Stable commutator length on free $\mathbb{Q}$-groups

Francesco Fournier-Facio

November 30, 2025

Abstract

We study stable commutator length on free $\mathbb{Q}$-groups. We prove that every non-identity element has positive stable commutator length, and that the corresponding free group embeds isometrically. We deduce that a non-abelian free $\mathbb{Q}$-group has an infinite-dimensional space of homogeneous quasimorphisms modulo homomorphisms, answering a question of Casals-Ruiz, Garreta, and de la Nuez González. We conjecture that stable commutator length is rational on free $\mathbb{Q}$-groups. This is connected to the long-standing problem of rationality on surface groups: indeed, we show that free $\mathbb{Q}$-groups contain isometrically embedded copies of non-orientable surface groups.

1 Introduction

A *quasimorphism* on a group G is a function $\varphi\colon G \to \mathbb{R}$ whose *defect*

$$D(\varphi) := \sup_{g,h \in G} |\varphi(g) + \varphi(h) - \varphi(gh)|,$$

is finite. A quasimorphism is *homogeneous* if moreover $\varphi(g^n) = n\varphi(g)$ for all $g \in G, n \in \mathbb{Z}$; the space of homogeneous quasimorphisms on G is denoted by $\mathrm{Q}(G)$. Quasimorphisms are central objects to the study of groups in relation to other areas of mathematics, in particular bounded cohomology [Fri17], knot theory [Mal04], symplectic geometry [PR14] and one-dimensional dynamics [Ghy87].

In most of these applications, what really matters is the quotient space $\mathrm{Q}(G)/\mathrm{Hom}(G)$ of homogeneous quasimorphisms modulo (real-valued) homomorphisms, which is sometimes called the space of *non-trivial quasimorphisms*. There are several conditions in the literature that ensure that there are no non-trivial quasimorphisms: for example, satisfying a law [Cal10]. Here we are interested in the dual property, namely, the existence of a *surjective word map*. An element w in a free group F_m defines a word map $w\colon G^m \to G$, obtained by substituting the basis of F_m with the input tuple in G^m. If $w \in [F_m, F_m]$, then the surjectivity of the corresponding word map implies that G is *uniformly perfect*, from which it follows easily that it has no non-trivial quasimorphisms. We prove that this is the only case in which such a criterion holds.

Theorem A (Quasimorphisms)**.** *There exists a countable group G with following properties:*

- *For every $w \in F_m \backslash [F_m, F_m]$, the word map $w \in G^m \to G$ is surjective;*
- *The space $\mathrm{Q}(G)/\mathrm{Hom}(G)$ is infinite-dimensional.*

This gives an example of a group with infinite-dimensional second bounded cohomology but non-trivial positive theory, answering positively [CRGdlNG21, Question 9.5]. We also give an example of a different flavour using Thompson groups (Proposition 5.2).

Recall that G is a $\mathbb{Q}$-*group* if every element admits a unique k-th root, for every $k \geqslant 1$. Such groups were introduced by Baumslag [Bau60], who was particularly intersted in *free* $\mathbb{Q}$-*groups* on a set S, denoted $F_S^{\mathbb{Q}}$. The structure of free $\mathbb{Q}$-groups was further studied by Myasnikov–Remeslennikov [MR94, MR96], and recently Jaikin-Zapirain proved that they are residually torsion-free nilpotent [JZ24]. The group in Theorem A can be taken to be a non-abelian free $\mathbb{Q}$-group.

We will prove Theorem A via the dual approach of *stable commutator length*, or scl for short. Given an element $g \in [G, G]$, we denote by $\mathrm{cl}_G(g)$ the minimal number of commutators $[x, y] : x, y \in G$ whose product equals g, and

$$\mathrm{scl}_G(g) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

If there exists $k \geqslant 1$ such that $g^k \in [G, G]$, we set $\mathrm{scl}_G(g) = \mathrm{scl}_G(g^k)/k$, and otherwise we set $\mathrm{scl}_G(g) = \infty$. In a natural way, we can extend the domain of definition of scl to *chains* on G (Section 2). This is the subject of a rich theory [Cal09a], especially thanks to its many incarnations: algebra (via the above definition), geometry (via efficient fillings of loops by surfaces), and most importantly for us: functional analysis. In this context Bavard Duality relates scl and quasimorphisms [Bav91], which allows to deduce Theorem A from the following statement, in the spirit of [CW11].

Theorem B (Isometry). *The embedding $F_S \to F_S^{\mathbb{Q}}$ is isometric for* scl*.*

Many landmark results on scl concern *spectral gaps*. The typical statement is of the form: there exists $\varepsilon > 0$ such that for all $g \in [G, G]$, either $\mathrm{scl}_G(g) \geqslant \varepsilon$ or $\mathrm{scl}_G(g) = 0$, and in the latter case g has to be of a special form (e.g. torsion, or conjugate to its own inverse) [DH91, CF10, BBF16, FFT19, CH]. In a non-abelian free $\mathbb{Q}$-group we cannot hope for a gap: taking $e \neq g \in [F_S, F_S]$, by Theorem B we have $\mathrm{scl}_{F_S^{\mathbb{Q}}}(g^{1/k}) = \mathrm{scl}_{F_S}(g)/k$, which is positive and arbitrarily small. Still, vanishing only holds for the identity element.

Theorem C (Positivity). *Every non-identity element in $F_S^{\mathbb{Q}}$ has positive* scl*.*

Theorems B and C hold more generally for A-completions of torsion-free non-cyclic hyperbolic groups, where A is a subring of $\mathbb{Q}$ (Corollary 4.2). The infinite-dimensionality of the space of quasimorphisms follows for A-completions of more general groups, such as non-abelian right-angled Artin groups (Corollary 4.4).

Another important class of results in scl is *rationality* theorems [Cal09b, Che18, Che20]. By analogy with the free group, the following conjecture is natural.

