Games on Graphs and Other Combinatorial Problems

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I would like to dedicate this thesis to my family.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and includes nothing which is the outcome of work done in collaboration, except where indicated in the text.

Chapter 2 is based on joint work with Adva Mond and Victor Souza.

Chapter 6 is based on joint work with Peter van Hintum and Marius Tiba.

Chapter 7 is based on joint work with Ohad Klein.

Chapter 8 is based on joint work with Peter van Hintum, Amy Shaw and Marius Tiba.

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Abstract

In this dissertation, we consider various combinatorial problems. The four chapters after the Introduction concern games on graphs, while latter on, we make progress on some questions in the settings of Rademacher sums and graph theory.

In Chapter 2, we study the (m,b) Maker-Breaker percolation game. This game, played by two players on the square lattice, was introduced by Day and Falgas-Ravry. The outcome of this game depends crucially on the parameters m and b. Day and Falgas-Ravry showed that Breaker wins whenever $b \geqslant 2m$, but their approach then faces a barrier. We introduce a new, more global approach to study this game and to improve their results: we show that Breaker can in fact guarantee victory whenever $b \geqslant (2 - 1/14 + o(1))m$. We also show that Breaker can win very fast in a different variant of this game as long as $b \geqslant 2m$.

In Chapters 3 and 4, we look at the Waiter-Client K_k-factor game, first studied by Clemens et al. Here, it is known that Waiter wins, and the question is how long the game will last if Waiter aims to win as fast as possible, Client tries to delay her as much as possible, and both players play optimally.

In Chapter 3, we determine the duration of the game under the optimal play of both players when $k = 3$, resolving the conjecture of Clemens et al. After that, we study the game for large k in Chapter 4, and obtain the first known non-trivial lower bound for its duration in this case.

In Chapter 5, we consider the so-called restricted online Ramsey numbers, which correspond to a certain colouring game in the Builder-Painter setup. We provide a tight lower bound for the restricted online Ramsey numbers of matchings as long as the number of the allowed colours is small, resolving the conjecture of Briggs and Cox.

The setting in the next two chapters is the following. Set $X = \sum_{i=1}^{n} a_i \varepsilon_i$, where ε_i are Rademacher random variables, i.e. independent, identically distributed random variables taking values ± 1 with probabilities $1/2$ each, and $\{a_i\}$ are arbitary real numbers.

In Chapter 6, we make progress towards an old conjecture of Tomaszewski, which concerns concentration of such random variables X. In Chapter 7, we study the reverse problem and build up a framework that allows us to show anti-concentration results for Rademacher sums, and in turn we significantly improve the known results in this setting.

In Chapter 8, we obtain the best possible bounds for the following problem, first studied by Erdős, Pach, Pollak and Tuza: given a connected, triangle-free graph on n vertices and of minimum degree at least δ, how large can the radius of such a graph be? We also study the variant of this problem in which the triangle-free condition is replaced by a condition about the girth of our graph.

In Chapter 9, we construct P_n-induced-saturated graph for each $n \geqslant 6$, answering the question of Axenovich and Csikós.

Finally, in Chapter 10, we obtain a result about the existence of the antipodal paths with few colour changes in a two-colouring of the edges of the hypercube graph.

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Chapter 1

Introduction

1.1 Structure

This dissertation consists of ten chapters. In the first chapter, we introduce the problems that we will consider, while each of the remaining nine chapters is about one of these problems.

The problems we study in Chapters 2, 3, 4 and 5 concern combinatorial games. Moreover, all of these games are played on graphs. Chapters 6 and 7 look on two variants of a question about sums of Rademacher random variables. In Chapter 6, we prove a certain concentration result in this setting, while in Chapter 7, we prove a similar anti-concentration result. Finally, in Chapters 8, 9 and 10, we obtain several results related to various parts of classical graph theory.

Below, we describe the contents in more detail.

1.2 Games on graphs

The four chapters after the Introduction deal with various combinatorial games on graphs, played by two players. We always try to investigate which player has a winning strategy, or, if that is already known, how fast the winning player can guarantee victory. The games are quite varied; in particular, our results concern well-known games like the Maker-Breaker, Waiter-Client and Builder-Painter games.

The unifying theme of Chapters 2, 3 and 4 is that they all consider so-called *positional games*. In a positional game we have a finite or infinite set Λ, called a *board*; a family of subsets of Λ, called *winning sets*; and a rule determining which player wins the game. These games attracted wide attention, starting with the papers of Hales and Jewett [43] and Erdős and Selfridge [33]. Probably the most studied positional games are the *Maker-Breaker games* we shall consider in Chapter 2.

The Maker-Breaker percolation game was introduced in two recent papers of Day and Falgas-Ravry [21, 22]. In the (m,b) Maker-Breaker percolation game, where $m,b \geqslant 1$ are fixed integers, two players called Maker and Breaker alternate in claiming the edges of $\mathbb{Z}^2$. Maker starts and in each turn claims m yet unclaimed edges, while in each of his turns, Breaker claims b yet unclaimed edges. If it ever happens that the connected component containing the origin and consisting of the edges of Maker and of the yet unclaimed edges becomes finite, Breaker wins; otherwise, Maker wins. One can use a pairing argument to show that Maker wins the $(m,1)$-game for any $m \geqslant 1$ and an argument involving perimeter to show that Breaker wins the $(1,b)$-game for any $b \geqslant 2$ (for more details, we refer the reader to Chapter 2), but what happens when $m,b \geqslant 2$ is lot more difficult to understand. Day and Falgas-Ravry showed that if $m \geqslant 2b$, Maker can guarantee victory; and if $b \geqslant 2m$, Breaker can guarantee victory. But the multiplicative constant 2 is tight for their arguments and in fact there is a perimetric barrier not allowing local arguments to improve it.

In Chapter 2, which is joint work with Adva Mond and Victor Souza and was adapted from parts of [29], we introduce a more global approach to study the game, which enables us to break this barrier and show that Breaker wins the $(m,(2-1/14+o(1))m)$-game. Addressing further questions of Day and Falgas-Ravry, we also show that with twice the power of Maker, Breaker can win very fast even if Maker is allowed to claim many edges before the game starts.

Next, we consider the *Waiter-Client games*. In particular, for a fixed graph H, we look at the unbiased H-factor Waiter-Client game on the edges of the complete graph, recently studied by Clemens et al. [15]. Waiter and Client play the following game on the edges of K_n (where n is divisible by the number of vertices of H). In each round, Waiter picks two edges that were not picked in any of the previous rounds. Client chooses one of these two edges to be added to the Waiter's graph and one to be added to the Client's graph. Waiter wins if she forces Client to create a H-factor in the Client's graph at some point; otherwise, Client wins.

Regardless of what graph H we pick, for n large enough (dependent on H), Waiter can win - this was previously observed in the literature, and it is also a consequence of one of the results that we prove. Clemens et al. considered the question how many rounds the game will last in the case when H is a complete graph K_k, Waiter aims to win as fast as possible, Client aims to delay her as much as possible, and both players play optimally. They conjectured that in the case $k=3$, i.e. when the winning sets are triangle-factors, the answer should be $\frac{7}{6}n+o(n)$, for which they obtained the corresponding upper bound. They also asked for non-trivial upper and lower bounds for large k.

In Chapter 3, which was adapted from [26], we verify the conjecture of Clemens et al. about the triangle-factor game and show that its duration when both players play optimally is indeed $\frac{7}{6}n + o(n)$ rounds. So far all the tight results for fast winning strategies for Waiter-Client games (and also for Maker-Breaker games) concern spanning structures which can be obtained *perfectly fast* (i.e. the number of wasted rounds is one) or *asymptotically fast* (i.e. the number of wasted rounds is of smaller order than the size of a smallest winning set). Our result provides the first non-trivial example of a game which is not won perfectly or asymptotically fast, but for which the asymptotic number of rounds under optimal play has been determined.

In Chapter 4, which was adapted from [27], we consider the k-clique-factor game for large k and obtain the first non-trivial lower bound. The strategy that Client uses is a simple random one, and we define certain carefully chosen probability events to carry out our analysis. The proof is rather technical, and hence for greater clarity, we first illustrate our method by deriving a somewhat weaker bound through similar techniques for which the proof is easier to motivate and understand.

Finally, we turn our attention to the following game between Builder and Painter. We take some families of graphs $\mathscr{G}_1, \dots, \mathscr{G}_t$ and an integer n such that $n \geqslant R(\mathscr{G}_1, \dots, \mathscr{G}_t)$. In each turn, Builder picks an edge of initially uncoloured K_n and Painter colours that edge with some colour $i \in \{1, \dots, t\}$ of her choice. The game ends when a graph G_i in colour i for some $G_i \in \mathscr{G}_i$ and some i is created. The *restricted online Ramsey number* $\tilde{R}(\mathscr{G}_1, \dots, \mathscr{G}_t; n)$ is the minimum number of turns that Builder needs to guarantee the game to end.

In a recent paper, Briggs and Cox [11] studied the restricted online Ramsey numbers of matchings and determined a general upper bound for them. They proved that for $n = 3r - 1 = R_2(rK_2)$ we have $\tilde{R}_2(rK_2; n) \leqslant n - 1$ and asked whether this was tight.

In Chapter 5, which was adapted from [25], we verify that the upper bound above is tight. We also obtain analogous such result for the case of three colours and resolve the case of four colours up to the precise value of the additive constant.

1.3 Rademacher sums

Chapters 6 and 7 are about the problems concerning *Rademacher sums*, which we now introduce.

Consider the following simple setting. We pick a natural number n and take arbitrary real numbers $a_1, \dots, a_n$. After, we set $X = \sum_{i=1}^{n} a_i \varepsilon_i$, where ε_i are *Rademacher random variables*, i.e. independent, identically distributed random variables taking values ± 1 with probabilities

$1/2$ each. There are several old conjectures about how concentrated or anti-concentrated such a random variable X must be in terms of the standard deviation of X.

The conjecture that received most attention is one of Tomaszewski (see [42]) which asserts that, regardless of our choice of n and $a_1, \ldots, a_n$, we always have probability at least $1/2$ that X is within one standard deviation of its mean (which is clearly zero). A particular difficulty of this conjecture is the broad variety of collections $\{a_i\}$ for which we have $\mathbb{P}\Big[|X| \leqslant \sqrt{\mathrm{Var}(X)}\Big] \geqslant 1/2$, but $\mathbb{P}\Big[|X| < \sqrt{\mathrm{Var}(X)}\Big] < 1/2$. Many probabilistic inequalities cannot differentiate between these two probabilities, and hence one must combine them with more analytical tools as well.

In Chapter 6, which is joint work with Peter van Hintum and Marius Tiba and was adapted from [31], we make some progress towards this conjecture, proving it for a slightly weaker constant 0.46. This improves the previous best known such bound of this type with constant 0.4276, proven independently by Boppana and by Hendrinks and van Zuijlen (later combined into one publication [9]). Our techniques, which differ from the ones used previously, enable us to bound the desired probability below by the solutions of certain linear programming problems. This new approach turns out very useful, as it enables us to use several of the methods handy for these sorts of problems, such as the second moment method or the mirroring arguments for random walks, at the same time and in their full power.

In Chapter 7, which is joint work with Ohad Klein and was adapted from [28], we stay in this setting but now look on how anti-concentrated X must be. There are fewer tools to use than in the concentration direction, and hence in this question the previous attempts at proving strong results were not very successful. Also, similarly to before, we face the difficulty of various collections $\{a_i\}$ for which $\mathbb{P}\Big[|X| \geqslant \sqrt{\mathrm{Var}(X)}\Big]$ is much greater than $\mathbb{P}\Big[|X| > \sqrt{\mathrm{Var}(X)}\Big]$. We build up a framework that enables us not only to prove much stronger results than previously known in this setting, but also would be useful when tackling similar such problems in the future. In particular, in order to help us, we estimate a certain more general function. Our tools are varied, ranging from the combinatorial ones such as relating our problem to the chains in the hypercube graph, to the probabilistic ones such as Prawitz's smoothing inequality.

1.4 Classical graph theory

In the last three chapters, we consider problems from various parts of classical graph theory.

Erdős, Pach, Pollack and Tuza [36] studied the following problem in the extremal graph theory. Fix integers n and $\delta \geqslant 2$. Given a connected, triangle-free graph on n vertices and of minimum degree at least δ, what is the largest possible value of r, the radius of our graph?

Erdős, Pach, Pollack and Tuza resolved the question, up to the value of the additive constant. Nonetheless, the exact answer is of interest, especially as there exists a simple family of graphs that is a natural candidate to be the extremal example.

In Chapter 8, which is joint work with Peter van Hintum, Amy Shaw and Marius Tiba and was adapted from [30], we determine the precise optimal value for all n, δ. In particular, this shows that the family mentioned above indeed is the extremal example. To prove our result, we have to engage in certain case analysis - most cases are quite easy to resolve with the methods we develop, but there are several more difficult cases where more elaborate arguments are necessary. We also consider more general version of the problem, where the triangle-free condition is replaced with the condition about the girth g of the graph. For several values of g, we derive essentially best possible results.

For a fixed graph H, we say that a graph G is *H-induced saturated* if G contains no induced copy of H, but either adding any non-edge to G or erasing any edge from G creates such a copy.

Denote a path graph on n vertices by P_n. Martin and Smith [57] showed that there exists no P_4-induced-saturated graph. Following this result, the question for which $n \geqslant 5$ there exist P_n-induced-saturated graphs was posed by Axenovich and Csikós [1] and received significant attention. Räty [66] constructed such a graph for $n = 6$, later Cho, Choi and Park [13] done so for $n = 3k$ for any $k \geqslant 2$, and Bonamy, Groenland, Johnston, Morrison and Scott [8] found such a graph for $n = 5$ by a computer search.

In Chapter 9, which was adapted from [24], we complete the classification by providing a construction of P_n-induced-saturated graphs for each $n \geqslant 6$. Unlike the previous constructions which were more involved and motivated by algebra, our construction is very simple, and we spend most of the chapter proving that it indeed works.

Finally, consider the following question in Ramsey theory, originating to Norine [60] and later asked in the present form by Leader and Long [54]. Colour the edges of the hypercube graph Q_n in two colours. Do there always exist two antipodal vertices joined by a monochromatic geodesic path with at most one colour change? Very little progress has been made on this question - it is not even known if there exists such a geodesic with $o(n)$ colour changes.

In Chapter 10, which was adapted from [23], we make a first small step towards such a result by improving the trivial bound of $(1/2 - o(1))n$ colour changes to $(3/8 + o(1))n$ colour changes. Our method uses a certain trade-off which we hope could be useful even for the arguments aiming for the $o(n)$ bound.

Chapter 2

Maker-Breaker percolation game

This chapter is joint work with Adva Mond and Victor Souza. The results of this chapter form a part of a currently submitted paper [29].

2.1 Introduction

2.1.1 Background

Positional games are two-player combinatorial games characterized by the following setting. We have a finite or infinite set Λ, called a *board*; a family of subsets of Λ, called *winning sets*; and a rule determining which player wins the game. These games attracted wide attention, starting with the papers of Hales and Jewett [43] and Erdős and Selfridge [33]. We refer the reader interested in positional games to the books of Beck [3], and of Hefetz, Krivelevich, Stojaković and Szabó [44].

The so-called *Maker-Breaker games* are well studied positional games. To define the simplest version of a Maker-Breaker game, we need a finite or infinite set Λ, our board, and a family $\mathscr{F}$ of subsets of Λ, the collection of winning sets. The game is played in *rounds*. In each round, Maker and Breaker respectively claim an as yet unclaimed element of Λ, where Maker is the first player. Breaker wins the game if he claims at least one element in each $F \in \mathscr{F}$ by any finite point of the game. Otherwise, Maker wins. On a finite board, this is equivalent to Maker claiming all elements of some $F \in \mathscr{F}$ by the end of the game, though the same is not true for infinite boards. This version of the game is also called the *unbiased Maker-Breaker game*. In the *biased Maker-Breaker game*, introduced by Chvátal and Erdős [14], the players may claim more elements. To be precise, given natural numbers m and b, in each round of the (m,b) Maker-Breaker game, Maker claims m elements of the board whereas Breaker claims b. For more information about Maker-Breaker games, we once again refer the reader to the books of Beck [3], and of Hefetz, Krivelevich, Stojaković and Szabó [44].

In this chapter, we address the following Maker-Breaker game played on an infinite board, introduced by Day and Falgas-Ravry [21, 22]. Let Λ be an infinite connected graph and let $v_0 \in V(\Lambda)$ be a vertex. In the scope of this chapter, we only have Λ being $\mathbb{Z}^2$. The (m,b) *Maker-Breaker percolation game on* (Λ, v_0) is the game with board $E(\Lambda)$ where the winning sets are all infinite connected subgraphs of Λ containing v_0. That is, Maker's goal is to ensure that v_0 is always contained in an infinite subgraph of Λ spanned by the edges that she claimed and the unclaimed edges. Note that Breaker wins the game by claiming at least one element in each winning set. In this game this means that Breaker's goal is to claim any set of edges separating v_0 from infinity. Notably, the $(1,1)$-game on Λ can be seen as a generalisation of the well-known Shannon switching game to an infinite board, see Lehmann [55] for a description and a solution of this game.

Throughout this chapter, we refer to the (m,b) Maker-Breaker percolation game on (Λ, v_0) as (m,b)-game on (Λ, v_0). If Λ is a transitive graph, we omit the vertex v_0 in our notation, as it does not change the analysis of the game.

Next, we summarise the main results of the paper [22] of Day and Falgas-Ravry.

Theorem 2.1.1 (Day and Falgas-Ravry [22]). *Let $m,b \in \mathbb{N}$. Then*

(i) Maker has a winning strategy for the $(1,1)$-game on $\mathbb{Z}^2$;

(ii) if $m \geqslant 2b$, then Maker has a winning strategy for the (m,b)-game on $\mathbb{Z}^2$;

(iii) if $b \geqslant 2m$, then Breaker has a winning strategy for the (m,b)-game on $\mathbb{Z}^2$.

Note that, clearly, neither player is harmed by having more moves on their turn, so if for instance, Breaker wins the (m,b)-game on a board, he also wins the (m,b')-game on that same board with $b' \geqslant b$. This property is called *bias monotonicity*.

Having proved Theorem 2.1.1, Day and Falgas-Ravry raised many interesting questions. Most strikingly, they asked if there is some critical ratio ρ^* such that, there exists a positive function $\varphi(m) = o(m)$ such that Breaker wins the $(m, \rho^* m + \varphi(m))$-game and Maker wins the $(m, \rho^* m - \varphi(m))$-game. Theorem 2.1.1 shows that if such ratio exists, then $1/2 \leqslant \rho^* \leqslant 2$. These bounds are associated with the fact that a set of k connected edges in $\mathbb{Z}^2$ has edge-boundary of size at most $2k+4$, see Lemma 2.2.1. By the *perimetric barrier* we refer to the limitation of either player being roughly twice as powerful as the the other player. Although the main problem Day and Falgas-Ravry suggest is to break the perimetric barrier, they set out as an open problem to show whether Breaker or Maker can win the $(m, 2m-1)$ or $(2b-1, b)$ games, respectively.

2.1.2 Our results

As our first result, we break the perimetric barrier on Breaker's side, making progress towards answering Question 5.5 from the paper [22] of Day and Falgas-Ravry about the critical ratio.

Theorem 2.1.2. *Consider the (m,b) Maker-Breaker percolation game on $\mathbb{Z}^2$, where $m \geqslant 29$ and $b \geqslant 2m-s$ for some $0 \leqslant s \leqslant \frac{m-22}{14}$. Then Breaker has a winning strategy, which moreover ensures that he wins within the first 3 rounds of the game.*

In particular, this shows that if ρ^* exists, then $1/2 \leqslant \rho^* \leqslant 27/14 \approx 1.93$, and thus, breaks the perimetric barrier as discussed previously. We do not believe this bound to be tight, and moreover, we did not attempt to optimise for this constant, as we also believe that the current method will not yield the optimal bound.

Theorem 2.1.2 improves the ratio on Breaker's side for $m \geqslant 36$, and it also shows that for $m \geqslant 29$, Breaker wins the $(m,2m)$-game rather fast. Nonetheless, it is also of interest to determine how powerful Breaker is in the $(m,2m)$-game for smaller values of m. In the proof of Theorem 2.1.1, Day and Falgas-Ravry show that Breaker can win the $(m,2m)$-game on $\mathbb{Z}^2$ within $m^{16m+O(1)}$ rounds, and ask [22, Question 5.7] how far this is from best possible. Concerning this question, we prove a slightly stronger result, showing that Breaker can win fast even when allowing Maker an initial boost in the form of an option to claim some edges before the game starts.

We consider the following variant of the game. For integers $m,b \geqslant 1$ and $c \geqslant 0$ define the *c-boosted (m,b) Maker-Breaker percolation game* on $\mathbb{Z}^2$ to be the same as the (m,b) percolation game, with the addition that only in her very first turn, Maker claims c extra edges (so overall in her first round she claims $m+c$ edges). Concerning this game, Day and Falgas-Ravry asked [22, Question 5.6] whether Breaker having a winning strategy for the (m,b)-game on (Λ, v_0) implies that he also has a winning strategy for the c-boosted version of the same game.

In view of [22, Questions 5.6, 5.7], we prove the following result.

Theorem 2.1.3. *Let $m \geqslant 1$ and $c \geqslant 0$ be integers, and let $b \geqslant 2m$. Then Breaker wins the c-boosted (m,b) Maker-Breaker percolation game on $\mathbb{Z}^2$, and moreover, he can ensure to win within the first $(2c+4)(2c+5)\left(\lceil \frac{2c+2}{m} \rceil + 2\right)$ rounds.*

This theorem tells us that Breaker can not only win the $(m,2m)$-game on $\mathbb{Z}^2$ quite fast, and can not only win the c-boosted game for any c, he can also win quite fast the c-boosted game. Moreover, the number of rounds Breaker needs is uniformly bounded in m and polynomial in c. Thus, for the $(m,2m)$-game, we answer the stronger combined version of Questions 5.6 and 5.7 of [22].

Applying Theorem 2.1.3 with $c = 0$, we get the following extension of Theorem 2.1.2 for the $(m,2m)$-game without an initial boost.

Corollary 2.1.4. *Let $m \geqslant 1$ and $b \geqslant 2m$ be integers. Then Breaker can guarantee to win the (m,b) percolation game on $\mathbb{Z}^2$ within the first 80 rounds of the game. Moreover, if $m \geqslant 29$ then Breaker wins within 3 rounds.*

Note that this cannot be extended for a win in 3 rounds for every m, as in fact, for $m = 1$, Maker can survive for 5 rounds. In the range $1 \leqslant m \leqslant 28$, the bound of 80 we obtain is not optimal, as the proof of Theorem 2.1.3 specialised to $c = 0$ could be greatly simplified. For $m \geqslant 29$, Maker can indeed survive for 3 rounds, as it becomes clear in the proof of Theorem 2.1.2.

2.1.3 Tools and strategy

Throughout this chapter we use two important tools. The first relates our game to an auxiliary game, where Maker has to keep her graph connected, or at least connected in some generalised sense. However, if we want to claim that it is enough to prove that Breaker wins against a restricted Maker, we have a certain price to pay. In particular, we consider only strategies of Breaker in which he claims edges from the edge-boundary of Maker's connected component, or a slightly generalised version of that. More importantly, we enable Maker to 'save' some edges for later, to make the auxiliary game resemble the original one. Despite these changes, this setting ends up being much easier to analyse, as one of the hard things to tackle when considering strategies for Breaker is handling different connected components in Maker's graph. Furthermore, it turns out that with these adjustments, analysing the auxiliary game is indeed sufficient to prove our results for the original game.

Our second tool is considering variations on Lemma 2.2.1 (Lemma 2.3 in [21]). This simple result tells us that the edge-boundary of any connected finite subgraph of $\mathbb{Z}^2$ is at most 'a bit' larger than twice the number of edges of this subgraph. Hence, for instance, when playing the $(m,2m)$-game, it is enough to force Maker to play several 'bad moves', not enlarging the edge-boundary of her graph by too much, so that Breaker can surround her connected component completely.

When we use a more general notion of connectivity, we need a slightly more general version of Lemma 2.2.1. This variant allows us to analyse the game in a global sense. This, in particular, is how we manage to break the perimetric barrier.

2.1.4 Organization

Firstly, we present our notation and the definitions we work with in Section 2.2. After that, we prove Theorem 2.1.2 in Section 2.3 and Theorem 2.1.3 in Section 2.4. Finally, we present several open problems and further directions in Section 2.5.

2.2 Preliminaries

For a graph G, we denote by $V(G)$ its vertex set, and by $E(G)$ its edge set. Furthermore, we set $e(G) := |E(G)|$. For a subset of vertices $U \subseteq V(G)$ we denote by $G[U]$ the subgraph of G induced by U.

The *edge boundary* ∂H of a finite subgraph H of a possibly infinite graph G is

$$\partial H := \big\{\{x,y\} \in E(G) \setminus E(H) :\ \{x,y\} \cap V(H) \neq \emptyset\big\}.$$

We usually abbreviate 'edge boundary' to 'boundary', as we do not consider any other type of boundary in this chapter.

We use the standard terminology where by the square lattice $\mathbb{Z}^2$, we mean the infinite graph with the following vertex and edge sets:

$$V(\mathbb{Z}^2) := \big\{(x,y) :\ x,y \in \mathbb{Z}\big\},$$
$$E(\mathbb{Z}^2) := \big\{\{(x,y),(x',y')\} \subseteq \mathbb{Z}^2 :\ |x-x'|+|y-y'| = 1\big\}.$$

2.2.1 Useful lemmas

Throughout this chapter, we use several times the following reverse isoperimetric inequality observed by Day and Falgas-Ravry [21].

Lemma 2.2.1 (Day and Falgas-Ravry [21])**.** *Let C be a finite connected subgraph of $\mathbb{Z}^2$, then*

$$|\partial C| \leqslant 2e(C)+4.$$

Proof. We follow the argument of Day and Falgas-Ravry [21]. The proof goes by induction on the number of the edges. If $e(C) = 1$, then we have $|\partial C| = 6$, and the result holds.

Now fix some $k > 1$ and assume that the result is true for any connected subgraph of $\mathbb{Z}^2$ with $k-1$ edges. Consider C, a connected subgraph of $\mathbb{Z}^2$ with $e(C) = k$. There must exist an edge $e \in E(C)$ such that $C' = C \setminus \{e\}$ is connected - indeed, if C contains some cycle, any edge of this cycle will have this property, and otherwise C is a tree and then any leaf will have

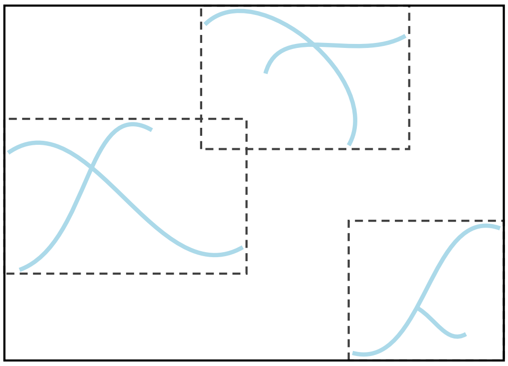

Fig. 2.1 Three connected components, their bounding boxes (dashed), and the bounding box of their box-component (solid).

this property. By induction, we have $|\partial C'| \leqslant 2(k-1)+4$. Note that adding e to C' erases one edge from the edge boundary of C' and it adds at most three new ones, since at most one endpoint of e is not in C'. Hence, $|\partial C| \leqslant |\partial C'|+2 \leqslant 2k+4$, completing the proof. □

We present two more versions of this lemma, for which we need some definitions.

Definition 2.2.2. *We say that a finite subgraph $B \subseteq \mathbb{Z}^2$ is a box, if it is induced by a set of vertices of the form*

$$\{(x,y):\ a \leqslant x \leqslant b,\, c \leqslant y \leqslant d\},$$

for some $a,b,c,d \in \mathbb{Z}$.

Definition 2.2.3. *Let $S \subseteq \mathbb{Z}^2$ be a finite set of edges and let V_S be the set vertices in the graph it spans. The bounding box of S, denoted* $\mathrm{bb}(S)$*, is the minimal box in $\mathbb{Z}^2$ containing S. To spell it out, let*

$$m_x(S) := \min\{x:\ (x,y) \in V_S\}, \qquad M_x(S) := \max\{x:\ (x,y) \in V_S\}.$$

Also, let $m_y(S), M_y(S)$ be defined analogously for the y-axis. Then $\mathrm{bb}(S)$ *is the box induced by the following set of vertices:*

$$\{(x,y):\ m_x(S) \leqslant x \leqslant M_x(S),\, m_y(S) \leqslant y \leqslant M_y(S)\}.$$

For a finite subgraph G of $\mathbb{Z}^2$, we write $\mathrm{bb}(G)$ for $\mathrm{bb}(E(G))$.

Lemma 2.2.4. *Let D be a finite connected subgraph of $\mathbb{Z}^2$, then*

$$|\partial\, \mathrm{bb}(D)| \leqslant |\partial D|.$$

Proof. Let ∂D consist of h horizontal and v vertical edges, so that $|\partial D| = h + v$. Let $m_x = m_x(D)$, $M_x = M_x(D)$, $m_y = m_y(D)$, and $M_y = M_y(D)$ be as in Definition 2.2.3. Note that for any $m_x \leqslant x_0 \leqslant M_x$ there are at least two vertical boundary edges in ∂D of the form $\{(x_0, y), (x_0, y+1)\}$. Analogously, for any $m_y \leqslant y_0 \leqslant M_y$ there are at least two horizontal boundary edges in ∂D of the form $\{(x, y_0), (x+1, y_0)\}$. In particular we have $h \geqslant 2(M_y - m_y + 1)$ and $v \geqslant (M_x - m_x + 1)$. Recall that the box $\mathrm{bb}(D)$ has sides of lengths $M_x - m_x$ and $M_y - m_y$, so

$$|\partial\, \mathrm{bb}(D)| = 2(M_x - m_x + M_y - m_y) + 4,$$

and the result follows. □

We define a generalisation of connected component that incorporates the notion of bounding box in Definition 2.2.3. For doing so we go through several definitions.

Definition 2.2.5. *Let $D_1, D_2 \subseteq \mathbb{Z}^2$ be boxes. We say that D_1, D_2 box-intersect if*

$$V(D_1) \cap V(D_2) \neq \emptyset.$$

Note that the relation in Definition 2.2.5 is clearly symmetric.

Definition 2.2.6. *Let $D \subseteq \mathbb{Z}^2$ be a finite subgraph. Let $C_1, \ldots C_t$ be the connected components of D, and let $\mathscr{R} := \{\mathrm{bb}(C_1), \ldots, \mathrm{bb}(C_t)\}$ be the collection of their bounding boxes. As long as possible, repeat the following process. If there exist $R_i, R_j \in \mathscr{R}$ which box-intersect, remove them from $\mathscr{R}$ and replace them by the bounding box of their union, that is, by $\mathrm{bb}(R_i \cup R_j)$. The final $\mathscr{R}$ obtained in the end of this process is called a collection of box-components of D. If the final $\mathscr{R}$ contains precisely one box-component, then we say that D is box-connected.*

Hence, for each subgraph $D \subseteq \mathbb{Z}^2$ we can consider its collection of box-components. This allows us to state a slightly generalised version of Lemma 2.2.1.

Lemma 2.2.7. *Let D be a finite box-connected subgraph of $\mathbb{Z}^2$, then*

$$|\partial\, \mathrm{bb}(D)| \leqslant 2e(D) + 4.$$

Proof. If D is connected, then the result follows immediately from combining Lemma 2.2.4 and Lemma 2.2.1.

Otherwise, it is enough to prove the statement for the case where is D a union of two graphs, C_1, C_2, for which the statement holds for both, and where $\mathrm{bb}(C_1), \mathrm{bb}(C_2)$ box-intersect. Indeed, we then apply this repeatedly for each step in the process defined in

Definition 2.2.6, which ends in a single box-component for D. Denote $R_i := \mathrm{bb}(C_i)$ for $i = 1, 2$, so by assumption we have

$$|\partial R_i| \leqslant 2e(C_i) + 4. \tag{2.1}$$

As R_1, R_2 box-intersect, we have that

$$V(R_1) \cap V(R_2) \neq \emptyset.,$$

If either $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$ then we either have

$$|\partial\, \mathrm{bb}(D)| = |\partial R_1| \leqslant 2e(C_1) + 4 < 2e(D) + 4,$$

or a similar relation holds with R_1 replaced by R_2.

Otherwise, we can easily argue that the set $(\partial(R_1) \cup \partial(R_2)) \setminus \partial(R_1 \cup R_2)$ forms a dual rectangle with strictly positive integer side lengths, and hence that it contains at least four elements. In particular,

$$|\partial(R_1 \cup R_2)| \leqslant |\partial R_1| + |\partial R_2| - 4.$$

Consequently,

$$\begin{aligned}
|\partial\, \mathrm{bb}(D)| &= |\partial\, \mathrm{bb}(C_1 \cup C_2)| \\
&= |\partial\, \mathrm{bb}(R_1 \cup R_2)| \\
&\leqslant |\partial(R_1 \cup R_2)| && \text{(By Lemma 2.2.4)} \\
&\leqslant |\partial R_1| + |\partial R_2| - 4 \\
&\leqslant 2e(C_1) + 2e(C_2) + 4 && \text{(By (2.1))} \\
&= 2e(D) + 4. && \square
\end{aligned}$$

Remark 2.2.8. *Note that the edge boundary ∂B of any box $B \subseteq \mathbb{Z}^2$, when regarded as a set of dual edges, forms a rectangle.*

2.3 Breaking the perimetric ratio

In this section, we prove Theorem 2.1.2. First, let us recall that by bias monotonicity, it is enough to prove Theorem 2.1.2 for $b = 2m - s$, as having more power cannot harm Breaker. As a first step, we describe an auxiliary game for which we show that a win of Breaker in this game implies a win of him in the original game. After that, we provide Breaker with an explicit strategy. The rest of this section is devoted to showing that Breaker, by following the suggested strategy, wins the auxiliary game within three rounds. We prove this by a geometric analysis, relying crucially on the introduced notion of box-connectivity and our tools from Section 2.2.

If Breaker plays only on the boundary, it is natural to arrive at the perimetric barrier of the ratio 2, because of Lemma 2.2.1. More precisely, when Breaker only claims edges from the boundary of Maker's graph, he cannot react to her future moves in advance. That is, in each turn, Maker is able to create as many new unclaimed boundary edges as possible, to which Breaker must respond. To get around this, it is helpful for Breaker to consider the global structure of Maker's graph. Indeed, in general terms, one could interpret our strategy as Breaker forcing Maker to claim edges in an already played region of the board. This extra power from previous turns will lead to the improvement on the ratio.

More particularly, in each round, Breaker will almost completely enclose Maker's graph from that round in a big rectangular box. After several rounds, the situation will inevitably occur when a new rectangle that Breaker wants to use shares a side with a rectangle already placed. At that point, Breaker does not need to use any edges to create this side of the rectangle, which is where the extra power that he needs comes from. Figures 2.2, 2.3 and 2.4 illustrate this.

When analysing the game from the point of view of Breaker, we wish to consider the graph of Maker as being always box-connected. Hence, we define the auxiliary game where we consider the box-component of Maker's graph containing the origin as her graph in each round. Consequently, we allow more flexibility in the number of edges that she can claim in each turn. Also, when defining Breaker's strategy later, we must insist that he can only play in a certain way for a result about an auxiliary game to translate into the result about the original game.

Definition 2.3.1 ((m,b) Maker-Breaker box-limited percolation game on $\mathbb{Z}^2$)**.** *Two players, Maker and Breaker alternate claiming yet unclaimed edges of a board $\mathbb{Z}^2$, starting in round 1 with Maker going first.*

- *In round i, Maker chooses a non-negative integer m_i such that for every i,*

$$\sum_{j=1}^{i} m_j \leqslant im, \tag{2.2}$$

and then claims m_i unclaimed edges from $E(\mathbb{Z}^2)$. Moreover, Maker must play in a way that in the end of each of her turns, her edges must be in the box-component of v_0 (see Definition 2.2.6).

- *In each round, Breaker claims at most b unclaimed edges.*
- *Breaker wins if the connected component of v_0 in the graph formed by Maker's edges and all unclaimed edges becomes finite. If Maker can ensure that this never happens, then she wins.*

A key result for us is the following proposition relating the two games.

Proposition 2.3.2. *Let $m,b \geqslant 1$ be integers. Assume that Breaker can ensure his win in the (m,b) box-limited percolation game on $\mathbb{Z}^2$ within the first k rounds by claiming only edges from the boundary of the bounding box of Maker's graph, or from inside the box itself. Then he can also ensure his win in the (m,b) percolation game on $\mathbb{Z}^2$ within the first k rounds.*

Proof. We show that if Maker has a strategy to ensure that Breaker will not win within the first k rounds of the (m,b) percolation game, then she can also ensure that Breaker will not win within the first k rounds of the box-limited percolation game, assuming that Breaker claims only edges from the boundary of the bounding box of her graph, or from inside it.

Assume that Maker has such a strategy for the (unlimited) percolation game. Then she can win the box-limited game by playing as follows. Denote by M the box-component spanned by the edges claimed by Maker. Maker follows her winning strategy for the percolation game, and whenever this includes playing some edge e that after the end of her turn would be in a box-component which is not M, she only marks this edge as an *imaginary* edge and does not play it in that round. However, she claims an imaginary edge e right after the first time she plays some edge that puts e in M.

Firstly, Maker can afford saving imaginary edges to claim later in the game, as the terms m_j only have to satisfy $\sum_{j=1}^{i} m_j \leqslant im$ for any $i \geqslant 1$. Furthermore, Breaker cannot claim an imaginary edge before Maker claims it, as we assume that Breaker wins by claiming only edges in $(\partial M) \cup E(M)$. The result follows. □

Now consider the $(m, 2m-s)$ *Maker-Breaker box-limited percolation game* on $\mathbb{Z}^2$, where $m \geqslant 36$ and $1 \leqslant s \leqslant \frac{m-22}{14}$.

We provide Breaker with the strategy below. While the description of the strategy may seem complicated at first, it is in fact very simple. For illustrations of the geometric content of this strategy, see Figures 2.2, 2.3 and 2.4.

Strategy 2.3.3 (Breaker's strategy for the (m,b) box-limited percolation game on $\mathbb{Z}^2$)**.** *For any $i \geqslant 1$, let M_i be the set of edges claimed by Maker in her i-th turn. Breaker plays according to the following steps. If at any point of the game Breaker cannot follow any particular step, he forfeits the game.*

First round

Set $B_1 := \mathrm{bb}(M_1)$.

(1) *If* $|\partial B_1| \leqslant 2m - s$, *claim all edges in* ∂B_1.

(2) *Otherwise, let* $g_1 := |\partial B_1| - 2m + s$. *Claim* $2m - s$ *edges from* ∂B_1, *leaving* g_1 *unclaimed boundary-edges in the middle (up to being possibly shifted by one edge, for parity reasons) of one of the longer sides of the box* B_1. *Denote by* G_1 *this set of* g_1 *unclaimed edges.*

Second round

(1) *If* $M_2 \cap G_1 = \emptyset$, *then claim all edges in* G_1 *if possible, or forfeit if not possible.*

(2) *Otherwise,* $M_2 \cap G_1 \neq \emptyset$. *Let* V_1 *be the set of vertices in* $\mathbb{Z}^2 \setminus B_1$ *which are contained in edges of* G_1. *Let* $P_1 := E\big(\mathbb{Z}^2[V_1]\big)$ *be the set of edges in the path induced by the vertices* V_1. *Let* $C_1 := E(B_1) \cup \partial B_1$ *and* $B_2 := \mathrm{bb}((M_2 \cup P_1) \setminus C_1)$.

(2.1) *If* $|(\partial B_2) \setminus C_1| \leqslant 2m - s$, *claim all edges in* $(\partial B_2) \setminus C_1$.

(2.2) *Otherwise, let* $g_2 := |(\partial B_2) \setminus C_1| - 2m + s$. *As* G_1 *is a set of boundary-edges in the middle of one of the longer sides of* B_1, *it splits the boundary edges adjacent to this side into two sets of consecutive boundary-edges. Denote these two sets by* L_1 *and* R_1, *such that* $|R_1 \cap (\partial B_2)| \leqslant |L_1 \cap (\partial B_2)|$. *Let* e *be an edge in* $(\partial B_2) \setminus C_1$ *of minimal distance to* G_1. *Let* G_2 *be* g_2 *consecutive edges in* $(\partial B_2) \setminus C_1$, *starting from* e *(see Figure 2.3 for an illustration). Claim all* $2m - s$ *edges in* $(\partial B_2) \setminus C_1$ *excluding those edges in* G_2.

Third round

(i) *If there is a set of at most* $2m - s$ *unclaimed edges such that claiming them ensures a win in this round, claim all edges in this set.*

(ii) *Otherwise, forfeit.*

We now show that Strategy 2.3.3 is enough to break the perimetric barrier for the box-limited game. Note that when using Strategy 2.3.3, Breaker only claims edges from the bounding box of Maker's graph or from its boundary. Hence, combining Proposition 2.3.2 with the following proposition gives us Theorem 2.1.2.

Proposition 2.3.4. *Let* $m \geqslant 36$ *and* $s \leqslant \frac{m-22}{14}$. *Then by following Strategy 2.3.3, Breaker wins the* $(m, 2m - s)$ *box-limited percolation game on* $\mathbb{Z}^2$ *in at most three rounds.*

Proof. We analyse the game by following Strategy 2.3.3 step by step, showing that Breaker can indeed follow it without forfeiting at any point, and thus to win the game by the end of the third round. In fact, we show that by the end of the third round, Breaker claims all edges in the boundary of the bounding box of Maker's graph, or a subgraph of it containing the origin.

Note that if during the game, Breaker grants Maker extra edges and wins when playing as if she claimed them, then he also wins the games without granting her those edges. We will use this assumption as it simplifies the analysis of the game.

Recall that for each $j \geqslant 1$, we denote by M_j the set of edges that Maker claimed in her j-th turn. Let $m_j := |M_j|$, so we have $\sum_{j=1}^{i} m_j \leqslant im$ for any $i \geqslant 1$.

Refer to Figures 2.2, 2.3 and 2.4 for a representation of Rounds 1, 2 and 3 respectively. We use in several points of this analysis that the game is invariant under translations, rotations by $\pi/4$ angles and horizontal and vertical reflections.

First Round

Maker plays all her m_1 edges M_1 in a box-component containing the origin. Set $B_1 := \mathrm{bb}(M_1)$, and let a_1 and b_1 be the number of vertices in the sides of B_1, with $a_1 \geqslant b_1$. Assume, without loss of generality, that the top and bottom sides of B_1 are at least as large as the left and right ones, that is, they consists of a_1 vertices. Note that as $|\partial B_1| = 2a_1 + 2b_1$, we get

$$a_1 \geqslant \tfrac{1}{4}|\partial B_1|. \tag{2.3}$$

Step (1) If $|\partial B_1| \leqslant 2m - s$, then by claiming all edges in ∂B_1, Breaker surrounds Maker's graph and wins the game.

Step (2) Assume otherwise, so we have

$$|\partial B_1| \geqslant 2m - s + 1. \tag{2.4}$$

Moreover, by Lemma 2.2.7 we get

$$|\partial B_1| \leqslant 2m_1 + 4, \tag{2.5}$$

so in particular,

$$m_1 \geqslant m - \tfrac{1}{2}s - \tfrac{3}{2}. \tag{2.6}$$

Denote

$$g_1 := |\partial B_1| - 2m + s. \tag{2.7}$$

Breaker chooses g_1 boundary-edges in the middle (up to being possibly shifted by one edge, for parity reasons) of the bottom side of ∂B_1, and denotes them by G_1. We refer to this set of edges as the 'gate' for the first round, see Figure 2.2. Then Breaker claims all edges in $(\partial B_1) \setminus G_1$.

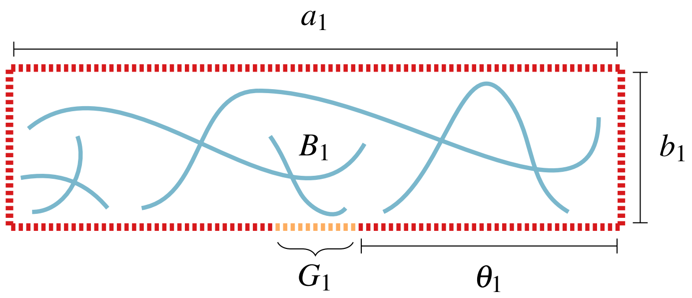

Fig. 2.2 End of Round 1, where Maker is in light blue and Breaker in dark red. The set G_1 of g_1 edges in the gate, in orange, is unclaimed.

This is possible because the bottom side of ∂B_1 contains at least g_1 edges, as we observe below.

$$\begin{aligned} g_1 &\leqslant s+4-2(m-m_1) \leqslant s+4 && \text{(By (2.7), (2.5), and } m_1 \leqslant m) && (2.8)\\ &\leqslant \frac{1}{4}(2m-s+1) \leqslant \frac{1}{4}|\partial B_1| && \left(\text{As } s \leqslant \tfrac{m-22}{14},\ m \geqslant 29 \text{ and (2.4)}\right) \\ &\leqslant a_1. && \text{(By (2.3))} && (2.9) \end{aligned}$$

Assume further, without loss of generality, that the box is fully contained in the top half-plane and that the origin $(0,0)$ is as close as possible to the centre of the bottom side of the box B_1. In particular, it is also in the centre of the gate G_1.

Second Round

First note that by (2.9), we have $|G_1| = g_1 \leqslant a_1 \leqslant m+1 \leqslant 2m-s$.

Step (1) If $M_2 \cap G_1 = \emptyset$, then by claiming all at most $2m-s$ edges in G_1, Breaker surrounds B_1 completely and thus wins the game.

Step (2) Otherwise, we have $M_2 \cap G_1 \neq \emptyset$. Let P_1, L_1 and R_1 be as in Strategy 2.3.3, and assume without loss of generality that L_1 and R_1 are sets of boundary-edges to the left and to the right of G_1, respectively.

Following the strategy, we set $C_1 := E(B_1) \cup \partial B_1$. In fact, Breaker can regard the edges in $B_1 \cup G_1$ as being played by Maker, and thus, regard C_1 as the set of already claimed edges. Note that as $M_2 \cap G_1 \neq \emptyset$, we get $M_2 \cap C_1 \neq \emptyset$.

Consider the graph spanned by the set of edges $M_2' := (M_2 \cup P_1) \setminus C_1$. As this is a proper subset of $M_2 \cup P_1$, it might not be box-connected. Consider its box-component containing P_1, and assume without loss of generality that it is M_2' itself, as otherwise it has only less edges. Indeed, we can 'return' the edges outside the box-component of P_1 back to Maker, since they will not be claimed by Breaker in this turn and Maker can reclaim them immediately in the next turn if she wishes to do so.

Denote by $B_2 := \mathrm{bb}(M_2')$ the new bounding box. Let a_2 and b_2 be the number of vertices in the top and bottom sides and in the left and right sides of B_2, respectively. Since $M_2 \cap G_1 \neq \emptyset$, we have $|M_2 \setminus C_1| \leqslant m_2 - 1$. Furthermore,

$$|P_1| = |V_1| - 1 = |G_1| - 1 = g_1 - 1,$$

so we have

$$|M_2'| \leqslant m_2 + g_1 - 2.$$

Therefore, Lemma 2.2.7 implies that

$$|\partial B_2| = 2a_2 + 2b_2 \leqslant 2m_2 + 2g_1. \tag{2.10}$$

Moreover, as $P_1 \subseteq B_2$, we get

$$g_1 \leqslant a_2. \tag{2.11}$$

Step (2.1) If $|(\partial B_2) \setminus C_1| \leqslant 2m - s$, then Breaker claims all edges in $(\partial B_2) \setminus C_1$. Note that $G_1 \subseteq \partial B_1 \cap (E(B_2) \cup \partial B_2)$. Thus, Breaker surrounds $B_1 \cup B_2$ completely and wins the game.

