

Electrical Network Synthesis: A Survey of Recent Work

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Dedicated to M. Vidyasagar on the occasion of his 70th birthday.

Abstract The field of electrical network synthesis has a number of long-standing unanswered questions. The most perplexing concern minimality in the context of resistor, inductor and capacitor (RLC) network synthesis. In particular, the famous Bott-Duffin networks appear highly non-minimal from a system theoretic perspective. We survey some recent developments on this topic. These include results establishing the minimality of the Bott-Duffin networks and their simplifications for realising certain impedances; enumerative approaches to the analysis of RLC networks within given classes of restricted complexity; and new necessary and sufficient conditions for a (not necessarily controllable) system to be passive. Finally, some remaining open questions are discussed.

1 Introduction

The purpose of this paper is to survey some recent developments in electrical network synthesis. Despite the rich history of this field, there remain several significant open questions which have never been fully resolved. This is particularly true for RLC (resistor, inductor, capacitor) network synthesis. Notably, it is not known how to design an RLC network to realise a given driving-point behavior while using the least possible number of elements; and classical methods of RLC network synthesis (such as the Bott-Duffin procedure [2]) appear highly non-minimal from a system theoretic perspective. Indeed, the need for a fresh examination of these issues, and an improved understanding of minimality in the context of passive systems, has recently been emphasised in [4, 5, 22, 39]. Additional practical motiva-

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tion for studying these questions follows from the recent invention of the *inverter*, which completed the electrical-mechanical analogy [33]. Using this analogy, there is a one-to-one correspondence between RLC networks and mechanical networks containing springs, dampers and inverters, which allows the results of RLC network synthesis to be directly applied to mechanical control. Applications of this approach to the design of vibration absorption systems, such as vehicle suspension, train suspension, motorcycle steering compensators, and building suspension, are described in [6, 7, 10, 21, 33, 36, 37, 38].

In this paper, we begin with a brief description of the objectives of electrical network synthesis (Section 2). We then discuss the classical methods of RLC network synthesis, notably the Bott-Duffin method [2] and its simplifications [8, 27, 30] (Section 3), and we present some newly discovered alternatives to Bott-Duffin [11]. In Section 4, we discuss some surprising results which establish the minimality of the Bott-Duffin networks and their simplifications for realising certain impedances [11, 14]. Sections 5 and 6 describe recent progress in the enumeration and analysis of RLC networks within a given class of restricted complexity [17, 18, 19, 22, 26]. Section 7 summarises results from [12] on passivity for systems which need not be controllable (as is the case for the Bott-Duffin networks). It is shown that the well known positive-real condition is not sufficient for a (not necessarily controllable) system to be passive, and an alternative necessary and sufficient condition is presented (Theorem 2). Finally, some open questions are discussed in Section 8.

The notation is as follows. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers. $\mathbb{C}_+$ and $\bar{\mathbb{C}}_+$ denote the open and closed right half plane. If $\lambda \in \mathbb{C}$, then $\Re(\lambda)$ and $\Im(\lambda)$ denote the real and imaginary parts of λ , and $\bar{\lambda}$ denotes its complex conjugate. $\mathbb{R}(s)$ and $\mathbb{R}[s]$ denote the rational functions and polynomials in the indeterminate s with real coefficients; and $\mathbb{R}^{m \times n}[s]$ denotes the $m \times n$ matrices with entries from $\mathbb{R}[s]$ (n is omitted if $n = 1$).

2 RLC networks and passivity

An RLC network comprises an interconnection of resistors, inductors and capacitors. The behaviors of these elements correspond to the set of solutions to the equations $v = iR$, $v = L \frac{di}{dt}$, and $i = C \frac{dv}{dt}$, respectively, where v denotes the voltage across and i the current through an element, and $R, L, C > 0$. Interconnection results in additional constraints corresponding to Kirchhoff's current law (the sum of currents entering any vertex is zero) and Kirchhoff's voltage law (the sum of voltages around any closed circuit is zero). In addition, energy can be exchanged with the environment at one or more ports, which correspond to pairs of terminals across which a voltage is applied and through which a current flows. Denoting the vector of port currents and corresponding voltages by $\mathbf{i}$ and $\mathbf{v}$, respectively, then the driving-point behavior of the network is the set of solutions to a differential equation of the form:

$$P\left(\frac{d}{dt}\right)\mathbf{i} = Q\left(\frac{d}{dt}\right)\mathbf{v}, \text{ with } \mathbf{i}, \mathbf{v} \text{ locally integrable, and } P, Q \in \mathbb{R}^{n \times n}[s]. \quad (1)$$

