_1\}$ satisfies*

$$\mathcal{E}_u^T[\Psi](v_0, v_1) = 2E_u^T[\Psi](v_0, v_1) - \int_{\mathbb{S}^2_{u,v}} F[\Psi](u, v, \theta, \varphi) \not{\epsilon}\Big|_{v_0}^{v_1} \tag{3.3.23}$$

$$\mathcal{E}_v^T[\Psi](u_0, u_1) = 2E_v^T[\Psi](u_0, u_1) + \int_{\mathbb{S}^2_{u,v}} F[\Psi](u, v, \theta, \varphi) \not{\epsilon}\Big|_{u_0}^{u_1} \tag{3.3.24}$$

with $F \doteq \frac{1}{4}\Psi(\partial_v \Psi - \partial_u \Psi)$. Hence,

$$E_{u_0}^T[\Psi](v_0, v_1) + E_{v_0}^T[\Psi](u_0, u_1) = E_{u_1}^T[\Psi](v_0, v_1) + E_{v_1}^T[\Psi](u_0, u_1). \tag{3.3.25}$$

Proof. Now since $\mathcal{L}_T \Psi = T(\Psi) = \frac{1}{2}(\partial_u \Psi + \partial_v \Psi)$ and $g^{uv} = -\frac{1}{2\Omega^2}$ one has

$$\mathcal{J}^X[\Psi]^u = \frac{1}{4\Omega^2}\Big(\partial_u \Psi \partial_v \Psi + |\partial_v \Psi|^2 - \Psi \partial_v^2 \Psi - \Psi \partial_u \partial_v \Psi\Big), \tag{3.3.26}$$

$$\mathcal{J}^X[\Psi]^v = \frac{1}{4\Omega^2}\Big(\partial_u \Psi \partial_v \Psi + |\partial_u \Psi|^2 - \Psi \partial_u^2 \Psi - \Psi \partial_u \partial_v \Psi\Big). \tag{3.3.27}$$

Using the decomposed wave equation (3.3.18) one finds

$$\begin{aligned}
\mathcal{J}^X[\Psi]^u + \frac{1}{\Omega^2 r^2}\partial_v(r^2 F) &= \frac{1}{2\Omega^2}\Big(|\partial_v \Psi|^2 - \Psi \partial_u \partial_v \Psi + \frac{\Omega^2}{r}\Psi(\partial_v \Psi - \partial_u \Psi)\Big) && (3.3.28)\\
&= \frac{1}{2\Omega^2}\Big(|\partial_v \Psi|^2 + \Omega^2 |\nabla\!\!\!/ \Psi|^2_{\not{g}}\Big) - \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}(\Psi(\not{d}\Psi)^\sharp) && (3.3.29)\\
\mathcal{J}^X[\Psi]^v - \frac{1}{\Omega^2 r^2}\partial_u(r^2 F) &= \frac{1}{2\Omega^2}\Big(|\partial_u \Psi|^2 - \Psi \partial_u \partial_v \Psi + \frac{\Omega^2}{r}\Psi(\partial_v \Psi - \partial_u \Psi)\Big) && (3.3.30)\\
&= \frac{1}{2\Omega^2}\Big(|\partial_u \Psi|^2 + \Omega^2 |\nabla\!\!\!/ \Psi|^2_{\not{g}}\Big) - \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}(\Psi(\not{d}\Psi)^\sharp). && (3.3.31)
\end{aligned}$$

Therefore,

$$\mathcal{J}^X[\Psi]^u = \frac{1}{2}\Big(|e_4(\Psi)|^2 + |\nabla\!\!\!/ \Psi|^2_{\not{g}}\Big) - \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}(\Psi(\not{d}\Psi)^\sharp) - \frac{1}{\Omega^2 r^2}\partial_v(F) \tag{3.3.32}$$

$$\mathcal{J}^X[\Psi]^v = \frac{1}{2}\Big(|e_3(\Psi)|^2 + |\nabla\!\!\!/ \Psi|^2_{\not{g}}\Big) - \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}(\Psi(\not{d}\Psi)^\sharp) + \frac{1}{\Omega^2 r^2}\partial_u(F). \tag{3.3.33}$$

If one integrates equations (3.3.32) and (3.3.33) over the sphere, the $\mathrm{d}\!\!\!/\mathrm{iv}$ terms vanish since $\partial \mathbb{S}^2 = \emptyset$. Hence, the result. $\square$

The key points that this proof illustrates is:

(i) The equation for the scalar field, i.e., the wave equation (3.2.6), in conjunction with integration by parts over $\mathbb{S}^2_{u,v}$ can be used to simplify the fluxes. In this case the wave equation can be used to remove $\partial_u \partial_v \Psi$ in exchange for first order derivative terms and an angular Laplacian of the solution, which can be integrated by parts.

(ii) If one adds $\frac{1}{\Omega^2 r^2}\partial_v(r^2\mathfrak{F})$ to $(\mathcal{J}^T)^u$ then subtracting $\frac{1}{\Omega^2 r^2}\partial_u(r^2\mathfrak{F})$ from $(\mathcal{J}^T)^v$ for some $\mathfrak{F}$ then one maintains a conservation law on hypersurfaces since the terms on the spheres at the corners of the characteristic rectangle cancel.

(iii) There are second order derivatives of Ψ that cannot be exchanged for first order derivative terms via the wave equation (as in point (i)). For example $\Psi\partial_v^2\Psi$ appears in $\mathcal{J}^T$. One can use point (ii) to remove these terms. This is precisely what allows one to identify $\mathfrak{F} = F$.

With this discussion of the wave equation in hand, some intuition for why the main result in theorem 3.1.7 is true can be given. First one should note that the Schwarzschild spacetime only has a limited number of symmetries so there cannot be arbitrarily many *independent* conservation laws. This means there is likely some relation between the canonical energy conservation law and Holzegel's conservation law (3.1.3). Further, observe that the linearised null structure equations of section 2.10.1 have the form

$$\nabla h = \overset{(1)}{\Gamma}, \tag{3.3.34}$$

$$\nabla\overset{(1)}{\Gamma} = \Gamma\overset{(1)}{\Gamma} + \overset{(1)}{W}, \tag{3.3.35}$$

where Γ is the background Ricci coefficients and $\overset{(1)}{\Gamma}$ is the linear perturbations to the Ricci coefficients and $\overset{(1)}{W}$ denotes the linearised Weyl curvature. Therefore, the flux densities $\mathcal{J}^T[h]$ involved in the canonical energy of h are of the schematic form

$$\mathcal{J}^T[h] = \mathcal{L}_T h\cdot\nabla h - h\cdot\nabla\mathcal{L}_T h = \overset{(1)}{\Gamma}\cdot\overset{(1)}{\Gamma} + \Gamma h\cdot\overset{(1)}{\Gamma} + h\cdot\overset{(1)}{W}. \tag{3.3.36}$$

It turns out that, in analogy with proposition 3.3.1 for solutions of the wave equation, by using <u>only</u> the linearised null structure equations (2.10.1), this last term involving curvature in equation (3.3.36) can be replaced (exactly like $\Psi\partial_u\partial_v\Psi$, $\Psi\partial_v^2\Psi$ and $\Psi\partial_u^2\Psi$ for the wave equation) by the boundary term $\pm\mathcal{A}$ (defined in theorem 3.1.7) on the spheres $\mathbb{S}^2_{u_0,v_0}$, $\mathbb{S}^2_{u_1,v_0}$, $\mathbb{S}^2_{u_0,v_1}$ and $\mathbb{S}^2_{u_1,v_1}$.

The intuition behind theorem 3.1.9 is the following. From the discussion in remark 2.10.3 combined with proposition C.1.4 that if h in double null gauge solves the linearised vacuum Einstein equation (I.5) then so does $\mathcal{L}_{\Omega_k}h$. So if one writes the conservation law for the modified T-canonical energy (see equations (3.1.1) and (3.1.2)) in terms of h then one can replace it everywhere with $\mathcal{L}_{\Omega_k}h$. Effectively due to $[T,\Omega_k]=0$ for all $k=1,2,3$ and therefore $\mathcal{L}_T\mathcal{L}_{\Omega_k}h = \mathcal{L}_{\Omega_k}\mathcal{L}_T h$, it turns out that this operation commutes through each term in equations (3.1.1) and (3.1.2) so one can replace each linearised Ricci coefficient $\overset{(1)}{\Gamma}$ with $\mathcal{L}_{\Omega_k}\overset{(1)}{\Gamma}$. Roughly speaking, $\sum_k \not{\mathcal{L}}_{\Omega_k}\overset{(1)}{\hat{\chi}}$ is similar to the divergence operator $\mathrm{d}\!\!/\mathrm{iv}$ on $\mathbb{S}^2_{u,v}$ acting on the linearised shear $\overset{(1)}{\hat{\chi}}$. From linearised

Codazzi equations in proposition 2.10.17 one can see that

$$\operatorname{d\!/iv}\overset{(1)}{\hat{\chi}} = -\overset{(1)}{\beta} + \ldots, \qquad \operatorname{d\!/iv}\overset{(1)}{\underline{\hat{\chi}}} = \overset{(1)}{\underline{\beta}} + \ldots. \tag{3.3.37}$$

Using the linearised null structure equations of propositions 2.10.7-2.10.17 in section 2.10.1, the linearised Bianchi equations of proposition 2.10.20 and integration by parts one can then establish theorem 3.1.9.

Finally, the intuition behind theorem 3.1.10 is the following. Following the same reasoning as discussed above for theorem 3.1.9 one can replace each metric coefficient h and each linearised Ricci coefficient $\overset{(1)}{\Gamma}$ in equations (3.1.1) and (3.1.2) with $\mathcal{L}_T h$ and $\mathcal{L}_T\overset{(1)}{\Gamma}$ respectively. From linearised shear equations in proposition 2.10.10 one can see that

$$\not{\mathcal{L}}_T\overset{(1)}{\hat{\chi}} = \not\nabla_3\overset{(1)}{\hat{\chi}} + \not\nabla_4\overset{(1)}{\hat{\chi}} = -\overset{(1)}{\alpha} + \ldots, \tag{3.3.38}$$

$$\not{\mathcal{L}}_T\overset{(1)}{\underline{\hat{\chi}}} = \not\nabla_3\overset{(1)}{\underline{\hat{\chi}}} + \not\nabla_4\overset{(1)}{\underline{\hat{\chi}}} = -\overset{(1)}{\underline{\alpha}} + \ldots. \tag{3.3.39}$$

Using the linearised null structure equations of propositions 2.10.7-2.10.17 in section 2.10.1, the linearised Bianchi equations of proposition 2.10.20 and integration by parts one can then establish theorem 3.1.10.

Remark 3.3.2. *The conservation law (3.1.23) in theorem 3.1.10 arises from the solution $\mathcal{L}_T h$ to the linearised vacuum Einstein equation (I.5). One can consider the conservation laws for the solution $\mathcal{L}_T^k h$. Following the same reasoning as discussed above for theorem 3.1.9 one can replace each metric coefficient h and each linearised Ricci coefficient $\overset{(1)}{\Gamma}$ in equations (3.1.1) and (3.1.2) with $\mathcal{L}_T^k h$ and $\mathcal{L}_T^k\overset{(1)}{\Gamma}$ respectively. Taking $k=4$, and by naively considering $\not{\mathcal{L}}_T^4\overset{(1)}{\hat{\chi}}$ and $\not{\mathcal{L}}_T^4\overset{(1)}{\underline{\hat{\chi}}}$ with equations* (3.3.38) *and* (3.3.39) *one has*

$$\not{\mathcal{L}}_T^4\overset{(1)}{\hat{\chi}} = -\not{\mathcal{L}}_T^3\overset{(1)}{\alpha} + \ldots = -(\not\nabla_3 + \not\nabla_4)^3\overset{(1)}{\alpha} + \ldots = \not\nabla_4 P + \ldots, \tag{3.3.40}$$

$$\not{\mathcal{L}}_T^4\overset{(1)}{\underline{\hat{\chi}}} = -\not{\mathcal{L}}_T^3\overset{(1)}{\underline{\alpha}} + \ldots = -(\not\nabla_3 + \not\nabla_4)^3\overset{(1)}{\underline{\alpha}} + \ldots = \not\nabla_3\underline{P} + \ldots, \tag{3.3.41}$$

where one $(P, \underline{P})$ are solutions to the Regge–Wheeler equation in proposition 2.10.28 with $n=4$. Therefore, by replacing $\overset{(1)}{\hat{\chi}}$ and $\overset{(1)}{\underline{\hat{\chi}}}$ by $\not{\mathcal{L}}_T^4\overset{(1)}{\hat{\chi}}$ and $\not{\mathcal{L}}_T^4\overset{(1)}{\underline{\hat{\chi}}}$ in equations (3.1.1) and (3.1.2), one sees that this will produce a conservation law that involves $|\not\nabla_4 P|^2$ in the flux on C_u and $|\not\nabla_3\underline{P}|^2$ in the flux on C_v at top order. It seems likely that this conservation law is related to the conservation law in proposition 11.1.1 in Dafermos–Holzegel–Rodnianski [28].

3.3.2 Preliminary Computations

One should note that from the definition of double null Eddington–Finkelstein coordinates in section 2.8 one has the following relations

$$\frac{1}{r^2}\slashed{\nabla}_3 r^2 = -(\mathrm{Tr}_{\slashed{g}}\chi), \quad \frac{1}{r^2}\slashed{\nabla}_4 r^2 = (\mathrm{Tr}_{\slashed{g}}\chi). \tag{3.3.42}$$

From the null structure and Bianchi equations in sections 2.6 and 2.7.3 one has the following relations for the background Ricci coefficients and curvature components of definitions 2.2.1 and 2.3.1 respectively:

$$\begin{aligned}
\slashed{\nabla}_4\Omega &= \omega, & \slashed{\nabla}_3\Omega &= -\omega,\\
\slashed{\nabla}_4(\Omega\mathrm{Tr}_{\slashed{g}}\chi) &= 2\omega\mathrm{Tr}_{\slashed{g}}\chi - \frac{\Omega}{2}(\mathrm{Tr}_{\slashed{g}}\chi)^2, & \slashed{\nabla}_3(\Omega\mathrm{Tr}_{\slashed{g}}\chi) &= \frac{\Omega}{2}(\mathrm{Tr}_{\slashed{g}}\chi)^2 - 2\omega\mathrm{Tr}_{\slashed{g}}\chi,\\
\slashed{\nabla}_4\rho &= -\frac{3}{2}\rho\mathrm{Tr}_{\slashed{g}}\chi, & \slashed{\nabla}_3\rho &= \frac{3}{2}\rho\mathrm{Tr}_{\slashed{g}}\chi,\\
\slashed{\nabla}_4\omega &= \Omega\rho, & \slashed{\nabla}_3\omega &= -\Omega\rho
\end{aligned} \tag{3.3.43}$$

and

$$\rho = -\hat{\omega}\mathrm{Tr}_{\slashed{g}}\chi. \tag{3.3.44}$$

These relations in equations (3.3.42), (3.3.43) and (3.3.44) will be used liberally throughout the rest of this section and the next.

For the canonical energy calculation one needs to compute $\nabla_\alpha h_{\beta\gamma}$ and $\nabla_\alpha(\mathcal{L}_T h)_{\beta\gamma}$ in the double null basis. The following lemma is useful for this.

Lemma 3.3.3. *Let* $\mathrm{S} \in \mathrm{sym}(T^\star M \otimes T^\star M)$ *on the Schwarzschild black hole exterior* Schw_4 *with*

$$\mathrm{S}_{44} = \mathrm{S}_{33} = \mathrm{S}_{3A} = 0, \tag{3.3.45}$$

in the normalised null basis (e_3, e_4, ∂_A) *associated to the double null Eddington–Finkelstein coordinates. Further, denote* $\mathrm{v}^{\mathrm{S}}_A \doteq \mathrm{S}_{4A}$ *and* $\slashed{\mathrm{S}}_{AB} \doteq \mathrm{S}_{AB}$ *which are considered as the components of a* $\mathbb{S}^2_{u,v}$*-covector and symmetric* $\mathbb{S}^2_{u,v}$ *2-tensor. Then the* <u>non-zero</u> *components of* $(\nabla_\alpha \mathrm{S})_{\beta\gamma}$ *have the following decomposition:*

$$\begin{aligned}
(\nabla_3\mathrm{S})_{43} &= e_3(\mathrm{S}_{43}), & (\nabla_4\mathrm{S})_{43} &= e_4(\mathrm{S}_{43}),\\
(\nabla_4\mathrm{S})_{A4} &= \slashed{\nabla}_4\mathrm{v}^{\mathrm{S}}_A - \hat{\omega}\mathrm{v}^{\mathrm{S}}_A, & (\nabla_3\mathrm{S})_{A4} &= \slashed{\nabla}_3\mathrm{v}^{\mathrm{S}}_A - \hat{\omega}\mathrm{v}^{\mathrm{S}}_A,\\
(\nabla_A\mathrm{S})_{44} &= -(\mathrm{Tr}_{\slashed{g}}\chi)\mathrm{v}^{\mathrm{S}}_A, & (\nabla_A\mathrm{S})_{34} &= \partial_A(\mathrm{S}_{34}) + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\mathrm{v}^{\mathrm{S}}_A,\\
(\nabla_4\mathrm{S})_{AB} &= (\slashed{\nabla}_4\slashed{\mathrm{S}})_{AB}, & (\nabla_3\mathrm{S})_{AB} &= (\slashed{\nabla}_3\slashed{\mathrm{S}})_{AB},
\end{aligned} \tag{3.3.46}$$

and

$$(\nabla_A \mathrm{S})_{3B} = \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big(\mathrm{S}_{AB} + \frac{1}{2}\mathrm{S}_{34}\slashed{g}_{AB}\Big), \tag{3.3.47}$$

$$(\nabla_A \mathrm{S})_{4B} = \slashed{\nabla}_A \mathrm{v}^{\mathrm{S}}_B - \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\Big(\mathrm{S}_{AB} + \frac{1}{2}\mathrm{S}_{34}\slashed{g}_{AB}\Big), \tag{3.3.48}$$

$$(\nabla_A \mathrm{S})_{BC} = (\slashed{\nabla}_A \slashed{\mathrm{S}})_{BC} + \frac{1}{4}(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{g}_{AC}\mathrm{v}^{\mathrm{S}}_B + \frac{1}{4}(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{g}_{AB}\mathrm{v}^{\mathrm{S}}_C. \tag{3.3.49}$$

Proof. One can calculate the above results using the formula

$$(\nabla_\mu \mathrm{S})_{\alpha\beta} = e_\mu(\mathrm{S}_{\alpha\beta}) - \mathrm{S}(\nabla_\mu e_\alpha, e_\beta) - \mathrm{S}(e_\alpha, \nabla_\mu e_\beta), \tag{3.3.50}$$

in conjunction with the proposition 2.8.2. □

It will turn out that for calculating the canonical energy in double null gauge only the following non-zero components of $\nabla_\alpha h_{\beta\gamma}$ will be required:

Proposition 3.3.4. *Let $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the 4-dimensional Schwarzschild exterior. Then, in the normalised null basis (e_3, e_4, ∂_A) associated to the double null Eddington–Finkelstein coordinates, one has*

$$\begin{aligned}
(\nabla_3 h)_{43} &= -\frac{4}{\Omega}\overset{(1)}{\underline{\omega}}, & (\nabla_4 h)_{43} &= -\frac{4}{\Omega}\overset{(1)}{\omega}, \\
(\nabla_4 h)_{A4} &= -\frac{1}{\Omega}(\slashed{\nabla}_4 \overset{(1)}{b})_A + 2\hat{\omega}\frac{\overset{(1)}{b}_A}{\Omega}, & (\nabla_3 h)_{A4} &= \frac{\overset{(1)}{b}_A}{2\Omega}(\mathrm{Tr}_{\slashed{g}}\chi) - 2(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})_A, \\
\widehat{(\nabla_4 h)}_{AB} &= 2\overset{(1)}{\hat{\chi}}_{AB} + \frac{2}{\Omega}(\slashed{\mathcal{D}}^\star_2 \overset{(1)}{b})_{AB}, & \widehat{(\nabla_3 h)}_{AB} &= 2\overset{(1)}{\hat{\underline{\chi}}}_{AB}.
\end{aligned} \tag{3.3.51}$$

Proof. To prove this statement note that for h in double null gauge

$$h_{44} = h_{33} = h_{3A} = 0, \quad h_{34} = -4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \quad \mathrm{v}^h_B = h_{4B} = -\frac{\overset{(1)}{b}_B}{\Omega}, \quad h_{AB} = \slashed{h}_{AB}. \tag{3.3.52}$$

The results follow directly from lemma 3.3.3 and the linearised null structure equations (in particular, proposition 2.10.7). The reader should note the decomposition

$$(\slashed{\nabla}_3 \slashed{h})_{AB} = \slashed{\nabla}_3(\mathrm{Tr}_{\slashed{g}}\slashed{h})\slashed{g}_{AB} + (\slashed{\nabla}_3 \hat{\slashed{h}})_{AB}, \tag{3.3.53}$$

$$(\slashed{\nabla}_4 \slashed{h})_{AB} = \slashed{\nabla}_4(\mathrm{Tr}_{\slashed{g}}\slashed{h})\slashed{g}_{AB} + (\slashed{\nabla}_4 \hat{\slashed{h}})_{AB}. \tag{3.3.54}$$

□

The following computation gives the components of $\mathcal{L}_T h$ in the normalised null frame.

Proposition 3.3.5. *Let $T \doteq \partial_t$ be the Killing field associated to stationarity of Schw_4. Further, let $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the 4-dimensional Schwarzschild exterior. Then, in the basis (e_3, e_4, ∂_A) associated to the double null Eddington–Finkelstein coordinates, $\mathcal{L}_T h$ has the following components*

$$(\mathcal{L}_T h)_{44} = 0, \quad (\mathcal{L}_T h)_{33} = 0, \quad (\mathcal{L}_T h)_{A3} = 0, \quad (\mathcal{L}_T h)_{34} = -2(\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}}), \tag{3.3.55}$$

$$(\mathcal{L}_T h)_{4A} = \mathrm{v}_A^{\mathcal{L}_T h} = -\frac{1}{2}(\nabla\!\!\!/_4 \overset{(1)}{b})_A - \Omega(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})_A + \frac{1}{4}(\mathrm{Tr}_{g\!\!\!/}\chi)\overset{(1)}{b}_A, \tag{3.3.56}$$

$$(\mathcal{L}_T h)_{AB} = (\mathcal{L}\!\!\!/_T h\!\!\!/)_{AB} = \frac{1}{2}(\mathcal{L}_T \mathrm{Tr}_{g\!\!\!/} h\!\!\!/) g\!\!\!/_{AB} + \Omega\overset{(1)}{\hat{\chi}}_{AB} + \Omega\overset{(1)}{\underline{\hat{\chi}}}_{AB} + (\mathcal{D}\!\!\!/_2^\star \overset{(1)}{b})_{AB}. \tag{3.3.57}$$

Proof. First note that since $t = u + v$ and $r_\star = v - u$ one has $T = \frac{\Omega}{2}(e_3 + e_4)$. From this one can compute $\nabla_4 T = \omega e_4$, $\nabla_3 T = -\omega e_3$ and $\nabla_A T = 0$. Also,

$$(\nabla_T h)_{\alpha\beta} = \frac{\Omega}{2}(\nabla_3 h)_{\alpha\beta} + \frac{\Omega}{2}(\nabla_4 h)_{\alpha\beta}. \tag{3.3.58}$$

Hence, one can use proposition 3.3.4 to compute $(\nabla_T h)_{\alpha\beta}$. Finally one can finish the calculations by using the formula

$$(\mathcal{L}_T h)_{\alpha\beta} = (\nabla_T h)_{\alpha\beta} + h_{\gamma\beta}(\nabla_\alpha T)^\gamma + h_{\gamma\alpha}(\nabla_\beta T)^\gamma \tag{3.3.59}$$

in conjunction with lemma 3.3.3 for h in double null gauge and proposition 3.3.4. □

Proposition 3.3.6. *Let $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the 4-dimensional Schwarzschild exterior expressed in double null Eddington–Finkelstein coordinates. Then, in the basis (e_3, e_4, ∂_A),*

$$(\nabla_3(\mathcal{L}_T h))_{43} = 2\Omega\Big(\overset{(1)}{\rho} + 2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) - \frac{2}{\Omega}\partial_u \overset{(1)}{\underline{\omega}}, \tag{3.3.60}$$

$$(\nabla_4(\mathcal{L}_T h))_{43} = -\frac{2}{\Omega}\partial_v \overset{(1)}{\omega} + 2\Omega\Big(\overset{(1)}{\rho} + 2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big), \tag{3.3.61}$$

$$\begin{aligned}(\nabla_3(\mathcal{L}_T h))_{A4} = {}& \frac{1}{4}(\mathrm{Tr}_{g\!\!\!/}\chi)\Big(\nabla\!\!\!/_4 \overset{(1)}{b} - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\chi)\overset{(1)}{b}\Big)_A + 2\nabla\!\!\!/_A(\overset{(1)}{\omega} - \overset{(1)}{\underline{\omega}}) + 2\Omega(\overset{(1)}{\underline{\beta}} + \overset{(1)}{\beta})_A \\ & - \frac{1}{2}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\overset{(1)}{\eta} + 3\overset{(1)}{\underline{\eta}})_A,\end{aligned} \tag{3.3.62}$$

$$(\widehat{\nabla_3(\mathcal{L}_T h)})_{AB} = (\widehat{\nabla\!\!\!/_3(\mathcal{L}\!\!\!/_T h)})_{AB} = \frac{1}{2}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\overset{(1)}{\underline{\hat{\chi}}} + \overset{(1)}{\hat{\chi}}) - 2\Omega\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\eta}} - 2\omega\overset{(1)}{\underline{\hat{\chi}}} - \Omega\overset{(1)}{\underline{\alpha}}, \tag{3.3.63}$$

$$(\widehat{\nabla_4(\mathcal{L}_T h)})_{AB} = (\widehat{\nabla\!\!\!/_4(\mathcal{L}\!\!\!/_T h)})_{AB} \tag{3.3.64}$$

$$= \mathcal{D}\!\!\!/_2^\star(\nabla\!\!\!/_4\overset{(1)}{b}) - \frac{1}{2}\mathrm{Tr}_{g\!\!\!/}\chi\mathcal{D}\!\!\!/_2^\star\overset{(1)}{b} - \frac{1}{2}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\overset{(1)}{\underline{\hat{\chi}}} + \overset{(1)}{\hat{\chi}}) - 2\Omega\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\eta}} + 2\omega\overset{(1)}{\hat{\chi}} - \Omega\overset{(1)}{\alpha}. \tag{3.3.65}$$

Proof. To prove these relations one uses lemma 3.3.3 and proposition 3.3.5 to double null decompose the above quantities. The results above then follow from an application of the commutation lemma 2.8.4 and the linearised null structure equations of section 2.10.1. □

Proposition 3.3.7. *Let $h \in \mathrm{sym}(T^\star M \otimes T^\star M)$ be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the 4-dimensional Schwarzschild exterior. Then one has the following relations*

$$\not{\mathcal{L}}_T \mathrm{Tr}_{\not g} \not h = (\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) + (\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\chi) - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}, \tag{3.3.66}$$

$$\begin{aligned}\not\nabla_3(\not{\mathcal{L}}_T \mathrm{Tr}_{\not g} \not h) = \frac{1}{\Omega}\Big(&2\Omega^2 \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}} + 2\Omega^2\Big(\overset{(1)}{\rho} + \frac{\overset{(1)}{\Omega}}{\Omega}\rho\Big) + \frac{1}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big((\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) + (\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\chi)\Big) \\ &- 2\omega(\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) - 2(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\underline{\omega}}\Big),\end{aligned} \tag{3.3.67}$$

$$\begin{aligned}\not\nabla_4(\not{\mathcal{L}}_T \mathrm{Tr}_{\not g} \not h) = \frac{1}{\Omega}\Big(&2\Omega^2 \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}} + 2\Omega^2\Big(\overset{(1)}{\rho} + \frac{\overset{(1)}{\Omega}}{\Omega}\rho\Big) - \frac{1}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big((\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) + (\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\chi)\Big) \\ &+ 2\omega(\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\chi) + 2(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\omega} - \Omega\not{\mathrm{d}}\mathrm{iv}(\not\nabla_4\overset{(1)}{b}) + \frac{1}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big).\end{aligned} \tag{3.3.68}$$

Proof. The first equation follows from proposition 2.10.7. The rest of the results then follow from the linearised Raychauduri equations in proposition 2.10.9 and propagation equations for the expansions in proposition 2.10.8. Note that for the last equation one uses the commutation lemma 2.8.4. □

3.3.3 Proof of Theorem 3.1.7

In the following subsection the main computation is performed. Many of the details are provided to leave the reader with no illusion as to the technical nature of the manipulations.

Proof of theorem 3.1.7. In this proof the following convention is adopted. The symbol $\equiv$ will denote equality under integration by parts on $\mathbb{S}^2_{u,v}$.

Recall that the T-canonical energy current $\mathcal{J}^T[h]^a$ is given by

$$\mathcal{J}^T[h]^a = P^{abcdef}\Big[(\mathcal{L}_T h)_{bc}\nabla_d h_{ef} - h_{bc}\nabla_d(\mathcal{L}_T h)_{ef}\Big], \tag{3.3.69}$$

with

$$P^{abcdef} \doteq g^{ae}g^{bf}g^{cd} - \frac{1}{2}g^{ad}g^{be}g^{cf} - \frac{1}{2}g^{ab}g^{ef}g^{cd} - \frac{1}{2}g^{ae}g^{df}g^{bc} + \frac{1}{2}g^{ad}g^{ef}g^{bc}. \tag{3.3.70}$$

The inverse metric has a very simple form in the dual basis to the normalised null frame $(e_3, e_4, \partial_\theta, \partial_\varphi)$ associated to double null Eddington–Finkelstein coordinates. In particular, its non-zero components are

$$g^{34} = -\frac{1}{2}, \qquad g^{AB} = \not{g}^{AB}. \tag{3.3.71}$$

Recall that solution h to the linearised vacuum Einstein equation (I.5) in double null gauge is given by

$$h_{44} = 0 = h_{33} = h_{3A}, \qquad h_{34} = -4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \qquad h_{4A} = -\frac{\overset{(1)}{b}_A}{\Omega}, \qquad h_{AB} = \not{h}_{AB}, \tag{3.3.72}$$

in the basis $(e_3, e_4, \partial_\theta, \partial_\varphi)$. Further by proposition 3.3.5, the vanishing components of $\mathcal{L}_T h$ are

$$(\mathcal{L}_T h)_{44} = 0, \quad (\mathcal{L}_T h)_{33} = 0, \quad (\mathcal{L}_T h)_{A3} = 0. \tag{3.3.73}$$

When calculating $\mathcal{J}^T[h]^4$ one should note that $\not{\nabla}_3 \not{g} = 0$ and hence $[\mathrm{Tr}_{\not{g}}, \not{\nabla}_3] = 0$. From lemma 3.3.3 one has

$$(\nabla_D \mathrm{S})_{3F} = \frac{1}{2}(\mathrm{Tr}_{\not{g}}\chi)(\not{\mathrm{S}}_{DF} + \frac{1}{2}\mathrm{S}_{34}\not{g}_{DF}) \tag{3.3.74}$$

for $\mathrm{S} = h$ or $\mathrm{S} = \mathcal{L}_T h$. Hence,

$$\not{g}^{DF}(\nabla_D \mathrm{S})_{3F} = \frac{1}{2}(\mathrm{Tr}_{\not{g}}\chi)(\mathrm{Tr}_{\not{g}}\not{\mathrm{S}} + \mathrm{S}_{34}), \tag{3.3.75}$$

for $\mathrm{S} = h$ or $\mathrm{S} = \mathcal{L}_T h$. Further, one can decompose $(\mathcal{L}_T h)_{AB}$ into its trace and symmetric traceless part as

$$(\mathcal{L}_T h)_{AB} = \frac{1}{2}(\mathcal{L}_T \mathrm{Tr}_{\not{g}}\not{h})\not{g} + \widehat{\not{\mathcal{L}}_T h}_{AB}, \tag{3.3.76}$$

$$\nabla_3(\mathcal{L}_T h)_{AB} = \frac{1}{2}\not{\nabla}_3(\mathcal{L}_T \mathrm{Tr}_{\not{g}}\not{h})\not{g} + \widehat{\not{\nabla}_3\not{\mathcal{L}}_T h}_{AB}, \tag{3.3.77}$$

where one uses that $(\mathcal{L}_T h)_{AB} = (\not{\mathcal{L}}_T \not{h})_{AB}$ and $\not{\mathcal{L}}_T \not{g} = 0$. Combining these facts gives that $\mathcal{J}^T[h]^4$ can be written in a decomposed form as

$$\begin{aligned} \mathcal{J}^T[h]^4 = &\frac{1}{4}\Big(\langle\widehat{\not{\mathcal{L}}_T\not{h}}, \widehat{\not{\nabla}_3\not{h}}\rangle - \langle\hat{\not{h}}, \widehat{\not{\nabla}_3\not{\mathcal{L}}_T\not{h}}\rangle\Big) + \frac{1}{8}\Big(\mathcal{L}_T(\mathrm{Tr}_{\not{g}}\not{h})e_3(h_{34}) - (\mathrm{Tr}_{\not{g}}\not{h})e_3((\mathcal{L}_T h)_{34})\Big) \\ &+ \frac{1}{8}\Big(\not{\nabla}_3(\mathcal{L}_T\mathrm{Tr}_{\not{g}}\not{h})\mathrm{Tr}_{\not{g}}\not{h} - (\mathcal{L}_T\mathrm{Tr}_{\not{g}}\not{h})\not{\nabla}_3(\mathrm{Tr}_{\not{g}}\not{h})\Big) - \frac{1}{8}\not{\nabla}_3\mathcal{L}_T(\mathrm{Tr}_{\not{g}}\not{h})h_{34} \\ &+ \frac{1}{8}\not{\nabla}_3(\mathrm{Tr}_{\not{g}}\not{h})(\mathcal{L}_T h)_{34} + \frac{1}{8}(\mathrm{Tr}_{\not{g}}\chi)\Big(h_{34}\mathcal{L}_T(\mathrm{Tr}_{\not{g}}\not{h}) - (\mathcal{L}_T h)_{34}(\mathrm{Tr}_{\not{g}}\not{h})\Big). \end{aligned} \tag{3.3.78}$$

Similarly, noting the relations derived above in lemma 3.3.3 and that

$$\frac{1}{4}\not{g}^{AB}\mathrm{v}_A^{\mathcal{L}_T h}\not{\nabla}_B\mathrm{Tr}_{\not{g}}\not{h} - \frac{1}{4}\not{g}^{AB}\mathrm{v}_A^{h}\not{\nabla}_B\mathcal{L}_T\mathrm{Tr}_{\not{g}}\not{h} \equiv -\frac{1}{4}\not{\mathrm{d}}\mathrm{ivv}^{\mathcal{L}_T h}\mathrm{Tr}_{\not{g}}\not{h} + \frac{1}{4}\not{\mathrm{d}}\mathrm{ivv}^{h}\mathcal{L}_T\mathrm{Tr}_{\not{g}}\not{h}, \tag{3.3.79}$$

the component $\mathcal{J}^T[h]^3$ can be calculated as

$$\begin{aligned}
\mathcal{J}^T[h]^3 \equiv{} & \frac{1}{4}\Big(\langle\widehat{\slashed{\mathcal{L}}_T\slashed{h}},\widehat{\slashed{\nabla}_4\slashed{h}}\rangle - \frac{1}{4}\langle\hat{\slashed{h}},\widehat{\slashed{\nabla}_4(\slashed{\mathcal{L}}_T\slashed{h})}\rangle\Big) + \frac{1}{8}\Big(\slashed{\nabla}_4(\mathcal{L}_T\mathrm{Tr}_{\slashed{g}}\slashed{h})\mathrm{Tr}_{\slashed{g}}\slashed{h} - (\mathcal{L}_T\mathrm{Tr}_{\slashed{g}}\slashed{h})\slashed{\nabla}_4(\mathrm{Tr}_{\slashed{g}}\slashed{h})\Big) \\
& + \frac{1}{8}\Big(\mathcal{L}_T(\mathrm{Tr}_{\slashed{g}}\slashed{h})e_4(h_{34}) - (\mathrm{Tr}_{\slashed{g}}\slashed{h})e_4((\mathcal{L}_T h)_{34}) - \slashed{\nabla}_4\mathcal{L}_T(\mathrm{Tr}_{\slashed{g}}\slashed{h})h_{34} + \slashed{\nabla}_4(\mathrm{Tr}_{\slashed{g}}\slashed{h})(\mathcal{L}_T h)_{34}\Big) \\
& - \frac{1}{8}(\mathrm{Tr}_{\slashed{g}}\chi)\Big(h_{34}\mathcal{L}_T(\mathrm{Tr}_{\slashed{g}}\slashed{h}) - (\mathcal{L}_T h)_{34}(\mathrm{Tr}_{\slashed{g}}\slashed{h})\Big) \qquad (3.3.80) \\
& - \frac{1}{4}(\mathcal{L}_T h)_{34}\slashed{\mathrm{div}}\mathrm{v}^h + \frac{1}{4}h_{34}\slashed{\mathrm{div}}\mathrm{v}^{\mathcal{L}_T h} - \frac{1}{2}\langle\widehat{\slashed{\mathcal{L}}_T h},\widehat{\slashed{\nabla}\mathrm{v}}^h\rangle + \frac{1}{2}\langle\hat{\slashed{h}},\widehat{\slashed{\nabla}\mathrm{v}}^{\mathcal{L}_T h}\rangle \\
& + \frac{1}{4}\Big(\slashed{g}^{AB}\mathrm{v}_A^{\mathcal{L}_T h}(\nabla_3 h)_{4B} - \slashed{g}^{AB}\mathrm{v}_A^h(\nabla_3\mathcal{L}_T h)_{4B} - \slashed{\mathrm{div}}\mathrm{v}^{\mathcal{L}_T h}\mathrm{Tr}_{\slashed{g}}\slashed{h} + \slashed{\mathrm{div}}\mathrm{v}^h\mathcal{L}_T\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big),
\end{aligned}$$

where $\mathrm{v}_B^{\mathrm{S}} = \mathrm{S}_{4B}$ (for $\mathrm{S} = h$ or $\mathrm{S} = \mathcal{L}_T h$) is considered as a covector.

Further, using proposition 2.10.7, one has

$$\slashed{\nabla}_A\mathrm{v}_B^h = -\frac{1}{\Omega}\slashed{\nabla}_A b_B, \qquad (3.3.81)$$

$$\slashed{\nabla}_A\mathrm{v}_B^{\mathcal{L}_T h} = -\frac{1}{2}\slashed{\nabla}_A(\slashed{\nabla}_4\overset{(1)}{b})_B - \Omega\slashed{\nabla}_A(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})_B + \frac{1}{2}\mathrm{Tr}_{\slashed{g}}\chi\slashed{\nabla}_A\overset{(1)}{b}_B. \qquad (3.3.82)$$

So,

$$\widehat{\slashed{\nabla}\mathrm{v}^h} = \frac{1}{\Omega}\slashed{\mathcal{D}}_2^\star\overset{(1)}{b}, \qquad \widehat{\slashed{\nabla}\mathrm{v}^{\mathcal{L}_T h}} = \frac{1}{2}\slashed{\mathcal{D}}_2^\star(\slashed{\nabla}_4\overset{(1)}{b}) + \Omega(\slashed{\mathcal{D}}_2^\star\overset{(1)}{\eta} - \slashed{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}}) - \frac{1}{2}\mathrm{Tr}_{\slashed{g}}\chi\slashed{\mathcal{D}}_2^\star\overset{(1)}{b}, \qquad (3.3.83)$$

$$\slashed{\mathrm{div}}\mathrm{v}^h = -\frac{1}{\Omega}\slashed{\mathrm{div}}\overset{(1)}{b}, \qquad \slashed{\mathrm{div}}\mathrm{v}^{\mathcal{L}_T h} = -\frac{1}{2}\slashed{\mathrm{div}}(\slashed{\nabla}_4\overset{(1)}{b}) - \Omega\slashed{\mathrm{div}}\overset{(1)}{\eta} + \Omega\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} + \frac{1}{2}\mathrm{Tr}_{\slashed{g}}\chi\slashed{\mathrm{div}}\overset{(1)}{b}. \qquad (3.3.84)$$

Therefore, exploiting these relations and propositions 3.3.4-3.3.7, one can write two complicated expressions for $\mathcal{J}^T[h]^4$ and $\mathcal{J}^T[h]^3$:

$$\begin{aligned}
\mathcal{J}^T[h]^4 \equiv{} & \frac{\Omega}{2}|\overset{(1)}{\hat{\underline{\chi}}}|^2 - \frac{\omega}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) - \frac{1}{2\Omega}\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi}) - \frac{1}{4\Omega}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})^2 + \frac{\Omega}{2}\langle\overset{(1)}{\hat{\chi}},\overset{(1)}{\hat{\underline{\chi}}}\rangle \qquad (3.3.85) \\
& - \frac{\Omega}{2}\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}\rangle - \frac{1}{4\Omega}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) - \frac{1}{2\Omega}\overset{(1)}{\underline{\omega}}\Big(2(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) - \slashed{\mathrm{div}}\overset{(1)}{b}\Big) - \frac{1}{2\Omega}\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) \\
& + \frac{\Omega}{2}\langle\slashed{\mathrm{div}}\hat{\slashed{h}},\overset{(1)}{\underline{\eta}}\rangle + \frac{\Omega}{4}\langle\hat{\slashed{h}},\overset{(1)}{\underline{\alpha}}\rangle + \frac{1}{2}\overset{(1)}{\underline{\beta}}(\overset{(1)}{b}) - \frac{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}{8}\langle\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}},\hat{\slashed{h}}\rangle + \frac{1}{2}\omega\langle\hat{\slashed{h}},\overset{(1)}{\hat{\underline{\chi}}}\rangle - \frac{1}{4}\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\underline{\eta}}(\overset{(1)}{b}) \\
& + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{16}\Big[(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) + (\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\Big]\Big(\mathrm{Tr}_{\slashed{g}}\slashed{h} - 4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) - \frac{1}{4\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(4\partial_u\overset{(1)}{\omega} + 4(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\underline{\omega}}\Big) \\
& + \frac{1}{16\Omega}\Big(4\Omega^2\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} - 4\omega(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) + 4(\partial_u\overset{(1)}{\underline{\omega}}) + 4(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\omega}\Big)\mathrm{Tr}_{\slashed{g}}\slashed{h},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{J}^T[h]^3 \equiv{}& \frac{\Omega}{2}|\overset{(1)}{\hat{\chi}}|^2 - \frac{1}{4\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)}{}^2 - \frac{1}{2\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\underline{\chi})}\overset{(1)}{\omega} + \frac{\omega}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)} + \frac{\Omega}{2}\langle \overset{(1)}{\hat{\underline{\chi}}}, \overset{(1)}{\hat{\chi}}\rangle \\
&+ \frac{\Omega}{2}|\overset{(1)}{\underline{\eta}}|^2 + \frac{1}{8}\mathrm{Tr}_{\not{g}}\chi\langle \overset{(1)}{b}, \overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}\rangle + \frac{1}{4}\langle \overset{(1)}{b}, \overset{(1)}{\underline{\beta}} - \overset{(1)}{\beta}\rangle + \frac{1}{8}(\Omega \mathrm{Tr}_{\not{g}}\chi)\langle \hat{\not{h}}, (\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}})\rangle_{\not{g}} - \frac{3}{2}\Omega\langle \overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle \\
&- \frac{1}{4\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\underline{\chi})}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)} - \frac{1}{\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)}\overset{(1)}{\omega} - \frac{1}{2\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)}\overset{(1)}{\underline{\omega}} + \frac{1}{2\Omega}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\overset{(1)}{\omega} \\
&- \frac{1}{2}\omega\langle \hat{\not{h}}, \overset{(1)}{\hat{\chi}}\rangle + \frac{\Omega}{4}\langle \hat{\not{h}}, \overset{(1)}{\alpha}\rangle + \frac{\Omega}{2}\langle \not{\mathrm{d}}\mathrm{iv}\hat{\not{h}}, \overset{(1)}{\eta}\rangle + \frac{1}{4}\Big(\langle(\not{\nabla}_4\overset{(1)}{b}), (\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})\rangle + \langle \overset{(1)}{b}, \not{\nabla}_4\overset{(1)}{\underline{\eta}}\rangle - \langle \overset{(1)}{b}, \not{\nabla}_3\overset{(1)}{\eta}\rangle\Big) \\
&+ \frac{1}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big((\Omega \mathrm{Tr}_{\not{g}}\chi)\overset{(1)}{\omega} - \partial_u\overset{(1)}{\omega}\Big) + \frac{\Omega \mathrm{Tr}_{\not{g}}\chi}{16}\Big[\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)} + \overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\underline{\chi})}\Big]\Big(4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big) \\
&+ \frac{1}{4\Omega}\Big(\Omega^2\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + \partial_v\overset{(1)}{\omega} + \omega\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)} - (\Omega \mathrm{Tr}_{\not{g}}\chi)\overset{(1)}{\underline{\omega}}\Big)\mathrm{Tr}_{\not{g}}\not{h},
\end{aligned}
\tag{3.3.86}
$$

where the following relation has been employed

$$
2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} \equiv -\langle \overset{(1)}{\eta}, \overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}\rangle
\tag{3.3.87}
$$

and similarly for $\overset{(1)}{\underline{\eta}}$. Also, note that in simplifying these expressions one uses the linearised Codazzi equations in proposition 2.10.17 to give that

$$
\langle \not{\mathcal{D}}_2^{\star}\overset{(1)}{b}, 2\overset{(1)}{\hat{\underline{\chi}}}\rangle \equiv 2\langle \overset{(1)}{b}, \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\hat{\underline{\chi}}}\rangle = \mathrm{Tr}_{\not{g}}\chi\langle \overset{(1)}{b}, \overset{(1)}{\eta}\rangle + 2\langle \overset{(1)}{b}, \overset{(1)}{\underline{\beta}}\rangle - \frac{1}{\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\underline{\chi})}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b},
\tag{3.3.88}
$$

$$
\langle \not{\mathcal{D}}_2^{\star}\overset{(1)}{b}, 2\overset{(1)}{\hat{\chi}}\rangle \equiv 2\langle \overset{(1)}{b}, \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\hat{\chi}}\rangle = -\mathrm{Tr}_{\not{g}}\chi\langle \overset{(1)}{b}, \overset{(1)}{\underline{\eta}}\rangle - 2\langle \overset{(1)}{b}, \overset{(1)}{\beta}\rangle - \frac{1}{\Omega}\overset{(1)}{(\Omega \mathrm{Tr}_{\not{g}}\chi)}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}.
\tag{3.3.89}
$$

The function $\mathcal{A}$ can be identified by the following observations:

(1) If one considers the wider problem of interest, namely the conservation law on the boundary of a characteristic rectangle on the exterior $\mathcal{E}_A$ of Schw_4 then if one can write

$$
\mathcal{J}^T[h]^3 = \overline{\mathcal{J}^T[h]}^3 - \frac{1}{r^2}\not{\nabla}_4 F, \qquad \mathcal{J}^T[h]^4 = \overline{\mathcal{J}^T[h]}^4 + \frac{1}{r^2}\not{\nabla}_3 F,
\tag{3.3.90}
$$

for some function F, then one has a cancellation of F at the spheres at the corners of the characteristic rectangle.

(2) There are terms in $\mathcal{J}^4$ and $\mathcal{J}^3$ which appear with the correct derivative (∂_u for $\mathcal{J}^4$ and ∂_v for $\mathcal{J}^3$) to integrate by parts but one has no expression for them in terms of the null structure equations. For example $\partial_u\overset{(1)}{\underline{\omega}}\mathrm{Tr}_{\not{g}}\not{h}$ and $\langle \not{\nabla}_4\overset{(1)}{b}, (\overset{(1)}{\eta} - \overset{(1)}{\eta})\rangle$. However, one has expressions for $\partial_v\overset{(1)}{\underline{\omega}}$ and $\partial_u\overset{(1)}{b}{}^A$ from propositions 2.10.13 and 2.10.7 respectively. So, with point (1) in mind, expressing such terms as total derivative and adding and subtracting such terms is advantageous to manipulate the expressions for $\mathcal{J}^3$ and $\mathcal{J}^4$.

The function $\mathcal{A}$ (defined in theorem 3.1.7) can be written as

$$\mathcal{A}[h] = \frac{1}{r^2}\Big(\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 - \mathcal{A}_4 + \mathcal{A}_5 - \mathcal{A}_6 - \mathcal{A}_7 + \mathcal{A}_8 + \mathcal{A}_9 - \mathcal{A}_{10}\Big) \tag{3.3.91}$$

with

$$\begin{aligned}
\mathcal{A}_1 &\doteq \frac{1}{4} r^2 \overset{(1)}{\underline{\omega}} \mathrm{Tr}_{\not g} \not h, & \mathcal{A}_2 &\doteq \frac{1}{4} r^2 \overset{(1)}{\omega} \mathrm{Tr}_{\not g} \not h, \\
\mathcal{A}_3 &\doteq \frac{r^2}{4} \langle \overset{(1)}{b}, \overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}} \rangle, & \mathcal{A}_4 &\doteq \frac{r^2}{8} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)} \mathrm{Tr}_{\not g} \not h, \\
\mathcal{A}_5 &\doteq \frac{r^2}{8} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\underline{\chi})} \mathrm{Tr}_{\not g} \not h, & \mathcal{A}_6 &\doteq \frac{r^2 \Omega}{4} \langle \overset{(1)}{\hat{\underline{\chi}}} - \overset{(1)}{\hat{\chi}}, \hat{\not h} \rangle, \\
\mathcal{A}_7 &\doteq \frac{3}{2} r^2 \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\underline{\chi})}, & \mathcal{A}_8 &\doteq \frac{3r^2}{2} \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)}, \\
\mathcal{A}_9 &\doteq \frac{r^2 (\Omega \mathrm{Tr}_{\not g}\chi)}{2} \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \mathrm{Tr}_{\not g} \not h, & \mathcal{A}_{10} &\doteq 2r^2 (\Omega \mathrm{Tr}_{\not g}\chi) \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2.
\end{aligned} \tag{3.3.92}$$

One can check that using proposition 2.10.7 that

$$\frac{1}{r^2} \not\nabla_3 \mathcal{A}_1 = -\frac{1}{4} (\mathrm{Tr}_{\not g}\chi) \overset{(1)}{\underline{\omega}} \mathrm{Tr}_{\not g} \not h + \frac{1}{4\Omega} \partial_u \overset{(1)}{\underline{\omega}} \mathrm{Tr}_{\not g} \not h + \frac{1}{2\Omega} \overset{(1)}{\underline{\omega}} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\underline{\chi})}, \tag{3.3.93}$$

$$\frac{1}{r^2} \not\nabla_4 \mathcal{A}_1 = \frac{1}{4}\Big((\mathrm{Tr}_{\not g}\chi) \overset{(1)}{\underline{\omega}} + \frac{1}{\Omega} \partial_v \overset{(1)}{\underline{\omega}}\Big) \mathrm{Tr}_{\not g} \not h + \frac{\overset{(1)}{\underline{\omega}}}{2\Omega}\Big(\overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)} - \not{\mathrm{d}}\mathrm{iv} \overset{(1)}{b}\Big), \tag{3.3.94}$$

$$\frac{1}{r^2} \not\nabla_3 \mathcal{A}_2 = -\frac{1}{4} (\mathrm{Tr}_{\not g}\chi) \overset{(1)}{\omega} \mathrm{Tr}_{\not g} \not h + \frac{1}{4\Omega} \partial_u \overset{(1)}{\omega} \mathrm{Tr}_{\not g} \not h + \frac{1}{2\Omega} \overset{(1)}{\omega} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\underline{\chi})}, \tag{3.3.95}$$

$$\frac{1}{r^2} \not\nabla_4 \mathcal{A}_2 = \frac{1}{4}\Big((\mathrm{Tr}_{\not g}\chi) \overset{(1)}{\omega} + \frac{1}{\Omega} \partial_v \overset{(1)}{\omega}\Big) \mathrm{Tr}_{\not g} \not h + \frac{\overset{(1)}{\omega}}{2\Omega}\Big(\overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)} - \not{\mathrm{d}}\mathrm{iv} \overset{(1)}{b}\Big). \tag{3.3.96}$$

Using propositions 2.10.7 and 2.10.11 one has

$$\frac{1}{r^2} \not\nabla_3 \mathcal{A}_3 = -\frac{1}{4} (\mathrm{Tr}_{\not g}\chi) \langle \overset{(1)}{b}, \overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}} \rangle + \frac{\Omega}{2} |\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}|^2 + \frac{1}{4} \langle \overset{(1)}{b}, \not\nabla_3 \overset{(1)}{\eta} \rangle - \frac{1}{4} \langle \overset{(1)}{b}, \overset{(1)}{\underline{\beta}} \rangle, \tag{3.3.97}$$

$$\frac{1}{r^2} \not\nabla_4 \mathcal{A}_3 = \frac{\mathrm{Tr}_{\not g}\chi}{8} (\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})(\overset{(1)}{b}) + \frac{1}{4}\Big[(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})(\not\nabla_4 \overset{(1)}{b}) - \langle \overset{(1)}{b}, \not\nabla_4 \overset{(1)}{\underline{\eta}} \rangle - \overset{(1)}{\beta}(\overset{(1)}{b})\Big]. \tag{3.3.98}$$

Using propositions 2.10.7, 2.10.8 and 2.10.9 gives

$$\begin{aligned}
\frac{1}{r^2} \not\nabla_3 \mathcal{A}_4 = {} & \frac{1}{4\Omega} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\underline{\chi})} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)} + \frac{\Omega}{4} \mathrm{Tr}_{\not g} \not h \not{\mathrm{d}}\mathrm{iv} \overset{(1)}{\eta} - \frac{1}{4\Omega} (\partial_u \overset{(1)}{\omega}) \mathrm{Tr}_{\not g} \not h \\
& - \frac{1}{16\Omega} \mathrm{Tr}_{\not g} \not h (\Omega \mathrm{Tr}_{\not g}\chi)\Big(\overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\underline{\chi})} + \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)}\Big),
\end{aligned} \tag{3.3.99}$$

$$\begin{aligned}
\frac{1}{r^2} \not\nabla_4 \mathcal{A}_4 = {} & \frac{1}{4\Omega} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)}^2 + \frac{1}{4\Omega} \mathrm{Tr}_{\not g} \not h \Big(\omega \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)} + (\Omega \mathrm{Tr}_{\not g}\chi) \overset{(1)}{\omega}\Big) \\
& - \frac{1}{4\Omega} \overset{(1)}{(\Omega \mathrm{Tr}_{\not g}\chi)} \not{\mathrm{d}}\mathrm{iv} \overset{(1)}{b}
\end{aligned} \tag{3.3.100}$$

and

$$\frac{1}{r^2}\slashed{\nabla}_3\mathcal{A}_5 = \frac{1}{4\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})}^2 - \frac{1}{4\Omega}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big(\omega\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} + (\Omega\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\underline{\omega}}\Big), \tag{3.3.101}$$

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{A}_5 = {} & \frac{1}{4\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} - \slashed{\mathrm{div}}\overset{(1)}{b}\Big) - \frac{1}{4\Omega}(\partial_v\overset{(1)}{\underline{\omega}})\mathrm{Tr}_{\slashed{g}}\slashed{h} \\ & + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{16}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} + \overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}\Big] + \frac{\Omega}{4}\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}}\mathrm{Tr}_{\slashed{g}}\slashed{h}.\end{aligned} \tag{3.3.102}$$

Using propositions 2.10.7 and 2.10.10 gives

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_3\mathcal{A}_6 = {} & \frac{1}{8}(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}}, \hat{\slashed{h}}\rangle - \frac{\Omega}{2}\langle\overset{(1)}{\hat{\underline{\chi}}}, \overset{(1)}{\hat{\chi}}\rangle + \frac{\Omega}{2}|\overset{(1)}{\hat{\underline{\chi}}}|^2 + \frac{\Omega}{2}\langle\overset{(1)}{\underline{\eta}}, \slashed{\mathrm{div}}\hat{\slashed{h}}\rangle \\ & - \frac{1}{2}\omega\langle\overset{(1)}{\hat{\underline{\chi}}}, \hat{\slashed{h}}\rangle - \frac{\Omega}{4}\langle\overset{(1)}{\underline{\alpha}}, \hat{\slashed{h}}\rangle,\end{aligned} \tag{3.3.103}$$

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{A}_6 \equiv {} & \frac{\Omega\mathrm{Tr}_{\slashed{g}}\chi}{8}\langle\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}}, \hat{\slashed{h}}\rangle + \frac{\Omega}{2}\langle\overset{(1)}{\hat{\underline{\chi}}}, \overset{(1)}{\hat{\chi}}\rangle - \frac{\Omega}{2}|\overset{(1)}{\hat{\chi}}|^2 - \frac{\Omega}{2}\langle\overset{(1)}{\underline{\eta}}, \slashed{\mathrm{div}}\hat{\slashed{h}}\rangle \\ & - \frac{1}{2}\omega\langle\overset{(1)}{\hat{\chi}}, \hat{\slashed{h}}\rangle + \frac{\Omega}{4}\langle\overset{(1)}{\alpha}, \hat{\slashed{h}}\rangle + \frac{1}{2}(\overset{(1)}{\underline{\beta}} + \overset{(1)}{\beta})(\overset{(1)}{b}) + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{4}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\overset{(1)}{b}) \\ & - \frac{1}{4\Omega}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} - \overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}\Big)\slashed{\mathrm{div}}\overset{(1)}{b}.\end{aligned} \tag{3.3.104}$$

With propositions 2.10.7, 2.10.8 and 2.10.9 one has

$$\frac{1}{r^2}\slashed{\nabla}_3\mathcal{A}_7 = \frac{3}{2\Omega}\overset{(1)}{\underline{\omega}}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} - \frac{3}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(\omega\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} + (\Omega\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\underline{\omega}}\Big), \tag{3.3.105}$$

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{A}_7 \equiv {} & \frac{3}{2\Omega}\overset{(1)}{\omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} - \frac{3}{2}\Omega\langle\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle - \frac{3\Omega}{2}\Omega|\overset{(1)}{\underline{\eta}}|^2 - \frac{3}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\partial_v\overset{(1)}{\underline{\omega}} \\ & + \frac{3}{4\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} + \overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}\Big).\end{aligned} \tag{3.3.106}$$

Analogously, with propositions 2.10.7, 2.10.8 and 2.10.9 one has

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_3\mathcal{A}_8 = {} & \frac{3}{2\Omega}\overset{(1)}{\underline{\omega}}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} - \frac{3}{2}\Omega|\overset{(1)}{\eta}|^2 - \frac{3}{2}\Omega\langle\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle - \frac{3}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\partial_u\overset{(1)}{\omega} \\ & - \frac{3}{4}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\mathrm{Tr}_{\slashed{g}}\chi)\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} + \overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})}\Big),\end{aligned} \tag{3.3.107}$$

$$\frac{1}{r^2}\slashed{\nabla}_4\mathcal{A}_8 = \frac{3}{2\Omega}\overset{(1)}{\omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} + \frac{3}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(\omega\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} + (\Omega\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\omega}\Big). \tag{3.3.108}$$

Using proposition 2.10.7 gives

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_3\mathcal{A}_9 = {}& -\frac{\Omega}{4}(\mathrm{Tr}_{\slashed{g}}\chi)^2\frac{\overset{(1)}{\Omega}}{\Omega}\mathrm{Tr}_{\slashed{g}}\slashed{h} - \omega\mathrm{Tr}_{\slashed{g}}\chi\frac{\overset{(1)}{\Omega}}{\Omega}\mathrm{Tr}_{\slashed{g}}\slashed{h} + \frac{1}{\Omega}(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\frac{\overset{(1)}{\Omega}}{\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})}\\ & + \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\underline{\omega}}\mathrm{Tr}_{\slashed{g}}\slashed{h},\end{aligned} \tag{3.3.109}$$

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{A}_9 = {}& \frac{1}{2}(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\omega}\mathrm{Tr}_{\slashed{g}}\slashed{h} + \frac{1}{4}\Big[\Omega\mathrm{Tr}_{\slashed{g}}\chi + 4\omega\Big]\mathrm{Tr}_{\slashed{g}}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\mathrm{Tr}_{\slashed{g}}\slashed{h}\\ & + \mathrm{Tr}_{\slashed{g}}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} + \frac{1}{2}\mathrm{Tr}_{\slashed{g}}\chi(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\overset{(1)}{b}),\end{aligned} \tag{3.3.110}$$

$$\frac{1}{r^2}\slashed{\nabla}_3\mathcal{A}_{10} = -\Omega(\mathrm{Tr}_{\slashed{g}}\chi)^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2 - 4\omega(\mathrm{Tr}_{\slashed{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2 + 4(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\underline{\omega}}, \tag{3.3.111}$$

$$\frac{1}{r^2}\slashed{\nabla}_4\mathcal{A}_{10} = \mathrm{Tr}_{\slashed{g}}\chi\Big[\Omega\mathrm{Tr}_{\slashed{g}}\chi + 4\omega\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2 + 4(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\omega}. \tag{3.3.112}$$

Therefore, denoting $(\overline{\mathcal{J}}^T[h])^3 = (\mathcal{J}^T[h])^3 + \frac{1}{r^2}\slashed{\nabla}_4(r^2\mathcal{A})$ and $(\overline{\mathcal{J}}^T[h])^4 = (\mathcal{J}^T[h])^4 - \frac{1}{r^2}\slashed{\nabla}_3(r^2\mathcal{A})$, one can calculate that

$$\begin{aligned}(\overline{\mathcal{J}}^T[h])^3 \equiv {}& \Omega|\overset{(1)}{\hat{\chi}}|^2 - \frac{1}{2\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}{}^2 - \frac{2}{\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})}\overset{(1)}{\omega} + \frac{4}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\omega\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} + 2\Omega|\overset{(1)}{\underline{\eta}}|^2\\ & + \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big\{\frac{2}{\Omega}\partial_v\overset{(1)}{\underline{\omega}} - \Omega\Big(\slashed{\mathrm{div}}\slashed{\mathrm{div}}\hat{\slashed{h}} - \frac{1}{2}\slashed{\Delta}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big) - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{2}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} - \overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}\Big)\\ & + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{4}\Big(\Omega\mathrm{Tr}_{\slashed{g}}\chi + 4\omega\Big)\Big(\mathrm{Tr}_{\slashed{g}}\slashed{h} - 4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)\Big\},\end{aligned} \tag{3.3.113}$$

$$\begin{aligned}(\overline{\mathcal{J}}^T[h])^3 \equiv {}& \Omega|\overset{(1)}{\hat{\underline{\chi}}}|^2 - \frac{4}{\Omega}\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} - \frac{2}{\Omega}\overset{(1)}{\underline{\omega}}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)} - \frac{1}{2\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})}{}^2 + 2\Omega|\overset{(1)}{\eta}|^2\\ & + \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big\{\frac{2}{\Omega}\partial_u\overset{(1)}{\omega} - \Omega\Big(\slashed{\mathrm{div}}\slashed{\mathrm{div}}\hat{\slashed{h}} - \frac{1}{2}\slashed{\Delta}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big) - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{2}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi})} - \overset{(1)}{(\Omega\mathrm{Tr}_{\slashed{g}}\chi)}\Big)\\ & + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{4}\Big(\Omega\mathrm{Tr}_{\slashed{g}}\chi + 4\omega\Big)\Big(\mathrm{Tr}_{\slashed{g}}\slashed{h} - 4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)\Big\},\end{aligned} \tag{3.3.114}$$

where one uses that

$$\Big\langle\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}, \slashed{\mathrm{div}}\hat{\slashed{h}} - \frac{1}{2}\slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big\rangle \equiv -2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big[\slashed{\mathrm{div}}\slashed{\mathrm{div}}\hat{\slashed{h}} - \frac{1}{2}\slashed{\Delta}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]. \tag{3.3.115}$$

Using the linearised Gauss equation in proposition 2.10.15 gives the result stated in theorem 3.1.7.