Step (2.2) Assume otherwise, and denote

$$g_2 := |(\partial B_2) \setminus C_1| - 2m + s. \tag{2.12}$$

First, note that by (2.10) we get

$$g_2 \leqslant 2g_1 + s + 2(m_2 - m). \tag{2.13}$$

Combining the assumption that $g_2 \geqslant 1$ and (2.10), it follows that

$$2m - s + 1 \leqslant |(\partial B_2) \setminus C_1| \leqslant 2m_2 + 2g_1 - |(\partial B_2) \cap C_1|,$$

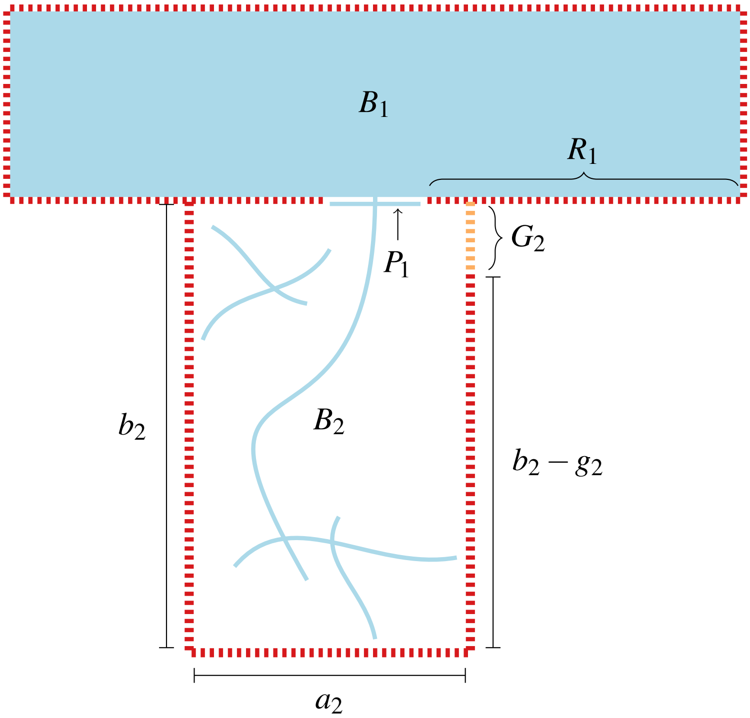

Fig. 2.3 End of Round 2, where Maker is in light blue and Breaker in dark red. In orange, the set G_2 of g_2 consecutive unclaimed edges, forms the gate for the second round.

and hence

$$|(\partial B_2) \cap C_1| \leqslant 2g_1 + s - 1 + 2(m_2 - m). \tag{2.14}$$

As L_1 and R_1 are sets of boundary-edges of B_1 on both sides of G_1 claimed by Breaker, define $\theta_1 := \min\{|L_1|, |R_1|\}$, and we have

$$\begin{aligned}
\theta_1 &= \lfloor \tfrac{1}{2}(a_1 - g_1) \rfloor \geqslant \tfrac{1}{2}(a_1 - g_1 - 1) \\
&\geqslant \tfrac{1}{2}\big(\tfrac{1}{4}|\partial B_1| - |\partial B_1| + 2m - s - 1\big) && \text{(By (2.3) and (2.7))} \\
&= \tfrac{1}{2}\big(2m - s - 1 - \tfrac{3}{4}|\partial B_1|\big) \\
&\geqslant \tfrac{1}{2}\big(2m - s - 1 - \tfrac{3}{4}(2m+4)\big) && \text{(By (2.5) and } m_1 \leqslant m) \\
&= \tfrac{1}{4}m - \tfrac{1}{2}s - 2.
\end{aligned} \tag{2.15}$$

Let x_0 be minimal such that

$$|x_0| \geqslant \max\big\{|x| : (x,y) \in V\big(\mathbb{Z}^2[\partial B_1]\big)\big\}.$$

Assume that Maker claimed any edge which is either to the right or to the left of B_1, that is, an edge which has a vertex $v = (x_0, y_0)$. In particular, as Maker plays box-connected, we get that $B_2 \setminus C_1$ contains a horizontal path of length at least $\theta_1 + g_1$, and thus $(\partial B_2) \cap C_1$ contains such a path as well. Hence, we get that

$$\begin{aligned}
|(\partial B_2) \cap C_1| &\geqslant \theta_1 + g_1 \\
&\geqslant \tfrac{1}{4}m - \tfrac{1}{2}s - 2 + g_1 && \text{(By (2.15))} \\
&\geqslant 2s + 4 + g_1 && \left(\text{As } s \leqslant \tfrac{m-22}{14} \text{ and } m \geqslant 29\right) \\
&\geqslant 2s + 4 - 2(m - m_1) + 2(m_2 - m) + g_1 && \text{(By (2.2))} \\
&\geqslant 2g_1 + s + 2(m_2 - m), && \text{(By (2.7))}
\end{aligned}$$

contradicting (2.14). Hence, Maker did not claim any such edge.

It follows that B_2 is completely contained in the infinite vertical stripe defined by the right and left sides of B_1. We remark that requiring $m \geqslant 29$ is necessary for this very step. More formally, we have

$$a_2 = |(\partial B_2) \cap C_1|. \tag{2.16}$$

Furthermore, note that we must have $B_1 \cap B_2 = \emptyset$, which implies $G_1 \subseteq \partial B_2$. In particular, by (2.14), we get

$$a_2 \leqslant 2g_1 + s - 1 + 2(m_2 - m), \tag{2.17}$$

and by (2.12), we get that

$$g_2 = |\partial B_2| - a_2 - 2m + s. \tag{2.18}$$

Recall that $|R_1 \cap (\partial B_2)| \leqslant |L_1 \cap (\partial B_2)|$, so in total we get

$$|R_1 \setminus (\partial B_2)| \geqslant \tfrac{1}{2}(a_1 - a_2). \tag{2.19}$$

Let G_2 be a set of g_2 consecutive edges starting at an edge $e \in (\partial B_2) \setminus C_1$ of minimal distance to G_1. Note that e is in fact the top-right horizontal edge in ∂B_2, and thus G_2 the set of g_2 most top boundary-edges on the right side of B_2. Following Strategy 2.3.3, Breaker claims all $2m - s$ edges of $(\partial B_2) \setminus C_1$, leaving the edges of G_2 unclaimed as in Figure 2.3, which we refer as the gate for the second round.

Third Round

We now show that there exists a set of at most $2m - s$ unclaimed edges such that by claiming them, Breaker wins the game in this round. We do this by a bit more careful geometric analysis, and by combining together all the information described above in the previous two rounds of the game.

We start by giving some notation analogous to those in the first two rounds in Strategy 2.3.3. Let M_3 be the set of edges that Maker claimed on her third turn. So we have $m_3 := |M_3|$. Let $C_2 := C_1 \cup E(B_2) \cup \partial B_2$. Again, we sometimes regard the edges in $B_2 \cup G_2$ as being played by Maker, and thus regard C_2 as the set of already claimed edges. Denote by D the set of boundary-edges adjacent to the right side of B_2. So we have $|D| = b_2$. Denote by A the box for which D is the set of boundary-edges of the left side of it, and $R_1 \setminus (\partial B_2)$ is the set of boundary-edges of the top side of it (see Figure 2.4). We consider three cases.

Case 1 $M_3 \cap G_2 = \emptyset$.

Similarly to the second round, by (2.13) and (2.8), and the fact that $m_1 + m_2 \leqslant 2m$, we have $|G_2| = g_2 \leqslant 3s + 8 \leqslant 2m - s$. Thus in this case we have $|M_3 \cap G_2| \leqslant 2m - s$, and by claiming all edges in the gate G_2, Breaker surrounds $B_1 \cup B_2$ completely and wins the game.

Case 2 $M_3 \cap G_2 \neq \emptyset$ and $(M_3 \setminus C_2) \cap (\partial A) = \emptyset$.

In this case, by claiming all unclaimed edges in ∂A, Breaker surrounds Maker's graph completely and wins the game. We show that he can indeed do so.

Note that there are $|\partial A| = 2|D| + 2|R_1 \setminus (\partial B_2)|$ edges in the boundary of A, from which only $|D| + |R_1 \setminus (\partial B_2)|$ are unclaimed. We get that,

$$\begin{aligned}
|(\partial A) \setminus C_2| &\leqslant |D| + |R_1 \setminus (\partial B_2)| \\
&\leqslant |D| + \tfrac{1}{2} m_1 && (\text{As } |R_1| \leqslant \tfrac{1}{2} m_1) \\
&\leqslant m_2 + \tfrac{1}{2} m_1 && (\text{As } |D| = b_2 \leqslant m_2) \\
&\leqslant 2m - \tfrac{1}{2} m_1 && (\text{As } m_1 + m_2 \leqslant 2m) \\
&\leqslant 2m - s, && (\text{As } m_1 \geqslant 2s \text{ by (2.6)}) \qquad (2.20)
\end{aligned}$$

and therefore, Breaker wins the game.

Case 3 $M_3 \cap G_2 \neq \emptyset$ and $(M_3 \setminus C_2) \cap (\partial A) \neq \emptyset$.

For this case we need some further notation. Similarly to the second round, let V_2 be the set of vertices in $\mathbb{Z}^2 \setminus B_2$ which are contained in edges of G_2. Let further $P_2 := E\big(\mathbb{Z}^2[V_2]\big)$ be the set of edges in the path induced by the vertices of V_2. Note that we have $P_2 \subseteq A$. Let $M_3' := (M_3 \cup P_2) \setminus C_2$. Similarly to the second round, we have $|P_2| = g_2 - 1$, and since $M_3 \cap G_2 \neq \emptyset$, we also have $|M_3 \setminus C_2| \leqslant m_3 - 1$. In total we get

$$|M_3'| \leqslant m_3 + g_2 - 2. \qquad (2.21)$$

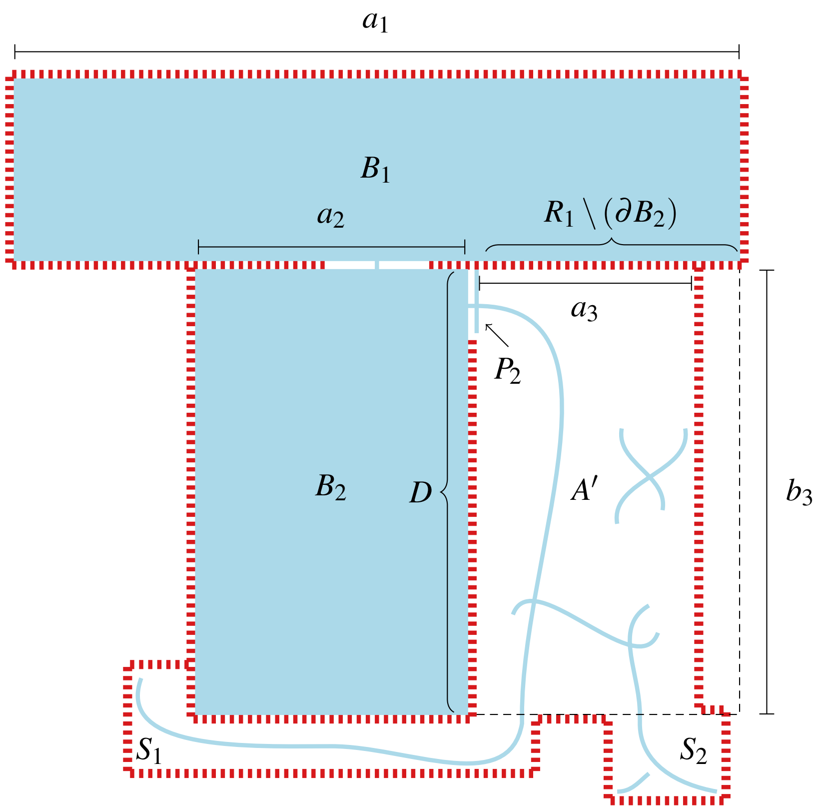

Fig. 2.4 End of Round 3, where Maker is in light blue and Breaker in dark red. The region bounded by the dashed lines is the box A.

Let $M_3^A := M_3' \cap E(A)$, and let $A' := \mathrm{bb}(M_3^A)$ be its bounding-box (see Figure 2.4). By box-connectivity, and since $(M_3 \setminus C_2) \cap (\partial A) \neq \emptyset$, we get that the box A' shares at least one full side with the box A. In Figure 2.4, they share the left side, for instance.

If $(\partial A') \cap M_3' = \emptyset$, then Breaker wins simply by claiming all the edges in $\partial A' \setminus C_2$. Indeed, as $A' \subseteq A$ and (2.20), we have

$$|(\partial A') \setminus C_2| \leqslant 2m - s.$$

Hence we may assume that $(\partial A') \cap M_3' \neq \emptyset$. Let $S_1, \ldots, S_t$ be the box-components of $M_3' \setminus M_3^A$ such that they intersect the boundary of A'. That is, for $i \in [t]$ we have $S_i \cap (\partial A') \neq \emptyset$. Further, denote $Q_i := \mathrm{bb}(S_i)$.

We now show that Breaker can claim all unclaimed edges that surround

$$A' \cup Q_1 \cup \cdots \cup Q_t,$$

which, in particular, implies that he wins the game. More precisely, we show that there at most $2m - s$ such edges, that is,

$$\big|\partial\big(A' \cup Q_1 \cup \cdots \cup Q_t\big) \setminus C_2\big| \leqslant 2m - s. \tag{2.22}$$

Let us emphasise that in the equation above we mean the edge-boundary of a union of boxes, rather than the edge-boundary of their joint bounding box, as in previous rounds.

To prove (2.22), first note that for any distinct $i, j \in [t]$, we have $Q_i \cap Q_j = \emptyset$ by box-connectivity. Therefore,

$$\begin{aligned} \big|\partial\big(A' \cup Q_1 \cup \cdots \cup Q_t\big) \setminus C_2\big| &\leqslant \big|(\partial A') \setminus C_2\big| + \sum_{i=1}^{t} \big|\partial(A' \cup Q_i) \setminus \partial A'\big| \\ &\leqslant \big|(\partial A') \setminus C_2\big| + \sum_{i=1}^{t} \big(|\partial(A' \cup Q_i)| - |\partial A'|\big). \end{aligned} \tag{2.23}$$

Now we bound each of these two terms separately.

We start with bounding the sum in (2.23). Similarly to the argument in the proof of Lemma 2.2.7, for each $i \in [t]$ we have

$$|\partial(A' \cup Q_i)| \leqslant |\partial A'| + |\partial Q_i| - 4.$$

Recall that by Lemma 2.2.7, we have $|\partial Q_i| \leqslant 2e(S_i)+4$ for each $i \in [t]$, so in total we get

$$\sum_{i=1}^{t}\big(|\partial(A'\cup Q_i)|-|\partial A'|\big) \leqslant 2\sum_{i=1}^{t} e(S_i). \tag{2.24}$$

As for the first term in (2.23), let a_3 and b_3 be the numbers of vertices in the top and bottom sides and in the left and right sides of A', respectively. Recall that A' shares at least one side with A, so in particular we have either $a_3 = |R_1 \setminus (\partial B_2)|$ or $b_3 = b_2$. It follows that there are at least $\min\{|R_1 \setminus (\partial B_2)|, b_2 - g_2\}$ edges in $\partial A'$ which are already claimed by Breaker. Moreover, we have $G_2 \subseteq \partial A'$, so we get a reduction of at least g_2 more edges from the amount that Breaker has to claim in $\partial A'$. In total, we get

$$\big|(\partial A')\cap C_2\big| \geqslant g_2 + \min\big\{|R_1 \setminus (\partial B_2)|, b_2 - g_2\big\}.$$

Denote $m_3^A := |M_3^A|$ and recall that $A' = \mathrm{bb}(M_3^A)$. By Lemma 2.2.7 and by the above, we get that

$$\begin{aligned}\big|(\partial A')\setminus C_2\big| &= \big|\partial A'\big| - \big|(\partial A')\cap C_2\big| \\ &\leqslant 2m_3^A + 4 - g_2 - \min\big\{|R_1 \setminus (\partial B_2)|, b_2 - g_2\big\}.\end{aligned}$$

To finish the proof, we need the following technical claim, which we prove later.

Claim 2.3.5. *We have*

$$\min\big\{|R_1 \setminus (\partial B_2)|, b_2 - g_2\big\} \geqslant g_2 + s + 2(m_3 - m). \tag{2.25}$$

Assume for now that Claim 2.3.5 holds. We get that

$$\big|(\partial A')\setminus C_2\big| \leqslant 2(m + m_3^A - m_3) - 2g_2 - s + 4. \tag{2.26}$$

Furthermore, by (2.21) we have

$$m_3 + g_2 - 2 \geqslant |M_3'| \geqslant m_3^A + \sum_{i=1}^{t} e(S_i),$$

and in particular

$$m_3^A - m_3 \leqslant g_2 - 2 - \sum_{i=1}^{t} e(S_i).$$

So by (2.26), we get

$$\left|(\partial A')\setminus C_2\right| \leqslant 2\Big(m+g_2-2-\sum_{i=1}^{t} e(S_i)\Big)-2g_2-s+4 = 2m-s-2\sum_{i=1}^{t} e(S_i).$$

Finally, by (2.24), (2.23), and by the above, we conclude

$$\Big|\partial\big(A'\cup Q_1\cup\cdots\cup Q_t\big)\setminus C_2\Big| \leqslant 2m-s,$$

proving (2.22), as required.

Hence, it is only left to prove Claim 2.3.5.

Proof of Claim 2.3.5. We start with the second term. Observe that

$$\begin{aligned} g_2 &= a_2+2b_2-2m+s && \text{(By (2.18) and (2.10))}\\ &\leqslant b_2+g_1+s-(2m-m_2). && \text{(Again by (2.10))} \end{aligned}$$

Therefore, we have

$$\begin{aligned} b_2-g_2 &\geqslant 2m-m_2-s-g_1\\ &\geqslant 2m-m_2-2s-4+2(m-m_1) && \text{(By (2.8))}\\ &\geqslant m-2s-4, && (2.27) \end{aligned}$$

where we used the fact that $m_1 \leqslant m$ and $m_1+m_2 \leqslant 2m$.

On the other hand, we have

$$\begin{aligned} g_2+s+2(m_3-m) &\leqslant 2g_1+2s+2(m_2-m)+2(m_3-m) && \text{(By (2.13))}\\ &\leqslant 4s+8+4m_1+2(m_2+m_3)-8m && \text{(By (2.8))}\\ &\leqslant 4s+8 && \text{(By (2.2))}\\ &\leqslant m-2s-4. && \big(\text{As } s\leqslant \tfrac{m-22}{14}\big). \end{aligned}$$

Combining this with (2.27), we get

$$b_2-g_2 \geqslant g_2+s+2(m_3-m),$$

proving the claim for the second term in (2.25).

As for the first term, we start by noticing that

$$\begin{aligned}
\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2 - 2g_1 &\geqslant \tfrac{1}{2}(a_1 - 3g_1) && \text{(By (2.11))}\\
&\geqslant \tfrac{1}{2}\Big(\tfrac{1}{4}|\partial B_1| - 3\big(|\partial B_1| - 2m + s\big)\Big) && \text{(By (2.3) and (2.7))}\\
&= \tfrac{1}{2}\big(6m - 3s - \tfrac{11}{4}|\partial B_1|\big)\\
&\geqslant \tfrac{1}{2}\big(6m - 3s - \tfrac{11}{4}(2m_1 + 4)\big). && \text{(By (2.5))} \qquad (2.28)
\end{aligned}$$

In addition, using (2.18) and (2.10), we can also write

$$g_2 \leqslant 2g_1 + s - a_2 - 2(m - m_2). \tag{2.29}$$

Hence we get

$$\begin{aligned}
|R_1 \setminus (\partial B_2)| - g_2 &\geqslant \tfrac{1}{2}(a_1 - a_2) - g_2 && \text{(By (2.19))}\\
&\geqslant \tfrac{1}{2}a_1 + \tfrac{1}{2}a_2 - 2g_1 - s + 2(m - m_2) && \text{(By (2.29))}\\
&\geqslant \tfrac{1}{2}\big(6m - 3s - \tfrac{11}{4}(2m_1 + 4)\big) - s + 2(m - m_2) && \text{(By (2.28))}\\
&= 7m - 2(m_1 + m_2 + m_3)\\
&\quad - \tfrac{3}{4}m_1 - \tfrac{5}{2}s - \tfrac{11}{2} + 2(m_3 - m)\\
&\geqslant m - \tfrac{3}{4}m_1 - \tfrac{5}{2}s - \tfrac{11}{2} + 2(m_3 - m) && \text{(By (2.2))}\\
&\geqslant \tfrac{1}{4}m - \tfrac{5}{2}s - \tfrac{11}{2} + 2(m_3 - m) && (\text{As } m_1 \leqslant m)\\
&\geqslant s + 2(m_3 - m) && \big(\text{As } s \leqslant \tfrac{m-22}{14}\big),
\end{aligned}$$

finishing the proof of the claim. □

This completes the proof. □

Note that a more careful analysis, in similar spirit to the one above, might produce a somewhat smaller ratio than $2 - \frac{1}{14} + o(1)$ in Theorem 2.1.2. However, as our main aim was to break the perimetric barrier, we did not attempt to optimise it in the benefit of clarity.

2.4 Fast win of Breaker in the boosted $(m, 2m)$-game

In this section we prove Theorem 2.1.3. Note that, as the game is bias monotone in b, it is enough to prove Theorem 2.1.3 for $b = 2m$.

As discussed in Section 2.1, the proof contains two important ideas. Firstly, assuming that Maker plays connected significantly simplifies the analysis of the game. Hence we consider a slight variation of the game for which the following hold.

(i) If Breaker wins this variant within the first k rounds using a strategy satisfying certain further condition, then he also wins the (m,b) percolation game within the first k rounds.

(ii) Maker must keep the graph spanned by the edges she claimed connected.

Note that there is a certain price we have to pay for this in the form of restrictions on Breaker's strategy; and in allowing Maker, instead of claiming precisely m edges in each round, to claim sometimes a bit more and sometimes a bit less.

The second important idea is to define a strategy of Breaker in such a way that he forces Maker to create a component of a 'bad' shape. More precisely, as we can see by Lemma 2.2.1, if we give Maker $km+c$ edges to build a connected component and Breaker $2km$ edges to place in the boundary of the said component, Breaker can ensure to claim 'almost' all edges in the boundary. Hence, if he can force Maker to create a component that is sufficiently far from equality in Lemma 2.2.1, Breaker can in fact claim all the edges in its boundary. In the rest of this section, we expand these ideas into a formal proof.

We start by defining the variation of the game that we mentioned before.

Definition 2.4.1 (c-boosted (m,b) Maker-Breaker limited percolation game on $\mathbb{Z}^2$)**.** *Two players, Maker and Breaker alternate claiming yet unclaimed edges of a board $\mathbb{Z}^2$, starting in round 1 with Maker going first.*

- *In round i, Maker chooses a non-negative integer m_i such that*

$$\sum_{j=1}^{i} m_j \leqslant im+c,$$

and then claims m_i unclaimed edges from $E(\mathbb{Z}^2)$. Moreover, Maker must play in a way that in the end of each of her turns, her edges must be in the component of v_0.

- *In each round, Breaker claims at most b unclaimed edges.*

- *Breaker wins if the connected component of v_0 consisting only of Maker's edges and unclaimed edges becomes finite. If Maker can ensure that this never happens, then she wins.*

The following proposition is a key result of the same spirit as Proposition 2.3.2, relating these two games.

Proposition 2.4.2. *Let $m,b \geqslant 1$ and $c \geqslant 0$ be integers. Assume that Breaker can ensure his win in the c-boosted (m,b) limited percolation game on $\mathbb{Z}^2$ within the first k rounds by claiming only edges from the boundary of the graph spanned by Maker's edges. Then he can also ensure his win in the c-boosted (m,b) percolation game on $\mathbb{Z}^2$ within the first k rounds.*

The proof of Proposition 2.4.2 is similar to the proof of Proposition 2.3.2 except for some minor details, so we do not include it.

Throughout the rest of this section, we consider only the c-boosted limited percolation game on $\mathbb{Z}^2$, and prove the following.

Proposition 2.4.3. *Let $m \geqslant 1$ and $c \geqslant 0$ be integers. Then Breaker can guarantee to win the c-boosted $(m,2m)$ limited percolation game on $\mathbb{Z}^2$ within the first $(2c+4)(2c+5)(\lceil\frac{2c+2}{m}\rceil+2)$ rounds of the game.*

Theorem 2.1.3 then follows by combining Proposition 2.4.2 and Proposition 2.4.3.

We start by providing various definitions and assumptions that we need in our proof. Firstly, without loss of generality, we may assume that Maker's graph is not only connected after each turn of hers, but also that she is adding edges to it, one by one, so that her graph, which we denote by C, is connected at any single point during her turn. We then let $C(\ell)$ be her graph after Maker claimed ℓ edges in total, and we denote by $C_k := C(\sum_{i=1}^{k} m_i)$ Maker's graph after she played k full turns.

Recall that we denote by $\partial C(\ell)$ the edge boundary of $C(\ell)$, and note that we include in this boundary also the edges that Breaker has already claimed. The set of those edges in the boundary of $C(\ell)$ that are yet unclaimed by Breaker at this point of the game is denoted by $\partial_F C(\ell)$, which stands intuitively for the 'free' boundary of $C(\ell)$. While this definition may be ambiguous as $\partial_F C(\ell)$ changes during the turn of Breaker, we will always make it clear to which particular point we refer to when using this notation.

Definition 2.4.4. *For any $\ell \geqslant 1$, we call an edge $e \in \partial C(\ell)$:*

- *Awful if $|\partial(C(\ell)+e)| - |\partial C(\ell)| \leqslant 1$.*
- *Bad if it is not awful, but by claiming e, Maker creates at least one new awful edge f in $\partial(C(\ell)+e)$ such that f touches e.*
- *Good otherwise.*

We now provide Breaker with a strategy.

Strategy 2.4.5 (Breaker's strategy for the (m,b) limited percolation game on $\mathbb{Z}^2$). *Let $k \geqslant 1$ and assume that by the end of her k-th turn Maker has claimed ℓ edges so far in the game for some $\ell \geqslant 0$. In his k-th turn, Breaker claims edges one by one from $\partial_F(C(\ell))$ in the following order of priority:*

(i) good edges,

(ii) bad edges,

(iii) awful edges.

Note that if at some point Breaker cannot follow Strategy 2.4.5, then it means that there are no unclaimed boundary edges in Maker's graph, which in particular means that Breaker has won the game.

We make the following straightforward observation that does not require proof.

Observation 2.4.6. *An awful edge stays awful until it is claimed by either player. A bad edge either stays bad or becomes awful until it is claimed by either player.*

Further, let

$$v_k := 2|E(C_k)| + 4 - |\partial C_k|,$$

and let w_k be the number of awful edges in $\partial_F C_k$ after k turns of both Maker and of Breaker, and set $v_0 = w_0 = 0$. Let $\partial_G C_k \subseteq \partial_F C_k$ denote the subset of good edges.

We consider the lexicographic order on ordered pairs $\{(x;y) : x,y \in \mathbb{R}\}$. That is, we have

$$(x;y) > (z;w) \leftrightarrow \{x > z\} \text{ or } \{x = z \text{ and } y > w\}.$$

When we write inequalities between ordered pairs, we always refer to this ordering.

Theorem 2.1.3 is implied by the following observation and proposition.

Observation 2.4.7. *Let $k \geqslant 1$ and assume that after k turns of both Maker and Breaker, Breaker has not yet won. Then we have $0 \leqslant v_k \leqslant 2c+3$ and $0 \leqslant w_k \leqslant 2c+4$.*

Proof. Firstly, recall that an awful edge is in particular an unclaimed edge, so by the definition of w_k and by Lemma 2.2.1 we have

$$0 \leqslant w_k \leqslant |\partial_F C_k| = |\partial C_k| - 2mk \leqslant 2c+4.$$

Secondly, note that $v_k \geqslant 0$ simply by Lemma 2.2.1. Since Breaker has not won the game yet by the end of his k-th turn, and as he claims only edges from the boundary of Maker's graph, we have $|\partial C_k| \geqslant 2mk+1$, and thus

$$v_k = 2|E(C_k)|+4-|\partial C_k| \leqslant 2c+3. \qquad \square$$

Proposition 2.4.8. *Let $k \geqslant 1$ and $c' = \lceil \frac{2c+2}{m} \rceil + 1$. Assume that after $k+c'$ turns of both Maker and Breaker, Breaker has not yet won the game. Then there exists some $1 \leqslant r \leqslant c'+1$, such that $(v_{k+r}; w_{k+r}) > (v_k; w_k)$.*

Before proving Proposition 2.4.8, let us see why it is useful for us.

Claim 2.4.9. *Proposition 2.4.3 is implied by Observation 2.4.7 and Proposition 2.4.8.*

Proof. Consider the set

$$S := \big\{(x;y):\ x \in \{0,1,2,\dots,2c+3\}, y \in \{0,1,2,\dots,2c+4\}\big\}.$$

We clearly have $|S| = (2c+4)(2c+5)$. By Observation 2.4.7 we have $(v_k; w_k) \in S$ for every k such that Breaker has not yet won after k rounds of the game. By Proposition 2.4.8 we can find strictly increasing sequence

$$(v_{i_1}; w_{i_1}) < (v_{i_2}; w_{i_2}) < \cdots$$

with $i_1 = 1$ and $i_{t+1} - i_t \leqslant c'+1$ for each $t \geqslant 1$. The result follows. $\square$

It is left to prove Proposition 2.4.8. We start by proving two easy lemmas.

Lemma 2.4.10. *Let $k \geqslant 0$ and assume that after k rounds of the game Breaker has not won yet. Assume that out of m_{k+1} edges that Maker claimed in her $(k+1)$-th turn, t were awful at the time they were claimed, for some $0 \leqslant t \leqslant m_{k+1}$. Then*

$$|\partial C_{k+1}| - |\partial C_k| \leqslant 2m_{k+1} - t.$$

In particular we get

$$v_{k+1} \geqslant v_k + t.$$

Proof. When claimed by Maker, an edge is removed from the edge boundary of the new component, and at most three new edges are added to it, which means that $|\partial C(\ell+1)| - |\partial C(\ell)| \leqslant 2$, for any $\ell \geqslant 1$. If the ℓ-th edge claimed by Maker is awful, then by definition we have $|\partial C(\ell+1)| - |\partial C(\ell)| \leqslant 1$. The first result follows by combining these two observations.

Then in particular, by the definition of v_k and by the above, we have

$$\begin{aligned} v_{k+1} &= 2|E(C_{k+1})| + 4 - |\partial C_{k+1}| \\ &\geqslant 2\big(|E(C_k)| + m_{k+1}\big) + 4 - |\partial C_k| - 2m_{k+1} + t = v_k + t. \end{aligned}$$ □

Lemma 2.4.11. *Let $k \geqslant 1$ and assume that after k rounds of the game, Breaker has not yet won. Then*

$$\sum_{i=1}^{k} m_i \geqslant km - 1.$$

Proof. Assume for contradiction that $\sum_{i=1}^{k} m_i \leqslant km - 2$. Then by Lemma 2.2.1, we have $|\partial C_k| \leqslant 2(km-2) + 4 = 2km$. However, within his first k turns, Breaker claims precisely $2km$ edges, all in ∂C_k. Thus he must have won at latest after k rounds, which is a desired contradiction. □

Next we show that we never create many good edges.

Lemma 2.4.12. *For any $\ell \geqslant 1$, the number of good edges in $\partial C(\ell+1)$ is at most one more than the number of good edges in $\partial C(\ell)$, that is*

$$|\partial_G C(\ell+1)| - |\partial_G C(\ell)| \leqslant 1.$$

Moreover, $|\partial_G C(1)|, |\partial_G C(2)| \leqslant 2$.

Proof. That $|\partial_G C(1)|, |\partial_G C(2)| \leqslant 2$ follows by inspection, so it only remains to prove the first assertion.

Let $\ell \geqslant 1$. By Observation 2.4.6, no edge that was bad or awful in $\partial C(\ell)$ can be good in $\partial C(\ell+1)$. So it is enough if we rule out the case that out of at most three edges in $\partial C(\ell+1) \setminus \partial C(\ell)$, two different ones would be good.

Let e be the lth edge claimed by Maker. By symmetry, we may assume that e is horizontal, i.e. that $e = \{(x,y),(x+1,y)\}$ for some $x,y \in \mathbb{Z}$. Further, since Maker always claims an edge in the edge boundary of her only connected component and due to symmetry again, we only need to consider the following two cases.

1. The edge $\big\{(x-1,y),(x,y)\big\}$ was already in $C(\ell)$.
2. The edge $\big\{(x,y+1),(x,y)\big\}$ was already in $C(\ell)$.

In either case, we have that $\partial C(\ell+1) \setminus \partial C(\ell)$ is contained in the set

$$\Big\{\big\{(x+1,y),(x+2,y)\big\},\big\{(x+1,y),(x+1,y-1)\big\},\big\{(x+1,y),(x+1,y+1)\big\}\Big\},$$

and out of these three edges, only $\{(x+1,y),(x+2,y)\}$ may be in $\partial_G C(\ell+1)$. Indeed, $f=\{(x+1,y),(x+1,y-1)\}$ cannot be in $\partial_G C(\ell+1)$ because the edge $\{(x,y-1),(x+1,y-1)\}$, if so far unclaimed, must be classed as awful once f is claimed, and if already claimed makes f awful. Analogous argument shows that $\{(x+1,y),(x+1,y+1)\}$ cannot be in $\partial_G C(\ell+1)$. □

Lemma 2.4.12 has the following corollary.

Corollary 2.4.13. *For any $k \geqslant 1$, at the end of Breaker's k-th turn, we have*

$$|\partial_G C_k| \leqslant (c-mk)_+,$$

where $n_+ := \max\{n,0\}$.

Proof. If $m \geqslant 3$, then we can assume $|E(C_1)| \geqslant 2$, otherwise Breaker wins already in the first round. And in the case when $|E(C_1)| \geqslant 2$, by Lemma 2.4.12, we get $|\partial_G C_1| \leqslant |E(C_1)|$ at the end of Maker's first turn. In his k-th turn, Breaker claims at least $\min\{2m, |\partial_G C_k|\}$ good edges. As this holds for any $k \geqslant 1$, at the end of Breaker's k-th turn we have

$$|\partial_G C_k| \leqslant \big(|E(C_k)|-2mk\big)_+ \leqslant (c-mk)_+.$$

If $m \in \{1,2\}$ and $m_1 \leqslant 1$, then although we can have $|\partial_G C_1| > |E(C_1)|$ at the end of Maker's first turn, we still have $|\partial_G C_1| = 0$ by the end of Breaker's turn, and the result follows similarly to the first case. □

We are now ready to prove Proposition 2.4.8.

Proof of Proposition 2.4.8. The crucial part of our proof is the following claim.

Claim 2.4.14. *There exists $1 \leqslant j \leqslant c'-1$, such that in his $(k+j)$-th turn Breaker claimed a bad or an awful edge.*

Proof of Claim 2.4.14. Assume for contradiction that in his $(k+j)$-th turn Breaker claimed only good edges for all $1 \leqslant j \leqslant c'-1$, meaning he claimed a total of $2m(c'-1)$ good edges during these rounds.

By Corollary 2.4.13, at the end of Breaker's k-th turn we have $|\partial_G C_k| \leqslant (c-mk)_+$. Moreover, by Lemma 2.4.11 Maker claimed at most

$$\sum_{i=k+1}^{k+c'-1} m_i \leqslant c+(k+c'-1)m-km+1 = c+1+m(c'-1)$$

edges after her k-th turn. Hence, by Lemma 2.4.12 at most $c+1+m(c'-1)$ new good edges were added to the boundary of Maker's graph in rounds $k+1,\dots,k+c'-1$. And by Observation 2.4.6 no awful or bad edge became good, so we get

$$2m(c'-1) \leqslant (c-mk)_+ + c + 1 + m(c'-1),$$

contradicting $c' = \lceil \frac{2c+2}{m} \rceil + 1$. □

Recall that by Lemma 2.4.10 the sequence v_i is non-decreasing. Pick the smallest such j from Claim 2.4.14. In $(k+j+1)$-st turn of Maker, Maker claims some bad or awful edge.

If Maker claims an awful edge (or claimed an awful edge in any other of the rounds $k+1,\dots,k+j+1$), we have $v_{k+j+1} > v_k$ and we are done.

If Maker claims no awful edge (and did not in any other of the rounds $k+1,\dots,k+j+1$) but claims a bad edge his $(k+j+1)$-th turn, note that Breaker also must have claimed no awful edge in his $(k+j)$-th turn. Next, there are two options. Either Breaker claims no awful edge in his $(k+j+1)$-th turn, and then $w_{k+j+1} > w_k$ and we are done (as $v_{k+j+1} = v_k$). Or Breaker claims some awful edge in his $(k+j+1)$-th turn, but then Maker claims some awful edge in her $(k+j+2)$-th turn, hence $v_{k+j+2} > v_k$ and we are again done.

Hence, the proof of Proposition 2.4.8 is finished. □

This now also concludes the proof of Theorem 2.1.3.

2.5 Concluding remarks

In this chapter we have made progress on some of the questions that Day and Falgas-Ravry [22] asked. Notably, we have improved the upper bound on the value of the ratio parameter ρ^* (if such ρ^* indeed exists).

However, some of their other questions still remain open, as well as some new problems, which we present below. There are two directions for further research that we consider to be of especially great interest.

The first of these concerns Maker's side. We still do not know any integers (m,b) with $b > 1$ and $m < 2b$ for which Maker has a winning strategy in the (m,b)-game on $\mathbb{Z}^2$. In fact, we do not know any such integers even in the boosted version of the game, where Maker is allowed to claim arbitrary but finite number of edges before her first turn. Maker's strategies in the papers of Day and Falgas-Ravry [21, 22] suggest that it may be beneficial for Maker to play as a dual Breaker in some auxiliary game. Hence, perhaps some of the techniques developed in this chapter could help in answering those questions.

Another direction would be to break the perimetric ratio in a stronger sense. In Section 2.3, we have shown that there exists $\delta > 0$ such that provided m is large enough and $b \geqslant (2-\delta)m$, Breaker has a winning strategy in the (m,b)-game on $\mathbb{Z}^2$. It would be interesting to study the boosted variant of this game, where Maker is allowed to claim finitely many edges before her first turn. There, we still do not know of any integers (m,b) with $b < 2m$ for which Breaker has a winning strategy in the boosted (m,b)-game (though as we have shown in Section 2.4, $b = 2m$ suffices even in this variant). Again, it is possible that some of the techniques of this chapter can be developed further to handle this case as well.

One could also try to improve the bound in Theorem 2.1.2 derived in this chapter. But while improving this constant significantly would be of some interest (note that improving Theorem 2.1.2 by little should not be very hard as we did not try to optimise the constant there fully), we believe that breaking the barriers mentioned above is even more important.

For a further overview of more open questions, see the paper of Day and Falgas-Ravry [22]. Note that many seemingly easy questions are still open, for instance, it is not known who wins the $(2,3)$ or $(3,2)$-game on $\mathbb{Z}^2$.

Chapter 3

Waiter-Client triangle-factor game

The results in this chapter were published in European Journal of Combinatorics [26].

3.1 Introduction

In the previous chapter, we introduced positional games as well as Maker-Breaker games, probably the most studied positional games.

Another rather similar type of well studied positional games are the *Waiter-Client games*. These games are played by two players, called Waiter and Client, in the following manner. We are given an integer $b \geqslant 1$, a set Λ and a family of winning sets $\mathscr{F}$ of the subsets of Λ. In each round, Waiter picks $b+1$ previously unclaimed elements of Λ and offers them to Client. Client chooses one of these elements and adds it to his graph, while the remaining b elements become a part of Waiter's graph. Waiter wins if she forces Client to create a winning set $F \in \mathscr{F}$ in Client's graph. If Client can prevent that, he wins.

Various papers studied, for given $\Lambda, b, \mathscr{F}$, which player has a winning strategy in the corresponding Waiter-Client games. Or, in the cases when we know that Waiter can win, how fast can she guarantee her victory to be. For instance, Bednarska-Bzdega, Hefetz, Krivelevich and Łuczak [4] studied such games with the winning sets being large components or long cycles. Hefetz, Krivelevich and Tan [45] looked on the Waiter-Client games involving planarity, colourability and minors, and later [46] the same authors studied a Hamiltonicity game with a board being a random graph. Krivelevich and Trumer [52] considered a maximum degree game. Yet more results were obtained by Tan [68] about colourability and k-SAT games.

Assume that for our triple $\Lambda, b, \mathscr{F}$, Waiter wins the corresponding Waiter-Client game. Then we will denote by $\tau_{WC}(\mathscr{F}, b)$ the number of rounds of the game when Waiter tries to win as fast as possible, Client tries to slow her down as much as possible, and they both play optimally. What the ground set Λ is will usually be clear from the context.

When $b = 1$, we call the corresponding Waiter-Client game *unbiased*. Recently, Clemens et al. [15] studied several unbiased Waiter-Client games played on the edges of the complete graph, i.e. with $\Lambda = E(K_n)$. For n divisible by 3, they considered the triangle-factor game, where the winning sets are the collections of $\frac{n}{3}$ vertex disjoint triangles. It is not hard to verify that for n large enough, Waiter can win this game. In the next two paragraphs, we provide a brief sketch of the strategy that Waiter can use - for full details, we refer the reader either to the paper of Clemens et al. [15] or to Section 4.4 (which contains the proof of this fact for a K_k-factor game for any $k \geqslant 3$).

The key observation is that given seven vertices $v_1, v_2, ..., v_7$ of an initially empty board K_7, Waiter can create a triangle $w_1w_2w_3$ in Client's graph and moreover keep the property that every edge placed so far in either Waiter's or Client's graph has at least one endpoint of the form w_i for some $1 \leqslant i \leqslant 3$. To achieve this, Waiter proceeds in two stages. First, Waiter offers Client one by one pairs of edges (v_1v_{2i}, v_1v_{2i+1}) for $i = 1,2,3$. This now gives a subset $\{a_1, a_2, a_3\}$ of three vertices connected to v_1 in Client's graph. Next, Waiter offers Client pair of edges (a_1a_2, a_2a_3), completing the triangle with the desired property in Client's graph.

Waiter can use the observation above to create $\frac{n}{3} - 2$ disjoint triangles by an iterative procedure, where at each step from an empty board on $3t$ vertices for some $t \geqslant 3$, we get a disjoint union of a triangle in Client's graph and an empty board on $3(t-1)$ vertices. To achieve that, Waiter simply picks a subset of seven vertices of the empty board and applies the procedure described in the previous paragraph. We want $\frac{n}{3}$ disjoint triangles, not just $\frac{n}{3} - 2$, so to turn our sketch into a formal proof, we need to modify the argument and create a large clique with a certain suitable property before starting the iterative procedure. The details are somewhat technical and described in Section 4.4.

Now that we see that Waiter can win, we can ask how fast. Clemens et al. obtained the following theorem giving the lower and upper bounds on the optimal duration of the game.

Theorem 3.1.1 (Clemens et al. [15])**.** *Assume n is divisible by* 3 *and large enough that Waiter wins the corresponding unbiased triangle-factor game on the edges of K_n. Then*

$$\frac{13}{12}n \leqslant \tau_{WC}(\mathscr{F}_{n,K_3-fac}, 1) \leqslant \frac{7}{6}n + o(n).$$

Further, they made a conjecture that $\tau_{WC}(\mathscr{F}_{n,K_3-\mathrm{fac}}, 1) = \frac{7}{6}n + o(n)$. Our aim in this chapter is to improve the lower bound from $\frac{13}{12}n$ to $\frac{7}{6}n$ and hence to verify their conjecture.

Theorem 3.1.2. *Assume n is divisible by 3 and large enough that Waiter wins the corresponding unbiased triangle-factor game on the edges of K_n. Then*

$$\tau_{WC}(\mathscr{F}_{n,K_3-fac},1) \geqslant \frac{7}{6}n.$$

Finally, let us note that unbiased triangle-factor game on the edges of K_n is an example of a more general phenomena that for a given board and parameters, Waiter can typically win the corresponding Waiter-Client game asymptotically at least as fast as Maker can win the corresponding Maker-Breaker game. Indeed, we know that in this case, Waiter needs $\frac{7}{6}n+o(n)$ rounds if she plays optimally, while it was observed by Krivelevich and Szabó that Maker cannot win the Maker-Breaker version of the game in less than $\frac{7}{6}n$ rounds (see [40]). In fact, using the framework that we build up, we can improve this lower bound to $\frac{4}{3}n$ rounds - see Observation 3.4.1. For a more thorough discussion of this relation between Maker-Breaker and Waiter-Client games, we refer reader to the paper of Clemens et al. [15].

The rest of the chapter is organized as follows: in rest of this section, we very briefly summarize some notation that we will use. Then we set up the necessary definitions and prove some easy results about these in Section 3.2. After that, we prove Theorem 3.1.2 in Section 3.3, by giving a strategy of Client and analyzing the game when Client uses this strategy.

We use the following standard notation throughout the chapter.

For a finite simple graph G, we denote by $V(G)$ its vertex set, and by $E(G)$ its edge set.

For $v,w \in V(G)$, we write $v \sim w$ to denote that v and w are connected by an edge in G, i.e. that $vw \in E(G)$.

Finally, we denote by $\delta(G)$ the minimum degree of G.

3.2 Good and bad connected components in the graph of Client

We need the following characterization of the connected graphs that contain a triangle-factor, yet have few edges.

Observation 3.2.1. *Let G be a connected graph with a triangle-factor and $|V(G)| = n_0$ (where n_0 clearly must be divisible by 3). Then $|E(G)| \geqslant \frac{4}{3}n_0 - 1$. Moreover if $|E(G)| = \frac{4}{3}n_0 - 1$, the triangle-factor is unique, and $\frac{n_0}{3}$ triangles in this triangle-factor are the only cycles in G.*

Proof. The proof of this observation that we present is different than the one in the published paper [26] and is due to the external examiner.

We know that G has at least one triangle-factor, consisting of the triangles

$$T_1 = a_1b_1c_1, ..., T_{\frac{n_0}{3}} = a_{\frac{n_0}{3}}b_{\frac{n_0}{3}}c_{\frac{n_0}{3}}.$$

If G contains multiple triangle-factors, pick one arbitrarily. We know we can remove one arbitrary edge from each of the triangles $T_1, ..., T_{\frac{n_0}{3}}$ while keeping the graph connected. But it is a well known fact that a connected graph on n_0 vertices contains at least $n_0 - 1$ edges. Taking into account the $\frac{n_0}{3}$ edges that we removed, we conclude that

$$|E(G)| \geqslant (n_0 - 1) + \frac{n_0}{3} = \frac{4}{3}n_0 - 1.$$

Next, assume we have equality. If G contained any cycle not of the form $a_ib_ic_i$, then we can first remove one edge of this cycle that is not of the form a_ib_i, b_ic_i or c_ia_i for any i. Such an edge must exist, because the sets $\{a_i, b_i, c_i\}$ are mutually disjoint. After, we still can remove one arbitrary edge from each of the triangles $T_1, ..., T_{\frac{n_0}{3}}$ while keeping the graph connected. By an analogous argument as in the first part, we conclude that

$$|E(G)| \geqslant (n_0 - 1) + \frac{n_0}{3} + 1 = \frac{4}{3}n_0.$$

Hence, in the case of equality, the only cycles in G are the ones of the form $a_ib_ic_i$ for $1 \leqslant i \leqslant \frac{n_0}{3}$. □

Taking into account Observation 3.2.1, the following definition is rather natural.

Definition 3.2.2. *Consider the connected components of Client's graph. We will make a distinction between good and bad ones. When a new connected component is created in Client's graph, initially it is called bad. Now assume that Client adds the edge ab to his graph. Then we update the state of the connected component of Client's graph containing the edge ab as follows.*

- *If at least one of the vertices a or b was part of a good component of Client's graph, then we consider Client's component containing the edge ab as good.*
- *If neither of the vertices a, b was a part of a good connected component before the edge ab was added, but after adding the edge ab, the connected component K of Client's graph containing the edge ab has a triangle-factor and satisfies $|E(K)| = \frac{4}{3}|V(K)| - 1$, we consider the connected component containing the edge ab to be a good component.*

Moreover, in this case (and only in this case), we say that this connected component of Client's graph was declared to be good at the time when we added the edge ab (or that a new good connected component of Client's graph was created).

- *In any other case, the connected component of Client's graph containing the edge ab is bad.*

Let us briefly comment why we use this precise definition. The most important fact to us is that if C is a connected components of the final graph of Client that is bad, then we have $|E(C)| \geqslant \frac{4}{3}|V(C)|$. To see that, note that if C is bad, then it also must have been bad before its last edge was added (or if it was disconnected at that point, both of its components must have been bad). But then, if we had $|E(C)| = \frac{4}{3}|V(C)| - 1$ (we cannot have any less edges in C because of Observation 3.2.1), adding this last edge would have made C good. It thus follows that $|E(C)| \geqslant \frac{4}{3}|V(C)|$.

Observation 3.2.3. *If Waiter won the game, and throughout the game, at most $\frac{n}{6}$ connected components of Client's graph were declared to be good, then Client's final graph contains at least $\frac{7}{6}n$ edges.*

Proof. Call $C_1, \ldots, C_k$ the connected components of the final graph of Client that are good, and $C_{k+1}, \ldots, C_l$ the connected components of the final graph of Client that are bad. Then if $1 \leqslant i \leqslant k$, we have $|E(C_i)| \geqslant \frac{4}{3}|V(C_i)| - 1$, while if $k+1 \leqslant i \leqslant l$, we have $|E(C_i)| \geqslant \frac{4}{3}|V(C_i)|$.