Conjecture D (Rationality). scl *is rational on $F_S^{\mathbb{Q}}$.*

This conjecture is likely to be hard. Indeed, one of the main open problems in scl is whether it is rational on fundamental groups of closed surfaces [Heu23, Question 7.5.4]. It turns out that Conjecture D is closely related to this problem.

Theorem E (Surfaces). *For all $m \geqslant 1$ there is an embedding $K_{2m+1} \to F_{2m}^{\mathbb{Q}}$ that is isometric for* scl*, where K_{2m+1} denotes the fundamental group of the closed non-orientable surface of demigenus $2m + 1$.*

Outline. We recall some generalities on scl in Section 2. Then in Section 3 we introduce rational extensions and iterated rational extensions, and prove some general results about their scl. In Section 4 we treat A-completions, which include free A-groups, and apply the general results to them, proving Theorems B and C. In Section 5 we address [CRGdlNG21, Question 9.5] and prove Theorem A. In Section 6 we discuss Conjecture D and prove Theorem E.

Acknowledgements. The author is supported by the Herchel Smith Postdoctoral Fellowship Fund. He thanks the Mathematisches Forschungsinstitut Oberwolfach for the hospitality, and Arman Darbinyan for the company. He thanks Montserrat Casals-Ruiz, Lvzhou Chen, Will Cohen, Alexis Marchand, Doron Puder and Henry Wilton for useful conversations, Rémi Coulon and Turbo Ho for the inspiration while working on [CFFH], and the anonymous referee for the valuable comments.

Notation. Subgroups of $\mathbb{Q}$ will play an important role. Hence we reserve 1 for the whole number, and instead use e to denote the identity element of a general discrete group. We will never explicitly write the group operation on $\mathbb{Q}$, to avoid confusion with the formal sums that appear when dealing with chains.

2 Generalities on scl

We start by extending the definition of scl to chains, following Calegari [Cal09a]. Let G be a discrete group. We denote by $\mathrm{C}_1(G)$ the space of real-valued chains on G, namely the real vector space with basis G. The subspace of boundaries is

$$\mathrm{B}_1(G) := \mathrm{span}\{g + h - gh : g, h \in G\}.$$

The quotient $\mathrm{C}_1(G)/\mathrm{B}_1(G)$ is the first real homology group $\mathrm{H}_1(G)$.

Consider an integral chain $c = \sum_i g_i$, where $g_i \in G$ (possibly with repetitions). Then $\mathrm{cl}_G(c)$ is the smallest number of commutators whose product is equal to an expression of the form $\prod_i t_i g_i t_i^{-1}$, where $t_i \in G$. We then define

$$\mathrm{scl}_G\left(\sum_i g_i\right) = \lim_{n\to\infty} \frac{\mathrm{cl}_G\left(\sum_i g_i^n\right)}{n},$$

and in fact the limit is an infimum [Cal09a, Lemma 2.73]. By homogeneity, scl can be extended to all rational chains, and then by continuity to all real chains. It takes finite values precisely on $\mathrm{B}_1(G)$.

Lemma 2.1. *Let $(G_j)_{j\in J}$ be a directed system of groups with colimit G_J. Let $c \in \mathrm{C}_1(G_{j_0})$, which we identify with its image in $\mathrm{C}_1(G_j), j_0 \leqslant j \in J$. Then*

$$\mathrm{scl}_{G_J}(c) = \inf_{j\in J} \mathrm{scl}_{G_j}(c).$$

Proof. Because scl is monotone, $\mathrm{scl}_{G_J}(c) \leqslant \inf \mathrm{scl}_{G_j}(c)$. For the converse inequality, we may assume that $\mathrm{scl}_{G_J}(c) < \infty$. Because scl extends uniquely from integral chains, we may also assume that $c = \sum_i g_i : g_i \in G_{j_0}$. The limit in the definition of scl_{G_J} is an infimum, so for all $\varepsilon > 0$ there exists n such that

$$\frac{\mathrm{cl}_{G_J}\left(\sum_i g_i^n\right)}{n} < \mathrm{scl}_{G_J}(c) + \varepsilon.$$

There are only finitely many elements involved in an expression witnessing this inequality, which therefore must eventually hold on G_j. This implies that $\mathrm{scl}_{G_j}(c) < \mathrm{scl}_{G_J}(c) + \varepsilon$ eventually holds. Letting $\varepsilon \to 0$, we conclude. □

We denote by $\mathrm{C}_1^H(G)$ the quotient of $\mathrm{C}_1(G)$ by the subspace spanned by elements of the form $g^n - ng : g \in G, n \in \mathbb{Z}$; and $hgh^{-1} - g : g, h \in G$. The map $\mathrm{C}_1(G) \to \mathrm{H}_1(G)$ factors through $\mathrm{C}_1^H(G)$. The kernel of the induced map $\mathrm{C}_1^H(G) \to \mathrm{H}_1(G)$ is denoted by $\mathrm{B}_1^H(G)$, and is the image of $\mathrm{B}_1(G)$ in $\mathrm{C}_1^H(G)$. Since scl vanishes on the elements we quotiented out, and it is subadditive on rational chains, it induces a seminorm on $\mathrm{B}_1^H(G)$. We denote by $\mathrm{S}(G) \subset \mathrm{B}_1^H(G)$ the subspace of elements with vanishing scl, so that scl is a norm on the quotient $\mathrm{B}_1^H(G)/\mathrm{S}(G)$.

Example 2.2. Let A be a non-trivial subgroup of $\mathbb{Q}$. Then every element of $\mathrm{C}_1^H(A)$ can be written uniquely as a real multiple of a fixed non-zero $a \in A$. It follows that $\mathrm{C}_1^H(A) \cong \mathrm{H}_1(A)$, and so every non-zero element of $\mathrm{C}_1^H(A)$ has infinite scl_A.

The above example is very basic, but it will be useful later on. For a less trivial example, scl is a genuine norm on $\mathrm{B}_1^H(G)$ whenever G is a free group [Cal09a, Proposition 2.84] or even a non-elementary hyperbolic group [Cal09a, Corollary 3.57].

