On the asymptotic properties of a canonical diffraction integral

Raphaël C. Assier* and I. David Abrahams†

* Department of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK

† Isaac Newton Institute, University of Cambridge, 20 Clarkson Road, Cambridge CB3 0EH, UK

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Abstract

We introduce and study a new canonical integral, denoted I^{ε}_{+-}, depending on two complex parameters α_1 and α_2. It arises from the problem of wave diffraction by a quarter-plane, and is heuristically constructed to capture the complex field near the tip and edges. We establish some region of analyticity of this integral in $\mathbb{C}^2$, and derive its rich asymptotic behaviour as $|\alpha_1|$ and $|\alpha_2|$ tend to infinity. We also study the decay properties of the function obtained from applying a specific double Cauchy integral operator to this integral. These results allow us to show that this integral shares all of the asymptotic properties expected from the key unknown function G_{+-} arising when the quarter-plane diffraction problem is studied via a two-complex-variables Wiener–Hopf technique (see Assier & Abrahams, arXiv:1905.03863, 2020). As a result, the integral I^{ε}_{+-} can be used to mimic the unknown function G_{+-} and to build an efficient 'educated' approximation to the quarter-plane problem.

1 Introduction and motivation

We propose to study the properties (asymptotic behaviour, analyticity and more) of the integral I^{ε}_{+-} that can be considered a function of two complex variables (α_1, α_2) and is defined by

$$I^{\varepsilon}_{+-}(\alpha_1, \alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2}, \varphi\right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu+3/2}} \mathrm{d}\varphi, \tag{1.1}$$

where the constants ε and ν and the function $f\left(\frac{\pi}{2}, \varphi\right)$ will be specified later. This work is directly motivated by the conclusions discussed in a recent article by the authors, [2], which focuses on a two-complex-variable investigation of the three-dimensional problem of wave diffraction by a quarter-plane with homogeneous Dirichlet boundary conditions subject to an incident plane wave.

The quarter-plane problem is an unsolved *Canonical Scattering Problem* (CSP). Typically, CSPs relate to wave diffraction by 'simple' geometries exhibiting some characteristic features such as curvature, edges or corners. They are the building blocks of most wave diffraction approximation techniques (such as the Geometrical Theory of Diffraction [7]) used for more complicated, real-life, geometries and are extremely important in applications including e.g. noise reduction and radar. Famous examples of solved CSPs are e.g. the Sommerfeld problem of diffraction by a half-plane [12] and wedge diffraction problems; both of these two-dimensional CSPs can be tackled by the

Wiener-Hopf technique in one complex variable [10, 8, 9]. The solutions are expressed as complex integrals whose far-field asymptotic behaviour can be written down exactly. The quarter-plane problem can be seen as the direct three-dimensional generalisation of the Sommerfeld problem.

Another well-known canonical integral, akin to (1.1) in the sense that it can be interpreted as a function of two complex variables, and with application to scattering amplitude in string theory [13], is the Beta function (also known as the Euler integral of the first kind).

The quarter-plane is occupying the $(x_1 > 0, x_2 > 0, x_3 = 0)$ subspace of a (x_1, x_2, x_3) Cartesian space. The total physical wave field is denoted $u(x_1, x_2, x_3)$ and the incident plane wave takes the form $e^{-i(a_1x_1+a_2x_2+a_3x_3)}$, where the constants $a_{1,2,3}$ depend solely on the incident direction and the wave number $k > 0$ such that $a_1^2 + a_2^2 + a_3^2 = k^2$. Note also that the time factor $e^{-i\omega t}$, where ω is the angular frequency of the wave, has been suppressed for brevity. This physical solution can be expressed in the form of an inverse Fourier transform

$$u(\boldsymbol{x}, x_3) \quad = \quad \frac{1}{(2\pi)^2} \int_{\mathcal{A}_1} \int_{\mathcal{A}_2} F_{++}(\boldsymbol{\alpha}) e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{x}} e^{i\frac{x_3}{K(\boldsymbol{\alpha})}} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2, \tag{1.2}$$

where $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{C}^2$ and the contours $\mathcal{A}_{1,2}$ defined in Figure 1 naturally lead to the definition of the upper and lower half planes $\mathrm{UHP}_{1,2}$ and $\mathrm{LHP}_{1,2}$ of the $\alpha_{1,2}$ complex planes. It is also useful to define the domains $\mathcal{D}_{++} = \mathrm{UHP}_1 \times \mathrm{UHP}_2$ and $\mathcal{D}_{+-} = \mathrm{UHP}_1 \times \mathrm{LHP}_2$. The crucial unknown function $F_{++}(\boldsymbol{\alpha})$ is to be determined and $K(\boldsymbol{\alpha}) = (k^2 - \alpha_1^2 - \alpha_2^2)^{-1/2}$ is the *kernel* of the problem.

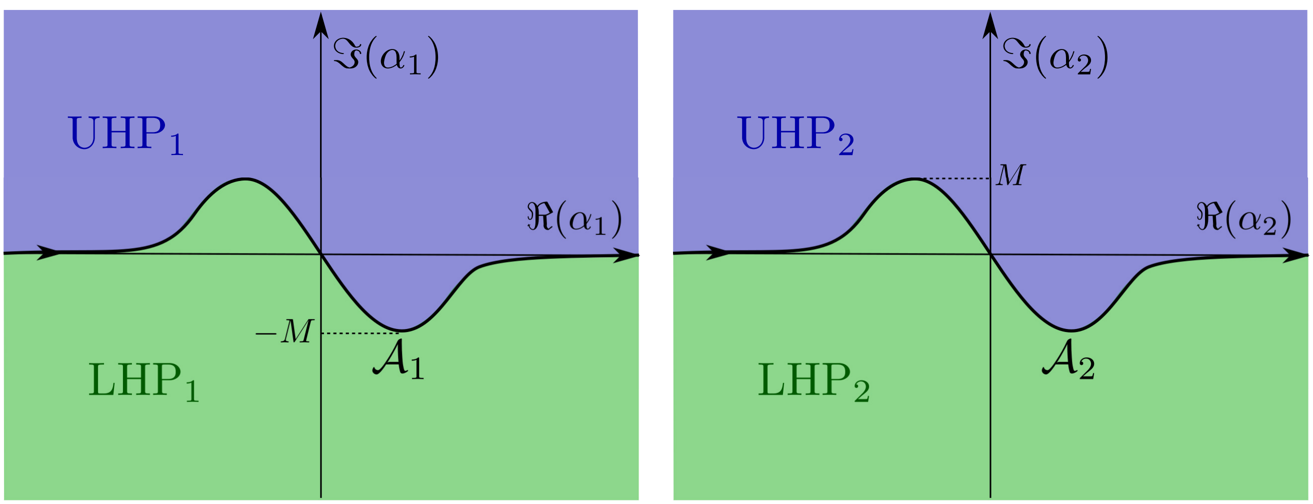

Figure 1: Description of the upper and lower-half planes $\mathrm{UHP}_{1,2}$ and $\mathrm{LHP}_{1,2}$ lying on and above and on and below the integration contours $\mathcal{A}_{1,2}$.

One of the achievements of [2] is the reduction of the complicated problem of diffraction by a quarter-plane to two equations in the two-complex-variables Fourier space. They relate the two unknown functions $F_{++}(\boldsymbol{\alpha})$ and $G_{+-}(\boldsymbol{\alpha})$, analytic on $\mathcal{D}_{++}$ and $\mathcal{D}_{+-}$ respectively, as follows

$$F_{++} = F_{++}^{\star} + \mathcal{I}[G_{+-}] \;\text{ on } \mathcal{D}_{++}, \tag{1.3}$$

$$0 = G_{+-}^{\star} + \mathcal{J}[G_{+-}] \text{ on } \mathcal{D}_{+-}, \tag{1.4}$$

where $\mathcal{I}$ and $\mathcal{J}$ are explicitly-known Cauchy integral operators depending on the kernel K, and the two functions $F_{++}^{\star}(\boldsymbol{\alpha})$ and $G_{+-}^{\star}(\boldsymbol{\alpha})$ are also known explicitly and can be written in terms of

K. It is remarkable that, in (1.3), the term $F^{\star}_{++}$ turns out to be exactly Radlow's erroneous ansatz [11], while (1.4), the so-called *compatibility equation*, only contains the unknown function G_{+-}. If (1.4) could somehow be inverted, then G_{+-} would be known, and hence F_{++} would be known by (1.3); see [2] for a more detailed interpretation of these equations.

