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Shadows of Teichmüller Discs in the Curve Graph

Robert Tang[1] and Richard C. H. Webb[2,*]

[1]Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA, and [2]DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK

[*]*Correspondence to be sent to: e-mail: webb@dpmms.cam.ac.uk*

We consider several natural sets of curves associated to a given Teichmüller disc, such as the systole set or cylinder set, and study their coarse geometry inside the curve graph. We prove that these sets are quasiconvex and agree up to uniformly bounded Hausdorff distance. We describe two operations on curves and show that they approximate nearest point projections to their respective targets. Our techniques can be used to prove a bounded geodesic image theorem for a natural map from the curve graph to the filling multi-arc graph associated to a Teichmüller disc.

1 Overview

In their groundbreaking paper [12], Masur and Minsky proved that the curve graph is hyperbolic by studying its interplay with the large-scale geometry of Teichmüller space. Their notion of *balance time* for curves on Teichmüller geodesics, in particular, proved useful for showing that Teichmüller geodesics "shadow" reparameterized quasigeodesics in the curve graph under the systole map.

Our paper studies the coarse geometry of *Teichmüller discs*, a natural generalization of Teichmüller geodesics, via various notions of "shadows" in the curve graph. We also generalize the balance time to *balance points* for curves on Teichmüller discs, and use it to extend results of [12].

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Throughout this article, let S be a closed, connected, orientable surface of genus $g \geq 2$. The *curve graph* $\mathcal{C}(S)$ associated to S has as its vertices the free homotopy classes of non-trivial simple closed curves on S, with edges spanning vertices if the corresponding curves can be realized disjointly. We shall equip the *Teichmüller space* $\mathrm{Teich}(S)$, the parameter space of marked conformal structures on S up to isotopy, with the *Teichmüller metric*. The unit cotangent bundle to $\mathrm{Teich}(S)$ is naturally identified with the space $\mathcal{QD}^1(S)$ of unit-norm holomorphic quadratic differentials. We primarily work with the *half-translation structure* on S naturally associated to such a quadratic differential: these consist of an atlas of charts from S to $\mathbb{C} \cong \mathbb{R}^2$ with transition functions of the form $z \mapsto \pm z + c$ (see Section 2.4 for details).

There is a natural action of $SL(2,\mathbb{R})$ on $\mathcal{QD}^1(S)$. Given a half-translation surface $q \in \mathcal{QD}^1(S)$, let Δ_q be its $SL(2,\mathbb{R})$–orbit. The projection of Δ_q to $\mathrm{Teich}(S)$, denoted D_q, is a geodesically embedded copy of the hyperbolic plane $\mathbb{H}^2$ (with curvature -4), called a *Teichmüller disc*. From a Teichmüller disc $D \subset \mathrm{Teich}(S)$, one can recover the $SL(2,\mathbb{R})$–orbit $\Delta \subset \mathcal{QD}^1(S)$ projecting to it by considering any Teichmüller geodesic in D. Thus, one can interpret the results of this article as pertaining to either an $SL(2,\mathbb{R})$–orbit or a Teichmüller disc.

Convention: We shall always write Δ for the $SL(2,\mathbb{R})$–orbit corresponding to a Teichmüller disc D, and vice versa.

A *systole* on a surface is an essential curve of minimal length. We consider several notions of length: *flat length* on a half-translation surface $q \in \Delta$, *extremal length* on a conformal structure $X \in D$, and the *hyperbolic length* on the unique hyperbolic metric in a given conformal class. Call $\alpha \in \mathcal{C}(S)$ a *cylinder curve* on $q \in \Delta$ if it can be realized as the core curve of a Euclidean cylinder (of positive width) on q.

We shall use the adjective *uniform* to describe constants depending only on (the topological type of) S, and *universal* if they can be chosen independently of S.

Theorem 1.1. For any Teichmüller disc $D \subset \mathrm{Teich}(S)$, the following sets agree up to universal Hausdorff distance in $\mathcal{C}(S)$.

- $V(\Delta)$ – the set of straight vertex cycles for Δ,
- $\mathsf{sys}(\Delta)$ – the flat systoles appearing on Δ,
- $\mathsf{sys}^{\mathsf{Ext}}(D)$ – the extremal length systoles appearing on D,
- $\mathsf{sys}^{\mathsf{Hyp}}(D)$ – the hyperbolic length systoles appearing on D.

The following agree with the aforementioned sets up to uniform Hausdorff distance.

- $\mathsf{cyl}(\Delta)$ – the cylinder curves on some (hence every) $q \in \Delta$,
- $\widehat{\mathsf{cyl}}(\Delta)$ – the curves with constant slope on some (hence every) $q \in \Delta$. □

Most of the above sets have been studied in some form in the literature. The only new definition, to our knowledge, are straight vertex cycles. Informally speaking, $V(\Delta)$ is the set of curves whose geodesic representatives on every $q \in \Delta$ run over each saddle connection at most once in each direction and cannot be further decomposed via curve surgery. It is worth emphasizing that curves in $V(\Delta)$ can have arbitrarily large lower bounds on their flat length (and therefore the extremal and hyperbolic lengths) over Δ – in particular, they are not systoles. We provide a brief description, and refer the reader to Section 3 for more details. See also Section 2.3 for background on train tracks.

A geodesic representative of a curve $\alpha \in \mathcal{C}(S)$ on a half-translation surface $q \in \Delta$ is typically a concatenation of saddle connections. One can "smooth" such a geodesic representative at each singular point to obtain a train track $\tau_q(\alpha)$ carrying α. This construction commutes with $SL(2, \mathbb{R})$–deformations of q, and so we can canonically define a train track $\tau_\Delta(\alpha)$. The set of *straight vertex cycles* for Δ is

$$V(\Delta) = \cup_{\alpha \in \mathcal{C}(S)} V(\tau_\Delta(\alpha)),$$

where $V(\tau_\Delta(\alpha))$ denotes the vertex cycles of $\tau_\Delta(\alpha)$.

It is worth emphasizing that this construction needs only the combinatorial pattern of the saddle connections used by geodesics, not their lengths or directions. Thus we can use combinatorial methods to show the following, with effective control on the constants.

Theorem 1.2. There is a universal constant Q_1 so that for any $SL(2, \mathbb{R})$–orbit $\Delta \subset \mathcal{QD}^1(S)$, the set $V(\Delta)$ is Q_1–quasiconvex in $\mathcal{C}(S)$. Moreover, the operation $\alpha \mapsto V(\tau_\Delta(\alpha))$ agrees with the nearest point projection relation from $\mathcal{C}(S)$ to $V(\Delta)$ up to universally bounded error. □

Corollary 1.3. For any Teichmüller disc D, the systole sets $\mathsf{sys}(\Delta)$, $\mathsf{sys}^{\mathsf{Ext}}(D)$, and $\mathsf{sys}^{\mathsf{Hyp}}(D)$ are universally quasiconvex. The sets $\mathsf{cyl}(\Delta)$ and $\widehat{\mathsf{cyl}}(\Delta)$ are uniformly quasiconvex. □

Given $q \in \Delta$, let $S' = S'(\Delta)$ be the underlying topological surface with the singularities of q considered as marked points. Any non-cylinder curve $\alpha \in \mathcal{C}(S)$ determines a multi-arc $\mathcal{A}_\Delta(\alpha)$ on S' by taking the set of saddle connections used by the geodesic representative of α on any $q \in \Delta$. In particular, if α is sufficiently far from $V(\Delta)$ in $\mathcal{C}(S)$ then $\mathcal{A}_\Delta(\alpha)$ fills S'. Let $\mathcal{FMA}(S')$ be the graph whose vertices are filling multi-arcs on S', with edges connecting pairs of filling multi-arcs if one contains the other. Let $\mathcal{FMA}(\Delta)$ be the subgraph spanned by the multi-arcs with component arcs realisable as saddle connections on q. There is a natural 1–Lipschitz retract $\mathcal{FMA}(S') \to \mathcal{FMA}(\Delta)$ which maps a multi-arc to the set of saddle connections appearing on its geodesic representative on q. It follows that $\mathcal{FMA}(\Delta)$ isometrically embeds into $\mathcal{FMA}(S')$. We prove the following bounded geodesic image theorem.

Theorem 1.4. There are universal constants A and B so that if G is a geodesic in $\mathcal{C}(S)$ disjoint from the A–neighbourhood of $V(\Delta)$, then $\mathcal{A}_\Delta(G) = \{\mathcal{A}_\Delta(\alpha) \mid \alpha \in G\}$ has diameter at most B in $\mathcal{FMA}(\Delta)$. □

The endpoints of a Teichmüller geodesic $\mathcal{G}$ correspond to a pair of projectivized measured transverse foliations which give the horizontal and vertical directions for every half-translation structure along $\mathcal{G}$. If a curve α is neither completely horizontal nor completely vertical along $\mathcal{G}$, then its *balance time* on $\mathcal{G}$ is the unique point on $\mathcal{G}$ where the horizontal and vertical lengths of α are equal. Masur and Minsky show that the operation of taking a systole at the balance time of a curve satisfies certain coarse retraction properties (Theorem 2.6 of reference [12]).

We introduce the notion of a *balance point* of a curve α on a Teichmüller disc D in Section 7. This is a point $X \in D$ such that for any $q \in \Delta$ which projects to X, the horizontal and vertical lengths of α on q agree up to a bounded ratio. (Any two balance points for α on D are universally close.) The choice of $q \in \Delta$ projecting to X is unique up to rotation, so it makes sense to speak of the flat length of α on $X \in D$ (so long as we restrict attention to a particular Teichmüller disc). Balance points can be used to approximate the balance times of α along *all* Teichmüller geodesics on D.

Theorem 1.5. Let D be a Teichmüller disc and $\alpha \in \mathcal{C}(S)$ a curve. If α does not have constant slope on Δ then there is a point $X \in D$, called a *balance point* of α on D, satisfying the following.

- For any Teichmüller geodesic $\mathcal{G}$ on D, the nearest point projection of X to $\mathcal{G}$ in D is at most a distance of $\log 2$ from the balance time of α on $\mathcal{G}$.
- Any flat systole on X is universally close to any nearest point projection of α to $\mathsf{sys}(\Delta)$ in $\mathcal{C}(S)$.

- If the flat length of $\alpha \in \mathcal{C}(S)$ is minimized at $m \in \Delta$, among all half-translation surfaces in Δ, then X is universally close to (the projection of) m in D. □

We also introduce the *auxiliary polygon* $P_q(\alpha)$, associated to a curve α on a half-translation surface q, as a useful tool in proving this theorem by reducing our arguments to elementary Euclidean geometry in the plane. For example, the auxiliary polygon can be used to explicitly find a balance point on $D = D_q$ given a geodesic representative of α on q. The area of $P_q(\alpha)$ also gives an estimate for the minimal flat length of α on Δ. The auxiliary polygon is used by Forester *et al.* [6] to give a characterization of Veech surfaces in terms of lengths of simple closed curves.

Organisation

In Section 2, we provide background on coarse geometry, curve graphs, train tracks, half-translation surfaces and Teichmüller discs. We then describe the construction of the train track $\tau_q(\alpha)$ in Section 3, where we also give a proof of Theorem 1.2 and Theorem 1.4.

Theorem 1.1 shall be proved through the following collection of coarse inclusions. In the diagram below, an arrow $A \xrightarrow{r} B$ indicates that A is contained in the r–neighbourhood of B in $\mathcal{C}(S)$, with arrows in both directions indicating coarse inclusions in both directions.

$$\begin{tikzcd}
\mathsf{sys}^{\mathsf{Ext}}(D) \arrow[d, leftrightarrow, "37"'] \arrow[dr, leftrightarrow, "37"] & & V(\Delta) \arrow[dl, Rightarrow, bend right, "54"'] & \\
\mathsf{sys}^{\mathsf{Hyp}}(D) \arrow[r, leftrightarrow, "37"'] & \mathsf{sys}(\Delta) \arrow[ur, Rightarrow, "0"'] \arrow[r, dashed, "\mathsf{h}_3(g)"] & \mathsf{cyl}(\Delta) \arrow[u, dotted, "0"'] \arrow[r, dotted, "0"] \arrow[l, dashed, bend left, "1"] & \widehat{\mathsf{cyl}}(\Delta) \arrow[l, dashed, bend left, "1"]
\end{tikzcd}$$

In Section 4, we discuss the various systole maps and prove the coarse inclusions indicated by solid arrows $\longrightarrow$ in Corollary 4.1. We prove in Section 5, the inclusions indicated by the double arrows $\Longrightarrow$ in Proposition 5.1 and Proposition 5.6; and dashed arrows $\dashrightarrow$ in Lemma 5.8 using a wide cylinder theorem of Vorobets [19]. Our arguments yield explicit constants $\mathsf{h}_3(g) = O(2^{32g})$. The inclusions indicated by dotted arrows $\cdots\rightarrow$ essentially follow by definition.

In Section 6, we describe the construction of the auxiliary polygon $P_q(\alpha)$ and prove some basic properties. These shall be utilized in Section 7 to prove Theorem 1.5.

Finally, we discuss some connections between balance points and curve decompositions in Section 8.

Convention for constants

We shall label constants as follows:

- K_i – coarse Lipschitz constant,
- Q_i – quasiconvexity constant,
- P_i – nearest point projection approximation constant,
- h_i – bound on Hausdorff distance,

where $i = 1$ when the constants relate to straight vertex cycles; $i = 2$ for systoles; and $i = 3$ for cylinders.

2 Background

We begin by establishing some definitions and conventions. Throughout this article, S will be a closed, connected, orientable surface of genus $g \geq 2$.

2.1 Coarse geometry

A constant is called *uniform* if it depends only on the topological type of a surface S. A constant is called *universal* if it can be chosen independently of S. Given $x, y \in \mathbb{R}$ and constants K, C write

- $x \prec_{\mathsf{K},\mathsf{C}} y \iff x \leq \mathsf{K}y + \mathsf{C}$,
- $x \asymp_{\mathsf{K},\mathsf{C}} y \iff x \prec_{\mathsf{K},\mathsf{C}} y$ and $y \prec_{\mathsf{K},\mathsf{C}} x$.

If K and C are universal constants, we will also write $x \prec y$ and $x \asymp y$, respectively. If X and Y are subsets of a metric space $(\mathcal{X}, d)$, write

- $X \subseteq_r Y \iff X \subseteq N_r(Y)$,
- $X \approx_r Y \iff X \subseteq_r Y$ and $Y \subseteq_r X$.