The fundamental result relating scl and quasimorphisms is the following.

Theorem 2.3 (Generalised Bavard Duality [Cal09a, Section 2.6])**.** *There is a natural isomorphism between the dual space of* $\mathrm{B}_1^H(G)/\mathrm{S}(G)$ *with the* scl *norm, and the space* $\mathrm{Q}(G)/\mathrm{Hom}(G)$*, with the norm* $2D(\cdot)$*.*

The way this duality is realised is by evaluating homogeneous quasimorphisms on chains. It is easy to see that the value only depends on the image of the chain in $\mathrm{C}_1^H(G)$, and that homomorphisms evaluate trivially on $\mathrm{B}_1^H(G)$.

From this we obtain the result connecting Theorems A and B from the introduction. Given a map $G_1 \to G_2$, we say that it is *isometric for* scl if the induced map $\mathrm{C}_1^H(G_1) \to \mathrm{C}_1^H(G_2)$ preserves scl.

Corollary 2.4. *Let* $G_1 \to G_2$ *be a map that is isometric for* scl*. Then for every* $\varphi_1 \in \mathrm{Q}(G_1)$ *there exists* $\varphi_2 \in \mathrm{Q}(G_2)$ *with the same defect whose pullback to* G_1 *coincides with* φ_1*. In particular, the pullback induces a surjection* $\mathrm{Q}(G_2)/\mathrm{Hom}(G_2) \to \mathrm{Q}(G_1)/\mathrm{Hom}(G_1)$*.*

Proof. Because the map preserves chains with infinite scl, it induces an embedding $\mathrm{H}_1(G_1) \to \mathrm{H}_1(G_2)$. The dual of first real homology is the space of real-valued homomorphisms, hence we obtain the result in case φ_1 is a homomorphism.

Now suppose that φ_1 is not a homomorphism. The map realises $\mathrm{B}_1^H(G_1)/\mathrm{S}(G_1)$ as a closed subspace of $\mathrm{B}_1^H(G_2)/\mathrm{S}(G_2)$ with the scl norm. Theorem 2.3 interprets φ_1 as a functional on $\mathrm{B}_1^H(G_1)/\mathrm{S}(G_1)$ with operator norm $2D(\varphi_1)$. Hahn–Banach gives a norm-preserving extension, that is, $\varphi_2 \in \mathrm{Q}(G_2)$ with $D(\varphi_2) = D(\varphi_1)$, whose pullback to G_1 is equal to $\varphi_1 + \psi_1$, for some $\psi_1 \in \mathrm{Hom}(G_1)$. By the first paragraph we can extend ψ_1 to $\psi_2 \in \mathrm{Hom}(G_2)$, and so $\varphi_2 - \psi_2$ is a defect-preserving extension of φ_1. □

When $G_1 \to G_2$ is an embedding, this is a statement about *extendability* of quasimorphisms. This problem has received much attention over the past few years, especially in two extreme cases: for hyperbolically embedded subgroups [HO13, FPS15], and for normal subgroups [KKM+24, FFMS]. However, in most of these results, the extension has controlled defect, but not *the same* defect. Corollary 2.4 can be applied to several examples of isometric embeddings between free groups [CW11, Mar25] or graphs of groups [Che18, Che20, Mar].

Remark 2.5. In the above citations, an embedding is called isometric for scl if it preserves scl of B_1^H: it is not required to preserve elements of infinite scl. Under this weaker assumption, the same argument as Corollary 2.4 shows that every quasimorphism admits a defect-preserving extension, but only up to modifying it by adding a homomorphism.

Another corollary of Theorem 2.3 is a criterion for positivity of scl.

Corollary 2.6. *For all $c \in \mathrm{B}_1^H(G)$, we have: $\mathrm{scl}_G(c) > 0$ if and only if there exists $\varphi \in \mathrm{Q}(G)$ such that $\varphi(c) > 0$. In particular, if $\mathrm{Q}(G) = \mathrm{Hom}(G)$, then every $c \in \mathrm{B}_1^H(G)$ has vanishing* scl.

We end this section with two results on scl of amalgamated products, which will be used in the proofs of Theorems B and C, respectively. The first one is a theorem of Chen and Heuer, specialised to this case.

Theorem 2.7 ([CH, Theorem 4.2])**.** *Let $P = G_1 *_Z G_2$ be an amalgamated product. Let $c_i \in \mathrm{C}_1^H(G_i)$. Then*

$$\mathrm{scl}_P(c_1 + c_2) = \inf\{\mathrm{scl}_{G_1}(c_1 + d) + \mathrm{scl}_{G_2}(c_2 - d)\},$$

where d runs over $\mathrm{C}_1^H(Z)$.

The second one is a positivity result. We start by recalling some notions. A subgroup $Z < G$ is *malnormal* if $Z \cap gZg^{-1} = \{e\}$ for all $g \in G \backslash Z$. An element $g \in G$ is *chiral* if there is no $n \geqslant 1$ such that g^n is conjugate to g^{-n}.

An action of a group G on a metric space X is *acylindrical* if for all $r \in \mathbb{N}$ there exist $R, N \in \mathbb{N}$ such that

$$d(x, y) \geqslant R \Rightarrow \#\{g \in G : \max\{d(x, gx), d(y, gy)\} \leqslant r\} \leqslant N.$$

A group G is *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a Gromov-hyperbolic space. If $g \in G$ is loxodromic for one such action, we say that g is a *generalised loxodromic* element. We isolate the following well-known fact, as it will be used twice.