In this article, we will focus on the function G_{+-}, since it leads to F_{++} directly via (1.3) and hence to the sought-after u via (1.2). Though unknown, this function can be expressed in terms of the wave field u by

$$G_{+-}(\boldsymbol{\alpha}) \;=\; \int_{-\infty}^{0}\int_{0}^{\infty} u(\boldsymbol{x},0)e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x}}\mathrm{d}x_1\mathrm{d}x_2. \tag{1.5}$$

The physical field u, restricted to the quadrant $(x_1 \geq 0, x_2 \leq 0, x_3 = 0)$, must obey the following edge and vertex conditions:

$$u(\boldsymbol{x},0) \underset{(x_1,x_2)\to(0^+,0^-)}{\overset{|x_2|\propto|x_1|}{\approx}} \mathcal{O}(r^{\nu-1/2}), \tag{1.6}$$

$$u(\boldsymbol{x},0) \underset{x_1\to 0^+}{\overset{x_2<0 \text{ fixed}}{\approx}} \mathcal{O}(1), \tag{1.7}$$

$$u(\boldsymbol{x},0) \underset{x_2\to 0^-}{\overset{x_1>0 \text{ fixed}}{\approx}} \mathcal{O}(|x_2|^{1/2}), \tag{1.8}$$

where $r = \sqrt{x_1^2 + x_2^2}$ is the distance to the vertex and

$$\nu = \sqrt{\ell_1 + 1/4} \approx 0.7967,$$

with ℓ_1 being the first eigenvalue of the Laplace-Beltrami operator (LBO) with Dirichlet conditions on the cut defined by $\left\{\theta = \frac{\pi}{2}, \varphi \in \left[0, \frac{\pi}{2}\right]\right\}$ in the usual spherical coordinates (see e.g. [3], [4]). Physically, the restrictions (1.6) and (1.8) come from imposing that the energy remains bounded in the neighbourhood of the vertex $(x_1 = 0, x_2 = 0, x_3 = 0)$ and the edge $(x_1 > 0, x_2 = 0, x_3 = 0)$ respectively. The exact value of the exponent comes from a separation of variable argument in spherical coordinates for (1.6) and cylindrical polar coordinates (around the edge) for (1.8). These conditions are often referred to as Meixner conditions. The condition (1.7) is representing the fact that the field is perfectly well behaved since no element of the scatterer is encountered in this limit.

In the Fourier space, using the Abelian theorems (see e.g. [10]), these conditions translate into the following asymptotic behaviour for G_{+-}:

$$G_{+-}(\alpha_1,\alpha_2) \underset{|\alpha_1|\to\infty}{\overset{|\alpha_2|\propto|\alpha_1|}{=}} \mathcal{O}\left(\frac{1}{|\alpha_1|^{\nu+3/2}}\right), \tag{1.9}$$

$$G_{+-}(\alpha_1,\alpha_2) \underset{|\alpha_1|\to\infty}{\overset{\alpha_2 \text{ fixed}}{=}} \mathcal{O}\left(\frac{1}{|\alpha_1|}\right), \tag{1.10}$$

$$G_{+-}(\alpha_1,\alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_1 \text{ fixed}}{=}} \mathcal{O}\left(\frac{1}{|\alpha_2|^{3/2}}\right), \tag{1.11}$$

for $\alpha_1 \in \mathrm{UHP}_1$ and $\alpha_2 \in \mathrm{LHP}_2$.

In order to derive the equations (1.3)–(1.4) rigorously in [2], we had to prove that G_{+-} satisfies an important property, i.e. that $\mathcal{I}[G_{+-}](\alpha_1,\alpha_2)$ tends to zero as $|\alpha_2| \to \infty$.

The purpose of the present article is two-fold; first to suggest an efficient approximation scheme to solve the quarter-plane problem and second to highlight a new canonical special function. The former will be done by introducing the explicitly defined integral I^{ε}_{+-}, and by showing that it mimics the behaviour of G_{+-}. By this we mean that I^{ε}_{+-} should be analytic on $\mathcal{D}_{+-}$, should have the asymptotic behaviour (1.9)–(1.11) and should satisfy $\mathcal{I}[I^{\varepsilon}_{+-}] \to 0$ as $|\alpha_2| \to \infty$.

The rest of the paper is organised as follows: in Section 2, we give the integral expression of I^{ε}_{+-} again and explain where it comes from; in Section 3 we highlight some important properties of the first Laplace-Beltrami eigenfunction; in Section 4 we prove that I^{ε}_{+-} does indeed have the correct asymptotic behaviour (1.9)–(1.11); and in Section 5, we prove that $\mathcal{I}[I^{\varepsilon}_{+-}]$ does tend to zero as $|\alpha_2|$ tends to infinity. Finally, we discuss the implications of our findings and conclude the paper in Section 6.

2 A canonical integral

In this section, we aim to derive and construct explicitly a function defined by an integral that satisfies all the conditions required of G_{+-}. Starting from the integral representation (1.5) and using the change of variable $x_1 = r\cos(\varphi)$, $x_2 = r\sin(\varphi)$, for $r \in \mathbb{R}^+$ and $\varphi \in [3\pi/2, 2\pi]$, we can write

$$G_{+-}(\boldsymbol{\alpha}) \;=\; \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\infty} \hat{u}(r,\varphi) e^{ir(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi))} r\mathrm{d}r\mathrm{d}\varphi, \tag{2.1}$$

where $\hat{u}(r,\varphi) = u(r\cos(\varphi), r\sin(\varphi), 0)$. Moreover, as explained in [2] for example, using separation of variables, it can be shown that

$$\hat{u}(r,\varphi) \underset{r\to 0}{\overset{\varphi\in\left[\frac{3\pi}{2},2\pi\right)}{\sim}} Af\left(\frac{\pi}{2},\varphi\right) r^{\nu-1/2}, \tag{2.2}$$

where $f(\theta,\varphi)$ is the eigenfunction of the LBO associated to the first eigenvalue λ_1 and A is a constant. For technical reasons that will become apparent later on, let us rewrite the asymptotic behaviour (2.2) in a slightly different form

$$\hat{u}(r,\varphi) \underset{r\to 0}{\sim} Af\left(\frac{\pi}{2},\varphi\right) r^{\nu-1/2} e^{-\varepsilon r}, \tag{2.3}$$

for some $\varepsilon > 0$. Note that (2.2) and (2.3) are equivalent to leading order since $e^{-\varepsilon r} \to 1$ as $r \to 0$. Because the asymptotic behaviour as $|\alpha_{1,2}| \to \infty$ in the Fourier space is intrinsically linked to the near-field behaviour in the physical space, we are interested in the integral I obtained by replacing $\hat{u}(r,\varphi)$ by its leading order behaviour (2.3) in (2.1):

$$I \;=\; A\int_{\frac{3\pi}{2}}^{2\pi} f\left(\frac{\pi}{2},\varphi\right)\int_0^{\infty} r^{\nu+1/2} e^{ir(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi))} e^{-\varepsilon r}\mathrm{d}r\mathrm{d}\varphi.$$

Note that the integral over r takes the form

$$\int_0^{\infty} r^{\lambda-1} e^{-\mu r}\mathrm{d}r, \tag{2.4}$$

for $\lambda = \nu + 3/2$ and $\mu = -i(\alpha_1\cos(\varphi) + \alpha_2\sin(\varphi) + i\varepsilon)$. Note (see [6] p317, 3.381.4.) that for $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\mu) > 0$, this integral is exactly equal to $\frac{\Gamma(\lambda)}{\mu^{\lambda}}$, where Γ is the Euler Gamma function. It is clear that $\mathrm{Re}(\lambda) > 0$ since $\nu \geqslant 0$. Moreover, we can choose ε such that $\mathrm{Re}(\mu) > 0$.

In order to do so, we refer to Figure 1, to see that for all $\boldsymbol{\alpha} \in \mathcal{D}_{+-}$, we have $\mathrm{Im}(\alpha_1) \geqslant -M$ and $\mathrm{Im}(\alpha_2) \leqslant M$ for some $M > 0$ depending on the choice of contour $\mathcal{A}_{1,2}$. Remembering that for $\varphi \in \left[\frac{3\pi}{2}, 2\pi\right]$ we have $0 \leqslant \cos(\varphi) \leqslant 1$ and $-1 \leqslant \sin(\varphi) \leqslant 0$, it is possible to show that upon choosing ε such that $\varepsilon > 2M$, we have $\mathrm{Re}(\mu) > 0$ for all $\boldsymbol{\alpha} \in \mathcal{D}_{+-}$, and hence I can be rewritten as

$$I = A\Gamma(\nu + 3/2) \int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2}, \varphi\right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu+3/2}} \mathrm{d}\varphi.$$

Because of our choice of ε the denominator of the integrand is never zero for $\boldsymbol{\alpha} \in \mathcal{D}_{+-}$; hence the integral is a '$+-$' function, i.e. it is analytic in $\mathcal{D}_{+-}$. This naturally leads to the definition of the canonical integral I^{ε}_{+-} to be studied in this paper:

$$I^{\varepsilon}_{+-}(\alpha_1, \alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2}, \varphi\right)}{(-i(\alpha_1 \cos(\varphi) + \alpha_2 \sin(\varphi) + i\varepsilon))^{\nu+3/2}} \mathrm{d}\varphi. \tag{2.5}$$

For the purpose of the present work, the values of k and M (and hence ε) can be any strictly positive numbers. For numerical illustration of our theoretical results, we will choose $k = 3$ and $\varepsilon = 1$.

3 A note on the Laplace-Beltrami eigenfunction

Before deriving various properties of the newly-introduced integral I^{ε}_{+-}, it is important to know the behaviour of the eigenfunction $f\left(\frac{\pi}{2}, \varphi\right)$. Its important properties are summarised in the following lemma, the proof of which (linked to the physical edge conditions) is omitted here for brevity.