Here, $N_r(Y)$ denotes the closed r–neighbourhood of Y in $\mathcal{X}$. The *Hausdorff distance* between closed sets X and Y in $\mathcal{X}$ is the smallest $r \geq 0$ such that $X \approx_r Y$.

Two relations $f_1 : \mathcal{X} \to \mathcal{Y}$ and $f_2 : \mathcal{X} \to \mathcal{Y}$ between metric spaces are said to *agree up to error* h if $f_1(x) \approx_{\mathsf{h}} f_2(x)$ for all $x \in \mathcal{X}$. We also say that f_1 and f_2 *coarsely agree*.

Let I be either an interval in $\mathbb{R}$, or the intersection of an interval with $\mathbb{Z}$. A relation $f : I \to \mathcal{X}$ is a (K,C)–*quasigeodesic* if

$$\mathrm{diam}_{\mathcal{X}}(f(s) \cup f(t)) \asymp_{\mathsf{K},\mathsf{C}} |s - t|$$

for all $s, t \in I$. We say $f : I \to \mathcal{X}$ is a *reparameterized quasigeodesic* if there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ and constants K, C and R such that $f \circ h$ is a (K,C)–quasigeodesic, and

$$\mathrm{diam}_{\mathcal{X}}(f([h(t), h(t+1)])) \leq \mathsf{R}$$

for all $t \in h^{-1}(I)$.

A relation $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces is called (K,C)–*coarsely Lipschitz* if for every $x, y \in \mathcal{X}$,

$$\mathrm{diam}_{\mathcal{Y}}(f(x) \cup f(y)) \prec_{\mathsf{K},\mathsf{C}} d_{\mathcal{X}}(x, y).$$

If, in addition, there is a constant $\mathsf{R} \geq 0$ such that $d_{\mathcal{X}}(y, f(y)) \leq \mathsf{R}$ for all $y \in \mathcal{Y}$ then we say f is a $(\mathsf{K},\mathsf{C},\mathsf{R})$–*coarse Lipschitz retract* onto $\mathcal{Y}$. We shall refer to $(\mathsf{K},\mathsf{K},\mathsf{K})$–coarse Lipschitz retracts as K–coarse Lipschitz retracts.

A geodesic space $\mathcal{X}$ is δ–*hyperbolic* if every geodesic triangle is δ–slim. That is, for all $x, y, z \in \mathcal{X}$ and geodesics $[x, y]$, $[x, z]$ and $[z, y]$, we have

$$[x, y] \subseteq_{\delta} [x, z] \cup [z, y].$$

A subset $\mathcal{Y}$ of a geodesic space $\mathcal{X}$ is Q–*quasiconvex* if every geodesic in $\mathcal{X}$ connecting a pair of points in $\mathcal{Y}$ is contained in the Q–neighbourhood of $\mathcal{Y}$.

Theorem 2.1 ([11] Lemma 3.3)**.** Let $\mathcal{X}$ be a δ–hyperbolic space and suppose a subset $\mathcal{Y} \subseteq \mathcal{X}$ admits a $(\mathsf{K},\mathsf{C},\mathsf{R})$–coarse Lipschitz retract $f : \mathcal{X} \to \mathcal{Y}$. Then $\mathcal{Y}$ is Q–quasiconvex, where $\mathsf{Q} = \mathsf{Q}(\delta, \mathsf{K}, \mathsf{C}, \mathsf{R})$. □

It is worth noting that a coarse Lipschitz retract to a subset $\mathcal{Y}$ of a δ–hyperbolic space $\mathcal{X}$ need not coarsely agree with a nearest point projection to $\mathcal{Y}$.

2.2 Curve graphs

By a *curve* on a surface S, we shall mean a free homotopy class of simple closed curves which are not null-homotopic. The *curve graph* $\mathcal{C}(S)$ of S is the graph whose vertices are curves on S, and whose edges span pairs of distinct curves which have disjoint representatives. We shall also write $\alpha \in \mathcal{C}(S)$ to mean that α is a curve on S.

Given curves $\alpha, \beta \in \mathcal{C}(S)$, define their distance $d_S(\alpha, \beta)$ to be the length of a shortest edge-path connecting them in $\mathcal{C}(S)$. The curve graph is locally infinite, and has infinite diameter. Let

$$i(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}$$

denote the *geometric intersection number* of α and β. Combining the distance bounds due to Hempel ([8] Lemma 2.1) for $g = 2, 3$, and Bowditch ([4] Corollary 2.2) for $g \geq 4$ gives the following.

Lemma 2.2. Let α and β be curves in $\mathcal{C}(S)$ where $i(\alpha, \beta) \geq 1$. Then

$$d_S(\alpha, \beta) \prec_{2,2} \log_{\mathsf{G}} i(\alpha, \beta),$$

where $\mathsf{G} = \max\{2, g - 2\}$. □

It follows that $\mathcal{C}(S)$ is connected. We shall be using the following observation several times throughout this article to obtain universal bounds on distances.

Lemma 2.3. For any $C > -2$, the quantity $\log_{\mathsf{G}}(g + C)$ achieves its maximum at $g = 4$ among all $g \geq 2$. □

Proof. By differentiation, it can be shown that $x \mapsto \log_{x-2}(x + C)$ is decreasing for $x \geq 4$. Similarly, $x \mapsto \log_2(x + C)$ is increasing for $2 \leq x \leq 4$. ■

Masur and Minsky prove the following fundamental result concerning the large-scale geometry of curve graphs.

Theorem 2.4 ([12])**.** The curve graph $\mathcal{C}(S)$ is δ–hyperbolic for some $\delta > 0$. □

Recent independent work of several authors shows that there is a universal hyperbolicity constant [1], [4], [5], [9].

2.3 Train tracks

We now review some background on train tracks. For a thorough introduction, see [18] and [16].

A *pretrack* τ is a properly embedded graph on S satisfying the following conditions. The edges of τ, called *branches*, are smoothly embedded. Each vertex s of τ, called a *switch*, is equipped with a preferred tangent $v_s \in T_s^1 S$. A half-branch incident to s is called incoming (or outgoing) if its derivative at s is v_s (or $-v_s$). For every switch s, we require each incident half-branch to be either incoming or outgoing, and that there is at least once of each type incident to s. We allow pretracks to have embedded loops as connected components.

A *train-route* (or *train-loop*) on τ is a smoothly immersed path (or loop) on S contained in τ. Say a simple closed curve $\alpha \in \mathcal{C}(S)$ is *carried* by τ, and write $\alpha \prec \tau$, if it can be realized as a train-loop on τ. A pretrack σ is *carried* by τ, also written $\sigma \prec \tau$, if it can be smoothly homotoped into τ.

Let τ be a pretrack on S. Any curve α realized as a train route on τ induces a function $w_\alpha : \mathcal{B}(\tau) \to \mathbb{R}_{\geq 0}$ on the branches of τ defined by setting $w_\alpha(b)$ to be the number of times α runs over the branch $b \in \mathcal{B}(\tau)$. In general, a function $w : \mathcal{B}(\tau) \to \mathbb{R}_{\geq 0}$ is called a *transverse measure* on τ if it satisfies the *switch conditions*: at every switch s of τ, we require

$$\sum_{b \in \mathcal{B}^+(s)} w(b) = \sum_{b \in \mathcal{B}^-(s)} w(b),$$

where $\mathcal{B}^+(s)$ and $\mathcal{B}^-(s)$ are respectively the sets of incoming and outgoing half-branches incident to s.

Integral measures on τ are in one-to-one correspondence with realizations of multicurves on τ, where we allow for multiple parallel copies of each component and trivial loops. One can recover a multicurve from an integral measure as follows: for each branch b, take $w(b)$ parallel strands within a *tie neighbourhood* of b on S. At each switch s, glue the incoming strands to the outgoing strands in pairs —there is exactly one pairing of the strands which does not lead to self-intersections. See Section 3.7 of [16] for more details.

A complementary region of a pretrack τ on S is called a *nullgon*, *monogon*, or *bigon* if it is a topological disc with zero, one, or two cusps on its boundary, respectively (see Figure 1). Call τ on S a *train track* if all its complementary regions have negative index: on closed surfaces, this is equivalent to saying that no complementary region is a nullgon, monogon, bigon, or annulus. This condition guarantees that every train-loop on a train track τ is essential on S, and that two train-loops on a train track τ are freely homotopic on S if and only if they agree up to reparameterization. In particular,

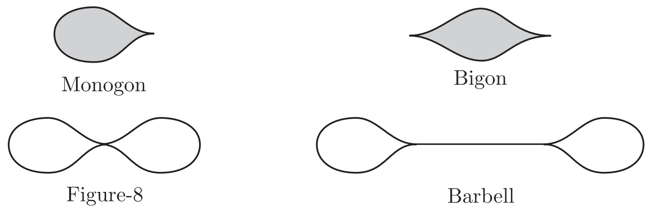

Fig. 1. Top: a monogon and bigon; Bottom: a figure–8 and barbell.

distinct integral measures on τ correspond to distinct homotopy classes of weighted multicurves.

The set $\mathbf{M}(\tau)$ of transverse measures on a train track τ defines a convex cone in $\mathbb{R}^{\mathcal{B}(\tau)}_{\geq 0}$ with finitely many extreme rays. Each extreme ray of $\mathbf{M}(\tau)$ contains a unique minimal integral transverse measure on τ, and such a measure arises from a simple closed curve carried by τ. We call such curves *vertex cycles* of τ, and write $V(\tau) \subset \mathcal{C}(S)$ for the set of vertex cycles of τ. (Vertex cycles are so named because they form vertices of the polytope obtained by projectivising $\mathbf{M}(\tau)$.)

Theorem 2.5 ([16] Proposition 3.11.3)**.** A simple closed curve α is a vertex cycle of a train track τ if and only if it realisable as either an embedded loop, a figure–8, or a barbell on τ (see Figure 1). □

Every transverse measure $w \in \mathbf{M}(\tau)$ can be written as a sum of transverse measures arising from vertex cycles of τ. Following a curve surgery argument of Masur, Mosher, and Schleimer with some extra care, one can show the following.

Theorem 2.6 ([15] Section 2.6)**.** Let $\alpha \in \mathcal{C}(S)$ be a multicurve carried by a train track τ. Then there are integers $m_v \geq 0$ for each $v \in V(\tau)$ such that $w_\alpha = \frac{1}{2}\sum_{v\in V(\tau)} m_v w_v$. □

Aougab shows that $V(\tau)$ has universally bounded diameter in $\mathcal{C}(S)$ by controlling the intersection number between any two curves carried by τ which run over each branch at most twice.

Theorem 2.7 ([1] Section 5)**.** There is a universal constant K_1 such that for any train track τ on S, the set $V(\tau)$ has diameter at most K_1 in $\mathcal{C}(S)$. □

If one follows Aougab's proof using Bowditch's distance bound [4], one can show that taking $\mathsf{K}_1 = 14$ suffices (even for surfaces with punctures). Masur and Minsky

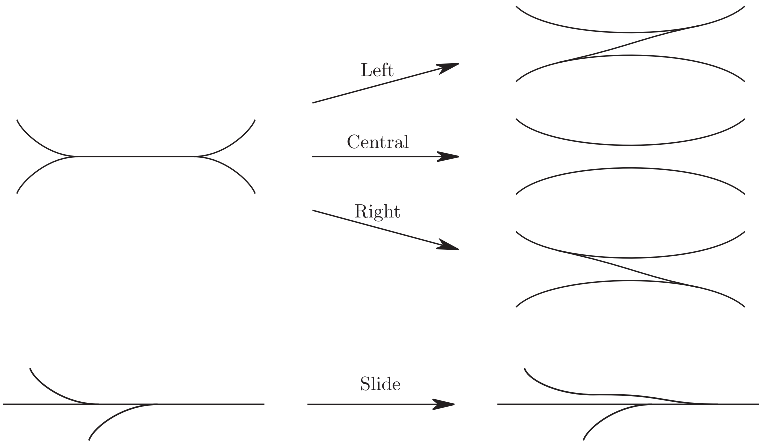

Fig. 2. Splitting a large branch; and sliding a mixed branch.

conjecture that K_1 can be taken to be equal to 3, and Aougab shows that this is true for sufficiently large genus.

A train track is called *generic* if all switches are trivalent. Any train track can be perturbed to a generic one which carries exactly the same curves. Suppose τ is a generic train track with at least one switch. Every switch of τ separates its three incident half-branches into one *large* half-branch on one side and a pair of *small* half-branches on the other. A branch b is called *large* (or *small*) if both of its half-branches are large (or small); if it has one of each type then we say it is *mixed*.

We now described splitting and sliding — refer to Sections 3.11 and 3.12 of [16] for more details.

To a generic train track τ, we may *split* along a large branch or *slide* along a mixed branch to obtain a new train track carried by τ – see Figure 2. We say $(\tau_i)_i$ is a *splitting and sliding sequence of train tracks* if each τ_{i+1} is obtained from τ_i by a split or slide. If α is a curve carried by τ, we may *split τ towards α* by splitting a large branch of τ to obtain a new train track carrying α, where the choice of left, right or central split is determined by the transverse measure on τ induced by α. (More specifically, α will either be carried by exactly one of the left or right splits; or both in which case we can choose the central split.) A splitting sequence carried out in this manner will eventually terminate in a train track with an embedded loop component isotopic to α.

Theorem 2.8. [14] Let $(\tau_i)_i$ be a splitting and sliding sequence of train tracks. Then $(V(\tau_i))_i$ forms a reparameterized quasigeodesic in $\mathcal{C}(S)$. □

Hamenstädt gives an alternative proof of this result which achieves universal quasiconvexity constants, and also removes the birecurrence assumption for splitting sequences ([7] Corollary 3.4 and Lemma 3.5).

2.4 Half-translation surfaces and Teichmüller discs

A *half-translation* structure q on a closed surface S consists of a finite set of singular points together with an atlas of charts to $\mathbb{C} \cong \mathbb{R}^2$ defined away from the singular set, where the transition maps are *half-translations*, i.e., of the form $z \mapsto \pm z + c$ where $c \in \mathbb{C}$. The singular points have Euclidean cone angle of the form $k\pi$ where $k \geq 3$. The atlas determines a preferred vertical slope on q. Half-translation surfaces can be constructed by taking a finite collection of disjoint Euclidean polygons in $\mathbb{R}^2$ with edges glued in pairs isometrically via half-translations. We shall consider half-translation structures up to isotopy.

