

An affine model of the dynamics of astrophysical discs

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ABSTRACT

Thin astrophysical discs are very often modelled using the equations of 2D hydrodynamics. We derive an extension of this model that describes more accurately the behaviour of a thin disc in the absence of self-gravity, magnetic fields, and complex internal motions. The ideal fluid theory is derived directly from Hamilton’s Principle for a 3D fluid after making a specific approximation to the deformation gradient tensor. We express the equations in Eulerian form after projection on to a reference plane. The disc is thought of as a set of fluid columns, each of which is capable of a time-dependent affine transformation, consisting of a translation together with a linear transformation in three dimensions. Therefore, in addition to the usual 2D hydrodynamics in the reference plane, the theory allows for a deformation of the mid-plane (as occurs in warped discs) and for the internal shearing motions that accompany such deformations. It also allows for the vertical expansions driven in non-circular discs by a variation of the vertical gravitational field around the horizontal streamlines, or by a divergence of the horizontal velocity. The equations of the affine model embody conservation laws for energy and potential vorticity, even for non-planar discs. We verify that they reproduce exactly the linear theories of 3D warped and eccentric discs in a secular approximation. However, the affine model does not rely on any secular or small-amplitude assumptions and should be useful in more general circumstances.

Key words: accretion, accretion discs – hydrodynamics.

1 INTRODUCTION

Astrophysical discs, consisting of continuous matter in orbital motion around a massive body, are found throughout the Universe on a variety of length-scales. They are usually thin, having a small aspect ratio $H/r \ll 1$, where H is a measure of the extent of the disc in the ‘vertical’ direction perpendicular to the orbital plane at radius r . The dynamics of thin discs is very often studied using 2D equations that neglect the vertical extent and vertical motion of the disc. However, this approximation is not generally valid, even in the limit $H/r \ll 1$.

Studies of wave propagation in astrophysical discs (Lubow & Ogilvie 1998), and of the dynamics of eccentric or tidally distorted discs (Ogilvie 2001, 2002), have shown that problems that have traditionally been studied using 2D models have quite different solutions when the internal vertical structure and vertical motion of the disc are taken into account, even when H/r is small. For example, Ogilvie (2008) found that the prograde precession of elliptical discs observed around Be stars has a natural explanation only when these effects are included.

The essential physics involved here is that the vertical structure of a thin disc is not generally hydrostatic except in the simplest situation of a steady, circular disc around a single mass. When the axial symmetry of this situation is broken by a free eccentricity of the disc or by the presence of an orbital companion, vertical oscillations are driven by the variation of the vertical gravitational force around the horizontal streamlines and by the divergence of the horizontal motion.

A complementary situation occurs if the reflectional symmetry of the disc about the mid-plane is broken. In particular, if the disc is warped such that the local orbital plane varies with r , horizontal oscillatory flows are driven within the disc. These are especially strong in Keplerian discs, leading to the remarkable properties of warp propagation in these systems (Papaloizou & Pringle 1983; Papaloizou & Lin 1995; Ogilvie 1999).

While fully 3D treatments of these situations are possible, and indeed necessary for the resolution of magnetohydrodynamic or other turbulence within the disc, global 3D simulations of realistically thin discs over the time-scales of interest remain extremely demanding. The purpose of this paper is to present and analyse a new model that augments the equations of 2D hydrodynamics to include the additional degrees of freedom that are needed for a description of discs that lack axial or reflectional symmetry. It takes into account

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the variable and non-hydrostatic thickness of astrophysical discs and allows a general displacement of the mid-plane of the disc from a reference plane. Although the method by which the equations are derived does not definitively establish their domain of applicability, the model does have a satisfying mathematical structure and internal consistency. In particular, we show that it implies conservation laws for energy and potential vorticity (PV) that generalize those of 2D hydrodynamics. Furthermore, we show that the equations correctly reproduce the linear hydrodynamics of eccentric and warped 3D discs in the secular approximation.

Relevant previous work was carried out by Stehle & Spruit (1999), who supplemented the 2D hydrodynamic equations with dynamical equations for the vertical velocity and scale height. Their equations are similar to those we derive below for the symmetric case (Section 10), but do not conserve energy or PV because the horizontal equation of motion was left in its 2D form. Earlier, Papaloizou & Lin (1988) wrote a dynamical equation for the thickness of an incompressible disc or ring, and Lubow (1981) studied in detail the resonances and wave emission associated with vertical compression of a disc in a binary system.

The remainder of this paper is structured as follows. Starting from a classical Lagrangian description of an ideal fluid (Section 2) in 3D, we restrict the degrees of freedom of the fluid to those of a continuum of extended fluid columns (Section 3). We introduce a thin-disc approximation (Section 4) and describe the vertical structure of the disc (Section 5). We then derive the equations of motion from the approximated Lagrangian (Section 6) and write them in Eulerian form by projecting them on to a reference plane (Section 7). We discuss the conservation laws (Section 8) and rotational symmetry (Section 9) of the model. The equations and their interpretation are considerably simplified in the case of a reflectionally symmetric disc (Section 10). We then derive the equilibrium conditions and linear perturbation theory for an axisymmetric disc (Section 11). We apply this to warped and eccentric discs (Sections 12 and 13) to show that the model exactly reproduces the previously known 3D secular theories in the appropriate limits. Our conclusions follow in Section 14.

2 LAGRANGIAN APPROACH

In this paper, we consider the case of an ideal fluid, which is inviscid and undergoes adiabatic thermodynamics. The equation of motion can then be derived from Hamilton's Principle using a Lagrangian analysis of the motion (e.g. Salmon 1988).

We label the fluid elements according to their position vectors $\mathbf{x}_0 = (x_0, y_0, z_0)$ in an arbitrary reference state. The reference state could be an initial condition or an equilibrium configuration, but this is not necessary. The quantities (x_0, y_0, z_0) are material or Lagrangian coordinates.

Let $\mathbf{x}(\mathbf{x}_0, t) = (x, y, z)$ be the position vector of a fluid element in the dynamical state at time t . The fluid velocity is

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}, \quad (1)$$

where

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} \right)_{\mathbf{x}_0} \quad (2)$$

is the Lagrangian time derivative. Let

$$J_{ij} = \frac{\partial x_i}{\partial x_{0j}} \quad (3)$$

be the Jacobian matrix of the time-dependent map from the reference state to the dynamical state, and let

$$J_3 = \det(J_{ij}) = \left| \frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)} \right| \quad (4)$$

be the Jacobian determinant of this 3D map. The quantity J_{ij} is known in continuum mechanics as the deformation gradient tensor.

A mass element of the fluid may be written as

$$dm = \rho d^3\mathbf{x} = \rho_0 d^3\mathbf{x}_0, \quad (5)$$

where $\rho(\mathbf{x}, t)$ is the mass density in the dynamical state and $\rho_0(\mathbf{x}_0)$ is the mass density in the reference state. Mass conservation implies

$$\rho = J_3^{-1} \rho_0 \quad (6)$$

and we require J_3 to be strictly positive.

The exact Lagrangian for a non-self-gravitating ideal fluid is

$$L = \int \left(\frac{1}{2} |\mathbf{u}|^2 - \Phi - e \right) dm, \quad (7)$$

where $\Phi(\mathbf{x}, t)$ is the (external) gravitational potential and $e(v, s)$ is the specific internal energy, which depends on the specific volume

$$v = \frac{1}{\rho} = J_3 v_0 \quad (8)$$

and the specific entropy

$$s = s_0, \quad (9)$$

$v_0(\mathbf{x}_0)$ and $s_0(\mathbf{x}_0)$ being the specific volume and entropy in the reference state. Fluid elements preserve their specific entropy in an ideal fluid flow. The differential of $e(v, s)$ satisfies the fundamental thermodynamic identity

$$de = T ds - p dv, \quad (10)$$

where T is the temperature and p is the pressure. In particular, for a perfect gas of constant adiabatic index γ , we have

$$\rho = J_3^{-1} \rho_0, \quad p = J_3^{-\gamma} p_0 \quad (11)$$

and

$$e = \frac{p}{(\gamma - 1)\rho} = J_3^{-(\gamma-1)} e_0. \quad (12)$$

Hamilton's Principle states that the action functional $S[\mathbf{x}] = \int L dt$ is stationary, leading to the Euler-Lagrange equation¹

$$\frac{\partial \mathcal{L}_3}{\partial x_i} - \frac{D}{Dt} \frac{\partial \mathcal{L}_3}{\partial u_i} - \frac{\partial}{\partial x_{0j}} \frac{\partial \mathcal{L}_3}{\partial J_{ij}} = 0, \quad (13)$$

where $L = \int \mathcal{L}_3 d^3\mathbf{x}_0$. After division by $-\rho_0$, this gives the desired equation of motion

$$\frac{Du_i}{Dt} = - \frac{\partial \Phi}{\partial x_i} - \frac{1}{\rho} \frac{\partial p}{\partial x_i}. \quad (14)$$

The pressure term in this equation deserves some comment. In the Lagrangian approach, this term emerges initially in the form

$$- \frac{1}{\rho_0} \frac{\partial}{\partial x_{0j}} (J_3^{-\gamma} p_0 C_{ij}), \quad (15)$$

¹ This is the standard Euler-Lagrange equation for several functions (\mathbf{x}) of several variables (\mathbf{x}_0, t). In our notation, D/Dt represents the derivative with respect to t when \mathbf{x}_0 is held constant. Also u_i corresponds to Dx_i/Dt and J_{ij} to $\partial x_i / \partial x_{0j}$.

where

$$C_{ij} = \frac{1}{2} \epsilon_{ikm} \epsilon_{jln} J_{kl} J_{mn} = \frac{\partial J_3}{\partial J_{ij}} = J_3 \frac{\partial x_{0j}}{\partial x_i} \quad (16)$$

is the cofactor of the element J_{ij} of the Jacobian matrix. Using the identity $\partial C_{ij} / \partial x_{0j} = 0$ to extract the cofactor from the bracket, and then the chain rule to convert derivatives with respect to the Lagrangian variable \mathbf{x}_0 to those with respect to the Eulerian variable \mathbf{x} , we obtain the form

$$-\frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad (17)$$

as given above.

3 COLUMNAR ELEMENTS AND AFFINE TRANSFORMATION

Our aim is to reduce the dynamics of a thin (but generally non-planar) 3D disc to a 2D description by applying certain assumptions and approximations. By doing this at the level of the Lagrangian function, we can ensure that the resulting theory is self-consistent and embodies the appropriate conservation laws. Our derivation is similar in spirit (although very different in detail) to the derivation of the shallow-water model of geophysical fluid dynamics by Miles & Salmon (1985).

Although our disc is generally not planar, we will describe it ultimately using a projection on to the plane $z = 0$, which we regard as horizontal and call the *reference plane*. In the case of a central force, our model will have complete rotational symmetry and the choice of reference plane is arbitrary. We consider the disc to be composed of extended fluid columns of infinitesimal width. The disc is therefore regarded as a 2D continuum of 1D elements (Fig. 1).

We envisage a convenient hypothetical reference state in which the disc has axial and reflectional symmetries and is in vertical hydrostatic equilibrium in a potential that has the same symmetries (and which may differ from the actual potential Φ). In the reference state, the columnar elements are vertical and centred on the reference plane $z = 0$. Let $H_0(r)$ be the vertical scale height (as defined in Section 5 below) of the column whose centre is at radius r from the symmetry axis.

To reach the dynamical state of the disc, each fluid column may undergo an arbitrary translation and an arbitrary linear transformation in 3D. The translation allows the centre of the column to be moved to any point, not necessarily in the plane $z = 0$. The linear transformation allows the column to be expanded or contracted and also rotated about its centre.

The combination of a translation and a linear transformation is known as an *affine transformation*, which explains the name of our model. Note that each column undergoes an independent affine transformation, the parameters of which will depend continuously on the column label and also on time.

In the Lagrangian viewpoint, we label the columnar elements by the horizontal position vectors $\bar{\mathbf{x}}_0 = (x_0, y_0, 0)$ of their centres in the reference state. Generally, we use an overbar to denote a planar quantity such as the horizontal projection of a 3D vector. The 3D fluid elements within each column are further identified by the dimensionless label

$$\zeta = \frac{z_0}{H_0}, \quad (18)$$

which runs from $-\infty$ to ∞ , with $\zeta = 0$ corresponding to the centre of the column and most of the mass being contained within $|\zeta| < 1$.

