

MANTON'S FIVE VORTEX EQUATIONS FROM SELF-DUALITY

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ABSTRACT. We demonstrate that the five vortex equations recently introduced by Manton arise as symmetry reductions of the anti-self-dual Yang–Mills equations in four dimensions. In particular the Jackiw–Pi vortex and the Ambjørn–Olesen vortex correspond to the gauge group $SU(1,1)$, and respectively the Euclidean or the $SU(2)$ symmetry groups acting with two-dimensional orbits. We show how to obtain vortices with higher vortex numbers, by superposing vortex equations of different types. Finally we use the kinetic energy of the Yang–Mills theory in 4+1 dimensions to construct a metric on vortex moduli spaces. This metric is not positive-definite in cases of non-compact gauge groups.

1. INTRODUCTION

The Abelian Higgs model at critical coupling admits static solutions modeling vortices which neither attract nor repel each other [13]. The mathematical content of the model consists of a Hermitian complex line bundle L over a Riemannian surface (Σ, g_Σ) , together with a $U(1)$ connection a and a complex Higgs field ϕ satisfying the Bogomolny equations

$$\bar{\mathcal{D}}\phi = 0, \quad f = \omega_\Sigma(1 - |\phi|^2).$$

Here ω_Σ is the Kähler form on Σ , $\bar{\mathcal{D}} = \bar{\partial} - ia^{(0,1)}$ is the covariant $\bar{\partial}$ -operator (the anti-holomorphic part of the covariant derivative D defined by a), $f = da$ is the Abelian Maxwell field and $a^{(0,1)}$ is the anti-holomorphic part of a . Setting $|\phi|^2 = \exp(h)$, and solving the first Bogomolny equation for the one-form a reduces the second equation to the Taubes equation [21]

$$\Delta h + 2(1 - e^h) = 0, \tag{1.1}$$

which is valid outside small discs enclosing the logarithmic singularities of h – the locations of vortices on Σ . Here Δ is the Laplace operator on (Σ, g_Σ) .

In [14] Manton has considered a two-parameter generalisation of the Taubes equation

$$\Delta h - 2(C_0 - Ce^h) = 0. \tag{1.2}$$

The constants C_0 and C can be rescaled to 0, 1 or -1 , and Manton has argued that only five combinations lead to non-singular vortex solutions

- $C = C_0 = -1$ corresponds to the Taubes equation [21]. The magnetic field f decays to zero away from vortex center.
- $C = C_0 = 1$ is the Popov equation [17, 18].
- $C = 0, C_0 = -1$ corresponds to the magnetic field with constant strength equal to 1. In [14] this was called the Bradlow vortex.

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- $C = 1, C_0 = -1$ is the Ambjørn–Olesen vortex. The magnetic field is enhanced away from the position of the vortex [1, 14].
- $C = 1, C_0 = 0$ is the Jackiw–Pi vortex equation, which arises in a first order Chern–Simons theory [11, 10]. In this case $|\phi|^2$ tends to zero at the position of the vortex and (non-compact surfaces) at ∞ .

The aim of this paper is to show (Theorem 2.1 in §2.3) that Manton’s equation (1.2) for all values of C_0, C arises as the symmetry reduction of the anti-self-dual Yang–Mills equations (ASDYM) on a four-manifold $M = \Sigma \times N$, where N is a surface of constant curvature, and the symmetry group is the group of local isometries of N . The value of C_0 in (1.2) is determined by the curvature of N , and C depends on the choice of the gauge group G_C . We shall demonstrate that $N = S^2$ if $C_0 = -1$, $N = \mathbb{H}^2$ if $C_0 = 1$, and $N = \mathbb{R}^2$ if $C_0 = 0$. The gauge group is $SU(2)$ if $C = -1$, $SU(1,1)$ if $C = 1$ and the Euclidean group $E(2)$ if $C = 0$. In the integrable cases the Gaussian curvatures of Σ and N add up to zero.

In §2.5 we shall show how the five vortex equations are related by a superposition principle which leads to a construction of vortices with higher vortex numbers.

The four-dimensional perspective allows us to derive a canonical expression for the resulting energy of vortices. By considering the kinetic energy of the dynamical Yang–Mills theory on $\mathbb{R} \times M$ in §3.1 we shall derive integral expressions for moduli space metrics corresponding to various choices of constants in (1.2). If the gauge group is non-compact, then the kinetic energy and the resulting moduli space metric are not positive definite and the moduli space may contain surfaces where they identically vanish. In the integrable cases of equation (1.2), the moduli space metric takes a simple form that generalises the one for integrable Taubes vortices described in [20].

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2. EQUIVARIANT ANTI-SELF-DUAL CONNECTIONS AND SYMMETRY REDUCTION

In this Section we shall formulate the main Theorem. Let us first introduce some notation.

2.1. The group G_C . The key role will be played by a Lie group $G_C \subset SL(2, \mathbb{C})$ which consists of matrices K such that

$$K \begin{pmatrix} 1 & 0 \\ 0 & -C \end{pmatrix} K^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -C \end{pmatrix}, \quad \text{where } C \in \mathbb{R}$$

or equivalently

$$G_C = \left\{ K = \begin{pmatrix} k_1 & k_2 \\ Ck_2 & k_1 \end{pmatrix}; k_1, k_2 \in \mathbb{C}, \quad \text{and } |k_1|^2 - C|k_2|^2 = 1 \right\}. \quad (2.3)$$

Therefore $G_{-1} = SU(2)$, $G_1 = SU(1, 1)$ and $G_0 = E(2)$ – the Euclidean group. The generators of the corresponding Lie algebra \mathfrak{g}_C ,

$$J_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -Ci & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -C & 0 \end{pmatrix}, \quad J_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (2.4)$$

satisfy commutation relations

$$[J_1, J_2] = -CJ_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2.$$

The explicit parametrisation of G_C as well as the left-invariant one-forms are constructed in the Appendix.