The space of half-translation structures on S can be naturally identified with $\mathcal{QD}(S)$, the space of non-zero *holomorphic quadratic differentials* on S up to isotopy. An element of $\mathcal{QD}(S)$ is a conformal structure X on S equipped with a differential locally of the form $q = q(w)dw^2$, where $q(w)$ is a holomorphic function in the local co-ordinate w. The corresponding half-translation structure can be obtained by defining charts near each regular point z_0 as

$$z \mapsto \int_{z_0}^{z} \sqrt{q(w)}\mathrm{dw}.$$

An order k zero of q corresponds to a singularity with cone angle $(k+2)\pi$. We shall write q as short-hand for S equipped with the half-translation structure arising from $q \in \mathcal{QD}(S)$.

A *marking* of a surface X is a homeomorphism $f : S \to X$ from the reference surface S. The *Teichmüller space* $\mathrm{Teich}(S)$ of S, the space of marked conformal (or complex) structures on S up to isotopy, is homeomorphic to an open ball $\mathbb{R}^{6g-6}$. The projection map $\mathcal{QD}(S) \to \mathrm{Teich}(S)$ defined by taking $q \in \mathcal{QD}(S)$ to its underlying conformal structure can be canonically identified with the cotangent bundle to $\mathrm{Teich}(S)$. Restricting this projection to the space $\mathcal{QD}^1(S)$ of *unit-area* half-translation structures on S gives the unit cotangent bundle to $\mathrm{Teich}(S)$.

There is a natural $SL(2,\mathbb{R})$–action on $\mathcal{QD}(S)$ defined as follows. Let $q \in \mathcal{QD}(S)$ be a half-translation surface and $A \in SL(2,\mathbb{R})$ be a real linear transformation on $\mathbb{R}^2$. The half-translation surface $A \cdot q$ has as its atlas the charts obtained by postcomposing each co-ordinate chart of q to $\mathbb{C} \cong \mathbb{R}^2$ with A. One can perform this action by deforming a defining set of polygons for q by A and observing that the gluing patterns are preserved. Also note that $SL(2,\mathbb{R})$–deformations preserve area, and so this action descends equivariantly to $\mathcal{QD}^1(S)$. We shall equip Teich(S) with the *Teichmüller metric* which measures the amount of quasiconformal distortion between two conformal structures. With this metric, Teich(S) is a complete geodesic space in which geodesics, called *Teichmüller geodesics*, are projections of orbits in $\mathcal{QD}(S)$ under the *Teichmüller geodesic flow* (or *diagonal action*)

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in SL(2,\mathbb{R})$$

for $t \in \mathbb{R}$. In particular, $t \mapsto g_t \cdot q$ gives a unit speed parameterization of a Teichmüller geodesic $\mathcal{G}_q$ through q.

A *saddle connection* on a half-translation surface is a geodesic segment which meets the singular set precisely at its endpoints. Let $\alpha \in \mathcal{C}(S)$ be a curve and $q \in \mathcal{QD}(S)$ a half-translation surface. Then either there is a unique maximal flat cylinder of positive width foliated by closed geodesic leaves isotopic to α, or α has a (unique) geodesic representative on q which is a concatenation of saddle connections. We call α a *cylinder curve* in the former case.

Write α^q for a geodesic representative of α on q, and define the *flat length* of α on q to be $l_q(\alpha) = l(\alpha^q)$. We can also define the *horizontal* and *vertical* lengths of α on q, denoted $l_q^H(\alpha)$ and $l_q^V(\alpha)$, respectively, by integrating α^q against the pullbacks of the infinitesimal metrics $|dx|$ and $|dy|$ on $\mathbb{R}^2$. Call α *completely horizontal* (or *completely vertical*) on q if $l_q^V(\alpha) = 0$ (or $l_q^H(\alpha) = 0$). We say that a curve α has *constant slope* on q if it is completely horizontal on $\rho_\theta^{-1} \cdot q$ for some rotation $\rho_\theta \in SO(2,\mathbb{R})$; otherwise it has *non-constant slope* on q.

Observe that

$$l^H_{g_t\cdot q}(\alpha) = e^t l^H_q(\alpha) \qquad \text{and} \qquad l^V_{g_t\cdot q}(\alpha) = e^{-t} l^V_q(\alpha).$$

Say α is *balanced* on q if $l_q^H(\alpha) = l_q^V(\alpha)$. We call $t \in \mathbb{R}$ the *balance time* of α along a Teichmüller geodesic $\mathcal{G} = \mathcal{G}_q$ if α is balanced on $g_t \cdot q$. The balance time exists, and is unique, whenever α is neither completely vertical nor horizontal along $\mathcal{G}$. Define $l_{\mathcal{G}}(\alpha) = \inf_{q\in\mathcal{G}} l_q(\alpha)$.

Lemma 2.9. The flat length of a curve α is strictly convex along any Teichmüller geodesic, and satisfies

$$l_{g_t\cdot q}(\alpha) \leq \sqrt{2}e^t l_q(\alpha)$$

for all $q \in \mathcal{QD}(S)$. If α is neither completely horizontal nor completely vertical on q then

$$l_{g_t \cdot q}(\alpha) \asymp l_{\mathcal{G}}(\alpha) \cosh (t - t_\alpha),$$

where t_α is the balance time α along $\mathcal{G} = \mathcal{G}_q$. Moreover, if the (unique) minimum of $l_{g_t \cdot q}(\alpha)$ occurs at $t = t_m$ then $|t_m - t_\alpha| \leq \cosh^{-1} 2$. □

A *Teichmüller disc* $D = D_q$ is the projection of the $SL(2,\mathbb{R})$–orbit Δ_q of a half-translation surface $q \in \mathcal{QD}^1(S)$ to Teich(S). Using the Teichmüller metric, D is an isometrically embedded copy of the hyperbolic plane $\mathbb{H}^2 \cong SO(2,\mathbb{R})\backslash SL(2,\mathbb{R})$ with curvature -4 in Teich(S). Write d_D for the Teichmüller metric restricted to D. Given $q, q' \in \Delta$, we shall also write $d_D(q, q')$ for the Teichmüller distance between their underlying conformal structures. Note that $q, q' \in \Delta$ project to the same point on D if and only if they differ by a rotation. Moreover, rotations do not alter the flat length function l_q. Define the *infimal length* of α with respect to Δ to be $l_\Delta(\alpha) = \inf_{q \in \Delta} l_q(\alpha)$. We shall discuss the relation between this quantity and the *auxiliary polygon* in Section 6.

Lemma 2.10. If α has non-constant slope on $q \in \Delta$, then the flat length of α has a unique minimizer on Δ up to rotation. □

Proof. Since α has non-constant slope, $l_q(\alpha)$ diverges to infinity as (the underlying conformal structure of) $q \in \Delta$ approaches the ideal boundary of D. Therefore a minimum must be attained. The minimizer must be unique (up to rotation), for otherwise we can connect distinct minimizers in D by a Teichmüller geodesic, a contradiction. ■

For a Teichmüller disc D, the boundary circle $\partial D \subset \mathcal{PMF}(S)$ corresponds to the projectivized measured foliations on S which can be realized as a constant slope foliation on some $q \in \Delta$, where the transverse measure is the Lebesgue measure (up to scale). Let $\mathcal{F}_q^H$ and $\mathcal{F}_q^V$, respectively denote the horizontal and vertical foliations on q.

3 Train tracks and multi-arcs on half-translation surfaces

3.1 Train tracks induced by geodesics on half-translation surfaces

Throughout this section, fix a half-translation surface $q \in \mathcal{QD}^1(S)$. Suppose a curve $\alpha \in \mathcal{C}(S)$ is not a cylinder curve on q, so that it has a unique geodesic representative α^q. We construct a train track $\tau_q(\alpha)$ on S by performing a "smoothing" operation at each singular point.

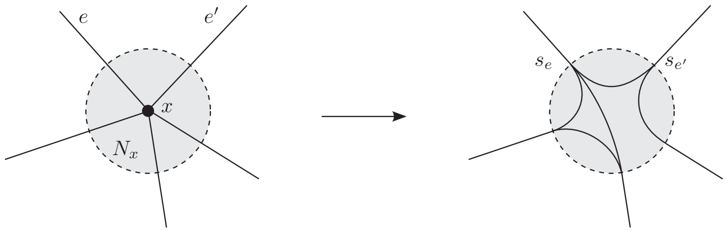

Fig. 3. Smoothing α^q at a singularity x to locally produce a train track.

Let $\Gamma = \Gamma_q(\alpha)$ be the embedded graph on q whose vertices and edges are respectively the singular points and saddle connections used by α^q. At each singular point x, delete from Γ a small open regular neighbourhood N_x. The *link* $\Gamma \cap \partial N_x$ of x is in natural bijection with the set of half-edges incident to x – write $s_e \in \Gamma \cap \partial N_x$ for the point associated to a half-edge e. Add a smoothly embedded arc, with interior inside N_x, connecting two such points s_e and $s_{e'}$ if and only if α^q, regarded as a closed path on q, passes through x by entering along e and exiting along e' (or vice versa) – see Figure 3. (Note that e and e' are necessarily distinct as α^q is a geodesic.) Such an arc can be realized so that at each endpoint, the incoming unit tangent vector along the arc coincides with the outgoing unit tangent vector along the corresponding half-edge. This gives a switch structure to the points s_e which are the endpoints of at least two new arcs. Since α is simple, we can arrange so that all the new arcs have disjoint interiors.

If α is a cylinder curve on q, we take $\tau_q(\alpha)$ to be any closed geodesic leaf representing α not passing through any singular points.

The above procedure produces a pretrack $\tau_q(\alpha)$ whose switches are the points s_e with at least three incident arcs. It may be that $\tau_q(\alpha)$ has no switches and is thus an embedded loop representing α. We can also naturally extend the definition of $\tau_q(\alpha)$ to multicurves, noting that it could possibly be disconnected.

A multicurve α can be realized as a train-loop on $\tau_q(\alpha)$ by perturbing it near the singularities: each time the closed path α^q passes through a singularity x along e and e', perturb it within N_x so that it instead runs along the arc on $\tau_q(\alpha)$ connecting s_e and $s_{e'}$.

Conversely, one can "straighten" $\tau_q(\alpha)$ to Γ: whenever there is an arc of $\tau_q(\alpha)$ connecting s_e and $s_{e'}$ within some N_x, homotope it to a geodesic path running along e and e' via x. This homotopy sends any train-route η on $\tau_q(\alpha)$ to the unique geodesic path on q connecting the endpoints of η in its relative homotopy class. In particular, any train-loop on $\tau_q(\alpha)$ is sent to a geodesic representative which lies completely within Γ. Note that $\tau_q(\alpha)$ is identical to its straightening if and only if every component of α is a cylinder curve.

Lemma 3.1. Let $\alpha \in \mathcal{C}(S)$ be a multicurve. Then $\tau_q(\alpha)$ is a train track carrying α. Furthermore, if β is a multicurve carried by $\tau_q(\alpha)$ then $\tau_q(\beta)$ is a subtrack of $\tau_q(\alpha)$. □

Proof. To show that $\tau_q(\alpha)$ is a train track we need to verify that every complementary region of $\tau_q(\alpha)$ on S has negative index. The remaining claims follow from the construction.

Suppose Y is a complementary region of $\tau_q(\alpha)$ of one of the following forbidden types. We apply the above "straightening" procedure to ∂Y. If Y is a nullgon or monogon, then ∂Y is a closed train-route which straightens to a geodesic on q homotopic to a constant path. If Y is a bigon, then ∂Y consists of two distinct train-routes connecting a pair of switches s and s'. Upon straightening, these train routes become distinct geodesics connecting s and s' in the same relative homotopy class. Either of these scenarios contradict the fact that there is a unique geodesic connecting a given pair of points on q in each relative homotopy class. Now suppose Y is an annulus. Then ∂Y consists of a pair of homotopic train-loops, say γ and γ', which straighten to a pair of distinct homotopic closed geodesics. Since these closed geodesics are distinct, they must be core curves of a common flat cylinder on q. The integral measure w_α on $\tau_q(\alpha)$ determined by α places positive weight on each branch – in particular, all branches which γ and γ' run over. As one reconstructs α from w_α using the process described in Section 2.3, we see that α contains two components which are parallel to γ and γ', respectively. This gives a contradiction. ■

3.2 Straight vertex cycles for Teichmüller discs

The construction of $\tau_q(\alpha)$ is completely determined by the sequence of directed saddle connections used by α^q. Since these data are preserved under $SL(2,\mathbb{R})$–deformations on q, we have the following observation.

Lemma 3.2. For any half-translation surface $q \in \mathcal{QD}^1(S)$, multicurve $\alpha \in \mathcal{C}(S)$ and $A \in SL(2,\mathbb{R})$, the train track $A \cdot \tau_q(\alpha)$ is isotopic to $\tau_{A\cdot q}(\alpha)$ on $A \cdot q$. □

This allows us to extend the construction to an $SL(2,\mathbb{R})$–orbit Δ. Let $\alpha \in \mathcal{C}(S)$ be a multicurve. Choose any $q \in \Delta$, and let $f_q : S \to q$ be a marking. Then

$$\tau_\Delta(\alpha) := f_q^{-1}(\tau_q(\alpha))$$

is a train track on S. By the above lemma, the isotopy class of $\tau_\Delta(\alpha)$ on S is independent of the choice of $q \in \Delta$. In addition, Lemma 3.1 also holds if one replaces τ_q with τ_Δ.

Given an $SL(2,\mathbb{R})$–orbit Δ, define its set of *straight train tracks*

$$\mathcal{TT}(\Delta) := \{\tau_\Delta(\alpha) \mid \alpha \text{ a multicurve on } S\} \subset \mathcal{TT}(S)$$

and its *straight vertex cycle* set

$$V(\Delta) := \bigcup_{\tau \in \mathcal{TT}(\Delta)} V(\tau).$$

Appealing to the characterization of vertex cycles, straight vertex cycles for Δ have geodesic representatives on any $q \in \Delta$ which run over each saddle connection at most once in each direction. One can then think of $V(\Delta)$ as the set of combinatorially short curves with respect to Δ. We shall also write $V_\Delta(\alpha) = V(\tau_\Delta(\alpha))$.