As $k \leqslant \frac{n}{6}$, we get

$$\sum_{i=1}^{l} |E(C_i)| \geqslant \frac{4}{3}\sum_{i=1}^{l} |V(C_i)| - k = \frac{4}{3}n - k \geqslant \frac{7}{6}n,$$

as required. □

3.3 Proof of Theorem 3.1.2

Throughout this section, denote by G_i Client's graph after i rounds.

We start with the definition that will be used when describing Client's strategy later.

Definition 3.3.1. *Any edge that Waiter offers to Client is called crucial if by choosing it to Client's graph, Client would create a new good connected component.*

Using Definition 3.2.2, it is trivial to check that it must be the case that the two endpoints of any crucial edge are in the same connected component of Client's graph at the time it is offered. Hence, we can make the following further definition.

Definition 3.3.2. *If Client is offered a crucial edge ab in the ith round, we say that the connected component of G_{i-1} containing ab is crucial in the ith round.*

Client will pick the edges according to the following strategy.

Strategy 3.3.3. *Client considers the following two possibilities.*

- *If one of the two edges that he is offered is crucial, while the other edge offered is not crucial, he picks to Client's graph the edge that is not crucial.*
- *In any other case, he picks one edge arbitrarily.*

Moreover, we call any round when Client is offered two crucial edges *difficult*.

In this section, we will work towards the following result.

Proposition 3.3.4. *If Client plays according to Strategy 3.3.3, he can ensure that, throughout the game, at most $\frac{n}{6}$ good connected components were created.*

Theorem 3.1.2 then follows by applying Observation 3.2.3.

We start by proving an easy lemma.

Lemma 3.3.5. *Assume that C is a bad connected component of Client's graph. Then there exists at most one (yet unclaimed) edge with its endpoints in C that, if offered by Waiter to Client, would be crucial.*

Proof. Assume such an edge exists. We will show it is unique.

If every vertex of C is already in some triangle of Client's graph, then by Definition 3.2.2 we can't have a crucial edge for C. If more than three vertices of C are not in any triangle of Client's graph, we can't have a crucial edge for C either, since by adding just one edge we can't create a triangle-factor of C. Finally, if precisely one or two vertices of C are not in any triangle of Client's graph yet, we can't have a crucial edge for C. Since if we did and this edge created some new triangles in C (which it must), at least one vertex of C would be in at least two triangles after adding this edge, contradicting Observation 3.2.1.

So precisely three vertices x_1, x_2, x_3 of C are not in any triangle of Client's graph yet. By Observation 3.2.1, any edge we add into C that will make it into a good connected component must complete a triangle $x_1x_2x_3$. But then it must be the case that the triangle $x_1x_2x_3$ misses precisely one edge in C, and this missing edge thus must be our crucial edge. □

The heart of our proof of Proposition 3.3.4 is Lemma 3.3.7. Before stating it, we need the following definition.

Definition 3.3.6. *Let A_i be the vertex set of all connected components of G_{i-1} that were crucial in round i. Let $B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j$.*

We emphasize that for $v \in A_i$, we genuinely need Waiter to offer Client in the ith round some crucial edge ab with the endpoints a,b being in the same connected component of G_{i-1} as v, and that it is not enough if such an edge simply exists but Waiter does not offer it to Client in the ith round.

Lemma 3.3.7. *Assume that Waiter offered Client some crucial edge in the ith round, and let C be the crucial connected component of G_{i-1} containing the endpoints of this edge. Then $|V(C) \cap B_i| \geqslant 3$.*

Proof. Let $C_1,\dots,C_r \subset V(C)$ be the following subsets. We include in this collection any $C_j \subset V(C)$ that at some point before round i was a vertex set of a crucial connected component of Client's graph (note that for all of these, Client did not choose the corresponding crucial edge to his graph though, and instead it went to the graph of Waiter - else C would be good by Definition 3.2.2).

Next, let $D_1,\dots,D_s \subset V(C)$ be the same collection as $C_1,\dots,C_r$, except that we delete any C_j for which we can find $k \neq j$ with $C_j \subset C_k$.

As the proof of Lemma 3.3.7 is rather long, we shall have several claims throughout to keep the structure of the proof of Lemma 3.3.7 as clear as possible.

Claim 3.3.8. *$D_1,\dots,D_s$ are disjoint subsets of $V(C)$.*

Proof of Claim 3.3.8. Assume $v \in D_j \cap D_k$ for some $v \in V(C)$ and some $1 \leqslant j,k \leqslant s$ with $j \neq k$. Assume also that Waiter offered the crucial edge for D_j before offering the crucial edge for D_k (she clearly could not have offered both at the same time, as then D_j, D_k would be the same set because their intersection is non-empty). Let w be any other vertex of D_j. Since v,w were in the same connected component of Client's graph at the time when D_j was a vertex set of a crucial component, they will stay in the same connected component forever after, and in particular as $v \in D_k$, we also have $w \in D_k$. As w was arbitrary, that implies $D_j \subset D_k$, which is a contradiction to our definition of the sets $D_1,\dots,D_s$. □

Let

$$X = V(C) \setminus \bigcup_{j=1}^{s} D_j.$$

Then clearly $X \subset V(C) \cap B_i$. Also, by the definition of a crucial connected component and by Claim 3.3.8, both $|V(C)|$ and $|\bigcup_{j=1}^{s} D_j| = \sum_{j=1}^{s} |D_j|$ are divisible by 3. So if we can show that $|X| > 0$, that immediately implies that $|X| \geqslant 3$ and proves Lemma 3.3.7.

Assume for contradiction that $X = \emptyset$ and hence $V(C) = \bigcup_{j=1}^{s} D_j$. First we rule out the case $s = 1$.

Claim 3.3.9. *If $X = \emptyset$, we have $s \geqslant 2$.*

Proof of Claim 3.3.9. Assume for contradiction that we have $s = 1$ and $V(C) = D_1$. At the first time that $V(C)$ was a vertex set of a crucial connected component C_0 of Client's graph, by Observation 3.2.1 we must have had $|E(C_0)| = \frac{4}{3}|V(C)| - 2$. But by Lemma 3.3.5, the crucial edge for C_0 was unique, and hence now we must have

$$|E(C)| \geqslant |E(C_0)| + 1 = \frac{4}{3}|V(C)| - 1.$$

But that contradicts C being a crucial connected component of Client's graph. □

For $j = 1, ..., s$, let $p_j q_j$ be the crucial edge offered when we had a crucial connected component with a vertex set D_j, and let r_j be the vertex that p_j, q_j would have formed a triangle with in Client's graph at that time, had Client taken $p_j q_j$ to his graph. Let C' be a connected component we obtain if Client picks a crucial edge with the endpoints in C to his graph in the ith round. Clearly $V(C) = V(C')$.

For $v \in V(C)$ belonging to some D_j, denote by $T(v)$ the set of all the sets D_k with $k \neq j$ that v is connected to by an edge in C'.

Claim 3.3.10. *Take p_j, q_j for any $1 \leqslant j \leqslant s$. Then $T(p_j), T(q_j) \neq \emptyset$ and moreover $T(p_j) \cap T(q_j) = \emptyset$.*

Proof of Claim 3.3.10. We know that p_j is a vertex of a triangle $p_j v_1 v_2$ in C', for some $v_1, v_2 \in V(C)$. We know that $q_j \notin \{v_1, v_2\}$, since the edge $p_j q_j$ belongs to Waiter's graph (as it was offered previously to Client, but Client did not take it). But we also know

$$\{v_1, v_2\} \cap D_j \subset \{q_j, r_j\},$$

since any other vertex of D_j was in some triangle in Client's graph already at the time when D_j was a vertex set of a crucial connected component, and by Observation 3.2.1 every vertex of C' is in precisely one triangle in Client's graph. Hence there must exist some $k \neq j$ with $\{v_1, v_2\} \cap D_k \neq \emptyset$, and $T(p_j) \neq \emptyset$ follows.

We derive $T(q_j) \neq \emptyset$ analogously.

Finally, assume for contradiction that $T(p_j) \cap T(q_j) \neq \emptyset$. Take D_k such that $D_k \in T(p_j) \cap T(q_j)$. Then we have $w_1, w_2 \in D_k$ such that $p_j \sim w_1$ and $q_j \sim w_2$ in C' (it may happen that w_1, w_2 are the same vertex). Let $w_1 z_1 \dots z_u w_2$ be a path between w_1 and w_2 in

D_k. Then $p_j w_1 z_1 \dots z_u w_2 q_j r_j$ is a cycle of length at least four in C', which by Observation 3.2.1 gives a contradiction to C' being a good connected component. $\square$

Now consider

$$S_0 = \{ab : ab \in E(C');\ a, b \text{ are in the different sets } D_j\}.$$

We will modify this set repeatedly as follows. As long as S_k contains any edge ab such that there are also both some edge ab' for $b' \neq b$ and some edge $a'b$ for $a' \neq a$ in S_k, erase some such edge ab from S_k to form the set S_{k+1}. This process eventually terminates with some final set S_{final}. Write $S = S_{\text{final}}$.

Let I be the following auxiliary graph. Its vertices are $D_1, \dots, D_s$ and $D_j \sim D_k$ in I if there is at least one edge going between D_j and D_k in S.

Claim 3.3.11. *The minimum degree of I satisfies $\delta(I) \geqslant 2$.*

Proof of Claim 3.3.11. Consider any j, $1 \leqslant j \leqslant s$. By Claim 3.3.10, S_0 contained at least one edge of the form $p_j a$ and at least one edge of the form $q_j b$ for some $a, b \in V(C)$. Moreover, by the definition of the process by which we obtained S from S_0, we can never erase the last edge of either of these two forms from S_k when creating S_{k+1}. Hence S also contains at least one edge of the form $p_j a'$ and at least one edge of the form $q_j b'$ for some $a', b' \in V(C)$. Finally, by Claim 3.3.10, a' and b' can not be in the same set D_k, giving $d_I(D_j) \geqslant 2$. As j was arbitrary, the result follows. $\square$

But Claim 3.3.11 in particular implies that I contains some cycle. That in turn implies that C' contains some cycle which contains at least three edges from S. But by the construction of S, any three different edges of S have at least four different endpoints in total. So C' contains a cycle of length at least four, contradicting Observation 3.2.1.

Thus the proof of Lemma 3.3.7 is finished. $\square$

Corollary 3.3.12. *If round i is a difficult round, then $|B_i| \geqslant 6$.*

Proof. By Lemma 3.3.5, we know that the two crucial edges offered in the difficult round could not have been in the same connected component of G_{i-1}. The result then follows by applying Lemma 3.3.7. $\square$

We are now ready to prove Proposition 3.3.4, and hence as discussed before also complete the proof of Theorem 3.1.2.

Proof of Proposition 3.3.4. Assume Client creates k good connected components in his graph throughout the game, and that overall the game lasted T rounds (where $T \geqslant k$, since Client can create at most one good connected component in each round). That implies there were at least k difficult rounds, as Client does not create good connected components in any other round. But then using Corollary 3.3.12 and the fact that $B_i \cap B_j = \emptyset$ whenever $i \neq j$, we get

$$n \geqslant \left| \bigcup_{i=1}^{T} B_i \right| = \sum_{i=1}^{T} |B_i| \geqslant 6k.$$

It follows that $k \leqslant \frac{n}{6}$, as required. □

3.4 Concluding remarks

Our main result resolves the triangle-factor game fully. As suggested by Clemens et al. [15], it may also be interesting to consider the general K_k-factor game instead of just the case $k = 3$, which is what we turn our attention to in the next chapter.

Another possible line of research is the unbiased Maker-Breaker triangle-factor game. As we mentioned in Section 3.1, it was observed by Krivelevich and Szabó that Maker cannot win this game in less than $\frac{7}{6}n$ rounds (see [40]). In fact, the framework of this chapter allows us to improve this bound.

Observation 3.4.1. *Assume n is divisible by 3. Breaker can ensure that the unbiased Maker-Breaker triangle-factor game on the edges of K_n lasts at least $\frac{4}{3}n$ rounds.*

Proof. We will call the connected components of Maker's graph good and bad according to precisely the same rules as we did for Client's graph in Definition 3.2.2. We claim that Breaker can ensure that no component of Maker's graph will ever be declared to be good - the result then follows by an argument analogous to the proof of Observation 3.2.3.

Breaker's strategy uses the definition of the crucial edges (see Definition 3.3.1), with a natural modification that now we call yet unclaimed edge crucial if by claiming it, Maker would create a good component. The strategy is very simple - if there exists at least one crucial edge, Breaker claims some such, and otherwise Breaker claims any edge. Initially, there are no crucial edges, and it is a consequence of Lemma 3.3.5 that in any round, Maker can create at most one crucial edge. But then because of the strategy of Breaker, Maker can never claim any crucial edge herself, which completes the proof. □

We suspect this lower bound is tight, up to the lower order terms.

Chapter 4

Waiter-Client clique-factor game

The results of this chapter are from a currently submitted paper [27].

4.1 Introduction

This setting in this chapter is the same as in the previous one - we are once again considering an unbiased Waiter-Client game played on the edges of the complete graph. Thus, for an introduction to the area, we refer reader to the introduction of Chapter 3.

In that chapter, we resolved the question of Clemens et al. [15] how long does the unbiased Waiter-Client triangle-factor game last when both players play optimally. Clemens et al. further pose a question what happens in the more general case of the unbiased Waiter-Client *k-clique-factor game*. Here, we insist that n, the number of vertices of our graph, is divisible by k, and the winning sets are K_k-factors, i.e. decompositions of the vertices of K_n into disjoint sets of size k, each of which forms a clique K_k. Once again, it is easy to see that for n large enough in terms of k, Waiter has a winning strategy (in particular, it is a consequence of our result in Section 4.4).

Addressing this question of Clemens et al., our main aim in this chapter is to give the first non-trivial lower bound on $\tau_{WC}(\mathscr{F}_{n,K_k-\mathrm{fac}},1)$. Combined with a simple upper bound (which we expect to be of the correct magnitude) following the strategy of Clemens et al. for the triangle-factor game, this gives the following result.

Theorem 4.1.1. *There exist functions $n_0(k), C(k)$ such that one has*

$$2^{k/3-o(k)}n \leqslant \tau_{WC}(\mathscr{F}_{n,K_k-fac},1) \leqslant 2^k\frac{n}{k}+C(k)$$

for $n \geqslant n_0(k)$ and divisible by k.

Most of this chapter deals with proving the more difficult lower bound. The strategy of Client that we use is a trivial random one - Client always picks either element with equal probability and independently of the other rounds. Once we show that with a positive probability Client can survive a certain number of rounds (no matter what strategy Waiter uses), that guarantees a deterministic strategy of Client that ensures he can always survive that many rounds.

The difficult part of our approach is finding a quantity which we can bound in the expected value when Client plays randomly, no matter what strategy Waiter uses, and suitably relate to a K_k-factor. Such a quantity cannot be simple like the number of vertices that are part of some k-clique in Client's graph. Indeed, Waiter can initially create a clique A with $2k-2$ vertices in Client's graph (for details how, see Lemma 4.4.1), and then one by one make each vertex v in the graph part of a k-clique consisting of v and $k-1$ elements of A, while using only about kn rounds in total. So one needs a lot more delicate definition of events whose probabilities we bound.

The structure of this chapter is as follows. To motivate our main proof, in Section 4.2 we present a simpler and easier to grasp lower bound using similar ideas. Building on this, we prove the lower bound in Theorem 4.1.1 in Section 4.3. In Section 4.4, we use a strategy of Clemens et al. to obtain the upper bound in Theorem 4.1.1. And in Section 4.5, we give some final remarks.

4.2 A simple exponential bound

In this section, we will motivate the proof of our main result by providing a simple proof along similar lines of a somewhat weaker lower bound (which still has a term exponential in k in front of n).

Theorem 4.2.1. *Fix $k \geqslant 100$ and consider n divisible by k and large enough that Waiter has a winning strategy for the corresponding Waiter-Client k-clique-factor game. Then $\tau_{WC}(\mathscr{F}_{n,K_k-fac},1) \geqslant 2^{k/6-\frac{k}{\log k}}n$.*

The strategy of Client will be random and extremely simple - Client always takes the two edges offered and chooses the one to add to his graph randomly, with equal probabilities for each. If we can show that with probability at least $1/2$, no matter what strategy Waiter uses, she does not win against randomly playing Client after $2^{k/6-\frac{k}{\log k}}n$ rounds, then that also implies Client has a deterministic strategy guaranteeing him to make the game last at least that long. Note that the choice of the probability $1/2$ for our purpose is arbitrary - we could replace it with any constant p such that $0<p<1$.

We shall colour the edges of Client's graph red and the edges of Waiter's graph blue. For a given vertex v, we shall also denote by $N_R(v)$ its red neighbourhood and by $N_B(v)$ its blue neighbourhood.

The proof is based on an idea that after $2^{k/6-\frac{k}{\log k}}n$ rounds, if there was a red k-clique factor already created, many of the cliques C in this k-clique factor will contain some vertex z_i such that, if we denote the vertices of C by $z_1,...,z_k$:

- z_i has red degree less than $2^{k/6-\frac{k}{2\log k}}$;
- for many pairs z_j, z_l, both the red edges z_iz_j and z_iz_l were added before the red edge z_jz_l (here many means $(\frac{1}{6}-o(1))k^2$).

We then bound the probability of this event happening at any vertex z_i and conclude by taking expectation that on average, there will not be many such vertices, which gives the desired contradiction. In the next few pages we shall formalize this idea and make it work.

Assume that $k \geqslant 100$, that n is divisible by k, that n is large enough that Waiter has a winning strategy for the corresponding Waiter-Client K_k-factor game and that $2^{k/6-\frac{k}{\log k}}n$ rounds have passed. If Waiter already won at this point, there are disjoint red k-cliques $C_1,...,C_{n/k}$ partitioning the vertices of the graph.

Call a vertex a *high degree vertex* if it has red degree at least $2^{k/6-\frac{k}{2\log k}}$, and call it a *low degree vertex* otherwise. Call a clique a *high degree clique* if it contains at least one high degree vertex, and call it a *low degree clique* otherwise.

Claim 4.2.2. *Among $C_1,...,C_{n/k}$, there are at most $\frac{n}{2k}$ high degree cliques.*

Proof. Assume for a contradiction that this is false, so we have at least $\frac{n}{2k}$ high degree vertices. As every red edge touches at most two high degree vertices (i.e. its two endpoints), that implies we have at least $2^{k/6-\frac{k}{2\log k}}\frac{n}{4k}$ red edges in our graph. Since only $2^{k/6-\frac{k}{\log k}}n$ rounds have passed, we have

$$2^{k/6-\frac{k}{\log k}}n < 2^{k/6-\frac{k}{2\log k}}\frac{n}{4k},$$

for $k \geqslant 100$, which is a contradiction. □

Next, we need to define several events. For a given vertex v, denote by $X(v)$ the event that v is not a high degree vertex after $2^{k/6-\frac{k}{\log k}}n$ rounds have passed. Denote by $Y(v)$ the event that:

- there exists a subset $w_1,...,w_{k-1}$ in $N_R(v)$ after $2^{k/6-\frac{k}{\log k}}n$ rounds have passed that forms a red clique;

- moreover, at least $\frac{(k-1)(k-2)}{6}$ edges w_iw_j within this subset were added after both edges vw_i and vw_j have been added.

Finally let $S(v) = X(v) \cap Y(v)$.

The following simple lemma explains why we care about the event $Y(v)$ - it is because having a red K_k-factor, we can guarantee this event to happen for many vertices.

Lemma 4.2.3. *Start with an empty graph on k vertices and keep adding edges in some order until our graph is complete. Then there exists a vertex v such that at least $\frac{(k-1)(k-2)}{6}$ edges w_iw_j were added after both the edges vw_i and vw_j were added.*

It is not hard to see that up to the smaller order terms, this result is best possible - indeed, just adding the edges in random order, it will typically happen that for each vertex v, about $\frac{k^2}{6}$ edges w_iw_j will be added after both the edges vw_i and vw_j were added.

Proof of Lemma 4.2.3. Call a pair (v, w_iw_j) consisting of a vertex v and an edge w_iw_j good if w_iw_j is the last edge added to triangle vw_iw_j. Clearly, there are $\binom{k}{3}$ good pairs, so some vertex v appears in at least $\frac{1}{k}\binom{k}{3} = \frac{(k-1)(k-2)}{6}$ good pairs.

□

Combining Claim 4.2.2 and Lemma 4.2.3, we can conclude that if Waiter won fast, there in fact must be many vertices v for which the event $S(v)$ occurred. This is crucial for us, as the probability of the event $S(v)$ happening is something we can bound efficiently.

Claim 4.2.4. *If Waiter already won after $2^{k/6-\frac{k}{\log k}}n$ rounds have passed, there must exist at least $\frac{n}{2k}$ vertices v such that the event $S(v)$ has occurred.*

Proof. Since by Claim 4.2.2, there can be at most $\frac{n}{2k}$ high degree cliques among $C_1, \ldots, C_{n/k}$, there must be at least $\frac{n}{2k}$ low degree cliques among them. But by Lemma 4.2.3 applied to an arbitrary low degree clique C_j, there exists a vertex v_j of C_j for which the event $Y(v_j)$ has occurred. As C_j is a low degree clique, that means the event $S(v_j)$ has occurred as well. Finally, as the cliques $C_1, \ldots, C_{n/k}$ are all disjoint, we conclude there must be at least $\frac{n}{2k}$ such vertices. □

So now, we know we can finish the proof of Theorem 4.2.1 if we can show the following.

Lemma 4.2.5. *No matter what strategy Waiter uses, after $2^{k/6-\frac{k}{\log k}}n$ rounds have passed, we have*

$$\mathbb{E}\Big[\sum_v 1_{S(v)}\Big] \leqslant \frac{n}{4k}.$$

In particular, note that Lemma 4.2.5 establishes that Waiter can win against randomly playing Client within $2^{k/6-\frac{k}{\log k}}n$ rounds with probability at most $1/2$.

To start the proof of Lemma 4.2.5, we define

$$Y = \{\mathbf{y} = (y_1, ..., y_{k-1}) : y_i \in \mathbb{N}, 1 \leqslant y_1 < ... < y_{k-1} \leqslant 2^{k/6-\frac{k}{2\log k}}\}.$$

Next, for $\mathbf{y} \in Y$ and a vertex v, we define the event $T(v,\mathbf{y})$ as follows. Label the vertices in $N_R(v)$ at the time when $2^{k/6-\frac{k}{\log k}}n$ rounds have passed as $w_1, ..., w_l$, where the vertices w_i are ordered by the time when the edge vw_i appeared. Then $T(v,\mathbf{y})$ is the event that:

- $v, w_{y_1}, ..., w_{y_{k-1}}$ is a red clique;
- moreover, at least $\frac{(k-1)(k-2)}{6}$ edges $w_{y_i}w_{y_j}$ were added after both the edges vw_{y_i} and vw_{y_j} were added.

We divide the rest of the proof into three claims.

Claim 4.2.6. *For any $\mathbf{y} \in Y$ and any v, we have $\mathbb{P}\big[T(v,\mathbf{y})\big] \leqslant 2^{-\frac{k^2}{6}+k}$, regardless of the strategy of Waiter.*

Proof. To complete a red clique $v, w_{y_1}, ..., w_{y_{k-1}}$ in such a way that the event $T(v,\mathbf{y})$ would occur, Waiter has to offer Client at least $\frac{(k-1)(k-2)}{6}$ edges $w_{y_i}w_{y_j}$ at the time when both the edges vw_{y_i} and vw_{y_j} were already added. If at any time Waiter offers Client two such edges at the same time, one of the edges receives a blue colour and then the probability of the clique $v, w_{y_1}, ..., w_{y_{k-1}}$ to end up all red is zero. Hence, Waiter can only offer Client one such edge at time, always succeeding with probability $1/2$ and independently of the other rounds. This gives the bound

$$\mathbb{P}\big[T(v,\mathbf{y})\big] \leqslant 2^{-\frac{(k-1)(k-2)}{6}} < 2^{-\frac{k^2}{6}+k},$$

as required.

Note that it is not an issue to us that throughout, we may not yet know what vertices will the latter ones, like $w_{y_{k-1}}$, be, or how many edges $w_{y_i}w_{y_j}$ precisely will be added after both the edges vw_{y_i} and vw_{y_j} were added. The definition of our event simply guarantees there must be at least $\frac{(k-1)(k-2)}{6}$ such edges, and at any time that any of these is offered, we know that if it is coloured blue, then the event $T(v,\mathbf{y})$ cannot occur anymore. □

Claim 4.2.7. *For any v, we have*

$$S(v) \subset \bigcup_{\mathbf{y}\in Y} T(v,\mathbf{y}).$$

Proof. For $S(v)$ to occur, v needs to be a low degree vertex. Hence, whatever subset of $k-1$ vertices in $N_R(v)$ sees the event $Y(v)$ happen must consist of the first $2^{k/6-\frac{k}{2\log k}}$ vertices connected in red to v, and hence also sees the corresponding event $T(v,\mathbf{y})$ happen for some $\mathbf{y} \in Y$. □

Combining Claim 4.2.6, Claim 4.2.7 and the union bound, we obtain the final result that we need.

Claim 4.2.8. *We have* $\mathbb{P}\big[S(v)\big] < \frac{1}{4k}$ *for any v.*

Proof. Using the results above, we have

$$\mathbb{P}\big[S(v)\big] \leqslant \mathbb{P}\big[\bigcup_{\mathbf{y}\in Y} T(v,\mathbf{y})\big] \leqslant \sum_{\mathbf{y}\in Y} \mathbb{P}\big[T(v,\mathbf{y})\big] \leqslant |Y|\,2^{-\frac{k^2}{6}+k}$$
$$= \binom{2^{k/6-\frac{k}{2\log k}}}{k-1} 2^{-\frac{k^2}{6}+k} < 2^{(k/6-\frac{k}{2\log k})k} 2^{-\frac{k^2}{6}+k} = 2^{k-\frac{k^2}{2\log k}} < \frac{1}{4k},$$

provided $k \geqslant 100$. □

But now, Lemma 4.2.5 follows immediately from Claim 4.2.8, and hence Theorem 4.2.1 follows as well.

4.3 An improved lower bound

In this section, we prove the lower bound in Theorem 4.1.1, with the $o(k)$ term being $\frac{k}{\log k}$ provided $k \geqslant 10^8$.

The strategy we will use will be like in the previous section, but this time with a more general definition of the events whose probabilities we are bounding. Roughly, we will use that if after $2^{k/3-\frac{k}{\log k}} n$ rounds, there was a red k-clique factor already created, many of the cliques C in this k-clique factor will contain a vertex z_i such that, if we denote the vertices of C by $z_1,\ldots,z_k$:

- z_i has red degree less than $2^{k/3-\frac{k}{2\log k}}$;
- for many pairs z_j, z_l, both the vertices z_j and z_l were in the red connected component of z_i in the graph spanned by $z_1,\ldots,z_k$ before the red edge $z_j z_l$ was added (here many means $(\frac{1}{3}-o(1))k^2$).

This improves the bound significantly, though the details get more technical.

As we only change several ingredients in the proof from the previous section while keeping the rest very similar, we omit the proofs of some claims and instead reference the proofs of the analogous claims in Section 4.2. We also use the same notation for the events playing the same role, in the hope that this will make the connections to the motivating simpler proof in Section 4.2 clearer.

Once again, the strategy of Client will be random - Client always takes the two edges offered and chooses the one to be added to his graph randomly, with equal probabilities for each and independently of the other rounds. The aim will also be as previously - to show that no matter what strategy Waiter uses, after $2^{k/3-\frac{k}{\log k}}n$ rounds have passed, she could have won with probability at most $1/2$ against randomly playing Client. This guarantees a desired deterministic strategy of Client to survive $2^{k/3-\frac{k}{\log k}}n$ rounds.

As before, we shall colour the edges of Client's graph red and the edges of Waiter's graph blue, and we shall denote by $N_R(v)$ and $N_B(v)$ the red and the blue neighbourhoods of a vertex v.

Assume that $k \geqslant 10^8$, that n is divisible by k and large enough that Waiter has a winning strategy for the corresponding Waiter-Client K_k-factor game, and that $2^{k/3-\frac{k}{\log k}}n$ rounds have passed. If Waiter already won at this point, there are disjoint red cliques $C_1,...,C_{n/k}$ partitioning the vertices of the graph.

Once again, call a vertex a *high degree vertex* if it has red degree at least $2^{k/3-\frac{k}{2\log k}}$, and a *low degree vertex* otherwise. And call a clique a *high degree clique* if it contains at least one high degree vertex, and a *low degree clique otherwise*.

Claim 4.3.1. *Among $C_1,...,C_{n/k}$, there are at most $\frac{n}{2k}$ high degree cliques.*

Proof. The proof is analogous to the proof of Claim 4.2.2. □

We once again define several events. For a given vertex v, denote by $X(v)$ the event that v is a low degree vertex after $2^{k/3-\frac{k}{\log k}}n$ rounds have passed. Denote by $Y(v)$ the event that:

- there exists a subset $w_1,...,w_{k-1}$ in $N_R(v)$ after $2^{k/3-\frac{k}{\log k}}n$ rounds have passed that forms a red clique;
- moreover, none of the vertices $w_1,...,w_{k-1}$ is a high degree vertex;
- finally, at least $\frac{k^2}{3}-\frac{k^2}{(\log k)^2}$ edges w_iw_j were added after both w_i and w_j were already in the red connected component of v in the graph spanned by $v,w_1,...,w_{k-1}$.

Let $S(v) = X(v) \cap Y(v)$.

The lemma that follows corresponds to Lemma 4.2.3 in the previous section - but as the result is stronger, we need more care proving it.

Lemma 4.3.2. *Start with an empty graph on k vertices and keep adding edges in some order until our graph is complete. Then there exists a vertex v such that at least $\frac{k^2}{3} - \frac{k^2}{(\log k)^2}$ edges $w_i w_j$ were added after both w_i and w_j were already in the connected component of v.*

Let us note that up to the smaller order terms, Lemma 4.3.2 is tight. Indeed, consider an initially empty graph on $k = 2^t$ vertices for some large t. First, create a matching in this graph. After this step we have 2^{t-1} connected components $A_1, ..., A_{2^{t-1}}$, each consisting of two vertices. Next, fill in all the edges between A_{2i-1} and A_{2i} for $i = 1, ..., 2^{t-2}$, leading to 2^{t-2} connected components of four vertices each, each forming a clique. Continue in this manner until the end, always halving the number of the connected components and doubling the number of vertices in each component in each step, and making sure that each connected component is a clique at the end of each step. It is not hard to see that for each vertex v, $(1+o(1))\frac{k^2}{3}$ edges $w_i w_j$ were added after both w_i and w_j were already in the connected component of v.

Proof of Lemma 4.3.2. We will count all the pairs $(v, w_i w_j)$ consisting of the vertex v and the edge $w_i w_j$ with the property that the edge $w_i w_j$ was added after both w_i and w_j were already in the connected component of v. If we show there are at least $\frac{k^3}{3} - \frac{k^3}{(\log k)^2}$ such pairs, we are done, as that means some vertex v_0 is counted in at least $\frac{k^2}{3} - \frac{k^2}{(\log k)^2}$ such pairs.

Call an edge *rare* if it is one of the first $4(\log k)^2$ edges added at some vertex z. Clearly, there are at most $4k(\log k)^2$ rare edges.

Further, call an edge *connective at z* if it has one endpoint at a vertex z and connects a vertex z to a connected component of at least $4(\log k)^2$ vertices that z was previously not connected to. Call an edge *connective* if there exists some vertex z_0 such that this edge is connective at z_0. Clearly, for any vertex z, there can be at most $\frac{k}{4(\log k)^2}$ edges connective at z used throughout, since no other vertex z' can be in two different connected components that z is connected to by an edge that is connective at z. Hence overall we have at most $\frac{k^2}{4(\log k)^2}$ connective edges over all the vertices.

Consider all the unordered triples of distinct vertices (v_1, v_2, v_3) such that none of the edges v_1v_2, v_1v_3 and v_2v_3 is rare or connective. Then we claim that if the edge v_av_b, $a, b \in \{1,2,3\}$, was added first out of the edges v_1v_2, v_1v_3 and v_2v_3 and $c \in \{1,2,3\}$ is such that $c \neq a, b$, then the pairs (v_a, v_bv_c) and (v_b, v_av_c) were counted. Once we show that, we are done, as summing over all the triples and using that we have less than

$$4k(\log k)^2 + \frac{k^2}{4(\log k)^2} < \frac{k^2}{3.5(\log k)^2}$$

rare or connective edges, this gives at least

$$2\left(\binom{k}{3} - k\frac{k^2}{3.5(\log k)^2}\right) > \frac{k^3}{3} - \frac{k^3}{(\log k)^2}$$

such pairs, provided $k \geqslant 10^8$.

So we are left to show that if the edge v_1v_2 was added before the edges v_1v_3 and v_2v_3 were added, and none of the edges v_1v_2, v_1v_3 and v_2v_3 is rare or connective, then the pair (v_1, v_2v_3) was counted (we can without loss of generality reduce just to this case, as for the rest we can just relabel as needed). If the pair (v_1, v_2v_3) was not counted, that would mean either v_1 was not in the same component as v_2 or v_1 was not in the same component as v_3 at the time when the edge v_2v_3 was added. As the edge v_1v_2 was added even before, it is clearly impossible that v_1 was not in the same component as v_2. But if v_2 and v_3 were not in the same component at the time when the edge v_2v_3 was added, then either this edge would have been one of the first $4(\log k)^2$ edges added at v_3, and hence rare, or it would have connected v_2 to a new connected component of at least $4(\log k)^2$ vertices (as in particular the connected component of v_3 contains the entire neighbourhood of v_3), and hence it would have been connective at v_2. As neither is true by assumption, we know this also could not have happened, v_2 and v_3 (and hence also v_1 and v_3) were in the same connected component at the time when the edge v_2v_3 was added, and we are done. □

Having now proven this result, we continue as in the previous section.

Claim 4.3.3. *If Waiter already won after $2^{k/3-\frac{k}{\log k}}n$ rounds have passed, there must exist at least $\frac{n}{2k}$ vertices v such that the event $S(v)$ has occurred.*

Proof. The proof is analogous to the proof of Claim 4.2.4. □

So now, once again, we know we can finish the proof if we can show the following lemma.

Lemma 4.3.4. *No matter what strategy Waiter uses, after $2^{k/3-\frac{k}{\log k}}n$ rounds have passed, we have*

$$\mathbb{E}\Big[\sum_v 1_{S(v)}\Big] \leqslant \frac{n}{4k}.$$

This time, we have to define a more involved set of labels which will let us take $S(v)$ as a subset of the union of events whose probabilities we can easily bound. Let

$$Z = \{\mathbf{y} = (y_1, z_1, y_2, z_2, ..., z_{k-2}, y_{k-1}) : y_i, z_i \in \mathbb{N},$$
$$1 \leqslant y_1, ..., y_{k-1} \leqslant 2^{k/3 - \frac{k}{2\log k}}, 1 \leqslant z_i \leqslant i+1\}.$$

Enumerate the vertices of our graph as $v_1, ..., v_n$. Moreover, for each v_i, enumerate its red neighbours after $2^{k/3 - \frac{k}{\log k}} n$ rounds have passed as $w_{i,1}, w_{i,2}, ..., w_{i,t_i}$, where the labels correspond to the order in which these red edges were added.

Next, for $\mathbf{y} \in Z$, let $T(v_s, \mathbf{y})$ be the following event:

- we have a red clique $x_1, ..., x_k$, consisting of low degree vertices only, where $x_1 = v_s$, $x_2 = w_{s,y_1}$ and we always obtain the next vertex as follows: if we have $x_1, ..., x_{i+1}$ already chosen, $z_i = m$ (for some $1 \leqslant m \leqslant i+1$) and $x_m = v_d$, then $x_{i+2} = w_{d,y_{i+1}}$;
- moreover, at least $\frac{k^2}{3} - \frac{k^2}{(\log k)^2}$ edges $x_i x_j$ were added after both x_i and x_j were already in the red connected component of v_s in the graph spanned by $x_1, ..., x_k$;
- further, if $z_i = m$, then x_m is the first vertex out of $x_1, ..., x_{i+1}$ that got connected to x_{i+2};
- finally, there is no pair x_{i_1}, x_{i_2} with $i_1 < i_2$ such that x_{i_2} appeared in the red connected component of v_s in the graph spanned by $x_1, ..., x_k$ strictly sooner than x_{i_1} (of course, both could have appeared at the same time, say when a new connected component consisting of several vertices gets added to the connected component of v_s in this graph).

As in the previous section, we divide the rest of the proof into three claims.

Claim 4.3.5. *For any $\mathbf{y} \in Z$, we have $\mathbb{P}\big[T(v, \mathbf{y})\big] \leqslant 2^{-\frac{k^2}{3} + \frac{k^2}{(\log k)^2}}$, regardless of the strategy of Waiter.*

Proof. The proof is very similar to the proof of Claim 4.2.6 and hence is omitted. The important thing that makes our argument still work is the fact that once again, we can identify the vertex x_i as soon as it appears in the connected component of v in the graph spanned by the final clique $x_1, ..., x_k$ that v is part of. So any edges inside the connected component of v within this clique that are offered and are counted among the at least $\frac{k^2}{3} - \frac{k^2}{(\log k)^2}$ edges crucial for our argument are known to have this role before they are offered. $\square$

Let us note that Claim 4.3.5 only establishes an upper bound. For many $\mathbf{y} \in Z$, the event $T(v, \mathbf{y})$ cannot happen at all. Say if for $i_1 < i_2$, we have $z_{i_1} = z_{i_2}$ but $y_{i_1} > y_{i_2}$, then the corresponding event cannot happen, as it is easy to check that it is then impossible to fulfill all the encoding rules.

The following claim is once again very similar to Claim 4.2.7, though this time bit less immediate.

Claim 4.3.6. *For any v, we have*

$$S(v) \subset \bigcup_{\mathbf{y} \in Z} T(v, \mathbf{y}).$$

Proof. It is easy to see that if a red clique $v, w_1, \ldots, w_{k-1}$ (consisting of low degree vertices only) was created, which moreover has the property that at least $\frac{k^2}{3} - \frac{k^2}{(\log k)^2}$ edges $w_i w_j$ were added after both w_i and w_j were already in the connected component of v in the graph spanned by $v, w_1, \ldots, w_{k-1}$, then we can encode this into a suitable event $T(v, \mathbf{y})$. Indeed, just start with v and keep adding the vertices in order in which they appear in its red connected component in the graph spanned by $v, w_1, \ldots, w_{k-1}$, following the rule during our encoding that if x_m is the first vertex out of $x_1, \ldots, x_{i+1}$ that got connected to x_{i+2}, then we set $z_i = m$ and choose y_{i+1} accordingly. When the entire new connected component is added in a single round to the connected component of v, add its vertices in such an order that at any point, the graph spanned by $x_1, \ldots, x_{i+1}$ at that point in time is connected. As all the vertices $v, w_1, \ldots, w_{k-1}$ are low degree ones (and hence the labels y_i will be in the required range), it is easy to check that such an encoding indeed works. □

Combining Claim 4.3.5, Claim 4.3.6 and the union bound, we once again obtain the final result.

Claim 4.3.7. *We have $\mathbb{P}\big[S(v)\big] < \frac{1}{4k}$ for any v.*

Proof. Using the results above, we have

$$\begin{aligned} \mathbb{P}\big[S(v)\big] &\leqslant \mathbb{P}\Big[\bigcup_{\mathbf{y} \in Z} T(v, \mathbf{y})\Big] \leqslant \sum_{\mathbf{y} \in Z} \mathbb{P}\big[T(v, \mathbf{y})\big] \\ &\leqslant |Z| 2^{-\frac{k^2}{3} + \frac{k^2}{(\log k)^2}} \leqslant k^k 2^{(k/3 - \frac{k}{2 \log k})k} 2^{-\frac{k^2}{3} + \frac{k^2}{(\log k)^2}} < \frac{1}{4k}, \end{aligned}$$

provided $k \geqslant 10^8$. □

But now, Lemma 4.3.4 follows immediately from Claim 4.3.7, and hence we are also done proving the lower bound in Theorem 4.1.1.

4.4 An upper bound

In this section, we prove the upper bound in Theorem 4.1.1. Roughly, we follow the approach of Clemens et al. [15], who used it for the case $k = 3$, i.e. the triangle-factor game, but we spell out the details for the convenience of the reader.

Assume $k \geqslant 4$ is fixed, n is divisible by k and is large enough (for instance $n \geqslant 4^{8^k}$ will do). Once again, colour the edges of Client's graph red and the edges of Waiter's graph blue.

The crucial ingredient of our proof is the following algorithm that Waiter has.

Lemma 4.4.1. *Given $l \geqslant 2$ and $2^l - 1$ vertices $v_1, v_2, ..., v_{2^l-1}$ of an initially empty board, Waiter can in at most 2^l rounds of playing on this board create a red clique $w_1, ..., w_l$ with $w_1 = v_1$, and moreover keep the property that every edge placed so far (whether red or blue) has at least one endpoint of the form w_i for some $1 \leqslant i \leqslant l$.*

Proof. To do that, Waiter uses the following approach. Waiter sets

$$w_1 = v_1, S_1 = \{v_1, v_2, v_3, ..., v_{2^l-1}\}.$$

After, Waiter offers Client one by one pairs of edges (v_1v_{2i}, v_1v_{2i+1}) for $i = 1, ..., 2^{l-1} - 1$. This now gives a subset of $2^{l-1} - 1$ vertices connected to v_1 in red, and we label this subset as S_2 and set w_2 to be a vertex v_j of the smallest index j in S_2 (so in this case, this smallest index j will be 2 or 3).

We continue iteratively in this manner. Given S_j, w_j (where $j < l$ and $|S_j| = 2^{l+1-j} - 1$), Waiter offers Client the edges from w_j to $S_j \setminus \{w_j\}$ in pairs, sets S_{j+1} to be the set of the vertices of $S_j \setminus \{w_j\}$ connected to w_j in red and sets w_{j+1} to be the vertex v_m of the smallest index m in S_{j+1}. Note that in particular this ensures that $|S_{j+1}| = 2^{l+1-(j+1)} - 1$.

Waiter can clearly continue like this until we obtain S_l, w_l and then $w_1, ..., w_l$ is our desired clique.

The number of rounds that had passed before our clique has been created is

$$(2^{l-1} - 1) + (2^{l-2} - 1) + ... + (2 - 1) < 2^l.$$

The property that every edge placed so far has at least one endpoint of the form w_i for some $1 \leqslant i \leqslant l$ trivially holds as well. □

Now we are ready to describe the strategy that Waiter will use to guarantee winning within the desired number of rounds.

Waiter will proceed in three stages. Throughout, Waiter will update a set F of vertices which is initially empty and has the property at any point in time that the vertices of F contain a red K_k-factor.

Stage I. In the first stage, Waiter creates a red clique R on 8^k vertices in the first 2^{8^k} rounds and ensures that every edge placed so far has at least one endpoint in R and the set of vertices S_0 defined as

$$S_0 = \{v : v \text{ is an endpoint of at least one red or blue edge}\}$$

satisfies $|S_0| < 2^{8^k}$.

To do that, Waiter simply picks an arbitrary set S_0 of $2^{8^k} - 1$ vertices and uses an algorithm from Lemma 4.4.1.

Denote by A the vertices of our graph not in S_0 and set $B = S_0 \setminus R$ (note that unlike F, the sets A, B, R will not be updated further). After this stage, Waiter still keeps $F = \emptyset$.

Stage II. In this stage, Waiter keeps picking $2^k - 1$ vertices at the time (with the first vertex v_1 always being from B and all the other ones from A until all the vertices of B are used, and after using just the vertices from A) and creating a new K_k using the algorithm from Lemma 4.4.1, insisting as mentioned that the one vertex we have from B is always in the resulting clique (until we run out of the vertices in B, which will happen before this stage ends, as we insisted that $n \geqslant 4^{8^k}$). Whenever Waiter creates such a clique, she puts its k vertices into F. Waiter does this until there are less than 2^k vertices in $A \setminus F$ left.

Stage III. Now we have some vertices $z_1, \ldots, z_t$ for $0 \leqslant t < 2^k$ in $A \setminus F$ left. One by one, Waiter offers for each z_i pairs of the edges between z_i and the vertices in what is left in $R \setminus F$ to Client until she creates a clique with one vertex z_i and $k-1$ vertices in $R \setminus F$. Then Waiter takes the k vertices of the resulting clique and puts them into F. Due to our constraints, we can see Waiter has enough time to do this for all the vertices $z_1, \ldots, z_t$, and what is left of the graph (i.e. not in F) after this process is a subset of R, which hence also decomposes into k-cliques and can be just put into F immediately. Thus, we have created a red K_k-factor.

In total, we see that Waiter has needed at most $2^{8^k} + 2^k \frac{n}{k} + 2^k k$ rounds, proving the upper bound in Theorem 4.1.1 with $C(k) = 2^{8^k} + 2^k k$.

4.5 Concluding remarks

The upper bound and the lower bound in Theorem 4.1.1 are still very far apart. We suspect that the upper bound is roughly of the correct magnitude, as it is hard to imagine Waiter could come up with a better strategy than some variant of the natural one from Section 4.4.

As a first step towards proving the tight result for the lower bound, we believe one could perhaps try to improve the lower bound in Theorem 4.1.1 to $2^{k/2-o(k)}n$. Indeed, the logic behind our belief that this should be easier than going past this barrier is as follows. In Section 4.2, we obtained the bound of the form $2^{k/6-o(k)}n = 2^{\frac{1}{3}(k/2)-o(k)}n$ as we managed to include approximately one third of the edges in the low degree cliques into our argument. We could then improve this to $2^{k/3-o(k)}n = 2^{\frac{2}{3}(k/2)-o(k)}n$ in Section 4.3 since using a more involved argument, we could make use of about two thirds of the edges in the low degree cliques (which, as mentioned in the relevant section, is tight for the particular argument that we use). So if one managed to come up with a different argument using almost all the edges in the low degree cliques (which does not seem inconceivable), then there would be a hope for the bound of the form $2^{k/2-o(k)}n$.

We expect getting past this barrier to be even more difficult and to require a different approach. Nonetheless, we presume it still should be sufficient to use a random play as the strategy of Client in this higher range.

Chapter 5

Restricted online Ramsey numbers of matchings

The results in this chapter were published in The Electronic Journal of Combinatorics [25].

5.1 Introduction

For families of graphs $\mathscr{G}_1,...,\mathscr{G}_t$, the *Ramsey number* $R(\mathscr{G}_1,...,\mathscr{G}_t)$ is the smallest integer n such that any colouring of edges of K_n with colours $1,...,t$ contains a graph G_i in colour i for some $G_i \in \mathscr{G}_i$ and some $i \in \{1,...,t\}$. When each family $\mathscr{G}_i$ contains a single graph G_i, we instead use the notation $R(G_1,...,G_t)$ for the corresponding Ramsey number. When moreover we have $G_1 = ... = G_t$, we use the notation $R_t(G_1)$ for $R(G_1,...,G_t)$. The Ramsey numbers of graphs have been studied extensively, see for instance a survey of Conlon, Fox and Sudakov [19].

Many variants of the Ramsey numbers have been considered. One of them are the so-called *online Ramsey numbers*, introduced by Beck [2] and later independently by Kurek and Ruciński [53]. For families of graphs $\mathscr{G}_1,...,\mathscr{G}_t$, the online Ramsey number $\tilde{R}(\mathscr{G}_1,...,\mathscr{G}_t)$ is the smallest integer k for which Builder can always guarantee a win within the first k moves of the following game between Builder and Painter. Initially, we are given an infinite set of vertices, with every two vertices connected by an uncoloured edge. In each turn, Builder picks an edge between some two vertices in our set and Painter chooses any colour out of $1,...,t$ and colours the edge with this colour. Builder wins once there is a graph G_i in colour i for some $G_i \in \mathscr{G}_i$ and some $i \in \{1,...,t\}$. From various results about online Ramsey numbers, probably the most notable ones are the papers of Conlon [17] and Conlon, Fox, Grinshpun and He [18] relating the online Ramsey numbers of cliques to the usual such Ramsey numbers.

In 2008, Prałat [64] also introduced the *restricted online Ramsey numbers* (note that he called these the generalized online Ramsey numbers, the name restricted online Ramsey numbers that is now standard was first used by Conlon, Fox, Grinshpun and He [18]). These correspond to the same game as the online Ramsey numbers, but this game is now instead played on a finite board. To define this formally, for families of graphs $\mathscr{G}_1,...,\mathscr{G}_t$ and an integer n such that $n \geqslant R(\mathscr{G}_1,...,\mathscr{G}_t)$, the restricted online Ramsey number $\tilde{R}(\mathscr{G}_1,...,\mathscr{G}_t;n)$ is the smallest integer l for which Builder can always guarantee a win within the first l moves of the following game between Builder and Painter. In each turn, Builder picks an edge of initially uncoloured K_n and Painter chooses any colour out of $1,...,t$ to colour this edge. Builder wins once there appears a graph G_i in colour i for some $G_i \in \mathscr{G}_i$ and some $i \in \{1,...,t\}$. We note that the definitions of $\tilde{R}(\mathscr{G}_1,...,\mathscr{G}_t;n)$ differ slightly between the previous papers on this topic [11, 18, 20, 41, 64], but it is easy to see that all are equivalent.