In what follows, G_{C_0} will denote the Lie group defined in the same way, but changing C into C_0 .

2.2. The four-manifold. Let M be the Riemannian four-manifold given by the Cartesian product $\Sigma \times N$ with a product metric

$$g = g_\Sigma + g_N, \quad (2.5)$$

where (Σ, g_Σ) is the Riemann surface introduced in §1, and (N, g_N) is a surface of constant Gaussian curvature $-C_0$. Let w be a local holomorphic coordinate on Σ , and z be a local holomorphic coordinate on N so that

$$g_\Sigma = \Omega dw d\bar{w}, \quad \text{and} \quad g_N = \frac{4dz d\bar{z}}{(1 - C_0|z|^2)^2},$$

where $\Omega = \Omega(w, \bar{w})$ is the conformal factor on Σ . The Kähler forms ω_Σ on Σ and ω_N on N are given by

$$\omega_\Sigma = \frac{i}{2} \Omega dw \wedge d\bar{w}, \quad \omega_N = \frac{2idz \wedge d\bar{z}}{(1 - C_0|z|^2)^2} = 2id\beta, \quad \text{where} \quad \beta = \frac{zd\bar{z} - \bar{z}dz}{2(1 - C_0|z|^2)}. \quad (2.6)$$

We shall choose an orientation on M by fixing the volume form $\text{vol}_M = \omega_\Sigma \wedge \omega_N$.

2.3. Equivariance. Let G_C and G_{C_0} be Lie groups corresponding, via (2.3) to two real constants C and C_0 . In the Theorem below G_C will play a role of a gauge group, and G_{C_0} will be the symmetry group.

Let $\mathcal{E} \rightarrow M$ be a vector bundle with a connection which, in a local trivialisation, is represented by a Lie-algebra valued one-form $A \in \mathfrak{g}_C \otimes \Lambda^1(M)$. The Lie group G_{C_0} is a subgroup of the conformal group on (M, g) , and acts on M isometrically by

$$(w, z) \mapsto \left(w, \frac{k_1 z + k_2}{C_0 \bar{k}_2 z + \bar{k}_1} \right). \quad (2.7)$$

We shall impose the symmetry equivariance condition on A : it is preserved up to a gauge transformation by the action (2.7) of G_{C_0} . The infinitesimal equivariance condition yields

$$\mathcal{L}_X A = DW, \quad (2.8)$$

where \mathcal{L} is the Lie derivative, X is any vector field on M generating the action (2.7) and $DW \equiv dW - [A, W]$ is the covariant derivative of a \mathfrak{g}_C -valued function on M .

In the coordinates (w, \bar{w}, z, \bar{z}) introduced above we have

$$A = A_\Sigma + A_N, \quad \text{where} \quad A_\Sigma = A_w dw + A_{\bar{w}} d\bar{w} \quad \text{and} \quad A_N = A_z dz + A_{\bar{z}} d\bar{z}.$$

Theorem 2.1. *Let $A \in \Lambda^1(M) \otimes \mathfrak{g}_C$ be G_{C_0} -equivariant. Then*

- (1) *There exists a gauge such that*

$$A = \begin{pmatrix} -C_0\beta + \frac{i}{2}a & -\frac{i}{1-C_0z\bar{z}}\phi d\bar{z} \\ \frac{iC}{1-C_0z\bar{z}}\bar{\phi} dz & C_0\beta - \frac{i}{2}a \end{pmatrix}, \quad (2.9)$$

where β is defined in (2.6), a is a $\mathfrak{u}(1)$ -valued one-form, and ϕ is a complex Higgs field on Σ .

- (2) *The ASDYM equations on (M, g) are*

$$\bar{D}\phi = 0, \quad f + \omega_\Sigma(C_0 - C|\phi|^2) = 0 \quad (2.10)$$

where $f = da$. Equivalently, setting $|\phi|^2 = e^h$,

$$\Delta_0 h - 2\Omega(C_0 - Ce^h) = 0, \quad \text{where } \Delta_0 = 4\partial_w\bar{\partial}_{\bar{w}}. \quad (2.11)$$

We shall split the proof of this theorem into two Propositions

Proposition 2.2. *Let G_{C_0} be the maximal group of isometries of (N, g_N) , where $N = \mathbb{R}^2, S^2$ or \mathbb{H}^2 . The most general G_{C_0} -equivariant G_C -connection is gauge equivalent to (2.9).*

and

Proposition 2.3. *The ASDYM equations on (2.9) are equivalent to (2.10) or (2.11).*

Proof of Proposition 2.2. Every vector field X generating the G_{C_0} action on M corresponds to a \mathfrak{g}_C -valued function Φ_X on M called the Higgs field. The symmetry group G_{C_0} does not act freely on N , which leads to a set of differential and algebraic constraints. These constraints are kinematical, as they arise purely from the symmetry requirement, and do not involve the ASDYM equations. Our analysis of the constraints, and the construction of A follows [9, 13], and we refer the reader to these works for details. Another method is presented in [15].