The driving-point behavior is *passive*, in accordance with the definition below.

Definition 1 (Passive system, [12]). The system in (1) is called *passive* if, for any given $(\mathbf{i}, \mathbf{v})$ satisfying (1) and $t_0 \in \mathbb{R}$, there exists a $K \in \mathbb{R}$ (dependent on $(\mathbf{i}, \mathbf{v})$ and t_0) such that, if $(\hat{\mathbf{i}}, \hat{\mathbf{v}})$ satisfies (1) and $(\hat{\mathbf{i}}(t), \hat{\mathbf{v}}(t)) = (\mathbf{i}(t), \mathbf{v}(t))$ for all $t < t_0$, then $-\int_{t_0}^{t_1} \hat{\mathbf{i}}^T(t) \hat{\mathbf{v}}(t) dt < K$ for all $t_1 \geq t_0$.

This definition formalises an important property of RLC networks: it is not possible to extract unlimited energy from the network from the present time (t_0) onward.

In the case of one-port networks (with the exception of an open circuit), the driving-point behavior takes the form $p(\frac{d}{dt})i = q(\frac{d}{dt})v$ with $p, q \in \mathbb{R}[s]$ and $q \neq 0$ [15]. The impedance $Z := p/q$ of the network is then positive-real, in accordance with the following definition:

Definition 2 (Positive-real). $Z \in \mathbb{R}(s)$ is called *positive-real* if (i) Z is analytic in $\mathbb{C}_+$; and (ii) $\Re(Z(\lambda)) \geq 0$ for all $\lambda \in \mathbb{C}_+$.

RLC network synthesis is concerned with variations on the general problem: *given a positive-real Z , find an RLC network whose impedance is equal to Z .* As many different RLC networks can realise the same impedance, then we also seek network realisations which are *minimal*, i.e., which use the least possible numbers of the various types of element.

3 Bott-Duffin and its simplifications

One of the most striking results of electric circuit theory is that a given $Z \in \mathbb{R}(s)$ can be realised as the impedance of an RLC network if and only if Z is positive-real. The necessity of this condition was first proved by Otto Brune in [3], where the positive-real concept was first defined. Sufficiency was later proved in a famous one page paper by Bott and Duffin [2]. That paper gave an algorithm which, for any given positive-real function Z , provides an RLC network whose impedance is Z .

The starting point for the Bott-Duffin procedure is the *Foster preamble*. This provides a sequence of simplifications to the given positive-real function, each corresponding to the series or parallel extraction of a single resistor, inductor or capacitor, or an inductor-capacitor pair. This sequence of simplifications terminates with a so-called *minimum function*:

Definition 3 (Minimum function). $Z \in \mathbb{R}(s)$ is called a *minimum function* if (i) Z is positive-real; (ii) Z has no poles or zeros on the extended imaginary axis; and (iii) there exists $\omega_0 > 0$ such that $\Re(Z(j\omega_0)) = 0$.

The contribution of Bott and Duffin was to show that any given minimum function Z can be realised by an RLC network containing precisely six reactive elements (three inductors and three capacitors) and two further networks whose impedances are positive-real functions with McMillan degrees at least two fewer than that of

Z. Thus, by the inductive application of this procedure (together with the Foster preamble), an RLC network is obtained to realise any given positive-real function.

The Bott-Duffin construction is based on the following theorem of Richards [31]:

Theorem 1. *If $Z \in \mathbb{R}(s)$ is positive-real and $\mu > 0$, then*

$$F(s) := \frac{\mu Z(s) - sZ(\mu)}{\mu Z(\mu) - sZ(s)}$$

is positive-real, and the McMillan degree of F does not exceed that of Z .