Analogously to the usual Ramsey numbers, when each family $\mathscr{G}_i$ contains a single graph G_i, we use the notation $\tilde{R}(G_1,...,G_t;n)$ for the corresponding restricted online Ramsey number. And when we further have $G_1 = ... = G_t$, we use the notation $\tilde{R}_t(G_1;n)$ for $\tilde{R}(G_1,...,G_t;n)$.

Briggs and Cox [11] studied the restricted online Ramsey numbers of matchings and trees. Before stating their results, recall the following well-known result of Cockayne and Lorimer [16] about the Ramsey numbers of matchings.

Theorem 5.1.1 (Cockayne and Lorimer [16]). *For any $t \geqslant 2$ and positive integers $r_1,...,r_t$, we have*

$$R(r_1K_2,...,r_tK_2) = \max_i r_i + 1 + \sum_{i=1}^{t}(r_i - 1).$$

Hence in particular, $R_t(rK_2) = r + 1 + t(r-1)$.

When r is fixed, we will denote by n_t for $t \geqslant 2$ the number $R_t(rK_2) = r+1+t(r-1)$. So in particular we have $n_2 = 3r-1$, $n_3 = 4r-2$ and $n_4 = 5r-3$. Now we are ready to state the result of Briggs and Cox [11].

Theorem 5.1.2 (Briggs and Cox [11]). *Fix $t \geqslant 2$ and positive integers $r_1,...,r_t$. If $n \geqslant R(r_1K_2,...,r_tK_2)$, then*

$$\tilde{R}(r_1K_2,...,r_tK_2;n) \leqslant \frac{2t-1+(t-3)\log_2(t-2)}{t+1}n$$

with the convention that $\log_2 0 = 0$.

Moreover, if we fix $r \geqslant 1$, then $\tilde{R}_2(rK_2;n_2) \leqslant 3r-2 = n_2-1$, $\tilde{R}_3(rK_2;n_3) \leqslant 5r-4$ and $\tilde{R}_4(rK_2;n_4) \leqslant 7r-5$.

They ask whether we have $\tilde{R}_2(rK_2;n_2) = n_2-1$. The aim of this short chapter is to verify that this indeed holds. We also show that the bound $\tilde{R}_3(rK_2;n_3) \leqslant 5r-4$ is tight and that the bound $\tilde{R}_4(rK_2;n_4) \leqslant 7r-5$ is tight except possibly for the exact value of the additive constant.

By describing a suitable strategy of Painter, we prove the following more general lower bound and a corollary about restricted online Ramsey numbers of matchings with few colours. This in particular answers the question of Briggs and Cox [11].

Theorem 5.1.3. *Fix $t \geqslant 2$ and positive integers $r_1,\dots,r_t$. If $n \geqslant R(r_1K_2,\dots,r_tK_2)$, then*

$$\tilde{R}(r_1K_2,\dots,r_tK_2;n) \geqslant 3\left(\sum_{i=1}^{t} r_i - t + 1\right) - n.$$

Hence if we fix $r \geqslant 1$, then $\tilde{R}_2(rK_2;n_2) = 3r-2 = n_2-1$, $\tilde{R}_3(rK_2;n_3) = 5r-4$ and $\tilde{R}_4(rK_2;n_4) \in \{7r-6, 7r-5\}$.

It remains unclear whether for t and r large, the magnitude of $\tilde{R}_t(rK_2;n_t)$ is closer to the upper bound from Theorem 5.1.2 or to the lower bound from Theorem 5.1.3.

5.2 Proof of Theorem 5.1.3

Consider the game played with t colours on the edges of an initially uncoloured K_n. To prove Theorem 5.1.3, we will describe a strategy of Painter that ensures that after $T = 3(\sum_{i=1}^{t} r_i - t + 1) - n - 1$ moves (where by a move we mean Builder choosing some still uncoloured edge and Painter colouring it), there is no r_iK_2 of colour i for $i = 1,\dots,t$.

While taking her turns (and to help her with her colouring decisions), Painter will moreover assign the following states to the coloured edges of K_n and to all the vertices of K_n. Coloured edges are either *free*, or *rooted*. Every rooted edge is characterized by its *root*, which is one of its endpoints. Painter will assign (and update) the states of the coloured edges according to the strategy described below.

Vertices are of three types, characterized in the following way.

- If a vertex v is a root of at least one coloured edge, it is of type I.
- If a vertex v is not of type I, but there is at least one free edge with endpoint v, it is of type II.

- If a vertex v is neither of type I nor of type II, it is of type III.

In particular, note that initially all the vertices are of type III, since no edges are coloured at the start of the game.

For $0 \leqslant j \leqslant \binom{n}{2}$ and $i = 1, \ldots, t$, let $A_j(i)$ be a number of type I vertices that are roots to at least one edge of colour i after j moves and let $B_j(i)$ be a number of free edges of colour i after j moves. Let $A_j = \sum_{i=1}^{t} A_j(i)$ and $B_j = \sum_{i=1}^{t} B_j(i)$.

Assume Builder chooses the edge ab in $(k+1)$st turn of his (where $0 \leqslant k \leqslant \binom{n}{2} - 1$). Without loss of generality (as we could otherwise switch a and b), we can assume that if b is of type I, then a is also of type I; and if b is of type II, then a is of type I or of type II. Painter chooses the colour of an edge and updates the states of the coloured edges as follows.

(i) If a is a vertex of type I, we declare the edge ab to be rooted at a. By definition, there exists at least one other edge rooted at a, of some colour c_1 (if there are more edges rooted at a, pick one arbitrarily). We colour ab by colour c_1.

(ii) If a is a vertex of type II, there exists by definition a free edge ac for some c, of some colour c_2 (if there are more free edges with endpoint a, pick one arbitrarily). We declare both edges ab, ac to be rooted at a and colour ab in c_2.

(iii) If a is a vertex of type III, then the edge ab is declared to be free. It is coloured in any colour c_3 such that $A_k(c_3) + B_k(c_3) \leqslant r_{c_3} - 2$ if at least one such colour exists, and if not in an arbitrary colour.

The next two observations are straightforward.

Observation 5.2.1. *The number of vertices of type III:*

- *stays the same during move (i)*
- *increases by* 1 *or stays the same during move (ii)*
- *decreases by* 2 *during move (iii)*

Observation 5.2.2. *If move j was (i) or (ii), we have $A_j(i) + B_j(i) = A_{j-1}(i) + B_{j-1}(i)$ for $i = 1, \ldots, t$. If move j was (iii) and Painter used colour c, we have $A_j(c) + B_j(c) = A_{j-1}(c) + B_{j-1}(c) + 1$ and for any $c' \neq c$ we have $A_j(c') + B_j(c') = A_{j-1}(c') + B_{j-1}(c')$.*

Using Observation 5.2.1 and Observation 5.2.2, we prove the key lemma.

Lemma 5.2.3. *We have $A_T + B_T \leqslant \sum_{i=1}^{t} r_i - t$.*

Proof. Let C_2 be the number of moves (ii) up to time T, and let C_3 be the number of moves (iii) up to time T. At time T, by Observation 5.2.1 we have at most $n + C_2 - 2C_3$ vertices of type III. That implies $n + C_2 - 2C_3 \geqslant 0$. Since we further have $C_2 + C_3 \leqslant T$, we must have $C_3 \leqslant \frac{n+T}{3}$.

Now by Observation 5.2.2,

$$A_T + B_T \leqslant C_3 \leqslant \frac{n+T}{3} = \sum_{i=1}^{t} r_i - t + \frac{2}{3},$$

and since $A_T + B_T$ is an integer, we have $A_T + B_T \leqslant \sum_{i=1}^{t} r_i - t$ as required. □

Continuing the proof of Theorem 5.1.3, we are now ready to show that after T moves, there is no $r_i K_2$ of colour i for $i = 1, \ldots, t$.

Note that the existence of $r_m K_2$ of colour m would in particular imply that $A_T(m) + B_T(m) \geqslant r_m$. Because of the strategy of Painter and Observation 5.2.2, that would imply that $A_T(i) + B_T(i) \geqslant r_i - 1$ for $i = 1, \ldots, t$. Hence we would have

$$A_T + B_T \geqslant (r_1 - 1) + \ldots + r_m + \ldots + (r_t - 1) = \sum_{i=1}^{t} r_i - t + 1,$$

contradicting Lemma 5.2.3. Thus the proof of Theorem 5.1.3 is finished.

This also concludes the part of the dissertation about games on graphs.

Chapter 6

Improved bound for Tomaszewski's problem

This chapter is joint work with Peter van Hintum and Marius Tiba and our results were published in SIAM Journal on Discrete Mathematics [31].

6.1 Introduction

Consider the so-called Rademacher sums, i.e. random variables X such that $X = \sum_{i=1}^{n} a_i \varepsilon_i$ for real numbers a_i and independently and uniformly distributed signs $\varepsilon_i \sim \{-1,1\}$. What can we say about $\mathbb{P}[|X| \leqslant t]$ for some positive t?

If $t < \sqrt{\operatorname{Var} X}$ we may have $\mathbb{P}[|X| \leqslant t] = 0$ - indeed, just consider the case when only a_1 is non-zero. If $t > \sqrt{\operatorname{Var} X}$, we obtain using Markov's inequality and the fact that X has mean zero that

$$\mathbb{P}[|X| > t] = \mathbb{P}[X^2 > t^2] \leqslant \frac{\operatorname{Var} X}{t^2},$$

and hence $\mathbb{P}[|X| \leqslant t] \geqslant 1 - \frac{\operatorname{Var} X}{t^2} > 0$.

What happens when $t = \sqrt{\operatorname{Var} X}$? In 1986, Tomaszewski (see [42]) conjectured that in that case, $\mathbb{P}\left[|X| \leqslant \sqrt{\operatorname{Var} X}\right] \geqslant \frac{1}{2}$. Note that this bound is tight for $n \geqslant 2$ as we can take for instance $a_1 = a_2 \neq 0$, $a_i = 0$ for $2 < i \leqslant n$.

Several papers have focused on showing bounds from below approaching $1/2$. Holzman and Kleitman [48] proved that $\mathbb{P}\left[|X| \leqslant \sqrt{\operatorname{Var} X}\right] \geqslant \frac{3}{8}$. In fact, they showed the stronger, tight result that $\mathbb{P}\left[|X| < \sqrt{\operatorname{Var} X}\right] \geqslant \frac{3}{8}$ as long as there is more than one non-zero term. Later, but independently and using different techniques, Ben-Tal, Nemirovski and Roos [6] obtained the weaker bound of $\frac{1}{3}$. Their method was later refined by Shnurnikov [67] to obtain the bound of 0.36, still weaker than the result of Holzman and Kleitman.

More recently, Boppana and Holzman [10] obtained a bound of 0.406259. Using a result of Bentkus and Dzindzalieta [7], their argument can be improved to actually give a better bound of approximately 0.4276, as was independently observed by Hendriks and van Zuijlen and by Boppana (later combined into one publication [9]). We make further progress on Tomaszewski's conjecture by using different techniques to prove our main theorem.

Theorem 6.1.1. *Any Rademacher sum $X = \sum_i a_i \varepsilon_i$ has*

$$\mathbb{P}[|X| \leqslant \sqrt{\mathrm{Var} X}] \geqslant 0.46.$$

Note that partial sums $\sum_{i=1}^{k} a_i \varepsilon_i$ can be interpreted as a random walk with prescribed step sizes. This interpretation suggests common techniques like mirroring, symmetry and second moment arguments, as have been used in previous papers on this problem [6, 9, 10, 48, 67]. We manage to set up a framework which allows for a tight interplay between all these techniques, by combining them with ideas from linear programming.

We normalize by insisting that $\sum_{i=1}^{n} a_i^2 = 1$, which means that in particular $\mathrm{Var} X = 1$. Further, depending on the size of $\max\{|a_i|\}$, we consider four cases: the intermediate ones represent the core of the proof and to tackle them we use a combination of mirroring, symmetry and second moment arguments to reduce the problem to an easily solvable linear program.

The efficacy of the techniques used in this chapter is dependent on the specific values of the a_i's. Our division into different cases allows us to push each of the ideas to their limit. Because of the variety of examples of values a_i's showing the tightness of the conjecture in the sense that $\mathbb{P}\left[|\sum_{i=1}^{n} a_i \varepsilon_i| < 1\right] < \frac{1}{2}$ (e.g. $\frac{1}{3}, \ldots, \frac{1}{3}$, and the infinite family $\frac{k-1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}$ for each $k \geqslant 2$), it seems inescapable to engage in case analysis.

6.2 Setup

Fix a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ with $\sum_{i=1}^{n} a_i^2 = 1$ and $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$. Let ε_i for $i = 1, \ldots, n$ be i.i.d. random variables with $\mathbb{P}[\varepsilon_i = +1] = \mathbb{P}[\varepsilon_i = -1] = \frac{1}{2}$, i.e. independent *Rademacher random variables*, and denote $X = \sum_{i=1}^{n} a_i \varepsilon_i$. To show that $\mathbb{P}[|X| \leqslant 1] \geqslant 0.46$, we consider the following four cases depending on the size of a_1, a_2, a_3:

- $a_1 + a_2 \leqslant 1$, $a_3 \leqslant 0.25$;
- $0.25 \leqslant a_3 \leqslant a_1 \leqslant 0.49$;
- $0.49 \leqslant a_1 \leqslant 0.67$;

- $a_1 \geqslant 0.67$.

We will use induction on the dimension n. Note that for $n = 1,2$ the result is trivial. For $n = 3$, it follows easily too, by noting that all of the sums $a_1 - a_2 + a_3, -a_1 + a_2 + a_3, -a_1 + a_2 - a_3, a_1 - a_2 - a_3$ have absolute value at most 1. Thus we will further assume $n \geqslant 4$. The only time we will appeal to the induction hypothesis is in the proof of Lemma 6.4.4.

We write $\mathbb{P}[N(0,1) \geqslant x]$ for the probability that a standard normal attains a value of at least x.

Several times, we will use the following result of Bentkus and Dzindzalieta [7].

Lemma 6.2.1. *Let $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$ be such that $\sum_{i=1}^n a_i^2 \leqslant 1$, and let ε_i for $i = 1,\ldots,n$ be i.i.d. Rademacher random variables. Then we have for any $x \in \mathbb{R}$*

$$\mathbb{P}\left[\sum_{i=1}^{n} a_i \varepsilon_i \geqslant x\right] \leqslant 3.18\ \mathbb{P}[N(0,1) \geqslant x].$$

6.3 Easy cases

In this section, we handle the more straightforward cases when either $a_1 + a_2 \leqslant 1, a_3 \leqslant 0.25$ or when $a_1 \geqslant 0.67$. Here we only need simple mirroring arguments, accompanied by the tail bound provided by Lemma 6.2.1.

Proposition 6.3.1. *If $a_1 + a_2 \leqslant 1$ and $a_3 \leqslant 0.25$, then $\mathbb{P}[|X| \leqslant 1] \geqslant 0.46$.*

Proof of Proposition 6.3.1. Define the following random process $(X_t)_{t=0}^n$. Let $X_0 = 0$, and for $1 \leqslant t \leqslant n$, let $X_t = \sum_{i=1}^t a_i \varepsilon_i$. Let

$$T = \begin{cases} \inf\{1 \leqslant t \leqslant n : |X_t| > 0.75\} & \text{if } \{1 \leqslant t \leqslant n : |X_t| > 0.75\} \neq \emptyset, \\ n+1 & \text{otherwise.} \end{cases}$$

Then T is a stopping time. Also define random process $(Y_t)_{t=0}^n$ by setting $Y_t = X_t$ for $0 \leqslant t \leqslant T$ and $Y_t = 2X_T - X_t$ for $n \geqslant t > T$. Now, Y_n has the same distribution as $X_n = \sum_{i=1}^n a_i \varepsilon_i$.

Claim 6.3.2. $\mathbb{P}\big[|X_n| > 1$ *and* $|Y_n| > 1\big] < 0.08$.

Proof of Claim 6.3.2. Consider the event $|X_n| > 1$, and $|Y_n| > 1$. We shall show that in this case we have either $|X_n| > 2.5$ or $|Y_n| > 2.5$. By construction it follows that $1 \leqslant T \leqslant n$. Furthermore, we have $0.75 \leqslant |X_T| \leqslant 1$, where the upper bound follows from the condition $a_1 + a_2 \leqslant 1$ in the case $T = 1,2$, and from the condition $a_3 \leqslant 0.25$ in the case $3 \leqslant T \leqslant n$. On one hand by construction we have $2 \geqslant 2|X_T| = |X_n + Y_n|$ and on the other hand by assumption we have $2 < |X_n| + |Y_n|$. It follows that $|X_n + Y_n| \neq |X_n| + |Y_n|$ which implies that X_n, Y_n have different signs which implies that $|X_n + Y_n| = ||X_n| - |Y_n||$. Therefore, putting all together we have that

$$\begin{aligned}
1.5 &\leqslant 2|X_T| \\
&= |X_n + Y_n| \\
&= ||X_n| - |Y_n|| \\
&= \max(|X_n|, |Y_n|) - \min(|X_n|, |Y_n|) \\
&< \max(|X_n|, |Y_n|) - 1.
\end{aligned}$$

We get that either $|X_n| > 2.5$ or $|Y_n| > 2.5$. We conclude with the following sequence of inequalities.

$$\begin{aligned}
\mathbb{P}\big[|X_n| > 1 \text{ and } |Y_n| > 1\big] &\leqslant \mathbb{P}\big[|X_n| > 2.5 \text{ or } |Y_n| > 2.5\big] \\
&\leqslant 2\,\mathbb{P}\big[|X_n| > 2.5\big] \\
&\leqslant 6.36\,\mathbb{P}\big[|N(0,1)| > 2.5\big] < 0.08,
\end{aligned}$$

where the second inequality follows from the union bound and from the fact that X_n, Y_n have the same distribution and the third inequality follows from Lemma 6.2.1.

□

Returning to the proof of the proposition, since $\mathbb{P}[|X| \leqslant 1] = \mathbb{P}\big[|X_n| \leqslant 1\big] = \mathbb{P}\big[|Y_n| \leqslant 1\big]$, we obtain

$$\begin{aligned}
\mathbb{P}[|X| \leqslant 1] &= \frac{1}{2}\mathbb{P}\big[|X_n| \leqslant 1\big] + \frac{1}{2}\mathbb{P}\big[|Y_n| \leqslant 1\big] \\
&\geqslant \frac{1}{2}\Big(1 - \mathbb{P}\big[|X_n| > 1 \text{ and } |Y_n| > 1\big]\Big) \\
&\geqslant \frac{1}{2}(1 - 0.08) \\
&= 0.46,
\end{aligned}$$

which concludes the proof of Proposition 6.3.1. □

Proposition 6.3.3. *If $a_1 \geqslant 0.67$ then $\mathbb{P}[|X| \leqslant 1] \geqslant 0.46$.*

Proof of Proposition 6.3.3. Note that

$$\mathbb{P}[|X| \leqslant 1] = \mathbb{P}\left[\left|\sum_{i=1}^{n} a_i \varepsilon_i\right| \leqslant 1\right] \geqslant \frac{1}{2}\mathbb{P}\left[\left|\sum_{i=2}^{n} a_i \varepsilon_i\right| \leqslant 1.67\right].$$

Consider the unit vector $(b_2,...,b_n)$ with $b_i = \dfrac{a_i}{\sqrt{1-a_1^2}}$ for $i = 2,...,n$, and apply Lemma 6.2.1 to conclude that

$$\mathbb{P}\left[\left|\sum_{i=2}^{n} a_i\varepsilon_i\right| \leqslant 1.67\right] \geqslant \mathbb{P}\left[\left|\sum_{i=2}^{n} b_i\varepsilon_i\right| \leqslant \frac{1.67}{\sqrt{1-0.67^2}}\right]$$
$$\geqslant 1-3.18\,\mathbb{P}[|N(0,1)| > 2.24] \approx 0.9202.$$

and hence that $\mathbb{P}[|X| \leqslant 1] \geqslant 0.46$. □

So far we resolved the case in which $a_1 \geqslant 0.67$ and the case in which $a_1 + a_2 \leqslant 1$ and $a_3 \leqslant 0.25$, so it is enough to consider the following two cases:

- $0.25 \leqslant a_3 \leqslant a_1 \leqslant 0.49$
- $0.49 \leqslant a_1 \leqslant 0.67$

Each of these cases shall be treated in a separate section.

6.4 First intermediate case - $0.25 \leqslant a_3 \leqslant a_1 \leqslant 0.49$

In this section we prove the following proposition.

Proposition 6.4.1. *If* $0.25 \leqslant a_3 \leqslant a_1 \leqslant 0.49$*, then* $\mathbb{P}[|X| \leqslant 1] \geqslant 0.46$.

The strategy is to produce a carefully designed partition of the probability space generated by the possible outcomes of $|\sum_{i\geqslant 3} a_i\varepsilon_i|$. In order to bound the probabilities of these events, the idea is to rely one some mirroring and reflection constructions. Finally, we reduce the problem to an easy linear program.

Assume throughout this section that $0.25 \leqslant a_3 \leqslant a_1 \leqslant 0.49$. Let $S = \sum_{i=3}^{n} a_i\varepsilon_i$. Consider the following seven intervals which partition the positive half-line in this order:

- $I_1 = [0, 1-a_1-a_2]$,
- $I_2 = (1-a_1-a_2, 1-a_1+a_2]$,
- $I_3 = (1-a_1+a_2, 1+a_1-a_2]$,
- $I_4 = (1+a_1-a_2, 1+a_1+a_2]$,
- $I_5 = (1+a_1+a_2, 3-3a_1+a_2]$,

- $I_6 = (3-3a_1+a_2, 3+3a_1-5a_2]$,
- $I_7 = (3+3a_1-5a_2, \infty)$.

For $i = 1, ..., 7$, denote $p_i = \mathbb{P}\big[|S| \in I_i\big]$.

Considering the four choices for $(\varepsilon_1, \varepsilon_2)$, by the way this intervals are constructed and by the restrictions on a_1, a_2, a_3 we have that

$$\mathbb{P}\left[\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| > 1 \;\middle|\; |S| \in I_j\right] = \mathbb{P}\Big[|a_1\varepsilon_1 + a_2\varepsilon_2 + S| > 1 \;\Big|\; |S| \in I_j\Big] = \begin{cases} 0 \text{ if } j = 1 \\ \frac{1}{4} \text{ if } j = 2 \\ \frac{1}{2} \text{ if } j = 3 \\ \frac{3}{4} \text{ if } j = 4 \\ 1 \text{ if } j \geqslant 5. \end{cases}$$

Thus we can express

$$\begin{aligned} \mathbb{P}\left[\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| > 1\right] &= \sum_{j=1}^{7} \mathbb{P}\left[\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| > 1 \;\middle|\; |S| \in I_j\right] \mathbb{P}\big[|S| \in I_j\big] \\ &= \frac{1}{4}p_2 + \frac{1}{2}p_3 + \frac{3}{4}p_4 + p_5 + p_6 + p_7. \end{aligned} \tag{6.1}$$

We shall bound from above this expression, by exploiting various constraints that the p_i's satisfy and reducing to a linear program. We collect the constraints into separate lemmas.

Firstly, as the events $\{|S| \in I_i\}$ for $i = 1, ..., 7$ partition our probability space, we know that

$$p_1 + ... + p_7 = 1. \tag{6.2}$$

Computing the second moment of S, we find

$$\begin{aligned} 1 - a_1^2 - a_2^2 &= \mathbb{E}[S^2] \\ &= \sum_{i=1}^{7} \mathbb{P}\big[|S| \in I_i\big] \mathbb{E}[S^2 \mid |S| \in I_i] \\ &\geqslant \sum_{i=1}^{7} p_i (\inf I_i)^2 \\ &= (1-a_1-a_2)^2 p_2 + (1-a_1+a_2)^2 p_3 + (1+a_1-a_2)^2 p_4 \\ &\quad + (1+a_1+a_2)^2 p_5 + (3-3a_1+a_2)^2 p_6 + (3+3a_1-5a_2)^2 p_7 \end{aligned} \tag{6.3}$$

Lemma 6.4.2. $p_3 + p_4 + p_5 \leqslant \frac{1}{2}$

Proof. Consider the random process $(S_t)_{t=3}^n$, given by $S_t = \sum_{i=3}^t a_i\varepsilon_i$ for $n \geqslant t \geqslant 3$. Let

$$T_1 = \begin{cases} \inf\{t \geqslant 3 : |S_t| > 1 - a_1\} & \text{if } \{t \geqslant 3 : |S_t| > 1 - a_1\} \neq \emptyset, \\ n+1 & \text{otherwise} \end{cases}$$

Then T_1 is a stopping time. Also define random process $(U_t)_{t=3}^n$ by setting $U_t = S_t$ for $3 \leqslant t \leqslant T_1$ and $U_t = 2S_{T_1} - S_t$ for $n \geqslant t > T_1$. Now, U_n has the same distribution as $S = S_n$. The conclusion of the claim follows if we show that at most one of $|S_n|, |U_n|$ can lie in the interval $I_3 \cup I_4 \cup I_5$.

Indeed, if $T_1 = n+1$, then $U_n = S_n \in I_1 \cup I_2$. Otherwise, if $T_1 \leqslant n$, then $|S_{T_1}| \in (1-a_1, 1-a_1+a_2]$. Assume for the sake of contradiction that we have both $|S_n|, |U_n| \in I_3 \cup I_4 \cup I_5$. On the one hand, by construction we have $2(1-a_1+a_2) \geqslant 2|S_{T_1}| = |S_n + U_n|$ and on the other hand, by assumption we have $2(1-a_1+a_2) < |S_n| + |U_n|$. It follows that $|S_n + U_n| \neq |S_n| + |U_n|$, which implies that S_n, U_n have different signs which implies that $|S_n + U_n| = ||S_n| - |U_n||$. Putting all together we have that

$$2(1-a_1) \leqslant 2|S_{T_1}| = |S_n + U_n| = ||S_n| - |U_n|| < \sup(I_3 \cup I_4 \cup I_5) - \inf(I_3 \cup I_4 \cup I_5) = 2(1-a_1),$$

which gives the desired contradiction.

$\square$

Lemma 6.4.3. $p_4 + p_5 + p_6 \leqslant \frac{1}{2}$

Proof. The proof is completely analogous to the proof of previous claim, with the stopping time T_2 defined by

$$T_2 = \begin{cases} \inf\{t \geqslant 3 : |S_t| > 1 + a_1 - 2a_2\} & \text{if } \{t \geqslant 3 : |S_t| > 1 + a_1 - 2a_2\} \neq \emptyset, \\ n+1 & \text{otherwise} \end{cases}$$

$\square$

Lemma 6.4.4. $p_1 \geqslant 0.115 \cdot \mathbb{1}\{a_1 + a_2 \leqslant 0.665\}$

Proof. Let

$$\widetilde{T} = \begin{cases} \inf\{t \geqslant 4 : |\sum_{i=4}^t a_i\varepsilon_i| > 0.335\} & \text{if } \{t \geqslant 4 : |\sum_{i=4}^t a_i\varepsilon_i| > 0.335\} \neq \emptyset, \\ n+1 & \text{otherwise} \end{cases}$$

Then $\widetilde{T}$ is a stopping time. Further write

$$S = \sum_{i=3}^{n} a_i \varepsilon_i = S_a + S_b + S_c, \text{ and}$$

$$S_{(\tau_a,\tau_b,\tau_c)} = \tau_a S_a + \tau_b S_b + \tau_c S_c \text{ for any } (\tau_a,\tau_b,\tau_c) \in \{\pm 1\}^3,$$

where $S_a = a_3\varepsilon_3$, $S_b = \sum_{i=4}^{\widetilde{T}} a_i\varepsilon_i$, and $S_c = \sum_{i=\widetilde{T}+1}^{n} a_i\varepsilon_i$ if $\widetilde{T} < n$ and $S_c = 0$ otherwise. Note that $S_{(\tau_a,\tau_b,\tau_c)}$ has the same distribution as S.

Assume $a_1 + a_2 \leqslant 0.665$ and recall $a_3 \geqslant 0.25$. In order to show that $\mathbb{P}\big[|S| \leqslant 1 - a_1 - a_2\big] \geqslant 0.115$ it is enough to show that $\mathbb{P}\big[|S| \leqslant 0.335\big] \geqslant 0.115$.

Observation 6.4.5. *The conclusion follows if we show that* $\mathbb{P}\big[|S_c| \leqslant 0.91\big] \geqslant 0.46$ *and that if* $|S_c| \leqslant 0.91$ *then there exists* $(\tau_a,\tau_b,\tau_c) \in \{\pm 1\}^3$ *such that* $|S_{(\tau_a,\tau_b,\tau_c)}| \leqslant 0.335$.

Indeed, let E be the event that $|S_c| \leqslant 0.91$; we have $\mathbb{P}[E] \geqslant 0.46$. For every point $p \in E$ there exists $(\tau_a^p, \tau_b^p, \tau_c^p) \in \{\pm 1\}^3$ such that $|S_{(\tau_a^p,\tau_b^p,\tau_c^p)}(p)| \leqslant 0.335$. Note that by construction $|S_{(-\tau_a^p,-\tau_b^p,-\tau_c^p)}(p)| \leqslant 0.335$. Therefore, there exists $(\tau_a,\tau_b,\tau_c) \in \{\pm 1\}^3$ and an event $F \subset E$ with $\mathbb{P}[F] \geqslant 0.115$ such that for every point $p \in F$ we have $|S_{(\tau_a,\tau_b,\tau_c)}(p)| \leqslant 0.335$. As $S_{(\tau_a,\tau_b,\tau_c)}$ has the same distribution as S, it follows that $\mathbb{P}\big[|S| \leqslant 0.335\big] \geqslant 0.115$.

Claim 6.4.6. $\mathbb{P}\big[|S_c| \leqslant 0.91\big] \geqslant 0.46$.

Proof. Note that the value of $\widetilde{T}$ is independent of the values of ε_i for $i > \widetilde{T}$, so fix a particular value of $\widetilde{T}$. If $\sum_{i=\widetilde{T}+1}^{n} a_i = 0$, of course the statement is trivial. Otherwise consider the unit vector $(b_i)_{i=\widetilde{T}+1}^{n}$ defined by $b_i = a_i\left(\sqrt{1 - \sum_{j=1}^{\widetilde{T}} a_j^2}\right)^{-1}$. By the induction hypothesis applied to this vector, we find

$$\mathbb{P}\big[|S_c| \leqslant 0.91\big] \geqslant \mathbb{P}\left[|S_c| \leqslant \sqrt{1 - \sum_{j=1}^{\widetilde{T}} a_j^2}\right] = \mathbb{P}\left[\left|\sum_{i=\widetilde{T}+1}^{n} b_i\varepsilon_i\right| \leqslant 1\right] \geqslant 0.46,$$

where the first inequality follows from the fact that $\widetilde{T} \geqslant 4$ and $a_1, a_2, a_3 \geqslant 0.25$. □

Claim 6.4.7. *If* $|S_c| \leqslant 0.91$*, then there exists* $(\tau_a,\tau_b,\tau_c) \in \{\pm 1\}^3$ *such that* $|S_{(\tau_a,\tau_b,\tau_c)}| \leqslant 0.335$.

Proof. Assume for the sake of contradiction that $|S_{(\tau_a,\tau_b,\tau_c)}| > 0.335$ for all $(\tau_a,\tau_b,\tau_c) \in \{\pm 1\}^3$. Furthermore, assume without loss of generality that $S_a, S_b, S_c \geqslant 0$. Recall that $S_a = a_3, S_b \in [0, 0.335 + a_3], S_c \in [0, 0.91]$, that $0.25 < a_3 \leqslant \frac{a_1+a_2}{2} \leqslant 0.3325$ and furthermore that if $S_c > 0$, then $S_b \in (0.335, 0.335 + a_3]$.

We have $S_a - S_b + S_c \geqslant a_3 - (0.335 + a_3) + 0 = -0.335$ and hence $S_a - S_b + S_c > 0.335$. Similarly, we have $S_a + S_b - S_c \geqslant -0.335$ by the following dichotomy; if $S_c = 0$, then $S_a + S_b - S_c \geqslant 0.25 + 0 - 0 \geqslant 0.25$, and if $S_c > 0$, then $S_a + S_b - S_c \geqslant 0.25 + 0.335 - 0.91 > -0.335$. Hence $S_a + S_b - S_c \geqslant 0.335$. Combining these inequalities we get $2S_a = 2a_3 \geqslant 0.67$ which contradicts the hypothesis that $a_3 \leqslant 0.3325$. The conclusion follows. □

The two claims combined with Observation 6.4.5 conclude the proof of Lemma 6.4.4. □

Lemma 6.4.8. *For any parameters a_1, a_2 such that $0.25 \leqslant a_2 \leqslant a_1 \leqslant 0.49$, the output $L(a_1, a_2)$ of the following linear program satisfies $L(a_1, a_2) \leqslant 0.54$.*

$$L(a_1,a_2) := \max\{\tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 + \tfrac{3}{4}x_4 + x_5 + x_6 + x_7 \text{ subject to}$$

$$\begin{gathered} x_1, \dots, x_7 \geqslant 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1 \\ x_3 + x_4 + x_5 \leqslant \frac{1}{2} \\ x_4 + x_5 + x_6 \leqslant \frac{1}{2} \\ (1 - a_1 - a_2)^2 x_2 + (1 - a_1 + a_2)^2 x_3 + (1 + a_1 - a_2)^2 x_4 + \\ +(1 + a_1 + a_2)^2 x_5 + (3 - 3a_1 + a_2)^2 x_6 + (3 + 3a_1 - 5a_2)^2 x_7 \leqslant 1 - a_1^2 - a_2^2 \\ x_1 \geqslant 0.115 \cdot \mathbb{1}\{a_1 + a_2 \leqslant 0.665\}\} \end{gathered}$$

Proof. While we could solve this linear program problem directly, we will instead reduce it to a finite number of cases as follows. Set the margin of error $e = 0.005$ and for parameters $a_1', a_2' \in \frac{1}{100}\mathbb{Z}$ of our choice consider the output $L'(a_1', a_2')$ of the following linear program.

$$L'(a_1', a_2') := \max\left\{\tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 + \tfrac{3}{4}x_4 + x_5 + x_6 + x_7 \text{ subject to}\right.$$

$$\begin{gathered} x_1, \dots, x_7 \geqslant 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1 \\ x_3 + x_4 + x_5 \leqslant \frac{1}{2} \\ x_4 + x_5 + x_6 \leqslant \frac{1}{2} \\ (1 - a_1' - a_2' - 2e)^2 x_2 + (1 - a_1' + a_2' - 2e)^2 x_3 + (1 + a_1' - a_2' - 2e)^2 x_4 \\ +(1 + a_1' + a_2' - 2e)^2 x_5 + (3 - 3a_1' + a_2' - 4e)^2 x_6 \\ +(3 + 3a_1' - 5a_2' - 8e)^2 x_7 \leqslant 1 - (a_1' - e)^2 - (a_2' - e)^2 \\ x_1 \geqslant 0.115 \cdot \mathbb{1}\{a_1' + a_2' + 2e \leqslant 0.665\}\}. \end{gathered}$$

Observation 6.4.9. *If we set a'_1 (a'_2 resp.) to be a_1 (a_2 resp.) rounded to the nearest one hundredth then we have $L'(a'_1,a'_2) \geqslant L(a_1,a_2)$, as every individual constraint in the linear program L' is at most as strict as its counterpart in the linear program L. Given the constraint $0.49 \geqslant a_1 \geqslant a_2 \geqslant 0.25$, we deduce the constraint $0.49 \geqslant a'_1 \geqslant a'_2 \geqslant 0.25$.*

A simple computer check shows that for all parameters $a'_1, a'_2 \in \frac{1}{100}\mathbb{Z}$ that satisfy $0.49 \geqslant a'_1 \geqslant a'_2 \geqslant 0.25$ we have $L'(a'_1,a'_2) \leqslant 0.54$. Using Observation 6.4.9 we conclude that $L(a_1,a_2) \leqslant 0.54$ as desired. □

We conclude this section with the proof of the main proposition.

Proof of Proposition 6.4.1. By 6.2, 6.3, Lemma 6.4.2, Lemma 6.4.3, and Lemma 6.4.4, the parameters p_i satisfy the constraints in Lemma 6.4.8, so that the set of **x**'s over which $L(a_1,a_2)$ is maximized includes **p**. Finally, by 6.1 and Lemma 6.4.8, we conclude that

$$\mathbb{P}\left[\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| > 1\right] = \frac{1}{4}p_2 + \frac{1}{2}p_3 + \frac{3}{4}p_4 + p_5 + p_6 + p_7 \leqslant L(a_1,a_2) \leqslant 0.54.$$

□

6.5 Second intermediate case - $0.49 \leqslant a_1 \leqslant 0.67$

In this section, we solve the last case we have not tackled yet.

Proposition 6.5.1. *If $0.49 \leqslant a_1 \leqslant 0.67$ then $\mathbb{P}[|X| \leqslant 1] \geqslant 0.46$.*

We shall follow a similar strategy to the previous section employing the same set of techniques. However, in this section we shall use the linear program only to further reduce the range of vectors **a** we are examining. We conclude the remaining cases using additional analytic arguments.

Assume throughout this section that $0.49 \leqslant a_1 \leqslant 0.67$. For $i > 1$, we call the term a_i *big* if $a_1 + a_i > 1$, and we call it *small* otherwise.

Lemma 6.5.2. *If we have any small term a_j such that $a_j \geqslant 0.25$, then $\mathbb{P}\left[\left|\sum_{i=1}^{n} a_i\varepsilon_i\right| > 1\right] \leqslant 0.54$.*

Proof. Assume we have such an a_j. Let $U = \sum_{\substack{2\leqslant i\leqslant n \\ i\neq j}} a_i\varepsilon_i$, i.e. the sum of all the signed terms except a_1 and a_j. Consider the following five intervals which partition the positive half-line in this order:

- $I_1 = [0, 1 - a_1 - a_j]$,

- $I_2 = (1 - a_1 - a_j, 1 - a_1 + a_j]$,
- $I_3 = (1 - a_1 + a_j, 1 + a_1 - a_j]$,
- $I_4 = (1 + a_1 - a_j, 1 + a_1 + a_j]$,
- $I_5 = (1 + a_1 + a_j, \infty)$.

For $i = 1, ..., 5$, write $p_i = \mathbb{P}\big[|U| \in I_i\big]$, so that, analogous to 6.1, we may write

$$\mathbb{P}\left[\left|\sum_{i=1}^{n} a_i \varepsilon_i\right| > 1\right] = \frac{1}{4}p_2 + \frac{1}{2}p_3 + \frac{3}{4}p_4 + p_5. \tag{6.4}$$

Analogous to the previous section we get (after noticing the events $\{|U| \in I_i\}$ form a partition of our probability space and after computing the second moment)

$$1 = p_1 + p_2 + p_3 + p_4 + p_5 \tag{6.5}$$

$$\begin{aligned} 1 - a_1^2 - a_j^2 \geqslant (1 - a_1 - a_j)^2 p_2 + (1 - a_1 + a_j)^2 p_3 \\ + (1 + a_1 - a_j)^2 p_4 + (1 + a_1 + a_j)^2 p_5 \end{aligned} \tag{6.6}$$

Claim 6.5.3. *For any parameters a_1, a_j such that $0.49 \leqslant a_1 \leqslant 0.67$ and $0.25 \leqslant a_j \leqslant 1 - a_1$, the output $M(a_1, a_j)$ of the following linear program satisfies $M(a_1, a_j) \leqslant 0.54$.*

$$M(a_1, a_j) := \max\{\tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 + \tfrac{3}{4}x_4 + x_5 \text{ subject to}$$

$$x_1, ..., x_5 \geqslant 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

$$(1 - a_1 - a_j)^2 x_2 + (1 - a_1 + a_j)^2 x_3 + (1 + a_1 - a_j)^2 x_4 + (1 + a_1 + a_j)^2 x_5 \leqslant 1 - a_1^2 - a_j^2\}$$

Proof. While we could solve this linear program problem directly, we will instead reduce it to a finite number of cases as follows. Set the margin of error $e = 0.005$ and for parameters $a_1', a_j' \in \frac{1}{100}\mathbb{Z}$ of our choice consider the output $M'(a_1', a_j')$ of the following linear program.

$$M'(a'_1,a'_j) := \max\{\tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 + \tfrac{3}{4}x_4 + x_5 \text{ subject to}$$

$$\begin{gathered} x_1,\ldots,x_5 \geqslant 0 \\ x_1+x_2+x_3+x_4+x_5 = 1 \\ g(a'_1,a'_j,e)^2x_2 + (1-a'_1+a'_j-2e)^2x_3+ \\ +(1+a'_1-a'_j-2e)^2x_4+(1+a'_1+a'_j-2e)^2x_5 \leqslant 1-(a'_1-e)^2-(a'_j-e)^2\} \end{gathered}$$

where $g(a'_1,a'_j,e) = 1-a'_1-a'_j-2e$ if $1-a'_1-a'_j-2e>0$, and $g(a'_1,a'_j,e)=0$ otherwise.

Observation 6.5.4. *If we set a'_1 (a'_j resp.) to be a_1 (a_j resp.) rounded to the nearest one hundredth then we have $M'(a'_1,a'_j) \geqslant M(a_1,a_j)$, as every individual constraint in the linear program M' is at most as strict as its counterpart in the linear program M. Given the constraint $0.49 \leqslant a_1 \leqslant 0.67$ and $0.25 \leqslant a_j \leqslant \min\{1-a_1,a_1\}$, we deduce the constraint $0.49 \leqslant a'_1 \leqslant 0.67$ and $0.25 \leqslant a'_j \leqslant \min\{1.01-a'_1,a'_1\}$.*

A simple computer check shows that for all parameters $a'_1,a'_j \in \frac{1}{100}\mathbb{Z}$ that satisfy $0.49 \leqslant a'_1 \leqslant 0.67$ and $0.25 \leqslant a'_j \leqslant \min\{1.01-a'_1,a'_1\}$ we have $M'(a'_1,a'_j) \leqslant 0.54$. Using Observation 6.5.4, we conclude that $M(a_1,a_j) \leqslant 0.54$ as desired.

□

We return to the proof of the lemma. By 6.5, and 6.6, the parameters p_i satisfy the constraints in Claim 6.5.3, so that the set of **x**'s over which $M(a_1,a_j)$ is maximized includes **p**. Finally, by 6.4 and Claim 6.5.3, we conclude that

$$\mathbb{P}\left[\left|\sum_{i=1}^n a_i\varepsilon_i\right| > 1\right] = \frac{1}{4}p_2 + \frac{1}{2}p_3 + \frac{3}{4}p_4 + p_5 \leqslant M(a_1,a_j) \leqslant 0.54.$$

□

Observation 6.5.5. *It was crucial that a_j was a small term. If it was big instead, for the interval $I'_1 = [0,a_1+a_j-1)$ around the origin, we have*

$$\mathbb{P}\left[\left|\sum_{i=1}^n a_i\varepsilon_i\right| > 1 \;\middle|\; |U| \in I'_1\right] = \frac{1}{2}.$$

This is in contrast with $\mathbb{P}\left[\left|\sum_i a_i\varepsilon_i\right| > 1 \;\middle|\; |U| \in I_1\right] = 0$, which we used in the proof of Lemma 6.5.2.

Henceforth we shall assume that there exist no small terms of size at least 0.25 and we shall use a mirroring argument to conclude. Let k be such that the terms $a_2,\dots,a_k$ are big and the terms $a_{k+1},\dots,a_n$ are small. We will need the following easy lemma.

Lemma 6.5.6. *If $2 \leqslant l \leqslant k$, then we have $a_2+a_3+\dots+a_{l-1}+2a_l \leqslant 2$.*

Proof. Using the fact that $\sum_{i=2}^{l} a_i^2 \leqslant 1-a_1^2$ and that a_l is the smallest term out of $a_2,\dots,a_l$, we get $a_l \leqslant \sqrt{\frac{1-a_1^2}{l-1}}$. Using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
a_2+a_3+\dots+a_{l-1}+2a_l &\leqslant (a_2+\dots+a_l)+\sqrt{\frac{1-a_1^2}{l-1}} \\
&\leqslant \sqrt{(l-1)(a_2^2+\dots+a_l^2)}+\sqrt{\frac{1-a_1^2}{l-1}} \\
&\leqslant \sqrt{l-1}\sqrt{1-a_1^2}+\sqrt{\frac{1-a_1^2}{l-1}}
\end{aligned}
$$

Next, note that as each big term is bigger than $1-a_1$, so

$$1-a_1^2 \geqslant a_2^2+a_3^2+\cdots+a_l^2 \geqslant (l-1)(1-a_1)^2$$

and thus $l-1 \leqslant \frac{1-a_1^2}{(1-a_1)^2} = \frac{1+a_1}{1-a_1}$.

Combining these two with the fact that the function $x+\frac{1}{x}$ is increasing on the interval $[1,\infty)$, we find

$$
\begin{aligned}
\sqrt{l-1}\sqrt{1-a_1^2}+\sqrt{\frac{1-a_1^2}{l-1}} &\leqslant \sqrt{1-a_1^2}\left(\sqrt{\frac{1+a_1}{1-a_1}}+\sqrt{\frac{1-a_1}{1+a_1}}\right) \\
&\leqslant (1+a_1)+(1-a_1)=2
\end{aligned}
$$

This concludes the proof of the lemma. $\square$

Proof of Proposition 6.5.1. Define the following random process $(A_t)_{t=0}^{n}$. We set $A_0=0$, $A_1=a_1\varepsilon_1$ and for $n \geqslant t \geqslant 2$, $A_t = a_1\varepsilon_1+\sum_{i=n-t+2}^{n} a_i\varepsilon_i$. Let

$$
T=\begin{cases}
\inf\{1 \leqslant t \leqslant n : |A_t| > 1-a_{n-t+1}\} & \text{if } \{1 \leqslant t \leqslant n : |A_t| > 1-a_{n-t+1}\} \neq \emptyset, \\
n+1 & \text{otherwise}
\end{cases}
$$

Then T is a stopping time. Note that if $T \leqslant n$, then $|A_T| \leqslant 1$. Also define the random process $(B_t)_{t=0}^n$ by setting $B_t = A_t$ for $t \leqslant T$ and $B_t = 2A_T - A_t$ for $n \geqslant t > T$. Note that B_n has the same distribution as $A_n = \sum_{i=1}^n a_i \varepsilon_i$.

Claim 6.5.7. *If $|A_n| > 1$ and $|B_n| > 1$, then $|A_n| > 2.5$ or $|B_n| > 2.5$.*

Proof. Assume $|A_n|, |B_n| > 1$. Clearly $T \leqslant n-1$ as otherwise if $T \in \{n, n+1\}$, then by construction we have $|A_n|, |B_n| \leqslant 1$. Now for $T \leqslant n-1$, note that we have $|A_T| \leqslant 1$ and hence

$$|A_n| + |B_n| > 2 \geqslant 2|A_T| = |A_n + B_n|$$

It follows that A_n and B_n must have opposite signs.

We argue $T < n-k+1$. Indeed, assume for the sake of contradiction that $n-k+1 \leqslant T < n$, and furthermore assume that $A_T > 1 - a_{n-T+1}$. As $n-T+1 \leqslant k$, by Lemma 6.5.6 we have that

$$A_n, B_n \geqslant A_T - (a_2 + \ldots + a_{n-T+1}) > 1 - (a_2 + \ldots + 2a_{n-T+1}) \geqslant -1.$$

This gives the desired contradiction as A_n and B_n have modulus strictly greater than 1 and opposite signs.

For $T < n-k+1$, we get that a_{n-T+1} is a small term, so $|A_T| > 0.75$. As A_n, B_n have opposite signs we have $|A_n + B_n| = ||A_n| - |B_n||$. Therefore, putting all together we have that

$$\begin{aligned}
1.5 &\leqslant 2|A_T| \\
&= |A_n + B_n| \\
&= ||A_n| - |B_n|| \\
&= \max(|A_n|, |B_n|) - \min(|A_n|, |B_n|) \\
&< \max(|A_n|, |B_n|) - 1.
\end{aligned}$$

This concludes the claim. □

Similarly to the proof of Proposition 6.3.1, we now have

$$\begin{aligned}
\mathbb{P}\big[|A_n| > 1 \text{ and } |B_n| > 1\big] &\leqslant \mathbb{P}\big[|A_n| > 2.5 \text{ or } |B_n| > 2.5\big] \\
&\leqslant 2\,\mathbb{P}\big[|A_n| > 2.5\big] \\
&\leqslant 6.36\,\mathbb{P}\big[|N(0,1)| > 2.5\big] < 0.08,
\end{aligned}$$

where the second inequality follows from the union bound and from the fact that A_n, B_n have the same distribution and the third inequality follows from Lemma 6.2.1.