□

3.3.4 Proof of Theorem 3.1.9

Proof of theorem 3.1.9. In this proof the following convention is adopted. The symbol $\equiv$ will denote equality under integration by parts of $\mathbb{S}^2_{u,v}$.

Now rather than go through the direct computation as in section 3.3.3 one can use the following idea to avoid (some of) the long computations. By propositions C.1.4 and the discussion in remark 2.10.3, if h satisfies the linearised null structure equations then $\mathcal{L}_{\Omega_k}h$ does. Note further that one can establish

$$(\mathcal{L}_{\Omega_k}h)_{33} = 0 = (\mathcal{L}_{\Omega_k}h)_{44} = (\mathcal{L}_{\Omega_k}h)_{3A}. \tag{3.3.116}$$

Therefore, if one expresses $(\mathcal{J}^T[h])^4 - \frac{1}{r^2}\nabla\!\!\!/_3(r^2\mathcal{A}[h])$ and $(\mathcal{J}^T[h])^3 + \frac{1}{r^2}\nabla\!\!\!/_4(r^2\mathcal{A}[h])$ in terms of h, then replacing with $\mathcal{L}_{\Omega_k}h$ will result in a simpler form for the components of $(\mathcal{J}^T[\mathcal{L}_{\Omega_k}h])$ (plus a boundary term). One can check that this yields

$$\begin{aligned}(\mathcal{J}^T[\mathcal{L}_{\Omega_k}h])^4 = \frac{1}{\Omega}\Big(&\Omega^2|\mathcal{L}\!\!\!/_{\Omega_k}\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2\Omega^2|\mathcal{L}\!\!\!/_{\Omega_k}\overset{(1)}{\eta}|^2 - 4\omega\Omega_k\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Omega_k((\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi})) \\ &- 2\Omega_k(\overset{(1)}{\underline{\omega}})\Omega_k((\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi)) - \frac{1}{2}(\Omega_k(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}))^2\Big) + \frac{1}{r^2}\nabla\!\!\!/_3(r^2\mathcal{A}[\mathcal{L}_{\Omega_k}h]),\end{aligned} \tag{3.3.117}$$

$$\begin{aligned}(\mathcal{J}^T[\mathcal{L}_{\Omega_k}h])^3 = \frac{1}{\Omega}\Big(&2\Omega^2|\mathcal{L}\!\!\!/_{\Omega_k}\overset{(1)}{\underline{\eta}}|^2 + \Omega^2|\mathcal{L}\!\!\!/_{\Omega_k}\overset{(1)}{\hat{\chi}}|^2 + 4\omega\Omega_k\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Omega_k((\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi))\Big) \\ &- 2\Omega_k((\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}))\Omega_k(\overset{(1)}{\omega}) - \frac{1}{2}(\Omega_k(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi))^2 - \frac{1}{r^2}\nabla\!\!\!/_4(r^2\mathcal{A}[\mathcal{L}_{\Omega_k}h]).\end{aligned} \tag{3.3.118}$$

For simplicity, denote $\mathcal{J}\!\!\!/^a \doteq \sum_k(\mathcal{J}^T[\mathcal{L}_{\Omega_k}h])^a$ and $\mathcal{A}\!\!\!/ \doteq \sum_k \mathcal{A}[\mathcal{L}_{\Omega_k}h]$. Now, note that $\{\Omega_k\}_k$ satisfy the following identities

$$\sum_k \Omega_k^A\Omega_k^B = r^2 g\!\!\!/^{AB}, \tag{3.3.119}$$

$$\mathcal{D}\!\!\!/^\star_2\Omega_k = 0 = \mathrm{d}\!\!\!/\mathrm{iv}\Omega_k, \tag{3.3.120}$$

where the latter two identities follow from the Killing property. Next from lemma 2.4.3, a covector $\xi \in \Omega^1(\mathbb{S}^2_{u,v})$ satisfies

$$|\mathcal{L}\!\!\!/_{\Omega_k}\xi|^2 = |\nabla\!\!\!/_{\Omega_k}\xi|^2 + \frac{1}{4}(\mathrm{c}\!\!\!/\mathrm{url}\Omega_k)^2|\xi|^2 + (\mathrm{c}\!\!\!/\mathrm{url}\Omega_k)\epsilon\!\!\!/(\xi, \nabla\!\!\!/_{\Omega_k}\xi). \tag{3.3.121}$$

One can check that

$$\sum_k(\mathrm{c}\!\!\!/\mathrm{url}\Omega_k)\Omega_k = 0. \tag{3.3.122}$$

Using equation (3.3.119) and that $\sum_k (\mathrm{c\slashed{u}rl}\Omega_k)^2 = 4$ one finds

$$\sum_k |\slashed{\mathcal{L}}_{\Omega_k}\xi|^2 = r^2|\slashed{\nabla}\xi|^2 + |\xi|^2. \tag{3.3.123}$$

Similarly, using lemma 2.4.3, for $\Theta \in \mathrm{symtr}(T^\star\mathbb{S}^2_{u,v} \otimes T^\star\mathbb{S}^2_{u,v})$ one has

$$\sum_k |\slashed{\mathcal{L}}_{\Omega_k}\Theta|^2 = r^2|\slashed{\nabla}\Theta|^2 + 4|\Theta|^2. \tag{3.3.124}$$

Then, using the above results with $\xi \in \{\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\}$ and $\Theta \in \{\overset{(1)}{\hat{\underline{\chi}}}, \overset{(1)}{\hat{\chi}}\}$ gives

$$\begin{aligned}
\mathscr{I}^4 \equiv{}& \frac{1}{\Omega}\Big(\Omega^2 r^2|\slashed{\nabla}\overset{(1)}{\hat{\underline{\chi}}}|^2 + 4\Omega^2|\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2\Omega^2 r^2|\slashed{\nabla}\overset{(1)}{\eta}|^2 + 2\Omega^2|\overset{(1)}{\eta}|^2 - \frac{1}{2}r^2\big|\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\big|^2 && (3.3.125)\\
&- 4r^2\omega\langle\slashed{\nabla}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \slashed{\nabla}((\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}))\rangle - 2r^2\langle\slashed{\nabla}(\overset{(1)}{\underline{\omega}}), \slashed{\nabla}((\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi}))\rangle\Big) + \frac{1}{r^2}\slashed{\nabla}_3(r^2\slashed{\mathcal{A}}),\\
\mathscr{I}^3 \equiv{}& \frac{1}{\Omega}\Big(2\Omega^2 r^2|\slashed{\nabla}\overset{(1)}{\underline{\eta}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2 + \Omega^2 r^2|\slashed{\nabla}\overset{(1)}{\hat{\chi}}|^2 + 4\Omega^2|\overset{(1)}{\hat{\chi}}|^2 - \frac{1}{2}r^2\big|\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\big|^2 && (3.3.126)\\
&- 2r^2\langle\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}), \slashed{\nabla}(\overset{(1)}{\omega})\rangle + 4r^2\omega\langle\slashed{\nabla}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\rangle\Big) - \frac{1}{r^2}\slashed{\nabla}_4(r^2\slashed{\mathcal{A}}).
\end{aligned}$$

Recall that, from propositions 2.10.7, 2.10.11, 2.10.14, and lemma 2.4.3

$$\begin{aligned}
\slashed{d}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) &= \frac{1}{2}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}), & \mathrm{c\slashed{u}rl}\overset{(1)}{\eta} &= \overset{(1)}{\sigma} = -\mathrm{c\slashed{u}rl}\overset{(1)}{\underline{\eta}},\\
\slashed{d}\overset{(1)}{\underline{\omega}} &= \frac{\Omega}{2}\Big(\slashed{\nabla}_3\overset{(1)}{\eta} - \mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\eta} + \overset{(1)}{\underline{\beta}}\Big), & \slashed{d}\overset{(1)}{\omega} &= \frac{\Omega}{2}\Big(\slashed{\nabla}_4\overset{(1)}{\underline{\eta}} + \mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\underline{\eta}} - \overset{(1)}{\beta}\Big), && (3.3.127)\\
|\slashed{\nabla}\Theta|^2 &\equiv 2|\mathrm{d\slashed{i}v}\Theta|^2 - \mathrm{Scal}(\slashed{g})|\Theta|^2, & |\slashed{\nabla}\xi|^2 &\equiv |\mathrm{c\slashed{u}rl}\xi|^2 + |\mathrm{d\slashed{i}v}\xi|^2 - \frac{\mathrm{Scal}(\slashed{g})}{2}|\xi|^2.
\end{aligned}$$

Combining these results with $\xi \in \{\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\}$ and $\Theta \in \{\overset{(1)}{\hat{\underline{\chi}}}, \overset{(1)}{\hat{\chi}}\}$ and the commutation lemma 2.8.4 gives

$$\begin{aligned}
\mathscr{I}^4 \equiv{}& \frac{1}{\Omega}\Big(2\Omega^2 r^2|\mathrm{d\slashed{i}v}\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2\Omega^2|\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2\Omega^2 r^2|\mathrm{d\slashed{i}v}\overset{(1)}{\eta}|^2 + 2\Omega^2 r^2|\overset{(1)}{\sigma}|^2 - \frac{1}{2}r^2\big|\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\big|^2 && (3.3.128)\\
&+ 2r^2\omega\,\mathrm{d\slashed{i}v}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}}) + r^2\Omega\slashed{\nabla}_3(\mathrm{d\slashed{i}v}\overset{(1)}{\eta})(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi}) - \frac{3}{2}r^2\Omega(\mathrm{Tr}_{\slashed{g}}\chi)\mathrm{d\slashed{i}v}(\overset{(1)}{\eta})(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\\
&+ r^2\Omega\,\mathrm{d\slashed{i}v}(\overset{(1)}{\underline{\beta}})(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\Big) + \frac{1}{r^2}\slashed{\nabla}_3(r^2\slashed{\mathcal{A}}),\\
\mathscr{I}^3 \equiv{}& \frac{1}{\Omega}\Big(2\Omega^2 r^2(\mathrm{d\slashed{i}v}\overset{(1)}{\underline{\eta}})^2 + 2\Omega^2 r^2|\overset{(1)}{\sigma}|^2 + \Omega^2 r^2|\mathrm{d\slashed{i}v}\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\hat{\chi}}|^2 - \frac{1}{2}r^2\big|\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\big|^2 && (3.3.129)\\
&+ r^2\Omega(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\slashed{\nabla}_4(\mathrm{d\slashed{i}v}\overset{(1)}{\underline{\eta}}) + \frac{3}{2}r^2\mathrm{Tr}_{\slashed{g}}\chi\Omega(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\mathrm{d\slashed{i}v}(\overset{(1)}{\underline{\eta}}) - r^2\Omega(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\mathrm{d\slashed{i}v}(\overset{(1)}{\beta})\\
&- 2r^2\omega\,\mathrm{d\slashed{i}v}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\Omega\overset{(1)}{\mathrm{Tr}_{\slashed{g}}\chi})\Big) - \frac{1}{r^2}\slashed{\nabla}_4(r^2\slashed{\mathcal{A}}).
\end{aligned}$$

Using the linearised Codazzi equations in proposition 2.10.17 one has

$$
\begin{aligned}
2\Omega^2 r^2|\slashed{\mathrm{div}}\overset{(1)}{\underline{\hat{\chi}}}|^2 \equiv{}& \frac{r^2}{2}|\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})|^2 + \frac{\Omega^2 r^2(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2}|\overset{(1)}{\eta}|^2 + 2\Omega^2 r^2|\overset{(1)}{\underline{\beta}}|^2 \\
&+ 2\Omega^2 r^2(\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\beta}}\rangle - \Omega r^2\mathrm{Tr}_{\slashed{g}}\chi(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\eta} - 2\Omega r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\underline{\beta}},
\end{aligned} \tag{3.3.130}
$$

$$
\begin{aligned}
2\Omega^2 r^2|\slashed{\mathrm{div}}\overset{(1)}{\hat{\chi}}|^2 \equiv{}& \frac{r^2}{2}|\slashed{\nabla}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)|^2 + \frac{\Omega^2 r^2(\mathrm{Tr}_{\slashed{g}}\chi)^2}{2}|\overset{(1)}{\underline{\eta}}|^2 + 2\Omega^2 r^2|\overset{(1)}{\beta}|^2 \\
&+ 2\Omega^2 r^2(\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\beta}\rangle + \Omega r^2\mathrm{Tr}_{\slashed{g}}\chi(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} + 2\Omega r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\beta}.
\end{aligned} \tag{3.3.131}
$$

Substituting into $\mathscr{J}^a$ gives

$$
\begin{aligned}
\mathscr{J}^4 \equiv{}& \frac{1}{\Omega}\Big(2\Omega^2|\overset{(1)}{\underline{\hat{\chi}}}|^2 + 2\Omega^2 r^2|\slashed{\mathrm{div}}\overset{(1)}{\eta}|^2 + 2\Omega^2 r^2|\overset{(1)}{\sigma}|^2 + \frac{\Omega^2 r^2}{2}(\mathrm{Tr}_{\slashed{g}}\chi)^2|\overset{(1)}{\eta}|^2 + 2\Omega^2 r^2|\overset{(1)}{\underline{\beta}}|^2 \\
&+ 2r^2\omega\slashed{\mathrm{div}}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) + r^2\Omega\slashed{\nabla}_3(\slashed{\mathrm{div}}\overset{(1)}{\eta})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + r^2\Omega\slashed{\mathrm{div}}(\overset{(1)}{\underline{\beta}})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) \\
&- \frac{3}{2}r^2\Omega(\mathrm{Tr}_{\slashed{g}}\chi)\slashed{\mathrm{div}}(\overset{(1)}{\eta})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + 2\Omega^2 r^2(\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\beta}}\rangle - \Omega r^2\mathrm{Tr}_{\slashed{g}}\chi(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\eta} \\
&- 2\Omega r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\underline{\beta}}\Big) + \frac{1}{r^2}\slashed{\nabla}_3(r^2\slashed{\mathcal{A}}),
\end{aligned} \tag{3.3.132}
$$

$$
\begin{aligned}
\mathscr{J}^3 \equiv{}& \frac{1}{\Omega}\Big(2\Omega^2 r^2(\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}})^2 + 2\Omega^2 r^2|\overset{(1)}{\sigma}|^2 + 2\Omega^2|\overset{(1)}{\hat{\chi}}|^2 + \frac{\Omega^2 r^2}{2}(\mathrm{Tr}_{\slashed{g}}\chi)^2|\overset{(1)}{\underline{\eta}}|^2 + 2\Omega^2 r^2|\overset{(1)}{\beta}|^2 \\
&+ r^2\Omega(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\nabla}_4(\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}}) + \frac{3}{2}r^2\mathrm{Tr}_{\slashed{g}}\chi\Omega(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}(\overset{(1)}{\underline{\eta}}) - r^2\Omega(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}(\overset{(1)}{\beta}) \\
&+ 2\Omega^2 r^2(\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\beta}\rangle + \Omega r^2\mathrm{Tr}_{\slashed{g}}\chi(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} + 2\Omega r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\beta} \\
&- 2r^2\omega\slashed{\mathrm{div}}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\Big) - \frac{1}{r^2}\slashed{\nabla}_4(r^2\slashed{\mathcal{A}}).
\end{aligned} \tag{3.3.133}
$$

By considering similar ideas to the points (1) and (2) around equation (3.3.90) in the proof of theorem 3.1.7 in section 3.3.3 one can identify a boundary term. To this end, define

$$
\begin{aligned}
\mathcal{B}_1 &\doteq r^4\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\eta} - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}}\Big), & \mathcal{B}_2 &\doteq r^4\overset{(1)}{\rho}\big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\big), \\
\mathcal{B}_3 &\doteq 2r^4\Omega\mathrm{Tr}_{\slashed{g}}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}, & \mathcal{B}_4 &\doteq r^4(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle, \\
\mathcal{B}_5 &\doteq \frac{r^4}{2\Omega}\mathrm{Tr}_{\slashed{g}}\chi(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi), & \mathcal{B}_6 &\doteq r^4\frac{\omega}{\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}).
\end{aligned} \tag{3.3.134}
$$

Computing $\slashed{\nabla}_3\mathcal{B}_1$ and $\slashed{\nabla}_4\mathcal{B}_1$ with propositions 2.10.9, 2.10.11 and 2.10.8 gives

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_3\mathcal{B}_1 \equiv{}& 2r^2\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\underline{\omega}}\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} - \frac{3}{2}r^2(\mathrm{Tr}_{\slashed{g}}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\eta} + 2r^2\Omega(\slashed{\mathrm{div}}\overset{(1)}{\eta})^2 + 2\Omega r^2\overset{(1)}{\rho}\slashed{\mathrm{div}}\overset{(1)}{\eta}\\ &- 2\Omega r^2\rho|\overset{(1)}{\eta}|^2 - 2\Omega r^2\rho\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle + r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\nabla}_3\slashed{\mathrm{div}}\overset{(1)}{\eta} - r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\underline{\beta}}\\ &+ \frac{2r^2\omega}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}},\end{aligned} \tag{3.3.135}$$

and

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{B}_1 \equiv{}& 2r^2\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\omega}\slashed{\mathrm{div}}\overset{(1)}{\eta} - \frac{3}{2}r^2(\mathrm{Tr}_{\slashed{g}}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} - 2r^2\Omega(\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}})^2 - 2\Omega r^2\overset{(1)}{\rho}\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}}\\ &+ 2\Omega r^2\rho|\overset{(1)}{\underline{\eta}}|^2 + 2\Omega r^2\rho\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle - r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\nabla}_4\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} - r^2(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\beta}\\ &+ \frac{2r^2\omega}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\slashed{\mathrm{div}}\overset{(1)}{\eta}.\end{aligned} \tag{3.3.136}$$

Again using propositions 2.10.9, 2.10.8 and the Bianchi equation for $\overset{(1)}{\rho}$ in proposition 2.10.20 one has that

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_3\mathcal{B}_2 \equiv{}& r^2\big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\big)\slashed{\mathrm{div}}\overset{(1)}{\underline{\beta}} + \frac{3}{2}r^2\frac{\rho}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})^2 + 2\Omega r^2|\overset{(1)}{\rho}|^2\\ &- \frac{3}{2}r^2\frac{\rho}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + 2\Omega r^2\overset{(1)}{\rho}\slashed{\mathrm{div}}\overset{(1)}{\eta} + 4\Omega r^2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}\\ &- r^2\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\rho}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) + 2r^2\frac{\omega}{\Omega}\overset{(1)}{\rho}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) + 2r^2\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\rho}\overset{(1)}{\underline{\omega}},\end{aligned} \tag{3.3.137}$$

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{B}_2 \equiv{}& r^2\big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\big)\slashed{\mathrm{div}}\overset{(1)}{\beta} - \frac{3}{2}r^2\frac{\rho}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)^2 - 2\Omega r^2|\overset{(1)}{\rho}|^2\\ &+ \frac{3}{2}r^2\frac{\rho}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - 2\Omega r^2\overset{(1)}{\rho}\slashed{\mathrm{div}}\overset{(1)}{\underline{\eta}} - 4\Omega r^2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}\\ &- r^2\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\rho}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + 2r^2\frac{\omega}{\Omega}\overset{(1)}{\rho}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + 2r^2\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\rho}\overset{(1)}{\omega}.\end{aligned} \tag{3.3.138}$$

Further, from propositions 2.10.7 and the Bianchi equation for $\overset{(1)}{\rho}$ in proposition 2.10.20 one has

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_3\mathcal{B}_3 \equiv{}& 2r^2(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\underline{\omega}}\overset{(1)}{\rho} + r^2(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\eta}\rangle + r^2(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\underline{\eta}}\rangle + 4r^2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}\\ &- 3r^2\rho(\mathrm{Tr}_{\slashed{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}),\end{aligned} \tag{3.3.139}$$

$$\begin{aligned}\frac{1}{r^2}\slashed{\nabla}_4\mathcal{B}_3 \equiv{}& 2r^2(\mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\omega}\overset{(1)}{\rho} - r^2(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\beta},\overset{(1)}{\eta}\rangle - r^2(\Omega\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{\beta},\overset{(1)}{\underline{\eta}}\rangle - 4r^2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}\\ &- 3r^2\rho(\mathrm{Tr}_{\slashed{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi).\end{aligned} \tag{3.3.140}$$

Noting that $r^2\rho\mathrm{Tr}_{g\!\!\!/}\chi = -4\Omega\omega = -\frac{4\omega}{\Omega}(1+r^2\rho)$ one can show, using proposition 2.10.11 that

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\mathcal{B}_4 \equiv{}& -2r^2(\mathrm{Tr}_{g\!\!\!/}\chi)\overset{(1)}{\underline{\omega}}\,\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\underline{\eta}} + 2r^2\rho\Omega(\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle - |\overset{(1)}{\eta}|^2) - 2\Omega|\overset{(1)}{\eta}|^2 \\ &+ r^2(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\langle\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}},\overset{(1)}{\underline{\beta}}\rangle,\end{aligned} \tag{3.3.141}$$

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_4\mathcal{B}_4 \equiv{}& -2r^2(\mathrm{Tr}_{g\!\!\!/}\chi)\overset{(1)}{\omega}\,\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta} - 2r^2\rho\Omega\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle + 2\Omega|\overset{(1)}{\underline{\eta}}|^2 + 2r^2\Omega\rho|\overset{(1)}{\underline{\eta}}|^2 \\ &+ r^2(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\langle\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}},\overset{(1)}{\beta}\rangle.\end{aligned} \tag{3.3.142}$$

Noting that $(\mathrm{Tr}_{g\!\!\!/}\chi)^2 = \frac{4}{r^2}+4\rho$ and $\omega\mathrm{Tr}_{g\!\!\!/}\chi = -\Omega\rho$ with propositions 2.10.9 and 2.10.8 gives

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\mathcal{B}_5 \equiv{}& r^2(\mathrm{Tr}_{g\!\!\!/}\chi)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})(\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho}) - 8\frac{\omega}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) - 8\frac{\omega}{\Omega}\rho r^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) \\ &- \frac{1}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})^2 - \frac{r^2\rho}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})^2 + \frac{r^2\rho}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) - \frac{4}{\Omega}\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) \\ &- \frac{4r^2}{\Omega}\rho\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}),\end{aligned} \tag{3.3.143}$$

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_4\mathcal{B}_5 \equiv{}& r^2(\mathrm{Tr}_{g\!\!\!/}\chi)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})(\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\underline{\eta}}+\overset{(1)}{\rho}) - 8\frac{\omega}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) - 8\frac{\omega}{\Omega}\rho r^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) \\ &+ \frac{1}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})^2 + \frac{r^2\rho}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})^2 - \frac{r^2\rho}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) + \frac{4}{\Omega}\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) \\ &+ \frac{4r^2}{\Omega}\rho\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}).\end{aligned} \tag{3.3.144}$$

Finally, one can check that using propositions 2.10.9 and 2.10.8 that

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\mathcal{B}_6 \equiv{}& -\frac{r^2}{2\Omega}\rho(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) + \frac{2\omega r^2}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta} + \frac{2r^2\omega}{\Omega}\overset{(1)}{\rho}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) \\ &+ 4r^2\frac{\omega\rho}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) + \frac{1}{2}r^2\frac{\rho}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}})^2 + \frac{2r^2\rho}{\Omega}\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}),\end{aligned} \tag{3.3.145}$$

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_4\mathcal{B}_6 \equiv{}& \frac{r^2}{2\Omega}\rho(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}) + \frac{2\omega r^2}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\underline{\eta}} + \frac{2r^2\omega}{\Omega}\overset{(1)}{\rho}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) \\ &+ 4r^2\frac{\omega\rho}{\Omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}) - \frac{1}{2}r^2\frac{\rho}{\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\chi})^2 - \frac{2r^2\rho}{\Omega}\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi}}).\end{aligned} \tag{3.3.146}$$

By writing that

$$\mathcal{B} = \frac{1}{r^2}(\mathcal{B}_1 - \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 - \mathcal{B}_5 + \mathcal{B}_6) + \sum_{k=1}^{3}\mathcal{A}[\mathcal{L}_{\Omega_k}h] \tag{3.3.147}$$

one can now calculate $\sum(\mathcal{J}[\mathcal{L}_{\Omega_i}h])^4 - \frac{1}{r^2}\nabla\!\!\!\!/_3(r^2\mathcal{B})$ and $\sum(\mathcal{J}[\mathcal{L}_{\Omega_i}h])^3 + \frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{B})$ to show the desired result. □

3.3.5 Proof of Theorem 3.1.10

Proof of Theorem 3.1.10. In this proof the following convention is adopted. The symbol $\equiv$ will denote equality under integration by parts of $\mathbb{S}^2_{u,v}$.

There are two methods to derive the conservation law of theorem 3.1.10. One can compute directly using the linearised Bianchi identities (proposition 2.10.20) and linearised null structure equations (propositions 2.10.7-2.10.17) that the fluxes appearing in the statement are conserved. More precisely, one can compute that

$$\begin{aligned} 0 \equiv\ & \frac{1}{r^2}\nabla\!\!\!/_3\Big[r^2\Big(\frac{\Omega^4}{4}|\overset{(1)}{\alpha}|^2 + \frac{3}{2}\Omega^4(|\overset{(1)}{\rho}|^2 + |\overset{(1)}{\sigma}|^2 + |\overset{(1)}{\beta}|^2) + \frac{\Omega^4}{2}|\overset{(1)}{\underline{\beta}}|^2 + f_2|\overset{(1)}{\hat{\underline{\chi}}}|^2 + f_1|\overset{(1)}{\hat{\chi}}|^2 \\ & + f_3|\overset{(1)}{\underline{\eta}}|^2 - \frac{1}{\Omega^2}f_3\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) + \frac{2}{\Omega^2}\Big(\omega f_3 + 2\Omega\mathrm{Tr}_{\not{g}}\chi f_2\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) \\ & - \frac{f_1}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)^2 - \frac{f_2}{2\Omega^2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) + 2(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big]^2\Big)\Big] \\ & + \frac{1}{r^2}\nabla\!\!\!/_4\Big[r^2\Big(\frac{\Omega^4}{4}|\overset{(1)}{\underline{\alpha}}|^2 + \frac{3}{2}\Omega^4(|\overset{(1)}{\rho}|^2 + |\overset{(1)}{\sigma}|^2 + |\overset{(1)}{\underline{\beta}}|^2) + \frac{\Omega^4}{2}|\overset{(1)}{\beta}|^2 + f_1|\overset{(1)}{\hat{\underline{\chi}}}|^2 + f_2|\overset{(1)}{\hat{\chi}}|^2 \\ & + f_3|\overset{(1)}{\eta}|^2 - \frac{f_3}{\Omega^2}\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) - \frac{2}{\Omega^2}\Big(\omega f_3 + 2\Omega\mathrm{Tr}_{\not{g}}\chi f_2\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) \\ & - \frac{f_1}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})^2 - \frac{f_2}{2\Omega^2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) - 2(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big]^2\Big)\Big]. \end{aligned} \tag{3.3.148}$$

Therefore, if one integrates over the spacetime region $[u_0, u_1] \times [v_0, v_1] \times \mathbb{S}^2_{u,v}$ then one obtains the conservation law in the statement of theorem 3.1.10. Given the fluxes, this is perhaps the simplest way to prove the conservation law.

The second way is more constructive proof and is completely analogous to the proof of theorem 3.1.9 where one computes the canonical energy of $\mathcal{L}_T h$ and then manipulates the fluxes into a 'satisfactory' form. This was how the author originally derived the conservation law. This is illustrated below, however, many explicit computations are left out for brevity in the main body of the work but can be found in appendix C.1.

Again rather than go through the direct computation of $(\mathcal{J}^T[\mathcal{L}_T h])^a$ as in section 3.3.3 one can use the same idea as in the proof of theorem 3.1.9 to avoid (some of) the long computations. Recall, by propositions C.1.4 and the discussion in remark 2.10.3, if h satisfies the linearised null structure equations then $\mathcal{L}_T h$ does. Therefore, if one expresses $(\mathcal{J}^T[h])^4 - \frac{1}{r^2}\nabla\!\!\!/_3(\mathcal{A}[h])$ and $(\mathcal{J}^T[h])^3 + \frac{1}{r^2}\nabla\!\!\!/_4(\mathcal{A}[h])$ in terms of h, then replacing with $\mathcal{L}_T h$ will result in a simpler form for the components of $(\mathcal{J}^T[\mathcal{L}_T h])$ (plus a boundary term). One can check that this yields

$$(\mathcal{J}^T[\mathcal{L}_T h])^4 \equiv \frac{1}{\Omega}\Big(\Omega^2|\not{\mathcal{L}}_T\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2\Omega^2|\not{\mathcal{L}}_T\overset{(1)}{\eta}|^2 - 4\omega T\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)T((\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})) \tag{3.3.149}$$
$$- 2T(\overset{(1)}{\underline{\omega}})T((\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})) - \frac{1}{2}\big(T(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\big)^2\Big) + \frac{1}{r^2}\not{\nabla}_3\mathcal{A}(\mathcal{L}_T h),$$

$$(\mathcal{J}^T[\mathcal{L}_T h])^3 \equiv \frac{1}{\Omega}\Big(2\Omega^2|\not{\mathcal{L}}_T\overset{(1)}{\underline{\eta}}|^2 + \Omega^2|\not{\mathcal{L}}_T\overset{(1)}{\hat{\chi}}|^2 + 4\omega T\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)T((\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})) \tag{3.3.150}$$
$$- 2T((\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}))T(\overset{(1)}{\omega}) - \frac{1}{2}\big(T(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\big)^2\Big) - \frac{1}{r^2}\not{\nabla}_4\mathcal{A}(\mathcal{L}_T h).$$

Calculating using the proposition 2.10.10 gives:

$$\not{\mathcal{L}}_T\overset{(1)}{\hat{\underline{\chi}}} = -\frac{\Omega}{2}\overset{(1)}{\underline{\alpha}} - \omega\overset{(1)}{\hat{\underline{\chi}}} - \Omega\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}} + \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}}\Big), \tag{3.3.151}$$
$$\not{\mathcal{L}}_T\overset{(1)}{\hat{\chi}} = -\frac{\Omega}{2}\overset{(1)}{\alpha} + \omega\overset{(1)}{\hat{\chi}} - \Omega\not{\mathcal{D}}_2^\star\overset{(1)}{\eta} - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}}\Big). \tag{3.3.152}$$

Further, from proposition 2.10.11

$$\not{\mathcal{L}}_T\overset{(1)}{\eta} = \not{\nabla}\overset{(1)}{\underline{\omega}} + \frac{1}{4}\Omega\mathrm{Tr}_{\not{g}}\chi(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}) - \frac{\Omega}{2}\Big(\overset{(1)}{\beta} + \overset{(1)}{\underline{\beta}}\Big), \tag{3.3.153}$$
$$\not{\mathcal{L}}_T\overset{(1)}{\underline{\eta}} = \not{\nabla}\overset{(1)}{\omega} - \frac{1}{4}\Omega\mathrm{Tr}_{\not{g}}\chi(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}) + \frac{\Omega}{2}\Big(\overset{(1)}{\beta} + \overset{(1)}{\underline{\beta}}\Big). \tag{3.3.154}$$

One can additionally compute $T((\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}))$, $T((\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}))$, $T(\overset{(1)}{\omega})$ and $T(\overset{(1)}{\underline{\omega}})$ from propositions 2.10.9, 2.10.8 and 2.10.13. Note that under integration by parts on the spheres and using the torsion equations in proposition 2.10.11:

$$|\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}}|^2 \equiv \frac{1}{2}|\overset{(1)}{\sigma}|^2 - \Big[\frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi) + \omega\Big]\frac{\mathrm{Tr}_{\not{g}}\chi}{\Omega}|\overset{(1)}{\underline{\eta}}|^2 + \frac{1}{2}|\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}}|^2, \tag{3.3.155}$$

and analogously for $|\not{\mathcal{D}}_2^\star\overset{(1)}{\eta}|^2$. This allows one to compute two (rather horrendous) expressions for $(\mathcal{J}^T[\mathcal{L}_T h])^4 - \frac{1}{r^2}\not{\nabla}_3(\mathcal{A}[\mathcal{L}_T h])$ and $(\mathcal{J}^T[\mathcal{L}_T h])^3 + \frac{1}{r^2}\not{\nabla}_4\mathcal{A}[\mathcal{L}_T h]$ where $\mathcal{A}[\mathcal{L}_T h]$ results from expressing $\mathcal{A}[h]$ above in terms of h and replacing it with $\mathcal{L}_T h$. These can be found in section C.2 of appendix C.1.

At this point one can then (arduously) identify the following boundary term to use in the manipulation of the resulting flux densities:

$$\begin{aligned}\mathcal{C}[h] \doteq{}& \frac{\Omega^2}{2}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)-\omega\Big)\Big[|\overset{(1)}{\hat{\chi}}|^2+|\overset{(1)}{\underline{\hat{\chi}}}|^2\Big]+\frac{\Omega^2}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\hat{\chi}}},\overset{(1)}{\hat{\chi}}\rangle-\Omega^3\Big[\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\underline{\eta}}\rangle+\langle\overset{(1)}{\beta},\overset{(1)}{\eta}\rangle\Big] \\ &+\overset{(1)}{\omega}T((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}))-\overset{(1)}{\underline{\omega}}T((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi))+(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}}-2\Omega^2\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}+\Omega^2\omega\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle \\ &+\frac{1}{4}\Omega^2(\Omega\mathrm{Tr}_{\not g}\chi)\Big[|\overset{(1)}{\eta}|^2+|\overset{(1)}{\underline{\eta}}|^2\Big]+\frac{1}{4}\Big(\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})^2+(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)^2\Big] \\ &-\frac{1}{8}(\Omega\mathrm{Tr}_{\not g}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})+\Big(2\omega^2-\frac{3}{2}\Omega^2\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})-(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)\Big] \\ &-\Big(4\Omega^2\omega\rho+\frac{3}{2}\Omega^2(\Omega\mathrm{Tr}_{\not g}\chi)\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2+\sum_k\mathcal{A}[\mathcal{L}_T h].\end{aligned} \tag{3.3.156}$$

Therefore, computing (as in section C.2 of appendix C.1) $(\mathcal{J}^T[\mathcal{L}_T h])^4-\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C})$ and $(\mathcal{J}^T[\mathcal{L}_T h])^3+\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C})$ allows one to show T-canonical energy of $\mathcal{L}_T h$ satisfies

$$\mathcal{E}^T_u[\mathcal{L}_T h](v_0,v_1)=2\dot{\bar{\mathcal{E}}}^T_u[h](v_0,v_1)-2\int_{\mathbb{S}^2_{u,v}}\mathcal{C}[h](u,v,\theta,\varphi)\not\epsilon\Big|_{v_0}^{v_1}, \tag{3.3.157}$$

$$\mathcal{E}^T_v[\mathcal{L}_T h](u_0,u_1)=2\dot{\bar{\mathcal{E}}}^T_v[h](u_0,u_1)+2\int_{\mathbb{S}^2_{u,v}}\mathcal{C}[h](u,v,\theta,\varphi)\not\epsilon\Big|_{u_0}^{u_1}. \tag{3.3.158}$$

By the conservation of the canonical energy and the cancellation of the boundary terms on the spheres $\mathbb{S}^2_{u,v}$ gives the result stated in theorem 3.1.10. □

3.4 Restrictions and Normalisation of Initial Data

In this section the restrictions on initial data required to prove the energy boundedness statements in theorems 3.1.3, 3.1.12 and 3.1.13 are discussed. In particular, this section reviews asymptotic flatness and the relation between restricting to $\ell \geq 2$ spherical harmonics and the linearised Schwarzschild and Kerr solutions. Additionally, particular gauge conditions at the future event horizon $\mathcal{H}^+$ are defined which prove useful in later sections.

3.4.1 Support on the $\ell = 0, 1$ Spherical Harmonics

The spherical harmonics functions Y_m^ℓ on the unit sphere where $\ell \in \mathbb{N}_0$ and $m \in \{-\ell, ..., \ell\}$ verify the equation

$$\mathring{\Delta} Y_m^\ell = -\ell(\ell+1) Y_m^\ell, \tag{3.4.1}$$

where $\mathring{\Delta}$ is the Laplacian on the unit sphere $\mathbb{S}^2$. Further, the explicit form of the $\ell = 0, 1$ spherical harmonics are

$$\begin{aligned} Y_{m=0}^{\ell=0} &= \frac{1}{\sqrt{4\pi}}, \\ Y_{m=-1}^{\ell=1} &= \sqrt{\frac{3}{4\pi}} \sin\theta \cos\varphi, \qquad Y_{m=0}^{\ell=1} = \sqrt{\frac{3}{4\pi}} \cos\theta, \qquad Y_{m=1}^{\ell=1} = \sqrt{\frac{3}{4\pi}} \sin\theta \sin\varphi. \end{aligned} \tag{3.4.2}$$

The spherical harmonics from an orthonormal basis for $L^2(\mathbb{S}^2)$. With this very brief review in hand one can make the following definition:

Definition 3.4.1 (Function Supported on $\ell \geq 2$)**.** *A function f on Schw_4 is said to be supported on $\ell \geq 2$ if*

$$\int_{\mathbb{S}^2} \sin\theta f Y_m^\ell d\theta d\varphi = 0, \tag{3.4.3}$$

for $\ell = 0, m = 0$ and $\ell = 1, m = -1, 0, 1$.

One can also extend this definition to one-forms. First, recall from the Hodge decomposition theorem that any one-form $\xi \in \Omega^1$ on a compact Riemannian manifold can be decomposed as

$$\xi = df + \delta\beta + \gamma, \tag{3.4.4}$$

where f is a function, $\beta \in \Omega^2$ and γ is a harmonic one-form. Now for $\mathbb{S}^2$ there are no non-trivial harmonic one-forms. Moreover, all two-forms are proportional to the volume form on the sphere, i.e., all two-forms can be written as $\beta = g\notin$ for $g \in C^\infty(\mathbb{S}^2)$. Hence, any one-form on $\mathbb{S}^2$ has a representation in terms of two functions f and g on the unit sphere as

$$\xi_A = -\nabla\!\!\!/_A f + \mathring{\epsilon\!\!/}_{AB}\mathring{\gamma\!\!/}_2^{BC}\nabla\!\!\!/_C g. \tag{3.4.5}$$

Note that f and g are uniquely defined up to constants so one can specify that their spherical means vanishes. With a rescaling, a $\mathbb{S}^2_{u,v}$ one-form ξ has a unique representation in terms of two functions f and g with vanishing spherical mean as

$$\xi = r\mathcal{D}\!\!\!/_1^\star(f, g), \tag{3.4.6}$$

This allows one to decompose any smooth $\mathbb{S}^2_{u,v}$ one-form ξ as

$$\xi = \xi_{\ell=1} + \xi_{\ell\geq 2}, \tag{3.4.7}$$

where

$$\xi_{\ell=1} \doteq r\mathcal{D}\!\!\!/_1^\star(f_{\ell=1}, g_{\ell=1}). \tag{3.4.8}$$

Note that

$$\mathrm{d\!\!/iv}\xi_{\ell=1} = -r\Delta\!\!\!/ f_{\ell=1}, \qquad \mathrm{cu\!\!/rl}\xi_{\ell=1} = r\Delta\!\!\!/ g_{\ell=1}. \tag{3.4.9}$$

So,

$$\begin{aligned} \int_{\mathbb{S}^2} r\mathrm{d\!\!/iv}\xi_{\ell=1} Y^\ell_m \sin\theta d\theta d\varphi &= -\int_{\mathbb{S}^2} r^2 \Delta\!\!\!/ f_{\ell=1} Y^\ell_m \sin\theta d\theta d\varphi \\ &= \ell(\ell+1)\int_{\mathbb{S}^2} f_{\ell=1} Y^\ell_m \sin\theta d\theta d\varphi \end{aligned} \tag{3.4.10}$$

and

$$\begin{aligned} \int_{\mathbb{S}^2} r\mathrm{cu\!\!/rl}\xi_{\ell=1} Y^\ell_m \sin\theta d\theta d\varphi &= \int_{\mathbb{S}^2} r^2 \Delta\!\!\!/ g_{\ell=1} Y^\ell_m \sin\theta d\theta d\varphi \\ &= -\ell(\ell+1)\int_{\mathbb{S}^2} g_{\ell=1} Y^\ell_m \sin\theta d\theta d\varphi. \end{aligned} \tag{3.4.11}$$

Hence the functions $r\mathrm{d\!\!/iv}\xi_{\ell=1}$ and $r\mathrm{cu\!\!/rl}\xi_{\ell=1}$ are in the span of the $\ell = 1$ spherical harmonics. This motivates the following definition:

Definition 3.4.2 (One-form supported on $\ell \geq 2$)**.** *A smooth one-form* $\xi \in \Omega^1(\mathbb{S}^2_{u,v})$ *on* Schw_4 *is supported on* $\ell \geq 2$ *if the functions* $(\mathrm{d\!\!/iv}\xi, \mathrm{cu\!\!/rl}\xi)$ *are supported on* $\ell \geq 2$.

One also has the following proposition concerning symmetric, traceless $\mathbb{S}^2_{u,v}$-tensors:

Proposition 3.4.1. *Let* $\Theta \in \mathrm{symtr}(T^\star\mathbb{S}^2_{u,v} \otimes T^\star\mathbb{S}^2_{u,v})$ *be a smooth symmetric, traceless* $\mathbb{S}^2_{u,v}$ 2*-tensor. Then* Θ *can be written as*

$$\Theta = r^2 \mathcal{D}_2^\star \mathcal{D}_1^\star (f, g), \tag{3.4.12}$$

where f *and* g *are supported on* $\ell \geq 2$*. In this sense, any symmetric traceless* 2 *tensor on* $\mathbb{S}^2_{u,v}$ *is supported on* $\ell \geq 2$*.*

Proof. One can find a proof of this statement in proposition 4.4.1 of section 4.4.2 of [28]. □

Definition 3.4.3 (Solution Supported on $\ell \geq 2$)**.** *A smooth solution* h *to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior is supported on* $\ell \geq 2$ *if any function, one-form or symmetric traceless* 2*-tensor constructed from* h *is supported on* $\ell \geq 2$*.*

It will become clear in the next subsection that solutions supported on $\ell \geq 2$ cannot be a linearised Kerr or Schwarzschild solution.

There is one last technical result that the reader should note for later use:

Proposition 3.4.2. *Let* h *be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior in double null Eddington–Finkelstein coordinates. Then* $\overset{(1)}{K}_{\ell=1} = 0$*.*

Proof. Note that from the computation in corollary 2.10.15,

$$2\overset{(1)}{K} = \mathrm{d\!\!/iv}\,\mathrm{d\!\!/iv}\,\hat{\not{h}} - \frac{1}{2}\not{\Delta}\mathrm{Tr}_{\not{g}}\not{h} + \Big[\rho - \frac{1}{4}(\mathrm{Tr}_{\not{g}}\chi)^2\Big]\mathrm{Tr}_{\not{g}}\not{h}. \tag{3.4.13}$$

Projecting onto $\ell = 1$ and noting proposition 3.4.1 gives

$$2\overset{(1)}{K}_{\ell=1} = \Big(\frac{1}{r^2} + \Big[\rho - \frac{1}{4}(\mathrm{Tr}_{\not{g}}\chi)^2\Big]\Big)(\mathrm{Tr}_{\not{g}}\not{h})_{\ell=1}. \tag{3.4.14}$$

Using the Schwarzschild values for ρ and $\mathrm{Tr}_{\not{g}}\chi$ one can compute

$$\Big[\rho - \frac{1}{4}(\mathrm{Tr}_{\not{g}}\chi)^2\Big] = -\frac{1}{r^2}. \tag{3.4.15}$$

Hence the result. □

3.4.2 The Linearised Schwarzschild and Kerr Solutions

In view of the Schwarzschild black hole family being a subfamily of the Kerr black hole family one cannot expect the stability statement that all linearised perturbations of the Schwarzschild exterior decay to a residual pure gauge solution. Indeed, the best that can be hoped for is that gravitational perturbations of the Schwarzschild black hole decay to a linearised Kerr solution plus a pure gauge solution. Therefore, it is paramount to identify the linearised Kerr solution so that the choice of data can be restricted appropriately. The following discussion is based upon section 6.2 of [28].

By rescaling the null coordinates as $u = 2M\hat{u}$ and $v = 2M\hat{v}$ (as well as $r = 2Mx$) one can write the metric on the Schwarzschild exterior as

$$g_{\mathrm{S}} = 4M^2\Big[-2\Big(1-\frac{1}{x}\Big)(d\hat{u}\otimes d\hat{v} + d\hat{v}\otimes d\hat{u}) + x^2\mathring{\gamma}_2\Big]. \tag{3.4.16}$$

Taking $M \mapsto M + \epsilon\mathrm{m}$ and expanding in ϵ gives a linearised metric h in double null gauge with non-vanishing double null components

$$\frac{\overset{(1)}{\Omega}}{\Omega} = \frac{\mathrm{m}}{M}, \qquad \mathrm{Tr}_{\not{g}}\not{h} = 4\frac{\mathrm{m}}{M}. \tag{3.4.17}$$

Proposition 3.4.3. *For every $\frac{\mathrm{m}}{M} \in \mathbb{R}$, the following is a spherically symmetric smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior. The non-vanishing metric components are:*

$$\frac{\overset{(1)}{\Omega}}{\Omega} = \frac{\overset{(1)}{\Omega}_{\mathrm{S}}}{\Omega} \doteq \frac{\mathrm{m}}{M}, \qquad \mathrm{Tr}_{\not{g}}\not{h} = \mathrm{Tr}_{\not{g}}\not{h}_{\mathrm{S}} \doteq 4\frac{\mathrm{m}}{M}. \tag{3.4.18}$$

The non-vanishing curvature components are

$$\overset{(1)}{\rho} = \overset{(1)}{\rho}_{\mathrm{S}} \doteq \frac{4\mathrm{m}}{r^3}, \qquad \overset{(1)}{K} = \overset{(1)}{K}_{\mathrm{S}} \doteq -\frac{2\mathrm{m}}{Mr^2}. \tag{3.4.19}$$

This 1-parameter family will be referred to as the reference $\ell = 0$ linearised Schwarzschild solutions.

The Kerr solution in double null gauge is written in equation (2.9.9) of section 2.9. To see the linearised metric components it is also useful to have the Boyer–Lindquist form (2.9.1) of the Kerr metric to $\mathcal{O}(a)$ in mind. Expanding to $\mathcal{O}(a)$ gives

$$g_{\mathrm{K}} = g_{\mathrm{S}} - \frac{2Ma\sin^2\theta}{r}(dt\otimes d\varphi + d\varphi\otimes dt) + \mathcal{O}(a^2), \tag{3.4.20}$$

where g_{S} is the standard metric on the exterior of Schw_4. This allows us to identify that there is no $\mathcal{O}(a)$ perturbation to the null lapse Ω or to the induced metric $\not{g}$, i.e.,

$$\overset{(1)}{\Omega}_{\mathrm{K}} = 0, \qquad \not{h}_{\mathrm{K}} = 0. \tag{3.4.21}$$

One can then read off from the Kerr metric in double null form (2.9.9) that the perturbation to b^A is

$$b^A = \Big(0, 0 + \frac{4Ma}{r^3} + \mathcal{O}(a^2)\Big). \tag{3.4.22}$$

So one has

$$\overset{(1)}{b} = \frac{4Ma}{r^3}\partial_\varphi. \tag{3.4.23}$$

Now the Schwarzschild spacetime is spherically symmetric so there is no preferred axis! This differs significantly from when one is linearising around a $a \neq 0$ Kerr solution, where one picks an axis to write down the solution. Hence, one picks up two other 'basis' solutions associated with using the $SO(3)$ symmetry group action on the $\mathbb{S}^2_{u,v}$-vector in equation (3.4.23):

$$\mathcal{L}_{\Omega_2}\overset{(1)}{b} = -\frac{4Ma}{r^3}\Omega_3, \tag{3.4.24}$$

$$\mathcal{L}_{\Omega_3}\overset{(1)}{b} = \frac{4Ma}{r^3}\Omega_2. \tag{3.4.25}$$

One can write this compactly as

$$\overset{(1)}{b}{}^A = \frac{4Ma}{r}\not{\epsilon}^{AB}\partial_B Y^{l=1}_m. \tag{3.4.26}$$

for $m = -1, 0, 1$. Note that change in overall constant is ineffectual due to linearity.

Proposition 3.4.4. *Let $Y^{\ell=1}_m$ for $m = -1, 0, 1$ denote the spherical harmonics in equation (3.4.2). For any $\mathrm{a} \in \mathbb{R}$, the following is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior. The non-vanishing metric coefficients are*

$$\overset{(1)}{b}{}^A = \overset{(1)}{b}{}^A_{\mathrm{K}} \doteq \frac{4M\mathrm{a}}{r}\not{\epsilon}^{AB}\partial_B Y^{l=1}_m. \tag{3.4.27}$$

The non-vanishing Ricci coefficients are

$$\overset{(1)}{\eta}{}^A = \overset{(1)}{\eta}{}^A_K \doteq \frac{3M\mathrm{a}}{r^2}\not{\epsilon}^{AB}\partial_B Y^{l=1}_m, \qquad \overset{(1)}{\underline{\eta}}{}^A = -\overset{(1)}{\eta}{}^A_K. \tag{3.4.28}$$

The non-vanishing curvature components are

$$\overset{(1)}{\beta} = \overset{(1)}{\beta}_{\mathrm{K}} = \frac{\Omega}{r}\overset{(1)}{\eta}_{\mathrm{K}}, \qquad \overset{(1)}{\underline{\beta}} = -\overset{(1)}{\beta}_{\mathrm{K}}, \qquad \overset{(1)}{\sigma} = \overset{(1)}{\sigma}_{\mathrm{K}} = \frac{6M\mathrm{a}}{r^4}Y^{\ell=1}_m. \tag{3.4.29}$$

The 3-parameter family spanned by the above solutions ($m = -1, 0, 1$) will be referred to as the reference $\ell = 1$ linearised Kerr solutions.

Remark 3.4.5. *Notice that, if one restricts attention to solutions of the linearised vacuum Einstein equation (I.5) in double null gauge which are supported on $\ell \geq 2$, one avoids the linearised Schwarzschild and Kerr solutions since these are supported purely on $\ell = 0, 1$ spherical harmonics. This relates directly to Hollands and Wald's restriction on the canonical energy that the linearised ADM charges must vanish.*

3.4.3 Asymptotic Flatness and Extendibility to Null Infinity

Under the asymptotic flatness assumptions imposed at the level of initial data of [28] (see section 8.3 in [28]) one gets a collection of r-weighted estimates for the quantities arising from a solution to the linearised vacuum Einstein equation (I.5) in double null gauge. In particular, one finds that the quantities in the following definition admit regular limits along any cone C_v as $v \to \infty$:

Definition 3.4.4 (Solution Extendible to Null Infinity)**.** *A smooth solution h to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild exterior is said to be extendible to null infinity if the following quantities resulting from a solution h have well-defined limits as $v \to \infty$ along any cone C_u with $u \geq u_0$ for some $s \in (0, 1]$*

$$\mathbb{E}^\infty \doteq \left\{ \frac{\overset{(1)}{\Omega}}{\Omega}, r^2(\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}} \chi), r(\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}} \underline{\chi}), r^{2+s}\overset{(1)}{\omega}, \overset{(1)}{\underline{\omega}}, r\overset{(1)}{\eta}, r^2\overset{(1)}{\underline{\eta}}, r^2\overset{(1)}{\hat{\chi}}, r\overset{(1)}{\underline{\hat{\chi}}} \right\} \\ \cup \left\{ r^3\overset{(1)}{\rho}, r^3\overset{(1)}{\sigma}, r^2\overset{(1)}{K}, r^{3+s}\overset{(1)}{\beta}, r^2\overset{(1)}{\underline{\beta}}, r^{3+s}\overset{(1)}{\alpha}, r\overset{(1)}{\underline{\alpha}}, r^2 \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta}, r^3 \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\underline{\eta}} \right\}. \tag{3.4.30}$$

Moreover, for any element $q \in \mathbb{E}^\infty$ and fixed $u_0 < u_f < \infty$ one has

$$\sup_{[u_0, u_f] \times \{v \geq v_0\} \times \mathbb{S}^2_{u,v}} |q| \leq K_{u_f} \tag{3.4.31}$$

for a constant K_{u_f} depending on u_f and initial data only.