Lemma 2.8. *Let G be a torsion-free acylindrically hyperbolic group, and let $g \in G$ be a generalised loxodromic element. Then g is chiral.*

Proof. Suppose otherwise. Let $f \in G, n \geqslant 1$ be such that $fg^nf^{-1} = g^{-n}$. By [DGO17, Corollary 6.6], f belongs to the elementary closure of g, which is the maximal virtually cyclic subgroup of G containing g. But G is torsion-free, and virtually cyclic torsion-free groups are cyclic, in particular abelian, which contradicts the relation $fg^nf^{-1} = g^{-n}$. □

Proposition 2.9. *Let $P = G_1 *_Z G_2$ be an amalgamated product of torsion-free groups. Suppose that Z is malnormal in G_1. Then every element in P with vanishing* scl *is conjugate into a G_i.*

Proof. The malnormality condition implies that P acts acylindrically on the Bass–Serre tree T: the stabiliser of a path of length 3 is trivial. Hence by [FFW23, Theorem 4.2], every chiral loxodromic element has positive scl. This applies to all elements that are not conjugate into a G_i, since by Lemma 2.8, loxodromic elements are automatically chiral. □

3 Iterated rational extensions

Let A be a subgroup of $\mathbb{Q}$. Let G be a group, and let $Z < G$ be isomorphic to a subgroup of A. The amalgamated product

$$G *_Z A$$

is called a *rational extension* of G.

Example 3.1. Let $z \in G$ be an infinite order element and let $p \geqslant 1$. Then the *root extension* $\langle G, t \mid t^p = z \rangle$ is a rational extension of G, where $A = \frac{1}{p}\mathbb{Z}$ and Z denotes the infinite cyclic groups $G > \langle z \rangle \cong \mathbb{Z} < A$.

A rational extension $G *_Z A$ is called *malnormal* if Z is a malnormal subgroup of G.

Proposition 3.2. *Let $E = G *_Z A$ be a rational extension of G.*

1. *The embedding $G \to E$ is isometric for* scl.
2. *Suppose that the rational extension is malnormal, and* $\mathrm{scl}_G(g) > 0$ *for all $e \neq g \in G$. Then* $\mathrm{scl}_E(g) > 0$ *for all $e \neq g \in E$.*

Proof. First, we notice that the inclusion $G \to E$ induces an embedding $\mathrm{H}_1(G) \to \mathrm{H}_1(E)$. In particular, if $c \in \mathrm{C}_1^H(G)$ has infinite scl_G, then it has infinite scl_E. Now suppose that $c \in \mathrm{B}_1^H(G)$, so that $\mathrm{scl}_E(c) \leqslant \mathrm{scl}_G(c) < \infty$. We write it as $c + 0$, seeing 0 as an element of $\mathrm{C}_1^H(A)$. Theorem 2.7 applies and gives

$$\mathrm{scl}_E(c) = \inf\{\mathrm{scl}_G(c + d) + \mathrm{scl}_A(d)\}.$$

But Example 2.2 shows that $\mathrm{scl}_A(d) = \infty$ unless $d = 0$. Since $\mathrm{scl}_E(c) < \infty$, the infimum must be attained at $d = 0$. This gives the first item.

Now with the assumptions of the second item, G must be torsion-free. Let $e \neq g \in E$; if g is not conjugate into either G or A, then $\mathrm{scl}_E(g) > 0$ by Proposition 2.9. If $g \in G$, then $\mathrm{scl}_E(g) = \mathrm{scl}_G(g) > 0$, by the first item, and the same follows for all conjugates. If $g \in A$, then there exists $z \in Z$ such that g and z have a common power. Since $\mathrm{scl}_E(z) = \mathrm{scl}_G(z) > 0$, this implies that $\mathrm{scl}_E(g) > 0$ as well, and the same follows for all conjugates. □

Remark 3.3. The first item of Proposition 3.2 can be proved alternatively using a result of Marchand [Mar, Corollary 4.2]. In the case of amalgamated products, it states that if $P = G_1 *_Z G_2$, where Z is amenable, and $\mathrm{H}_2(G_1; \mathbb{Q}) \to \mathrm{H}_2(P; \mathbb{Q})$ is surjective, then the embedding $G_1 \to P$ preserves scl of boundaries.

Now let $E = G *_Z A$ be a rational extension. As in the proof of Proposition 3.2, it is easy to prove directly that the embedding $G \to E$ preserves elements of infinite scl, so by the above it remains to show that $\mathrm{H}_2(G; \mathbb{Q}) \to \mathrm{H}_2(E; \mathbb{Q})$ is surjective. The Mayer–Vietoris sequence gives:

$$\cdots \to \mathrm{H}_2(Z; \mathbb{Q}) \to \mathrm{H}_2(G; \mathbb{Q}) \oplus \mathrm{H}_2(A; \mathbb{Q}) \to \mathrm{H}_2(E; \mathbb{Q}) \\ \to \mathrm{H}_1(Z; \mathbb{Q}) \to \mathrm{H}_1(G; \mathbb{Q}) \oplus \mathrm{H}_1(A; \mathbb{Q}) \to \cdots$$

Because Z and A are locally cyclic, they have homological dimension 1. Moreover, $\mathrm{H}_1(Z; \mathbb{Q}) \to \mathrm{H}_1(A; \mathbb{Q})$ is injective, so $\mathrm{H}_2(G; \mathbb{Q}) \to \mathrm{H}_2(E; \mathbb{Q})$ is an isomorphism, and we conclude.

We now push this process by transfinite induction. An *iterated rational extension* is a directed union of groups indexed by ordinals $(G_j)_{j\leqslant\alpha}$, such that G_{j+1} is a rational extension of G_j, and for a limit ordinal β, G_β is the directed union of its subgroups $(G_j)_{j<\beta}$. If all of the rational extensions involved are malnormal, we call this an *iterated malnormal rational extension*.

Theorem 3.4. *Let $(G_j)_{j\leqslant\alpha}$ be an iterated rational extension.*

1. *The embedding $G_0 \to G_\alpha$ is isometric for* scl.
2. *Suppose that the iterated rational extension is malnormal, and $\mathrm{scl}_{G_0}(g) > 0$ for all $e \neq g \in G_0$. Then $\mathrm{scl}_{G_\alpha}(g) > 0$ for all $e \neq g \in G_\alpha$.*

Proof. This follows by transfinite induction, using Proposition 3.2 for successor ordinals, and Lemma 2.1 for limit ordinals. □

4 A-groups

Let A be an associative ring. An A-*group* is a group endowed with a map $G \times A \to G$: $(g, a) \mapsto g^a$ with the following properties, for all $g, h \in G, a, b \in A$:

1. $g^0 = e, g^1 = g, e^a = e$;
2. $g^{a+b} = g^a g^b, (g^a)^b = g^{ab}$;
3. $(hgh^{-1})^a = hg^a h^{-1}$;
4. If $gh = hg$, then $(gh)^a = g^a h^a$.