We conclude that, since $\mathbb{P}[|X| \leqslant 1] = \mathbb{P}\big[|A_n| \leqslant 1\big] = \mathbb{P}\big[|B_n| \leqslant 1\big]$, we obtain

$$\begin{aligned}\mathbb{P}[|X| \leqslant 1] &= \frac{1}{2}\mathbb{P}\big[|A_n| \leqslant 1\big] + \frac{1}{2}\mathbb{P}\big[|B_n| \leqslant 1\big] \\ &\geqslant \frac{1}{2}\big(1 - \mathbb{P}\big[|A_n| > 1 \text{ and } |B_n| > 1\big]\big) \\ &\geqslant \frac{1}{2}(1 - 0.08) \\ &= 0.46.\end{aligned}$$

This finishes the proof of Proposition 6.5.1. □

6.6 Concluding remarks

There was an important recent development regarding the problem we study in this chapter. Several months after submission of our paper [31], Keller and Klein [50] presented a preprint proving Tomaszewski's conjecture. Note that their argument uses techniques very similar to the ones introduced in our paper (though independently discovered). On top of that, they also develop some other important tools. One of them is a clever argument proving the conjecture in the case when $a_1 + a_2 \geqslant 1$, provided one can prove it for all the cases with $a_1 + a_2 < 1$. Another tool are improved Berry-Esseen type inequalities for Rademacher sums, which were subsequently also used by the present author and Klein in the work presented in the next chapter about the anti-concentration problem for Rademacher sums. Finally, let us note that the paper of Keller and Klein is very long (76 pages in its current arxiv version) - hence the question whether there exists some short and elegant proof of the conjecture still remains of interest.

Chapter 7

Anti-concentration of Rademacher sums

This chapter is joint work with Ohad Klein. Many results of this chapter are from a currently submitted paper [28].

7.1 Introduction

7.1.1 Background

Tail inequalities characterize the possible values of $\mathbb{P}[X \geqslant t]$ for various thresholds t and random variables X with mean 0. Here, same as in the previous chapter about the concentration result, we consider the case of Rademacher sums $X = \sum_{i=1}^{n} a_i \varepsilon_i$ for real numbers a_i and independently and uniformly distributed signs $\varepsilon_i \sim \{-1, 1\}$. We further focus on lower bounds to $\mathbb{P}[X \geqslant t]$.

If $t > \sqrt{\operatorname{Var} X}$ we may have $\mathbb{P}[X \geqslant t] = 0$. If $t \leqslant 0$, clearly $\mathbb{P}[X \geqslant t] \geqslant \frac{1}{2}$ because of the symmetry, and if $0 < t < \sqrt{\operatorname{Var} X}$, the Paley-Zygmund inequality gives

$$\mathbb{P}[X \geqslant t] \geqslant \mathbb{P}[X > t] = \frac{1}{2}\mathbb{P}[X^2 > t^2] \geqslant \frac{1}{2}(1 - \frac{t^2}{\operatorname{Var} X})^2 \frac{(\operatorname{Var} X)^2}{\mathbb{E}[X^4]} > 0.$$

Once again, the most interesting question is, what happens when $t = \sqrt{\operatorname{Var} X}$? This case was studied in 1967 by Burkholder [12] with the conclusion that if $C_s = \inf_X \mathbb{P}[X \geqslant s\sqrt{\operatorname{Var} X}]$, where the infimum is taken over all Rademacher sums, then $C_1 > 0$. It was then improved by Hitczenko and Kwapień [47] to $C_1 \geqslant e^{-4}/8$, and then in 1996 by Oleszkiewicz [61] to $C_1 \geqslant 1/20$. Hitczenko and Kwapień [47] conjectured that $C_1 = 7/64$, having the tightness example $a_1 = \cdots = a_6 > 0$.

We point out that this problem is a natural counterpart to the Tomaszewski's problem [42], studied in the previous chapter.

7.1.2 Our results

The main result of this chapter is the following.

Theorem 7.1.1. *Any Rademacher sum $X = \sum_i a_i \varepsilon_i$ has*

$$\mathbb{P}[X \geqslant \sqrt{\operatorname{Var} X}] \geqslant 6/64.$$

This theorem improves on the previously best known bound by Oleszkiewicz [61], who derived an analogous result with the constant $\frac{1}{20} = 0.05$ instead of our constant $\frac{6}{64} = 0.09375$. We believe that our tools could be useful in order to prove the conjectured optimal bound of $\frac{7}{64}$. We make some progress toward this goal by handling certain difficult, near-extremal, classes of Rademacher sums. See further Section 7.1.4.

While already $\mathbb{P}[X > \sqrt{\operatorname{Var} X}]$ might be 0, as demonstrated by $X = 1 \cdot \varepsilon_1$, the aforementioned proof by Oleszkiewicz [61] in fact shows that $\mathbb{P}[X > \sqrt{\operatorname{Var} X}] \geqslant 1/20$ whenever X is not of the form $a_i \varepsilon_i$. This bound is quite tight due to the example $a_1 = \cdots = a_4 > 0$ having $\mathbb{P}[X > \sqrt{\operatorname{Var} X}] = 1/16$. We show that this is indeed the extremal case.

Theorem 7.1.2. *Any Rademacher sum $X = \sum_i a_i \varepsilon_i$ with $a_1, a_2 > 0$ has*

$$\mathbb{P}[X > \sqrt{\operatorname{Var} X}] \geqslant 1/16.$$

Another inequality in this vein was conjectured by Lowther [56] to be $C_{1/\sqrt{7}} = 1/4$, which is saturated by $a_1 = \cdots = a_7 > 0$. We prove the following slightly weaker result.

Theorem 7.1.3. *Any Rademacher sum $X = \sum_i a_i \varepsilon_i$ has*

$$\mathbb{P}[X > 0.35\sqrt{\operatorname{Var} X}] \geqslant 1/4.$$

In the paper of Ben-Tal, Nemirovski and Roos [6], the higher-dimensional analogue of the $C_1 = 7/64$ problem first appeared. In this setting, $X = \sum_i a_i \varepsilon_i$ with $a_i \in \mathbb{R}^d$ and we are concerned with the probability $P(X) := \mathbb{P}\left[\|X\|_2^2 \geqslant \mathbb{E}[\|X\|_2^2]\right]$. The best result in this framework is due to Veraar [69] who showed that $P(X) \geqslant (\sqrt{12} - 3)/15 \approx 0.031$. We remark that the following holds.

Theorem 7.1.4. *Any $X = \sum_i a_i \varepsilon_i$ with $a_i \in \mathbb{R}^d$ (for any $d \geqslant 1$) has*

$$\mathbb{P}\left[\|X\|_2^2 \geqslant \mathbb{E}[\|X\|_2^2]\right] \geqslant \frac{1 - \sqrt{1 - 1/e^2}}{2} > 0.035.$$

Interestingly, we are not aware of any example that would demonstrate that the constant in Theorem 7.1.4 could not be as large as $\frac{7}{32}$ (which is the best one could hope for, since the result does not hold for any constant larger than that even when we only consider the case $d = 1$, as commented previously).

7.1.3 Overview of techniques

A prevalent method for understanding the distribution of Rademacher sums is to partition their weights $\{a_i\}$ into two parts ($X = L + S$): large weights and small weights. Such partitioning is efficient, as the Rademacher sum having small weights is easy to analyze using quantitative versions of the Central Limit Theorem, while the Rademacher sum having large weights can be analyzed by enumeration over all the possibilities. In high level, this is the approach we take, but let us dive a little further into the details.

Consider a Rademacher sum X with $\mathrm{Var}X = 1$. The problem addressed in Theorem 7.1.1 concerns with bounding $\mathbb{P}[X \geqslant 1]$ below. It turns out to be instructive to generalize this problem in two different ways:

- Enable a more flexible threshold t, and not only $t = 1$.
- Impose a restriction on the weights: $|a_i| \leqslant a$ for a parameter $a \leqslant 1$.

Denote by $G(a,t)$ the answer to this more general problem: the infimum of $\mathbb{P}[X \geqslant t]$, assuming $|a_i| \leqslant a$ ($a \in (0,1]$, $t \in \mathbb{R}$). Ultimately, Theorem 7.1.1 is encapsulated in the statement $G(1,1) \geqslant 6/64$, but we study $G(a,t)$ for all parameters a,t at once.

The crucial point is that using the decomposition of our Rademacher sum to its large and small parts $X = L + S$, we can bound $G(a,t)$ below by

$$G(a,t) \geqslant \inf_{L} \mathop{\mathbb{E}}_{l \sim L}[G(a'/\sigma, (t-l)/\sigma)] \tag{7.1}$$

where the infimum is taken over all possible values of L induced by decompositions $X = L + S$ (for example, if we decompose $X = L + S$ with $L = a_1\varepsilon_1 + a_2\varepsilon_2$ whenever $a_1 + a_2 \geqslant 1$ and $L = a_1\varepsilon_1$ otherwise, the infimum is taken over all $L = a_1\varepsilon_1 + a_2\varepsilon_2$ with $a_1 + a_2 \geqslant 1$ and with $a' = \min(a_2, \sqrt{1 - a_1^2 - a_2^2})$ and $L = a_1\varepsilon_1$ with $a' = \min(a_1, 1 - a_1)$), the expectation is taken over l being a realization of the random variable L, σ is the standard deviation of S (that is, $\sqrt{1 - \mathrm{Var}L}$), and a' is an upper bound on the weights of S (whose value depends on the notion of how we decompose $X = L + S$).

Equation (7.1) enables one to recursively compute lower bounds on $G(a,t)$, and ultimately on $G(1,1)$. Roughly speaking, considering the decompositions $X = L + S$ with L containing at most the three largest weights of X, we almost deduce Theorem 7.1.1. However, using solely this method, we run into the following problem: In order to concretely define $G(a,t)$

through the recursive (7.1), we have to propose an initial lower estimate for $G(a,t)$. The initial estimate we use is 'continuous' in nature (the Berry-Esseen inequality), and is unable to differentiate between bounds on $\mathbb{P}[X \geqslant t]$ and on $\mathbb{P}[X > t]$. However, there are various instances X, detailed in Section 7.1.4, for which the stronger bound $\mathbb{P}[X > 1] \geqslant \frac{7}{64}$ (or even the bound $\mathbb{P}[X > 1] \geqslant \frac{6}{64}$, that we prove) does not hold! (e.g. the aforementioned $a_1 = \cdots = a_4 > 0$.)

To handle these more tight cases, we take a completely different approach toward bounding $\mathbb{P}[X \geqslant 1]$ below (i.e. Theorem 7.1.1). That is, we *upper bound* $\mathbb{P}[X \in (-1,1)]$ (recall that X is symmetric). To do that, we take the advantage of the following trade-off that usually arises. The collections $\{a_1,\dots,a_n\}$ that either contain large mass of their variance in the small weights, or have their large weights very non-uniform, are harder to describe precisely, but are nevertheless easy to analyze, since usually stronger bounds hold for these. And the collections $\{a_1,\dots,a_n\}$ that contain only very small mass of their variance in the small weights and have their large weights quite uniform are easier to describe precisely, so despite only more tight bounds being true for these, we can derive those bounds.

In various tight cases that arise, we commonly want to bound above $\mathbb{P}[X \in I]$ for some particular interval $I \subset \mathbb{R}$. To do that, we use a chain lemma, and a few related observations.

In the chain lemma, we assume X has some weights $a_1,\dots,a_l$ which are 'large' compared to the length of I and consider the signed sums $\pm a_1 \pm \ldots \pm a_l$ – ignoring the remaining 'small' weights. We then associate the set of these 2^l signed sums with a hypercube graph in a natural way and then use a famous result of Erdős [34] to show that these sums are not very tightly concentrated. That in turn implies an upper bound on $\mathbb{P}[X \in I]$.

Occasionally, we have to consider the case when I is a very short interval (much smaller than $(-1,1)$). In such a case we divide the small weights into disjoint parts (a method introduced by Montgomery-Smith [59]), so that each part has a substantial probability to be large compared to I, and apply the chain lemma on these 'large' parts to deduce that $\mathbb{P}[X \in I]$ is small enough.

7.1.4 Difficult cases

As described in the previous subsection, similarly to Tomaszewski's problem, the particular difficulty we are facing when trying to prove the conjecture $C_1 = 7/64$, are the cases when $\mathbb{P}[X > 1] < 7/64$ despite $\mathbb{P}[X \geqslant 1] \geqslant 7/64$ (and their 'neighborhoods', i.e. the collections with the few largest weights being roughly of the same sizes as in these cases). Notably, we have

- for $a_1 = 1$, $\mathbb{P}[X > 1] = 0$;

- for $a_1 = \ldots = a_4 = \frac{1}{2}$, $\mathbb{P}[X > 1] = \frac{1}{16}$;
- for $a_1 = \ldots = a_9 = \frac{1}{3}$, $\mathbb{P}[X > 1] = \frac{23}{256} \approx 0.0898\ldots < \frac{6}{64}$;
- for $a_1 = \frac{2}{3}, a_2 = \ldots = a_6 = \frac{1}{3}$, $\mathbb{P}[X > 1] = \frac{6}{64}$;
- for $a_1 = a_2 = \frac{1}{2}, a_3 = \ldots = a_{10} = \frac{1}{4}$, $\mathbb{P}[X > 1] = \frac{55}{512} < \frac{7}{64}$.

We have to deal with the first three cases even when proving our bound of $6/64$, and the last two cases are further hurdles on the way to the optimal bound.

In our proof of the $6/64$ bound, the big part of the argument is spent dealing with a subcase presented in Section 7.4.1, which corresponds to the collections 'close to' the third case from above (which is the most intricate of the first three 'barriers').

In Section 7.6, we discuss these difficulties in more detail and make progress toward proving the $7/64$ bound, by proving it for families corresponding to the 'neighbourhoods' of all the cases above except the third one.

7.1.5 Organization

In Section 7.2, we introduce the setting and the notation. In Section 7.3, we describe our main tools and prove Theorem 7.1.3. We then use these tools in Section 7.4 to prove Theorem 7.1.1, the main result of the chapter. Section 7.5 contains the proof of Theorem 7.1.2. In Section 7.6, we discuss the deficiency of our $6/64$ proof and propose how to advance toward $7/64$, proving the result in two out of three 'difficult' cases. In Section 7.7, we discuss the high dimensional version of the problem as well as of the problem of Tomaszewski and prove Theorem 7.1.4. Finally in Section 7.8, we summarize the open problems arising in the chapter.

Some of the more technical proofs from various parts of the chapter are in Appendix 7.A and Appendix 7.B.

7.2 Background and definitions

In this section, we describe our setting, notation and assumptions that we are working with.

Throughout, we will consider $X = \sum_{i=1}^{n} a_i \varepsilon_i$, where ε_i are independent Rademacher random variables (i.e. independent random variables such that $\mathbb{P}\left[\varepsilon_i = +1\right] = \mathbb{P}\left[\varepsilon_i = -1\right] = \frac{1}{2}$) and a_i are real numbers with $\sum_{i=1}^{n} a_i^2 = 1$. Moreover, we will always, without loss of generality, assume that

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0.$$

Sometimes, we will work with variables $\{b_i\}$ or $\{c_i\}$ instead of $\{a_i\}$. For these, we do not assume any conditions on their ordering unless so stated.

At some points, we will also write $\mathbf{a}$ to denote $\{a_1, \ldots, a_n\}$.

Our central aim will be to bound below

$$\mathbb{P}\big[X \geqslant 1\big] = \frac{1}{2}\mathbb{P}\big[|X| \geqslant 1\big]. \tag{7.2}$$

At some points, we will work with $\mathbb{P}\big[X \geqslant 1\big]$, while at other points, we will work with $\mathbb{P}\big[|X| \geqslant 1\big]$. As expressed by (7.2), working with these two forms is of course equivalent and the entire proof could be rewritten using just one of these. We use both quantities in order to streamline the proof.

The function $D(a,x) : (0,1] \times \mathbb{R} \to \mathbb{R}$ appears repeatedly throughout the proof. This is a particular function that we construct in subsection 7.3.3 and it has a property that for any $a \in (0,1], x \in \mathbb{R}$, if we have $a_1 \leqslant a$, then $\mathbb{P}\big[X \geqslant x\big] \geqslant D(a,x)$. While its computation is computer-aided, we emphasize that by writing 'D', we always refer to its exact value, and not to its approximation.

7.3 Tools

7.3.1 Stopped random walks and a chain argument

We start with an observation (following trivially from a well known result of Erdős [34]) which we will use repeatedly.

Observation 7.3.1. *Let $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_t > 0$ be such that $b_{t-k+1} + \ldots + b_t \geqslant \alpha$ for some $\alpha > 0$ and $0 < k \leqslant t$. Then, for any x and any $b_{t+1}, \ldots, b_s$, we have*

$$\mathbb{P}\Big[\sum_{i=1}^{s} b_i\varepsilon_i \in (x-\alpha, x+\alpha)\Big] \leqslant f(k,t)/2^t$$

where $f(k,t)$ denotes the sum of k largest binomial coefficients of the form $\binom{t}{i}$ for some i, $0 \leqslant i \leqslant t$.

Proof. If the probability was more than $f(k,t)/2^t$ for some fixed x, then in particular we can choose signs $\varepsilon_{t+1} = \varepsilon'_{t+1}, \ldots, \varepsilon_s = \varepsilon'_s$ in such a way that at least $f(k,t)+1$ of the sums

$$\pm b_1 \pm \ldots \pm b_t + b_{t+1}\varepsilon'_{t+1} + \ldots + b_s\varepsilon'_s$$

are all within less than 2α of each other. Let

$$T = \{\pm b_1 \pm \ldots \pm b_t + b_{t+1}\varepsilon'_{t+1} + \ldots + b_s\varepsilon'_s\}.$$

Consider the bijection $g : T \to Q_t \simeq \{\pm 1\}^t$ given by

$$b_1\varepsilon_1 + \ldots + b_t\varepsilon_t + b_{t+1}\varepsilon'_{t+1} + \ldots + b_s\varepsilon'_s \to (\varepsilon_1, \ldots, \varepsilon_t).$$

Let $S \subset T$ be the set of $f(k,t)+1$ elements of T that are all within 2α of each other. Then by the result of Erdős [34, Theorem 5], $g(S)$ contains a chain of length at least k. But that contradicts the assumption that $b_{t-k+1} + \ldots + b_t \geqslant \alpha$. □

Occasionally, we will only check the stronger condition that (in the cases $k = 2, 3$) no two out of the sums $x_0 \pm b_1 \pm \ldots \pm b_k$ are within less than 2δ of each other, which in particular implies no two hit any interval of the form $(x - \delta, x + \delta)$. For the special cases we need, we will use the following two straightforward observations to verify that.

Observation 7.3.2. *Fix $\delta > 0$ and $b_1, b_2 \geqslant \delta$ such that $|b_1 - b_2| \geqslant \delta$. Then for any x and any $b_3, \ldots, b_l$, we have*

$$\mathbb{P}\Big[\sum_{i=1}^{l} b_i\varepsilon_i \in (x - \delta, x + \delta)\Big] \leqslant \frac{1}{4}.$$

Proof. If the probability was more than $\frac{1}{4}$ for some fixed x, then in particular we can choose signs $\varepsilon_3 = \varepsilon'_3, \ldots, \varepsilon_l = \varepsilon'_l$ in such a way that at least two of the four sums

$$\pm b_1 \pm b_2 + b_3\varepsilon'_3 + \ldots + b_l\varepsilon'_l$$

are within less than 2δ of each other. Looking at differences of this set, it can only happen if the set

$$D = \{b_1 + b_2,\, b_1,\, b_2,\, |b_1 - b_2|\}$$

contains some element smaller than δ, and our assumptions guarantee that can not happen. □

Observation 7.3.3. *Fix $\delta > 0$ and $c_1 \geqslant c_2 \geqslant c_3 \geqslant \delta$ such that $c_1 - c_2, c_2 - c_3 \geqslant \delta$, $|c_1 - c_2 - c_3| \geqslant \delta$. Then for any x and any $c_4, \ldots, c_m$, we have*

$$\mathbb{P}\Big[\sum_{i=1}^{m} c_i\varepsilon_i \in (x - \delta, x + \delta)\Big] \leqslant \frac{1}{8}.$$

Proof. If the probability was more than $\frac{1}{8}$ for some fixed x, then in particular we can choose signs $\varepsilon_4 = \varepsilon_4', \ldots, \varepsilon_m = \varepsilon_m'$ in such a way that at least two of the eight sums

$$\pm c_1 \pm c_2 \pm c_3 + c_4\varepsilon_4' + \ldots + c_m\varepsilon_m'$$

are within less than 2δ of each other. Looking at differences of this set, it can only happen if the set

$$D = \{c_1, c_2, c_3, c_1 \pm c_2, c_1 \pm c_3, c_2 \pm c_3, c_1 + c_2 \pm c_3, c_1 - c_2 + c_3, |c_1 - c_2 - c_3|\}$$

contains some element smaller than δ; our assumptions guarantee it is impossible. □

In the easy cases, we are already given enough large weights as a part of our collection $\{a_i\}$ and can use these weights in the anti-concentration observations above. But if that is not true and we instead have a lot of very small weights, we can 'generate' larger weights from them, as described in the subsection that follows.

7.3.2 The random process $W(S;x)$ and its success probability

For a set of real numbers $S = \{d_1, \ldots, d_n\}$ and a real number $x > 0$, we denote by $W(S;x)$ (or by $W(d_1, \ldots, d_n;x)$) the following random process. We first fix a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ which maximizes the probability that the process is successful (what it means for this process to be successful will be defined in due course). Next, we set $W_0 = 0$. After choosing W_j for some $j < n$, if $|W_j| \geqslant x$, we set

$$W_{j+1} = \ldots = W_n = W_j.$$

While if $|W_j| < x$, we let $\varepsilon_{i_{j+1}}$ be a Rademacher random variable independent of the previous part of the process, and set

$$W_{j+1} = W_j + d_{i_{j+1}}\varepsilon_{i_{j+1}}.$$

We denote by $r(S;x)$ (or by $r(d_1, \ldots, d_n;x)$) the final value of this process, i.e. W_n. We call it *successful* if $|r(S;x)| \geqslant x$, and *unsuccessful* otherwise.

We denote by $p(S;x)$ (or by $p(d_1, \ldots, d_n;x)$) the probability that the process is successful. In particular, if we have $|d_i| \geqslant x$ for any $i \in \{1, \ldots, n\}$, clearly the corresponding process will always be successful because of our condition on ordering.

The following lemma is crucial for us when working with such random processes.

Lemma 7.3.4. *Assume we have positive reals $b_1,\dots,b_k$ such that $\sum_{i=1}^k b_i^2 \geqslant c\alpha^2$ for some fixed $c>1$ and fixed $\alpha>0$. Then*

$$p(b_1,\dots,b_k;\alpha) \geqslant \frac{c-1}{c+3}.$$

Moreover, if for some $\eta \in (0,1)$, we have $b_1,\dots,b_k \in (0,\eta\alpha]\cup[\alpha,\infty)$, then

$$p(b_1,\dots,b_k;\alpha) \geqslant \frac{c-1}{c+\eta^2+2\eta}.$$

Proof. If any term out of $b_1,\dots,b_k$ has size at least α, then clearly $p(b_1,\dots,b_k;\alpha)=1$. So further assume none of the terms has size at least α.

Run the random process $W(b_1,\dots,b_k;\alpha)$. Without loss of generality (and for notational convenience), we can assume that the ordering $b_1,\dots,b_k$ maximizes the probability that the process is successful. We define the stopping time T as follows. Let T be the first time i such that $|W_i|\geqslant\alpha$ if this time is at most k, and let $T=k$ otherwise. Let $p=p(b_1,\dots,b_k;\alpha)$ be the probability that the process $W(b_1,..,b_k;\alpha)$ is successful, i.e. that it hits absolute value at least α.

Now we will bound $\mathbb{E}\big[W_T^2\big]$ above and below.

Clearly $|W_T|\leqslant 2\alpha$ (as every term has size at most α and T is the first time we reach absolute value at least α), and $|W_T|\leqslant\alpha$ in the case when we never hit absolute value at least α. This gives

$$\mathbb{E}\big[W_T^2\big] \leqslant 4p\alpha^2+(1-p)\alpha^2. \tag{7.3}$$

But also, writing $A=b_1\varepsilon_1+\ldots+b_T\varepsilon_T$ and $B=b_{T+1}\varepsilon_{T+1}+\ldots+b_k\varepsilon_k$ (setting $B=0$ if $T=k$), we collect the following easy observations. Firstly

$$\mathbb{E}\big[AB\big] = \sum_{T_0,x}\mathbb{P}\big[T=T_0,A=x\big]\mathbb{E}\big[AB|T=T_0,A=x\big]=0, \tag{7.4}$$

since for any T_0,x, we have

$$\mathbb{E}\big[AB|T=T_0,A=x\big]=x\mathbb{E}\big[b_{T_0+1}\varepsilon_{T_0+1}+\ldots+b_k\varepsilon_k\big]=0.$$

Furthermore, noting that if $T=k$, then $B=0$, we obtain

$$\mathbb{E}\big[B^2\big]=\sum_{i=1}^{k-1}\mathbb{P}\big[T=i\big]\mathbb{E}\big[B^2|T=i\big]=\sum_{i=1}^{k-1}\mathbb{P}\big[T=i\big]\big(\sum_{j=i+1}^{k}b_j^2\big)\leqslant p\sum_{i=1}^{k}b_i^2. \tag{7.5}$$

Using (7.4) we conclude

$$\sum_{i=1}^{k} b_i^2 = \mathbb{E}\big[(A+B)^2\big] = \mathbb{E}\big[A^2\big] + \mathbb{E}\big[B^2\big] + 2\mathbb{E}\big[AB\big] = \mathbb{E}\big[A^2\big] + \mathbb{E}\big[B^2\big]. \tag{7.6}$$

Overall, combining (7.5) and (7.6) we conclude

$$\mathbb{E}\big[A^2\big] \geqslant (1-p)\sum_{i=1}^{k} b_i^2 \geqslant (1-p)c\alpha^2. \tag{7.7}$$

Combining (7.3) and (7.7), we obtain

$$(1-p)c\alpha^2 \leqslant \mathbb{E}\big[W_T^2\big] \leqslant 4p\alpha^2 + (1-p)\alpha^2.$$

Rearranging gives the first result.

For the second result, just note that with our additional condition $b_1,\dots,b_k \in (0,\eta\alpha]$, we can replace the inequality

$$\mathbb{E}\big[W_T^2\big] \leqslant 4p\alpha^2 + (1-p)\alpha^2$$

by the stronger inequality

$$\mathbb{E}\big[W_T^2\big] \leqslant p(1+\eta)^2\alpha^2 + (1-p)\alpha^2,$$

and conclude in exactly the same way as before. □

7.3.3 Dynamic Programming bound

Denote by $\widetilde{G}(a_1,x)$ the quantity $\inf_X \mathbb{P}[X > x]$ where the infimum is taken over all Rademacher sums X with $\operatorname{Var} X = 1$, and whose largest weight is at most a_1.

For the proof, it is useful to understand the function $\widetilde{G}$. Evaluating the function $\widetilde{G}(a_1,x)$ is in general harder than the problem we are concerned with in Theorem 7.1.1; the latter is nonrigorously encapsulated in $\widetilde{G}(1,1-\varepsilon)$.

The goal of the dynamic-programming approach is to derive a lower bound on $\widetilde{G}$ by first obtaining some lower bound on $\widetilde{G}(a_1,x)$ for many values of a_1,x, and then using an iterative procedure to improve this bound further. The key tool enabling us to iterate is elimination of the largest weight (see Section 7.3.4 for more details about elimination).

Prawitz's smoothing inequality

We will use a smoothing inequality of Prawitz [65]. This inequality is a useful tool, providing bounds on the values of the cumulative distribution function of a random variable, in terms of a partial information regarding its characteristic function. Specifically, given the characteristic function of a random variable, it is possible to determine its distribution via the Gil-Pelaez formula. In the case of a Rademacher sum $X = \sum_i a_i \varepsilon_i$, we have the characteristic function $\varphi_X(t) = \prod_i \cos(a_i t)$. Assuming that we know the largest weight a_1, it is possible to estimate the value of $\varphi_X(t)$ for $t \ll 1/a_1$. Although for $t \gg 1/a_1$, we have no information regarding $\varphi_X(t)$, Prawitz' inequality is still capable of providing a decent estimate for the cumulative distribution function of X.

While the inequality is applicable to all random variables, it was shown in [50] that its specialization to Rademacher sums gives tighter estimates.

Prawitz' bound gives a lower bound on $\widetilde{G}(a_1,x)$, for all parameters $q \in [0,1],\ T > 0$:

$$\forall q \in [0,1], T > 0: \qquad \widetilde{G}(a_1,x) \geqslant F(a_1,x,T,q). \tag{7.8}$$

Specifically, a formula for F may be derived from [50, Proposition 4.2] (which is derived from [65]):

$$\begin{aligned} F(a,x,T,q) = 1/2 - \int_0^q |k(u,x,T)|\, g(Tu,a)\mathrm{d}u - \int_q^1 |k(u,x,T)|\, h(Tu,a)\mathrm{d}u \\ - \int_0^q k(u,x,T)\exp(-(Tu)^2/2)\mathrm{d}u, \end{aligned} \tag{7.9}$$

where $k(u,x,T) = \frac{(1-u)\sin(\pi u + Tux)}{\sin(\pi u)} + \frac{\sin(Tux)}{\pi}$ (note that k can be smoothly continued to the range $u \in [0,1]$ by setting $k(0,x,T) = 1 + Tx/\pi$ and $k(1,x,T) = 0$),

$$g(v,a) = \begin{cases} \exp(-v^2/2) - \cos(av)^{1/a^2}, & av \leqslant \frac{\pi}{2} \\ \exp(-v^2/2) + 1, & \text{otherwise} \end{cases}, \quad h(v,a) = \begin{cases} \exp(-v^2/2), & av \leqslant \theta \\ (-\cos(av))^{1/a^2}, & \theta \leqslant av \leqslant \pi \\ 1, & \text{otherwise} \end{cases},$$

$Z \sim N(0,1)$ is a standard Gaussian and $\theta = 1.778 \pm 10^{-4}$ is the unique solution of $\exp(-\theta^2/2) = -\cos(\theta)$ in the interval $[0,\pi]$. We note that $F(a,x,T,q)$ is a function (weakly) decreasing in a.

Recursion

Note that as in (7.1), by considering the two values that the sign of the largest weight can take (see subsection 7.3.4 for more details), we have

$$\widetilde{G}(a_1,x) \geqslant \frac{1}{2}\inf_{a\in(0,a_1]}\left(\widetilde{G}\left(\frac{a}{\sqrt{1-a^2}},\frac{x-a}{\sqrt{1-a^2}}\right)+\widetilde{G}\left(\frac{a}{\sqrt{1-a^2}},\frac{x+a}{\sqrt{1-a^2}}\right)\right). \tag{7.10}$$

Hence, $\widetilde{G}$ is bounded below by the lowest function satisfying both inequalities (7.8), (7.10). Computationally, to obtain a concrete lower bound on $\widetilde{G}$, we iteratively define the functions

$$D_i\colon (0,1)\times\mathbb{R}\to\mathbb{R}$$

by $D_0(a_1,x) = \max(F(a_1,x),\ \mathbb{1}\{x<0\}/2)$ with $F(a_1,x) = \sup_{T,q}\{F(a_1,x,T,q)\}$ and

$$D_{i+1}(a_1,x) = \max\left(D_i(a_1,x), \frac{1}{2}\inf_{a\in(0,a_1]}\left(D_i\left(\frac{a}{\sqrt{1-a^2}},\frac{x-a}{\sqrt{1-a^2}}\right)+D_i\left(\frac{a}{\sqrt{1-a^2}},\frac{x+a}{\sqrt{1-a^2}}\right)\right)\right), \tag{7.11}$$

and observe that $\widetilde{G}(a_1,x)\geqslant D_i(a_1,x)$ for all i. Choosing a large I ($I=10$ suffices) and writing

$$D(a_1,x) = D_I(a_1,x)$$

we derive

$$\forall X\in\mathscr{X}\colon \left(X=\sum b_i\varepsilon_i \wedge |b_i|\leqslant a_1\right) \implies \mathbb{P}[X>x]\geqslant D(a_1,x). \tag{7.12}$$

Note that D is a function depending on two continuous variables, which cannot be stored programmatically. We compute $D_i(a_1,x)$ for $a_1\in[0,1]$ and $x\in[-3,3]$ with granularity of $\delta = 1/400$ (a_1 starting from 0 and x starting from -3). Correspondingly, we replace (7.11) with a variant that feeds D_{i+1} with arguments rounded up (to a multiple of δ), hence underestimating D_{i+1}; This enables considering a finite set of $a\in[0,a_1]$ in the infimum at (7.11). We apply this rounding-up to both the $\frac{a}{\sqrt{1-a^2}}$ and the $\frac{x\pm a}{\sqrt{1-a^2}}$ arguments. Moreover, in any computation of $D(a_1,x)$ we round the arguments up to multiples of δ. When $x<-3$ we round x to -3, and when $x\geqslant 3$ we round x to ∞ and set $D(a_1,\infty)=0$. This results in a dynamic-programming method for computing $D_i(a_1,x)$.

Our implementation of this computation can be found at [32].

Several concrete values. Along the chapter, we use the following lower bounds for values of D, derived by the described computation.

$$\begin{aligned} &D(0.35,0.35) > \frac{1}{4}, && D(0.3,1) > \frac{3}{32}, \\ &D(0.3/\sqrt{0.51},0.3/\sqrt{0.51}) > \frac{3}{16}, && D(0.4,1) > \frac{1}{12}, \\ &D(0.5,0.5) > \frac{1}{6}, && D(0.34,1.42) > 0.04, \\ &D(0.43,1.42) > 0.03, && D(0.51,1.01) = \frac{1}{16}. \end{aligned} \tag{7.13}$$

Note that $D(0.51,1.01) = 1/16$ is a precise value (unlike the other values mentioned for which we just have lower bounds). On the one hand, we clearly see that $D(0.51,1.01) \leqslant 1/16$, as saturated by the weights $a_1 = \ldots = a_4 = 1/2$. On the other hand, to derive $D(0.51,1.01) \geqslant 1/16$, it is crucial that we set $D_0(a_1,x) = \max(F(a_1,x), \mathbb{1}\{x<0\}/2)$ instead of just using $F(a_1,x)$. Our iterative procedure and the lower bounds on $F(a_1,x)$ are then enough to prove $D(0.51,1.01) \geqslant 1/16$.

Precision. As described, the lower bound $D(a,x)$ we numerically get for $\widetilde{G}(a,x)$ is precise. The only detail disregarded so far is the computation of $F(a_1,x)$. Programmatically we replace $F(a_1,x)$ by $F(a_1,x,\pi/a_1,0.5)$, that is, we do not compute the maximum of $F(a_1,x,T,q)$ over all values of T,q, but set $T=\pi/a_1$ and $q=0.5$. Since we use $F(a_1,x)$ as a lower bound, this underestimation of $F(a_1,x)$ is valid. We further note that this choice of T,q simplifies the first integrand in $F(a_1,x,T,q)$ to be continuous (specifically, $g(v,a)$ is applied only when $av \leqslant \pi/2$). Finally, to numerically estimate the integrals appearing in the definition of $F(a,x,T,q)$ we take two approaches.

In the first approach we compute the integrals appearing in (7.9) verbatim by using the standard Python integrator `scipy.integrate.quad`, and check that the integrator estimates that its error is well below some constant (0.01) that we discount from $F(a,x,T,q)$. We also split the domains of integration so that the integrands are smooth in each subdomain. This evaluation of F is simple, but requires relying on the accuracy of `scipy.integrate.quad`.

In the second approach we compute the integrals with the trapezoid rule, using explicit bounds B on the derivatives of the integrands (more accurately, we use that these are B-Lipschitz functions), to get an explicit estimation of the integrals, together with a provable error estimates. The bounds B are computed in [50, Appendix B.2].

While the first approach is neat and simple, the second approach is transparent and reviewable. The accompanied code is available at [32].

7.3.4 Elimination

Elimination is the process of replacing a probabilistic inequality in $X = \sum_{i=1}^{n} a_i\varepsilon_i$, by an inequality involving $Z = \sum_{i=m}^{n} a_i\varepsilon_i$ with $m > 1$. For example, the inequality

$$\mathbb{P}[X \geqslant 1] \geqslant 3/32$$

is equivalent to the following inequality, which involves $Z = \sum_{i=2}^{n} a_i\varepsilon_i$ (i.e. $m = 2$),

$$\mathbb{P}[Z \geqslant 1 - a_1] + \mathbb{P}[Z \geqslant 1 + a_1] \geqslant 3/16.$$

via the law of total probability. A more elaborate derivation can be found at [50, Lemma 2.1].

7.3.5 A $1/\sqrt{7}$-type inequality

Lowther [56] conjectured that $\mathbb{P}[|X| \geqslant 1/\sqrt{7}] \geqslant 1/2$ is true for all Rademacher sums X with $\operatorname{Var} X = 1$. In the proof of Theorem 7.1.1 we make use of Theorem 7.1.3, i.e. $\mathbb{P}[|X| > 0.35] \geqslant 1/2$, which we henceforth prove.

We split into two cases. If $a_1 > 0.35$, and $\varepsilon' = (-\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, then at least one of $X(\varepsilon)$ and $X(\varepsilon')$ has absolute value more than a_1, hence $\mathbb{P}[|X| > 0.35] \geqslant 1/2$. If $a_1 \leqslant 0.35$, then we conclude using (7.13) since

$$D(0.35, 0.35) > 1/4.$$

7.4 Proof of $\mathbb{P}[X \geqslant 1] \geqslant 3/32$

In this section we show that for any Rademacher sum X with $\operatorname{Var} X = 1$,

$$\mathbb{P}[X \geqslant 1] \geqslant 3/32, \tag{7.14}$$

that is, Theorem 7.1.1. The proof splits into two main cases - the case when $a_1 + a_2 + a_3 \leqslant 1$ and the case when $a_1 + a_2 + a_3 > 1$.

In the case $a_1 + a_2 + a_3 \leqslant 1$, the tools we have developed in subsections 7.3.3 and 7.3.4 enable us to handle most of the subcases. Nevertheless, as discussed before, one can not hope for these tools to work in the subcase $(a_1, a_2, a_3) \approx (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $a_1 + a_2 + a_3 \leqslant 1$. Thus, we spend majority of this subsection dealing with the subcase $a_3 \geqslant 0.325$ and $a_1 + a_2 + a_3 \leqslant 1$. To do that, we use the tools developed in subsection 7.3.1. Our strategy is to show that the family of such collections $\{a_i\}$ with $a_3 \geqslant 0.325$ and $a_1 + a_2 + a_3 \leqslant 1$ is contained in the union of several subfamilies, for each of which we can obtain the desired bound.

In the case $a_1 + a_2 + a_3 > 1$, the proof is less lengthy. We divide it into several subcases and use the tools from subsections 7.3.3 and 7.3.4 and crucially also Theorem 7.1.3, to resolve these cases.

7.4.1 Case $a_1 + a_2 + a_3 \leqslant 1$

Subcase $a_1 \leqslant 0.3$

Using (7.13) we have

$$D(0.3, 1) > 3/32$$

implying the assertion (7.14) through (7.12).

Subcase $a_1 \geqslant 0.7$

Using elimination, in order to deduce (7.14) regarding $X = \sum_{i=1}^n a_i\varepsilon_i$ it suffices to check

$$\mathbb{P}\left[X' \geqslant \frac{1-a_1}{\sqrt{1-a_1^2}}\right] \geqslant 3/16 \tag{7.15}$$

with $X' = \frac{1}{\sqrt{1-a_1^2}} \sum_{i=2}^n a_i\varepsilon_i$ the a_1-eliminated version of X. Using (7.13) we deduce (7.15) from

$$D(0.3/\sqrt{0.51}, 0.3/\sqrt{0.51}) > 3/16,$$

since $a_1 + a_2 \leqslant 1$. This argument does not rely on $a_1 + a_2 + a_3 \leqslant 1$, but only assumes $a_1 + a_2 \leqslant 1$ (and $a_1 \geqslant 0.7$). This is used in subsection 7.4.2.

Subcase $a_3 \leqslant 0.325$ **and** $a_1 \in [0.3, 0.7]$

Under the conditions $a_1 \geqslant 0.3$ and $a_3 \leqslant 0.325$ (and $a_2 \in [a_3, a_1]$), denote $\sigma_2 = \sqrt{1 - a_1^2 - a_2^2}$, and note that $a = \min(1 - a_1 - a_2, a_2, 0.325)$ is an upper bound on a_3. We show in Appendix 7.A.1 that

$$\mathop{\mathbb{E}}_{\varepsilon \in \{-1,1\}^2}\left[D\left(\frac{a}{\sigma_2}, \frac{1 + a_1\varepsilon_1 + a_2\varepsilon_2}{\sigma_2}\right)\right] \geqslant 3/32, \tag{7.16}$$

verifying (7.14) in this case, via elimination of a_1, a_2.

Subcase $a_3 \geqslant 0.325$

Let $Y = \sum_{i=4}^{n} a_i \varepsilon_i$ and denote:

$$\begin{aligned} q_1 &= \mathbb{P}\big[|Y| \geqslant 1 - a_1 - a_2 - a_3\big], & q_2 &= \mathbb{P}\big[|Y| \geqslant 1 - a_1 - a_2 + a_3\big], \\ q_3 &= \mathbb{P}\big[|Y| \geqslant 1 - a_1 + a_2 - a_3\big], & q_4 &= \mathbb{P}\big[|Y| \geqslant 1 + a_1 - a_2 - a_3\big], \\ q_5 &= \mathbb{P}\big[|Y| \geqslant 1 - a_1 + a_2 + a_3\big], & q_6 &= \mathbb{P}\big[|Y| \geqslant 1 + a_1 - a_2 + a_3\big], \\ q_7 &= \mathbb{P}\big[|Y| \geqslant 1 + a_1 + a_2 - a_3\big], & q_8 &= \mathbb{P}\big[|Y| \geqslant 1 + a_1 + a_2 + a_3\big]. \end{aligned}$$

Then using elimination, we have $\mathbb{P}[X \geqslant 1] = \frac{1}{16}(q_1 + \ldots + q_8)$. Hence we are required to show

$$q_1 + \ldots + q_8 \geqslant 3/2. \tag{7.17}$$

The key lemma which lets us handle this case, is the following.

Lemma 7.4.1. *Let $\mathscr{A}$ be the family of the collections $\mathbf{a} = (a_1, \ldots, a_n)$ with $n \geqslant 4$, $a_1 \geqslant \ldots \geqslant a_n > 0$, $\sum_{i=1}^{n} a_i^2 = 1$, $a_1 + a_2 + a_3 \leqslant 1$ and $a_3 \geqslant 0.325$. Then $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$, where $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are the subsets of $\mathscr{A}$ characterized by the following additional conditions:*

- *$\mathscr{A}_1$: $a_4 \leqslant 7/40$,*
- *$\mathscr{A}_2$: $q_1 \geqslant \frac{793}{1024}$,*
- *$\mathscr{A}_3$: $q_2, q_3 \geqslant \frac{37}{128}$.*

Proof. Firstly, if we had $a_1 + a_2 + a_3 = 1$, then clearly $q_1 = 1$. So further consider only the case $a_1 + a_2 + a_3 < 1$. Write $a_3 = \frac{1}{3} - \delta$, and assume that $a_1 + a_2 + a_3 < 1$ and $0 < \delta \leqslant \frac{1}{120}$ (which is equivalent to $a_3 \geqslant 0.325$). Note that

$$1 - a_1 - a_2 - a_3 \leqslant 1 - 3a_3 = 3\delta, \tag{7.18}$$

and that

$$1 - a_1 - a_2 + a_3, 1 - a_1 + a_2 - a_3 \leqslant 1 - a_3 = \frac{2}{3} + \delta. \tag{7.19}$$

If $a_4 \leqslant 21\delta \leqslant 7/40$, we have $\mathbf{a} \in \mathscr{A}_1$. So further assume that $a_4 \geqslant 21\delta$, in which case we have to show that $\mathbf{a} \in \mathscr{A}_2 \cup \mathscr{A}_3$.

Let k be the smallest integer such that $a_k < 1 - a_1 - a_2 - a_3$ (if $a_n \geqslant 1 - a_1 - a_2 - a_3$, set $k = n + 1$). Note that $k \geqslant 5$, since

$$a_4 \geqslant 21\delta > 3\delta \geqslant 1 - a_1 - a_2 - a_3,$$

where the last inequality follows by (7.18).

Claim 7.4.2. *If* $\sum_{i=k}^n a_i^2 \geqslant 450\delta^2$*, then* $\mathbf{a} \in \mathscr{A}_2$.

Proof of Claim 7.4.2. Note that $a_k, \dots, a_n < 3\delta$. We can find disjoint subsets $S, T_1, \dots, T_4$ of $\{a_k, \dots, a_n\}$ with the following properties. We have

$$234\delta^2 \geqslant \sum_{i \in S} a_i^2 \geqslant 225\delta^2$$

and for $j = 1, \dots, 4$, we have

$$54\delta^2 \geqslant \sum_{i \in T_j} a_i^2 \geqslant 45\delta^2.$$

Now consider the corresponding random processes $W(S;9\delta)$ and $W(T_j;3\delta)$ for $j = 1, \dots, 4$.

We consider three events partitioning our probability space. The first event is the event C_1 that $W(S;9\delta)$ is successful and also at least one out of $W(T_j;3\delta)$ for $j = 1, \dots, 4$ is successful. The second event is the event $C_2 = C_2' \cap C_1^C$, where C_2' is the event that at least one out of

$$W(S;9\delta),\ W(T_1;3\delta), \dots,\ W(T_4;3\delta)$$

is successful. And the last event is $C_3 = C_1^C \cap C_2^C$.

By independence of the processes $W(S;9\delta), W(T_1;3\delta), \dots, W(T_4;3\delta)$ and Lemma 7.3.4, we have

$$\mathbb{P}\big[C_1\big] \geqslant \frac{15}{32}, \qquad \mathbb{P}\big[C_3\big] \leqslant \frac{1}{32}. \tag{7.20}$$

We start by assessing the probability $\mathbb{P}\big[|Y| \geqslant 1 - a_1 - a_2 - a_3\big]$ conditioned on C_1. We look at

$$\mathbb{P}\Big[\Big|\sum_{i=4}^n a_i \varepsilon_i\Big| < 1 - a_1 - a_2 - a_3 \big| C_1, x_1, i_1, j, x_2, i_2, r(T_1;3\delta), \dots, r(T_{j-1};3\delta)\Big],$$

for fixed $x_1, i_1, j, x_2, i_2, r(T_1;3\delta), \dots, r(T_{j-1};3\delta)$, where x_1, i_1, j, x_2, i_2 are reals such that both $|r(S;9\delta)| = x_1 \in [9\delta, 12\delta]$, and the processes $W(T_1;3\delta), \dots, W(T_{j-1};3\delta)$ are not successful, but the process $W(T_j;3\delta)$ is successful for some fixed j, $1 \leqslant j \leqslant 4$, and

$$|r(T_j;3\delta)| = x_2 \in [3\delta, 6\delta].$$

Moreover, for the process $W(S;9\delta)$ it took i_1 terms to be successful, and for the process $W(T_j;3\delta)$ it took i_2 terms to be successful. Note that the value of $W(S;9\delta)$ is $\pm x_1$ with equal probabilities, and the value of $W(T_j;3\delta)$ is $\pm x_2$ with equal probabilities, independently both of each other and of all the other information.