Lemma 1 Let $f(\theta, \varphi)$ be the first eigenfunction of the LBO. Then $f\left(\frac{\pi}{2}, \varphi\right)$ is equal to zero on the cut $\varphi \in \left[0, \frac{\pi}{2}\right]$, and is smooth and such that $0 < f\left(\frac{\pi}{2}, \varphi\right) \leqslant 1$ for $\varphi \in \left(\frac{\pi}{2}, 2\pi\right)$. Moreover, its behaviour at the edge of the non-zero region is given by

$$f\left(\frac{\pi}{2}, \varphi\right) \underset{\varphi \to \pi/2}{\overset{\varphi > \pi/2}{=}} \mathcal{O}\left(\sqrt{\varphi - \pi/2}\right) \quad \text{and} \quad f\left(\frac{\pi}{2}, \varphi\right) \underset{\varphi \to 2\pi}{\overset{\varphi < 2\pi}{=}} \mathcal{O}\left(\sqrt{2\pi - \varphi}\right).$$

In particular, there exists a constant β, such that

$$f\left(\frac{\pi}{2}, 2\pi - \psi\right) \underset{\psi \to 0}{\overset{\psi > 0}{\sim}} \beta\sqrt{\psi}.$$

In addition, it transpires that $f\left(\frac{\pi}{2}, \varphi\right)$ is strictly decreasing for $\varphi \in \left[\frac{3\pi}{2}, 2\pi\right]$.

Though not an exact result, it seems that $f\left(\frac{\pi}{2}, \varphi\right)$ is approximated very well by the function $g(\varphi)$ defined by

$$g(\varphi) = \begin{cases} 0 & \text{if} \quad \varphi \in \left[0, \frac{\pi}{2}\right], \\ \sqrt{\sin\left(\frac{2\varphi - \pi}{3}\right)} & \text{if} \quad \varphi \in \left[\frac{\pi}{2}, 2\pi\right]. \end{cases}$$

Note that g trivially satisfies the conclusions of Lemma 1, with $\beta = \sqrt{\frac{2}{3}}$. In Figure 2, we compare numerical results obtained using a surface finite element method developed in [4] and the function g, showing excellent agreement.

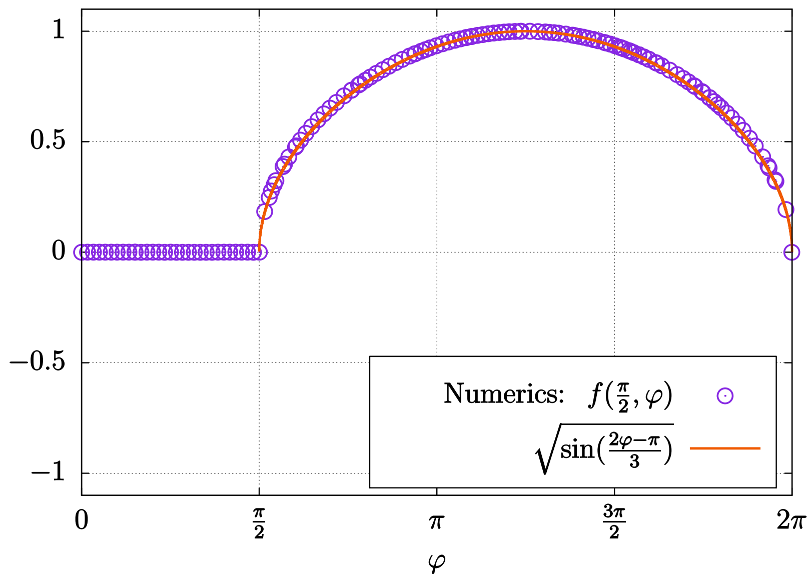

Figure 2: Comparison between a numerical approximation of $f\left(\frac{\pi}{2},\varphi\right)$ and the function $g(\varphi)$.

4 Asymptotic behaviour of I^{ε}_{+-}

In this section, we show that the integral $I^{\varepsilon}_{+-}(\alpha_1,\alpha_2)$ shares the same rich asymptotic behaviour as $G_{+-}(\alpha_1,\alpha_2)$; that is, it should behave like (1.9)–(1.11), as $|\alpha_{1,2}|\to\infty$ within $\mathcal{D}_{+-}$. We will summarise the key results here, the detailed proofs being given in Appendix A. Since $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\mathcal{D}_{+-}$, whenever $|\alpha_1|$ (resp. $|\alpha_2|$) tends to infinity, we can write $\alpha_1=|\alpha_1|e^{i\phi_1}$ (resp. $\alpha_2=|\alpha_2|e^{i\phi_2}$) for $\phi_1\in(0,\pi)$ (resp. $\phi_2\in(-\pi,0)$). Note that if $\alpha_{1,2}$ are not assumed large, we cannot write them in this way, since they may lie within the indented part of the contours $\mathcal{A}_{1,2}$.

Asymptotic behaviour when both $|\alpha_1|$ and $|\alpha_2|$ tend to infinity within $\mathcal{D}_{+-}$ This is the simplest of the three different cases to be considered. We will take both $|\alpha_1|$ and $|\alpha_2|\to\infty$ within $\mathcal{D}_{+-}$, in such a way that there exists an $m>0$ such that $|\alpha_2|=m|\alpha_1|$. In this case, we can write $\alpha_{1,2}=|\alpha_{1,2}|e^{i\phi_{1,2}}$ and we have

$$I^{\varepsilon}_{+-}(\alpha_1,\alpha_2)\underset{|\alpha_1|\to\infty}{\overset{|\alpha_2|=m|\alpha_1|}{\sim}}\frac{1}{|\alpha_1|^{\nu+3/2}}I^{0}_{+-}(e^{i\phi_1},me^{i\phi_2}).\tag{4.1}$$

The validity of this asymptotic behaviour is illustrated in Figure 3.

Asymptotic behaviour when $|\alpha_1|\to\infty$ within UHP_1 and α_2 is fixed in LHP_2 In this case, we can write $\alpha_1=|\alpha_1|e^{i\phi_1}$, and we have

$$I^{\varepsilon}_{+-}(\alpha_1,\alpha_2)\underset{|\alpha_1|\to\infty}{\overset{\alpha_2\text{ fixed}}{\sim}}\frac{\Lambda_1(\alpha_2,\varepsilon)}{\alpha_1},\tag{4.2}$$

where

$$\Lambda_1(\alpha_2,\varepsilon)=\frac{2if\left(\frac{\pi}{2},\frac{3\pi}{2}\right)}{(1+2\nu)}\times\frac{1}{(\varepsilon+i\alpha_2)^{\nu+1/2}}.$$

As can be seen in Appendix A, the proof is slightly more subtle than the previous case, and one needs to split the φ integral into two parts, one where $\cos(\varphi)$ is very small, and one where it is bounded away from zero. The validity of the asymptotic behaviour (4.2) is illustrated in Figure 4.

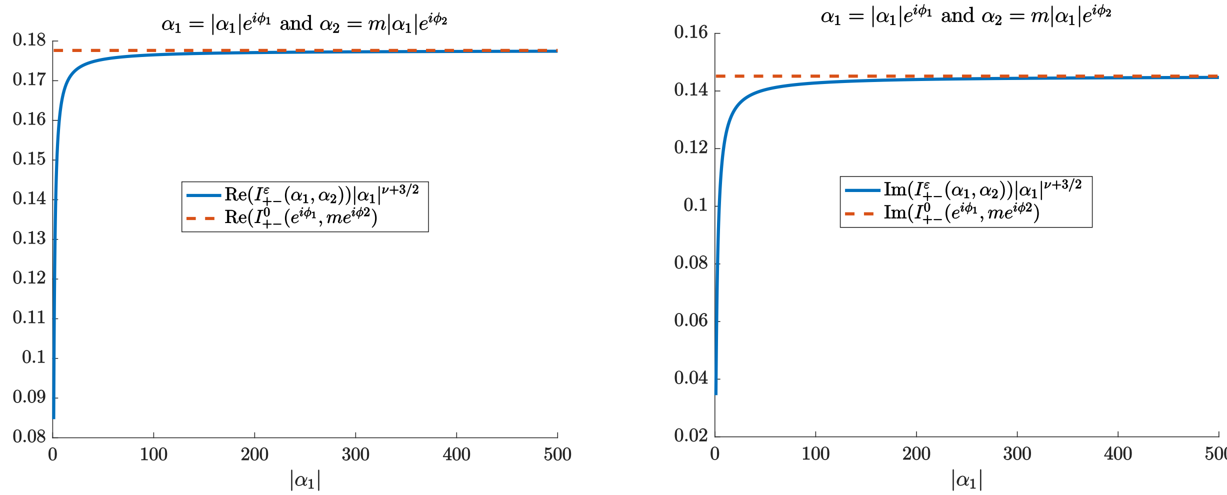

Figure 3: Numerical illustration of the asymptotic behaviour (4.1) as both $|\alpha_{1,2}| \to \infty$ for $\phi_1 = \frac{\pi}{4}$, $\phi_2 = -\frac{\pi}{2}$, $\varepsilon = 1$ and $m = 2$, using $g(\varphi)$ instead of $f\left(\frac{\pi}{2}, \varphi\right)$ in the definition of I^{ε}_{+-}.

