The Phase of the Cosmological Wavefunction

Ayngaran Thavanesan

Supervisor: Prof. Aron Clark Wall

Department of Applied of Mathematics & Theoretical Physics
University of Cambridge

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted, or is being concurrently submitted, for any degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

Ayngaran Thavanesan
May 2025

தமிழரின் தாகம் தமிழீழ தாயகம்

“Tamilarin Thaagam, Tamil Eela Thayagam”
translates to “Tamils thirst for Tamil Eelam”.

As the first Eelam Tamil to pursue the field of theoretical physics, I am honoured that this PhD thesis will be housed in the University of Cambridge Library. The sacrifices of our fallen Maaveerar (great heroes) and the ongoing struggles of my fellow Eelam Tamils in the face of genocide by the Sri Lankan state will not be forgotten.

Tamil Eelam: the homeland of Eelam Tamils.

I particularly would like to dedicate this thesis to my uncles Kuganesan Kanapathipillai and Santhosam Master Umainesan Kanapathipillai, my sister, parents and extended family.

Left: Kuganesan Kanapathipillai (05.04.1956 - 18.05.2009)
Right: Santhosam Master Umainesan Kanapathipillai (04.01.1959 - 21.10.1987)

From each and every one of you, I have learned some of the most important lessons a man can. I hope to one day be a man worthy of the many sacrifices you have all made.

"We all change. When you think about it, we're all different people, all through our lives. And that's ok, that's good, you've got to keep moving, so long as you remember all the people that you used to be. I will not forget one line of this, not one day, I swear."
— The Eleventh Doctor

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To anybody reading this thesis, know that no physicist is the same and that everybody has the ability to make their own contribution towards our understanding of the universe. Don't give up.

I don't want to regret anything... I never want to think I should've...
Naruto Uzumaki

Abstract

The Phase of the Cosmological Wavefunction

Ayngaran Thavanesan

The principles of symmetry, unitarity, and locality form a vital foundation for our descriptions of fundamental physics. Their implications for particle physics are well established, through, for instance, the spin–statistics theorem, the optical theorem, and the CPT theorem. In contrast, their role in quantum interactions in curved spacetimes is only now being systematically explored. This is essential for understanding the inflationary universe, where spacetime is approximately de Sitter.

This thesis investigates how such principles constrain cosmological correlators and the wavefunction of the universe (also referred to as the cosmological wavefunction) in inflationary spacetimes. Using the Effective Field Theory of Inflation (EFToI), the first part of the thesis classifies all parity-even graviton bispectra from local interactions to all orders in derivatives. We identify a minimal set of operators contributing to tree-level bispectra and show how diffeomorphism invariance links contact and exchange terms. Despite the complexity of bulk interactions, the resulting boundary correlators have a remarkably simple analytic structure, with poles only in the total energy. These results provide a complete description of graviton non-Gaussianities in single-clock inflation and new tools for interpreting upcoming cosmic microwave background and large-scale structure data.

The second part establishes a non-perturbative framework for understanding CPT symmetry in cosmology. We show that in both flat and de Sitter space, CRT arises from a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group structure relating discrete rotations (from Lorentz invariance), reflection reality (implied by unitarity), and CRT. In inflationary de Sitter, these must be understood as local Lagrangian symmetries. This leads to the discovery of a novel Cosmological CPT theorem, linking discrete scale invariance and reflection reality, to derive a non-perturbative unitarity constraint in a single Poincaré patch. Applying this to the boundary wavefunction yields universal constraints valid to all loop orders.

Finally, we apply these constraints to establish a cosmological analogue of Furry's theorem and a no-go theorem for parity violation: in single-field inflation with a Bunch–Davies vacuum and scale-invariant operators, late-time parity-odd correlators vanish. This rules out parity-violating signals, such as chiral gravitational waves, for a broad class of inflationary models.

Together, these results provide a symmetry-based foundation for understanding the quantum structure of the early universe, answering long-standing questions by establishing non-perturbative constraints on the wavefunction of the universe derived from bulk unitarity. In doing so, they pave the way for identifying viable dS/CFT candidates in holographic cosmology.

Publications

This thesis is based on the following publications:

[1] G. Cabass, D. Stefanyszyn, J. Supeł and A. Thavanesan, *On graviton non-Gaussianities in the Effective Field Theory of Inflation*, *JHEP* **10** (2022) 154 [2209.00677].

[2] H. Goodhew, A. Thavanesan and A.C. Wall, *The Cosmological CPT Theorem*, 2408.17406.

[3] A. Thavanesan, *No-go Theorem for Cosmological Parity Violation*, 2501.06383.

Contribution of the author to the publications:

The publications on which this thesis is based are the result of collaborative work. Many of the ideas presented in them resulted from group discussion, and most calculations were performed or checked by more than one member of the collaboration. The paragraphs below summarise the contribution of the author to the publications.

The work presented in Chapter 2 is based on a paper written in collaboration with Giovanni Cabass, David Stefanyszyn and Jakub Supeł [1]. The author calculated the curvature and connection data presented in Section 2, as well as the calculations of contact and single-exchange diagrams in Sections 4.1, 4.2, 5.1 and 5.2. Section 5.1 was written mainly by the author. The author also produced Mathematica code that was used to check the agreement of the closed-form expressions with explicit calculations.

The work presented in Chapters 3 to 6 is based on a paper written in collaboration with Harry Goodhew and Aron Wall [2], although the original idea and starting point of the project was due to the author. Sections 1, 2.1, 3.1, 3.2, 3.5, 7.0, 7.3, 7.4, 7.5, 8 were written mainly by the author. The author also produced Mathematica code that was used to check the agreement of the constraints with explicit calculations.

The work presented in Chapter 7 is based on a paper conceptualised and written entirely by the author [3], but benefited from valuable discussions with Harry Goodhew, Juan Maldacena, David Stefanyszyn, Xi Tong, and Aron Wall. The ideas presented also evolved through many interactions within the author's research group led by Aron Wall and Enrico Pajer's research group, as well as during the Parity Violation from Home 2024 conference and numerous other conversations. These collective exchanges have inevitably shaped the author's understanding and interpretations reflected in the final work.

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"No matter what you look at, if you look at it closely enough, you are involved in the entire universe"
— *Michael Faraday*

1

Introduction

Quantum Cosmology

Quantum cosmology is the study of the universe's origin and evolution using the principles of quantum field theory (QFT) applied to the universe as a whole. Unlike classical cosmology, which treats the geometry of the universe as a fixed and deterministic background spacetime governed by general relativity (GR), quantum cosmology treats the entire spacetime geometry — and its fluctuations — as dynamical quantum degrees of freedom. This framework becomes essential when considering the very early universe, which, from compelling experimental evidence such as the cosmic microwave background (CMB), suggests that the universe began in a hot, dense, and rapidly expanding phase known as inflation.

At the heart of quantum cosmology is the Wheeler–DeWitt (WDW) equation [6], a generalisation of the Schrödinger equation to the entire spacetime geometry. This equation defines the *Wavefunction of the Universe*, $\Psi[h_{ij}, \phi]$, which encodes all possible configurations of the spatial metric h_{ij} and matter fields ϕ on constant-time slices (see e.g. [6–25]). The wavefunction satisfies quantum constraints reflecting diffeomorphism invariance and the dynamics of GR, expressed through the *Hamiltonian constraint* and *momentum constraints* in $D = d + 1$ spacetime dimensions. Throughout this thesis, we use the terms cosmological wavefunction and Wavefunction of the Universe (WFU) interchangeably. Both refer to the wavefunctional $\Psi[h_{ij}, \phi]$ of quantum fields in a cosmological background.

The *Hamiltonian constraint* operator $\hat{\mathcal{H}}$ imposes that physical states obey

$$\hat{\mathcal{H}}\Psi[h_{ij}, \phi] = 0\,, \tag{1.1}$$

where, in the Arnowitt–Deser–Misner (ADM) formalism [26, 27], the classical Hamiltonian constraint takes the form

$$\mathcal{H} = \frac{16\pi G_N}{\sqrt{h}} \left(\Pi^{ij}\Pi_{ij} - \frac{1}{d-1}\Pi^2 \right) - \frac{\sqrt{h}}{16\pi G_N} \left(R^{(d)} - 2\Lambda \right) + \mathcal{H}_{\text{matter}} = 0\,. \tag{1.2}$$

where h_{ij} is the spatial metric on the d-dimensional spatial slice, $h = \det(h_{ij})$, Π^{ij} is the canonical momentum conjugate to h_{ij}, $\Pi = h_{ij}\Pi^{ij}$, $R^{(d)}$ is the Ricci scalar curvature of the spatial slice, Λ

is the cosmological constant, and $\mathcal{H}_{\text{matter}}$ is the Hamiltonian density of matter fields on the slice. Quantisation replaces Π^{ij} by functional derivatives with respect to h_{ij}, yielding a functional differential equation that the wavefunctional $\Psi[h_{ij}, \phi]$ must satisfy.

The *momentum constraints* express invariance under spatial diffeomorphisms and read

$$\hat{\mathcal{H}}^i \Psi[h_{ij}, \phi] = 0\,, \quad \text{where} \quad \mathcal{H}^i = -2\nabla_j \Pi^{ij} + \mathcal{H}^i_{\text{matter}}\,. \tag{1.3}$$

where, ∇_j is the covariant derivative compatible with h_{ij}, and $\mathcal{H}_{i,\text{matter}}$ encodes the momentum density contributions of matter fields. These constraints ensure the wavefunction is invariant under spatial coordinate transformations.

Together, the Hamiltonian and momentum constraints encode the dynamics and gauge invariances of the gravitational field and matter in a canonical formulation, enforcing that $\Psi[h_{ij}, \phi]$ only depends on physically meaningful degrees of freedom. Solving the WDW equation subject to these constraints is a major challenge in quantum cosmology, but it is crucial for understanding the universe's initial conditions, the emergence of classical spacetime, and the quantum origin of cosmic structure.

Quantum cosmology aims to provide a consistent description of the universe at the Planck scale[1], and in doing so, raises profound questions: *what is the precise quantum state/WFU? What boundary conditions, if any, should be imposed on this wavefunction? And how do classical spacetime and matter emerge from an underlying quantum gravitational description?* By synthesising concepts from quantum gravity, QFT in curved spacetime, and statistical mechanics, quantum cosmology offers a conceptual and mathematical laboratory for addressing fundamental issues about time, causality, and the universe's initial conditions. Moreover, it connects directly with observational cosmology, since primordial quantum fluctuations seeded the temperature anisotropies observed in the CMB, which later evolved into the large-scale structure (LSS) of the universe seen today.

Another key question in quantum cosmology concerns the fate of discrete symmetries — such as charge conjugation (**C**), parity (**P**), and time reversal (**T**) — in the quantum gravitational regime. For local, Lorentz-invariant QFTs in flat spacetime the combined symmetry of **CPT** (or **CRT**[2] in general spacetime dimensions) is guaranteed by the CPT theorem. However, its status in cosmological settings, i.e. spacetimes with time-dependent backgrounds such as de Sitter space, is subtle. In particular, inflationary cosmology introduces a preferred arrow of time

[1]The Planck mass and Planck length, denoted by M_{Pl} and ℓ_{Pl} respectively, are defined as

$$M_{\text{Pl}} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\,\text{GeV}\,, \quad \ell_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\text{m}\,, \tag{1.4}$$

which quantify the scale where the classical description of spacetime breaks down and quantum effects of the early universe become dominant.

[2]**CRT** denotes the discrete symmetry relevant in general $D = d+1$-spacetime dimensions, where **R** reflects a single spatial coordinate. This differs from the standard **CPT** symmetry in flat space QFT in four spacetime dimensions, where **P** denotes *parity*, i.e. the inversion of *all* spatial coordinates. While **CPT** is a well-defined symmetry in *even* spacetime dimensions due to the possibility of using rotational invariance to relate full parity to **R**, this is *not* true in *odd* spacetime dimensions. Specifically, a rotation by π radians (i.e., 180°) in the $x_i - x_j$ plane is equivalent to flipping both coordinates $x_i \to -x_i$, $x_j \to -x_j$, whilst keeping all other x_k for $k \neq i, j$ held fixed. In *odd* spacetime dimensions, it is impossible to flip an even number of spatial directions via rotations alone, and so one cannot define **CPT**, by combining **CRT** with rotational symmetry, and it is the more general, dimension-independent reflection symmetry **R** that appears in **CRT**.

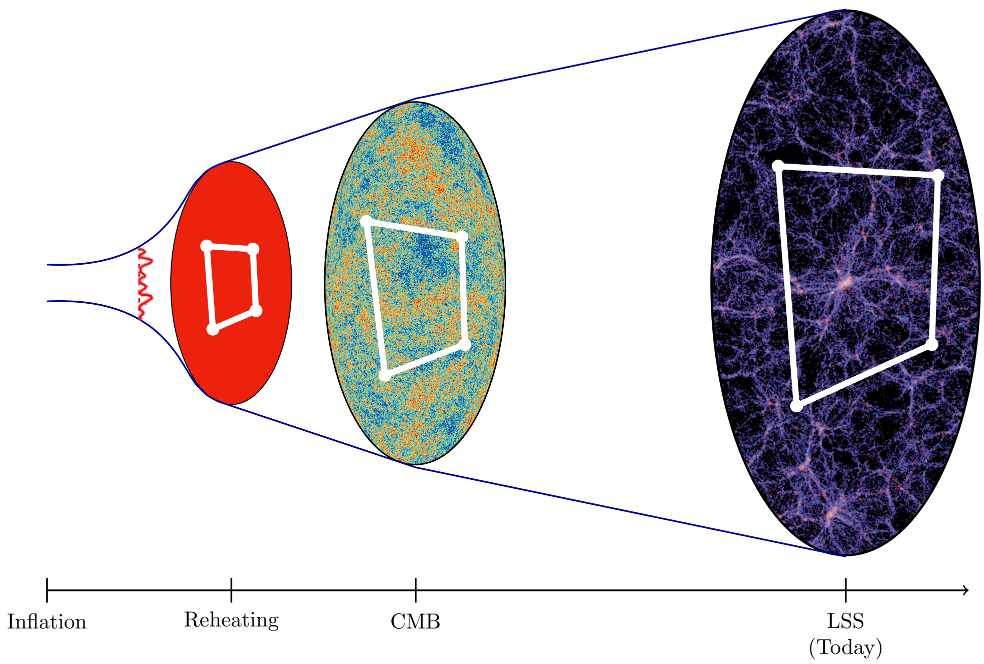

Fig. 1.1 Fluctuations (red) generated during inflation are stretched and evolve from the reheating surface to form the large-scale structure (LSS) observed today. These correlations represent the key observable imprint of inflationary physics. CMB image courtesy of ESA and the Planck Collaboration [4]; LSS image courtesy of V.Springel, Max-Planck Institut für Astrophysik, Garching bei München [5].

through the exponential expansion of the early universe, and a choice of initial vacuum state — typically the Bunch-Davies vacuum — introduces a preferred time orientation. This apparent tension with time-reversal invariance raises the question: does the universe as a whole, or its wavefunction, obey **CPT** symmetry?

This thesis advances our understanding of fundamental symmetries and physical principles, such as unitary time evolution in cosmology, by establishing a version of the CPT theorem applicable to time-dependent, cosmological spacetimes: **the Cosmological CPT Theorem** [2]. This result shows that **CPT** symmetry constrains quantum cosmology at both global — through properties of the entire spacetime manifold — and locally at the level of the Lagrangian. As in particle physics, where **CPT** symmetry imposes strong constraints on scattering amplitudes and correlation functions, its cosmological counterpart imposes non-trivial restrictions on the analytic structure of the cosmological wavefunction and on possible holographic dual descriptions of cosmology.

These theoretical insights have potential observational implications. In particular, the Cosmological CPT Theorem can be used to derive no-go theorems for parity-violating signals in the CMB and LSS [3], enabling stringent constraints on classes of inflationary models and their UV completions. As such, this result provides a rare opportunity to test aspects of quantum gravity through cosmological observations, bridging fundamental theory with empirical data. It offers a unifying perspective on unitarity, time-reversal symmetry, and quantum dynamics in the early universe, and forms a central pillar of this thesis.

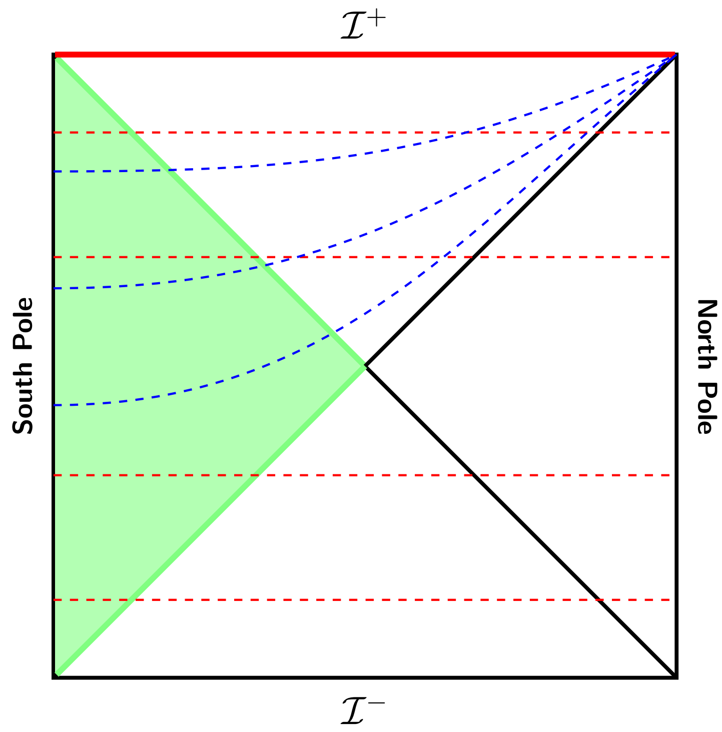

Fig. 1.2 Penrose diagram of de Sitter spacetime illustrating multiple foliations and key geometric features (refer to Chapter 4 for more details). The blue dashed lines represent the Poincaré slicing, which is conformally flat, covers only half of the spacetime, and asymptotes to the future boundary (shared with the global slicing of de Sitter); this slicing is most relevant for inflationary cosmology, as it naturally describes an expanding universe with flat spatial sections. In this framework, we are metaobservers at $\mathcal{I}^+$, where late-time cosmological correlators are measured. The red dashed lines correspond to the global slicing, which foliates the entire spacetime into spatial $(D-1)$-spheres of constant global time. The thick red horizontal line at the top denotes the future de Sitter boundary $\mathcal{I}^+$, where the holographic dual in the dS/CFT correspondence is proposed to reside and encode information about cosmological correlators. The green shaded region indicates the static patch accessible to an observer at the South Pole. Each point in the interior of the diagram corresponds to a $(D-2)$-sphere whose radius varies across the diagram: it shrinks to zero at the left and right edges, representing the South Pole and North Pole respectively, which are the poles of the spatial slices. This shrinking radius captures the spherical geometry of spatial sections in global coordinates.

Cosmology

Cosmology is the scientific study of the origin, evolution, LSS, and ultimate fate of the universe. It seeks to understand the dynamics that have governed the cosmos from its earliest moments to the present day, drawing on both theoretical models and empirical observations. Rooted in GR, modern cosmology treats spacetime itself as a dynamical entity whose geometry evolves in response to the distribution of matter and energy. Observational breakthroughs — such as the discovery of the CMB, the accelerating expansion of the universe, and the large-scale distribution of galaxies — have established the ΛCDM model as the standard cosmological paradigm. However, this model leaves many foundational questions unanswered, including the nature of dark matter and dark energy, the mechanism driving cosmic inflation, and the role of quantum effects in the early universe. As a result, contemporary cosmology lies at the intersection of astrophysics, high-energy particle physics, and quantum gravity, using the universe itself as a natural laboratory to probe fundamental physics.

A key observational tool in cosmology is the study of correlation functions of primordial fluctuations — known as *cosmological correlators*. The Fourier transform of the two-point correlation function is called the *power spectrum*, which describes the variance of fluctuations at different scales. The three-point and four-point correlation functions in momentum space are similarly called the *bispectrum* and *trispectrum*[3], respectively:

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P(k_1)\,, \tag{1.5}$$

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3), \tag{1.6}$$

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle_c = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\,, \tag{1.7}$$

where, ζ denotes the comoving curvature perturbation, which is gauge-invariant and conserved on superhorizon scales in single-field inflation models; the subscript c in the connected trispectrum indicates that disconnected contributions have been subtracted, isolating the genuinely connected four-point correlations. These correlators measure deviations from a purely Gaussian distribution of primordial fluctuations: any non-zero bispectrum or connected trispectrum signals non-Gaussianities, which provide powerful constraints on inflationary dynamics, interactions, and potential new physics beyond the simplest slow-roll models.

This terminology naturally generalises to *tensor perturbations*, such as primordial gravitational waves. The corresponding two-point function of the graviton field γ_{ij} defines the *tensor power spectrum*, and higher-point functions define the tensor *bispectrum*, *trispectrum*, and so on:

$$\langle\gamma^{s_1}(\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_T(k_1)\delta^{s_1s_2}\,, \tag{1.8}$$

$$\langle\gamma^{s_1}(\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_T^{s_1s_2s_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\,, \tag{1.9}$$

$$\langle\gamma^{s_1}(\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\gamma^{s_4}(\mathbf{k}_4)\rangle_c = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)T_T^{s_1s_2s_3s_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\,, \tag{1.10}$$

[3]These terms mirror usage in signal processing and are particularly useful in the study of non-Gaussianities in the CMB and LSS.

where s_i denote the helicities of the graviton modes. These graviton correlators are crucial probes of the inflationary epoch, potentially revealing signatures of parity violation, non-standard spin interactions, and other imprints of quantum gravity[4].

Observationally, tensor perturbations imprint a unique signature in the CMB polarisation: the decomposition of the polarisation field on the sky yields two orthogonal modes — the curl-free gradient-like E-mode and the divergence-free curl-like B-mode. The B-mode polarisation pattern is parity-odd and does not arise from scalar perturbations at linear order, making it a distinctive signature of primordial gravitational waves. A detection of primordial B-mode polarisation would thus provide compelling evidence for inflationary tensor modes.

In addition to pure scalar and tensor statistics, *cross-correlations* between scalar and tensor modes[5] offer powerful complementary information. These include:

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B^{s_3}_{\zeta\zeta\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\,, \tag{1.11}$$

$$\langle \zeta(\mathbf{k}_1)\gamma^{s_2}(\mathbf{k}_2)\gamma^{s_3}(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B^{s_2 s_3}_{\zeta\gamma\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\,, \tag{1.12}$$

which are the scalar–scalar–tensor and scalar–tensor–tensor bispectra respectively. Such mixed correlators are important probes of interactions during inflation that go beyond the simplest single-field, parity-conserving scenarios, offering additional windows into the dynamics and particle content of the early universe.

From an observational perspective, primordial fluctuations imprint themselves on the CMB through temperature anisotropies and polarisation patterns. Scalar perturbations predominantly source the E-mode polarisation, which has even parity, whereas tensor perturbations generate both E-modes and distinctive B-modes — the latter having odd parity and no scalar analogue at linear order. Thus, a detection of B-mode polarisation at large angular scales would provide compelling evidence for primordial gravitational waves. Cross-correlations between scalar and tensor modes can also induce characteristic temperature–polarisation and E–B cross-spectra, offering additional avenues for constraining the physics of inflation and potential signatures of parity violation in the early universe.

The study of higher-point functions — including the scalar [28, 29] and graviton bispectra [30], as well as the trispectrum in exact de Sitter [31–35] — within the framework of the EFT of Inflation (EFToI) [1, 36–38], provides a powerful diagnostic of the symmetries and principles underlying the early universe. The EFToI offers a systematic low-energy description of fluctuations around an inflating background, capturing the most general interactions consistent with the breaking of time translations by the background evolution. By parameterising the effects of heavy fields and allowing for a wide class of non-minimal couplings, the EFToI enables model-independent predictions for inflationary correlators. In this thesis, we develop a mathematical framework to constrain higher-point functions in this setting and explicitly derive the analytic form of graviton non-bispectra within the EFToI.

[4]In quantum gravity, such correlation functions of the metric perturbations encodes important dynamical information of the energy–momentum tensor in quantum gravity.

[5]In standard single-field slow-roll inflation, the scalar–tensor *power spectrum* $\langle\zeta(\mathbf{k})\gamma_{ij}(\mathbf{k}')\rangle$ typically vanishes due to rotational and parity symmetries. However, higher-order *mixed bispectra* such as scalar–scalar–tensor $\langle\zeta\zeta\gamma\rangle$ and scalar–tensor–tensor $\langle\zeta\gamma\gamma\rangle$ are generally non-zero and arise naturally from the coupling between scalar and tensor modes. Non-zero scalar–tensor power spectra can appear in models with parity violation, anisotropy, or non-trivial initial states, offering probes of physics beyond the simplest inflationary scenarios.

Inflation

The inflationary epoch provides a unique window into physics at the interface of QFT and gravity. Although Einstein's equations successfully describe the universe's large-scale dynamics, they offer little guidance regarding its initial conditions. Inflation — a brief period of accelerated expansion in the early universe — addresses many puzzles of the standard Big Bang model, such as the flatness, horizon, and monopole[6] problems (see e.g. [39–44] for more details), while also seeding the primordial fluctuations that give rise to the LSS.

A pivotal insight in inflationary cosmology is that cosmological correlators serve as powerful probes of ultra-high-energy physics, far beyond the reach of terrestrial colliders. Whereas particle physicists infer fundamental interactions from the outcomes of scattering events, cosmologists extract early-universe information from the correlation functions of primordial fluctuations, imprinted in the CMB and the distribution of galaxies. In this way, cosmological correlators play an analogous role to scattering amplitudes in flat spacetime, but in a time-dependent, curved background. Evaluated at fixed time — typically on the future boundary of inflation — they encode quantum dynamics in curved spacetime and provide a rare glimpse into the ultraviolet (UV) structure of quantum gravity.

Inflation is believed to occur at energy scales up to $\sim 10^{15}$ GeV, far exceeding those accessible by current or foreseeable colliders, which are limited to the TeV scale. Just as particle physicists use collision debris to reconstruct high-energy processes, cosmologists use cosmological correlators as "detectors" of early-universe physics. In this sense, inflation offers an unparalleled observational window into quantum gravity. Recent efforts have focused on extracting signatures of these correlators from LSS data, driven by the advent of next-generation telescopes such as *Euclid* and *JWST*. These developments have in turn spurred progress in the Effective Field Theory of LSS (EFToLSS) and in computational methods for higher-point correlation functions.

A central focus of this thesis is the development and application of the **Cosmological CPT Theorem** [2], which provides a profound constraint on the analytic structure and symmetry properties of late-time cosmological correlators. Analogous in spirit to the celebrated CPT theorem in QFT, this result establishes that the cosmological wavefunction in de Sitter-like inflationary backgrounds must be invariant under **CPT**, provided certain basic assumptions are met: bulk unitary time evolution, locality, and analyticity of the wavefunction coefficients ψ_n in the lower-half complex η-plane. The boundary manifestation of **CPT** symmetry leads to non-trivial constraints on the allowed structure of cosmological observables, especially those generated by parity violation or **CPT**-odd interactions.

[6]The *monopole problem* arises in grand unified theories (GUTs), which generically predict the production of magnetic monopoles during symmetry-breaking phase transitions in the early universe. Standard Big Bang cosmology suggests a monopole abundance of roughly one per Hubble volume at the GUT scale, corresponding to a present-day monopole number density $n_M \sim 10^{-15}\,\mathrm{cm}^{-3}$. This is in stark contrast to the observational upper bounds of the searches for monopole-induced catalysis of proton decay and magnetic monopole detectors, which constrain $n_M \lesssim 10^{-29}\,\mathrm{cm}^{-3}$. Inflation resolves this discrepancy by exponentially diluting the monopole density: any monopoles formed before or during the early stages of inflation are redshifted away, leaving an effectively monopole-free observable universe.