Under the affine transformation, the column maps to

$$\mathbf{x} = \mathbf{X}(\bar{\mathbf{x}}_0, t) + \mathbf{H}(\bar{\mathbf{x}}_0, t)\zeta, \quad (19)$$

where $\mathbf{X} = (X, Y, Z)$ is the position vector of the centre of the column in the dynamical state and $\mathbf{H} = (H_x, H_y, H_z)$ is a *scale vector* with the dimensions of length. For example, the fluid elements labelled by $\zeta = \pm 1$ are separated by $2H_0 \mathbf{e}_z$ in the reference state but by $2\mathbf{H}$ in the dynamical state.

The fluid velocity is then

$$\mathbf{u} = \mathbf{v} + \mathbf{w}\zeta, \quad (20)$$

where

$$\mathbf{v} = \frac{D\mathbf{X}}{Dt} \quad (21)$$

is the velocity of the centre of the column and

$$\mathbf{w} = \frac{D\mathbf{H}}{Dt} \quad (22)$$

is the rate of change of the scale vector. Note that $D\zeta/Dt = 0$ because ζ is a Lagrangian coordinate labelling fluid elements.

Each columnar element has six degrees of freedom (\mathbf{X}, \mathbf{H}) . The variables X and Y give the fluid all the potentialities of (compressible) 2D hydrodynamics. In addition, the variable Z allows the mid-plane of the disc to be deformed away from the plane $z = 0$, as occurs for example in warped discs. We refer to the surface $z = Z$, which is the locus of column centres $\zeta = 0$, as the *deformed mid-plane*.

The variable H_z allows the disc to undergo vertical expansion or contraction, as occurs for example in eccentric or tidally distorted discs. Finally, the variables H_x and H_y allow the columns to be tilted so that the disc undergoes internal shearing motions as in warped discs.

It can be helpful to think of the map from the reference state to the dynamical state as a composition of two stages: $(x_0, y_0, z_0) \mapsto (X, Y, \zeta) \mapsto (x, y, z)$. The Jacobian matrix J_{ij} and determinant J_3 of the composite map are the products of those of the two stages. The intermediate variables (X, Y, ζ) represent a system of ‘columnar’ coordinates, with $\bar{\mathbf{X}} = (X, Y)$ identifying a column by means of the horizontal position vector of its centre and ζ labelling the fluid elements within a column. In the Eulerian viewpoint, we will regard quantities such as Z , \mathbf{H} , \mathbf{v} , and \mathbf{w} as functions of $(\bar{\mathbf{X}}, t)$ rather than functions of $(\bar{\mathbf{x}}_0, t)$.

The first stage $(x_0, y_0, z_0) \mapsto (X, Y, \zeta)$ of the map has the Jacobian matrix

$$\frac{\partial(X, Y, \zeta)}{\partial(x_0, y_0, z_0)} = \begin{pmatrix} \partial X / \partial x_0 & \partial X / \partial y_0 & 0 \\ \partial Y / \partial x_0 & \partial Y / \partial y_0 & 0 \\ -\zeta \partial \ln H_0 / \partial x_0 & -\zeta \partial \ln H_0 / \partial y_0 & 1/H_0 \end{pmatrix}, \quad (23)$$

with determinant

$$\left| \frac{\partial(X, Y, \zeta)}{\partial(x_0, y_0, z_0)} \right| = \frac{J_2}{H_0}, \quad (24)$$

where

$$J_2 = \left| \frac{\partial(\bar{\mathbf{X}})}{\partial(\bar{\mathbf{x}}_0)} \right| = \left| \frac{\partial(X, Y)}{\partial(x_0, y_0)} \right| \quad (25)$$

is the Jacobian determinant of the 2D map $(x_0, y_0) \mapsto (X, Y)$ and the factor of $1/H_0$ comes from $z_0 \mapsto \zeta = z_0/H_0$.

In considering the second stage $(X, Y, \zeta) \mapsto (x, y, z)$ of the map, it is helpful in preparation for an Eulerian viewpoint to regard Z and \mathbf{H} as functions of $\bar{\mathbf{X}}$ rather than $\bar{\mathbf{x}}_0$, as mentioned above. The Jacobian matrix of the second stage is then

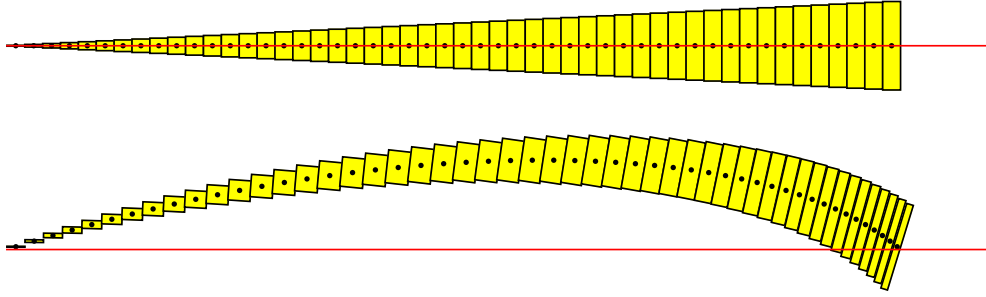


Figure 1. Illustration of the affine model, viewed in a cut perpendicular to the reference plane (red line). The disc is thought of as a continuum of extended fluid columns. In the reference state (top), the columns are vertical and their centres lie in the reference plane. To reach the dynamical state (bottom), the columns undergo time-dependent translations and linear transformations.

$$\frac{\partial(x, y, z)}{\partial(X, Y, \zeta)} = \begin{pmatrix} 1 + H_{x,X}\zeta & H_{x,Y}\zeta & H_x \\ H_{y,X}\zeta & 1 + H_{y,Y}\zeta & H_y \\ Z_X + H_{z,X}\zeta & Z_Y + H_{z,Y}\zeta & H_z \end{pmatrix}, \quad (26)$$

where $H_{x,X} = \partial H_x / \partial X$, $Z_X = \partial Z / \partial X$, etc. Its determinant will not generally be positive for all ζ because of the clashing of neighbouring columns. In other words, the columnar coordinate system generally breaks down sufficiently far from the (deformed) mid-plane. However, for a thin disc with moderate deformations, we expect there to be negligible mass in these distant regions. Indeed, in the next section, we will make an approximation that prohibits the Jacobian determinant from changing sign far from the disc.

The deformed mid-plane at any instant of time can now be thought of as the surface $z = Z(X, Y)$ or $\mathbf{x} = \mathbf{X}(\bar{\mathbf{X}})$. We assume that the deformation of the disc is sufficiently moderate that Z is a single-valued function of (X, Y) , and that $J_2 > 0$. In some cases, this may require the reference plane to be chosen judiciously. The vector area element of the deformed mid-plane is

$$d\mathbf{S} = \left(\frac{\partial \mathbf{X}}{\partial X} \times \frac{\partial \mathbf{X}}{\partial Y} \right) dX dY = \mathbf{n} dX dY, \quad (27)$$

where the vector

$$\mathbf{n} = \mathbf{e}_z - \bar{\nabla} Z = (-Z_X, -Z_Y, 1) \quad (28)$$

is normal to the surface, and generally of greater than unit length. The notation

$$\bar{\nabla} = \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, 0 \right) \quad (29)$$

represents the planar gradient operator for quantities that depend on $(\bar{\mathbf{X}}, t)$.

4 THIN-DISC APPROXIMATION

For a thin disc with large-scale deformations, we approximate the Jacobian matrices (23) and (26) of the two stages by evaluating them at $\zeta = 0$ and neglecting their dependence on ζ . Thus

$$\frac{\partial(X, Y, \zeta)}{\partial(x_0, y_0, z_0)} = \begin{pmatrix} \partial X / \partial x_0 & \partial X / \partial y_0 & 0 \\ \partial Y / \partial x_0 & \partial Y / \partial y_0 & 0 \\ 0 & 0 & 1/H_0 \end{pmatrix}, \quad (30)$$

with determinant

$$\left| \frac{\partial(X, Y, \zeta)}{\partial(x_0, y_0, z_0)} \right| = \frac{J_2}{H_0} \quad (31)$$

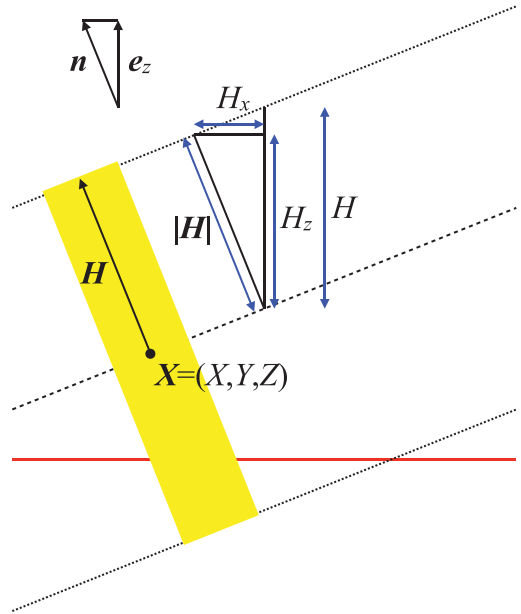


Figure 2. Illustration of the projected vertical scale height H of the disc in the simple case of a flat disc that is tilted with respect to the reference plane (red solid line). The dashed and dotted lines represent the deformed mid-plane $\zeta = 0$ and the surfaces $\zeta = \pm 1$ within which most of the mass is contained, respectively. The yellow rectangle represents a single columnar element.

as before, and

$$\frac{\partial(x, y, z)}{\partial(X, Y, \zeta)} = \begin{pmatrix} 1 & 0 & H_x \\ 0 & 1 & H_y \\ Z_X & Z_Y & H_z \end{pmatrix}, \quad (32)$$

with determinant

$$\left| \frac{\partial(x, y, z)}{\partial(X, Y, \zeta)} \right| = H_z - \bar{\mathbf{H}} \cdot \bar{\nabla} Z = \mathbf{H} \cdot \mathbf{n}. \quad (33)$$

We define the important quantity

$$\mathbf{H} = \mathbf{H} \cdot \mathbf{n}, \quad (34)$$

which is the projected vertical scale height of the disc (Fig. 2).

This approximation results in a deformation gradient tensor that is uniform within each column, and equal to the exact expression at the centre of each column. It can be justified on scaling grounds if $\|\bar{\nabla} \mathbf{H}\| \ll 1$, i.e. if $|\mathbf{H}|$ is small compared to the length-scale on which \mathbf{H} varies. This condition should be satisfied in a thin disc if the deformations are of large scale.

Under this approximation, the Jacobian determinant of the composite map is

$$J_3 = J_2 \frac{H}{H_0}. \quad (35)$$

The (3,3) element of the approximated inverse Jacobian matrix is

$$\left(\frac{\partial z_0}{\partial z} \right)_{x,y} = \frac{J_2}{J_3} = \frac{H_0}{H}. \quad (36)$$

Therefore, a vertical integration through the disc at constant (x, y) becomes

$$\int_{-\infty}^{\infty} \cdot dz = \int_{-\infty}^{\infty} \cdot \frac{H}{H_0} dz_0 = H \int_{-\infty}^{\infty} \cdot d\zeta. \quad (37)$$

In other words, in order to remain at constant x and y as we increase ζ , we must sample different fluid columns if they are tilted. Equation (19) tells us that X and Y must change such that $dX = -H_x d\zeta$ and $dY = -H_y d\zeta$. Therefore, $dz = dZ + H_z d\zeta = H d\zeta$ with $H = H_z - \vec{H} \cdot \vec{\nabla} Z$.

5 VERTICAL STRUCTURE

Let Σ and P denote the density and pressure integrated vertically (i.e. with respect to the coordinate perpendicular to the reference plane). In the reference state, their values are

$$\Sigma_0(\bar{x}_0) = \int \rho_0 dz_0, \quad P_0(\bar{x}_0) = \int p_0 dz_0. \quad (38)$$

The hydrostatic reference state may be written as

$$\rho_0 = \frac{\Sigma_0}{H_0} F_\rho(\zeta), \quad (39)$$

$$p_0 = \frac{P_0}{H_0} F_p(\zeta), \quad (40)$$

where the dimensionless functions F_ρ and F_p satisfy the dimensionless equations of vertical structure,

$$\frac{dF_p}{d\zeta} = -F_\rho \zeta, \quad (41)$$

$$\int_{-\infty}^{\infty} F_\rho d\zeta = 1, \quad (42)$$

$$\int_{-\infty}^{\infty} F_p d\zeta = 1, \quad (43)$$

The first of these equations is a dimensionless form of hydrostatic balance in any gravitational field that is proportional to the height above the mid-plane, which is generic for a non-self-gravitating thin disc. The second and third equations are normalization conditions required for equation (38). The first and second dimensionless moments of the density are

$$\int_{-\infty}^{\infty} F_\rho \zeta d\zeta = 0, \quad (44)$$

which follows from the reflectional symmetry about the mid-plane, and

$$\int_{-\infty}^{\infty} F_\rho \zeta^2 d\zeta = 1, \quad (45)$$

which follows from the equations of vertical structure after an integration by parts. In dimensional terms, we have

$$H_0^2 = \int \rho_0 z_0^2 dz_0 / \int \rho_0 dz_0, \quad (46)$$

which gives a precise meaning to the scale height as the standard deviation of the density distribution.