By inverting the relationship in Theorem 1, we obtain

$$Z(s) = \left(\frac{F(s)}{Z(\mu)} + \frac{\mu}{Z(\mu)s} \right)^{-1} + \left(\frac{1}{Z(\mu)F(s)} + \frac{s}{Z(\mu)\mu} \right)^{-1}, \quad (2)$$

whereupon a network realisation for Z is immediately obtained from network realisations for $F(s)/Z(\mu)$ and $1/(Z(\mu)F(s))$.

If Z is a minimum function, then there exists $\omega_0 > 0$ such that $Z(j\omega_0) = j\omega_0 X$, where $0 \neq X \in \mathbb{R}$. Bott and Duffin then considered the two cases: (i) $X > 0$; and (ii) $X < 0$. In case (i), it is easily shown that there exists a $\mu > 0$ such that $X = Z(\mu)/\mu$. For this value of μ , then F in Theorem 1 has the property that $1/F$ is positive-real and has poles at $\pm j\omega_0$. By a well known property of positive-real functions, it follows that there exists a positive-real G and an $\alpha > 0$ such that

$$\frac{1}{F(s)} = \frac{1}{G(s)} + \frac{2\alpha s}{s^2 + \omega_0^2}, \quad (3)$$

and the McMillan degree of G is two fewer than that of F . In this manner, network realisations for $F(s)/Z(\mu)$ and $1/(Z(\mu)F(s))$ are immediately obtained from network realisations for $G(s)/Z(\mu)$ and $1/(Z(\mu)G(s))$. Thus, from (2)–(3), if $X > 0$, then Z is realised by the network on the left of Fig. 1. A similar argument for case (ii) concludes that, if $X < 0$, then Z is realised by the network on the right of Fig. 1.

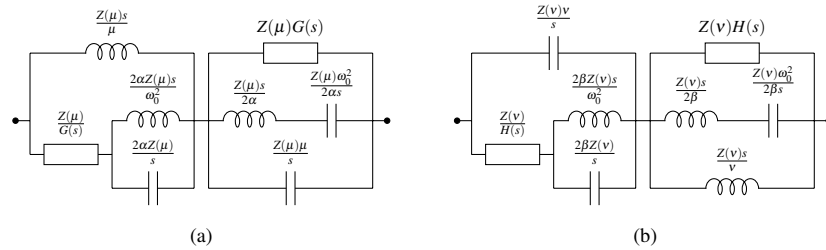


Fig. 1 Bott-Duffin networks for realising a minimum function Z with $Z(j\omega_0) = Xj$: (a) $X > 0$, and (b) $X < 0$. In case (a), μ , α and G are as defined in Section 3. In case (b), then $v > 0$ satisfies $-\omega_0^2 X = vZ(v)$, and $\beta > 0$ and H satisfy $(vZ(s) - sZ(v))/(vZ(v) - sZ(s)) = 1/H(s) + 2\beta s/(s^2 + \omega_0^2)$ with H positive-real and the McMillan degree of H at least two fewer than that of Z .

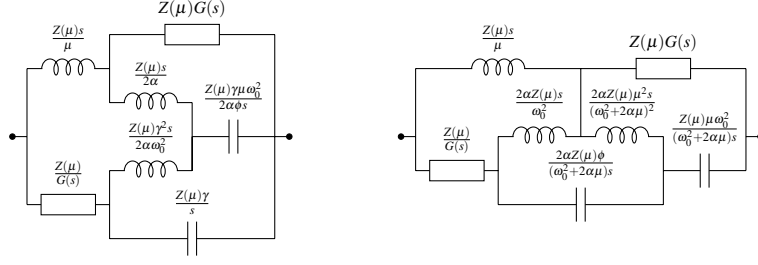


Fig. 2 Realisations of a minimum function Z with $Z(j\omega_0) = Xj$, $X > 0$. Here, μ , α and G are as defined in Section 3, $\gamma = \mu + 2\alpha$, and $\phi = \omega_0^2 + 2\alpha\mu + \mu^2$.