Let K_1 be a universal upper bound on the diameter of the vertex set of any train track.

Lemma 3.3. The relation $V_\Delta : \mathcal{C}(S) \to V(\Delta)$ is a K_1–coarsely Lipschitz retract. □

Proof. Let α and β be disjoint curves. By Lemma 3.1, $\tau_\Delta(\alpha)$ and $\tau_\Delta(\beta)$ are both subtracks of $\tau_\Delta(\alpha \cup \beta)$, from which it follows that $V_\Delta(\alpha) \cup V_\Delta(\beta) \subseteq V_\Delta(\alpha \cup \beta)$. Since the diameter of $V_\Delta(\alpha \cup \beta)$ in $\mathcal{C}(S)$ is at most K_1, it follows that V_Δ is K_1–coarsely Lipschitz.

Now suppose $\gamma \in V(\Delta)$, which means there is some multicurve α such that $\gamma \in V(\tau_\Delta(\alpha))$. By Lemma 3.1, $\tau_\Delta(\gamma)$ is a subtrack of $\tau_\Delta(\alpha)$ which also carries γ. By the characterization of vertex cycles, γ is also a vertex cycle of $\tau_\Delta(\gamma)$. Therefore $\gamma \in V_\Delta(\gamma)$ and so V_Δ is a coarse retract. ■

It immediately follows, via Theorem 2.1, that $V(\Delta)$ is quasiconvex.

Corollary 3.4. There is a universal constant Q_1 such that for any Teichmüller disc $\Delta \subset \mathrm{Teich}(S)$, the set $V(\Delta)$ is Q_1–quasiconvex in $\mathcal{C}(S)$. □

Proposition 3.5. The operation $V_\Delta : \mathcal{C}(S) \to V(\Delta)$ agrees with the nearest point projection map from $\mathcal{C}(S)$ to $V(\Delta)$ up to a universal error P_1. □

Proof. Let $\alpha \in \mathcal{C}(S)$. Perturb $\tau_\Delta(\alpha)$ to a generic train track τ_0, and consider a train track splitting and sliding sequence $\tau_0 \succ \tau_1 \succ \ldots \succ \tau_n = \alpha$. Choose a vertex cycle $\gamma_i \in V(\tau_i)$ for $0 \le i \le n$. By Theorem 2.8, the sequence $(\gamma_i)_i$ forms a reparameterized quasigeodesic from $V_\Delta(\tau)$ to α in $\mathcal{C}(S)$ with universal quasiconvexity constants.

Let β be a nearest point projection of α to $V(\Delta)$. Since $V(\Delta)$ is Q_1–quasiconvex, β must lie within a universal distance of any geodesic connecting α to $V_\Delta(\alpha)$, and hence the quasigeodesic $(\gamma_i)_i$. Therefore, there is some γ_k which is universally close to β and hence $V(\Delta)$. By Lemma 3.3, V_Δ is a K_1–coarse Lipschitz retract which means $V_\Delta(\gamma_k)$ is universally close to γ_k. But by Lemma 3.1, $\tau_\Delta(\gamma_k)$ is a subtrack of $\tau_\Delta(\alpha)$, and so $V_\Delta(\gamma_k) \subseteq V_\Delta(\alpha)$. It follows that β is within a universally bounded distance of $\tau_\Delta(\alpha)$. ■

3.3 Filling multi-arc graphs for Teichmüller discs

Given $q \in \Delta$, we consider the underlying surface S with marked points at the singularities of q. More precisely, let $S' = (S, Z)$ be the reference surface S whose set of marked points Z is the preimage of the singular points on q via the marking $f_q : S \to q$. After isotopy, we may also assume that for every $q' \in \Delta$, the marking $f_{q'} : S \to q'$ maps Z bijectively to the singular points of q'. Thus, we may consider the reference surface with marked points $S' = S'(\Delta)$ associated to Δ.

The *arc complex* $\mathcal{A}(S')$ of S' has as vertices proper isotopy classes of embedded arcs on S whose endpoints lie in Z. The simplices of $\mathcal{A}(S')$ correspond to *multi-arcs*: a set of arcs which can be properly isotoped to have pairwise disjoint interiors. Let $\mathcal{A}(\Delta)$ be the subcomplex of $\mathcal{A}(S')$ spanned by the arcs which can be realized as a saddle connection on some (hence any) $q \in \Delta$. The arc complex is connected, has infinite diameter, and is locally infinite. Moreover, its 1–skeleton is universally hyperbolic [9].

The 1–skeleton of the first barycentric subdivision of $\mathcal{A}(S')$ is the *multi-arc graph* $\mathcal{MA}(S')$. This naturally contains the subgraph $\mathcal{MA}(\Delta)$ spanned by multi-arcs realisable as saddle connections on some $q \in \Delta$. Observe that the geodesic representative of any multi-arc (or multicurve) on q forms a collection of disjoint saddle connections, which in turn gives a vertex in $\mathcal{MA}(\Delta)$. So there is a 1–Lipschitz map $\mathcal{MA}(S') \to \mathcal{MA}(\Delta)$ that coincides with the identity when restricted to $\mathcal{MA}(\Delta)$.

Lemma 3.6. There is a 1-Lipschitz retract $\mathcal{MA}(S') \to \mathcal{MA}(\Delta)$. In particular, $\mathcal{MA}(\Delta)$ is connected and isometrically embedded in $\mathcal{MA}(S')$. □

Let $\mathcal{A}_\Delta : \mathcal{C}(S) \to \mathcal{MA}(\Delta)$ be the 1–Lipschitz map that sends a curve α to the set of saddle connections appearing on α^q for any $q \in \Delta$. If α is a cylinder curve, we set $\mathcal{A}_\Delta(\alpha)$ to be the saddle connections appearing on the boundary of the maximal flat cylinder on q with core curve α.

Lemma 3.7. Suppose that $\mathcal{A}_\Delta(\alpha)$ does not fill S'. Then there exists $\beta \in V(\Delta)$ such that $d_S(\alpha, \beta) \le 2$. □

Proof. Choose any $\beta \in V_\Delta(\alpha)$. Then by Lemma 3.1, $\tau_\Delta(\beta)$ is a subtrack of $\tau_\Delta(\alpha)$, and so upon straightening it follows that $\mathcal{A}_\Delta(\beta) \subseteq \mathcal{A}_\Delta(\alpha)$ (viewed as sets of saddle connections). Since $\mathcal{A}_\Delta(\alpha)$ does not fill S', there is some $\gamma \in \mathcal{C}(S)$ disjoint from $\mathcal{A}_\Delta(\alpha)$ and hence $\mathcal{A}_\Delta(\beta)$. It follows that γ is disjoint from both α and β. ∎

A multi-arc on S' is *filling* if every complementary region is topologically a disc with at most one marked point. Define the *filling multi-arc graph* $\mathcal{FMA}(S')$ to be the subgraph of $\mathcal{MA}(S')$ spanned by filling multi-arcs on S'. In contrast with $\mathcal{MA}(S')$, this graph is locally finite. Let $\mathcal{FMA}(\Delta)$ be the subgraph of $\mathcal{FMA}(S')$ spanned by the filling multi-arcs realizable as saddle connections on any $q \in \Delta$. Restricting the retraction $\mathcal{MA}(S') \to \mathcal{MA}(\Delta)$ to $\mathcal{FMA}(S')$, we deduce the following.

Lemma 3.8. There is a 1–Lipschitz retract $\mathcal{FMA}(S') \to \mathcal{FMA}(\Delta)$. In particular, $\mathcal{FMA}(\Delta)$ is connected, locally finite, and isometrically embedded in $\mathcal{FMA}(S')$. □

We are now ready to prove a bounded geodesic image theorem for the projection $\mathcal{A}_\Delta$.

Theorem 3.9. There are universal constants A and B so that if G is a geodesic in $\mathcal{C}(S)$ disjoint from the A–neighbourhood of $V(\Delta)$, then $\mathcal{A}_\Delta(G) = \{\mathcal{A}_\Delta(\alpha) \mid \alpha \in G\}$ has diameter at most B in $\mathcal{FMA}(\Delta)$. □

Proof. Let $\beta \in G$ be a curve which is closest to $V(\Delta)$ among all curves on G. We claim that $\mathcal{A}_\Delta(G)$ lies in a universally bounded neighbourhood of $\mathcal{A}_\Delta(\beta)$ in $\mathcal{FMA}(\Delta)$.

Choose any curve α on G, and let G' be a geodesic connecting α to $V_\Delta(\alpha)$ in $\mathcal{C}(S)$. Appealing to δ–hyperbolicity of $\mathcal{C}(S)$ and Q_1–quasiconvexity of $V(\Delta)$, we deduce that G' must pass within a distance $\mathsf{C} = \mathsf{C}(\delta, \mathsf{Q}_1)$ of β, so long as G does not come within a distance $\mathsf{A} = \mathsf{A}(\delta, \mathsf{Q}_1)$ of $V(\Delta)$.

Now consider the train track $\tau_\Delta(\alpha)$. After perturbing to a generic train track τ_0, we may produce a splitting and sliding sequence $\tau_0 \succ \tau_1 \succ \ldots \succ \tau_n = \alpha$. By Theorem 2.8, the vertex cycles along this splitting sequence form a quasigeodesic which agrees with G' up to universal Hausdorff distance. It follows that there is some vertex cycle γ along this splitting sequence within a universal distance C' of β. Let G'' be a geodesic connecting γ to β. Taking A to be larger if necessary, we may assume every vertex along G'' is at least distance 3 from $V(\Delta)$. Thus, by Lemma 3.7, every curve on G'' maps to a filling multi-arc under $\mathcal{A}_\Delta$. Since $\mathcal{A}_\Delta$ is 1–Lipschitz, $\mathcal{A}_\Delta(G)$ is a path of length at most C' in $\mathcal{FMA}(\Delta)$ connecting $\mathcal{A}_\Delta(\gamma)$ to $\mathcal{A}_\Delta(\beta)$.

Since γ is carried by $\tau_\Delta(\alpha)$, we may apply Lemma 3.1 and argue as in Lemma 3.7 to show that $\mathcal{A}_\Delta(\gamma) \subseteq \mathcal{A}_\Delta(\alpha)$. In particular, $\mathcal{A}_\Delta(\gamma)$ and $\mathcal{A}_\Delta(\alpha)$ are equal or adjacent in $\mathcal{FMA}(\Delta)$. Thus $d_{\mathcal{FMA}(\Delta)}(\mathcal{A}_\Delta(\alpha), \mathcal{A}_\Delta(\beta)) \leq \mathsf{C}' + 1$, and so $\mathcal{A}_\Delta(G)$ has diameter at most $\mathsf{B} = 2\mathsf{C}' + 2$ in $\mathcal{FMA}(\Delta)$. ∎

Remark 3.10. This theorem is inspired by, but does not seem to follow readily from, Masur and Minsky's bounded geodesic image theorem for subsurface projections —see Theorem 3.1 of [13]. □

The valency of each vertex of $\mathcal{FMA}(\Delta)$ can be bounded in terms of the topology of S'. In particular, Theorem 3.9 shows that there is a uniform bound on the number of distinct saddle connections that appear on the geodesic representatives of curves along G. Moreover, one can uniformly bound the pairwise intersection number between arcs in $\mathcal{A}_\Delta(G)$. In particular, if the geodesic representatives of curves α and β use saddle connections $a \in \mathcal{A}_\Delta(\alpha)$ and $b \in \mathcal{A}_\Delta(\beta)$ such that $i(a, b)$ is sufficiently large, then any geodesic G connecting α and β in $\mathcal{C}(S)$ must come close to $V(\Delta)$.

Corollary 3.11. There exists a uniform constant $\mathsf{N} = \mathsf{N}(S)$ such that for any geodesic G in $\mathcal{C}(S)$ which does not meet the A–neighbourhood of $V(\Delta)$, we have $\#\{a \in \mathcal{A}_\Delta(\alpha) \mid \alpha \in G\} \leq \mathsf{N}$. □

We note that $\mathcal{A}_\Delta(\alpha)$ and $\tau_\Delta(\alpha)$ may also be viewed as *markings* on S. (This is not the same notion as a marking from a reference surface S.) Say an embedded graph *fills* S if all complementary regions are topologically discs. We may define a marking to be (the isotopy class of) an embedded graph on S that fills, with a fixed uniform bound on the numbers of vertices and edges. It then follows that there are only finitely many such homeomorphism classes of markings on S.

Define the intersection number between two (isotopy classes of) embedded graphs Γ, Γ' on S as follows. Consider all realizations of Γ, Γ' on S so that all their intersections are transverse and occur on the interior of edges. Set $i(\Gamma, \Gamma')$ to be the minimal number of intersections taken over all such realizations.

The *marking graph* $\mathcal{M}(S)$ can be defined by taking the set of markings as vertices, with an edge spanning two markings if they have bounded intersection number (the bound should be taken large enough to ensure that the marking graph is connected). The mapping class group $\mathrm{Mod}(S)$ acts properly and cocompactly on $\mathcal{M}(S)$ by isometries and so, by the Švarc–Milnor Lemma, it is quasi-isometric to the marking graph. The same proof as the above theorem gives:

Corollary 3.12. Let G be a geodesic in $\mathcal{C}(S)$ disjoint from the A–neighbourhood of $V(\Delta)$. Then the sets $\mathcal{A}_\Delta(G)$ and $\tau_\Delta(G) = \{\tau_\Delta(\alpha) \mid \alpha \in G\}$ have uniformly bounded diameter in $\mathcal{M}(S)$. $\square$

4 Systole sets

A *systole* on a metric surface is an essential curve of shortest length. Define the *flat systole map* to be the relation $\mathsf{sys} : \mathcal{QD}^1(S) \to \mathcal{C}(S)$ which assigns a half-translation surface its set of systoles.

We can similarly define the systole maps $\mathsf{sys}^{\mathsf{Ext}} : \mathrm{Teich}(S) \to \mathcal{C}(S)$ and $\mathsf{sys}^{\mathsf{Hyp}} : \mathrm{Teich}(S) \to \mathcal{C}(S)$ which assigns a conformal structure X its systoles with respect to extremal length and hyperbolic length respectively. By abuse of notation, we shall also write $\mathsf{sys}^{\mathsf{Ext}}(q)$ and $\mathsf{sys}^{\mathsf{Hyp}}(q)$ to respectively mean the extremal and hyperbolic systole maps applied to the underlying conformal structure of $q \in \mathcal{QD}^1(S)$.