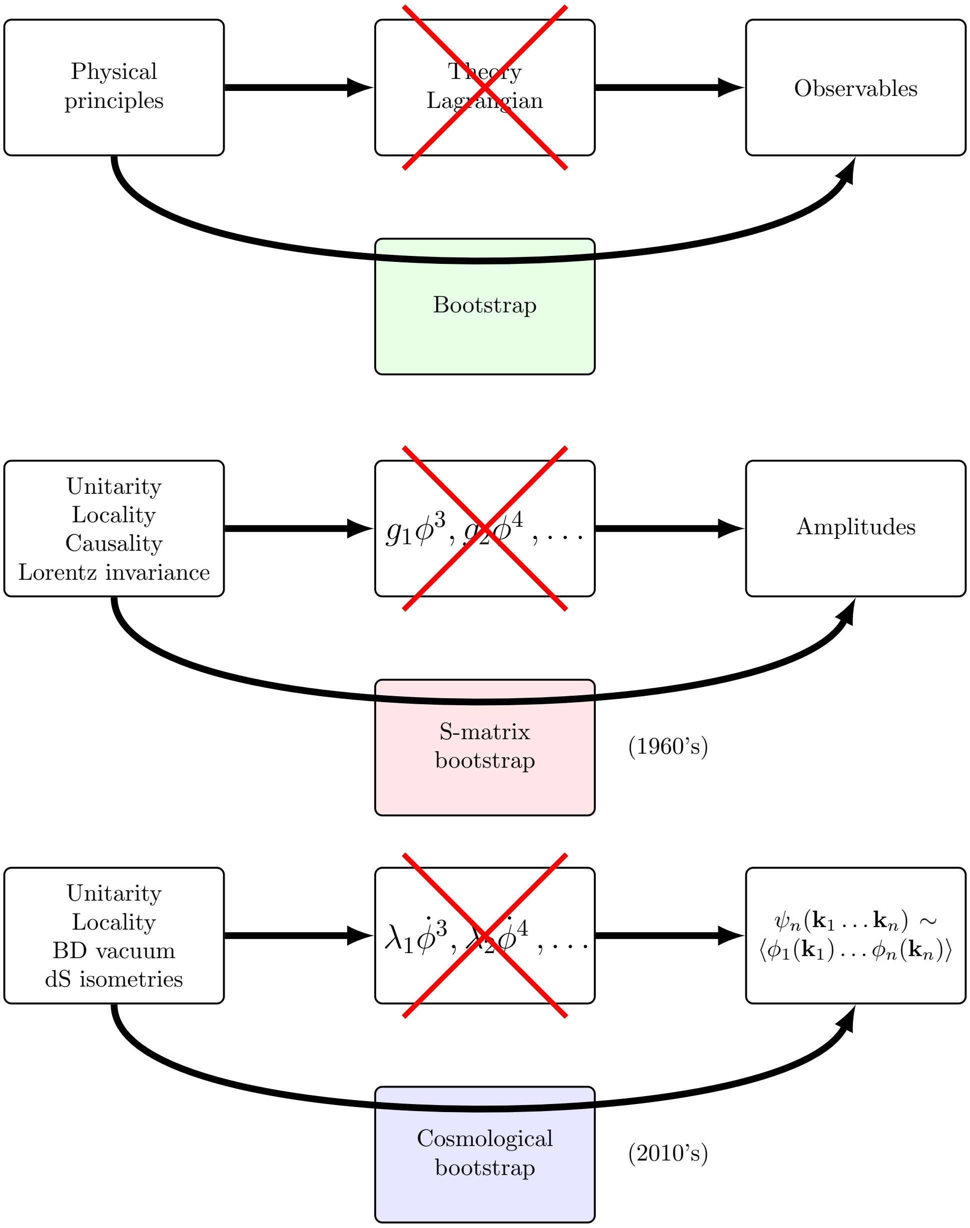

Fig. 1.3 Schematic of the bootstrap approach to amplitudes and cosmological correlators. The input is the symmetry, locality, and unitarity structure of the bulk theory; the output is a constrained space of viable observables.

Cosmological Bootstrap

The inflationary paradigm provides a framework in which small quantum fluctuations are stretched to cosmic scales and ultimately source all structure in the universe. The key observables are the correlation functions of scalar (curvature) and tensor (gravitational wave) perturbations evaluated on a constant-time hypersurface at the end of inflation (typically a de Sitter-like expansion, see Figure 1.2). Unlike scattering amplitudes in flat space, these correlators are computed as "in-in" expectation values rather than "in-out" amplitudes, due to the absence of a future vacuum state in de Sitter spacetime. Traditionally, these correlators are computed using time-dependent perturbation theory, evolving quantum fields from an initial Bunch-Davies vacuum and integrating over bulk interaction vertices. However, this approach is often technically cumbersome and conceptually opaque, as it obscures the role of physical principles such as unitarity, locality, and symmetry, while also introducing gauge redundancies and ambiguities from field redefinitions and time parameterisations.

The *cosmological bootstrap* [1, 29, 30, 45–58] as well as a recent Snowmass white paper [59]) seeks to bypass these issues by constructing correlators directly from their analytic and symmetry properties, inspired by the success of the amplitude bootstrap in flat space [60–65] (see Figure 1.3). One typically assumes a functional ansatz for the correlators that respects the conformal symmetries of $D = d+1$-dimensional de Sitter space, $SO(d+1, 1)$. In the *boostless bootstrap*, this is further reduced to invariance under translations, rotations, and dilatations, i.e. $ISO(d) \times \mathrm{D}$, where D denotes dilatations. While the breaking of Special Conformal Transformations (SCTs) is phenomenologically important [66], the reduced symmetry requires additional consistency conditions to fully constrain the correlators. This motivated the introduction of unitarity-based constraints via the Cosmological Optical Theorem (COT) [50–52, 67] and locality-based constraints via the Manifestly Local Test (MLT) [49]. These tools have enabled the complete bootstrap of the tree-level scalar [29] and tensor [1, 38] bispectra in both the Effective Field Theory of Inflation (EFToI) [68] and Solid Inflation [37, 69], including all orders in derivatives of the bulk interactions.

The Cosmological CPT Theorem imposes powerful constraints on the analytic structure of cosmological correlators. Specifically, it requires that the complex phase of the boundary wavefunction coefficients ψ_n is completely determined by the mass of the boundary fields, as well as under complex conjugation and momentum reversal. These conditions follow from fundamental properties of the bulk theory — namely, unitary time evolution, locality, analyticity and scale invariance — and significantly constrain the allowed form of late-time observables.

Implications for Parity Violation

One of the striking consequences of the Cosmological CPT Theorem is its constraint on the generation of parity-odd correlation functions. In even spacetime dimensions, it can be shown that parity-odd scalar and tensor correlators must vanish under these key assumptions:

- the bulk theory contains only massless scalar and even-spin fields (possibly with an even number of external conformally coupled scalar fields and massless odd-spin fields such as photons), and the initial state is the Bunch–Davies vacuum;

- the interactions are local and **CRT**-invariant (or, equivalently, unitary and scale-invariant);
- the boundary wavefunction coefficients ψ_n are analytic and both infrared (IR)- and ultraviolet (UV)-finite.

This result rules out a wide class of proposed parity-violating signals in inflationary cosmology, unless these assumptions are explicitly violated — e.g. by introducing massive spinning fields, non-local interactions, or breaking scale invariance.

Notably, recent observational claims of parity violation in the CMB and LSS [70–72] provide compelling motivation to revisit these assumptions. The **CPT**-based no-go theorems derived in this thesis clarify the precise theoretical conditions under which parity-violating observables can arise, and guide the construction of physically consistent models that evade the constraints. These include scenarios involving ghost condensates [73], time-dependent couplings [74], and UV-divergent loop corrections [75] in which a parity-odd correlator can be generated by a particular choice of renormalisation scheme [3, 67, 75, 76].

Finally, the broader significance of this work lies in its potential to connect inflationary cosmology to quantum gravity. Just as the flat-space CPT theorem reflects the deep axiomatic structure of QFT, the Cosmological CPT Theorem suggests that any consistent quantum theory of gravity in de Sitter space must obey similar analyticity and symmetry conditions at its boundary. In this sense, the boundary correlators from inflation encode imprints of microscopic gravitational physics, and the constraints derived in this thesis serve as diagnostic tools for testing candidate UV completions of quantum gravity.

Quantum Gravity and Holography

Quantum gravity seeks to unify the principles of quantum mechanics with those of GR into a single, consistent framework. While quantum mechanics governs the behaviour of particles and fields at the smallest scales, and GR describes the dynamics of spacetime and gravity at macroscopic scales, these two foundational theories are fundamentally incompatible in regimes where both quantum effects and strong gravitational fields are significant — such as near black hole singularities or in the earliest moments of the universe. A successful theory of quantum gravity must not only reconcile this tension, but also provide a coherent description of spacetime at the Planck scale, where classical notions of geometry are expected to break down. Moreover, it should explain the microscopic origin of gravitational entropy, resolve the black hole information paradox, and clarify the nature of spacetime singularities. Although no complete theory has yet been experimentally confirmed, several promising candidates have emerged, including string theory, loop quantum gravity, and approaches based on holography and emergent spacetime. Understanding quantum gravity is one of the deepest and most ambitious goals of modern theoretical physics.

One of the most far-reaching principles to emerge from the study of quantum gravity is holography. At its core, the holographic principle posits that all the information contained within a region of spacetime can be described by degrees of freedom on its boundary. This radically non-local feature — wherein the bulk is encoded on a lower-dimensional boundary —

is consistent with black hole thermodynamics and appears to be a fundamental property of any UV-complete theory of gravity.

The original motivation for holography stems from the study of black holes. The Bekenstein–Hawking entropy of a black hole is proportional to the area of its event horizon, not its volume

$$S_{\rm BH} = \frac{A}{4G_N}\,, \tag{1.13}$$

which suggests that the number of fundamental degrees of freedom inside a gravitational system is bounded by its surface area. This starkly contrasts with local QFT, where entropy typically scales with volume. Such a discrepancy signals a breakdown of locality in quantum gravity and motivates the need for a new boundary-based description.

The AdS/CFT Correspondence

The first precise realisation of a duality between anti-de Sitter (AdS) space and a conformal field theory (CFT) was written explicitly by Maldacena in 1997 [77]

$$Z_{\text{strings on AdS}_5\times S^5}[\phi] = Z_{\rm CFT}[\phi]\,. \tag{1.14}$$

On the left-hand side, we consider the partition function of closed strings propagating in the $\mathrm{AdS}_5 \times S^5$ background. This is a functional of the boundary conditions (Dirichlet data) for all bulk supergravity fields, collectively denoted by ϕ, which includes the graviton supermultiplet: the spacetime metric, the dilaton, various gauge fields, and their supersymmetric partners. On the right-hand side, we have the partition function of $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory with gauge group $U(N)$, where the bulk boundary values ϕ act as sources for local, gauge-invariant operators in the CFT. Turning on these sources deforms the theory away from the conformal fixed point. For instance, the boundary value of the dilaton couples to a marginal operator that determines the Yang–Mills coupling.

Our primary interest lies in the 't Hooft limit, where the 't Hooft coupling $\sqrt{\lambda} = g_{\rm YM}\sqrt{N} = \sqrt{4\pi g_{\rm string} N}$ becomes large. In the limit $\sqrt{\lambda} \gg 1$ (whilst keeping $N \gg 1$ and fixed), α' (stringy) corrections are suppressed. This means the curvature radius L_{AdS} of $AdS_5 \times S^5$ is much larger than the string length l_s. The radius L_{AdS} of both AdS_5 and S^5 is related to the string scale $l_s = \sqrt{\alpha'}$ and the 't Hooft coupling λ via:

$$\frac{L_{AdS}^2}{\alpha'} = \sqrt{\lambda} = \left(\frac{L_{AdS}}{l_s}\right)^2\,. \tag{1.15}$$

This means that taking $\sqrt{\lambda} \gg 1$ or equivalently $\lambda \gg 1$ (the 't Hooft limit) corresponds to a large AdS radius in string units: $L_{AdS} \gg l_s$, so stringy corrections are suppressed. Thus, one can safely use the low-energy SUGRA approximation instead of full string theory. allowing us to approximate the full string theory by its low-energy supergravity (SUGRA) limit. Taking $N \to \infty$ further suppresses quantum gravity corrections, rendering the SUGRA action classical. Thus, we obtain the following sequence of approximations on the gravity side

$$Z_{\text{strings on AdS}_5\times S^5}[\phi] \xrightarrow[\text{fixed } N\gg 1]{\sqrt{\lambda}\gg 1} Z_{\rm SUGRA}[\phi] \xrightarrow{N\to\infty} e^{iI^{\rm cl}_{\rm SUGRA}[\phi]}\,, \tag{1.16}$$

where $I^{\text{cl}}_{\text{SUGRA}}[\phi]$ denotes the classical, on-shell action of type IIB supergravity.

In this limit, computing connected correlation functions in the strongly coupled boundary theory reduces to taking functional derivatives of the classical bulk action with respect to the sources

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\text{conn},\phi} \sim \frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}\log Z_{\text{CFT}}[\phi] \sim \frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}I^{\text{cl}}_{\text{SUGRA}}[\phi]\,, \tag{1.17}$$

where $\mathcal{O}$ is the single-trace operator dual to the bulk field ϕ. This relation illustrates the essence of the AdS/CFT correspondence: strongly-coupled CFT correlators can be computed via classical, tree-level processes in the dual gravitational theory. The resulting observables exhibit generalised free field behaviour, characteristic of a weakly-coupled bulk description.

Beyond computational tractability, the AdS/CFT correspondence reveals that bulk locality and spacetime geometry are emergent concepts. At finite N, quantum gravity is nonlocal, and locality only emerges as an effective property in the classical limit. The correspondence therefore provides a mechanism to explain several puzzles in quantum gravity:

- the unitarity of the boundary theory, where information is preserved holographically, provides strong evidence for the unitarity of black hole evaporation in the context of the *black hole information paradox*.
- the *failure of Hilbert space factorisation* in gravitational systems becomes natural: the bulk Hilbert space does not factorise because spacetime itself is emergent.
- the *UV/IR correspondence* links short-distance (UV) physics in the CFT to large-radius (IR) behaviour in the bulk, with the AdS radial coordinate acting as a scale parameter.

One of the most striking developments in this context is the connection between entanglement and geometry. According to the Ryu–Takayanagi formula [78, 79], the entanglement entropy of a region A in the boundary theory is given by the area of a minimal surface in the bulk:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}\,, \tag{1.18}$$

where γ_A is a codimension-2 minimal surface anchored to the boundary of A. This relation implies that spacetime geometry is, at a fundamental level, woven from quantum entanglement in the boundary theory. Recent advances even suggest that Einstein's equations themselves may emerge from consistency conditions on entanglement entropy.

Maldacena's original proposal [77] sketched the outlines of a broader landscape of holographic dualities linking various backgrounds in string/M-theory to different CFTs:

- $\mathbf{AdS}_4 \times S^7$: this arises from the near-horizon limit of M2-branes in M-theory and is dual to the IR limit of a strongly-coupled $\mathcal{N} = 8$ superconformal field theory in 2+1 dimensions. A more tractable example is the ABJM theory [80], a particular type of Chern-Simons-Matter theory that captures the low-energy physics of multiple M2-branes.
- $\mathbf{AdS}_7 \times S^4$: this geometry arises from M5-branes in M-theory and is conjectured to be dual to the mysterious six-dimensional $(2,0)$ superconformal theory. Though it lacks a

Lagrangian description, its existence is supported by compactification arguments and anomaly matching [81, 82].

- **AdS**$_3 \times S^3 \times \mathcal{M}_4$: this background comes from the near-horizon limit of D1-D5 branes in type IIB string theory, where $\mathcal{M}_4$ is either T^4 or $K3$. The dual is a two-dimensional $\mathcal{N} = (4, 4)$ SCFT, often realised as the symmetric product orbifold $\mathrm{Sym}^N(\mathcal{M}_4)$ in certain limits [77, 83–85].

These examples span diverse spacetime dimensions and probe different corners of the string/M-theory landscape, yet they are all unified by a common holographic structure: a gravitational theory in $D = d + 1$-dimensional AdS space is fully equivalent to a conformal field theory in d dimensions. At the most general level, this duality takes the schematic form

$$Z_{\mathrm{AdS}_{d+1}}[\phi_{\mathrm{bdry}}] = Z_{\mathrm{CFT}_d}[\phi] = e^{i \int \mathrm{d}^d x\, \phi(x) \mathcal{O}(x)} \,, \tag{1.19}$$

where ϕ_{bdry} is the boundary value of a bulk field, acting as a source for a CFT operator $\mathcal{O}(x)$. Functional differentiation then yields correlation functions in the CFT:

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta^n Z_{\mathrm{AdS}}[\phi_{\mathrm{bdry}}]}{\delta\phi(x_1) \cdots \delta\phi(x_n)}\bigg|_{\phi=0} \,. \tag{1.20}$$

In Euclidean signature, the path integral becomes exponentially suppressed:

$$Z_{\mathrm{EAdS}_{d+1}}[\phi_{\mathrm{bdry}}] = Z_{\mathrm{ECFT}_d}[\phi] = e^{- \int \mathrm{d}^d x\, \phi(x) \mathcal{O}(x)} \,. \tag{1.21}$$

The AdS/CFT correspondence provides the most complete realisation of the holographic principle to date. It offers a non-perturbative definition of quantum gravity in AdS and has transformed our understanding of space, time, and information in gravitational systems.

Up to this point, we have expressed CFT correlators and generating functionals in position space. However, for cosmological applications, especially in inflationary perturbation theory and wavefunction-based approaches, it is often more convenient to work in Fourier space. This is because cosmological observables, such as power spectra and non-Gaussianities, are most naturally defined in terms of spatial momentum variables. Moreover, the analytic structure of wavefunction coefficients and the implementation of symmetry constraints (such as scale invariance) are often more transparent when expressed in momentum space. We now adopt this Fourier-space perspective as we turn to the dS/CFT correspondence and its interpretation in terms of the cosmological wavefunction.

Holographic Cosmology (dS/CFT)

Inspired by the AdS/CFT correspondence, holographic cosmology posits that quantum gravity in cosmological spacetimes — particularly in de Sitter (dS) space — can be described by a lower-dimensional conformal field theory (CFT), i.e. a dS/CFT correspondence. The dS/CFT correspondence was first proposed by Strominger in [86]. Strominger conjectured that quantum gravity in asymptotically Lorentzian de Sitter space is dual to a Euclidean CFT living on the spacelike future boundary $\mathcal{I}^+$ of dS (see Figure 1.2). This conjecture was based on the matching of isometries between de Sitter space and the Euclidean conformal group, and supported by the asymptotic behaviour of bulk fields near $\mathcal{I}^+$, which Strominger interpreted as defining boundary operators in the dual CFT. However, he did not frame the correspondence in terms of the WFU or a bulk gravitational path integral.

In contrast, Maldacena [87] later proposed a refinement of dS/CFT, where the Hartle-Hawking (or Bunch-Davies) WFU is computed via an analytic continuation of a Euclidean AdS path integral. This approach provides a more concrete realisation of dS/CFT by leveraging the better-understood gravitational path integral in AdS, providing a more precise bulk gravitational definition of the duality. In this framework, the WFU is interpreted as the generating functional for the dual Euclidean CFT. That is, in Fourier space, the wavefunction $\Psi[\varphi]$ for a bulk field with boundary value φ is given by

$$\Psi[\varphi] \equiv Z_{\mathrm{dS}}[\varphi] = \exp\left\{ -\sum_{n=2}^{\infty} \left[\prod_{a=1}^{n} \int \frac{\mathrm{d}^d \mathbf{k}_a}{(2\pi)^d} \varphi(\mathbf{k}_a) \right] \psi_n(\mathbf{k}) \right\}, \tag{1.22}$$

where the coefficients $\psi_n(\mathbf{k}) \equiv \psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ are the boundary wavefunction coefficients at future infinity. The resulting wavefunctional $\Psi[\varphi]$ is typically taken to be the Bunch–Davies state [88], which corresponds to the inflationary vacuum in QFT on curved spacetime, but also arises naturally from the gravitational path integral in quantum cosmology. In particular, it arises as the semiclassical limit of the Hartle–Hawking "no-boundary" wavefunction [8, 9], which is a proposed solution to the WDW equation in minisuperspace[7], and defined as a Euclidean path integral over compact 4-geometries with a specified boundary 3-metric γ (which of course can be generalised to higher dimensions),

$$\Psi_{\mathrm{HH}}[\gamma, \phi] = \int \mathrm{D}g\, \mathrm{D}\varphi\, e^{-S_E[g,\varphi]}\,. \tag{1.23}$$

This defines a wavefunctional over spatial geometries and field profiles on a final spacelike slice. In the semiclassical (WKB) regime where the path integral is dominated by classical saddles, one can write

$$\Psi[h_{ij}, \phi] \approx \exp\left(\frac{i}{\hbar} W[h_{ij}, \phi] \right), \tag{1.24}$$

where W is the Hamilton–Jacobi functional evaluated on the classical solution [17, 89]. This connects the no-boundary proposal to the Bunch–Davies wavefunctional, providing a conceptual bridge between quantum cosmology and quantum field theory in curved spacetime (see also [90] for a recent attempt to make this equivalence more precise). The Bunch–Davies wavefunctional is evaluated in terms of a field basis φ[8] usually living at the late-time inflationary boundary at $\eta_0 = 0^-$ (but in principle can be defined at any finite time slice η_0), i.e. it is given by the following QFT in curved spacetime path integral

$$\Psi_{\mathrm{BD}}[\eta_0; \varphi] = \int_{\phi_{\mathrm{BD}}(-\infty)=0}^{\phi(\eta_0)=\varphi} \mathrm{D}\phi\, e^{iS[\phi]} = \exp\left\{ -\sum_{n=2}^{\infty} \left[\prod_{a=1}^{n} \int \frac{\mathrm{d}^d \mathbf{k}_a}{(2\pi)^d} \varphi(\mathbf{k}_a) \right] \psi_n(\eta_0; \mathbf{k}) \right\}, \tag{1.25}$$

[7] Minisuperspace is an approximation that simplifies the infinite-dimensional space of possible geometries in GR to a finite-dimensional space. This is achieved by restricting the form of the spatial metric in superspace, focusing on a limited number of degrees of freedom, often functions of time only, i.e. $\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Omega_d^2$.

[8] Here φ can be any spinning field, hence for the case of pure gravity φ would correspond to boundary gravitons/metric fluctuations, with their indices suppressed.

and is conjectured to have the form of a generating functional of a Euclidean CFT, as expressed in the right hand side of (1.22). Notice that this parameterisation does not require any saddle-point approximation of the bulk path integral that defines Ψ. In fact, the wavefunction coefficients ψ_n can be found non-perturbatively from

$$\psi_n(\eta_0;\mathbf{k}) \equiv \psi_n'(\eta_0;\mathbf{k})(2\pi)^d\delta^d\left(\sum_{a=1}^{n}\mathbf{k}_a\right) = -\frac{\delta^n \log\Psi[\eta_0;\varphi]}{\delta\varphi_{\mathbf{k}_1}\cdots\delta\varphi_{\mathbf{k}_n}}\bigg|_{\varphi=0} = \langle\mathcal{O}(\mathbf{k}_1)\cdots\mathcal{O}(\mathbf{k}_n)\rangle_{\eta_0}\,, \quad (1.26)$$

where the prime in ψ_n' is used to explicitly highlight that these wavefunction coefficients contain the momentum-conserving δ-function[9] whilst those without the δ-function will be denoted by ψ_n. Upon renormalisation, ψ_n can be computed to any desired order in perturbation theory including any number of loops. Hence, in the context of dS/CFT, the wavefunction coefficients are the dS/CFT correlators, e.g. for the 2-point stress tensor correlator we have

$$\langle T_{ij}(\mathbf{k}_1)T_{lm}(\mathbf{k}_2)\rangle_{\eta_0} = -\frac{\delta^2\log\Psi[\eta_0;g_{ij}]}{\delta g_{ij}(\mathbf{k}_1)\delta g_{lm}(\mathbf{k}_2)}\bigg|_{g_{ij}=0} = \psi_2'(\eta_0;\mathbf{k})(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2)\,. \quad (1.33)$$

While this formulation bears a striking formal resemblance to the AdS/CFT dictionary, it diverges in crucial conceptual and technical ways. The most prominent distinction arises from the causal structure: in AdS, the boundary is timelike and Lorentzian, whereas in dS it is spacelike and Euclidean. This change alters the nature of the dual CFT, which is no longer unitary in the usual sense. Moreover, while the AdS partition function is related to boundary

[9]This is a result of translation invariance, which in position space, implies that correlation functions are invariant under spatial translations, i.e.,

$$\langle\mathcal{O}_1(\mathbf{x}_1+\mathbf{a})\cdots\mathcal{O}_n(\mathbf{x}_n+\mathbf{a})\rangle = \langle\mathcal{O}_1(\mathbf{x}_1)\cdots\mathcal{O}_n(\mathbf{x}_n)\rangle \quad (1.27)$$

for any constant vector $\mathbf{a}$. Now, if we consider the Fourier transform of an n-point function

$$\langle\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle = \int\prod_{a=1}^{n}\mathrm{d}^d x_a\, e^{-i\mathbf{k}_a\cdot\mathbf{x}_a}\langle\mathcal{O}_1(\mathbf{x}_1)\cdots\mathcal{O}_n(\mathbf{x}_n)\rangle\,, \quad (1.28)$$

and perform a simultaneous shift of all positions $\mathbf{x}_a \to \mathbf{x}_a + \mathbf{a}$. Translation invariance ensures that the correlator is unchanged, while the plane waves pick up a phase:

$$\prod_{a=1}^{n} e^{-i\mathbf{k}_a\cdot(\mathbf{x}_a+\mathbf{a})} = e^{-i\mathbf{a}\cdot\sum_a\mathbf{k}_a}\prod_{a=1}^{n} e^{-i\mathbf{k}_a\cdot\mathbf{x}_a}\,. \quad (1.29)$$

Since the position-space correlator is invariant, the only change is this overall phase. But this implies that the momentum-space correlator satisfies

$$\langle\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle = e^{-i\mathbf{a}\cdot\sum_a\mathbf{k}_a}\langle\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle\,, \quad (1.30)$$

which can only hold for all $\mathbf{a}$ if

$$\sum_{a=1}^{n}\mathbf{k}_a = 0\,. \quad (1.31)$$

Therefore, translation invariance implies that *momentum is conserved*, and this is encoded as a delta function enforcing momentum conservation

$$\langle\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle = (2\pi)^d\delta_{\mathrm{D}}^{(d)}\left(\sum_{a=1}^{n}\mathbf{k}_a\right)\times\langle\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle'\,, \quad (1.32)$$

where the reduced correlator $\langle\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle'$ contains the dynamical information not fixed by translation invariance.

correlators through a well-controlled semiclassical expansion, the dS wavefunction defines a probability distribution $|\Psi|^2 \equiv \Psi\Psi^*$ which is naturally related to "in-in" observables rather than an "in-out" amplitude. Consequently, the analytic continuation often employed to relate dS wavefunctions to Euclidean AdS partition functions complicates the physical interpretation, particularly when defining observables relevant to cosmology.

Another fundamental challenge is the status of unitarity. The boundary theory in dS/CFT is generally non-unitary due to its Euclidean signature, yet the bulk dynamics of de Sitter space appears to be unitary. It remains unclear how this bulk unitarity is encoded in the non-unitary boundary theory. Some proposals invoke reflection positivity [91–93] or other non-obvious analytic properties, but a complete understanding is lacking. This difficulty is compounded by the fact that the dual CFT is not fully specified in known examples: we do not yet have a definitive candidate dual theory of de Sitter gravity in the same sense as $\mathcal{N} = 4$ SYM for AdS_5.

Several open questions loom large over the dS/CFT programme. Foremost among them is the problem of definition: Is there a complete and consistent example of a holographic dual to de Sitter space? If so, what are its degrees of freedom, and how do they encode time evolution in a theory without an explicit time variable? The emergence of time from a Euclidean theory remains a profound puzzle. Relatedly, the operational interpretation of the WFU is not fully resolved: does it define physical probabilities, and if so, how does one extract them in a fully gauge-invariant way?

Other structural issues include the lack of a standard holographic renormalisation procedure for dS — a critical ingredient for matching correlators and RG flows in AdS — as well as the longstanding puzzle of de Sitter entropy. Since dS space has a finite Gibbons-Hawking entropy, it is difficult to reconcile this with the apparent infinite-dimensional nature of the boundary CFT. Furthermore, attempts to realise dS/CFT in string theory have met with resistance: despite much effort, there is no clear, stable example of a de Sitter vacuum in known string constructions. These obstacles raise the possibility that dS/CFT may be more of an effective or approximate framework, rather than an exact duality.

Nevertheless, a number of recent developments have revitalised interest in the dS/CFT paradigm. A prominent development is the cosmological bootstrap program [1, 29, 30, 45–59], which aims to reconstruct correlators in inflationary cosmology from first principles using symmetries, consistency conditions, and the analytic structure of the wavefunction, without appealing to time evolution. This approach has uncovered powerful constraints on allowed interactions and non-Gaussianities in the primordial universe, and dovetails naturally with the wavefunctional formulation of dS/CFT. Parallel progress has been made in formulating precise dictionaries between wavefunction coefficients and CFT correlators, including work on identifying conformal blocks and OPE structures in late-time cosmological data [91, 92, 94–98].

A particularly intriguing line of development involves higher-spin gravity, where concrete proposals exist for a dS/CFT duality between Vasiliev-like higher-spin theories in de Sitter and non-unitary $Sp(N)$-type models [24, 25, 99–102]. Although these models are far from realistic cosmologies, they provide rare examples where both sides of the correspondence can be studied explicitly, and serve as valuable laboratories for understanding how dS holography might work.

Taken together, these developments suggest that while the dS/CFT correspondence remains incomplete and speculative in many respects, it provides a powerful conceptual framework that may help illuminate the deep structure of quantum gravity in cosmological spacetimes. Whether or not a precise dual theory of de Sitter space exists, the formalism of the WFU, and its interpretation through the lens of holography, continues to offer fertile ground for both foundational and phenomenological insight.