Simple examples of solutions of these equations (Ogilvie & Barker 2014) are the isothermal structure,

$$F_\rho(\zeta) = F_p(\zeta) = (2\pi)^{-1/2} \exp\left(-\frac{\zeta^2}{2}\right), \quad (47)$$

the homogeneous structure,

$$F_\rho(\zeta) = \frac{1}{2\sqrt{3}}, \quad (48)$$

$$F_p(\zeta) = \frac{3 - \zeta^2}{4\sqrt{3}} \quad (49)$$

(for $\zeta^2 < 3$ only), and the polytropic structure,

$$F_\rho(\zeta) = C_n \left(1 - \frac{\zeta^2}{2n+3}\right)^n, \quad (50)$$

$$F_p(\zeta) = \frac{2n+3}{2(n+1)} C_n \left(1 - \frac{\zeta^2}{2n+3}\right)^{n+1} \quad (51)$$

(for $\zeta^2 < 2n+3$ only), where $n > 0$ (not necessarily an integer) is the polytropic index and

$$C_n = [(2n+3)\pi]^{-1/2} \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n+1)}$$

is a normalization constant. It can be shown that the polytropic structure approaches the isothermal structure in the limit $n \rightarrow \infty$, and approaches the homogeneous structure in the limit $n \rightarrow 0$. The reason for the multiplicity of possible solutions is that either the vertical temperature profile, or the vertical entropy profile, can be freely chosen in the case of an ideal fluid. In a dissipative disc, these profiles would be determined from a balance between heating and cooling in the thermal energy equation.

An important property of the affine transformation is that each columnar element undergoes a uniform expansion or compression, because (in the thin-disc approximation explained in Section 4) the Jacobian determinant J_3 is independent of ζ . Therefore, the dimensionless profile of density is preserved, and so are those of pressure and other thermodynamic variables if (as we assume here) the gas is perfect and behaves adiabatically.

The density, pressure, and specific internal energy of a perfect gas in the dynamical state are therefore

$$\rho = J_3^{-1} \rho_0, \quad p = J_3^{-\gamma} p_0, \quad e = J_3^{-(\gamma-1)} e_0. \quad (52)$$

It follows from equations (37) and (35) that the vertically integrated density and pressure are

$$\Sigma = J_2^{-1} \Sigma_0, \quad P = J_2^{-1} J_3^{-(\gamma-1)} P_0. \quad (53)$$

We can then write

$$\rho = \bar{\rho} F_\rho(\zeta), \quad p = \bar{p} F_p(\zeta), \quad (54)$$

where

$$\bar{\rho} = \frac{\Sigma}{H}, \quad \bar{p} = \frac{P}{H} \quad (55)$$

are the representative density and pressure of each column. Since $F_\rho(0)$ varies between $1/2\sqrt{3} \approx 0.289$ and $1/\sqrt{2\pi} \approx 0.399$, while $F_p(0)$ varies between $\sqrt{3}/4 \approx 0.433$ and $1/\sqrt{2\pi} \approx 0.399$, depending on the polytropic index, the representative density and pressure are larger by a factor of about 2 or 3 than the density and pressure on the deformed mid-plane $\zeta = 0$.

The scale height in the dynamical state is defined by

$$H^2 = \int \rho(z - Z)^2 dz \Big/ \int \rho dz \quad (56)$$

(where the integrals are carried out at constant x and y), so it is again the standard deviation of the density distribution perpendicular to the reference plane.

Let \mathcal{R} be the gas constant and μ the mean molecular weight. Then, the temperature is

$$T = \frac{\mu}{\mathcal{R}} \frac{p}{\rho} = \bar{T} F_T(\zeta), \quad (57)$$

where

$$\bar{T} = \frac{\mu}{\mathcal{R}} \frac{\bar{p}}{\bar{\rho}} = \frac{\mu}{\mathcal{R}} \frac{P}{\Sigma} \quad (58)$$

and

$$F_T = \frac{F_p}{F_\rho}. \quad (59)$$

The specific entropy is (apart from an unimportant additive constant)

$$s = \frac{\mathcal{R}}{\mu} \frac{1}{\gamma - 1} \ln(p \rho^{-\gamma}) = \bar{s} + \frac{\mathcal{R}}{\mu} \ln F_s(\zeta), \quad (60)$$

where

$$\bar{s} = \frac{\mathcal{R}}{\mu} \frac{1}{\gamma - 1} \ln(\bar{p} \bar{\rho}^{-\gamma}) = \frac{\mathcal{R}}{\mu} \frac{1}{\gamma - 1} \ln(P \Sigma^{-\gamma} H^{\gamma-1}) \quad (61)$$

and

$$F_s = F_p^{1/(\gamma-1)} F_\rho^{-\gamma/(\gamma-1)}. \quad (62)$$

6 LAGRANGIAN AND EQUATIONS OF MOTION

We now express the Lagrangian (7) of the ideal fluid in terms of the variables we have introduced.

The 3D mass element is

$$dm = \rho_0 d^3 \mathbf{x}_0 = \rho_0 d^2 \bar{\mathbf{x}}_0 dz_0 = \Sigma_0 d^2 \bar{\mathbf{x}}_0 F_\rho d\zeta. \quad (63)$$

For the kinetic energy, we have

$$\begin{aligned} & \int \frac{1}{2} |\mathbf{u}|^2 dm \\ &= \iint \left(\frac{1}{2} |\mathbf{v}|^2 + \mathbf{v} \cdot \mathbf{w} \zeta + \frac{1}{2} |\mathbf{w}|^2 \zeta^2 \right) \Sigma_0 d^2 \bar{\mathbf{x}}_0 F_\rho d\zeta \\ &= \int \frac{1}{2} (|\mathbf{v}|^2 + |\mathbf{w}|^2) \Sigma_0 d^2 \bar{\mathbf{x}}_0. \end{aligned} \quad (64)$$

For the gravitational energy, we expand the gravitational potential in a Taylor series about the centre of the fluid column:

$$\begin{aligned} \Phi(\mathbf{x}) &= \Phi(\mathbf{X} + \mathbf{H}\zeta) \\ &= \Phi(\mathbf{X}) + \zeta \mathbf{H} \cdot \nabla \Phi + \frac{1}{2} \zeta^2 \mathbf{H} \mathbf{H} : \nabla \nabla \Phi + \dots \end{aligned} \quad (65)$$

(in which the derivatives are evaluated at \mathbf{X} , and we have suppressed any explicit time-dependence of the potential). For a thin disc, we accept the (quadrupolar) truncation

$$\int \Phi dm = \int \left[\Phi(\mathbf{X}) + \frac{1}{2} \mathbf{H} \mathbf{H} : \nabla \nabla \Phi \right] \Sigma_0 d^2 \bar{\mathbf{x}}_0. \quad (66)$$

Finally, for the internal energy of a perfect gas, we have

$$\begin{aligned} \int e dm &= \iint J_3^{-(\gamma-1)} e_0 \Sigma_0 d^2 \bar{\mathbf{x}}_0 F_\rho d\zeta \\ &= \int J_3^{-(\gamma-1)} \frac{P_0}{(\gamma-1)\rho_0} \Sigma_0 d^2 \bar{\mathbf{x}}_0 F_\rho d\zeta \\ &= \iint \frac{J_3^{-(\gamma-1)} P_0}{(\gamma-1)} d^2 \bar{\mathbf{x}}_0 F_\rho d\zeta \\ &= \int \frac{J_3^{-(\gamma-1)} P_0}{(\gamma-1)} d^2 \bar{\mathbf{x}}_0. \end{aligned} \quad (67)$$

Thus, we obtain the Lagrangian

$$\begin{aligned} L &= \int \left[\frac{1}{2} (|\mathbf{v}|^2 + |\mathbf{w}|^2) - \Phi(\mathbf{X}, t) - \frac{1}{2} \mathbf{H} \mathbf{H} : \nabla \nabla \Phi \right. \\ &\quad \left. - \frac{J_3^{-(\gamma-1)} P_0}{(\gamma-1)\Sigma_0} \right] \Sigma_0 d^2 \bar{\mathbf{x}}_0. \end{aligned} \quad (68)$$

Writing this as $L = \int \mathcal{L}_2 d^2 \bar{\mathbf{x}}_0$, where the Lagrangian density \mathcal{L}_2 depends on \mathbf{X} and \mathbf{H} and their derivatives with respect to t and $\bar{\mathbf{x}}_0$, we identify the Euler–Lagrange equations as

$$\frac{\partial \mathcal{L}_2}{\partial X_i} - \frac{D}{Dt} \frac{\partial \mathcal{L}_2}{\partial v_i} - \frac{\partial}{\partial \bar{x}_{0j}} \frac{\partial \mathcal{L}_2}{\partial (\partial X_i / \partial \bar{x}_{0j})} = 0, \quad (69)$$

$$\frac{\partial \mathcal{L}_2}{\partial H_i} - \frac{D}{Dt} \frac{\partial \mathcal{L}_2}{\partial w_i} - \frac{\partial}{\partial \bar{x}_{0j}} \frac{\partial \mathcal{L}_2}{\partial (\partial H_i / \partial \bar{x}_{0j})} = 0, \quad (70)$$

where summation over $j = \{1, 2\}$ is implied. After division by $-\Sigma_0$ and application of algebraic identities, these give the desired equations of motion

$$\frac{D^2 \mathbf{X}}{Dt^2} = -\nabla \Phi - \frac{1}{2} \mathbf{H} \mathbf{H} : \nabla \nabla \nabla \Phi - \frac{1}{\Sigma} \bar{\nabla} P + \frac{1}{\Sigma} \bar{\nabla} \cdot \left(\frac{P \bar{\mathbf{H}} \mathbf{n}}{H} \right),$$

$$\frac{D^2 \mathbf{H}}{Dt^2} = -\mathbf{H} \cdot \nabla \nabla \Phi + \frac{P \mathbf{n}}{\Sigma H}. \quad (72)$$

The terms in equation (71) involving the vertically integrated pressure P are written here in terms of derivatives with respect to the Eulerian coordinates (X, Y) on the reference plane, rather than the Lagrangian coordinates (x_0, y_0) ; this involves operations similar to those leading to equation (14). In the last term, the divergence is taken on the first index (belonging to $\bar{\mathbf{H}}$). The terms involving \mathbf{n} in these equations, which are not present in 2D hydrodynamics, come from the property that J_3 is proportional to $H = \mathbf{H} \cdot \mathbf{n}$.