We will only be concerned with A-groups when A is a subring of $\mathbb{Q}$. The two extreme cases are $\mathbb{Z}$-groups, which are just groups, and $\mathbb{Q}$-groups, which are groups in which every element admits a unique k-th root for every $k \geqslant 1$.

Given a group G, every homomorphism from G to an A-group factors through its A-*completion*, denoted G^A: existence and uniqueness of G^A are proved in [MR94, Theorems 1 and 2]. In general, this map needs not be injective, and even when it is, it may not be easy to describe explicitly. We will focus on a case where an explicit description is possible. We are only concerned with subrings of $\mathbb{Q}$, in which case this was already achieved by Baumslag [Bau60, Sections V–VIII], but we follow the more concise exposition of Myasnikov–Remeslennikov [MR94, MR96]. Recall that a group is *conjugately separated abelian (CSA)* if all centralisers of all non-identity elements are abelian and malnormal.

Let A be a subring of $\mathbb{Q}$. Suppose that G is a torsion-free CSA group. We choose a collection $\mathcal{Z}$ of centralisers that are not already A-modules, and such that every centraliser that is not an A-module is conjugate to a unique $Z \in \mathcal{Z}$. Because every $Z \in \mathcal{Z}$ is torsion-free, we can define the amalgamated product $G *_Z (Z \otimes A)$. This group is still torsion-free, moreover it is CSA [MR96, Theorem 5]. Choosing a well-ordering of $\mathcal{Z}$, we apply this process inductively to obtain a group G^*, which is again torsion-free and CSA [MR96, Lemma 6]. Moreover, for every $e \neq g \in G$, if Z is its centraliser in G, then its centraliser in G^* is isomorphic to $Z \otimes A$ [MR96, Lemma 4].

We define $G^{(0)} := G$, and by induction $G^{(n+1)} := (G^{(n)})^*$. Then the directed union

$$G = G^{(0)} \to G^{(1)} \to \cdots \to G^A$$

coincides with the A-completion of G [MR96, Theorem 8].

Let us assume moreover that G does not contain a copy of $\mathbb{Z}^2$. For a torsion-free CSA group, this is equivalent to saying that all centralisers are locally cyclic, hence isomorphic to subgroups of $\mathbb{Q}$. Then an amalgamated product $G *_Z (Z \otimes A)$ is a malnormal rational extension, and $G \to G^*$ is an iterated malnormal rational extension. By [MR96, Lemmas 4 and 6], the group G^* satisfies the same hypotheses as G, and so we can extend the inductive process to the directed union $G^{(0)} \to G^{(1)} \to \cdots \to G^A$. Therefore G^A is an iterated malnormal rational extension of G_0, and Theorem 3.4 applies.

Theorem 4.1. *Let A be a subring of $\mathbb{Q}$, and let G be a torsion-free CSA group that does not contain a copy of $\mathbb{Z}^2$.*

1. *The embedding $G \to G^A$ is isometric for* scl.
2. *Suppose that $\mathrm{scl}_G(g) > 0$ for all $e \neq g \in G$. Then $\mathrm{scl}_{G^A}(g) > 0$ for all $e \neq g \in G^A$.* □

Corollary 4.2. *Let A be a subring of $\mathbb{Q}$, and let G be a non-cyclic torsion-free hyperbolic group, or a non-abelian free group (of any rank).*

1. *The embedding $G \to G^A$ is isometric for* scl.
2. $\mathrm{scl}_{G^A}(g) > 0$ *for all $e \neq g \in G^A$.*
3. *The space $\mathrm{Q}(G^A)/\mathrm{Hom}(G^A)$ is infinite-dimensional.*

Proof. With these assumptions, G is CSA and does not contain a copy of $\mathbb{Z}^2$, hence the first item follows from Theorem 4.1.

If G is a non-elementary hyperbolic group, then all chiral elements have positive scl [CF10]. If G is moreover torsion-free, then all elements are chiral (Lemma 2.8). So Theorem 4.1 gives the second item for non-cyclic torsion-free hyperbolic groups, in particular for free groups of finite rank. For free groups of infinite rank, it follows from this, and the fact that every element is contained in a retract isomorphic to F_2, hence also in this case all non-identity elements have positive scl.

Non-abelian free groups [Bro81] and non-elementary hyperbolic groups [EF97] admit an infinite-dimensional space of homogeneous quasimorphisms modulo homomorphisms. Hence the third item follows from the first and Corollary 2.4. □

Remark 4.3. The quasimorphism spaces from Corollary 4.2 have uncountable dimension. What is more, they are not separable, when seen as Banach spaces endowed with the defect norm. Indeed, this is true for free groups [Cal09a, Example 2.62], from which it follows for all acylindrically hyperbolic groups via extensions from hyperbolically embedded subgroups [HO13], from which it follows for any isometric embedding of such a group by Corollary 2.4, in particular to the A-completions from Corollary 4.2.

When $G = F_S$ is a free group, the A-completion F_S^A is called the *free A-group on S*.

Proof of Theorems B and C. If $|S| \leqslant 1$, then $F_S \to F_S^{\mathbb{Q}}$ is either the identity on the trivial group, or the embedding $\mathbb{Z} \to \mathbb{Q}$, and then the statements obviously hold. If $|S| > 1$, Corollary 4.2 applies. □

The first step towards Corollary 4.2 was the fact that rational extensions preserve scl. Both proofs (Proposition 3.2 and Remark 3.3) rely on the fact that we extend along a locally cyclic abelian group. There are other groups where A-completions have been described as iterated extensions by centralisers, for example coherent RAAGs [CRDK23], but the presence of higher rank centralisers is an obstacle to adapting our approach. Nevertheless, we can obtain the infinite-dimensionality result for such groups as well.