Consider $N = G_{C_0}/U(1)$ as the homogeneous space, where $U(1) \subset G_{C_0}$ stabilises a point with coordinates $z = 0$ in N . This $U(1)$ action is generated by $J_3 \in \mathfrak{g}_{C_0}$ defined in (2.4), and corresponds to a vector field X^3 on M . The constraint equations are

$$D\Phi_3 = 0, \quad [\Phi_3, \Phi_m]_{\mathfrak{g}_C} + \Phi_{[X^3, X^m]_{\mathfrak{g}_{C_0}}} = 0, \quad (2.12)$$

where the vector fields $X^m, m = 1, 2, 3$ generate the G_{C_0} action, and we set $\Phi_m \equiv \Phi_{X^m}$. The map $X^3 \mapsto \Phi_3$ is a homomorphism from $\mathfrak{u}(1) \subset \mathfrak{g}_{C_0}$ to \mathfrak{g}_C , and equations (2.12) imply that the Σ -components of the potential $A_\Sigma = A_w dw + A_{\bar{w}} d\bar{w}$ belong to the image of this homomorphism in the gauge Lie algebra, i.e. $\mathfrak{u}(1) \subset \mathfrak{g}_C$. We also find that the components $(A_w, A_{\bar{w}})$ are functions on Σ which do not depend on the coordinates on N . Therefore we have arrived at the Abelian Maxwell potential a on Σ given by

$$a = a_w dw + a_{\bar{w}} d\bar{w} \equiv A_\Sigma.$$

To solve (2.12) for the Higgs fields, we chose a gauge such that Φ_3 is diagonal. This is always possible if $C = -1$ as Hermitian matrices are diagonalisable by unitary changes of basis, but needs to be verified by a direct calculation which uses the constraint equations if $C = 0$ or $C = 1$. In general we find $\Phi_3 = \epsilon J_3$, where $\epsilon = \pm 1$. The choice of ϵ amounts

to a choice of an orientation¹ on Σ , and we shall take $\epsilon = 1$. The general solution of (2.12) now becomes

$$\Phi_1 = \phi_1 J_1 + \phi_2 J_2, \quad \Phi_2 = \phi_2 J_1 - \phi_1 J_2, \quad \Phi_3 = J_3,$$

where $\phi = \phi_2 - i\phi_1$ is a complex valued function on Σ which we shall later identify with the Abelian Higgs field in the vortex equation.

We shall now construct the remaining part of the gauge potential A – its components on the surface N . Following [9] define a gauge potential on G_{C_0} by

$$\mathcal{A} = \sum_{m=1}^3 \Phi_m \otimes \chi_m,$$

where χ_1, χ_2, χ_3 are right invariant one-forms on G_{C_0} given by (A2). In what follows, we use coordinates $(\kappa_1, \kappa_2, \kappa_3)$ for G_{C_0} described in the Appendix.

We can perform a gauge transformation on G_{C_0} such that

$$\rho^3 \lrcorner \mathcal{A} = 0, \quad \mathcal{L}_{\rho^3} \mathcal{A} = 0,$$

where ρ^1, ρ^2, ρ^3 are right-invariant vector fields on G_{C_0} such that $\rho^m \lrcorner \chi_l = \delta^m_l$, and $\rho^3 = \partial/\partial\kappa_3$. This yields a gauge potential defined on the quotient $G_{C_0}/U(1)$, where the $U(1)$ subgroup is generated by J_3 . It is given by

$$\mathcal{A} = \mathcal{A}_1 d\kappa_1 + \mathcal{A}_2 d\kappa_2,$$

where

$$\begin{aligned} \mathcal{A}_1 &= -\phi_2 J_1 + \phi_1 J_2 \\ \mathcal{A}_2 &= \frac{1}{\sqrt{-C_0}} \sin(\sqrt{-C_0}\kappa_1) \phi_1 J_1 + \frac{1}{\sqrt{-C_0}} \sin(\sqrt{-C_0}\kappa_1) \phi_2 J_2 + \cos(\sqrt{-C_0}\kappa_1) J_3. \end{aligned}$$

Notice that \sin and \cos above become hyperbolic functions for $C_0 > 0$. This one-form Lie derives along $\partial/\partial\kappa_3$ and thus we can use the diffeomorphism between N and $G_{C_0}/U(1)$ to pull back \mathcal{A} to N . This gives $A_N = \mathcal{A}$, and finally $A = A_\Sigma + A_N$.

A direct calculation shows that this gauge potential is indeed G_{C_0} -equivariant. Indeed, it satisfies (2.8) where the vector fields $X^m, m = 1, 2, 3$, are push forwards of the left-invariant vector fields (A3) on G_{C_0} by the projection $G_{C_0} \rightarrow N$, and

$$W_1 = -\frac{\sqrt{-C_0}}{\sin(\sqrt{-C_0}\kappa_1)} \cos \kappa_2 J_3, \quad W_2 = -\frac{\sqrt{-C_0}}{\sin(\sqrt{-C_0}\kappa_1)} \sin \kappa_2 J_3, \quad W_3 = 0.$$

In order to obtain a neater final expression, perform another gauge transformation by $\text{diag}(e^{-\frac{i}{2}\kappa_2}, e^{\frac{i}{2}\kappa_2})$ and change into z -coordinates using the local coordinate formula (A1) to find the G_{C_0} -equivariant G_C -gauge potential on M given by (2.9). □

¹A calculation analogous to the one in this proof shows that if we choose $\epsilon = -1$, then the resulting form of A will be

$$A = \begin{pmatrix} C_0\beta + \frac{i}{2}a & -\frac{i}{1-C_0z\bar{z}}\phi dz \\ \frac{iC}{1-C_0z\bar{z}}\phi d\bar{z} & -C_0\beta - \frac{i}{2}a \end{pmatrix},$$

which is equal to (2.9) up to a change $z \leftrightarrow \bar{z}, w \leftrightarrow \bar{w}$.