7 PROJECTED EULERIAN REPRESENTATION

We now interpret equations (71) and (72) fully in an Eulerian sense, projected on to the reference plane $z = 0$. The projected Eulerian form of the equations is

$$\frac{D\mathbf{v}}{Dt} = -\nabla \Phi - \frac{1}{2} \mathbf{H} \mathbf{H} : \nabla \nabla \nabla \Phi - \frac{1}{\Sigma} \bar{\nabla} P + \frac{1}{\Sigma} \bar{\nabla} \cdot \left(\frac{P \bar{\mathbf{H}} \mathbf{n}}{H} \right), \quad (73)$$

$$\frac{D\mathbf{w}}{Dt} = -\mathbf{H} \cdot \nabla \nabla \Phi + \frac{P \mathbf{n}}{\Sigma H}, \quad (74)$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{\mathbf{v}} \cdot \bar{\nabla}, \quad (75)$$

$$\mathbf{v} = \bar{\mathbf{v}} + v_z \mathbf{e}_z, \quad (76)$$

$$v_z = \frac{DZ}{Dt}, \quad (77)$$

$$\mathbf{w} = \frac{D\mathbf{H}}{Dt}, \quad (78)$$

and again $H = \mathbf{H} \cdot \mathbf{n}$ with $\mathbf{n} = \mathbf{e}_z - \bar{\nabla}Z$. In addition, we need evolutionary equations for Σ and P . From $\Sigma = J_2^{-1}\Sigma_0$ we obtain, as in 2D hydrodynamics,

$$\frac{D \ln \Sigma}{Dt} = -\bar{\nabla} \cdot \bar{\mathbf{v}}. \quad (79)$$

From $P = J_2^{-1} J_3^{-(\gamma-1)} P_0$, we find

$$\frac{D \ln P}{Dt} = -\gamma \bar{\nabla} \cdot \bar{\mathbf{v}} - (\gamma - 1) \frac{D \ln H}{Dt}, \quad (80)$$

in which

$$\frac{D \ln H}{Dt} = \frac{1}{H} \frac{D}{Dt} (\mathbf{H} \cdot \mathbf{n}) = \frac{1}{H} \mathbf{n} \cdot (\mathbf{w} - \bar{\mathbf{H}} \cdot \bar{\nabla} \mathbf{v}). \quad (81)$$

These equations have numerous alternative forms such as

$$\frac{\partial \Sigma}{\partial t} + \bar{\nabla} \cdot (\Sigma \bar{\mathbf{v}}) = 0, \quad (82)$$

$$\frac{D \bar{\rho}}{Dt} = -\bar{\rho} \left(\bar{\nabla} \cdot \bar{\mathbf{v}} + \frac{D \ln H}{Dt} \right), \quad (83)$$

$$\frac{D \bar{p}}{Dt} = -\gamma \bar{p} \left(\bar{\nabla} \cdot \bar{\mathbf{v}} + \frac{D \ln H}{Dt} \right), \quad (84)$$

$$\frac{D \bar{s}}{Dt} = 0, \quad (85)$$

etc.

A full set of equations is written out explicitly in Cartesian coordinates in Appendix B. Polar coordinates would of course be more appropriate for many applications.

An Eulerian representation of the fluid variables, valid within a few scale heights of the deformed mid-plane, is

$$\mathbf{u} \approx \mathbf{v}(\bar{\mathbf{x}}, t) + \mathbf{w}(\bar{\mathbf{x}}, t)\zeta, \quad (86)$$

$$\rho \approx \bar{\rho}(\bar{\mathbf{x}}, t)F_\rho(\zeta), \quad (87)$$

$$p \approx \bar{p}(\bar{\mathbf{x}}, t)F_p(\zeta), \quad (88)$$

where

$$\zeta = \frac{z - Z}{H}, \quad \bar{\rho} = \frac{\Sigma}{H}, \quad \bar{p} = \frac{P}{H}. \quad (89)$$

Some care is needed with the notation of derivatives. In the terms of equation (73) involving P , the operator $\bar{\nabla}$ acts on planar quantities that are functions of (X, Y, t) only, and there is no ambiguity concerning these derivatives. In contrast, Φ is generally a function of (x, y, z, t) ; the horizontal components of $\nabla\Phi$ in equation (73) are obtained by differentiating Φ with respect to x or y and then setting $\mathbf{x} = \mathbf{X}$, rather than by first evaluating the potential at $z = Z(X, Y, t)$ and then differentiating with respect to X or Y , which would introduce further terms via the chain rule.

While equation (73) contains all the terms present in 2D hydrodynamics, it differs from that model in several respects. First, the equation has a vertical component, describing how the mid-plane of the disc moves vertically in situations lacking reflectional symmetry (e.g. a warped disc). Secondly, the second term on the right-hand side is the gravitational quadrupolar force acting on the extended fluid column; as seen in Section 11 below, this term is active even in a hydrostatic situation. Thirdly, the last term on the

right-hand side is a novel force arising from pressure and a deformation of the mid-plane; this term conserves momentum but leads to an anisotropic stress in the reference plane. It may seem puzzling that an anisotropic stress can arise from pressure. For example, the vertical component of equation (73) indicates that there is a horizontal flux density of vertical momentum equal to $-P\bar{\mathbf{H}}/H$ within the reference plane. In fact, the flux density of vertical momentum in 3D is just $p\mathbf{e}_z$; however, if the columns are tilted, then the pressure transmits vertical momentum from one column to its neighbours, resulting in an apparent horizontal flux within the reference plane.

Equation (74) is relatively novel, although the vertical component describes breathing oscillations of the disc and has been considered in previous work (e.g. Stehle & Spruit 1999). The horizontal components capture the shearing horizontal oscillations driven by pressure gradients in warped discs or other situations lacking reflectional symmetry.

The thin-disc approximation introduced in Section 4 was justified on the grounds that $\|\bar{\nabla}\mathbf{H}\| \ll 1$, i.e. that $|\mathbf{H}|$ is small compared to the length-scale on which \mathbf{H} varies. This approximation results in a Lagrangian that does not depend on the spatial derivatives of \mathbf{H} and gives rise to the equations in the form presented above. In Appendix B, we present the form of the equations for a more general model in which the Jacobian determinant J_3 can depend on the spatial derivatives of \mathbf{H} . We will see in Section 11 below that particular extensions of this type are desirable to improve the accuracy and stability of the model at small scales comparable to $|\mathbf{H}|$.

8 CONSERVATION OF ENERGY AND POTENTIAL VORTICITY

The equations of the previous section imply the local conservation of total energy in the Eulerian form

$$\begin{aligned} \frac{\partial}{\partial t} (\Sigma \mathcal{E}) + \bar{\nabla} \cdot \left[(\Sigma \mathcal{E} + P) \bar{\mathbf{v}} - \frac{P}{H} (\mathbf{v} \cdot \mathbf{n}) \bar{\mathbf{H}} \right] \\ = \Sigma \left(\dot{\Phi} + \frac{1}{2} \mathbf{H} \mathbf{H} : \nabla \nabla \Phi \right), \end{aligned} \quad (90)$$

with specific total energy

$$\mathcal{E} = \frac{1}{2} (|\mathbf{v}|^2 + |\mathbf{w}|^2) + \Phi + \frac{1}{2} \mathbf{H} \mathbf{H} : \nabla \nabla \Phi + \frac{P}{(\gamma - 1)\Sigma}. \quad (91)$$

This expression for \mathcal{E} has a clear interpretation: the first two terms are kinetic energy, the next two are gravitational potential energy (again in the quadrupolar approximation for extended fluid columns), and the last term is internal energy. The source term on the right-hand side of equation (90) involves $\dot{\Phi} = \partial\Phi/\partial t$, which vanishes in the case of a time-independent potential.

The fact that there is an exact form of energy conservation in the affine model is reassuring and implies a certain self-consistency. It is not surprising, however, because we derived the model from Hamilton's Principle and the conservation of energy is directly related to the symmetry of the Lagrangian under time translation.

Less obvious is the conservation of PV. Also known as *vortensity* in the context of astrophysical discs, this is a modified version of the vertical component of vorticity that is conserved in ideal, barotropic 2D hydrodynamics and has been found to play an important role in numerous problems in astrophysical discs. In geophysical fluid dynamics, the theory of PV is highly developed. Expressions for the PV take a variety of forms depending on the model (shallow water, quasi-geostrophic, etc.) being employed, but PV conservation can always be related to Kelvin's circulation theorem and derived from

the symmetry of the Lagrangian under the continuous relabelling of fluid elements (e.g. Miles & Salmon 1985; Badin & Crisciani 2018).

We define the PV in the affine model as

$$q = \frac{1}{\Sigma} \mathbf{n} \cdot [\bar{\nabla} \mathbf{x} (\mathbf{v} + w_i \bar{\nabla} H_i)], \quad (92)$$

where there is an implied summation over Cartesian indices $i = \{1, 2, 3\}$. Note that \mathbf{v} , and therefore $\bar{\nabla} \mathbf{x} \mathbf{v}$, are generally three-component vectors. It can then be shown from the equations of the preceding section that

$$\frac{Dq}{Dt} = \frac{S_q}{\Sigma} \quad (93)$$

or, in Eulerian conservative form,

$$\frac{\partial}{\partial t} (\Sigma q) + \bar{\nabla} \cdot (\Sigma q \bar{\mathbf{v}}) = S_q, \quad (94)$$

where

$$S_q = \left[\bar{\nabla} \left(\frac{P}{H} \right) \times \bar{\nabla} \left(\frac{H}{\Sigma} \right) \right]_z \quad (95)$$

is a baroclinic source of PV per unit area. The source term can be written in various ways, e.g.

$$S_q = [\bar{\nabla} \bar{p} \times \bar{\nabla} \bar{v}]_z = [\bar{\nabla} \bar{T} \times \bar{\nabla} \bar{s}]_z. \quad (96)$$

Since the gradient vectors are horizontal and $n_z = 1$, these expressions are equivalent to

$$S_q = \mathbf{n} \cdot [\bar{\nabla} \mathbf{x} (\bar{p} \bar{\nabla} \bar{v})] = \mathbf{n} \cdot [\bar{\nabla} \mathbf{x} (\bar{T} \bar{\nabla} \bar{s})]. \quad (97)$$

Consider a simple, closed material curve C that lies in the deformed mid-plane and moves with the velocity field \mathbf{v} . Let S be the open material surface consisting of the region of the deformed mid-plane enclosed by C . The projections of C and S on the reference plane are the planar curve \bar{C} and the planar area \bar{S} . Integration of equation (93) over \bar{S} with respect to the invariant mass element $dm = \Sigma dX dY$ results in

$$\frac{d}{dt} \int q dm = \int_{\bar{S}} S_q dX dY, \quad (98)$$

i.e.

$$\begin{aligned} \frac{d}{dt} \int_{\bar{S}} \mathbf{n} \cdot [\bar{\nabla} \mathbf{x} (\mathbf{v} + w_i \bar{\nabla} H_i)] dX dY \\ = \int_{\bar{S}} \mathbf{n} \cdot [\bar{\nabla} \mathbf{x} (\bar{T} \bar{\nabla} \bar{s})] dX dY. \end{aligned} \quad (99)$$

Using expression (27) for the vector area element, we may write this as

$$\frac{d}{dt} \int_S [\bar{\nabla} \mathbf{x} (\mathbf{v} + w_i \bar{\nabla} H_i)] \cdot d\mathbf{S} = \int_S [\bar{\nabla} \mathbf{x} (\bar{T} \bar{\nabla} \bar{s})] \cdot d\mathbf{S}. \quad (100)$$

By Stokes's theorem, this implies

$$\frac{d}{dt} \oint_C (\mathbf{v} + w_i \bar{\nabla} H_i) \cdot d\mathbf{X} = \oint_C (\bar{T} \bar{\nabla} \bar{s}) \cdot d\mathbf{X}. \quad (101)$$

In particular, if C is an isentropic material curve on which \bar{s} is constant, then we verify Kelvin's circulation theorem in the form

$$\frac{d}{dt} \oint_C (\mathbf{v} + w_i \bar{\nabla} H_i) \cdot d\mathbf{X} = 0. \quad (102)$$

The conserved circulation can also be written as

$$\oint (\mathbf{v} \cdot d\mathbf{X} + \mathbf{w} \cdot d\mathbf{H}). \quad (103)$$

This can be interpreted as the action integral

$$\oint \sum_i p_i dq_i \quad (104)$$

of Hamiltonian dynamics, where q_i are the generalized coordinates (in our case, \mathbf{X} and \mathbf{H}) and p_i are the conjugate momenta per unit mass (in our case, \mathbf{v} and \mathbf{w}). It can also be related to the conserved circulation in 3D ideal hydrodynamics, which is the line integral $\oint \mathbf{u} \cdot d\mathbf{x}$ around a closed material curve within an isentropic surface. Given the expression (60) for the specific entropy in our disc of non-zero thickness, if \bar{s} is constant around C then the isentropic material curves are those displaced from C by any constant value of ζ . On these curves, $\mathbf{x} = \mathbf{X} + \mathbf{H}\zeta$, $d\mathbf{x} = d\mathbf{X} + \zeta d\mathbf{H}$, and $\mathbf{u} = \mathbf{v} + \mathbf{w}\zeta$. Expanding the differential $\mathbf{u} \cdot d\mathbf{x} = (\mathbf{v} + \mathbf{w}\zeta) \cdot (d\mathbf{X} + \zeta d\mathbf{H})$ and replacing ζ and ζ^2 with their mass-weighted averages of 0 and 1, respectively, we plausibly obtain the above expression $\mathbf{v} \cdot d\mathbf{X} + \mathbf{w} \cdot d\mathbf{H}$.