Relation to the Cosmological CPT Theorem

The Cosmological CPT Theorem provides a foundational framework for understanding the interplay of the discrete symmetry of **CRT** with Scale Invariance and Unitarity in realistic cosmological settings. The key insight underlying the theorem is that the Poincaré patch of de Sitter allows for a natural analytic continuation between expanding and contracting patches. A global $SO^+(1,1) \subset SO^+(d+1,1)$ Lorentz boost in de Sitter becomes, when restricted to a single Poincaré patch, a scaling transformation $\mathbf{D}_\lambda$ acting as

$$(\eta, \mathbf{y}) \mapsto (\lambda\eta, \lambda\mathbf{y})\,. \tag{1.34}$$

The discrete 180° rotation corresponds to an analytic continuation $\mathbf{D}_{-1} : (\eta, \mathbf{y}) \mapsto (e^{\pm i\pi}\eta, e^{\pm i\pi}\mathbf{y})$, which we term Discrete Scale Invariance. In a realistic inflationary context where full de Sitter invariance (specifically SCTs), is broken, this discrete remnant survives and becomes central to identifying symmetry constraints on observables.[10] Remarkably, we find a $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure generated by **CRT** and $\mathbf{D}_{-1}$, which extends for the wavefunction to $\mathrm{Aut}(\mathbb{Z})$.

The Cosmological CPT Theorem relates these discrete symmetries via the following logic:

$$\text{Scale Invariance} + \text{Unitarity} \implies \textbf{CRT}\ \text{invariance}\quad ,$$

$$\text{Discrete Scale Invariance} + \text{Reflection Reality} \implies \textbf{CRT}\ \text{invariance}\quad ,$$

$$\textbf{CRT}\ \text{invariance} + \text{Discrete Scale Invariance} \implies \text{Reflection Reality}\quad ,$$

$$\textbf{CRT}\ \text{invariance} + \text{Reflection Reality} \implies \text{Discrete Scale Invariance}\quad .$$

Here, Reflection Reality (**RR**) is a fundamental requirement of bulk unitarity that all physically consistent theories must satisfy, and remarkably holds for wavefunctions of any flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology.

Non-perturbative Constraints on the Cosmological Wavefunction

We examine how these symmetries act on the WFU Ψ, and particularly on its Taylor-expanded coefficients ψ_n in (1.25), which encode bulk correlation functions (see e.g. [87, 89, 110–119]). A key result of the Cosmological CPT framework is that these constraints of ψ_n hold non-perturbatively, allowing us to go beyond previous results reliant on perturbative expansions [50, 52]. Throughout this thesis, we use "non-perturbative" to refer to results that hold non-perturbatively within a bulk QFT on a fixed curved spacetime background. Of course these results can be expressed

[10] In [103–109] the authors assume the existence of a CPT-reflected universe to derive observational constraints. Our Cosmological CPT theorem not only proves that **CRT** *is* a symmetry of the expanding universe, but also enables us to determine non-trivial constraints in the expanding Poincaré patch without any need for analytical continuation to a contracting one.

at each order in perturbation theory, but they have additional content if the amplitude is non-analytic in the coupling constant. That is, our constraints are not derived by truncating at any particular loop order, but rather follow from symmetry principles or the analytic structure of QFT in curved spacetime. However, we do not claim to include genuinely non-perturbative *quantum gravitational* effects — such as gravitational instantons, vacuum tunnelling, or large-N resummation — unless explicitly stated.

The action of the discrete symmetries on the wavefunction coefficients is:

$$\mathbf{CRT}: \quad [\psi_n(\eta; \mathbf{y}; \Omega)]^* = \psi_n(e^{-i\pi}\eta; -\mathbf{y}; e^{-i\pi}\Omega)\,, \tag{1.35}$$

$$\mathbf{D}^{\pm}_{-1}: \quad \psi_n(\eta; \mathbf{y}; \Omega) = \psi_n(e^{\pm i\pi}\eta; e^{\pm i\pi}\mathbf{y}; \Omega)\,, \tag{1.36}$$

$$\mathbf{RR}: \quad [\psi_n(\eta; \mathbf{y}; \Omega)]^* = \psi_n(\eta; -e^{i\pi}\mathbf{y}; e^{-i\pi}\Omega)\,, \tag{1.37}$$

where the complex analytic continuation of the Weyl factor Ω ensures these transformations stay within the original dS space (or, for the case of **RR**, any general flat FLRW spacetime). Importantly, because ψ_n arises from a single-sheeted path integral (unlike in-in correlators B_n), it can carry non-trivial phases under these continuations due to monodromies, especially around branch points in the complex η- and $\mathbf{y}$-planes. Given that **RR** represents a non-perturbative constraint that *any wavefunction arising from bulk unitary time evolution is required to satisfy*, this is the cosmological or dS/CFT analogue of reflection positivity, and thus answers one of the outstanding fundamental questions of how bulk unitary time evolution manifests itself in holographic cosmology, where the boundary theory is seemingly "non-unitary"!

Perturbative Constraints

We can also expand the non-perturbative identities above to arbitrary loop order in perturbation theory. In this context, we write the loop-expanded wavefunction coefficient as

$$\psi_n = \sum_{L=0}^{\infty} \psi_n^{(L)}\,, \tag{1.38}$$

where $\psi_n^{(L)}$ denotes the contribution from L-loop diagrams in the bulk. Taking the Fourier transform, we find:

$$\mathbf{CRT}: \quad \left[\psi_n^{(L)}(\eta_0; \mathbf{k})\right]^* = e^{i\pi(d+1)(L-1)}\psi_n^{(L)}(e^{-i\pi}\eta_0; \mathbf{k})\,, \tag{1.39}$$

$$\mathbf{D}^{\pm}_{-1}: \quad \psi_n^{(L)}(\eta_0; \mathbf{k}) = e^{\pm i\pi d}\psi_n^{(L)}(e^{\pm i\pi}\eta_0; e^{\mp i\pi}\mathbf{k})\,, \tag{1.40}$$

$$\mathbf{RR}: \quad \left[\psi_n^{(L)}(\eta_0; \mathbf{k})\right]^* = e^{i\pi((d+1)L-1)}\psi_n^{(L)}(\eta_0; e^{-i\pi}\mathbf{k})\,. \tag{1.41}$$

These relations hold for UV-finite amplitudes, but can be applied directly in the case of logarithmic divergences, by using dimensional regularisation (dim-reg); however, just as in flat space QFT, one must be careful and consistent with the choice of renormalisation scheme. Crucially, for odd d, the Ω-dependence in the non-perturbative symmetries becomes trivial, and tree-level diagrams respect the above identities exactly.

These results provide a precise and universal classification of the analytic continuation phases picked up by cosmological wavefunction coefficients under discrete symmetries. They can be used to predict interference patterns, identify allowed EFT interactions, and exclude certain non-local or non-unitary theories at the level of the wavefunction.

Implications for Holographic Cosmology

Phase of the Wavefunction. The **CRT** formula (1.39) has significant implications for holographic cosmology. By pushing the wavefunction coefficients to the future boundary where, for light fields, the time dependence factorises straightforwardly, and the resulting boundary wavefunction coefficient, $\overline{\psi}_n^{(L)}$, becomes independent of η. Thus, we can then deduce the phase of an arbitrary wavefunction coefficient

$$\arg(\overline{\psi}_n^{(L)}) = -\frac{\pi}{2}\left((d+1)(L-1) + dn - \sum_\alpha \Delta_\alpha\right) + \pi\mathbb{N}\,, \tag{1.42}$$

where the last term indicates that **CRT** can only fix the phase up to an overall $\pm$ sign. Δ is the conformal dimension of the n operators $\overline{\phi}_+$ used to differentiate the boundary wavefunction $\overline{\Psi}[\overline{\phi}_-]$, which is written in the basis of the conjugate operators $\overline{\phi}_-$ with dimension $d-\Delta$.

From a holographic cosmology or dS/CFT perspective (1.42) allows us to determine the phases of arbitrary n-point functions in the dual CFT. In dS/CFT, the objects with dimension $d-\Delta$ are the "sources" of the CFT partition function $Z_{\text{CFT}} = \overline{\Psi}$, while the ones with dimension Δ are the "operators". But from a cosmology perspective, both are operators in the bulk Hilbert space, and we have to choose a basis. In cosmology, one normally chooses this basis by writing $\overline{\Psi}$ as a function of whichever field component falls off more slowly, which would imply that $\Delta \geq d/2$, but the phase formula above would still be valid if we make the opposite choice.[11] This result holds for any boundary wavefunction coefficients that are UV and IR-finite (i.e. free from logarithmic divergences in η or the UV cutoff ϵ), as well as the coefficient in front of the leading $\log(\eta)$ divergence for any IR-divergent wavefunction coefficients, or the leading $\log(\epsilon)$ divergence for any UV-divergent diagram.[12]

When interactions are present, the conformal dimensions Δ_α of the boundary operators can acquire anomalous contributions, $\Delta_\alpha = \Delta_\alpha^{(0)} + \gamma_\alpha$, where γ_α denotes the anomalous dimension. In Chapters 6 and 7, we compute explicit examples of these corrections, and demonstrate that they modify the phase of the boundary wavefunction coefficient via (1.42)[13]. This leads to a precise and predictive relationship between the anomalous dimensions of operators in the dual theory and the phase of the wavefunction coefficients. In this sense, the imaginary parts of the wavefunction coefficients can serve as sensitive probes of the scale-dependence introduced by interactions, offering a novel observable window into the renormalisation structure of de Sitter holography.

It is important to emphasise that this constraint on the phase of boundary wavefunction coefficients is a genuinely cosmological phenomenon. In flat space, the S-matrix is invariant

[11] While an extension of these results to heavy fields in the principal series, where $\Delta = d/2 + i\nu$, we leave this for future work. Doing so would require making the unpleasant decision to either sacrifice self-adjointness, or conformal invariance of the boundary conditions. See e.g. [120–129] for discussions regarding the principal series.

[12] There is also an extra correction for diagrams with spinor propagators, see footnote 4.

[13] This observation is particularly useful in cases where the anomalous dimension arises at loop level, such as those computed in Chapters 6 and 7, and where the bulk UV divergences map to anomalous scaling in the boundary CFT.

under **CRT** symmetry, but it does not fix the relative phases of individual scattering amplitudes. This is because Lorentz boosts in flat spacetime, including the $SO^+(1,1)$ subgroup, act non-trivially on the asymptotic boundary at null infinity: they shift the support of wavepackets and change the definition of in/out states. By contrast, in cosmology, the analogue of a boost acts as a conformal rescaling that preserves the future boundary of de Sitter space. This crucial geometric difference implies that the Cosmological CPT theorem is able to constrain the phase of wavefunction coefficients in a way that is not possible in flat space. The future boundary provides a fixed hypersurface on which **CRT** acts internally, rather than mapping between disconnected asymptotic regions. This makes the Cosmological CPT theorem a more refined diagnostic of unitarity and scale-dependence in quantum gravity than its flat space counterpart.

If we restrict attention to the case where all particles have $\Delta = d$, as for the stress-tensor and massless scalar fields, we find the phase is independent of n:

$$\arg(\overline{\psi}_n^{(L)}) = -\frac{\pi}{2}(d+1)(L-1) + \pi\mathbb{N}\,. \tag{1.43}$$

This then tells us that massless scalar fields and gravitons have real boundary wavefunction coefficients to all loop order in even spacetime dimension provided they don't diverge in the UV. When UV divergences are present, one finds in dim-reg a generic complex phase proportional to the dim-reg parameter δ, i.e. $\arg(\overline{\psi}_n^{(L)}) \propto \pi\delta$, but this is dependent on the scheme one chooses [3, 67, 75, 76]. For tree-level, IR-finite and scale-invariant Feynman-Witten diagrams in cosmology, this was recently found independently [130, 131] where the authors proved this reality condition by direct analysis of the bulk time integrals.[14] The implications of this for cosmology in terms of no-go theorems for parity violations has previously been explored for the specific case of the tree-level scalar trispectrum in [74, 132], and will be treated for more general tree-level and loop diagrams in Chapter 7 which is based on the results of [3]. This is also consistent with what is found in flat space amplitudes which can be shown to be real at tree-level, see e.g. [133].[15]

This lends hope that **CRT** symmetry may help resolve some of the deeper puzzles in quantum gravity in de Sitter space.

Central charge of dS/CFT. The **CRT** phase formula also gives further insight into the manifestation of bulk unitarity in de Sitter quantum gravity (dS/CFT), where previous attempts to identify imprints of bulk unitarity in the boundary theory have proven elusive (see e.g. [86, 99]), as it determines the phase of the required number of degrees of freedom, defined by the central charge c_{dS}, of the holographic CFT in de Sitter. Given the close relationship between **CRT** and unitarity, and the fact that **CRT** simplifies at the future boundary $\mathcal{I}^+$, this gives some reason to think that bulk unitarity corresponds to some equally natural restriction on the boundary CFT side. At the very least, if somebody claims to have an example of dS/CFT holography, the phase formula provides a simple test for ruling out bulk theories that violate Reflection Reality (**RR**), without requiring any calculations that probe the interior bulk dynamics. Then, to ensure that the bulk theory is fully unitary, the remaining task is to check by hand that the

[14]The author would like to express his gratitude to David Stefanyszyn, Xi Tong and Yuhang Zhu for various extensive and helpful discussions on reality in cosmology.
[15]The author thanks Carlos Duaso Pueyo for making him aware of this result in amplitudes.

bulk Lagrangian contains no negative norm fields. Success at this task would not require any additional fine-tuning of continuous parameters, although it might require making good discrete choices.[16]

There are a number of possible definitions of c_{dS}, including:

1. various coefficients of the trace anomaly $\langle T^i_i \rangle$ in $\overline{\psi}_1^{(L)}$, when $d =$ even, or
2. the coefficient of the 2-point stress tensor correlator $\langle T_{ij}(\mathbf{k}) T_{lm}(-\mathbf{k}) \rangle$ in $\overline{\psi}_2^{(L)}$ for general d,

where L is the loop order in the bulk theory, not the boundary CFT. These definitions coincide with the Virasoro central charge c when $d = 2$, but differ in arbitrary d. Happily, the phase formula agrees for all these indicators,[17] so regardless of the definition of c_{dS}, one finds that

$$\arg(c_{\text{dS}}) = \arg(\overline{\psi}_n^{(L)}) = -\frac{\pi}{2}(d+1)(L-1) + \pi\mathbb{N}\,. \tag{1.44}$$

From (1.44) we can infer that for $d = 2, 4, \ldots$, the central charge is imaginary at tree level (but real at odd-loop order), while for $d = 1, 3, 5, \ldots$ it is real at all orders (see Section 6.4.5). This matches what has been found in the literature:

- For $d = 2$ at tree-level, $c = i\frac{3\ell}{2G_N}$ (e.g. from applying Friedel's analysis to dS [10]); while for the case of pure gravity at 1-loop order we have $c = i\frac{3\ell}{2G_N} + 13 + \mathcal{O}(G_N)$[18] [135];[19]
- For $d = 3$ at tree-level, $c_{\text{dS}} \propto -\frac{\ell^2}{G_N} + \mathcal{O}(1)$ (e.g. in higher-spin dS/CFT [24] or Maldacena [87]);
- For $d = 4$ at tree level, $c_{\text{dS}} \propto -i\frac{\ell^3}{G_N} + \mathcal{O}(1)$ (e.g. Maldacena [87]).
- For $d = 5$ we have $c_{\text{dS}} \propto +\frac{\ell^4}{G_N} + \mathcal{O}(1)$, the same sign as a normal unitary CFT [136].

Although the phase formula does not determine the real sign of c_{dS}, this can be fixed by the stability requirement that $G_N > 0$, and one finds that the sign of the tree-level c_{dS} has a repeating pattern in $d \bmod 4$ [136], as represented by the four cases above. This compatibility with previous results, as well as the results from more non-trivial explicit checks (forthcoming in [137] and summarised in Section 6.4.5), confirm the validity of the phase formula (1.42).

Crucially, the phase formula provides a natural method to reconstruct the complex de Sitter wavefunction from its modulus-squared $|\Psi|^2$, which is often interpreted in terms of a

[16] Of course, there are other consistency conditions that dS/CFT should obey besides bulk unitarity. This includes normalisability conditions on the wavefunction Ψ, which can be used to fix the sign of the real part of typical ψ_2 coefficients. A particularly strong constraint is the absence of bulk tachyons, which seems to require the absence of irrelevant single-trace operators $\Delta > d$ in the boundary field theory.

[17] The phase also agrees with the sphere free energy $F = \log Z$ in $d =$ odd, see [134] for a pedagogical review.

[18] The renormalisation of Newton's constant G_N is scheme dependent, meaning its exact running or loop corrections depend on the choice of renormalisation procedure. In (A)dS$_3$, when one reformulates the gravitational path integral (GPI) using a boundary Schwarzian-type description, there exists a localisation result (analogous to the Duistermaat–Heckman theorem) which implies that, in that specific scheme, the renormalisation of G_N stops at one loop. However, this one-loop exactness is not universal and might not hold in other renormalisation schemes. Importantly, the physical quantity is the boundary central charge (which appears in the dual CFT), not the bare G_N itself. Therefore, as is standard in QFT, one should treat G_N as a scheme-dependent parameter that is adjusted (renormalised) to keep physical observables—like the central charge—fixed.

[19] The author thanks Jordan Cotler and Kristan Jensen for making him aware of their previous result for the 1-loop contribution to the central charge in dS_3 as well as upcoming work for loop calculations in dS_2 and dS_4. The more observant reader will notice that the real 1-loop contribution is positive in the convention chosen by the authors as predicted from the combination of the normalisability constraint and the phase formula.

doubled Euclidean CFT. We will return to a more detailed discussion of this point, including its implications for dS_3/CFT_2 dualities and complex central charges, in Section 6.4.5.

In summary, the Cosmological CPT Theorem provides a symmetry-based foundation for understanding the analytic structure of cosmological wavefunctions. Its implications range from the consistency of the dS/CFT correspondence, to constraints on the phase structure of cosmological correlators, to no-go theorems for certain parity-violating interactions. This framework therefore offers a robust and testable approach to quantum gravity in spacetimes with a positive cosmological constant.

Structure of Thesis

Natural units with $\hbar = c = 1$ will be used throughout this thesis for simplicity.[20] Unless otherwise stated, we will work in a real basis of fields and focus on bosonic fields with integer spin. Spinors and fermions will not be considered in general, though they will be referenced explicitly at certain points. The term "non-perturbative" will be used in the sense defined earlier — namely, to refer to results that hold non-perturbatively within QFT on a fixed curved spacetime background (and which can be expressed to all orders in perturbation theory), without implying inclusion of genuinely quantum gravitational effects such as gravitational instantons or or large-N resummation.

The remainder of this thesis is organised as follows.

Chapter 2 lays the groundwork by introducing the wavefunction formalism for computing cosmological boundary observables. Within this framework, we systematically classify all cubic interactions that contribute to the tree-level graviton bispectrum in single-field inflation. We show that, up to field redefinitions, these interactions are fully captured by Effective Field Theory of Inflation (EFToI) operators containing up to three powers of the extrinsic curvature and its covariant derivatives. Despite the apparent complexity, the resulting bispectra simplify into elegant closed-form expressions.

In Chapter 3, we revisit the CPT theorem in flat spacetime from a fresh perspective. We challenge the standard identification of the combined **CRT** symmetry with a 180° Euclidean rotation, emphasising that an antilinear symmetry cannot arise solely from a linear geometric transformation. This leads us to identify a previously unrecognised discrete antiunitary symmetry, Reflection Reality (**RR**), which emerges from the unitarity structure of quantum theory. Together, **CRT**, **RR**, and the Euclidean rotation $\mathbf{R}_\pi$ form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. This group-theoretic viewpoint not only clarifies the role of unitarity and Lorentz invariance in the CPT theorem but also yields new converse results connecting these symmetries in ways not previously explored.

Building on this, Chapter 4 extends this discrete symmetry framework into de Sitter spacetime. By exploiting the embedding of dS into a higher-dimensional Minkowski space, we precisely characterise the action of **CRT**, **RR**, and $\mathbf{R}_\pi$ on both global and Poincaré patches. This uncovers a universal $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group that constrains late-time cosmological correlators, with $\mathbf{R}\pi$ interpreted as an analytic continuation of dilation symmetry relevant for inflationary physics. Importantly, this symmetry structure acts differently at the level of the Wavefunction

[20]Setting $\hbar = c = 1$ simplifies equations by measuring mass, length, and time in compatible units; where necessary, factors of $\hbar$ and c will be restored for clarity.

of the Universe, revealing a subtle tension between spacetime symmetries and local Lagrangian symmetries.

This tension is the focus of Chapter 5, where we carefully distinguish global spacetime symmetries from local symmetries of the inflationary Lagrangian. We show how discrete symmetries such as **CRT** and a discrete scale inversion symmetry $\mathbf{D}_{-1}$ can be realised as local Lagrangian symmetries in the Poincaré patch, thereby constraining the wavefunction coefficients ψ_n. Remarkably, we prove that Reflection Reality (**RR**) alone suffices to guarantee bulk unitarity in many cases. We then establish new converse results, showing that **CRT** invariance combined with discrete scale invariance implies **RR**, deepening the interplay between these fundamental symmetries in cosmology.

The central contribution of this thesis appears in Chapter 6, where we formulate and prove the *Cosmological CPT Theorem*. This theorem provides a non-perturbative characterisation of CPT invariance in an inflationary universe, establishing that **CRT** symmetry combined with discrete scale invariance is equivalent to the unitarity of the Wavefunction of the Universe, and vice versa. The result generalises traditional CPT arguments to a single Poincaré patch, and highlights the unique role of **RR** as the only symmetry automatically guaranteed by any unitary quantum field theory on a flat FLRW background.

Chapter 7 then explores the phenomenological implications of these symmetries, focusing on parity violation in cosmological correlators. We prove a no-go theorem demonstrating that, in even spacetime dimensions, **CRT** symmetry forbids parity-odd correlators for a large class of inflationary models; along the way we establish a cosmological analogue of Furry's theorem. However, the situation differs markedly in odd dimensions, where parity-violating correlators can exist. This chapter thereby delineates precise conditions under which parity violation can or cannot arise, sharpening predictions for observational signatures of fundamental physics.

Finally, Chapter 8 summarises the key findings and discusses potential avenues for future research.

In appendix A, we determine all the relevant curvature and connection data within the EFToI, which will then use in Section 2.2 to show — for the first time — that all EFToI operators contributing to graviton bispectra can be constructed purely from operators containing two or three powers of the extrinsic curvature and its covariant derivatives.

"The effort to understand the universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy."
— Steven Weinberg, The First Three Minutes

2

Graviton non-Gaussianities in the Effective Field Theory of Inflation

In this chapter, we derive parity-even graviton bispectra (3-point functions) in the Effective Field Theory of Inflation (EFToI), working to all orders in derivatives. Using perturbation theory, we systematically construct all possible cubic interactions that can contribute to the tree-level graviton bispectrum. We show that, up to field redefinitions, these interactions originate solely from EFToI operators involving two or three powers of the extrinsic curvature and its covariant derivatives; all other operators either contribute at higher orders in perturbations or can be removed by field redefinitions.

For the case of operators cubic in the extrinsic curvature — which preserve the single-clock consistency relations without modifying the graviton two-point function — we use the Manifestly Local Test (MLT) to efficiently extract the boundary correlators from bulk evolution. Despite the complexity of the bulk interactions, the resulting graviton bispectra take a particularly compact and elegant form.

For operators quadratic in the extrinsic curvature, the leading-order graviton bispectrum consists of a sum of contact and single-exchange diagrams, whose structure is constrained by spatial diffeomorphism invariance. We derive these bispectra to all orders in derivatives by explicitly computing the relevant bulk time integrals. In the case of single-exchange diagrams, we exploit the factorisation properties of the bulk-bulk propagator for massless gravitons and express the final result as a finite sum over residues. Somewhat surprisingly, we find that these single-exchange contributions exhibit only total-energy poles and also satisfy the MLT.

2.1 Bootstrapping Graviton non-Gaussianities

2.1.1 Bootstrapping cosmological correlators

In cosmology the observables in our night sky are correlation functions known as "cosmological correlators". We aim to measure these through the CMB and LSS, which we can trace back to the boundary from the end of a period of exponential expansion in the early universe known as inflation. The energy scale during inflation is orders of magnitude larger than those we can probe on Earth with the LHC or any other conceivable colliders in the foreseeable future ∼TeV with an upper bound of $\leq 10^{15}$ GeV. Particle physicists use the final readings after particle collisions to infer properties of high energy particle physics. As cosmologists we can use cosmological correlators as our final readings to understand the physics of the early universe and to possibly constrain quantum gravity in ways which would otherwise be inaccessible to us through the low-energy experiments run on Earth. Therefore, understanding inflation is currently our best avenue at probing quantum gravity. It is also worth noting that there have been a lot of push towards finding the imprints of these correlation function within the LSS thanks to upcoming telescopes, such as Euclid and JWST, which have motivated advancements both in the EFToLSS and the efficiency of computation of higher-point correlation functions of galaxies.

For cosmological observables we do not compute "in-out" scattering amplitudes as we do for flat space, instead we are interested in "in-in" expectation values which live at the boundary of an approximate de Sitter space which we identify with the end of inflation, where the limit for conformal time is saturated $\eta \to \eta_0 \approx 0$. This a consequence of the fact that at late times, cosmological perturbations in general evolve and interact with each other, hence we cannot assume that the state of the universe at late times is a superposition of free states, as we do for particle scattering.

Our usual methods of writing down a Lagrangian for the scalar field which drove inflation, commonly referred to as the inflaton, allows for a plethora of phenomenological models for inflation. This results in a huge amount of redundancy when going from theory to observables, for example different Lagrangians can be related to each other by gauge transformations or field redefinitions. In addition, we introduce time evolution in the bulk of the inflationary spacetime during the standard perturbative calculation, despite the fact we only have access to correlators at the end of inflation and never observe time evolution of perturbations during inflation directly. The traditional computational process requires complicated (nested) time integrals to be performed. Although these can be computed, they can obscure the origin of analytic properties of the final answer, and prevents us from understanding the role of fundamental principles, such as unitarity, locality and symmetry. For the purposes of the paper [1], which this chapter is based on we use a combination of these "bulk" time integrals and bootstrap methods, giving us a partial understanding of the correlators' analytical structure. Future work will involve deriving the results using purely bootstrap methods to make this more precise.

An analogous issue arose in particle physics. The role of fundamental principles is hard to observe when calculating scattering amplitudes from Lagrangians in perturbation theory, and once again there are many redundancies. As a result, the on-shell method was developed to describe all physically consistent S-matrices [60–65]. Recently, progress has been made in using fundamental principles as constraints on the late-time correlators which live at the boundary from the end of inflation - The Cosmological Bootstrap [59]. In the cosmological bootstrap program, we use symmetries to constrain the form of the cosmological correlators which have no reference to time in them, i.e. "bootstrapping time". Any physically consistent correlator should obey locality, unitarity and the symmetry of the inflationary spacetime. This is a more powerful way to obtain the necessary observables without having to compute them explicitly, and has produced many interesting results in the past few years. For example, it is possible to completely constrain the form of the three-point correlator of a massless scalar by Bose symmetry, manifest locality and unitarity [29, 49].

2.1.2 Wavefunction of the Universe

Inflation is believed to be a *quasi-de Sitter* expansion. The fact that we have a boundary at $\eta = \eta_0$, gives us the opportunity to treat these calculations in a holographic way, where we have the correlators as the boundary observables related to the physics happening in the bulk spacetime. This approach is motivated by our knowledge of the AdS/CFT correspondence, where one can relate dynamical quantum gravity in AdS (anti-de Sitter) space to CFT correlators in a one-lower-dimensional space living at the boundary (such as AdS_3 and CFT_2). In the case of inflationary cosmology we work in a limit where the inflaton has decoupled from gravity, i.e. the dynamical modes of gravity have been integrated out from the Wavefunction of the Universe (WFU) solution to the Wheeler-DeWitt (WDW) equation [89]. The overall goal of quantum cosmology is to obtain a non-perturbative formulation of quantum gravity, i.e. a dS/CFT correspondence, where it is suspected that the emergent dimension would be time, in the same way there is an emergent spatial dimension in the AdS/CFT correspondence.

The WFU is a helpful object to work with in the cosmological bootstrap program. Given that in cosmology we are doing QFT in dS for a generic field ϕ evaluated at conformal time η_0, we can express the wavefunction as the path integral

$$\Psi[\bar{\phi};\eta_0] = \int_{\phi_{\rm BD}(-\infty)=0}^{\phi(\eta_0)=\bar{\phi}} \mathrm{D}\phi \mathrm{D}\pi e^{i\int \mathrm{d}^4x\, [\phi'\pi - \mathcal{H}(\phi,\pi)]} \,, \tag{2.1.1}$$

where for spinning fields indices are suppressed. For the path integral we impose the boundary conditions that the field in the infinite past started from a Bunch-Davies vacuum and takes on a value $\bar{\phi}$ at conformal time η_0. The Bunch-Davies vacuum state initial condition proves to have powerful consequences [29, 50–52, 67] and comes from the assumption that at very early times the mode functions are those of the flat-space theory. Physically this is because at very high energies the modes do not feel the expansion of the universe.