Asymptotic behaviour when $|\alpha_2| \to \infty$ within LHP_2 and α_1 is fixed in UHP_1 In this case, we can write $\alpha_2 = |\alpha_2|e^{i\phi_2}$, and we have

$$I^{\varepsilon}_{+-}(\alpha_1, \alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_1 \text{ fixed}}{\sim}} \frac{\Lambda_2(\alpha_1, \varepsilon)}{\alpha_2^{3/2}}, \tag{4.3}$$

where

$$\Lambda_2(\alpha_1, \varepsilon) \quad = \quad \frac{\beta\sqrt{\pi}\Gamma(\nu)e^{-3i\frac{\pi}{4}}}{2\Gamma(\nu + 3/2)} \times \frac{1}{(\varepsilon - i\alpha_1)^{\nu}}.$$

Here again, as can be seen in Appendix A, the proof is subtle and requires a particular split of the φ integral, this time according to whether or not $\sin(\varphi)$ is close to being zero. The validity of the asymptotic behaviour (4.3) is illustrated in Figure 5.

Remark 1 Even though the results in this section have been derived for $\phi_1 \in (0, \pi)$ and $\phi_2 \in (-\pi, 0)$, they remain valid for $\phi_1 = 0, \pi$ and $\phi_2 = -\pi, 0$; this is easily checked numerically. Hence these asymptotic results can be used when $|\alpha_{1,2}| \to \infty$ along $\mathcal{A}_{1,2}$.

Remark 2 If we let $|\alpha_2| \to \infty$ in (4.2), we obtain a quantity behaving like $\mathcal{O}(\alpha_1^{-1}/\alpha_2^{\nu+1/2})$,while if we let $|\alpha_1| \to \infty$ in (4.3), we obtain a quantity behaving like $\mathcal{O}(\alpha_1^{-\nu}/\alpha_2^{3/2})$, both expressions being compatible with the behaviour (4.1), for $|\alpha_1| \propto |\alpha_2|$.

We have hence shown that I^{ε}_{+-} and G_{+-} have the same asymptotic behaviour at infinity. We mentioned in the introduction that in order to prove the main result of [2], that is (1.3)–(1.4), it was crucial to show that $\mathcal{I}[G_{+-}] \to 0$ as $|\alpha_2| \to \infty$. We hence expect that a good approximation to G_{+-} should have the same property. In the following section we will precisely define the integral operator $\mathcal{I}$ and show that I^{ε}_{+-} does indeed satisfy this crucial property.

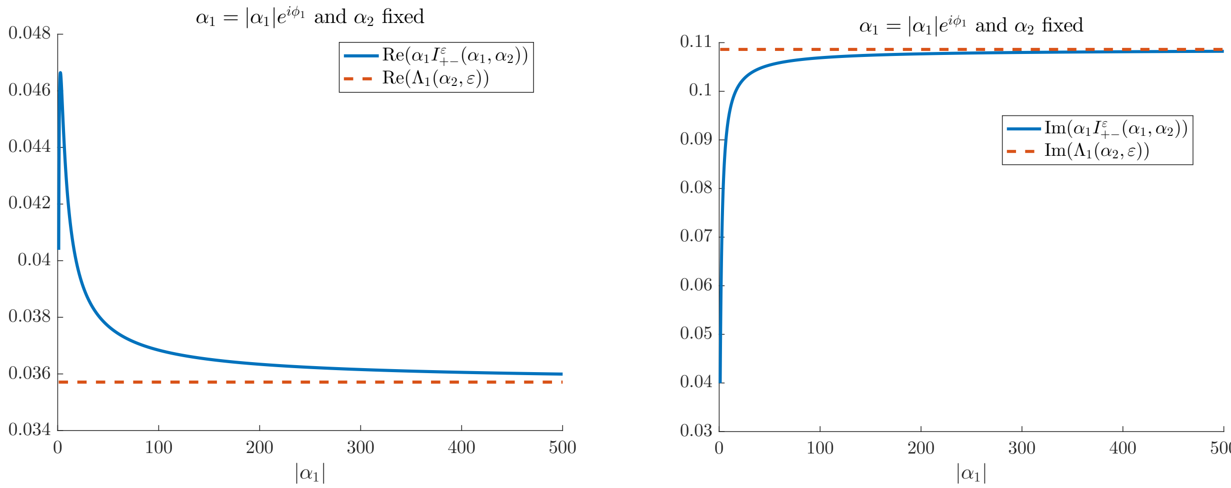

Figure 4: Numerical illustration of the asymptotic behaviour (4.2) as $|\alpha_1| \to \infty$ for $\phi_1 = \frac{\pi}{4}$, fixed $\alpha_2 = 1 - 3i$ and $\varepsilon = 1$, using $g(\varphi)$ instead of $f\left(\frac{\pi}{2}, \varphi\right)$ in the definition of I^{ε}_{+-}.

5 On the behaviour of $\mathcal{I}[I^{\varepsilon}_{+-}]$ as $|\alpha_2| \to \infty$

5.1 The integral operator $\mathcal{I}$

For a generic function $\Phi(\alpha_1, \alpha_2)$ analytic on $\mathcal{A}_1 \times \mathcal{A}_2$, the Cauchy integral operator $\mathcal{I}$ is defined as follows:

$$\mathcal{I}[\Phi] \quad = \quad \left[\frac{K_{-+}(a_1, \alpha_2)}{K_{+-}(\boldsymbol{\alpha})} \left[\frac{\Phi}{K_{-\circ}} \right]_{+\circ} \right]_{\circ +}, \tag{5.1}$$

where the functions K_{-+}, K_{+-} and $K_{-\circ}(\boldsymbol{\alpha}) = K_{--}(\boldsymbol{\alpha})K_{-+}(\boldsymbol{\alpha})$ are directly related to the four-way factorisation of the kernel $K(\alpha_1, \alpha_2)$ discussed in [2]. On $\mathcal{A}_1 \times \mathcal{A}_2$, the kernel K can be written as

$$K(\boldsymbol{\alpha}) \quad = \quad K_{++}(\boldsymbol{\alpha})K_{+-}(\boldsymbol{\alpha})K_{-+}(\boldsymbol{\alpha})K_{--}(\boldsymbol{\alpha}),$$

where $K_{++}(\boldsymbol{\alpha})$ is analytic on $\mathcal{D}_{++}$, etc. Explicit integral expressions for these factors, which may be evaluated very rapidly, are given in [2], and the following asymptotic behaviour is valid

$$K_{\pm\pm}(\alpha_1, \alpha_2) \underset{|\alpha_2| \to \infty}{\overset{\alpha_1 \text{ fixed}}{=}} \mathcal{O}(1/|\alpha_2|^{1/4}) \text{ for } \boldsymbol{\alpha} \in \mathcal{D}_{\pm\pm}. \tag{5.2}$$

The brackets $[\ \ ]_{+\circ}$ and $[\ \ ]_{\circ+}$ are Cauchy integral sum-split operators in the α_1 and α_2 complex planes respectively, defined for a generic function Φ by

$$[\Phi]_{+\circ}(\alpha_1, \alpha_2) = \frac{1}{2\pi i} \int_{\mathcal{A}^b} \frac{\Phi(z, \alpha_2)}{(z - \alpha_1)} \mathrm{d}z \quad \text{and} \quad [\Phi]_{\circ+}(\alpha_1, \alpha_2) = \frac{1}{2\pi i} \int_{\mathcal{A}^b} \frac{\Phi(\alpha_1, z)}{(z - \alpha_2)} \mathrm{d}z,$$

where $\mathcal{A}^b$ is a contour that lies just below $\mathcal{A}_1$ or $\mathcal{A}_2$ as appropriate. This ensures that $[\Phi]_{+\circ}$ and $[\Phi]_{\circ+}$ can be freely evaluated (and are analytic) on $\mathcal{A}_{1,2}$.

As discussed for example in [2] and [5], the function $K_{-\circ}$ can be written analytically from the α_1 factorisation of $K(\boldsymbol{\alpha})$, and is

$$K_{-\circ}(\alpha_1, \alpha_2) \quad = \quad \frac{1}{\sqrt{\sqrt{k^2 - \alpha_2^2} - \alpha_1}}, \tag{5.3}$$

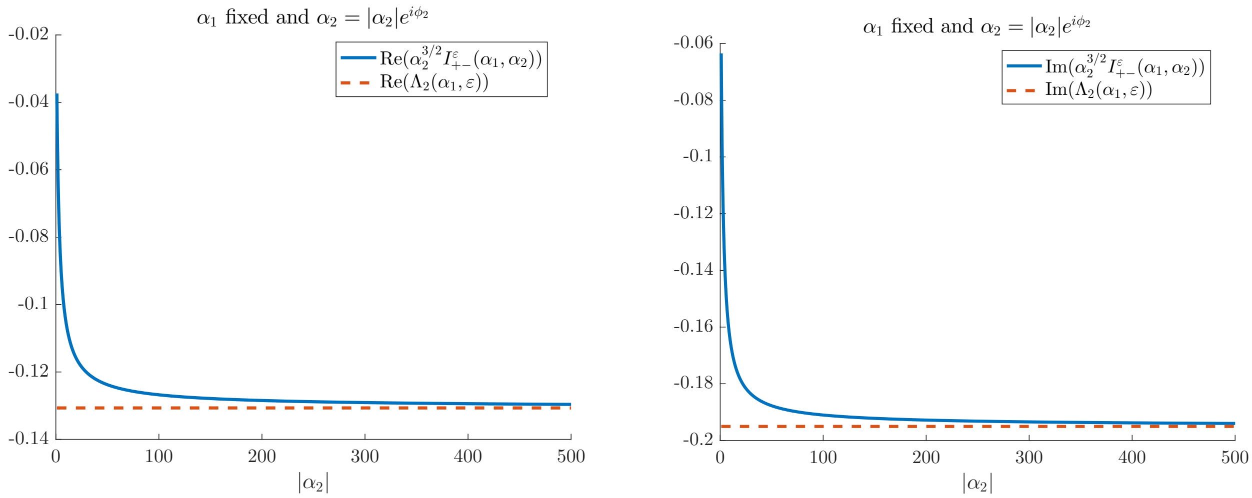

Figure 5: Numerical illustration of the asymptotic behaviour of (4.3) as $|\alpha_2| \to \infty$ for $\phi_2 = -\frac{\pi}{4}$, fixed $\alpha_1 = 1 + 3i$ and $\varepsilon = 1$, using $g(\varphi)$ instead of $f\left(\frac{\pi}{2}, \varphi\right)$ in the definition of I^{ε}_{+-}.

with a careful choice of branch-cut location (in UHP_1 when the function is seen as a function of α_1). We also remind the reader that a_1 is a constant depending on the incident angles.