9 CASE OF A CENTRAL FORCE

For a central force deriving from a spherically symmetric potential $\Phi(R)$, where $R = |\mathbf{x}|$, we have

$$\nabla \Phi = \frac{1}{R} \frac{d\Phi}{dR} \mathbf{x}, \quad (105)$$

$$\mathbf{H} \cdot \nabla \nabla \Phi = \frac{1}{R} \frac{d}{dR} \left(\frac{1}{R} \frac{d\Phi}{dR} \right) (\mathbf{H} \cdot \mathbf{x}) \mathbf{x} + \frac{1}{R} \frac{d\Phi}{dR} \mathbf{H}, \quad (106)$$

$$\begin{aligned} \mathbf{H} \mathbf{H} : \nabla \nabla \nabla \Phi &= \frac{1}{R} \frac{d}{dR} \left[\frac{1}{R} \frac{d}{dR} \left(\frac{1}{R} \frac{d\Phi}{dR} \right) \right] (\mathbf{H} \cdot \mathbf{x})^2 \mathbf{x} \\ &+ \frac{1}{R} \frac{d}{dR} \left(\frac{1}{R} \frac{d\Phi}{dR} \right) [2(\mathbf{H} \cdot \mathbf{x}) \mathbf{H} + |\mathbf{H}|^2 \mathbf{x}]. \end{aligned} \quad (107)$$

In particular, a Newtonian point-mass potential has

$$\Phi = -\frac{GM}{R}, \quad (108)$$

$$\frac{1}{R} \frac{d\Phi}{dR} = \frac{GM}{R^3}, \quad (109)$$

$$\frac{1}{R} \frac{d}{dR} \left(\frac{1}{R} \frac{d\Phi}{dR} \right) = -\frac{3GM}{R^5}, \quad (110)$$

$$\frac{1}{R} \frac{d}{dR} \left[\frac{1}{R} \frac{d}{dR} \left(\frac{1}{R} \frac{d\Phi}{dR} \right) \right] = \frac{15GM}{R^7}. \quad (111)$$

Even though the equations of Section 7 are projected on to a reference plane, they do possess complete rotational symmetry in the case of a central force, and would have the same form for any choice of the reference plane. We will verify this in Section 12 below through the demonstration of a rigid-tilt mode of zero frequency.

10 THE SYMMETRIC CASE

An important special case occurs when the gravitational potential has reflectional symmetry about the reference plane and the disc also shares this symmetry. It is helpful to introduce the notation

$$\Psi = \left. \frac{\partial^2 \Phi}{\partial z^2} \right|_{z=0}. \quad (112)$$

Reflectional symmetry of the disc implies that $Z = 0$, so $\mathbf{n} = \mathbf{e}_z$ and $\mathbf{H} = H_z \mathbf{e}_z$. The vectors \mathbf{H} and \mathbf{w} are purely vertical, while \mathbf{v} is purely

horizontal. We can simplify the notation by writing w for w_z and \mathbf{v} for $\bar{\mathbf{v}}$. The equations then reduce to

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \bar{\nabla}\right) \mathbf{v} = -\bar{\nabla}\Phi - \frac{1}{2}H^2\bar{\nabla}\Psi - \frac{1}{\Sigma}\bar{\nabla}P, \quad (113)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \bar{\nabla}\right) w = -H\Psi + \frac{P}{\Sigma H}, \quad (114)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \bar{\nabla}\right) H = w, \quad (115)$$

together with appropriate equations for Σ and P (or equivalent variables), e.g.

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \bar{\nabla}\right) \Sigma = -\Sigma\bar{\nabla} \cdot \mathbf{v}, \quad (116)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \bar{\nabla}\right) P = -\gamma P\bar{\nabla} \cdot \mathbf{v} - \frac{(\gamma-1)Pw}{H}. \quad (117)$$

We may write an explicit 3D Eulerian representation of the fluid variables in this case as

$$\mathbf{u} \approx \mathbf{v}(\bar{\mathbf{x}}, t) + w(\bar{\mathbf{x}}, t)\zeta \mathbf{e}_z, \quad (118)$$

$$\rho \approx \bar{\rho}(\bar{\mathbf{x}}, t)F_\rho(\zeta), \quad (119)$$

$$p \approx \bar{p}(\bar{\mathbf{x}}, t)F_p(\zeta), \quad (120)$$

where

$$\zeta = \frac{z}{H}, \quad \bar{\rho} = \frac{\Sigma}{H}, \quad \bar{p} = \frac{P}{H}. \quad (121)$$

The specific energy and PV simplify to

$$\mathcal{E} = \frac{1}{2}(|\mathbf{v}|^2 + w^2) + \Phi + \frac{1}{2}H^2\Psi + \frac{P}{(\gamma-1)\Sigma}, \quad (122)$$

$$\mathbf{q} = \frac{1}{\Sigma}\mathbf{e}_z \cdot [\bar{\nabla} \times (\mathbf{v} + w\bar{\nabla}H)]. \quad (123)$$

11 AXISYMMETRIC EQUILIBRIUM AND LINEARIZED EQUATIONS

If the potential is also steady and axisymmetric, such that Φ and Ψ are functions of cylindrical radius r in the plane $z=0$, then the simplest solution shares these symmetries, having $\mathbf{v} = r\Omega(r)\mathbf{e}_\phi$, $\Sigma = \Sigma(r)$, $P = P(r)$, and $H = H(r)$, as well as $Z=0$, $\mathbf{n} = \mathbf{e}_z$, and $\mathbf{H} = H\mathbf{e}_z$ as in the previous section. Equations (113) and (114) give

$$-r\Omega^2 = -\frac{d\Phi}{dr} - \frac{1}{2}H^2\frac{d\Psi}{dr} - \frac{1}{\Sigma}\frac{dP}{dr}, \quad (124)$$

$$0 = -H\Psi + \frac{P}{\Sigma H}. \quad (125)$$

The second of these equations corresponds to the vertical hydrostatic equilibrium of the disc. The first equation represents the radial force balance, showing how the rotation of the disc differs from that of a particle orbit because of the thickness and pressure of the disc. The second term on the right-hand side of this equation does not appear in 2D hydrodynamics, although it is generally comparable to the third term; it represents the dilution of the radial gravitational force due to the thickness of the disc, and can be interpreted as the quadrupolar gravitational force acting on a fluid column. Let $\Omega_0(r)$

be the angular velocity of a circular particle orbit of radius r , given by

$$r\Omega_0^2 = \left.\frac{d\Phi}{dr}\right|_{z=0}. \quad (126)$$

The equilibrium conditions then reduce to

$$P = \Sigma H^2\Psi, \quad (127)$$

$$\frac{d}{dr}(\Sigma H^2\Psi) + \frac{1}{2}\Sigma H^2\frac{d\Psi}{dr} = \Sigma r(\Omega^2 - \Omega_0^2). \quad (128)$$

The linearized equations in case of small departures from this basic state separate into two decoupled subsystems. The first is relevant for perturbations that preserve the reflectional symmetry of the disc, and takes the form

$$Dv'_r - 2\Omega v'_\phi = -HH'\frac{d\Psi}{dr} - \frac{1}{\Sigma}\frac{\partial P'}{\partial r} + \frac{\Sigma'}{\Sigma^2}\frac{dP}{dr}, \quad (129)$$

$$Dv'_\phi + \frac{v'_r}{r}\frac{d}{dr}(r^2\Omega) = -\frac{1}{\Sigma r}\frac{\partial P'}{\partial \phi}, \quad (130)$$

$$D\Sigma' + v'_r\frac{d\Sigma}{dr} = -\frac{\Sigma}{r}\left[\frac{\partial}{\partial r}(rv'_r) + \frac{\partial v'_\phi}{\partial \phi}\right], \quad (131)$$

$$DP' + v'_r\frac{dP}{dr} = -\frac{\gamma P}{r}\left[\frac{\partial}{\partial r}(rv'_r) + \frac{\partial v'_\phi}{\partial \phi}\right] - \frac{(\gamma-1)Pw'}{H}, \quad (132)$$

$$Dw' = -H'\Psi + \left(\frac{P}{\Sigma H}\right)', \quad (133)$$

$$DH' + v'_r\frac{dH}{dr} = w', \quad (134)$$

where (r, ϕ) are polar coordinates on the reference plane and

$$D = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}. \quad (135)$$

Note that

$$\left(\frac{P}{\Sigma H}\right)' = \frac{P}{\Sigma H}\left(\frac{P'}{P} - \frac{\Sigma'}{\Sigma} - \frac{H'}{H}\right). \quad (136)$$

The second subsystem describes perturbations that break the reflectional symmetry of the disc, and takes the form

$$D^2Z' = -\Psi Z' - \frac{1}{2}\Xi H^2Z' - \frac{d\Psi}{dr}HH'_r + \frac{1}{\Sigma r}\left[\frac{\partial}{\partial r}\left(\frac{rPH'_r}{H}\right) + \frac{\partial}{\partial \phi}\left(\frac{PH'_\phi}{H}\right)\right], \quad (137)$$

$$(D^2 - \Omega^2)H'_r - 2\Omega DH'_\phi = -\frac{d}{dr}(r\Omega_0^2)H'_r - \frac{d\Psi}{dr}HZ' - \frac{P}{\Sigma H}\frac{\partial Z'}{\partial r}, \quad (138)$$

$$(D^2 - \Omega^2)H'_\phi + 2\Omega DH'_r = -\Omega_0^2 H'_\phi - \frac{P}{\Sigma Hr}\frac{\partial Z'}{\partial \phi}, \quad (139)$$

where

$$\Xi = \left.\frac{\partial^4\Phi}{\partial z^4}\right|_{z=0}. \quad (140)$$

We will discuss special slowly varying solutions of the linearized equations representing warped and eccentric discs in the following two sections. A complementary situation is one in which the perturbations have a short radial wavelength comparable to $H \ll r$. In this limit, the dominant variation of the perturbations is through the phase factor

$$\exp \left\{ i \left[\int k(r) dr + m\phi - \omega t \right] \right\}, \quad (141)$$

where $k(r)$ is a local radial wavenumber satisfying $|k|r \gg 1$, m (an integer of order unity) is an azimuthal wavenumber and ω is an angular frequency. Let $\hat{\omega} = \omega - m\Omega$ be the intrinsic wave frequency in the frame locally moving with the fluid. After some algebra, we find that the local dispersion relation is

$$\left(\hat{\omega}^2 - \kappa^2 - \frac{\gamma P}{\Sigma} k^2 \right) [\hat{\omega}^2 - (\gamma + 1)v^2] = \left[\frac{(\gamma - 1)Pk}{\Sigma H} \right]^2 \quad (142)$$

for symmetric modes and

$$(\hat{\omega}^2 - \kappa^2) (\hat{\omega}^2 - v^2) = \left(\frac{Pk}{\Sigma H} \right)^2 \quad (143)$$

for antisymmetric modes, where κ and v are the epicyclic and vertical frequencies given by

$$\kappa^2 = \frac{1}{r^3} \frac{d(r^4 \Omega_0^2)}{dr}, \quad (144)$$

$$v^2 = \Psi. \quad (145)$$

Each case admits two solutions for $\hat{\omega}^2$. The symmetric case involves a mixture of the classical density wave $\hat{\omega}^2 = \kappa^2 + \frac{\gamma P}{\Sigma} k^2$ with the breathing mode $\hat{\omega}^2 = (\gamma + 1)v^2$; these are coupled when $\gamma > 1$. The antisymmetric case involves a coupling of the epicyclic oscillation $\hat{\omega}^2 = \kappa^2$ with the vertical oscillation $\hat{\omega}^2 = v^2$. Typical dispersion relations for the case $\gamma = 5/3$ are shown in Fig. 3, where they are compared with the corresponding modes in a 3D polytropic disc (calculated as in Korycansky & Pringle 1995 or Ogilvie 1998). The polytropic disc is neutrally stratified in order to eliminate internal gravity waves. It can be seen from the figure that the affine model is accurate in describing this type of motion for $kH \ll 1$ and useful for $kH \lesssim 1$.