Since $a_4 \geqslant 21\delta$, we can apply Observation 7.3.3 with a_4, x_1, x_2 to conclude that

$$\mathbb{P}\big[|\sum_{i=4}^{n} a_i\varepsilon_i| < 1-a_1-a_2-a_3|C_1, x_1, i_1, j, x_2, i_2, r(T_1;3\delta), \ldots, r(T_{j-1};3\delta)\big] \leqslant \frac{1}{8}. \quad (7.21)$$

As $x_1, i_1, j, x_2, i_2, r(T_1;3\delta), \ldots, r(T_{j-1};3\delta)$ were arbitrary and we have finitely many possibilities for them, we conclude from (7.21) that

$$\mathbb{P}\big[|\sum_{i=4}^{n} a_i\varepsilon_i| < 1-a_1-a_2-a_3|C_1\big] \leqslant \frac{1}{8}. \quad (7.22)$$

We can furthermore estimate the probability $\mathbb{P}\big[|Y| \geqslant 1-a_1-a_2-a_3\big]$, conditioned on C_2, using Observation 7.3.2:

$$\mathbb{P}\big[|\sum_{i=4}^{n} a_i\varepsilon_i| < 1-a_1-a_2-a_3|C_2\big] \leqslant \frac{1}{4}. \quad (7.23)$$

Analogously, the probability $\mathbb{P}\big[|Y| \geqslant 1-a_1-a_2-a_3\big]$ conditioned on C_3, is significant, as shown by Observation 7.3.1:

$$\mathbb{P}\big[|\sum_{i=4}^{n} a_i\varepsilon_i| < 1-a_1-a_2-a_3|C_3\big] \leqslant \frac{1}{2}. \quad (7.24)$$

Combining (7.20), (7.22), (7.23) and (7.24), we get $q_1 \geqslant \frac{205}{256} > \frac{793}{1024}$, and hence $\mathbf{a} \in \mathscr{A}_2$. $\square$

We turn to investigating the case $\sum_{i=k}^{n} a_i^2 < 450\delta^2$. We record a property that will repeatedly be used in the sequel

$$\begin{aligned} a_1^2+a_2^2+a_3^2 &= (a_1+a_2+a_3-2a_3)^2+2a_3^2-2(a_1-a_3)(a_2-a_3) \\ &\leqslant (\frac{1}{3}+2\delta)^2+2(\frac{1}{3}-\delta)^2 = \frac{1}{3}+6\delta^2. \end{aligned} \quad (7.25)$$

Claim 7.4.3. *If $\sum_{i=k}^{n} a_i^2 < 450\delta^2$, then $k \geqslant 11$.*

Proof of Claim 7.4.3. Assume that we had $\sum_{i=k}^{n} a_i^2 < 450\delta^2$ and $k \leqslant 10$. Then using (7.25), we get

$$1 = \sum_{i=1}^{n} a_i^2 < (\frac{1}{3}+2\delta)^2 + 8(\frac{1}{3}-\delta)^2 + 450\delta^2 = 1+462\delta^2-4\delta,$$

being a contradiction, as $1+462\delta^2-4\delta<1$ for $\delta \in (0,1/120]$. $\square$

Claim 7.4.4. *If* $k \geqslant 11$ *and* $a_8+a_9+a_{10} \geqslant \frac{2}{3}+\delta$*, then* $\mathbf{a} \in \mathscr{A}_3$.

Proof of Claim 7.4.4. Assume that we had $a_8+a_9+a_{10} \geqslant \frac{2}{3}+\delta$ and $k \geqslant 11$. Then by (7.19) and Observation 7.3.1 applied to $a_4,\dots,a_{10}$, we obtain $q_2,q_3 \geqslant \frac{37}{128}$. $\square$

Claim 7.4.5. *If* $\sum_{i=k}^n a_i^2 < 450\delta^2$ *and* $a_5-a_{10} \geqslant 3\delta$*, then* $\mathbf{a} \in \mathscr{A}_2$.

Proof of Claim 7.4.5. Consider the events D_1, D_2, where

$$D_1 = \{\varepsilon_4 = \varepsilon_6 = \varepsilon_7\}$$

and $D_2 = D_1^C$. Note that

$$\mathbb{P}\big[D_1\big] = \frac{1}{4} \qquad \mathbb{P}\big[D_2\big] = \frac{3}{4}. \tag{7.26}$$

In the case when D_1 occurs, let $c_1 = a_4+a_6+a_7$, $c_2 = a_5$, $c_3 = a_{10}$. Since the conditions of Observation 7.3.3 hold for c_1,c_2,c_3 (by Claim 7.4.3, $a_{10} \geqslant 1-a_1-a_2-a_3$), we deduce that

$$\mathbb{P}\big[|\sum_{i=4}^n a_i\varepsilon_i| \leqslant 1-a_1-a_2-a_3\,|\,D_1\big] \leqslant \frac{1}{8}. \tag{7.27}$$

In the case when D_2 occurs, Observation 7.3.2 applied on $b_1 = a_5$, $b_2 = a_{10}$ implies that

$$\mathbb{P}\big[|\sum_{i=4}^n a_i\varepsilon_i| \leqslant 1-a_1-a_2-a_3\,|\,D_2\big] \leqslant \frac{1}{4}. \tag{7.28}$$

Combining (7.26), (7.27) and (7.28), we get

$$q_1 \geqslant \frac{25}{32} \geqslant \frac{793}{1024}.$$

$\square$

Claim 7.4.6. *If* $\sum_{i=k}^n a_i^2 < 450\delta^2$*,* $a_5-a_{10} < 3\delta$ *and* $a_8+a_9+a_{10} < \frac{2}{3}+\delta$*, then* $k \geqslant 15$.

Proof of Claim 7.4.6. Assume that all of the conditions above hold, yet $k \leqslant 14$. We clearly have

$$a_4 \leqslant a_3 = \frac{1}{3}-\delta, \tag{7.29}$$

and the combination of $a_5-a_{10} < 3\delta$ and $a_8+a_9+a_{10} < \frac{2}{3}+\delta$ gives

$$a_{10},\dots,a_{13} \leqslant \frac{2}{9}+\frac{\delta}{3} \qquad a_5,\dots,a_9 < \frac{2}{9}+\frac{10}{3}\delta. \tag{7.30}$$

Using $\sum_{i=k}^n a_i^2 < 450\delta^2$, (7.25), (7.29) and (7.30), we get

$$\begin{aligned}1 &= \sum_{i=1}^n a_i^2\\ &= \sum_{i=1}^3 a_i^2 + a_4^2 + \sum_{i=5}^9 a_i^2 + \sum_{i=10}^{k-1} a_i^2 + \sum_{i=k}^n a_i^2\\ &\leqslant \left(\frac{1}{3}+6\delta^2\right) + \left(\frac{1}{3}-\delta\right)^2 + 5\left(\frac{2}{9}+\frac{10}{3}\delta\right)^2 + 4\left(\frac{2}{9}+\frac{\delta}{3}\right)^2 + 450\delta^2\\ &= \frac{8}{9} + 513\delta^2 + \frac{22}{3}\delta,\end{aligned}$$

being a contradiction, as the ultimate expression is strictly smaller than 1 for any $\delta \in (0,1/120]$. Hence $k \geqslant 15$. □

Claim 7.4.7. *If $k \geqslant 15$, then $\mathbf{a} \in \mathscr{A}_2$.*

Proof of Claim 7.4.7. Applying Observation 7.3.1 with $a_4,\dots,a_{14}$ gives $q_1 \geqslant \frac{793}{1024}$, as required. □

The combination of the above claims concludes the proof of Lemma 7.4.1. □

We are now ready to complete the proof of (7.14) in the case $a_3 \geqslant 0.325$ and $a_1+a_2+a_3 \leqslant 1$; that is, we verify (7.17).

We note that combining $\delta \leqslant \frac{1}{120}$ with (7.25), we get

$$a_1^2+a_2^2+a_3^2 \leqslant \frac{801}{2400}. \tag{7.31}$$

First, consider the family $\mathscr{A}_1$ with $a_4 \leqslant \frac{7}{40}$. In this case, (7.31) implies

$$\frac{a_4}{\sqrt{\sum_{i=4}^n a_i^2}} \leqslant \frac{\frac{7}{40}}{\sqrt{\frac{1599}{2400}}} < 0.216. \tag{7.32}$$

Moreover, using (7.18):

$$\frac{1-a_1-a_2-a_3}{\sqrt{\sum_{i=4}^n a_i^2}} \leqslant \frac{\frac{3}{120}}{\sqrt{\frac{1599}{2400}}} < 0.032. \tag{7.33}$$

Finally, (7.32) and (7.33) imply

$$q_1 \geqslant 2D(0.216, 0.032). \tag{7.34}$$

Analogously to (7.34), we have

$$\begin{aligned} q_2 &\geqslant 2D(0.216, 0.828), \qquad & q_3 &\geqslant 2D(0.216, 0.828), \\ q_4 &\geqslant 2D(0.216, 0.858), \qquad & q_5 &\geqslant 2D(0.216, 1.634), \\ q_6 &\geqslant 2D(0.216, 1.654), \qquad & q_7 &\geqslant 2D(0.216, 1.654), \end{aligned}$$
$$q_8 \geqslant 2D(0.216, 2.452).$$

Using the following estimate,

$$D(0.216, 0.032) + D(0.216, 0.828) + D(0.216, 0.828) + D(0.216, 0.858) + D(0.216, 1.634) + \\ D(0.216, 1.654) + D(0.216, 1.654) + D(0.216, 2.452) \geqslant \frac{3}{4}$$

we deduce (7.17) for any $\mathbf{a} \in \mathscr{A}_1$.

Next, we consider an $\mathbf{a}$ in the families $\mathscr{A}_2, \mathscr{A}_3$. Using $a_4 \leqslant \frac{1}{3}$ and (7.31), we obtain

$$\frac{a_4}{\sqrt{\sum_{i=4}^n a_i^2}} \leqslant \frac{\frac{1}{3}}{\sqrt{\frac{1599}{2400}}} < 0.41, \tag{7.35}$$

and we note that (7.33) still holds. Using (7.33) and (7.35), we obtain

$$q_1 \geqslant 2D(0.41, 0.032). \tag{7.36}$$

Analogously to (7.36), we derive

$$\begin{aligned} q_2 &\geqslant 2D(0.41, 0.828), \qquad & q_3 &\geqslant 2D(0.41, 0.828), \\ q_4 &\geqslant 2D(0.41, 0.858), \qquad & q_5 &\geqslant 2D(0.41, 1.634), \\ q_6 &\geqslant 2D(0.41, 1.654), \qquad & q_7 &\geqslant 2D(0.41, 1.654), \end{aligned}$$
$$q_8 \geqslant 2D(0.41, 2.452).$$

Note that we only mention the bound for q_8 above for the sake of completeness, since we have $D(0.41, 2.452) = 0$.

When $\mathbf{a} \in \mathscr{A}_2$ we can easily verify that

$$\frac{793}{2048} + D(0.41, 0.828) + D(0.41, 0.828) + D(0.41, 0.858) + D(0.41, 1.634) + \\ D(0.41, 1.654) + D(0.41, 1.654) + D(0.41, 2.452) \geqslant \frac{3}{4}$$

and hence (7.17) follows for all $\mathbf{a} \in \mathscr{A}_2$. For the family $\mathscr{A}_3$ we can verify that

$$D(0.41, 0.032) + \frac{37}{256} + \frac{37}{256} + D(0.41, 0.858) + D(0.41, 1.634) + \\ D(0.41, 1.654) + D(0.41, 1.654) + D(0.41, 2.452) \geqslant \frac{3}{4}$$

and hence (7.17) follows for all $\mathbf{a} \in \mathscr{A}_3$. Proof of this subcase is thus finished.

7.4.2 Case $a_1 + a_2 + a_3 > 1$

Subcase $a_1 + a_2 \geqslant 1$

Using Observation 7.3.1, we have $\mathbb{P}[|X| \geqslant 1] \geqslant 1/4$.

Subcase $a_1 \geqslant 0.7$ and not the previous subcase

The proof is the same as in subsection 7.4.1.

Setting for the rest of the subcases

Assume $a_1 + a_2 < 1$ and $a_1 + a_2 + a_3 > 1$. The required inequality (7.14), involves $\mathbb{P}[|X| \geqslant 1]$, and may be re-written using elimination in terms of $Y = \sum_{i=4}^{n} a_i \varepsilon_i$ as

$$\frac{2}{8}\mathbb{P}[|Y| \leqslant a_1 + a_2 + a_3 - 1] + \frac{1}{8}\mathbb{P}[|Y| > a_1 + a_2 + a_3 - 1] + \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 - a_1 - a_2 + a_3] + \\ \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 - a_1 + a_2 - a_3] + \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 + a_1 - a_2 - a_3] + \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 - a_1 + a_2 + a_3] + \\ \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 + a_1 - a_2 + a_3] + \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 + a_1 + a_2 - a_3] + \frac{1}{8}\mathbb{P}[|Y| \geqslant 1 + a_1 + a_2 + a_3] \geqslant 3/16.$$

Denote

$$L_1, L_2, L_3, L_4 = a_1 + a_2 + a_3 - 1, 1 - a_1 - a_2 + a_3, 1 - a_1 + a_2 - a_3, 1 + a_1 - a_2 - a_3.$$

The inequality we are proving follows by rearranging and multiplying the following inequality by $1/8$:

$$\mathbb{P}[|Y| \in (L_1, L_2)] \leqslant 1/2 + \mathbb{P}[|Y| \geqslant L_3] + \mathbb{P}[|Y| \geqslant L_4]. \tag{7.37}$$

Write $\sigma_j^2 = 1 - \sum_{i=1}^{j} a_i^2$. Recall the variance of Y is σ_3^2 and its largest weight is a_4.

Subcase $a_4 \geqslant 1 - a_1 - a_3$ and (either $a_4 \notin (L_1, L_2)$ or $\max(L_2 - a_4, a_4 - L_1) \leqslant 0.35\sigma_4$) and not the previous subcases

Let us prove (7.37), i.e. $\mathbb{P}[|Y| \in (L_1, L_2)] \leqslant 1/2 + \mathbb{P}[|Y| \geqslant L_3] + \mathbb{P}[|Y| \geqslant L_4]$.

Since this inequality is symmetric with respect to Y, we may assume without loss of generality that $\varepsilon_4 = 1$, in which case it is clearly sufficient to prove

$$\mathbb{P}[Y \in (L_1, L_2) | \varepsilon_4 = 1] + \mathbb{P}[Y \in (-L_2, -L_1) | \varepsilon_4 = 1] \leqslant 1/2 + \mathbb{P}[Y > L_3 | \varepsilon_4 = 1].$$

To this end, note that $\mathbb{P}[Y \in (-L_2, -L_1) | \varepsilon_4 = 1] \leqslant \mathbb{P}[Y > L_3 | \varepsilon_4 = 1]$, which follows by (recall $L_3 - a_4 \leqslant a_4 + L_1$ by assumption):

$$\mathbb{P}[Y' + a_4 < -L_1] = \mathbb{P}[Y' > L_1 + a_4] \leqslant \mathbb{P}[Y' > L_3 - a_4]$$

with $Y' = Y - a_4 \varepsilon_4$.

Hence our task is to verify $\mathbb{P}[Y \in (L_1, L_2) | \varepsilon_4 = 1] \leqslant 1/2$. There are two subcases. If $a_4 \leqslant L_1$ or $a_4 \geqslant L_2$, then we conclude with a general $\mathbb{P}[Y' > 0] \leqslant 1/2$ bound. If $a_4 \in [L_1, L_2]$, we conclude with the inequality from Section 7.3.5, recalling that $\max(L_2 - a_4, a_4 - L_1) \leqslant 0.35\sigma_4$.

Subcase not previous cases

Note that (7.37) follows from

$$\mathbb{P}[|Y| > L_1] \leqslant 1/2 + \mathbb{P}[|Y| \geqslant L_2] + \mathbb{P}[|Y| \geqslant L_3] + \mathbb{P}[|Y| \geqslant L_4].$$

As the left hand side is a probability, it is sufficient we show the right hand side is at least 1. This in turn follows from (see Appendix 7.A.2)

$$D(a_4/\sigma_3, L_2/\sigma_3) + D(a_4/\sigma_3, L_3/\sigma_3) + D(a_4/\sigma_3, L_4/\sigma_3) \geqslant 1/4. \tag{7.38}$$

7.5 Proof of $\mathbb{P}[X > 1] \geqslant 1/16$ unless $X = \varepsilon_1$

In this section, we prove Theorem 7.1.2 (which is the best possible). Note that unlike for Theorem 7.1.1 where significant further work was required, most of the work toward proving Theorem 7.1.2 was done when we developed our tools in 7.3.3 and now we can just conclude pretty easily.

7.5.1 Case $a_1 + a_2 + a_3 > 1$

Clearly,

$$\mathbb{P}[X > 1] \geqslant \mathbb{P}\left[\sum_{i=1}^{3} a_i\varepsilon_i > 1 \wedge \sum_{j=4}^{n} a_j\varepsilon_j \geqslant 0\right] \geqslant 1/8 \cdot 1/2 = 1/16.$$

7.5.2 Case $a_1 + a_2 + a_3 \leqslant 1$

In this case we actually show $\mathbb{P}[X > 1] \geqslant 1/12$, and the proof is analogous to that of Section 7.4.1.

Subcase $a_1 \leqslant 0.4$

We conclude using (7.12) and (7.13) with $D(0.4, 1) > \frac{1}{12}$.

Subcase $a_1 \geqslant 0.6$

Let $a = 0.6$. We conclude using elimination, (7.12) and (7.13) with

$$D\left(\frac{1-a}{\sqrt{1-a^2}}, \frac{1-a}{\sqrt{1-a^2}}\right) = D(1/2, 1/2) > 1/6.$$

Notice that in this case a_1 might be 1, which forbids elimination by a_1. This is where the assumption $X \neq \varepsilon_1$ is used.

Subcase $a_1 \in [0.4, 0.6]$

We write $\sigma_2 = \sqrt{1 - a_1^2 - a_2^2}$ and recall that a_3 is bounded above by $a = \min(a_2, 1 - a_1 - a_2)$, so that $\mathbb{P}[X > 1] \geqslant 1/12$ follows from (see Appendix 7.A.3):

$$\mathop{\mathbb{E}}_{\varepsilon \in \{-1,1\}^2}\left[D\left(\frac{a}{\sigma_2}, \frac{1 + a_1\varepsilon_1 + a_2\varepsilon_2}{\sigma_2}\right)\right] \geqslant 1/12. \tag{7.39}$$

7.6 Toward the $7/64$ bound

We strongly believe that $C_1 = 7/64$. Further to the brief discussion in subsection 7.1.4, we will comment in this section what the next steps would be and what hurdles one would face if we try to continue further to this bound using the methods of this chapter, i.e. combining lower bounds of the type 7.3.3 with separate arguments for some difficult cases. While somewhat tedious, we note that a similar approach was recently used by Keller and Klein to resolve the problem of Tomaszewski [50]. Nevertheless, the tools needed here would be rather different than the ones used in the proof of Tomaszewski's conjecture, since we are now dealing with an anti-concentration inequality instead of a concentration one.

Continuing further to the $7/64$ bound using our methods (or similar ones), there are two particular classes of the collections $\{a_i\}$ one has to be very careful about.

The first such class are the collections $\{a_i\}$ for which we have precisely $\mathbb{P}\big[X \geqslant 1\big] = \frac{7}{64}$ and thus we can not afford to obtain any suboptimal bound. As an example of the collection in the first class, one can consider $a_1 = \ldots = a_6 = \frac{1}{\sqrt{6}}$. For this particular collection, the bound follows trivially from Observation 7.3.1, since $a_3 + a_4 + a_5 \geqslant 1$. We suspect that in fact all the collections in this class satisfy $a_3 + a_4 + a_5 \geqslant 1$, making it not too difficult to handle.

The second such class are the collections $\{a_i\}$ with

$$\mathbb{P}\big[X > 1\big] < \frac{7}{64},$$

since for these one can't verify the conjecture by only assuming that the few largest weights lie in some, however narrow, ranges. Five examples of the collections in the second class are mentioned in the subsection 7.1.4 and we believe these are only such examples.

The collections 'close to' $a_1 = 1$ are not a big problem for us, since Lemma 7.3.4 allows us to show that the bound of $7/64$ holds for collections with a_1 large.

Proposition 7.6.1. *If $a_1 \geqslant \frac{14}{15}$, then $\mathbb{P}\big[X \geqslant 1\big] \geqslant \frac{7}{64}$.*

Proof. Note that it is enough to argue that $p(a_2, \ldots, a_n; 1 - a_1) \geqslant \frac{7}{8}$. For that, by Lemma 7.3.4 we know that it suffices if $1 - a_1^2 \geqslant 29(1 - a_1)^2$. We can easily check that this is satisfied whenever $a_1 \geqslant \frac{14}{15}$. □

'Neighbourhoods' of the remaining problematic collections are more difficult (though luckily note that the family $F_1(\delta)$ below covers the 'neighbourhood' of both the second and the fifth collection). For fixed $\delta > 0$, consider the families

$$F_1(\delta) = \{a_1 + a_2 < 1; a_2 \geqslant \frac{1}{2} - \delta\},$$

$$F_2(\delta) = \{a_1 + a_2 < 1; |a_1 - \frac{2}{3}|, |a_2 - \frac{1}{3}| \leqslant \delta\},$$

$$F_3(\delta) = \{a_1 + a_2 + a_3 < 1; a_3 \geqslant \frac{1}{3} - \delta\}.$$

If we want to verify that $C_1 = 7/64$ with the help of computational methods similar to the ones used in this chapter, we must be able to find some $\delta > 0$ for which we can verify by different means that the conjecture holds for all the collections in $F_1(\delta), F_2(\delta), F_3(\delta)$. We hope this could be done in somewhat similar way as the proof of $6/64$ bound within $F_3(\frac{1}{120})$ in 7.4.1 when proving Theorem 7.1.1.

We make a progress in that direction by using stopped random walks and chain arguments to prove the following.

Proposition 7.6.2. *For $\delta_0 = 10^{-9}$, we have $\mathbb{P}\big[X \geqslant 1\big] \geqslant \frac{7}{64}$ for all collections $\{a_i\}$ in $F_1(\delta_0), F_2(\delta_0)$.*

Our value δ_0 is extremely small, but that is because we have not tried to optimize it at all (as that would result in an even more tedious argument). We believe with some effort, our solution could be improved to work for much larger value of δ which could actually be used in practice.

The arguments for $F_1(\delta_0)$ and $F_2(\delta_0)$ are rather similar in style and are somewhat tedious. Hence in this section, we only include the argument for the family $F_1(\delta_0)$ and the argument for the family $F_2(\delta_0)$ is placed in Appendix 7.B.

Surprisingly, while we were able to improve the bound closer to $\frac{7}{64}$ in that case too, we were not able to prove the bound of $7/64$ for the family $F_3(\delta)$ for any $\delta > 0$, so we pose this as an open problem to the reader. We believe even verifying the conjecture just in this narrow range of parameters would be of interest.

In subsection 7.6.1 and in Appendix 7.B, we sometimes sketch the proofs instead of going through all the details of the calculations. That is because the calculations would otherwise be very long and it is easy to see that the sketch could indeed be turned into a rigorous proof.

7.6.1 Family $F_1(\delta_0)$

In this subsection, we prove the following result.

Proposition 7.6.3. *For $\delta_0 = 10^{-9}$, we have $\mathbb{P}\big[X \geqslant 1\big] \geqslant \frac{7}{64}$ for all collections $\{a_i\}$ in $F_1(\delta_0)$.*

Together with Proposition 7.B.1, this implies Proposition 7.6.2.

Assume $a_1 + a_2 < 1$ and $a_2 = \frac{1}{2} - \delta$ for some $\delta \leqslant 10^{-9}$. Also assume our collection $\{a_1, \ldots, a_n\}$ has $\mathbb{P}\big[|X| \geqslant 1\big] < \frac{7}{32}$. We will derive a contradiction.

Note that $1-a_1-a_2 \leqslant 2\delta$. Denote $Y=\sum_{i=3}^{n} a_i\varepsilon_i$ and

$$p_1=\mathbb{P}\big[|Y|\geqslant 1-a_1-a_2\big], \qquad p_2=\mathbb{P}\big[|Y|\geqslant 1-a_1+a_2\big], \qquad p_3=\mathbb{P}\big[|Y|\geqslant 1+a_1-a_2\big].$$

Then, in particular, we have

$$\mathbb{P}\big[|X|\geqslant 1\big]\geqslant \frac{1}{4}(p_1+p_2+p_3).$$

So, it is enough to show

$$p_1+p_2+p_3\geqslant \frac{7}{8}. \tag{7.40}$$

We can also assume that

$$a_3+a_4+a_5<1, \tag{7.41}$$

else we would be done by Observation 7.3.1. We will make consecutive claims about $\{a_1,\dots,a_n\}$, characterizing it more and more precisely until we are ready to obtain a contradiction.

Call a_i big if $a_i\geqslant 1-a_1-a_2$, and small otherwise. So in particular if $a_i\geqslant 2\delta$, it must be big. Let k be the smallest integer such that $a_k<1-a_1-a_2$ (if $a_n\geqslant 1-a_1-a_2$, set $k=n+1$).

Claim 7.6.4. *Let k be the smallest integer such that $a_k<1-a_1-a_2$. Then we have $\sum_{i=k}^{n} a_i^2\leqslant 240000\delta^2<\frac{\delta}{1000}$.*

Proof. Assume for contradiction that this is not true. Then we can take disjoint subsets $S_1,\dots,S_{10000}$ of $\{a_k,\dots,a_n\}$ with

$$24\delta^2\geqslant \sum_{i\in S_j} a_i^2\geqslant 20\delta^2$$

for $j=1,\dots 10000$. Now considering the random processes $W(S_i;2\delta)$ for $i=1,\dots,10000$, with probability at least $\frac{99}{100}$, at least 1000 of these are successful, and conditional on that, we obtain $\mathbb{P}\big[|Y|\geqslant 1-a_1-a_2\big]\geqslant\frac{19}{20}$ by Observation 7.3.1. Hence we overall get

$$p_1\geqslant\frac{1881}{2000}>\frac{7}{8},$$

and (7.40) holds. □

Claim 7.6.5. *a_5 and a_6 are big terms, that is, $a_6\geqslant 1-a_1-a_2$.*

Proof. Assume for contradiction that a_6 is a small term (i.e. that $k \leqslant 6$). Combining Claim 7.6.4 with (7.41), we arrive at a contradiction for all sufficiently small $\delta > 0$:

$$\begin{aligned} 1 &= \sum_{i=1}^{5} a_i^2 + \sum_{i=6}^{n} a_i^2 \leqslant a_1^2 + a_2 \sum_{i=2}^{5} a_i + \frac{\delta}{1000} \\ &\leqslant (1/2+\delta)^2 + (1/2-\delta)(3/2-\delta) + \frac{\delta}{1000} = 1 - \frac{999}{1000}\delta + 2\delta^2. \end{aligned}$$

□

At this point, we split our proof into two cases, the uniform and the non-uniform one, both of which we handle separately.

The uniform case - $a_3 - a_{k-1} \leqslant 20\delta$

Claim 7.6.6. *Let k be the smallest integer such that $a_k < 1 - a_1 - a_2$. Then we have $k \leqslant 11$.*

Proof. Assume we had $k \geqslant 12$. Note that $a_3 - a_{k-1} \leqslant 20\delta$ would then in particular imply

$$a_3 \leqslant \sqrt{0.5/9} + O(\delta) < 0.24, \tag{7.42}$$

and we also know

$$\sum_{i=3}^{n} a_i^2 \geqslant 0.4999 \tag{7.43}$$

and

$$1 + a_1 - a_2 \leqslant 1.00001. \tag{7.44}$$

Using Observation 7.3.1 for $a_3, \ldots, a_{11}$, we get

$$p_1 \geqslant \frac{386}{512} > \frac{3}{4}.$$

Combining (7.42), (7.43) and (7.44), and using (7.13), we get

$$p_2, p_3 \geqslant 2D(0.34, 1.42) > 0.08 > \frac{1}{16}$$

and hence (7.40) holds. □

The next corollary follows by combining Chebychev inequality with Claim 7.6.4, using that δ is small.

Corollary 7.6.7. *Let k be the smallest integer such that $a_k < 1 - a_1 - a_2$. Then we have* $\mathbb{P}\big[|\sum_{i=k}^{n} a_i \varepsilon_i| \geqslant 0.0001\big] < \frac{1}{1000}$.

We now sketch how we finish our argument in the subcase $a_3 - a_{k-1} \leqslant 20\delta$, using Corollary 7.6.7. We consider five separate cases depending on the particular value of k which we know is at least 7 and at most 11 (and in fact, we can rule out the case $k = 7$ as then we would have $a_3 + a_4 + a_5 \geqslant 1$). Due to our restrictions on the value of δ and Corollary 7.6.7, we know that $\sum_{i=3}^{n} a_i \varepsilon_i$ behaves 'essentially' like $\sum_{i=3}^{k-1} \varepsilon_i \frac{1}{\sqrt{2k-6}}$. So for instance in the case $k = 8$, we argue that $p_1 \geqslant \frac{999}{1000}$, as due to our restrictions on $a_3, \ldots, a_7$, we know we can only have $|\sum_{i=3}^{n} a_i \varepsilon_i| < 2\delta$ if $|\sum_{i=8}^{n} a_i \varepsilon_i| \geqslant 0.0001$; further, in this case $k = 8$, we analogously argue that $p_2, p_3 \geqslant \frac{999}{1000} \cdot \frac{1}{32}$.

Similarly, in the case $k = 9$, we argue that $p_1 \geqslant \frac{11}{16}$, $p_2, p_3 \geqslant \frac{999}{1000} \cdot \frac{7}{32}$.

The reader can easily verify that such arguments indeed work in all the cases considered. □

The non-uniform case - $a_3 - a_{k-1} > 20\delta$

In this case, we first notice that Observation 7.3.2 applied to a_3, a_{k-1} immediately implies the following.

Claim 7.6.8. *We have $p_1 \geqslant \frac{3}{4}$.*

Next we obtain.

Claim 7.6.9. *We have $a_3 + a_4 + a_5 + a_6 < 1 + 2\delta$.*

Proof. Assume not. Then by Observation 7.3.1, we have $p_2, p_3 \geqslant \frac{1}{16}$, and combining this with Claim 7.6.8 gives (7.40). □

Claim 7.6.10. *a_7 is a big term.*

Proof. Assume for contradiction that a_7 is a small term (i.e. that $k \leqslant 7$), and recall that $a_3 + a_4 + a_5 + a_6 < 1 + 2\delta$ and $\sum_{i=7}^{n} a_i^2 < 240000\delta^2 < \frac{\delta}{1000}$. Write $A = a_3 + a_4$, and arrive at a contradiction for all sufficiently small $\delta > 0$:

$$\begin{aligned}
1 = \sum_{i=1}^{6} a_i^2 + \sum_{i=7}^{n} a_i^2 &\leqslant a_1^2 + a_2^2 + a_2(a_3+a_4) + a_5(a_5+a_6) + \frac{\delta}{1000} \\
&\leqslant \Big(\frac{1}{2}+\delta\Big)^2 + \Big(\frac{1}{2}-\delta\Big)^2 + \Big(\frac{1}{2}-\delta\Big)A + \frac{1+2\delta}{3}(1+2\delta-A) + \frac{\delta}{1000} \\
&= \frac{1}{2} + 2\delta^2 + A\Big(\frac{1}{6}-\frac{5}{3}\delta\Big) + \frac{1}{3}(1+2\delta)^2 + \frac{\delta}{1000} \\
&\leqslant \frac{1}{2} + 2\delta^2 + (1-2\delta)\Big(\frac{1}{6}-\frac{5}{3}\delta\Big) + \frac{1}{3}(1+2\delta)^2 + \frac{\delta}{1000} \\
&= 1 - \frac{2}{3}\delta + \frac{20}{3}\delta^2 + \frac{\delta}{1000} < 1.
\end{aligned}$$

where we used the estimates $a_5 + a_6 \leqslant 1 + 2\delta - A$, and $a_5 \leqslant (1+2\delta)/3$ and $A \leqslant 2a_2 \leqslant 1 - 2\delta$. □

Claim 7.6.11. *We have $a_4 \geqslant 0.07$.*

Proof. If not, we can use Claim 7.6.4 to argue that we have at least 44 big terms, otherwise we would have

$$\sum_{i=3}^{k-1} a_i^2 < 0.49.$$

But Observation 7.3.1 then implies $p_1 \geqslant \frac{7}{8}$, and (7.40) follows. □

Claim 7.6.12. *We have $p_2, p_3 \geqslant \frac{3}{64}$.*

Proof. We consider two cases. If $a_3 \leqslant 0.3$, the result follows using the bounds (7.43) and (7.44) as well as (7.13) by

$$p_2, p_3 \geqslant 2D(0.43, 1.42) > 0.06 > \frac{3}{64}.$$

If on the other hand $a_3 > 0.3$, we may argue (using Claim 7.6.11 and argument much along the same lines as the proofs of Claim 7.6.5 and Claim 7.6.10) that

$$a_3 + a_4 + \sqrt{\sum_{i=5}^{n} a_i^2} \geqslant 1 + a_1 - a_2.$$

But then let ε' be a sign of $\sum_{i=5}^{n} a_i \varepsilon_i$, and consider the events

$$A = \{\varepsilon\colon \varepsilon_3 = \varepsilon_4 = \varepsilon'\}, \qquad B = \Big\{\varepsilon\colon |\sum_{i=5}^{n} a_i\varepsilon_i| \geqslant \Big(\sum_{i=5}^{n} a_i^2\Big)^{1/2}\Big\}.$$

We have $\mathbb{P}\big[A\cap B\big] \geqslant \frac{3}{64}$ (using our bound from the previous sections), and clearly

$$|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant 1+a_1-a_2$$

whenever event $A\cap B$ occurs. The result follows. □

Claim 7.6.13. *Let k be the smallest integer such that $a_k < 1-a_1-a_2$. Then we have $a_4-a_{k-1} \leqslant 2\delta$.*

Proof. Assume for contradiction that $a_4-a_{k-1} \geqslant 2\delta$. Then $a_3+a_5+a_6$ is not within 2δ neither from a_4 nor from a_4+a_{k-1}. Using Observation 7.3.2 for a_4, a_{k-1} in the case when we do not have $\varepsilon_3=\varepsilon_5=\varepsilon_6$, and Observation 7.3.3 for $a_4, a_{k-1}, a_3+a_5+a_6$ in the case when we have $\varepsilon_3=\varepsilon_5=\varepsilon_6$ (which happens with probability $\frac{1}{4}$) gives

$$p_1 \geqslant \frac{3}{4}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{7}{8}=\frac{25}{32}. \tag{7.45}$$

Combining Claim 7.6.12 with (7.45) gives (7.40). □

Now we are ready to reach the contradiction. First, if $a_3 \notin (2a_{k-1}-8\delta, 2a_{k-1}+8\delta)$, let $f_1=a_3+a_{k-1}$ and $f_2=a_4+a_5$. Let $A_1=\{\varepsilon_3=\varepsilon_{k-1}\}$ and $A_2=\{\varepsilon_4=\varepsilon_5\}$. Then conditional on $A_1\cap A_2$, we have $\mathbb{P}\big[|Y|\geqslant 1-a_1-a_2\big] \geqslant \frac{7}{8}$ by Observation 7.3.3 for f_1, f_2, a_6; conditional on $A_1\cap A_2^C$, we have $\mathbb{P}\big[|Y|\geqslant 1-a_1-a_2\big] \geqslant \frac{3}{4}$ by Observation 7.3.2 for a_3+a_{k-1}, a_6; and conditional on A_1^C, we have $\mathbb{P}\big[|Y|\geqslant 1-a_1-a_2\big] \geqslant \frac{3}{4}$ by Observation 7.3.2 for a_3-a_{k-1}, a_6. So we conclude $p_1 \geqslant \frac{25}{32}$, and hence (7.40) holds.

So next assume $a_3 \in (2a_{k-1}-8\delta, 2a_{k-1}+8\delta)$. Here, we observe that we can assume $k \leqslant 15$, else we could conclude $p_1 \geqslant \frac{25}{32}$ from Observation 7.3.1. But now, we proceed analogously to how we did at the end of the argument for the uniform case, again using Corollary 7.6.7 and detailed analysis of each of the several cases we have depending on the value of k. Carrying out such analysis is made possible by Claim 7.6.13.

So the proof of Proposition 7.6.3 is complete. □

7.7 The high-dimensional version of the problem

The following (non-tight) result constitutes a high-dimensional variant of Tomaszewski's problem as well as of the problem studied in this chapter. The result is merely a consequence of the combination of [69, Proposition 2.2] and [49, Theorem 2]. Nevertheless, for the sake of completeness, we prove it here.

Proposition 7.7.1. *Let $v_1, \dots, v_n \in \mathbb{R}^d$ be vectors with $\sum_i \|v_i\|_2^2 = 1$. The random variable $X = \sum v_i \varepsilon_i$ with $\varepsilon_i \sim \{-1,1\}$ uniformly and independently distributed, satisfies*

$$\mathbb{P}[\|X\|_2 \geqslant 1] \geqslant \frac{1-\sqrt{1-1/e^2}}{2} > 0.035, \qquad \mathbb{P}[\|X\|_2 \leqslant 1] \geqslant \frac{1-\sqrt{1-1/e^2}}{2}.$$

Proof. The function $f(\varepsilon) = \|X(\varepsilon)\|_2^2 - 1 = \sum_{i,j} \varepsilon_i \varepsilon_j \langle v_i, v_j \rangle$ is a *homogenuous* polynomial of degree 2 in the ε_i's. We wish to bound below the probabilities $\mathbb{P}[f(\varepsilon) \geqslant 0]$ and $\mathbb{P}[f(\varepsilon) \leqslant 0]$. Recall [49, Theorem 2]:

$$\|f\|_2 \leqslant e\|f\|_1. \tag{7.46}$$

Since $\mathbb{E}[f] = 0$, we can derive (see below)

$$\|f\|_1^2 \leqslant 4\mathbb{P}[f > 0]\mathbb{P}[f \leqslant 0]\|f\|_2^2. \tag{7.47}$$

Plugging (7.46) into (7.47) we get

$$\|f\|_1^2 \leqslant 4e^2\mathbb{P}[f > 0]\mathbb{P}[f \leqslant 0]\|f\|_1^2.$$

When $f \equiv 0$, we have $\mathbb{P}[f = 0] = 1$. Otherwise, dividing by $\|f\|_1^2$ we obtain

$$\mathbb{P}[f > 0]\mathbb{P}[f \leqslant 0] \geqslant e^{-2}/4,$$

which means both $\mathbb{P}[f > 0]$ and $\mathbb{P}[f \leqslant 0]$ are at least $\frac{1-\sqrt{1-1/e^2}}{2}$, through $\mathbb{P}[f > 0] + \mathbb{P}[f \leqslant 0] = 1$.

To see (7.47), notice that by the Cauchy-Schwarz inequality,

$$\|f\|_2^2 = \mathbb{E}[f^2 \cdot \mathbb{1}\{f > 0\}] + \mathbb{E}[f^2 \cdot \mathbb{1}\{f \leqslant 0\}] \geqslant \mathbb{P}[f > 0]\,\mathbb{E}[f|f > 0]^2 + \mathbb{P}[f \leqslant 0]\,\mathbb{E}[f|f \leqslant 0]^2. \tag{7.48}$$

As $\mathbb{E}[f] = 0$, we have

$$\mathbb{E}[f \cdot \mathbb{1}\{f > 0\}] = -\mathbb{E}[f \cdot \mathbb{1}\{f \leqslant 0\}] = \frac{1}{2}\|f\|_1.$$

Likewise, we may assume $\mathbb{P}[f > 0]$ and $\mathbb{P}[f \leqslant 0]$ are both positive as otherwise (7.47) trivially holds. Under this assumption, (7.48) yields

$$\|f\|_2^2 \geqslant \frac{1}{4}\|f\|_1^2 \left(\frac{1}{\mathbb{P}[f > 0]} + \frac{1}{\mathbb{P}[f \leqslant 0]}\right),$$

being (7.47), using again $\mathbb{P}[f > 0] + \mathbb{P}[f \leqslant 0] = 1$. $\square$

Denote by T_d the maximum constant for which $\mathbb{P}[\|X\|_2 \leqslant 1] \geqslant T_d$ for all X of dimension d as in Proposition 7.7.1, and denote by O_d the maximum constant for which $\mathbb{P}[\|X\|_2 \geqslant 1] \geqslant O_d$ for all X of dimension d as in Proposition 7.7.1. Clearly, T_d and O_d are non-increasing in d. We know $T_1 = \frac{1}{2}$ [50], while in this chapter we prove that $\frac{6}{32} \leqslant O_1 \leqslant \frac{7}{32}$. Proposition 7.7.1 establishes that $T_d, O_d \geqslant 0.035$ for any d. There are two directions for further research here.

The first is to find tighter bounds for T_d, O_d for small values of $d > 1$. We know that $T_2 \leqslant \frac{1}{4}$, as demonstrated by

$$v_1 = (\frac{1}{\sqrt{3}}, 0), \qquad v_2 = (-\frac{1}{2\sqrt{3}}, \frac{1}{2}), \qquad v_3 = (-\frac{1}{2\sqrt{3}}, -\frac{1}{2}),$$

and $T_3 \leqslant \frac{3}{16}$, as demonstrated by

$$\begin{aligned} &v_1 = (\sqrt{\frac{7}{30}}, \frac{1}{3}, \frac{1}{5}), \quad v_2 = (\sqrt{\frac{7}{30}}, -\frac{1}{3}, -\frac{1}{5}), \quad v_3 = (0, \frac{1}{3}, -\frac{1}{5}), \\ &v_4 = (0, 0, \frac{1}{5}), \quad v_5 = (0, 0, \frac{1}{5}). \end{aligned}$$

Interestingly, we have not been able to find any examples demonstrating that $O_2 < \frac{7}{32}$ (or even that $O_{d_0} < \frac{7}{32}$ for any $d_0 \in \mathbb{N}$).

The second possible direction is to investigate how T_d, O_d behave for large d, and in particular to find better bounds for $\inf_d T_d$ and $\inf_d O_d$. It appears that Proposition 7.7.1 is far from being tight. Also, as just mentioned, it does not seem completely unthinkable that $O_d = \frac{7}{32}$ for every $d \in \mathbb{N}$ could hold.

7.8 Concluding remarks

As mentioned previously, we would hope that mixed with some new ideas, the methods developed in this chapter could be used to prove the conjectured optimal bound of $7/64$ in Theorem 7.1.1. That is the main open problem left, and even some progress toward that (like improving Theorem 7.1.1 to hold for some constant between $6/64$ and $7/64$) would be of interest.

Another, perhaps easier step one could take in this direction would be to prove the bound of $7/64$ for the 'difficult' family $F_3(\delta)$ for some $\delta > 0$. The significance of this is discussed in more detail in Section 7.6.

In a bit different direction, it is likely that one could improve the multiplicative factor in front of $\sqrt{\operatorname{Var} X}$ in Theorem 7.1.3 from 0.35 to the optimal conjectured [56] value of $1/\sqrt{7}$. That would not only be of interest on its own, but as demonstrated by this chapter and our use of Theorem 7.1.3 when deriving Theorem 7.1.1, also a useful tool when attacking similar problems.

Finally, let us mention two interesting generalizations of our main problem that one can consider.

Firstly, same as Keller and Klein [50], we ask what is the behaviour of the function

$$F(x) = \sup_X \mathbb{P}[X > x],$$

where the supremum is taken over all the Rademacher sums with variance 1. Theorem 7.1.1 establishes that $F(-1) \leqslant \frac{58}{64}$. We know some asymptotic results about the behaviour of $F(x)$ [63] and we also know the precise value of $F(x)$ for some x [7, 50, 63], but much remains to be understood. It would be tempting to conjecture that $F(x) = F^{=}(x)$, where for $F^{=}(x)$, we take the supremum over all the the Rademacher sums with variance 1 and all the weights equal. Nevertheless, this conjecture turns out not to be true, see [62].

Further, one can also study the various multi-dimensional questions that arise, as discussed in Section 7.7. We find it especially intriguing that we have not managed to find any $d_0 \in \mathbb{N}$ for which we could show that $O_{d_0} < 7/32$. If there is no such d_0, that would be a beautiful generalization of the result for the one dimensional version of the problem.

Appendix 7.A Proofs of real numbers inequalities

7.A.1 Proof of (7.16)

We consider only these a_1, a_2 with $a_1 + a_2 \leqslant 1$ and $a_2 \leqslant a_1 \in [0.3, 0.7]$. We denote $a = \min(1 - a_1 - a_2, a_2, 0.325)$ (being an upper bound on a_3), and $\sigma_2 = \sqrt{1 - a_1^2 - a_2^2}$. We note that both a/σ_2 and $(1 + a_1\varepsilon_1 + a_2\varepsilon_2)/\sigma_2$ for any choice of $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}^2$ are 10-Lipschitz in our domain (e.g., by checking that all partial derivatives $< \sqrt{50}$ in absolute value), so it suffices we check

$$\mathop{\mathbb{E}}_{\varepsilon \in \{-1,1\}^2}\left[D\left(\frac{a}{\sigma_2} + \delta, \frac{1 + a_1\varepsilon_1 + a_2\varepsilon_2}{\sigma_2} + \delta\right)\right] \geqslant 3/32 \tag{7.49}$$

on a mesh of $\{(a_1, a_2) : a_1 + a_2 \leqslant 1, a_2 \leqslant a_1 \in [0.3, 0.7]\}$ of granularity $\delta/10$ in both axes. Inequality (7.49) can easily be verified [32] for $\delta = 0.005$ on such a mesh.

7.A.2 Proof of (7.38)

In order to verify (7.38) for all relevant a_1, a_2, a_4, a_4, we confirm

$$D(a_4/\sigma_3+\delta, L_2/\sigma_3+\delta)+D(a_4/\sigma_3+\delta, L_3/\sigma_3+\delta)+D(a_4/\sigma_3+\delta, L_4/\sigma_3+\delta) \geqslant 1/4, \tag{7.50}$$

on a fine enough mesh of a_1, a_2, a_3 (which induce an upper bound on a_4). Notice the other subcases in the proof handle cases in which $a_4 \geqslant 1-a_1-a_3$ and

$$L_2 - L_1 \leqslant 0.35\sqrt{1-a_1^2-a_2^2-2a_3^2}. \tag{7.51}$$

All expressions L_i/σ_3 and a_3/σ_3 and $(1-a_1-a_3)/\sigma_3$ have partial derivatives < 10, hence considering a mesh of $\{(a_1,a_2,a_3) : a_3 \leqslant a_2 \leqslant a_1 \leqslant 0.7, a_1+a_2+a_3 \geqslant 1, a_1+a_2 \leqslant 1\}$, with granularity $\delta/15$ in every axis, we may verify (7.38) by checking (7.50) on the mesh points. One detail is that on the mesh points we bound a_4 by $1-a_1-a_3$ (instead of $\min(a_3, \sigma_3)$) only if $L_2 - L_1 + \delta/2 < 0.35\sqrt{1-a_1^2-a_2^2-2a_3^2}$, ensuring that if (7.51) is not satisfied for a point, then its nearest mesh point will not use the improved bound $a_4 \leqslant 1-a_1-a_3$ (introducing 'discontinuity'); this behavior is overridden to the points $(a_1,a_2,a_3) = (0.5\pm 0.02, 0.5\pm 0.02, 0.5\pm 0.02)$, since there (7.51) is always satisfied. Choosing $\delta = 0.03$, (7.50) can be verified [32] to all the described mesh points.

7.A.3 Proof of (7.39)

Instead of checking (7.39), we will check that

$$\mathop{\mathbb{E}}_{\varepsilon\in\{-1,1\}^2}\left[D\left(\frac{\min(a_2, 1-a_1-a_2)}{\sigma_2}+\delta, \frac{1+a_1\varepsilon_1+a_2\varepsilon_2}{\sigma_2}+\delta\right)\right] \geqslant 1/12 \tag{7.52}$$

with $\sigma_2 = \sqrt{1-a_1^2-a_2^2}$ is satisfied on a mesh of points in $\{(a_1,a_2) : a_1+a_2 \leqslant 1, a_2 \leqslant a_1 \in [0.4, 0.6]\}$. Since all the involved arguments fed to D are 10-Lipschitz, it suffices we verify (7.52) on a mesh with $\delta/10$ granularity in every axis. Verification [32] can be done with $\delta = 0.01$.

Appendix 7.B Family $F_2(\delta_0)$

In this appendix, we prove the following result, which together with Proposition 7.6.3, implies Proposition 7.6.2.

Proposition 7.B.1. *For $\delta_0 = 10^{-9}$, we have $\mathbb{P}\big[X \geqslant 1\big] \geqslant \frac{7}{64}$ for all collections $\{a_i\}$ in $F_2(\delta_0)$.*

To prove Proposition 7.B.1, take smallest possible $\delta > 0$ such that $a_1 \in \big[\frac{2}{3}-\delta, \frac{2}{3}+\delta\big]$ and $a_2 \in \big[\frac{1}{3}-\delta, \frac{1}{3}+\delta\big]$. Assume $\delta \leqslant 10^{-9}$. Assume our collection $\{a_1,\dots,a_n\}$ has $\mathbb{P}\big[|X| \geqslant 1\big] < \frac{7}{32}$. We will derive a contradiction.

Note that

$$1 - a_1 - a_2 \leqslant 2\delta. \tag{7.53}$$

Denote

$$p_1 = \mathbb{P}\Big[|\sum_{i=3}^n a_i\varepsilon_i| \geqslant 1-a_1-a_2\Big], \quad p_2 = \mathbb{P}\Big[|\sum_{i=3}^n a_i\varepsilon_i| \geqslant 1-a_1+a_2\Big], \quad p_3 = \mathbb{P}\Big[|\sum_{i=3}^n a_i\varepsilon_i| \geqslant 1+a_1-a_2\Big].$$

Note that $\mathbb{P}\big[|X| \geqslant 1\big] \geqslant \frac{1}{4}(p_1+p_2+p_3)$, so it is enough to show that

$$p_1+p_2+p_3 \geqslant \frac{7}{8}. \tag{7.54}$$

We can assume

$$a_3+a_4+a_5 < 1, \tag{7.55}$$

else we would be done by Observation 7.3.1. We will make consecutive claims about the collection $\{a_1,\dots,a_n\}$, characterizing it more and more precisely until we are ready to obtain a contradiction.