Now that we have defined properly what was meant by $\mathcal{I}[I^{\varepsilon}_{+-}]$, we will prove in the two next sections that it does indeed tend to zero as $|\alpha_2|$ tends to infinity.

5.2 A sufficient condition

Because of (5.2), it is clear that

$$\frac{K_{-+}(a_1,\alpha_2)}{K_{+-}(\boldsymbol{\alpha})} \underset{|\alpha_2|\to\infty}{\overset{\alpha_1\in\mathrm{UHP}_1}{=}} \mathcal{O}(1). \tag{5.4}$$

It is well-known (see e.g. Lemma B.1 of [2]) that if $\Phi(\alpha_1, \alpha_2)$ tends to zero as a power of $|\alpha_2|$ as $|\alpha_2| \to \infty$ on $\mathcal{A}_2$, then the sum-split bracket $[\Phi]_{\circ+}$ tends to zero as $|\alpha_2| \to \infty$. Hence for $\mathcal{I}[I^{\varepsilon}_{+-}]$ to tend to zero as $|\alpha_2| \to \infty$, using (5.4), it is enough to show that

$$\left[\frac{I^{\varepsilon}_{+-}}{K_{-\circ}}\right]_{+\circ}(\alpha_1,\alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_2\in\mathcal{A}_2}{=}} \mathcal{O}(1/|\alpha_2|^{\gamma}), \tag{5.5}$$

for some $\gamma > 0$, which remains a non-trivial task that will be completed in what follows.

5.3 Proof strategy

By definition of the Cauchy bracket, and the integral I^{ε}_{+-}, we have

$$\left[\frac{I^{\varepsilon}_{+-}}{K_{-\circ}}\right]_{+\circ}(\boldsymbol{\alpha}) \;=\; \int_{\mathcal{A}^b} \Psi(z,\alpha_1,\alpha_2) I^{\varepsilon}_{+-}(z,\alpha_2)\mathrm{d}z, \tag{5.6}$$

where

$$\Psi(z,\alpha_1,\alpha_2) \;=\; \frac{1}{2\pi i(z-\alpha_1)K_{-\circ}(z,\alpha_2)}.$$

It is interesting to note that the singularities of the integrand of (5.6) in the z UHP are exclusively those of $\Psi(z, \alpha_1, \alpha_2)$ since $I^{\varepsilon}_{+-}(z, \alpha_2)$ is analytic there. Hence, in the z UHP, the integrand of (5.6) has one simple pole at $z = \alpha_1$ and one branch point at $z = \sqrt{k^2 - \alpha_2^2}$, with a branch-cut going vertically upwards as depicted in Figure 6 (left).

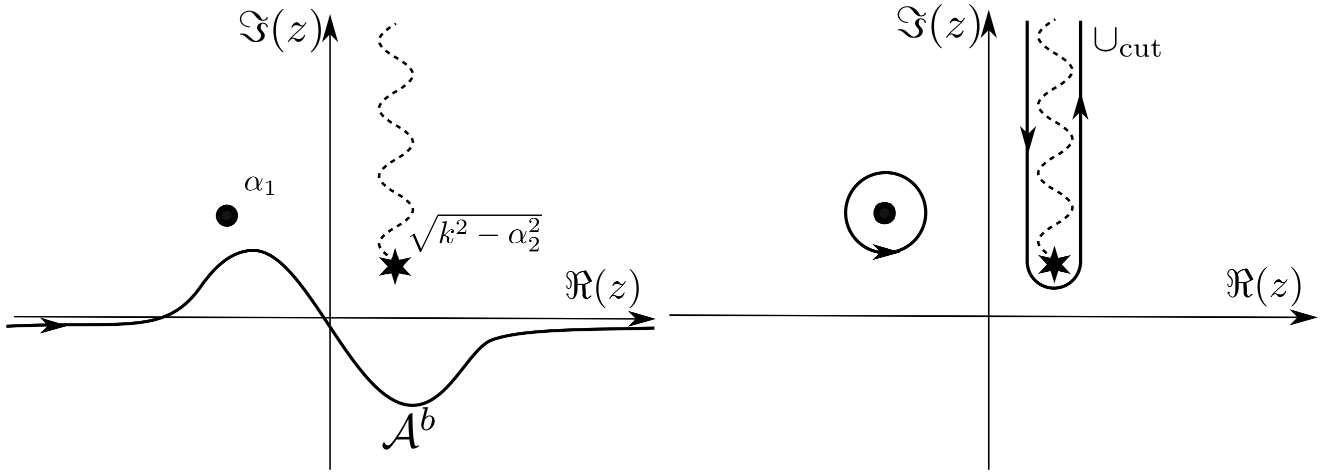

Figure 6: Singularity map of the integrand of (5.6) in the z UHP (left) and contour deformation around the pole and branch-cut (right).

It is hence possible to deform the contour from $\mathcal{A}^b$ to a contour $\cup_{\text{cut}}$ surrounding the branch cut. Assuming[1] for now that $\alpha_1 \neq \sqrt{k^2 - \alpha_2^2}$, the pole at $z = \alpha_1$ is picked up in the process and its contribution must be accounted for, see Figure 6 (right).

Noting that

$$2\pi i \operatorname*{Res}_{z=\alpha_1}(\Psi(z, \alpha_1, \alpha_2) I^{\varepsilon}_{+-}(z, \alpha_2)) \quad = \quad \frac{I^{\varepsilon}_{+-}(\alpha_1, \alpha_2)}{K_{-\circ}(\alpha_1, \alpha_2)},$$

we can write

$$\left[\frac{I^{\varepsilon}_{+-}}{K_{-\circ}}\right]_{+\circ}(\boldsymbol{\alpha}) \quad = \quad \frac{I^{\varepsilon}_{+-}(\alpha_1, \alpha_2)}{K_{-\circ}(\alpha_1, \alpha_2)} + I_{\text{cut}}(\alpha_1, \alpha_2), \tag{5.7}$$

where

$$I_{\text{cut}}(\alpha_1, \alpha_2) \quad = \quad \int_{\cup_{\text{cut}}} \Psi(z, \alpha_1, \alpha_2) I^{\varepsilon}_{+-}(z, \alpha_2) \mathrm{d}z.$$

Now using the fact that Ψ changes sign across the cut, and that it is equal to zero at the branch point, we can rewrite this integral in the slightly simpler form

$$I_{\text{cut}}(\alpha_1, \alpha_2) \quad = \quad 2i \int_0^{\infty} \Psi\left(\sqrt{k^2 - \alpha_2^2} + it, \alpha_1, \alpha_2\right) I^{\varepsilon}_{+-}\left(\sqrt{k^2 - \alpha_2^2} + it, \alpha_2\right) \mathrm{d}t,$$

where Ψ is only evaluated on the right side of its cut.

[1]Since we are interested in the behaviour of this bracket for fixed α_1 as $|\alpha_2| \to \infty$, we can make this assumption without loss of generality.

At this stage, it is useful to note that by (5.3), we have

$$K_{-\circ}(\alpha_1, \alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_1 \text{ fixed}}{=}} \mathcal{O}(|\alpha_2|^{-1/2}). \tag{5.8}$$

Using this and the asymptotic result (4.3), we find that the first term in the RHS of (5.7) behaves like

$$\frac{I^{\varepsilon}_{+-}(\alpha_1, \alpha_2)}{K_{-\circ}(\alpha_1, \alpha_2)} \underset{|\alpha_2|\to\infty}{\overset{\alpha_1 \text{ fixed}}{=}} \mathcal{O}\left(\frac{1}{|\alpha_2|}\right), \tag{5.9}$$

which satisfies the condition (5.5). Moreover, we show in Appendix B that we also have

$$I_{\text{cut}}(\alpha_1, \alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_1 \text{ fixed}}{=}} \mathcal{O}\left(\frac{1}{|\alpha_2|^{\nu+1}}\right), \tag{5.10}$$

which also satisfies the condition (5.5). Numerical evaluation of this integral confirms this finding, as illustrated on Figure 7.