It is hardly surprising that the dispersion relation is inaccurate for $kH \gg 1$. In this limit, the higher frequency ($|\hat{\omega}| > \kappa$) modes of a polytropic disc become concentrated near the surfaces of the disc and the vertical structure of the velocity field is far removed from the simple linear profile assumed in the affine model. More concerning is the behaviour of the low-frequency antisymmetric mode. The smaller root for $\hat{\omega}^2$ vanishes at $kH = \kappa/v$ and becomes negative for larger kH , indicating instability on wavelengths smaller than a few H . This instability is unphysical and needs to be suppressed in numerical implementations unless they are of sufficiently low resolution. Its origin can be traced to the assumption made in Section 4 that the deformation of the disc is of large scale, leading to an approximation that makes the internal energy insensitive to spatial derivatives of \mathbf{H} . The simple modification F_1 proposed in Appendix B restores such a dependence; when it is applied, we find that the symmetric modes are unaffected, while the dispersion relation for antisymmetric modes becomes

$$\left(\hat{\omega}^2 - \kappa^2 - \frac{P}{\Sigma} k^2 \right) (\hat{\omega}^2 - v^2) = \left(\frac{Pk}{\Sigma H} \right)^2. \quad (146)$$

This modification stabilizes the low-frequency antisymmetric mode at large wavenumbers and in fact gives excellent agreement with

the 3D dispersion relation of the polytropic disc (Fig. 4); indeed it agrees exactly with the dispersion relation of $n = 1$ modes in a strictly isothermal disc. It is possible to improve the accuracy of the symmetric modes by making a similar modification involving derivatives of H_r . Including the term F_2 proposed in Appendix B changes the dispersion relation for symmetric modes to

$$\begin{aligned} & \left(\hat{\omega}^2 - \kappa^2 - \frac{\gamma P}{\Sigma} k^2 \right) \left[\hat{\omega}^2 - (\gamma + 1)v^2 - \frac{P}{\Sigma} k^2 \right] \\ & = \left[\frac{(\gamma - 1)Pk}{\Sigma H} \right]^2, \end{aligned} \quad (147)$$

which gives better agreement with the 3D dispersion relation of the polytropic disc (Fig. 4).

12 LINEAR THEORY OF WARPS

In this section, we assume that the potential is spherically symmetric, which implies

$$\Psi = \Omega_0^2, \quad \Xi = \frac{3}{r} \frac{d\Omega_0^2}{dr} \quad (148)$$

and eliminates nodal precession of inclined orbits.

For the antisymmetric perturbations, and assuming the azimuthal dependence $e^{-i\phi}$, we have

$$\begin{aligned} D^2 Z' &= -\Omega_0^2 Z' - \frac{d\Omega_0^2}{dr} \left(\frac{3H^2 Z'}{2r} + HH_r' \right) \\ &+ \frac{1}{\Sigma r} \left[\frac{\partial}{\partial r} \left(\frac{rPH_r'}{H} \right) - \frac{iPH_\phi'}{H} \right], \end{aligned} \quad (149)$$

$$\begin{aligned} (D^2 - \Omega^2)H_r' - 2\Omega DH_\phi' &= -\frac{d(r\Omega_0^2)}{dr} H_r' - \frac{d\Omega_0^2}{dr} HZ' \\ &- \frac{P}{\Sigma H} \frac{\partial Z'}{\partial r}, \end{aligned} \quad (150)$$

$$(D^2 - \Omega^2)H_\phi' + 2\Omega DH_r' = -\Omega_0^2 H_\phi' + \frac{iPZ'}{\Sigma Hr}, \quad (151)$$

with now

$$D = \frac{\partial}{\partial t} - i\Omega. \quad (152)$$

It is easily verified, using equations (127) and (128), that these equations are exactly satisfied by a time-independent rigid-tilt mode

$$Z' = -r, \quad H_r' = H, \quad H_\phi' = -iH, \quad (153)$$

which corresponds to an infinitesimal change in the orientation of the disc. This property is to be expected because of the complete rotational symmetry of the problem.

Slowly varying warps in an inviscid disc have been treated by Papaloizou & Lin (1995) and Ogilvie (1999), among others. The behaviour is complicated by a resonance that occurs in Keplerian discs owing to the coincidence of the orbital and epicyclic frequencies. In the non-resonant case, the secular scalings for slowly varying warps in a thin disc lead us to approximate Ω as Ω_0 and to neglect time derivatives except where the leading terms cancel in equation (149):

$$\begin{aligned} -2i\Omega_0 \frac{\partial Z'}{\partial t} &= \frac{Z'}{\Sigma r \Omega_0} \frac{d}{dr} (\Sigma H^2 \Omega_0^3) - \frac{d\Omega_0^2}{dr} \left(\frac{3H^2 Z'}{2r} + HH_r' \right) \\ &+ \frac{1}{\Sigma r} \left[\frac{\partial}{\partial r} \left(\frac{rPH_r'}{H} \right) - \frac{iPH_\phi'}{H} \right], \end{aligned} \quad (154)$$

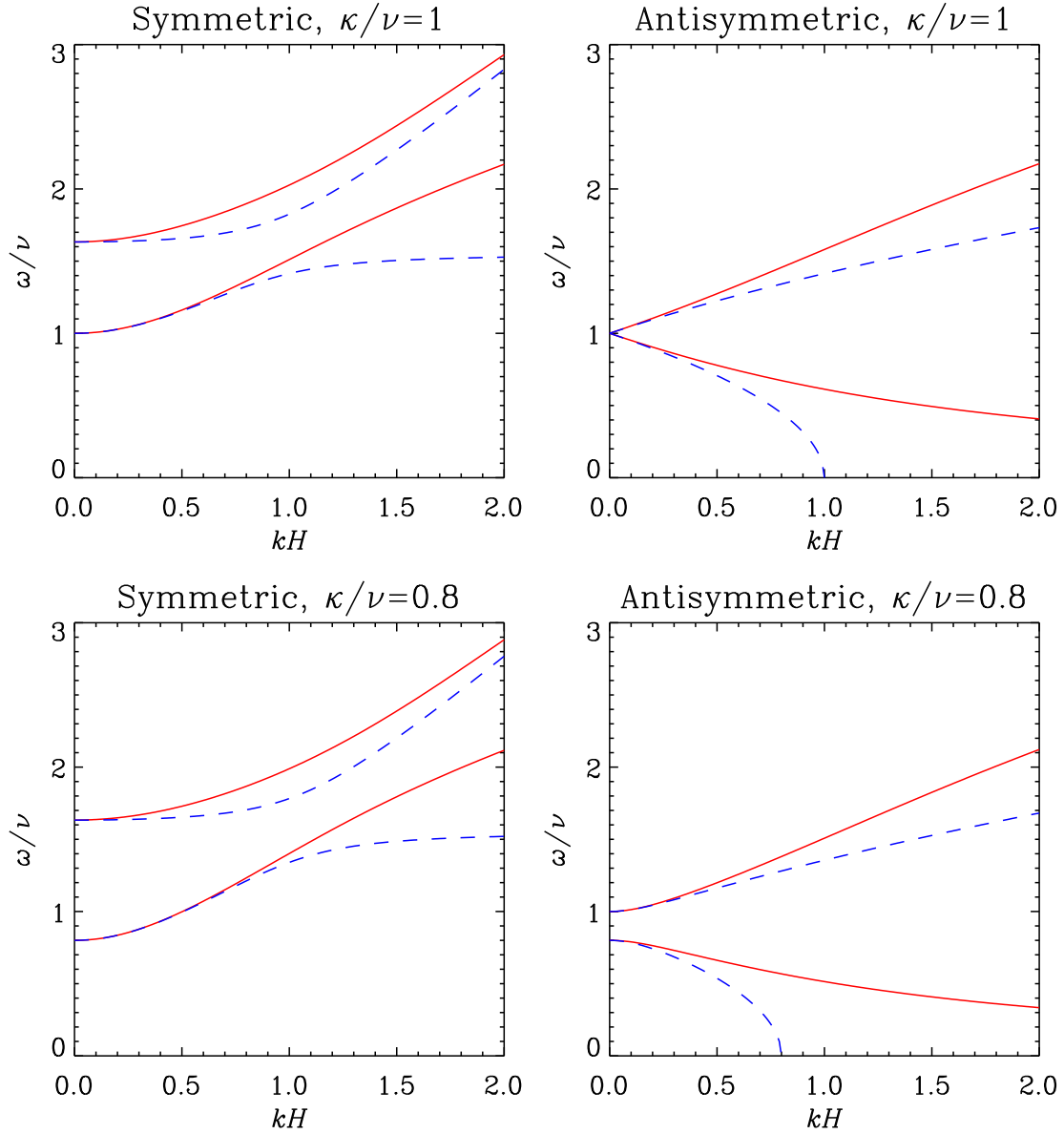


Figure 3. Local dispersion relation for a 3D polytropic disc (red solid lines) and in the unmodified affine model (blue dashed lines). In each case, $\gamma = 5/3$. The top panels are for a Keplerian disc and the lower two are for a non-Keplerian disc with $\kappa < \nu$. The left-hand panels show the two symmetric modes and the right-hand panels show the two antisymmetric modes. Other modes of the polytropic disc with higher vertical mode numbers are not plotted.

$$-\Omega_0^2 H'_r + 2i\Omega_0^2 H'_\phi = -\frac{d\Omega_0^2}{dr}(rH'_r + HZ') - \Omega_0^2 H \frac{\partial Z'}{\partial r}, \quad (155)$$

$$-\Omega_0^2 H'_\phi - 2i\Omega_0^2 H'_r = \frac{i\Omega_0^2 HZ'}{r}. \quad (156)$$

H'_ϕ can be eliminated to obtain

$$-2i\Omega_0 \frac{\partial Z'}{\partial t} = \frac{\Omega_0^2 Z'}{\Sigma} \frac{d}{dr} \left(\frac{\Sigma H^2}{r} \right) + \frac{r\Omega_0^2}{\Sigma} \frac{\partial}{\partial r} \left(\frac{\Sigma H H'_r}{r} \right), \quad (157)$$

$$\frac{d(r^3 \Omega_0^2)}{dr} \frac{H'_r}{H} = -\frac{\partial}{\partial r} \left(r^3 \Omega_0^2 \frac{Z'}{r} \right). \quad (158)$$

These combine into

$$2i\Sigma r^3 \Omega_0 \frac{\partial W}{\partial t} = \frac{\partial}{\partial r} \left[\frac{\Sigma H^2 r^5 \Omega_0^4}{d(r^3 \Omega_0^2)/dr} \frac{\partial W}{\partial r} \right], \quad (159)$$

where $W = -Z/r$ is the dimensionless tilt variable (related to the inclination angle) used by Papaloizou & Pringle (1983), Papaloizou & Lin (1995), and others. We see again that a stationary rigid tilt ($W = \text{constant}$) is a possible solution. This Schrödinger-like dispersive wave equation for the warp is exactly equivalent to equation (131) derived by Ogilvie (1999) from a global asymptotic analysis.

In the resonant case for a Keplerian disc ($\Omega_0 \propto r^{-3/2}$), equations (150) and (151) become degenerate, both reducing to $H'_\phi \approx -2iH'_r$ at leading order. Taking a (1, 2i) linear combination of these equations to eliminate the dominant terms, we obtain the approximation

$$-2i\Omega_0 \frac{\partial H'_r}{\partial t} = -r\Omega_0^2 H \frac{\partial}{\partial r} \left(\frac{Z'}{r} \right) \quad (160)$$

as well as

$$-2i\Omega_0 \frac{\partial Z'}{\partial t} = \frac{r\Omega_0^2}{\Sigma} \frac{\partial}{\partial r} \left(\frac{\Sigma H H'_r}{r} \right). \quad (161)$$