One notable feature of the Bott-Duffin networks is the apparently excessive number of reactive elements. For example, if Z is biquadratic (i.e., the McMillian degree of Z is two), then the Bott-Duffin procedure uses six reactive elements. Yet, as will be discussed in Sections 5–6, many positive-real functions of McMillan degree two can be realised with only two reactive elements. This motivates the search for simpler alternatives to the Bott-Duffin networks.

Somewhat surprisingly, the best known improvements to the Bott-Duffin networks (for realising a minimum function) only save a single reactive element in each inductive step. The first such improvement was discovered simultaneously by several authors in the 1950s [8, 27, 30]. More recently, a second class of networks was discovered [11], which contain the same number of reactive elements and resistors as the networks in [8, 27, 30]. For a single step in the realisation of a general minimum function Z with $\Im(Z(j\omega_0)) > 0$, these networks are shown in Fig. 2.

4 Electric networks and minimality

From a systems and control perspective, it is particularly interesting to ask: *what is the least number of reactive elements required to realise a given biquadratic positive-real function?* The behavior of an electric circuit is determined by a first order linear differential algebraic equation whose degree is equal to the number of reactive elements in the circuit. More specifically, if $\mathbf{x}$ comprises the currents and voltages in the circuit's inductors, capacitors and resistors, and i and v are the port current and voltage, then there exist real matrices A, B, C, D and E such that $E \frac{d\mathbf{x}}{dt} = A\mathbf{x} + Bi$ and $v = C\mathbf{x} + Di$, and the rank of E is equal to the number of reactive elements. It might therefore be assumed that only two reactive elements are required to realise a biquadratic positive-real function. But it has been proven that this is not the case. In fact, the following two surprising results have recently been shown:

1. The Bott-Duffin networks contain the least possible number of reactive elements for realising a biquadratic minimum function with *series-parallel* networks [14];

2. The networks in [8, 11, 27, 30] contain the least possible number of reactive elements for realising almost all biquadratic minimum functions with arbitrary RLC networks [11].

These results were inspired by [32], which aimed to show that the networks in [8, 27, 30] contain the least possible number of elements for realising certain biquadratic minimum functions. However, it is shown in [9, 11] that [32] overlooked certain networks, and did not determine the minimality of these networks in the number of reactive elements used.

In [14], the proof of 1 exploits the special form of the impedance of a series-parallel network and the properties of positive-real functions. A key observation is that the impedance of a series-parallel network is a composition of sums and inverses of positive-real functions. For example, any series-parallel network N is either a series or a parallel connection of two series-parallel networks N_1 and N_2 . In the first case, the impedance Z satisfies $Z = Z_1 + Z_2$, where Z_1 and Z_2 are the impedances of N_1 and N_2 , respectively. Since Z_1 and Z_2 are positive-real, then: (a) since $\Re(Z(j\omega_0)) = 0$, then $\Re(Z_1(j\omega_0)) = 0$ and $\Re(Z_2(j\omega_0)) = 0$; (b) since Z has no imaginary axis poles, then neither Z_1 nor Z_2 has any imaginary axis poles; and (c) since $\Im(Z(j\omega_0)) \neq 0$, then either $\Im(Z_1(j\omega_0)) \neq 0$ or $\Im(Z_2(j\omega_0)) \neq 0$. It can then be shown that N must contain at least five reactive elements. Also, in [14], all of the series-parallel networks containing exactly five reactive elements which satisfy conditions (a)–(c) are identified, and it is shown that none of these can realise a biquadratic minimum function. A similar argument holds in the case that N is a parallel connection of two series-parallel networks, and proves 1.

An entirely different argument is required to prove 2. The proof in [11] considers a general RLC network N whose impedance Z is a minimum function, and a sinusoidal trajectory of N at the minimum frequency. For such a trajectory, it is shown that the net amount of energy dissipated by N over a single period must be zero, so only the reactive elements can transmit current. Further constraints are placed on N by virtue of the fact that Z has no imaginary axis poles or zeros. These observations were used in [11] to identify all of the RLC networks which realise a minimum function and contain fewer than five reactive elements. It was then shown that these RLC networks can only realise a small subset of the biquadratic minimum functions.