It is well known that the hyperbolic and extremal systole maps coarsely agree. We show that the flat systole map also coarsely agrees with these maps when restricted to Δ.

Proposition 4.1. There is a universal constant h_2 such that the maps sys, $\mathsf{sys}^{\mathsf{Ext}}$ and $\mathsf{sys}^{\mathsf{Hyp}}$ agree up to error h_2 when restricted to Δ. Consequently, we have

$$\mathsf{sys}(\Delta) \approx_{\mathsf{h}_2} \mathsf{sys}^{\mathsf{Ext}}(D) \approx_{\mathsf{h}_2} \mathsf{sys}^{\mathsf{Hyp}}(D) \approx_{\mathsf{h}_2} \mathsf{sys}(\Delta),$$

where the Hausdorff distance is measured in $\mathcal{C}(S)$. Choosing $\mathsf{h}_2 = 37$ suffices. $\square$

Let us first consider flat systoles. One can use a standard argument by bounding the injectivity radius to show the following.

Lemma 4.2. Any systole on a unit-area half-translation surface has length at most $\frac{2}{\sqrt{\pi}}$. $\square$

The main ingredient in showing that the relation $\mathsf{sys} : \Delta \to \mathcal{C}(S)$ is coarsely well-defined is the existence of annuli of definite width – a key step in both Masur and Minsky's [12], and Bowditch's [3] proofs of hyperbolicity of the curve graph. Masur and Minsky used a limiting argument, whereas Bowditch's method can produce explicit bounds assuming only an isoperimetric inequality on a given singular Riemannian surface.

Proposition 4.3 ([4] Lemma 2.3)**.** For any $q \in \mathcal{QD}^1(S)$, there is a topological annulus on q whose width is at least $\mathsf{W} = \frac{\sqrt{2\pi}}{4(2g-1)(2g+6)} \asymp \frac{1}{g^2}$. $\square$

Following their arguments with extra care in computing the constants, we deduce the following. Recall that $\mathsf{G} = \max\{2, g-2\}$.

Lemma 4.4. For any $q \in \mathcal{QD}^1(S)$ and any pair of intersecting curves $\alpha, \beta \in \mathcal{C}(S)$, we have

$$d_S(\alpha, \beta) \prec_{2,\mathsf{D}} \log_{\mathsf{G}} l_q(\alpha) + \log_{\mathsf{G}} l_q(\beta)$$

for some universal constant $\mathsf{D} = 33.2$. In particular, for any $L \geq 0$, the set of curves whose length on q is at most L has diameter in $\mathcal{C}(S)$ at most $4\log_{\mathsf{G}} L + \mathsf{D}$. □

Proof. Let γ be the core curve of an annulus of width at least W on q. Applying Lemma 2.2, we get

$$\begin{aligned} d_S(\alpha, \gamma) &\leq 2\log_{\mathsf{G}} i(\alpha, \gamma) + 2 \\ &\leq 2\log_{\mathsf{G}} \left(\frac{l_q(\alpha)}{\mathsf{W}}\right) + 2 \\ &= 2\log_{\mathsf{G}} l_q(\alpha) + 2\log_{\mathsf{G}} \left(\frac{4(2g-1)(2g+6)}{\sqrt{2\pi}}\right) + 2. \end{aligned}$$

Applying Lemma 2.3, we see that the function

$$g \mapsto \log_{\mathsf{G}} \{(2g-1)(2g+6)\} = \log_{\mathsf{G}}(g - \frac{1}{2}) + \log_{\mathsf{G}}(g+3) + \log_{\mathsf{G}} 4.$$

is maximized, when $g = 4$ among all $g \geq 2$. Thus

$$\begin{aligned} d_S(\alpha, \gamma) &\leq 2\log_{\mathsf{G}} l_q(\alpha) + 2\log_2 \left(\frac{392}{\sqrt{2\pi}}\right) + 2 \\ &\leq 2\log_{\mathsf{G}} l_q(\alpha) + 16.6. \end{aligned}$$

Applying the triangle inequality to $d_S(\alpha, \beta)$ completes the proof. ■

Lemma 4.5. The systole map $\mathsf{sys} : (\Delta, d_D) \to \mathcal{C}(S)$ is $(\mathsf{K}_2, \mathsf{C}_2)$–coarsely Lipschitz, where $\mathsf{K}_2 = \frac{2}{\log \mathsf{G}}$ and $\mathsf{C}_2 = 36$. □

Proof. Given $q, q' \in \Delta$, let $\alpha \in \mathsf{sys}(q)$ and $\beta \in \mathsf{sys}(q')$ be respective systoles. After rotation, we may assume $q = g_t \cdot q'$, where $t = d_D(q, q')$. By Lemmas 2.9 and 4.2, we have

$$l_q(\beta) \leq \sqrt{2}e^t l_{q'}(\beta) \leq \frac{2\sqrt{2}}{\sqrt{\pi}} e^t.$$

Applying Lemma 4.4 gives

$$\begin{aligned} d_S(\alpha,\beta) &\le 2\log_{\mathsf{G}}\left(\frac{2\sqrt{2}}{\sqrt{\pi}}\right) + 2\log_{\mathsf{G}}\left(\frac{2\sqrt{2}}{\sqrt{\pi}}\mathrm{e}^t\right) + 33.2 \\ &\le \frac{2t}{\log \mathsf{G}} + 2\log_2\left(\frac{8}{\pi}\right) + 33.2 \\ &\le \frac{2}{\log \mathsf{G}} d_D(q,q') + 36 \end{aligned}$$

and we are done. ■

Let $X \in \mathrm{Teich}(S)$ be a conformal class of metrics on S. The *extremal length* of $\alpha \in \mathcal{C}(S)$ on X is

$$\mathsf{Ext}_X(\alpha) = \sup_{\rho\in X} \frac{l_\rho(\alpha)^2}{\mathrm{area}(\rho)},$$

where $l_\rho(\alpha)$ denotes the geodesic length of α on ρ. It immediately follows that for any $q \in \mathcal{QD}^1(S)$, we have $l_q(\alpha) \le \sqrt{\mathsf{Ext}_X(\alpha)}$, where $X \in \mathrm{Teich}(S)$ is the conformal class of q. Let $\mathsf{Hyp}_X(\alpha)$ denote the geodesic length of α on the unique hyperbolic metric in the conformal class X. A theorem of Maskit [10] states that

$$\frac{\mathsf{Hyp}_X(\alpha)}{\pi} \le \mathsf{Ext}_X(\alpha) \le \frac{1}{2}\mathsf{Hyp}_X(\alpha)\mathrm{e}^{\frac{1}{2}\mathsf{Hyp}_X(\alpha)}.$$

Using standard hyperbolic geometry, any hyperbolic systole α on X satisfies $\mathsf{Hyp}_X(\alpha) \le 2\log(4g-2)$.

Lemma 4.6. If α is an extremal or hyperbolic systole on $X \in \mathrm{Teich}(S)$ then

$$l_q(\alpha)^2 \le (4g-2)\log(4g-2)$$

for any $q \in \mathcal{QD}^1(S)$ in the conformal class X. □

Proposition 4.1 is a consequence of the following.

Lemma 4.7. Let $q \in \mathcal{QD}^1(S)$. Then

$$\mathrm{diam}_{\mathcal{C}(S)}\{\mathsf{sys}(q) \cup \mathsf{sys}^{\mathsf{Ext}}(q) \cup \mathsf{sys}^{\mathsf{Hyp}}(q)\} \le 37.$$

Consequently, the maps sys, $\mathsf{sys}^{\mathsf{Ext}}$ and $\mathsf{sys}^{\mathsf{Hyp}}$ are coarsely well defined and agree up to universally bounded error. □

Proof. Fix a surface $q \in \Delta$, and choose curves $\alpha, \beta \in \mathsf{sys}(q) \cup \mathsf{sys}^{\mathsf{Ext}}(q) \cup \mathsf{sys}^{\mathsf{Hyp}}(q)$. By the above corollary and Lemma 4.4, we have

$$d_S(\alpha, \beta) \leq 2 \log_{\mathsf{G}}((4g-2)\log(4g-2)) + \mathsf{D}.$$

A computation shows that this quantity is maximized at $g = 4$ among all integers $g \geq 2$, and so

$$d_S(\alpha, \beta) \leq 2 \log_2(14 \log 14) + \mathsf{D} \leq 37.3$$

which completes the proof. ■

5 Bounding Hausdorff distance

In this section, we prove that the systole set and the set of vertex cycles for Δ agree up to universally bounded Hausdorff distance through a sequence of coarse inclusions. We will also show that the cylinder set for Δ agrees with the systole set up to uniformly bounded Hausdorff distance.

5.1 Systoles are straight vertex cycles

Fix an $SL(2, \mathbb{R})$–orbit Δ and choose some $q \in \Delta$. Any curve $\alpha \in \mathcal{C}(S)$ induces a minimal integral transverse measure w_α on $\tau = \tau_q(\alpha)$. By Theorem 2.6, we can write $w_\alpha = \frac{1}{2}\sum_{v \in V(\tau)} m_v w_v$, for some non-negative integers m_v. By straightening $\tau_q(\alpha)$ and counting, the number of times α^q and each v^q run over each saddle connection, we can deduce

$$l_q(\alpha) = \frac{1}{2} \sum_{v \in V(\tau)} m_v l_q(v).$$

Proposition 5.1. Suppose $\alpha \in \mathsf{sys}(\Delta)$. Then α is a vertex cycle of $\tau = \tau_\Delta(\alpha)$. □

Proof. If α is a systole on some $q \in \Delta$, then

$$l_q(\alpha) = \frac{1}{2} \sum_{v \in V(\tau)} m_v l_q(v) \geq \frac{1}{2} \sum_{v \in V(\tau)} m_v l_q(\alpha),$$

and hence the m_v must sum to at most 2. In particular, at most two of the m_v terms can be non-zero. If exactly one m_v is non-zero then α is a vertex cycle of τ and we are done. We now suppose otherwise, and work towards a contradiction. There are distinct

$v, v' \in V(\tau)$ such that $m_v = m_{v'} = 1$, and so $w_\alpha = \frac{1}{2}(w_v + w_{v'})$. Furthermore, since α is a systole on q we have $l_q(\alpha) = l_q(v) = l_q(v')$. Recall from Theorem 2.5 that vertex cycles on τ are either embedded loops, figure–8's, or barbells. In particular, they use each branch at most twice.

If v is an embedded loop or figure–8 on τ, then $w_v(b) = 1$ on every branch $b \in \mathcal{B}(\tau)$ used by v. If $b \in \mathcal{B}(\tau)$ is a branch used by v then

$$w_\alpha(b) = \frac{1}{2}(w_v(b) + w_{v'}(b)) = \frac{1}{2} + \frac{1}{2}w_{v'}(b) \leq \frac{1}{2} + 1.$$

Since $w_\alpha(b)$ is an integer, it follows that $w_{v'}(b) = 1$. This means that v' must use every branch used by v. In particular, v' must be an embedded loop or a figure–8, as a barbell cannot contain an embedded loop or figure–8 as a subtrack. We can similarly deduce that v uses all branches used by v'. Therefore $v = v'$, a contradiction.

Now suppose v is a barbell on τ, and consider the two closed train-routes η, η' on v which remain after deleting the branches that v runs over twice (see Figure 1). The curves β, β' formed by these closed train-routes may not necessarily be carried by τ. We claim that they must be essential on S. If not, then η, say, bounds a monogon (assuming S is closed). This is not possible since τ is a train track. (There may be other branches of τ inside this monogon, but τ will still have a forbidden complementary region.) Now straighten the train track τ to an embedded graph Γ on q. Doing so straightens η to a closed geodesic path in Γ representing β (which is not necessarily a geodesic representative of β). It follows that $l_q(\beta) < l_q(v) = l_q(\alpha)$, contradicting the assumption that α is a systole on q. ■

Remark 5.2. If S is not assumed to be closed, one can still obtain a weaker result. The above proof shows that any curve $\alpha \in \mathsf{sys}(\Delta)$ must run over each branch of $\tau_\Delta(\alpha)$ at most twice. By Theorem 2.7 (see also the surrounding remarks), α is at most a distance $\mathsf{K}_1 \leq 14$ from any vertex cycle of τ. □

5.2 Straight vertex cycles are near systoles

Before we prove the reverse inclusion, we first state some technical lemmas.

Lemma 5.3. If b is a vertical arc with no singular points on its interior on a unit-area half-translation surface q, then there exists a horizontal arc of length at most $\frac{1}{l_q(b)}$ whose endpoints lie on b with no singular points on its interior. □

Proof. We use a Poincaré recurrence argument for the geodesic flow on q in the horizontal direction.

Define the *horizontal neighbourhood* $N_r^H(b)$ of b with radius $r \geq 0$ to be the set of points x on q for which there exists a horizontal arc of length at most r starting from a point on b and ending at x with no singular points on its interior. Observe that $N_r^H(b)$ is the image of a locally isometric immersion $\iota : R \to q$, where R is a Euclidean rectangle $[-r, r] \times b$ with finitely many horizontal intervals of the form $(t, r] \times \{p\}$ or $[-r, t) \times \{p\}$ removed. These removed intervals correspond to when a horizontal arc emanating from b hits a singularity and cannot be uniquely extended. Moreover, the immersion is locally area preserving. If ι is an embedding, then

$$\operatorname{area}(N_r^H(b)) = \operatorname{area}(R) = 2rl_q(b).$$

Since q is assumed to have unit area, ι cannot be an embedding when $r \geq \frac{1}{2l_q(b)}$. Thus there is a non-singular point $x \in q$ which can be connected to b using two distinct horizontal arcs of length at most $\frac{1}{2l_q(b)}$. Taking the union of the two horizontal arcs produces the desired horizontal arc with endpoints on b. ■

We introduce Δ–*bicorns* as an intermediate step. By definition, these are curves which have representatives formed by taking the union of two straight line segments on some (hence all) $q \in \Delta$. Such curves are non-trivial since geodesics segments on a non-positively curved surface are unique in their homotopy class relative to their endpoints.