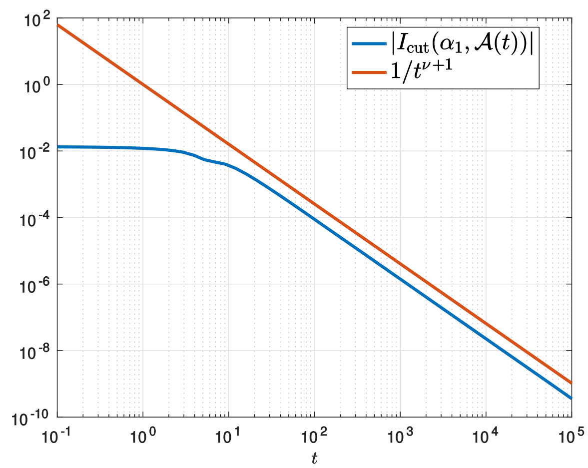

Figure 7: Log log plot of the absolute value of I_{cut} for $\alpha_1 = 1 + 3i$, and α_2 tends to infinity on the contour $\mathcal{A}$. Here, $\mathcal{A}$ is parametrised by a real parameter t, hence the $\mathcal{A}(t)$ notation. $\mathcal{A}(t)$ here should be understood to take complex values, but be such that $\mathcal{A}(t) \sim t$ as $t \to \infty$. The decay is compared to that of $1/t^{\nu+1}$.

In conclusion, using (5.7), (5.9) and (5.10), it is quite clear that as $|\alpha_2| \to \infty$ on $\mathcal{A}_2$, we have

$$\left[\frac{I^{\varepsilon}_{+-}}{K_{-\circ}}\right]_{+\circ}(\alpha_1, \alpha_2) = \mathcal{O}\left(\frac{1}{|\alpha_2|}\right),$$

meaning that the condition (5.5) is fulfilled for $\gamma = 1$. We can hence conclude this section by saying that

$$\mathcal{I}[I^{\varepsilon}_{+-}] \underset{|\alpha_2|\to\infty}{\longrightarrow} 0.$$

6 Significance and perspectives

In this work we have introduced an explicit canonical integral, $I^{\varepsilon}_{+-}(\alpha_1, \alpha_2)$, that is very similar to the key unknown function $G_{+-}(\alpha_1, \alpha_2)$ in the quarter-plane problem, in the sense that they share the same asymptotic behaviour as $|\alpha_{1,2}| \to \infty$, and they share the same crucial property $\mathcal{I}[G_{+-}] \to 0$ and $\mathcal{I}[I^{\varepsilon}_{+-}] \to 0$. The latter was used in [2] to prove the two formulae (1.3) (including Radlow's ansatz) and (1.4) (coined the *compatibility equation*).

The exact form of the integral I^{ε}_{+-} has the potential to be of great assistance to the design of a scheme to accurately approximate the key unknown function G_{+-}. One could for example consider an approximation of the type

$$G^{(N)}_{+-}(\boldsymbol{\alpha}) \;=\; \mathcal{C} I^{\varepsilon}_{+-}(\boldsymbol{\alpha}) T_{+-}(\boldsymbol{\alpha}) \left(1 + \sum_{j=1}^{N} g^{(j)}_{+-}(\boldsymbol{\alpha}) \right),$$

where $\mathcal{C}$ is a constant, $T_{+-}(\boldsymbol{\alpha})$ is a bounded function (but not decaying to zero) at infinity, and the functions $g^{(j)}_{+-}(\boldsymbol{\alpha})$ are a set of simple functions (possibly with simple poles at given locations, but with unknown residues) that decay to zero at infinity. We must choose T_{+-} and $g^{(j)}_{+-}$ to be analytic in $\mathcal{D}_{+-}$. Note that the aim here is similar to that of [1] say, where functions analytic in a half-plane were approximated via Padé approximants.

We could for example choose the T_{+-} function to take the form

$$T_{+-}(\boldsymbol{\alpha}) \;=\; \frac{\mathcal{L}\alpha_1 - \alpha_2}{\alpha_1 - \alpha_2},$$

for some unknown constant $\mathcal{L}$, while the $g^{(j)}_{+-}$ could be chosen to take the form

$$g^{(j)}_{+-} \;=\; \frac{\mathcal{R}^{(j)}}{(\alpha_1 - a^{(j)}_1)(\alpha_2 - a^{(j)}_2)},$$

for some specified $(a^{(j)}_1, a^{(j)}_2) \in \mathcal{D}_{-+}$, and some unknown residues $\mathcal{R}^{(j)}$.

The correcting function T_{+-} and the constant $\mathcal{C}$ are included to compensate for the fact that the pre-factors (depending on ε) in the asymptotic results (4.1)–(4.3) on I^{ε}_{+-} may not be exactly equal to those of G_{+-}.

For a given N, we will hence have $N + 2$ unknowns: $(\mathcal{C}, \mathcal{L}, \mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(N)})$, which will be determined by ensuring that the compatibility equation is satisfied at a set of $N + 2$ collocation points. The implementation of such scheme is beyond the scope of the present work and will constitute the basis of further investigations by the authors.

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A Proof of the asymptotic behaviour of I^{ε}_{+-}

A.1 Proof of the asymptotic form (4.1)

This is the simplest of the three different cases to be considered. We will consider that both $|\alpha_1|$ and $|\alpha_2| \to \infty$ within $\mathcal{D}_{+-}$, in such a way that there exists an $m > 0$ such that $|\alpha_2| = m|\alpha_1|$. As discussed in Section 4, since both α_1 and α_2 are large, we can write $\alpha_{1,2} = |\alpha_{1,2}|e^{i\phi_{1,2}}$ for $\phi_1 \in (0, \pi)$ and $\phi_2 \in (-\pi, 0)$. In this case, starting from (1.1), we have

$$\begin{aligned}
I^{\varepsilon}_{+-}(\alpha_1,\alpha_2) &\underset{\alpha_2=|\alpha_2|e^{i\phi_2}}{\overset{\alpha_1=|\alpha_1|e^{i\phi_1}}{=}} \int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(|\alpha_1|e^{i\phi_1}\cos(\varphi)+|\alpha_2|e^{i\phi_2}\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi \\
&\underset{|\alpha_2|=m|\alpha_1|}{=} \frac{1}{|\alpha_1|^{\nu+3/2}}\int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{\left(-i\left(e^{i\phi_1}\cos(\varphi)+me^{i\phi_2}\sin(\varphi)+\frac{i\varepsilon}{|\alpha_1|}\right)\right)^{\nu+3/2}}\mathrm{d}\varphi \\
&\underset{|\alpha_1|\to\infty}{\sim} \frac{1}{|\alpha_1|^{\nu+3/2}}\int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(e^{i\phi_1}\cos(\varphi)+me^{i\phi_2}\sin(\varphi)))^{\nu+3/2}}\mathrm{d}\varphi \\
&\underset{|\alpha_1|\to\infty}{\sim} \frac{1}{|\alpha_1|^{\nu+3/2}} I^0_{+-}(e^{i\phi_1}, me^{i\phi_2}) \underset{|\alpha_1|\to\infty}{=} \mathcal{O}\left(\frac{1}{|\alpha_1|^{\nu+3/2}}\right),
\end{aligned}$$

since the quantity $e^{i\phi_1}\cos(\varphi) + me^{i\phi_2}\sin(\varphi)$ can never be equal to zero and hence the last integral is well defined and independent of $|\alpha_1|$.

This last statement can be proven as follows. Let us assume that $e^{i\phi_1}\cos(\varphi) + me^{i\phi_2}\sin(\varphi)=0$ for some $\varphi \in \left[\frac{3\pi}{2}, 2\pi\right]$. If $\varphi = 3\pi/2$, then we have $-me^{i\phi_2} = 0$, which is impossible since $m > 0$. If $\varphi = 2\pi$, then we have $e^{i\phi_1} = 0$, which is impossible. Hence our quantity cannot be zero at the end points of the integration domain. We can hence assume that $\cos(\varphi) \neq 0$ and $\sin(\varphi) \neq 0$ and rewrite the equality as $e^{i(\phi_1-\phi_2)} = -m\tan(\varphi) > 0$. Now taking the imaginary part on both sides, we get $\sin(\phi_1 - \phi_2) = 0$, implying that $\phi_1 = \phi_2 + n\pi$ for some $n \in \mathbb{Z}$. Clearly, from the restriction on $\phi_{1,2}$, we can only have $n = -1, 0, 1, 2$. For $n = -1, 1$, we would get $-1 = -m\tan(\varphi) > 0$, which is impossible. Hence we have $n = 0$ or 2, i.e. $\phi_1 = \phi_2$ or $\phi_1 = \phi_2 + 2\pi$. Because of the restriction on $\phi_{1,2}$ this imposes $\phi_{1,2} = 0$ or $\phi_1 = \pi$ and $\phi_2 = -\pi$, but these values are excluded according to the restriction on $\phi_{1,2}$, which contradicts our initial assumption and proves that $e^{i\phi_1}\cos(\varphi) + me^{i\phi_2}\sin(\varphi) \neq 0$.