5 Enumeration approach

As discussed in Section 3, the realisation method introduced by Bott and Duffin in 1949 can prove to be apparently extravagant in the number of elements used. Following this result, many questions in passive network synthesis were still unanswered, and no better theory to solve the problem of minimal realisation of positive-real functions was found. An entirely different research strategy for this problem consists in enumerating and analysing all RLC networks within a given restricted class. Following the renewed interest in minimal realisations for passive mechanical control [33], Jiang and Smith adopted this approach to study the class of all

five-element networks and six-element series-parallel networks [17, 18]. In [17], the concept of regularity was introduced, which greatly facilitates the classification of impedances, as follows:

Definition 4 (Regular). A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\Re(Z(j\omega))$ or $\Re(Z^{-1}(j\omega))$ occurs at $\omega = 0$ or $\omega = \infty$.

The *Ladenheim Catalogue*, i.e. the set of all essentially distinct RLC networks with at most two reactive elements and at most three resistors, was studied in [17]. There are 108 essentially distinct such networks, which all realise biquadratic (or possibly bilinear or constant) functions of the form

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}, \quad (4)$$

where $A, B, C, D, E, F \geq 0$. These networks were first enumerated and studied by Edward Ladenheim, Foster's last student, in [23]. For each network in the class, the value of the impedance is first computed in [23], then the inverse process is performed, i.e. given the impedance an expression for each element of the network is found. There is, however, no derivation in [23] of conditions which ensure positivity of the values computed in the last step.

It was shown in [17] that all but two of the Ladenheim networks can only realise regular impedances, and that the remaining two cover some, but not all, of the non-regular positive-real biquadratics. It was also proven that just six networks from the catalogue are needed to realise any regular biquadratic. By considering the class of all five-element networks instead, with an arbitrary number of reactive elements, it was finally shown that only another six networks, with three reactive elements, can realise some, but again not all, non-regular biquadratics.

A well-known result in circuit theory is that the class of impedances that can be realised using only two reactive elements is not increased by using more than three resistors [19]. This fundamental result is known as Reichert's theorem, and its first proof, which uses a complicated topological argument, can be found in Reichert's original German language publication [29]. This proof was later reworked in [19], and some parts of the argument were expanded. Recently, an alternative proof based on a result in [5] was provided in [40].

The class of six-element series-parallel networks was studied in [18], following earlier work [34, 35]. Based on Reichert's theorem, it is not useful to consider two-reactive six-element networks, as they do not expand the set of impedances realised by the Ladenheim class. A set of networks which can realise all the non-regular biquadratics that may be realised by the class was found in [18], consisting of networks with either three or four reactive elements.

Based on the results in [18], a four-reactive seven-element network was proposed in [20] which is very effective in realising most, but not all, non-regular biquadratics.

6 The Ladenheim Catalogue revisited

The analysis carried out in [17] provided a complete understanding of the realisation power of the Ladenheim class of networks, through the concept of regularity. Such an analysis however offers no insight as to how structured the set is and whether further results or patterns might emerge from a more complete study of the realisability conditions for each network in the class.

The need for a fresh study of the catalogue was independently advocated in 2010 by Kalman, who considered the catalogue “valuable experimental data, to be checked for accuracy and explained by theory” [22]. In his latest work, Kalman defined the concept of generic (or minimal) networks and approached the problem of electrical network synthesis using algebraic invariant theory [24].