Lemma 5.4. Let Γ be an embedded graph on some $q \in \Delta$, with vertices at singular points and whose edges are saddle connections. Then there is a cylinder curve or a Δ–bicorn which intersects each edge of Γ at most once. □

Proof. Let e be an edge of Γ, which we may assume to be vertical. Applying the Poincaré recurrence argument as in the above lemma, there is a horizontal arc a with endpoints on e with no singularities in its interior. If a intersects each edge of Γ at most once, we may concatenate it with a (possibly degenerate) subarc $e' \subseteq e$ to form a Δ–bicorn intersecting each edge of Γ at most once as desired. If not, we may choose an innermost subarc $a' \subseteq a$ with the property that its endpoints are on the same edge of Γ. The arc a' intersects each edge of Γ at most once, and so we may argue as above. ■

Recall that the infimal length of α with respect to Δ is $l_\Delta(\alpha) = \inf_{q \in \Delta} l_q(\alpha)$.

Lemma 5.5. Given any Δ–bicorn β, there is some $\gamma \in \mathcal{C}(S)$ with $l_\Delta(\gamma) \leq 2$ such that $d_S(\beta, \gamma) \leq 2$. □

Proof. Applying a suitable $SL(2, \mathbb{R})$–deformation, we may assume that β has a bicorn representative on some $q \in \Delta$ where the two line segments are in the horizontal and vertical directions. These line segments shall be denoted b^H and b^V, respectively. By Lemma 5.3, there is a horizontal arc c^H with endpoints on b^V satisfying

$$l_q(b^V)l_q(c^H) \leq 1.$$

Let $\gamma = c^H \cup c^V$, where $c^V \subseteq b^V$ is the subarc of b^V connecting the endpoints of c^H. Note that the quantity $l_q(c^H)l_q(c^V)$ remains constant under the Teichmüller geodesic flow g_t. Therefore we may, in addition, choose $q \in \Delta$ to satisfy $l_q(c^H) = l_q(c^V) \leq l_q(b^V)$. Combined with the above inequality, it follows that $l_q(c^H)$ and $l_q(c^V)$ are both at most 1 and hence

$$l_\Delta(\gamma) \leq l_q(\gamma) \leq l_q(c^H) + l_q(c^V) \leq 2.$$

Finally, observe that $i(\beta, \gamma) \leq 1$ which implies $d_S(\beta, \gamma) \leq 2$. ■

Proposition 5.6. The set $V(\Delta)$ is contained in the 54–neighbourhood of $\mathsf{sys}(\Delta)$ in $\mathcal{C}(S)$. □

Proof. Let $\alpha \in V(\Delta)$ be a straight vertex cycle for Δ. Recall from the proof of Lemma 3.3 that α must be a vertex cycle of $\tau_\Delta(\alpha)$. Choose some $q \in \Delta$, and let Γ be the straightening of $\tau_q(\alpha)$ on q, i.e. the embedded graph whose edges are exactly the saddle connections used by α^q. Note that α^q runs over each edge of Γ at most twice. Using an Euler characteristic argument with the Gauss–Bonnet theorem, one can show that Γ has at most $18(g-1)$ edges. Applying Lemma 5.4, there exists some cylinder curve or Δ–bicorn β satisfying $i(\alpha, \beta) \leq 36(g-1)$, and so by Lemma 2.2 we have $d_S(\alpha, \beta) \leq 15$. By Lemma 5.5, there is some curve $\gamma \in \mathcal{C}(S)$ satisfying $l_\Delta(\gamma) \leq 2$ within distance 2 of β. Using Lemma 4.4 with $L = 2$, we deduce that γ is in the 37–neighbourhood of $\mathsf{sys}(\Delta)$. Finally, applying the triangle inequality completes the proof. ■

5.3 Cylinder curves

We conclude this section by showing that the set of cylinder curves for an $SL(2, \mathbb{R})$–orbit Δ agrees with the systole set up to uniformly bounded Hausdorff distance in the curve graph. The key ingredient for our proof is a stronger version of Bowditch's wide annulus result (Proposition 4.3) for half-translation surfaces due to Vorobets.

Proposition 5.7 ([19] Theorem 1.3)**.** Let q be unit-area half-translation surface of genus $g \geq 2$. Then there is a flat cylinder on q with width at least $\mathsf{W}' = (4\sqrt{2}(g-1)2^{2^{32(g-1)}})^{-1}$. $\square$

Vorobets' statement is originally for translation surfaces, but it can be generalized to half-translation surfaces by taking a branched double cover the singular points.

Let $\mathsf{cyl}(\Delta)$ and $\widehat{\mathsf{cyl}}(\Delta)$, respectively denote the set of cylinder curves and constant slope curves for any (hence all) $q \in \Delta$.

Lemma 5.8. For any $SL(2,\mathbb{R})$–orbit $\Delta \subset \mathcal{QD}^1(S)$, we have

$$\widehat{\mathsf{cyl}}(\Delta) \approx_1 \mathsf{cyl}(\Delta) \approx_{\mathsf{h}_3} \mathsf{sys}(\Delta)$$

where $\mathsf{h}_3 \asymp \frac{2^{32g}}{\log g}$. $\square$

Proof. We will proceed by showing the following chain of coarse inclusions:

$$\mathsf{cyl}(\Delta) \subseteq \widehat{\mathsf{cyl}}(\Delta) \subseteq_1 \mathsf{cyl}(\Delta) \subseteq_1 \mathsf{sys}(\Delta) \subseteq_{\mathsf{h}_3} \mathsf{cyl}(\Delta),$$

where h_3 shall be determined later.

The first inclusion holds since cylinders have constant slope. For the second inclusion, suppose α has constant slope (which we may assume to be vertical) on some $q \in \Delta$. By shrinking the half-translation structure in the vertical and expanding in the horizontal slope, we can make α arbitrarily short and hence disjoint from a flat cylinder of width at least W'.

If α is a (vertical) cylinder curve, then we can make this cylinder arbitrarily wide by expanding in the horizontal and shrinking in the vertical slope. Since systoles have length at most $\frac{2}{\sqrt{\pi}}$, it follows that α is disjoint from a systole on some $q \in \Delta$, giving us the third inclusion.

Finally, suppose $\alpha \in \mathsf{sys}(q)$ for some $q \in \Delta$, and let β be a cylinder with width at least W' on q. Then $l_q(\beta) \leq \frac{1}{\mathsf{W}'}$ and so by Lemma 4.4

$$d_S(\alpha,\beta) \prec_{2,\mathsf{D}} \log_{\mathsf{G}}\left(\frac{2}{\sqrt{\pi}}\right) + \log_{\mathsf{G}}\left(\frac{1}{\mathsf{W}'}\right) \asymp \frac{2^{32g}}{\log g},$$

which gives the fourth inclusion. One can choose $\mathsf{h}_3 \asymp \frac{2^{32g}}{\log g}$. $\blacksquare$

Our bound h_3 depends on the genus of S, though we do not expect this to be sharp. This motivates the following.

Question 1. What are the optimal asymptotics for the Hausdorff distance between $\mathsf{cyl}(\Delta)$ and $\mathsf{sys}(\Delta)$ in terms of genus? Does there exist a universal bound? □

Similar questions can also be posed when restricting to certain classes of half-translation surfaces, such as square-tiled surfaces, Veech surfaces, or completely periodic surfaces.

6 Auxiliary polygons

To a curve $\alpha \in \mathcal{C}(S)$ and a half-translation surface $q \in \mathcal{QD}(S)$, we associate a convex Euclidean polygon $P_q(\alpha) \subset \mathbb{R}^2$ called its *auxiliary polygon*. Geometric properties of α^q under $SL(2,\mathbb{R})$–deformations of q can be observed by performing the same deformations on $P_q(\alpha)$. This allows us to simplify many of our arguments in Section 7. The construction works equally well if α is a multicurve on S, or a multi-arc on $S'(\Delta)$, but we shall focus only on curves to simplify the exposition.

Consider a geodesic representative α^q of α on q which is a concatenation of saddle connections. Suppose the saddle connections $e_1, \ldots, e_k$ of α^q appear with multiplicities $w_1, \ldots, w_k$. Each saddle connection e_i has a well-defined length and slope on q. To each e_i we associate a parallel vector $\mathbf{u}_i \in \mathbb{R}^2$ of magnitude $w_i \times \text{length}(e_i)$.

Definition 6.1. The *auxiliary polygon* of α with respect to q is

$$P_q(\alpha) = \left\{ \sum_i t_i \mathbf{u}_i \mid -\frac{1}{2} \leq t_i \leq \frac{1}{2} \right\} \subset \mathbb{R}^2.$$ □

Note that this definition does not depend on the choice of orientation for each $\mathbf{u}_i$. Observe that $P_q(\alpha)$ is centrally symmetric, and is the convex hull of the finite set $\left\{\sum_i \epsilon_i \mathbf{u}_i \mid \epsilon_i = \pm\frac{1}{2}\right\}$ in $\mathbb{R}^2$. Thus this construction produces a convex Euclidean polygon unless all the $\mathbf{u}_i$ are parallel, in which case the polygon $P_q(\alpha)$ degenerates to a straight line segment. This situation occurs exactly when all saddle connections of α^q are parallel, i.e., when α^q has constant slope on q. We call $P_q(\alpha)$ *degenerate* when this happens. Furthermore, if α is a cylinder curve then $P_q(\alpha)$ is a line segment whose slope and length is that of any geodesic representative of α on q.

Without loss of generality, we may assume the directions of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k, -\mathbf{u}_1, \ldots, -\mathbf{u}_k$ appear in increasing anticlockwise order. (In general, this order will not agree with the order in which the saddle connections appear on α^q.) If all of the saddle connections of α^q are non-parallel, then as one follows $\partial P_q(\alpha)$ in an anti-clockwise direction, its edges (viewed as oriented line segments) coincide exactly with $\mathbf{u}_1, \ldots, \mathbf{u}_k, -\mathbf{u}_1, \ldots, -\mathbf{u}_k$ up to cyclic permutation. In general, an edge of $\partial P_q(\alpha)$ coincides with the sum of consecutive parallel vectors. In the case where $P_q(\alpha)$ degenerates to a single straight line segment, we view $\partial P_q(\alpha)$ as a closed path which traverses the line segment once in each direction.

We may also think of $P_q(\alpha)$ as being constructed as follows: take two copies of each saddle connection of α^q on q (counting multiplicity) and place them on the Euclidean plane (maintaining their slopes) as line segments with opposing orientations. Then translate the oriented line segments in $\mathbb{R}^2$ so that they are positioned head to tail in order of increasing direction. This yields a closed polygonal path forming the boundary of $P_q(\alpha)$ (up to translation).

For any Euclidean polygon $P \subseteq \mathbb{R}^2$, let area(P) denote its Euclidean area. We also define

$$\text{width}(P) = \sup\{|x_1 - x_2| \mid (x_1, y_1), (x_2, y_2) \in P\}$$

and

$$\text{height}(P) = \sup\{|y_1 - y_2| \mid (x_1, y_1), (x_2, y_2) \in P\}.$$

We may pull back several notions of length on $\mathbb{R}^2$ to S via q: let $l_q^H(\alpha)$, $l_q^V(\alpha)$ and $l_q(\alpha)$, respectively denote the length of α^q with respect to $|dx|$, $|dy|$ and the Euclidean metric. The following is immediate.

Lemma 6.2. Let q, α and $P_q(\alpha)$ be as above. Then

- $\text{width}(P_q(\alpha)) = l_q^H(\alpha)$,
- $\text{height}(P_q(\alpha)) = l_q^V(\alpha)$,
- $l(\partial P_q(\alpha)) = 2\, l_q(\alpha)$, and
- for any $A \in SL(2, \mathbb{R})$, we have $A \cdot P_q(\alpha) = P_{A \cdot q}(\alpha)$. $\square$

For any curve $\alpha \in \mathcal{C}(S)$ and $SL(2, R)$–orbit Δ, define its *polygonal area* with respect to Δ to be

$$\text{area}_\Delta(\alpha) = \text{area}(P_q(\alpha))$$

where q is any unit area half-translation surface in Δ. This is well defined since Euclidean area is preserved under $SL(2,\mathbb{R})$–deformations. Recall the definition of infimal length $l_\Delta(\alpha) = \inf_{q\in\Delta} l_q(\alpha)$ of α with respect to Δ.

Proposition 6.3. With α and Δ as above, we have

$$\pi \operatorname{area}_\Delta(\alpha) \leq (l_\Delta(\alpha))^2 \leq 8 \operatorname{area}_\Delta(\alpha).$$

In particular, $l_\Delta(\alpha) = 0$ if and only if $\alpha \in \widehat{\mathsf{cyl}}(\Delta)$. Furthermore, the infimal length of α is realized if and only if it is positive. □

Thus, one can estimate the infimal length of a curve α over Δ by computing the area of $P_q(\alpha)$ for any $q \in \Delta$.

To prove this result, we require the following *Round Polygon Lemma*. Let $B^1(r)$ and $B^\infty(r)$, respectively denote the balls of radius r about the origin in $\mathbb{R}^2$ with respect to the l^1 and l^∞ norms.

Lemma 6.4. Suppose P is a convex (non-degenerate) Euclidean polygon in $\mathbb{R}^2$ with π–rotational symmetry about the origin. Then there exists $A \in SL(2,\mathbb{R})$ such that

$$B^1(r) \subseteq A \cdot P \subseteq B^\infty(r)$$

for some $r > 0$. □

Proof. Consider the set of parallelograms spanned by a pair of diagonals of P which pass through the origin, and choose one such parallelogram $U \subseteq P$ with largest area (see Figure 4). Deform P using some $A \in SL(2,\mathbb{R})$ so that $A \cdot U = B^1(r)$ for some $r > 0$. We claim that $A \cdot P \subseteq B^\infty(r)$. Supposing otherwise, $A \cdot P$ must have a corner $(x, y) \in \mathbb{R}^2$ lying outside $B^\infty(r)$. Without loss of generality, we may assume $|x| > r$. Since $A \cdot P$ has π–rotational symmetry about its centre, the point $(-x, -y)$ is also a corner of $A \cdot P$. Let $U' \subseteq A \cdot P$ be the parallelogram spanned by the points $\pm(x, y)$ and $\pm(0, r)$, the latter pair being corners of $B^1(r)$ and hence $A \cdot P$. Now

$$\operatorname{area}(A^{-1} \cdot U') = \operatorname{area}(U') = 2r|x| > 2r^2 = \operatorname{area}(A \cdot U) = \operatorname{area}(U),$$

contradicting the maximality assumption on U. ■

***Proof of Proposition* 6.3 .** To prove the lower bound on $l_\Delta(\alpha)$, we apply the well-known isoperimetric inequality for the plane to $P_q(\alpha)$ for every $q \in \Delta$: A planar region $U \subset \mathbb{R}^2$ enclosed by an embedded curve of length $L = 2l_q(\alpha)$ must satisfy $4\pi \operatorname{area}(U) \leq L^2$.