Corollary 4.4. *Let A be a subring of $\mathbb{Q}$. If G^A is an A-group that surjects onto a free A-group, then $\mathrm{Q}(G^A)/\operatorname{Hom}(G^A)$ is infinite-dimensional.*

In particular, this is true for the A-completion of a non-abelian right-angled Artin group.

Proof. If $G^A \to F^A$ is a surjection, then the pullback $\mathrm{Q}(F^A)/\operatorname{Hom}(F^A) \to \mathrm{Q}(G^A)/\operatorname{Hom}(G^A)$ is injective, so the first statement follows from Corollary 4.2.

If G is a non-abelian RAAG, then it retracts onto a parabolic subgroup F that is free of rank 2. So the universal property of G^A gives induces a map:

$$\begin{array}{ccc} G & \longrightarrow & G^A \\ \downarrow\!\!\downarrow & \searrow & \vdots\downarrow \\ F & \longrightarrow & F^A \end{array}$$

Because the image is a $\mathbb{Q}$-group and contains the image of $F \to F^A$, the induced map must be surjective. □

5 Positive theory

The *positive theory* of a group G is the collection of first-order sentences without negations that hold in G. The positive theory of every group contains the positive theory of a non-abelian free group [Mer66, Mak82]; if there are no additional positive sentences, then the group is said to have *trivial positive theory.*

Many non-trivial positive sentences imply that every homogeneous quasimorphism is a homomorphism. This is true for example for groups satisfying a law [Cal10], uniformly perfect groups, and groups with commuting conjugates [Kot08, FFL23]. In [CRGdlNG21, Section 9.3], the authors speculate on a relation between positive theory and second bounded cohomology. In particular, they ask whether there exist groups with infinite-dimensional second bounded cohomology and non-trivial positive theory [CRGdlNG21, Question 9.5] (the converse [CRGdlNG21, Question 9.4] is addressed in [CFFH]). Note that the space of homogeneous quasimorphisms modulo homomorphisms embeds in the second bounded cohomology [Cal09a, Theorem 2.50]. In this section, we give two examples.

Lemma 5.1. *Let G be a $\mathbb{Q}$-group. Then for all $w \in F_m \backslash [F_m, F_m]$, the word map $w\colon G^m \to G$ is surjective.*

Proof. Write

$$w = x_1^{i_1} \cdots x_m^{i_m} v;$$

where $v \in [F_m, F_m]$. Because $w \notin [F_m, F_m]$, one of the exponents i_k must be non-zero; up to reordering the basis, we may assume that $i_1 \neq 0$. No $w(g, 1, \ldots, 1) = g^{i_1}$, so $w(G^m)$ contains the image of $x \mapsto x^{i_1}$, which is surjective on a $\mathbb{Q}$-group. □

Proof of Theorem A. Let G be the $\mathbb{Q}$-completion of a torsion-free non-cyclic hyperbolic group. Then G satisfies both statements, by Lemma 5.1 and Corollary 4.2. □

Surjectivity of a word map is a positive sentence, but not always a non-trivial one. Indeed, there are certain words, called *silly* by Segal [Seg09, Section 3.1], that define a surjective word map on all groups. These are the ones that can be written as $w = x_1^{i_1} \cdots x_m^{i_m} v$, as in the proof of Lemma 5.1, where $\gcd(i_1, \ldots, i_m) = 1$.

Taking a non-silly word map (e.g. $w = y^2$) we get a non-trivial positive sentence (e.g. $\forall x \exists y : y^2 = x$). This shows that $\mathbb{Q}$-groups have non-trivial positive theory, and the ones from Theorem A have infinite-dimensional second bounded cohomology, which gives a positive answer to [CRGdlNG21, Question 9.5].

We also give a different example, which is less natural but more direct. Consider Thompson's group T [CFP96]: this is a finitely presented infinite simple group of homeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$. The group $\widetilde{T}$ is the group of homeomorphisms of $\mathbb{R}$ that commute with integer translations and induce an element of T on $\mathbb{R}/\mathbb{Z}$.

Proposition 5.2. *The group $\widetilde{T}^{\mathbb{N}}$ has non-trivial positive theory, and $\mathrm{Q}(\widetilde{T}^{\mathbb{N}})/\mathrm{Hom}(\widetilde{T}^{\mathbb{N}})$ is infinite-dimensional.*

Proof. The group $\widetilde{T}$ is perfect, and has a one-dimensional space of homogeneous quasimorphisms, spanned by the rotation quasimorphism (see e.g. [Cal09a, Chapter 5]). It follows that $\widetilde{T}^{\mathbb{N}}$ has an infinite-dimensional space of homogeneous quasimorphisms, and no non-zero homomorphisms.

Moreover, T is uniformly perfect [GL23], and hence so is $T^{\mathbb{N}}$, which in particular has non-trivial positive theory. Now $\widetilde{T}^{\mathbb{N}}$ is a central extension of $T^{\mathbb{N}}$, and hence it also has non-trivial positive theory [CRGdlNG21, Theorem G]. □

Note that both examples are countable but infinitely generated. We do not know of a finitely generated example.

6 Rationality

We say that scl *is rational* on a group G, if $\mathrm{scl}_G(c) \in \mathbb{Q}$ for all $c \in \mathrm{C}_1^H(G)$ with rational coefficients. Rationality on free groups is one of the most influential results on scl [Cal09b], which motivated our Conjecture D. To approach it, one would have to start by showing that malnormal rational extensions of free groups have rational scl. Indeed, this is not only a reasonable first step, but also necessary.

Proposition 6.1. *Let G be a torsion-free CSA group that does not contain a copy of $\mathbb{Z}^2$. Let $z \in G$ be an element that generates its own centraliser. Then every rational extension of the form $G *_{z=a} A$ embeds into $G^{\mathbb{Q}}$, isometrically for* scl.