Proof of Proposition 2.3. Let (w, z) be holomorphic coordinates on M . The basis of self-dual two forms is spanned by

$$\Re(dw \wedge dz), \quad \Im(dw \wedge dz), \quad \omega_N + \omega_\Sigma.$$

The ASDYM equations $(\omega_N + \omega_\Sigma) \wedge F = 0$, and $dw \wedge dz \wedge F = 0$ take the form

$$F_{wz} = 0, \quad F_{\bar{w}\bar{z}} = 0, \quad \Omega^{-1}F_{w\bar{w}} + \frac{(1 - C_0|z|^2)^2}{4}F_{z\bar{z}} = 0. \quad (2.13)$$

The components of the gauge field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ are given by

$$\begin{aligned} F_{z\bar{z}} &= \frac{-C_0 + C\phi\bar{\phi}}{(1 - C_0z\bar{z})^2} \sigma_3, & F_{w\bar{w}} &= \frac{i}{2} f_{w\bar{w}} \sigma_3, \\ F_{z\bar{w}} &= \frac{i}{1 - C_0z\bar{z}} D_{\bar{w}} \phi \sigma_+, & F_{z\bar{w}} &= -\frac{i}{1 - C_0z\bar{z}} D_{\bar{w}} \bar{\phi} \sigma_-, \\ F_{zw} &= -\frac{i}{1 - C_0z\bar{z}} D_y \bar{\phi} \sigma_-, & F_{z\bar{w}} &= \frac{i}{1 - C_0z\bar{z}} D_w \phi \sigma_+, \end{aligned}$$

where $\phi = \phi_2 - i\phi_1$, $f_{w\bar{w}} = \partial_w a_{\bar{w}} - \partial_{\bar{w}} a_w$, D is the covariant derivative with respect to the $U(1)$ -connection a and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Set

$$D\phi = d\phi - ia\phi, \quad D\bar{\phi} = d\bar{\phi} + ia\bar{\phi}$$

and

$$\mathcal{D} = dw \otimes (\partial_w - ia_w), \quad \bar{\mathcal{D}} = d\bar{w} \otimes (\partial_{\bar{w}} - ia_{\bar{w}}), \quad \text{so that } D = \mathcal{D} + \bar{\mathcal{D}}.$$

The ASDYM equations (2.13) yield vortex-type equations

$$D_{\bar{w}} \phi = \partial_{\bar{w}} \phi - ia_{\bar{w}} \phi = 0, \quad (2.14)$$

$$\frac{i}{2} f_{w\bar{w}} + \frac{\Omega}{4} (-C_0 + C\phi\bar{\phi}) = 0. \quad (2.15)$$

This system of non-linear PDEs can be reduced to a single second order equation for one scalar function. In fact, solve the first equation (2.14) for $a_{\bar{w}}$ so that $a_{\bar{w}} = -i\partial_{\bar{w}} \ln(\phi)$ and, using the reality of a , $a_w = i\partial_w \ln(\bar{\phi})$. Using these expressions for the components of a , calculate the Abelian Maxwell field $f_{w\bar{w}}$ and the second equation (2.15) yields (2.11). \square

2.4. Integrable cases. Following the integrability dogma [15, 7, 3], a symmetry reduction of ASDYM is integrable if the ASDYM equations are defined on a background (M, g) with anti-self-dual Weyl curvature. Computing the Weyl tensor of (2.5) shows that conformal anti-self-duality is equivalent to the vanishing of the scalar curvature of g . Thus in the integrable cases the Riemann surface (Σ, g_Σ) on which the vortex equations are defined must have constant Gaussian curvature equal to minus the Gaussian curvature of (N, g_N) , i.e. locally,

$$g_\Sigma = \frac{4dw d\bar{w}}{(1 + C_0|w|^2)^2}, \quad g = \frac{4dw d\bar{w}}{(1 + C_0|w|^2)^2} + \frac{4dz d\bar{z}}{(1 - C_0|z|^2)^2}. \quad (2.16)$$

The local solutions of integrable vortex equations are given explicitly, in a suitable gauge, by [14]

$$\phi = \frac{1 + C_0|w|^2}{1 + C|s(w)|^2} \frac{ds}{dw}, \quad (2.17)$$

where $s = s(w)$ is a holomorphic map from Σ to a surface of curvature C . The vortices are located at zeros of ϕ , which are the zeros of ds/dw and the poles² of s of order at least 2.

The integrable cases on simply-connected Riemann surfaces under the anti-self-duality framework are the following:

- The Taubes vortex ($C = C_0 = -1$) is integrable on \mathbb{H}^2 , in which case it is a symmetry reduction from ASDYM on $\mathbb{H}^2 \times S^2$. In this case, s is a Blaschke function

$$s(w) = \frac{(w - c_0) \dots (w - c_{\mathbf{N}})}{(w - \bar{c}_0) \dots (w - \bar{c}_{\mathbf{N}})},$$

where $|c_k| < 0$. This is the original integrable reduction of Witten [22].