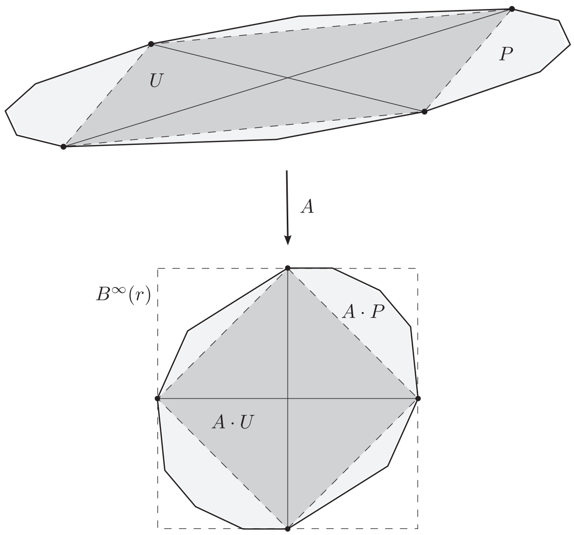

Fig. 4. Applying $A \in SL(2,\mathbb{R})$ to the polygon P to make it “round”.

For the other direction, we may apply Lemma 6.4 to find some $q \in \Delta$ so that $B^1(r) \subseteq P_q(\alpha) \subseteq B^\infty(r)$ for some $r > 0$. We divide the boundary of $P = P_q(\alpha)$ into four subpaths connecting adjacent corners of $B^1(r)$. The subpath η in the first quadrant is a concatenation of straight line segments connecting consecutive points with co-ordinates $(r,0) = (x_0,y_0),(x_1,y_1),\dots,(x_m,y_m) = (0,r)$. Since P is convex, it follows that $x_0 \geq x_1 \geq \dots \geq x_m$ and $y_0 \leq y_1 \leq \dots \leq y_m$, and hence

$$l(\eta) \leq l^1(\eta) = \sum_{i=1}^{m} |x_i - x_{i-1}| + |y_i - y_{i-1}| = 2r.$$

Applying this to the other subpaths, we deduce $2l_\Delta(\alpha) = l(\partial P) \leq 8r$. We finally combine this with the inequality $\text{area}(P) \geq \text{area}(B^1(r)) = 2r^2$ to obtain the desired upper bound.

Observe that $l_\Delta(\alpha) = 0$ if and only if $\text{area}_\Delta(\alpha) = 0$, which occurs precisely when $P_q(\alpha)$ is degenerate for any (hence all) $q \in \Delta$. ■

7 Balance points on Teichmüller discs

In this section, we generalize Masur and Minsky’s notion of balance time on Teichmüller geodesics to Teichmüller discs. In particular, we prove the existence of a *balance point* which coarsely determines the balance time for *every* Teichmüller geodesic contained inside a common Teichmüller disc. The auxiliary polygon from Section 6 plays a key role in our proofs.

Let D be a Teichmüller disc and $\alpha \in \mathcal{C}(S)$ be a curve. If $\alpha \in \widehat{\mathsf{cyl}}(\Delta)$, we define the *balance point* of α with respect to D to be the projectivized measured foliation $\mathcal{F} \in \partial D \subset \mathcal{PMF}(S)$ with the same slope as α. If $\alpha \notin \widehat{\mathsf{cyl}}(\Delta)$, we say $X \in D$ is a balance point of α with respect to Δ if for any $q \in \Delta$ projecting to X, the auxiliary polygon $P_q(\alpha)$ is round in the sense of Lemma 6.4: there is some $r > 0$ such that $B^1(r) \subseteq P_q(\alpha) \subseteq B^\infty(r)$. Balance points always exist by Lemma 6.4. Write $\mathcal{G}_\alpha$ for the balance time of α along a Teichmüller geodesic $\mathcal{G}$. If α is completely horizontal or completely vertical with respect to $\mathcal{G}$, then we set $\mathcal{G}_\alpha$ to be the endpoint of $\mathcal{G}$ corresponding to the foliation with the same slope as α.

Proposition 7.1. Let $X \in D \cup \partial D$ be a balance point for a curve $\alpha \in \mathcal{C}(S)$ on a Teichmüller disc D. For any Teichmüller geodesic $\mathcal{G} \subset D$, let $Y \in \mathcal{G}$ be the nearest point projection of X to $\mathcal{G}$ in D. Then $d_D(Y, \mathcal{G}_\alpha) \leq \log 2$. In particular, if $\mathcal{G}$ passes through X then $d_D(X, \mathcal{G}_\alpha) \leq \log 2$. □

It follows that balance points are coarsely unique: If X and X' are balance points for α on D, then $d_D(X, X') \leq 2\log 2$. If a balance point is in ∂D, then it is unique by definition. Combining the above with Lemma 2.9, we deduce the following.

Corollary 7.2. Suppose the minimal length of $\alpha \in \mathcal{C}(S)$ on Δ is attained at $m \in \Delta$. Let $Y \in D$ be the projection of m, and X be any balance point of α to D. Then $d_D(X, m) \leq \cosh^{-1} 2 + \log 2$. □

Theorem 7.3. Let X be a balance point for a curve $\alpha \in \mathcal{C}(S)$ on a Teichmüller disc D. Then any systole on X is universally close to any nearest point projection of α to $\mathsf{sys}(\Delta)$ in $\mathcal{C}(S)$. □

7.1 Proof of Proposition 7.1

For this section, fix an $SL(2,\mathbb{R})$–orbit Δ and a curve $\alpha \in \mathcal{C}(S)$. We will deal with two separate cases, corresponding to whether the geodesic representative α^q on any (hence all) $q \in \Delta$ has constant slope. We shall be keeping track of the half-translation structure on q.

For a Teichmüller geodesic $\mathcal{G} \subset D$, let $\mathcal{G}^+, \mathcal{G}^- \in \partial D$, respectively denote the vertical and horizontal foliations associated to $\mathcal{G}$. We shall regard $\mathcal{G}$ as a directed geodesic with the forwards direction pointing towards $\mathcal{G}^+$. If $\mathcal{G}_1, \mathcal{G}_2 \subset D$ are geodesic rays emanating from a common point $Y \in D$, let $\angle_Y(\mathcal{G}_1, \mathcal{G}_2) \in [0, \pi]$ be the smaller angle between them under the Poincaré disc model for $\mathbb{H}^2 \cong D$. Define $\angle_q(\mathcal{F}_1, \mathcal{F}_2) \in [0, \frac{\pi}{2}]$ to be the smaller angle between two foliations $\mathcal{F}_1, F_2 \in \partial D$ on a half-translation surface $q \in \Delta$. More precisely, realize $\mathcal{F}_1, \mathcal{F}_2$ as constant slope foliations on q, and take the smaller angle between respective leaves l_1, l_2 intersecting at any non-singular point x on q.

Lemma 7.4. Suppose $q \in \Delta$ is a half-translation surface projecting to $Y \in D$. Let $\mathcal{G}_1, \mathcal{G}_2 \subset D$ be Teichmüller geodesics emanating from Y. Then $\angle_Y(\mathcal{G}_1, \mathcal{G}_2) = 2\angle_q(\mathcal{G}_1^+, \mathcal{G}_2^+)$. □

Case 1: Suppose $\alpha \in \widehat{\mathsf{cyl}}(\Delta)$. The balance point of α is the foliation $\mathcal{F} \in \partial D$ which has the same slope as α on every $q \in \Delta$. Any Teichmüller geodesic $\mathcal{G} \subset D$ with an endpoint at $\mathcal{F} \in \partial D$ can be oriented so that $\mathcal{G}^+ = \mathcal{F}$. Since α is completely vertical along $\mathcal{G}$ then, by definition, $\mathcal{F}$ is also the balance point of α on $\mathcal{G}$. Now suppose $\mathcal{G} \subset D$ is a Teichmüller geodesic with $\mathcal{G}^\pm \neq \mathcal{F}$. Let $Y \in \mathcal{G}$ be the nearest point projection of $\mathcal{F}$ to $\mathcal{G}$ in D, and let $p \in \Delta$ be the quadratic differential projecting to Y whose vertical foliation is $\mathcal{G}^+$. Let $\mathcal{G}_1$ be the geodesic ray from Y to $\mathcal{F}$. Using basic hyperbolic geometry, we have $\angle_Y(\mathcal{G}, \mathcal{G}_1) = \frac{\pi}{2}$. Therefore, by the previous lemma, $\angle_p(\mathcal{F}, \mathcal{F}_p^V) = \frac{\pi}{4} = \angle_p(\mathcal{F}, \mathcal{F}_p^H)$. Since α has the same slope as $\mathcal{F}$, it follows that $l_p^H(\alpha) = \frac{1}{\sqrt{2}} l_p(\alpha) = l_p^V(\alpha)$ and thus $\mathcal{G}_\alpha = p$.

Case 2: Now assume α has non-constant slope on Δ. Let $X \in D$ be a balance point of α on D, and let $q \in \Delta$ be a half-translation surface projecting to X. Let $\mathcal{G} \subset D$ be a Teichmüller geodesic. Apply a rotation $\rho_\theta \in SO(2, \mathbb{R})$ to q so that

$$\angle_{\rho_\theta \cdot q}(\mathcal{G}^+, \mathcal{F}^V_{\rho_\theta \cdot q}) = \phi = \angle_{\rho_\theta \cdot q}(\mathcal{G}^-, \mathcal{F}^V_{\rho_\theta \cdot q})$$

for some $0 < \phi \leq \frac{\pi}{4}$. Using the Teichmüller geodesic flow to respectively stretch and shrink the horizontal and vertical directions on $\rho_\theta \cdot q$ by a factor of $\sqrt{\cot \phi}$, we obtain a new half-translation surface $p \in \mathcal{G}$ on which $\mathcal{G}^+$ and $\mathcal{G}^-$ are perpendicular. Moreover, $\angle_p(\mathcal{G}^\pm, \mathcal{F}_p^V) = \frac{\pi}{4} = \angle_p(\mathcal{G}^\pm, \mathcal{F}_p^H)$. Let $Y \in D$ be the projection of $p \in \Delta$. Applying the previous lemma, the Teichmüller geodesic in D connecting X to Y is perpendicular to $\mathcal{G}$, and so Y is the nearest point projection of X to $\mathcal{G}$ in D. Let $l^H(\alpha)$ and $l^V(\alpha)$ be the horizontal and vertical lengths of α on $\rho_{\frac{\pi}{4}} \cdot p$. These are exactly the lengths of α on p with respect to the foliations $\mathcal{G}^\pm$. Thus, to prove Y is near the balance point of α to $\mathcal{G}$ we must show that these lengths are almost equal.

Our strategy is to apply appropriate $SL(2, \mathbb{R})$–deformations to $P = P_q(\alpha)$, as illustrated in Figure 5, in order to obtain estimates for $l^H(\alpha)$ and $l^V(\alpha)$ via Lemma 6.2. By Proposition 6.3, the auxiliary polygon $P = P_q(\alpha)$ is non-degenerate and satisfies

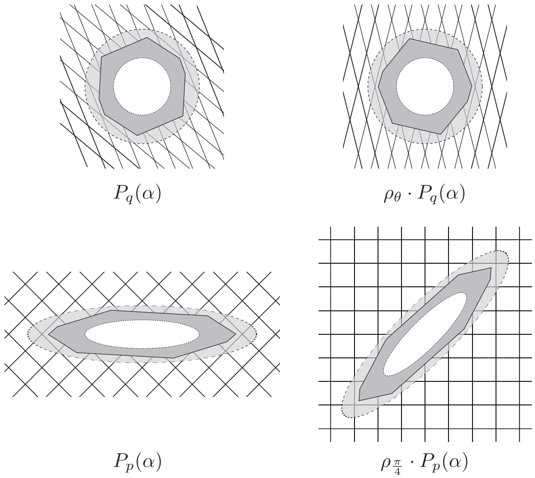

Fig. 5. The auxiliary polygons for α, corresponding to the half-translation structures q, $\rho_\theta \cdot q$, p, and $\rho_{\frac{\pi}{4}} \cdot p$ respectively, are nested between pairs of ellipses. The grid lines indicate the pair of transverse slopes corresponding to $\mathcal{G}$.

$B^1(r) \subseteq P \subseteq B^\infty(r)$ for some $r > 0$. Observe that $\rho_\theta \cdot P$ can be nested between a pair of concentric Euclidean circles

$$B^2(R) \subseteq \rho_\theta \cdot B^1(r) \subseteq \rho_\theta \cdot P \subseteq \rho_\theta \cdot B^\infty(r) \subseteq B^2(2R),$$

where $R = \frac{r}{\sqrt{2}}$. The polygon $P_p(\alpha)$ can be obtained by respectively stretching and shrinking $\rho_\theta \cdot P$ by a factor of $\sqrt{\cot\phi}$ in the horizontal and vertical directions. By applying the same deformations to the circles, $P_p(\alpha)$ can be nested between a pair of ellipses

$$E_{inner} \subseteq P_p(\alpha) \subseteq E_{outer}$$

defined by the equations

$$\frac{x^2}{\cot\phi} + (\cot\phi)y^2 \leq R^2 \qquad \text{and} \qquad \frac{x^2}{\cot\phi} + (\cot\phi)y^2 \leq 4R^2,$$

respectively.