A.2 Proof of the asymptotic form (4.2)

We have

$$I^{\varepsilon}_{+-}(\alpha_1,\alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi.$$

We can see that as $|\alpha_1| \to \infty$, the denominator is dominated by the term involving $\alpha_1\cos(\varphi)$ for all φ, except when $\varphi \approx \frac{3\pi}{2}$, where $\cos(\varphi)$ approaches zero. We can hence split the integral into two parts as $I^{\varepsilon}_{+-}(\alpha_1,\alpha_2) = I^{\varepsilon,\delta}_1(\alpha_1,\alpha_2) + I^{\varepsilon,\delta}_2(\alpha_1,\alpha_2)$, where

$$I^{\varepsilon,\delta}_1(\alpha_1,\alpha_2) = \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}+\delta} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi, \tag{A.1}$$

$$I^{\varepsilon,\delta}_2(\alpha_1,\alpha_2) = \int_{\frac{3\pi}{2}+\delta}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi, \tag{A.2}$$

for some small $\delta = \delta(|\alpha_1|) > 0$, chosen such that $|\alpha_1|\delta \to \infty$ as $|\alpha_1| \to \infty$ and $\delta \to 0$ as $|\alpha_1| \to \infty$. Let us select $\delta = 1/\sqrt{|\alpha_1|}$ as it will prove to work. Upon making the change of variable $\psi = \varphi - \frac{3\pi}{2}$, these integrals become

$$\begin{aligned} I_1^{\varepsilon,\delta}(\alpha_1,\alpha_2) &= \int_0^{\delta} \frac{f\left(\frac{\pi}{2}, \frac{3\pi}{2}+\psi\right)}{(-i(\alpha_1\sin(\psi)-\alpha_2\cos(\psi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\psi, \\ I_2^{\varepsilon,\delta}(\alpha_1,\alpha_2) &= \int_{\delta}^{\frac{\pi}{2}} \frac{f\left(\frac{\pi}{2}, \psi+\frac{3\pi}{2}\right)}{(-i(\alpha_1\sin(\psi)-\alpha_2\cos(\psi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\psi. \end{aligned}$$

For the first integral, we have

$$I_1^{\varepsilon,\delta}(\alpha_1,\alpha_2) \underset{\delta\to 0}{\sim} \int_0^{\delta} \frac{f\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(-i(\alpha_1\psi-\alpha_2+i\varepsilon))^{\nu+3/2}}\mathrm{d}\psi. \tag{A.3}$$

Since $|\alpha_1|\to\infty$, remember that we can write $\alpha_1=|\alpha_1|e^{i\phi_1}$ with $\phi_1\in(0,\pi)$, and making the change of variable $\theta=|\alpha_1|\psi$ in (A.3) we get

$$\begin{aligned} I_1^{\varepsilon,\delta}(\alpha_1,\alpha_2) &\sim \int_0^{|\alpha_1|\delta} \frac{f\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(-i(\theta e^{i\phi_1}-\alpha_2+i\varepsilon))^{\nu+3/2}}\frac{\mathrm{d}\theta}{|\alpha_1|} \\ &\underset{\delta|\alpha_1|\to\infty}{\sim} \frac{f\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{|\alpha_1|}\int_0^{\infty} \frac{1}{(-i(\theta e^{i\phi_1}-\alpha_2+i\varepsilon))^{\nu+3/2}}\mathrm{d}\theta. \end{aligned}$$

The latter integral can be recast in the form $\int_0^\infty \frac{\mathrm{d}\theta}{(A-B\theta)^\lambda}$, for $A=i\alpha_2+\varepsilon$, $B=ie^{i\phi_1}$ and $\lambda=\nu+3/2$. Since we have $\mathrm{Re}(A)>0$, $A/B\notin\mathbb{R}$ and $\lambda>1$, this integral can easily be shown to be equal to $\frac{A^{1-\lambda}}{B(1-\lambda)}$, and hence

$$I_1^{\varepsilon,\delta}(\alpha_1,\alpha_2) \underset{|\alpha_1|\to\infty}{\overset{\alpha_2\text{ fixed}}{\sim}} \frac{1}{\alpha_1}\times\frac{2if\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(i\alpha_2+\varepsilon)^{\nu+1/2}(2\nu+1)}. \tag{A.4}$$

For the second integral, we can use the fact that $|\alpha_1\sin(\psi)|>|\alpha_1\sin(\delta)|\to\infty$ to simplify the denominator and obtain

$$I_2^{\varepsilon,\delta}(\alpha_1,\alpha_2) \underset{|\alpha_1|\to\infty}{\overset{\alpha_1=|\alpha_1|e^{i\phi_1}}{\sim}} \frac{1}{|\alpha_1|^{\nu+3/2}}\int_{\delta}^{\frac{\pi}{2}} \frac{f\left(\frac{\pi}{2}, \psi+\frac{3\pi}{2}\right)}{(-ie^{i\phi_1}\sin(\psi))^{\nu+3/2}}\mathrm{d}\psi.$$

Hence, since by Lemma 1, $0\leqslant f\left(\frac{\pi}{2},\psi+\frac{3\pi}{2}\right)\leqslant f\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, and because $|\sin(\delta)|\leqslant|\sin(\psi)|$, we have

$$\begin{aligned} |I_2^{\varepsilon,\delta}(\alpha_1,\alpha_2)| &\underset{|\alpha_1|\to\infty}{\sim} \frac{1}{|\alpha_1|^{\nu+3/2}}\int_{\delta}^{\frac{\pi}{2}} \frac{f\left(\frac{\pi}{2}, \psi+\frac{3\pi}{2}\right)}{|-ie^{i\phi_1}\sin(\psi)|^{\nu+3/2}}\mathrm{d}\psi \\ &\leqslant \frac{\frac{\pi}{2}f\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(|\sin(\delta)||\alpha_1|)^{\nu+3/2}} \underset{\delta\to 0}{\sim} \frac{\frac{\pi}{2}f\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(\delta|\alpha_1|)^{\nu+3/2}} \underset{\delta=\frac{1}{\sqrt{\alpha_1}}}{=} \mathcal{O}\left(\frac{1}{|\alpha_1|^{\frac{\nu}{2}+\frac{3}{4}}}\right) = o\left(\frac{1}{|\alpha_1|}\right) \end{aligned}$$

since $\frac{\nu}{2}+\frac{3}{4}\approx 1.15>1$. Hence, overall, $I_2^{\varepsilon,\delta}$ can be neglected to leading order and, using (A.4), we obtain

$$I_{+-}^{\varepsilon}(\alpha_1,\alpha_2) \underset{|\alpha_1|\to\infty}{\overset{\alpha_2\text{ fixed}}{\sim}} \frac{\Lambda_1(\alpha_2,\varepsilon)}{\alpha_1}, \quad \text{where} \quad \Lambda_1(\alpha_2,\varepsilon)=\frac{2if\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}{(\varepsilon+i\alpha_2)^{\nu+1/2}(1+2\nu)},$$

as required.

A.3 Proof of the asymptotic form (4.3)

We have

$$I_{+-}^{\varepsilon}(\alpha_1,\alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi,$$

and we can see that as $|\alpha_2|\to\infty$, the denominator is dominated by the term involving $\alpha_2\sin(\varphi)$ for all φ, except when $\varphi\sim 2\pi$, in which region $\sin(\varphi)$ approaches zero. We should hence split the integral into two parts as $I_{+-}^{\varepsilon}(\alpha_1,\alpha_2) = I_3^{\varepsilon,\delta}(\alpha_1,\alpha_2) + I_4^{\varepsilon,\delta}(\alpha_1,\alpha_2)$, where

$$I_3^{\varepsilon,\delta}(\alpha_1,\alpha_2) = \int_{2\pi-\delta}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi, \tag{A.5}$$

$$I_4^{\varepsilon,\delta}(\alpha_1,\alpha_2) = \int_{\frac{3\pi}{2}}^{2\pi-\delta} \frac{f\left(\frac{\pi}{2},\varphi\right)}{(-i(\alpha_1\cos(\varphi)+\alpha_2\sin(\varphi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\varphi, \tag{A.6}$$

for some small $\delta=\delta(|\alpha_2|)>0$, chosen such that $|\alpha_2|\delta\to\infty$ as $|\alpha_2|\to\infty$ and $\delta\to 0$ as $|\alpha_2|\to\infty$. For specificity, let us select $\delta = 1/|\alpha_2|^{1/4}$. Upon making the change of variable $\psi=2\pi-\varphi$, these integrals become

$$I_3^{\varepsilon,\delta}(\alpha_1,\alpha_2) = \int_0^{\delta} \frac{f\left(\frac{\pi}{2},2\pi-\psi\right)}{(-i(\alpha_1\cos(\psi)-\alpha_2\sin(\psi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\psi,$$

$$I_4^{\varepsilon,\delta}(\alpha_1,\alpha_2) = \int_{\delta}^{\frac{\pi}{2}} \frac{f\left(\frac{\pi}{2},2\pi-\psi\right)}{(-i(\alpha_1\cos(\psi)-\alpha_2\sin(\psi)+i\varepsilon))^{\nu+3/2}}\mathrm{d}\psi.$$

For the first integral, using Lemma 1, we have

$$\begin{aligned} I_3^{\varepsilon,\delta}(\alpha_1,\alpha_2) &\underset{\delta\to 0}{\sim} \int_0^{\delta} \frac{\beta\sqrt{\psi}}{(-i(\alpha_1-\alpha_2\psi+i\varepsilon))^{\nu+3/2}}\mathrm{d}\psi \\ &\underset{\theta=|\alpha_2|\psi}{\overset{\alpha_2=|\alpha_2|e^{i\phi_2}}{=}} \int_0^{|\alpha_2|\delta} \frac{\beta\sqrt{\frac{\theta}{|\alpha_2|}}}{(-i(\alpha_1-e^{i\phi_2}\theta+i\varepsilon))^{\nu+3/2}}\frac{\mathrm{d}\theta}{|\alpha_2|} \\ &\underset{|\alpha_2|\delta\to\infty}{\sim} \frac{\beta}{|\alpha_2|^{3/2}}\int_0^{\infty} \frac{\sqrt{\theta}}{(-i(\alpha_1-e^{i\phi_2}\theta+i\varepsilon))^{\nu+3/2}}\mathrm{d}\theta. \end{aligned} \tag{A.7}$$