- The Popov vortex ($C = C_0 = 1$) is integrable on S^2 , in which case it is a symmetry reduction from $S^2 \times \mathbb{H}^2$. In this case, $s : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ is a rational function $p(w)/q(w)$, where p and q are polynomials of the same degree with no common root.
- The Bradlow vortex ($C = 0, C_0 = -1$) is integrable on \mathbb{H}^2 , in which case it is a symmetry reduction from $\mathbb{H}^2 \times S^2$.
- The Ambjörn–Olesen vortex ($C = 1, C_0 = -1$) is integrable on \mathbb{H}^2 , in which case it is a symmetry reduction from $\Sigma \times S^2$.
- The Jackiw–Pi vortex ($C = 1, C_0 = 0$) is integrable on \mathbb{R}^2 , in which case it is a symmetry reduction from $\mathbb{R}^2 \times \mathbb{R}^2$.

In each case the symmetry group is G_{C_0} and the gauge group is G_C .

These integrable cases of (1.2) do not exhaust the list of all integrable vortices: there are other integrable cases related to the sinh-Gordon and the Tzitzeica equations [8, 5, 4], where the vortex is interpreted as a surface with conical singularities.

2.5. Superposition of vortices. Given a solution to the vortex equation (1.2) define the vortex number to be

$$\mathbf{N} = \frac{1}{2\pi} \int_{\Sigma} f. \quad (2.18)$$

This is an integer equal to the first Chern number of the vortex line bundle $L \rightarrow \Sigma$, and we shall assume that this integer is non-negative.

Let us now explain why there exist only five vortex equations among the nine possible combinations of values of C and C_0 . Equation (2.14) implies that the vortex number \mathbf{N} coincides with the number of zeros of ϕ counted with multiplicities [13]. Since ϕ is holomorphic, \mathbf{N} is necessarily non-negative. Since \mathbf{N} is proportional to the magnetic flux, the magnetic field $B \equiv -2if_w\bar{w}/\Omega$ can not be negative everywhere on Σ , but (2.15) implies that this is only possible for the choice of constants

$$(C, C_0) = (-1, -1), (0, -1), (1, -1), (1, 0), (1, 1), (0, 0),$$

²Notice that $|\phi|^2$ is invariant under $s \mapsto 1/s$.

where the last possibility means that the magnetic field is null everywhere and (1.2) is the Laplace equation. We shall call this the Laplace vortex.

The resulting six equations are not disconnected from one another. We shall show that it is possible to construct higher order vortex solutions of one type by superposing two other types of vortices. Let h be a vortex solution on Σ with vortex number \mathbf{N} satisfying

$$\Delta_0 h + 2\Omega(-C_0 + C_1 e^h) = 0, \quad (2.19)$$

so that $|\phi|^2 = e^h$ has \mathbf{N} isolated zeros, counting multiplicities. We say that this vortex is of type (C_1, C_0) .

Consider a metric on Σ

$$\tilde{g}_\Sigma = e^h g_\Sigma$$

which has conical singularities at zeros of $|\phi|^2$. Let \tilde{h} be a vortex solution with vortex number $\tilde{\mathbf{N}}$ on $(\Sigma, \tilde{g}_\Sigma)$ satisfying

$$\Delta_0 \tilde{h} + 2e^h \Omega(-C_1 + C e^{\tilde{h}}) = 0, \quad (2.20)$$

so that $|\phi|^2 = e^{\tilde{h}}$ has $\tilde{\mathbf{N}}$ zeros. We shall say that this vortex is of type (C, C_1) with a rescaled metric. Adding both PDEs (2.19) and (2.20), we find that

$$\Delta_0(h + \tilde{h}) + 2\Omega(-C_0 + C e^{h+\tilde{h}}) = 0$$

and that $e^{h+\tilde{h}}$ has $\mathbf{N} + \tilde{\mathbf{N}}$ zeros. Therefore the superposition of a vortex of type (C, C_1) with a rescaled metric on a vortex of type (C_1, C_0) gives rise to a vortex of type (C, C_0) with vortex number being the sum of the first two vortex numbers, $\mathbf{N} + \tilde{\mathbf{N}}$. In the case $C = C_0 = -1$ this is the Baptista superposition rule [2].

Taking into account the six possible vortex equations, we make a list of all possible superpositions

$$\begin{aligned} \text{Taubes} + \text{Taubes} &= \text{Taubes} \\ \text{Bradlow} + \text{Taubes} &= \text{Bradlow} \\ \text{Laplace} + \text{Bradlow} &= \text{Bradlow} \\ \text{Ambj\o{r}n-Olesen} + \text{Taubes} &= \text{Ambj\o{r}n-Olesen} \\ \text{Jackiw-Pi} + \text{Bradlow} &= \text{Ambj\o{r}n-Olesen} \\ \text{Popov} + \text{Ambj\o{r}n-Olesen} &= \text{Ambj\o{r}n-Olesen} \\ \text{Popov} + \text{Jackiw-Pi} &= \text{Jackiw-Pi} \\ \text{Jackiw-Pi} + \text{Laplace} &= \text{Jackiw-Pi} \\ \text{Popov} + \text{Popov} &= \text{Popov}, \end{aligned}$$

where the non-commutative summation $+$ means that the first vortex (of type (C, C_1) with a rescaled metric) is superposed on the second one (of type (C_1, C_0)) to result in the vortex on the right hand side of the equality (of type (C, C_0)) with higher vortex number.