Note that for $\eta = 10^{-5}$, the following two lemmas hold.

Lemma 7.B.2. *Assume $b_1 \geqslant \dots \geqslant b_m > 0$, $\sum_{i=1}^m b_i^2 = 1$ and $b_1 \leqslant \frac{1}{2}+\eta$. Then*

$$\mathbb{P}\Big[|\sum_{i=1}^m b_i\varepsilon_i| \geqslant 4\delta\Big] \geqslant \frac{5}{8}.$$

Proof. Note that if $b_3 \geqslant 4\delta$, we are done by Observation 7.3.1. So we only need to consider the case when $b_3 < 4\delta$. First, we argue that we have

$$\sum_{i=3}^m b_i^2 \geqslant 960\delta^2. \tag{7.56}$$

Since we know that

$$\sum_{i=3}^m b_i^2 \geqslant 1-2(\frac{1}{2}+\eta)^2 = \frac{1}{2}-2\eta-2\eta^2,$$

to prove (7.56) holds, it is enough to show

$$960\delta^2 \leqslant \frac{1}{2} - 2\eta - 2\eta^2. \tag{7.57}$$

But (7.57) trivially holds as $\delta \leqslant 10^{-9}, \eta = 10^{-5}$.

Now using $4\delta > b_3, \dots, b_m > 0$ and (7.56), we know that we can choose 10 disjoint subsets $S_1, \dots, S_{10}$ of $\{b_3, \dots, b_m\}$ such that for $1 \leqslant i \leqslant 10$, we have

$$96\delta^2 \geqslant \sum_{b_j \in S_i} b_j^2 \geqslant 80\delta^2.$$

Then for each of these sets S_i, we consider the random process $W(S_i; 4\delta)$. By Lemma 7.3.4, each of these is successful with probability at least $\frac{1}{2}$ and independently of the other ones. If for some t, $1 \leqslant t \leqslant 10$, we condition on the event E_t that precisely t of these processes are successful, Observation 7.3.1 ensures that

$$\mathbb{P}\big[|\sum_{i=1}^{m} b_i \varepsilon_i| < 4\delta | E_t\big] \leqslant \binom{t}{\lfloor t/2 \rfloor} 2^{-t}.$$

So we can bound

$$\mathbb{P}\big[|\sum_{i=1}^{m} b_i \varepsilon_i| < 4\delta\big] \leqslant 2^{-10} + 2^{-10} \sum_{t=1}^{10} \binom{10}{t} \binom{t}{\lfloor t/2 \rfloor} 2^{-t} \leqslant \frac{3}{8}.$$

This now finishes the proof of Lemma 7.B.2. □

Lemma 7.B.3. *Assume $b_1 \geqslant \ldots \geqslant b_m > 0$, $\sum_{i=1}^m b_i^2 = 1$ and $b_1 \leqslant \frac{1}{2} + \eta$. Then*

$$\mathbb{P}\big[|\sum_{i=1}^{m} b_i \varepsilon_i| \geqslant 1 + 4\delta\big] \geqslant \frac{1}{8}.$$

Proof. This follows directly using (7.13) by $D(0.51, 1.01) = \frac{1}{16}$. □

Now we can use these lemmas to prove the following corollary.

Corollary 7.B.4. *We have $p_1 \geqslant \frac{5}{8}$ and $p_2 \geqslant \frac{1}{8}$.*

Proof. Note that $a_3 \leqslant a_2 \leqslant \frac{1}{3} + \delta$ and

$$\sum_{i=3}^{n} a_i^2 \geqslant 1 - (\frac{2}{3} + \delta)^2 - (\frac{1}{3} - \delta)^2 = \frac{4}{9} - \frac{2}{3}\delta - 2\delta^2.$$

So for $\delta \leqslant 10^{-9}, \eta = 10^{-5}$, we see that we have

$$a_3 \leqslant (\frac{1}{2} + \eta)\sqrt{\sum_{i=3}^{n} a_i^2}.$$

Thus we conclude from Lemma 7.B.2 that

$$\begin{aligned} p_1 &= \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant 1 - a_1 - a_2\big] \\ &\geqslant \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant 2\delta\big] \\ &\geqslant \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant 4\delta\sqrt{\sum_{i=3}^{n} a_i^2}\big] \\ &\geqslant \frac{5}{8}. \end{aligned}$$

Analogously, we conclude from Lemma 7.B.3 that

$$\begin{aligned} p_2 &= \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant 1 - a_1 + a_2\big] \\ &\geqslant \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant \frac{2}{3} + 2\delta\big] \\ &\geqslant \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| \geqslant (1+4\delta)\sqrt{\sum_{i=3}^{n} a_i^2}\big] \\ &\geqslant \frac{1}{8}. \end{aligned}$$

□

Claim 7.B.5. *We have $a_3 > 14\delta$.*

Proof. Assume we had $a_3 \leqslant 14\delta$. By our choice of δ, we can trivially check that

$$\sum_{i=3}^{n} a_i^2 \geqslant 1 - (\frac{2}{3} + \delta)^2 - (\frac{1}{3} - \delta)^2 \geqslant 3920\delta^2. \tag{7.58}$$

Using (7.58), we can choose 20 disjoint subsets (possibly containing a single element) $S_1, \ldots, S_{20}$ of $\{a_3, \ldots, a_n\}$ such that for $1 \leqslant i \leqslant 20$, either S_i contains a single element $a_i \geqslant 2\delta$, or all its elements are smaller than 2δ and we have

$$24\delta^2 \geqslant \sum_{b_j \in S_i} b_j^2 \geqslant 20\delta^2.$$

Then for each of these sets S_i, consider the random process $W(S_i;2\delta)$. By Lemma 7.3.4, each of these is successful with probability at least $\frac{1}{2}$ and independently of the other ones. If for some t, $1 \leqslant t \leqslant 20$, we condition on the event F_t that precisely t of these processes are successful, Observation 7.3.1 ensures that

$$\mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| < 2\delta | F_t\big] \leqslant \binom{t}{\lfloor t/2 \rfloor} 2^{-t}.$$

So we can bound

$$1 - p_1 \leqslant \mathbb{P}\big[|\sum_{i=3}^{n} a_i\varepsilon_i| < 2\delta\big] \leqslant 2^{-20} + 2^{-20}\sum_{t=1}^{20}\binom{20}{t}\binom{t}{\lfloor t/2 \rfloor} 2^{-t} \leqslant \frac{1}{4}.$$

Combining $p_1 \geqslant \frac{3}{4}$ with $p_2 \geqslant \frac{1}{8}$ that we have proven before, this verifies (7.54). □

Let k be an integer such that $a_{k-1} \geqslant 1 - a_1 - a_2$, but $a_k < 1 - a_1 - a_2$ (if $a_n \geqslant 1 - a_1 - a_2$, set $k = n+1$).

Claim 7.B.6. *Let k be the smallest integer such that $a_k < 1 - a_1 - a_2$. Then we have $\sum_{i=k}^{n} a_i^2 < 328\delta^2$.*

Proof. Assume that we had $\sum_{i=k}^{n} a_i^2 \geqslant 328\delta^2$. Then we can find disjoint subsets $S_1, S_2, T_1, \ldots, T_5$ of $\{a_k, \ldots, a_n\}$ such that the following holds. For $x = 1, 2$ we have

$$104\delta^2 \geqslant \sum_{i \in S_x} a_i^2 \geqslant 100\delta^2$$

and for $y = 1, \ldots, 5$, we have

$$24\delta^2 \geqslant \sum_{i \in T_y} a_i^2 \geqslant 20\delta^2.$$

Now consider the random processes

$$W(S_1;6\delta), W(S_2;6\delta), W(T_1;2\delta), \ldots, W(T_5;2\delta).$$

By Lemma 7.3.4, each of these is successful with probability at least $\frac{1}{2}$ and independently of the other ones. We apply Observations 7.3.2 and 7.3.3, using a_3 and $r(S_1;6\delta), \ldots, r(T_5;2\delta)$, to bound p_1. With probability at least $\frac{93}{128}$, both some process corresponding to S_x and some process corresponding to T_y are successful, and conditional on that we get the lower bound of $\frac{7}{8}$ on $\mathbb{P}\big[|\sum_{i=3}^n a_i\varepsilon_i| \geqslant 1-a_1-a_2\big]$. Further, we get the lower bound of $\frac{3}{4}$ on $\mathbb{P}\big[|\sum_{i=3}^n a_i\varepsilon_i| \geqslant 1-a_1-a_2\big]$ if either some process corresponding to S_x or some process corresponding to T_y are successful, and the lower bound of $\frac{1}{2}$ otherwise (this last case happens at most with probability $\frac{1}{128}$). So overall, we obtain $p_1 > \frac{3}{4}$. Combining that with $p_2 \geqslant \frac{1}{8}$ that we have proven in Corollary 7.B.4, we verify that (7.54) holds. $\square$

Claim 7.B.7. *Let k be the smallest integer such that $a_k < 1-a_1-a_2$. Then we have $a_3 - a_{k-1} < 1-a_1-a_2 < 2\delta$.*

Proof. If we had two terms $a_s, a_t > 1-a_1-a_2$ such that $|a_s - a_t| \geqslant 1-a_1-a_2$, Observation 7.3.2 for a_s, a_t gives $p_1 \geqslant \frac{3}{4}$. Combining that with $p_2 \geqslant \frac{1}{8}$ verifies (7.54). $\square$

Claim 7.B.8. *Let k be the smallest integer such that $a_k < 1-a_1-a_2$. Then we have $k < 12$.*

Proof. If we had $k \geqslant 12$, by Observation 7.3.1, we have $p_1 \geqslant \frac{193}{256} > \frac{3}{4}$, and combining that with $p_2 \geqslant \frac{1}{8}$ verifies (7.54). $\square$

Claim 7.B.9. *Let k be the smallest integer such that $a_k < 1-a_1-a_2$. Then we have $k < 8$.*

Proof. We have already shown that $k \leqslant 11$. If $11 \geqslant k \geqslant 8$, note that using our choice of δ, Claim 7.B.6 and Claim 7.B.7, we get

$$a_5 + a_6 + a_7 \geqslant \frac{2}{3} + 2\delta \geqslant 1 - a_1 + a_2.$$

That gives $p_2 \geqslant \frac{7}{32}$.

Since $k \geqslant 8$, we also have $p_1 \geqslant \frac{11}{16}$ by Observation 7.3.1. Hence we verify (7.54). $\square$

Claim 7.B.10. *Let k be the smallest integer such that $a_k < 1-a_1-a_2$. Then we have $k \geqslant 7$ (and hence as also $k \leqslant 7$, we have $k = 7$).*

Proof. By our choice of δ, we have $\sum_{i=6}^n a_i^2 \geqslant 328\delta^2$, and result thus follows by Claim 7.B.6. $\square$

We will now show that $\sum_{i=7}^n a_i^2 \geqslant 328\delta^2$ (which together with Claim 7.B.6 gives a desired contradiction). By our definition of δ and assumption that $a_1 + a_2 < 1$, we either have $a_1 = \frac{2}{3} - \delta$ or $a_2 = \frac{1}{3} - \delta$.

First consider the case $a_2 = \frac{1}{3} - \delta$. Then

$$\sum_{i=1}^{6} a_i^2 \leqslant (\frac{2}{3} + \delta)^2 + 5(\frac{1}{3} - \delta)^2 = 1 - 2\delta + 6\delta^2,$$

and hence

$$\sum_{i=7}^{n} a_i^2 \geqslant 2\delta - 6\delta^2 > 328\delta^2$$

for every $0 < \delta < \frac{1}{167}$.

So we can assume that instead $a_1 = \frac{2}{3} - \delta$. But now we use (7.55) to bound

$$\sum_{i=1}^{6} a_i^2 \leqslant (\frac{2}{3} - \delta)^2 + (\frac{1}{3} + \delta)^2 + 4(\frac{1}{3})^2 \leqslant 1 - \frac{2}{3}\delta + 2\delta^2,$$

and hence

$$\sum_{i=7}^{n} a_i^2 \geqslant \frac{2}{3}\delta + 2\delta^2 > 328\delta^2$$

for every $0 < \delta < \frac{1}{489}$.

Thus we reached a desired contradiction, and the proof of Proposition 7.B.1 is complete.

□

Chapter 8

Radius, girth and minimum degree

This chapter is joint work with Peter van Hintum, Amy Shaw and Marius Tiba. Our results were published in Journal of Graph Theory [30].

8.1 Introduction

The girth of a graph G is the length of the shortest cycle in G; we set the girth to be ∞ if no cycle exists. The radius r of a connected graph G is the smallest integer such that there exists some $v \in V(G)$ with $d(v,w) \leqslant r$ for every $w \in V(G)$.

Consider the following question: given a connected graph G on n vertices, with minimum degree $\delta \geqslant 2$ and girth at least $g \geqslant 4$, what is the maximum radius r this graph can have (note that the connectedness condition is superfluous if we let r be the biggest radius of a connected component)? Denote this maximum value of the radius as $r(n,\delta,g)$.

Erdős, Pach, Pollack and Tuza [36] studied $r(n,\delta,4)$, and proved that it is at most $\frac{n-2}{\delta} + 12$. They also noted that, up to the additive constant 12, this bound is tight. We improve this to a best possible bound.

Theorem 8.1.1. *Fix integer $\delta \geqslant 2$.*

- *If $2\delta \leqslant n \leqslant 2\delta + 1$, then $r(n,\delta,4) = 2$.*
- *If $2\delta + 2 \leqslant n < 4\delta$, then $r(n,\delta,4) = 3$.*
- *If $n \geqslant 4\delta$, then*

$$r(n,\delta,4) = \begin{cases} \frac{n}{\delta} - 1 & \text{if } \delta \text{ is odd and } n = k\delta \text{ for } k \text{ odd,} \\ \left\lfloor \frac{n}{\delta} \right\rfloor & \text{otherwise.} \end{cases}$$

Observe that every graph of order n and minimum degree greater than $n/2$ has a triangle, so in the study of $r(n,\delta,4)$, we may assume that $n \geqslant 2\delta$.

Next we consider the case when the girth g is bigger than 4. We shall prove the following upper bound.

Theorem 8.1.2. *Let $n, \delta \geqslant 2$, and $g = 2k \geqslant 4$. Then*

$$r(n,\delta,2k) \leqslant \frac{nk(\delta-2)}{2((\delta-1)^k-1)} + 3k.$$

In the cases $g = 6, 8, 12$, we shall prove the following lower bound.

Theorem 8.1.3. *Let $\delta \geqslant 2$ be such that $\delta - 1$ is a prime power.*
Then there exists sequences $(n_i), (n_i'), (n_i'')$ with $n_i, n_i', n_i'' \to \infty$ such that

- $r(n_i,\delta,6) \geqslant \frac{3n_i}{2(\delta^2-\delta+1)} - 3 = \frac{3n_i(\delta-2)}{2((\delta-1)^3-1)} - 3,$
- $r(n_i',\delta,8) \geqslant \frac{2n_i'}{\delta^3-2\delta^2+2\delta} - 4 = \frac{2n_i'(\delta-2)}{(\delta-1)^4-1} - 4,$
- $r(n_i'',\delta,12) \geqslant \frac{3n_i''}{((\delta-1)^3+1)(\delta^2-\delta+1)} - 6 = \frac{3n_i''(\delta-2)}{(\delta-1)^6-1} - 6.$

We note that the results for the girth 6, 8 and 12 are optimal up to the value of the additive constant, as established by Theorem 8.1.2. We are very grateful to the anonymous referee for pointing out that our previous approach can be optimized to obtain the correct bounds even for all lower order terms.

It would be interesting to see whether the upper bound from Theorem 8.1.2 is tight, at least up to some constant factor. We believe it is and hence make a following conjecture.

Conjecture 8.1.4. *Let $g = 2k \geqslant 4$. Then there exists infinitely many values δ for which the following holds.*

There exists a sequence $(n(\delta)_i)$ with $n(\delta)_i \to \infty$ and a positive constant $c(\delta)$ such that $r(n(\delta)_i,\delta,2k) \geqslant c(\delta)\frac{n(\delta)_i}{\delta^{k-1}}$.

As our final result, we obtain the following theorem.

Theorem 8.1.5. *Let $r, c > 0$, $g = 2k$ and $n \leqslant c(r+1)\delta^{k-1}$, so that $r(n,\delta,g) \geqslant r$. Then there exists a connected graph of girth at least $2k$ on at most $(2k+1)c\delta^{k-1}$ vertices with at least $\frac{1}{2}\delta^2(\delta-1)^{k-2}$ edges.*

This theorem is related to the following girth conjecture of Erdős from [35].

Conjecture 8.1.6 (Erdős [35])**.** *For any positive integers l, n, there exists a graph with girth $2l+1$, n vertices and $\Omega(n^{1+\frac{1}{l}})$ edges.*

If the upper bound from Theorem 8.1.2 was tight up to a constant factor for some fixed $g = 2k$, then we could find graphs G_i with $\delta_i \to \infty$ and $n_i \leqslant c(r(n_i, \delta_i, 2k) + 1)\delta_i^{k-1}$ for some fixed c. By Theorem 8.1.5, that would verify the girth conjecture of Erdős for $l = k-1$.

We note that some similar problems relating various parameters in a graph have been studied in the literature - for instance the analogous problem for the diameter instead of the radius [36, 51], and problems involving more detailed information about the degree sequence of the graph [58].

The structure of the chapter is as follows: in Lemma 8.2.1, we establish a general tool that gives a lower bound on n in terms of r and δ that is tight in many cases. This lemma unfortunately is not strong enough to handle all cases, so we prove the additional Lemma 8.3.6. We use these two lemmas (the key ingredients of the proof) to prove Lemma 8.3.4 which establishes the upper bound on r in Theorem 8.1.1. Together with Lemma's 8.3.1 and 8.3.3, which establish the lower bound on r, this completes our proof. Finally, in Section 8.4, we consider the case of general girth.

Throughout the chapter, for a vertex v in a graph, we will denote by $N(v)$ its open neighbourhood, and by $N[v]$ its closed neighbourhood. Difference between closed and open neighbourhood is that the closed one contains also the vertex v itself.

8.2 Strategy

The following lemma will be used throughout the chapter. It tells us that if we can find a large collection of vertices in our graph such that any two vertices are either neighbours or sufficiently far away from each other, then our graph must in fact have many vertices.

We thank the anonymous referee for pointing out how to improve the lemma.

Lemma 8.2.1. *Assume G is a graph on n vertices of girth $g \geqslant 2k$ (where $k \geqslant 2$) with minimum degree δ. Let $T \subset V(G)$ be such that all pairs of non-adjacent vertices in T have distance at least $2k-1$ from each other. Then we have $n \geqslant |T| \left(\frac{(\delta-1)^k-1}{\delta-2}\right)$. Moreover if $|T|$ is odd, this inequality is strict.*

Note that for $k = 2$, i.e. the triangle-free case, this means that $n \geqslant |T|\delta$ and if $|T|$ is odd, then $n > |T|\delta$.

Proof. Since G contains no triangles, we know we can label

$$T = \{x_1, y_1, \ldots, x_i, y_i, z_1, \ldots, z_j\},$$

with $d(x_l, y_l) = 1$ for $1 \leqslant l \leqslant i$, while all other pairs of vertices in T have distance at least $2k-1$.

For $v \in T$, let

$$S(v) = \{w \in V(G) \mid d(v,w) = k-1\}$$

and

$$B(v) = \{w \in V(G) \mid d(v,w) \leqslant k-1\}.$$

Consider the sets $B(x_1), S(y_1) \setminus B(x_1), \ldots, B(x_i), S(y_i) \setminus B(x_i), B(z_1), \ldots, B(z_j)$.

Note that by the distance condition, all these sets are disjoint. Moreover, by the girth condition, for any $v \in T$, we have

$$|B(v)| \geqslant 1 + \delta\left(1 + (\delta-1) + \ldots + (\delta-1)^{k-2}\right) = 1 + \delta\frac{(\delta-1)^{k-1}-1}{\delta-2}$$

and that for any $1 \leqslant l \leqslant i$,

$$|S(y_l) \setminus B(x_l)| \geqslant (\delta-1)^{k-1}.$$

We conclude

$$\begin{aligned}
n = |V(G)| &\geqslant \left| \bigcup_{l=1}^{i} B(x_l) \cup \bigcup_{l=1}^{i} S(y_l) \setminus B(x_l) \cup \bigcup_{l=1}^{j} B(z_l) \right| \\
&= \sum_{l=1}^{i} |B(x_l)| + \sum_{l=1}^{i} |S(y_l) \setminus B(x_l)| + \sum_{l=1}^{j} |B(z_l)| \\
&\geqslant (|T| - i)\left(1 + \delta\frac{(\delta-1)^{k-1}-1}{\delta-2}\right) + i(\delta-1)^{k-1} \\
&\geqslant \frac{|T|}{2}\left(1 + \delta\frac{(\delta-1)^{k-1}-1}{\delta-2} + (\delta-1)^{k-1}\right) \\
&= |T|\left(\frac{(\delta-1)^{k}-1}{\delta-2}\right).
\end{aligned}$$

Moreover, when $|T|$ is odd, the third inequality above is strict (as $|B(v)| > |S(v)|$ for any v and we cannot have $j = 0$ in the odd case). Hence, the proof is finished. □

To find such large collections of points T with restricted distances, we shall use several observations. We formulate these observations used throughout the proof in the following general setting.

Let G be a connected graph with n vertices and radius r. We take v_0 to be some fixed centre of G, i.e. a vertex of G with distance at most r from any other vertex. We let v_r be a vertex such that $d(v_0, v_r) = r$, and let $v_0, v_1, \ldots, v_r$ be a path of length r from v_0 to v_r.

Fix an integer $m \in \{1, \ldots, r-1\}$, and let v' be a vertex such that $d(v_m, v') \geqslant r$. Then let $t \geqslant 0$ be such that $d(v_0, v') = r - t$, and let $v_0 = v'_0, v'_1, ..., v'_{r-t} = v'$ be a path of length $r-t$ from v_0 to $v' = v'_{r-t}$.

Observation 8.2.2. *We have $t \leqslant m$.*

Proof. Assume for contradiction that we had $t > m$. Then by a triangle inequality

$$d(v_m, v'_{r-t}) \leqslant d(v_m, v_0) + d(v_0, v'_{r-t}) = m + (r-t) < r$$

which is a contradiction. □

Observation 8.2.3. *For any $m \leqslant i \leqslant r$ and any $0 \leqslant j \leqslant r-t$, we have*

$$d(v_i, v'_j) \geqslant d(v_m, v'_{r-t}) + m + t + j - r - i$$

and for any $i < m$ and any $0 \leqslant j \leqslant r-t$, we have

$$d(v_i, v'_j) \geqslant d(v_m, v'_{r-t}) + i + j + t - m - r.$$

Moreover, in either of these cases, we also have

$$d(v_i, v'_j) \geqslant |i - j|.$$

Proof. For the case $m \leqslant i \leqslant r$, note that

$$d(v_m, v'_{r-t}) \leqslant d(v_m, v_i) + d(v_i, v'_j) + d(v'_j, v'_{r-t}) = (i-m) + d(v_i, v'_j) + (r-t-j).$$

Rearranging gives the result.

For the case $i < m$, note that

$$d(v_m, v'_{r-t}) \leqslant d(v_m, v_i) + d(v_i, v'_j) + d(v'_j, v'_{r-t}) = (m-i) + d(v_i, v'_j) + (r-t-j).$$

Rearranging gives the result.

For the last claim, note that by triangle inequality

$$d(v_i, v'_j) \geqslant |d(v_i, v_0) - d(v_0, v'_j)| = |i - j|.$$

□

Observation 8.2.4. *We can not have $v_i = v'_i$ for any $i > \frac{m+r-t-d(v_m, v'_{r-t})}{2}$, and we can not have $v_i = v'_j$ for any $i \neq j$.*

Proof. Assume that $v_i = v'_i$ for some $r-t \geqslant i > \frac{m+r-t-d(v_m, v'_{r-t})}{2}$. Then we obtain a contradiction, as $d(v_i, v'_i) > 0$ by Observation 8.2.3.

We can not have $v_i = v'_j$ for any $i \neq j$, since

$$d(v_0, v_i) = i \neq j = d(v_0, v'_j).$$

□

Now we are ready to move on to the case of the triangle-free graphs.

8.3 Triangle-free graphs

To prove Theorem 8.1.1, we will establish the following four lemmas.

Lemma 8.3.1. *Fix integers $n \geqslant 4, \delta \geqslant 2$. If $n \geqslant 2\delta$, there exists a connected triangle-free graph on n vertices with minimum degree δ and radius 2. If $n \geqslant 2\delta + 2$, there exists a connected triangle-free graph on n vertices with minimum degree δ and radius 3.*

Lemma 8.3.2. *Every connected triangle-free graph on n vertices with minimum degree $\delta \geqslant 2$ and radius r satisfies $r \geqslant 2$ and $n \geqslant 2\delta$. Moreover, if $r = 3$, we have $n \geqslant 2\delta + 2$.*

Lemma 8.3.3. *Fix integers $r \geqslant 4$, $\delta \geqslant 2$, $c \geqslant 0$. There exists a connected triangle-free graph with $2\lceil \frac{r\delta}{2} \rceil + c$ vertices, minimum degree δ and radius r.*

Lemma 8.3.4. *Every connected triangle-free graph on n vertices with minimum degree $\delta \geqslant 2$ and radius $r \geqslant 4$ satisfies $n \geqslant 2\lceil \frac{r\delta}{2} \rceil$.*

Let us first see how Theorem 8.1.1 follows from these.

Proof of Theorem 8.1.1 assuming Lemma 8.3.1, 8.3.2, 8.3.3, 8.3.4.

Case. $2\delta \leqslant n \leqslant 2\delta + 1$.

As $2\delta \leqslant n$, Lemma 8.3.1 shows $r(n, \delta, 4) \geqslant 2$. As $n < 2\delta + 2 \leqslant 4\delta$, Lemma 8.3.4 shows $r(n, \delta, 4) < 4$ and Lemma 8.3.2 shows $r(n, \delta, 4) \neq 3$. We conclude $r(n, \delta, 4) = 2$.

Case. $2\delta + 2 \leqslant n \leqslant 4\delta - 1$.

As $2\delta + 2 \leqslant n$, Lemma 8.3.1 shows that $r(n, \delta, 4) \geqslant 3$. As $n < 4\delta$, Lemma 8.3.4 shows $r < 4$. We conclude $r(n, \delta, 4) = 3$.

Case. $4\delta \leqslant n$.

In this case, we consider two subcases depending on the precise form of n.

Subcase. *$n = k\delta$ with δ and k both odd.*

We set $r = \frac{n}{\delta} - 1, c = \delta$ and we show that $r(n,\delta,4) = r$.

By Lemma 8.3.3, there exists a connected triangle-free graph with $2\lceil \frac{r\delta}{2} \rceil + c = n$ vertices, minimum degree δ and radius r, and hence $r(n,\delta,4) \geqslant r$.

First consider the case $r(n,\delta,4) < 4$. As $n \geqslant 4\delta$, we also have $r(n,\delta,4) < \frac{n}{\delta}$ and hence $r(n,\delta,4) \leqslant r$, finishing this case. So further assume $r(n,\delta,4) \geqslant 4$. By Lemma 8.3.4, every connected triangle-free graph on n vertices and of minimum degree $\delta \geqslant 2$ and radius $r(n,\delta,4) \geqslant 4$ satisfies $n \geqslant 2\left\lceil \frac{r(n,\delta,4)\delta}{2} \right\rceil$. As n is odd integer and $2\left\lceil \frac{r(n,\delta,4)\delta}{2} \right\rceil$ is an even integer, we must even have $n - 1 \geqslant 2\left\lceil \frac{r(n,\delta,4)\delta}{2} \right\rceil$. So we get $n - 1 \geqslant r(n,\delta,4)\delta$. Therefore, $r(n,\delta,4) < \frac{n}{\delta}$ and hence $r(n,\delta,4) \leqslant r$.

Subcase. *n is not of the form $k\delta$ with δ and k both odd.*

We set $r = \lfloor \frac{n}{\delta} \rfloor$ and $c = n - 2\lceil \frac{r\delta}{2} \rceil$ and show that $r(n,\delta,4) = r$.

By Lemma 8.3.3, there exists a connected triangle-free graph with $2\lceil \frac{r\delta}{2} \rceil + c = n$ vertices, minimum degree δ and radius r, and hence $r(n,\delta,4) \geqslant r$.

First once again consider the case $r(n,\delta,4) < 4$. As $n \geqslant 4\delta$, we also have $r(n,\delta,4) \leqslant \frac{n}{\delta}$ and hence $r(n,\delta,4) \leqslant r$, completing the proof in this case. Hence further assume $r(n,\delta,4) \geqslant 4$. By Lemma 8.3.4, every connected triangle-free graph on n vertices and of minimum degree $\delta \geqslant 2$ and radius $r(n,\delta,4) \geqslant 4$ satisfies $n \geqslant 2\left\lceil \frac{r(n,\delta,4)\delta}{2} \right\rceil$. Therefore, $r(n,\delta,4) \leqslant \frac{n}{\delta}$ and hence $r(n,\delta,4) \leqslant r$. □

In the rest of the section, we will prove Lemmas 8.3.1, 8.3.2, 8.3.3 and 8.3.4 and thus prove Theorem 8.1.1. The section will be divided into five subsections - in the first subsection we prove Lemma 8.3.3; in the second subsection we prove a technical lemma we will need to prove Lemma 8.3.4; in the third subsection we prove Lemmas 8.3.1, 8.3.2; in the fourth subsection we prove Lemma 8.3.4 when $r \in \{4k, 4k+1, 4k+2\}$; and in the final subsection we prove Lemma 8.3.4 when $r = 4k+3$.

8.3.1 Proof of Lemma 8.3.3

It suffices to prove the lemma for $c = 0$. Indeed, given a triangle-free graph G, we can add a vertex while preserving both the radius and the minimum degree: if $v \in V(G)$ is such that $d(v) = \delta$, then add a vertex v' to $V(G)$, which is connected precisely to the same vertices as v is.

For $c = 0$, consider the following example.

Partition $V(G)$ into $2r$ sets labelled $B_0, \ldots, B_{2r-1}$ such that

$$|B_i| = \begin{cases} \left\lceil \frac{\delta}{2} \right\rceil & \text{if } i \equiv 0, 1 \pmod 4 \\ \left\lfloor \frac{\delta}{2} \right\rfloor & \text{if } i \equiv 2, 3 \pmod 4. \end{cases}$$

Connect all vertices in B_i to all vertices in B_j whenever $i - j \equiv \pm 1 \pmod{2r}$. An example of such a graph with $r = 5$ and $\delta = 6$ is depicted in Figure 8.1.

It is easy to see that this is a connected triangle-free graph with $2\lceil \frac{r\delta}{2} \rceil$ vertices, minimum degree δ and radius r. □

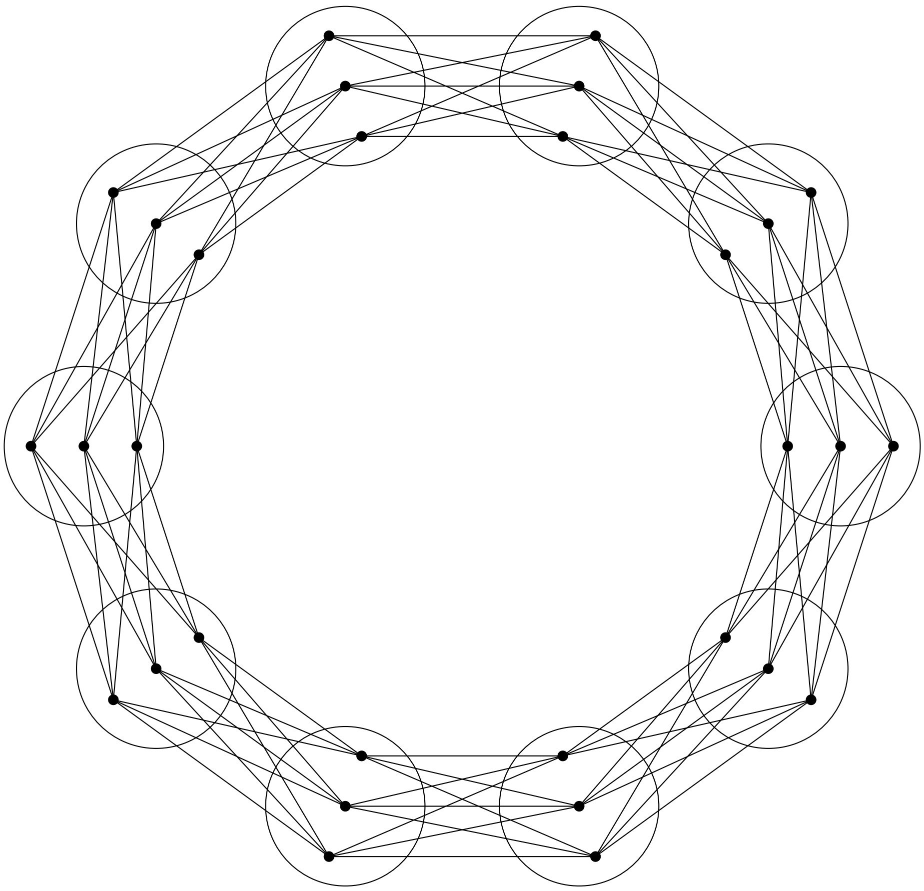

Fig. 8.1 Construction from Lemma 8.3.3 for $r = 5, \delta = 6$.

8.3.2 Technical lemma

First, recall Lemma 8.2.1 which implies the following result for triangle-free graphs.

Lemma 8.3.5. *Let G be a triangle-free graph on n vertices and with minimum degree δ. Then for any subset $T \subset V(G)$ such that no two vertices of T are at distance 2, we have $n \geqslant 2\left\lceil \frac{\delta|T|}{2} \right\rceil$.*

We will also need another lemma of similar flavour here.

Lemma 8.3.6. *Let G be a triangle-free graph on n vertices and with minimum degree δ. Assume for some $r \geqslant 4$, we have a subset $U \subset V(G)$ such that $|U| = 2r$ and U is as follows: if we consider auxiliary graph H such that $V(H) = U$ and in which we connect two vertices if their distance in G is precisely 2, then H is disjoint union of two cycles of length r. Then we have $n \geqslant 2\left\lceil \frac{r\delta}{2} \right\rceil$.*

Proof. Let $c_1, \dots, c_r$ and $d_1, \dots, d_r$ be our two cycles of length r in H. Consider the open neighbourhoods $N(c_1), \dots, N(c_r)$. On the one hand, we have $|N(c_i)| \geqslant \delta$ for $1 \leqslant i \leqslant r$. On the other hand, each $v \in V(G)$ can be contained in the neighbourhood of at most two vertices from $\{c_1, \dots, c_r\}$ by our triangle-free condition.

Further, as no c_i and d_j are at distance 2, and G is triangle-free, no vertex can be contained both in a $N(c_i)$ and in a $N(d_j)$.

Let $B = \bigcup_i N(c_i)$, so that by the above discussion

$$2|B| \geqslant |N(c_1)| + \dots + |N(c_r)| \geqslant r\delta.$$

Since $|B|$ is integer, we have $|B| \geqslant \lceil \frac{r\delta}{2} \rceil$.

Let $B' = \bigcup_j N(d_j)$, so that similarly we get $|B'| \geqslant \lceil \frac{r\delta}{2} \rceil$. As B, B' are disjoint, we conclude $n \geqslant 2\lceil \frac{r\delta}{2} \rceil$. □

8.3.3 Proof of Lemma 8.3.1, 8.3.2

Here we handle the small radius cases.

Proof of Lemma 8.3.1. If $n \geqslant 2\delta$, note that $K_{\delta,n-\delta}$ is a connected triangle-free graph on n vertices of radius 2 and minimum degree δ.

If $n \geqslant 2\delta + 2$, start with a complete bipartite graph $K_{\delta+1,n-\delta-1}$ with vertex classes $\{v_1, \dots, v_{\delta+1}\}$ and $\{w_1, \dots, w_{n-\delta-1}\}$. Erase the edges

$$v_1w_1, v_2w_2, \dots, v_{\delta+1}w_{\delta+1}, v_{\delta+1}w_{\delta+2}, \dots, v_{\delta+1}w_{n-\delta-1}.$$

The resulting graph is a connected triangle-free graph on n vertices of radius 3 and minimum degree δ. □

Proof of Lemma 8.3.2. Consider a connected triangle-free graph G on n vertices of radius r and minimum degree $\delta \geqslant 2$. We must have $r \geqslant 2$, since the only connected triangle-free graphs of radius 1 are star graphs, which have minimum degree 1.

Consider adjacent vertices $a, b \in V(G)$. It follows from Lemma 8.3.5 applied to $T = \{a, b\}$ that $n \geqslant 2\delta$.

If $r = 3$, then we can take a, b which instead satisfy $d(a, b) \geqslant 3$. But then even their closed neighbourhoods are disjoint, which implies

$$|V(G)| \geqslant |N[a] \cup N[b]| = |N[a]| + |N[b]| \geqslant 2\delta + 2.$$

□

8.3.4 Proof of Lemma 8.3.4 for $r = 4k, 4k+1, 4k+2$

In this subsection, we prove the following.

Lemma 8.3.7. *If G is a connected triangle-free graph on n vertices with minimum degree $\delta \geqslant 2$ and radius $r \geqslant 4$ such that $r = 4k + i$ for some k and some $i \in \{0, 1, 2\}$, then we have $n \geqslant 2\left\lceil \frac{r\delta}{2} \right\rceil$.*

Proof. Let v_0 be a centre of our graph G with a minimal number of vertices at distance r. Let v_r be any vertex such that $d(v_0, v_r) = r$. Let $v_0, v_1, \ldots, v_r$ be a path of length r from v_0 to v_r.

Let v'_{r-t} be a following vertex: if v_3 is not a centre of G, then let v'_{r-t} be any vertex such that $d(v_3, v'_{r-t}) \geqslant r+1$. If v_3 is a centre of G, then let v'_{r-t} be a vertex such that $d(v_3, v'_{r-t}) = r$ and $d(v_0, v'_{r-t}) < r$ (such a vertex exists by a choice of v_0).

Let t be so that $d(v_0, v'_{r-t}) = r - t$. Let $v_0 = v'_0, v'_1, \ldots, v'_{r-t}$ be a path of length $r - t$ from v_0 to v'_{r-t}. It follows from Observation 8.2.2 that $t \leqslant 3$.

Claim 8.3.8. *If $r = 4k + 1$ and $0 \leqslant t \leqslant 2$, or $r \in \{4k, 4k+2\}$, then $n \geqslant 2\lceil \frac{r\delta}{2} \rceil$.*

Proof of Claim 8.3.8. We show that in each of these cases, we can find a collection C of r vertices in G such that no two are at distance 2. The result then follows from Lemma 8.3.5.

Depending on the values of r and t, choose C to be the following collection.

$r=4k$	$t=0$	$v_3,v_4,v_7,v_8,\dots,v_{4k-1},v_{4k},v'_3,v'_4,v'_7,v'_8,\dots,v'_{4k-1},v'_{4k}$
$r=4k$	$t=1$	$v_0,v_3,v_4,v_7,v_8,\dots,v_{4k-1},v_{4k},v'_3,v'_4,v'_7,v'_8,\dots,v'_{4k-5},v'_{4k-4},v'_{4k-1}$
$r=4k$	$t=2$	$v_3,v_4,v_7,v_8,\dots,v_{4k-1},v_{4k},v'_1,v'_2,v'_5,v'_6,\dots,v'_{4k-3},v'_{4k-2}$
$r=4k$	$t=3$	$v_0,v_3,v_4,v_7,v_8,\dots,v_{4k-1},v_{4k},v'_1,v'_4,v'_5,v'_8,v'_9,\dots,v'_{4k-4},v'_{4k-3}$
$r=4k+1$	$t=0$	$v_0,v_4,v_5,v_8,v_9,\dots,v_{4k},v_{4k+1},v'_4,v'_5,v'_8,v'_9,\dots,v'_{4k},v'_{4k+1}$
$r=4k+1$	$t=1$	$v_0,v_3,v_4,v_7,v_8,\dots,v_{4k-1},v_{4k},v'_3,v'_4,v'_7,v'_8,\dots,v'_{4k-1},v'_{4k}$
$r=4k+1$	$t=2$	$v_0,v_1,v_4,v_5,v_8,v_9,\dots,v_{4k},v_{4k+1},v'_3,v'_4,v'_7,v'_8,\dots,v'_{4k-5},v'_{4k-4},v'_{4k-1}$
$r=4k+2$	$t=0$	$v_0,v_1,v_5,v_6,v_9,v_{10},\dots,v_{4k+1},v_{4k+2},v'_5,v'_6,v'_9,v'_{10},\dots,v'_{4k+1},v'_{4k+2}$
$r=4k+2$	$t=1$	$v_0,v_1,v_4,v_5,v_8,v_9,\dots,v_{4k},v_{4k+1},v'_4,v'_5,v'_8,v'_9,\dots,v'_{4k},v'_{4k+1}$
$r=4k+2$	$t=2$	$v_0,v_1,v_5,v_6,v_9,v_{10},\dots,v_{4k+1},v_{4k+2},v'_3,v'_4,v'_7,v'_8,\dots,v'_{4k-1},v'_{4k}$
$r=4k+2$	$t=3$	$v_1,v_2,v_5,v_6,v_9,v_{10},\dots,v_{4k+1},v_{4k+2},v'_2,v'_3,v'_6,v'_7,\dots,v'_{4k-2},v'_{4k-1}$

Subclaim 8.3.9. *$|C|=r$, and if $v_i,v_j,v'_i,v'_j\in C$, then $d(v_i,v_j),d(v'_i,v'_j)\neq 2$.*

Proof of subclaim. None of the collections above contains both v_1 and v'_1. For all other pairs v_i,v'_j, it follows from Observation 8.2.4 that $v_i\neq v'_j$. Hence, C consists of r distinct vertices.

Note that $v_0,\dots,v_r$ is a path of length r and $v'_0,\dots,v'_{r-t}$ is an induced path of length $r-t$. Hence, C contains no two vertices of the form v_i,v_j such that $d(v_i,v_j)=2$ and no two vertices of the form v'_i,v'_j such that $d(v'_i,v'_j)=2$. □

Subclaim 8.3.10. *If $v_i,v'_j\in C$, then $d(v_i,v'_j)\neq 2$.*

Proof of subclaim. If $|i-j|>2$, the claim follows from Observation 8.2.4. Henceforth assume $|i-j|\leqslant 2$.

Case. $i=1$.

It follows from Observation 8.2.3 that it suffices to ensure that if our collection contains v_1, then it does not contain:

- v'_1 in the case $v_1\neq v'_1$;
- v'_2 in the case $d(v_3,v'_{r-t})+t\leqslant r+2$;
- v'_3 in the case $d(v_3,v'_{r-t})+t\leqslant r+1$.

Recall that if $t=0$, then $d(v_3,v'_{r-t})\geqslant r+1$. Hence, we easily verify that C satisfies these conditions.

Case. $i=2$.

It follows from Observation 8.2.3 that it suffices to ensure that if our collection contains v_2, then it does not contain:

- v'_1 in the case $d(v_3, v'_{r-t}) + t \leqslant r + 2$;
- v'_2 in the case $d(v_3, v'_{r-t}) + t \leqslant r + 1$.

We again can verify easily that all of the collections above satisfy these conditions.

Case. $i \geqslant 3$.

Note that by our choice of v_0, v'_{r-t}, we always have either $t \geqslant 1$ or $d(v_3, v'_{r-t}) \geqslant r+1$. If $j \geqslant i-1$, it follows from Observation 8.2.3 that $d(v_i, v'_j) \geqslant 3$. If $j = i-2$, $d(v_i, v'_j) \geqslant 3$ follows from Observation 8.2.3 under additional assumption that $d(v_3, v'_{r-t}) + t \geqslant r+2$. Hence, for $i \geqslant 3$, it is enough if C does not contain both v_i and v'_{i-2} in the case when we have $d(v_3, v'_{r-t}) + t \leqslant r+1$. But it is easy to check this condition is satisfied for all of the collections above. □

The claim follows. □

Claim 8.3.11. *If $r = 4k+1$ and $t = 3$, we have $n \geqslant 2\left\lceil \frac{r\delta}{2} \right\rceil$.*

Proof of Claim 8.3.11. We let v''_{r-s} be such a vertex that $d(v_1, v''_{r-s}) \geqslant r$, then $d(v_0, v''_{r-s}) = r - s$ for some $0 \leqslant s \leqslant 1$. We consider two cases based on the value of $d(v''_{r-s}, v'_{4k-2})$.

Case. $d(v''_{r-s}, v'_{4k-2}) \geqslant 3$.

Let

$$T = \left\{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v'_1, v'_2, v'_5, v'_6, \ldots, v'_{4k-3}, v'_{4k-2}, v''_{r-s}\right\}.$$

Assume for a contradiction two vertices of T have distance 2. It follows from Observation 8.2.3 that one of them has to be v''_{r-s}. Since for any $v, w \in V(G)$, we have

$$d(v,w) \geqslant |d(v,v_0) - d(w,v_0)|$$

and as also $d(v''_{r-s}, v'_{4k-2}) \geqslant 3$ by assumption, it further follows that the other vertex would have to be v_{4k-2} or v_{4k-1}. Note that if $d(v_i, v''_{r-s}) \leqslant 2$ for some $1 \leqslant i \leqslant 4k-1$, then

$$d(v_1, v''_{r-s}) \leqslant d(v_1, v_i) + d(v_i, v''_{r-s}) \leqslant (4k-2) + 2 < r$$

yielding a desired contradiction. Hence, no two vertices of T have distance 2 while $|T| = r$. The result then follows from Lemma 8.3.5.

Case. $d(v''_{r-s}, v'_{4k-2}) < 3$.

Since

$$d(v''_{r-s}, v'_{4k-2}) \geqslant |d(v''_{r-s}, v_0) - d(v'_{4k-2}, v_0)| \geqslant 3 - s \geqslant 2,$$

this means $s = 1$ and $d(v''_{r-1}, v'_{4k-2}) = 2$. Hence there exists a vertex a, such that a is neighbour of both v''_{r-1} and v'_{4k-2}. Moreover, clearly $d(a, v_0) = r - 2$.

Consider two cases based on the value of $d(a, v_{4k+1})$.

Subcase. $d(a, v_{4k+1}) \geqslant 3$.

Take

$$T = \{v_1, v_2, v_5, v_6, \dots, v_{4k-3}, v_{4k-2}, v_{4k+1}, v'_2, v'_3, v'_6, v'_7, \dots v'_{4k-6}, v'_{4k-5}, v'_{4k-2}, a\}.$$

Assume for a contradiction two vertices of T are at distance 2. It follows from Observation 8.2.3 one of them has to be a. Since for any v, w in G, we have

$$d(v, w) \geqslant |d(v, v_0) - d(w, v_0)|$$

as well as $d(a, v_{4k+1}) \geqslant 3$ and $d(a, v'_{4k-2}) = 1$, the other vertex has to be v_{4k-3} or v_{4k-2}. Note that if $d(a, v_i) \leqslant 2$ for some $3 \leqslant i \leqslant 4k-2$, then

$$d(v_3, v'_{4k-2}) \leqslant d(v_3, v_i) + d(v_i, a) + d(a, v_{4k-2}) \leqslant (4k-5) + 2 + 1 < r,$$

a contradiction. Hence, no two vertices of T are at distance 2 and $|T| = r$. The result follows from Lemma 8.3.5.

Subcase. $d(a, v_{4k+1}) < 3$.

By the triangle inequality, we have

$$d(a, v_{4k+1}) \geqslant |d(a, v_0) - d(v_0, v_{4k+1})| = 2,$$

so that $d(a, v_{4k+1}) = 2$. Hence, there exists a vertex b such that b is neighbour of both a and v_{4k+1}. Consider

$$U = \{v_0, v_1, v_2, v_3, \dots, v_{4k+1}, v'_1, v'_2, \dots, v'_{4k-2}, a, b\}.$$

We have $|U| = 8k + 2 = 2r$. Consider auxiliary graph H on $V(H) = U$ in which we connect two vertices if their distance in G is precisely 2. H is union of two disjoint cycles of length r, first being $v_0, v_2, \dots, v_{4k}, b, v'_{4k-2}, \dots, v'_2$, and second being $v_1, v_3, \dots, v_{4k+1}, a, v'_{4k-3}, \dots, v'_1$.