This condition is the analogue of the finiteness and regularity conditions at infinity of point (iv) Hollands and Wald's admissible data (see section 1.1.5). This assumption on regularity will allow one to take the limit of the canonical energies to the future null infinity $\mathcal{I}^+$.

3.4.4 Gauge Conditions

To study the linear stability of the Schwarzschild black hole exterior, it is very useful to impose specific gauge conditions on the solution h to the linearised vacuum Einstein equation (I.5) in double null gauge. For the following discussion, one should have in mind the following Penrose diagram depicting the characteristic initial value problem:

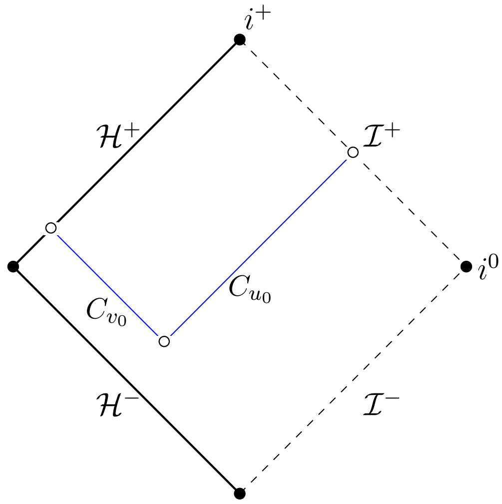


Fig. 3.3 The characteristic initial value problem on the exterior of Schw_4.

Definition 3.4.5 (Partially Initial Data Normalised Gauge)**.** *Let $C_{u_0} \cup C_{v_0}$ be the initial data cone depicted in blue in the Penrose diagram 3.3. A smooth solution h to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild exterior satisfies*

(i) the first horizon gauge condition if

$$(\Omega \overset{(1)}{\mathrm{Tr}_{\not g} \chi})(\infty, v_0, \theta, \varphi) = 0, \tag{3.4.32}$$

(ii) the second horizon gauge condition if

$$(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + \overset{(1)}{\rho})(\infty, v_0, \theta, \varphi) = \overset{(1)}{\rho}_{\ell=0}(\infty, v_0, \theta, \varphi), \tag{3.4.33}$$

(iii) and the basic round sphere condition at infinity if

$$\lim_{v \to \infty} r^2 \overset{(1)}{K}_{\ell \geq 2}(u_0, v, \theta, \varphi) = 0 \tag{3.4.34}$$

where $\overset{(1)}{K}_{\ell \geq 2}$ is the restriction of the linearised Gauss curvature to its $\ell \geq 2$ spherical harmonics.

A h that satisfies (i), (ii) and (iii) will be referred to as in partially initial data normalised gauge.

Remark 3.4.6. *The use of the word 'partially' in definition 3.4.5 stems from the fact that there is still residual gauge freedom after the above conditions have been imposed. In contrast, the initial data gauge of definition 9.1 of section 9.1 in [28] fixes the gauge completely.*

The rest of this section is devoted to the implementation of conditions (i) and (ii) of definition 3.4.5 by using lemmas 2.10.25 and 2.10.26 of section 2.10.23. This is achieved in the following lemma:

Lemma 3.4.7. *Let h be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the exterior of Schw_4. There exists a residual pure gauge solution h_{pg} such that $h' \doteq h - h_{\mathrm{pg}}$ is in double null gauge and the solution satisfies the partially initial data normalised gauge of definition 3.4.5.*

Proof. The implementation of point (iii) of definition 3.4.5 is deferred to [28]: under the asymptotic flatness assumptions of [28], the basic round sphere condition at infinity can consistently be imposed, see definition 9.1 and theorem 9.1 of [28].

From lemmas 2.10.25 and 2.10.26, a general residual pure gauge solution with $f^1 = 0 = f^2$ has expansion given by

$$(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}} \chi)_{\mathrm{pg}} = (\Omega \mathrm{Tr}_{\not g} \chi)\partial_v f^4 + 2\Omega^2 \not\Delta f^3 + \frac{1}{2}(\Omega \mathrm{Tr}_{\not g}\chi)\Big(4\omega - (\Omega \mathrm{Tr}_{\not g}\chi)\Big)(f^4 - f^3). \tag{3.4.35}$$

Therefore,

$$(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}} \chi)_{\mathrm{pg}}(\infty, v) = \frac{1}{2M^2}(\mathring{\not\Delta} - 1)(\Omega^2 f^3)\Big|_{u=\infty}. \tag{3.4.36}$$

Now one can check, from lemmas 2.10.25 and 2.10.26, that for a general residual pure gauge solution with $f^1 = 0 = f^2$ one also has

$$(\not{\mathrm{div}} \overset{(1)}{\eta} + \overset{(1)}{\rho})_{\mathrm{pg}} = \frac{\Omega^2}{r^3}\mathring{\not\Delta} f^4 - \frac{1}{r\Omega^2}\mathring{\not\Delta}\Big[\partial_u\Big(\frac{\Omega^2}{r} f^3\Big)\Big] + \frac{6M\Omega^2}{r^4}(f^4 - f^3). \tag{3.4.37}$$

Let $h_1 = h_1(\theta, \varphi)$ and $h_2 = h_2(\theta, \varphi)$ be smooth functions on the unit sphere, $\hat{\mathrm{m}}$ be a constant and pick

$$\begin{aligned} f^3(u, \theta, \varphi) = {} & \Big[r(u, v_0)\Big(1 - \frac{r(u_0, v_0)}{r(u, v_0)}\Big) + \frac{r(u_0, v_0)^2}{2M}\Big(1 - \frac{2M}{r(u_0, v_0)}\Big)\Big]\frac{h_2(\theta, \varphi)}{\Omega^2(u, v_0)} \\ & + h_1(\theta, \varphi) - \frac{\hat{\mathrm{m}}}{2\Omega(u, v_0)^2}\big(r(u, v_0) - r(u_0, v_0)\big), \end{aligned} \tag{3.4.38}$$

$$\begin{aligned} f^4(v, \theta, \varphi) = {} & r(u_0, v_0)\Big(1 - \frac{r(u_0, v_0)}{r(u_0, v)}\Big)\frac{h_2(\theta, \varphi)}{\Omega^2(u_0, v)} + h_1(\theta, \varphi) \\ & + \frac{\hat{\mathrm{m}}}{2\Omega(u_0, v)^2}\big(r(u_0, v) - r(u_0, v_0)\big). \end{aligned} \tag{3.4.39}$$

With this choice one can check that the residual pure gauge expansion at $\mathbb{S}^2_{\infty, v_0}$ is given by

$$(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}} \chi)_{\mathrm{pg}}(\infty, v_0) = \frac{1}{4M^3}(r(u_0, v_0) - 2M)^2(\mathring{\not\Delta} - 1)h_2 - \frac{\hat{\mathrm{m}}}{4M^2}(r(u_0, v_0) - 2M) \tag{3.4.40}$$

and

$$(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})_{\mathrm{pg}}(\infty,v_0,\theta,\varphi)=\frac{r(u_0,v_0)}{4M^3}\Big(\frac{r(u_0,v_0)}{4M}-1\Big)\mathring{\not{\Delta}}h_2-\frac{1}{8M^3}\mathring{\not{\Delta}}h_1 \\ -\frac{3}{4M^2}\Big(\frac{r(u_0,v_0)}{2M}-1\Big)^2h_2-\frac{3\hat{\mathrm{m}}}{16M^3}(r(u_0,v_0)-2M). \tag{3.4.41}$$

Since $r(u_0,v_0)>2M$, one can take h_2 supported on $\ell\geq 1$ and then decompose in terms of spherical harmonics to solve the equation

$$(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})_{\mathrm{pg}}(\infty,v_0)=F(\theta,\varphi), \tag{3.4.42}$$

for an arbitrary smooth function $F(\theta,\varphi)$ on the sphere $\mathbb{S}^2_{\infty,v_0}$, where one uses $\hat{\mathrm{m}}$ to solve for the $\ell=0$ part of $F(\theta,\varphi)$. Having solved for $\hat{\mathrm{m}}$ and h_2, one can solve

$$(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})_{\mathrm{pg}}(\infty,v_0,\theta,\varphi)=G(\theta,\varphi)-\frac{3\hat{\mathrm{m}}}{16M^3}(r(u_0,v_0)-2M), \tag{3.4.43}$$

for any arbitrary smooth function $G(\theta,\varphi)$ on the sphere $\mathbb{S}^2_{\infty,v_0}$ supported on $\ell\geq 1$ by taking using h_1. Note that the $\ell=0$ projection of the pure gauge solution picked here is

$$(\overset{(1)}{\rho}_{\mathrm{pg}})_{\ell=0}(\infty,v_0)=-\frac{3\hat{\mathrm{m}}}{16M^3}(r(u_0,v_0)-2M). \tag{3.4.44}$$

Given a solution h to the linearised vacuum Einstein equation (I.5) in double null gauge one can consider $h'=h-h_{\mathrm{pg}}$, where h_{pg} is the pure gauge solution defined by f^3 and f^4 above. Therefore, picking

$$F=(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\infty,v_0), \tag{3.4.45}$$
$$G=(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})_{\ell\geq 1}(\infty,v_0), \tag{3.4.46}$$

gives conditions (i) and (ii) in definition 3.4.7. □

Proposition 3.4.8. *The partially initial data normalised gauge conditions of definition 3.4.5 are evolutionary. More precisely, if the basic round sphere condition holds on C_{u_0} as $v\to\infty$ it holds along any cone C_u. If $(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\infty,v_0,\theta,\varphi)=0$ then*

$$(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\infty,v,\theta,\varphi)=0, \tag{3.4.47}$$

for all $v\in[v_0,\infty)$. Similarly, if $(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})(\infty,v_0,\theta,\varphi)=0$ then

$$(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})(\infty,v,\theta,\varphi)=0, \tag{3.4.48}$$

for all $v\in[v_0,\infty)$.

Proof. The statement about the basic round sphere condition can be found in corollary A.1 of [28].

The Raychauduri equation for $(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})$ on the future event horizon $\mathcal{H}^+$ gives

$$\partial_v(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi}) = \frac{1}{2M}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi}), \tag{3.4.49}$$

since

$$(\Omega\mathrm{Tr}_{\not g}\chi)|_{\mathcal{H}^+} = \frac{2}{r}\Big(1-\frac{2M}{r}\Big)\Big|_{\mathcal{H}^+} = 0 \quad \text{and} \quad \omega|_{\mathcal{H}^+} = \frac{M}{r^2}\Big|_{\mathcal{H}^+} = \frac{1}{4M}. \tag{3.4.50}$$

One can check that

$$(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi}) = A(\infty,\theta,\varphi)e^{\frac{1}{2M}v}, \tag{3.4.51}$$

is the solution to equation (3.4.49). The initial condition gives that

$$A(\infty,\theta,\varphi)e^{\frac{1}{2M}v_0} = 0 \implies A(\infty,\theta,\varphi) \equiv 0. \tag{3.4.52}$$

Note that from the commutation lemma 2.8.4, the linearised transport equations for torsion (proposition 2.10.11) and the linearised Bianchi equations (proposition 2.10.20) one finds

$$\Omega\not\nabla_4(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho}) = -(\Omega\mathrm{Tr}_{\not g}\chi)\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\frac{1}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}}-\frac{3}{2}\rho(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})-\frac{3}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\rho}.$$

Evaluating on the future event horizon gives

$$\partial_v(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})|_{\mathcal{H}^+} = 0, \tag{3.4.53}$$

since $(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})|_{\mathcal{H}^+} = 0$ and $(\Omega\mathrm{Tr}_{\not g}\chi)(2M) = 0$. The unique solution for the initial data $(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})(\infty,v_0,\theta,\varphi) = 0$ is $(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})(\infty,v,\theta,\varphi) = 0$ for all $v \in [v_0,\infty)$. □

3.4.5 Extendibility to the Future Event Horizon

Whilst not strictly speaking a restriction on the data one has to understand the regular quantities at the future event horizon $\mathcal{H}^+$ when working with double null Eddington–Finkelstein coordinates. As noted in section 2.8 the double null Eddington–Finkelstein coordinates do not cover the future event horizon of the Schwarzschild spacetime. This means that the frame $(e_3 = \frac{1}{\Omega}\partial_u, e_4 = \frac{1}{\Omega}\partial_v, e_A)$ does not extend regularly to the horizon. However, by transforming to Kruskal–Szekeres coordinates [103, 104] one can show that the re-scaled frame $(\frac{1}{\Omega}e_3, \Omega e_4, e_1, e_2)$ does extend regularly to a non-vanishing null frame on $\mathcal{H}^+$. It will be necessary to take the limit of the canonical energy to the future event horizon $\mathcal{H}^+$ to understand the regular quantities there.

Hence, one has the following proposition for the regular linearised quantities at the future event horizon $\mathcal{H}^+$:

Proposition 3.4.9. *For a smooth solution h to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild exterior. Define*

$$\mathbb{E}^+ \doteq \Big\{\frac{\overset{(1)}{\Omega}}{\Omega}, \mathrm{Tr}_{\not g}\not h, \overset{(1)}{b}, \hat{\not h}, (\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi}), \frac{(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})}{\Omega^2}, \overset{(1)}{\omega}, \frac{\overset{(1)}{\underline{\omega}}}{\Omega^2}, \overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}, \Omega\overset{(1)}{\hat{\chi}}, \frac{\overset{(1)}{\hat{\underline{\chi}}}}{\Omega}, \overset{(1)}{\rho}, \overset{(1)}{\sigma}, \overset{(1)}{K}, \Omega\overset{(1)}{\beta}, \frac{\overset{(1)}{\underline{\beta}}}{\Omega}, \Omega^2\overset{(1)}{\alpha}, \frac{\overset{(1)}{\underline{\alpha}}}{\Omega^2}\Big\}. \tag{3.4.54}$$

Then any $q \in \mathbb{E}^+$ extends smoothly to $\mathcal{H}^+$ in the sense that for any $n_1, n_2, n_3 \in \mathbb{N}\cup\{0\}$

$$\Big(\frac{1}{\Omega^2}\partial_u\Big)^{n_1}\partial_v^{n_2}\partial_A^{n_3} q \tag{3.4.55}$$

extends continuously to $\mathcal{H}^+$.

Proof. The method to identify the regularity is the following. From the discussion on the formal linearisation of the metric in double null form (see section 2.10) around Schwarzschild one has

$$h\Big(\frac{1}{\Omega}e_3, \Omega e_4\Big) = -4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \quad h(\Omega e_4, e_A) = -\overset{(1)}{b}_A, \quad h(e_A, e_B) = \hat{\not h}_{AB} + \frac{1}{2}\mathrm{Tr}_{\not g}\not h \not g_{AB}. \tag{3.4.56}$$

The quantities on the LHS extend regularly to $\mathcal{H}^+$ since they are written in the frame that extends regularly, so the quantities on the RHS must extend regularly to $\mathcal{H}^+$.

Next from the linearised equations for the metric coefficients (proposition 2.10.7) around Schwarzschild one has:

$$\begin{aligned}
&\Omega e_4(\mathrm{Tr}_{\not g}\not h) = 2\Big((\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi}) - \mathrm{d\not{i}v}\overset{(1)}{b}\Big), && \frac{1}{\Omega}e_3(\mathrm{Tr}_{\not g}\not h) = \frac{2}{\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}}),\\
&\frac{1}{\Omega}\widehat{(\not\nabla_3\not h)}_{AB} = \frac{2}{\Omega}\overset{(1)}{\hat{\underline{\chi}}}_{AB}, && \Omega\widehat{(\not\nabla_4\not h)}_{AB} = 2\Omega\overset{(1)}{\hat{\chi}}_{AB} + 2(\not{\mathcal{D}}_2^\star\overset{(1)}{b})_{AB},\\
&\frac{1}{\Omega^2}\partial_u\overset{(1)}{b}{}^A = 2(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})^A, && \not\nabla_A\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \frac{1}{2}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})_A,\\
&\Omega e_4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \overset{(1)}{\omega}, && \frac{1}{\Omega}e_3\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \frac{1}{\Omega^2}\overset{(1)}{\underline{\omega}}.
\end{aligned} \tag{3.4.57}$$

Again, the quantities on the LHS extend regularly to $\mathcal{H}^+$ since they are written in the frame that extends regularly, so the quantities on the RHS must also extend regularly to $\mathcal{H}^+$.

Next from the linearised null structure equations (propositions 2.10.10, 2.10.14, 2.10.11, 2.10.13 and 2.10.15) around Schwarzschild one has:

$$\begin{aligned}
&\frac{\overset{(1)}{\underline{\alpha}}}{\Omega^2} = -\frac{1}{\Omega}\nabla\!\!\!\!/_3\Big(\frac{\overset{(1)}{\hat{\underline{\chi}}}}{\Omega}\Big) - \frac{(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}{\Omega}\frac{\overset{(1)}{\hat{\underline{\chi}}}}{\Omega}, && \Omega^2\overset{(1)}{\alpha} = -\Omega\nabla\!\!\!\!/_4(\Omega\overset{(1)}{\hat{\chi}}) + 2\omega\Omega\overset{(1)}{\hat{\chi}} - (\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\Omega\overset{(1)}{\hat{\chi}},\\
&\Omega\overset{(1)}{\beta} = -\Omega\nabla\!\!\!\!/_4\overset{(1)}{\eta} + \frac{1}{2}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\overset{(1)}{\underline{\eta}} - \overset{(1)}{\eta}), && \frac{1}{\Omega}\overset{(1)}{\underline{\beta}} = \frac{1}{\Omega}\nabla\!\!\!\!/_3\overset{(1)}{\underline{\eta}} + \frac{1}{2\Omega}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})(\overset{(1)}{\underline{\eta}} - \overset{(1)}{\eta}),\\
&\overset{(1)}{\rho} = -2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho - \frac{1}{\Omega}e_3(\overset{(1)}{\omega}), && \overset{(1)}{\sigma} = -\mathrm{curl}\overset{(1)}{\eta},
\end{aligned} \tag{3.4.58}$$

and

$$\overset{(1)}{K} = \frac{\overset{(1)}{\Omega}}{2\Omega}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\frac{(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}{\Omega} - \overset{(1)}{\rho} - \frac{1}{4}\Big((\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\frac{\overset{(1)}{(\Omega\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}}{\Omega^2} + \overset{(1)}{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)}\frac{(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})}{\Omega}\Big). \tag{3.4.59}$$

The RHS of these equations extends regularly to $\mathcal{H}^+$ and, therefore, so does the LHS. □

3.4.6 The Generality of Solutions

In the next section all solutions of the linearised vacuum Einstein equation (I.5) will be assumed to be in partially initial data normalised gauge, supported on $\ell \geq 2$ and extendible to null infinity. The reader may worry about a loss of generality of the solutions considered here. However, one should note the following theorem from [28]:

Theorem 3.4.10 ([28]). *Under a suitable assumption for asymptotic flatness on the initial data for a smooth solution h to the linearised vacuum Einstein equation (I.5), one can construct a residual pure gauge solution h_{pg} to the linearised vacuum Einstein equation (I.5) in double null gauge and a linearised Kerr solution h_{K}, both explicitly computable and controllable from initial data, such that $h' = h + h_{\mathrm{pg}} - h_{\mathrm{K}}$ is a (partially) initial data normalised solution (see definition 9.1 in [28]) supported on $\ell \geq 2$ which is extendible to null infinity.*

3.4.7 The Limits of the Canonical Energy Fluxes for Restricted Data

The Future Event Horizon Limit

Proposition 3.4.11. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior that satisfies the first horizon gauge condition (see definition 3.4.5). Let*

$$\overline{\mathcal{E}}_{\mathcal{H}^+}[h](v_0,v) \doteq \lim_{u\to\infty}\overline{\mathcal{E}}_u[h](v_0,v), \tag{3.4.60}$$

$$\overline{\mathcal{E}\!\!\!/}_{\mathcal{H}^+}[h](v_0,v) \doteq \lim_{u\to\infty}\overline{\mathcal{E}\!\!\!/}_u[h](v_0,v), \tag{3.4.61}$$

$$\dot{\overline{\mathcal{E}}}_{\mathcal{H}^+}[h](v_0,v) \doteq \lim_{u\to\infty}\dot{\overline{\mathcal{E}}}_u[h](v_0,v). \tag{3.4.62}$$

Then for all $v > v_0$, the modified (higher order) T-canonical energy on the future event horizon $\mathcal{H}^+$ is given by

$$\overline{\mathcal{E}}_{\mathcal{H}^+}[h](v_0,v) = \int_{v_0}^{v}\int_{\mathbb{S}^2_{\infty,v}} |\Omega\overset{(1)}{\hat{\chi}}|^2 dv\not{\epsilon}, \tag{3.4.63}$$

$$\overline{\not{\mathcal{E}}}_{\mathcal{H}^+}[h](v_0,v) = \int_{v_0}^{v}\int_{\mathbb{S}^2_{\infty,v}} \frac{r^2}{2}|\Omega\overset{(1)}{\beta}|^2 dv\not{\epsilon}, \tag{3.4.64}$$

$$\dot{\overline{\mathcal{E}}}_{\mathcal{H}^+}[h](v_0,v) = \int_{v_0}^{v}\int_{\mathbb{S}^2_{\infty,v}} \frac{1}{4}|\Omega^2\overset{(1)}{\alpha}|^2 dv\not{\epsilon}. \tag{3.4.65}$$

Proof. Recall from proposition 3.4.9 in section 3.4.5 that the relevant regular weighted quantities on the future event horizon $\mathcal{H}^+$ are

$$\Big\{\frac{\overset{(1)}{\Omega}}{\Omega}, (\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}), \frac{1}{\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}), \overset{(1)}{\omega}, \overset{(1)}{\underline{\eta}}, \Omega\overset{(1)}{\hat{\chi}}, \frac{1}{\Omega}\overset{(1)}{\underline{\hat{\chi}}}, \overset{(1)}{\rho}, \overset{(1)}{\sigma}, \Omega\overset{(1)}{\beta}, \Omega^2\overset{(1)}{\alpha}\Big\}. \tag{3.4.66}$$

Writing the $\overline{\mathcal{E}}^T_u[h](v_0,v_1)$ flux in terms of regular quantities gives

$$\begin{aligned}\overline{\mathcal{E}}^T_u[h](v_0,v_1) = \int_{v_0}^{v_1}\int_{\mathbb{S}^2_{u,v}} \Big(&|\Omega\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2 - \frac{1}{2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})^2 \\ &+ 4\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) - 2\Omega^2\overset{(1)}{\omega}\frac{(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})}{\Omega^2}\Big)dv\not{\epsilon}.\end{aligned} \tag{3.4.67}$$

Now the flux is written in terms of smooth functions which smoothly extend to the future event horizon $\mathcal{H}^+$. Hence, the integrand is bounded by some constant depending on v_0, v_1 but independent of u. Since $[v_0,v_1]\times\mathbb{S}^2_{u,v}$ is compact, one can apply Lebesgue's bounded convergence theorem (see chapter 2, theorem 1.4 of [126]) to pass the limit through the integral and conclude

$$\overline{\mathcal{E}}^T_{\mathcal{H}^+}[h](v_0,v_1) = \int_{v_0}^{v_1}\int_{\mathbb{S}^2_{u,v}} \Big[2|\Omega\overset{(1)}{\hat{\chi}}|^2 - (\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})^2 + 8\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\Big]\Big|_{\mathcal{H}^+} dv\not{\epsilon}. \tag{3.4.68}$$

The result then follows by recalling that from proposition 3.4.7 that if a smooth solution satisfies the first initial horizon gauge condition then $(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})|_{\mathcal{H}^+} = 0$. The limit of the other fluxes is analogous. □

The Null Infinity Limit

Proposition 3.4.12. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior that is extendible to null infinity, satisfies the partial initial data normalised gauge conditions (see definition 3.4.5) and is supported on $\ell \geq 1$. Let $\overline{\mathcal{E}}_{\mathcal{I}^+}[h](u_0,u) \doteq \lim_{v\to\infty}\overline{\mathcal{E}}_v[h](u_0,u)$. Then for all $u > u_0$ one has that* *the modified T-canonical energy on null infinity is given by*

$$\overline{\mathcal{E}}_{\mathcal{I}^+}[h](u_0,u_f)=\lim_{v\to\infty}\int_{u_0}^{u_f}\int_{\mathbb{S}^2_{u,v}}|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2 du\not{\epsilon}+\lim_{v\to\infty}\int_{\mathbb{S}^2_{u,v}}\frac{r}{2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\not{\epsilon}\Big|_{u_0}^{u_1}. \tag{3.4.69}$$

Proof. In this proof the notation $\equiv$ will be used to denote a formula that holds under integration over $\mathbb{S}^2_{u,v}$.

Fix $u_0\leq u_1<\infty$ a v large. Now note that using the linearised torsion equations and linearised metric equations 2.10.7 of propositions 2.10.11 and 2.10.7 respectively one has

$$r^2\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\equiv\partial_u\Big[r^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\Big]+r\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})+r^2|\overset{(1)}{\eta}|^2+\mathcal{O}\Big(\frac{1}{r}\Big), \tag{3.4.70}$$

using the definition of extendibility to null infinity. Therefore, using this manipulation and making further use of the extendibility to null infinity one finds

$$\begin{aligned}\overline{\mathcal{E}}_v[h](u_0,u_1)=&\int_{u_0}^{u_1}\int_{\mathbb{S}^2_{u,v}}\Big[|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2-\frac{2}{r}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})-\frac{1}{2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})^2+\mathcal{O}\Big(\frac{1}{r^3}\Big)\Big]du\not{\epsilon}\\&-2\int_{\mathbb{S}^2_{u,v}}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\not{\epsilon}\Big|_{u_0}^{u_1}.\end{aligned} \tag{3.4.71}$$

From the linearised Gauss equations (proposition 2.10.15) one has

$$r^2\overset{(1)}{K}=-r^2\overset{(1)}{\rho}-\frac{r}{2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})-(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\Big]-2\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big). \tag{3.4.72}$$

Using that the solution is extendible to null infinity and the round sphere condition gives:

$$4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)=r^2\overset{(1)}{K}_{\ell=0}-r(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})+\mathcal{O}\Big(\frac{1}{r}\Big), \tag{3.4.73}$$

since $\overset{(1)}{K}_{\ell=1}=0$ identically (see proposition 3.4.2). Substituting this relation above and using that the solution has support on $\ell\geq 1$ gives

$$\begin{aligned}\overline{\mathcal{E}}_v[h](u_0,u_1)=&\int_{u_0}^{u_1}\int_{\mathbb{S}^2_{u,v}}\Big[|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2+\mathcal{O}\Big(\frac{1}{r^3}\Big)\Big]du\not{\epsilon}\\&+\int_{\mathbb{S}^2_{u,v}}\Big[\frac{r}{2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})+\mathcal{O}\Big(\frac{1}{r}\Big)\Big](\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\not{\epsilon}\Big|_{u_0}^{u_1}.\end{aligned} \tag{3.4.74}$$

Since, $c_{u_0,u_1,v_0}r\leq v\leq C_{u_0,u_1,v_0}r$ in the region $\mathcal{E}_A\cap\{v\geq v_0\}\cap\{u_0\leq u\leq u_1\}$ the limit is as stated. □

Proposition 3.4.13. *Suppose h is a solution to the linearised vacuum Einstein equation (I.5) in double null gauge that is extendible to null infinity on the Schwarzschild black hole exterior. Let*

$$\overline{\not{\mathcal{E}}}_{\mathcal{I}^+}[h](u_0,u) \doteq \lim_{v\to\infty} \overline{\not{\mathcal{E}}}_v[h](u_0,u), \tag{3.4.75}$$

$$\dot{\overline{\mathcal{E}}}_{\mathcal{I}^+}[h](u_0,u) \doteq \lim_{v\to\infty} \dot{\overline{\mathcal{E}}}_v[h](u_0,u). \tag{3.4.76}$$

Then for all $u > u_0$, the modified higher order T-canonical energies on null infinity are given by

$$\overline{\not{\mathcal{E}}}_{\mathcal{I}^+}[h](u_0,u) = \lim_{v\to\infty} \int_{u_0}^{u_1} \int_{\mathbb{S}^2_{u,v}} \frac{r^2}{2} |\Omega \overset{(1)}{\underline{\beta}}|^2 du \not{\epsilon}, \tag{3.4.77}$$

$$\dot{\overline{\mathcal{E}}}_{\mathcal{I}^+}[h](u_0,u) = \lim_{v\to\infty} \int_{u_0}^{u_1} \int_{\mathbb{S}^2_{u,v}} \frac{1}{4} |\Omega^2 \overset{(1)}{\underline{\alpha}}|^2 du \not{\epsilon}. \tag{3.4.78}$$

Proof. Fix $u_0 \leq u_f < \infty$ a v large. Note that $v \sim r$ in the region $\mathcal{E}_A \cap \{v \geq v_0\} \cap \{u_0 \leq u \leq u_f\}$. Then writing the fluxes in terms of quantities extendible to null infinity (see definition 3.4.4), one has

$$\begin{aligned}\overline{\not{\mathcal{E}}}^T_v[h](u_0,u_1) = \int_{u_0}^{u_1} \int_{\mathbb{S}^2_{u,v}} & \Big(\frac{\Omega^2}{2} |r^2 \overset{(1)}{\underline{\beta}}|^2 + \frac{3\Omega^2 r^2 \rho}{2} |r \overset{(1)}{\eta}|^2 + \frac{\Omega^2 r^4}{2r^6} (|r^3 \overset{(1)}{\sigma}|^2 + |r^3 \overset{(1)}{\rho}|^2) \\ & - \frac{3r^2 \rho}{2} \overset{(1)}{\underline{\omega}} (r^2 (\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}} \chi})) + \frac{3r^3 \rho}{2} \Big[\frac{1}{2} (\Omega \mathrm{Tr}_{\not{g}} \chi) - 2\omega \Big] \Big(\frac{\overset{(1)}{\Omega}}{\Omega} \Big) \big(r (\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}} \underline{\chi}}) \big) \Big) du \mathring{\not{\epsilon}},\end{aligned} \tag{3.4.79}$$

where $\mathring{\not{\epsilon}}$ is the volume form on the unit sphere. Doing the same for the $\dot{\overline{\mathcal{E}}}^T_v[h](u_0,u_1)$ gives the limits as stated. □

3.4.8 Boundary Conditions for Mode Solutions to the Teukolsky ODE

In this section mode solutions for the traditional Teukolsky equation (2.10.150) in the Newman–Penrose formalism are studied. The reader should see section 2.10.5 for further details on the connection between the Newman–Penrose formalism of the Teukolsky equation and the Teukolsky equation for $(\overset{(1)}{\alpha}, \overset{(1)}{\underline{\alpha}})$.

The equation (2.10.150) is fully separable so one can study fixed frequency fully separated mode solutions of the form

$$\upalpha^{[s]} = e^{-i\omega t} e^{im\varphi} S^{[s]}_{m\uplambda^{[s]}}(\theta) R^{[s]}(r), \tag{3.4.80}$$

where $S^{[s]}_{m\uplambda^{[s]}}$ is a smooth spin s-weighted spheroidal harmonic which satisfies the angular ODE

$$-\frac{1}{\sin\theta} \frac{d}{d\theta} \Big(\sin\theta \frac{dS^{[s]}_{m\uplambda^{[s]}}}{d\theta} \Big)(\theta) + \Big(\frac{(m + s\cos\theta)^2}{\sin^2\theta} - s \Big) S^{[s]}_{m\uplambda^{[s]}}(\theta) = \uplambda^{[s]} S^{[s]}_{m\uplambda^{[s]}}(\theta). \tag{3.4.81}$$

The reader should consult proposition 2.1 in the work [32] for further information on the angular ODE and the smooth spin s-weighted spheroidal harmonics $S^{[s]}_{m\lambda^{[s]}}$. The Teukolsky equation (2.10.150) then implies that $R^{[s]}(r)$ satisfies the radial ODE

$$\frac{1}{(r^2D(r))^s}\frac{d}{dr}\Big((r^2D(r))^{s+1}\frac{dR^{[s]}}{dr}\Big)(r)+\Big(\frac{\omega^2r^2-2i\omega s(r-M)}{D(r)}\Big)R^{[s]}(r) \\ +\Big(4is\omega r-\lambda^{[s]}\Big)R^{[s]}(r)=0, \tag{3.4.82}$$

for $R^{[s]}(r)$ where $\lambda^{[s]}=\ell(\ell+1)-s(s+1)$ (from section 6.2.1 in [29]) is the separation constant.[c] The section now embarks upon the asymptotic analysis of the radial ODE (3.4.82). In particular, a basis of solution to the ODE (3.4.82) associated with the points $r=2M$ and $r\to\infty$ are identified (see proposition 3.4.14).

The radial ODE (3.4.82) can be written in a more practical form for asymptotic ODE analysis as

$$\frac{d^2R^{[s]}}{dr^2}(r)+\mathrm{P}^{[s]}(r)\frac{dR^{[s]}}{dr}(r)+\mathrm{Q}^{[s]}_{\omega,m,\lambda}(r)R^{[s]}(r)=0, \tag{3.4.83}$$

with

$$\mathrm{P}^{[s]}(r)\doteq\frac{2(s+1)(r-M)}{r^2D(r)},\quad \mathrm{Q}^{[s]}_{\omega,m,\lambda}(r)\doteq\frac{\omega^2r^2-2i\omega s(r-M)}{r^2D(r)^2}+\frac{4is\omega r-\lambda^{[s]}}{r^2D(r)}. \tag{3.4.84}$$

The function $(r-2M)\mathrm{P}^{[s]}(r)$ has a power series expansion,

$$(r-2M)\mathrm{P}^{[s]}(r)=\sum_{n=0}^{\infty}\mathrm{p}_n(r-2M)^n, \tag{3.4.85}$$

with $\mathrm{p}_0=(s+1)$ and

$$\mathrm{p}_n=\frac{(s+1)(-1)^{n-1}}{(2M)^n}, \tag{3.4.86}$$

for $n\geq 1$ which converges for all $r<4M$ (by the ratio test). The function $(r-2M)^2\mathrm{Q}^{[s]}_{\omega,m,\lambda}(r)$ has the power series expansion

$$(r-2M)^2\mathrm{Q}^{[s]}_{\omega,m,\lambda}(r)=\sum_{n=0}^{\infty}\mathrm{q}_n(r-2M)^n, \tag{3.4.87}$$

[c] This form of the radial ODE here and in [32] differs from the original paper of Whiting [17]. The relation between the $R^{[s]}$ here and the $\tilde{R}^{[s]}$ in [17] is $R^{[s]}=(r(r-2M))^{-\frac{s}{2}}\tilde{R}^{[s]}$.

with

$$\mathrm{q}_0 = 4M^2\omega^2 - 2iMs\omega, \qquad \mathrm{q}_1 = \frac{8M^2\omega^2 + 4iMs\omega - \lambda^{[s]}}{2M}, \qquad \mathrm{q}_3 = \frac{\lambda^{[s]}}{4M^2} + \omega^2 \quad (3.4.88)$$

and

$$\mathrm{q}_n = \frac{\lambda^{[s]}(-1)^n}{(2M)^n}, \quad (3.4.89)$$

for $n \geq 4$ which converges for $r < 4M$.

The indicial equation (see appendix A.2) for this ODE is

$$I^{[s]}(\alpha) \doteq \alpha(\alpha - 1) + (s + 1)\alpha + 2M\omega(2M\omega - is). \quad (3.4.90)$$

The polynomial $I^{[s]}$ has roots

$$\alpha_1 = -2iM\omega - s, \qquad \alpha_2 = 2iM\omega. \quad (3.4.91)$$

An exceptional set of roots of $I^{[s]}$ for applying Frobenius's theorem A.2.3 occurs when

$$\alpha_1 - \alpha_2 = -4iM\omega - s \in \mathbb{Z}. \quad (3.4.92)$$

This requires $\Re(\omega) = 0$, and $4M\Im(\omega) \in \mathbb{Z}$. Therefore, provided

$$\omega \notin \Upsilon \doteq \Big\{\frac{ik}{4M} : k \in \mathbb{Z}\Big\}, \quad (3.4.93)$$

Frobenius's theorem A.2.3 gives a basis of solution to the ODE (3.4.83) associated to $r = 2M$ of the form

$$\rho^{[s]}_{2M,+}(r) = (r - 2M)^{-2iM\omega - s} \sum_{j=0}^{\infty} b^{[s]}_{j,+}(r - 2M)^j, \quad (3.4.94)$$

$$\rho^{[s]}_{2M,-}(r) = (r - 2M)^{2iM\omega} \sum_{j=0}^{\infty} b^{[s]}_{j,-}(r - 2M)^j, \quad (3.4.95)$$

where $b^{[s]}_{j,\pm}$ can be calculated recursively (from equation (A.2.10)) and $b^{[s]}_{0,\pm} = 1$ without loss of generality. If $\omega \in \Upsilon$ then theorem A.2.5 gives an altered form for the basis solution $\rho^{[s]}_{2M,\pm}$. This needs to be examined case by case. The exceptional cases for $\Im(\omega) = \frac{k}{4M} \geq 0$ are then the following:

(1) If $s=-2$ and $\Im(\omega)\geq 0$, then, $\alpha_1=\frac{k}{2}+2>-\frac{k}{2}=\alpha_2$. So $\alpha_+=\alpha_1$ and $\alpha_-=\alpha_2$ and theorem A.2.5 gives

$$\rho^{[-2]}_{2M,+}(r)=(r-2M)^{2+\frac{k}{2}}\sum_{j=0}^{\infty}b^{[-2]}_{j,+}(r-2M)^j, \tag{3.4.96}$$

$$\rho^{[-2]}_{2M,-}(r)=(r-2M)^{-\frac{k}{2}}\sum_{j=0}^{\infty}b^{[-2]}_{j,-}(r-2M)^j+C_{k+2}\rho^{[-2]}_{2M,+}(r)\ln(r-2M), \tag{3.4.97}$$

where $a_0^{[-2],\pm}=1$.

(2) If $s=+2$, $\Im(\omega)\geq 0$ and $k>1$, then, $\alpha_1=\frac{k}{2}-2\geq-\frac{k}{2}=\alpha_2$. So $\alpha_+=\alpha_1$ and $\alpha_-=\alpha_2$ and theorem A.2.5 gives

$$\rho^{[+2]}_{2M,+}(r)=(r-2M)^{\frac{k}{2}-2}\sum_{j=0}^{\infty}b^{[+2]}_{j,+}(r-2M)^j, \tag{3.4.98}$$

$$\rho^{[+2]}_{2M,-}(r)=(r-2M)^{\delta_{2,k}-\frac{k}{2}}\sum_{j=0}^{\infty}b^{[+2]}_{j,-}(r-2M)^j+C_{k-2}\rho^{[+2]}_{2M,+}(r)\ln(r-2M), \tag{3.4.99}$$

where $C_0\neq 0$ (see remark A.2.6 and end of page 155 in [97]).

(3) If $s=+2$, $\Im(\omega)\geq 0$ and $k=1$, then, $\alpha_1=-\frac{3}{2}\leq-\frac{1}{2}=\alpha_2$. So $\alpha_+=\alpha_2$ and $\alpha_-=\alpha_1$ and theorem A.2.5 gives[d]

$$\rho^{[+2]}_{2M,+}(r)=(r-2M)^{-\frac{1}{2}}\sum_{j=0}^{\infty}b^{[+2]}_{j,+}(r-2M)^j, \tag{3.4.100}$$

$$\rho^{[+2]}_{2M,-}(r)=(r-2M)^{-\frac{3}{2}}\sum_{j=0}^{\infty}b^{[+2]}_{j,-}(r-2M)^j+C_1\rho^{[+2]}_{2M,-}(r)\ln(r-2M). \tag{3.4.101}$$

From the remark A.2.6 following theorem A.2.5, the co-efficient C_1 vanishes if, and only if,

$$\Big(-\frac{3}{2}\mathrm{p}_1+\mathrm{q}_1\Big)=0 \implies -\frac{\lambda^{[+2]}+7}{2M}=0 \tag{3.4.102}$$

Since $\lambda^{[+2]}=\ell(\ell+1)-6$ and $\ell\geq 2$, $C_1\neq 0$.

The point $r=\infty$ is a irregular singular point of the ODE (3.4.83) since $\mathrm{P}^{[s]}(r)$ and $\mathrm{Q}^{[s]}_{\omega,m,\lambda}(r)$ admit convergent power series expansions in some annulus (thinking of r as a complex variable)

$$\mathrm{P}^{[s]}(r)=\sum_{n=0}^{\infty}\frac{\mathrm{p}_n}{r^n},\qquad \mathrm{Q}^{[s]}_{\omega,m,\lambda}(r)=\sum_{n=0}^{\infty}\frac{\mathrm{q}_n}{r^n}, \tag{3.4.103}$$

[d] One should note that definition 2.3 in the work [32] uses the opposite notation for these solutions (and drops the logarithmic term), i.e., in the notation of [32] $R^{[+2]}_{\mathcal{H}^\pm}=\rho^{[+2]}_{2M,\mp}$ for $\omega=\frac{i}{4M}$. However, as will be shown in proposition 3.4.14, if one wants the $\pm$ notation to denote the basis element that is smoothly extendible to $\mathcal{H}^\pm$ then this is the correct convention.

with

$$\mathrm{p}_0 = 0, \qquad \mathrm{p}_1 = 2(1+s), \qquad \mathrm{q}_0 = \omega^2, \qquad \mathrm{q}_1 = 2i\omega s + 4M\omega^2. \tag{3.4.104}$$

The constants $\lambda_\pm$ and $\mu_\pm$ appearing in theorem A.2.7 can be calculated to be

$$\begin{aligned} \lambda_+ &= i\omega, & \lambda_- &= -i\omega, \\ \mu_+ &= -1-2s+2Mi\omega, & \mu_- &= -1-2Mi\omega. \end{aligned} \tag{3.4.105}$$

Now provided $\frac{1}{4}\mathrm{p}_0^2 \neq q_0$, which is equivalent to $\omega \neq 0$, one can use theorem A.2.7 to construct a basis of solution $(\rho^{[s]}_{\infty,+}, \rho^{[s]}_{\infty,-})$ to the ODE (3.4.83) associated to the irregular singular point $r = \infty$.[e] Theorem A.2.7 gives that these solutions satisfy

$$\rho^{[s]}_{\infty,+} = e^{i\omega r} r^{-1-2s+2Mi\omega}\Big(1 + \mathcal{O}\Big(\frac{1}{r}\Big)\Big), \tag{3.4.106}$$

$$\rho^{[s]}_{\infty,-} = e^{-i\omega r} r^{-1-2Mi\omega}\Big(1 + \mathcal{O}\Big(\frac{1}{r}\Big)\Big). \tag{3.4.107}$$

With the asymptotic analysis in hand, the remaining issue to address is what boundary conditions to impose at $r = 2M$ and $r = \infty$ on $\alpha^{[s]}$ for $s = \pm 2$. The key proposition is the following:

Proposition 3.4.14 (Admissible Boundary Conditions for a Real/Growing Mode)**.** *Let $s = \pm 2$, $\mathfrak{I}(\omega) \geq 0$ and $\omega \neq 0$. Let $\rho^{[s]}_{2M,\pm}$ be the basis for the space of solutions to the radial ODE (3.4.83) as defined in equations (3.4.94)–(3.4.101) and $\rho^{[s]}_{\infty,\pm}$ be the basis for the space of solutions to the radial ODE (3.4.83) as defined in equations (3.4.106) and (3.4.107). In particular, to any solution $R^{[s]}$ of the radial ODE (3.4.83) one can ascribe four numbers $a^{[s]}_{2M,+}, a^{[s]}_{2M,-}, a^{[s]}_{\infty,+}, a^{[s]}_{\infty,-} \in \mathbb{C}$ defined by*

$$R^{[s]} = a^{[s]}_{2M,+}\rho^{[s]}_{2M,+} + a^{[s]}_{2M,-}\rho^{[s]}_{2M,-}, \tag{3.4.108}$$

$$R^{[s]} = a^{[s]}_{\infty,+}\rho^{[s]}_{\infty,+} + a^{[s]}_{\infty,-}\rho^{[s]}_{\infty,-}. \tag{3.4.109}$$

Let $\alpha^{[s]}$ be a solution of the traditional Teukolsky equation (2.10.150) associated to $R^{[s]}(r)$ a solution of the radial ODE (3.4.83) through equation (3.4.80). Further, let $(\overset{(1)}{\alpha}, \overset{(1)}{\underline{\alpha}})$ be associated to $\alpha^{[s]}$ through the equations (2.10.160) and (2.10.161). Then the admissible boundary conditions for the radial ODE (3.4.83) are defined by the requirement that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ extend regularly to $\mathcal{H}^+$ and are extendible to null infinity $\mathcal{I}^+$. One can characterise this requirement with the following statements:

(i) If $\mathfrak{I}(\omega) > 0$ then $(\overset{(1)}{\alpha}, \overset{(1)}{\underline{\alpha}})$ that are extendible to null infinity if, and only if, $a^{[s]}_{\infty,-} = 0$.

[e] In the case where $\omega = 0$, $\mathrm{q}_1 = 0$ also. This, in fact, turns $r = \infty$ into a regular singular point (see Olver [97] chapter 7 section 1.3) One can construct a basis of convergent series solutions associated to $r = \infty$.

(ii) If $\Im(\omega) = 0$ *then* $\overset{(1)}{\underline{\alpha}}$ *is extendible to null infinity if* $a^{[-2]}_{\infty,-} = 0$.

(iii) If $\Im(\omega) = 0$ *then* $\overset{(1)}{\alpha}$ *is extendible to null infinity if, and only if,* $a^{[+2]}_{\infty,-} = 0$.

(iv) $\frac{1}{\Omega^2}\overset{(1)}{\underline{\alpha}}$ *smoothly extends to the future event horizon if, and only if,* $a^{[-2]}_{2M,-} = 0$.

(v) $\Omega^2\overset{(1)}{\alpha}$ *smoothly extends to the future event horizon if, and only if,* $a^{[+2]}_{2M,-} = 0$.

Henceforth, a solution to traditional Teukolsky equation (2.10.150) obeying the boundary conditions in (i)-(v) will be referred to as an 'outgoing mode solution'.[f]

Proof. Starting with $r = 2M$ and restricting for the moment to $\omega \notin \Upsilon$, recall that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ do not extend regularly to the future event horizon $\mathcal{H}^+$ due to the use of the irregular frame $(e_3 = \frac{1}{\Omega}\partial_u, e_4 = \frac{1}{\Omega}\partial_v)$. However, as discussed in section 3.4.5 one can show that $\Omega^2\overset{(1)}{\alpha}$ and $\frac{\overset{(1)}{\underline{\alpha}}}{\Omega^2}$ extend regularly to the future event horizon. Using the relations (2.10.158) and (2.10.159), this translates to the conditions that $D(r)^2\alpha^{[+2]}$ and $\frac{\alpha^{[-2]}}{D(r)^2}$ should extend regularly to the future event horizon. At the level of mode solutions this imposes that

$$D(r)^2\alpha^{[+2]} = \frac{e^{i\omega r}(r-2M)^{2Mi\omega+2}}{r^2}R^{[+2]}(r)e^{-i\omega\tilde{v}}e^{im\varphi}S^{[+2]}(\theta), \tag{3.4.110}$$

$$\frac{\alpha^{[-2]}}{D(r)^2} = r^2e^{i\omega r}(r-2M)^{2Mi\omega-2}R^{[-2]}(r)e^{-i\omega\tilde{v}}e^{im\varphi}S^{[-2]}(\theta), \tag{3.4.111}$$

should be smooth at the future event horizon, where $(\tilde{v}, r, \theta, \varphi)$ with $\tilde{v} = t + r_\star(r)$ and $r_\star(r) \doteq r + 2M\ln|r-2M|$ are ingoing Eddington–Finkelstein coordinates (these coordinates are well defined at the future event horizon). By substituting $\rho^{[s]}_{2M,\pm}$ from equations (3.4.94) and (3.4.95) into the equations (3.4.110) and (3.4.111) one can see that for $a^{[s]}_{2M,-} = 0$ and $a^{[s]}_{2M,+} \neq 0$,

$$D(r)^2\alpha^{[+2]} = a^{[+2]}_{2M,+}\frac{e^{i\omega r}}{r^2}e^{-i\omega\tilde{v}}e^{im\varphi}S^{[+2]}(\theta)\sum_{j=0}^{\infty}b^{[+2]}_{j,+}(r-2M)^j, \tag{3.4.112}$$

$$\frac{\alpha^{[-2]}}{D(r)^2} = a^{[-2]}_{2M,+}r^2e^{i\omega r}e^{-i\omega\tilde{v}}e^{im\varphi}S^{[-2]}(\theta)\sum_{j=0}^{\infty}b^{[-2]}_{j,+}(r-2M)^j \tag{3.4.113}$$

and for $a^{[s]}_{2M,-} \neq 0$ and $a^{[s]}_{2M,+} = 0$,

$$D(r)^2\alpha^{[+2]} = a^{[+2]}_{2M,-}\frac{e^{i\omega r}(r-2M)^{4Mi\omega+2}}{r^2}e^{-i\omega\tilde{v}}e^{im\varphi}S^{[+2]}(\theta)\sum_{j=0}^{\infty}b^{[+2]}_{j,-}(r-2M)^j, \tag{3.4.114}$$

$$\frac{\alpha^{[-2]}}{D(r)^2} = a^{[-2]}_{2M,-}\frac{r^2e^{i\omega r}(r-2M)^{4Mi\omega}}{(r-2M)^2}e^{-i\omega\tilde{v}}e^{im\varphi}S^{[-2]}(\theta)\sum_{j=0}^{\infty}b^{[-2]}_{j,-}(r-2M)^j. \tag{3.4.115}$$

Equations (3.4.112) and (3.4.113) allow one to conclude that, for $a_{2M,-} = 0$ and $a_{2M,+} \neq 0$, $D(r)^2\alpha^{[+2]}$ and $\frac{\alpha^{[-2]}}{D(r)^2}$ smoothly extend to $r = 2M$. In contrast, equations (3.4.114) and (3.4.115)

[f] For further motivation see section 3.3 of [127], definition 2.4 of [32] and appendix D of [26].

allow one to conclude that, for $a_{2M,-} \neq 0$ and $a_{2M,+} = 0$, $D^2\alpha^{[+2]}$ extends smoothly if $2 + 4Mi\omega \in \mathbb{Z}_{\geq 0}$ and $\frac{1}{D^2}\alpha^{[-2]}$ extends smoothly if $4Mi\omega - 2 \in \mathbb{Z}_{\geq 0}$. If $\mathfrak{I}(\omega) \geq 0$ and $\omega \notin \Upsilon$ then this gives $D^2\alpha^{[+2]}$ and $\frac{1}{D^2}\alpha^{[-2]}$ never extend smoothly for $a_{2M,-} \neq 0$ and $a_{2M,+} = 0$.

For the exceptional set $\omega \in \Upsilon \cap \{\mathfrak{I}(\omega) \geq 0\}$ one has $\omega = \frac{ik}{4M}$ with $k \in \mathbb{Z}_+ \cup \{0\}$. This means

$$D(r)^2\alpha^{[+2]} = \frac{e^{-\frac{kr}{4M}}(r-2M)^{2-\frac{k}{2}}}{r^2} R^{[+2]}(r) e^{-i\omega\tilde{v}} e^{im\varphi} S^{[+2]}(\theta), \tag{3.4.116}$$

$$\frac{\alpha^{[-2]}}{D(r)^2} = r^2 e^{-\frac{k}{4M}r}(r-2M)^{-\frac{k}{2}-2} R^{[-2]}(r) e^{-i\omega\tilde{v}} e^{im\varphi} S^{[-2]}(\theta), \tag{3.4.117}$$

should be smooth at the future event horizon, where $(\tilde{v}, r, \theta, \varphi)$ with $\tilde{v} = t + r_\star(r)$ and $r_\star(r) \doteq r + 2M \ln|r-2M|$ are ingoing Eddington–Finkelstein coordinates (these coordinates are well defined at the future event horizon). The exceptional cases are then the following:

(1) If $s = -2$ and $\omega = \frac{ik}{4M}$ with $k \geq 0$ then from equations (3.4.96) and (3.4.97) one has

$$\frac{\alpha^{[-2]}}{D(r)^2} = a^{[-2]}_{2M,+} r^2 e^{-\frac{k}{4M}r} e^{-i\omega\tilde{v}} e^{im\varphi} S^{[-2]}(\theta) \sum_{j=0}^{\infty} b^{[-2]}_{j,+}(r-2M)^j, \tag{3.4.118}$$

$$\begin{aligned}\frac{\alpha^{[-2]}}{D(r)^2} = a^{[-2]}_{2M,-} \frac{r^2 e^{-\frac{k}{4M}r}}{(r-2M)^{\frac{k}{2}+2}} e^{-i\omega\tilde{v}} e^{im\varphi} S^{[-2]}(\theta) \Big[\sum_{j=0}^{\infty} b^{[-2]}_{j,-}(r-2M)^{j-\frac{k}{2}} \\ + C_{k+2}\rho^{[-2]}_{2M,+}(r)\ln(r-2M) \Big].\end{aligned} \tag{3.4.119}$$

Therefore, to extend smoothly (and be non-trivial) $a^{[-2]}_{2M,+} \neq 0$ and $a^{[-2]}_{2M,-} = 0$.

(2) If $s = +2$ and $\omega = \frac{ik}{4M}$ with $k > 1$ then from equations (3.4.98) and (3.4.99) one has

$$D(r)^2\alpha^{[+2]} = \frac{e^{-\frac{kr}{4M}}}{r^2} e^{-i\omega\tilde{v}} e^{im\varphi} S^{[+2]}(\theta) \sum_{j=0}^{\infty} b^{[+2]}_{j,+}(r-2M)^j, \tag{3.4.120}$$

$$\begin{aligned}D(r)^2\alpha^{[+2]} = a^{[+2]}_{2M,-} \frac{(r-2M)^{2-\frac{k}{2}}}{e^{\frac{kr}{4M}} r^2} e^{-i\omega\tilde{v}} e^{im\varphi} S^{[+2]}(\theta) \Big[\sum_{j=0}^{\infty} \frac{b^{[+2]}_{j,-}(r-2M)^j}{(r-2M)^{\frac{k}{2}-\delta_{2,k}}} \\ + C_{k-2}\rho^{[+2]}_{2M,+}(r)\ln(r-2M) \Big].\end{aligned} \tag{3.4.121}$$

If $k = 2$ then $C_0 \neq 0$ and therefore $D(r)^2\alpha^{[+2]}$ has an essential singularity if $a^{[+2]}_{2M,-} \neq 0$ (by virtue of the $\ln$ term). If $k > 2$ then the leading order behaviour in $(r-2M)$ is at least $(r-2M)^{-1}$ and therefore, not smoothly extendible if $a^{[+2]}_{2M,-} \neq 0$.

(3) If $s=+2$ and $\omega=\frac{i}{4M}$ then from equations (3.4.100) and (3.4.101) one has

$$D(r)^2\alpha^{[+2]} = a^{[+2]}_{2M,+}\frac{e^{-\frac{r}{4M}}(r-2M)}{r^2}e^{-i\omega\tilde{v}}e^{im\varphi}S^{[+2]}(\theta)\sum_{j=0}^{\infty}b^{[+2]}_{j,+}(r-2M)^j, \quad (3.4.122)$$

$$D(r)^2\alpha^{[+2]} = a^{[+2]}_{2M,-}\frac{(r-2M)^{\frac{3}{2}}}{e^{\frac{r}{4M}}r^2}e^{-i\omega\tilde{v}}e^{im\varphi}S^{[+2]}(\theta)\Big[\sum_{j=0}^{\infty}b^{[+2]}_{j,-}(r-2M)^{j-\frac{3}{2}} + C_1\rho^{[+2]}_{2M,+}(r)\ln(r-2M)\Big]. \quad (3.4.123)$$

Above it was noted that $C_1\neq 0$, so to extend smoothly to the future event horizon $a^{[+2]}_{2M,-}=0$.