3. ENERGY AND MODULI SPACE METRIC

The energy functional E of pure Yang-Mills theory on M can be reduced to the energy function of an Abelian-Higgs model on Σ using the ansatz (2.9). This can be done by direct calculation,

$$\begin{aligned} E &= -\frac{1}{8\pi^2} \int_M \text{Tr}(F \wedge \star_g F) \\ &= \frac{1}{4\pi^2} \int_N \omega_N \int_\Sigma \left[\frac{1}{4} (B^2 + (C_0 - C\phi\bar{\phi})^2) - \frac{C}{\Omega} (|\mathcal{D}\phi|^2 + |\overline{\mathcal{D}}\phi|^2) \right] \omega_\Sigma \end{aligned}$$

where

$$B = -2if_{w\bar{w}}/\Omega$$

is the magnetic field on Σ . This expression for the energy is proportional to the one given in [14].

If we assume that N is compact³ then the first integral is the area of N given by

$$\text{Area}_N = \int_N \omega_N = \begin{cases} \frac{4\pi}{-C_0}, & \text{if } C_0 < 0 \\ 4\pi, & \text{if } C_0 = 0 \\ \frac{4\pi}{C_0}(\mathfrak{g} - 1) & \text{if } C_0 > 0, \end{cases}$$

where \mathfrak{g} is the genus of N and we normalised the area of the torus ($C_0 = 0$) to 4π . Thus the energy can be written, using the Bogomolny argument along with $[D_w, D_{\bar{w}}]\phi = -if_{w\bar{w}}\phi$ and the integration by parts (with an additional boundary condition $D\phi = 0$ if Σ is not compact) as

$$E = \frac{\text{Area}_N}{16\pi^2} \int_\Sigma \left[(B + C_0 - C|\phi|^2)^2 - \frac{8C}{\Omega} |\overline{\mathcal{D}}\phi|^2 \right] \omega_\Sigma - C_0 \frac{\text{Area}_N}{4\pi} \mathbf{N},$$

where \mathbf{N} is the vortex number defined by (2.18). If the vortex equations (2.14) and (2.15) are satisfied, then the energy is proportional to the vortex number, characterising a non-interacting theory, and this value is the global minimum of E if $C \leq 0$. However, equation (2.14) cannot be naturally derived from this argument if $C = 0$. In fact the theory corresponding to this energy functional does not involve any Higgs field in this case even though the symmetry reduction of ASDYM necessarily gives rise to a holomorphic Higgs field satisfying (2.14). This is a counter-example to the principle of symmetric criticality, proved under certain conditions in [16].

If N is not compact we can still make sense of *energy density* (or energy per unit of area of N).

Originally, E is the energy functional of pure Yang-Mills theory in four dimensions, which under the ASDYM condition is

$$E = \frac{1}{8\pi^2} \int_{\mathcal{M}} \text{Tr}(\mathcal{F} \wedge \mathcal{F}) \equiv k,$$

³Notice that this is already the case if $C_0 = -1$, as $N = S^2$. Otherwise we assume that the corresponding surfaces are quotiented out by a discrete subgroup of G_{C_0} . If $C_0 = 0$, N is a two-torus \mathbb{T}^2 (\mathbb{R}^2 quotiented out by a lattice) and if $C_0 = 1$ N is a compact Riemann surface of genus $\mathfrak{g} > 1$ (\mathbb{H}^2 quotiented out by a Fuchsian group). The ansatz (2.9) must then admit this further discrete symmetry.

where k is the instanton number. Comparing both expressions for the energy, we derive a relation between the vortex and instanton numbers,

$$k = -C_0 \frac{\text{Area}_N}{4\pi} \mathbf{N} = (1 - \mathfrak{g})\mathbf{N}, \quad \mathfrak{g} = 0, 1, 2, \dots$$

3.1. Dynamical theory. Yang–Mills instantons on M can be regarded as static solitons on $\mathbb{R} \times M$ with a product metric $-dt^2 + g$. Implementing the symmetry reduction of Theorem 2.1, but from five dimensions, leads to vortices on Σ interpreted as stationary solitons in a dynamical theory on $\mathbb{R} \times \Sigma$. We shall use this approach to find the kinetic term in the total energy functional on $\mathbb{R} \times \Sigma$, and use it to read-off the metric on the moduli space of static vortices. Let \mathcal{F} be a \mathfrak{g}_C -valued Yang–Mills field on $\mathbb{R} \times M$. The action functional of pure YM theory with t -dependence is

$$S = -\frac{1}{8\pi^2} \int_{\mathbb{R} \times M} \text{Tr}(\mathcal{F} \wedge \star_5 \mathcal{F}) = \int_{\mathbb{R}} \mathcal{L} dt,$$

where \mathcal{L} is defined by the second equality and involves the integral on M alone. Under the symmetry reduction of Theorem 2.1, \mathcal{L} becomes a Lagrangian on $\mathbb{R} \times \Sigma$,

$$\mathcal{L} = -\frac{\text{Area}_N}{4\pi^2} \int_{\Sigma} \left(\frac{1}{4}(B^2 + (C_0 - C\phi\bar{\phi})^2) - \frac{C}{\Omega}(D_{\bar{w}}\bar{\phi}D_w\phi + D_{\bar{w}}\phi D_w\bar{\phi}) - \frac{1}{\Omega}f_{0w}f_{0\bar{w}} + \frac{C}{2}|D_0\phi|^2 \right) \omega_{\Sigma}.$$