In particular, if scl *is rational on $G^{\mathbb{Q}}$, then it is rational on $G *_{z=a} A$.*

Proof. Recall the construction of $G^{\mathbb{Q}}$ from Section 4. The first step (that is, the embedding $G = G^{(0)} \to G^{(1)}$) consists in choosing a set $\mathcal{Z}$ of representatives of conjugacy classes of centralisers in G, and performing the single rational extension $G *_Z (Z \otimes \mathbb{Q})$ where $Z \in \mathcal{Z}$ is the first element in a well-ordering of $\mathcal{Z}$. By assumption $z \in G$ generates its own centraliser, so we may choose it to generate the first centraliser in $\mathcal{Z}$. Hence

$$G \to G *_{\langle z \rangle} (\langle z \rangle \otimes \mathbb{Q}) = G *_{z=1} \mathbb{Q}$$

is the first step of the iterated rational extension that constructs $G^{\mathbb{Q}}$. This shows that $G^{\mathbb{Q}}$ is an iterated rational extension of $G *_{z=1} \mathbb{Q}$.

Now consider a rational extension $G *_{z=a} A$. Up to replacing A by an isomorphic subgroup of $\mathbb{Q}$, we may assume that $a = 1$. But then

$$G *_{z=1} A \to (G *_{z=1} A) *_A \mathbb{Q} \cong G *_{z=1} \mathbb{Q}$$

is a rational extension. Therefore $G^{\mathbb{Q}}$ is an iterated rational extension of $G *_{z=1} A$. We conclude by Theorem 3.4. $\square$

It turns out that it suffices to look at root extensions.

Lemma 6.2. *Let G be a group and $z \in G$. Suppose that every root extension of the form $\langle G, t \mid t^p = z \rangle$ has rational* scl. *Then every rational extension of the form $G *_{z=a} A$ has rational* scl.

Proof. We start by proving the lemma for $E = G *_{z=1} \mathbb{Q}$. For all $n \geqslant 2$, define

$$E_n := G *_{z=1} \left(\mathbb{Z}\left[\frac{1}{n!}\right] \right) \cong \langle G, t \mid t^{n!} = z \rangle.$$

By assumption, scl is rational on E_n. Moreover,

$$E_{n+1} \cong E_n *_{\mathbb{Z}\left[\frac{1}{n!}\right]} \left(\mathbb{Z}\left[\frac{1}{(n+1)!}\right] \right).$$

This shows that E_{n+1} is a rational extension of E_n, and so by Proposition 3.2 the embedding $E_n \to E_{n+1}$ is isometric. Finally, E is the directed union of the E_n, so by Lemma 2.1 we conclude.

In general, let $E = G *_{z=a} A$, where $a \in A$. Up to changing A to an isomorphic subgroup of $\mathbb{Q}$, we may assume that $a = 1$. Now consider

$$E' = E *_A \mathbb{Q} \cong G *_{z=1} \mathbb{Q}.$$

It is a rational extension of E, so by Proposition 3.2 the embedding $E \to E'$ is isometric. Moreover, scl is rational on E' by the previous paragraph, and so scl is rational on E. $\square$

However, square root extensions of the free group already present a major obstacle. Let $m \geqslant 1$. Then the following square root extension of F_{2m}:

$$K_{2m+1} = \langle a_1, b_1, \ldots, a_m, b_m, t \mid t^2 = [a_1, b_1] \cdots [a_m, b_m] \rangle$$

is the fundamental group of a closed non-orientable surface. Its orientable double cover corresponds to the index-2 subgroup

$$\langle a_i, b_i \rangle *_{\prod [a_i, b_i] = \prod t[a_i, b_i]t^{-1}} \langle t a_i t^{-1}, t b_i t^{-1} \rangle,$$

which is the fundamental group of a closed surface of genus $2m$. It follows that the Euler characteristic of the non-orientable surface is $(2-4m)/2 = 1-2m$, and so its demigenus is $2-(1-2m) = 2m+1$, hence the notation. When $m = 1$, we recover the classical presentation of Dyck's surface [Dyc88].

Proof of Theorem E. K_{2m+1} is a root extension of $F_{2m} = \langle a_1, b_1, \ldots, a_m, b_m \rangle$ over the element $z = [a_1, b_1] \cdots [a_m, b_m]$. Since z generates its own centraliser, Proposition 6.1 shows that K_{2m+1} embeds into $F_{2m}^{\mathbb{Q}}$, isometrically for scl. □

In particular, if scl were rational on $F_{2m}^{\mathbb{Q}}$, then it would be rational on K_{2m+1}. Rationality of scl on surface groups is a major open problem [Heu23, Question 7.5.4], which motivated many of the recent advances in stable commutator length [Che20, Mar25].

References

[Bau60] G. Baumslag, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303. MR 122859

[Bav91] C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) **37** (1991), no. 1-2, 109–150. MR 1115747

[BBF16] M. Bestvina, K. Bromberg, and K. Fujiwara, *Stable commutator length on mapping class groups*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 3, 871–898. MR 3494163

[Bro81] R. Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., No. 97, Princeton Univ. Press, Princeton, NJ, 1981, pp. 53–63. MR 624804

[Cal09a] D. Calegari, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009. MR 2527432

[Cal09b] ———, *Stable commutator length is rational in free groups*, J. Amer. Math. Soc. **22** (2009), no. 4, 941–961. MR 2525776

[Cal10] ———, *Quasimorphisms and laws*, Algebr. Geom. Topol. **10** (2010), no. 1, 215–217. MR 2580432

[CF10] D. Calegari and K. Fujiwara, *Stable commutator length in word-hyperbolic groups*, Groups Geom. Dyn. **4** (2010), no. 1, 59–90. MR 2566301

[CFFH] R. Coulon, F. Fournier-Facio, and M.-C. Ho, *First-order theory of torsion-free Tarski monsters*, arXiv preprint arXiv:2508.21244.