Like the wavefunction of any system, the WFU contains all the information of our universe, and at late-times, the wavefunction has an expansion in the late-time value of the spin-2 field $\gamma_{ij} \equiv \gamma_{ij}(x,\eta_0)$, we can parameterise Ψ as

$$\Psi[\gamma_{ij};\eta_0] = \exp\left[-\sum_{n=2}^{\infty}\frac{1}{n!}\int\left(\prod_a^n \sum_{h_i=\pm} \mathrm{d}^3\mathbf{k}_a\, \psi_n^{h_1,\dots,h_n}(\mathbf{k_1}\dots\mathbf{k_n})\, (2\pi)^3\delta^3\left(\sum \mathbf{k}_a\right)\gamma^{h_i}(\mathbf{k}_a)\right)\right], \tag{2.1.2}$$

where $\gamma^h(\mathbf{k})$ is the spin-h Fourier mode of the graviton with helicity $h = \pm 2$

$$\gamma_{ij}(x,\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{h=\pm} e_{ij}^h(\mathbf{k})\gamma_h(\mathbf{k},\eta)\,. \tag{2.1.3}$$

The *wavefunction coefficients* $\psi_n(\mathbf{k}_1\dots\mathbf{k}_n)$ which contain the dynamical information about the bulk processes have an implicit helicity h-dependence. Note that translation invariance ensures that the $\psi_n(\mathbf{k}_1\dots\mathbf{k}_n)$ always contain a momentum conserving delta function and so we can write

$$\psi_n(\mathbf{k}_1 \ldots \mathbf{k}_n) = \psi_n'(\mathbf{k}_1 \ldots \mathbf{k}_n)(2\pi)^3\delta^3(\mathbf{k}_1 + \ldots + \mathbf{k}_n). \tag{2.1.4}$$

We will often drop the prime even when we do not explicitly include the delta function. Given that we are doing QFT in curved spacetime, we can do a saddle point approximation of the path integral where the wavefunction is completely fixed by the value of the action evaluated on classical solutions

$$\Psi[\eta_0, \phi(\mathbf{k})] \approx e^{iS_{\rm cl}[\phi(\mathbf{k})]}. \tag{2.1.5}$$

The free action of a massless graviton is

$$S_{\gamma_{ij},\text{free}} = \left(\frac{M_{\rm pl}}{2}\right)^2 \int \mathrm{d}\eta\, \mathrm{d}^3\mathbf{x}\, a^2(\eta) \frac{1}{2}\left[\gamma_{ij}'^2 - \partial_l\gamma_{ij}\partial_l\gamma_{ij}\right], \tag{2.1.6}$$

which for the case of a flat FLRW universe is the same as the action for a massless scalar field ϕ, up to a normalisation factor of $M_{\rm pl}/2$. Working in momentum space, we write the quantum free field operator as

$$\hat{\gamma}_h(\mathbf{k},\eta) = \gamma_h^-(k,\eta)a_{\mathbf{k}} + \gamma_h^+(k,\eta)a_{-\mathbf{k}}^\dagger, \tag{2.1.7}$$

where the mode functions $\gamma_h^\pm(k,\eta)$ correspond to solutions of the free classical equation of motion and are given by

$$\gamma_h^\pm(\mathbf{k},\eta) = e_{ij}^h(\mathbf{k})\frac{H}{2M_{\rm pl}}\phi^\pm(k,\eta) = e_{ij}^h(\mathbf{k})\frac{H}{2M_{\rm pl}\sqrt{2k^3}}(1 \mp ik\eta)e^{\pm ik\eta}. \tag{2.1.8}$$

The polarisation tensors $e_{ij}^h(\mathbf{k})$ distinguish graviton bispectra from scalar bispectra, and satisfy the usual properties:

$$e_{ii}^h(\mathbf{k}) = k^i e_{ij}^h(\mathbf{k}) = 0 \qquad \text{(transverse and traceless)}, \tag{2.1.9}$$
$$e_{ij}^h(\mathbf{k}) - e_{ji}^h(\mathbf{k}) = 0 \qquad \text{(symmetric)}, \tag{2.1.10}$$
$$e_{ij}^h(\mathbf{k})e_{jk}^h(\mathbf{k}) = 0 \qquad \text{(lightlike)}, \tag{2.1.11}$$
$$e_{ij}^h(\mathbf{k})e_{ij}^{h'}(\mathbf{k})^* - 4\delta_{hh'} = 0 \qquad \text{(normalisation)}, \tag{2.1.12}$$
$$e_{ij}^h(\mathbf{k}) - [e_{ij}^h(-\mathbf{k})]^* = 0 \qquad (\gamma_{ij}(x) \text{ is real}). \tag{2.1.13}$$

The mode functions for graviton fluctuations take the same form as massless scalar mode functions (with sound speed $c_s = 1$[1]) with the addition of a normalisation factor of $M_{\rm pl}/2$ and polarisation tensors $e_{ij}^h(\mathbf{k})$, with $h = \pm 2$, as required by little group scaling. This is because for each polarisation mode the equation of motion is that of a massless scalar.

$$S_{\gamma_{ij},\text{free}} = \sum_{h=\pm 2}\left(\frac{M_{\rm pl}}{2}\right)^2 \int \mathrm{d}\eta\, \mathrm{d}^3\mathbf{k}\, a^2(\eta)\left[(\gamma_{\mathbf{k}}^h)'(\gamma_{\mathbf{k}}^h)' - k^2\gamma_{\mathbf{k}}^h\gamma_{\mathbf{k}}^h\right], \tag{2.1.14}$$

[1]Gravitons have a sound speed $c_s = 1$, but for a scalar field we can allow for an arbitrary, constant speed of sound c_s by sending $k \to c_s k$ which signals the fact we are allowing for dS boosts to be spontaneously or explicitly broken. When the speed of sound differs from the speed of light appearing in the metric, $c_s \neq 1$, the sound cone is not invariant under de Sitter boosts, a fact which can be simply seen in the flat-space limit, where de Sitter boosts reduce to Lorentz boosts.

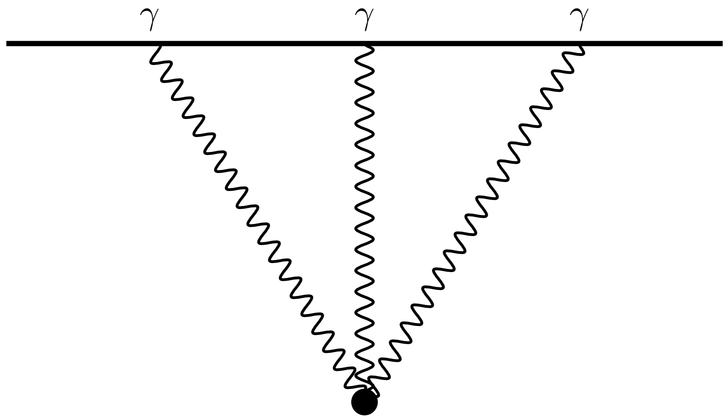

Fig. 2.1 Cubic contact diagram.

where we have introduced the notation $\gamma_k^h(\eta) \equiv \gamma^h(k,\eta)$ to shorten the expressions. Traditionally, one computes $S_{\rm cl}[\phi(\mathbf{k})]$ from (2.1.5) in perturbation theory using Feynman diagrams which involve bulk interaction vertices, bulk-boundary propagators $K(k,\eta)$ and bulk-bulk propagators $G(k,\eta,\eta')$. To illustrate the Wavefunction of the Universe method, let us focus on a single massless scalar ϕ. The generalisation to gravitons simply requires the addition of $SO(3)$ indices where appropriate and we will elaborate on this more in Section 2.3.1. If we denote the scalar's free equation of motion as $\mathcal{O}(k,\eta)\phi = 0$, then these propagators satisfy

$$\mathcal{O}(k,\eta)K(k,\eta) = 0, \tag{2.1.15}$$

$$\mathcal{O}(k,\eta)G(k,\eta,\eta') = -\delta(\eta-\eta'), \tag{2.1.16}$$

with boundary conditions

$$\lim_{\eta\to\eta_0} K(k,\eta) = 1, \qquad \lim_{\eta\to-\infty(1-i\epsilon)} K(k,\eta) = 0 \tag{2.1.17}$$

$$\lim_{\eta,\eta'\to\eta_0} G(k,\eta,\eta') = 0, \qquad \lim_{\eta,\eta'\to-\infty(1-i\epsilon)} G(k,\eta,\eta') = 0. \tag{2.1.18}$$

We can then write both propagators in terms of the positive and negative frequency mode functions as

$$K(k,\eta) = \frac{\phi_k^+(\eta)}{\phi_k^+(\eta_0)}\,, \tag{2.1.19}$$

$$\begin{aligned} G(p,\eta,\eta') &= i\left[\theta(\eta-\eta')\left(\phi_p^+(\eta')\phi_p^-(\eta) - \frac{\phi_p^-(\eta_0)}{\phi_p^+(\eta_0)}\phi_p^+(\eta)\phi_p^+(\eta')\right) + (\eta\leftrightarrow\eta')\right] \\ &= iP(p)\left[\theta(\eta-\eta')\frac{\phi_p^+(\eta')}{\phi_p^+(\eta_0)}\left(\frac{\phi_p^-(\eta)}{\phi_p^-(\eta_0)} - \frac{\phi_p^+(\eta)}{\phi_p^+(\eta_0)}\right) + (\eta\leftrightarrow\eta')\right], \end{aligned} \tag{2.1.20}$$

where $P(p)$ is the power spectrum of ϕ. Note that for our purposes we are interested in the graviton power spectrum $P_\gamma(k)$ which is related to the quadratic wavefunction coefficient ψ_2 computed using the two-derivative quadratic action coming from GR in Eqn (2.1.14)

$$P_\gamma(k) = \frac{H^2}{M_{\rm pl}^2}\frac{1}{2k^3}\delta_{hh'} = \frac{1}{2{\rm Re}\psi_{2,{\rm GR}}^{hh'}(k)}\,. \tag{2.1.21}$$

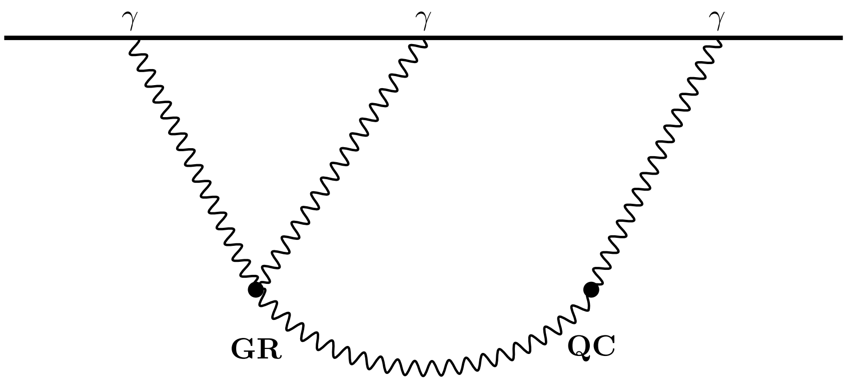

Fig. 2.2 Single exchange diagram consisting of a single cubic interaction connected to a quadratic correction (QC) vertex by a bulk-bulk propagator.

Now to extract the wavefunction coefficients ψ_n one follows the usual Feynman rules. For our purposes, we will be working to tree-level in perturbation theory, i.e. contact and exchange Feynman diagrams, specifically cubic contact and single-exchange diagrams of the form shown in Figs. 2.1 and 2.2.[2] When computing the time integrals for these Feynman diagrams, we integrate from the far past at $\eta = -\infty$ to the future boundary at $\eta = \eta_0$. The boundary condition on the fields, that they vanish in the infinite past ($\eta \to -\infty$) leads to attributing an $i\epsilon$-prescription to the magnitude of the spatial momenta, i.e. $k = \sqrt{k \cdot k} = \tilde{k}(1 - i\epsilon) \in \mathbb{C}$, where $\tilde{k} \in \mathbb{R} \geq 0$. This $i\epsilon$ prescription ensures that there is a short period of evolution that dampens the exponential factors appearing in the integral, thereby projecting the theory onto the vacuum state [8, 87, 88, 90].

The wavefunction coefficients ψ_n correspond to the usual cosmological correlators B_n.

$$B_n(\mathbf{k}_1 \ldots \mathbf{k}_n)(2\pi)^3\delta^3\left(\sum \mathbf{k}_a\right) \equiv \langle \phi(\mathbf{k}_1) \ldots \phi(\mathbf{k}_n) \rangle = \frac{\int \mathrm{D}\phi \Psi \Psi^* \phi(\mathbf{k}_1) \ldots \phi(\mathbf{k}_n)}{\int \mathrm{D}\phi \Psi \Psi^*}. \tag{2.1.22}$$

For example, we hope to use the CMB to see tensor modes soon, and from this measure the graviton bispectrum which is what this chapter focuses on and is related to the 3-point wavefunction coefficient ψ_3 by

$$B_3(\{k\}, \{\mathbf{k}\}) = -\frac{\psi_3(\{k\}, \{\mathbf{k}\}) + \psi_3^\star(\{k\}, \{-\mathbf{k}\})}{\prod_{a=1}^3 2\mathrm{Re}\ \psi_2(k_a)}. \tag{2.1.23}$$

We can write a general three-point wavefunction coefficient ψ_3 in terms of a polarisation factor multiplied by a "trimmed" wavefunction coefficient ψ_3^{trimmed} which is an $SO(3)$ scalar and can be computed from bulk time integrals (and thus can be treated separately)

$$\psi_3^{h_1,h_2,h_3}(\{k\}, \{\mathbf{k}\}) = \sum_{\text{contractions}} [e^{h_1}(\mathbf{k}_1) e^{h_2}(\mathbf{k}_2) e^{h_3}(\mathbf{k}_3) \mathbf{k}_1^{\alpha_1} \mathbf{k}_2^{\alpha_2} \mathbf{k}_3^{\alpha_3}] \times \psi_3^{\text{trimmed}}(\{k\}) \quad . \tag{2.1.24}$$

Here we define the total number of spatial momenta contracted with polarisation factors as $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, and index contractions between the momenta and polarisation tensors are left

[2] For brevity, we will not review the Feynman rules here and instead refer the reader to [38, 49, 50, 67] for more details and examples, as well as a Mathematica notebook written by the author to compute Feynman diagrams in cosmology.

implicit. There can be additional spatial momenta but they are contracted amongst themselves. In order to construct an SO(3)-invariant object, we need to contract momenta with one of the red or blue tensor structures in (2.1.25).

$$e^{h_1}_{i_1 i_2} e^{h_2}_{i_3 i_4} e^{h_3}_{i_5 i_6} \text{ (even)} \quad \text{or} \quad \epsilon_{i_1 i_2 i_3} e^{h_1}_{i_4 i_5} e^{h_2}_{i_6 i_7} e^{h_3}_{i_8 i_9} \text{ (odd)} \quad . \tag{2.1.25}$$

For parity-even $\alpha \in 2\mathbb{Z} \leq 6$ and for parity-odd $\alpha \in 2\mathbb{Z}+1 \leq 7$, e.g. $e^{h_1}_{lk} e^{h_2}_{mk} e^{h_3}_{ij} k^i_1 k^j_2 k^l_3 k^m_3$ for $\alpha = 4$ and $\epsilon_{ijk} e^{h_1}_{mn} e^{h_2}_{qp} e^{h_3}_{lk} k^i_1 k^j_2 k^m_2 k^n_2 k^q_1 k^p_1 k^l_1$ for $\alpha = 7$. where the presence of a Levi-Civita tensor in the tensor structure tells us that the resulting graviton bispectrum will violate parity - parity-odd. All remaining contractions are made with δ_{ij} and we generally omit the dependence of polarisation tensors on momenta for simplicity of notation. We can see that α can be at most 6 in the parity-even case, with all six polarisation indices contracted with momenta, and 7 in the parity-odd case since we can have at most two spatial momenta contracted with the Levi-Civita tensor due to momentum conservation. We will deal with the parity-even in this chapter (and parity-odd cases were considered separately in [38]). When deriving the full ψ_3 wavefunction coefficient we use the spinor-helicity methods outlined in Section 2.3.1 to account for the tensor structures from (2.1.25) and compute ψ_3^{trimmed} separately.

2.1.3 Graviton bispectra in de Sitter

An important previous result is that of Maldacena and Pimentel from 2011 [30]; assuming invariance under the full isometry group of de Sitter $SO(4,1)$, it was shown that for gravitons only three cubic cosmological wavefunctions are allowed by the constraint equations arising from the de Sitter isometries: two preserving parity (parity-even) and one violating parity (parity-odd), and of those only the two parity-even ones can lead to a non-vanishing bispectrum. These isometries imply that these correlation functions should be conformally invariant. One of the shapes is produced by the ordinary gravity action. The other shape is produced by a higher derivative correction and could be as large as the gravity contribution. The parity violating shape comes from an interaction of the form $W^2\tilde{W}$, where $\tilde{W}$ is the Weyl tensor with two indices contracted with ϵ_{ijk} which will have at least 6 derivatives; this does not contribute to the bispectrum, even though it is present in the wavefunction, as a parity-odd correlator can only arise when there is a logarithmic IR-divergence in the associated wavefunction coefficient, which may only happen when $2n_{\partial_\eta} + n_{\partial_i} \leq 3$, where n_{∂_η} and n_{∂_i} are respectively the number of time and space derivatives in the parity-odd interaction [38].

2.1.4 Graviton bispectra in de Sitter without boosts

De Sitter boosts are actually broken in phenomenologically relevant models of inflation, and the observed symmetries in the sky are those of statistical homogeneity, isotropy and scale invariance, i.e. *ISO(3)* with dilations. The most general tree-level three-point functions (bispectra) for a massless graviton to all orders in derivatives were derived for these isometries in [38]. Instead of working with explicit Lagrangians, [38] took a bootstrap approach and obtained results using the recently derived constraints from unitarity, i.e. the Cosmological Optical Theorem (COT) [50–52], locality, i.e. the Manifestly Local Test (MLT) [49] and the choice of a Bunch-Davies (BD) vacuum. Since no assumptions about de Sitter boosts are made, the results capture the phenomenology of large classes of models such as the Effective Field Theory of Inflation (EFToI) and solid inflation. Through this approach [38] obtained formulae for an infinite number of parity-even bispectra. For parity-odd bispectra, they showed that the constraints of unitarity from the COT allows for only a handful of possible shapes: three for graviton-graviton-graviton, three for scalar-graviton-graviton and one for scalar-scalar-graviton, which were bootstrapped explicitly. These parity-odd non-Gaussianities can be large, for example in solid inflation, and therefore constitute a concrete and well-motivated target for future observations.

Of the three parity-odd graviton bispectra only one belongs to the EFToI. Hence, we can now ask whether we can further constrain the parity-even graviton bispectra to identify those that belong to the EFToI. This is what we will explore in this chapter. In Section 2.2, we begin by reviewing the EFToI and then show that we only need consider a single building block to construct operators contributing to EFToI graviton bispectra. This proves to be a powerful result when it comes to deriving the parity-even graviton bispectra belonging to the EFToI in Sections 2.3 to 2.5. Finally in Section 2.6, we conclude with some potential future directions that we may pursue in the future.

2.2 Effective Field Theory of Inflation

The Effective Field Theory of Inflation (EFToI) encapsulates a large class of single-field inflationary models, and is based upon the spontaneous breaking of global time translation by the dynamics of matter fields in a spacetime[3] [68]. The almost scale invariant primordial power spectrum for scalar fluctuations of the CMB indicates that inflation was well-described by an expanding *quasi-de Sitter* spacetime. This is characterised by a time dependence of the inverse de Sitter radius, the Hubble parameter H, with deviation from exact de Sitter controlled by the slow-roll parameter $\epsilon = -\dot{H}/H^2$. The EFToI then describes scalar fluctuations around this background FLRW evolution. In particular this framework is agnostic as to what mechanism generated inflation: a single scalar field acquires a time dependent vacuum expectation value (VEV) which then appears as an effective Goldstone degree of freedom encoding the spontaneously broken time diffeomorphisms in a way very similar to a spontaneously broken gauge theory. In this section, we briefly review the EFToI and show, for the first time, that all graviton bispectra can be derived from extrinsic curvature operators only.

2.2.1 Building blocks of the Effective Field Theory of Inflation

In analogy with gauge theory one can define a unitary gauge where the perturbations corresponding to the matter part of the theory are set to zero and the dynamics is governed by the metric perturbations with both scalar and tensor modes. The action can be constructed by writing all terms compatible with the remaining unbroken (linearly realised) symmetries of the background evolution. This, in general, involves both scalar and tensor part of the metric.

[3] In General Relativity time re-parametrisation invariance is a gauge symmetry and unphysical. In the context of the EFToI, it is the *global* part of time translation, characterised by a timelike Killing vector, which is spontaneously broken. We refer the reader to [138] for a more precise discussion.

However in this work we will only be interested in graviton interactions and thus we will work in the limit where the scalar and tensor fluctuations decouple, and thus take the metric to be

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)(e^{\gamma})_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.2.1}$$

where the graviton is transverse and traceless: $\gamma_{ii} = 0$, $\partial_i\gamma_{ij} = 0$. Here we have solved the constraint equations and used the fact that this does not affect the graviton action at cubic order in graviton fluctuations so we can set the lapse to unity, $N = 1$, and the shift together with the curvature perturbation to zero $\zeta = 0 = N^i$. Given that γ_{ij} is a symmetric tensor, the transverse components have the following properties $\gamma_{12} = \gamma_{21}$, $\gamma_{13} = \gamma_{31}$, $\gamma_{23} = \gamma_{32}$, leaving only six independent components. The traceless property $\gamma_{ii} = 0$ relates the trace components by $\gamma_{11} + \gamma_{22} + \gamma_{33} = 0$ reducing the number of independent components by one. The transverse property $\partial^i\gamma_{ij} = 0$ relates the components by three simultaneous equations of the form $\partial_1\gamma_{1j} + \partial_2\gamma_{2j} + \partial_3\gamma_{3j} = 0$ (where j takes on a fixed value of 1, 2 or 3 for each simultaneous equation) reducing the number of independent components by three. Hence, γ_{ij} has indeed only $9 - (3 + 1 + 3) = 2$ independent components. We define the scale factor $a = a(t)$ and Hubble parameter $H = \dot{a}/a$. For now we work in cosmological time but later on we will convert to conformal time which is more suitable for computing late-time cosmological correlators. In unitary gauge the most general action that we can write down is [68]

$$S = \int \mathrm{d}^4x\sqrt{-g}\,F(R_{\mu\nu\rho\sigma}, g^{00}, K_{\mu\nu}, \nabla_\mu, t), \tag{2.2.2}$$

where all free indices are, in general, upper 0's. Here the theory is invariant under spatial diffeomorphisms and time diffeomorphisms are non-linearly realised by a time scalar evolving scalar $\phi(t, \vec{x}) = \phi_0(t)$. For our purposes, however, there are simplifications. We work in the scale invariant approximation (scale invariance will be most transparent when we convert to conformal time), i.e. we assume that H and $\dot{H}$ vary slowly and restrict all t-dependence to be that coming from the metric. In practice this means that we work with a fixed de Sitter background metric, so all correlators we compute are valid up to small slow-roll corrections [87]. This corresponds to an approximate shift symmetry for the inflaton ϕ. We also take $g^{00} = -1$, since we don't include the lapse, and this means we can write all temporal indices downstairs. Hence our general action reduces to

$$S = \int \mathrm{d}^4x\sqrt{-g}\,F(R_{\mu\nu\rho\sigma}, K_{\mu\nu}, \nabla_\mu), \tag{2.2.3}$$

Furthermore we don't need to use the full four-dimensional Riemann tensor $R^i{}_{jkl}$, as it is well-known that the full 20 components of $R^i{}_{jkl}$ are not independent of the extrinsic curvature K_{ij}, but only the three-dimensional part $\tilde{R}^i{}_{jkl}$[4] is independent (see e.g. [139]). Rather we can use $\tilde{R}_{ijkl}$ and the perturbed extrinsic curvature $\delta K_{ij} = K_{ij} - Hh_{ij}$, where $h_{\mu\nu} = g_{\mu\nu} + \delta^0_\mu\delta^0_\nu$. Note that throughout we will actually work with the perturbed extrinsic curvature without loss of generality. A further simplification we can make is by expressing the three-dimensional Riemann tensor $\tilde{R}_{ijkl}$ in terms of its trace-free part known as the Weyl tensor $\tilde{C}_{ijkl}$ which vanishes in three dimensions and the Ricci tensor $\tilde{R}_{ij}$. One way to see this is by counting the number of degrees of

[4] Throughout we use a tilde to represent three-dimensional objects.

freedom of the Weyl tensor C_{ijkl}. The Riemann tensor R_{ijkl} in D dimensions has $\frac{1}{12}D^2(D^2-1)$ degrees of freedom its trace has $\frac{1}{2}D(D+1)$ degrees of freedom, hence in three dimensions the Weyl tensor C_{ijkl} has $6-6=0$ components. Hence the three-dimensional Riemann tensor is completely fixed by the three-dimensional Ricci tensor $\tilde{R}_{ij}$.

Our general action now reduces to

$$S = S_0 + \int \mathrm{d}^4x\sqrt{-g}\, F(\tilde{R}_{ij}, \delta K_{ij}, \nabla_0, \nabla_i), \tag{2.2.4}$$

where we have separated out S_0 which includes the Einstein-Hilbert (EH) action plus the terms required to make the unperturbed metric a consistent solution. Out of the non-vanishing Christoffel symbols only the three-dimensional one $\tilde{\Gamma}^i_{jk}$ cannot be expressed in terms of the extrinsic curvature. Thus all temporal covariant derivatives $\nabla_0\delta K_{ij} = \partial_t\delta K_{ij} + \mathcal{O}(\delta K^2)$ and we can treat them as partial ones $\nabla_0 \to \partial_t$, and we only need to use the three-dimensional covariant derivative $\tilde{\nabla}_i$ since $\tilde{R}_{ij}$ and δK_{ij} are three-dimensional objects, simplifying our action to

$$S = S_0 + \int \mathrm{d}^4x\sqrt{-g}\, F(\tilde{R}_{ij}, \delta K_{ij}, \partial_t, \tilde{\nabla}_i). \tag{2.2.5}$$

The S_0 part of the action describes the minimal set-up of slow-roll inflation, with all other operators describing higher derivative corrections. Any operators not contained in S_0 start at quadratic order in γ, so do not affect the tadpole cancellation [68]: they capture all different theories of cosmological perturbations on the same FLRW background. Indeed, δK_{ij} is a perturbed object by construction, and $\tilde{R}_{ij}$ vanishes on the background since it is insensitive to time dependence. We have [68]

$$S_0 = \frac{M_{\rm pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2(\dot{H}+3H^2) + 2\dot{H}g^{00}\right], \tag{2.2.6}$$

and we remind the reader that we are working in the limit where H and $\dot{H}$ do not vary significantly in one Hubble time: all time dependence of the action is slow-roll suppressed. We note that the perturbed action $S - S_0$ is derivatively coupled: in the general EFToI the only terms without derivatives are polynomials in g^{00}, and for us, these are all trivial.

In some cases, it is advantageous to retain covariant derivatives and postpone simplifications until the action is explicitly expanded. We have therefore shown that all EFToI operators contributing to graviton bispectra can be constructed from a minimal set of building blocks: $\tilde{R}ij$, δKij, and their covariant derivatives. Further technical details of these simplifications are provided in Appendix A. In Section 2.2.2, we will take this analysis one step further and demonstrate — for the first time — that all graviton bispectra can, in fact, be derived purely from operators built from the extrinsic curvature.

2.2.2 Field redefinitions and redundancy of $\tilde{R}_{ij}$

Now we will show that to construct graviton bispectra it is sufficient to work with the restricted action that only depends on the extrinsic curvature δK_{ij}. Since both $\tilde{R}_{ij} \sim \mathcal{O}(\gamma)$ and $\delta K_{ij} \sim \mathcal{O}(\gamma)$, i.e. they start at linear order in perturbations, the graviton action up to cubic order, that comes in addition to the EH part, comes from EFToI operators that are at most cubic in these building blocks.

Let us first consider operators that are constructed out of three building blocks i.e. are of the schematic form: $\tilde{R}^3$, $\tilde{R}^2\delta K$, $\tilde{R}\delta K^2$ and δK^3 where we have suppressed indices and derivatives. In Section 2.2.1 we argued that $\nabla_0 \delta K_{ij} = \partial_t \delta K_{ij} + \mathcal{O}(\delta K^2)$ and thus we can always use partial time derivatives $\nabla_0 \to \partial_t$. For spatial derivatives the difference between using a partial one and a covariant one is captured by $\tilde{\Gamma}^i_{jk} \sim \mathcal{O}(\gamma)$, from which it follows that $\tilde{\nabla}_i \delta K_{ij} = \partial_i \delta K_{ij} + \mathcal{O}([\tilde{\Gamma}^i_{jk}]^2) = \partial_i \delta K_{ij} + \mathcal{O}(\gamma^2)$. For three building block operators this will only introduce differences at $\mathcal{O}(\gamma^4)$. For our interests we can therefore treat all derivatives as partial ones for three building block operators.