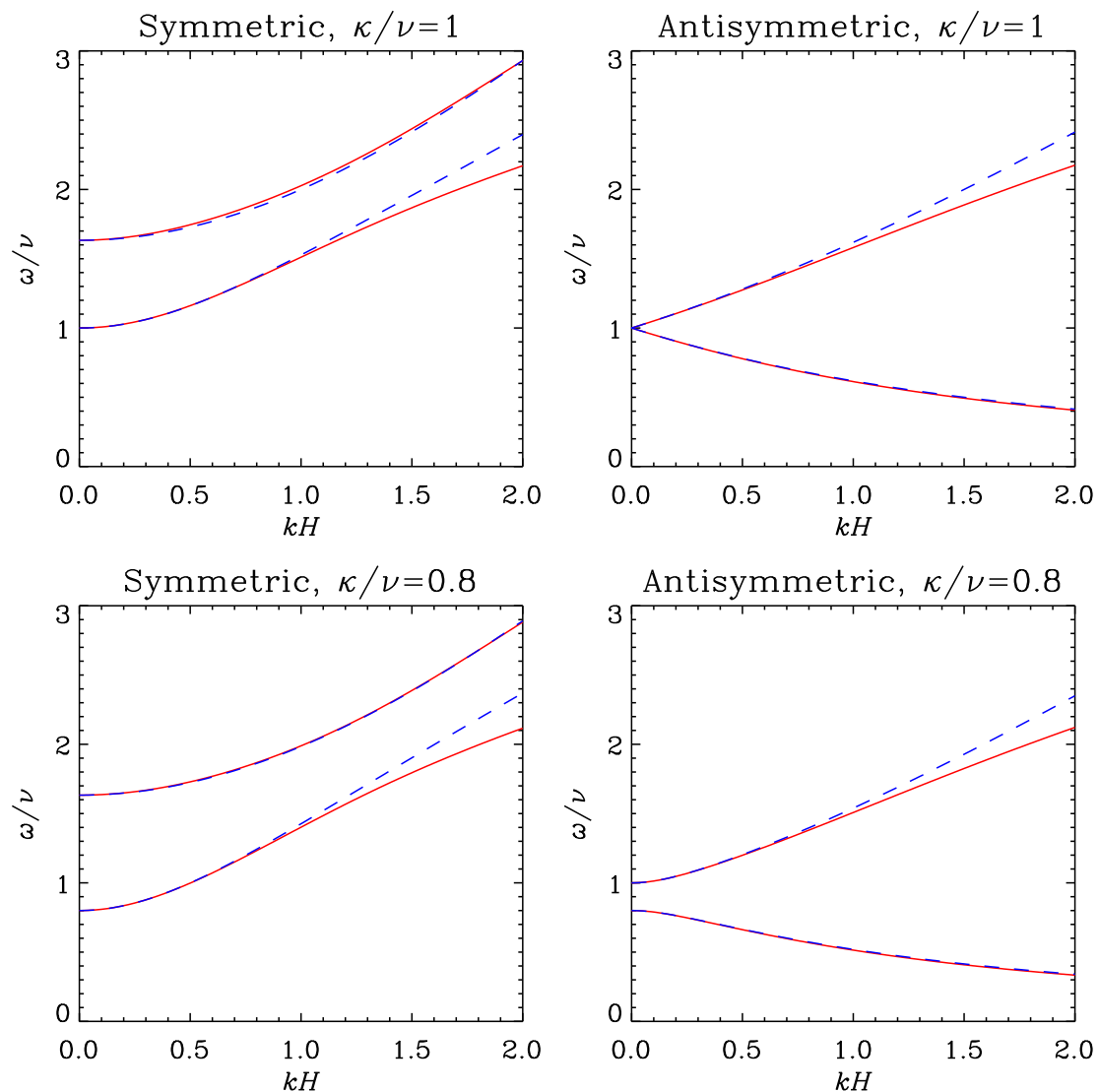


Figure 4. As for Fig. 3, except that the affine model (blue dashed lines) has been modified as described in the text to improve the accuracy and stability of the model on small scales.

Identifying Z with $-rW$ (as above) and $\Sigma H \Omega_0^2 r^2 H'_r$ with $2iG$, where G is a complex internal torque variable, we obtain exactly equations (5) and (6) of Lubow, Ogilvie & Pringle (2002) for an inviscid Keplerian disc, i.e.

$$\Sigma r^2 \Omega_0 \frac{\partial W}{\partial t} = \frac{1}{r} \frac{\partial G}{\partial r}, \quad (162)$$

$$\frac{\partial G}{\partial t} = \frac{1}{4} \Sigma H^2 r^3 \Omega_0^3 \frac{\partial W}{\partial r}, \quad (163)$$

which combine into a non-dispersive wave equation for W , with wave speed $H\Omega_0/2$.

13 LINEAR THEORY OF ECCENTRIC DISCS

We return to the linearized equations of Section 11 in the case of a point-mass potential for which $\Omega_0^2 = \Psi = GM/r^3$. To make a comparison between the affine model and the known secular theory of eccentric discs, we introduce the small parameter $\epsilon \ll 1$ such that $H/r = O(\epsilon)$, and use it to expand the quantities of the basic

state as

$$\Omega = \Omega_0 + \epsilon^2 \Omega_2 + \dots, \quad (164)$$

$$\Sigma = \Sigma_0 + \epsilon^2 \Sigma_2 + \dots, \quad (165)$$

$$P = \epsilon^2 (P_0 + \epsilon^2 P_2 + \dots), \quad (166)$$

$$H = \epsilon (H_0 + \epsilon^2 H_2 + \dots). \quad (167)$$

The equilibrium conditions (127) and (128) reduce at leading order to

$$P_0 = \Sigma_0 H_0^2 \Omega_0^2, \quad (168)$$

$$\frac{dP_0}{dr} - \frac{3P_0}{2r} = 2\Sigma_0 r \Omega_0 \Omega_2. \quad (169)$$

We describe a small eccentricity by considering reflectionally symmetric perturbations proportional to $e^{-i\phi}$. The linearized equations are

$$Dv'_r - 2\Omega v'_\phi = \frac{3\Sigma_0^2 H H'}{r} - \frac{1}{\Sigma} \frac{\partial P'}{\partial r} + \frac{\Sigma'}{\Sigma^2} \frac{dP}{dr}, \quad (170)$$

$$Dv'_\phi + \frac{v'_r}{r} \frac{d(r^2\Omega)}{dr} = \frac{iP'}{\Sigma r}, \quad (171)$$

$$D\Sigma' + v'_r \frac{d\Sigma}{dr} = -\frac{\Sigma}{r} \left[\frac{\partial}{\partial r}(rv'_r) - iv'_\phi \right], \quad (172)$$

$$DP' + v'_r \frac{dP}{dr} = -\frac{\gamma P}{r} \left[\frac{\partial}{\partial r}(rv'_r) - iv'_\phi \right] - \frac{(\gamma-1)Pw'}{H}, \quad (173)$$

$$Dw' = -\Omega_0^2 H' + \left(\frac{P}{\Sigma H} \right)', \quad (174)$$

$$DH' + v'_r \frac{dH}{dr} = w', \quad (175)$$

with

$$D = \frac{\partial}{\partial t} - i\Omega. \quad (176)$$

We then expand

$$v'_r = v'_{r0} + \epsilon^2 v'_{r2} + \dots, \quad (177)$$

$$v'_\phi = v'_{\phi0} + \epsilon^2 v'_{\phi2} + \dots, \quad (178)$$

$$w' = \epsilon w'_0 + \dots, \quad (179)$$

$$\Sigma' = \Sigma'_0 + \dots, \quad (180)$$

$$P' = \epsilon^2 (P'_0 + \dots), \quad (181)$$

$$H' = \epsilon (H'_0 + \dots), \quad (182)$$

where the perturbations depend on time through a slow variable $\tau = \epsilon^2 t$. The horizontal components of the equation of motion at leading order are

$$-i\Omega_0 v'_{r0} - 2\Omega_0 v'_{\phi0} = 0, \quad (183)$$

$$-i\Omega_0 v'_{\phi0} + \frac{1}{2}\Omega_0 v'_{r0} = 0, \quad (184)$$

with solution

$$v'_{r0} = ir\Omega_0 E(r, \tau), \quad v'_{\phi0} = \frac{1}{2}r\Omega_0 E(r, \tau), \quad (185)$$

representing a small eccentricity in the orbital motion. Here, E is the complex eccentricity used by Ogilvie (2001) and others. The remaining equations at leading order are

$$-i\Omega_0 \Sigma'_0 + ir\Omega_0 E \frac{d\Sigma_0}{dr} = -\Sigma_0 ir\Omega_0 \frac{\partial E}{\partial r}, \quad (186)$$

$$-i\Omega_0 P'_0 + ir\Omega_0 E \frac{dP_0}{dr} = -\gamma P_0 ir\Omega_0 \frac{\partial E}{\partial r} - \frac{(\gamma-1)P_0 w'_0}{H_0}, \quad (187)$$

$$-i\Omega_0 w'_0 = -\Omega_0^2 H'_0 + \frac{P_0}{\Sigma_0 H_0} \left(\frac{P'_0}{P_0} - \frac{\Sigma'_0}{\Sigma_0} - \frac{H'_0}{H_0} \right), \quad (188)$$

$$-i\Omega_0 H'_0 + ir\Omega_0 E \frac{dH_0}{dr} = w'_0, \quad (189)$$

which have the solution

$$\Sigma'_0 = r \frac{\partial(\Sigma_0 E)}{\partial r}, \quad (190)$$

$$P'_0 = Er \frac{dP_0}{dr} + \frac{P_0}{\gamma} \left[3(\gamma-1)E + (2\gamma-1)r \frac{\partial E}{\partial r} \right], \quad (191)$$

$$w'_0 = \frac{i\Omega_0 H_0}{\gamma} \left[3E - (\gamma-1)r \frac{\partial E}{\partial r} \right], \quad (192)$$

$$H'_0 = Er \frac{dH_0}{dr} - \frac{H_0}{\gamma} \left[3E - (\gamma-1)r \frac{\partial E}{\partial r} \right]. \quad (193)$$

Finally, the horizontal components of the equation of motion at $O(\epsilon^2)$ are

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} - i\Omega_2 \right) v'_{r0} - i\Omega_0 v'_{r2} - 2\Omega_2 v'_{\phi0} - 2\Omega_0 v'_{\phi2} \\ = \frac{3\Omega_0^2 H_0 H'_0}{r} - \frac{1}{\Sigma_0} \frac{\partial P'_0}{\partial r} + \frac{\Sigma'_0}{\Sigma_0^2} \frac{dP_0}{dr}, \end{aligned} \quad (194)$$

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} - i\Omega_2 \right) v'_{\phi0} - i\Omega_0 v'_{\phi2} + \frac{1}{2}\Omega_0 v'_{r2} + \frac{v'_{r0}}{r} \frac{d}{dr}(r^2\Omega_2) \\ = \frac{iP'_0}{\Sigma_0 r}. \end{aligned} \quad (195)$$

We eliminate v'_{r2} and $v'_{\phi2}$ by taking the (1, 2i) linear combination of these equations:

$$\begin{aligned} 2ir\Omega_0 \frac{\partial E}{\partial \tau} = 2r^{1/2}\Omega_0 E \frac{d}{dr}(r^{3/2}\Omega_2) + \frac{3\Omega_0^2 H_0 H'_0}{r} \\ - \frac{1}{\Sigma_0 r^2} \frac{\partial}{\partial r}(P'_0 r^2) + \frac{\Sigma'_0}{\Sigma_0^2} \frac{dP_0}{dr}. \end{aligned} \quad (196)$$

Substituting for Ω_2 , Σ'_0 , P'_0 , H'_0 , and multiplying by $-\Sigma_0 r$, we obtain

$$\begin{aligned} -2i\Sigma_0 r^2 \Omega_0 \frac{\partial E}{\partial \tau} = \Sigma_0 E \frac{d}{dr} \left(\frac{3rP_0}{2\Sigma_0} - \frac{r^2}{\Sigma_0} \frac{dP_0}{dr} \right) \\ - 3\Sigma_0 \Omega_0^2 Er H_0 \frac{dH_0}{dr} + \frac{3}{\gamma} \Sigma_0 \Omega_0^2 H_0^2 \left[3E - (\gamma-1)r \frac{\partial E}{\partial r} \right] \\ + \frac{1}{r} \frac{\partial}{\partial r} \left\{ Er^3 \frac{dP_0}{dr} + \frac{P_0 r^2}{\gamma} \left[3(\gamma-1)E + (2\gamma-1)r \frac{\partial E}{\partial r} \right] \right\} \\ - \frac{r^2}{\Sigma_0} \frac{dP_0}{dr} \frac{\partial(\Sigma_0 E)}{\partial r}, \end{aligned} \quad (197)$$

which simplifies to

$$\begin{aligned} -2i\Sigma_0 r^2 \Omega_0 \frac{\partial E}{\partial \tau} = \frac{1}{r} \frac{\partial}{\partial r} \left[\left(2 - \frac{1}{\gamma} \right) P_0 r^3 \frac{\partial E}{\partial r} \right] \\ + \left(4 - \frac{3}{\gamma} \right) r \frac{dP_0}{dr} E + 3 \left(1 + \frac{1}{\gamma} \right) P_0 E. \end{aligned} \quad (198)$$

This Schrödinger-like dispersive wave equation agrees exactly with the linear equation for the secular evolution of eccentricity in a 3D adiabatic disc, as found in equation (2) of Teyssandier & Ogilvie (2016) or equation (176) of Ogilvie & Barker (2014).

14 CONCLUSIONS

In this paper, we have presented an affine model of the dynamics of astrophysical discs. It extends the 2D hydrodynamic equations that are often applied without adequate justification to thin discs. The additional degrees of freedom included here allow the disc to expand and contract in the vertical direction, to undergo deformation of the mid-plane and to develop the internal shearing motions that accompany such deformations. All of these are necessary to describe eccentric and warped discs and we have shown that the model exactly reproduces the linear secular theory of such discs in an appropriate limit. However, it does not rely on any secular or

small-amplitude approximation and so should be useful in describing discs with general combinations of tidal deformations, density waves, eccentricity, and warping. The equations of the affine model are 2D partial differential equations that can be seen as a useful and generally more applicable extension of the 2D hydrodynamic model.