[CFP96] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) **42** (1996), no. 3-4, 215–256. MR 1426438

[CH] L. Chen and N. Heuer, *Spectral gap of scl in graphs of groups and 3-manifolds*, arXiv preprint arXiv:1910.14146.

[Che18] L. Chen, *Scl in free products*, Algebr. Geom. Topol. **18** (2018), no. 6, 3279–3313. MR 3868221

[Che20] ———, *Scl in graphs of groups*, Invent. Math. **221** (2020), no. 2, 329–396. MR 4121154

[CRDK23] M. Casals-Ruiz, A. Duncan, and I. Kazachkov, *Limit groups over coherent right-angled Artin groups*, Publ. Mat. **67** (2023), no. 1, 199–257. MR 4522934

[CRGdlNG21] M. Casals-Ruiz, A. Garreta, and J. de la Nuez González, *On the positive theory of groups acting on trees*, Int. Math. Res. Not. IMRN (2021), no. 3, 1837–1918. MR 4206600

[CW11] D. Calegari and A. Walker, *Isometric endomorphisms of free groups*, New York J. Math. **17** (2011), 713–743. MR 2851070

[DGO17] F. Dahmani, V. Guirardel, and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. **245** (2017), no. 1156, v+152. MR 3589159

[DH91] A. J. Duncan and J. Howie, *The genus problem for one-relator products of locally indicable groups*, Math. Z. **208** (1991), no. 2, 225–237. MR 1128707

[Dyc88] W. Dyck, *Beiträge zur Analysis situs*, Math. Ann. **32** (1888), no. 4, 457–512. MR 1510522

[EF97] D. B. A. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36** (1997), no. 6, 1275–1289. MR 1452851

[FFL23] F. Fournier-Facio and Y. Lodha, *Second bounded cohomology of groups acting on 1-manifolds and applications to spectrum problems*, Adv. Math. **428** (2023), Paper No. 109162, 42. MR 4604795

[FFMS] F. Fournier-Facio, G. Mangioni, and A. Sisto, *Bounded cohomology, quotient extensions, and hierarchical hyperbolicity*, arXiv preprint arXiv:2505.20462.

[FFT19] T. Fernós, M. Forester, and J. Tao, *Effective quasimorphisms on right-angled Artin groups*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 4, 1575–1626. MR 4010865

[FFW23] F. Fournier-Facio and R. D. Wade, Aut*-invariant quasimorphisms on groups*, Trans. Amer. Math. Soc. **376** (2023), no. 10, 7307–7327. MR 4636691

[FPS15] R. Frigerio, M. B. Pozzetti, and A. Sisto, *Extending higher-dimensional quasi-cocycles*, J. Topol. **8** (2015), no. 4, 1123–1155. MR 3431671

[Fri17] R. Frigerio, *Bounded cohomology of discrete groups*, Mathematical Surveys and Monographs, vol. 227, American Mathematical Society, Providence, RI, 2017. MR 3726870

[Ghy87] E. Ghys, *Groupes d'homéomorphismes du cercle et cohomologie bornée*, The Lefschetz centennial conference, Part III (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1987, pp. 81–106. MR 893858

[GL23] N. Guelman and I. Liousse, *Uniform simplicity for subgroups of piecewise continuous bijections of the unit interval*, Bull. Lond. Math. Soc. **55** (2023), no. 5, 2341–2362, With an appendix by Pierre Arnoux. MR 4672899

[Heu23] N. Heuer, *Stable commutator length*, Bounded cohomology and simplicial volume, London Math. Soc. Lecture Note Ser., vol. 479, Cambridge Univ. Press, Cambridge, 2023, pp. 65–76. MR 4496346

[HO13] M. Hull and D. Osin, *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol. **13** (2013), no. 5, 2635–2665. MR 3116299

[JZ24] A. Jaikin-Zapirain, *Free* $\mathbb{Q}$*-groups are residually torsion-free nilpotent*, Ann. Sci. Éc. Norm. Supér. (4) **57** (2024), no. 4, 1101–1133. MR 4773301

[KKM$^+$24] M. Kawasaki, M. Kimura, S. Maruyama, T. Matsushita, and M. Mimura, *Survey on invariant quasimorphisms and stable mixed commutator length*, Topology Proc. **64** (2024), 129–174. MR 4773366

[Kot08] D. Kotschick, *Stable length in stable groups*, Groups of diffeomorphisms, Adv. Stud. Pure Math., vol. 52, Math. Soc. Japan, Tokyo, 2008, pp. 401–413. MR 2509718

[Mak82] G. S. Makanin, *Equations in a free group*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 6, 1199–1273, 1344. MR 682490

[Mal04] A. V. Malyutin, *Writhe of (closed) braids*, Algebra i Analiz **16** (2004), no. 5, 59–91. MR 2106667

[Mar] A. Marchand, *Bavard duality for the relative Gromov seminorm*, arXiv:2305.02149. To appear in Enseign. Math.

[Mar25] ———, *Isometric embeddings of surfaces for scl*, Groups Geom. Dyn. **19** (2025), no. 4, 1445–1478. MR 4957790

[Mer66] Ju. I. Merzljakov, *Positive formulae on free groups*, Algebra i Logika Sem. **5** (1966), no. 4, 25–42. MR 222149

[MR94] A. G. Myasnikov and V. N. Remeslennikov, *Degree groups. I. Foundations of the theory and tensor completions*, Sibirsk. Mat. Zh. **35** (1994), no. 5, 1106–1118, iii. MR 1308240

[MR96] ———, *Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups*, Internat. J. Algebra Comput. **6** (1996), no. 6, 687–711. MR 1421886

[PR14] L. Polterovich and D. Rosen, *Function theory on symplectic manifolds*, CRM Monograph Series, vol. 34, American Mathematical Society, Providence, RI, 2014. MR 3241729

[Seg09] D. Segal, *Words: notes on verbal width in groups*, London Mathematical Society Lecture Note Series, vol. 361, Cambridge University Press, Cambridge, 2009. MR 2547644

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK
E-mail address: ff373@cam.ac.uk