Now such operators do not contribute to the quadratic action for the graviton so to compute the cubic wavefunction coefficient we only need to consider contact diagrams where all external lines are on-shell in which case any appearances of $\partial^2\gamma_{ij}$ are degenerate with $\dot{\gamma}_{ij}$, and its time derivatives, by the graviton equation of motion:

$$\ddot{\gamma}_{ij} + 3H\dot{\gamma}_{ij} - a^{-2}\partial^2\gamma_{ij} = 0. \tag{2.2.7}$$

As we will discuss below, the two-derivative quadratic action can be brought into the Einstein-Hilbert form without loss of generality [140], with higher-derivative corrections that we treat perturbatively such that (2.2.7) is always the on-shell relation. Once we remove all copies of $\partial^2\gamma_{ij}$, all remaining interactions can be derived from δK_{ij} and its derivatives since at linear order we have $\delta K_{ij} \sim \dot{\gamma}_{ij}$. So for three building blocks operators we can safely ignore operators containing $\tilde{R}_{ij} \sim \partial^2\gamma_{ij}$, at least when computing the on-shell cubic vertices. We can also make the redundancy of any operators containing $\tilde{R}_{ij}$ manifest with field redefinitions, as we will show below.

Two and single building block operators are slightly more complicated but nevertheless we can still reduce the action to one constructed from δK_{ij} only using field redefinitions, for which it will be useful to define the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Let's start by considering which operators can contribute to the action up to cubic order. Since the graviton is transverse all vector quantities start at quadratic order in perturbations: $\nabla_i \tilde{R}_{ij} \sim \mathcal{O}(\gamma^2)$ and $\nabla_i \delta K_{ij} \sim \mathcal{O}(\gamma^2)$. Given that there are no other vector quantities, any operator that involves covariant derivatives contracted with one of the building blocks will start at $\mathcal{O}(\gamma^4)$. It follows that ∇_i must always appear as ∇^2. The action for two building block operators is therefore

$$S = S_0 + M_{\rm pl}^2 \int {\rm d}^4x\sqrt{-g}\,\left(\delta K^{ij}\mathcal{O}_{(0)}\delta K_{ij} + \tilde{R}^{ij}\mathcal{O}_{(1)}\delta K_{ij} + \tilde{R}^{ij}\mathcal{O}_{(2)}\tilde{R}_{ij} + c\,\delta K^{ij}\delta K_{ij} + d\,\tilde{R}\right), \tag{2.2.8}$$

where c and d are constants, $\mathcal{O}_{(i)}$ are derivative operators constructed out of ∇_0 and $\tilde{\nabla}^2 = \tilde{\nabla}^i\tilde{\nabla}_i$, and we have integrated by parts to move all derivatives onto a single building block[5]. For $\mathcal{O}_{(i)}$, i

[5]One might worry about boundary terms that could affect the wavefunction, but it can be shown that there are always enough derivatives for the boundary terms to vanish.

counts its negative mass dimension which we can fix using factors of $M_{\rm pl}$:

$$\mathcal{O}_{(i)} = \frac{1}{M_{\rm pl}^i} \sum_{m,n} b_{m,n} \nabla_0^m \tilde{\nabla}^{2n}, \tag{2.2.9}$$

where $b_{m,n}$ are constant couplings with mass dimension $[-(m+2n)]$ and we remind the reader that we are working with a dimensionless γ_{ij}. Note that we have not included any terms that depend on the trace of the extrinsic curvature since this object vanishes to all orders in γ (we refer the reader to Appendix A for further details). For the purposes of this derivation we have included terms that depend on the three-dimensional Ricci scalar $\tilde{R}$, but it is well-known that a linear term can be eliminated by Gauss-Codazzi relations [140][6], while non-linear terms start at $\mathcal{O}(\gamma^4)$ since $\tilde{R} \sim \mathcal{O}(\gamma^2)$. We will see that the operators $\delta K^{ij}\delta K_{ij}$ and $\tilde{R}$ can be removed by conformal transformations of the metric. These are the operators that, in addition to pure δK_{ij} operators, contribute to γ^2 and/or γ^3. Any other operator we can write down using at most two building blocks starts at $\mathcal{O}(\gamma^4)$ or higher.

Now the variation of the EH action to linear order is

$$\delta S_0 = \int \mathrm{d}^4x \sqrt{-g} M_{\rm pl}^2 \hat{G}_{\mu\nu} \delta g^{\mu\nu}, \tag{2.2.10}$$

which we can use to cancel some of these operators. Here we have written $\hat{G}_{\mu\nu}$ which satisfies the equations of motion, i.e. it includes the cosmological constant term, and therefore starts at linear order in γ since the zeroth order part is zero by the equations of motion. Since $\hat{G}_{\mu\nu}$ and the building blocks start at $\mathcal{O}(\gamma)$, shifts that are quadratic or higher in building blocks are not useful, as they can not be used to cancel operators which start at $\mathcal{O}(\gamma^2)$. The possible shifts are therefore

$$\delta g^{00} = g^{00}(a + \mathcal{O}\tilde{R}), \tag{2.2.11}$$

$$\delta g^{ij} = g^{ij}(b + \mathcal{O}\tilde{R}) + \mathcal{O}\tilde{R}^{ij} + \mathcal{O}\delta K^{ij}. \tag{2.2.12}$$

Now only the last two terms in δg^{ij} can generate terms of the form $\tilde{R}^{ij}\mathcal{O}\tilde{R}_{ij}$, $\tilde{R}^{ij}\mathcal{O}\delta K_{ij}$ and $\delta K^{ij}\mathcal{O}\delta K_{ij}$ so we cannot remove all operators of this form, as we don't have enough freedom in the transformations, and thus choose to keep $\delta K^{ij}\mathcal{O}_{(0)}\delta K_{ij}$. The two building block operators we want to get rid of are therefore

$$M_{\rm pl}^2 \int \mathrm{d}^4x \sqrt{-g} \left(\tilde{R}^{ij}\mathcal{O}_{(1)}\delta K_{ij} + \tilde{R}^{ij}\mathcal{O}_{(2)}\tilde{R}_{ij} + c\, \delta K^{ij}\delta K_{ij} + d\tilde{R} \right). \tag{2.2.13}$$

First consider the shift

$$\delta g^{ij} = \mathcal{O}_{(2)}\tilde{R}_{ij} \tag{2.2.14}$$

which can be used to remove

$$\tilde{R}^{ij}\mathcal{O}_{(2)}\tilde{R}_{ij} \tag{2.2.15}$$

[6] Eliminating a linear term in $\tilde{R}$ is more involved if we allow for time-dependent couplings but it can still be done [140, 141].

while renormalising the other operators and introducing non-linear terms in $\tilde{R}$ which start at $\mathcal{O}(\gamma^4)$ and hence don't contribute to the action up to cubic order. Now we can do a similar shift

$$\delta g^{ij} = \mathcal{O}_{(1)}\delta K_{ij} \tag{2.2.16}$$

to remove

$$\tilde{R}^{ij}\mathcal{O}_{(1)}\delta K_{ij} \tag{2.2.17}$$

while renormalising the other operators and introducing other operators that don't contribute to the cubic action. We are therefore left with

$$c\,\delta K^{ij}\delta K_{ij} + d\tilde{R}. \tag{2.2.18}$$

Now consider the conformal transformations

$$\delta g^{00} = \tilde{c}\, g^{00}, \tag{2.2.19}$$

$$\delta g^{ij} = \tilde{d}\, g^{ij}. \tag{2.2.20}$$

By choosing $\tilde{c}$ and $\tilde{d}$ we can cancel these final two terms. This means that for two building block operators we only need to consider δK_{ij} and its derivatives. We can use this procedure iteratively to cancel all such operators appearing in (2.2.13). As previously mentioned, we cannot also eliminate the $\delta K^{ij}\mathcal{O}_{(0)}\delta K_{ij}$ operators since there is not enough freedom in (2.2.10). Note that in terms of γ_{ij}, these field redefinitions shift γ_{ij} by terms with derivatives and then scale invariance ensures that the redefinitions vanish at late-times. So wavefunction coefficients and cosmological correlators evaluated at the end of inflation are not affected by these field redefinitions.

Now it is clear how we can generalise this discussion to manifestly remove all copies of $\tilde{R}_{ij}$ from three building block operators. We simply need to use field redefinitions of the schematic form $\delta g = \tilde{R}^2 + \tilde{R}\delta K + \delta K \delta K$ where we have suppressed indices and derivatives. Acting on the EH action with these field redefinitions allows us to reduce three building block operators to ones cubic in δK_{ij}. It follows that the most general action that can contribute to graviton bispectra at tree-level is

$$S = S_0 + M_{\rm pl}^2 \int {\rm d}^4x\sqrt{-g}\,[\delta K^{ij}\mathcal{O}_{(0)}\delta K_{ij} + \mathcal{O}(\delta K^3)]. \tag{2.2.21}$$

Therefore in this section we have shown for the first time that to compute the bispectra for gravitons, in addition to the minimal action, we only need to consider EFToI actions constructed out of the extrinsic curvature δK_{ij}. Time derivatives can always be taken to be partial ones and spatial derivatives only need to be covariant for two building block operators and they only need to be contracted amongst themselves. We refer to the non-Gaussianities that arise from operators quadratic in δK_{ij} as Type-I and those that come from operators cubic in δK_{ij} as Type-II. The former only satisfy the consistency relations with a correction to the power spectrum, while the latter satisfy the consistency relations without such a correction. We study these two cases in more detail in Section 2.3.

2.3 Graviton non-Gaussianities in the EFToI

We can now compute the general action up to cubic order in γ_{ij}. The Einstein-Hilbert part of the action yields

$$S_{\gamma,\mathrm{GR}} = \frac{M_{\mathrm{pl}}^2}{8} \int \mathrm{d}t\, \mathrm{d}^3 x a^3(t)[\dot{\gamma}_{ij}\dot{\gamma}_{ij} - a^{-2}\partial_k \gamma_{ij}\partial_k \gamma_{ij} + a^{-2}(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl})\partial_k\partial_l \gamma_{ij}] + \mathcal{O}(\gamma^4), \quad (2.3.1)$$

and we remind the reader that for γ_{ij} operators, spatial indices are raised and lowered with δ_{ij} so we will not be so careful about the placement of indices. Throughout we have been working in cosmological time but when we come to compute the late-time wavefunction we would like to do so in conformal time η where we evolve perturbations from the far past at $\eta = -\infty$ to the boundary of quasi-de Sitter space, or the end of inflation, at $\eta = 0$. We will therefore change coordinates in the actions we derive so that the background metric is

$$\mathrm{d}s^2 = a^2(\eta)[-\,\mathrm{d}\eta^2 + dx^2], \quad (2.3.2)$$

and we will approximate the scale factor as $a(\eta) = -1/(H\eta)$ which is the de Sitter one in Poincaré (also known as flat-slicing) coordinates.

2.3.1 Cosmological spinor-helicity formalism

In this section we provide a brief overview of the spinor-helicitly formalism used in cosmology developed in [30]. As mentioned in Section 2.1.2, we can write a general three-point wavefunction coefficient ψ_3 in terms of a polarisation factor multiplied by a "trimmed" wavefunction coefficient $\psi_3^{\mathrm{trimmed}}$ which is an $SO(3)$ scalar and can be computed from bulk time integrals (and thus can be treated separately)

$$\psi_3^{h_1,h_2,h_3}(\{k\},\{\mathbf{k}\}) = \sum_{\text{contractions}} [e^{h_1}(\mathbf{k}_1)e^{h_2}(\mathbf{k}_2)e^{h_3}(\mathbf{k}_3)\mathbf{k}_1^{\alpha_1}\mathbf{k}_2^{\alpha_2}\mathbf{k}_3^{\alpha_3}] \times \psi_3^{\mathrm{trimmed}}(\{k\}) \quad . \quad (2.3.3)$$

where h_i are the helicities of the external fields, we define $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, and $\{k\}$ and $\{\mathbf{k}\}$ collectively denote the external energies and momenta, respectively. Since we are only interested in massless spin-2 modes, we have $h_i = \pm 2$, and we will concentrate on the $+++$ and $++-$ configurations as the others can be extracted from these by parity transformations [38]. Once we have the final form of the various time integrals, the last step is to account for the tensor structure. To compute the tensor structure we work with the de Sitter spinor-helicity formalism, and here we discuss the subject only very briefly and refer the reader to [24, 30, 38, 48, 50, 52, 116, 142] for further details.

A null four-vector k_μ can be represented by a pair of two-component, non-Grassmanian spinors $(\lambda, \tilde{\lambda})$

$$k_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} k_\mu = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}\,, \quad (2.3.4)$$

where we defined $k_\mu = (k, \mathbf{k})$ and $\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma})$ are the Pauli matrices. Here we have constructed a null four-vector from the spatial momenta with $k = |\mathbf{k}|$. Conservation of spatial momenta for three particles $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{0}$ yields

$$\langle ab\rangle [ab] = k_T I_c, \qquad \text{for} \quad a \neq b \neq c, \tag{2.3.5}$$

$$\sum_{a=1}^{3} \lambda_\alpha^{(a)} \tilde{\lambda}_{\dot{\alpha}}^{(a)} = k_T (\sigma^0)_{\alpha\dot{\alpha}}, \tag{2.3.6}$$

where we have defined the spinor-helicity brackets

$$\langle ij\rangle = \epsilon^{\alpha\beta} \lambda_\alpha^{(i)} \lambda_\beta^{(j)}, \tag{2.3.7}$$

$$[ij] = \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{\dot{\alpha}}^{(i)} \tilde{\lambda}_{\dot{\beta}}^{(j)}, \tag{2.3.8}$$

$$(ij) = (\sigma^0)^{\alpha\dot{\alpha}} \lambda_\alpha^{(i)} \tilde{\lambda}_{\dot{\alpha}}^{(j)}, \tag{2.3.9}$$

which satisfy the following properties

$$\langle ab\rangle [ab] = k_T I_c \quad \text{for} \quad a \neq b \neq c, \tag{2.3.10}$$

$$(ab)[ac] = I_b [bc] \quad \text{for} \quad a \neq b \neq c, \tag{2.3.11}$$

$$(ab)\langle bc\rangle = I_a \langle ac\rangle \quad \text{for} \quad a \neq b \neq c\,. \tag{2.3.12}$$

$I_a \equiv k_T - 2k_a$ and k_T is one of the elementary symmetric polynomials $k_T \equiv k_1 + k_2 + k_3$, $e_2 \equiv k_1 k_2 + k_1 k_3 + k_2 k_3$, $e_3 \equiv k_1 k_2 k_3$ [7]. The object $(\bar{\sigma}^0)^{\dot{\alpha}\alpha}$ can be used to pick out of the time component of a vector e.g. $(\bar{\sigma}^0)^{\dot{\alpha}\alpha} k_{\alpha\dot{\alpha}} = 2k$. This would not be allowed in a Lorentz invariant theory but is perfectly fine in our cosmological setting. This allows us to define polarisation vectors which have a vanishing time component. If we write the polarisation tensors for γ_{ij} as $e^{\pm}_{ij}(\mathbf{k}) = e^{\pm}_i(\mathbf{k}) e^{\pm}_j(\mathbf{k})$, then we have

$$e^{+}_{\alpha\dot{\alpha}}(\mathbf{k}) \;=\; \frac{(\sigma^0)_{\alpha\dot{\beta}}\, \tilde{\lambda}^{\dot{\beta}} \tilde{\lambda}_{\dot{\alpha}}}{k}\,, \tag{2.3.13}$$

$$e^{-}_{\alpha\dot{\alpha}}(\mathbf{k}) \;=\; \frac{(\sigma^0)_{\beta\dot{\alpha}}\, \lambda^{\beta} \lambda_{\alpha}}{k}\,. \tag{2.3.14}$$

The simplicity of these expressions motivates this particular normalisation $e^{h}_{ij}(\mathbf{k}) e^{h'}_{ij}(\mathbf{k})^* = 4\delta_{hh'}$. For parity-even interactions at three-points, useful formulae for going from polarisations to spinors are

$$e^{a+} \cdot e^{b+} = -\frac{[ab]^2}{2k_a k_b}, \qquad e^{a-} \cdot e^{b-} = -\frac{\langle ab\rangle^2}{2k_a k_b}, \qquad e^{a+} \cdot e^{b-} = \frac{I_b^2}{2k_a k_b} \frac{\langle cb\rangle^2}{\langle ca\rangle^2} = \frac{I_a^2}{2k_a k_b} \frac{[ca]^2}{[cb]^2}, \tag{2.3.15}$$

$$k^a \cdot e^{b+} = \frac{I_b}{2k_b} \frac{[ab][bc]}{[ac]}, \qquad k^a \cdot e^{b-} = \frac{I_b}{2k_b} \frac{\langle ab\rangle \langle bc\rangle}{\langle ac\rangle}. \tag{2.3.16}$$

[7] In flat-space there is time translation invariance (i.e. energy is conserved which implies $k_T = 0$), hence a number of flat-space spinor-helicity identities are altered in de Sitter.

The wavefunction coefficients can then always be expressed in terms of coupling constants, the square and angle brackets, and the energies k_a, with the linear combinations I_a playing a special role. The tensor structure part of $\psi_{3,+++}$ always takes the following form [38]

$$\left(e^+(\mathbf{k}_1)e^+(\mathbf{k}_2)e^+(\mathbf{k}_3)\mathbf{k}_1^{\alpha_1}\mathbf{k}_2^{\alpha_2}\mathbf{k}_3^{\alpha_3}\right) = \frac{[12]^2[23]^2[31]^2}{e_3^2}h_\alpha(k_a) \equiv \mathrm{SH}_{+++}h_\alpha(k_a)\,, \tag{2.3.17}$$

where h_α is a polynomial in the energies, of dimension $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. For parity even interactions, which we are considering here, possible values are $\alpha = 0, 2, 4, 6$. For each interaction, h_α must be found by an explicit calculation. For example, looking at (2.3.1) where the Einstein-Hilbert action is written up to cubic order implies that the GR vertex can be represented by $h_{\alpha=2}$.

The simplicity of the method lies in the fact that once the $+++$ computation is performed, all other helicity configurations come almost automatically. The time integral part of the wavefunction coefficient is the same as for the $+++$, while the tensor structure part can be found as follows:

$$\begin{aligned}
\left(e^+(\mathbf{k}_1)e^+(\mathbf{k}_2)e^-(\mathbf{k}_3)\mathbf{k}_1^{\alpha_1}\mathbf{k}_2^{\alpha_2}\mathbf{k}_3^{\alpha_3}\right) &= \frac{[12]^6}{[23]^2[31]^2}\frac{I_1^2I_2^2}{e_3^2}h_\alpha(k_1,k_2,-k_3) \equiv \mathrm{SH}_{++-}h_\alpha(k_1,k_2,-k_3)\,,\\
\left(e^-(\mathbf{k}_1)e^-(\mathbf{k}_2)e^+(\mathbf{k}_3)\mathbf{k}_1^{\alpha_1}\mathbf{k}_2^{\alpha_2}\mathbf{k}_3^{\alpha_3}\right) &= \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\frac{I_1^2I_2^2}{e_3^2}h_\alpha(k_1,k_2,-k_3) \equiv \mathrm{SH}_{--+}h_\alpha(k_1,k_2,-k_3)\,,\\
\left(e^-(\mathbf{k}_1)e^-(\mathbf{k}_2)e^-(\mathbf{k}_3)\mathbf{k}_1^{\alpha_1}\mathbf{k}_2^{\alpha_2}\mathbf{k}_3^{\alpha_3}\right) &= \frac{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}{e_3^2}h_\alpha(k_1,k_2,k_3) \equiv \mathrm{SH}_{---}h_\alpha(k_a)\,,
\end{aligned} \tag{2.3.18}$$

We refer the reader to [38] for a derivation and a more in-depth discussion of this construction.

In Section 2.2.2 it was shown that there are only two types of bispectra that we can construct arising from corrections to the minimal action: Type-I which arise from two building block operators (i.e. quadratic in δK_{ij}) and Type-II which come from three building block operators (i.e. cubic in δK_{ij}). We will consider the Type-I and Type-II bispectra in Sections 2.4 and 2.5 respectively, and present their final form using these spinor variables.

2.4 Type-I bispectra

After a lot of manipulation the resulting cubic interactions in conformal time we have for the two building block operators are

$$\begin{aligned}
S_{\gamma,2\mathrm{BB}} = \frac{M_{\mathrm{pl}}^2}{4}\int \mathrm{d}\eta\,\mathrm{d}^3x \sum_{n=1} g_{n,0}a^{2-2n}(\eta)(\gamma_{ij})^{2n+1}\gamma'_{ij} + \\
\frac{M_{\mathrm{pl}}^2}{4}\int \mathrm{d}\eta\,\mathrm{d}^3x \sum_{n=0,m=1} g_{n,m}a^q(\eta)(\gamma_{ij})^{n+1}\left(\partial^{2m}\gamma'_{ij} - m\gamma^{lk}\partial^{2m-2}\partial_l\partial_k\gamma'_{ij} + \sum_p^{m-1}\partial^{2p}Q_s[\gamma,\partial^{2m-2-2p}\gamma']_{ij}\right),
\end{aligned} \tag{2.4.1}$$

where the symmetric function Q_s is given by

$$Q_s[\gamma,\gamma']_{ij} = -\partial_i\gamma_{kl}\partial_k\gamma'_{lj} + \partial_l\gamma_{jk}\partial_k\gamma'_{il} + (i \leftrightarrow j), \tag{2.4.2}$$

and $q = 2 - 2m - n$. This is the most general action, from two building block operators, that can contribute to the graviton bispectra in the EFToI. It contains a sum of quadratic and cubic terms with their relative coefficients tied together by the linear realisation of spatial diffeomorphisms and nonlinear realisation of time diffeomorphisms. Note that in order to obtain this expression we have freely added and subtracted three building block operators. We were allowed to do this, because we are going to consider three building block operators in their full generality in Section 2.5. Therefore for Type-I bispectra that arise from two building block operators, both single exchange and contact diagrams contribute and we deal with them in turn.

2.4.1 Contact diagrams

For gravitons, contact diagrams (see Figure 2.1) factorise into a *trimmed* wavefunction ψ_3^{trimmed} and a polarisation factor which contains the three polarisation tensors for the external gravitons. Traditionally ψ_3^{trimmed} would be computed by integrating the three bulk-boundary propagators, but another way of computing ψ_3^{trimmed} was derived in [49] which doesn't require any time evolution, and is referred to as the Manifestly Local Test (MLT)

$$\frac{\partial}{\partial k_c}\psi_3^{\text{trimmed}}\Big|_{k_c=0} = 0\,, \qquad \forall\, c = 1,2,3\,. \tag{2.4.3}$$

Requiring ψ_3^{trimmed} to satisfy this test follows from the fact that the bulk-boundary propagator for massless gravitons (and scalars) does not contain any terms linear in any of the k_c

$$K_\gamma(k,\eta) = (1 - ik\eta)e^{ik\eta}, \qquad \frac{\partial K_\gamma}{\partial k}\Big|_{k=0} = 0, \tag{2.4.4}$$

and this property is inherited by ψ_3^{trimmed} since it holds for all η. Locality also forces there to be no inverse powers of the energies, and at cubic order there are no inverse Laplacians ∇^{-2} to worry about. Note that as we send one $k_c \to 0$, we do so while holding the others fixed which distinguishes the MLT from soft theorems. The MLT can be used to construct cosmological correlators as was done in [38, 49, 143], but also to verify that the often complicated final results have the correct structure as was done in [144].

In the literature (see e.g. [46]), it has been shown that the assumption of a Bunch-Davies vacuum, which we impose throughout, tells us that ψ_3^{trimmed} is a rational function with poles only occurring at $k_T = 0$[8]. There can also be logs in the trimmed wavefunction, for low derivative operators, but no other kinematic singularities unless there is some form of non-locality or different vacuum conditions [29]. In the EFToI we do not encounter logs since there are too many derivatives in the graviton interactions. Indeed the absence of logs in contact diagrams requires $2n_{\partial_\eta} + n_{\partial_i} > 3$ [29], where n_{∂_η} and n_{∂_i} are respectively the number of conformal time and space derivatives, and we saw previously that this condition is always satisfied in the EFToI.

Now to solve the MLT we simply write down an ansatz and organise the solutions in terms of the order of the leading total-energy pole which we denote by p and which counts the number

[8]This is the kinematical limit where energy is conserved, and interestingly, the residue of the leading total-energy pole contains the flat-space scattering amplitude for the same process [30, 31, 50] [145]. In some cases the leading total-energy pole can be purely cosmological [146].

of derivatives in the corresponding vertex [29],

$$\psi_3^{\text{trimmed}}(\{k\}) = \frac{\overbrace{\text{Poly}_{p+3-\alpha}(k_T, e_2, e_3)}^{\text{polynomial of degree } p+3-\alpha}}{\underbrace{k_T^p}_{\text{BD vacuum}}} \,. \tag{2.4.5}$$

Guidance comes from Bose symmetry: we fix ψ_3^{trimmed} to have the same symmetry as the polarisation part and sum over the remaining permutations once we have constructed ψ_3^{trimmed}. This is a consistent thing to do since the MLT is satisfied for all permutations. With the cubic wavefunction coefficient in hand we can compute the corresponding bispectrum B_3 [38][9]

$$B_3(\{k\}, \{\mathbf{k}\}) = -\frac{\psi_3(\{k\}, \{\mathbf{k}\}) + \psi_3^{\star}(\{k\}, \{-\mathbf{k}\})}{\prod_{a=1}^{3} 2\text{Re}\ \psi_2(k_a)}, \tag{2.4.6}$$

where ψ_2 is the quadratic wavefunction coefficient which is perturbatively fixed by the two-derivative quadratic action coming from GR for which we have

$$\psi_{2,\text{GR}}^{hh'} = \frac{M_{\text{pl}}^2}{H^2} k^3 \delta_{hh'}. \tag{2.4.7}$$

We have dropped imaginary terms in this expression, which are actually divergent at late-times, since they never contribute to correlators. For parity-even interactions, which are the ones of interest here, the numerator is 2Re ψ_3 which follows from having an even number of spatial momenta. We can also use $e_{ij}^h(\mathbf{k})^* = e_{ij}^h(-\mathbf{k})$, which follows from the reality of $\gamma_{ij}(x)$, to see that the polarisation factor is a common factor on both the left and right hand side of this equation. We refer the reader to [38] for more details on deriving this relationship.

We have to compute such contact diagrams for both Type-I and Type-II bispectra. In Section 2.4 we will directly compute the necessary bulk time integrals for Type-I, since we can see from (2.4.1), the time dependence for two building block operators is quite specific so computing the integrals is more straightforward than finding the necessary subset of MLT solutions. For Type-II, we will explain in Section 2.5 that we can use the MLT to very efficiently write down all allowed wavefunction coefficients by taking into account the fact that each graviton is differentiated with respect to time at least once.

2.4.2 Single exchange diagrams

Single exchange diagrams are more complicated to compute compared to contact ones since there are nested time integrals. Note that since we have corrections to the quadratic action, there are other diagrams that can contribute to the cubic wavefunction coefficient at tree-level. However, we are treating these corrections perturbatively in which case the only exchange diagram which contributes at leading order in the field theory couplings is Figure 2.2 for which we need to take the cubic vertex to be independent of this small coupling: the GR vertex scales like $1/P_\gamma$, while the bulk-bulk propagator scales as P_γ, so the product of the two is $\mathcal{O}(1)$. At this

[9] The COT for cubic contact diagrams is $\psi_3(\{k\}, \{\mathbf{k}\}) + \psi_3^{\star}(\{-k\}, \{-\mathbf{k}\} = 0$ [50] and this dictates which part of the wavefunction contributes to expectation values [38].

order, this single exchange diagram and contact diagram that comes from these two building block operators are of comparable size.