The affine model is derived here, in the case of an ideal fluid, from Hamilton's Principle after making a specific approximation to the deformation gradient tensor. It naturally incorporates conservation laws for total energy and PV, even for non-planar discs. We have therefore shown that PV or vortensity can be defined for thin discs with variable thickness and with deformable mid-planes.

Future work should consider the numerical implementation of the equations of the affine model and their application to various astrophysical problems of interest. It would be valuable to include non-ideal effects such as viscous or other shear stresses, heating and cooling. It may also be possible to incorporate self-gravity and magnetic fields in some approximation.

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APPENDIX A: EQUATIONS OF THE AFFINE MODEL IN CARTESIAN COORDINATES

In Cartesian coordinates, the equations of Section 7 read

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) v_x - \Phi_x - \left(\frac{1}{2} H_x^2 \Phi_{xxx} + \frac{1}{2} H_y^2 \Phi_{xyy} + \frac{1}{2} H_z^2 \Phi_{xzz} + H_x H_y \Phi_{xxy} + H_x H_z \Phi_{xxz} + H_y H_z \Phi_{xyz}\right) - \frac{1}{\Sigma} \frac{\partial P}{\partial X} - \frac{1}{\Sigma} \frac{\partial}{\partial X} \left(\frac{P H_x}{H} \frac{\partial Z}{\partial X}\right) - \frac{1}{\Sigma} \frac{\partial}{\partial Y} \left(\frac{P H_y}{H} \frac{\partial Z}{\partial X}\right), \quad (\text{A1})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) v_y = -\Phi_y - \left(\frac{1}{2} H_x^2 \Phi_{xxy} + \frac{1}{2} H_y^2 \Phi_{yyy} + \frac{1}{2} H_z^2 \Phi_{yzz} + H_x H_y \Phi_{xyy} + H_x H_z \Phi_{xyz} + H_y H_z \Phi_{yyz}\right) - \frac{1}{\Sigma} \frac{\partial P}{\partial Y} - \frac{1}{\Sigma} \frac{\partial}{\partial X} \left(\frac{P H_x}{H} \frac{\partial Z}{\partial Y}\right) - \frac{1}{\Sigma} \frac{\partial}{\partial Y} \left(\frac{P H_y}{H} \frac{\partial Z}{\partial Y}\right), \quad (\text{A2})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) v_z = -\Phi_z - \left(\frac{1}{2} H_x^2 \Phi_{xxz} + \frac{1}{2} H_y^2 \Phi_{yyz} + \frac{1}{2} H_z^2 \Phi_{zzz} + H_x H_y \Phi_{xyz} + H_x H_z \Phi_{xzz} + H_y H_z \Phi_{yzz}\right) + \frac{1}{\Sigma} \frac{\partial}{\partial X} \left(\frac{P H_x}{H}\right) + \frac{1}{\Sigma} \frac{\partial}{\partial Y} \left(\frac{P H_y}{H}\right), \quad (\text{A3})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) w_x = -H_x \Phi_{xx} - H_y \Phi_{xy} - H_z \Phi_{xz} - \frac{P}{\Sigma H} \frac{\partial Z}{\partial X}, \quad (\text{A4})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) w_y = -H_x \Phi_{xy} - H_y \Phi_{yy} - H_z \Phi_{yz} - \frac{P}{\Sigma H} \frac{\partial Z}{\partial Y}, \quad (\text{A5})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) w_z = -H_x \Phi_{xz} - H_y \Phi_{yz} - H_z \Phi_{zz} + \frac{P}{\Sigma H}, \quad (\text{A6})$$

with

$$H = H_z - H_x \frac{\partial Z}{\partial X} - H_y \frac{\partial Z}{\partial Y}, \quad (\text{A7})$$

$$v_z = \left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) Z, \quad (\text{A8})$$

$$w_x = \left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) H_x, \quad (\text{A9})$$

$$w_y = \left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y}\right) H_y, \quad (\text{A10})$$

$$w_z = \left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) H_z. \quad (\text{A11})$$

Here

$$\Phi_{xyz} = \frac{\partial^3 \Phi}{\partial x \partial y \partial z} \Big|_{x=X, y=Y, z=Z} \quad (\text{A12})$$

etc. The remaining equations can be written as, e.g.

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) \Sigma = -\Sigma \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y} \right), \quad (\text{A13})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) P = -\gamma P \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y} \right) - \frac{(\gamma - 1)P}{H} \frac{DH}{Dt}, \quad (\text{A14})$$

where

$$\frac{DH}{Dt} = w_z - H_x \frac{\partial v_z}{\partial X} - H_y \frac{\partial v_z}{\partial Y} - \frac{\partial Z}{\partial X} \left(w_x - H_x \frac{\partial v_x}{\partial X} - H_y \frac{\partial v_x}{\partial Y} \right) - \frac{\partial Z}{\partial Y} \left(w_y - H_x \frac{\partial v_y}{\partial X} - H_y \frac{\partial v_y}{\partial Y} \right). \quad (\text{A15})$$

In the symmetric case discussed in Section 10, these equations reduce to

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) v_x = -\Phi_x - \frac{1}{2} H^2 \Psi_x - \frac{1}{\Sigma} \frac{\partial P}{\partial X}, \quad (\text{A16})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) v_y = -\Phi_y - \frac{1}{2} H^2 \Psi_y - \frac{1}{\Sigma} \frac{\partial P}{\partial Y}, \quad (\text{A17})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) w = -\Psi H + \frac{P}{\Sigma H}, \quad (\text{A18})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) H = w, \quad (\text{A19})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) \Sigma = -\Sigma \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y} \right), \quad (\text{A20})$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial X} + v_y \frac{\partial}{\partial Y} \right) P = -\gamma P \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y} \right) - \frac{(\gamma - 1)Pw}{H}, \quad (\text{A21})$$

where

$$\Psi = \frac{\partial^2 \Phi}{\partial z^2} \Big|_{z=0}. \quad (\text{A22})$$

APPENDIX B: ADDITIONAL TERMS RESULTING FROM EXTENSIONS OF THE THIN-DISC APPROXIMATION

The exact Jacobian determinant of the second stage of the map (equation 26) is a quadratic function of ζ in which the terms dependent on ζ involve spatial derivatives of \mathbf{H} . The approximate expression H is subject to the correction factor

$$1 + \left\{ \bar{\mathbf{V}} \cdot \bar{\mathbf{H}} - \frac{1}{H} \mathbf{n} \cdot [(\bar{\mathbf{H}} \cdot \bar{\mathbf{V}}) \mathbf{H}] \right\} \zeta + \frac{\mathbf{H}}{H} \cdot \left(\frac{\partial \mathbf{H}}{\partial X} \times \frac{\partial \mathbf{H}}{\partial Y} \right) \zeta^2. \quad (\text{B1})$$

These terms become important when the condition $\|\bar{\mathbf{V}} \mathbf{H}\|$ is not satisfied, and the assumption of a uniform expansion or contraction of the fluid columns is violated. In principle, the exact internal energy associated with the affine transformation could be computed, as a function of $\bar{\mathbf{V}} \mathbf{H}$, by raising this expression to the power $-(\gamma - 1)$ and integrating over ζ , weighted by $F_p(\zeta)$. We will not pursue this approach because the complexity it introduces is not justified by the simplicity of our assumption regarding the affine transformation.

Let us write the correction factor (after averaging over ζ) as e^F , where F depends on the spatial derivatives of \mathbf{H} ; it might also depend on H itself and (through \mathbf{n}) on the spatial derivatives of Z . We then have

$$J_3 = J_2 \frac{H}{H_0} e^F, \quad (\text{B2})$$

and the new factor of $e^{-(\gamma - 1)F}$ in the internal energy contribution to the Lagrangian gives rise to the following additional terms in the equations of motion:

$$\frac{D^2 \bar{X}_i}{Dt^2} = \dots + \frac{1}{\Sigma} \frac{\partial}{\partial \bar{X}_j} \left[P \frac{\partial H_k}{\partial \bar{X}_i} \frac{\partial F}{\partial (\partial H_k / \partial \bar{X}_j)} \right] + \frac{1}{\Sigma} \frac{\partial}{\partial \bar{X}_j} \left[P \frac{\partial Z}{\partial \bar{X}_i} \frac{\partial F}{\partial (\partial Z / \partial \bar{X}_j)} \right], \quad (\text{B3})$$

$$\frac{D^2 Z}{Dt^2} = \dots - \frac{1}{\Sigma} \frac{\partial}{\partial \bar{X}_j} \left[P \frac{\partial F}{\partial (\partial Z / \partial \bar{X}_j)} \right], \quad (\text{B4})$$

$$\frac{D^2 H_i}{Dt^2} = \dots + \frac{P}{\Sigma} \frac{\partial F}{\partial H_i} - \frac{1}{\Sigma} \frac{\partial}{\partial \bar{X}_j} \left[P \frac{\partial F}{\partial (\partial H_i / \partial \bar{X}_j)} \right], \quad (\text{B5})$$

where summation over $j = \{1, 2\}$ is implied. The evolutionary equation for P is also modified to

$$\frac{DP}{Dt} = \dots - (\gamma - 1)P \frac{DF}{Dt}, \quad (\text{B6})$$

which, together with the modified equations for \bar{X}_i and \bar{H}_i , conserves the total energy in the same form as equation (90).²

A useful model is

$$F_1 = -\frac{1}{2}(\bar{\nabla} \cdot \bar{\mathbf{H}})^2. \quad (\text{B7})$$

This is motivated by the first correction term $(\bar{\nabla} \cdot \bar{\mathbf{H}})\zeta$ in equation (B1), which is relatively easy to understand. If H_x increases with X , for example, then the variable tilt of the fluid columns rarifies the disc above the mid-plane and compresses it below. The increase in net internal energy at second order is here modelled by the factor $e^{-(\gamma-1)F_1}$. This model produces the following additional terms:

$$\frac{D^2 \bar{X}_i}{Dt^2} = \dots - \frac{1}{\Sigma} \frac{\partial}{\partial \bar{X}_j} \left[P(\bar{\nabla} \cdot \bar{\mathbf{H}}) \frac{\partial \bar{H}_j}{\partial \bar{X}_i} \right], \quad (\text{B8})$$

$$\frac{D^2 \bar{\mathbf{H}}}{Dt^2} = \dots + \frac{1}{\Sigma} \bar{\nabla} [P(\bar{\nabla} \cdot \bar{\mathbf{H}})], \quad (\text{B9})$$

$$\frac{DP}{Dt} = \dots + (\gamma - 1)P(\bar{\nabla} \cdot \bar{\mathbf{H}}) \left(\bar{\nabla} \cdot \bar{\mathbf{w}} - \frac{\partial \bar{v}_i}{\partial \bar{X}_j} \frac{\partial \bar{H}_j}{\partial \bar{X}_i} \right). \quad (\text{B10})$$

A second useful modification is to add

$$F_2 = -\frac{1}{2}|\bar{\nabla} H_z|^2. \quad (\text{B11})$$

Although harder to justify based on equation (B1), this model produces the following additional terms that are found to improve the dispersion relation of symmetric modes at short wavelengths:

$$\frac{D^2 \bar{X}_i}{Dt^2} = \dots - \frac{1}{\Sigma} \frac{\partial}{\partial \bar{X}_j} \left(P \frac{\partial H_z}{\partial \bar{X}_j} \frac{\partial H_z}{\partial \bar{X}_i} \right), \quad (\text{B12})$$

$$\frac{D^2 H_z}{Dt^2} = \dots + \frac{1}{\Sigma} \bar{\nabla} \cdot (P \bar{\nabla} H_z), \quad (\text{B13})$$

$$\frac{DP}{Dt} = \dots + (\gamma - 1)P(\bar{\nabla} H_z) \cdot \left(\bar{\nabla} w_z - \frac{\partial H_z}{\partial \bar{X}_i} \bar{\nabla} v_i \right). \quad (\text{B14})$$

² Note that the ‘novel’ terms involving \mathbf{n} , already present in equations (71) and (72), derive from the above rules applied to the function $F = \ln H = \ln(\mathbf{H} \cdot \mathbf{n})$, which describes the departure of J_3 from J_2 in our standard affine model.

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