Turning to $r=\infty$, recall that section 3.4.3 on extendibility to null infinity imposed that $r|\overset{(1)}{\underline{\alpha}}|$ and $r^{3+w}|\overset{(1)}{\alpha}|$ for $w\in(0,1]$ should have a limit on any outgoing cone. Using the equations (2.10.158) and (2.10.159), one can compute that

$$|\overset{(1)}{\underline{\alpha}}|^2 = \frac{2}{\Omega^4 r^8}|\alpha^{[-2]}|^2, \qquad |\overset{(1)}{\alpha}|^2 = 2\Omega^4|\alpha^{[+2]}|^2. \quad (3.4.124)$$

Hence, the extendiblity to null infinity condition, translates to the conditions that $r^{3+w}|\alpha^{[+2]}|$ and $\frac{|\alpha^{[-2]}|}{r^3}$ should have finite limits on null infinity. At the level of mode solutions this imposes that

$$r^{3+w}\alpha^{[+2]} = \frac{r^{3+w}e^{-i\omega r}}{(r-2M)^{2Mi\omega}}R^{[+2]}(r)e^{-i\omega\tilde{u}}e^{im\varphi}S^{[+2]}(\theta), \quad (3.4.125)$$

$$\frac{\alpha^{[-2]}}{r^3} = \frac{e^{-i\omega r}}{r^3(r-2M)^{2Mi\omega}}R^{[-2]}(r)e^{-i\omega\tilde{u}}e^{im\varphi}S^{[-2]}(\theta), \quad (3.4.126)$$

should extend to future null infinity, where $(\tilde{u},r,\theta,\varphi)$ with $\tilde{u}=t-r_\star(r)$ and $r_\star(r)\doteq r+2M\ln|r-2M|$ are outgoing Eddington–Finkelstein coordinates. By substituting the asymptotic behaviour $\rho^{[s]}_{\infty,+}$ from equations (3.4.106) and (3.4.107) into the equations (3.4.125) and (3.4.126) one can see that for $a^{[s]}_{\infty,-}=0$ and $a^{[s]}_{\infty,+}\neq 0$,

$$r^{3+w}\alpha^{[+2]} = a^{[+2]}_{\infty,+}r^{-2+w}e^{-i\omega\tilde{u}}e^{im\varphi}S^{[+2]}(\theta)\Big(1+\mathcal{O}\Big(\frac{1}{r}\Big)\Big), \quad (3.4.127)$$

$$\frac{\alpha^{[-2]}}{r^3} = a^{[-2]}_{\infty,+}e^{-i\omega\tilde{u}}e^{im\varphi}S^{[-2]}(\theta)\Big(1+\mathcal{O}\Big(\frac{1}{r}\Big)\Big) \quad (3.4.128)$$

and for $a^{[s]}_{\infty,-}\neq 0$ and $a^{[s]}_{\infty,+}=0$,

$$r^{3+w}\alpha^{[+2]} = a^{[+2]}_{\infty,-}\frac{r^{2+w}e^{-2i\omega r}}{r^{4Mi\omega}}e^{-i\omega\tilde{u}}e^{im\varphi}S^{[+2]}(\theta)\Big(1+\mathcal{O}\Big(\frac{1}{r}\Big)\Big), \quad (3.4.129)$$

$$\frac{\alpha^{[-2]}}{r^3} = a^{[-2]}_{\infty,-}\frac{e^{-2i\omega r}}{r^{4+4Mi\omega}}e^{-i\omega\tilde{u}}e^{im\varphi}S^{[-2]}(\theta)\Big(1+\mathcal{O}\Big(\frac{1}{r}\Big)\Big). \quad (3.4.130)$$

Therefore, for $a^{[s]}_{\infty,-} = 0$ and $a^{[s]}_{\infty,+} \neq 0$, $r^{3+w}\alpha^{[+2]}$ and $\frac{1}{r^3}\alpha^{[-2]}$ always has finite limit at null infinity. In contrast, if $\Im(\omega) > 0$ then, for $a^{[s]}_{\infty,-} \neq 0$ and $a^{[s]}_{\infty,+} = 0$ one can see that $r^{3+w}\alpha^{[+2]}$ and $\frac{1}{r^3}\alpha^{[-2]}$ have exponential growth in r.

For the case where $\Im(\omega) = 0$ then $r^{3+w}\alpha^{[+2]}$ grows only if $a^{[+2]}_{\infty,-} = 0$ and is extendible to null infinity if $a^{[+2]}_{\infty,-}$. For $\frac{|\alpha^{[-2]}|}{r^3}$ one can readily see that both branches allow $\frac{|\alpha^{[-2]}|}{r^3}$ to extend to null infinity. Hence, the lack of 'only if' in the statement of (ii) in the proposition. □

One of the most important things to notice about the proof of this proposition 3.4.14 is that, if there exist non-trivial outgoing mode solutions to the traditional Teukolsky equation (2.10.150), the limits of (the regular versions of) $\alpha^{[+2]}$ are non-zero on the future event horizon, except for $\omega = \frac{i}{4M}$. For this frequency one has could have growing outgoing mode solution for which (the regularised version of) $\alpha^{[+2]}$ vanishes at the future event horizon. This will be slightly problematic later in this chapter where L^2-boundedness of $\overset{(1)}{\alpha}$ on the future event horizon is exploited to rule out the existence of outgoing mode solutions for $\overset{(1)}{\alpha}$. The proof runs into difficulty for $\omega \neq \frac{i}{4M}$ since this could in theory give rise to a *zero flux mode*.

Fortunately, there is a correspondence between solutions of the $s = \pm 2$ radial Teukolsky ODE (3.4.83) given by the so-called Teukolsky–Starobinsky identites. These were originally proved in [128, 129] for $s = \pm 1, \pm 2$ and extended to general $s \in \frac{1}{2}\mathbb{Z}$ in [130]. This in conjunction with a proof that the L^2-boundedness of $\overset{(1)}{\underline{\alpha}}$ on null infinity rules out the existence of outgoing mode solutions $\overset{(1)}{\underline{\alpha}}$ for all $\Im(\omega) \geq 0$ such that $\omega \neq 0$ saves the proof of mode stability. The relevant lemma required is the following:

Lemma 3.4.15. *Let $s = +2$ and $\omega = \frac{i}{4M}$. Let $\rho^{[s]}_{2M,\pm}$ be the basis for the space of solutions to the radial ODE (3.4.83) as defined in equations (3.4.94)–(3.4.101) and $\rho^{[s]}_{\infty,\pm}$ be the basis for the space of solutions to the radial ODE (3.4.83) as defined in equations (3.4.106) and (3.4.107). Let $R^{[+2]}$ be a solution to the radial ODE (3.4.83) such that $R^{[+2]}$ gives rise to a outgoing mode solution (as identified in proposition 3.4.14), i.e., $R^{[+2]}$ has a representation as*

$$R^{[+2]}(r) = a^{[+2]}_{\infty,+}\rho^{[+2]}_{\infty,+}(r) = a^{[+2]}_{2M,+}\rho^{[+2]}_{2M,+}. \tag{3.4.131}$$

Define

$$\mathscr{R}^{[-2]} \doteq \Delta^s(\mathscr{D}_0^+)^{2s}(\Delta^s R^{[+2]}), \tag{3.4.132}$$

where

$$\mathscr{D}_0^+ \doteq \frac{d}{dr} + \frac{i\omega r^2}{\Delta}, \tag{3.4.133}$$

and $\Delta \doteq r(r-2M)$*. Then* $\mathscr{R}^{[-2]}$ *solves the radial ODE (3.4.14) with* $s=-2$*. Moreover,* $\mathscr{R}^{[-2]}$ *has the following representation*

$$\mathscr{R}^{[-2]} = \mathfrak{C}_{\frac{i}{4M}} a^{[+2]}_{2M,+} \rho^{[-2]}_{2M,+} = \mathfrak{C}^{(1)}_2 a^{[-2]}_{\infty,+} \rho^{[-2]}_{\infty,+}, \tag{3.4.134}$$

where

$$\mathfrak{C}_{\frac{i}{4M}} \doteq \frac{M}{3}\Big(81 + 57\lambda^{[+2]} + 13(\lambda^{[+2]})^2 + (\lambda^{[+2]})^3\Big), \qquad \mathfrak{C}^{(1)}_2 \doteq (2i\omega)^4. \tag{3.4.135}$$

Proof. A proof of the fact that $\mathscr{R}^{[-2]}$ solves the radial ODE (3.4.14) with $s=-2$ follows from the lemmas 2.16 and 2.17 in [32] and were originally proved in [128–130]. The proof of the statement that $\mathscr{R}^{[-2]} = \mathfrak{C}^{(1)}_2 a_{\infty,+} \rho^{[-2]}_{\infty,+}$ can be found in proposition 2.14 in [32].[g] The representation formula (3.4.134) at the future event horizon will be proved here.

Start by recalling that, for $\omega = \frac{i}{4M}$,

$$\rho^{[+2]}_{2M,+}(r) = (r-2M)^{-\frac{1}{2}} \sum_{j=0}^{\infty} b^{[+2]}_{j,+}(r-2M)^j, \tag{3.4.136}$$

where the superfluous subscripts and superscripts on $b^{[+2]}_{j,+}$ will be dropped henceforth, i.e., $b_j \doteq b^{[+2]}_{j,+}$ with $b_0 = 1$. Using the recursion relation (A.2.10) combined with equations (3.4.86), (3.4.88) and (3.4.89) one can compute that

$$b_1 = \frac{4+\lambda^{[+2]}}{4M}, \qquad b_2 = \frac{3 + 6\lambda^{[+2]} + 2(\lambda^{[+2]})^2}{96M^2}, \tag{3.4.137}$$

$$b_3 = \frac{57 - 6\lambda^{[+2]} - 5(\lambda^{[+2]})^2 + (\lambda^{[+2]})^3}{1152M^3}. \tag{3.4.138}$$

One can compute that

$$\widehat{\mathscr{R}}^{[-2]} \doteq \Delta^2(\mathscr{D}^+_0)^k(\Delta^2\rho^{[+2]}_{2M,-}) = \sum_{j=0}^{\infty} b_j \upsilon_k(r,j) r^2 (r-2M)^{j+\frac{7}{2}-k}, \tag{3.4.139}$$

with $\upsilon_k(r,j)$ for $k \geq 1$ defined via the recursive formula,

$$\upsilon_k(r,j) \doteq (r-2M)\frac{d\upsilon_{k-1}(r,j)}{dr} + \Big[\Big(j + \frac{3}{2} - (k-1)\Big) - \frac{r}{4M}\Big]\upsilon_{k-1}(r,j), \tag{3.4.140}$$

[g]Note that there are exceptional cases involving the horizon representation formulas not covered by proposition 2.14 in [32] For $a=0$, these were identified above. These cases persist for the $a \neq 0$. In particular, the exceptional set Υ identified above where $\omega = \frac{ik}{4M}$ for $k \in \mathbb{Z}$ generalises to $\mathfrak{I}(\omega) = \frac{i(r_+ - r_-)k}{4Mr_+}$ with $\mathfrak{R}(\omega) = \frac{ma}{2Mr_+}$, where $r_\pm = M \pm \sqrt{M^2 - a^2}$.

with $\upsilon_0(r,j) \doteq r^2$. One can check that

$$\upsilon_1(r,j) = \Big(j+\frac{7}{2}\Big)r^2 - 4Mr - \frac{r^3}{4M}, \tag{3.4.141}$$

$$\upsilon_2(r,j) = \frac{r^4}{(4M)^2} - \Big(2j+7\Big)\frac{r^3}{4M} - 4M\Big(2j+5\Big)r + 8M^2 + \Big(j^2+6j+\frac{45}{4}\Big)r^2, \tag{3.4.142}$$

$$\upsilon_3(r,j) = 12(3+2j)M^2 - 3(19+16j+4j^2)Mr + \frac{1}{8}(259+202j+60j^2+8j^3)r^2 - \frac{3(41+24j+4j^2)r^3}{16M} + \frac{3(7+2j)r^4}{32M^2} - \frac{r^5}{64M^3}, \tag{3.4.143}$$

$$\upsilon_4(r,j) = 12(5+8j+4j^2)M^2 - 2(55+70j+36j^2+8j^3)Mr + \Big(\frac{1225}{16}+84j+\frac{73j^2}{2}+8j^3+j^4\Big)r^2 - \frac{(199+178j+60j^2+8j^3)r^3}{8M} + \frac{(119+72j+12j^2)r^4}{32M^2} - \frac{(7+2j)r^5}{32M^3} + \frac{r^6}{256M^4}. \tag{3.4.144}$$

Writing $\xi(r,j) = r^2\upsilon_4(r,j)$ and expressing $\xi(r,j)$ as a polynomial in $(r-2M)$ gives

$$\xi(r,j) = \sum_{k=0}^{8}[\xi(j)]_k(r-2M)^k, \tag{3.4.145}$$

where the first few coefficients $[\xi(j)]_k$ are given by

$$[\xi(j)]_0 \doteq 16(j-2)(j-1)j(1+j)M^4, \quad [\xi(j)]_1 \doteq 16(j-1)j(1+j)(2j-1)M^3,$$
$$[\xi(j)]_2 \doteq 2j(1+j)(1+2j)(6j-5)M^2, \quad [\xi(j)]_3 \doteq (1+j)(8j^3-20j-13). \tag{3.4.146}$$

Now expressing $\widehat{\mathscr{R}}^{[-2]}$ as a series in $(r-2M)$ one has

$$\widehat{\mathscr{R}}^{[-2]} = \Delta^2(\mathscr{D}_0^+)^4(\Delta^2\rho^{[+2]}_{2M,-}) = \sum_{j=0}^{\infty} c_j(r-2M)^{j-\frac{1}{2}}. \tag{3.4.147}$$

One can compute the first few coefficients, c_i, of this series as

$$c_0 = b_0[\xi(0)]_0 = 0, \tag{3.4.148}$$

$$c_1 = b_0[\xi(0)]_1 + b_1[\xi(1)]_0 = 0, \tag{3.4.149}$$

$$c_2 = b_0[\xi(0)]_2 + b_1[\xi(1)]_1 + b_2[\xi(2)]_0 = 0 \tag{3.4.150}$$

and

$$c_3 = b_0[\xi(0)]_3 + b_1[\xi(1)]_2 + b_2[\xi(2)]_1 + b_3[\xi(3)]_0 \tag{3.4.151}$$

$$= \frac{M}{3}\Big(81 + 57\lambda^{[+2]} + 13(\lambda^{[+2]})^2 + (\lambda^{[+2]})^3\Big). \tag{3.4.152}$$

Since $\lambda^{[+2]} = \ell(\ell+1) - s(s+1) \geq 0$, $c_3 \neq 0$. Hence,

$$\widehat{\mathscr{R}}^{[-2]} = \frac{M}{3}\Big(81 + 57\lambda^{[+2]} + 13(\lambda^{[+2]})^2 + (\lambda^{[+2]})^3\Big)\Big[(r-2M)^{\frac{5}{2}} + \mathcal{O}(r-2M)\Big]. \quad (3.4.153)$$

Therefore, since $\mathscr{R}^{[-2]}$ is a solution to the radial ODE with $s = -2$, it must be in the span of the basis elements $\rho^{[-2]}_{2M,+}$ and $\rho^{[-2]}_{2M,-}$ given by (3.4.96) and (3.4.97). By direct inspection one sees that $\mathscr{R}^{[-2]}$ cannot be in the span of $\rho^{[-2]}_{2M,-}$. Therefore,

$$\mathscr{R}^{[-2]} = a^{[+2]}_{2M,+}\mathfrak{C}_{\frac{i}{4M}}\rho^{[-2]}_{2M,+}. \quad (3.4.154)$$

□

3.5 Weak Stability Statements from the Canonical Energy

In this section three weak stability theorems are proved. Define the following initial data energies for a solution h of the linearised vacuum Einstein equation (I.5) in double null gauge that is supported on $\ell \geq 2$, extendible to null infinity and satisfies the partially initial data normalised gauge conditions:

$$\overline{\mathcal{E}}^T_{\mathrm{data}}[h](u) \doteq \overline{\mathcal{E}}^T_{v_0}[h](u_0,u) + \overline{\mathcal{E}}^T_{u_0}[h](v_0,\infty) + \lim_{v\to\infty}\int_{\mathbb{S}^2_{u,v}} r(\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\not\epsilon\Big|_{u_0}, \quad (3.5.1)$$

$$\overline{\not\mathcal{E}}_{\mathrm{data}}[h](u) \doteq \overline{\not\mathcal{E}}_{v_0}[h](u_0,u) + \overline{\not\mathcal{E}}_{u_0}[h](v_0,\infty) + \frac{3}{2}\overline{\mathcal{E}}^T_{\mathrm{data}}[h](u). \quad (3.5.2)$$

By the fundamental theorem of calculus these energies are continuous. Further, $\overline{\mathcal{E}}_{\mathrm{data}}[h](u)$ is uniformly bounded in u by the estimate

$$\overline{\mathcal{E}}^T_{v_0}[h](u_0,u) \leq C_M \mathrm{E}(u) \leq C_M\mathrm{E}(\infty), \quad (3.5.3)$$

where

$$\mathrm{E}(u) \doteq \int_{u_0}^{u}\int_{\mathbb{S}^2_{u',v_0}} \Omega^2\Big[|\overset{(1)}{\hat{\underline{\chi}}}|^2 + |\overset{(1)}{\eta}|^2 + \Big(\frac{(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})}{\Omega^2}\Big)^2 + \Big(\frac{\overset{(1)}{\underline{\omega}}}{\Omega^2}\Big)^2 + (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)^2 + \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2\Big]du'\not\epsilon. \quad (3.5.4)$$

Since $\mathrm{E}(u)$ is written in terms of quantities that are smoothly extendible to the event horizon, one has uniform boundedness of $\overline{\mathcal{E}}_{\mathrm{data}}[h](u)$ in u (assuming the L^2-norms of the above quantities are finite in the initial data). Similarly, $\overline{\not\mathcal{E}}_{\mathrm{data}}[h](u)$ is uniformly bounded in u. Define also

$$\overline{\mathcal{E}}^T_{\mathrm{data}}[h] \doteq \lim_{u\to\infty}\overline{\mathcal{E}}^T_{\mathrm{data}}[h](u), \quad (3.5.5)$$

$$\overline{\not\mathcal{E}}^T_{\mathrm{data}}[h] \doteq \lim_{u\to\infty}\overline{\not\mathcal{E}}^T_{\mathrm{data}}[h](u). \quad (3.5.6)$$

Having established the (local) equivalence of the canonical energy to Holzegel's conservation laws [90], the stability theorem from the canonical energy arising from Holzegel's work is the following:

Theorem 3.5.1. *Suppose h is a smooth solution to the linearised Einstein equation in double null gauge on the Schwarzschild black hole exterior supported on $\ell \geq 2$, extendible to null infinity and satisfies the partially initial data normalised gauge conditions. Then, for all $u_f > u_0$.*

$$\int_{v_0}^{\infty}|\Omega\overset{(1)}{\hat{\chi}}(u_f,v)|^2 dv\not\epsilon + \int_{u_0}^{u_f}|\overset{(1)}{\hat{\underline{\chi}}}(u,\infty)|^2 du\not\epsilon \leq \sup_{u_f\in[u_0,\infty)}\mathrm{E}_{\mathrm{data}}[h](u_f) < \infty, \quad (3.5.7)$$

where $\mathrm{E}_{\mathrm{data}}[h](u_f) \doteq \overline{\mathcal{E}}^T_{\mathrm{data}}[h](u_f) + \mathcal{R}(u_f, v_0)$ *and* $\mathcal{R}(u_f, v_0)$ *is defined in terms of initial data as*

$$\mathcal{R}(u_f, v_0) \doteq \frac{1}{6M}\int_{\mathbb{S}^2_{u_f,v_0}} \left| r^3(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta}) - \frac{r^3}{2\Omega^2}\Delta\!\!\!/\Big[4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - r(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})\Big]\right|^2 \mathring{\epsilon\!\!/} \tag{3.5.8}$$
$$- \int_{\mathbb{S}^2_{u_f,v_0}} \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\mathring{\epsilon\!\!/}.$$

Additionally,

$$\lim_{u_f\to\infty}\mathcal{R}(u_f, v_0) = 0. \tag{3.5.9}$$

Moreover,

$$\int_{v_0}^{\infty} |\Omega\overset{(1)}{\hat{\chi}}(\infty, v)|^2 dv\epsilon\!\!/ + \int_{u_0}^{\infty} |\overset{(1)}{\underline{\hat{\chi}}}(u, \infty)|^2 du\epsilon\!\!/ \leq \overline{\mathcal{E}}^T_{\mathrm{data}}[h]. \tag{3.5.10}$$

Hence, $\overline{\mathcal{E}}^T_{\mathrm{data}}[h] \geq 0.$

For completeness, theorem 3.5.1 is reproved in section 3.5.2 of this work. The key ingredient to the proof of this theorem is the conservation law (3.1.3) for the modified T-canonical energy. However, as is evident from considering the fluxes in equations (3.1.1) and (3.1.2) the coercivity properties are very obscure. There are two key ideas that allow one to gain some coercivity for the fluxes and hence produce the energy estimates stated in theorem 3.5.1. First notice from section 3.4.7 on the limits of the canonical energy fluxes that in taking the limit to null infinity of the flux $\overline{\mathcal{E}}^T_{v_f}[h]$ in equation (3.1.2) that (up to the boundary terms on $\mathbb{S}^2_{u_f,\infty}$ and $\mathbb{S}^2_{u_0,\infty}$) the flux is now positive. Second is that one can manipulate the gauge on the final outgoing cone C_{u_f} using the theory developed in section 2.10.3. Using the residual gauge freedom in double null gauge one can impose that the modified T-canonical energy of a solution $h' \doteq h - h_{\mathrm{pg}}$ on the final cone is given by

$$\overline{\mathcal{E}}^T_{u_f}[h'](v_0, v_f) = \int_{v_0}^{v_f} \Big(|\Omega\overset{(1)}{\hat{\chi}}{}'|^2 + 2|\Omega\overset{(1)}{\underline{\eta}}{}'|^2\Big) dv\epsilon\!\!/, \tag{3.5.11}$$

where $\overset{(1)}{\hat{\chi}}{}'$ and $\overset{(1)}{\underline{\eta}}{}'$ are associated to h'. On the face of it, this looks unhelpful since one now has two solutions entering the problem, h and h'. However, it turns out that the change of gauge can be chosen such that $\overset{(1)}{\hat{\chi}}{}' = \overset{(1)}{\hat{\chi}}$. Additionally, it also turns out that the difference between $\overline{\mathcal{E}}^T_{u_f}[h'](v_0, v_f)$ and $\overline{\mathcal{E}}^T_{u_f}[h](v_0, v_f)$ is a boundary term on the spheres $\mathbb{S}^2_{u_f,v_f}$ and $\mathbb{S}^2_{u_f,v_0}$. Miraculously, the former boundary term cancels the boundary term coming from taking the limit of $\overline{\mathcal{E}}^T_{v_f}[h]$ to null infinity.

Through a similar process, one can use the conservation law arising in theorem 3.1.9 to bound $|\overset{(1)}{\beta}|^2$ on any outgoing cone C_u and $|\overset{(1)}{\underline{\beta}}|^2$ at null infinity $\mathcal{I}^+$ by initial data. In particular, one has the following theorem:

Theorem 3.5.2. *Suppose h is a smooth solution to the linearised Einstein equation in double null gauge on the Schwarzschild black hole exterior supported on $\ell \geq 2$, extendible to null infinity that is in partially initial data normalised gauge. Then, for all $u_f > u_0$*

$$\int_{v_0}^{\infty} \frac{r^2}{2}\Big(|\Omega\overset{(1)}{\beta}(u_f)|^2 + |\Omega\overset{(1)}{\sigma}(u_f)|^2\Big) dv\not{\epsilon} + \int_{u_0}^{u_f} \frac{r^2}{2}|\overset{(1)}{\underline{\beta}}(u,\infty)|^2 du\not{\epsilon} \leq \sup_{u_f\in[u_0,\infty)} \not{E}_{\text{data}}[h](u_f) < \infty, \tag{3.5.12}$$

where $\not{E}_{\text{data}}[h](u_f) \doteq \not{\mathcal{E}}_{\text{data}}[h](u_f) + \not{\mathcal{R}}(u_f, v_0)$ and $\not{\mathcal{R}}(u_f, v_0)$ is defined in terms of initial data as

$$\begin{aligned}\not{\mathcal{R}}(u_f, v_0) \doteq{}& \frac{3}{2M}\int_{\mathbb{S}^2_{u_f,v_0}} \Big(1 - \frac{M}{r}\Big)\Big|r^3(\overset{(1)}{\rho} + \not{d}\text{iv}\overset{(1)}{\eta}) - \frac{r^3}{2\Omega^2}\not{\Delta}\mathfrak{F} - \frac{2M}{\Omega^2}\mathfrak{F}\Big|^2 \mathring{\not{\epsilon}} \\ &+ \int_{\mathbb{S}^2_{u_f,v_0}} \Big[\frac{3M}{4\Omega^4}\mathfrak{F}^2 + \frac{3}{2r}\Big|r^3(\overset{(1)}{\rho} + \not{d}\text{iv}\overset{(1)}{\eta}) - \frac{3M}{\Omega^2}\mathfrak{F} - \frac{r^3}{2\Omega^2}\not{\Delta}\mathfrak{F}\Big|^2 + 3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\text{Tr}_{\not{g}}\underline{\chi}})\Big]\mathring{\not{\epsilon}} \\ &+ \frac{3M}{2}\int_{\mathbb{S}^2_{u_f,v_0}} \Big|\mathring{\not{\nabla}}\Big(\frac{\mathfrak{F}}{2\Omega^2}\Big)\Big|^2_{\mathring{\not{\gamma}}}\mathring{\not{\epsilon}} - \int_{\mathbb{S}^2_{u_f,v_0}} \frac{3r^3}{4\Omega^2}(\Omega\overset{(1)}{\text{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\text{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}},\end{aligned} \tag{3.5.13}$$

with

$$\mathfrak{F}(u_f, \theta, \varphi) \doteq \Big(4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - r(\Omega\overset{(1)}{\text{Tr}_{\not{g}}\chi})\Big)\Big|_{(u_f,v_0)}. \tag{3.5.14}$$

Additionally, $\not{\mathcal{R}}(u_f, v_0)$ satisfies the limit

$$\lim_{u_f\to\infty} \not{\mathcal{R}}(u_f, v_0) = 0. \tag{3.5.15}$$

Moreover,

$$\int_{v_0}^{\infty} \frac{r^2}{2}|\Omega\overset{(1)}{\beta}(\infty, v)|^2 dv\not{\epsilon} + \int_{u_0}^{\infty} \frac{r^2}{2}|\overset{(1)}{\underline{\beta}}(u,\infty)|^2 du\not{\epsilon} \leq \overline{\not{\mathcal{E}}}^T_{\text{data}}[h]. \tag{3.5.16}$$

With the success of the conservation laws in theorems 3.1.7 and 3.1.9 producing L^2-boundedness statements for the shears $(\overset{(1)}{\hat{\chi}}, \overset{(1)}{\underline{\hat{\chi}}})$ and $(\overset{(1)}{\beta}, \overset{(1)}{\underline{\beta}})$, the reader may be wondering about if one can use the local conservation law for $(\overset{(1)}{\alpha}, \overset{(1)}{\underline{\alpha}})$ to produce the analogue of theorems 3.5.1 and 3.5.2? In particular, can the conservation law arising in theorem 3.1.10 produce a boundedness statement for $|\overset{(1)}{\alpha}|^2$ on any outgoing cone C_u and $|\overset{(1)}{\underline{\alpha}}|^2$ at null infinity $\mathcal{I}^+$. However, at the time of writing, any attempt to gain coercivity of the conservation law in theorem 3.5.2 by a limiting argument and a change of gauge to has failed. However, one does have a commuted estimate arising from theorem 3.5.1:

Theorem 3.5.3. *Suppose h is a solution to the linearised Einstein equation in double null gauge on the Schwarzschild black hole exterior supported on $\ell \geq 2$, extendible to null infinity and satisfies the partially initial data normalised gauge conditions. Then, $\exists C > 0$ such that*

$$\int_{v_0}^{\infty} |\Omega^2 \overset{(1)}{\alpha}(\infty, v)|^2 dv \not\epsilon + \int_{u_0}^{\infty} |\overset{(1)}{\underline{\alpha}}(u, \infty)|^2 du \not\epsilon \leq C\Big(\overline{\mathcal{E}}^T_{\mathrm{data}}[\mathcal{L}_T h] + \overline{\mathcal{E}}^T_{\mathrm{data}}[h]\Big). \tag{3.5.17}$$

It is this estimate that allows one to prove the following mode stability statement:

Corollary 3.5.4. *Let h be a smooth solution to the linearised vacuum Einstein equation (I.5) of the form*

$$h_{\mu\nu} = e^{-i\omega t} e^{im\varphi} S_{\mu\nu}(\theta) R_{\mu\nu}(r) \tag{3.5.18}$$

with $\Im(\omega) \geq 0$ and let h_{pg} be the pure gauge solution such that $h' \doteq h - h_{\mathrm{pg}}$ is in double null gauge (as defined in definition 2.10.1). Construct $\alpha^{[\pm 2]}$ via proposition 2.10.7, proposition 2.10.10 and equations (2.10.158) and (2.10.159). If $\alpha^{[\pm 2]}$ defines outgoing mode solution to the Teukolsky equation (2.10.150) as in proposition 3.4.14, then h has to be the sum of a pure gauge solution and a linearised Kerr solution.

Remark 3.5.5. *Its interesting to entertain the possibility that the estimate in theorem 3.5.3 could be used to prove a spacetime integral estimate for the Teukolsky equation directly.*

Theorems 3.5.2 and 3.5.3 are proved in section 3.5.2.

3.5.1 Manipulating the Double Null Gauge

As the discussion following theorem 3.5.1 eluded to, understanding how to manipulate the gauge on the final outgoing cone C_{u_f} using the theory developed in section 2.10.3 will be essential to establishing this result. The relevant lemma is the following:

Lemma 3.5.6 (Change of Double Null Gauge [90])**.** *Let h be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the Schwarzschild black hole exterior. Let $f(v, \theta, \varphi)$ be a smooth function generating a residual pure gauge solution h_{pg} as in lemma 2.10.25. Let $h' = h - h_{\mathrm{pg}}$ which defines a new smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge. Then,*

$$\overline{\mathcal{E}}^T_u[h'](v_0, v) - \overline{\mathcal{E}}^T_u[h](v_0, v) = -\int_{\mathbb{S}^2_{u,v}} \mathcal{G}(u, v, \theta, \varphi) \not\epsilon \Big|_{v_0}^{v}, \tag{3.5.19}$$

$$\overline{\not{\mathcal{E}}}_u[h'](v_0, v) - \overline{\not{\mathcal{E}}}_u[h](v_0, v) = -\int_{\mathbb{S}^2_{u,v}} \not{\mathcal{G}}(u, v, \theta, \varphi) \not\epsilon \Big|_{v_0}^{v}, \tag{3.5.20}$$

with

$$\mathcal{G} \doteq \frac{6M(\Omega^2 f)^2}{r^4} - \frac{r}{2\Omega^2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)'\Big](\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) + 2\Omega^2 f(\overset{(1)}{\rho}{}' - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}}{}') \\ - \frac{f}{r^2}(r-4M)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)' \tag{3.5.21}$$

and

$$\not{\mathcal{G}} \doteq -\frac{6M^2(\Omega^2 f)^2}{r^5} + \frac{3M}{r}\Big[\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})' - \Omega^2 f(\overset{(1)}{\rho}{}' + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}{}')\Big] \\ - \frac{3M^2 f}{r^3}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)' + \frac{3M}{2r^2}\Big|\mathring{\not\nabla}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2 + \frac{12M^2}{r^4}f\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)', \tag{3.5.22}$$

where 'primed' quantities are associated to h' and 'unprimed' quantities are associated to h.

Proof. In this proof the notation $\equiv$ will denote equality under integration over $\mathbb{S}^2_{u,v}$.

A proof of the first statement can be found in [90] as proposition 5.1. The second is stated as proposition 8.2 in [90] but it is proved here for completeness. Recall the relevant fluxes are the following:

$$\overline{\not{\mathcal{E}}}{}^T_u[h'](v_0,v_1) = \int_{u_0}^{u_1}\int_{\mathbb{S}^2_{u,v}}\Big(\frac{\Omega^2 r^4}{2}|\overset{(1)}{\beta}|^2 - 3M\Omega^2 r|\overset{(1)}{\underline{\eta}}{}'|^2 + \frac{\Omega^2 r^4}{2}(|\overset{(1)}{\sigma}|^2 + |\overset{(1)}{\rho}{}'|^2) \\ + 3Mr\overset{(1)}{\omega}{}'(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})' + 3M\Big(1-\frac{4M}{r}\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)'\Big)du\mathring{\not\epsilon}, \tag{3.5.23}$$

$$\overline{\not{\mathcal{E}}}{}^T_u[h](v_0,v_1) = \int_{u_0}^{u_1}\int_{\mathbb{S}^2_{u,v}}\Big(\frac{\Omega^2 r^4}{2}|\overset{(1)}{\beta}|^2 - 3M\Omega^2 r|\overset{(1)}{\underline{\eta}}|^2 + \frac{\Omega^2 r^4}{2}(|\overset{(1)}{\sigma}|^2 + |\overset{(1)}{\rho}|^2) \\ + 3Mr\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) + 3M\Big(1-\frac{4M}{r}\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)\Big)du\mathring{\not\epsilon}, \tag{3.5.24}$$

where one uses that $\overset{(1)}{\beta} = \overset{(1)}{\beta}{}'$ and $\overset{(1)}{\sigma} = \overset{(1)}{\sigma}{}'$. Now

$$\overset{(1)}{\omega} = \overset{(1)}{\omega}{}' + \partial_v\Big(\frac{1}{2\Omega^2}\partial_v(\Omega^2 f)\Big), \qquad \overset{(1)}{\underline{\eta}}_A = \overset{(1)}{\underline{\eta}}{}'_A + \frac{r}{\Omega^2}\not\nabla_A\Big[\partial_v\Big(\frac{\Omega^2}{r}f\Big)\Big], \\ \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' + \frac{1}{2\Omega^2}\partial_v(\Omega^2 f), \qquad \overset{(1)}{\rho} = \overset{(1)}{\rho}{}' + \frac{6M\Omega^2}{r^4}f, \tag{3.5.25}$$

and

$$(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) = (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)' + 2\partial_v\Big(\frac{\Omega^2}{r}f\Big), \tag{3.5.26}$$

$$(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) = (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})' + \frac{2\Omega^2}{r^2}\Big(1-\frac{4M}{r}\Big)f + 2\Omega^2\not\Delta f. \tag{3.5.27}$$

Denote $\delta\overline{\mathcal{E}} = \overline{\mathcal{E}}_u^T[h'](v_0, v_1) - \overline{\mathcal{E}}_u^T[h](v_0, v_1)$ then decompose into pure gauge terms and mixed terms as $\delta\overline{\mathcal{E}} = \delta\overline{\mathcal{E}}_M + \delta\overline{\mathcal{E}}_P$ with

$$\delta\overline{\mathcal{E}}_M = \int_{v_0}^{v_1} \Big(-3Mr\Big\{\partial_v\Big(\frac{1}{2\Omega^2}\partial_v(\Omega^2 f)\Big)(\Omega \overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})' + \overset{(1)}{\omega}{}'\frac{2\Omega^2}{r^2}\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\not\Delta} f\Big]\Big\} \tag{3.5.28}$$
$$- 6M\Big[\Omega^2\overset{(1)}{\rho}{}' - \frac{\Omega^2}{r^2}\Big(\Big(1-\frac{4M}{r}\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' + r^2 \mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\eta}}{}'\Big)\Big](\Omega^2 f)$$
$$- \frac{3M}{2\Omega^2}\Big(1-\frac{4M}{r}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)'\partial_v(\Omega^2 f) - \frac{6M}{r}\Big[\Big(1-\frac{4M}{r}\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' + r^2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\eta}}{}'\Big]\partial_v(\Omega^2 f)\Big)dv\mathring{\not\epsilon},$$
$$\delta\overline{\mathcal{E}}_P = \int_{v_0}^{v_1} \Big(\frac{3Mr^3}{\Omega^2}\Big|\not\nabla\Big[\partial_v\Big(\frac{\Omega^2}{r}f\Big)\Big]\Big|^2 - \frac{3M}{\Omega^2}\Big(1-\frac{4M}{r}\Big)\partial_v(\Omega^2 f)\partial_v\Big(\frac{\Omega^2}{r}f\Big) \tag{3.5.29}$$
$$- \frac{\Omega^2 r^4}{2}\Big|\frac{6M\Omega^2}{r^4}f\Big|^2 - 3Mr\partial_v\Big(\frac{1}{2\Omega^2}\partial_v(\Omega^2 f)\Big)\Big[\frac{2\Omega^2}{r^2}\Big(1-\frac{4M}{r}\Big)f + \frac{2\Omega^2}{r^2}\mathring{\not\Delta} f\Big]\Big)dv\mathring{\not\epsilon}.$$

Now first consider terms appearing in $\delta\mathcal{E}_M$ and denote

$$\delta\overline{\mathcal{E}}_M^1 \doteq 3Mr\partial_v\Big(\frac{1}{2\Omega^2}\partial_v(\Omega^2 f)\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})', \tag{3.5.30}$$
$$\delta\overline{\mathcal{E}}_M^2 \doteq 3Mr\overset{(1)}{\omega}{}'\Big[\frac{2\Omega^2}{r^2}\Big(1-\frac{4M}{r}\Big)f + \frac{2\Omega^2}{r^2}\mathring{\not\Delta} f\Big]. \tag{3.5.31}$$

Then

$$\delta\overline{\mathcal{E}}_M^1 = \partial_v\Big(\frac{3Mr}{2\Omega^2}\partial_v(\Omega^2 f)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})'\Big) - \frac{3Mr}{4\Omega^2}\partial_v(\Omega^2 f)(\Omega\mathrm{Tr}_{\not g}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)' \tag{3.5.32}$$
$$- \frac{3Mr}{2\Omega^2}\partial_v(\Omega^2 f)\Big(2\Omega^2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\eta}}{}' + 2\Omega^2\overset{(1)}{\rho}{}' + 4\Omega^2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\Big)$$

and

$$\delta\overline{\mathcal{E}}_M^2 \equiv \partial_v\Big(\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{6M\Omega^2}{r}\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\not\Delta} f\Big]\Big) \tag{3.5.33}$$
$$+ \Big[\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{6M\Omega^2}{r^2}\Big(1-\frac{8M}{r}\Big) + \mathrm{d}\not{\mathrm{i}}\mathrm{v}(\overset{(1)}{\eta}{}' + \overset{(1)}{\underline{\eta}}{}')3M\Omega^2\Big](\Omega^2 f)$$
$$- \Big[\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\Big(\frac{6M}{r}\Big)\Big(1-\frac{4M}{r}\Big) + 3Mr\mathrm{d}\not{\mathrm{i}}\mathrm{v}(\overset{(1)}{\eta}{}' + \overset{(1)}{\underline{\eta}}{}')\Big]\partial_v(\Omega^2 f).$$

Therefore,

$$
\begin{aligned}
\delta\overline{\not{E}}_M = \int_{v_0}^{v_1}\int_{\mathbb{S}^2_{u,v}} &\Big(-6M\Omega^2\Big[\overset{(1)}{\rho} - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{4M}{r^3} + \frac{1}{2}\not{\mathrm{d}}\mathrm{iv}(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})\Big](\Omega^2 f) \\
&+ 3M\Big[r\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + r\overset{(1)}{\rho} - \frac{4M}{r^2}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' + \frac{M}{\Omega^2 r}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'\Big]\partial_v(\Omega^2 f) \\
&- \partial_v\Big(\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{6M\Omega^2}{r}\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\not{\Delta}}f\Big]\Big) - \partial_v\Big(\frac{3Mr}{2\Omega^2}\partial_v(\Omega^2 f)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})'\Big)\Big)dv\mathring{\not{\epsilon}}.
\end{aligned}
\tag{3.5.34}
$$

Note that from proposition 2.10.11 and linearised Bianchi identities in proposition 2.10.20 one has

$$
\partial_v\Big(r\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + r\overset{(1)}{\rho}\Big) = -\Omega^2\not{\mathrm{d}}\mathrm{iv}(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}) - 2\Omega^2\overset{(1)}{\rho} - \frac{3}{2}\rho r(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'. \tag{3.5.35}
$$

Therefore using linearised Raychauduri in proposition 2.10.9 one has

$$
\begin{aligned}
\delta\overline{\not{E}}_M &= \int_{v_0}^{v_1}\int_{\mathbb{S}^2_{u,v}} \Big(\partial_v\Big(3M\Big[r\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + r\overset{(1)}{\rho} - \frac{4M}{r^2}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' + \frac{M}{\Omega^2 r}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'\Big]\Omega^2 f\Big) \qquad (3.5.36) \\
&\quad - \partial_v\Big(\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{6M\Omega^2}{r}\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\not{\Delta}}f\Big]\Big) - \partial_v\Big(\frac{3Mr}{2\Omega^2}\partial_v(\Omega^2 f)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})'\Big)\Big)dv\mathring{\not{\epsilon}} \\
&= \int_{\mathbb{S}^2_{u,v}} 3M\Big[\frac{M}{\Omega^2 r}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})' - r\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}} + r\overset{(1)}{\rho} - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{2\Omega^2}{r}\Big]\Omega^2 f\mathring{\not{\epsilon}}\Big|_{v_0}^{v_1} \qquad (3.5.37) \\
&\quad - \int_{\mathbb{S}^2_{u,v}} \frac{3Mr}{2\Omega^2}\partial_v(\Omega^2 f)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})'\Big)\mathring{\not{\epsilon}}\Big|_{v_0}^{v_1}.
\end{aligned}
$$

Now consider the terms appearing in $\delta\overline{\not{E}}_P$. Denote

$$
\begin{aligned}
\delta\overline{\not{E}}^1_P &\doteq \frac{3M}{r}\partial_v\Big(\frac{1}{\Omega^2}\partial_v(\Omega^2 f)\Big)\Big[\Big(1-\frac{4M}{r}\Big)\Omega^2 f + \mathring{\not{\Delta}}(\Omega^2 f)\Big], && (3.5.38) \\
\delta\overline{\not{E}}^2_P &\doteq \frac{3M}{\Omega^2}\Big(1-\frac{4M}{r}\Big)\partial_v(\Omega^2 f)\partial_v\Big(\frac{\Omega^2}{r}f\Big), && (3.5.39) \\
\delta\overline{\not{E}}^3_P &\doteq \frac{3Mr^3}{\Omega^2}\Big|\not{\nabla}\Big[\partial_v\Big(\frac{\Omega^2}{r}f\Big)\Big]\Big|^2. && (3.5.40)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\delta\overline{\not{E}}^1_P \equiv\ & \partial_v\Big(\frac{3M}{r}\partial_v(\Omega^2 f)\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\not{\Delta}}f\Big]\Big) + \frac{3M}{\Omega^2 r}\Big|\partial_v(\Omega^2\mathring{\not{\nabla}}f)\Big|^2_{\mathring{\not{\gamma}}} \\
&- \frac{3M}{r\Omega^2}\Big(1-\frac{4M}{r}\Big)|\partial_v(\Omega^2 f)|^2 + \frac{6M\Omega^2}{2r^3}\Big(1-\frac{12M}{r}\Big)|\Omega^2 f|^2 \\
&- \frac{3M\Omega^2}{r^3}\Big|\mathring{\not{\nabla}}(\Omega^2 f)\Big|^2_{\mathring{\not{\gamma}}} + \partial_v\Big(\frac{3M}{2r^2}\Big(1-\frac{8M}{r}\Big)|\Omega^2 f|^2 + \frac{3M}{2r^2}\Big|\mathring{\not{\nabla}}(\Omega^2 f)\Big|^2_{\mathring{\not{\gamma}}}\Big),
\end{aligned}
\tag{3.5.41}
$$

$$\delta\overline{\not{\mathfrak{C}}}^2_P = \frac{3M}{\Omega^2 r}\Big(1-\frac{4M}{r}\Big)|\partial_v(\Omega^2 f)|^2 - \partial_v\Big(\frac{3M}{2r^2}\Big(1-\frac{4M}{r}\Big)(\Omega^2 f)^2\Big) + \partial_v\Big(\frac{3M}{2r^2}\Big(1-\frac{4M}{r}\Big)\Big)(\Omega^2 f)^2 \tag{3.5.42}$$

and

$$\delta\overline{\not{\mathfrak{C}}}^3_P = \frac{3M}{\Omega^2 r}\Big|\partial_v(\Omega^2\mathring{\nabla\!\!\!/} f)\Big|^2_{\mathring{\gamma\!\!\!/}} - \frac{3M\Omega^2}{r^3}\Big|\mathring{\nabla\!\!\!/}(\Omega^2 f)\Big|^2_{\mathring{\gamma\!\!\!/}}. \tag{3.5.43}$$

Combining gives

$$\begin{aligned}\delta\overline{\not{\mathfrak{C}}}_P = \int_{v_0}^{v_1} &\Big(\partial_v\Big(\frac{3M}{2r^2}\Big(1-\frac{4M}{r}\Big)(\Omega^2 f)^2 - \frac{3M}{r}\partial_v(\Omega^2 f)\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\Delta\!\!\!/} f\Big]\Big) \\ &- \partial_v\Big(\frac{3M}{2r^2}\Big(1-\frac{8M}{r}\Big)|\Omega^2 f|^2 + \frac{3M}{2r^2}\Big|\mathring{\nabla\!\!\!/}(\Omega^2 f)\Big|^2_{\mathring{\gamma\!\!\!/}}\Big) \\ &- \partial_v\Big(\frac{3M}{2r^2}\Big(1-\frac{4M}{r}\Big)\Big)(\Omega^2 f)^2 - \frac{3M\Omega^2}{r^3}\Big(1-\frac{6M}{r}\Big)(\Omega^2 f)^2\Big)dv\mathring{\not{\epsilon}}.\end{aligned} \tag{3.5.44}$$

One can check the last line cancels and hence,

$$\begin{aligned}\delta\overline{\not{\mathfrak{C}}}_P = \int_{\mathbb{S}^2}\Big(&\frac{6M^2}{r^3}(\Omega^2 f)^2 - \frac{3M}{2r^2}\Big|\mathring{\nabla\!\!\!/}(\Omega^2 f)\Big|^2_{\mathring{\gamma\!\!\!/}} \\ &- \frac{3M}{r}\partial_v(\Omega^2 f)\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\Delta\!\!\!/} f\Big]\Big)\mathring{\not{\epsilon}}\Big|^{v_1}_{v_0}.\end{aligned} \tag{3.5.45}$$

Combining one has

$$\begin{aligned}\delta\overline{\not{\mathfrak{C}}} = &\int_{\mathbb{S}^2_{u,v}}\Big(3M\Big[\frac{M}{\Omega^2 r}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)' - r\mathrm{d\!\!\!/iv}\overset{(1)}{\underline{\eta}}' + r\overset{(1)}{\rho}' - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'\frac{2\Omega^2}{r}\Big]\Omega^2 f + \frac{6M^2}{r^3}(\Omega^2 f)^2\Big)\mathring{\not{\epsilon}}\Big|^{v_1}_{v_0} \\ &+ \int_{\mathbb{S}^2_{u,v}}\Big(3Mr\Big[\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big](\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) - \frac{3M}{2r^2}\Big|\mathring{\nabla\!\!\!/}(\Omega^2 f)\Big|^2_{\mathring{\gamma\!\!\!/}}\Big)\mathring{\not{\epsilon}}\Big|^{v_1}_{v_0},\end{aligned} \tag{3.5.46}$$

where one uses that

$$(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) = (\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})' + \frac{2\Omega^2}{r^2}\Big[\Big(1-\frac{4M}{r}\Big)f + \mathring{\Delta\!\!\!/} f\Big], \quad \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)_{\mathrm{pg}} = \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'. \tag{3.5.47}$$

Note that to obtain the precise expression for $\not{\mathcal{G}}$ stated one can substitute

$$\begin{aligned}3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) \equiv 3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})' + \frac{6M\Omega^2}{r}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})'\Big(1-\frac{4M}{r}\Big)f \\ + 3Mr\Omega^2\mathrm{d\!\!\!/iv}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})'f.\end{aligned} \tag{3.5.48}$$

□

In this work a different choice of gauge function to that of Holzegel (see section 6 of [90]) is made to establish the desired weak stability statement. The reasons for this are two-fold: (1) this choice generalises to the Kerr case, (2) it can be employed along with theorem 3.1.9 to prove the stability statement in theorem 3.5.2. The relevant lemma is the following:

Lemma 3.5.7 (Choice of Gauge Function). *Fix an outgoing null cone C_{u_f} and an incoming null cone C_{v_f} for some $u_f > u_0$ and $v_f > v_0$. Let h be a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge. Let $f(v, \theta, \varphi)$ be a function generating a residual pure gauge solution $h_{\rm pg}$ as in lemma 2.10.25 given by*

$$f(v,\theta,\varphi) = \frac{2}{\Omega(u_f,v)^2}\int_{v_0}^{v} \Omega(u_f,w)\overset{(1)}{\Omega}(u_f,w,\theta,\varphi)dw + \frac{F(\theta,\varphi)}{\Omega^2(u_f,v)} \tag{3.5.49}$$

with

$$F(\theta,\varphi) = 2r(u_f,v_0)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(u_f,v_0) - \frac{r^2(u_f,v_0)}{2\Omega^2(u_f,v_0)}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})(u_f,v_0). \tag{3.5.50}$$

Let $h' \doteq h - h_{\rm pg}$ be the new smooth solution of the linearised vacuum Einstein equation (I.5) in double null gauge. Then,

$$\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'(u_f,v,\theta,\varphi) = 0, \qquad \overset{(1)}{\omega}{}'(u_f,v,\theta,\varphi) = 0, \qquad (\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})'(u_f,v) = 0 \tag{3.5.51}$$

and

$$r^3(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + \overset{(1)}{\rho})'(u_f,v) = r^2(\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} + \overset{(1)}{\rho})'(u_f,v_0). \tag{3.5.52}$$

Proof. Recall that from lemma 2.10.25

$$\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)_{\rm pg} = \frac{1}{2\Omega^2}\partial_v(\Omega^2 f). \tag{3.5.53}$$

Then with f defined as in the statement one has (suppressing the (θ, φ) dependence)

$$\frac{\partial_v(\Omega(u,v)^2 f)}{2\Omega^2(u,v)} = \frac{\overset{(1)}{\Omega}(u_f,v)}{\Omega(u_f,v)} + \frac{\Omega^2(u_f,v)}{2\Omega^2(u,v)}\partial_v\Big(\frac{\Omega(u,v)^2}{\Omega(u_f,v)^2}\Big)f(v,\theta,\varphi). \tag{3.5.54}$$

Hence,

$$\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)'(u_f,v,\theta,\varphi) = 0, \qquad \overset{(1)}{\omega}{}'(u_f,v,\theta,\varphi) = 0. \tag{3.5.55}$$

Now, from lemma 2.10.25,

$$(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)' = (\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + \frac{2\Omega^2}{r^2}(\Omega^2 f) - \frac{4\Omega^2}{r}\frac{1}{2\Omega^2}\partial_v(\Omega^2 f). \tag{3.5.56}$$

Evaluating on C_{u_f} gives

$$(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'(u_f, v) = (\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)(u_f, v) + \frac{2\Omega^2}{r^2}(\Omega^2 f)(u_f, v) - \frac{4\Omega^2}{r}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(u_f, v). \tag{3.5.57}$$

So inserting f and evaluating on the sphere $\mathbb{S}^2_{u_f, v_0}$ gives $(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'(u_f, v_0) = 0$.

Now one has that $(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'$ satisfies the linearised Raychauduri equation of proposition 2.10.9 along C_{u_f} so

$$\frac{\Omega^2}{r^2}\partial_v\Big(\frac{r^2}{\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'\Big)\Big|_{u=u_f} = 2(\Omega \mathrm{Tr}_{\slashed{g}}\chi)\overset{(1)}{\omega}'|_{u=u_f} = 0. \tag{3.5.58}$$

Hence, $\frac{r^2}{\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'(u_f, v) = \frac{r^2}{\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'(u_f, v_0) = 0$. Therefore,

$$(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'(u_f, v) = 0, \tag{3.5.59}$$

on C_{u_f}.

Using the propagation equations for the linearised torsions (proposition 2.10.11) and the Bianchi identities (proposition 2.10.20), one has the propagation equation for the combined quantity:

$$\partial_v\big(r^3(\slashed{\mathrm{div}}\overset{(1)}{\eta}' + \overset{(1)}{\rho}')\big) = r^3(\Omega \mathrm{Tr}_{\slashed{g}}\chi)\slashed{\Delta}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)' - \frac{3r^3}{2}\rho(\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'. \tag{3.5.60}$$

Evaluating on C_{u_f} gives

$$\partial_v\big(r^3(\slashed{\mathrm{div}}\overset{(1)}{\eta} + \overset{(1)}{\rho})'\big) = 0, \tag{3.5.61}$$

since $(\frac{\overset{(1)}{\Omega}}{\Omega})'(u_f, v) = (\Omega \overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)'(u_f, v) = 0$. Hence, integrating (suppressing angular dependence) gives

$$r^3(\slashed{\mathrm{div}}\overset{(1)}{\eta}' + \overset{(1)}{\rho}')(u_f, v) = r^3(\slashed{\mathrm{div}}\overset{(1)}{\eta}' + \overset{(1)}{\rho}')(u_f, v_0), \tag{3.5.62}$$

as stated. □

3.5.2 Proof of the Weak Stability Statements

Proof of theorem 3.5.1. Fix $u_f > u_0$ and $v_f > v_0$. Let $h' = h - h_{\mathrm{pg}}$ where h_{pg} is the residual pure gauge solution generated (through lemma 2.10.3) by the residual gauge function f defined in equation (3.5.49). The canonical energy conservation law implies the modified canonical energy conservation law

$$\overline{\mathcal{E}}^T_{v_0}[h](u_0,u_f) + \overline{\mathcal{E}}^T_{u_0}[h](v_0,v_f) = \overline{\mathcal{E}}^T_{v_f}[h](u_0,u_f) + \overline{\mathcal{E}}^T_{u_f}[h](v_0,v_f). \tag{3.5.63}$$

Start by writing

$$\overline{\mathcal{E}}^T_{v_f}[h](u_0,u_f) = \int_{u_0}^{u_f}\int_{\mathbb{S}^2_{u,v_f}} |\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2 du\not{\epsilon} + \int_{\mathbb{S}^2_{u,v_f}} \frac{r}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\not{\epsilon}\Big|_{u_0}^{u_f} + \mathcal{V}(v_f), \tag{3.5.64}$$

where $\lim_{v_f\to\infty}\mathcal{V}(v_f) = 0$. Note that

$$\overline{\mathcal{E}}^T_{u_f}[h](v_0,v_f) = \int_{v_0}^{v_f}\int_{\mathbb{S}^2_{u_f,v}} \Big(|\Omega\overset{(1)}{\hat{\chi}}(u_f,v)|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}{}'|^2(u_f,v)\Big)dv\not{\epsilon}, \tag{3.5.65}$$

where one uses that $\overset{(1)}{\hat{\chi}}{}' = \overset{(1)}{\hat{\chi}}$ from lemma 2.10.25 and $(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'(u_f,v) = 0$ and $\overset{(1)}{\omega}{}'(u_f,v) = 0$ from lemma 3.5.7.