The open questions regarding the structure of the Ladenheim class motivated a fresh study of the catalogue, which was carried out recently [25, 26]. A rigorous and complete classification of the catalogue was performed, by defining a group action and equivalence relation on the set. Equivalence relations include certain well-known network transformations (namely $Y-\Delta$ and *Zobel* transformations). More generally two networks are defined to be equivalent if they realise the same set of impedance functions. Group actions are based on the following transformations on the impedance $Z(s)$: frequency inversion $s \rightarrow s^{-1}$ and impedance inversion $Z \rightarrow Z^{-1}$. These notions allow the catalogue to be partitioned into 24 *groups* of networks, each comprising a certain number of *equivalence classes* and *orbits*. The set of realisable impedances for one network in each group is derived in explicit form as a semialgebraic set, in terms of necessary and sufficient conditions. This leads to a systematic derivation of the realisability conditions for every network of the Ladenheim class. By way of illustration, Fig. 3 shows one of the five-element groups of networks, with the structure that emerged from the classification of the catalogue.

The new study of the Ladenheim catalogue in [26] demonstrates that there are no new equivalence relations which were not known classically. Yet, the question remains open whether further patterns might still be revealed which might shed some light on the question of realisability for higher-order networks.

7 Realisation of behaviors

The necessity of the apparently extravagant numbers of reactive elements in the Bott-Duffin and related networks motivates a fresh examination of passivity for systems which are nonminimal in a system theoretic sense. A natural framework for examining these issues is the behavioral approach [28], and the associated concept of (behavioral) controllability [28, Definition 5.2.2]. Indeed, the driving-point behaviors of the Bott-Duffin networks and their simplifications are examples of uncontrollable behaviors. For example, from [15, Section 7], the driving-point behavior of the Bott-Duffin network on the left of Fig. 1 for realising the biquadratic minimum function $Z(s) = (s^2 + s + 1)/(s^2 + s + 4)$ is the set of solutions to the equation:

$$\left(\frac{d}{dt} + 1\right) \left(\frac{d^2}{dt^2} + \frac{d}{dt} + 4\right) v = \left(\frac{d}{dt} + 1\right) \left(\frac{d^2}{dt^2} + \frac{d}{dt} + 1\right) i. \quad (5)$$

That this behavior is uncontrollable is a result of the common differential operator $\left(\frac{d}{dt} + 1\right)$ acting on both sides of the above equation [28, Theorem 5.2.10]. Note that the set of solutions to (5) contains the set of solutions to

$$\left(\frac{d^2}{dt^2} + \frac{d}{dt} + 4\right) v = \left(\frac{d^2}{dt^2} + \frac{d}{dt} + 1\right) i. \quad (6)$$

However, there are solutions to (5) which are not solutions to (6). It follows that the passivity of (5) guarantees the passivity of (6), but the converse implication does not hold. In particular, for a behavior $p\left(\frac{d}{dt}\right)i = q\left(\frac{d}{dt}\right)v$ ($q \neq 0$) to be passive, it is necessary *but not sufficient* for the transfer function p/q to be positive-real.

By focusing on the driving-point behavior of an electric network, as opposed to its impedance, we obtain another class of unanswered network synthesis problems (see [4] and [39, Section 12]). For example, it is not known *what are the necessary and sufficient conditions (akin to the positive-real condition) for a (not necessarily controllable) behavior to be realised as the driving-point behavior of an RLC network*. A more fundamental question is *what are the necessary and sufficient con-*