The Euler-Lagrange equations, resulting from calculating the variation with respect to ϕ , a_w and a_0 , are, respectively,

$$D_w D_{\bar{w}}\bar{\phi} + D_{\bar{w}} D_w\bar{\phi} - \frac{\Omega}{2} D_0 D_0\bar{\phi} + \frac{\Omega}{2} (-C_0 + C\phi\bar{\phi})\bar{\phi} = 0, \quad (3.21)$$

$$-2\partial_{\bar{w}} \left(\frac{1}{\Omega} f_{w\bar{w}} \right) + \partial_0 f_{0\bar{w}} - iC(D_{\bar{w}}\bar{\phi}\phi - D_{\bar{w}}\phi\bar{\phi}) = 0, \quad (3.22)$$

$$\partial_w f_{0\bar{w}} + \partial_{\bar{w}} f_{0w} + \frac{iC\Omega}{2} (\phi D_0\bar{\phi} - \bar{\phi} D_0\phi) = 0. \quad (3.23)$$

The equations resulting from varying $\bar{\phi}$ and $a_{\bar{w}}$ are the complex conjugate of equations (3.21) and (3.22), respectively. The third equation is usually referred to as Gauss' law.

This system of second order dynamical equations is satisfied by static solutions to the first order vortex equations (2.14–2.15) in the temporal gauge $a_0 = 0$. In fact, Gauss' law (3.23) is automatically satisfied. To see that (3.21) is satisfied, use (the complex conjugate of) (2.14) to write $D_w D_{\bar{w}}\bar{\phi} = [D_w, D_{\bar{w}}]\bar{\phi} = i f_{w\bar{w}}\bar{\phi}$ and eliminate $f_{w\bar{w}}$ with (2.15). Finally, equation (3.22) is satisfied upon eliminating $f_{w\bar{w}}$ in the same way and using $\partial_{\bar{w}}(\phi\bar{\phi}) = (D_{\bar{w}}\phi)\bar{\phi} + \phi D_{\bar{w}}\bar{\phi} = \phi D_{\bar{w}}\bar{\phi}$.

The kinetic energy T can be read off from \mathcal{L} . In the temporal gauge when $a_0 = 0$ it takes the form

$$T = \frac{\text{Area}_N}{8\pi^2} \int_{\Sigma} \left[\frac{2}{\Omega} \dot{a}_w \dot{a}_{\bar{w}} - C \dot{\phi} \dot{\bar{\phi}} \right] \omega_{\Sigma}, \quad (3.24)$$

where the dots denote t -derivatives. This generalises the known kinetic energy for the Taubes vortex ($C, C_0 < 0$) [20].

In the usual Abelian Higgs model in the critical coupling (yielding Taubes vortices), there is a moduli space \mathcal{M}_N of static \mathbf{N} vortex solutions. All these solutions have the same potential energy, so there are no static forces. The moduli space acquires a metric from the kinetic energy of the theory, and the geodesics of this metric model slow

motion of \mathbf{N} -vortices. There are several ways to obtain the metric for both flat and curved backgrounds [13, 6]. One way to proceed for the integrable vortices is to assume that the vortex positions depend on time, and substitute the explicit solution (2.17) into the kinetic energy (3.24). This, when quotiented out by the gauge equivalence (which is equivalent to imposing Gauss' law [13]), gives a quadratic form on \mathcal{M}_N . In case of Taubes vortices the resulting metric is positive definite, but we see that (3.24) is not positive definite if $C > 0$, which is the case for Jackiw–Pi, Popov and Ambjørn–Olesen vortices.

Samols derived a localised expression for the metric of the moduli space of Taubes vortices [19] (see also [20] and [12] for the metric of moduli space of hyperbolic vortices), where the moduli are the vortex positions (or zeros of the Higgs field). The moduli space metric of Taubes vortices with simple zeros $\{W_1, \dots, W_N\}$ is, from (3.24),

$$\frac{Area_N}{8\pi} \sum_{i,j=1}^N (\Omega(W_k)\delta_{ij} + 2\partial_{W_i}b_j) dW_i d\bar{W}_j,$$

where $b_j = \partial_{\bar{w}}(h - 2 \log |w - W_j|)|_{w=W_j, \bar{w}=\bar{W}_j}$ and $\Omega(W_i)$ means that the conformal factor Ω is being evaluated at the point $(w, \bar{w}) = (W_i, \bar{W}_i)$.

A similar calculation as the one performed by Samols (cf. also [13]) adapted to vortices defined by (1.2) gives the following result for the metric on the moduli subspace associated to the simple zeros of the Higgs field

$$\frac{Area_N}{8\pi} \sum_{i,j=1}^N (-C_0\Omega(W_k)\delta_{ij} + 2\partial_{W_i}b_j) dW_i d\bar{W}_j, \quad (3.25)$$

where the constant C_0 appears because the calculation involves the use of equation (1.2).

In the integrable case, the background metric is locally given by g_Σ in (2.16) and the Higgs field is (2.17) in a particular gauge. As above, we assume that $|\phi|$ admits only simple zeros and that each of them is as zero of ds/dw and not a pole, which is always the case up to performing the transformation $s \mapsto 1/s$. In this case, b_j can be calculated directly and is given by

$$b_j = C_0 \frac{W_j}{1 + C_0 |W_j|^2} + \frac{3}{2}\bar{\beta}_j,$$

where $\beta_j = \frac{s_3^{(j)}}{s_2^{(j)}}$ and $s_k^{(j)} = k! \frac{d^k s}{dw^k}|_{w=W_j}$, $k = 0, 1, 2, 3, \dots$, that is to say, the $s_k^{(j)}$'s are defined by

$$s(w) = s_0^{(j)} + s_2^{(j)}(w - W_j)^2 + s_3^{(j)}(w - W_j)^3 + \dots,$$

where the linear term is absent because we assumed that W_j is a simple zero of s .