By lemma 3.5.6 one has

$$\overline{\mathcal{E}}^T_{u_f}[h'](v_0,v_f) = \overline{\mathcal{E}}^T_{u_f}[h'](v_0,v_f) + \int_{\mathbb{S}^2_{u_f,v}} \mathcal{G}(u_f,v,\theta,\varphi)\not{\epsilon}\Big|_{v_0}^{v_f}, \tag{3.5.66}$$

with

$$\begin{aligned}\mathcal{G} = \frac{6M(\Omega^2 f)^2}{r^4} &- \frac{r}{2\Omega^2}\Big[(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) - (\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'\Big](\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}) + 2\Omega^2 f(\overset{(1)}{\rho}{}' - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}}{}')\\ &- \frac{(r-4M)f}{r^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'.\end{aligned} \tag{3.5.67}$$

Noting lemma 3.5.7, one has for all $v \in [v_0,v_f]$ and for all $\delta \in (0,1]$ that

$$\begin{aligned}r^2\mathcal{G}(u_f,v) = \Big(\delta\frac{\sqrt{6M}(\Omega^2 f)}{r} + \frac{1}{\delta\sqrt{6M}}[r^3(\overset{(1)}{\rho}{}' + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}{}')](v_0)\Big)^2 + \frac{6M(1-\delta^2)(\Omega^2 f)^2}{r^2}\\ - \frac{1}{6M\delta^2}[r^3(\overset{(1)}{\rho}{}' + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}{}')]^2(v_0) - \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}),\end{aligned} \tag{3.5.68}$$

where one uses that $\not{\Delta}(\frac{\overset{(1)}{\Omega}}{\Omega})' = \not{\mathrm{d}}\mathrm{iv}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})' = 0$. Hence,

$$\overline{\mathcal{E}}^T_{u_f}[h](v_0,v_f) + \overline{\mathcal{E}}^T_{v_f}[h](u_0,u_f) = \int_{v_0}^{v_f}\int_{\mathbb{S}^2_{u_f,v}} \Big(|\Omega\overset{(1)}{\hat{\chi}}(u_f,v)|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}'|^2(u_f,v)\Big) dv\not{\epsilon} + \int_{u_0}^{u_f}\int_{\mathbb{S}^2_{u,v_f}} |\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2(u,v_f) du\not{\epsilon} + \mathcal{Q}_\delta, \tag{3.5.69}$$

where $\mathcal{Q}_\delta$ is defined as

$$\begin{aligned}\mathcal{Q}_\delta \doteq \mathcal{V}(v_f) &- \int_{\mathbb{S}^2_{u_0,v_f}} \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}} + \int_{\mathbb{S}^2_{u_f,v_0}} \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}} \\ &+ \int_{\mathbb{S}^2_{u_f,v_f}} \Big\{\Big(\delta\frac{\sqrt{6M}(\Omega^2 f)}{r} + \frac{1}{\delta\sqrt{6M}}[r^3(\overset{(1)}{\rho}' + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}')](v_0)\Big)^2 + \frac{6M(1-\delta^2)(\Omega^2 f)^2}{r^2}\Big\}\mathring{\not{\epsilon}} \\ &- \int_{\mathbb{S}^2_{u_f,v_0}} \Big\{\Big(\delta\frac{\sqrt{6M}(\Omega^2 f)}{r} + \frac{1}{\delta\sqrt{6M}}[r^3(\overset{(1)}{\rho}' + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}')](v_0)\Big)^2 + \frac{6M(1-\delta^2)(\Omega^2 f)^2}{r^2}\Big\}\mathring{\not{\epsilon}}.\end{aligned} \tag{3.5.70}$$

First, one should note that

$$\mathcal{Q}_\delta \geq \bar{\mathcal{Q}}_\delta \doteq -\mathcal{R}_\delta(u_f,v_0) + \mathcal{V}(v_f) - \int_{\mathbb{S}^2_{u_0,v_f}} \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}}, \tag{3.5.71}$$

with

$$\begin{aligned}\mathcal{R}_\delta(u_f,v_0) \doteq &\int_{\mathbb{S}^2_{u_f,v_0}} \Big(\delta\frac{\sqrt{6M}(\Omega^2 f)}{r} + \frac{1}{\delta\sqrt{6M}}[r^3(\overset{(1)}{\rho}' + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}')](v_0)\Big)^2\mathring{\not{\epsilon}} \\ &+ \int_{\mathbb{S}^2_{u_f,v_0}} \frac{6M(1-\delta^2)(\Omega^2 f)^2}{r^2}\mathring{\not{\epsilon}} - \int_{\mathbb{S}^2_{u_f,v_0}} \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}}.\end{aligned} \tag{3.5.72}$$

Noting that

$$f(v_0) = \frac{2r}{\Omega^2}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \frac{r^2}{2\Omega^4}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) \tag{3.5.73}$$

gives

$$\overset{(1)}{\rho}' = \overset{(1)}{\rho} - \frac{3M}{\Omega^2 r^3}\Big[4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - r(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\Big], \tag{3.5.74}$$

$$\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}' = \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta} - \frac{1}{2\Omega^2}\not{\Delta}\Big[4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - r(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})\Big], \tag{3.5.75}$$

on $\mathbb{S}^2_{u_f,v_0}$ by lemma 2.10.25. So taking $\delta = 1$ gives

$$\mathcal{R}_1(u_f, v_0) = \frac{1}{6M}\int_{\mathbb{S}^2_{u_f,v_0}} \Big| r^3 \big(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta}\big) - \frac{r^3}{2\Omega^2}\Delta\!\!\!/\Big[4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - r(\Omega\overset{(1)}{\mathrm{Tr}_{g\!\!/}\chi})\Big]\Big|^2 \mathring{\epsilon\!\!/} \qquad (3.5.76)$$
$$- \int_{\mathbb{S}^2_{u_f,v_0}} \frac{r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{g\!\!/}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{g\!\!/}\underline{\chi}})\mathring{\epsilon\!\!/},$$

which is precisely $\mathcal{R}(u_f, v_0)$ as stated in theorem 3.1.7. Therefore, one can estimate that

$$\int_{v_0}^{v_f}\Big[|\Omega\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2\Big]dv\epsilon\!\!/\Big|_{u_f} + \int_{u_0}^{u_f}|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2 du\epsilon\!\!/\Big|_{v_f} \leq \overline{\mathcal{E}}^T_{u_0}[h](v_0, v_f) + \overline{\mathcal{E}}^T_{v_0}[h](u_0, u_f) - \overline{\mathcal{Q}}_1, \qquad (3.5.77)$$

where the integral over $\mathbb{S}^2_{u,v}$ on the left hand side is implicit. Since $\mathcal{V}(v_f)$ is a continuous function that vanishes at infinity it is bounded. The other terms on the RHS of (3.5.77) are defined in the initial data, so are bounded. Hence, $\exists C > 0$ independent of v_f such that

$$\int_{v_0}^{v_f}\int_{\mathbb{S}^2_{u_f,v}}\Big(2|\Omega\overset{(1)}{\hat{\chi}}|^2 + 4\Omega^2|\overset{(1)}{\underline{\eta}}|^2\Big)dv\epsilon\!\!/ \leq C. \qquad (3.5.78)$$

So, since the integrand is positive its integral defines a bounded monotone sequence of real numbers and, therefore, by monotone convergence

$$\limsup_{v_f\to\infty}\int_{v_0}^{v_f}\int_{\mathbb{S}^2_{u_f,v}}\Big(|\Omega\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2\Big)dv\epsilon\!\!/ = \int_{v_0}^{\infty}\int_{\mathbb{S}^2_{u_f,v}}\Big(|\Omega\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2\Big)dv\epsilon\!\!/. \qquad (3.5.79)$$

Note further that $|r\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2(v_f, u)$ is a smooth function with finite limit as $v_f \to \infty$ by extendiblity to null infinity. Additionally, recall that extendiblity to null infinity gives that

$$\sup_{[u_0,u_f]\times\{v\geq v_0\}\times\mathbb{S}^2_{u,v}}|r\Omega\overset{(1)}{\hat{\underline{\chi}}}|(v_f, u) \leq C_{u_f} \qquad (3.5.80)$$

for some $C_{u_f} > 0$ independent of v_f. Therefore, since $[u_0, u_f] \times \mathbb{S}^2_{u,v}$ is a compact set, Lebesgue's bounded convergence theorem (see chapter 2, theorem 1.4 of [126]) lets one pass the limit through the integral to give

$$\limsup_{v_f\to\infty}\int_{u_0}^{u_f}\int_{\mathbb{S}^2_{u,v_f}}|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2 du\epsilon\!\!/ = \int_{u_0}^{u_f}\int_{\mathbb{S}^2_{u,\infty}}|\Omega\overset{(1)}{\hat{\underline{\chi}}}(u,\infty)|^2 du\epsilon\!\!/. \qquad (3.5.81)$$

Taking the limit superior with respect to $v_f \to \infty$ of the right-hand side of (3.5.77) gives

$$\int_{v_0}^{\infty}\int_{\mathbb{S}^2_{u_f,v}}\Big[|\Omega\overset{(1)}{\hat{\chi}}|^2+2\Omega^2|\overset{(1)}{\underline{\eta}}|^2\Big]dv\epsilon\!\!\!/+\int_{u_0}^{u_f}\int_{\mathbb{S}^2_{u,\infty}}|\Omega\overset{(1)}{\underline{\hat{\chi}}}|^2(u,\infty)du\epsilon\!\!\!/\le\overline{\mathcal{E}}^T_{\text{data}}[h](u_f)+\mathcal{R}(u_f,v_0). \tag{3.5.82}$$

To establish that $\mathcal{R}(u_f,v_0)$ vanishes as $u_f\to\infty$, note that along the cone C_{v_0} one has that

$$\partial_u\Big(r(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\chi})-4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)=-4M\Omega^2\overset{(1)}{\underline{\omega}}+2\Omega^2r(\text{d}\!\!\!/\text{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})-\Omega^2(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\underline{\chi}}). \tag{3.5.83}$$

Integrating in a region close to the future event horizon $\mathcal{H}^+$, i.e., from $u\gg u_0$ to infinity and using the horizon gauge conditions gives

$$\Big[4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)-r(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\chi})\Big](u,v_0)=\int_u^{\infty}\Omega^4\Big[\frac{2r(\text{d}\!\!\!/\text{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})}{\Omega^2}-\Big(4M\frac{\overset{(1)}{\underline{\omega}}}{\Omega^2}+\frac{(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\underline{\chi}})}{\Omega^2}\Big)\Big]du. \tag{3.5.84}$$

The horizon gauge condition $(\text{d}\!\!\!/\text{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})(\infty,v_0)=\overset{(1)}{\rho}_{\ell=0}=0$ and the smoothness of the solution implies that for $u\gg u_0$ close to infinity

$$(\text{d}\!\!\!/\text{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})(u,v_0)=\mathcal{O}(\Omega^2)\implies\frac{(\text{d}\!\!\!/\text{iv}\overset{(1)}{\eta}+\overset{(1)}{\rho})}{\Omega^2}(u,v_0)=\mathcal{O}(1). \tag{3.5.85}$$

So, for $u\gg u_0$ close to infinity

$$\Big[4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)-r(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\chi})\Big](u,v_0)=\mathcal{O}(\Omega^4). \tag{3.5.86}$$

Moreover, commuting with angular operators yields

$$\Big[4\Omega^2\mathring{\Delta\!\!\!/}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)-r\mathring{\Delta\!\!\!/}(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\chi})\Big](u,v_0)=\mathcal{O}(\Omega^4), \tag{3.5.87}$$

for $u\gg u_0$ close to infinity. Additionally, the gauge conditions on the future event horizon in definition 3.4.5 give

$$\lim_{u_f\to\infty}\frac{1}{\Omega^2}(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\chi})(\Omega\overset{(1)}{\text{Tr}_{g\!\!\!/}\underline{\chi}})\Big|_{(u_f,v_0)}=0. \tag{3.5.88}$$

Hence, one has $\lim_{u_f\to 0}\mathcal{R}(u_f,v_0)=0$. □

Proof of theorem 3.5.2. Fix $u_f>u_0$ and $v_f>v_0$. Let $h'=h-h_{\text{pg}}$ where h_{pg} is the residual pure gauge solution generated (through lemma 2.10.3) by the residual gauge function f defined in equation (3.5.49). The canonical energy conservation law implies the modified canonical energy conservation law

$$\overline{\not{\mathfrak{E}}}^T_{v_0}[h](u_0,u_f)+\overline{\not{\mathfrak{E}}}^T_{u_0}[h](v_0,v_f)=\overline{\not{\mathfrak{E}}}^T_{v_f}[h](u_0,u_f)+\overline{\not{\mathfrak{E}}}^T_{u_f}[h](v_0,v_f). \tag{3.5.89}$$

Note that

$$\overline{\not{\mathfrak{E}}}^T_{u_f}[h'](v_0,v_1)=\int_{v_0}^{v_1}\Big(\frac{\Omega^2r^2}{2}|\overset{(1)}{\beta}|^2-\frac{3M\Omega^2}{r}|\overset{(1)}{\underline{\eta}}{}'|^2+\frac{\Omega^2r^2}{2}\big(|\overset{(1)}{\sigma}|^2+|\overset{(1)}{\rho}{}'|^2\big)\Big)dv\not{\epsilon}, \tag{3.5.90}$$

where one uses that $\overset{(1)}{\beta}$ and $\overset{(1)}{\sigma}$ are gauge invariant (by lemma 2.10.25) and $(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})'(u_f,v)=0$ and $\overset{(1)}{\omega}{}'(u_f,v)=0$ from lemma 3.5.7. Observe that one immediately runs into an issue, namely $\overline{\not{\mathfrak{E}}}^T_u[h'](v_0,v_1)$ is not necessarily positive. However, one can add $\overline{\mathcal{E}}^T_u[h'](v_0,v_1)$ to achieve positivity. In particular, let $\kappa\in[0,\infty)$ and define

$$\overline{\not{\mathfrak{E}}}^T_{\kappa,u}[h](v_0,v_1):=\overline{\not{\mathfrak{E}}}^T_u[h](v_0,v_1)+\kappa\overline{\mathcal{E}}^T_u[h](v_0,v_1) \tag{3.5.91}$$

and similarly for $\overline{\not{\mathfrak{E}}}^T_{\kappa,v}[h](u_0,u_1)$. From the conservation laws for the modified canonical energies one the following conservation law

$$\overline{\not{\mathfrak{E}}}^T_{\kappa,v_0}[h](u_0,u_f)+\overline{\not{\mathfrak{E}}}^T_{\kappa,u_0}[h](v_0,v_f)=\overline{\not{\mathfrak{E}}}^T_{\kappa,v_f}[h](u_0,u_f)+\overline{\not{\mathfrak{E}}}^T_{\kappa,u_f}[h](v_0,v_f). \tag{3.5.92}$$

One can compute that

$$\overline{\not{\mathfrak{E}}}^T_{\kappa,u_f}[h'](v_0,v_1)=\int_{v_0}^{v_1}\Big(\frac{r^2}{2}|\Omega\overset{(1)}{\beta}|^2+\Big(2\kappa-\frac{3M}{r}\Big)|\Omega\overset{(1)}{\underline{\eta}}{}'|^2+\frac{\Omega^2r^2}{2}\big(|\overset{(1)}{\sigma}|^2+|\overset{(1)}{\rho}{}'|^2\big)+\kappa|\Omega\overset{(1)}{\hat{\chi}}|^2\Big)dv\not{\epsilon}. \tag{3.5.93}$$

Hence, $\overline{\not{\mathfrak{E}}}^T_{\kappa,u}[h'](v_0,v_1)\geq 0$ for $\kappa\geq\frac{3}{4}$. Additionally, as with the proof of theorem 3.1.7, one has

$$\overline{\not{\mathfrak{E}}}^T_{\kappa,v_f}[h](u_0,u_f)=\int_{u_0}^{u_f}\Big(\frac{\Omega^2r^2}{2}|\overset{(1)}{\underline{\beta}}|^2+\kappa|\Omega\overset{(1)}{\underline{\hat{\chi}}}|^2\Big)du\not{\epsilon}+\int_{\mathbb{S}^2_{u,v_f}}\frac{\kappa r}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\not{\epsilon}\Big|_{u_0}^{u_f}+\not{V}(v_f), \tag{3.5.94}$$

where $\lim_{v_f\to\infty}\not{V}(v_f)=0$.

By lemma 3.5.1 one has

$$\overline{\not{\mathfrak{E}}}^T_{\kappa,u_f}[h](v_0,v_1)=\overline{\not{\mathfrak{E}}}^T_{\kappa,u_f}[h'](v_0,v_1)+\int_{\mathbb{S}^2_{u_f,v}}\not{\mathfrak{G}}_\kappa(u_f,v,\theta,\varphi)\not{\epsilon}\Big|_{v_0}^{v_f}, \tag{3.5.95}$$

with $\not{\mathfrak{G}}_\kappa = \not{\mathcal{G}} + \kappa\mathcal{G}$ or explicitly for the gauge function in lemma 3.5.7, one has for all $v \in [v_0, v_f]$ that

$$r^2\not{\mathfrak{G}}_\kappa = 3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) + [r^3(\overset{(1)}{\rho} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta})](v_0)\Big(2\kappa - \frac{3M}{r}\Big)\frac{\Omega^2 f}{r} \\ + \frac{3M}{2}\Big|\mathring{\not{\nabla}}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2_{\mathring{\not{\gamma}}} + \Big(3\kappa - \frac{3M}{r}\Big)\frac{2M(\Omega^2 f)^2}{r^2} - \frac{\kappa r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}). \tag{3.5.96}$$

Manipulation of this function is slightly subtle since the first term is merely bounded by extendibility to null infinity so naively gives a boundary term (without sign) on $\mathbb{S}^2_{u_f,v_f}$. Recall that from the expression for the Gauss curvature (proposition 2.10.15) and extendibility to null infinity

$$r(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) = -4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \mathcal{O}\Big(\frac{1}{r}\Big), \tag{3.5.97}$$

so that for large v_f

$$3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) = -12M\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2 + \mathcal{O}\Big(\frac{1}{r}\Big), \tag{3.5.98}$$

by extendibility to null infinity. Now from lemma 2.10.25 recall that

$$(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)' = (\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) - \frac{2}{r}\partial_v(\Omega^2 f) + \frac{2\Omega^2}{r^2}(\Omega^2 f). \tag{3.5.99}$$

Evaluating this on C_{u_f}, noting, by lemma 3.5.7, $(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)' = 0$, gives

$$(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) - \frac{4\Omega^2}{r}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{2\Omega^2}{r^2}(\Omega^2 f) = 0, \tag{3.5.100}$$

where one uses that

$$\partial_v(\Omega^2 f) = 2\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)_{\mathrm{pg}} = 2\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \tag{3.5.101}$$

where the last equality follows from lemma 3.5.7. Additionally, this tells one that $f = \mathcal{O}(r)$ as $v_f \to \infty$. On C_{u_f} one can now solve equation (3.5.100) for $\frac{\overset{(1)}{\Omega}}{\Omega}$ in terms of f which gives

$$\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) = \frac{\Omega^2 f}{2r} + \frac{r}{4\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi). \tag{3.5.102}$$

Substituting into equation (3.5.98) gives

$$3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}}) = -\frac{3M}{r^2}(\Omega^2 f)^2 + \mathcal{O}\Big(\frac{1}{r}\Big), \tag{3.5.103}$$

by the assumption of extendibility to null infinity. Hence,

$$\begin{aligned} r^2\not{\mathfrak{C}}_\kappa(v_f) = {}& [r^3(\overset{(1)}{\rho} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta})](v_0)\Big(2\kappa - \frac{3M}{r}\Big)\frac{\Omega^2 f}{r} + \Big(3\kappa - \frac{3}{2} - \frac{3M}{r}\Big)\frac{2M(\Omega^2 f)^2}{r^2} \\ &+ \frac{3M}{2}\Big|\mathring{\not\nabla}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2_{\mathring{\not\gamma}} - \frac{\kappa r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}}) + \mathcal{O}\Big(\frac{1}{r}\Big). \end{aligned} \tag{3.5.104}$$

Equivalently for $\delta \in (0, 1]$

$$\begin{aligned} r^2\not{\mathfrak{C}}_\kappa(v_f) = {}& \Big(2\kappa - \frac{3M}{r}\Big)\Big(\frac{1}{\delta\sqrt{2M}}[r^3(\overset{(1)}{\rho} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta})](v_0) + \frac{\delta\sqrt{2M}\Omega^2 f}{r}\Big)^2 \\ &- \Big(2\kappa - \frac{3M}{r}\Big)\frac{1}{2M\delta^2}[r^3(\overset{(1)}{\rho} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta})]^2(v_0) \\ &+ \Big(2\kappa - \frac{3M}{r}\Big)\frac{2M(1-\delta^2)(\Omega^2 f)^2}{r^2} + \Big(\kappa - \frac{3}{2}\Big)\frac{2M(\Omega^2 f)^2}{r^2} \\ &+ \frac{3M}{2}\Big|\mathring{\not\nabla}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2_{\mathring{\not\gamma}} - \frac{\kappa r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}}) + \mathcal{O}\Big(\frac{1}{r}\Big). \end{aligned} \tag{3.5.105}$$

Additionally, one can write $r^2\not{\mathfrak{C}}_\kappa(v_0)$ in a comparable form

$$\begin{aligned} r^2\not{\mathfrak{C}}_\kappa(v_0) = {}& \Big(2\kappa - \frac{3M}{r}\Big)\Big(\frac{1}{\delta\sqrt{2M}}[r^3(\overset{(1)}{\rho} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta})](v_0) + \frac{\delta\sqrt{2M}\Omega^2 f}{r}\Big)^2 \\ &+ \kappa\frac{2M(\Omega^2 f)^2}{r^2} - \Big(2\kappa - \frac{3M}{r}\Big)\frac{1}{2M\delta^2}[r^3(\overset{(1)}{\rho} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta})]^2(v_0) \\ &+ \Big(2\kappa - \frac{3M}{r}\Big)\frac{2M(1-\delta^2)(\Omega^2 f)^2}{r^2} + \frac{3M}{2}\Big|\mathring{\not\nabla}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2_{\mathring{\not\gamma}} \\ &+ 3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}}) - \frac{\kappa r^3}{2\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega\overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}}). \end{aligned} \tag{3.5.106}$$

Taking $\kappa = \frac{3}{2}$, $\delta = 1$ and defining

$$\begin{aligned} \not{E}^T[h](u_f, v_f) \doteq {}& \int_{v_0}^{v_f}\Big(\frac{r^2}{2}|\Omega\overset{(1)}{\beta}|^2 + \frac{\Omega^2 r^2}{2}|\overset{(1)}{\sigma}|^2\Big)dv\not\epsilon + \int_{u_0}^{u_f}\Big(\frac{\Omega^2 r^2}{2}|\overset{(1)}{\underline{\beta}}|^2 + \frac{3}{2}|\Omega\overset{(1)}{\hat{\underline{\chi}}}|^2\Big)du\not\epsilon \\ &+ \int_{v_0}^{v_f}\Big(2\Big(1 - \frac{M}{r}\Big)|\Omega\overset{(1)}{\underline{\eta}}|^2 + \frac{\Omega^2 r^2}{2}|\overset{(1)}{\rho}|^2 + \frac{3}{2}|\Omega\overset{(1)}{\hat{\chi}}|^2\Big)dv\not\epsilon, \end{aligned} \tag{3.5.107}$$

one has

$$\overline{\not{\mathfrak{C}}}^T_{\frac{3}{2},u_f}[h](v_0, v_f) + \overline{\not{\mathfrak{C}}}^T_{\frac{3}{2},v_f}[h](u_0, u_f) = \not{E}^T[h](u_f, v_f) + \not{\mathcal{Q}}, \tag{3.5.108}$$

where

$$
\begin{aligned}
\mathcal{Q} \doteq \mathcal{V}(v_f) &- \int_{\mathbb{S}^2_{u_0,v_f}} \frac{3r^3}{4\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\mathring{\not\epsilon} + \int_{\mathbb{S}^2_{u_f,v_0}} \frac{3r^3}{4\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\mathring{\not\epsilon} \\
&+ 3\int_{\mathbb{S}^2_{u_f,v_f}} \Big(1-\frac{M}{r}\Big)\Big(\frac{1}{\sqrt{2M}}[r^3(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta})](v_0) + \frac{\sqrt{2M}\Omega^2 f}{r}\Big)^2\mathring{\not\epsilon} \\
&- 3\int_{\mathbb{S}^2_{u_f,v_0}} \Big(1-\frac{M}{r}\Big)\Big(\frac{1}{\sqrt{2M}}[r^3(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta})] + \frac{\sqrt{2M}\Omega^2 f}{r}\Big)^2\mathring{\not\epsilon} \\
&+ \frac{3M}{2}\int_{\mathbb{S}^2_{u_f,v_f}} \Big|\mathring{\not\nabla}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2_{\mathring{\not\gamma}}\mathring{\not\epsilon} - \frac{3M}{2}\int_{\mathbb{S}^2_{u_f,v_0}} \Big|\mathring{\not\nabla}\Big(\frac{\Omega^2 f}{r}\Big)\Big|^2_{\mathring{\not\gamma}}\mathring{\not\epsilon} \\
&- \int_{\mathbb{S}^2_{u_f,v_0}} \Big[\frac{3M(\Omega^2 f)^2}{r^2} + \frac{3}{2r}[r^3(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta})]^2 + 3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\Big]\mathring{\not\epsilon}.
\end{aligned} \tag{3.5.109}
$$

Now

$$
\mathcal{Q} \geq -\mathcal{R}(u_f,v_0) + \mathcal{V}(v_f) - \int_{\mathbb{S}^2_{u_0,v_f}} \frac{3r^3}{4\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\mathring{\not\epsilon}, \tag{3.5.110}
$$

where recall that $\mathcal{R}(u_f, v_0)$ is defined as

$$
\begin{aligned}
\mathcal{R}(u_f,v_0) \doteq \frac{3}{2M}\int_{\mathbb{S}^2_{u_f,v_0}} &\Big(1-\frac{M}{r}\Big)\Big|r^3(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta}) - \frac{r^3}{2\Omega^2}\not\Delta\mathfrak{F} - \frac{2M}{\Omega^2}\mathfrak{F}\Big|^2\mathring{\not\epsilon} \\
&+ \int_{\mathbb{S}^2_{u_f,v_0}} \Big[\frac{3M}{4\Omega^4}\mathfrak{F}^2 + \frac{3}{2r}\Big|r^3(\overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta}) - \frac{3M}{\Omega^2}\mathfrak{F} - \frac{r^3}{2\Omega^2}\not\Delta\mathfrak{F}\Big|^2 + 3Mr\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\Big]\mathring{\not\epsilon} \\
&+ \frac{3M}{2}\int_{\mathbb{S}^2_{u_f,v_0}} \Big|\mathring{\not\nabla}\Big(\frac{\mathfrak{F}}{2\Omega^2}\Big)\Big|^2_{\mathring{\not\gamma}}\mathring{\not\epsilon} - \int_{\mathbb{S}^2_{u_f,v_0}} \frac{3r^3}{4\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\chi})(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\underline{\chi}})\mathring{\not\epsilon},
\end{aligned} \tag{3.5.111}
$$

with

$$
\mathfrak{F}(u_f,\theta,\varphi) \doteq \Big(4\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - r(\Omega \overset{(1)}{\mathrm{Tr}_{\not g}\chi})\Big)\Big|_{(u_f,v_0)}. \tag{3.5.112}
$$

Note that to produce $\mathcal{R}(u_f, v_0)$ uses that on $\mathbb{S}^2_{u_f,v_0}$

$$
f(v_0) = \frac{r}{2\Omega^4}\mathfrak{F}, \qquad \overset{(1)}{\rho}{}' + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta}{}' = \overset{(1)}{\rho} + \mathrm{d}\!\!/\mathrm{iv}\overset{(1)}{\eta} - \frac{3M}{\Omega^2 r^3}\mathfrak{F} - \frac{\not\Delta\mathfrak{F}}{2\Omega^2}. \tag{3.5.113}
$$

Therefore, the key estimate is

$$\not{E}^T[h](u_f,v_f) \leq \overline{\not{\mathfrak{E}}}^T_{\frac{3}{2},u_0}[h](v_0,v_f) + \overline{\not{\mathfrak{E}}}^T_{\frac{3}{2},v_0}[h](u_0,u_f) + \not{\mathcal{R}}(u_f,v_0) + \int_{\mathbb{S}^2_{u_0,v_f}} \frac{3r^3}{4\Omega^2}(\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}\chi})(\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}} - \not{V}(v_f). \tag{3.5.114}$$

Arguing similarly to the proof of theorem 3.1.7,

$$\int_{v_0}^{\infty}\Big(\frac{r^2}{2}|\Omega\overset{(1)}{\beta}|^2 + \frac{\Omega^2 r^2}{2}|\overset{(1)}{\sigma}|^2\Big)dv + \int_{u_0}^{u_f}\frac{\Omega^2 r^2}{2}|\overset{(1)}{\underline{\beta}}(u_f,\infty)|^2 du \leq \not{\mathfrak{E}}^T_{\mathrm{data}}(u_f) + \not{\mathcal{R}}(u_f,v_0). \tag{3.5.115}$$

The conclusion about the limit of $\not{\mathcal{R}}(u_f,v_0)$ follows analogously to the proof of theorem 3.5.1 since (from the proof of theorem 3.5.1) $\mathfrak{F} \sim \mathcal{O}(\Omega^4)$ for $u_f >> u_0$ close to infinity. The only new term arising is $3Mr(\frac{\overset{(1)}{\Omega}}{\Omega})(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})$ integrated over $\mathbb{S}^2_{u_f,v_0}$. One just observes that

$$3M\int_{\mathbb{S}^2_{u_f,v_0}} r\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})\mathring{\not{\epsilon}} = 3M\int_{\mathbb{S}^2_{u_f,v_0}} r\Omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\frac{(\Omega\overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}})}{\Omega^2}\mathring{\not{\epsilon}} = \mathcal{O}(\Omega^2) \tag{3.5.116}$$

by considering regular quantities at the future event horizon $\mathcal{H}^+$ (see section 3.4.5). □

Proof of theorem 3.5.3. Since h solves the linearised vacuum Einstein equation (I.5) so does $\mathcal{L}_T h$. Therefore, one has the commuted estimate from theorem 3.5.1

$$\int_{u_0}^{\infty}\Omega^2|\not{\mathcal{L}}_T\overset{(1)}{\hat{\chi}}|^2 dv\not{\epsilon} + \int_{v_0}^{\infty}\Omega^2|\not{\mathcal{L}}_T\overset{(1)}{\hat{\underline{\chi}}}|^2 du\not{\epsilon} \leq \overline{\mathcal{E}}^T_{\mathrm{data}}[\mathcal{L}_T h]. \tag{3.5.117}$$

From the linearised null structure equations of section 2.10.1 (in particular propositions 2.10.10)

$$r\not{\mathcal{L}}_T\overset{(1)}{\hat{\underline{\chi}}} = -\frac{\Omega r}{2}\overset{(1)}{\underline{\alpha}} - r\omega\overset{(1)}{\hat{\underline{\chi}}} - \Omega r\not{\mathcal{D}}^{\star}_2\overset{(1)}{\underline{\eta}} + \frac{r}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}}\Big), \tag{3.5.118}$$

$$\Omega\not{\mathcal{L}}_T\overset{(1)}{\hat{\chi}} = -\frac{\Omega^2}{2}\overset{(1)}{\alpha} + \Omega\omega\overset{(1)}{\hat{\chi}} - \Omega^2\not{\mathcal{D}}^{\star}_2\overset{(1)}{\eta} - \frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}}\Big). \tag{3.5.119}$$

Therefore, restricting to null infinity and the horizon respectively one has

$$r\not{\mathcal{L}}_T\overset{(1)}{\hat{\underline{\chi}}}\Big|_{v=\infty} = -\frac{r}{2}\overset{(1)}{\underline{\alpha}}, \qquad \Omega\not{\mathcal{L}}_T\overset{(1)}{\hat{\chi}}\Big|_{u=\infty} = -\frac{\Omega^2}{2}\overset{(1)}{\alpha} + \frac{\Omega\overset{(1)}{\hat{\chi}}}{4M}, \tag{3.5.120}$$

by the definition of extendiblity to future null infinity, the proposition 3.4.9 on extendiblity to the future event horizon and that $\omega = \frac{1}{4M}$ on the future event horizon. Hence, one has the bound for $\overset{(1)}{\underline{\alpha}}$,

$$\int_{u_0}^{\infty}\frac{1}{4}|\overset{(1)}{\underline{\alpha}}|^2 du\not{\epsilon} \leq \overline{\mathcal{E}}^T_{\mathrm{data}}[\mathcal{L}_T h]. \tag{3.5.121}$$

Turning to the horizon and using Young's inequality with $0 < \delta < \frac{1}{2}$ gives

$$\frac{1-2\delta}{4}\int_{v_0}^{\infty} |\Omega^2 \overset{(1)}{\alpha}|^2 du\not\epsilon \leq \mathcal{E}_{\text{data}}[\mathcal{L}_T h] + \frac{1-2\delta}{2\delta}\frac{1}{(4M)^2}\int_{v_0}^{\infty} |\Omega \overset{(1)}{\hat{\chi}}|^2 du\not\epsilon. \tag{3.5.122}$$

The term involving $\overset{(1)}{\hat{\chi}}$ can now be bounded by $\overline{\mathcal{E}}^T_{\text{data}}[h]$ which establishes the desired estimate. □

3.5.3 Mode Stability from Canonical Energy

Mode Type for Metric to Mode Type for Teukolsky

Lemma 3.5.8. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) on the Schw_4 exterior is of mode type in usual Schwarzschild (t, r, θ, φ) coordinates, i.e.,*

$$h_{\alpha\beta} = e^{-i\omega t} H_{\alpha\beta}(r, \theta, \varphi), \tag{3.5.123}$$

then $(\overset{(1)}{\alpha}, \overset{(1)}{\underline{\alpha}})$ constructed from h by changing to double null gauge and then constructing $\overset{(1)}{\hat{\chi}}$, $\overset{(1)}{\underline{\hat{\chi}}}$ from proposition 2.10.7 and $\overset{(1)}{\alpha}$, $\overset{(1)}{\underline{\alpha}}$ from proposition 2.10.10 is of mode type, i.e.,

$$\overset{(1)}{\alpha} = e^{-i\omega t}\overset{(1)}{\mathfrak{A}}(r, \theta, \varphi), \qquad \overset{(1)}{\underline{\alpha}} = e^{-i\omega t}\overset{(1)}{\underline{\mathfrak{A}}}(r, \theta, \varphi). \tag{3.5.124}$$

Proof. Set $\mu = -i\omega$. Start by expressing h in double null Eddington–Finkelstein coordinates (u, v, θ, φ):

$$h_{\alpha\beta} = e^{\mu(u+v)} H_{\alpha\beta}(v - u, \theta, \varphi). \tag{3.5.125}$$

An abuse of notation has been used here where one uses $h_{\alpha\beta}$ and $H_{\alpha\beta}$ in both coordinate bases. Let ξ be the one-form that generates the pure gauge solution h_{pg} that takes h to double null gauge, i.e., $h' = h - h_{\text{pg}}$ satisfies definition 2.10 where $(h_{\text{pg}})_{ab} = 2\nabla_{(a}\xi_{b)}$.

Now in double null gauge $h'_{33} = 0 = h'_{44}$, $h'_{3A} = 0$ and $h'_{4A} = -\frac{\overset{(1)}{b}_A}{\Omega}$ which implies (using proposition 2.8.2) the relations

$$e_3(\xi_3) = -\hat{\omega}\xi_3 - \frac{1}{2}e^{\mu(u+v)}H_{33}, \qquad e_4(\xi_4) = \hat{\omega}\xi_4 - \frac{1}{2}e^{\mu(u+v)}H_{44}, \tag{3.5.126}$$

$$(\nabla\!\!\!/_3 \xi\!\!\!/)_A = -e^{\mu(u+v)}\underline{\gamma\!\!\!/}^H_A - \partial_A(\xi_3) - \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)\xi\!\!\!/_A, \tag{3.5.127}$$

$$(\nabla\!\!\!/_4 \xi\!\!\!/)_A = -\frac{\overset{(1)}{b}_A}{\Omega} - e^{\mu(u+v)}\gamma\!\!\!/^H_A - \partial_A(\xi_4) + \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)\xi\!\!\!/_A, \tag{3.5.128}$$

where $\not{\xi}_A \doteq \xi_A$ and $\underline{\not{v}}^H$ and $\not{v}^H$ are $\mathbb{S}^2_{u,v}$ co-vectors defined component-wise as $\underline{\not{v}}^H_A \doteq H_{3A}$ and $\not{v}^H_A \doteq H_{4A}$. One can combine these results using the commutation lemma 2.8.4 to show

$$\not{\nabla}_3\not{\nabla}_3\not{\xi} = e^{\mu(u+v)}\Big(\frac{1}{2}\not{d}(H_{33}) + \frac{1}{2}\mathrm{Tr}_{\not{g}}\chi\underline{\not{v}}^H - \frac{\mu}{\Omega}\underline{\not{v}}^H - (\not{\nabla}_3\underline{\not{v}}^H) - \hat{\omega}\underline{\not{v}}^H\Big) - \hat{\omega}(\not{\nabla}_3\not{\xi}), \tag{3.5.129}$$

$$\not{\nabla}_4\not{\nabla}_4\not{\xi} = e^{\mu(u+v)}\Big(\frac{1}{2}\not{d}(H_{44}) - \frac{1}{2}\mathrm{Tr}_{\not{g}}\chi\not{v}^H - \frac{\mu}{\Omega}\not{v}^H - (\not{\nabla}_4\not{v}^H) + \hat{\omega}\not{v}^H\Big) + \hat{\omega}(\not{\nabla}_4\not{\xi}) - \frac{1}{\Omega}(\not{\nabla}_4\overset{(1)}{b})_A + \frac{2\hat{\omega}}{\Omega}\overset{(1)}{b}_A - \frac{\mathrm{Tr}_{\not{g}}\chi}{2\Omega}\overset{(1)}{b}_A \tag{3.5.130}$$

One can turn to computing the linearised shears $(\overset{(1)}{\underline{\hat{\chi}}}, \overset{(1)}{\hat{\chi}})$ and the linearised null curvature components $(\overset{(1)}{\underline{\alpha}}, \overset{(1)}{\alpha})$. Start by writing that

$$\hat{\not{h}}'_{AB} = e^{\mu(u+v)}\hat{\not{H}}_{AB} - 2(\not{\mathcal{D}}_2^\star\not{\xi})_{AB} \tag{3.5.131}$$

where $\hat{\not{H}}_{AB} \doteq H_{(AB)} - \frac{1}{2}(\mathrm{Tr}_{\not{g}}H)\not{g}_{AB}$. Now, proposition 2.10.7 gives

$$\overset{(1)}{\underline{\hat{\chi}}} \doteq \frac{1}{2}(\not{\nabla}_3\hat{\not{h}}'), \tag{3.5.132}$$

$$\overset{(1)}{\hat{\chi}} \doteq \frac{1}{2}(\not{\nabla}_4\hat{\not{h}}') - \frac{1}{\Omega}\not{\mathcal{D}}_2^\star\overset{(1)}{b}. \tag{3.5.133}$$

So, using the commutation lemma gives

$$\overset{(1)}{\underline{\hat{\chi}}} = \frac{1}{2}\Big(\frac{\mu}{\Omega}e^{\mu(u+v)}\hat{\not{H}} + e^{\mu(u+v)}\not{\nabla}_3\hat{\not{H}} - 2(\not{\mathcal{D}}_2^\star\not{\nabla}_3\not{\xi}) - \mathrm{Tr}_{\not{g}}\chi(\not{\mathcal{D}}_2^\star\not{\xi})\Big), \tag{3.5.134}$$

$$\overset{(1)}{\hat{\chi}} = \frac{1}{2}\Big(\frac{\mu}{\Omega}e^{\mu(u+v)}\hat{\not{H}} + e^{\mu(u+v)}\not{\nabla}_4\hat{\not{H}} - 2(\not{\mathcal{D}}_2^\star\not{\nabla}_4\not{\xi}) + \mathrm{Tr}_{\not{g}}\chi(\not{\mathcal{D}}_2^\star\not{\xi})\Big) - \frac{1}{\Omega}\not{\mathcal{D}}_2^\star\overset{(1)}{b}. \tag{3.5.135}$$

Further, proposition 2.10.10 gives

$$-\overset{(1)}{\underline{\alpha}} \doteq \not{\nabla}_3\overset{(1)}{\underline{\hat{\chi}}} + \Big(\hat{\omega} - \mathrm{Tr}_{\not{g}}\chi\Big)\overset{(1)}{\underline{\hat{\chi}}}, \tag{3.5.136}$$

$$-\overset{(1)}{\alpha} \doteq \not{\nabla}_4\overset{(1)}{\hat{\chi}} - \Big(\hat{\omega} - \mathrm{Tr}_{\not{g}}\chi\Big)\overset{(1)}{\hat{\chi}}. \tag{3.5.137}$$

So,

$$-\overset{(1)}{\underline{\alpha}} \doteq \frac{1}{2}e^{\mu(u+v)}\Big(\frac{\mu^2}{\Omega^2}\hat{\not{H}} + \frac{2\mu}{\Omega}\not{\nabla}_3\hat{\not{H}} + \not{\nabla}_3\not{\nabla}_3\hat{\not{H}} + \Big(\hat{\omega} - \mathrm{Tr}_{\not{g}}\chi\Big)\Big(\frac{\mu}{\Omega}\hat{\not{H}} + \not{\nabla}_3\hat{\not{H}}\Big)\Big) - (\not{\mathcal{D}}_2^\star\not{\nabla}_3\not{\nabla}_3\not{\xi}) - \hat{\omega}(\not{\mathcal{D}}_2^\star\not{\nabla}_3\not{\xi}), \tag{3.5.138}$$

$$-\overset{(1)}{\alpha} \doteq \frac{1}{2}e^{\mu(u+v)}\Big(\frac{\mu^2}{\Omega^2}\hat{\not{H}} + \frac{2\mu}{\Omega}\not{\nabla}_4\hat{\not{H}} + \not{\nabla}_4\not{\nabla}_4\hat{\not{H}} - \Big(\hat{\omega} - \mathrm{Tr}_{\not{g}}\chi\Big)\Big(\frac{\mu}{\Omega}\hat{\not{H}} + \not{\nabla}_4\hat{\not{H}}\Big)\Big) - (\not{\mathcal{D}}_2^\star\not{\nabla}_4\not{\nabla}_4\not{\xi}) + \hat{\omega}(\not{\mathcal{D}}_2^\star\not{\nabla}_4\not{\xi}) + \Big(\frac{2\hat{\omega}}{\Omega} - \frac{\mathrm{Tr}_{\not{g}}\chi}{2\Omega}\Big)\not{\mathcal{D}}_2^\star\overset{(1)}{b} - \frac{1}{\Omega}\mathcal{D}_2^\star\not{\nabla}_4\overset{(1)}{b}, \tag{3.5.139}$$

Therefore, using equations (3.5.129) and (3.5.130) one has

$$\overset{(1)}{\underline{\alpha}} \doteq -\frac{1}{2}e^{\mu(u+v)}\Big(\frac{\mu^2}{\Omega^2}\hat{\not{H}} + \frac{2\mu}{\Omega}\not{\nabla}_3\hat{\not{H}} + \not{\nabla}_3\not{\nabla}_3\hat{\not{H}} + \Big(\hat{\omega} - \mathrm{Tr}_{\not{g}}\chi\Big)\Big(\frac{\mu}{\Omega}\hat{\not{H}} + \not{\nabla}_3\hat{\not{H}}\Big)\Big) \tag{3.5.140}$$
$$+ e^{\mu(u+v)}\not{\mathcal{D}}_2^\star\Big(\frac{1}{2}\not{d}(H_{33}) + \frac{1}{2}\mathrm{Tr}_{\not{g}}\chi\underline{\not{v}}^H - \frac{\mu}{\Omega}\underline{\not{v}}^H - (\not{\nabla}_3\underline{\not{v}}^H) - \hat{\omega}\underline{\not{v}}^H\Big),$$
$$\overset{(1)}{\alpha} \doteq -\frac{1}{2}e^{\mu(u+v)}\Big(\frac{\mu^2}{\Omega^2}\hat{\not{H}} + \frac{2\mu}{\Omega}\not{\nabla}_4\hat{\not{H}} + \not{\nabla}_4\not{\nabla}_4\hat{\not{H}} - \Big(\hat{\omega} - \mathrm{Tr}_{\not{g}}\chi\Big)\Big(\frac{\mu}{\Omega}\hat{\not{H}} + \not{\nabla}_4\hat{\not{H}}\Big)\Big) \tag{3.5.141}$$
$$+ e^{\mu(u+v)}\not{\mathcal{D}}_2^\star\Big(\frac{1}{2}\not{d}(H_{44}) - \frac{1}{2}\mathrm{Tr}_{\not{g}}\chi\not{v}^H - \frac{\mu}{\Omega}\not{v}^H - (\not{\nabla}_4\not{v}^H) + \hat{\omega}\not{v}^H\Big),$$

as claimed. □

Proof of Mode Stability

Proof of corollary 3.5.4. If h is a solution to the linearised vacuum Einstein equation (I.5) on the Schwarzschild black hole exterior then by lemma 3.5.8 if h is of the form

$$h = e^{-i\omega t}e^{im\varphi}H(r,\theta), \tag{3.5.142}$$

then $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ are of the form

$$\overset{(1)}{\alpha} = e^{-i\omega t}e^{im\varphi}\overset{(1)}{\mathfrak{A}}(r,\theta), \qquad \overset{(1)}{\underline{\alpha}} = e^{-i\omega t}e^{im\varphi}\overset{(1)}{\underline{\mathfrak{A}}}(r,\theta). \tag{3.5.143}$$

Let h_{pg} be the pure gauge solution that takes h to double null gauge. By the equations (2.10.158) and (2.10.159) one has that the complex scalars $\alpha^{[+2]}$ and $\alpha^{[-2]}$ are of the form

$$\alpha^{[+2]} = e^{-i\omega t}e^{im\varphi}\mathscr{A}^{[+2]}(r,\theta), \tag{3.5.144}$$
$$\alpha^{[-2]} = e^{-i\omega t}e^{im\varphi}\mathscr{A}^{[-2]}(r,\theta). \tag{3.5.145}$$

Any solution of the above form can be spanned by solutions of the form (see remark 2.3 of [32])

$$\alpha^{[+2]} = e^{-i\omega t}e^{im\varphi}R^{[+2]}(r)S^{[+2]}(\theta), \tag{3.5.146}$$
$$\alpha^{[-2]} = e^{-i\omega t}e^{im\varphi}R^{[-2]}(r)S^{[-2]}(\theta). \tag{3.5.147}$$

Therefore, by the traditional Teukolsky equation (2.10.150) one has that $R^{[s]}(r)$ solves the ODE (3.4.83). If $\mathfrak{I}(\omega) \geq 0$ and $\omega \neq 0$ proposition 3.4.14 gives that the $s = \pm 2$ solution should be of the form,

$$R^{[s]} = a^{[s]}_{2M,+}\rho^{[s]}_{2M,+} = a^{[s]}_{\infty,+}\rho^{[s]}_{\infty,+}. \tag{3.5.148}$$

Theorem 3.5.3 proves that

$$\int_{u_0}^{\infty} |\overset{(1)}{\underline{\alpha}}|^2 du\not{g} + \int_{v_0}^{\infty} |\Omega^2 \overset{(1)}{\alpha}|^2 dv\not{g} \leq C(\overline{\mathcal{E}}^T_{\text{data}}[\mathcal{L}_T h] + \overline{\mathcal{E}}^T_{\text{data}}[h]). \tag{3.5.149}$$

Using the relations (2.10.160) and (2.10.161) this boundedness statement translates to the following statement for the complex scalars $\alpha^{[+2]}$ and $\alpha^{[-2]}$:

$$\int_{u_0}^{\infty} \frac{2}{\Omega^4 r^8} |\alpha^{[-2]}|^2 du\not{g} + \int_{v_0}^{\infty} 2\Omega^8 |\alpha^{[+2]}|^2 dv\not{g} \leq C(\overline{\mathcal{E}}^T_{\text{data}}[\mathcal{L}_T h] + \overline{\mathcal{E}}^T_{\text{data}}[h]). \tag{3.5.150}$$

For a outgoing mode solution this bound implies that

$$|a^{[-2]}_{\infty,+}|^2 \int_{\tilde{u}(u_0)}^{\infty} e^{2\Im(\omega)\tilde{u}} \left| S^{[-2]}(\theta) \right|^2 d\tilde{u}\mathring{\not{g}} \lesssim \overline{\mathcal{E}}^T_{\text{data}}[\mathcal{L}_T h] + \overline{\mathcal{E}}^T_{\text{data}}[h] \tag{3.5.151}$$

and, for $\omega \neq \frac{i}{4M}$,

$$|a^{[+2]}_{2M,+}|^2 \int_{\tilde{v}(v_0)}^{\infty} e^{2\Im(\omega)\tilde{v}} \left| S^{[+2]}(\theta) \right|^2 d\tilde{v}\mathring{\not{g}} \lesssim \overline{\mathcal{E}}^T_{\text{data}}[\mathcal{L}_T h] + \overline{\mathcal{E}}^T_{\text{data}}[h], \tag{3.5.152}$$

where one uses equations (3.4.112), (3.4.118), (3.4.120) and (3.4.128).

With $\Im(\omega) \geq 0$ and $\omega \neq 0$, the bound (3.5.151) is violated unless $|a^{[-2]}_{\infty,+}| = 0$ which implies that $a^{[-2]}_{\infty,+} = 0$. Therefore, $R^{[-2]} \equiv 0 \ \forall \omega \neq 0$ with $\Im(\omega) \geq 0$ in some annulus $A \doteq \{r > a > 2M\}$. This means that for any $r_0 \in A$, $R^{[-2]}(r_0) = 0$ and $\frac{dR^{[-2]}}{dr}(r_0) = 0$. By ODE uniqueness theory (for example, see theorem 1.1 of chapter 5 of [97]), $R^{[-2]} \equiv 0$, for all $r \in (2M, \infty)$.

With $\Im(\omega) \geq 0$ and $\omega \notin \{0, \frac{i}{4M}\}$, the bound (3.5.152) is violated unless $|a^{[+2]}_{2M,+}| = 0$ which implies $a^{[+2]}_{2M,+} = 0$. Therefore, $R^{[+2]} \equiv 0 \ \forall \omega$ such that $\Im(\omega) \geq 0$ and $\omega \notin \{0, \frac{i}{4M}\}$ in some open set $r \in (2M, R)$ with $R > 2M$. This means that for any $r_0 \in (2M, R)$, $R^{[+2]}(r_0) = 0$ and $\frac{dR^{[+2]}}{dr}(r_0) = 0$. By ODE uniqueness theory (again, see theorem 1.1 of chapter 5 of [97]), $R^{[+2]} \equiv 0$, for all $r \in (2M, \infty)$.

For $\omega = \frac{i}{4M}$, one <u>does not</u> have that the bound (3.5.152) is violated. This is because as demonstrated by equation (3.4.122), $\Omega^4 \alpha^{[+2]} = 0$ at $r = 2M$ if one has an outgoing mode solution. However, lemma 3.4.15 gives one that the solution

$$R^{[+2]} = a^{[+2]}_{2M,+} \rho^{[+2]}_{2M,+} = a^{[+2]}_{\infty,+} \rho^{[+2]}_{\infty,+} \tag{3.5.153}$$

maps to a the solution

$$\mathscr{R}^{[-2]} = \mathfrak{C}_{\frac{i}{4M}} a^{[+2]}_{2M,+} \rho^{[-2]}_{2M,+} = \mathfrak{C}^{(1)}_2 a^{[+2]}_{\infty,+} \rho^{[-2]}_{\infty,+} \tag{3.5.154}$$

to the radial ODE (3.4.83) with $s = -2$ and $\omega = \frac{i}{4M}$. Since the existence of such solutions were ruled out two paragraphs above, one must have $a^{[+2]}_{2M,+} = 0 = a^{[+2]}_{\infty,+}$. Hence, $R^{[+2]} \equiv 0 \;\forall \omega \neq 0$ with $\mathfrak{I}(\omega) \geq 0$.

Since $R^{[-2]} \equiv 0 \equiv R^{[+2]}$ one has that $\Omega^4 \alpha^{[+2]} \equiv 0$ and $\frac{1}{\Omega^2 r^3} \alpha^{[-2]} \equiv 0$ globally on the exterior of Schw_4 which translates through equations (2.10.160) and (2.10.161) to $\overset{(1)}{\alpha} = 0 = \overset{(1)}{\underline{\alpha}}$ on the exterior. Theorem B.1 in appendix B.1 of [28] proves that if $\overset{(1)}{\alpha} = 0 = \overset{(1)}{\underline{\alpha}}$ then the solution $h' = h - h_{\mathrm{pg}}$ is the sum of a residual pure gauge solution and a linearised Kerr solution. Therefore, h itself must be sum of a pure gauge solution and a linearised Kerr solution. $\square$

Chapter 4

An Alternative Energy for the Linearised Vacuum Einstein Equation

4.1 Introduction

The canonical energy of Hollands and Wald is undoubtedly a appealing construction to allow for the study of linear stability of a black hole spacetime. One may wonder if there is an alternative method of defining a useful energy in linearised theory. As discussed at the beginning of section 3.2, for many field theories one can construct currents and therefore energies associated to the energy–momentum tensor, $\mathbb{T}_{ab}$, of the theory. However, recall that for the linearised vacuum Einstein equation (I.5) one immediately encounters the issue that there is no energy–momentum tensor associated to a solution h. In this chapter, a resolution to this issue is suggested: a current for the linearised vacuum Einstein equation (I.5) is constructed that is analogous to the usual current

$$J^X[\Phi]_a \doteq \mathbb{T}[\Phi]_{ab}X^b = X(\Phi)\nabla_a\Phi - \frac{X_a}{2}|\nabla\Phi|_g^2, \tag{4.1.1}$$

constructed for a solution Ψ to the wave equation (3.2.6). It should be stressed that no symmetric 2-tensor $\mathbb{T}[h]_{ab}$ is constructed for h.

A way around the problem of a lack of energy–momentum tensor is simply to abandon the view point that these currents arise from a energy–momentum tensor and approach with a 'vector field multiplier' method. This method proceeds as follows. Let X be a vector field and suppose f is a scalar or a tensor on a spacetime (M, g) which solves some linear equation

$$\mathfrak{D}_g f = 0, \tag{4.1.2}$$

where $\mathfrak{D}_g$ is some differential operator depending on the metric. Then one can try to construct an 'X-energy' for the equation (4.1.2) by multiplying the equation by $\mathcal{L}_X(f)$ and trying to write the expression as a total divergence plus terms that vanish if X is a Killing symmetry of the spacetime. For example, one can consider constructing an energy for solution $\Phi \in C^\infty(M)$ to the wave equation

$$\Box_g \Phi = 0 \tag{4.1.3}$$

in this manner. Let $Y^a \doteq X^a \nabla_a \Phi \nabla^c \Phi$. Then multiplying the equation by $\mathcal{L}_X(\Phi) = X(\Phi)$ gives

$$0 = X(\Phi)\Box_g \Phi = X^a \nabla_a \Phi g^{bc} \nabla_b \nabla_c \Phi = \mathrm{div} Y - (\nabla^b X^a) \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} \nabla_X |\nabla \Phi|_g^2 \tag{4.1.4}$$

$$= \mathrm{div}(J^X[\Phi]) - (\Pi^X)^{ab} \Big(\nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} |\nabla \Phi|_g^2 \Big), \tag{4.1.5}$$

where $J^X[\Phi]_a$ is the usual current arising from the energy–momentum tensor in equation (4.1.1) and Π^X_{ab} is the deformation tensor.

It turns out that for the linearised vacuum Einstein equation (I.5) one can perform an analogous computation to this vector field multiplier view point by expressing the equation (plus its trace) as in proposition 3.2.7, i.e.,

$$P^a{}_{(bc)}{}^{def} \nabla_a \nabla_d h_{ef} = 0, \tag{4.1.6}$$

contracting with $(\mathcal{L}_X h)^{bc}$ and trying to write the expression as a total divergence plus terms that vanish if X is a Killing symmetry of the spacetime. As will be proved in this chapter, this procedure results in the following current (see proposition 4.2.1 of section 4.2):

$$(\mathfrak{J}^X[h])^a \doteq P^{abcdef} (\mathcal{L}_X h)_{bc} \nabla_d h_{ef} - \frac{1}{2} X^a P(\nabla h, \nabla h), \tag{4.1.7}$$

with

$$P(\nabla h, \nabla h) \doteq P^{abcdef} \nabla_a h_{bc} \nabla_d h_{ef}, \tag{4.1.8}$$

where P is defined in equation (3.2.7).[a] This has a divergence of the form

$$\mathrm{div}(\mathfrak{J}^X[h]) = P \cdot \Pi^X \cdot (\nabla h)^2 + P \cdot \nabla \Pi^X \cdot h \cdot \nabla h. \tag{4.1.9}$$

Hence, like the canonical energy, it also gives rise to a conservation law for a general solution to the linearised vacuum Einstein equation (I.5) if X is Killing.

[a] To the best of the author's knowledge this current has not appeared in the literature prior to this chapter.

There are a two main advantages to this current over the canonical energy current of definition (3.2.6):

1. On hypersurface Σ with normal n_Σ the flux density $n_\Sigma(\mathfrak{J}^X[h])$ always gives a conservation law at the level of linearised Ricci coefficients, i.e., $n_\Sigma(\mathfrak{J}^X[h])$ never gives linearised curvature since there are no second derivatives of h appearing in its definition. Note that in stark contrast to the canonical energy current which will always yield linearised curvature due to the second term term which is of the form

$$P^{abcdef}h_{bc}\nabla_d(\mathcal{L}_X h)_{ef}. \tag{4.1.10}$$

2. From a practical standpoint $\mathfrak{J}^X[h]$ is easier to compute since only first covariant and Lie derivatives of the linearised metric h are required to decompose the current. In the same vein, if one wants to compute $\mathfrak{J}^X[h]$ for many different vector fields $X \in \mathfrak{X}(M)$, one only has to compute $P(\nabla h, \nabla h)$ once. This is in contrast to the canonical energy current which has to be re-computed from scratch for a new vector field.

Remark 4.1.1. *The reader may wonder about attempting to prove a spacetime integral estimate using either this current or the canonical energy current. The bulk resulting by taking the divergence of the canonical energy current $\mathcal{J}[h]^X$ for a general vector field X is the following:*

$$\mathrm{div}(\mathcal{J}[h]^X) = -P^{abcdef}h_{bc}\nabla_a\nabla_d(\mathcal{L}_X h)_{ef}. \tag{4.1.11}$$

The form of this divergence is slightly unfortunate since it will not lead directly to estimates involving spacetime integrals of $(\partial_\alpha h)^2$ but rather $h \cdot \partial^2_{\alpha\beta}h$. Therefore, the resulting estimates are undesirable for proving a Morawetz-type estimate [124]. Now it is reasonable to expect with some integration by parts that this could be rectified. As noted above the bulk arising from the current $\mathfrak{J}[h]^X$ is of the form

$$\mathrm{div}(\mathfrak{J}^X[h]) = P \cdot \Pi^X \cdot (\nabla h)^2 + P \cdot \nabla\Pi^X \cdot h \cdot \nabla h \tag{4.1.12}$$

which on the face of things seems more desirable. However, the reader should note that proving a Morawetz-type estimate using either of these currents seems to be a tall order. Indeed, as was examined in great detail in chapter 3, even the energies on hypersurfaces were not manifestly positive. It seems unlikely then that a spacetime integral will yield anything easier to work with.

The rest of this chapter is structured as follows. In section 4.2 it is proven that the current $\mathfrak{J}^X[h]$ in equation (4.1.7) is divergence free if X is a Killing symmetry of the background vacuum spacetime and h is a solution the linearised vacuum Einstein equation (I.5). Section 4.3 derives the explicit relation between the current $\mathfrak{J}^X[h]$ in equation (4.1.7) and the canonical energy current $\mathcal{J}^X[h]$ of definition 3.2.6. This chapter ends with section 4.4 which gives evidence of the usefulness of the current $\mathfrak{J}^X[h]$ in equation (4.1.7) by using it to derive the generalisation of the local conservation law in theorem 3.1.7 to the n-dimensional Schwarzschild–Tangherlini spacetime. The reader should note that the current $\mathfrak{J}^X[h]$ in equation (4.1.7) and the following manipulations can also be used on the Kerr_4 spacetime to produce the generalisation of the local conservation law in theorem 3.1.7 to $a \neq 0$.