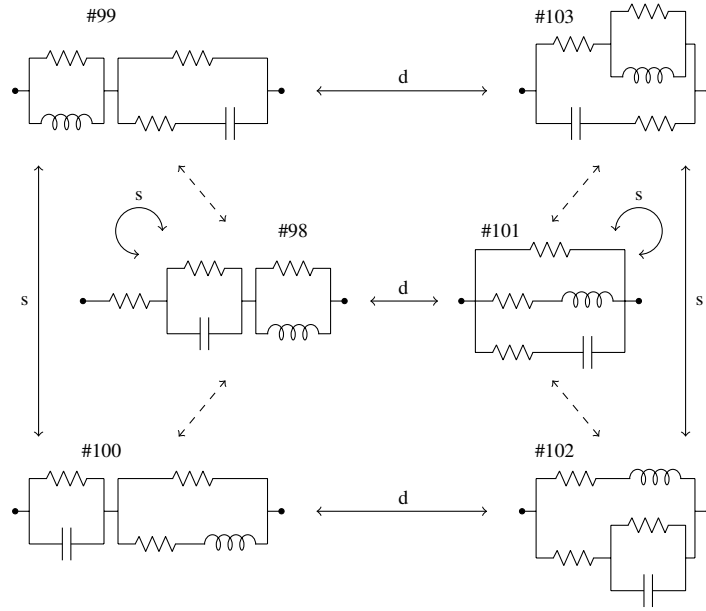


Fig. 3 Networks of group V_F in the catalogue arranged according to the group structure. *Zobel* transformations are represented through dashed arrows and define two equivalence classes. Duality and frequency inversion relations (d and s , respectively) are indicated through arrows, while the composition of the two and the identity (which complete the Klein four-group) are not shown. Here, the group action induces two *orbits* which comprise four and two networks respectively.

ditions for a (not necessarily controllable) behavior to be passive. This question was recently answered in [12], in the context of multi-port networks, as follows:

Theorem 2 ([12], Theorem 11). *The system in (1) is passive if and only if the following three conditions hold:*

1. $P(\lambda)Q(\bar{\lambda})^T + Q(\lambda)P(\bar{\lambda})^T \geq 0$ for all $\lambda \in \bar{\mathbb{C}}_+$.
2. $\text{rank}([P \ -Q](\lambda)) = n$ for all $\lambda \in \bar{\mathbb{C}}_+$.
3. If $\mathbf{p} \in \mathbb{R}^n[s]$ and $\lambda \in \mathbb{C}$ satisfy $\mathbf{p}(s)^T(P(s)Q(-s)^T + Q(s)P(-s)^T) = 0$ and $\mathbf{p}(\lambda)^T[P \ -Q](\lambda) = 0$, then $\mathbf{p}(\lambda) = 0$.

It was shown in [12] that if any one of the conditions in Theorem 2 does not hold then an arbitrarily large amount of energy can be extracted from the system. Thus, these conditions are necessary for a system to be passive. That these conditions are sufficient to guarantee passivity was shown using an algebraic argument to construct a *storage function* for the system, which provided an upper bound to the energy that can be extracted. Moreover, this storage function can be used in the *reactance extraction* method to network analysis and synthesis [1] to prove that the system in (1) is passive if and only if it can be realised as the driving-point behavior of a network containing resistors, inductors, capacitors, transformers and gyrators [13].

8 Open questions

The discussion in the preceding sections motivates the following three questions (of which, 1 and 2 were first posed in [16]):

1. Among RLC realisations, what is the minimum number of reactive elements required to realise all positive-real functions of McMillan degree n ?
2. In order to synthesise the entire class of impedances realisable by RLC networks containing n reactive elements, what is the least number of resistors required?
3. What is the class of impedances realisable by RLC networks containing at most n reactive elements and $n + 1$ resistors?

All of these questions have been answered in the case $n = 2$. Five reactive elements are required to realise all positive-real functions of McMillan degree 2 (see Sections 3–4). Three resistors are required to synthesise the entire class of impedances realisable by RLC networks containing 2 reactive elements (see Section 5). The class of impedances realisable by RLC networks containing at most 2 reactive elements and 3 resistors (namely the class corresponding to the Ladenheim catalogue) is stated explicitly in [17, 26] (see Section 6). However, these three questions remain unanswered for $n > 2$.

We also restate the following open problem from Section 7 on the realisation of behaviors, which is a slight variation on [4, Problems 1 and 2]:

4. What are the necessary and sufficient conditions for the behavior $p(\frac{d}{dt})i = q(\frac{d}{dt})v$ to be realised as the driving-point behavior of an RLC network?

9 Conclusions

It is now known that minimality in the context of RLC network synthesis differs from minimality in a system theoretic sense. In the absence of a general theory for the minimal realisation of RLC network behaviors, the enumeration approach provides a valuable research strategy for investigating networks of restricted complexity. To date, this has proved productive for investigating RLC networks whose impedances have McMillan degree less than or equal to two. A remaining challenge is to extend these results to higher-order networks.

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