Now, the moduli space metric (3.25) of simple vortices can be calculated and is given by

$$\frac{3 Area_N}{16\pi} \sum_{i,j=1}^N \frac{\partial \bar{\beta}_j}{\partial W_i} dW_i d\bar{W}_j. \quad (3.26)$$

This is in agreement with the formula derived for integrable Taubes vortices in [20]. In particular, (3.26) tells us that the metric is zero when s depends only holomorphically on the vortex positions, which is the case for some Popov and Jackiw–Pi vortices. In

fact, this is what happens for the $\mathbf{N} = 2$ Popov vortex on $\Sigma = S^2$ corresponding to $s = (w - W_1)^2 / (w - W_2)^2$, and to the $\mathbf{N} = 1$ Jackiw–Pi vortex on $\Sigma = \mathbb{R}^2$ corresponding to $s = (w - W_1)^2$.

APPENDIX. THE GROUP G_{C_0}

In this Appendix, we denote by $J_m^{C_0}$, $m = 1, 2, 3$ the generators of the Lie-algebra \mathfrak{g}_{C_0} given by (2.4) with C replaced by C_0 .

A parametrisation of G_{C_0} is given by

$$K = \begin{pmatrix} e^{i(\kappa_3 - \kappa_2)/2} \cos(\sqrt{-C_0}\kappa_1/2) & -\frac{1}{\sqrt{-C_0}} e^{i(\kappa_2 + \kappa_2)/2} \sin(\sqrt{-C_0}\kappa_1/2) \\ \sqrt{-C_0} e^{-i(\kappa_3 + \kappa_2)/2} \sin(\sqrt{-C_0}\kappa_1/2) & e^{-i(\kappa_3 - \kappa_2)/2} \cos(\sqrt{-C_0}\kappa_1/2) \end{pmatrix},$$

where $0 \leq \kappa_3 \leq 4\pi$, $0 \leq \kappa_2 \leq 4\pi$, $0 \leq \kappa_1 < \pi/\sqrt{-C_0}$ if $C_0 < 0$ and $\kappa_1 \geq 0$ if $C_0 \geq 0$.

The coordinate κ_3 parametrises the $U(1)$ fibres of the fibration $G_{C_0} \rightarrow N = G_{C_0}/U(1)$. In the proof of Proposition 2.2 we need expressions relating the local coordinates (z, \bar{z}) on N to (κ_1, κ_2) on $G_{C_0}/U(1)$. Let $p \in N$ be a point corresponding to the coordinate $z = 0$. Consider the group action (2.7) such that the RHS is 0. This gives a system of two equations for (z, \bar{z}) with a solution

$$z = \frac{1}{\sqrt{-C_0}} \tan(\sqrt{-C_0}\kappa_1/2) e^{i\kappa_2}. \quad (\text{A1})$$

Note that $\frac{1}{\sqrt{-C_0}} \tan(\sqrt{-C_0}\kappa_1/2) \geq 0$ regardless of the sign of C_0 . The formula (A1) is well defined for $C_0 = 0$ upon taking the limit $C_0 \rightarrow 0$. The coordinate κ_3 of G_{C_0} parametrises the stabiliser of $p \in N$, which is a $U(1)$ subgroup generated by $J_3^{C_0}$.

The right-invariant one-forms χ_1, χ_2, χ_3 such that $(dK)K^{-1} + \sum_{m=1}^3 \chi_m \otimes J_m^{C_0} = 0$, and the left-invariant vector fields η^1, η^2, η^3 on G_{C_0} are given by

$$\begin{aligned} \chi_1 &= \left(\frac{1}{\sqrt{-C_0}} \sin(\sqrt{-C_0}\kappa_1) \cos \gamma d\kappa_2 + \sin \kappa_3 d\kappa_1 \right) \\ \chi_2 &= \left(-\cos \kappa_3 d\alpha + \frac{1}{\sqrt{-C_0}} \sin \gamma \sin(\sqrt{-C_0}\kappa_1) d\kappa_2 \right) \\ \chi_3 &= \left(-d\kappa_3 + \cos(\sqrt{-C_0}\kappa_1) d\kappa_2 \right), \end{aligned} \quad (\text{A2})$$

and

$$\begin{aligned} \eta_1 &= -\sin \kappa_2 \partial_{\kappa_1} - \frac{\sqrt{-C_0}}{\tan(\sqrt{-C_0}\kappa_1)} \cos \kappa_2 \partial_{\kappa_2} - \frac{\sqrt{-C_0}}{\sin(\sqrt{-C_0}\kappa_1)} \cos \kappa_2 \partial_{\kappa_3}, \\ \eta_2 &= \cos \beta \partial_{\kappa_1} - \frac{\sqrt{-C_0}}{\tan(\sqrt{-C_0}\kappa_1)} \sin \kappa_2 \partial_{\kappa_2} - \frac{\sqrt{-C_0}}{\sin(\sqrt{-C_0}\kappa_1)} \sin \kappa_2 \partial_{\kappa_3} \\ \eta_3 &= -\partial_{\kappa_2}. \end{aligned} \quad (\text{A3})$$

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