4.2 The Current for the Linearised Vacuum Einstein Equation

The main proposition of this section is the following:

Proposition 4.2.1. *Let g solve the vacuum Einstein equation (I.2) and h be a solution to the linearised vacuum Einstein equation (I.5) and X a vector field. Let $(\mathfrak{J}^T[h])^a$ be defined as in equation (4.1.7). Then*

$$\begin{aligned}\operatorname{div}(\mathfrak{J}^X[h]) = P^{abcdef}\Big(\nabla_a h^p{}_c \Pi^X_{bp} + \nabla^p h_{bc}\Pi^X_{ap} + \nabla_a h_b{}^p \Pi^X_{cp}\Big)\nabla_d h_{ef} \\ - \frac{1}{2} g^{ab}\Pi^X_{ab} P(\nabla h, \nabla h) + P^{abcdef}\Big(\mathrm{K}^X_{abp} h^p{}_c \nabla_d h_{ef} + \mathrm{K}^X_{acp} h^p{}_b \nabla_d h_{ef}\Big),\end{aligned} \tag{4.2.1}$$

where

$$\mathrm{K}^X_{abc} \doteq \nabla_a \Pi^X_{bc} + \nabla_b \Pi^X_{ac} - \nabla_c \Pi^X_{ab}, \qquad \Pi^X_{ab} \doteq \frac{1}{2}(\mathcal{L}_X g)_{ab}. \tag{4.2.2}$$

In particular, when X is Killing then $\mathfrak{J}^X[h]$ is divergence-free.

Proof. Note that from proposition 3.2.7, $P^{a(bc)def}\nabla_a \nabla_d h_{ef} = 0$ if h satisfies the linearised Einstein equation. Further, from appendix C.1,

$$\nabla_a(\mathcal{L}_X h)_{bc} = \mathcal{L}_X(\nabla h)_{abc} + \mathrm{K}^X_{acd} h_b{}^d + \mathrm{K}^X_{abd} h^d{}_c. \tag{4.2.3}$$

This using the definition of the Lie derivative and writing $\nabla_a X_b = \Pi^X_{ab} + \nabla_{[a} X_{b]}$ gives

$$\begin{aligned}\nabla_a(\mathcal{L}_X h)_{bc} = \nabla_X(\nabla_a h)_{bc} + \nabla_a h_b{}^p \Pi^X_{cp} + \nabla^p h_{bc}\Pi^X_{ap} + \nabla_a h_c{}^p \Pi^X_{bp} + \mathrm{K}^X_{acd} h_b{}^d \\ + \mathrm{K}^X_{abd} h^d{}_c + \nabla_a h_b{}^p \nabla_{[c} X_{p]} + \nabla^p h_{bc}\nabla_{[a} X_{p]} + \nabla_a h_c{}^p \nabla_{[b} X_{p]}.\end{aligned} \tag{4.2.4}$$

One can calculate that

$$P^{abcdef}\nabla_a h_b{}^p \nabla_{[c} X_{p]} \nabla_d h_{ef} = \Big(\nabla^a h^{bc}\nabla_c h^p{}_b - \frac{1}{2}(\operatorname{div} h)^p \nabla^a \operatorname{Tr} h\Big)\nabla_{[a} X_{p]}, \tag{4.2.5}$$

$$\begin{aligned}P^{abcdef}\nabla^p h_{bc}\nabla_{[a} X_{p]}\nabla_d h_{ef} = \Big(\nabla^p h^{bc}\nabla_c h^a{}_b - \frac{1}{2}\nabla^p h^{ad}\nabla_d \operatorname{Tr} h\Big)\nabla_{[a} X_{p]} \\ - \frac{1}{2}(\operatorname{div} h)^a \nabla^p \operatorname{Tr} h \nabla_{[a} X_{p]},\end{aligned} \tag{4.2.6}$$

$$P^{abcdef}\nabla_a h_c{}^p \nabla_{[b} X_{p]} \nabla_d h_{ef} = -\frac{1}{2}\nabla^a h^{pd}\nabla_d \operatorname{Tr} h \nabla_{[a} X_{p]}. \tag{4.2.7}$$

Therefore, by symmetry, the sum of these terms vanishes. Hence, denoting

$$Z^a = P^{abcdef}(\mathcal{L}_X h)_{bc}\nabla_d h_{ef}, \tag{4.2.8}$$

one has

$$\begin{aligned}\mathrm{div}(Z) = {} & P^{abcdef}\nabla_X(\nabla_a h)_{bc}\nabla_d h_{ef} + P^{abcdef}\Big(\mathrm{K}^X_{acd}h_b{}^d + \mathrm{K}^X_{abd}h^d{}_c\Big)\nabla_d h_{ef} \\ & + P^{abcdef}\Big(\nabla_a h_b{}^p\Pi^X_{cp} + \nabla^p h_{bc}\Pi^X_{ap} + \nabla_a h_c{}^p\Pi^X_{bp}\Big)\nabla_d h_{ef}.\end{aligned} \tag{4.2.9}$$

Now, using that $P^{abcdef}\nabla_a h_{bc}\nabla_d h_{ef} = P^{defabc}\nabla_a h_{bc}\nabla_d h_{ef}$ one has

$$\begin{aligned}P^{abcdef}\mathcal{L}_X(\nabla h)_{abc}\nabla_d h_{ef} = {} & \mathrm{div}\Big(\frac{X}{2}P(\nabla h,\nabla h)\Big) - \frac{1}{2}(\mathrm{div}X)P(\nabla h,\nabla h) \\ & + P^{abcdef}\Big(\mathrm{K}^X_{acd}h_b{}^d + \mathrm{K}^X_{abd}h^d{}_c\Big)\nabla_d h_{ef} \\ & + P^{abcdef}\Big(\nabla_a h^p{}_c\Pi^X_{bp} + \nabla^p h_{bc}\Pi^X_{ap} + \nabla_a h_b{}^p\Pi^X_{cp}\Big)\nabla_d h_{ef}.\end{aligned} \tag{4.2.10}$$

□

4.3 Relation to the Canonical Energy Current

For a general vector field X, it seems reasonable to expect that the X-canonical energy for the linearised vacuum Einstein equation (I.5) on a spacetime is related to the X-energy associated to the current $\mathfrak{J}^X[h]$ constructed here. The following proposition confirms this expectation.

Proposition 4.3.1. *Suppose X is a Killing field for a vacuum spacetime (M,g) and h solves the linearised vacuum equation (I.5). Then the X-canonical energy current defined in definition 3.2.6 can be expressed as*

$$(\mathcal{J}^X[h])^a = 2(\mathfrak{J}^X[h])^a + (j^X[h])^a, \tag{4.3.1}$$

where $(\mathfrak{J}^X[h])_a$ is defined in equation (4.1.7) and

$$(j^X[h])^a \doteq (\nabla_h A)^{ah}, \qquad A^{ah} \doteq X^{[a}P^{h]bcdef}h_{bc}\nabla_d h_{ef}, \tag{4.3.2}$$

i.e., $\mathcal{J}^X[h]$ and $\mathfrak{J}^X[h]$ are related by a divergence. Moreover, $(j^X[h])_a$ is divergence free.

Proof. Using that X is Killing in proposition C.1.2 in appendix C.1 gives

$$(\mathcal{J}^X[h])^a = P^{abcdef}\Big[(\mathcal{L}_X h)_{bc}(\nabla_d h)_{ef} - h_{bc}\mathcal{L}_X(\nabla_d h)_{ef}\Big]. \tag{4.3.3}$$

Using the Leibniz rule for the Lie derivative (and that X is Killing so $\mathcal{L}_X P = 0$) gives

$$(\mathcal{J}^X[h])^a = 2P^{abcdef}(\mathcal{L}_X h)_{bc}(\nabla_d h)_{ef} - (\nabla_X Y - \nabla_Y X)^a, \tag{4.3.4}$$

with $Y^a = P^{abcdef} h_{bc} \nabla_d h_{ef}$. Therefore,

$$(\mathcal{J}^X[h])^a = 2P^{abcdef}(\mathcal{L}_X h)_{bc}(\nabla_d h)_{ef} - \nabla_h (X \otimes Y - Y \otimes X)^{ha} - (\mathrm{div}Y)X^a. \tag{4.3.5}$$

where one uses that $\mathrm{div}X = 0$ since X is Killing. Now one can compute that

$$\mathrm{div}Y = P^{abcdef}\nabla_a h_{bc}\nabla_d h_{ef} + P^{abcdef} h_{bc}\nabla_a\nabla_d h_{ef} = P^{abcdef}\nabla_a h_{bc}\nabla_d h_{ef}, \tag{4.3.6}$$

where the last equality is by the linearised vacuum Einstein equation (I.5). Now $X \otimes Y - Y \otimes X = A$ as defined in the proposition statement so then

$$(\nabla_a \nabla_h A)^{ha} = (\nabla_{[a}\nabla_{h]} A)^{ha} = R^h{}_{bah} A^{ba} + R^a{}_{bah} A^{hb} = -2(\mathrm{Ric}(g))_{ab} A^{ab} = 0. \tag{4.3.7}$$

□

4.4 Application: A Conservation Law for Schwarzschild–Tangherlini

This section is concerned with the linear stability problem for the Schwarzschild–Tangherlini black hole solution. In particular, this section provide a first step toward generalising the linear stability results of section 3.5.2 to the n-dimensional case by providing the generalisation of the conservation law in theorem 3.1.3 as theorem 4.4.3 below.

Recall from sections 1.1.1 and 2.8.3 that the Schwarzschild–Tangherlini spacetime is a n-dimensional black hole solution to the vacuum Einstein equation (I.2) with metric on its exterior region in double null Eddington–Finkelstein coordinates (u, v, θ, φ) given by

$$g = -2\Omega(u,v)^2(du \otimes dv + dv \otimes du) + r(u,v)^2 \mathring{\gamma}_{n-2}, \quad \Omega(u,v)^2 = 1 - \frac{2M}{r(u,v)^{n-3}}, \tag{4.4.1}$$

where $(u, v) \in \mathbb{R}^2$ and (θ, φ) are the standard coordinates on the $(n-2)$-sphere $(\mathbb{S}^{n-2}, \mathring{\gamma}_{n-2})$. The non-vanishing Ricci coefficients are

$$(\Omega \mathrm{Tr}_{\not{g}} \chi) = \frac{(n-2)D_n(r)}{r} = -(\Omega \mathrm{Tr}_{\not{g}} \underline{\chi}), \qquad \omega = \frac{(n-3)M}{r^{n-2}} = -\underline{\omega}, \tag{4.4.2}$$

and the non-vanishing curvature components are

$$\begin{aligned} \rho &= -\frac{(n-2)(n-3)M}{r^{n-1}}, & \slashed{\mathrm{Ric}}(\slashed{g}) &= \frac{(n-3)}{r^2}\slashed{g}, \\ \slashed{\mathrm{Scal}}(\slashed{g}) &= \frac{(n-2)(n-3)}{r^2}, & \slashed{R}_{ABCD} &= \frac{1}{r^2}(\slashed{g}_{AC}\slashed{g}_{BD} - \slashed{g}_{AD}\slashed{g}_{BC}) \end{aligned} \tag{4.4.3}$$

and

$$R_{ABCD} = -\frac{2\rho}{(n-2)(n-3)}(\slashed{g}_{AC}\slashed{g}_{BD} - \slashed{g}_{AD}\slashed{g}_{BC}). \tag{4.4.4}$$

In this chapter, the current $(\mathfrak{J}[h]^T)^a$ as defined in equation (4.1.7) is evaluated in a region $\mathcal{R}$ of exterior of the Schwarzschild–Tangherlini black hole spacetime bounded by a characteristic rectangle, as shown in blue in the following Penrose diagram:

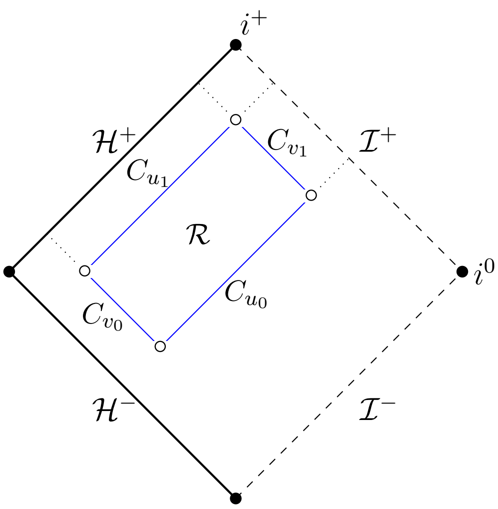


Fig. 4.1 The Penrose diagram depicting the setup up for the computation of the $(\mathfrak{J}[h]^T)^a$ current on the exterior of the Schw_4 spacetime.

Here C_u and C_v are the null hypersurfaces given by the level sets of the double null Eddington–Finkelstein coordinates, i.e., $\{u = \text{const.}\}$ and $\{v = \text{const.}\}$ respectively. From proposition 4.2.1, the divergence theorem then gives the following conservation law

$$\mathrm{E}^T_{u_1}[h](v_0, v_1) + \mathrm{E}^T_{v_1}[h](u_0, u_1) = \mathrm{E}^T_{u_0}[h](v_0, v_1) + \mathrm{E}^T_{v_0}[h](u_0, u_1), \tag{4.4.5}$$

where

$$\mathrm{E}^T_u[h](v_0, v_1) \doteq 2\int_{v_0}^{v_1} (\mathfrak{J}[h]^T)^3 \Omega dv\slashed{\epsilon}, \quad \mathrm{E}^T_v[h](u_0, u_1) \doteq 2\int_{u_0}^{u_1} (\mathfrak{J}[h]^T)^4 \Omega du\slashed{\epsilon}, \tag{4.4.6}$$

will, henceforth, be referred to as the 'T-energies' on subsets of the null hypersurfaces C_u and C_v. Evaluating $\mathrm{E}^T_u[h]$ and $\mathrm{E}^T_v[h]$ for h in double null gauge on the Schwarzschild–Tangherlini black hole exterior is (as was true in the canonical energy case of chapter 3) an involved computation and yields two expressions for the flux densities $(\mathfrak{J}[h]^T)^3$ and $(\mathfrak{J}[h]^T)^4$ in terms of the double null decomposition which have obscure coercivity properties. As in section 3.3 in chapter 3, one can simplify matters by integrating by parts on the spheres $\mathbb{S}^2_{u,v}$, and using the fact that if

$$(\mathfrak{J}[h]^T)^3 = \overline{(\mathfrak{J}[h]^T)}^3 + \frac{1}{r^{n-2}} e_4(r^{n-2}\mathcal{A}), \qquad (\mathfrak{J}[h]^T)^4 = \overline{(\mathfrak{J}[h]^T)}^4 - \frac{1}{r^{n-2}} e_3(r^{n-2}\mathcal{A}), \quad (4.4.7)$$

then, conservation law (4.4.5), the modified T-energies

$$\overline{\mathrm{E}}^T_u[h](u_0,u_1) \doteq 2\int_{v_0}^{v_1} \overline{(\mathfrak{J}[h]^X)}^3 \Omega dv\not{\epsilon} = \mathrm{E}^T_u[h](v_0,v_1) - 2\int_{\mathbb{S}^2_{u,v}} \mathcal{A}(u,v,\theta,\varphi)\not{\epsilon}\Big|_{v_0}^{v_1}, \qquad (4.4.8)$$

$$\overline{\mathrm{E}}^T_v[h](v_0,v_1) \doteq 2\int_{u_0}^{u_1} \overline{(\mathfrak{J}[h]^X)}^4 \Omega du\not{\epsilon} = \mathrm{E}^T_v[h](u_0,u_1) + 2\int_{\mathbb{S}^2_{u,v}} \mathcal{A}(u,v,\theta,\varphi)\not{\epsilon}\Big|_{u_0}^{u_1}, \qquad (4.4.9)$$

satisfy the *equivalent* conservation law,

$$\overline{\mathrm{E}}^T_{u_1}[h](v_0,v_1) + \overline{\mathrm{E}}^T_{v_1}[h](u_0,u_1) = \overline{\mathrm{E}}^T_{u_0}[h](v_0,v_1) + \overline{\mathrm{E}}^T_{v_0}[h](u_0,u_1). \qquad (4.4.10)$$

Using the above two points in conjunction with the linearised null structure equations and linearised Bianchi equations in propositions 2.10.7-2.10.20 allows one to prove the following theorem (which is proved in section 4.4.3):

Theorem 4.4.1. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on (Schw_n, g_s) and T is a Killing field for (Schw_n, g_s). Then one has the following conservation law for linear perturbations of the Schwarzschild–Tangherlini spacetime:*

$$\overline{\mathrm{E}}^T_{u_0}[h](v_0,v_1) + \overline{\mathrm{E}}^T_{v_0}[h](u_0,u_1) = \overline{\mathrm{E}}^T_{u_1}[h](v_0,v_1) + \overline{\mathrm{E}}^T_{v_1}[h](u_0,u_1), \qquad (4.4.11)$$

with

$$\overline{\mathrm{E}}^T_v[h](u_0,u_1) \equiv \int_{u_0}^{u_1} \Big[\Omega^2|\overset{(1)}{\hat{\underline{\chi}}}|^2 + 2\Omega^2|\overset{(1)}{\eta}|^2 - \Big(\frac{2(n-4)}{(n-2)}\Omega\mathrm{Tr}_{\not{g}}\chi + 4\omega\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) \qquad (4.4.12)$$
$$- 2\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) - \frac{(n-3)}{(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi})^2 + \frac{\Omega^2}{2}\langle\hat{\not{h}}, \widehat{\overset{(1)}{\not{\mathrm{Ric}}}}\rangle - \frac{\Omega^2(n-4)}{4(n-2)}\overset{(1)}{\not{\mathrm{Scal}}}\mathrm{Tr}_{\not{g}}\not{h}\Big]du$$

and

$$\overline{\mathrm{E}}^T_u[h](v_0,v_1) \equiv \int_{v_0}^{v_1} \Big[\Omega^2|\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2 + \Big(\frac{2(n-4)}{(n-2)}\Omega\mathrm{Tr}_{\not{g}}\chi + 4\omega\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi) \qquad (4.4.13)$$
$$- 2\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\underline{\chi}) - \frac{(n-3)}{(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not{g}}\chi)^2 + \frac{\Omega^2}{2}\langle\hat{\not{h}}, \widehat{\overset{(1)}{\not{\mathrm{Ric}}}}\rangle - \frac{(n-4)\Omega^2}{4(n-2)}\overset{(1)}{\not{\mathrm{Scal}}}\mathrm{Tr}_{\not{g}}\not{h}\Big]dv$$

where $\equiv$ denotes an implicit integration over $\mathbb{S}^{n-2}_{u,v}$.

Remark 4.4.2. *There are a few interesting points to note about this theorem:*

1. *As with the proof of theorem 3.1.9, the linearised Gauss and Codazzi constraint equations are key to proving this result.*

2. *As the reader would expect this conservation law can be proved directly in a manner analogous to Holzegel [90] from the linearised null structure equations in propositions 2.10.7-2.10.17 and linearised Bianchi identities in proposition 2.10.20. The computation much more involved than Holzegel's* 4*-dimensional case and therefore, the conservation law would be much more difficult to spot by eye. This is where having a current, such as* $\mathfrak{J}^T$ *in equation (4.1.7) or the canonical energy current* $\mathcal{J}^T$ *in definition 3.2.6 is invaluable; one starts with a conservation law associated to* $\mathfrak{J}^T$ *or* $\mathcal{J}^T$ *and one can compute and manipulate the fluxes that arise from this current into a desirable form. However, for completeness, there is a sketch of the direct proof of theorem 4.4.1 in section 4.4.3.*

3. *For* $n = 4$, $\overset{(1)}{\widehat{\not{\mathrm{R}}\mathrm{ic}}} = 0$ *so the fluxes* $\overline{\mathrm{E}}^T_u[h](v_0, v_1)$ *and* $\overline{\mathrm{E}}^T_v[h](u_0, u_1)$ *appearing here are precisely the fluxes appearing in Holzegel's conservation law in equation* (3.1.3).

The rest of this section is structured as follows. Previous results on the stability problem are reviewed briefly in the next section (section 4.4.1). Section 4.4.2 gives some technical computations in preparation for the computation of the current $\mathfrak{J}^T[h]$ (defined in equation (4.1.7)) in double null gauge in section 4.4.3. Section 4.4.3 contains two proofs of the theorem 4.4.3.

4.4.1 Background on the Stability Problem for $\mathrm{Schw_n}$

Stability questions concerning the Schwarzschild–Tangherlini spacetime have been studied in the works [54, 105, 131–134]. Of most relevance to the discussion here is the work of Ishibashi and Kodama [131], which studied the linear stability of the Schwarzschild–Tangherlini spacetime. By exploiting the spherical symmetry of the spacetime, they derive decoupled master equations for the *mode decomposed* linear perturbations. By studying the spectral properties of these master equations they prove that there are no growing modes. The reliance on spherical symmetry is slightly unattractive. Indeed, outside of spherical symmetry, there are a number of very interesting black hole solutions, for example, the Myers–Perry black hole [41] or Emparan–Reall black ring solutions [80], whose stability problems remain largely open. Moreover, apart from the decreased symmetry of these solutions, the decoupling of linear perturbations typically fails (see [87, 88] and references therein). In deriving the T-energy associated to the current $\mathfrak{J}^T[h]$ in equation (4.1.7) one does not need to decompose in modes or exploit spherical symmetry. Therefore, the T-energy gives an attractive avenue to pursue in higher dimensions where decoupling fails and symmetry is decreased. The natural place to initiate this investigation is with the 'simplest' higher-dimensional black hole, the Schwarzschild–Tangherlini black hole.

A related issue is that the true linear stability proof constructed in [28] does not extend readily from the $4D$ Schwarzschild spacetime to the general Schwarzschild–Tangherlini spacetime. As mentioned in the introduction of chapter 3, the linear stability proof of [28] relies upon exploiting the existence of the Teukolsky equation for the null curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ which decouple from the full system *and* are gauge invariant. However, even for this highly symmetric case of Schw_n the celebrated Teukolsky/Bardeen–Press equation fails to be a completely decoupled equation for $n > 4$; see section 2.10.4. To produce the fluxes arising from the T-energy one *does not* have to make use of the Teukolsky equations. Therefore, by using the decomposition of section 2 and the linearisation of the null structure equations in higher dimensions (see section 2.10.1), it is reasonable to expect that the fluxes arising in the modified T-canonical energy for the Schwarzschild–Tangherlini spacetime will be very similar to those appearing in theorem 3.1.7. In this section, this expectation is confirmed. It seems likely that a stability (boundedness) statement from the T-energy also holds but at the time of writing this is still speculation. Additionally, if one can understand how to improve the strength of such a stability statement, the method has the potential to produce a result such as [28] for higher-dimensional Schwarzschild.

4.4.2 Preliminary Computations

Proposition 4.4.3. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the exterior of (Schw_n, g_s) in double null Eddington–Finkelstein coordinates, i.e., in the basis (e_3, e_4, e_A), $h_{33} = 0 = h_{44}$ and*

$$h_{34} = -4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \qquad h_{4A} = -\frac{\overset{(1)}{b}_A}{\Omega}, \qquad h_{AB} = \not{h}_{AB}. \tag{4.4.14}$$

Then the non-zero components of $(\nabla_\alpha h)_{\beta\gamma}$ are

$$\begin{aligned}
(\nabla_3 h)_{34} &= -\frac{4}{\Omega}\overset{(1)}{\underline{\omega}}, & (\nabla_4 h)_{34} &= -\frac{4}{\Omega}\overset{(1)}{\omega}, \\
(\nabla_A h)_{44} &= \frac{2(\mathrm{Tr}_{\not{g}}\chi)}{(n-2)\Omega}\overset{(1)}{b}_A, & (\nabla_A h)_{34} &= -2(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})_A - \frac{\mathrm{Tr}_{\not{g}}\chi}{(n-2)\Omega}\overset{(1)}{b}_A, \\
(\nabla_3 h)_{4A} &= -2(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})_A + \frac{(\mathrm{Tr}_{\not{g}}\chi)}{(n-2)\Omega}\overset{(1)}{b}_A, & (\nabla_4 h)_{4A} &= -\frac{1}{\Omega}(\not{\nabla}_4 \overset{(1)}{b})_A + \frac{2\omega}{\Omega^2}\overset{(1)}{b}_A,
\end{aligned} \tag{4.4.15}$$

and

$$(\nabla_B h)_{3A} = -\frac{\mathrm{Tr}_{\not{g}}\chi}{(n-2)^2}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big)\not{g}_{AB} + \frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\hat{\not{h}}_{AB}, \tag{4.4.16}$$

$$(\nabla_B h)_{4A} = \frac{1}{\Omega}(\not{\mathcal{D}}_2^\star \overset{(1)}{b})_{AB} + \frac{1}{\Omega}\not\nabla_{[A}\overset{(1)}{b}_{B]} - \frac{\mathrm{Tr}_{\not{g}}\chi}{n-2}\hat{\not{h}}_{AB} - \frac{1}{(n-2)\Omega}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{b}\not{g}_{AB} \tag{4.4.17}$$
$$+ \frac{\mathrm{Tr}_{\not{g}}\chi}{(n-2)^2}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big)\not{g}_{AB},$$

$$(\nabla_3 h)_{BC} = 2\overset{(1)}{\hat{\underline{\chi}}}_{BC} + \frac{2}{(n-2)\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\not{g}}\underline{\chi}})\not{g}_{BC}, \tag{4.4.18}$$

$$(\nabla_4 h)_{BC} = 2\overset{(1)}{\hat{\chi}}_{BC} + \frac{2}{\Omega}(\not{\mathcal{D}}_2^\star \overset{(1)}{b})_{BC} + \frac{2}{(n-2)\Omega}\Big((\overset{(1)}{\Omega\mathrm{Tr}_{\not{g}}\chi}) - \mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{b}\Big)\not{g}_{BC}, \tag{4.4.19}$$

$$(\nabla_A h)_{BC} = (\not\nabla_A \hat{\not{h}})_{BC} + \frac{1}{(n-2)}\not\nabla_A(\mathrm{Tr}_{\not{g}}\not{h})\not{g}_{BC} - \frac{\mathrm{Tr}_{\not{g}}\chi}{2(n-2)\Omega}\Big(\not{g}_{AC}\overset{(1)}{b}_B + \not{g}_{AB}\overset{(1)}{b}_C\Big). \tag{4.4.20}$$

Proof. Follows from a direct computation using propositions 2.2.3 and 2.10.7 and

$$(\nabla_\alpha h)_{\beta\gamma} = e_\alpha(h_{\beta\gamma}) - h(\nabla_\alpha e_\beta, e_\gamma) - h(e_\beta, \nabla_\alpha e_\gamma). \tag{4.4.21}$$

□

Proposition 4.4.4. *Let T be the stationary Killing field on (Schw_n, g_s) where g_s is given in double null Eddington–Finkelstein coordinates. Then one has the following relations:*

$$\begin{aligned} &(\nabla_3 T)^4 = 0, && (\nabla_4 T)^3 = 0, \\ &(\nabla_3 T)^3 = -\omega, && (\nabla_4 T)^4 = \omega, \\ &(\nabla_3 T)^A = 2\not{g}^{AB}(\nabla_B T)^4 = 0, && (\nabla_4 T)^A = 2\not{g}^{AB}(\nabla_B T)^3 = 0 \end{aligned} \tag{4.4.22}$$

and $(\nabla_A T)^B = 0.$

Proof. This follows from writing

$$(\nabla_\alpha T)^\beta = g^{\beta\gamma}(\nabla_\alpha T)_\gamma = g^{\beta\gamma}(\nabla_{[\alpha} T)_{\gamma]} \tag{4.4.23}$$

by the Killing property of T. □

Proposition 4.4.5. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the exterior of (Schw_n, g_s) in double null Eddington–Finkelstein coordinates* *and T is a Killing field for (Schw_n, g_s). Then $(\mathcal{L}_T h)_{44} = 0$, $(\mathcal{L}_T h)_{3A} = 0$ and*

$$(\mathcal{L}_T h)_{34} = -2(\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}}), \tag{4.4.24}$$

$$(\mathcal{L}_T h)_{4A} = -\Omega(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}) + \frac{1}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi \overset{(1)}{b}_A - \frac{1}{2}(\not{\nabla}_4 \overset{(1)}{b})_A, \tag{4.4.25}$$

$$(\mathcal{L}_T h)_{AB} = \Omega(\overset{(1)}{\hat{\underline{\chi}}} + \overset{(1)}{\hat{\chi}})_{AB} + (\not{\mathcal{D}}_2^{\star} \overset{(1)}{b})_{AB} + \frac{1}{(n-2)}\mathcal{L}_T \mathrm{Tr}_{\not{g}}\not{h}\not{g}_{AB}, \tag{4.4.26}$$

$$\mathcal{L}_T \mathrm{Tr}_{\not{g}}\not{h} = (\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}) + (\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}. \tag{4.4.27}$$

Proof. This follows from writing

$$(\mathcal{L}_T h)_{\alpha\beta} = \frac{\Omega}{2}(\nabla_3 h)_{\alpha\beta} + \frac{\Omega}{2}(\nabla_4 h)_{\alpha\beta} + (\nabla_\alpha T)^\gamma h_{\gamma\beta} + (\nabla_\beta T)^\gamma h_{\gamma\alpha} \tag{4.4.28}$$

and using propositions 4.4.3 and 4.4.4. □

Proposition 4.4.6. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the exterior of (Schw_n, g_s) in double null Eddington–Finkelstein coordinates. Then*

$$\nabla_3 \mathrm{Tr}_g h = \frac{4}{\Omega}\overset{(1)}{\underline{\omega}} + \frac{2}{\Omega}(\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}\underline{\chi}}), \tag{4.4.29}$$

$$\nabla_4 \mathrm{Tr}_g h = \frac{4}{\Omega}\overset{(1)}{\omega} + \frac{2}{\Omega}\Big((\Omega \overset{(1)}{\mathrm{Tr}_{\not{g}}\chi}) - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big), \tag{4.4.30}$$

$$\nabla_A \mathrm{Tr}_g h = 2(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})_A + \not{\nabla}_A \mathrm{Tr}_{\not{g}}\not{h}, \tag{4.4.31}$$

$$(\mathrm{div}h)_3 = \frac{2}{\Omega}\overset{(1)}{\underline{\omega}} + \frac{1}{n-2}(\mathrm{Tr}_{\not{g}}\chi)\Big(\mathrm{Tr}_{\not{g}}\not{h} - 2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big), \tag{4.4.32}$$

$$(\mathrm{div}h)_4 = \frac{2}{\Omega}\overset{(1)}{\omega} - \frac{1}{n-2}(\mathrm{Tr}_{\not{g}}\chi)\Big(\mathrm{Tr}_{\not{g}}\not{h} - 2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) - \frac{1}{\Omega}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}, \tag{4.4.33}$$

$$(\mathrm{div}h)_A = (\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})_A - \frac{n}{2(n-2)\Omega}(\mathrm{Tr}_{\not{g}}\chi)\overset{(1)}{b}_A + (\not{\mathrm{d}}\mathrm{iv}\hat{\not{h}})_A + \frac{1}{n-2}\not{\nabla}_A \mathrm{Tr}_{\not{g}}\not{h}. \tag{4.4.34}$$

Proof. For the first three relations one writes

$$\nabla_\alpha(\mathrm{Tr}_g h) = e_\alpha\Big(4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \mathrm{Tr}_{\not{g}}\not{h}\Big) \tag{4.4.35}$$

and uses the equations for the linearised metric coefficients 2.10.7. The second three follow from proposition 4.4.3. □

Lemma 4.4.7. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the exterior of (Schw_n, g_s) in double null Eddington–Finkelstein coordinates.*

Then $\hat{\not{h}}$ satisfies

$$\not\nabla^A \hat{\not{h}}^{BC} \not\nabla_C \hat{\not{h}}_{BA} - \frac{1}{2}|\not\nabla \hat{\not{h}}|^2 \equiv -\frac{1}{n-2}\mathrm{S}\not{c}\mathrm{al}(\not{g})|\hat{\not{h}}|^2 - \langle \hat{\not{h}}, \overset{(1)}{\hat{\tau}}\rangle + \frac{n-4}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi\langle \overset{(1)}{\underline{\hat{\chi}}} - \overset{(1)}{\hat{\chi}}, \hat{\not{h}}\rangle + \frac{n-4}{2(n-2)}\langle \mathrm{d}\not{\mathrm{i}}\mathrm{v}\hat{\not{h}}, \not\nabla \mathrm{Tr}_{\not{g}}\not{h}\rangle, \tag{4.4.36}$$

where $\equiv$ denotes an implicit integration over $\mathbb{S}^{n-2}_{u,v}$.

Proof. To prove this one integrates by parts on $\mathbb{S}^{n-2}_{u,v}$, applies the Ricci identity and the corollary of the linearised Gauss equations in proposition 2.10.15. □

Proposition 4.4.8. *Suppose h is a smooth solution to the linearised vacuum Einstein equation (I.5) in double null gauge on the exterior of (Schw_n, g_s) in double null Eddington–Finkelstein coordinates. Then*

$$\begin{aligned}
|\nabla h|^2_P \equiv{}& 2|\overset{(1)}{\eta}|^2 + 2|\overset{(1)}{\underline{\eta}}|^2 + \frac{1}{\Omega}\mathrm{Tr}_{\not{g}}\chi(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\overset{(1)}{b}) + 2\langle \overset{(1)}{\hat{\chi}}, \overset{(1)}{\underline{\hat{\chi}}}\rangle + \frac{2(3-n)}{\Omega^2(n-2)}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} \\
&+ \frac{1}{\Omega^2}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{b} + \frac{2\mathrm{Tr}_{\not{g}}\chi}{n-2}\langle \hat{\not{h}}, \overset{(1)}{\underline{\hat{\chi}}} - \overset{(1)}{\hat{\chi}} - \frac{1}{\Omega}\not{\mathcal{D}}^{\star}_2\overset{(1)}{b}\rangle + \frac{\mathrm{Tr}_{\not{g}}\chi}{2\Omega}\overset{(1)}{b}(\mathrm{Tr}_{\not{g}}\not{h}) - \frac{\mathrm{S}\not{c}\mathrm{al}(\not{g})}{n-2}|\hat{\not{h}}|^2 \\
&- \langle \hat{\not{h}}, \overset{(1)}{\hat{\tau}}\rangle + \frac{n-4}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi\langle \overset{(1)}{\underline{\hat{\chi}}} - \overset{(1)}{\hat{\chi}}, \hat{\not{h}}\rangle - \langle \overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}, \not\nabla\mathrm{Tr}_{\not{g}}\not{h}\rangle - \frac{2}{\Omega^2}\overset{(1)}{\omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} \\
&- \frac{2}{\Omega^2}\overset{(1)}{\underline{\omega}}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} + \frac{2\mathrm{Tr}_{\not{g}}\chi}{(n-2)\Omega}(\overset{(1)}{\omega} - \overset{(1)}{\underline{\omega}})\Big(\mathrm{Tr}_{\not{g}}\not{h} - 2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) \\
&- \frac{n-4}{(n-2)^2\Omega}\mathrm{Tr}_{\not{g}}\chi\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big)\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} - \mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{b}\Big) \\
&+ \Big[4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\not{g}}\not{h}\Big]\Big(\frac{n-3}{(n-2)\Omega}\mathrm{Tr}_{\not{g}}\chi\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\Big] - 2\overset{(1)}{\rho}\Big) \\
&+ \Big[4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\not{g}}\not{h}\Big]\Big(\frac{\mathrm{S}\not{c}\mathrm{al}(\not{g})}{n-2}\mathrm{Tr}_{\not{g}}\not{h} - \frac{2(n-3)}{n-2}(\mathrm{Tr}_{\not{g}}\chi)^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big).
\end{aligned} \tag{4.4.37}$$

Proof. One should note that $|\nabla h|^2_P$ can be written more explicitly as

$$P(\nabla h, \nabla h) = g^{ae}g^{bf}g^{cd}\nabla_a h_{bc}\nabla_d h_{ef} - \frac{1}{2}|\nabla h|^2_g - \langle \mathrm{div}h, \nabla\mathrm{Tr}_g h\rangle_g + \frac{1}{2}|\nabla\mathrm{Tr}_g h|^2_g. \tag{4.4.38}$$

Denote the first two terms as P_1. Then one can decompose P_1 as

$$\begin{aligned}
P_1 ={}& \frac{1}{4}\not{g}^{AB}(\nabla_3 h_{4A})(\nabla_3 h_{4B}) - \not{g}^{BF}\not{g}^{AD}\nabla_3 h_{BA}\nabla_D h_{4F} - \not{g}^{AE}\not{g}^{BF}\nabla_A h_{3B}\nabla_4 h_{EF} \\
&+ \frac{1}{2}\langle \nabla_3 h, \nabla_4 h\rangle_{\not{g}} + \frac{1}{2}\Big[\nabla_3 h_{4A} - \frac{1}{2}\nabla_A h_{34}\Big]\not{g}^{AB}(\nabla_B h)_{34} - \frac{1}{2}|\nabla h|^2_{\not{g}} \\
&+ \not{g}^{AE}\not{g}^{BF}\not{g}^{CD}\nabla_A h_{BC}\nabla_D h_{EF}.
\end{aligned} \tag{4.4.39}$$

So, naively calculating using propositions 4.4.3 and 4.4.6, one finds

$$
\begin{aligned}
|\nabla \mathrm{Tr}_g h|_g^2 = & -\frac{16}{\Omega^2}\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} - \frac{4}{\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - \slashed{\mathrm{div}}\overset{(1)}{b}\Big) - \frac{8}{\Omega^2}\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) \qquad (4.4.40)\\
& - \frac{8}{\Omega^2}\overset{(1)}{\underline{\omega}}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - \slashed{\mathrm{div}}\overset{(1)}{b}\Big) + 4|\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}}|^2 + |\slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}|^2 + 4\langle\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle,\\
\langle \mathrm{div} h, \nabla \mathrm{Tr}_g h\rangle_g = & \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)\Omega}(\overset{(1)}{\underline{\omega}} - \overset{(1)}{\omega})\Big(\mathrm{Tr}_{\slashed{g}}\slashed{h} - 2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) + \langle \slashed{\mathrm{div}}\hat{\slashed{h}}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle \qquad (4.4.41)\\
& + \frac{1}{n-2}|\slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}|^2 - \frac{8}{\Omega^2}\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} - \frac{2\overset{(1)}{\underline{\omega}}}{\Omega^2}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - 2\slashed{\mathrm{div}}\overset{(1)}{b}\Big) + 2\langle\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}}, \slashed{\mathrm{div}}\hat{\slashed{h}}\rangle\\
& + \frac{n}{n-2}\langle\overset{(1)}{\eta}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle + \frac{4-n}{n-2}\langle\overset{(1)}{\underline{\eta}}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle - \frac{n}{2(n-2)\Omega}(\mathrm{Tr}_{\slashed{g}}\chi)\langle\overset{(1)}{b}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle\\
& - \frac{1}{(n-2)\Omega}\mathrm{Tr}_{\slashed{g}}\chi\Big(\mathrm{Tr}_{\slashed{g}}\slashed{h} - 2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - (\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) - \slashed{\mathrm{div}}\overset{(1)}{b}\Big)\\
& - \frac{2}{\Omega^2}\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) + \frac{1}{\Omega^2}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\slashed{\mathrm{div}}\overset{(1)}{b} + 2|\overset{(1)}{\eta}|^2 - 2|\overset{(1)}{\underline{\eta}}|^2 - \frac{n\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)\Omega}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})(\overset{(1)}{b}),
\end{aligned}
$$

and

$$
\begin{aligned}
P_1 = & \,2|\overset{(1)}{\eta}|^2 - 2|\overset{(1)}{\underline{\eta}}|^2 - 4\langle\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle - \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)\Omega}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})(\overset{(1)}{b}) + 2\langle\overset{(1)}{\hat{\chi}}, \overset{(1)}{\hat{\underline{\chi}}}\rangle + \frac{2}{\Omega^2(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})\\
& + \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{n-2}\langle\hat{\slashed{h}}, \overset{(1)}{\hat{\underline{\chi}}} - \overset{(1)}{\hat{\chi}} - \frac{1}{\Omega}\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{b}\rangle - \frac{2\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)^2\Omega}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\slashed{g}}\slashed{h}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) \qquad (4.4.42)\\
& + \frac{2}{(n-2)^2\Omega}\mathrm{Tr}_{\slashed{g}}\chi\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\slashed{g}}\slashed{h}\Big)\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - \slashed{\mathrm{div}}\overset{(1)}{b}\Big) - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)\Omega}\overset{(1)}{b}(\mathrm{Tr}_{\slashed{g}}\slashed{h})\\
& + \slashed{\nabla}^A\hat{\slashed{h}}^{BC}\slashed{\nabla}_C\hat{\slashed{h}}_{BA} - \frac{1}{2}|\slashed{\nabla}\hat{\slashed{h}}|^2 + \frac{4-n}{2(n-2)^2}|\slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}|^2 + \frac{2}{n-2}\langle\slashed{\mathrm{div}}\hat{\slashed{h}}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle.
\end{aligned}
$$

Combining and using lemma 4.4.7 and the linearised Gauss equation corollaries gives the result. □

4.4.3 Proof of the Conservation Law

Proof of Theorem 4.4.1. Let the symbol $\equiv$ denote equality under integration by parts of $\mathbb{S}^{n-2}_{u,v}$. The currents decompose as

$$
\begin{aligned}
(\mathfrak{J}^T)^4 = {} & \frac{1}{4}\langle\widehat{\mathcal{L}_T h}, \widehat{(\not{\nabla}_3 \not{h})}\rangle_{\not{g}} - \frac{1}{2}\not{g}^{BF}\not{g}^{CD}\widehat{(\mathcal{L}_T h)}_{BC}\widehat{(\nabla_D h)}_{3F} - \frac{\Omega}{4}|\nabla h|^2_P \\
& + \frac{1}{8}\mathcal{L}_T \mathrm{Tr}_{\not{g}}\not{h}\Big[e_3(h_{34}) - \frac{2(n-3)}{(n-2)}\mathrm{Tr}_{\not{g}}(\nabla_3 h) + \frac{2(n-4)}{(n-2)}\not{g}^{AB}(\nabla_D h)_{3B}\Big] \\
& + \frac{1}{8}(\mathcal{L}_T h)_{34}\Big[\not{\nabla}_3 \mathrm{Tr}_{\not{g}}\not{h} - 2\not{g}^{DF}(\nabla_D h)_{3F}\Big],
\end{aligned} \tag{4.4.43}
$$

$$
\begin{aligned}
(\mathfrak{J}^T)^3 = {} & \frac{1}{4}\langle\widehat{\mathcal{L}_T h}, \widehat{(\nabla_4 h)}\rangle_{\not{g}} - \frac{1}{2}\not{g}^{BF}\not{g}^{CD}\widehat{(\mathcal{L}_T h)}_{BC}(\nabla_D h)_{4F} - \frac{\Omega}{4}|\nabla h|^2_P \\
& + \frac{1}{8}\mathcal{L}_T \mathrm{Tr}_{\not{g}}\not{h}\Big[(\nabla_4 h)_{34} - \frac{2(n-3)}{(n-2)}\mathrm{Tr}_{\not{g}}(\nabla_4 h) + \frac{2(n-4)}{(n-2)}\not{g}^{AB}(\nabla_D h)_{4B}\Big] \\
& + \frac{1}{8}(\mathcal{L}_T h)_{34}\Big[\mathrm{Tr}_{\not{g}}(\nabla_4 h) - 2\not{g}^{DF}(\nabla_D h)_{4F}\Big] + \frac{1}{4}(\mathcal{L}_T h)_4^A\Big[(\nabla_3 h)_{4A} + \mathrm{Tr}_{\not{g}}(\nabla_A h)\Big].
\end{aligned} \tag{4.4.44}
$$

So, using propositions 4.4.3 and 4.4.5 one can compute

$$
\begin{aligned}
(\mathfrak{J}^T)^4 \equiv {} & \frac{\Omega}{2}|\overset{(1)}{\hat{\underline{\chi}}}|^2 + \frac{\Omega}{2}\langle\overset{(1)}{\hat{\chi}}, \overset{(1)}{\hat{\underline{\chi}}}\rangle - \frac{\mathrm{Tr}_{\not{g}}\chi}{2(n-2)}\langle\Omega(\overset{(1)}{\hat{\chi}} + \overset{(1)}{\hat{\underline{\chi}}}) + \not{\mathcal{D}}^{\star}_2\overset{(1)}{b}, \hat{\not{h}}\rangle - \frac{1}{2\Omega}(\overset{(1)}{\omega} + 2\overset{(1)}{\underline{\omega}})\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} \\
& - \frac{1}{4\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}^2 - \frac{\mathrm{Tr}_{\not{g}}\chi}{2(n-2)}(\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}})\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big) - \frac{1}{2\Omega}\overset{(1)}{\underline{\omega}}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big) \\
& - \frac{1}{4\Omega}\Big[\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} + \frac{n-4}{n-2}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big]\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} + \frac{n-3}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{\eta}(\overset{(1)}{b}) + \frac{1}{2}\langle\overset{(1)}{b}, \overset{(1)}{\underline{\beta}}\rangle - \frac{\Omega}{4}|\nabla h|^2_P \\
& - \frac{n-4}{8(n-2)}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} + \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big)\Big(\frac{2\mathrm{Tr}_{\not{g}}\chi}{(n-2)}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big) + \frac{2}{\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\Big)
\end{aligned} \tag{4.4.45}
$$

and

$$
\begin{aligned}
(\mathfrak{J}^T)^3 \equiv {} & \frac{1}{2}\Omega|\overset{(1)}{\hat{\chi}}|^2 + \frac{1}{2}\Omega\langle\overset{(1)}{\hat{\chi}}, \overset{(1)}{\hat{\underline{\chi}}}\rangle + \frac{1}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi\langle\Omega(\overset{(1)}{\hat{\chi}} + \overset{(1)}{\hat{\underline{\chi}}}) + \not{\mathcal{D}}^{\star}_2\overset{(1)}{b}, \hat{\not{h}}\rangle - \frac{1}{2\Omega}\overset{(1)}{\omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} \\
& + \frac{1}{2\Omega}\overset{(1)}{\omega}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b} - \frac{1}{2\Omega}(2\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}})\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} + \frac{\mathrm{Tr}_{\not{g}}\chi}{2(n-2)}(\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}})\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big) \\
& + \frac{1}{8(n-2)}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{b}(\mathrm{Tr}_{\not{g}}\not{h}) - \frac{1}{4\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big) - \frac{1}{4\Omega}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big)^2 \\
& - \frac{1}{4}\Omega\langle\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}, \not{\nabla}\mathrm{Tr}_{\not{g}}\not{h}\rangle - \frac{\mathrm{Tr}_{\not{g}}\chi}{4(n-2)}(\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}})(\overset{(1)}{b}) + \frac{1}{4}\langle\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}, \not{\nabla}_4\overset{(1)}{b}\rangle - \frac{1}{8}\langle\not{\nabla}_4\overset{(1)}{b}, \not{\nabla}\mathrm{Tr}_{\not{g}}\not{h}\rangle \\
& + \frac{n-4}{8(n-2)}\Big(\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)} + \overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\underline{\chi})} - \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b}\Big)\Big(\frac{2\mathrm{Tr}_{\not{g}}\chi}{(n-2)}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\not{g}}\not{h}\Big) - \frac{2}{\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\Big) \\
& - \frac{n-3}{2(n-2)}\mathrm{Tr}_{\not{g}}\chi\overset{(1)}{\underline{\eta}}(\overset{(1)}{b}) - \frac{n-3}{2(n-2)\Omega}\overset{(1)}{(\Omega\mathrm{Tr}_{\not{g}}\chi)}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{b} - \frac{1}{2}\langle\overset{(1)}{b}, \overset{(1)}{\beta}\rangle + \frac{1}{2}\Omega|\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}|^2 - \frac{\Omega}{4}|\nabla h|^2_P,
\end{aligned} \tag{4.4.46}
$$

where one integrates by parts on $\mathbb{S}^{n-2}_{u,v}$ and uses the linearised Codazzi (proposition 2.10.17) for the terms $\langle \overset{(1)}{\hat{\chi}}, \slashed{\mathcal{D}}_2^\star \overset{(1)}{b}\rangle$ and $\langle \overset{(1)}{\hat{\underline{\chi}}}, \slashed{\mathcal{D}}_2^\star \overset{(1)}{b}\rangle$.

Adding the contribution for $|\nabla h|^2_P$ gives

$$
\begin{aligned}
(\mathfrak{J}^T)^4 \equiv\ & \frac{\Omega}{2}|\overset{(1)}{\hat{\underline{\chi}}}|^2 - \frac{\Omega \mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}\langle \overset{(1)}{\hat{\underline{\chi}}}, \hat{\slashed{h}}\rangle - \frac{1}{\Omega}\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega \mathrm{Tr}_{\slashed{g}}\underline{\chi}}) - \frac{\Omega}{2}|\overset{(1)}{\eta}|^2 - \frac{\Omega}{2}|\overset{(1)}{\underline{\eta}}|^2 + \frac{1}{2\Omega}\overset{(1)}{\underline{\omega}}\slashed{\mathrm{div}}\overset{(1)}{b} \\
& - \frac{n-3}{2\Omega(n-2)}(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi}})^2 - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}\overset{(1)}{\underline{\omega}}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\slashed{g}}\slashed{h}\Big) + \frac{\Omega}{4}\langle \hat{\slashed{h}}, \overset{(1)}{\hat{\tau}}\rangle \\
& - \frac{(n-4)}{8(n-2)}\Omega\mathrm{Tr}_{\slashed{g}}\chi\langle \overset{(1)}{\hat{\underline{\chi}}} - \overset{(1)}{\hat{\chi}}, \hat{\slashed{h}}\rangle + \frac{\Omega}{4}\langle \overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}, \slashed{\nabla} \mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle + \frac{\Omega \mathrm{S\slashed{c}al}(\slashed{g})}{4(n-2)}|\hat{\slashed{h}}|^2 \\
& + \frac{n-4}{4(n-2)}\mathrm{Tr}_{\slashed{g}}\chi \overset{(1)}{\eta}(\overset{(1)}{b}) - \frac{1}{4}\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\underline{\eta}}(\overset{(1)}{b}) + \frac{1}{2}\langle\overset{(1)}{b},\overset{(1)}{\underline{\beta}}\rangle - \frac{1}{4\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\slashed{\mathrm{div}}\overset{(1)}{b} \\
& - \frac{n-4}{2(n-2)^2}\mathrm{Tr}_{\slashed{g}}\chi(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\slashed{g}}\slashed{h}\Big) - \frac{\mathrm{Tr}_{\slashed{g}}\chi}{8}\overset{(1)}{b}(\mathrm{Tr}_{\slashed{g}}\slashed{h}) \\
& - \frac{\Omega}{4}\Big[4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]\Big(\frac{n-3}{(n-2)\Omega}\mathrm{Tr}_{\slashed{g}}\chi\Big[(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi}) - (\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\Big] - 2\overset{(1)}{\rho}\Big) \\
& - \frac{\Omega}{4}\Big[4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]\Big(\frac{\mathrm{S\slashed{c}al}(\slashed{g})}{n-2}\mathrm{Tr}_{\slashed{g}}\slashed{h} - \frac{2(n-3)}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)
\end{aligned}
\tag{4.4.47}
$$

and

$$
\begin{aligned}
(\mathfrak{J}^T)^3 \equiv\ & \frac{1}{2}\Omega|\overset{(1)}{\hat{\chi}}|^2 + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}\langle \Omega\overset{(1)}{\hat{\chi}} + \slashed{\mathcal{D}}_2^\star\overset{(1)}{b}, \hat{\slashed{h}}\rangle + \frac{1}{2\Omega}\overset{(1)}{\omega}\slashed{\mathrm{div}}\overset{(1)}{b} - \frac{1}{\Omega}\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi}) - \Omega\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle \\
& + \frac{\Omega \mathrm{S\slashed{c}al}(\slashed{g})}{4(n-2)}|\hat{\slashed{h}}|^2 + \frac{\Omega}{4}\langle\hat{\slashed{h}}, \overset{(1)}{\hat{\tau}}\rangle - \frac{1}{2}\langle \overset{(1)}{b}, \overset{(1)}{\beta}\rangle - \frac{n-4}{8(n-2)}\Omega \mathrm{Tr}_{\slashed{g}}\chi\langle \overset{(1)}{\hat{\underline{\chi}}} - \overset{(1)}{\hat{\chi}}, \hat{\slashed{h}}\rangle \\
& - \frac{(n-1)\mathrm{Tr}_{\slashed{g}}\chi}{4(n-2)}\overset{(1)}{\eta}(\overset{(1)}{b}) - \frac{3(n-3)}{4(n-2)}\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{\underline{\eta}}(\overset{(1)}{b}) + \frac{1}{4}\langle\overset{(1)}{\eta} - \overset{(1)}{\underline{\eta}}, \slashed{\nabla}_4\overset{(1)}{b}\rangle - \frac{1}{8}\langle\slashed{\nabla}_4\overset{(1)}{b}, \slashed{\nabla}\mathrm{Tr}_{\slashed{g}}\slashed{h}\rangle \\
& - \frac{n-3}{8(n-2)}\mathrm{Tr}_{\slashed{g}}\chi\overset{(1)}{b}(\mathrm{Tr}_{\slashed{g}}\slashed{h}) - \frac{1}{4\Omega}\Big((\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi}) - \slashed{\mathrm{div}}\overset{(1)}{b}\Big)^2 - \frac{n-4}{4(n-2)\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi})^2 \\
& + \frac{(n-4)\mathrm{Tr}_{\slashed{g}}\chi}{2(n-2)^2}(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi})\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\slashed{g}}\slashed{h}\Big) + \frac{(n-4)\mathrm{Tr}_{\slashed{g}}\chi}{2(n-2)}(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\overset{(1)}{b}) \\
& - \frac{(n-4)\mathrm{Tr}_{\slashed{g}}\chi}{2(n-2)^2}\overset{(1)}{b}(\mathrm{Tr}_{\slashed{g}}\slashed{h}) - \frac{1}{4\Omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi})\slashed{\mathrm{div}}\overset{(1)}{b} + \frac{\mathrm{Tr}_{\slashed{g}}\chi}{(n-2)}\overset{(1)}{\omega}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) - \mathrm{Tr}_{\slashed{g}}\slashed{h}\Big) \\
& - \frac{\Omega}{4}\Big[4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]\Big(\frac{n-3}{(n-2)\Omega}\mathrm{Tr}_{\slashed{g}}\chi\Big[(\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\chi}) - (\overset{(1)}{\Omega\mathrm{Tr}_{\slashed{g}}\underline{\chi}})\Big] - 2\overset{(1)}{\rho}\Big) \\
& - \frac{\Omega}{4}\Big[4\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) + \frac{(n-4)}{2(n-2)}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]\Big(\frac{\mathrm{S\slashed{c}al}(\slashed{g})}{n-2}\mathrm{Tr}_{\slashed{g}}\slashed{h} - \frac{2(n-3)}{n-2}(\mathrm{Tr}_{\slashed{g}}\chi)^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)
\end{aligned}
\tag{4.4.48}
$$

where one uses that

$$\text{d}\!\!/\text{iv}\overset{(1)}{b}\Big(2(n-2)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)-\text{Tr}_{\not g}\not h\Big)=\overset{(1)}{b}(\text{Tr}_{\not g}\not h)-(n-2)(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})(\overset{(1)}{b}). \tag{4.4.49}$$

Denote

$$\mathcal{A}\doteq\mathcal{A}_1-\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4-\mathcal{A}_5-\frac{1}{n-2}\mathcal{A}_6+2\mathcal{A}_7-\mathcal{A}_8, \tag{4.4.50}$$

where

$$\begin{aligned}
&\mathcal{A}_1\doteq\frac{r^{n-2}}{4}(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}})(\overset{(1)}{b}), && \mathcal{A}_2\doteq\frac{r^{n-2}}{8}\overset{(1)}{b}{}^A\not\nabla_A\text{Tr}_{\not g}\not h,\\
&\mathcal{A}_3\doteq\frac{r^{n-2}}{4(n-2)}(\Omega\text{Tr}_{\not g}\chi)|\hat{\not h}|^2, && \mathcal{A}_4\doteq r^{n-2}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\text{Tr}}_{\not g}\underline{\chi}),\\
&\mathcal{A}_5\doteq r^{n-2}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\text{Tr}}_{\not g}\chi), && \mathcal{A}_6\doteq r^{n-2}(\Omega\text{Tr}_{\not g}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\text{Tr}_{\not g}\not h,\\
&\mathcal{A}_7\doteq r^{n-2}(\Omega\text{Tr}_{\not g}\chi)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2, && \mathcal{A}_8\doteq\frac{r^{n-2}}{8}\frac{n-4}{(n-2)^2}(\Omega\text{Tr}_{\not g}\chi)(\text{Tr}_{\not g}\not h)^2.
\end{aligned} \tag{4.4.51}$$

One can then compute that using the linearised null structure equations of section 2.10.1 the $\not\nabla_4$ and $\not\nabla_3$ derivatives of each term. Explicitly, using the linearised torsion equations of proposition 2.10.11 and linearised metric equations of proposition 2.10.7 one can compute that

$$\begin{aligned}
\frac{1}{r^{n-2}}\not\nabla_4\mathcal{A}_1=&\frac{1}{4}\Big(\frac{n-3}{n-2}\Big)\text{Tr}_{\not g}\chi(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}})(\overset{(1)}{b})+\frac{1}{4}(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}})(\not\nabla_4\overset{(1)}{b})-\frac{1}{2}\overset{(1)}{\beta}(\overset{(1)}{b})+\frac{1}{2}\overset{(1)}{\omega}\text{d}\!\!/\text{iv}\overset{(1)}{b} \\
&+\frac{\text{Tr}_{\not g}\chi}{2(n-2)}\overset{(1)}{\underline{\eta}}(\overset{(1)}{b}),
\end{aligned} \tag{4.4.52}$$

$$\frac{1}{r^{n-2}}\not\nabla_3\mathcal{A}_1=\frac{\text{Tr}_{\not g}\chi}{4}\Big(\overset{(1)}{\underline{\eta}}(\overset{(1)}{b})-\frac{(n-4)}{(n-2)}\overset{(1)}{\eta}(\overset{(1)}{b})\Big)-\frac{1}{2}\overset{(1)}{\beta}(\overset{(1)}{b})+\frac{\Omega}{2}|\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}}|^2-\frac{1}{2\Omega}\overset{(1)}{\underline{\omega}}\text{d}\!\!/\text{iv}\overset{(1)}{b}. \tag{4.4.53}$$