Let us illustrate the Feynman rules for computing Figure 2.2 by taking the cubic vertex to be GR, and the quadratic mixing vertex to be one of the terms in (2.4.1) with $m = 0$, for which we have

$$\begin{aligned}\psi^{(n,m=0)}_{3,\text{exchange}} &= -i \times \frac{M^2_{\text{pl}}}{8}(-2e^{h_1}_{ik}e^{h_2}_{jl}k^3_k k^3_l e^{h_3}_{ij} + e^{h_1}_{ij}e^{h_2}_{kl}k^3_k k^3_l e^{h_3}_{ij} + 5 \text{ perms}) \times 4 \times \frac{M^2_{\text{pl}}g_{n,0}}{4} \\ &\times \int \mathrm{d}\eta\, \mathrm{d}\eta' a^2(\eta)a^{2-2n}(\eta')K_\gamma(k_1,\eta)K_\gamma(k_2,\eta) \\ &\times \left[K^{(2n+1)}_\gamma(k_3,\eta')\frac{\partial}{\partial\eta'}G(k_3,\eta,\eta') + K'_\gamma(k_3,\eta')\frac{\partial^{2n+1}}{\partial\eta'^{2n+1}}G(k_3,\eta,\eta')\right] + 2 \text{ perms},\end{aligned} \tag{2.4.8}$$

where the factor of 4 in the first line comes from applying (2.1.12) to the quadratic mixing, and the bulk-bulk propagator is given by (see e.g. [50])

$$G(k,\eta,\eta') = 2P_\gamma(k)[\theta(\eta-\eta')K(k,\eta')\text{Im}K(k,\eta) + (\eta \leftrightarrow \eta')], \tag{2.4.9}$$

where $\text{Im}K(k,\eta)$ is the imaginary part of the bulk-boundary propagator which takes the form

$$\text{Im}K(k,\eta) = -\frac{i}{2}[(1-ik\eta)e^{ik\eta} - (1+ik\eta)e^{-ik\eta}]. \tag{2.4.10}$$

Computing and analysing such time integrals is far simpler when there are no time derivatives on the bulk-bulk propagator. We can guarantee this by integrating by parts and, as always, we can drop all boundary terms. We then have

$$\psi^{(n,m=0)}_{3,\text{exchange}} = -\frac{iM^4_{\text{pl}}g_{n,0}}{8}(2e^{h_1}_{ik}e^{h_2}_{jl}k^3_k k^3_l e^{h_3}_{ij} - e^{h_1}_{ij}e^{h_2}_{kl}k^3_k k^3_l e^{h_3}_{ij} + 5 \text{ perms})\mathcal{I}_{n,0}(k_1,k_2,k_3) + 2 \text{ perms}, \tag{2.4.11}$$

where we have defined

$$\begin{aligned}\mathcal{I}_{n,0}(k_1,k_2,k_3) &= \int \mathrm{d}\eta\, \mathrm{d}\eta' a^2(\eta)K_\gamma(k_1,\eta)K_\gamma(k_2,\eta)G(k_3,\eta,\eta') \\ &\times \left[\frac{\partial}{\partial\eta'}(a^{2-2n}(\eta')K^{(2n+1)}_\gamma(k_3,\eta')) + \frac{\partial^{2n+1}}{\partial\eta'^{2n+1}}(a^{2-2n}(\eta')K'_\gamma(k_3,\eta'))\right].\end{aligned} \tag{2.4.12}$$

A very similar expression also applies when we take quadratic mixing vertices with $m \neq 0$ from (2.4.1), and is given by

$$\begin{aligned}\psi^{(n,m\geq 1)}_{3,\text{exchange}} =& \frac{(-1)^{m+1}iM^4_{\text{pl}}g_{n,m}}{8}(2e^{h_1}_{ik}e^{h_2}_{jl}k^3_k k^3_l e^{h_3}_{ij} - e^{h_1}_{ij}e^{h_2}_{kl}k^3_k k^3_l e^{h_3}_{ij} + 5 \text{ perms})k^{2m}_3\mathcal{I}_{n,m}(k_1,k_2,k_3) \\ &+ 2 \text{ perms},\end{aligned} \tag{2.4.13}$$

where

$$\mathcal{I}_{n,m}(k_1,k_2,k_3) = \int \mathrm{d}\eta\,\mathrm{d}\eta' a^2(\eta) K_\gamma(k_1,\eta) K_\gamma(k_2,\eta) G(k_3,\eta,\eta') \times \left[\frac{\partial}{\partial \eta'}(a^q(\eta') K_\gamma^{(n+1)}(k_3,\eta')) + (-1)^n \frac{\partial^{n+1}}{\partial \eta'^{n+1}}(a^q(\eta') K'_\gamma(k_3,\eta'))\right]. \quad (2.4.14)$$

We remind the reader that $q = 2 - 2m - n \leq 0$. In all cases it is easy to see that every time integral we need to compute for these single exchange diagrams is of the form

$$\mathcal{M}(\alpha,\beta) = \int \mathrm{d}\eta\,\mathrm{d}\eta'\; a^2(\eta) K_\gamma(k_1,\eta) K_\gamma(k_2,\eta) G(k_3,\eta,\eta') a^\alpha(\eta') K_\gamma^{(\beta)}(k_3,\eta'), \quad (2.4.15)$$

with $\alpha \leq 0$ and $\beta \geq 1$. We will refer to this integral as the *master time integral*. For $m = 0$ we have

$$\mathcal{I}_{n,0}(k_1,k_2,k_3) = \int \mathrm{d}\eta\,\mathrm{d}\eta' a^2(\eta) K_\gamma(k_1,\eta) K_\gamma(k_2,\eta) G(k_3,\eta,\eta') a^{2-2n}(\eta') \times \left[K_\gamma^{(2n+2)}(k_3,\eta') + (2-2n) H a(\eta') K_\gamma^{(2n+1)}(k_3,\eta') + \sum_{k=0}^{2n-2} c_{n,k} H^k a^k(\eta') K_\gamma^{(2+2n-k)}(k_3,\eta')\right], \quad (2.4.16)$$

where we have defined

$$c_{n,k} = (-1)^k k! \binom{2n+1}{k} \binom{2n-2}{k}, \quad (2.4.17)$$

and from this expression we can write

$$\mathcal{I}_{n,0}(k_1,k_2,k_3) = \mathcal{M}(2-2n, 2n+2) + (2-2n) H \mathcal{M}(3-2n, 2n+1) + \sum_{k=0}^{2n-2} c_{n,k} H^k \mathcal{M}(k+2-2n, 2+2n-k). \quad (2.4.18)$$

Similarly, for $m \neq 0$ we can write

$$\mathcal{I}_{n,1}(k_1,k_2,k_3) = \mathcal{M}(-n, n+2) - nH\mathcal{M}(1-n, n+1) + \sum_{k=0}^{n} d_{n,1,k} H^k \mathcal{M}(k-n, n+2-k), \quad (2.4.19)$$

$$\mathcal{I}_{n,m}(k_1,k_2,k_3) = \mathcal{M}(2-2m-n, n+2) + (2-2m-n) H \mathcal{M}(3-2m-n, n+1) + \sum_{k=0}^{n+1} d_{n,m,k} H^k \mathcal{M}(k+2-2m-n, n+2-k), \qquad m \geq 2, \quad (2.4.20)$$

where we have defined

$$d_{n,m,k} = (-1)^{n+k} k! \binom{n+1}{k} \binom{n+2m-2}{k}. \quad (2.4.21)$$

In Section 2.4.3 we will show that $\mathcal{M}(\alpha,\beta)$ is only singular at $k_T = 0$, and satisfies the MLT for each external leg. We will then dedicate Section 2.4.4 to computing $\mathcal{M}(\alpha,\beta)$ and finding the final form of these single exchange diagrams.

2.4.3 Proof of only k_T poles in single exchange diagrams

In the literature (see e.g. [48]), it has been shown that exchange diagrams at tree-level have a restricted set of singularities: they can be singular when the energy of all external legs sums to zero, and when the energy of an individual vertex sums to zero. This is a consequence of having local vertices and Bunch-Davies initial conditions. Hence for $\mathcal{M}(\alpha, \beta)$, in general the allowed singularities are at $k_T = 0$ and $k_3 = 0$. We will now show that the leading single exchange diagrams in the EFToI only have poles as $k_T \to 0$ and they always satisfy the MLT for all three legs.

Let us study the integral

$$\mathcal{M}(\alpha, 0) = \int_{-\infty}^{0} \mathrm{d}\eta \int_{-\infty}^{0} \mathrm{d}\eta' \;\; a^2(\eta) K(k_1, \eta) K(k_2, \eta) G(k_3, \eta, \eta') K(k_3, \eta'). \tag{2.4.22}$$

which is the *master time integral* $\mathcal{M}(\alpha, \beta)$ for the case of $\beta = 0$. It is important to note that actually $\beta \geq 1$, as this will play an important role in the proof. We can perform the η' vertex first, since the only propagators that depend on η' are $G(k_3, \eta, \eta')$ and $K(k_3, \eta')$,

$$\begin{aligned} \mathcal{M}(\alpha, 0) &= \int_{-\infty}^{0} \mathrm{d}\eta \;\; a^2(\eta) K(k_1, \eta) K(k_2, \eta) \int_{-\infty}^{0} \mathrm{d}\eta' \;\; G(k_3, \eta, \eta') K(k_3, \eta') & (2.4.23) \\ &= \int_{-\infty}^{0} \mathrm{d}\eta \;\; a^2(\eta) K(k_1, \eta) K(k_2, \eta) M(k_3, \eta). & (2.4.24) \end{aligned}$$

where a new combined propagator $M(k_3, \eta)$ has been introduced, which is just a mixed propagator, like those mentioned in [54].[10] Thus, we can focus on the integral of the mixed propagator, and see why it gives us a contact like contribution

$$\begin{aligned} M(k_3, \eta) &= a^2(\eta) \int_{-\infty}^{0} \mathrm{d}\eta' \;\; G(k_3, \eta, \eta') K(k_3, \eta') \\ &= a^2(\eta) \frac{H^2}{24 k_3^4} e^{-ik_3\eta} (15(1 + ik_3\eta)\theta(\eta) - e^{2ik_3\eta}[6ik_3\eta((-1+i) + k_3\eta)((1+i) + k_3\eta) \\ &\quad + 4k_3\eta(i + k_3\eta)(3 + k_3^2\eta^2)\theta(-\eta) + 3(5 + k_3\eta(-i + 2k_3\eta(2 - ik_3\eta)))\theta(\eta))] \, . \end{aligned} \tag{2.4.25}$$

We know that

$$\int_{-\infty}^{0} \mathrm{d}\eta \;\; P(\eta)\theta(\eta) = 0 \, , \tag{2.4.26}$$

where $P(\eta)$ is a complex polynomial of η, hence we can drop terms with $\theta(\eta)$, leaving us with

$$M(k_3, \eta) = \frac{1}{12 k_3^3} \frac{1}{\eta} e^{ik_3\eta} [-6i(1 - ik_3\eta) + 3ik_3^2\eta^2 + 2i(1 - ik_3\eta)(3 + k_3^2\eta^2)\theta(-\eta)] \, . \tag{2.4.27}$$

[10] A mixed propagator combines a bulk-bulk propagator with a bulk-boundary propagator to give a pseudo "bulk-boundary" propagator.

We can then drop the $\theta(-\eta)$, since we are now just integrating over negative η

$$M(k_3,\eta) = \frac{1}{12k_3^3}\frac{1}{\eta}e^{ik_3\eta}(5ik_3^2\eta^2 + 2k_3^3\eta^3) = \frac{1}{12k_3}e^{ik_3\eta}(5i\eta + 2k_3\eta^2) = \frac{1}{12k_3}e^{ik_3\eta}P(\eta^{3-1}) , \tag{2.4.28}$$

which we can rewrite the mixed propagator $M(k_3,\eta)$ in terms of bulk-boundary propagators

$$M(k_3,\eta) = \frac{i}{12k_3^3}\left[\dot{K}(k_3,\eta) + 16\dot{K}\left(\frac{k_3}{2},\eta\right)K\left(\frac{k_3}{2},\eta\right)\right] . \tag{2.4.29}$$

Therefore the exchange diagram can be rewritten as a sum of contact diagrams, and thus there must be a field redefinition to rewrite the quadratic mixing as a contact contribution. The mixed propagator $M(k_3,\eta)$ gives the k_3^{-1} pole found for $\mathcal{M}(\alpha,0)$. This means we can only get this pole and the k_T pole for $\mathcal{M}(\alpha,0)$, since

$$\int_{-\infty}^{0} \mathrm{d}\eta \;\; P(\eta^{p-1})e^{ik_a\eta} = \frac{1}{k_a^p} + \mathcal{O}(k_a^{p-1}), \qquad p \neq 0 , \tag{2.4.30}$$

where $P(\eta^{p-1})$ is a polynomial of order $p-1$. Hence, an $\eta^4 e^{ik_T\eta}$ term gives an order five k_T pole. Now we can show this explicitly by including the contribution from the other bulk-boundary propagators in the cubic GR vertex to give the full form of $\mathcal{M}(\alpha,0)$,

$$\mathcal{M}(\alpha,0) \sim \int_{-\infty}^{0} \mathrm{d}\eta \;\; \frac{1}{k_3}e^{i(k_1+k_2+k_3)\eta}P(\eta^{2-1})P(\eta^{2-1})P(\eta^{3-1}) = \frac{1}{k_3}e^{ik_T\eta}P(\eta^{5-1}) , \tag{2.4.31}$$

hence we only get a k_T pole of order five and k_3 pole of order one. This is removed provided we have at least one spatial or time derivative in the quadratic mixing, i.e. $\beta \geq 1$, which we do and this can be generalised to general β as we do in [1]. Therefore, we can never get k_3 poles.

We conclude that single exchange diagrams in the EFToI only have k_T poles, and the final answer is given by a tensor structure multiplied by a function of the energies that satisfies the MLT for each leg, summed over permutations. It follows that the final result of these diagrams is captured by the analysis of [38]. It would be interesting to investigate if this holds for other cubic diagrams too i.e. those with more bulk-bulk propagators.

2.4.4 Type-I: Single exchange diagrams

We begin with the exchange contributions to the wavefunction which arise due to the quadratic interactions in (2.4.1)

$$S_{\gamma,2\mathrm{BB}} \supset \frac{M_{\mathrm{pl}}^2}{4}\int \mathrm{d}\eta\, \mathrm{d}^3x \sum_{n=1} g_{n,0}a^{2-2n}(\eta)(\gamma_{ij})^{2n+1}\gamma'_{ij} \;+ \sum_{n=0,m=1} g_{n,m}a^q(\eta)(\gamma_{ij})^{n+1}\left(\partial^{2m}\gamma'_{ij}\right) . \tag{2.4.32}$$

where $q = 2 - 2m - n$ and $m \geq 1$. We have computed these contributions and they are given by Eqns. (2.4.11) and (2.4.13). If we write the polarisation tensors in such a way that we have full symmetry in particles 1 and 2, then for the $+++$ configuration we have

$$4e^{h_1}_{ik}e^{h_2}_{jl}k^3_k k^3_l e^{h_3}_{ij} - e^{h_1}_{ij}e^{h_2}_{kl}k^3_k k^3_l e^{h_3}_{ij} - e^{h_2}_{ij}e^{h_1}_{kl}k^3_k k^3_l e^{h_3}_{ij} = -\mathrm{SH}_{+++}\left(2I_1I_2 + \frac{1}{2}(I_1^2+I_2^2)\right). \quad (2.4.33)$$

Using this expression in Eqns. (2.4.11) and (2.4.13), for the $+++$ configuration, we find

$$\psi^{+++}_{\text{2BB, exchange}} = \sum_{n=1} \frac{iM^4_{\rm pl}g_{n,0}}{16}\mathrm{SH}_{+++}k_T^2\mathcal{I}_{n,0}(k_1,k_2,k_3) + 2 \text{ perms}, \quad (2.4.34)$$

$$+ \sum_{n=0,m=1} \frac{(-1)^m iM^4_{\rm pl}g_{n,m}}{16}\mathrm{SH}_{+++}k_T^2\mathcal{I}_{n,m}(k_1,k_2,k_3) + 2 \text{ perms}. \quad (2.4.35)$$

Notice how once we summed over the remaining two permutations in the tensor structure, the contribution reduced simply to k_T^2. Recall that here we are using the tensor structure of GR where this factor of k_T^2 is familiar [30] and cancels the $1/k_T^2$ that comes from the time integral when computed in pure gravity. This means that the $+++$ configuration for pure gravity in de Sitter space does not have a total-energy pole which can be traced back to the fact that the corresponding amplitude for this configuration is zero. As we explained above we can now easily extract the $++-$ configuration which is given by

$$\psi^{++-}_{\text{2BB, exchange}} = \sum_{n=1} \frac{iM^4_{\rm pl}g_{n,0}}{16}\mathrm{SH}_{++-}I_3^2\left(\mathcal{I}_{n,0}(k_1,k_2,k_3) + 2 \text{ perms}\right), \quad (2.4.36)$$

$$+ \sum_{n=0,m=1} \frac{(-1)^m iM^4_{\rm pl}g_{n,m}}{16}\mathrm{SH}_{++-}I_3^2\left(\mathcal{I}_{n,m}(k_1,k_2,k_3) + 2 \text{ perms}\right). \quad (2.4.37)$$

where we have used the fact that $k_T^2 \to I_3^2$ as we send $k_3 \to -k_3$. We remind the reader that $\mathcal{I}_{n,0}$ and $\mathcal{I}_{n,m}$ are defined in Eqns. (2.4.12) and (2.4.14), which can be compactly written as Eqns. (2.4.18), (2.4.19) and (2.4.20) in terms of the master time integral (2.4.15). Our focus now is on computing this master time integral

$$\mathcal{M}(\alpha,\beta) = \int \mathrm{d}\eta\,\mathrm{d}\eta'\; a^2(\eta)K_\gamma(k_1,\eta)K_\gamma(k_2,\eta)G(k_3,\eta,\eta')a^\alpha(\eta')K_\gamma^{(\beta)}(k_3,\eta'). \quad (2.4.38)$$

To compute this integral we can use the formalism developed in [144] to write down a dispersion formula in terms of discontinuities of the bulk-boundary propagator as shown below. This is equivalent to factorizing the diagram in terms of three-point and two-point functions, i.e. "cutting rules". We refer the reader to [52, 67] for more discussion of these "cosmological cutting rules". In order to ensure a Bunch-Davies vacuum in the infinite past requires the use of the $i\epsilon$ prescription, $k \to k - i\epsilon$, where the norm of k is given a negative imaginary part, i.e. $-k \to -k + i\epsilon$. In polar coordinates, $k^2 = e^{i\theta}$, this becomes the condition $\theta \in (-2\pi, 0)$. If θ is in this interval, then the Feynman integrals converge. It is therefore natural to place the k^2 branch cut on the positive real axis. We then define the following monodromy operation

$$\mathrm{disc}_{p^2} f(k^2) = f((e^{-i\pi}k)^2) - f(k^2) = f((-k+i\epsilon)^2) - f((k-i\epsilon)^2) = f((-k^*)^2) - f(k^2), \quad (2.4.39)$$

i.e. the argument of the term on the left hand side comes from rotating k in the complex plane by $\theta = -\pi$. By utilising the Hermitian analyticity of the bulk-boundary propagator, we can express the discontinuity of the bulk-boundary propagator as[11]

$$\mathrm{disc}_{p^2} K(p,\eta) = K(e^{-i\pi}p,\eta) - K(p,\eta) = K^*(p,\eta) - K(p,\eta) = -2i\,\mathrm{Im}K(p,\eta). \tag{2.4.40}$$

We can write a general bulk-bulk propagator $G_\nu(k,\eta,\eta')$ in terms of a dispersion formula

$$G_\nu(k,\eta,\eta') = \frac{1}{2\pi i}\int_0^{+\infty} \frac{dp^2}{p^2-k^2+i\epsilon}\mathrm{disc}_{p^2} G_\nu(p,\eta,\eta'), \tag{2.4.41}$$

where ν is the usual order of the Hankel function related to the mass of the bulk field by

$$\nu = \sqrt{\frac{9}{4} - m^2}\,. \tag{2.4.42}$$

The discontinuity of the bulk-bulk propagator can then be expressed in terms of the discontinuities of the bulk-boundary propagators

$$\mathrm{disc}_{p^2} G_\nu(p,\eta,\eta') = iP_\nu(p)\mathrm{disc}_{p^2}K(p,\eta)\mathrm{disc}_{p^2}K(p,\eta'), \tag{2.4.43}$$

where P_ν is the power spectrum. Given that we are working with massless gravitons in the EFToI, we can take $\nu = 3/2$, allowing us to rewrite Eq. (2.4.38) as

$$\mathcal{M}(\alpha,\beta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dp}{p^2-k_3^2+i\epsilon}\frac{iH^2}{M_{\mathrm{pl}}^2 p^2}\int_{-\infty}^0 \mathrm{d}\eta\,(-H\eta)^{-2}K(k_1,\eta)K(k_2,\eta)[K^*(p,\eta)-K(p,\eta)]$$
$$\int_{-\infty}^0 \mathrm{d}\eta'\,(-H\eta')^{\alpha}K^{(\beta)}(k_3,\eta')[K^*(p,\eta')-K(p,\eta')], \tag{2.4.44}$$

Here, we have switched the order of the integrals. This is typically not allowed: the behavior of $\mathrm{disc}_{p^2}K(p,\eta)$ in the infinite past is such that the η integral converges only for particular values of p^2. One can carry out the integral for these values and then analytically continue: the final result is that the nested integral becomes an integral in p^2 of the product of the discontinuities of single integrals which is an even function of p. We can then change variables from p^2 to p, and take advantage of the fact that the whole integrand is even in p to extend the range of integration from $-\infty$ to $+\infty$ [144]. Carrying out the integrals in $\mathrm{d}\eta$ and $\mathrm{d}\eta'$ we are left with an integral in dp

$$\mathcal{M}(\alpha,\beta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dp\,\mathcal{N}(k_1,k_2,k_3,p)\,, \tag{2.4.45}$$

[11]A function is described to be Hermitian analytic provided it satisfies the relation $f^*(-k^*) = f(k)$. We refer the reader to [50, 52] for further discussion on this topic.

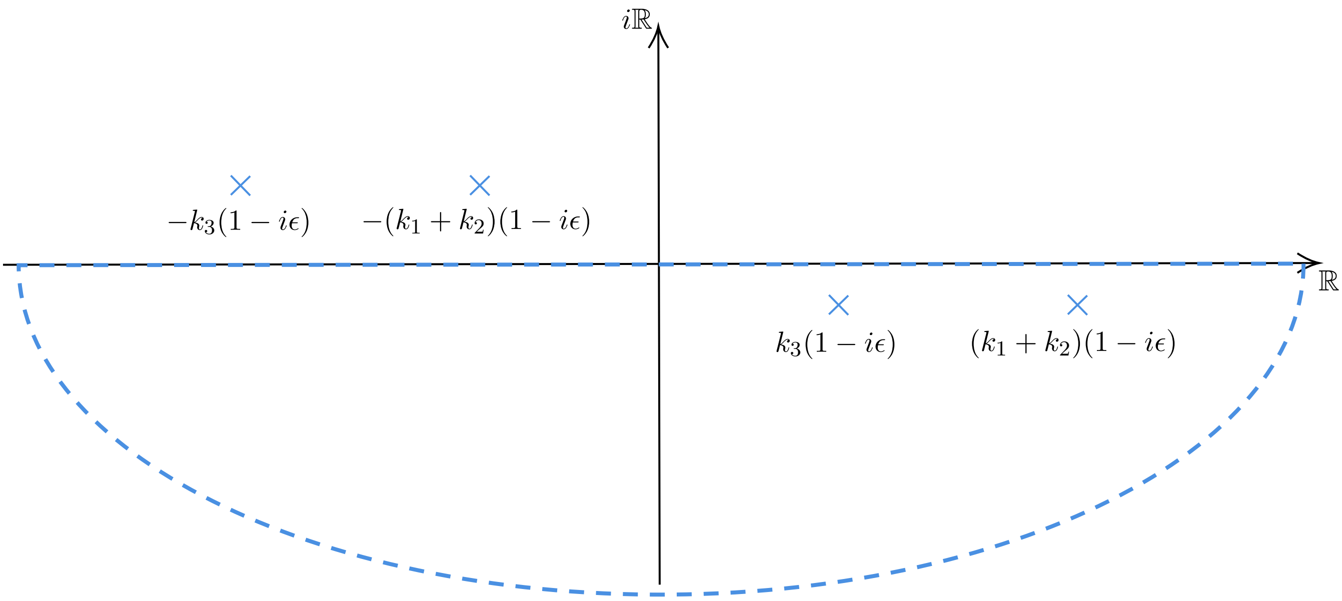

Fig. 2.3 Contour taken to compute ψ_3 time integral for single exchange diagrams.

where we have dropped the $i\epsilon$ for simplicity of notation and defined the integrand of the form

$$\begin{aligned}\mathcal{N}(k_1,k_2,k_3,p) =& -\frac{iH^{-\alpha}}{M_{\rm pl}^2}\frac{p(k_1^2+4k_1k_2+k_2^2-p^2)}{(p^2-k_3^2)(k_1+k_2-p)^2(k_1+k_2+p)^2}\\ &\times (ik_3)^{\beta}(-\alpha)!i^{\alpha}\Bigg[\frac{((k_3-p)(k_3+p)^{\alpha}-(k_3+p)(k_3-p)^{\alpha})\,(\beta-1)}{k_3^2-p^2}\\ &+(-\alpha+1)\big((k_3-p)^{\alpha-2}(k_3+\beta p-p)-(k_3+p)^{\alpha-2}(k_3-\beta p+p)\big)\\ &-(-\alpha+2)(-\alpha+1)k_3p\big((k_3-p)^{\alpha-3}+(k_3+p)^{\alpha-3}\big)\Bigg],\end{aligned} \tag{2.4.46}$$

which is manifestly symmetric under $p\to -p$. We also see that the integrand Eq. (2.4.46) vanishes as $p\to\infty$ in the complex plane, and therefore we can close the p-contour in either the upper or lower half-plane.

The poles in p are located at

$$\begin{aligned}p &= \pm k_3\\ p &= \pm(k_1+k_2).\end{aligned} \tag{2.4.47}$$

When evaluating the contour integral, we should recall that the Bunch-Davies boundary condition implies that the k_a have a small, negative imaginary part. Finally, we can close the contour in the lower half-plane to find

$$\mathcal{M}(\alpha,\beta) = -\left(\mathrm{Res}_{p=(k_1+k_2)}\Big[\mathcal{N}(k_1,k_2,k_3,p)\Big]+\mathrm{Res}_{p=k_3}\Big[\mathcal{N}(k_1,k_2,k_3,s)\Big]\right). \tag{2.4.48}$$

The residue at $(k_1 + k_2)$ can be computed for generic n, m but the same is not true for the residue at k_3, for which we cannot find a closed-form expression. In any case we notice that, taken separately, the residues at k_3 and $(k_1 + k_2)$ both present a divergence at $k_1 + k_2 = k_3$. It is only when combined that such divergence cancels, as it was expected from the analysis of Section 2.4.2. Let us end this section by writing down some expressions for the time integrals of interest so we can see the expected properties explicitly. Since in all cases the result must be symmetric in the exchange of k_1 and k_2, we write the integrals in terms of the symmetric polynomials in two variables: $\hat{e}_1 = k_1 + k_2$, $\hat{e}_2 = k_1 k_2$. We still use $k_T = \hat{e}_1 + k_3$. We have (note that the factors of H and $M_{\rm pl}$ come from the scale factors and the graviton power spectrum):

$$\mathcal{I}_{1,0} = \frac{ik_3^3}{2M_{\rm pl}^2 k_T^5}(7\hat{e}_1^3 + 14\hat{e}_1\hat{e}_2 + 11\hat{e}_1^2 k_3 - 2\hat{e}_2 k_3 + 5\hat{e}_1 k_3^2 + k_3^3), \tag{2.4.49}$$

$$\mathcal{I}_{2,0} = \frac{3iH^2 k_3^3}{2M_{\rm pl}^2 k_T^7}[15\hat{e}_1^5 + 30\hat{e}_1^3\hat{e}_2 + k_3(25\hat{e}_1^4 - 30\hat{e}_1^2\hat{e}_2) \tag{2.4.50}$$

$$+ k_3^2(26\hat{e}_1^3 + 34\hat{e}_1\hat{e_2}) + k_3^3(22\hat{e}_1^2 - 2\hat{e}_2) + 7k_3^4\hat{e}_1 + k_3^5], \tag{2.4.51}$$

$$\mathcal{I}_{0,1} = \frac{ik_3}{2M_{\rm pl}^2 k_T^5}(-\hat{e}_1^3 - 2\hat{e}_1\hat{e}_2 + 3\hat{e}_1^2 k_3 + 14\hat{e}_2 k_3 + 5\hat{e}_1 k_3^2 + k_3^3), \tag{2.4.52}$$

$$\mathcal{I}_{1,1} = \frac{iHk_3}{4M_{\rm pl}^2 k_T^5}(-\hat{e}_1^3 - 2\hat{e}_1\hat{e}_2 + 3\hat{e}_1^2 k_3 + 14\hat{e}_2 k_3 + 5\hat{e}_1 k_3^2 + k_3^3), \tag{2.4.53}$$

$$\mathcal{I}_{2,1} = \frac{12iH^2 k_3^2}{M_{\rm pl}^2 k_T^7}(\hat{e}_1^4 + 3\hat{e}_1^2\hat{e}_2 - 7\hat{e}_1\hat{e}_2 k_3 - \hat{e}_1^2 k_3^2 + 2\hat{e}_2 k_3^2). \tag{2.4.54}$$

We note that the expressions for $\mathcal{I}_{0,1}$ and $\mathcal{I}_{1,1}$ are proportional which was to be expected. This is a consequence of the fact that for $m \neq 0$ and odd n, we can integrate by parts to write the quadratic mixing in terms of operators with lower values of n. As we explained before, it is still useful to keep our current labelling and definition of n however, since we expect the contact contributions to be different for all n. For larger values of n, it will remain the case that for odd n we can write the result in terms of expressions with lower and even n. As another example, we have $\mathcal{I}_{1,3} = \frac{9H}{2}\mathcal{I}_{1,2} - 3H^2\mathcal{I}_{1,1}$. As expected, we see that only total-energy poles arise, and once we include the factors of k_3^{2m} the MLT will be satisfied for each k.