The latter integral can be recast in the form $\int_0^\infty \frac{\sqrt{\theta}}{(A-B\theta)^\lambda}\mathrm{d}\theta$, with $A=(-i\alpha_1+\varepsilon)$, $B=-ie^{i\phi_2}$ and $\lambda=\nu+3/2$. Since we have $\mathrm{Re}(A)>0$, $A/B\notin\mathbb{R}$ and $\lambda>3/2$, this integral can be shown to be equal to $\frac{A^{-\lambda}\sqrt{\pi}\Gamma(\lambda-3/2)}{2(-B/A)^{3/2}\Gamma(\lambda)}$ (see [6] p285, 3.194.3.), and hence

$$I_3^{\varepsilon,\delta}(\alpha_1,\alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_1\text{ fixed}}{\sim}} \frac{1}{\alpha_2^{3/2}}\times\frac{\beta(-i)^{3/2}\sqrt{\pi}\Gamma(\nu)}{2(\varepsilon-i\alpha_1)^{\nu}\Gamma(\nu+3/2)}. \tag{A.8}$$

For the second integral, since $|\alpha_2\sin(\psi)|>|\alpha_2\sin(\delta)|\to\infty$, we have

$$I_4^{\varepsilon,\delta}(\alpha_1,\alpha_2) \underset{|\alpha_2|\to\infty}{\sim} \frac{1}{(i\alpha_2)^{\nu+3/2}}\int_{\delta}^{\frac{\pi}{2}} \frac{f\left(\frac{\pi}{2},2\pi-\psi\right)}{(\sin(\psi))^{\nu+3/2}}\mathrm{d}\psi.$$

Hence, using Lemma 1, we have

$$|I_4^{\varepsilon,\delta}(\alpha_1,\alpha_2)| \quad < \quad \frac{f\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\frac{\pi}{2}}{|\alpha_2\sin(\delta)|^{\nu+3/2}} \underset{\delta=|\alpha_2|^{-1/4}}{=} \frac{\pi f\left(\frac{\pi}{2},\frac{3\pi}{2}\right)}{2|\alpha_2|^{\frac{3\nu}{4}+\frac{9}{8}}} = \mathcal{O}\left(\frac{1}{|\alpha_2|^{3/2}}\right),$$

since $\frac{3\nu}{4}+\frac{9}{8}\approx 1.72 > \frac{3}{2}$.

Hence, overall, $I_4^{\varepsilon,\delta}$ can be neglected to leading order and, using (A.8), we obtain

$$I_{+-}^{\varepsilon}(\alpha_1,\alpha_2) \underset{|\alpha_2|\to\infty}{\overset{\alpha_1 \text{ fixed}}{\sim}} \frac{\Lambda_2(\alpha_1,\varepsilon)}{\alpha_2^{3/2}}, \quad \text{where} \quad \Lambda_2(\alpha_1,\varepsilon) = \frac{\beta e^{-3i\frac{\pi}{4}}\sqrt{\pi}\Gamma(\nu)}{2(\varepsilon - i\alpha_1)^{\nu}\Gamma(\nu+3/2)},$$

as required.

B Asymptotic behaviour of I_{cut}

Since we are only interested in the behaviour of I_{cut} when $|\alpha_2|\to\infty$ and $\alpha_2\in\mathcal{A}_2$, we can consider α_2 to be real here. Let us assume that $\alpha_2>0$ and $\alpha_2\to\infty$; the α_2 negative case can be dealt with in a similar fashion. Using the definitions of the functions $K_{-\circ}(\alpha_1,\alpha_2)$ and $\sqrt{k^2-\alpha_2^2}$ given in [2], one can show that, on the right side of the cut, $1/K_{-\circ}\left(\sqrt{k^2-\alpha_2^2}+it,\alpha_2\right) = \sqrt{t}e^{\frac{3i\pi}{4}}$ and that, as $\alpha_2\to\infty$, $\sqrt{k^2-\alpha_2^2}\sim i\alpha_2$, we obtain

$$\begin{aligned} I_{\text{cut}}(\alpha_1,\alpha_2) \quad &= \quad 2i\int_0^{\infty}\Psi\left(\sqrt{k^2-\alpha_2^2}+it,\alpha_1,\alpha_2\right) I_{+-}^{\varepsilon}\left(\sqrt{k^2-\alpha_2^2}+it,\alpha_2\right)\mathrm{d}t \\ &\underset{\alpha_2\to\infty}{\approx} \quad \frac{e^{\frac{3i\pi}{4}}}{\pi}\int_0^{\infty}\frac{\sqrt{t}}{(i\alpha_2+it-\alpha_1)}\int_{\frac{3\pi}{2}}^{2\pi}\frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi\mathrm{d}t}{(\alpha_2 e^{-i\varphi}+t\cos(\varphi)+\varepsilon)^{\nu+3/2}}. \end{aligned}$$

Now make the substitution $t=\alpha_2 u$ to get

$$I_{\text{cut}}(\alpha_1,\alpha_2) \underset{\alpha_2\to\infty}{\approx} \frac{e^{\frac{3i\pi}{4}}}{\pi}\frac{1}{\alpha_2^{\nu+1}}\int_0^{\infty}\frac{\sqrt{u}}{\left(i(u+1)-\frac{\alpha_1}{\alpha_2}\right)}\int_{\frac{3\pi}{2}}^{2\pi}\frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi\mathrm{d}u}{\left(e^{-i\varphi}+u\cos(\varphi)+\frac{\varepsilon}{\alpha_2}\right)^{\nu+3/2}}.$$

Now the denominators are never zero, even when neglecting the small terms involving $1/\alpha_2$, so we have

$$I_{\text{cut}}(\alpha_1,\alpha_2) \underset{\alpha_2\to\infty}{\approx} \frac{e^{\frac{3i\pi}{4}}}{\pi}\frac{1}{\alpha_2^{\nu+1}}\int_0^{\infty}\frac{\sqrt{u}}{(i(u+1))}\underbrace{\int_{\frac{3\pi}{2}}^{2\pi}\frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi}{(e^{-i\varphi}+u\cos(\varphi))^{\nu+3/2}}}_{J(u)}\mathrm{d}u. \tag{B.1}$$

We need to check that the double integral actually exists. There are no convergence problems as $u\to 0$ and no singularity of the integrand for $u\in\mathbb{R}^+$. So, we just need to ensure that the integral converges at ∞. For this purpose, it is enough to study the behaviour of $J(u)$ as $u\to\infty$.

The only possible issue occurs when $\cos(\varphi)\approx 0$; to resolve this, consider a small fixed constant $\delta>0$, and write

$$\begin{aligned} J(u) \quad &= \quad \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}+\delta}\frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi}{(e^{-i\varphi}+u\cos(\varphi))^{\nu+3/2}} + \int_{\frac{3\pi}{2}+\delta}^{2\pi}\frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi}{(e^{-i\varphi}+u\cos(\varphi))^{\nu+3/2}} \\ &= \quad J_1(u)+J_2(u) \end{aligned}$$

Let us start by studying $J_2(u)$:

$$\begin{aligned}
|J_2(u)| &= \left| \int_{\frac{3\pi}{2}+\delta}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi}{(e^{-i\varphi}+u\cos(\varphi))^{\nu+3/2}} \right| \\
&\underset{u\to\infty}{\approx} \left| \int_{\frac{3\pi}{2}+\delta}^{2\pi} \frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi}{(u\cos(\varphi))^{\nu+3/2}} \right| \\
&< \frac{1}{u^{\nu+3/2}} \frac{f\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\frac{\pi}{2}}{\delta^{\nu+3/2}} = \mathcal{O}\left(\frac{1}{u^{\nu+3/2}}\right).
\end{aligned}$$

Now consider the slightly more subtle case of J_1, and using Lemma 1, we obtain

$$\begin{aligned}
J_1(u) &= \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}+\delta} \frac{f\left(\frac{\pi}{2},\varphi\right)\mathrm{d}\varphi}{(e^{-i\varphi}+u\cos(\varphi))^{\nu+3/2}} \\
&\underset{\delta\ll 1}{\overset{z=\varphi-\frac{3\pi}{2}}{\approx}} f\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\int_0^{\delta} \frac{\mathrm{d}z}{(i+uz)^{\nu+3/2}} \\
&\underset{\delta u\to\infty}{\overset{x=uz}{\approx}} \frac{f\left(\frac{\pi}{2},\frac{3\pi}{2}\right)}{u}\int_0^{\infty} \frac{\mathrm{d}x}{(i+x)^{\nu+3/2}} = \mathcal{O}\left(\frac{1}{u}\right).
\end{aligned}$$

Hence, since $\nu+3/2>1$, $J_2(u)$ can be neglected to leading order, and, overall $J(u)=\mathcal{O}(1/u)$. This ensures that the double integral in (B.1) converges. Since it is independent from α_2, it is quite clear that $I_{\text{cut}}=\mathcal{O}\left(\frac{1}{|\alpha_2|^{\nu+1}}\right)$, as claimed in Section 5.3, and as confirmed numerically.