Using the linearised metric equations of proposition 2.10.7 one has

$$\begin{aligned}
\frac{1}{r^{n-2}}\not\nabla_4\mathcal{A}_2=&\frac{1}{8}\Big(\frac{n-3}{n-2}\Big)\text{Tr}_{\not g}\chi\overset{(1)}{b}(\text{Tr}_{\not g}\not h)-\frac{1}{4\Omega}\Big((\Omega\overset{(1)}{\text{Tr}}_{\not g}\chi)-\text{d}\!\!/\text{iv}\overset{(1)}{b}\Big)\text{d}\!\!/\text{iv}\overset{(1)}{b} \\
&+\frac{1}{8}\langle\not\nabla_4\overset{(1)}{b},\not\nabla\text{Tr}_{\not g}\not h\rangle,
\end{aligned} \tag{4.4.54}$$

$$\frac{1}{r^{n-2}}\not\nabla_3\mathcal{A}_2=-\frac{\text{Tr}_{\not g}\chi}{8}\overset{(1)}{b}(\text{Tr}_{\not g}\not h)+\frac{\Omega}{4}\langle(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}}),\not\nabla\text{Tr}_{\not g}\not h\rangle-\frac{1}{4\Omega}(\Omega\overset{(1)}{\text{Tr}}_{\not g}\underline{\chi})\text{d}\!\!/\text{iv}\overset{(1)}{b}. \tag{4.4.55}$$

Similarly, using the linearised metric equation of proposition 2.10.7 one can compute that

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_4\mathcal{A}_3 = \frac{\Omega}{4(n-2)}\mathrm{S\!\!\!/cal}(g\!\!\!/)|\hat{h\!\!\!/}|^2 + \frac{\mathrm{Tr}_{g\!\!\!/}\chi}{n-2}\langle\hat{h\!\!\!/}, \Omega\overset{(1)}{\hat{\chi}} + \mathcal{D\!\!\!/}_2^{\star}\overset{(1)}{b}\rangle, \tag{4.4.56}$$

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_3\mathcal{A}_3 = -\frac{\Omega}{4(n-2)}\mathrm{S\!\!\!/cal}(g\!\!\!/)|\hat{h\!\!\!/}|^2 + \frac{\Omega\mathrm{Tr}_{g\!\!\!/}\chi}{n-2}\langle\hat{h\!\!\!/}, \overset{(1)}{\underline{\hat{\chi}}}\rangle. \tag{4.4.57}$$

Using propositions 2.10.7, 2.10.8 and 2.10.9 one can compute

$$\begin{aligned}\frac{1}{r^{n-2}}\nabla\!\!\!/_4\mathcal{A}_4 = {} & \frac{1}{\Omega}\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}) + 2\Omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big) - \Omega\langle\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle - \Omega|\overset{(1)}{\underline{\eta}}|^2 \\ & + \Big(\frac{n-3}{n-2}\Big)\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}) + \frac{1}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi),\end{aligned} \tag{4.4.58}$$

$$\begin{aligned}\frac{1}{r^{n-2}}\nabla\!\!\!/_3\mathcal{A}_4 = {} & \frac{1}{\Omega}\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) + 2\Omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big) - \Omega\langle\overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle - \Omega|\overset{(1)}{\eta}|^2 \\ & - \Big(\frac{n-3}{n-2}\Big)\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) - \frac{1}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}).\end{aligned} \tag{4.4.59}$$

Similarly, using propositions 2.10.7, 2.10.8 and 2.10.9 one has

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_4\mathcal{A}_5 = \frac{1}{\Omega}\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) + \Big[\frac{n-4}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi + \frac{2\omega}{\Omega}\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) + 2\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\omega}, \tag{4.4.60}$$

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_3\mathcal{A}_5 = \frac{1}{\Omega}\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}) - \Big[\frac{n-4}{n-2}\mathrm{Tr}_{g\!\!\!/}\chi + \frac{2\omega}{\Omega}\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}) - 2\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\underline{\omega}}. \tag{4.4.61}$$

Using propositions 2.10.7 one has

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_4\mathcal{A}_6 = \Omega\mathrm{S\!\!\!/cal}(g\!\!\!/)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\mathrm{Tr}_{g\!\!\!/}h\!\!\!/ + \mathrm{Tr}_{g\!\!\!/}\chi\Big(\overset{(1)}{\omega}\mathrm{Tr}_{g\!\!\!/}h\!\!\!/ + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) + (\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})(\overset{(1)}{b})\Big), \tag{4.4.62}$$

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_3\mathcal{A}_6 = -\Omega\mathrm{S\!\!\!/cal}(g\!\!\!/)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\mathrm{Tr}_{g\!\!\!/}h\!\!\!/ + \mathrm{Tr}_{g\!\!\!/}\chi\overset{(1)}{\underline{\omega}}\mathrm{Tr}_{g\!\!\!/}h\!\!\!/ + 2\mathrm{Tr}_{g\!\!\!/}\chi\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}). \tag{4.4.63}$$

Again, using propositions 2.10.7 one has

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_4\mathcal{A}_7 = \Omega\mathrm{S\!\!\!/cal}(g\!\!\!/)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2 + 2\mathrm{Tr}_{g\!\!\!/}\chi\overset{(1)}{\omega}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big), \tag{4.4.64}$$

$$\frac{1}{r^{n-2}}\nabla\!\!\!/_3\mathcal{A}_7 = -\Big(\frac{n-3}{n-2}\frac{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2}{\Omega} + 2\omega\mathrm{Tr}_{g\!\!\!/}\chi\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2 + 2\mathrm{Tr}_{g\!\!\!/}\chi\overset{(1)}{\underline{\omega}}\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big) \tag{4.4.65}$$

and

$$\frac{1}{r^{n-2}}\slashed{\nabla}_4\mathcal{A}_8 = \frac{(n-4)\Omega\mathrm{S\slashed{c}al}(\slashed{g})}{8(n-2)^2}(\mathrm{Tr}_{\slashed{g}}\slashed{h})^2 + \frac{(n-4)\mathrm{Tr}_{\slashed{g}}\chi}{2(n-2)^2}\Big(\mathrm{Tr}_{\slashed{g}}\slashed{h}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) + \overset{(1)}{b}(\mathrm{Tr}_{\slashed{g}}\slashed{h})\Big), \quad (4.4.66)$$

$$\frac{1}{r^{n-2}}\slashed{\nabla}_3\mathcal{A}_8 = -\frac{n-4}{8(n-2)^2}\Omega\mathrm{S\slashed{c}al}(\slashed{g})(\mathrm{Tr}_{\slashed{g}}\slashed{h})^2 + \frac{n-4}{2(n-2)^2}\mathrm{Tr}_{\slashed{g}}\chi\mathrm{Tr}_{\slashed{g}}\slashed{h}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}). \quad (4.4.67)$$

Therefore, denoting

$$\mathfrak{F}^3 = (\mathfrak{J}^T)^3 - \frac{1}{r^{n-2}}\slashed{\nabla}_4\mathcal{A}, \quad (4.4.68)$$

$$\mathfrak{F}^4 = (\mathfrak{J}^T)^4 + \frac{1}{r^{n-2}}\slashed{\nabla}_3\mathcal{A}, \quad (4.4.69)$$

gives

$$\begin{aligned}\mathfrak{F}^3 \equiv{}& \frac{\Omega}{2}|\overset{(1)}{\hat{\chi}}|^2 + \Omega|\overset{(1)}{\underline{\eta}}|^2 - \frac{1}{\Omega}\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) + \Big(\frac{n-4}{(n-2)}\mathrm{Tr}_{\slashed{g}}\chi + \frac{2\omega}{\Omega}\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) \quad (4.4.70)\\ &- \frac{(n-3)}{2\Omega(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)^2 + \frac{\Omega}{4}\langle\hat{\slashed{h}}, \overset{(1)}{\widehat{\mathrm{R\slashed{i}c}}}(\slashed{g})\rangle - \frac{(n-4)\Omega}{8(n-2)}\overset{(1)}{\mathrm{S\slashed{c}al}}(\slashed{g})\mathrm{Tr}_{\slashed{g}}\slashed{h},\\ \mathfrak{F}^4 \equiv{}& \frac{\Omega}{2}|\overset{(1)}{\underline{\hat{\chi}}}|^2 + \Omega|\overset{(1)}{\eta}|^2 - \frac{1}{\Omega}\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - \Big(\frac{n-4}{n-2}\mathrm{Tr}_{\slashed{g}}\chi + \frac{2\omega}{\Omega}\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) \quad (4.4.71)\\ &- \frac{n-3}{2\Omega(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})^2 + \frac{\Omega}{4}\langle\hat{\slashed{h}}, \overset{(1)}{\widehat{\mathrm{R\slashed{i}c}}}(\slashed{g})\rangle - \frac{\Omega(n-4)}{8(n-2)}\overset{(1)}{\mathrm{S\slashed{c}al}}(\slashed{g})\mathrm{Tr}_{\slashed{g}}\slashed{h},\end{aligned}$$

where the linearised Gauss relations of proposition 2.10.15 have been used. □

Sketch of Alternative Proof of Theorem 4.4.1. Let the symbol $\equiv$ denote equality under integration by parts of $\mathbb{S}^{n-2}_{u,v}$. The direct approach involves using propositions 2.10.7-2.10.20 to show that

$$\begin{aligned}0 \equiv{}& \partial_v\Big(r^{n-2}\Big[\Omega^2|\overset{(1)}{\underline{\hat{\chi}}}|^2 + 2\Omega^2|\overset{(1)}{\eta}|^2 - \Big(\frac{2(n-4)}{(n-2)}\Omega\mathrm{Tr}_{\slashed{g}}\chi + 4\omega\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) \quad (4.4.72)\\ &- 2\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi) - \frac{(n-3)}{(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi})^2 + \frac{\Omega^2}{2}\langle\hat{\slashed{h}}, \overset{(1)}{\widehat{\mathrm{R\slashed{i}c}}}\rangle - \frac{\Omega^2(n-4)}{4(n-2)}\overset{(1)}{\mathrm{S\slashed{c}al}}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]\Big)\\ &+ \partial_u\Big(r^{n-2}\Big[\Omega^2|\overset{(1)}{\hat{\chi}}|^2 + 2\Omega^2|\overset{(1)}{\underline{\eta}}|^2 + \Big(\frac{2(n-4)}{(n-2)}\Omega\mathrm{Tr}_{\slashed{g}}\chi + 4\omega\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)\\ &- 2\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\underline{\chi}) - \frac{(n-3)}{(n-2)}(\Omega\overset{(1)}{\mathrm{Tr}}_{\slashed{g}}\chi)^2 + \frac{\Omega^2}{2}\langle\hat{\slashed{h}}, \overset{(1)}{\widehat{\mathrm{R\slashed{i}c}}}\rangle - \frac{(n-4)\Omega^2}{4(n-2)}\overset{(1)}{\mathrm{S\slashed{c}al}}\mathrm{Tr}_{\slashed{g}}\slashed{h}\Big]\Big).\end{aligned}$$

Whilst many of the linearised null structure equations and linearised Bianchi equations in propositions 2.10.7-2.10.20 are required to show this, the corollary 2.10.18 of the linearised Codazzi equations and the constraint in proposition 2.10.22 are key to this direct approach. As corollary 2.10.18 relates $\widehat{\operatorname{d\!/iv}\overset{(1)}{\nu}}$ and $\widehat{\operatorname{d\!/iv}\overset{(1)}{\underline{\nu}}}$ to $\Delta\!\!\!/\overset{(1)}{\hat{\chi}}$ and $\Delta\!\!\!/\overset{(1)}{\hat{\underline{\chi}}}$ respectively, this allows one to show, after using propositions 2.10.7, 2.10.8, 2.10.9, 2.10.10, 2.10.20, significant amounts of algebra and integration by parts on $\mathbb{S}^{n-2}_{u,v}$, that

$$T\Big(\frac{1}{2}\langle \hat{h\!\!/}, \overset{(1)}{\widehat{\mathrm{R\!\!/ic}}}\rangle - \frac{n-4}{4(n-2)}\overset{(1)}{\mathrm{S\!\!\!/cal}}\mathrm{Tr}_{g\!\!/}h\!\!/\Big) \equiv 2\langle \overset{(1)}{\hat{\chi}} + \overset{(1)}{\hat{\underline{\chi}}}, \overset{(1)}{\widehat{\mathrm{R\!\!/ic}}}\rangle - \frac{n-4}{n-2}\frac{\overset{(1)}{\mathrm{S\!\!\!/cal}}}{\Omega}\Big((\Omega \overset{(1)}{\mathrm{Tr}_{g\!\!/}\chi}) + (\Omega \overset{(1)}{\mathrm{Tr}_{g\!\!/}\underline{\chi}})\Big), \tag{4.4.73}$$

where $T \doteq \partial_t = \frac{\Omega}{2}(e_3 + e_4)$ and $\equiv$ denotes equality under integration over $\mathbb{S}^{n-2}_{u,v}$. One requires the constraint in proposition 2.10.22 here to compute that

$$\langle \mathcal{D\!\!\!/}^{\star}_2 \overset{(1)}{b}, \overset{(1)}{\widehat{\mathrm{R\!\!/ic}}}\rangle \equiv -\frac{n-4}{2(n-2)}\overset{(1)}{\mathrm{S\!\!\!/cal}}\,\mathrm{d\!\!/iv}\overset{(1)}{b}. \tag{4.4.74}$$

At this point, the equation (4.4.72) can be shown fairly straight forwardly by computing directly with propositions 2.10.7-2.10.17 and equation (4.4.73). □

Appendix A

Appendix for Chapter 1

A.1 Christoffel and Riemann Tensor Components for the $\mathrm{Schw}_4 \times \mathbb{R}$

To compute $\Box_g h_{ab}$ one requires the Christoffel symbols and the Riemann tensor components; the non-zero Christoffel symbols are listed below:

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{M(r-2M)}{r^3}, & \Gamma^t_{tr} &= \frac{M}{r(r-2M)}, \\
\Gamma^r_{rr} &= \frac{-M}{r(r-2M)}, & \Gamma^\theta_{r\theta} &= \frac{1}{r} = \Gamma^\varphi_{r\varphi}, \\
\Gamma^r_{\theta\theta} &= (2M-r), & \Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta, \\
\Gamma^r_{\varphi\varphi} &= (2M-r)\sin^2\theta, & \Gamma^\varphi_{\theta\varphi} &= \cot\theta.
\end{aligned} \tag{A.1.1}$$

The others are obtained from symmetry of lower indices. Note, $R^z{}_{\mu\alpha\beta} = R^\mu{}_{z\alpha\beta} = R^\mu{}_{\alpha z\beta} = R^\mu{}_{\alpha\beta z} = 0$. So the Riemann tensor components that are relevant are the ones with spacetime indices $\mu \in \{0, ..., 3\}$ which are just the usual Schwarzschild Riemann tensor components; the non-zero ones are listed below for completeness,

$$\begin{aligned}
R^t{}_{rtr} &= \frac{2M}{r^2(r-2M)}, & R^t{}_{\theta t\theta} &= -\frac{M}{r}, & R^t{}_{\varphi t\varphi} &= -\frac{M\sin^2\theta}{r}, \\
R^r{}_{trt} &= -\frac{2M(r-2M)}{r^4}, & R^r{}_{\theta r\theta} &= -\frac{M}{r}, & R^r{}_{\varphi r\varphi} &= -\frac{M\sin^2\theta}{r}, \\
R^\theta{}_{t\theta t} &= \frac{M(r-2M)}{r^4}, & R^\theta{}_{r\theta r} &= -\frac{M}{r^2(r-2M)}, & R^\theta{}_{\varphi\theta\varphi} &= \frac{2M\sin^2\theta}{r}, \\
R^\varphi{}_{t\varphi t} &= \frac{M(r-2M)}{r^4}, & R^\varphi{}_{r\varphi r} &= -\frac{M}{r^2(r-2M)}, & R^\varphi{}_{\theta\varphi\theta} &= \frac{2M}{r}.
\end{aligned} \tag{A.1.2}$$

Any others can be found from the $R^a{}_{b(cd)} = 0$ symmetry.

A.2 Singularities in Second Order ODE

This section is heavily based on the book of Olver [97]. In particular, see chapter 5 sections 4 and 5 and chapter 7 section 2.

Definition A.2.1 (Ordinary Point/Regular Singularity/Irregular Singularity)**.** *Let p and q be meromorphic functions on a subset of $\mathbb{C}$. Consider the linear 2^{nd} order ODE*

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0. \tag{A.2.1}$$

Then $z_0 \in \mathbb{C}$ is an ordinary point of this differential equation if both $p(z)$ and $q(z)$ are analytic there. If z_0 is not an ordinary point and both

$$(z - z_0)p(z) \quad \textit{and} \quad (z - z_0)^2 q(z) \tag{A.2.2}$$

are analytic at z_0 then z_0 is a regular singularity, otherwise z_0 is an irregular singularity.

Remark A.2.1. *The singular behavior of $z = \infty$ is determined by making the change of variables $\tilde{z} = \frac{1}{z}$ in the ODE (A.2.1). This case will be considered explicitly in section A.2.2.*

In the following, general results for ODE are presented.

A.2.1 Regular Singularities

In this thesis solutions of a second order ODE in a neighbourhood $|z - z_0| < r$ of a regular singular point are required. The classical method is to search for a convergent series solution in such a neighbourhood.

Definition A.2.2 (Indicial Equation)**.** *Let p and q be meromorphic functions on a subset of $\mathbb{C}$. Consider the following 2^{nd}-order ODE with a regular singularity at $z_0 \in \mathbb{C}$*

$$\frac{d^2 f}{dz^2}(z) + p(z)\frac{df}{dz}(z) + q(z)f(z) = 0. \tag{A.2.3}$$

Assume that there exist a convergent power series,

$$(z - z_0)p(z) = \sum_{j=0}^{\infty} p_j (z - z_0)^j, \qquad (z - z_0)^2 q(z) = \sum_{j=0}^{\infty} q_j (z - z_0)^j \qquad \forall |z - z_0| < r. \tag{A.2.4}$$

The indicial equation is defined as

$$I(\alpha) \doteq \alpha(\alpha - 1) + p_0\alpha + q_0 = 0. \tag{A.2.5}$$

Remark A.2.2. *The indicial equation arises by considering the a solution of the form* $f(z) = (z - z_0)^\alpha$ *to the ODE*

$$\frac{d^2 f}{dz^2}(z) + \frac{p_0}{z - z_0}\frac{df}{dz}(z) + \frac{q_0}{(z - z_0)^2} f(z) = 0. \tag{A.2.6}$$

The ODE (A.2.6) is the leading order approximation of the ODE (A.2.3). The function $f(z) = (z - z_0)^\alpha$ *solves the ODE (A.2.6) if the* α *satisfies the indicial equation.*

The following two theorems deal with the asymptotic behaviour of solutions in the neighbourhood of a regular singularity.

Theorem A.2.3 (Frobenius)**.** *Let* p *and* q *be meromorphic functions on a subset of* $\mathbb{C}$*. Consider the following* 2^{nd}*-order ODE with a regular singularity at* $z_0 \in \mathbb{C}$

$$\frac{d^2 f}{dz^2}(z) + p(z)\frac{df}{dz}(z) + q(z) f(z) = 0, \tag{A.2.7}$$

where

$$(z - z_0)p(z) = \sum_{j=0}^{\infty} p_j (z - z_0)^j, \qquad (z - z_0)^2 q(z) = \sum_{j=0}^{\infty} q_j (z - z_0)^j \tag{A.2.8}$$

converge for all $|z - z_0| < r$*, where* $r > 0$*. Let* $\alpha_\pm$ *be the two roots of the indicial equation. Suppose further that* $\alpha_- \neq \alpha_+ + s$*, where* $s \in \mathbb{Z}$*. Then there exists a basis of solution to the ODE (A.2.7) of the form*

$$f^+(z) = (z - z_0)^{\alpha_+} \sum_{j=0}^{\infty} a_j^+ (z - z_0)^j, \qquad f^-(z) = (z - z_0)^{\alpha_-} \sum_{j=0}^{\infty} a_j^- (z - z_0)^j \tag{A.2.9}$$

where these series converge for all z *such that* $|z - z_0| < r$*. Moreover,* a_j^+ *and* a_j^- *can be calculated recursively by the formula*

$$I(\alpha_\pm + j)a_j^\pm + (1 - \delta_{j,0}) \sum_{s=0}^{j-1} ((\alpha_\pm + s)p_{j-s} + q_{j-s})a_s^\pm = 0. \tag{A.2.10}$$

Remark A.2.4. *If the roots of the indicial equation do not differ by an integer then theorem A.2.3 gives a basis of solutions for the ODE in a neighbourhood of the singular point. Equation (A.2.10) determines the coefficients of the series expansion recursively from an arbitrarily assigned* $a_0 \neq 0$*, which can be taken to be* 1*. This process runs into difficulty if, and only if, the two roots differ by a positive integer. To see this, let* α_+ *be the root of the indicial equation with largest real part, the other root is then* $\alpha_+ - N$ *for some* $N \in \mathbb{Z}_+$*. Then since* $I((\alpha_+ - N) + N) = 0$ *one cannot determine* a_N *via equation (A.2.10) for this power series. In this case, one solution can be found with the above method by taking the root of the indicial equation with largest real part.*

The following theorem investigates the case where the roots differ by an integer. Let α_+ be the root of the indicial equation with largest real part, the other root is then $\alpha_+ - N$ for some $N \in \mathbb{Z}_+ \cup \{0\}$.

Theorem A.2.5. *Consider the ODE (A.2.7) as in theorem A.2.3 again satisfying (A.2.8). Let α_+ and $\alpha_- = \alpha_+ - N$, with $N \in \mathbb{Z}_+ \cup \{0\}$, be roots of the indicial equation. Then there exists a basis of solutions of the form*

$$f^+(z) = (z - z_0)^{\alpha_+} \sum_{j=0}^{\infty} a_j^+ (z - z_0)^j, \tag{A.2.11}$$

$$f^-(z) = (z - z_0)^{\gamma} \sum_{j=0}^{\infty} a_j^+ (z - z_0)^j + C_N f^+(z) \ln(z - z_0) \tag{A.2.12}$$

with $\gamma = \alpha_+ + 1$ if $N = 0$ and $\gamma = \beta_-$ if $N \neq 0$, where these power series are convergent for all z such that $|z - z_0| < r$. Moreover, the coefficients a_j^+, a_j^- and C_N can be calculated recursively.

Remark A.2.6. *If $N = 0$ then $C_0 \neq 0$. However, if $N > 0$, it can occur that $C_N = 0$. This happens if, and only if, the second term of equation (A.2.10) vanishes for $j = N$. At this point one can take $a_N^- = 0$ to construct another convergent series solution with no logarithmic singularity. See section 5.2 of chapter 5 of [97] for more detail.*

A.2.2 Irregular Singularities

This section summaries the key result for constructing a basis of solutions to the ODE (1.3.7) associated to $r \to \infty$. (The results presented can in fact be applied to any irregular singular point of an ODE (A.2.1) since without loss of generality, the irregular singularity can be assumed to be at infinity after a change of coordinates.) The following definition makes precise the notion of a irregular singularity at infinity.

Definition A.2.3 (Irregular Singularity at Infinity)**.** *Let p and q be meromorphic functions on a subset of $\mathbb{C}$ which includes the set $\{z \in \mathbb{C} : |z| > a\}$. Consider the following 2^{nd}-order ODE*

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z) f = 0. \tag{A.2.13}$$

Assume for $|z| > a$, p and q may be expanded as convergent power series

$$p(z) = \sum_{n=0}^{\infty} \frac{p_n}{z^n}, \qquad q(z) = \sum_{n=0}^{\infty} \frac{q_n}{z^n} \tag{A.2.14}$$

The ODE (A.2.13) has an irregular singular point at infinity if one of p_0, q_0 and q_1 do not vanish.

The main theorem A.2.7 of this section can be motivated by the following discussion. Consider a formal power series

$$w = e^{\lambda z} z^{\mu} \sum_{n=0}^{\infty} \frac{a_n}{z^n}. \tag{A.2.15}$$

Substituting the expansions into the ODE and equating coefficients yields

$$\lambda^2 + p_0\lambda + q_0 = 0 \tag{A.2.16}$$
$$(p_0 + 2\lambda)\mu = -(p_1\lambda + q_1) \tag{A.2.17}$$

and

$$(p_0 + 2\lambda)na_n = (n-\mu)(n-1-\mu)a_{n-1} + \sum_{j=1}^{n}(\lambda p_{j+1} + q_{j+1} - (j-n-\mu)p_j)a_{n-j}. \tag{A.2.18}$$

Now, equation (A.2.16) has two roots

$$\lambda_\pm = \frac{1}{2}\Big(-p_0 \pm \sqrt{p_0^2 - 4q_0}\Big). \tag{A.2.19}$$

These give rise to

$$\mu_\pm = -\frac{p_1\lambda_\pm + q_1}{p_0 + 2\lambda_\pm}. \tag{A.2.20}$$

The two values of a_0, $a_0^\pm$ can be, without loss of generality, set to 1 and the higher order coefficients determined iteratively from equation (A.2.18) unless one is in the exceptional case where $p_0^2 = 4q_0$ (for further information on this case see section 1.3 of chapter 7 in [97]). The issue that arises is that in most cases the formal series solution (A.2.15) does not converge. However, the following theorem characterises when (A.2.15) provides an asymptotic expansion for the solution for sufficently large $|z|$.

Theorem A.2.7. *Let $p(z)$ and $q(z)$ be meromorphic functions with convergent series expansions*

$$p(z) = \sum_{n=0}^{\infty} \frac{p_n}{z^n}, \qquad q(z) = \sum_{n=0}^{\infty} \frac{q_n}{z^n} \tag{A.2.21}$$

for $|z| > a$ with $p_0^2 \neq 4q_0$. Then the second order ODE

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0 \tag{A.2.22}$$

has unique solutions $f^{\pm}(z)$, such that in the regions

$$\begin{cases} \{|z| > a\} \cap \{|\mathrm{Arg}((\lambda_- - \lambda_+)z)| \leq \pi\} & (\textit{for } f^+) \\ \{|z| > a\} \cap \{|\mathrm{Arg}((\lambda_+ - \lambda_-)z)| \leq \pi\} & (\textit{for } f^-) \end{cases} \tag{A.2.23}$$

of the complex plane, $f^{\pm}$ is holomorphic, where $\lambda_{\pm}$ and $\mu_{\pm}$ are defined in equations (A.2.19) and (A.2.20). Moreover, for all $N > 1$, $f^{\pm}(z)$ satisfies

$$f^{\pm}(z) = e^{\lambda_{\pm} z} z^{\mu_{\pm}} \Big(\sum_{n=0}^{N-1} \frac{a_n^{\pm}}{z^n} + \mathcal{O}\Big(\frac{1}{z^N}\Big)\Big) \tag{A.2.24}$$

in the regions given in equation (A.2.23).

A.3 Tranformation to Schrödinger Form

Proposition A.3.1. *Consider the second order homogeneous linear ODE*

$$\frac{d^2u}{dr^2} + p(r)\frac{du}{dr} + q(r)u = 0, \qquad p, q \in C^1(I), \qquad I \subset \mathbb{R}. \tag{A.3.1}$$

Suppose that there exists a sufficiently regular coordinate transformation $s(r)$ and a function $w(r)$ such that

$$\frac{dw}{dr} + \frac{1}{2}\Big(\frac{1}{(\frac{ds}{dr})}\frac{d^2s}{dr^2} + p\Big)w = 0. \tag{A.3.2}$$

Then the ODE (A.3.1) can be reduced to the form

$$-\frac{d^2z}{ds^2}(s) + V(s)z(s) = 0, \tag{A.3.3}$$

with

$$V(s) = \frac{1}{2(\frac{ds}{dr})^2}\Big(\frac{dp}{dr} - \frac{3}{2(\frac{ds}{dr})^2}\Big(\frac{d^2s}{dr^2}\Big)^2 + \frac{1}{(\frac{ds}{dr})}\frac{d^3s}{dr^3} + \frac{p^2}{2} - 2q\Big). \tag{A.3.4}$$

Proof. The proof is a straight-forward calculation. Take $u(s) = w(s)z(s)$, then

$$\Big(\frac{ds}{dr}\Big)^2 w\frac{d^2z}{ds^2} + \Big(2\Big(\frac{ds}{dr}\Big)^2\frac{dw}{ds} + w\frac{d^2s}{dr^2} + pw\frac{ds}{dr}\Big)\frac{dz}{ds} + \bar{q}z = 0 \tag{A.3.5}$$

where

$$\bar{q} \doteq \Big(\Big(\frac{ds}{dr}\Big)^2\frac{d^2w}{ds^2} + \frac{dw}{ds}\frac{d^2s}{dr^2} + p\frac{dw}{ds}\frac{ds}{dr} + qw\Big). \tag{A.3.6}$$

To reduce this to symmetric form one can set

$$2\Big(\frac{ds}{dr}\Big)^2\frac{dw}{ds} + w\frac{d^2s}{dr^2} + pw\frac{ds}{dr} = 0, \tag{A.3.7}$$

which is equivalent to $w(r)$ satisfying

$$\frac{dw}{dr} + \frac{1}{2}\Big(\frac{1}{(\frac{ds}{dr})}\frac{d^2s}{dr^2} + p\Big)w = 0. \tag{A.3.8}$$

Hence,

$$\frac{d^2w}{dr^2} = -\frac{1}{2}\Big(\frac{df}{dr} - \frac{1}{(\frac{ds}{dr})^2}\Big(\frac{d^2s}{dr^2}\Big)^2 + \frac{1}{(\frac{ds}{dr})}\frac{d^3s}{dr^3} - \frac{1}{2}\Big(\frac{1}{(\frac{ds}{dr})}\frac{d^2s}{dr^2} + p\Big)^2\Big)w. \tag{A.3.9}$$

Notice $\bar{q}$ in the ODE for z reduces to

$$\bar{q} = \frac{d^2w}{dr^2} + p\frac{dw}{dr} + qw. \tag{A.3.10}$$

Reducing this with the expressions for the derivatives of w gives the potential for $-\frac{d^2z}{ds^2} + V(s)z = 0$ as

$$V(s) = \frac{1}{2(\frac{ds}{dr})^2}\Big(\frac{dp}{dr} - \frac{3}{2(\frac{ds}{dr})^2}\Big(\frac{d^2s}{dr^2}\Big)^2 + \frac{1}{(\frac{ds}{dr})}\frac{d^3s}{dr^3} + \frac{p^2}{2} - 2q\Big). \tag{A.3.11}$$

□

Remark A.3.2. *Applying this to* $s = r_*(r) = r + 2M \log |r - 2M|$ *gives*

$$V(r(r_*)) = \frac{(r-2M)^2}{2r^2}\Big(\frac{df}{dr} + \frac{2M(2r-3M)}{r^2(r-2M)^2} + \frac{p^2}{2} - 2q\Big). \tag{A.3.12}$$

A.4 Useful Results From Analysis

A.4.1 Sobolev Embedding

Theorem A.4.1 (Local Compactness of the H^s Sobolev Injection)**.** *Let $d \geq 1$, $s > 0$ and*

$$p_c = \begin{cases} \frac{2d}{d-2s} & s < \frac{d}{2} \\ \infty & \textit{otherwise.} \end{cases} \tag{A.4.1}$$

Then the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^d)$ is compact $\forall 1 \leq p < p_c$. In otherwords, for $(f_n)_n \subset H^s(\mathbb{R}^d)$ bounded, there exists $f \in H^s(\mathbb{R}^d)$ and a subsequence $(f_{n_m})_m$ such that

$$f_{n_m} \rightharpoonup f \qquad H^s(\mathbb{R}^d), \tag{A.4.2}$$

$$f_{n_m} \to f \qquad L^p_{\text{loc}}(\mathbb{R}^d) \quad \forall 1 \leq p < p_c. \tag{A.4.3}$$

Proof. This result can be found in any text on Sobolev spaces, for example Brezis [135]. □

A.4.2 The Multiplication Operator is Compact from H^s to L^2

Proposition A.4.2. *Let $q \in C^0(\mathbb{R}^n, \mathbb{R})$ with $\lim_{|x|\to\infty} q(x) = 0$ and $s > 0$. Then $M_q : u \to qu$ is a compact operator from $H^s(\mathbb{R}^n, \mathbb{R})$ to $L^2(\mathbb{R}^n, \mathbb{R})$.*

Proof. The function q is continuous and decays, hence it is bounded. Let $\epsilon > 0$, then, by assumption, $\exists R > 0$ such that

$$|q(x)| \leq \epsilon \qquad \text{if } |x| \geq R. \tag{A.4.4}$$

Define, $\chi_R : \mathbb{R} \to \mathbb{R}$ smooth by

$$\chi_R(x) = \begin{cases} 1 & |x| \leq R \\ 0 & |x| \geq R + 1. \end{cases} \tag{A.4.5}$$

Let $(f_n)_n \subset H^s(\mathbb{R}^n, \mathbb{R})$ be bounded, so local compactness of the Sobolev embedding (theorem A.4.1) gives weak convergence in $H^s(\mathbb{R}^n, \mathbb{R})$ and strong convergence in $L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$ up to a subsequence. Let the limit be $f \in H^s(\mathbb{R}^n, \mathbb{R})$. Therefore,

$$||\chi_R q f_{m_n} - \chi_R q f||^2_{L^2(\mathbb{R}^n)} = ||\chi_R q f_{m_n} - \chi_R q f||^2_{L^2(B_{R+1}(0))} \tag{A.4.6}$$

$$\leq C \sup_{x\in\mathbb{R}} |q(x)|^2 ||f_{m_n} - f||^2_{L^2(B_{R+1}(0))} \leq \epsilon^2. \tag{A.4.7}$$

Further, consider the set $S_R \doteq \{\chi_R q f : f \in H^s(\mathbb{R}^n, \mathbb{R}), ||f||_{H^s(\mathbb{R}^n)} \leq 1\}$. Then

$$||(1 - \chi_R) q f||_{L^2(\mathbb{R}^n)} \leq \epsilon^2 ||f||_{L^2(\mathbb{R}^n)} \leq \epsilon^2. \tag{A.4.8}$$

Hence, S_∞ is within an ϵ-neighbourhood of S_R, which is compact, therefore S_∞ is compact. By the characterisation of compactness through weak convergence, $qf_m \to qf$ in $L^2(\mathbb{R}^n, \mathbb{R})$ up to a subsequence. □

A.4.3 A Regularity Result

Theorem A.4.3 (Regularity for the Schrödinger Equation)**.** *Let $u \in H^1(\mathbb{R})$ be a weak solution of the equation $(-\Delta + V)u = \lambda u$ where V is a measurable function and $\lambda \in \mathbb{C}$. Then, if $V \in C^\infty(\Omega)$ with $\Omega \subset \mathbb{R}$ open, not necessarily bounded, then $u \in C^\infty(\Omega)$ also.*

Proof. Reed and Simon volume II page 55 [136]. Note one can argue this from standard elliptic regularity results and Sobolev embeddings. In this thesis, only the one-dimensional case of this is applied, which is completely elementary. □

A.5 A Result on Stability in Spherical Gauge

This section contains a few technical results on where the instability may lie in frequency space. This helped guide the search for a suitable test function and the subsequent instability.

Proposition A.5.1. *Consider the quartic polynomial*

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e. \tag{A.5.1}$$

Let Δ denote its discriminant and define

$$\Delta_0 = 64a^3e - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4. \tag{A.5.2}$$

If $\Delta < 0$, then $P(x)$ has two distinct real roots and two complex conjugate roots with non-zero imaginary part. If $\Delta > 0$ and $\Delta_0 > 0$, then there are two pairs of complex conjugate roots with non-zero imaginary part.

Proof. See reference [137]. □

Proposition A.5.2 (Regions of Stability in Frequency Space)**.** *Let $\mu > 0$ and $k \neq 0$. There does not exist a solution $\mathfrak{h}$ of the ODE (1.3.7) with $c_1 = 0$, $k_2 = 0$ and $\hat{k} \in \mathbb{R} \setminus (-\sqrt{2}, \sqrt{2})$ or $\hat{\mu} \geq \frac{3}{8}\sqrt{\frac{3}{2}}$.*

Proof. From proposition 1.3.15, the admissible boundary conditions for the solution are $\mathfrak{h}(r) = k_1 \mathfrak{h}^{2M,+}(r)$ at the future event horizon and $\mathfrak{h}(r) = c_2 \mathfrak{h}_z^{\infty,-}(r)$ at spacelike infinity. Without loss of generality, take $k_1 > 0$. Now, since the solution must decay exponentially towards infinity, there must be maxima $a \in (1, \infty)$. At such a point, one has

$$\frac{d^2\mathfrak{h}}{dr^2}(a) = \frac{a(\hat{\mu}^2 a + \hat{k}^2(\hat{\mu}^2 a^4 - 2a + 2) + \hat{k}^4 a^3(a-1))}{(\hat{k}^2 a^3 + 1)(a-1)^2}\mathfrak{h}(a), \tag{A.5.3}$$

with $\mathfrak{h}(a) > 0$. To derive a contradition, one must have

$$\frac{a(\hat{\mu}^2 a + \hat{k}^2(\hat{\mu}^2 a^4 - 2a + 2) + \hat{k}^4 a^3(a-1))}{(\hat{k}^2 a^3 + 1)(a-1)^2} > 0. \tag{A.5.4}$$

A sufficient condition for the numerator to be positive is

$$\hat{\mu}^2 a^4 - 2a + 2 \geq 0. \tag{A.5.5}$$

This has discriminant

$$\Delta = 16\hat{\mu}^4(128\hat{\mu}^2 - 27), \qquad \Delta_0 = 128\hat{\mu}^2. \tag{A.5.6}$$

Hence, if $\hat{\mu}^2 > \frac{27}{128}$, then there are no real roots. Thus, because the polynomial is positive at a point, say $a = 1$, it is positive everywhere. If $\Delta = 0$, there is a double real root and two complex conjugate roots. The real roots can only occur at a stationary point of the polynomial and therefore the polynomial cannot be negative anywhere. Since all other terms in the numerator are positive, the prefactor of $\mathfrak{h}$ also is. Hence, there can be no solution with the conditions $k_2 = 0$ and $c_1 = 0$ if $\hat{\mu} \geq \frac{3}{8}\sqrt{\frac{3}{2}}$.

Another sufficient condition for positivity of the numerator is

$$\hat{k}^2 a^3 - 2 \geq 0. \tag{A.5.7}$$

This polynomial has a single real root at $a = (\frac{2}{\hat{k}^2})^{\frac{1}{3}}$. For positivity on $a \in (1, \infty)$, one requires $\frac{2}{\hat{k}^2} \leq 1$ or $\hat{k}^2 \geq 2$. Note that if $\hat{\mu} = 0$ then this is precisely the polynomial that governs positivity. Hence, this bound for $\hat{k}$ is sharp. □

Remark A.5.3. *By an almost identical argument one can make the bound for $\hat{\mu}$ even sharper and show that $\hat{\mu} < \frac{1}{4}$ and $\hat{\mu} \leq \sqrt{2}|\hat{k}|$.*

Appendix B

Appendix for Chapter 2

B.1 Derivation of the Null Structure Equations

In this section the results of section 2.6 are derived. Further, there is a useful trick when deriving most of these equations. Namely, one can always swap e_3 to e_4 in the equations which results in changes from 'barred' to 'unbarred' quantities and visa versa. One should note that a few quantities also pick up a sign change. These are recorded here:

$$\beta \mapsto -\underline{\beta}, \tag{B.1.1}$$

$$\underline{\beta} \mapsto -\beta, \tag{B.1.2}$$

$$\sigma \mapsto -\sigma, \tag{B.1.3}$$

$$\zeta \mapsto -\zeta. \tag{B.1.4}$$

The general strategy for deriving these equations is as follows: denote $\check{\Gamma}_{A_1 \ldots A_p}$ $(p \leq 2)$ as a member of

$$\Big\{\chi, \underline{\chi}, \eta, \underline{\eta}, \omega, \underline{\omega}\Big\}. \tag{B.1.5}$$

One then computes as follows

$$(\nabla\!\!\!/_4 \check{\Gamma})_{A_1 \ldots A_p} = e_4(\check{\Gamma}_{A_1 \ldots A_p}) - \check{\Gamma}(\nabla\!\!\!/_4 e_1, \ldots, e_p) - \ldots - \check{\Gamma}(e_1, \ldots, \nabla\!\!\!/_4 e_p). \tag{B.1.6}$$

Now by the definition 2.2.1 one can express any $\check{\Gamma}_{A_1 \ldots A_p}$ in terms of $g(\nabla_{e_\alpha} e_\beta, e_\gamma)$ for appropriate $\alpha, \beta, \gamma, \in \{3, 4, A\}$. So the first term on the right-hand side of (B.1.6) can be expressed in the form

$$e_4(\check{\Gamma}_{A_1 \ldots A_p}) = g(\nabla_4 \nabla_{e_\alpha} e_\beta, e_\gamma) + g(\nabla_{e_\alpha} e_\beta, \nabla_4 e_\gamma), \tag{B.1.7}$$

for appropriate $\alpha, \beta, \gamma, \in \{3, 4, A\}$. One can then use the definition of the Riemann curvature tensor,

$$R(X,Y)Z \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{B.1.8}$$

to write the first term of the right-hand side of equation (B.1.7) as a

$$g(\nabla_4 \nabla_{e_\alpha} e_\beta, e_\gamma) = g(R(e_4, e_\alpha)e_\beta, e_\gamma) + g(\nabla_{e_\alpha} \nabla_4 e_\beta, e_\gamma) + g(\nabla_{[e_4, e_\alpha]} e_\beta, e_\gamma). \tag{B.1.9}$$

One can then compute $[e_4, e_\alpha]$ using that proposition 2.2.3 and the torsion-free condition on the Levi-Civita connection

$$[e_4, e_\alpha] = \nabla_4 e_\alpha - \nabla_\alpha e_4. \tag{B.1.10}$$

The first term of equation (B.1.9) can be decomposed using definition 2.3.1 and the second can also be computed via proposition 2.2.3.

One more general remark before turning to proofs, its is very useful to spot quickly the instances when $g(e_a, e_b)$ vanishes for the double null frame:

$$g_{33} = g_{44} = g_{3A} = g_{4A} = 0. \tag{B.1.11}$$

Proof of proposition 2.6.2. Start by writing

$$(\nabla\!\!\!/_4 \chi)_{AB} = e_4(\chi_{AB}) - \chi(\nabla\!\!\!/_4 e_A, e_B) - \chi(e_A, \nabla\!\!\!/_4 e_B). \tag{B.1.12}$$

Using the definition of χ (see definition 2.2.1) and propositions 2.2.3 and 2.5.1 one has

$$(\nabla\!\!\!/_4 \chi)_{AB} = g(\nabla_4 \nabla_A e_4, e_B) - \chi(\nabla\!\!\!/_4 e_A, e_B) \tag{B.1.13}$$

$$= g(R(e_4, e_A)e_4 + \nabla_A \nabla_4 e_4 + \nabla_{[e_4, e_A]} e_4, e_B) - \chi(\nabla\!\!\!/_4 e_A, e_B), \tag{B.1.14}$$

where one uses the definition of the Riemann tensor in equation (B.1.8). Using propositions 2.2.3 and the definition 2.3.1 of the curvature components along with $[e_4, e_A] = \nabla_4 e_A - \nabla_A e_4$ one has

$$(\nabla\!\!\!/_4 \chi)_{AB} = -\alpha_{AB} + g(\hat{\omega} \nabla_A e_4 + \nabla_{\nabla\!\!\!/_4 e_A + \underline{\eta}_A e_4 - \chi_A^C e_C - \zeta_A e_4} e_4, e_B) - \chi(\nabla\!\!\!/_4 e_A, e_B), \tag{B.1.15}$$

which simplifies to

$$(\nabla\!\!\!/_4 \chi)_{AB} = -\alpha_{AB} + \hat{\omega} \chi_{AB} - \chi_A^C \chi_{BC}. \tag{B.1.16}$$

Taking the trace and the trace-free parts whilst noting proposition 2.3.2 gives the results. □

Proof of proposition 2.6.3. Start by writing

$$(\nabla\!\!\!/_4\eta)_A = e_4(\eta_A) - \eta(\nabla\!\!\!/_4 e_A). \tag{B.1.17}$$

Using the definition of η (see definition 2.2.1) and propositions 2.2.3 and 2.5.1 one has

$$(\nabla\!\!\!/_4\eta)_A = \frac{1}{2}g(\nabla_4\nabla_3 e_4, e_A) = \frac{1}{2}g(R(e_4,e_3)e_4 + \nabla_3\nabla_4 e_4 + \nabla_{[e_4,e_3]}e_4, e_A), \tag{B.1.18}$$

where one uses the definition of the Riemann tensor in equation (B.1.8). Using propositions 2.2.3 and the definition 2.3.1 of the curvature components one has

$$(\nabla\!\!\!/_4\eta)_A = -\beta_A + \frac{1}{2}g(e_A, \nabla_3(\hat{\omega}e_4)) + \frac{1}{2}g(e_A, \nabla_{\nabla_4 e_3}e_4) - \frac{1}{2}g(e_A, \nabla_{\nabla_3 e_4}e_4) \tag{B.1.19}$$

$$= -\beta_A + \chi_{AB}(\underline{\eta} - \eta)^B. \tag{B.1.20}$$

□

Proof of proposition 2.6.4 and 2.6.6. Using the strategy of the previous two proofs one can show

$$(\nabla\!\!\!/_4\underline{\chi})_{AB} = 2(\nabla_A\eta)_B - \hat{\omega}\underline{\chi}_{AB} - R_{3B4A} + 2(\underline{\eta}\otimes\underline{\eta})_{AB} - (\chi\times\underline{\chi})_{AB}. \tag{B.1.21}$$

Antisymmetrising gives the results in proposition 2.6.4. Symmetrising gives

$$(\nabla\!\!\!/_4\underline{\chi})_{AB} = 2(\nabla_{(A}\eta)_{B)} - \hat{\omega}\underline{\chi}_{AB} - R_{3(B|4|A)} + 2(\underline{\eta}\otimes\underline{\eta})_{AB} - \frac{1}{2}(\chi\times\underline{\chi} + \underline{\chi}\times\chi)_{AB}.$$

Taking the trace and trace-free parts whilst noting proposition 2.3.2 gives the results in proposition 2.6.6. □

Proof of proposition 2.6.5. Using the strategy outlined in the above proofs. One has

$$\nabla_3\omega = \underline{\hat{\omega}}\omega - \frac{\Omega}{2}g(\nabla_3\nabla_4 e_4, e_3) - \frac{\Omega}{2}g(\nabla_4 e_4, \nabla_3 e_3). \tag{B.1.22}$$

Using the definition of the Riemann tensor and proposition 2.2.3 gives

$$\nabla_3\omega = 2\underline{\hat{\omega}}\omega - \frac{\Omega}{2}g(R(e_3,e_4)e_4 + \nabla_{2(\eta-\underline{\eta})^A e_A + \hat{\omega}e_3 - \underline{\hat{\omega}}e_4}e_4 + \nabla_4\nabla_3 e_4, e_3). \tag{B.1.23}$$

Repeated application of proposition 2.2.3 and definition 2.3.1 gives

$$\nabla_3\omega = -2\Omega\rho + 4\Omega\langle\eta,\underline{\eta}\rangle - \Omega|\eta|^2 - \Omega|\underline{\eta}|^2 - \hat{\omega}\underline{\omega} - \Omega\nabla_4\underline{\hat{\omega}} \tag{B.1.24}$$

which can be rearranged to show

$$2\nabla_3\omega = 4\Omega\langle\eta,\underline{\eta}\rangle - 2\Omega\rho - \Omega|\eta|^2 - \Omega|\underline{\eta}|^2 - [e_4,e_3](\Omega). \tag{B.1.25}$$

By noting $[e_4, e_3] = -2(\eta - \underline{\eta})^A e_A - \hat{\omega} e_3 + \hat{\underline{\omega}} e_4$ and proposition 2.6.1 gives the result. □

Before proving the Gauss constraint equations, the following definition is useful.

Definition B.1.1 (Second Fundamental Form). *Let $p \in \mathcal{S}$ where $\mathcal{S}$ is a closed embedded codimension-2 submanifold of a Lorentzian manifold M. Then the second fundamental form $\mathrm{II}_p : T_p\mathcal{S} \times T_p\mathcal{S} \to (T_p\mathcal{S})^\perp$ of $\mathcal{S}$ at p is defined as*

$$\mathrm{II}_p(X, Y) = (\nabla_X Y)^\perp, \qquad \forall X, Y \in T_p\mathcal{S}. \tag{B.1.26}$$

Remark B.1.1. *At $p \in \mathcal{S}$ one can decompose the covariant derivative as follows*

$$\begin{aligned}(\nabla_X Y)_p &= (\nabla\!\!\!/_X Y)_p - \frac{1}{2} g_p(\nabla_X Y, e_4) e_3 - \frac{1}{2} g_p(\nabla_X Y, e_3) e_4 &\text{(B.1.27)}\\ &= (\nabla\!\!\!/_X Y)_p + \mathrm{II}_p(X, Y), &\text{(B.1.28)}\end{aligned}$$

where one extends II is extended to T_pM by zero on (e_3, e_4). Therefore,

$$\begin{aligned}\mathrm{II}_p(X, Y) &= -\frac{1}{2} g_p(\nabla_X Y, e_4) e_3 - \frac{1}{2} g_p(\nabla_X Y, e_3) e_4 &\text{(B.1.29)}\\ &= \frac{1}{2} g_p(Y, \nabla_X e_4) e_3 + \frac{1}{2} g_p(Y, \nabla_X e_3) e_4 &\text{(B.1.30)}\\ &= \frac{1}{2}\chi(X, Y) e_3 + \frac{1}{2}\underline{\chi}(X, Y) e_4. &\text{(B.1.31)}\end{aligned}$$

This leads one to refer to χ and $\underline{\chi}$ as the null second fundamental forms of $S_{u,v}$.

Proof of proposition 2.6.7. Let $R\!\!\!/$ denote the induced Riemann curvature of $\mathcal{S}_{u,v}$. Let $X, Y, W, Z \in T\mathcal{S}_{u,v}$ then note that

$$\begin{aligned}R\!\!\!/(X, Y)Z &= \nabla\!\!\!/_X \nabla\!\!\!/_Y Z - \nabla\!\!\!/_Y \nabla\!\!\!/_X Z - \nabla\!\!\!/_{[X,Y]} Z &\text{(B.1.32)}\\ \nabla\!\!\!/_X Y &= \nabla_X Y - \mathrm{II}(X, Y). &\text{(B.1.33)}\end{aligned}$$

Further

$$\begin{aligned}g(R(X, Y)Z, W) &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W) &\text{(B.1.34)}\\ &= g(\nabla_X \nabla\!\!\!/_Y Z + \nabla_X(\mathrm{II}(Y, Z)) - \nabla_Y \nabla\!\!\!/_X Z - \nabla_Y(\mathrm{II}(X, Z)), W) &\text{(B.1.35)}\\ &\quad - g(\nabla\!\!\!/_{[X,Y]} Z, W),\end{aligned}$$

where one recalls that $\mathrm{II} : T\mathcal{S} \times \mathcal{S} \to (T\mathcal{S})^\perp$. Therefore,

$$\begin{aligned}g(R(X, Y)Z, W) &= g(\nabla\!\!\!/_X \nabla\!\!\!/_Y Z - \nabla_Y \nabla\!\!\!/_X Z - \nabla\!\!\!/_{[X,Y]} Z, W) + g(\mathrm{II}(Y, Z), \nabla_X W) &\text{(B.1.36)}\\ &\quad - g(\mathrm{II}(X, Z), \nabla_Y W)\\ &= g(R\!\!\!/(X, Y)Z, W) + g(\mathrm{II}(Y, Z), \mathrm{II}(X, W)) - g(\mathrm{II}(X, Z), \mathrm{II}(Y, W)).\end{aligned}$$

Recall that

$$\mathrm{II}(X,Y) = \frac{1}{2}\chi(X,Y)e_3 + \frac{1}{2}\underline{\chi}(X,Y)e_4. \tag{B.1.37}$$

So,

$$\begin{aligned} g(R(X,Y)Z,W) = g(\not{R}(X,Y)Z,W) &- \frac{1}{2}\chi(Y,Z)\underline{\chi}(X,W) - \frac{1}{2}\underline{\chi}(Y,Z)\chi(X,W) \\ &+ \frac{1}{2}\chi(X,Z)\underline{\chi}(Y,W) + \frac{1}{2}\underline{\chi}(X,Z)\chi(Y,W). \end{aligned} \tag{B.1.38}$$

Hence, one has the first Gauss constraint equation of proposition 2.6.7:

$$\not{R}_{ABCD} = R_{ABCD} - \frac{1}{2}\chi_{DB}\underline{\chi}_{CA} - \frac{1}{2}\underline{\chi}_{DB}\chi_{CA} + \frac{1}{2}\chi_{CB}\underline{\chi}_{DA} + \frac{1}{2}\underline{\chi}_{CB}\chi_{DA}. \tag{B.1.39}$$

One can trace this once and then twice to get the other two Gauss constraints whilst noting proposition 2.3.2whilst noting proposition 2.3.2. □

Before proving the Codazzi constraint equations, the following definition is useful.

Definition B.1.2 (Normal Connection)**.** *The normal connection of a submanifold $\mathcal{S} \subset M$ is a map $\nabla^\perp : T\mathcal{S} \times (T\mathcal{S})^\perp \to (T\mathcal{S})^\perp$ given by*

$$\nabla^\perp_X Y = \mathrm{nor}(\nabla_X Y) \qquad \forall X \in T\mathcal{S}, Y \in (T\mathcal{S})^\perp \tag{B.1.40}$$

where $\mathrm{nor}(Z)$ denotes the normal projection of Z onto $(T\mathcal{S})^\perp$.