2.4.5 Type-I: Contact diagrams

In addition to the exchange contributions we have just computed, for Type-I bispectra there are also contact diagrams that contribute to the bispectra at the same order in perturbation theory. These arise due to the cubic interactions in (2.4.1) which take the form

$$S_{\gamma,2\rm BB} \supset \frac{M_{\rm pl}^2}{4}\int {\rm d}\eta\, {\rm d}^3x a^q(\eta) \sum_{n,m=1} g_{n,m}(\gamma_{ij})^{n+1}\left(-m\gamma^{lk}\partial^{2m-2}\partial_l\partial_k\gamma'^i{}_j + \sum_{p=0}^{m-1}\partial^{2p}Q_s[\gamma, \partial^{2m-2-2p}\gamma']^i{}_j\right), \tag{2.4.55}$$

where $q = 2 - 2m - n$ and $m \geq 1$. Each contribution contains one γ_{ij} with $n+1$ time derivatives, one with a single time derivative, and one without any. Up to permutations and overall factors, the time integral that we therefore need to compute is therefore

$$\mathcal{J}_{n,m}(k_1,k_2,k_3) = \int \mathrm{d}\eta\ \eta^{n+2m-2} K^{(n+1)}(k_1,\eta)K'(k_2,\eta)K(k_3,\eta). \tag{2.4.56}$$

With the result of this integral we can multiply it by the tensor structure dictated by the cubic interactions, then sum over permutations. Given that $n+2m-2 \geq 0$, the result of this integral will only have poles at $k_T = 0$, and no logs. Using the general expression for the derivatives of the bulk-boundary propagator

$$K^{(\beta)}(k,\eta) = \frac{\partial^\beta}{\partial\eta^\beta}K(k,\eta) = (ik)^\beta(1-\beta-ik\eta)e^{ik\eta}. \tag{2.4.57}$$

and

$$\lim_{\eta\to 0}\int \mathrm{d}\eta\ \eta^z e^{ik_T\eta} = \frac{z!(-i)^{1-z}}{k_T^{1+z}} + \mathcal{O}(\eta^{z+1}), \tag{2.4.58}$$

for $z \geq 0$, we find that at late times the bulk time integral yields

$$\begin{aligned}\mathcal{J}^{123}_{n,m} = (-1)^{m-n} in(n+2m-1)!\frac{k_1^{n+1}k_2^2}{k_T^{n+2m}} \\ + (-1)^{1+m-n}i(n+2m)!\frac{k_1^{n+1}k_2^2(k_1-nk_3)}{k_T^{n+2m+1}} \\ + (-1)^{1+m-n}i(n+2m+1)!\frac{k_1^{n+2}k_2^2k_3}{k_T^{n+2m+2}},\end{aligned} \tag{2.4.59}$$

where we have introduced a more compact notation with $\mathcal{J}^{123}_{n,m} \equiv \mathcal{J}_{n,m}(k_1,k_2,k_3)$. We see that the degree of the leading total-energy pole is equal to the number of derivatives in the cubic vertices, as expected [29], and one can check that this expression satisfies the MLT for each external energy.

Now for the tensor structure we have

$$Q_s[\gamma,\gamma']_{ij} = -\partial_i\gamma_{kl}\partial_k\gamma'_{lj} + \partial_l\gamma_{jk}\partial_k\gamma'_{il} + (i \leftrightarrow j), \tag{2.4.60}$$

and in the final term in (2.4.55) we can integrate by parts to move all ∂^{2p} terms onto the first γ_{ij} such that

$$S_{\gamma,2\mathrm{BB}} \supset -\frac{M_{\mathrm{pl}}^2}{4}\int \mathrm{d}\eta\,\mathrm{d}^3x a^q(\eta) \sum_{n,m=1} m g_{n,m}(\gamma_{ij})^{n+1}\gamma^{lk}\partial^{2m-2}\partial_l\partial_k\gamma'_{ij} \tag{2.4.61}$$

$$+\frac{M_{\mathrm{pl}}^2}{4}\int \mathrm{d}\eta\,\mathrm{d}^3x a^q(\eta) \sum_{n,m=1}\sum_{p=0}^{m-1} g_{n,m}\partial^{2p}(\gamma_{ij})^{n+1}Q_s[\gamma,\partial^{2m-2-2p}\gamma']_{ij}. \tag{2.4.62}$$

From the first line, once we convert to momentum space using, we will find tensor structures of the form

$$k_2^{2m-2}e^{h_1}_{ij}(\mathbf{k}_1)e^{h_2}_{ij}(\mathbf{k}_2)k_l^2k_k^2e^{h_3}_{lk}(\mathbf{k}_3) + \text{perms}, \tag{2.4.63}$$

and if we convert this expression into one with spinors, for the $+++$ configuration we find

$$k_2^{2m-2}e_{ij}^{+}(\mathbf{k}_1)e_{ij}^{+}(\mathbf{k}_2)k_l^2k_k^2e_{lk}^{+}(\mathbf{k}_3) = \frac{1}{2}\frac{k_2^{2m-2}I_3^2}{e_3^2}[12]^2[23]^2[31]^2 = \frac{1}{2}k_2^{2m-2}I_3^2\text{SH}_{+++}. \tag{2.4.64}$$

We remind the reader that the $++-$ configuration can be extracted from this expression, as we explained above. For the second line, the tensor structures are

$$k_1^{2p}k_2^{2m-2-2p}e_{ij}^{h_1}(\mathbf{k}_1)[e_{il}^{h_2}(\mathbf{k}_2)e_{jk}^{h_3}(\mathbf{k}_3)k_k^2k_l^3 - e_{lj}^{h_2}(\mathbf{k}_2)e_{kl}^{h_3}(\mathbf{k}_3)k_k^2k_i^3] + \text{perms}, \tag{2.4.65}$$

which when converted to spinors yields

$$k_1^{2p}k_2^{2m-2-2p}e_{ij}^{+}(\mathbf{k}_1)[e_{il}^{+}(\mathbf{k}_2)e_{jk}^{+}(\mathbf{k}_3)k_k^2k_l^3 - e_{lj}^{+}(\mathbf{k}_2)e_{kl}^{+}(\mathbf{k}_3)k_k^2k_i^3] = -k_1^{2p}k_2^{2m-2-2p}k_3I_3\text{SH}_{+++}. \tag{2.4.66}$$

We can now collect everything together. By multiplying these tensor structures by the result of the time integral, and including all constant factors as dictated by the Feynman rules, for the $+++$ helicity configuration we find

$$\begin{aligned}\psi_{\text{2BB,contact}}^{+++} &= \frac{iM_{\text{pl}}^2H^{n+2m-2}}{8}\text{SH}_{+++}\sum_{n,m=1} g_{n,m}(-1)^{n+m}m\left[k_2^{2m-2}I_3^2\mathcal{J}_{n,m}^{123} + 5\text{ perms}\right]\\ &+ \frac{iM_{\text{pl}}^2H^{n+2m-2}}{2}\text{SH}_{+++}\sum_{n,m=1} g_{n,m}(-1)^{n+m}\sum_{p=0}^{m-1}\left[k_1^{2p}k_2^{2m-2p-2}k_3I_3\mathcal{J}_{n,m}^{123} + 5\text{ perms}\right].\end{aligned} \tag{2.4.67}$$

We have summed over the remaining permutations to find an object with the correct symmetry. Now to extract the wavefunction coefficient for the $++-$ helicity configuration we need to send $k_3 \to -k_3$ and $\text{SH}_{+++} \to \text{SH}_{++-}$, while keeping $\mathcal{J}_{n,m}$ fixed. This yields

$$\begin{aligned}\psi_{\text{2BB,contact}}^{++-} &= \frac{iM_{\text{pl}}^2H^{n+2m-2}}{8}\text{SH}_{++-}\sum_{n,m=1} g_{n,m}(-1)^{n+m}m[k_2^{2m-2}k_T^2\mathcal{J}_{n,m}^{123} + k_1^{2m-2}k_T^2\mathcal{J}_{n,m}^{213}\\ &+ k_1^{2m-2}I_1^2\mathcal{J}_{n,m}^{312} + k_3^{2m-2}I_1^2\mathcal{J}_{n,m}^{132} + k_2^{2m-2}I_2^2\mathcal{J}_{n,m}^{321} + k_3^{2m-2}I_2^2\mathcal{J}_{n,m}^{231}]\\ &+ \frac{iM_{\text{pl}}^2H^{n+2m-2}}{2}\text{SH}_{++-}\sum_{n,m=1} g_{n,m}(-1)^{n+m}\sum_{p=0}^{m-1}[-k_1^{2p}k_2^{2m-2p-2}k_3k_T\mathcal{J}_{n,m}^{123}\\ &- k_2^{2p}k_1^{2m-2p-2}k_3k_T\mathcal{J}_{n,m}^{213} - k_1^{2p}k_3^{2m-2p-2}k_2I_1\mathcal{J}_{n,m}^{132}\\ &- k_3^{2p}k_1^{2m-2p-2}k_2I_1\mathcal{J}_{n,m}^{312} - k_2^{2p}k_3^{2m-2p-2}k_1I_2\mathcal{J}_{n,m}^{231} - k_3^{2p}k_2^{2m-2p-2}k_1I_2\mathcal{J}_{n,m}^{321}].\end{aligned} \tag{2.4.68}$$

We note that for both helicity configurations these contributions to the wavefunction are real since the overall factor of i is cancelled by the i in $\mathcal{J}_{n,m}$. This ensures that these contributions do indeed contribute to the bispectra since for parity-even interactions only the real part contributes (c.f. (2.1.23)). In the following section we will combine these contributions to the cubic wavefunction with the exchange contributions we computed above to derive the full form of the two building block bispectra.

2.4.6 Type I: Putting everything together

Now that we have all contributions to the cubic wavefunction coefficient, to leading order in new couplings, we can now convert these into expressions for the bispectra. In perturbation theory expectation values are algebraically related to wavefunction coefficients with the relations derived in a number of places e.g. [38, 115, 116]. When there is a small correction to the two-point function, as is the case here, the expression for parity-even interactions is given by

$$B_3^{\{\lambda_i\}} = \frac{1}{\Pi_{i=1}^3 2\text{Re}\left(\psi_{2,\text{GR}}\right)}\left[-2\text{Re}\left(\psi_{3,\text{total}}^{\{\lambda_i\}}\right) + 2\text{Re}\left(\psi_{3,\text{GR}}^{\{\lambda_i\}}\right)\left(\frac{\text{Re}\left(\delta\psi_2^{\lambda_1}\right)}{\text{Re}\left(\psi_{2,\text{GR}}\right)} + 2\text{ perms}\right)\right], \tag{2.4.69}$$

where $\psi_{2,\text{GR}}$ is the GR contribution to the two-point function, while $\delta\psi_2$ is a small correction due to our higher-derivative corrections to the quadratic action. Since we are working up to linear order in new couplings, we take the first term in the square brackets to be all contributions we have computed in this section i.e. $\psi_{3,\text{total}} = \psi_{2\text{BB,exchange}} + \psi_{2\text{BB,contact}}$, while the ψ_3 in the second term must be the GR contribution since $\delta\psi_2$ is already linear the new couplings. In addition to $\psi_{3,\text{total}}$, we now also need the GR wavefunction up to cubic order, and the small corrections to ψ_2.

We have essentially already computed the GR cubic wavefunction coefficient when we computed the Type-I exchange diagrams so let us simply write the result here. We have

$$\psi_{3,\text{GR}}^{+++} = \frac{M_{\text{pl}}^2}{8H^2}\text{SH}_{+++}\frac{k_T^2}{k_T^2}(e_3 + k_T e_2 - k_T^3), \tag{2.4.70}$$

$$\psi_{3,\text{GR}}^{++-} = \frac{M_{\text{pl}}^2}{8H^2}\text{SH}_{++-}\frac{I_3^2}{k_T^2}(e_3 + k_T e_2 - k_T^3). \tag{2.4.71}$$

For the corrections to the two-point function, we need to compute the Feynman diagram in Figure 2.2. Since this is a small correction to the quadratic wavefunction, we compute it in the way we compute any contact diagram: we insert a bulk-boundary propagator for each external line, add tensor structures and time derivatives as dictated by corresponding the bulk vertex, and use the Feynman rules we discussed above. For example, for $m = 0$ we have

$$\delta\psi_2^{hh'} = -i\sum_{n=1}\frac{g_{n,0}M_{\text{pl}}^2 H^{2n-2}}{4}\int d\eta\ \eta^{2n-2}K_\gamma^{(2n+1)}(k,\eta)K_\gamma'(k,\eta)\times 2\delta_{hh'}\times 2 \tag{2.4.72}$$

$$= \sum_{n=1}\frac{(2n)!}{2^{2n+1}}g_{n,0}M_{\text{pl}}^2 H^{2n-2}k^3\delta_{hh'}, \tag{2.4.73}$$

where we have used momentum conversation and summed over the two possible permutations. The computation for $m \neq 0$ is very similar and in total we have

$$\delta\psi_2^{hh'} = \sum_{n=1}\frac{(2n)!}{2^{2n+1}}g_{n,0}M_{\text{pl}}^2 H^{2n-2}k^3\delta_{hh'} + \sum_{n=0,m=1}\frac{(-1)^n(2m+n-1)!(n-2m)}{2^{n+2m+1}}g_{n,m}M_{\text{pl}}^2 H^{n+2m-2}k^3\delta_{hh'}. \tag{2.4.74}$$

The final forms of the bispectra at linear order in the new couplings for a few n and m:

$$n=1,m=0: \quad \delta B_3^{+++} = -g_{1,0}\frac{H^6}{4M_{\rm pl}^4 e_3^3}\mathrm{SH}_{+++}\frac{k_T^6 - k_T^4 e_2 - e_3 k_T^3 + 24 e_3^2}{k_T^3}, \tag{2.4.75}$$

$$\delta B_3^{++-} = -g_{1,0}\frac{H^6}{4M_{\rm pl}^4 e_3^3}\mathrm{SH}_{++-}I_3^2\frac{k_T^6 - k_T^4 e_2 - e_3 k_T^3 + 24 e_3^2}{k_T^5}, \tag{2.4.76}$$

$$\delta P^{hh'} = -g_{1,0}\frac{H^4}{2M_{\rm pl}^2 k^3}\delta_{hh'}. \tag{2.4.77}$$

$$n=2,m=0: \quad \delta B_3^{+++} = -g_{2,0}\frac{3H^8}{4M_{\rm pl}^4 e_3^3}\mathrm{SH}_{+++}k_T^{-5}(k_T^8 - e_2 k_T^6 - e_3 k_T^5 + 80 e_3^2 k_T^2 - 240 e_2 e_3^2), \tag{2.4.78}$$

$$\delta B_3^{++-} = -g_{2,0}\frac{3H^8}{4M_{\rm pl}^4 e_3^3}\mathrm{SH}_{++-}I_3^2 k_T^{-7}(k_T^8 - e_2 k_T^6 - e_3 k_T^5 + 80 e_3^2 k_T^2 - 240 e_2 e_3^2), \tag{2.4.79}$$

$$\delta P^{hh'} = -g_{2,0}\frac{3H^6}{2M_{\rm pl}^2 k^3}\delta_{hh'}. \tag{2.4.80}$$

$$n=3,m=0: \quad \delta B_3^{+++} = -g_{3,0}\frac{45H^{10}}{8M_{\rm pl}^4 e_3^3}\mathrm{SH}_{+++}k_T^{-7}(k_T^{10} - k_T^8 e_2 - k_T^7 e_3 + 448 k_T^4 e_3^2 \tag{2.4.81}$$

$$- 2240 k_T^2 e_2 e_3^2 - 560 k_T e_3^3 + 1792 e_2^2 e_3^2), \tag{2.4.82}$$

$$\delta B_3^{++-} = -g_{3,0}\frac{45H^{10}}{8M_{\rm pl}^4 e_3^3}\mathrm{SH}_{++-}I_3^2 k_T^{-9}(k_T^{10} - k_T^8 e_2 - k_T^7 e_3 + 448 k_T^4 e_3^2$$

$$- 2240 k_T^2 e_2 e_3^2 - 560 k_T e_3^3 + 1792 e_2^2 e_3^2), \tag{2.4.83}$$

$$\delta P^{hh'} = -g_{3,0}\frac{45H^8}{4M_{\rm pl}^2 k^3}\delta_{hh'}. \tag{2.4.84}$$

$$n=0,m=1: \quad \delta B_3^{+++} = g_{0,1}\frac{H^6}{4M_{\rm pl}^4 e_3^3}\mathrm{SH}_{+++}\frac{k_T^6 - k_T^4 e_2 - k_T^3 e_3 + 72 e_3^2}{k_T^3} \tag{2.4.85}$$

$$\delta B_3^{++-} = g_{0,1}\frac{H^6}{4M_{\rm pl}^4 e_3^3}\mathrm{SH}_{++-}I_3^2 k_T^{-5}(e_3^2(-24 I_3^2 + 80 k_T)$$

$$+ e_3(-9 I_3^2 k_T^3 + 8 k_T^5 + 12 e_2 k_T (I_3^2 + k_T^2)) \tag{2.4.86}$$

$$+ k_T(-e_2 I_3^2 k_T^3 + I_3^2 k_T^5 + 4 e_2^2 k_T (I_3^2 - k_T^2) - 16 e_3 k_3 (k_T - k_3)(k_T^2 - 3k_1 k_2))), \tag{2.4.87}$$

$$\delta P^{hh'} = g_{0,1}\frac{H^4}{2M_{\rm pl}^2 k^3}\delta_{hh'}. \tag{2.4.88}$$

$$n=1,m=1: \quad \delta B_3^{+++} = g_{1,1}\frac{H^7}{8M_{\rm pl}^4 e_3^3}\mathrm{SH}_{+++}\frac{384 e_2 e_3^2 + 24 e_3^2 k_T^2 - e_3 k_T^5 - e_2 k_T^6 + k_T^8}{k_T^5}, \tag{2.4.89}$$

$$\delta B_3^{++-} = g_{1,1}\frac{H^7}{16M_{\rm pl}^4 e_3^3}\mathrm{SH}_{++-}k_T^{-5}\Big(- 2I_3^2 \left(24 e_3^2 - 12 e_2 e_3 k_T - 4 e_2^2 k_T^2 + 9 e_3 k_T^3 + e_2 k_T^4 - k_T^6\right)$$

$$- 8\big(- 96 k_1 k_2 e_3^2 + k_3^2 (k_1 + k_2)^2 k_T^4 + e_3 k_3 k_T^3 (k_T - 3k_3)$$

$$+ k_1^2 k_2^2 (48 k_3^4 - 36 k_3^3 k_T - 20 k_3^2 k_T^2 - k_3 k_T^3 + k_T^4)\big)\Big), \tag{2.4.90}$$

$$\delta P^{hh'} = g_{1,1}\frac{H^5}{4M_{\rm pl}^2 k^3}\delta_{hh'}. \tag{2.4.91}$$

2.4.7 Type-I: Checking consistency relations

For Type-I bispectra that arise from two building block operators, at tree-level, and to leading order in the field theory couplings, both single exchange and contact diagrams contribute to ψ_3. The single exchange diagram perturbatively accounts for possible corrections to the graviton power spectrum, for which the leading order contribution comes from taking the cubic vertex to be that of GR with the quadratic vertex coming from expanding two building block operators to quadratic order. The contact diagram that comes from these two building block operators is of a comparable size to this exchange diagram, and indeed cancellations are required between these two diagrams to satisfy the EFToI consistency relations which relate the soft limit of a three-point function to the two-point function, which now receives corrections [147].

2.5 Type-II bispectra

Now consider three building block operators which all start at $\mathcal{O}(\gamma^3)$. As we discussed previously we can treat all covariant derivatives as partial ones and since $\delta K_{ij} \sim \gamma'_{ij}$ at linear order, we simply need to write down all independent contractions of three copies of γ'_{ij}, and spatial derivatives. This problem was tackled in [38] at the level of polarisation tensors and spatial momenta but it is simple to convert it into a Lagrangian statement. There are five independent tensor structures once we use the fact that the graviton is transverse and traceless, and integrate by parts (which is equivalent to momentum conservation). These structures are organised by the number of spatial derivatives that are contracted with a graviton and we denote this number by α. For parity-even interactions α is an even number and for $\alpha = 0, 4, 6$ there is a single structure, while for $\alpha = 2$ there are two. We cannot have $\alpha \geq 8$ since there are only 6 free indices across the three gravitons. For each γ_{ij} we can add additional time derivatives, while additional spatial derivatives can be restricted to two fields only by integration by parts, and two derivatives that are contracted with each other should act on different fields. Any other contractions can be removed by the graviton's equation of motion in favour of time derivatives which we are already adding. We have

$$S_{\gamma,3\text{BB}} = \sum_{\alpha} \int \mathrm{d}\eta\, \mathrm{d}^3x \mathcal{L}_{\gamma,\alpha}, \tag{2.5.1}$$

where

$$\mathcal{L}_{\gamma,\alpha=0} = \sum_{n_1,n_2,n_3,p} a^r(\eta) h^0_{n_1,n_2,n_3,p} (\gamma_{ij})^{1+n_1} \partial_{i_1\ldots i_p} (\gamma_{jk})^{1+n_2} \partial_{i_1\ldots i_p} (\gamma_{ki})^{1+n_3}, \tag{2.5.2}$$

$$\mathcal{L}_{\gamma,\alpha=2} = \sum_{n_1,n_2,n_3,p} a^r(\eta) h^2_{n_1,n_2,n_3,p} (\gamma_{ij})^{1+n_1} \partial_{i_1\ldots i_p} \partial_i (\gamma_{lm})^{1+n_2} \partial_{i_1\ldots i_p} \partial_j (\gamma_{lm})^{1+n_3} \tag{2.5.3}$$

$$+ \sum_{n_1,n_2,n_3,p} a^r(\eta) \hat{h}^2_{n_1,n_2,n_3,p} (\gamma_{ij})^{1+n_1} \partial_{i_1\ldots i_p} (\gamma_{lm})^{1+n_2} \partial_{i_1\ldots i_p} \partial_i \partial_l (\gamma_{jm})^{1+n_3}, \tag{2.5.4}$$

$$\mathcal{L}_{\gamma,\alpha=4} = \sum_{n_1,n_2,n_3,p} a^r(\eta) h^4_{n_1,n_2,n_3,p} \partial_i (\gamma_{lk})^{1+n_1} \partial_{i_1\ldots i_p} \partial_j (\gamma_{mk})^{1+n_2} \partial_{i_1\ldots i_p} \partial_l \partial_m (\gamma_{ij})^{1+n_3}, \tag{2.5.5}$$

$$\mathcal{L}_{\gamma,\alpha=6} = \sum_{n_1,n_2,n_3,p} a^r(\eta) h^6_{n_1,n_2,n_3,p} \partial_m \partial_k (\gamma_{il})^{1+n_1} \partial_{i_1\ldots i_p} \partial_i \partial_n (\gamma_{jm})^{1+n_2} \partial_{i_1\ldots i_p} \partial_j \partial_l (\gamma_{kn})^{1+n_3}, \tag{2.5.6}$$

with constant couplings $h^{\alpha}_{n_1,n_2,n_3,p}$ and $r = 1 - 2p - n_1 - n_2 - n_3 - \alpha$ which is fixed by scale invariance. These are the actions we will use to compute the graviton bispectra. Despite the complicated looking nature of these interactions, we will now show that the resulting bispectra take a very compact form, which we will not actually compute using these explicit Lagrangian expressions and instead we will use the MLT [49] to efficiently write down all possibilities.

2.5.1 Bootstrapping Type-II bispectra

As we have seen, the building block for EFToI interactions for which the consistency relations are satisfied without a correction to the two-point function is γ'_{ij} rather than γ_{ij}. The bulk-boundary propagator for the graviton, with the polarisation factor stripped away, is given by

$$K_\gamma = (1 - ik\eta)e^{ik\eta}, \tag{2.5.7}$$

and its first derivative is given by

$$K'_\gamma = k^2 \eta e^{ik\eta}. \tag{2.5.8}$$

It therefore follows that all graviton bispectra arising from cubic interactions constructed out of γ'_{ij} contain an overall factor of $e_3^2 = (k_1 k_2 k_3)^2$. We can therefore update our ansatz from (2.4.5) to

$$\psi_3^{\text{trimmed}}(\{k\}) = \frac{\overbrace{e_3^2}^{\delta K_{ij} \propto \gamma'_{ij}}}{\underbrace{k_T^p}_{\text{BD vacuum}}} \underbrace{\text{Poly}_{p-3-\alpha}(k_1, k_2, k_3)}_{\text{polynomial of degree } p-3-\alpha}\,, \tag{2.5.9}$$

which by virtue of the factor of e_3^2 satisfies the MLT for each external energy and for all choices of $\text{Poly}_{p-3-\alpha}(k_T, e_2, e_3)$. One can also notice that the form of K'_γ is related to the bulk-boundary propagator of a conformally coupled scalar K_φ bar factors of η_0, for which the MLT is trivial

$$\frac{\partial}{\partial k_c}\psi_n(k_1, ..., k_n; \{p\}; \{\mathbf{k}\})|_{k_c=0} < \infty, \quad \textit{i.e.} \text{ finite}, \quad \forall\, c = 1, ..., n\,, \tag{2.5.10}$$

which can be seen from the series expansion of the bulk-boundary propagator for K_φ.

Now wavefunction coefficients were constructed in [38] by multiplying a tensor structure by a solution to the MLT with the symmetries of the MLT solution fixed by the symmetries of the tensor structure. Without having to compute any time integrals we can now construct ψ_3^{trimmed} for each α, and use these expressions to compute the bispectra $B^{+++}_{3\text{BB},\alpha}$ and $B^{++-}_{3\text{BB},\alpha}$. Note that when we convert wavefunction coefficients to correlators we pick up a factor of $1/e_3^3$ from the inverse powers of ψ_2 in (2.4.6). We can absorb all constant factors into the arbitrary polynomials. Let's go order by order in even α.

$\underline{\alpha = \mathbf{0}}$ For $\alpha = 0$, all-plus bispectra with order p total-energy poles are given by

$$B^{+++}_{\alpha=0,p} = \frac{1}{e_3^3}\frac{[12]^2[23]^2[31]^2}{e_3^2}\text{MLT}_{\alpha=0,p} \equiv \frac{\text{SH}_{+++}}{e_3^3}\text{MLT}_{\alpha=0,p}\,, \tag{2.5.11}$$

$$B^{++-}_{\alpha=0,p} = \frac{1}{e_3^3}\frac{[12]^6}{[23]^2[31]^2}\frac{I_1^2 I_2^2}{e_3^2}\text{MLT}_{\alpha=0,p} \equiv \frac{\text{SH}_{++-}}{e_3^3}\text{MLT}_{\alpha=0,p}\,. \tag{2.5.12}$$

Given that SH_{+++} and SH_{+++} are fully symmetric, $\text{MLT}_{\alpha=0,p}$ must also be fully symmetric and so built out of the elementary symmetric polynomials k_T, e_2, e_3. The EFToI sub-set is

$$B^{+++}_{\text{3BB},0} = \frac{e_3^2\text{SH}_{+++}}{e_3^3 k_T^p}\text{Poly}_{p-3}(k_T, e_2, e_3)\,, \tag{2.5.13}$$

$$B^{++-}_{\text{3BB},0} = \frac{e_3^2\text{SH}_{++-}}{e_3^3 k_T^p}\text{Poly}_{p-3}(k_T, e_2, e_3)\,, \tag{2.5.14}$$

where the degree of the polynomial is fixed by scale invariance and in all cases must be a non-negative number, so for $\alpha = 0$ we need $p \geq 3$.

$\underline{\alpha = \mathbf{2}}$ Now consider $\alpha = 2$. In general the bispectra are

$$B^{+++}_{\alpha=2,p} = \frac{\text{SH}_{+++}}{e_3^3}\sum_{\text{perms}} k_1 k_2 \text{MLT}_{\alpha=2,p}\,, \tag{2.5.15}$$

where now $\text{MLT}_{\alpha=2,p}$ is symmetric in k_1 and k_2 only. The EFToI sub-set is

$$B^{+++}_{\text{3BB},2} = \frac{e_3^2\text{SH}_{+++}}{e_3^3 k_T^p}[k_1k_2\text{Poly}_{p-5}(k_T, e_2, k_3) + k_1k_3\text{Poly}_{p-5}(k_T, e_2, k_2) + k_2k_3\text{Poly}_{p-5}(k_T, e_2, k_1)]\,, \tag{2.5.16}$$

$$B^{++-}_{\text{3BB},2} = \frac{e_3^2\text{SH}_{++-}}{e_3^3 k_T^p}[k_1k_2\text{Poly}_{p-5}(k_T, e_2, k_3) - k_1k_3\text{Poly}_{p-5}(k_T, e_2, k_2) - k_2k_3\text{Poly}_{p-5}(k_T, e_2, k_1)]\,, \tag{2.5.17}$$

The arguments of the polynomial are fixed as follows: since the polynomial only needs to be symmetric in k_1 and k_2, it can always be written as a function of $k_1 + k_2 = k_T - k_3$, k_1k_2 and k_3. Now if we want bispectra with total energy poles of degree p, we are left with k_1k_2 and k_3 as the independent variables.