We then rotate $P_p(\alpha)$ through an angle of $\frac{\pi}{4}$ to obtain $\rho_{\frac{\pi}{4}} \cdot P_p(\alpha)$. By Lemma 6.2, we know that

$$l^H(\alpha) = \text{width}(\rho_{\frac{\pi}{4}} \cdot P_p(\alpha)) \qquad \text{and} \qquad l^V(\alpha) = \text{height}(\rho_{\frac{\pi}{4}} \cdot P_p(\alpha)).$$

Rotating the ellipses E_{inner} and E_{outer} through an angle of $\frac{\pi}{4}$ yields the following estimates:

$$\text{width}(\rho_{\frac{\pi}{4}} \cdot E_{inner}) \leq l^H(\alpha) \leq \text{width}(\rho_{\frac{\pi}{4}} \cdot E_{outer}) = 2\,\text{width}(\rho_{\frac{\pi}{4}} \cdot E_{inner})$$

and

$$\text{height}(\rho_{\frac{\pi}{4}} \cdot E_{inner}) \leq l^V(\alpha) \leq \text{height}(\rho_{\frac{\pi}{4}} \cdot E_{outer}) = 2\,\text{height}(\rho_{\frac{\pi}{4}} \cdot E_{inner}).$$

Note that the ellipses $\rho_{\frac{\pi}{4}} \cdot E_{inner}$ and $\rho_{\frac{\pi}{4}} \cdot E_{outer}$ have reflective symmetry about the line $y = x$. This implies

$$\text{width}(\rho_{\frac{\pi}{4}} \cdot E_{inner}) = \text{height}(\rho_{\frac{\pi}{4}} \cdot E_{inner}).$$

Combining this with the preceding inequalities yields

$$\frac{1}{2} l^H(\alpha) \leq l^V(\alpha) \leq 2 l^H(\alpha),$$

and therefore p is within a distance of $\log 2$ of the balance time of α to $\mathcal{G}$. □

7.2 Projecting to the systole set

In this section, we prove Theorem 7.3 which generalizes the following theorem for Teichmüller geodesics to Teichmüller discs. Let $\mathcal{G}_\alpha$ denote the balance time of a curve α to a Teichmüller geodesic $\mathcal{G}$.

Theorem 7.5 ([12] Theorem 2.6)**.** For any Teichmüller geodesic $\mathcal{G} \subset \text{Teich}(S)$, the systole set $\mathsf{sys}(\mathcal{G})$ is a uniform reparameterized quasigeodesic in $\mathcal{C}(S)$. Furthermore, the relation $\alpha \mapsto \mathsf{sys}(\mathcal{G}_\alpha)$ is a uniformly coarse Lipschitz retract from $\mathcal{C}(S)$ to $\mathsf{sys}(\mathcal{G})$. □

In [4], Bowditch gives universal bounds on the quasiconvexity and Lipschitz constants in the case of Teichmüller geodesics $\mathcal{G}(\beta, \beta')$ arising from a half-translation structure dual to a filling pair of weighted multicurves β and β'. He defines a "line" between pairs of weighted multicurves β and β' in $\mathcal{C}(S)$ in terms of intersection numbers as follows ([3] Section 4). Set

$$\mathcal{L}_t(\beta, \beta') = \{\gamma \in \mathcal{C}(S) \mid e^t i(\beta, \gamma) + e^{-t} i(\beta', \gamma) \leq \mathsf{L}_0 \sqrt{i(\beta, \beta')}\}$$

and $\mathcal{L}(\beta', \beta) = \bigcup_{t \in \mathbb{R}} \mathcal{L}_t(\beta, \beta')$, where L_0 is a suitable universal constant. Let q be the half-translation structure dual to β and β', where its horizontal and vertical directions, respectively agree with β and β'. Then $\mathcal{L}_t(\beta,\beta')$ is precisely the set of bounded length curves on $g_t \cdot q$ under the L_1–metric. Applying Lemma 4.4, we see that $\mathcal{L}_t(\beta,\beta') \approx \mathsf{sys}(g_t \cdot q)$ and hence $\mathcal{L}(\beta,\beta') \approx \mathsf{sys}(\mathcal{G}(\beta,\beta'))$.

Bowditch proves that this system of lines satisfies a "slim triangles" condition with universal constants which, by a criterion of Masur and Schleimer ([17] Theorem 3.15), implies the curve graph is universally hyperbolic and that the line $\mathcal{L}(\beta,\beta')$ is a universal reparameterized quasigeodesic between β and β'. Furthermore, he approximates "coarse centres" of geodesic triangles using balance times. We can reinterpret this in terms of nearest point projections: If $\mathcal{G}_\alpha$ is the balance time of α along $\mathcal{G} = \mathcal{G}(\beta,\beta')$, then $\mathsf{sys}(\mathcal{G}_\alpha)$ is universally close to any nearest point projection of α to $\mathsf{sys}(\mathcal{G})$ in $\mathcal{C}(S)$.

We now outline a proof extending Bowditch's results to arbitrary Teichmüller geodesics via a limiting argument.

Proposition 7.6. For any Teichmüller geodesic $\mathcal{G} \subset \mathrm{Teich}(S)$, the systole set $\mathsf{sys}(\mathcal{G})$ is Q_2–quasiconvex for some universal constant Q_2. Furthermore, the operation $\alpha \mapsto \mathsf{sys}(\mathcal{G}_\alpha)$ agrees with the nearest point projection from $\mathcal{C}(S)$ to $\mathsf{sys}(\mathcal{G})$ up to a universal error P_2. □

Proof. Suppose $\mathcal{G}$ is an arbitrary Teichmüller geodesic whose endpoints correspond to a pair of transverse measured foliations λ and λ'. Define $\mathcal{L}_t(\lambda,\lambda')$ and $\mathcal{L}(\lambda,\lambda')$ using the same intersection number conditions as above. Using a suitable parameterization, these sets respectively agree with $\mathsf{sys}(\mathcal{G}_t)$ and $\mathsf{sys}(\mathcal{G})$ up to universal Hausdorff distance. Let β_n and β'_n be sequences of weighted multicurves converging in the space of measured foliations $\mathcal{MF}(S)$ to λ and λ' respectively. Appealing to continuity of intersection number on $\mathcal{MF}(S) \times \mathcal{MF}(S)$ [2] and Lemma 4.4, for each $t \in \mathbb{R}$ we have $\mathcal{L}_t(\beta_n,\beta'_n) \approx \mathcal{L}_t(\lambda,\lambda')$ for all n sufficiently large. Since each $\mathcal{L}(\beta_n,\beta'_n)$ is universally quasiconvex [4], it follows that $\mathcal{L}(\lambda,\lambda')$ and hence $\mathsf{sys}(\mathcal{G})$ are also universally quasiconvex.

To prove the second claim, one can again appeal to continuity of intersection number to show that $\mathsf{sys}(\mathcal{G}(\beta_n,\beta'_n)_\alpha)$ will eventually agree with $\mathsf{sys}(\mathcal{G}_\alpha)$ up to universal Hausdorff distance for all sufficiently large n. Applying Bowditch's nearest point projection result completes the proof. ■

Using the above results, we can give an alternative proof that systole sets are universally quasiconvex, though with weaker effective control over the constants.

Corollary 7.7. For any $SL(2,\mathbb{R})$–orbit Δ, the systole set $\mathsf{sys}(\Delta)$ is Q_2–quasiconvex. □

Proof. Given any pair of curves $\alpha, \beta \in \mathsf{sys}(\Delta)$, let $q, q' \in \Delta$ be points such that $\alpha \in \mathsf{sys}(q)$ and $\beta \in \mathsf{sys}(q')$. Let $\mathcal{G} \subset D$ be a Teichmüller geodesic connecting the projections of q, q' to D. Then any geodesic in $\mathcal{C}(S)$ connecting α to β must lie within a distance Q_2 of $\mathsf{sys}(\mathcal{G}) \subseteq \mathsf{sys}(\Delta)$. ■

***Proof of Theorem* 7.3 .** Let $\alpha \in \mathcal{C}(S)$ be a curve and X a balance point with respect to a Teichmüller disc D. Let γ be a nearest point projection of α to $\mathsf{sys}(\Delta)$ in $\mathcal{C}(S)$. Then γ is a systole for some $q \in \Delta$. Let Y be a projection of q to D. If Y coincides with X then we are done, so suppose otherwise. Let $\mathcal{G}$ be an infinite Teichmüller geodesic passing through X and Y. By Theorem 7.5 and Proposition 7.6, $\mathsf{sys}(\mathcal{G}_\alpha)$ is universally close to γ. By Theorem 7.1, we have $d_D(X, \mathcal{G}_\alpha) \leq \log 2$ and so by Lemma 4.5, it follows that $\mathsf{sys}(X)$ is uniformly close to $\mathsf{sys}(\mathcal{G}_\alpha)$. ■

This proof also works if $X \in D$ is the projection of some $m \in \Delta$ which minimizes the flat length of α.

8 Balance points and curve decompositions

Fix a Teichmüller disc $D \subset \mathrm{Teich}(S)$. Let $\tau \in \mathcal{TT}(\Delta)$ be a straight train track with respect to Δ, and suppose α is a multicurve carried by τ. By Theorem 2.6, we can write $w_\alpha = \frac{1}{2}\sum_{v \in V(\tau)} m_v w_v$ where the m_v are non-negative integers. Let X and X_v respectively be balance points for α and v on D. Let $H = H(\tau)$ be the convex hull of the X_v's in D.

The goal of this section is to show the following connection between balance points and straight vertex cycle decompositions.

Proposition 8.1. The sets $\mathsf{sys}(X)$, $\mathsf{sys}(H)$, and $V(\tau)$ agree up to universally bounded Hausdorff distance in $\mathcal{C}(S)$. In particular, $\mathsf{sys}(H)$ has universally bounded diameter. □

Combining this result with Proposition 5.1, Proposition 5.6, and Theorem 7.3, we can give an alternative proof to Proposition 3.5, albeit with weaker control over the constants.

Lemma 8.2. Let $\mathcal{G}$ be a Teichmüller geodesic in D. Let q and q_v be the respective balance times of α and v on $\mathcal{G}$, for all $v \in V(\tau)$. If $I \subseteq \mathcal{G}$ is the minimal subinterval which contains all the q_v's, then q is also contained in I. □

Proof. We may assume the horizontal directions on q and all the q_v's are the same, i.e., they correspond to the same endpoint of $\mathcal{G}$. Suppose the claim is false. Then the q_v's must all lie on the same component of $\mathcal{G} - \{q\}$. This means that either $l_q^H(v) > l_q^V(v)$ for all v; or $l_q^H(v) < l_q^V(v)$ for all v. It follows that either

$$l_q^H(\alpha) = \frac{1}{2}\sum_v m_v l_q^H(v) > \frac{1}{2}\sum_v m_v l_q^V(v) = l_q^V(\alpha);$$

or $l_q^H(\alpha) < l_q^V(\alpha)$. Then α is not balanced at q, a contradiction. ■

Lemma 8.3. Given $\tau \in \mathcal{TT}(\Delta)$, let $H = H(\tau)$ be as above. Then for any multicurve α carried by τ, any balance point X of α on D is in the $2\log 2$–neighbourhood of H. □

Proof. If X is in H then we are done, so suppose otherwise. Let Y be the unique closest point projection of X to H, and $\mathcal{G}$ be the infinite Teichmüller passing through X and Y. Let q, q_v and I be as in the previous lemma for $\mathcal{G}$. By Proposition 7.1,

$$d_D(X,H) = d_D(X,Y) \leq d_D(X,q) + d_D(q,Y) \leq \log 2 + d_D(q,Y)$$

and so it suffices to prove $d_D(q,Y) \leq \log 2$.

Let J and J' be the two closed rays contained in $\mathcal{G}$ which have Y as their endpoint. Assume $X \in J'$. Let $\pi_{\mathcal{G}} : D \to \mathcal{G}$ be the closest point projection map, and let $p_v = \pi_{\mathcal{G}}(X_v)$. Using elementary hyperbolic geometry, one can show that $\pi_{\mathcal{G}}(H) \subseteq J$. By Proposition 7.1, we have $d_D(q_v,p_v) \leq \log 2$ which implies $I \subseteq_{\log 2} J$. Applying the previous lemma, we have $q \in I$ and so

$$d_D(q,J) \leq \log 2.$$

On the other hand,

$$d_D(q,J') \leq d_D(q,X) \leq \log 2$$

by Proposition 7.1. Combining these two bounds, we deduce that q is within distance $\log 2$ of the common endpoint of J and J', namely Y. ■

Lemma 8.4. Let $v \in V(\Delta)$ be a straight vertex cycle for Δ and let $X_v \in D$ be a balance point of v. Then v is universally close to $\mathrm{sys}(X_v)$ in $\mathcal{C}(S)$. □

Proof. Let γ be any nearest point projection of v to $\mathsf{sys}(\Delta)$. By Proposition 5.6 and Theorem 7.3 respectively, we deduce

$$d_S(v,\gamma) = d_S(v,\mathsf{sys}(\Delta)) \leq 54$$

and

$$d_S(\gamma,\mathsf{sys}(X_v)) \leq \mathsf{P}_2.$$

The result follows by applying the triangle inequality. ∎

***Proof of Proposition* 8.1 .** By Lemmas 4.5 and 8.3, $\mathsf{sys}(X)$ lies in a universally bounded neighbourhood of $\mathsf{sys}(H)$. Recall from Theorem 2.7 that $V(\tau)$ has diameter at most $\mathsf{K}_1 \leq 14$ in $\mathcal{C}(S)$. Therefore, to universally bound the pairwise Hausdorff distances between $\mathsf{sys}(X)$, $\mathsf{sys}(H)$ and $V(\tau)$ in $\mathcal{C}(S)$, it suffices to show that $\mathsf{sys}(H)$ is contained in a universally bounded neighbourhood of $V(\tau)$.

Recall that D is isometric to the hyperbolic plane with curvature -4. Thus H is contained in a universally bounded neighbourhood of the union of all Teichmüller geodesics connecting X_v and $X_{v'}$, for all pairs $v, v' \in V(\tau)$. By Lemma 4.5, Theorem 7.5, and Proposition 7.6, the relation $\mathsf{sys} : D \to \mathcal{C}(S)$ is coarsely Lipschitz and sends Teichmüller geodesics to reparameterized quasigeodesics with universal quasiconvexity constants. Therefore $\mathsf{sys}(H)$ is contained in a universally bounded neighbourhood of the set $\{\mathsf{sys}(X_v) \mid v \in V(\tau)\}$. By the previous lemma, $\mathsf{sys}(X_v)$ is universally close to v and we are done. ∎

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