Remark B.1.2. *The following properties are immediate from the definition*

$$\nabla^\perp_{fX+gY} Z = f\nabla^\perp_X Z + g\nabla^\perp_Y Z, \qquad \forall f \in C^\infty(M), X, Y \in T\mathcal{S}, Z \in (T\mathcal{S})^\perp. \tag{B.1.41}$$

$$\nabla^\perp_X fY = X(f)Y + f\nabla^\perp_X Y, \qquad \forall f \in C^\infty(M), X \in T\mathcal{S}, Y \in (T\mathcal{S})^\perp \tag{B.1.42}$$

$$X(g(Y,Z)) = g(\nabla^\perp_X Y, Z) + g(Y, \nabla^\perp_X Z), \qquad \forall X \in T\mathcal{S}, Y \in (T\mathcal{S})^\perp, Z \in (T\mathcal{S})^\perp. \tag{B.1.43}$$

One can extend this to general normal tensor fields on $\mathcal{S}$. In particular, for the second fundamental form one has

$$(\nabla^\perp_X \mathrm{II})(Y,Z) = \nabla^\perp_X(\mathrm{II}(Y,Z)) - \mathrm{II}(\not{\nabla}_X Y, Z) - \mathrm{II}(Y, \not{\nabla}_X Z). \tag{B.1.44}$$

Proof of proposition 2.6.8. Let $X, Y, Z \in T\mathcal{S}_{u,v}$. Taking the normal projection of the Riemann tensor gives

$$\mathrm{nor}(R(X,Y)Z) = \mathrm{nor}(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z). \tag{B.1.45}$$

Note that

$$\mathrm{nor}(\nabla_X \nabla_Y Z) = \mathrm{nor}(\nabla\!\!\!/_X \nabla\!\!\!/_Y Z + \mathrm{II}(X, \nabla\!\!\!/_Y Z) + \nabla_X(\mathrm{II}(Y,Z))) \tag{B.1.46}$$

$$= \mathrm{II}(X, \nabla\!\!\!/_Y Z) + \nabla^{\perp}_X(\mathrm{II}(Y,Z)) \tag{B.1.47}$$

$$\mathrm{nor}(\nabla_{[X,Y]} Z) = \mathrm{II}(\nabla\!\!\!/_X Y, Z) - \mathrm{II}(\nabla\!\!\!/_Y X, Z). \tag{B.1.48}$$

Hence,

$$\begin{aligned}\mathrm{nor}(R(X,Y)Z) &= \mathrm{II}(X, \nabla\!\!\!/_Y Z) + \nabla^{\perp}_X(\mathrm{II}(Y,Z)) - \mathrm{II}(Y, \nabla\!\!\!/_X Z) - \nabla^{\perp}_Y(\mathrm{II}(X,Z)) \qquad (B.1.49)\\ &\quad - \mathrm{II}(\nabla\!\!\!/_X Y, Z) + \mathrm{II}(\nabla\!\!\!/_Y X, Z)\\ &= (\nabla^{\perp}_X \mathrm{II})(Y,Z) - (\nabla^{\perp}_Y \mathrm{II})(X,Z). \qquad (B.1.50)\end{aligned}$$

Now

$$\begin{aligned}(\nabla^{\perp}_X \mathrm{II})(Y,Z) &= \nabla^{\perp}_X(\mathrm{II}(Y,Z)) - \mathrm{II}(\nabla\!\!\!/_X Y, Z) - \mathrm{II}(Y, \nabla\!\!\!/_X Z) \qquad (B.1.51)\\ &= \frac{1}{2}\nabla^{\perp}_X(\chi(Y,Z)e_3 + \underline{\chi}(Y,Z)e_4) - \chi(\nabla\!\!\!/_X Y, Z)e_3 - \underline{\chi}(\nabla\!\!\!/_X Y, Z)e_4 \qquad (B.1.52)\\ &\quad - \chi(Y, \nabla\!\!\!/_X Z)e_3 - \underline{\chi}(Y, \nabla\!\!\!/_X Z)e_4\\ &= \frac{1}{2}X(\chi(Y,Z))e_3 + \frac{1}{2}X(\underline{\chi}(Y,Z))e_4 + \frac{1}{2}\chi(Y,Z)\nabla^{\perp}_X e_3 \qquad (B.1.53)\\ &\quad + \frac{1}{2}\underline{\chi}(Y,Z)\nabla^{\perp}_X e_4 - \frac{1}{2}\chi(\nabla\!\!\!/_X Y, Z)e_3 - \frac{1}{2}\underline{\chi}(\nabla\!\!\!/_X Y, Z)e_4\\ &\quad - \frac{1}{2}\chi(Y, \nabla\!\!\!/_X Z)e_3 - \frac{1}{2}\underline{\chi}(Y, \nabla\!\!\!/_X Z)e_4,\end{aligned}$$

since $\chi(X,Y) \in C^{\infty}(\mathcal{S})$. Hence,

$$\begin{aligned}(\nabla^{\perp}_X \mathrm{II})(Y,Z) &= \frac{1}{2}\nabla\!\!\!/_X(\chi(Y,Z))e_3 + \frac{1}{2}\nabla\!\!\!/_X(\underline{\chi}(Y,Z))e_4 + \frac{1}{2}\chi(Y,Z)\nabla^{\perp}_X e_3 \qquad (B.1.54)\\ &\quad + \frac{1}{2}\underline{\chi}(Y,Z)\nabla^{\perp}_X e_4 - \frac{1}{2}\chi(\nabla\!\!\!/_X Y, Z)e_3\\ &\quad - \frac{1}{2}\underline{\chi}(\nabla\!\!\!/_X Y, Z)e_4 - \frac{1}{2}\chi(Y, \nabla\!\!\!/_X Z)e_3 - \frac{1}{2}\underline{\chi}(Y, \nabla\!\!\!/_X Z)e_4\\ &= \frac{1}{2}(\nabla\!\!\!/_X \chi)(Y,Z)e_3 + \frac{1}{2}(\nabla\!\!\!/_X \underline{\chi})(Y,Z)e_4 + \frac{1}{2}\chi(Y,Z)\nabla^{\perp}_X e_3 \qquad (B.1.55)\\ &\quad + \frac{1}{2}\underline{\chi}(Y,Z)\nabla^{\perp}_X e_4.\end{aligned}$$

Now,

$$\nabla^{\perp}_X e_3 = -\frac{1}{2}g(\nabla_X e_3, e_3)e_4 - \frac{1}{2}g(\nabla_X e_3, e_4)e_3 = \zeta(X)e_3, \tag{B.1.56}$$

$$\nabla^{\perp}_X e_4 = -\zeta(X)e_4. \tag{B.1.57}$$

So,

$$(\nabla^{\perp}_{X}\mathrm{II})(Y,Z) = \frac{1}{2}(\nabla\!\!\!/_{X}\chi)(Y,Z)e_3 + \frac{1}{2}(\nabla\!\!\!/_{X}\underline{\chi})(Y,Z)e_4 + \frac{1}{2}\chi(Y,Z)\zeta(X)e_3 - \frac{1}{2}\underline{\chi}(Y,Z)\zeta(X)e_4. \tag{B.1.58}$$

Hence,

$$\begin{aligned}\mathrm{nor}(R(X,Y)Z) = {}& \frac{1}{2}(\nabla\!\!\!/_{X}\chi)(Y,Z)e_3 + \frac{1}{2}(\nabla\!\!\!/_{X}\underline{\chi})(Y,Z)e_4 + \frac{1}{2}\chi(Y,Z)\zeta(X)e_3 \\ & - \frac{1}{2}\underline{\chi}(Y,Z)\zeta(X)e_4 - \frac{1}{2}(\nabla\!\!\!/_{Y}\chi)(X,Z)e_3 - \frac{1}{2}(\nabla\!\!\!/_{Y}\underline{\chi})(X,Z)e_4 \\ & - \frac{1}{2}\chi(X,Z)\zeta(Y)e_3 + \frac{1}{2}\underline{\chi}(X,Z)\zeta(Y)e_4.\end{aligned} \tag{B.1.59}$$

Therefore, since

$$g(\mathrm{nor}(R(X,Y)Z),e_3) = g(R(X,Y)Z,e_3), \tag{B.1.60}$$

$$g(\mathrm{nor}(R(X,Y)Z),e_4) = g(R(X,Y)Z,e_4), \tag{B.1.61}$$

one has

$$g(R(X,Y)Z,e_4) = -(\nabla\!\!\!/_{X}\chi)(Y,Z) - \chi(Y,Z)\zeta(X) + (\nabla\!\!\!/_{Y}\chi)(X,Z) + \chi(X,Z)\zeta(Y), \tag{B.1.62}$$

$$g(R(X,Y)Z,e_3) = -(\nabla\!\!\!/_{X}\underline{\chi})(Y,Z) + \underline{\chi}(Y,Z)\zeta(X) + (\nabla\!\!\!/_{Y}\underline{\chi})(X,Z) - \underline{\chi}(X,Z)\zeta(Y). \tag{B.1.63}$$

Alternatively in indices

$$R_{4CAB} = -2(\nabla\!\!\!/_{[A}\chi)_{B]C} - \chi_{BC}\zeta_A + \chi_{AC}\zeta_B, \tag{B.1.64}$$

$$R_{3CAB} = -2(\nabla\!\!\!/_{[A}\underline{\chi})_{B]C} + \underline{\chi}_{BC}\zeta_A - \underline{\chi}_{AC}\zeta_B. \tag{B.1.65}$$

Using that $R_{4CAB} = -R_{ABC4}$ (and similarly for R_{3CAB}), the definitions of the curvature components 2.3.1 and decomposing χ into its trace and trace-free part gives the result. To get the equations for $(\mathrm{d}\!\!\!/\mathrm{iv}\hat{\chi}, \mathrm{d}\!\!\!/\mathrm{iv}\hat{\underline{\chi}})$ one simply traces over the (B,C) indices above and uses proposition 2.3.2. □

B.2 The Bianchi Identities in Double Null Gauge

Proof of proposition 2.7.1. One should note that this proof assumes that (M,g) satisfies the vacuum Einstein equation (I.2).

The general strategy is as follows: denote $\check{\mathrm{W}}_{A_1\ldots A_p}$ $(p \leq 4)$ as a member of

$$\Big\{\beta, \underline{\beta}, \tau, \sigma, \alpha, \underline{\alpha}, \nu, \underline{\nu}, R_{ABCD}\Big\}. \tag{B.2.1}$$

The reader should note that if one computes an equation for τ then tracing gives the relevant equation for ρ from proposition 2.3.2. One then computes as follows

$$(\nabla\!\!\!/_4 \check{\mathrm{W}})_{A_1\ldots A_p} = e_4(\check{\mathrm{W}}_{A_1\ldots A_p}) - \check{\mathrm{W}}(\nabla\!\!\!/_4 e_1, ..., e_p) - ... - \check{\mathrm{W}}(e_1, ..., \nabla\!\!\!/_4 e_p). \tag{B.2.2}$$

Now by the definition 2.3.1 one can express any $\check{\mathrm{W}}_{A_1\ldots A_p}$ in terms of $R_{\alpha\beta\gamma\delta}$ for appropriate $\alpha, \beta, \gamma, \delta \in \{3, 4, A\}$. So the first term on the right-hand side of (B.2.2) can be expressed in the form

$$e_4(\check{\mathrm{W}}_{A_1\ldots A_p}) = (\nabla_4 R)_{\alpha\beta\gamma\delta} + R(\nabla_4 e_\alpha, e_\beta, e_\gamma, e_\delta) + ... + R(e_\alpha, e_\beta, e_\gamma, \nabla_4 e_\delta), \tag{B.2.3}$$

for appropriate $\alpha, \beta, \gamma, \delta \in \{3, 4, A\}$. Now the first term on the right-hand side of this equation (B.2.3) can be manipulated via either the second Bianchi identity

$$(\nabla_a R)_{bcde} + (\nabla_d R)_{bcea} + (\nabla_e R)_{bcad} = 0, \tag{B.2.4}$$

or its first contracted version (assuming the vacuum Einstein equation (I.2))

$$(\mathrm{div} R)_{abc} = 0. \tag{B.2.5}$$

One then decomposes either identity with proposition 2.2.3 and the definition 2.3.1 by writing

$$(\nabla_\alpha R)_{\beta\gamma\delta\mu} = e_\alpha(R_{\beta\gamma\delta\gamma}) - R(\nabla_\alpha e_\beta, e_\gamma, e_\delta, e_\mu) - ... - R(e_\beta, e_\gamma, e_\delta, \nabla_\alpha e_\mu), \tag{B.2.6}$$

for appropriate $\alpha, \beta, \gamma, \delta \in \{3, 4, A\}$. The other terms on the right-hand side of equation (B.2.3) can also be decomposed using proposition 2.2.3 and the definition 2.3.1 of the curvature components.

The equation for $\nabla\!\!\!/_3 R_{ABCD}$: Using the outlined strategy one has

$$\nabla\!\!\!/_3 R_{ABCD} = \nabla_3 R_{ABCD} - \eta_A \underline{\nu}_{CDB} + \eta_B \underline{\nu}_{CDA} - \eta_C \underline{\nu}_{ABD} + \eta_D \underline{\nu}_{ABC}. \tag{B.2.7}$$

Note the Bianchi identity

$$\nabla_3 R_{ABCD} = \nabla_D R_{ABC3} - \nabla_C R_{ABD3} \tag{B.2.8}$$

and compute

$$\nabla_C R_{ABD3} = \nabla\!\!\!/_C \underline{\nu}_{ABD} + \chi_{C[A}\underline{\alpha}_{B]D} + \underline{\chi}_{C[A}(\tau_{B]D} - \sigma_{B]D}) - \underline{\chi}_C^E R_{ABDE} - \zeta_C \underline{\nu}_{ABD} + \underline{\chi}_{CD}\sigma_{AB}. \tag{B.2.9}$$

Antisymmetrise over (C, D) and plug back in to give the stated equation:

$$\nabla\!\!\!/_3 R_{ABCD} = 2(\zeta - \eta)_{[C}\underline{\nu}_{|AB|D]} - 2\eta_{[A}\underline{\nu}_{|CD|B]} - 2\nabla\!\!\!/_{[C}\underline{\nu}_{|AB|D]} + 2\underline{\chi}_{[C}^E R_{|AB|D]E} - \chi_{A[C}\underline{\alpha}_{D]B} + \chi_{B[C}\underline{\alpha}_{D]A} - \underline{\chi}_{A[C}(\tau_{D]B} + \sigma_{D]B}) + \underline{\chi}_{B[C}(\tau_{D]A} + \sigma_{D]A}). \tag{B.2.10}$$

The equation for $\nabla\!\!\!/_3\hat{\tau}$ and $\nabla\!\!\!/_3\rho$: By expressing (via proposition 2.3.2 under the assumption of the vacuum Einstein equation (I.2))

$$\tau_{AB} = g\!\!\!/^{CD} R_{CADB}, \tag{B.2.11}$$

one can use the equation (B.2.11) for $\nabla\!\!\!/_3 R_{ABCD}$ in conjunction with proposition 2.3.2 to give

$$(\nabla\!\!\!/_3\tau)_{AB} = -2\eta^D \underline{\nu}_{D(BA)} + 2(\eta \otimes \underline{\beta})_{(AB)} + \underline{\chi}_{[A}^C \sigma_{B]C} + \underline{\chi}_{[A}^C \tau_{B]C} + (\nabla\!\!\!/_B\underline{\beta})_A + (\underline{\alpha} \times \chi)_{(AB)} + \underline{\chi}_{AB}\rho - (\underline{\beta} \otimes \zeta)_{AB} - (\mathrm{d}\!\!\!/\mathrm{iv}\underline{\nu})_{AB} + \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})(\sigma_{AB} - \tau_{AB}) - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\chi)\underline{\alpha}_{AB} + \underline{\chi}^{CE} R_{CABE} + \zeta^C \underline{\nu}_{CAB}. \tag{B.2.12}$$

Tracing this equation and using proposition 2.3.2 gives the equation for $\nabla\!\!\!/_3\rho$ and taking the symmetric trace-free part gives the equation for $\nabla\!\!\!/_4\hat{\tau}$. Antisymmetrising gives the constraint

$$(\mathrm{d}\!\!\!/\mathrm{iv}\underline{\nu})_{[AB]} = \underline{\chi}_{[A}^C(\sigma + \tau)_{B]C} - \frac{1}{2}(d\!\!\!/\underline{\beta})_{AB} - \frac{1}{2}(\underline{\beta} \wedge \zeta)_{AB} + \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\sigma_{AB} + \zeta^C \underline{\nu}_{C[AB]}. \tag{B.2.13}$$

The equation for $\nabla\!\!\!/_3\underline{\nu}$ and $\nabla\!\!\!/_3\underline{\beta}$: Using the outlined strategy one has

$$(\nabla\!\!\!/_3\underline{\nu})_{ABC} = \nabla_3 R_{C3AB} - \eta_A\underline{\alpha}_{BC} + \eta_B\underline{\alpha}_{AC} + \hat{\underline{\omega}}\underline{\nu}_{ABC}. \tag{B.2.14}$$

The second Bianchi identity gives

$$\nabla_3 R_{C3AB} = \nabla_B R_{C3A3} - \nabla_A R_{C3B3}. \tag{B.2.15}$$

One can compute that

$$\nabla_B R_{C3A3} = \nabla\!\!\!/_B\underline{\alpha}_{CA} + \underline{\chi}_{BC}\underline{\beta}_A + \underline{\chi}_{BA}\underline{\beta}_C - \underline{\chi}_B^D \underline{\nu}_{ADC} - \underline{\chi}_B^D \underline{\nu}_{CDA} - 2\zeta_B\underline{\alpha}_{CA}. \tag{B.2.16}$$

Antisymmetrising on (B, A) and substituting gives the result. Tracing the resulting equation on (B, C) then gives the equation and using proposition 2.3.2 for $\not\nabla_3\underline{\beta}$.

The equation for $\not\nabla_4\underline{\nu}$ and $\not\nabla_4\underline{\beta}$: Proceeding as above one finds

$$(\not\nabla_4\underline{\nu})_{ABC} = 2\underline{\eta}_C\sigma_{BA} + 2(\tau+\sigma)_{C[A}\underline{\eta}_{B]} + 2\underline{\eta}^D R_{ABCD} - \hat{\omega}\underline{\nu}_{ABC} + g(e_A, [\nabla_4 R](e_C, e_3)e_B). \tag{B.2.17}$$

Now,

$$g(e_A, [\nabla_4 R](e_C, e_3)e_B) = g(e_C, [\nabla_4 R](e_A, e_B)e_3) = 2g(e_C, [\nabla_{[A} R](e_4, e_{B]})e_3). \tag{B.2.18}$$

Then

$$\nabla_A R_{C34B} = \chi_{AB}\underline{\beta}_C - \underline{\chi}_{AC}\beta_B - \chi_A^D\underline{\nu}_{DBC} + \underline{\chi}_A^D\nu_{CDB} - (\not\nabla_A(\sigma+\tau))_{CB}. \tag{B.2.19}$$

Hence,

$$2\nabla_{[A}R_{|C34|B]} = -2(\not\nabla_{[A}(\tau-\sigma))_{B]C} - 2\underline{\chi}_{C[A}\beta_{B]} + 2\chi^D_{[A}\underline{\nu}_{B]DC} + 2\nu_{CD[B}\underline{\chi}^D_{A]}. \tag{B.2.20}$$

Combining gives the result. Tracing over (B, C) and using proposition 2.3.2 (in conjunction with the vacuum Einstein equation (I.2)) gives the equation for $\not\nabla_4\underline{\beta}$.

The equation for $\not\nabla_3\alpha$ and $\not\nabla_4\sigma$: By expressing α as in definition 2.3.1 one can use the strategy outlined above to show

$$(\not\nabla_3\alpha)_{AB} = 2(\eta\otimes\beta)_{AB} + g(e_A, [\nabla_3 R](e_B, e_4)e_4) + 2(\beta\otimes\eta)_{AB} - 2\underline{\hat{\omega}}\alpha_{AB} + 2\eta^C(\nu_{ACB} + \nu_{BCA}). \tag{B.2.21}$$

The second Bianchi identities give

$$g(e_A, [\nabla_3 R](e_B, e_4)e_4) + g(e_A, [\nabla_B R](e_4, e_3)e_4) + g(e_A, [\nabla_4 R](e_3, e_B)e_4) = 0. \tag{B.2.22}$$

Using the relations in proposition 2.2.3 and the definition 2.3.1 of curvature one has

$$g(e_A, [\nabla_B R](e_4, e_3)e_4) = 2\rho\chi_{AB} - 2(\not\nabla_B\beta)_A - (\tau\times\chi)_{AB} + (\alpha\times\underline{\chi})_{AB} + 3(\sigma\times\chi)_{AB} - 2(\beta\otimes\zeta)_{AB}. \tag{B.2.23}$$

and

$$g(e_A, [\nabla_4 R](e_3, e_B)e_4) = (\nabla\!\!\!/_4(\sigma - \tau))_{AB} - 2\underline{\eta}^C \nu_{CBA} - 2(\beta \otimes \underline{\eta})_{AB}. \tag{B.2.24}$$

Combining these results gives

$$\begin{aligned}(\nabla\!\!\!/_3\alpha)_{AB} &= 2(\eta \otimes \beta)_{AB} + 2(\beta \otimes \eta)_{AB} - 2\underline{\hat{\omega}}\alpha_{AB} - 4\eta^C \nu_{C(AB)} - (\nabla\!\!\!/_4(\sigma - \tau))_{AB} \\ &+ 2\underline{\eta}^C \nu_{CBA} + 2(\beta \otimes \underline{\eta})_{AB} - 2\rho\chi_{AB} + 2(\nabla\!\!\!/_B\beta)_A + (\tau \times \chi)_{AB} - (\alpha \times \underline{\chi})_{AB} \\ &- 3(\sigma \times \chi)_{AB} + 2(\beta \otimes \zeta)_{AB}.\end{aligned} \tag{B.2.25}$$

Antisymmetrising (B.2.26) gives and using proposition 2.3.2 gives

$$\begin{aligned}(\nabla\!\!\!/_4\sigma)_{AB} &= \underline{\eta}^C \nu_{ABC} + (\beta \wedge (\underline{\eta} + \zeta))_{AB} - (\mathrm{d}\!\!\!/\beta)_{AB} + (\hat{\tau} \wedge \hat{\chi})_{AB} - (\alpha \wedge \hat{\underline{\chi}})_{AB} \\ &- 3(\sigma \wedge \hat{\chi})_{AB} - \frac{3}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\sigma_{AB},\end{aligned} \tag{B.2.26}$$

where one defines $(\sigma \wedge \hat{\chi})_{AB} := (\sigma \times \hat{\chi})_{[AB]}$. Tracing (B.2.26) and using proposition 2.3.2 gives

$$\nabla\!\!\!/_4\rho = \langle 2\underline{\eta} + \zeta, \beta\rangle - \Big(\frac{n-1}{n-2}\Big)\rho\mathrm{Tr}_{g\!\!\!/}\chi + \mathrm{d}\!\!\!/\mathrm{iv}\beta + \frac{1}{2}\langle\hat{\tau}, \hat{\chi}\rangle - \frac{1}{2}\langle\alpha, \hat{\underline{\chi}}\rangle. \tag{B.2.27}$$

Finally, taking the symmetric-traceless part of (B.2.26) gives

$$\begin{aligned}(\nabla\!\!\!/_3\alpha)_{AB} - (\nabla\!\!\!/_4\hat{\tau})_{AB} &= ((2\eta + \zeta)\hat{\otimes}\beta)_{AB} - 4\eta^C\big(\nu_{C(AB)} - \frac{1}{n-2}\beta_C g\!\!\!/_{AB}\big) - 2\underline{\hat{\omega}}\alpha_{AB} \\ &- 2(\mathcal{D}\!\!\!\!/_2^\star\beta)_{AB} + \widehat{(\hat{\tau} \times \hat{\chi})}_{AB} + \frac{1}{n-2}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\tau}_{AB} - 2\rho\frac{(n-1)}{(n-2)}\hat{\chi}_{AB} \\ &- \widehat{(\alpha \times \hat{\underline{\chi}})}_{AB} - \frac{1}{n-2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\alpha_{AB} - 3\widehat{(\sigma \times \hat{\chi})}_{AB} \\ &+ 2\underline{\eta}^C\Big(\nu_{C(BA)} - \frac{1}{n-2}\beta_C g\!\!\!/_{AB}\Big) + (\beta\hat{\otimes}\underline{\eta})_{AB}.\end{aligned} \tag{B.2.28}$$

Plugging in the equation for $\nabla\!\!\!/_4\hat{\tau}$ gives

$$\begin{aligned}(\nabla\!\!\!/_3\alpha)_{AB} &= \Big(\Big(2\eta + \frac{1}{2}\zeta\Big)\hat{\otimes}\beta\Big)_{AB} - 4\eta^C\big(\nu_{C(AB)} - \frac{1}{n-2}\beta_C g\!\!\!/_{AB}\big) - 2\underline{\hat{\omega}}\alpha_{AB} \\ &- (\mathcal{D}\!\!\!\!/_2^\star\beta)_{AB} + \widehat{(\hat{\tau} \times \hat{\chi})}_{AB} - \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\chi)\hat{\tau}_{AB} - \rho\frac{n}{(n-2)}\hat{\chi}_{AB} \\ &- \frac{1}{2}(\mathrm{Tr}_{g\!\!\!/}\underline{\chi})\alpha_{AB} + \hat{\chi}^{CE}R_{CABE} - 3\widehat{(\sigma \times \hat{\chi})}_{AB} - \widehat{(\mathrm{d}\!\!\!/\mathrm{iv}\nu)}_{AB} \\ &- \zeta^C\Big(\nu_{C(AB)} - \frac{1}{n-2}\beta_C g\!\!\!/_{AB}\Big) + \frac{1}{n-2}\langle\hat{\tau}, \hat{\chi}\rangle g\!\!\!/_{AB}.\end{aligned} \tag{B.2.29}$$

□

Appendix C

Appendix for Chapter 3 and 4

C.1 Useful Identities

Proposition C.1.1. *Let (M, g) be a spacetime which satisfies the vacuum Einstein equation (I.2) and let X be a vector field. Then*

$$\nabla_a \nabla_b X_c = \mathrm{K}^X_{abc} + R_{cba}{}^d X_d. \tag{C.1.1}$$

where

$$\mathrm{K}^X_{abc} \doteq \nabla_a \Pi^X_{bc} + \nabla_b \Pi^X_{ac} - \nabla_c \Pi^X_{ab}, \tag{C.1.2}$$

$$\Pi^X_{ab} \doteq \frac{1}{2}(\mathcal{L}_X g)_{ab}. \tag{C.1.3}$$

Proof. First we note that by definition:

$$2\mathrm{K}^X_{abc} = \nabla_a \nabla_b X_c + \nabla_a \nabla_c X_b + \nabla_b \nabla_a X_c + \nabla_b \nabla_c X_a - \nabla_c \nabla_a X_b - \nabla_c \nabla_b X_a. \tag{C.1.4}$$

Further by the Ricci identity,

$$2\mathrm{K}^X_{abc} = 2\nabla_a \nabla_b X_c - R^d{}_{cba} X_d - R^d{}_{bac} X_d - R^d{}_{abc} X_d. \tag{C.1.5}$$

Using the first algebraic Bianchi identity and the symmetries of the Riemann tensor one has

$$\nabla_a \nabla_b X_c = \mathrm{K}^X_{abc} + R_{cba}{}^d X_d. \tag{C.1.6}$$

□

Proposition C.1.2. *Let $T \in \bigotimes_{i=1}^n T^*M$ and X a vector field. Then*

$$\nabla_a (\mathcal{L}_X T)_{b_1 \ldots b_n} = \mathcal{L}_X (\nabla T)_{ab_1 \ldots b_n} + \mathrm{K}^X_{ab_1 c} T^c{}_{b_2 \ldots b_n} + \ldots + \mathrm{K}^X_{ab_n c} T_{b_1 \ldots b_{n-1}}{}^c. \tag{C.1.7}$$

Proof. The proof of this follows from proposition C.1.1 and repeated use of the Ricci identity. □

In the following, it will be established that if h is a solution to the linearised vacuum Einstein equation I.5 and the background spacetime has a Killing field k then $\mathcal{L}_k h$ is also a solution to the linearised vacuum Einstein equation (I.5). First the following lemma is established to simplify the proof.

Lemma C.1.3. *Let (M, g) be a spacetime with a Killing field k. Then*

$$(\mathcal{L}_k R)_{abcd} = 0. \tag{C.1.8}$$

Proof. The formula for the Lie derivative of the Riemann tensor is the following:

$$\begin{aligned}(\mathcal{L}_k R)_{abcd} = k^e(\nabla_e R)_{abcd} + R_{ebcd}(\nabla_a k)^e + R_{aecd}(\nabla_b k)^e + R_{abed}(\nabla_c k)^e \\ + R_{abce}(\nabla_d k)^e.\end{aligned} \tag{C.1.9}$$

Using the second Bianchi identity one has

$$\begin{aligned}(\mathcal{L}_k R)_{abcd} = -k^e(\nabla_c R)_{abde} - k^e(\nabla_d R)_{abec} + R_{ebcd}(\nabla_a k)^e + R_{aecd}(\nabla_b k)^e \\ + R_{abed}(\nabla_c k)^e + R_{abce}(\nabla_d k)^e.\end{aligned} \tag{C.1.10}$$

Now using the antisymmetry of the Riemann tensor in its last two indices one has

$$(\mathcal{L}_k R)_{abcd} = \nabla_c(k^e R_{abed}) + \nabla_d(k^e R_{abce}) + R_{ebcd}(\nabla_a k)^e + R_{aecd}(\nabla_b k)^e. \tag{C.1.11}$$

Now since k is Killing by using the Ricci identity one can establish

$$(\nabla_a \nabla_b k)^c = R^c{}_{bad} k^d. \tag{C.1.12}$$

Therefore,

$$(\mathcal{L}_k R)_{abcd} = -(\nabla_c \nabla_d \nabla_b k)_a + (\nabla_d \nabla_c \nabla_b k)_a + R_{ebcd}(\nabla_a k)^e + R_{aecd}(\nabla_b k)^e, \tag{C.1.13}$$

which vanishes by the Ricci identity and the symmetries of the Riemann tensor. □

Proposition C.1.4. *Suppose h solves the linearised vacuum Einstein equation (I.5) on a vacuum spacetime background (M, g) with Killing vector field k then $\mathcal{L}_k h$ is also a solution to the linearised vacuum Einstein equation (I.5).*

Proof. Using the Ricci identity and

$$\nabla_a \nabla_b k^c = R^c{}_{bad} k^d, \tag{C.1.14}$$

one can establish that for arbitrary tensor ${T_{a_1 \ldots a_m}}^{b_1 \ldots b_n}$ one has

$$\nabla_c (\mathcal{L}_k T)_{a_1 \ldots a_m}{}^{b_1 \ldots b_n} = \mathcal{L}_k \nabla_c T_{a_1 \ldots a_m}{}^{b_1 \ldots b_n}. \tag{C.1.15}$$

Therefore, using $\mathcal{L}_k g^{ab} = 0$, one finds

$$\Delta_L (\mathcal{L}_k h)_{ab} = -2 (\mathcal{L}_k R)_a{}^c{}_b{}^d h_{cd} = 0, \tag{C.1.16}$$

by lemma C.1.3. □

C.2 Details for Proof Theorem 3.1.10

For the enthusiastic reader, here are explicit details for the proof of theorem 3.1.10. Let $\dot{\mathcal{J}} = (\mathcal{J}^T[\mathcal{L}_T h])$, then using equations (3.3.149-3.3.155) and the discussion following them one can compute

$$\begin{aligned}
\dot{\mathcal{J}}^4 \equiv{}& \frac{\Omega^3}{4}|\overset{(1)}{\underline{\alpha}}|^2 + \frac{\Omega^3}{2}|\overset{(1)}{\sigma}|^2 + \Omega^2\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)\langle\overset{(1)}{\underline{\hat{\chi}}},\overset{(1)}{\underline{\alpha}}\rangle + \Omega^3\langle\overset{(1)}{\underline{\eta}},\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\underline{\alpha}}\rangle - \frac{\Omega^2}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\hat{\chi}},\overset{(1)}{\underline{\alpha}}\rangle \\
&+ \Omega\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)^2|\overset{(1)}{\underline{\hat{\chi}}}|^2 + \frac{\Omega}{16}(\Omega\mathrm{Tr}_{\not g}\chi)^2|\overset{(1)}{\hat{\chi}}|^2 - \frac{\Omega}{2}\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\hat{\chi}}},\overset{(1)}{\hat{\chi}}\rangle \\
&- \frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\beta}+\overset{(1)}{\underline{\beta}},\overset{(1)}{\eta}\rangle + 2\Omega^2\Big(\omega - (\Omega\mathrm{Tr}_{\not g}\chi)\Big)\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\underline{\eta}}\rangle + \Omega^3\overset{(1)}{\rho}\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta} + 2\Omega^2\overset{(1)}{\underline{\omega}}\mathrm{d}\!\!\!/\mathrm{iv}(\overset{(1)}{\beta}+\overset{(1)}{\underline{\beta}}) \\
&+ 2\Omega|\nabla\!\!\!/\overset{(1)}{\underline{\omega}}|^2 + \frac{\Omega^3}{2}\Big(|\overset{(1)}{\beta}|^2 + |\overset{(1)}{\underline{\beta}}|^2\Big) + \Omega^3\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\beta}\rangle - \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\underline{\omega}}\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta} - 2\Omega\omega(\overset{(1)}{\omega}+\overset{(1)}{\underline{\omega}})\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\underline{\eta}} \\
&+ \frac{\mathrm{Tr}_{\not g}\chi}{4}\Big((\Omega\mathrm{Tr}_{\not g}\chi) - 2\omega\Big)\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi}) + \frac{1}{\Omega}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)^2 - \frac{3}{2}\omega(\Omega\mathrm{Tr}_{\not g}\chi) + 2\omega^2\Big)\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}}) \\
&- \frac{\omega}{2\Omega}(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi}) - \frac{2\omega}{\Omega}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi) - \omega\Big)\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}}) + \frac{2\omega}{\Omega}(\Omega\mathrm{Tr}_{\not g}\chi)(\overset{(1)}{\omega}+\overset{(1)}{\underline{\omega}})\overset{(1)}{\underline{\omega}} \\
&- \frac{1}{2\Omega}(\Omega\mathrm{Tr}_{\not g}\chi)^2\overset{(1)}{\underline{\omega}}{}^2 - \Big[(\nabla\!\!\!/_3\overset{(1)}{\underline{\omega}})T((\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi})) - (\nabla\!\!\!/_4\overset{(1)}{\underline{\omega}})T((\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}}))\Big] + \frac{3}{2}\Omega^3\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)^2 \\
&+ \Omega\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi}) - \frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}}) \\
&- 2\omega\Omega(\overset{(1)}{\omega}+\overset{(1)}{\underline{\omega}})\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big) + \Omega\Omega\mathrm{Tr}_{\not g}\chi\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)\overset{(1)}{\omega} + \Big(\frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not g}\chi)^2 - \Omega^3\rho\Big)|\overset{(1)}{\eta}|^2 \\
&- 2\Omega^3\rho\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\eta}\rangle + \Big(\frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not g}\chi)^2 + \Omega^3\rho\Big)|\overset{(1)}{\underline{\eta}}|^2 - \frac{\mathrm{Tr}_{\not g}\chi}{4}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi) - \omega\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}})(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi}) \\
&- \frac{1}{32\Omega}(\Omega\mathrm{Tr}_{\not g}\chi)^2(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi})^2 - \frac{1}{2\Omega}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi) - \omega\Big)^2(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}})^2,
\end{aligned} \tag{C.2.1}$$

and

$$
\begin{aligned}
\dot{\mathcal{J}}^3 \equiv\ & \frac{\Omega^3}{4}|\overset{(1)}{\alpha}|^2 + \frac{\Omega^3}{2}|\overset{(1)}{\sigma}|^2 - \Omega^2\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)\langle \overset{(1)}{\hat{\chi}}, \overset{(1)}{\alpha}\rangle + \Omega^3\langle \overset{(1)}{\eta}, \not{\mathrm{div}}\overset{(1)}{\alpha}\rangle + \frac{\Omega^2}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\langle \overset{(1)}{\underline{\hat{\chi}}}, \overset{(1)}{\alpha}\rangle \\
&+ \Omega\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)^2|\overset{(1)}{\hat{\chi}}|^2 + \frac{\Omega}{16}(\Omega\mathrm{Tr}_{\not{g}}\chi)^2|\overset{(1)}{\underline{\hat{\chi}}}|^2 - \frac{\Omega}{2}\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)(\Omega\mathrm{Tr}_{\not{g}}\chi)\langle \overset{(1)}{\underline{\hat{\chi}}}, \overset{(1)}{\hat{\chi}}\rangle \\
&- \frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not{g}}\chi)\langle \overset{(1)}{\beta} + \overset{(1)}{\underline{\beta}}, \overset{(1)}{\underline{\eta}}\rangle + 2\Omega^2\Big(\omega - (\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)\langle \overset{(1)}{\beta}, \overset{(1)}{\eta}\rangle + \Omega^3\overset{(1)}{\rho}\not{\mathrm{div}}\overset{(1)}{\underline{\eta}} - 2\Omega^2\overset{(1)}{\omega}\not{\mathrm{div}}(\overset{(1)}{\beta} + \overset{(1)}{\underline{\beta}}) \\
&+ 2\Omega|\not{\nabla}\overset{(1)}{\omega}|^2 + \frac{\Omega^3}{2}\Big(|\overset{(1)}{\beta}|^2 + |\overset{(1)}{\underline{\beta}}|^2\Big) + \Omega^3\langle \overset{(1)}{\underline{\beta}}, \overset{(1)}{\beta}\rangle + \Omega(\Omega\mathrm{Tr}_{\not{g}}\chi)\overset{(1)}{\omega}\not{\mathrm{div}}\overset{(1)}{\underline{\eta}} + 2\Omega\omega(\overset{(1)}{\underline{\omega}} + \overset{(1)}{\omega})\not{\mathrm{div}}\overset{(1)}{\eta} \\
&+ \frac{\mathrm{Tr}_{\not{g}}\chi}{4}\Big((\Omega\mathrm{Tr}_{\not{g}}\chi) - 2\omega\Big)\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi}) + \frac{1}{\Omega}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)^2 - \frac{3}{2}\omega(\Omega\mathrm{Tr}_{\not{g}}\chi) + 2\omega^2\Big)\overset{(1)}{\omega}(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi) \\
&- \frac{\omega}{2\Omega}(\Omega\mathrm{Tr}_{\not{g}}\chi)\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi}) - \frac{2\omega}{\Omega}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi) - \omega\Big)\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi) + \frac{2\omega}{\Omega}(\Omega\mathrm{Tr}_{\not{g}}\chi)(\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}})\overset{(1)}{\omega} \\
&- \frac{1}{2\Omega}(\Omega\mathrm{Tr}_{\not{g}}\chi)^2\overset{(1)}{\omega}{}^2 - \Big[(\not{\nabla}_4\overset{(1)}{\omega})T((\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi})) - (\not{\nabla}_3\overset{(1)}{\omega})T((\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi))\Big] + \frac{3}{2}\Omega^3\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)^2 \\
&+ 2\omega\Omega(\overset{(1)}{\omega} + \overset{(1)}{\underline{\omega}})\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big) - \Omega\Omega\mathrm{Tr}_{\not{g}}\chi\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)\overset{(1)}{\underline{\omega}} + \Big(\frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not{g}}\chi)^2 + \Omega^3\rho\Big)|\overset{(1)}{\eta}|^2 \\
&- 2\Omega^3\rho\langle \overset{(1)}{\underline{\eta}}, \overset{(1)}{\eta}\rangle + \Big(\frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not{g}}\chi)^2 - \Omega^3\rho\Big)|\overset{(1)}{\underline{\eta}}|^2 - \Omega\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi}) \\
&+ \frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi) - \frac{1}{4\Omega}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi) - \omega\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi})(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi) \\
&- \frac{1}{32\Omega}(\Omega\mathrm{Tr}_{\not{g}}\chi)^2(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi})^2 - \frac{1}{2\Omega}\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi) - \omega\Big)^2(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi)^2, \qquad \text{(C.2.2)}
\end{aligned}
$$

where the Codazzi constraint equations in proposition 2.10.17 have been used. Now define:

$$
\begin{aligned}
&\mathcal{C}_1 \doteq \frac{\Omega^2}{2}\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)|\overset{(1)}{\hat{\chi}}|^2, && \mathcal{C}_2 \doteq \frac{\Omega^2}{2}\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)|\overset{(1)}{\underline{\hat{\chi}}}|^2, \\
&\mathcal{C}_3 \doteq \frac{\Omega^2}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\langle \overset{(1)}{\underline{\hat{\chi}}}, \overset{(1)}{\hat{\chi}}\rangle, && \mathcal{C}_4 \doteq \Omega^3\langle \overset{(1)}{\underline{\eta}}, \overset{(1)}{\underline{\beta}}\rangle, \\
&\mathcal{C}_5 \doteq \Omega^3\langle \overset{(1)}{\eta}, \overset{(1)}{\beta}\rangle, && \mathcal{C}_6 \doteq \overset{(1)}{\underline{\omega}}T((\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi)), \\
&\mathcal{C}_7 \doteq \overset{(1)}{\omega}T((\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi})), && \mathcal{C}_8 \doteq (\Omega\mathrm{Tr}_{\not{g}}\chi)\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}}, \\
&\mathcal{C}_9 \doteq \Omega^2\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho}, && \mathcal{C}_{10} \doteq \Omega^2\omega\langle \overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}\rangle, \\
&\mathcal{C}_{11} \doteq \frac{1}{4}\Omega^2(\Omega\mathrm{Tr}_{\not{g}}\chi)|\overset{(1)}{\eta}|^2, && \mathcal{C}_{12} \doteq \frac{1}{4}\Omega^2(\Omega\mathrm{Tr}_{\not{g}}\chi)|\overset{(1)}{\underline{\eta}}|^2, \\
&\mathcal{C}_{13} \doteq \Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi})^2, && \mathcal{C}_{14} \doteq \Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not{g}}\chi)\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi)^2, \\
&\mathcal{C}_{15} \doteq (\Omega\mathrm{Tr}_{\not{g}}\chi)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi})(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi), && \mathcal{C}_{16} \doteq \Big(2\omega^2 - \frac{3}{2}\Omega^2\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\chi), \\
&\mathcal{C}_{17} \doteq \Big(2\omega^2 - \frac{3}{2}\Omega^2\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}}_{\not{g}}\underline{\chi}), && \mathcal{C}_{18} \doteq \Big[4\Omega^2\omega\rho + \frac{3}{2}\Omega^2(\Omega\mathrm{Tr}_{\not{g}}\chi)\rho\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2.
\end{aligned}
\qquad \text{(C.2.3)}
$$

So, for $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ one uses the linearised shear equation in proposition 2.10.10 and the linearised Codazzi constraints in proposition 2.10.17 to give

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_1) &= \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\eta}\rangle+2\Omega^2\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\beta},\overset{(1)}{\eta}\rangle \\ &\quad+\frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big(3\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)|\overset{(1)}{\hat{\chi}}|^2-\frac{\Omega}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\hat{\chi}},\overset{(1)}{\underline{\hat{\chi}}}\rangle \\ &\quad+\Omega\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big](\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\eta},\end{aligned}\tag{C.2.4}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_1) &= \Omega\Big(2\omega^2-\frac{7}{4}\omega(\Omega\mathrm{Tr}_{\not g}\chi)+\frac{3}{16}(\Omega\mathrm{Tr}_{\not g}\chi)^2\Big)|\overset{(1)}{\hat{\chi}}|^2 \\ &\quad-\Omega^2\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\hat{\chi}},\overset{(1)}{\alpha}\rangle,\end{aligned}\tag{C.2.5}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_2) &= \Omega\Big(\frac{7}{4}\omega(\Omega\mathrm{Tr}_{\not g}\chi)-2\omega^2-\frac{3}{16}(\Omega\mathrm{Tr}_{\not g}\chi)^2\Big)|\overset{(1)}{\underline{\hat{\chi}}}|^2 \\ &\quad-\Omega^2\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\underline{\hat{\chi}}},\overset{(1)}{\underline{\alpha}}\rangle,\end{aligned}\tag{C.2.6}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_2) &= -\Omega(\Omega\mathrm{Tr}_{\not g}\chi)\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\eta}\rangle-2\Omega^2\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\underline{\eta}}\rangle \\ &\quad+\frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big(\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)-3\omega\Big)|\overset{(1)}{\underline{\hat{\chi}}}|^2+\frac{\Omega}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\hat{\chi}},\overset{(1)}{\underline{\hat{\chi}}}\rangle \\ &\quad+\Omega\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big](\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\eta}},\end{aligned}\tag{C.2.7}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_3) &= -\frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)^2|\overset{(1)}{\eta}|^2-\frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\eta}\rangle+\frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\eta} \\ &\quad-\Omega(\Omega\mathrm{Tr}_{\not g}\chi)\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\underline{\hat{\chi}}},\overset{(1)}{\hat{\chi}}\rangle-\frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not g}\chi)^2|\overset{(1)}{\underline{\hat{\chi}}}|^2-\frac{\Omega^2}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\alpha}},\overset{(1)}{\hat{\chi}}\rangle,\end{aligned}\tag{C.2.8}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_3) &= \frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)^2|\overset{(1)}{\underline{\eta}}|^2+\frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\beta}\rangle+\frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\eta}} \\ &\quad+\Omega(\Omega\mathrm{Tr}_{\not g}\chi)\Big[\omega-\frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\langle\overset{(1)}{\underline{\hat{\chi}}},\overset{(1)}{\hat{\chi}}\rangle+\frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not g}\chi)^2|\overset{(1)}{\hat{\chi}}|^2-\frac{\Omega^2}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\alpha},\overset{(1)}{\underline{\hat{\chi}}}\rangle.\end{aligned}\tag{C.2.9}$$

For $\mathcal{C}_4$, $\mathcal{C}_5$ one uses the linearised torsion equations in proposition 2.10.11 and the linearised Bianchi equations in proposition 2.10.20 to show

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_4) &= \Omega^3|\overset{(1)}{\underline{\beta}}|^2-\frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\beta}}\rangle+\Omega^2\Big(\frac{3}{2}(\Omega\mathrm{Tr}_{\not g}\chi)-4\omega\Big)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\underline{\beta}}\rangle \\ &\quad-\Omega^3\langle\overset{(1)}{\underline{\eta}},\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\alpha}}\rangle,\end{aligned}\tag{C.2.10}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_4) &= -\Omega^3|\overset{(1)}{\sigma}|^2-3\Omega^3\rho|\overset{(1)}{\underline{\eta}}|^2+\Omega^3\langle\overset{(1)}{\beta},\overset{(1)}{\underline{\beta}}\rangle-\Omega^2\Big((\Omega\mathrm{Tr}_{\not g}\chi)-2\omega\Big)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\underline{\beta}}\rangle \\ &\quad+2\Omega^2\langle\not\nabla\overset{(1)}{\omega},\overset{(1)}{\underline{\beta}}\rangle+\Omega^3\overset{(1)}{\rho}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\underline{\eta}},\end{aligned}\tag{C.2.11}$$

$$\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_5) = \Omega^3|\overset{(1)}{\sigma}|^2 + 3\Omega^3\rho|\overset{(1)}{\eta}|^2 - \Omega^3\langle\overset{(1)}{\beta},\overset{(1)}{\underline{\beta}}\rangle + \Omega^2\Big((\Omega\mathrm{Tr}_{\not g}\chi) - 2\omega\Big)\langle\overset{(1)}{\eta},\overset{(1)}{\beta}\rangle$$
$$+ 2\Omega^2\langle\not\nabla\overset{(1)}{\underline{\omega}},\overset{(1)}{\beta}\rangle - \Omega^3\overset{(1)}{\rho}\,\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}, \tag{C.2.12}$$

$$\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_5) = -\Omega^3|\overset{(1)}{\beta}|^2 + \Omega^2\Big(4\omega - \frac{3}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big)\langle\overset{(1)}{\beta},\overset{(1)}{\eta}\rangle + \Omega^3\langle\overset{(1)}{\eta},\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\alpha}\rangle$$
$$+ \frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\beta}\rangle. \tag{C.2.13}$$

For $\mathcal{C}_6$ and $\mathcal{C}_7$ one uses propositions 2.10.13 and propositions 2.10.9 and 2.10.8 to show

$$\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_6) \equiv (\not\nabla_3\overset{(1)}{\underline{\omega}})T((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)) + \Omega\Big(2\rho - \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)^2\Big)\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} - 2\Omega^2\overset{(1)}{\underline{\omega}}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\beta}}$$
$$- 2\Omega|\not\nabla\overset{(1)}{\underline{\omega}}|^2 + \Omega\Big(2\rho + \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)^2\Big)\overset{(1)}{\underline{\omega}}^2 - \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\underline{\omega}}\Big(\overset{(1)}{\rho} + 2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)$$
$$- \Omega\rho\overset{(1)}{\underline{\omega}}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) + 2(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\Big), \tag{C.2.14}$$

$$\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_6) = \not\nabla_3\overset{(1)}{\omega}T((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi)) + \frac{1}{2}\rho\overset{(1)}{\underline{\omega}}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) + (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\Big) + 2\Omega\omega\overset{(1)}{\underline{\omega}}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\eta}$$
$$+ \frac{2}{\Omega}\omega^2\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) - 2\rho\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} + \Omega\Big[2\omega - (\Omega\mathrm{Tr}_{\not g}\chi)\Big]\overset{(1)}{\underline{\omega}}\Big(\overset{(1)}{\rho} + 2\Omega\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) + \mathrm{Tr}_{\not g}\chi\overset{(1)}{\underline{\omega}}\partial_v\overset{(1)}{\omega}, \tag{C.2.15}$$

$$\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_7) \equiv \not\nabla_4\overset{(1)}{\underline{\omega}}T((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})) + \frac{1}{2}\rho\overset{(1)}{\omega}\Big((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) + (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\Big) - 2\Omega\omega\overset{(1)}{\omega}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\underline{\eta}}$$
$$+ \frac{2}{\Omega}\omega^2\overset{(1)}{\omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi}) - 2\rho\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} + \Omega\Big[(\Omega\mathrm{Tr}_{\not g}\chi) - 2\omega\Big]\overset{(1)}{\omega}\Big(\overset{(1)}{\rho} + 2\Omega\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big) - \mathrm{Tr}_{\not g}\chi\overset{(1)}{\omega}\partial_u\overset{(1)}{\underline{\omega}}, \tag{C.2.16}$$

$$\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_7) \equiv (\not\nabla_4\overset{(1)}{\omega})T((\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})) + \Omega\Big(2\rho - \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)^2\Big)\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} + 2\Omega^2\overset{(1)}{\omega}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\beta}$$
$$- 2\Omega|\not\nabla\overset{(1)}{\omega}|^2 + \Omega\Big(2\rho + \frac{1}{2}(\mathrm{Tr}_{\not g}\chi)^2\Big)\overset{(1)}{\omega}^2 + \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\omega}\Big(\overset{(1)}{\rho} + 2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)$$
$$- \Omega\rho\overset{(1)}{\omega}\Big(2(\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\chi) + (\Omega\overset{(1)}{\mathrm{Tr}}_{\not g}\underline{\chi})\Big). \tag{C.2.17}$$

Using propositions 2.10.13 one has

$$\frac{1}{r^2}\not\nabla_3(r^2\mathcal{C}_8) = -(\Omega\mathrm{Tr}_{\not g}\chi)\Big[2\omega + \frac{1}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} - \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\underline{\omega}}\Big(\overset{(1)}{\rho} + 2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)$$
$$+ (\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\omega}\not\nabla_3\overset{(1)}{\underline{\omega}}, \tag{C.2.18}$$

$$\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_8) = (\Omega\mathrm{Tr}_{\not g}\chi)\Big[2\omega + \frac{1}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\overset{(1)}{\omega}\overset{(1)}{\underline{\omega}} - \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\omega}\Big(\overset{(1)}{\rho} + 2\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big)$$
$$+ (\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\underline{\omega}}\not\nabla_4\overset{(1)}{\omega}. \tag{C.2.19}$$

Using propositions 2.10.7 and the linearised Bianchi equations in proposition 2.10.20 one can show

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3\Big(r^2\mathcal{C}_9\Big) &= \Omega\omega\overset{(1)}{\underline{\omega}}\overset{(1)}{\rho} - 2\Omega\omega^2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho} - \frac{3}{2}\rho\Omega^3\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho} + \frac{\Omega^2\omega}{2}\langle\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}},\overset{(1)}{\underline{\beta}}\rangle \\ &\quad - \frac{3}{2}\Omega\rho\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}}),\end{aligned} \tag{C.2.20}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_4\Big(r^2\mathcal{C}_9\Big) &= \Omega\omega\overset{(1)}{\omega}\overset{(1)}{\rho} + \Big[2\Omega\omega^2 + \frac{3}{2}\rho\Omega^3\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\rho} - \frac{\Omega^2\omega}{2}\langle\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}},\overset{(1)}{\beta}\rangle \\ &\quad - \frac{3}{2}\Omega\rho\omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\chi}).\end{aligned} \tag{C.2.21}$$

Using the torsion propagation equations of proposition 2.10.11 one has

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3\Big(r^2\mathcal{C}_{10}\Big) &\equiv \Omega^2\omega\langle\overset{(1)}{\underline{\beta}},\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}}\rangle - \frac{\Omega}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\omega|\overset{(1)}{\eta}|^2 - 2\omega\Omega\overset{(1)}{\underline{\omega}}\not{\mathrm{div}}\overset{(1)}{\underline{\eta}} \\ &\quad + \Omega\Big(\frac{3}{2}\omega(\Omega\mathrm{Tr}_{\not g}\chi) - 2\omega^2\Big)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle,\end{aligned} \tag{C.2.22}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_4\Big(r^2\mathcal{C}_{10}\Big) &\equiv \Omega^2\omega\langle\overset{(1)}{\beta},\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}}\rangle + \frac{\Omega}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\omega|\overset{(1)}{\underline{\eta}}|^2 - 2\omega\Omega\overset{(1)}{\omega}\not{\mathrm{div}}\overset{(1)}{\eta} \\ &\quad + \Omega\Big(2\omega^2 - \frac{3}{2}\omega(\Omega\mathrm{Tr}_{\not g}\chi)\Big)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle,\end{aligned} \tag{C.2.23}$$

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3\Big(r^2\mathcal{C}_{11}\Big) &= \Omega(\Omega\mathrm{Tr}_{\not g}\chi)\Big(\frac{3}{8}(\Omega\mathrm{Tr}_{\not g}\chi) - \omega\Big)|\overset{(1)}{\eta}|^2 - \frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\beta}}\rangle \\ &\quad - \Omega^2(\Omega\mathrm{Tr}_{\not g}\chi)\overset{(1)}{\underline{\omega}}\not{\mathrm{div}}\overset{(1)}{\eta},\end{aligned} \tag{C.2.24}$$

$$\frac{1}{r^2}\not\nabla_3\Big(r^2\mathcal{C}_{12}\Big) = \Big[\Omega^3\rho + \frac{\Omega}{8}(\Omega\mathrm{Tr}_{\not g}\chi)^2\Big]|\overset{(1)}{\underline{\eta}}|^2 - \frac{\Omega}{4}(\Omega\mathrm{Tr}_{\not g}\chi)^2\langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle + \frac{\Omega^2}{2}(\Omega\mathrm{Tr}_{\not g}\chi)\langle\overset{(1)}{\underline{\eta}},\overset{(1)}{\underline{\beta}}\rangle, \tag{C.2.25}$$

with similar expressions (resulting from 'barring' these results and accounting for sign changes) for $\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_{11})$ and $\frac{1}{r^2}\not\nabla_4(r^2\mathcal{C}_{12})$. Using the linearised expansion equations and Raychauduri (propositions 2.10.8 and 2.10.9) gives

$$\begin{aligned}\frac{1}{r^2}\not\nabla_3\Big(r^2\mathcal{C}_{13}\Big) &= \frac{1}{\Omega}\Big[\frac{7}{2}\omega(\Omega\mathrm{Tr}_{\not g}\chi) - 4\omega^2 - \frac{3}{8}(\Omega\mathrm{Tr}_{\not g}\chi)^2\Big](\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}})^2 \\ &\quad - 4\mathrm{Tr}_{\not g}\chi\Big[\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{\not g}\chi)\Big]\overset{(1)}{\underline{\omega}}(\overset{(1)}{\Omega\mathrm{Tr}_{\not g}\underline{\chi}}),\end{aligned} \tag{C.2.26}$$

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\Big(r^2\mathcal{C}_{14}\Big) &= \Big[\frac{3}{2}\omega\mathrm{Tr}_{g\!\!\!/}\chi - \frac{1}{8\Omega}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2\Big](\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi)^2 \\ &+ 4\Omega\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi)\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta} \\ &+ 4\Omega\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\Big)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) \\ &- \mathrm{Tr}_{g\!\!\!/}\chi\Big(\omega - \frac{1}{4}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}),\end{aligned} \tag{C.2.27}$$

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\Big(r^2\mathcal{C}_{15}\Big) &= \Big(\frac{(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2}{\Omega} - 4\omega\mathrm{Tr}_{g\!\!\!/}\chi\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi})(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) \\ &- \frac{1}{2\Omega}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi})^2 - \frac{2}{\Omega}(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2\overset{(1)}{\underline{\omega}}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) \\ &+ 2\Omega(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi})\mathrm{d}\!\!\!/\mathrm{iv}\overset{(1)}{\eta} + 2\Omega(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi})\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big),\end{aligned} \tag{C.2.28}$$

with similar expressions (resulting from 'barring' these results and accounting for sign changes) for $\frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C}_{13})$, $\frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C}_{14})$ and $\frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C}_{15})$. Finally, using the propositions 2.10.7, 2.10.8 and 2.10.9 gives

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\Big(r^2\mathcal{C}_{16}\Big) &= -\Big(2\Omega\omega^2 - \frac{3}{2}\Omega^3\rho\Big)\Big(|\overset{(1)}{\eta}|^2 + \langle\overset{(1)}{\eta},\overset{(1)}{\underline{\eta}}\rangle\Big) - \frac{3}{2}\Omega(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\rho\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) \\ &+ \Big(2\omega^2 - \frac{3}{2}\Omega^2\rho\Big)\frac{\overset{(1)}{\underline{\omega}}}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\chi) - \frac{1}{2}\mathrm{Tr}_{g\!\!\!/}\chi\Big(2\omega^2 - \frac{3}{2}\Omega^2\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}) \\ &+ 2\Omega\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\Big(2\omega^2 - \frac{3}{2}\Omega^2\rho\Big)\Big(\overset{(1)}{\rho} + 2\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\rho\Big),\end{aligned} \tag{C.2.29}$$

$$\begin{aligned}\frac{1}{r^2}\nabla\!\!\!\!/_3\Big(r^2\mathcal{C}_{17}\Big) &= \Big(2\omega^2 - \frac{3}{2}\Omega^2\rho\Big)\frac{\overset{(1)}{\underline{\omega}}}{\Omega}(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}) + \Big(4\Omega\omega\rho + 3\mathrm{Tr}_{g\!\!\!/}\chi\Omega^2\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\underline{\omega}} \\ &+ \frac{1}{\Omega}\Big(2\Omega^2\omega\rho - 4\omega^3 - \frac{9}{4}\Omega^2(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\rho\Big)\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)(\Omega\overset{(1)}{\mathrm{Tr}}_{g\!\!\!/}\underline{\chi}),\end{aligned} \tag{C.2.30}$$

$$\frac{1}{r^2}\nabla\!\!\!\!/_3\Big(r^2\mathcal{C}_{18}\Big) = \Big[8\omega\rho\Omega + 3(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)\rho\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)\overset{(1)}{\underline{\omega}} + \Big[\frac{3}{2}\Omega(\Omega\mathrm{Tr}_{g\!\!\!/}\chi)^2\rho - 8\Omega^2\omega^2\rho\Big]\Big(\frac{\overset{(1)}{\Omega}}{\Omega}\Big)^2, \tag{C.2.31}$$

with similar expressions (resulting from 'barring' these results and accounting for sign changes) for $\frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C}_{16})$, $\frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C}_{17})$ and $\frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C}_{18})$. At this point one can check explicitly the computation of $(\mathcal{J}^T[\mathcal{L}_T h])^4 - \frac{1}{r^2}\nabla\!\!\!\!/_3(r^2\mathcal{C})$ and $(\mathcal{J}^T[\mathcal{L}_T h])^3 + \frac{1}{r^2}\nabla\!\!\!\!/_4(r^2\mathcal{C})$. Note that

$$\begin{aligned}\mathcal{C} &= \mathcal{C}_3 + \mathcal{C}_7 + \mathcal{C}_8 + \mathcal{C}_{10} + \mathcal{C}_{11} + \mathcal{C}_{12} + \mathcal{C}_{17} + \frac{1}{4}(\mathcal{C}_{13} + \mathcal{C}_{14}) \\ &- \Big(\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_{16} + \mathcal{C}_{18} + 2\mathcal{C}_9 + \frac{1}{8}\mathcal{C}_{15}\Big).\end{aligned} \tag{C.2.32}$$

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