The result then follows from Lemma 8.3.6. The only non trivial relationships needed to prove that H is union of two disjoint cycles of length r are

$$d(b,v_{4k-1}),d(b,v_{4k-2}),d(a,v_{4k}),d(a,v_{4k-1}),d(a,v_{4k-2}),d(a,v_{4k-3})\geqslant 3.$$

If any of these distances was at most 2, we could find a path of length at most $r-1$ from v_3 to v'_{4k-2}. That would be a contradiction. □

Putting Claim 8.3.8 and Claim 8.3.11 together now finishes the proof of Lemma 8.3.7. □

8.3.5 Proof of Lemma 8.3.4 for $r=4k+3$

In this subsection, we prove the following.

Lemma 8.3.12. *If G is a connected triangle-free graph on n vertices with minimum degree $\delta\geqslant 2$ and radius $r\geqslant 4$ such that $r=4k+3$ for some k, then we have $n\geqslant 2\lceil\frac{r\delta}{2}\rceil$.*

We use a slightly weaker and more general set-up than we did in the proof of Lemma 8.3.7. This will have the advantage that we have more freedom in our choice of a centre v_0 as well as in the choice of v'_{r-t}.

Proof. Take v_0 to be any centre of our graph G. Let v_r be any vertex such that $d(v_0,v_r)=r$. Let $v_0,v_1,\dots,v_r$ be any path of length r from v_0 to v_r. Let v'_{r-t} be any vertex such that $d(v_3,v'_{r-t})\geqslant r$. Then we have $d(v_0,v'_{r-t})=r-t$ for some $t\geqslant 0$. Let $v_0=v'_0,v'_1,\dots,v'_{r-t}$ be a path of length $r-t$ from v_0 to v'_{r-t}. By Observation 8.2.2, we have $t\leqslant 3$. Moreover, consider a vertex v''_{r-s} such that $d(v_4,v''_{r-s})\geqslant r$.

As before, we have $d(v_0,v''_{r-s})=r-s$ for some $s\geqslant 0$. Let $v_0=v''_0,v''_1,\dots,v''_{r-s}$ be a path of length $r-s$ form v_0 to v''_{r-s}. By Observation 8.2.2, we have $s\leqslant 4$.

We will consider four cases depending on the value of t.

Case. $t=s=0$.

Let

$$T=\left\{v_0,v_3,v'_3,v_6,v''_6,v_7,v''_7,v_{10},v''_{10},v_{11},v''_{11},\dots,v_{r-5},v''_{r-5},v_{r-4},v''_{r-4},v_{r-1},v''_{r-1},v_r,v''_r\right\}.$$

By Observation 8.2.3, no two vertices in T have distance 2. The result follows from Lemma 8.3.5.

Case. $t=0$, $1\leqslant s\leqslant 4$.

We claim that we can find four vertices z_1,z_2,z_3,z_4 such that no two out of z_1,z_2,z_3,z_4 have distance 2, and for $i = 1,2,3,4$;

$$r-4 \leqslant d(v_0,z_i) \leqslant r-3.$$

Set $z_1 = v_{r-4}$, $z_2 = v_{r-3}$, and $z_3 = v''_{r-4}$. By Observation 8.2.3, we immediately see $d(v_{r-4},v''_{r-4}) \geqslant 5$, and $d(v_{r-3},v''_{r-4}) \geqslant 4$. If we have a vertex x such that x is neighbour of v''_{r-4} and $d(v_0,x) \geqslant r-4$, we can set $z_4 = x$ and are done. If on the other hand there exists no such x, that implies $d(v'_{r-3},v''_{r-4}) \geqslant 3$. By Observation 8.2.3 we have $d(v_{r-4},v'_{r-3}) \geqslant 4$, $d(v_{r-3},v'_{r-3}) \geqslant 3$, so we can set $z_4 = v'_{r-3}$. Hence, we can always find suitable z_1,z_2,z_3,z_4.

Let

$$T = \left\{v_0,v_3,v''_3,v_4,v''_4,v_7,v''_7,v_8,v''_8,\dots,v_{r-8},v''_{r-8},v_{r-7},v''_{r-7},z_1,z_2,z_3,z_4,v_r,v'_r\right\}.$$

It follows from Observation 8.2.3 that no two vertices in T have distance 2. The result follows from Lemma 8.3.5.

Case. $t = 2$.

Let

$$T = \left\{v_0,v_3,v_4,v_7,v_8,\dots,v_{4k-1},v_{4k},v_{4k+3},v'_1,v'_4,v'_5,v'_8,v'_9,\dots,v'_{4k}.v'_{4k+1}\right\}.$$

It follows from Observation 8.2.3 that no two vertices in T have distance 2. The result follows from Lemma 8.3.5.

Case. $t = 3$.

Let w_{r-u} be so that $d(v_1,w_{r-u}) \geqslant r$ and $d(v_0,w_{r-u}) = r-u$ for some $0 \leqslant u \leqslant 1$. We consider subcases based on the value of $d(w_{r-u},v'_{4k})$.

Subcase. $d(w_{r-u},v'_{4k}) \geqslant 3$.

Let

$$T = \left\{v_0,v_1,v_4,v_5,\dots,v_{4k},v_{4k+1},v'_3,v'_4,v'_7,v'_8,\dots,v'_{4k-1},v'_{4k},w_{r-u}\right\}.$$

Assume for a contradiction that two vertices of T are at distance 2. It follows from Observation 8.2.3 that one of them has to be w_{r-u}. Since for any v,w in G, we have

$$d(v,w) \geqslant |d(v,v_0)-d(w,v_0)|, \qquad \text{and} \qquad d(w_{r-u},v'_{4k}) \geqslant 3,$$

it further follows that the other vertex would have to be v_{4k} or v_{4k+1}. If we had $d(v_i, w_{r-u}) \leqslant 2$ for some $1 \leqslant i \leqslant 4k+1$, then

$$d(v_1, w_{r-u}) \leqslant d(v_1, v_i) + d(v_i, w_{r-u}) \leqslant 4k+2 < r,$$

a contradiction. Hence, no two vertices of T are at distance 2, and $|T| = r$. The result follows from Lemma 8.3.5.

Subcase. $d(w_{r-u}, v'_{4k}) < 3$.

Since

$$d(w_{r-u}, v'_{4k}) \geqslant |d(w_{r-u}, v_0) - d(v'_{4k}, v_0)| \geqslant 3 - u \geqslant 2,$$

we have $u = 1$ and $d(w_{r-1}, v'_{4k}) = 2$. Hence there exists a vertex a such that a is neighbour of both w_{r-1} and v'_{4k}. Moreover, clearly $d(a, v_0) = r-2$.

Consider two cases based on the value of $d(a, v_{4k+3})$.

Subsubcase. $d(a, v_{4k+3}) \geqslant 3$

Let

$$T = \left\{v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v_{4k+3}, v'_1, v'_4, v'_5, v'_8, v'_9, \ldots, v'_{4k-4}, v'_{4k-3}, v'_{4k}, a\right\}.$$

Assume for a contradiction that two vertices of T are at distance 2. It follows from Observation 8.2.3 that one of them has to be a. Since for any v, w in G, we have

$$d(v,w) \geqslant |d(v,v_0) - d(w,v_0)|, \qquad d(a, v_{4k+3}) \geqslant 3, \qquad \text{and} \qquad d(a, v'_{4k}) = 1,$$

the other has to be v_{4k-1} or v_{4k}. Since $d(a, v_i) \leqslant 2$ for some $3 \leqslant i \leqslant 4k$, we find

$$d(v_3, v'_{4k}) \leqslant d(v_3, v_i) + d(v_i, a) + d(a, v_{4k}) \leqslant (4k-3) + 2 + 1 < r,$$

a contradiction. Hence, no two vertices of T are at distance 2 while $|T| = r$. The result follows from Lemma 8.3.5.

Subsubcase. $d(a, v_{4k+3}) < 3$.

By the triangle inequality, we have $d(a, v_{4k+3}) \geqslant 2$, so that $d(a, v_{4k+3}) = 2$. Hence, there exists a vertex b, such that b is neighbour of both a and v_r. Consider

$$U = \left\{v_0, v_1, v_2, v_3, \ldots, v_{4k+3}, v'_1, v'_2, \ldots, v'_{4k}, a, b\right\}.$$

We have $|U| = 8k+6 = 2r$. Consider the auxiliary graph H on $V(H) = U$ in which two vertices are connected if their distance in G is precisely 2. H is the union of two disjoint cycles of length r, the first being $v_0, v_2, \dots, v_{4k+2}, b, v'_{4k}, \dots, v'_2$, and the second being $v_1, v_3, \dots, v_{4k+3}, a, v'_{4k-1}, \dots, v'_1$. Indeed, the only non trivial relationships needed to prove that H is the union of two disjoint cycles of length r are

$$d(b, v_{4k+1}), d(b, v_{4k}), d(a, v_{4k+2}), d(a, v_{4k+1}), d(a, v_{4k}), d(a, v_{4k-1}) \geqslant 3.$$

If any of these distances was at most 2, we could find a path of length at most $r-1$ from v_3 to v'_{4k}. The result follows from Lemma 8.3.6. This concludes the case $t = 3$.

Case. $t = 1$.

We start with the following useful claim.

Claim 8.3.13. *Assume $r \geqslant 4$, $r = 4k+3$ and $t = 1$. Further assume there are four distinct vertices y_1, y_2, y_3, y_4 such that no two out of them have distance 2, and $d(v_0, y_i) \leqslant 3$ for $i = 1,2,3,4$. Then we have $n \geqslant 2\lceil \frac{r\delta}{2} \rceil$.*

Proof of Claim 8.3.13. Let

$$T = \left\{y_1, y_2, y_3, y_4, v_6, v_7, v_{10}, v_{11}, \dots, v_{4k+2}, v_{4k+3}, v'_6, v'_7, v'_{10}, v'_{11}, \dots, v'_{4k-2}, v'_{4k-1}, v'_{4k+2}\right\}.$$

It follows by Observation 8.2.3 that no two vertices of T have distance 2. Moreover, we have $|T| = r$. The result follows by Lemma 8.3.5. □

We return to the proof of Lemma 8.3.12 in the case $t = 1$.

Subcase. v_2 *is not a centre of* G.

There exists a vertex c such that $d(v_2, c) \geqslant r+1$, and by the triangle inequality $d(v_0, c) \geqslant r-1$ and $d(v_3, c) \geqslant r$. We consider two cases: if $d(v_0, c) = r$, we could have chosen c in place of v'_{r-t} (as $d(v_3, c) \geqslant r$) and pass to a case $t = 0$ which we already solved. If, on the other hand, $d(v_0, c) = r-1$, then let $v_0 = v'''_0, v'''_1, \dots, v'''_{r-1} = c$ be a path of length $r-1$ from v_0 to c. No two out of v_3, v_2, v'''_3, v'''_2 can have distance 2 by Observation 8.2.3, using that $d(v_2, c) \geqslant r+1$. Hence, we conclude by using Claim 8.3.13 for $y_1 = v_3$, $y_2 = v_2$, $y_3 = v'''_3$, $y_4 = v'''_2$.

Subcase. v_2 *is a centre of* G.

The vertices v_0, v_1, v_4, v_5 all have distance at most three to v_2 and no two have distance 2. Now start the proof again with $v^{\ddagger}_0 := v_2$ instead of v_0 (choosing some vertices $v^{\ddagger}_r$ and $(v'_{r-t^{\ddagger}})^{\ddagger}$

in place of v_r, and v'_{r-t}). If $t^\ddagger \neq 1$, then the conclusion follows as before. If $t^\ddagger = 1$, then we can find four distinct vertices

$$y_1 = v_0, \qquad y_2 = v_1, \qquad y_3 = v_4, \qquad y_4 = v_5$$

such that no two out of them have distance 2, and $d(v_2, y_i) \leqslant 3$ for $i = 1,2,3,4$. We conclude with Claim 8.3.13.

This finishes the proof of Lemma 8.3.12. $\square$

8.4 General problem for girth $g \geqslant 5$

We first establish Theorem 8.1.2 using Lemma 8.2.1.

Proof of Theorem 8.1.2. We will find a large enough collection of vertices T such that no two non-adjacent vertices of T are at distance less than $2k-1$. The result then follows by Lemma 8.2.1.

Let v_0 be a centre of G, v_r a vertex with $d(v_0, v_r) = r$, and $v_0, v_1, \dots, v_{r-1}, v_r$ a path of length r in G from v_0 to v_r. If $r \leqslant 2k$, we know the inequality holds, so assume $r \geqslant 2k$. We let v'_{r-t} be a vertex such that $d(v_{2k}, v'_{r-t}) \geqslant r$ and denote $d(v_0, v'_{r-t}) = r - t$ for some $0 \leqslant t \leqslant 2k$. Further, let $v_0 = v'_0, v'_1, \dots, v'_{r-t}$ be a path of length $r-t$ from v_0 to v'_{r-t}.

Let

$$\begin{aligned} T = &\left\{ v_{2ki} \mid 0 \leqslant i \leqslant \left\lfloor \frac{r}{2k} \right\rfloor \right\} \cup \left\{ v_{2ki+1} \mid 0 \leqslant i \leqslant \left\lfloor \frac{r}{2k} \right\rfloor - 1 \right\} \\ &\cup \left\{ v'_{2ki} \mid 1 \leqslant i \leqslant \left\lfloor \frac{r}{2k} \right\rfloor - 1 \right\} \cup \left\{ v'_{2ki+1} \mid 1 \leqslant i \leqslant \left\lfloor \frac{r}{2k} \right\rfloor - 2 \right\}. \end{aligned}$$

It follows from Observation 8.2.4 that the above is a disjoint union. It follows from Observation 8.2.3 that no two non-adjacent vertices of T are at distance less than $2k-1$. Hence, we conclude by Lemma 8.2.1 that

$$n \geqslant |T| \left(\frac{(\delta-1)^k - 1}{\delta - 2} \right) \geqslant \left(\frac{2r}{k} - 6 \right) \left(\frac{(\delta-1)^k - 1}{\delta - 2} \right).$$

$\square$

We prove the next lemma using an idea similar to one of Erdős, Pollack, Pach and Tuza [36]. Its most important corollary is Theorem 8.1.3.

Lemma 8.4.1. *Denote by $f(g,\delta)$ for $\delta \geqslant 2$, $g \geqslant 3$ the minimum number of vertices in the graph of girth at least g and minimum degree δ. Then for any $r > \frac{g}{2}$, there exists a connected graph G on $n = \lceil \frac{2r}{g} \rceil f(g,\delta)$ vertices of girth at least g, minimum degree δ and radius at least r.*

Proof. Let H be a connected graph with $f(g,\delta)$ vertices, minimum degree δ and girth at least g. As $\delta > 1$, we know H contains a cycle. Let v, w be two neighbouring vertices of H such that the edge vw is part of a cycle. Let H' be the (still connected) graph obtained by deleting the edge vw from H. By the girth condition, we have $d_{H'}(v,w) \geqslant g-1$.

Take $\lceil \frac{2r}{g} \rceil$ identical disjoint copies of H', called $H'_1, \ldots, H'_{\lceil \frac{2r}{g} \rceil}$, with vertices $v_1, \ldots, v_{\lceil \frac{2r}{g} \rceil}$ and $w_1, \ldots, w_{\lceil \frac{2r}{g} \rceil}$, and connect v_i to w_{i+1}, where $w_{\lceil \frac{2r}{g} \rceil + 1} = w_1$. The resulting graph has $\lceil \frac{2r}{g} \rceil f(g,\delta)$ vertices, radius at least r, girth at least g, and minimum degree δ. □

Theorem 8.1.3 follows easily.

Proof of Theorem 8.1.3. We know (see [37]) that when $\delta - 1$ is a prime power, then

$$\begin{aligned} f(6,\delta) &\leqslant 2(\delta^2 - \delta + 1), \\ f(8,\delta) &\leqslant 2(\delta^3 - 2\delta^2 + 2\delta), \\ f(12,\delta) &\leqslant 2((\delta-1)^3 + 1)(\delta^2 - \delta + 1). \end{aligned}$$

Hence the result follows directly from Lemma 8.4.1 by taking

$$n_i = \left\lceil \frac{i}{3} \right\rceil f(6,\delta), \qquad n'_i = \left\lceil \frac{i}{4} \right\rceil f(8,\delta), \qquad n''_i = \left\lceil \frac{i}{6} \right\rceil f(12,\delta).$$

□

Finally, we prove Theorem 8.1.5.

Proof of Theorem 8.1.5. Let v_0 be a centre of our graph, v_r a vertex with $d(v_0, v_r) = r$ and $v_0, \ldots, v_r$ a path of length r.

For $0 \leqslant i \leqslant r$, let

$$Q(v_i) := \{v \in V(G) : d(v, v_i) \leqslant k\}.$$

Every vertex in our graph is in at most $2k+1$ of these sets, so in particular there is an i_0 so that

$$|Q(v_{i_0})| \leqslant (2k+1)c\delta^{k-1}.$$

We easily find using the girth condition that v_{i_0} has at least $\delta(\delta-1)^{k-2}$ vertices at distance at most $k-1$ from it. Hence, as all edges adjacent to these vertices are inside $Q(v_{i_0})$, we get that the subgraph induced by $Q(v_{i_0})$ has at least $\frac{1}{2}\delta^2(\delta-1)^{k-2}$ edges.

We conclude that the subgraph induced by $Q(v_{i_0})$ is a connected graph of girth at least $2k$ on at most $(2k+1)c\delta^{k-1}$ vertices with at least $\frac{1}{2}\delta^2(\delta-1)^{k-2}$ edges. $\square$

Chapter 9

Induced saturation for paths

The results in this chapter were published in The Electronic Journal of Combinatorics [24].

9.1 Introduction

Given graphs G,H, we say that G is *H-saturated* if G contains no subgraph isomorphic to H, but adding any edge from G^c to G creates a subgraph isomorphic to H (where by G^c, we mean a graph with the same vertex set as G, and an edge ab present in G^c if and only if it is not present in G for all $a,b \in V(G)$). Related problems have been extensively studied, see for instance a survey of Faudree, Faudree and Schmitt [38].

In 2012, Martin and Smith [57] introduced the notion of induced saturation on trigraphs. As a special case of this more general framework, there arises the notion of induced-saturated graphs, first studied in its own right by Behrens et al. [5] and later also by Axenovich and Csikós [1]. Given graphs G,H, we say G is *H-induced-saturated* if G contains no induced subgraph isomorphic to H, but deleting any edge of G creates an induced subgraph isomorphic to H, and adding any new edge to G from G^c also creates an induced subgraph isomorphic to H. Throughout, we will abbreviate a H-induced-saturated graph as a H-IS graph.

While for any graph H, there exist H-saturated graphs, the same is not true for H-IS graphs. Indeed, for instance for a path on 4 vertices P_4, Martin and Smith [57] showed that there exists no P_4-IS graph.

On the other hand, it is easy to see that there do exist P_2-IS and P_3-IS graphs. This leads to a question, asked by Axenovich and Csikós [1], for what integers $n \geqslant 5$ do there exist P_n-IS graphs. Räty [66] was the first to make a progress on this question, showing by an algebraic construction that there exists a P_6-IS graph. Cho, Choi and Park [13] later showed that in fact for any $k \geqslant 2$, there exists a P_{3k}-IS graph. We use a different construction to settle the question completely, with the exception of the case $n = 5$.

Theorem 9.1.1. *For each $n \geqslant 6$, there is a P_n-induced-saturated graph.*

As it turns out, Bonamy, Groenland, Johnston, Morrison and Scott [8] found a P_5-IS graph by a computer search. Hence, the original question of Axenovich and Csikós [1] is now fully answered.

9.2 Construction

We will construct, for each $n \geqslant 6$, a P_n-IS graph G_n. Our construction has been inspired by the observation of Cho, Choi and Park [13] that the Petersen graph is P_6-IS. We let

$$V(G_n) = \{v_1, \ldots, v_{n-1}, w_1, \ldots, w_{n-1}\}$$

Further, the edge set $E(G_n)$ of G_n is defined as follows. For $1 \leqslant i, j \leqslant n-1$, we have $v_i w_j \in E(G_n)$ if and only if $i = j$. For $1 \leqslant i, j \leqslant n-1$, we have $v_i v_j \in E(G_n)$ if and only if $i - j \equiv \pm 1 \pmod{n-1}$. And finally for $1 \leqslant i, j \leqslant n-1$, we have $w_i w_j \in E(G_n)$ if and only if $i \neq j$ and $i - j \not\equiv \pm 1 \pmod{n-1}$.

Note that the graph G_6 is isomorphic to the Petersen graph. A labelled graph G_7 is illustrated in the Figure 9.1 below, and (unlabelled) graphs G_6, G_7, G_8 are illustrated in the Figure 9.2 below.

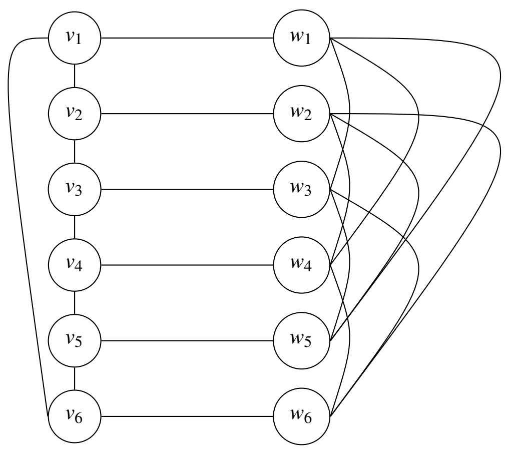

Fig. 9.1 Labelled graph G_7.

In the rest of the chapter, we will prove that for each $n \geqslant 6$, G_n is P_n-IS, by checking the three properties that we need by the definition of an induced saturation.

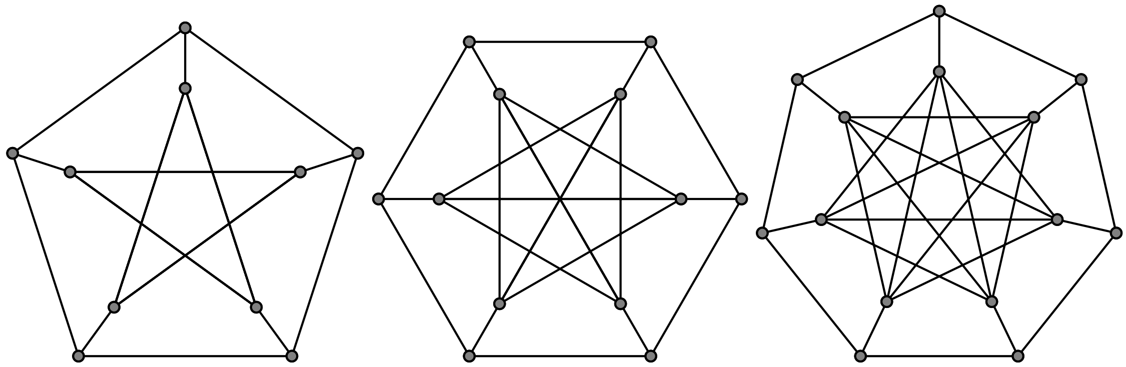

Fig. 9.2 Graphs G_6, G_7, G_8.

9.3 Proof that the construction works

We divide the proof into three claims.

Claim 9.3.1. *For each $n \geqslant 6$, G_n contains no induced copy of P_n.*

Proof. For $n = 6$, the result is easy to check by hand. So throughout rest of the proof assume that $n \geqslant 7$. Also assume for contradiction that we have an induced copy of P_n in G_n.

First we claim that, since $n \geqslant 7$, among any five mutually disjoint vertices of the form w_i, w_j, w_k, w_l, w_m for some $1 \leqslant i < j < k < l < m \leqslant n-1$, some three form a triangle K_3 in G_n. To see that, note that G_n necessarily contains at least one of the edges $w_iw_j, w_jw_k, w_kw_l, w_lw_m, w_mw_i$ and due to the symmetry, we may without loss of generality assume that G_n contains an edge w_iw_j. But then $w_iw_jw_l$ forms a triangle.

Write W for $\{w_1, ..., w_{n-1}\} \subset V(G_n)$. Since P_n is acyclic, we must have at most four vertices from W in our induced copy of P_n. We also must have at least one vertex from W in our induced copy of P_n, since $|V(G_n) \setminus W| = n-1 < n = |V(P_n)|$. We will consider four cases depending on the number of vertices of W in our induced copy of P_n.

If we have one vertex from W in our induced copy of P_n, then we know our induced copy contains all of the vertices $v_1, ..., v_{n-1}$, but these form a cycle, which gives a contradiction.

If we have two vertices from W in our induced copy of P_n, we may without loss of generality assume that our induced copy contains all of the vertices $v_1, ..., v_{n-2}$, but not the vertex v_{n-1}. Since P_n contains no vertex of degree more than two, we know our copy of P_n can not contain any of the vertices $w_2, ..., w_{n-3}$. But looking at all three two-element subsets of the set $\{w_{n-2}, w_{n-1}, w_1\}$, we see that adding none of these subsets to the set $\{v_1, v_2, ..., v_{n-2}\}$ will create an induced copy of P_n.

Next assume we have three vertices w_i, w_j, w_k from W in our induced copy of P_n. Since P_n is acyclic, we know we must have at least one of the relations $i - j \equiv \pm 1 \pmod{n-1}, i - k \equiv$

$\pm 1 \pmod{n-1}, j-k \equiv \pm 1 \pmod{n-1}$ to hold, else w_i, w_j, w_k would form a triangle. Consider two subcases depending on if one or two of the relations above hold (since $n > 4$, we know all three can not hold simulateneously).

If two of the relations above hold, we may without loss of generality assume that we have precisely the vertices w_1, w_2, w_3 from W in our induced copy of P_n. Then note that we must have v_2 in our copy of P_n too, else the degree of w_2 in this copy would be zero. Also, w_2 has degree one in our copy of P_n, hence it forms one of the endpoints of P_n. Since P_n is connected, our copy of it must also contain one of the vertices v_1 or v_3, due to the symmetry we may without loss of generality assume it contains v_1. But then it can not contain v_3, else it would contain a cycle $v_1v_2v_3w_3w_1$, hence w_3 also has degree one in our copy of P_n and it forms another of the endpoints of P_n. But then we conclude $n \leqslant 5$, since distance of the endpoints of P_n in our copy of it is at most four as we have a path $w_3w_1v_1v_2w_2$ connecting them, giving us a desired contradiction.

If just one of the relations above holds, we may without loss of generality assume that we have precisely the vertices w_1, w_2, w_j for some j such that $4 \leqslant j \leqslant n-2$ from W in our induced copy of P_n. We can not have v_j in our copy, else w_j would have degree three in the copy. As P_n is connected and $n > 3$, we must have either v_1 or v_2 in our copy, and we can not have both, as then it would contain a cycle $v_1v_2w_2w_jw_1$. If we have v_1 but not v_2 in our copy of P_n, we can easily see that as P_n is connected and $j \geqslant 4$, it can contain none of the vertices $v_2, v_3, ..., v_j$, and hence contains at most $n-1$ vertices, giving us a contradiction. If we have v_2 but not v_1 in our copy, we conclude analogously by noting our copy of P_n contains none of the vertices $v_1, v_{n-1}, ..., v_j$ and $j \leqslant n-2$.

Finally assume we have four vertices w_i, w_j, w_k, w_l from W in our induced copy of P_n. If one of these four vertices is connected to all of the others, we must have a triangle in our copy of P_n (since at least one of the three pairs of the other three vertices is connected too) and reach a contradiction. So due to this observation and the symmetry, it is enough to consider configurations w_1, w_2, w_l, w_{l+1} where $3 \leqslant l \leqslant n-2$.

First consider the case $l = 3$ (the case $l = n-2$ is analogous). In that case, we can not have v_1 or v_4 included in our copy of P_n, since that would mean degree of w_1 or w_4 respectively in the copy would be at least three. But then as P_n is connected, none of the vertices $v_4, v_5, ..., v_{n-2}, v_{n-1}, v_1$ can be in the copy, so our path P_n has at most six vertices and hence $n \leqslant 6$, which is a contradiction.

Finally consider the case $3 < l < n-2$. In this case $w_1w_lw_2w_{l+1}$ is a cycle, contradicting that P_n is acyclic. $\square$

Claim 9.3.2. *For each $n \geqslant 6$, deleting any edge of G_n creates an induced copy of P_n.*

Proof. The edge we delete can be one of three types: v_iv_j, v_iw_i or w_iw_j for some $1 \leqslant i, j \leqslant n-1$; we consider these cases separately.

First assume we delete an edge of the form v_iv_j. Then we must have $i-j \equiv \pm 1 \pmod{n-1}$ and due to the symmetry, we may without loss of generality assume that the edge we deleted was v_1v_{n-1}.

Then for

$$S_1 = \{w_1\} \cup \{v_i : 1 \leqslant i \leqslant n-1\},$$

$G_n[S_1]$ is isomorphic to P_n.

Next assume we delete an edge of the form v_iw_i. Then due to the symmetry, we may without loss of generality assume that the edge we deleted was v_1w_1.

Then for

$$S_2 = \{w_1, w_{n-2}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\},$$

$G_n[S_2]$ is isomorphic to P_n.

Finally assume we delete an edge of the form w_iw_j for some i, j such that $i \neq j$, $i-j \not\equiv \pm 1 \pmod{n-1}$. Due to the symmetry, we may without loss of generality assume that the edge we deleted was w_1w_j for some j such that $3 \leqslant j \leqslant n-2$.

Then if $3 < j < n-2$, for

$$S_3 = \{w_1, w_{j-1}, w_j, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant j-2\} \cup \{v_i : j \leqslant i \leqslant n-3\},$$

$G_n[S_3]$ is isomorphic to P_n, if $j = 3$, for

$$S_3' = \{w_1, w_3\} \cup \{v_1\} \cup \{v_i : 3 \leqslant i \leqslant n-1\},$$

$G_n[S_3']$ is isomorphic to P_n, and if $j = n-2$, for

$$S_3'' = \{w_1, w_{n-2}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\},$$

$G_n[S_3'']$ is isomorphic to P_n. □

Claim 9.3.3. *For each $n \geqslant 6$, adding any edge of G_n^c to G_n creates an induced copy of P_n.*

Proof. The edge we add can be one of three types: v_iv_j, v_iw_j or w_iw_j for some $1 \leqslant i, j \leqslant n-1$; we consider these cases separately.

First assume we add an edge of the form w_iw_j. Then we must have $i-j \equiv \pm 1 \pmod{n-1}$ and due to the symmetry, we may without loss of generality assume that the edge we added was w_1w_{n-1}.

Then for

$$T_1 = \{w_1, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\},$$

$G_n[T_1]$ is isomorphic to P_n.

Next assume we add an edge of the form $v_i v_j$ for some i, j such that $i \neq j$, $i - j \not\equiv \pm 1 \pmod{n-1}$. Due to the symmetry, we may without loss of generality assume that the edge we added was $v_1 v_j$ for some j such that $3 \leqslant j \leqslant n-2$.

Then if $3 < j \leqslant n-2$, for

$$T_2 = \{w_{j-2}, w_{j-1}, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant j-2\} \cup \{v_i : j \leqslant i \leqslant n-2\},$$

$G_n[T_2]$ is isomorphic to P_n, while if $j = 3$, for

$$T_2' = \{w_2, w_{n-2}, w_{n-1}\} \cup \{v_1\} \cup \{v_i : 3 \leqslant i \leqslant n-2\},$$

$G_n[T_2']$ is isomorphic to P_n.

Finally assume we add an edge of the form $v_i w_j$ for some $i \neq j$. Due to the symmetry, we may without loss of generality assume that the edge we added was $v_1 w_j$ for some j such that $2 \leqslant j \leqslant n-1$.

Then if $2 \leqslant j \leqslant n-3$, for

$$T_3 = \{w_{j-1}, w_j, w_{j+1}\} \cup \{v_i : 1 \leqslant i \leqslant j-1\} \cup \{v_i : j+1 \leqslant i \leqslant n-2\},$$

$G_n[T_3]$ is isomorphic to P_n, if $j = n-2$, for

$$T_3' = \{w_{n-3}, w_{n-2}, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant n-3\},$$

$G_n[T_3']$ is isomorphic to P_n and if $j = n-1$, for

$$T_3'' = \{w_{n-2}, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\},$$

$G_n[T_3'']$ is isomorphic to P_n. □

This now completes the proof that the construction indeed works. □

9.4 Concluding remarks

Firstly, let us note that for $n = 5$, our construction would give a graph that does contain an induced copy of P_5. So the only examples of P_5-IS graphs known so far are the ones found by computer search [8]. It would be interesting to find a more motivated example of P_5-IS graph.

Further, we know there exists no P_4-IS graph. It is a natural question to ask for other graphs H with the property that there exists no H-IS graph. Clearly, cliques and empty graphs have this property, but we are not aware of any more interesting examples. In particular, it would be interesting to see whether with high probability a random graph has such a property or not.

Chapter 10

Norine's antipodal-colouring conjecture

The results in this chapter were published in The Electronic Journal of Combinatorics [23].

10.1 Introduction

The hypercube graph Q_n has vertex set $\{0,1\}^n$, with two vertices joined by an edge if they differ in a single coordinate. We call two vertices of Q_n antipodal if their graph distance is n. We call a pair of edges of Q_n v_1w_1 and v_2w_2 antipodal if either v_1 and v_2 are antipodal vertices and w_1 and w_2 are also antipodal vertices, or if v_1 and w_2 are antipodal vertices and v_2 and w_1 are also antipodal vertices. A 2-colouring of the edges of Q_n is called *antipodal* if no pair of antipodal edges has the same colour. Norine [60] conjectured the following.

Conjecture 10.1.1 (Norine [60])**.** *In any antipodal 2-colouring of the edges of Q_n, there exists a pair of antipodal vertices which are joined by a monochromatic path.*

Feder and Subi [39] later made the following conjecture.

Conjecture 10.1.2 (Feder and Subi [39])**.** *In any 2-colouring of the edges of Q_n, we can find a pair of antipodal vertices and a path joining them with at most one colour change.*

If true, this implies the conjecture of Norine. Indeed, consider an antipodal 2-colouring of the edges of Q_n. By Conjecture 10.1.2, we can now find an antipodal path P_1P_2 such that both paths P_1 and P_2 are monochromatic. If they have the same colour we are done; if not the path $P_2^C P_1$ will work, where P_2^C is the antipodal path to path P_2.

Call a path in Q_n a *geodesic* if no two of its edges have the same direction. Leader and Long [54] proved the following result.

Theorem 10.1.3 (Leader and Long [54])**.** *In any 2-colouring of the edges of Q_n, we can find a monochromatic geodesic of length at least $\lceil \frac{n}{2} \rceil$.*

Leader and Long proposed a conjecture that strengthens that of Feder and Subi.

Conjecture 10.1.4 (Leader and Long [54]). *In any 2-colouring of the edges of Q_n, we can find a pair of antipodal vertices and a geodesic joining them with at most one colour change.*

Theorem 10.1.3 implies that we can always find a pair of antipodal vertices and a geodesic joining them with at most $\frac{n}{2}$ colour changes. Moreover, as Theorem 10.1.3 is sharp, there is no hope of improving the result by finding longer monochromatic geodesic. In this chapter, we establish the following result.

Theorem 10.1.5. *In any 2-colouring of the edges of Q_n, we can find a pair of antipodal vertices and a geodesic joining them with at most $(\frac{3}{8}+o(1))n$ colour changes.*

To prove the theorem, we employ the strategy of dividing the Q_n graph into small pieces (Q_3 graphs in fact) and finding a collection of geodesics with certain properties within each piece. The conditions we impose on these local geodesics let us glue them together into a collection of geodesics in Q_n in such a way that on average these long geodesics will have not too many colour changes. From that we in particular conclude that at least one of the long geodesics must not have too many colour changes.

10.2 Good and bad Q_3 graphs

In this short section we collect together some facts about 2-colourings of the 3-dimensional cube.

From now on, we call a geodesic connecting two antipodal points simply an *antipodal geodesic*.

We call a colouring of Q_3 by two colours *good* if we can find 4 antipodal geodesics, with each vertex being an endpoint of exactly one of these, such that these 4 geodesics have in total at most two colour changes. If a colouring of Q_3 is not good we call it *bad*.

The terms good and bad Q_3 will be sometimes used instead of good and bad colouring of Q_3, and it is understood that we refer to a particular colouring.

When showing that Conjecture 10.1.4 holds for $n=5$, Feder and Subi [39] proved the following simple lemma which we will use too.

Lemma 10.2.1 (Feder and Subi [39]). *Assume in a 2-colouring of Q_3, there are antipodal vertices v and v' such that all the geodesics connecting v and v' have two colour changes. Then the other three pairs of antipodal points are connected by geodesics without colour changes.*

In particular, note that we can deduce the following easy claim from the lemma above.

Claim 10.2.2. *Assume we have a bad 2-colouring of Q_3. Then at most one pair of antipodal points in this Q_3 is connected by a geodesic without a colour change.*

Proof. Lemma 10.2.1 implies that in any bad 2-colouring of Q_3, any pair of antipodal vertices is connected by a geodesic with at most one colour change. So if we had two pairs of antipodal vertices connected by a geodesic without a colour change, then we could find 4 antipodal geodesics, with each vertex being an endpoint of exactly one of these, such that these 4 geodesics have in total at most two colour changes. But that would by definition imply we have a good 2-colouring of Q_3. □

Our first lemma gives us an easy way to identify many Q_3 graphs as good.

Lemma 10.2.3. *Assume in a 2-colouring of Q_3, all three edges at some vertex of Q_3 have the same colour. Then it is a good colouring.*

Proof. Assume this colouring is bad. Without loss of generality take the vertex where all edges have the same colour to be 000 and this colour to be blue. If all the edges with neither of their endpoints being 000 or 111 are red, it is a good colouring, as the other three pairs of antipodal vertices are connected by the antipodal geodesics with no colour changes. So assume some edge with neither endpoint being 000 or 111 is blue. Without loss of generality it is (100, 110).

From 001, we have the antipodal geodesic with no colour change (001, 000, 100, 110). So if the colouring is bad, then we know by Claim 10.2.2 that for no other pair of antipodal points can we have an antipodal geodesic with no colour change. So the edge (100, 101) must be red by considering the geodesic (010, 000, 100, 101), the edge (001, 101) must be red by considering the geodesic (010, 000, 001, 101) and the edge (001, 011) must be red by considering the geodesic (100, 000, 001, 011).

But that gives the red antipodal geodesic with no colour change (100, 101, 001, 011), thus a contradiction. □

Next, note that one particular example of a bad Q_3 graph occurs when we colour all the edges in one direction by one colour, and all the edges in the other two directions by the other colour. Lemma 10.2.4 that follows tells us that any bad colouring behaves very much like this example in a sense we will need in our proof.

Lemma 10.2.4. *Consider any bad colouring of Q_3 and any vertex v. Then there exists an antipodal geodesic from v to v' with exactly one colour change, a red edge at v and a blue edge at v'.*

Proof. Without loss of generality let v be 000. Assume no such antipodal geodesic exists. By Lemma 10.2.3, we have at least one red edge from 000, without loss of generality to 100. Also we have at least one blue edge from 111. If this edge went to 110 or 101, we would be done immediately, so it must go to 011 and the other two edges from 111 must be red. Furthermore, the other two edges from 000 must be blue, or else we would be done, so assume they are blue.

As the colouring of Q_3 we consider is bad, at most one pair of antipodal points can be connected by an antipodal geodesic without a colour change. Whichever colour the edge (001, 101) has, it creates an antipodal geodesic without a colour change, either between 010 and 101 or between 001 and 110. So 000 cannot be connected to 111 by an antipodal geodesic without a colour change, forcing the edges (100, 110) and (100, 101) to be blue and the edges (010, 011) and (001, 011) to be red.

But now we see that both 010 and 001 are connected to their antipodals by geodesics without a colour change, which is a contradiction. $\square$

Analogously, in any bad Q_3, there exists such an antipodal geodesic with exactly one colour change, a blue edge at v and a red edge at v'.

10.3 Proof of the main result

We will now prove Theorem 10.1.5.

As we have the $o(n)$ term included in our bound, clearly it suffices to prove the theorem for n divisible by 3, so assume $n = 3k$. For the vertices v, w of distance 3, let $G(v, w)$ denote the subgraph of Q_n spanned by the geodesics between v and w (so $G(v, w) \cong Q_3$). Call two such subgraphs $G_1 \cong Q_3$ and $G_2 \cong Q_3$ of Q_n *neighbours* if they share exactly one vertex. If this vertex is v, call them *v-neighbours*. Consider a set A of all the ordered pairs (v, w) of the vertices of Q_n such that $d(v, w) = 3$. Assume $f : A \to V(Q_n)$ satisfies the following three conditions for all the vertices v, w:

(i) $d(v, f(v, w)) = 1$

(ii) $d(w, f(v, w)) = 2$

(iii) $d(f(w, v), f(v, w)) = 1$

In other words, this is equivalent to $(v, f(v, w), f(w, v), w)$ being an antipodal geodesic in $G(v, w)$.

Now, given the antipodal geodesic $(v_0, v_1, v_2, v_3, v_4, \ldots, v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}, \ldots, v_{3k})$, we will modify it into the antipodal geodesic

$$(v_0, f(v_0, v_3), f(v_3, v_0), v_3, f(v_3, v_6), \ldots, v_{3i}, f(v_{3i}, v_{3i+3}), f(v_{3i+3}, v_{3i}), v_{3i+3}, \ldots, v_{3k}).$$

We will show that for every fixed colouring, we can define such f in a way that the expected number of colour changes on a geodesic obtained by this modification from a uniformly random antipodal geodesic is no more than $(\frac{3}{8} + o(1))n$, where the $o(1)$ term depends on n only, not on the colouring. More precisely, we will define f_1 and f_2 (depending on the colouring) and show that at least one of these two must always work.

If $G(v,w)$ is a good Q_3, let $f_1(v,w), f_1(w,v)$ be such that no other geodesic between v and w has strictly less colour changes than $(v, f_1(v,w), f_1(w,v), w)$, and set $f_2(v,w) = f_1(v,w), f_2(w,v) = f_1(w,v)$.

Call the vertex v of Q_n *even* if its distance from $000\ldots000$ is even and call it *odd* otherwise. Every geodesic of length 3 connects an odd and an even vertex. For $G(v,w)$ bad with v even and w odd, define $f_1(v,w)$ and $f_1(w,v)$ such that $(v, f_1(v,w), f_1(w,v), w)$ has exactly one colour change, $(v, f_1(v,w))$ is blue and $(f_1(w,v), w)$ is red. Also define $f_2(v,w)$ and $f_2(w,v)$ such that $(v, f_2(v,w), f_2(w,v), w)$ has exactly one colour change, $(v, f_2(v,w))$ is red and $(f_2(w,v), w)$ is blue. By Lemma 10.2.4, there exist such functions.

Denote by p the proportion of good Q_3 subgraphs of Q_n in this colouring and the proportion of bad ones is thus $1-p$. Picking two Q_3 subgraphs that are neighbours uniformly at random, denote the probability that both are good by a, the probability that one is good and one is bad by b, and thus the probability that both are bad is $1-a-b$. We clearly must have $p = a + \frac{b}{2}$.

How large can b be? Suppose at any vertex v, of all Q_3 graphs containing v, there are s good ones and $\binom{n}{3} - s$ bad ones. There are $\frac{1}{2}\binom{n}{3}\binom{n-3}{3}$ pairs of v-neighbours, and of them at most $s(\binom{n}{3} - s) \leqslant \frac{1}{4}\binom{n}{3}^2$ are good-bad pairs. As this applies to every vertex and is independent of s, we have

$$b \leqslant \frac{1}{2}\binom{n}{3}\binom{n-3}{3}^{-1} = \frac{1}{2} + o(1).$$

Now, choose an antipodal geodesic uniformly at random and modify it as described. Due to the symmetry, and the properties of good and bad Q_3 graphs, for any $j : 0 \leqslant j \leqslant n-1$ and for either value of i, the expected number of colour changes inside the geodesic $(v_{3j}, f_i(v_{3j}, v_{3j+3}), f_i(v_{3j+3}, v_{3j}), v_{3j+3})$ is at most $\frac{1}{2}$ if we condition on the graph $G(v_{3j}, v_{3j+3})$ being good, and at most 1 if we condition on the graph $G(v_{3j}, v_{3j+3})$ being bad. Since the proportion of good Q_3 subgraphs of Q_n is p, we obtain that the expected number of colour changes inside the geodesic $(v_{3j}, f_i(v_{3j}, v_{3j+3}), f_i(v_{3j+3}, v_{3j}), v_{3j+3})$ is at most $\frac{1}{2}p + (1-p) = 1 - \frac{p}{2}$.

What is the probability that, for some fixed $j : 1 \leqslant j \leqslant k-1$, we have a colour change between the edges $(f_i(v_{3j}, v_{3j-3}), v_{3j})$ and $(v_{3j}, f_i(v_{3j}, v_{3j+3}))$ (due to the symmetry this is same for all such j)? With probability $1-a-b$, both $G(v_{3j}, v_{3j-3})$ and $G(v_{3j}, v_{3j+3})$ are bad, and then we do not have a colour change by definition of f_i. If one is good and one is bad, exactly one of f_1 and f_2 has a colour change between these two edges. So choose as our f that f_i for which the probability of a change in this case is at most $\frac{1}{2}$.

Finally, with probability a, both graphs are good. Consider any fixed vertex v. Choosing a random subgraph Q_3 containing v, by Lemma 10.2.3, the probability that it is good is at least the probability that choosing 3 random distinct edges from v, they all have the same colour. So we conclude there are at least $(\frac{1}{4} - o(1))\binom{n}{3} \geqslant \frac{1}{8}\binom{n}{3}$ good subgraphs containing v for n large enough. Suppose precisely t good subgraphs contain v. Clearly, the number of pairs of neighbours of good subgraphs that have a colour change at v is at most $\frac{1}{4}t^2$. Also for any good subgraph G_1 containing v, the number of good subgraphs that share v and at least one other vertex with G_1 is less than $3n^2$. So the number of pairs of two good graphs that are v-neighbours is at least $\frac{1}{2}t(t-3n^2)$. So the probability that a uniform random pair of good v-neighbours switches colour there is at most $\frac{1}{2}\frac{t}{t-3n^2}$. This is a decreasing function of t for $t > 3n^2$, so using $t \geqslant \frac{1}{8}\binom{n}{3}$, and as this applies to any vertex, we get that in this case, the probability of a colour switch is no more than $\frac{1}{2} + o(1)$.

Hence we get that for our chosen value of i, the probability that for any fixed $j : 1 \leqslant j \leqslant k-1$, we have a colour change between the edges $(f_i(v_{3j}, v_{3j-3}), v_{3j})$ and $(v_{3j}, f_i(v_{3j}, v_{3j+3}))$ is at most $\frac{b}{2} + (\frac{1}{2} + o(1))a$. Putting this together with the fact that for any $j : 0 \leqslant j \leqslant n-1$ and for our chosen value of i, the expected number of colour changes inside the geodesic $(v_{3j}, f_i(v_{3j}, v_{3j+3}), f_i(v_{3j+3}, v_{3j}), v_{3j+3})$ is at most $1 - \frac{p}{2}$, we obtain that on average our modified antipodal geodesic has at most

$$(1 - \frac{p}{2})\frac{n}{3} + (\frac{b}{2} + (\frac{1}{2} + o(1))a)\frac{n}{3} = (\frac{1}{3} + \frac{b}{12} + o(1))n$$

colour changes (using $p = a + \frac{b}{2}$). But $b \leqslant \frac{1}{2} + o(1)$, giving the result.

Thus, the proof of Theorem 10.1.5 is finished.

10.4 Concluding remarks

We hope that ideas similar to the ones used in the proof above could be used to obtain the stronger bound of the order $o(n)$. There are two particular parts of our strategy that we believe could help with this.

Firstly, we introduce the idea of finding the antipodal geodesics in Q_n with a few colour changes by taking a uniformly random antipodal geodesic, fixing both of its endpoints as well as some points on the geodesic and then modifying the geodesic suitably between these points. This does seem to be a useful framework to think about the problem and considering Q_n divided into the bits of the size say $Q_{\log n}$ instead of the size Q_3 may help together with some new ideas. Indeed, by certain arguments of this sort, we can infer various properties of the worst case colouring in the case that at least δn colour changes are needed between the typical pair of antipodal points in the worst case colouring.

Further, if we try for the inductive proof, it seems very helpful to consider what proportion of the pairs of the points (a,b) of certain distance d in Q_n has the property that we have two geodesics joining a and b, one with the red edge at a and the other with the blue edge at a, and both these geodesics have same many colour changes and at most as many colour changes as any other geodesic joining a and b. If the proportion is large, that helps us build longer geodesics with a few colour changes. If the proportion is small, we obtain more information about our colouring, which we hope could be used to bound the number of colour changes on optimal geodesics in different ways. Definition of good and bad Q_3 graphs in this chapter is motivated precisely by this idea, and it shows that in this particular case, the trade-off can be formalized very nicely.

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