$\underline{\alpha = \mathbf{4}}$ For $\alpha = 4$ we generally have

$$B^{+++}_{\alpha=4,p} = \frac{\text{SH}_{+++}}{e_3^3}\sum_{\text{perms}} I_3^2 I_1 I_2 \text{MLT}_{\alpha=4,p}\,, \tag{2.5.18}$$

where again $\text{MLT}_{\alpha=4,p}$ is symmetric in k_1 and k_2 only. The EFToI sub-set is

$$B^{+++}_{3\text{BB},4} = \frac{e_3^2 \text{SH}_{+++}}{e_3^3 k_T^p}[I_3^2 I_1 I_2 \text{Poly}_{p-7}(k_T, e_2, k_3) + I_1^2 I_2 I_3 \text{Poly}_{p-7}(k_T, e_2, k_1) + I_2^2 I_1 I_3 \text{Poly}_{p-7}(k_T, e_2, k_2)]\,, \tag{2.5.19}$$

$$B^{++-}_{3\text{BB},4} = \frac{e_3^2 \text{SH}_{++-}}{e_3^3 k_T^p}[k_T^2 I_1 I_2 \text{Poly}_{p-7}(k_T, e_2, k_3) - I_2^2 I_1 k_T \text{Poly}_{p-7}(k_T, e_2, k_1) - I_1^2 I_2 k_T \text{Poly}_{p-7}(k_T, e_2, k_2)]\,, \tag{2.5.20}$$

where we have used that under $k_3 \to -k_3$ we have $I_3 \to k_T$, $I_1 \to -I_2$, $I_2 \to -I_1$.

<u>$\alpha = \mathbf{6}$</u> Finally for $\alpha = 6$ we generally have

$$B^{+++}_{\alpha=6,p} = \frac{\text{SH}_{+++}}{e_3^3}(I_1 I_2 I_3)^2 \text{MLT}_{\alpha=6,p}\,, \tag{2.5.21}$$

where $\text{MLT}_{\alpha=6,p}$ is now again symmetric in all external energies. The EFToI sub-set is

$$B^{+++}_{3\text{BB},6} = \frac{e_3^2 \text{SH}_{+++}}{e_3^3 k_T^p} I_1^2 I_2^2 I_3^2 \text{Poly}_{p-9}(k_T, e_2, e_3)\,, \tag{2.5.22}$$

$$B^{++-}_{3\text{BB},6} = \frac{e_3^2 \text{SH}_{++-}}{e_3^3 k_T^p} I_1^2 I_2^2 k_T^2 \text{Poly}_{p-9}(k_T, e_2, e_3)\,. \tag{2.5.23}$$

The above structures give the most general graviton bispectra coming from three building block operators in the EFToI, to all orders in derivatives (or equivalently to all orders in p). Compared to the general Lagrangian in (2.5.1), the resulting bispectra take a very compact form.

2.5.2 Counting Type-II bispectra

Let's now count how many bispectra we have for each α. We will consider $\alpha = 0, 6$ and $\alpha = 2, 4$ separately.

$\alpha = 0, 6$ Here we construct polynomials of the form $\text{Poly}_{p-3-\alpha}(e_2, e_3)$. The number of such polynomials is equivalent to the number of solutions $N(n)$ to the linear diophantine equation

$$2x + 3y = p - 3 - \alpha \equiv n, \tag{2.5.24}$$

where x, y, n are non-negative integers. Let $\{x, y\}$ be the set of solutions to this equation and consider the solutions to $2X + 3Y = n + 6$. Some of these solutions are given by $\{x, y + 2\}$. We are missing solutions corresponding to $Y = 0, 1$ for which there is only a single solution for a given n. So we have

$$N(n + 6) = N(n) + 1. \tag{2.5.25}$$

We therefore only need to count the number of solutions for $n = 0, \ldots, 5$ then we can easily read off the total number. We have

$$N(0) = 1, \tag{2.5.26}$$
$$N(1) = 0, \tag{2.5.27}$$
$$N(2) = 1, \tag{2.5.28}$$
$$N(3) = 1, \tag{2.5.29}$$
$$N(4) = 1, \tag{2.5.30}$$
$$N(5) = 1. \tag{2.5.31}$$

The total number of solutions is therefore

$$N(n) = \lfloor n/6 \rfloor \qquad \text{if} \qquad 6|n-1 \tag{2.5.32}$$
$$N(n) = \lfloor n/6 \rfloor + 1 \qquad \text{otherwise.} \tag{2.5.33}$$

$\alpha = 2, 4$ Here we construct polynomials of the form $\text{Poly}_{p-3-\alpha}(k_1 k_2, k_3)$. The number of such polynomials is equivalent to the number of solutions $N(n)$ to the diophantine equation

$$x + 2y = p - 3 - \alpha \equiv n, \tag{2.5.34}$$

where x, y, n are non-negative integers. Let $\{x, y\}$ be the set of solutions to this equation and consider the solutions to $X + 2Y = n + 2$. Some of these solutions are given by $\{x, y + 1\}$. We are missing solutions corresponding to $Y = 0$ for which there is only a single solution. So we have

$$N(n + 2) = N(n) + 1. \tag{2.5.35}$$

We therefore only need to count the number of solutions for $n = 0, 1$ then we can easily read off the total number. We have

$$N(0) = 1, \tag{2.5.36}$$
$$N(1) = 1. \tag{2.5.37}$$

The total number of solutions is therefore

$$N(n) = \lfloor n/2 \rfloor + 1. \tag{2.5.38}$$

Total Let's now consider all α and write $N_{\text{total}}(p)$. To write a general formula we need $p \geq 9$ such that we always have $n \geq 0$. For low p we have

$$N_{\text{total}}(3) = 1 \tag{2.5.39}$$
$$N_{\text{total}}(4) = 0 \tag{2.5.40}$$
$$N_{\text{total}}(5) = 2 \tag{2.5.41}$$
$$N_{\text{total}}(6) = 2 \tag{2.5.42}$$
$$N_{\text{total}}(7) = 4 \tag{2.5.43}$$
$$N_{\text{total}}(8) = 4, \tag{2.5.44}$$

and for $p \geq 9$ we have

$$N_{\text{total}}(p) = \left(2\lfloor\frac{p-5}{2}\rfloor + 1\right) + \left(2\lfloor\frac{p-3}{6}\rfloor - 1\right) \quad \text{if} \quad 6|p-4 \tag{2.5.45}$$
$$N_{\text{total}}(p) = \left(2\lfloor\frac{p-5}{2}\rfloor + 1\right) + \left(2\lfloor\frac{p-3}{6}\rfloor + 1\right) \quad \text{otherwise} \tag{2.5.46}$$

which we can simplify to

$$N_{\text{total}}(p) = 2\left(\lfloor\frac{p-5}{2}\rfloor + \lfloor\frac{p-3}{6}\rfloor\right) \quad \text{if} \quad 6|p-4 \tag{2.5.47}$$
$$N_{\text{total}}(p) = 2\left(\lfloor\frac{p-5}{2}\rfloor + \lfloor\frac{p-3}{6}\rfloor + 1\right) \quad \text{otherwise.} \tag{2.5.48}$$

It is unclear why the number of bispectra scale linearly with p, but this would be interesting to investigate.

2.6 Summary and Outlook

In this chapter, we have shown for the first time, using field redefinitions, that the quadratic and cubic action for massless gravitons, which appear in addition to the Einstein-Hilbert part, can be derived by considering covariant operators constructed out of the extrinsic curvature only. Operators quadratic in the extrinsic curvature contribute to both the quadratic and cubic operators for the transverse-traceless fluctuation, and we refer to the corresponding bispectra as *Type-I*. Covariant operators that are cubic in the extrinsic curvature only contribute to the cubic operators, and we refer to these bispectra as *Type-II*.

For Type-I bispectra, both single exchange and contact Feynman diagrams contribute and they are tied together by spatial diffeomorphisms and the non-linear realisation of time diffeomorphisms. Type-II bispectra receive only contributions from contact diagrams, since there are no corrections to the quadratic action. In both cases, we computed these bispectra to all orders in derivatives and have shown that our results are a consistent subset of the general graviton bispectra constructed in [38].

There are avenues for future investigation:

- In this work, we combined bulk and bootstrap techniques. It would be valuable to derive this full collection of EFToI graviton bispectra using bootstrap methods alone. In particular, consistency relations derived from soft theorems could provide a powerful tool for directly constructing them. Since leading-order soft theorems are insufficient to fully constrain the bispectra, subleading soft theorems must be employed. These, in turn, require knowledge of mixed correlators involving both gravitons and the curvature perturbation ζ [147].

- In light of this, constructing mixed correlators is a natural next step. However, deriving these correlators from the general EFToI action would be computationally inefficient (see [148] for recent attempts at this). Instead, new bootstrap techniques must be developed. Unlike graviton correlators, ζ correlators are expected to violate the manifest locality test (MLT), since their self-interactions are not manifestly local. One strategy is to use soft [29]; another is to seek a generalisation of the MLT that applies directly to ζ correlators. Such a generalisation seems plausible, as the time integrals involved are similar to those of a spectator scalar field. However, dealing with the inverse Laplacians —— which arise when integrating out the non-dynamical components of the metric — requires a deeper understanding of locality in the presence of dynamical gravity.
- More ambitiously, one might hope to constrain three-point functions by requiring consistency with higher-point correlators, such as the trispectrum. This mirrors techniques in scattering amplitudes, where cubic couplings and the spectrum are constrained by demanding that four-point amplitudes exhibit only simple poles and factorise consistently [60, 61, 149]. Thanks to recent progress, the analytic structure of four-point cosmological correlators is now well understood [31–35]. These consistency conditions could thus constrain the three-point functions that contribute to four-point functions. For the EFToI, one would need to impose that the spectrum contains a single scalar and a massless graviton as the dynamical modes, and the Cosmological Optical Theorem [50] could provide a useful tool to yield constraints on three-point functions.

"The CPT theorem is a theorem. It's not just good advice."
— Edward Witten, PiTP 2010 Program on "Aspects in Supersymmetry" - Seiberg Witten Theory

3

The Flat Space CPT Theorem

The CPT theorem in flat space states that a Unitary and Lorentz-invariant theory will also be **CRT** invariant. The usual proof within standard QFT textbooks and lecture notes (see e.g. [150–153]) of **CRT** invariance involves Wick rotating to Euclidean signature and proving invariance under the 180° rotation in the $\tau - x$-plane, where $\tau = it$ is the Euclidean time. Hence, this is working off the assumption that for the case of integer-spin fields, **CRT** and the 180° rotation are two discrete $\mathbb{Z}_2$[1] symmetries which are equivalent. However, a simple yet fundamental question arises in this statement: "how can **CRT** be an antilinear symmetry when the 180° rotation (as well as any other rotation) is clearly a linear symmetry?".

To address this question, this chapter identifies an additional discrete antiunitary $\mathbb{Z}_2$ symmetry that plays a fundamental role in the proof of the CPT theorem. Interestingly, this symmetry arises from the "Unitarity" aspect of quantum field theory: more precisely, it is a discrete $\mathbb{Z}_2$ symmetry inherent to all vector spaces with a real norm e.g. Krein spaces (which are orthogonal direct sums of positive- and negative-norm Hilbert spaces); hence we will refer to this symmetry as Reflection Reality **RR**[2]. The combination of this antilinear $\mathbb{Z}_2$ symmetry with the linear 180° rotation yields the third antilinear symmetry of **CRT**.

Notably, most proofs of the CPT theorem do not admit converse statements, and part of the novelty of this work is to allow for results in the converse diraction. Remarkably, the three $\mathbb{Z}_2$ symmetries combine to form the identity, resulting in a closed $\mathbb{Z}_2 \times \mathbb{Z}_2$ group structure (also known as the Klein 4-group). This chapter thus offers a new perspective on the proof of the CPT theorem by illuminating the group-theoretic structure underlying a set of discrete symmetries that follow from **CRT**, Unitarity, and Lorentz Invariance. This structure clarifies the limitations of potential converse statements, highlighting which aspects of Unitarity or Lorentz Invariance

[1] Of course the 180° rotation is not a $\mathbb{Z}_2$ symmetry for the case of spinors, and in fact becomes a $\mathbb{Z}_4$ symmetry where applying the rotation twice, i.e. a 360° rotation, gives a factor of $(-1)^{2s}$, where s is the spin of the field; and for spin-statistics-satisfying fields this is of course the same as the fermion parity $(-1)^F$. This subtle interplay between discrete spacetime symmetries and spin-statistics will be discussed in detail in Section 3.6, where we review the connection between CPT invariance and the Spin–Statistics theorem.

[2] It is well known that the Euclidean manifestation of one aspect of Unitarity, specifically the positivity of norm of states in a Hilbert space, is Reflection Positivity. **RR** is a weaker constraint which is a property of all complex vector spaces with states real norm and not necessarily positive.

are absent in such cases. In Chapter 5, we show that in many field theories, full Unitarity is, in fact, *accidentally* implied, subject to certain discrete choices.

By exploiting this $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure, we not only recover the standard CPT theorem as a consequence of invariance under the other two discrete symmetries, but also, for the first time in the literature, explicitly derive two non-trivial converse directions to the CPT theorem. These results have profound implications not only for QFT in flat space, but also for quantum field theory in cosmological settings. In particular, in Chapters 4 to 6 we will demonstrate that an analogous $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure emerges in cosmology, which enables us to identify how Lorentzian bulk Unitarity manifests in a holographically dual Euclidean field theory at $\mathcal{I}^+$ (see Figure 1.2). This provides new insight into a longstanding open question in holographic cosmology and the dS/CFT correspondence.

3.1 Antiunitarity of CRT

Before identifying how **CRT** and the 180° rotation differ, let us first clarify how **CRT** is an antilinear symmetry, by looking at each component of **CRT** individually:

- Charge conjugation, **C**, can act trivially on field operators when one works in a basis of real fields, i.e. we can always choose to write any complex field $\phi \equiv \phi_{\mathbb{R}} + i\phi_{\text{Im}}$, which means that it simply acts on a state or wavefunction as the identity, i.e.

$$\mathbf{C} : \Psi(t, x) \to \Psi(t, x)\,. \tag{3.1.1}$$

Charge conjugation, **C**, exchanges particles with their antiparticles. It acts non-trivially when fields carry complex structure — for example, a complex scalar field ϕ or a Dirac spinor ψ — typically involving complex conjugation and, in some cases, transformations on internal symmetry indices. However, if one is working in a real basis of fields — meaning your field variables are real-valued functions (like a real scalar ϕ or a Majorana spinor) — then **C** simply maps a real field to itself:

$$\mathbf{C}\phi(\mathbf{x})\mathbf{C}^{-1} = \phi(\mathbf{x})\,. \tag{3.1.2}$$

So the action of **C** is trivial on both the field operators and their eigenstates. When working in the Schrödinger picture, the wavefunctional $\Psi[\phi]$ assigns amplitudes to configurations of real field values $\phi(\vec{x})$. Since **C** leaves these field configurations invariant,

$$\mathbf{C} : \Psi[\phi] = \Psi[\phi]\,, \tag{3.1.3}$$

then $\Psi[\phi]$ is invariant under **C** up to an overall phase (which depends on the choice of state, e.g. in a definite **C**-eigenstate). If one later switches to a complex field basis, **C** will act by taking complex conjugates and changing the functional arguments accordingly.

For gauge fields, charge conjugation reverses the sign of the gauge potential:

$$\mathbf{C}A^\mu(\mathbf{x})\mathbf{C}^{-1} = -A^\mu(\mathbf{x})\,, \tag{3.1.4}$$

reflecting the fact that the gauge field mediates interactions between particles of opposite charges. In the Schrödinger picture, this means that under **C**, the wavefunctional transforms as

$$\mathbf{C} : \Psi[A^\mu] = \Psi[-A^\mu]\,, \tag{3.1.5}$$

again, possibly up to an overall phase. Thus, even though A^μ may be real-valued, it transforms non-trivially under charge conjugation because it couples to charged matter fields. Thus, even though A^μ is a real field, it transforms non-trivially under **C** due to its role as a mediator of interactions between particles and antiparticles and will be discussed in more detail in Section 7.4.2 of Chapter 7. Note that the precise implementation of **C** may differ depending on the conventions adopted for the field content and charge assignments, especially in theories with non-Abelian gauge groups or additional global symmetries;

- Reflection, **R**, which simply acts on a state or wavefunction as

$$\mathbf{R} : \Psi(t,x) \to \Psi(t,-x)^{3} \tag{3.1.7}$$

  and thus is clearly a linear symmetry;

- Time-reversal, **T**, is an antiunitary symmetry, as shown by Wigner in 1932 [154], which can be seen most simply by identifying that the Schrödinger equation

$$i\frac{\partial \Psi(t,x)}{\partial t} = \nabla^2 \Psi(t,x) \tag{3.1.8}$$

  is not invariant under $t \to -t$. Since there's no increase in entropy for a single quantum particle, we do expect the physics, and thus the Schrödinger equation to be invariant under time reversal. The key is the factor of i in the Schrödinger equation. Suppose that $\Psi(t,x)$ is a solution to the Schrödinger equation, then $\Psi(t,x)$ is not a solution, but due to the factor of i, $\Psi^*(-t,x)$ is a solution. Viewed as an operator acting on the Hilbert space, this complex conjugation translates into the requirement that **T** is an antiunitary operator, i.e. it is an antilinear and unitary operator, rather than the more familiar unitary operator. This means that, acting on states or wavefunctions, we have

$$\mathbf{T} : \Psi(t,x) \to \Psi^*(-t,x)\,. \tag{3.1.9}$$

By combining these together we find that **CRT** acts on a state or wavefunction as

$$\mathbf{CRT} : \Psi(t,x) \to \Psi^*(-t,-x)\,. \tag{3.1.10}$$

We therefore have identified how **CRT** is an antilinear symmetry and its action on a state or wavefunction in Lorentzian signature.

[3] For conciseness, we have omitted writing the explicit dependence on other spatial coordinates, hence this is really shorthand for

$$\mathbf{R} : \Psi(t,x_1,x_2,\ldots,x_n) \to \Psi(t,-x_1,x_2,\ldots,x_n)\,. \tag{3.1.7}$$

3.2 Lorentz Invariance

A common confusion in the literature is interchangeable labelling of the proper orthochronous Lorentz group (also known as the restricted Lorentz group) $SO^+(d,1)$ and the proper Lorentz group $SO(d,1)$. The precise statement of Lorentz Invariance in a $d+1$-dimensional Minkowski spacetime is invariance under the proper orthochronous Lorentz group $SO^+(d,1)$ which consists of those Lorentz transformations that preserve both the orientation of space and the direction of time. Table 3.1 provides a classification of all the subgroups of the Lorentz group which contain **at least one** of the four connected components $\{1, P, T, PT\}$. [4] The collection of the four interconnected parts can be organised into a group as the quotient group $O(d,1)/SO^+(d,1)$ which is also isomorphic to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. Any Lorentz transformation can be defined through a correct, proper orthochronous Lorentz transformation combined with two additional pieces of data, which distinguish one of the four interconnected parts. This structure is characteristic of finite-dimensional Lie groups.

Connected components	Name	Group label
1	Proper Orthochronous Lorentz group	$SO^+(d,1)$
1, PT	Proper Lorentz group	$SO(d,1)$
1, P	Orthochronous Lorentz group	$O^+(d,1)$
1, T	Orthochiral Lorentz group	$O^{\times}(d,1)$
1, P, T, PT	Lorentz group	$O(d,1)$

Table 3.1 Subgroups of the Lorentz group which contain **at least one** of the four connected components $\{1, P, T, PT\}$, where d is the number of spatial dimensions. In applications to de Sitter, one should replace d with $D = d + 1$.

3.3 Outline of CPT Theorem

The CPT theorem states that any QFT that is:

1. Unitary,
2. Lorentz invariant, and has
3. Energy bounded from below,

must also be invariant under the combined **CRT** symmetry. We will now dissect each assumption one by one.

Assumption 3 allows us to analytically continue to Euclidean signature, using the fact that $e^{-H\Delta\tau}$ is suppressed for positive Euclidean times $\Delta\tau > 0$. Let us adopt coordinates in Euclidean signature where $\tau = it$ and x_I, $I = 1 \ldots d$ are the spatial coordinates. We highlight here (as it will become important in the cosmological case) that the **P** in **CRT** reflects a single spatial coordinate, say $x_1 \to -x_1$.

[4] In [155] the authors refer to the $O^{\times}(d,1)$ subgroup which contains the connected components $\{1, T\}$ as the "orthochorous" group, after the Greek word χωρικοσ meaning a spatial territory, but we feel that the name "orthochiral" is more intuitive and distinct, given that this is the group of transformations which preserve the spatial orientation, i.e. the chirality.

The $SO^+(d,1)$ Lorentz Invariance in $d+1$-dimensional Minkowski spacetime analytically continues to $SO(d+1)$ rotational invariance in Euclidean signature. On the other hand, Unitarity implies that Euclidean correlators satisfy Reflection Positivity, which we will define in the following section. However, it is not necessary to enforce either of these relationships in their entirety to ensure **CRT** symmetry. We do not require full $SO^+(d,1)$ Lorentz Invariance; we simply need the subgroup $SO^+(1,1)$ boost [5] (this will become important when studying the implications of the CPT theorem for cosmology). In fact, all that is required is a discrete, $\mathbb{Z}_2$ symmetry from each of them:

$$\begin{array}{rcccl} \text{Unitarity} & \longleftrightarrow & \text{Reflection Positivity} & \Longrightarrow & \text{Reflection Reality} \\ \text{Lorentz-Invariance} & \longleftrightarrow & \text{Rotational Invariance} & \Longrightarrow & 180^\circ \text{ Rotation} \end{array}$$

where in each row, the first arrow represents the Wick rotation to Euclidean Signature and the second arrow is an implication. The CPT theorem is then simply the statement that the combination of these two symmetries is a **CRT** transformation. Therefore, requiring the symmetry of correlators under both Reflection Reality and this 180° rotation enforces their invariance under the **CRT** transformation.

Suppose, for concreteness, we start by considering vacuum correlators of the form

$$B_n = \langle\, \phi_1(\tau,x)\phi_2(\tau,x)\ldots\phi_n(\tau,x)\,\rangle\,, \tag{3.3.5}$$

where ϕ is some generic, real, scalar field and we have suppressed the dependence on the other spatial coordinates that we do not transform. Without loss of generality, we will choose to work in a real basis for all fields, so that the transformation **CT** just becomes **T** and we have an **RT** theorem (this will become important when discussing heavy fields in cosmology). To obtain the discrete transformations for complex fields, one must additionally consider the action of charge conjugation **C** whenever complex conjugating. Then the $\mathbb{Z}_2$ transformations of interest act in the following way on Euclidean field theory correlators:

[5] Consider a Lorentz boost in the t-x^d plane of $d+1$-dimensional Minkowski spacetime. The infinitesimal transformation of coordinates is given by

$$\begin{pmatrix} t' \\ x^{D\prime} \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} t \\ x^D \end{pmatrix}, \tag{3.3.1}$$

where ξ is the rapidity parameter of the boost, and we have omitted dependence on the other $d-1$ spatial coordinates for simplicity. To Wick rotate to Euclidean signature, we set

$$t = -i\tau \qquad \text{and} \qquad \xi = -i\theta\,, \tag{3.3.2}$$

which leads to the transformation

$$\begin{pmatrix} -i\tau' \\ x^{D\prime} \end{pmatrix} = \begin{pmatrix} \cosh(-i\theta) & \sinh(-i\theta) \\ \sinh(-i\theta) & \cosh(-i\theta) \end{pmatrix} \begin{pmatrix} -i\tau \\ x^D \end{pmatrix}. \tag{3.3.3}$$

Using the identities $\cosh(-i\theta) = \cos\theta$ and $\sinh(-i\theta) = -i\sin\theta$, this becomes

$$\begin{pmatrix} \tau' \\ x^{D\prime} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tau \\ x^D \end{pmatrix}, \tag{3.3.4}$$

which is a *clockwise rotation* in the τ-x^D plane by angle θ. Hence, under the Wick rotation $t=-i\tau$ and $\xi=-i\theta$, the non-compact Lorentz boosts $SO^+(1,1)$ in Minkowski space analytically continue to compact Euclidean rotations $SO(2)$ in the τ-x^D plane.

- As we will show in Sections 3.4.1 and 3.4.2, Euclidean Reflection Positivity implies a weaker principle know as Reflection Reality. We can enforce this by insisting that our correlators are invariant under the transformation:

$$\mathbf{RR} : B(\tau, x) \to B^*(-\tau, x)\,, \tag{3.3.6}$$

$$\mathbf{RR}(B_n) = \langle\, \phi_1(-\tau, x)\phi_2(-\tau, x) \ldots \phi_n(-\tau, x)\,\rangle^*\,; \tag{3.3.7}$$

- A 180° ($= \pi$) rotation in the τ-x plane corresponds to the transformation:[6]

$$\mathbf{R}_\pi : B(\tau, x) \to B(-\tau, -x)\,, \tag{3.3.8}$$

$$\mathbf{R}_\pi(B_n) = \langle\, \phi_1(-\tau, -x)\phi_2(-\tau, -x) \ldots \phi_n(-\tau, -x)\,\rangle\,; \tag{3.3.9}$$

- Finally, **CRT** is the transformation:

$$\mathbf{CRT} : B(\tau, x) \to B^*(\tau, -x)\,, \tag{3.3.10}$$

$$\mathbf{CRT}(B_n) = \langle\, \phi_1(\tau, -x)\phi_2(\tau, -x) \ldots \phi_n(\tau, -x)\,\rangle^*\,. \tag{3.3.11}$$

  It might seem surprising that there is no transformation of the τ coordinate here. However, this is simply a reflection of the fact that these correlators are a function of Euclidean time, $\tau = it$, so simultaneously complex conjugating and sending $t \to -t$ leaves τ unchanged.

It is straightforward to see that combining any two of these transformations produces the third and therefore (together with the identity) they form a group which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The consequence of this is that a theory respecting any two of these discrete symmetries will automatically respect the third.[7]

If we exploit this group structure to derive **CRT** from invariance under the other two transformations we recover the standard CPT theorem. However, there also exist two converse directions, where we use

$$\mathbf{CRT} + \mathbf{RR} \implies \mathbf{R}_\pi\,,$$

or else

$$\mathbf{CRT} + \mathbf{R}_\pi \implies \mathbf{RR}\,.$$

[6]Some authors refer to this transformation as "CPT". The difference in convention goes back to the earliest papers on the CPT theorem, see [156] for a nice review. In the language of [156], we are defining **T** as a "Wigner time reversal" (which involves a complex conjugation) while other authors use "Schwinger time reversal" (which instead reverses the order of operators. For example, the recent paper by Susskind [157] and [153], as well as Harlow and Numasawa [152] on gauging "CRT", is actually defining **CRT** as what we call the 180° rotation, at least in their Euclidean signature construction (see also [158], where the authors find that in order to construct physically consistent bulk local states, it is necessary to take a CPT-invariant linear combination of the two primary states of the dual CFT). It is unclear whether it makes sense to gauge our Wignerian **CRT**, as this would raise the question of what it means to have a non-trivial complex conjugation of amplitudes when going around a non-contractible cycle. (Perhaps it just means the whole amplitude vanishes.)

[7]One might at first glance think that it is required to have a 90° rotation $r : \tau \to x, x \to -\tau$ to relate Reflection Reality to **CRT**. This would be true if the proof required a conjugation $r(\mathrm{RR})r^{-1} = \mathbf{CRT}$. While this would be necessary to establish an isomorphism between RR and **CRT**, proving the CPT theorem requires only taking the group product with the 180° rotation, as the product of any 2 symmetries is another symmetry.

This last implication is particularly interesting as, in many field theories, Reflection Reality is strong enough to *accidentally* imply the full principle of Unitarity, despite being weaker that it. This will be discussed further in Section 5.2.

To generalise to correlators of operators that carry spin, it is of course also necessary to specify how the fields transform under the coordinate transformations above, e.g. by defining them as tensor fields. In the case of a tensor field with an odd number of τ indices, the transformation $\tau \to -\tau$ will reverse its sign; this is to ensure that e.g. the time component of a real vector field V_t remains self adjoint: $(V_t)^\dagger = V_t \implies (V_\tau)^\dagger = -V_\tau$. (The spatial polarisations are unaffected by this transformation.)

For theories involving spinor fields, it is also necessary to consider projective representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this paragraph, let us assume for simplicity that the theory is fully rotationally covariant. Then a 360° rotation gives a factor of $(-1)^{2s}$ (where s is the spin); this is of course the same as the fermion parity $(-1)^F$ for fields that satisfy spin-statistics. Hence, the 180° rotation R_π has order 4 when acting on half-integer spin fields, whilst Unitarity requires that $\mathbf{RR}^2 = +1$ and hence in a rotationally covariant theory we also need $\mathbf{CRT}^2 = +1$. This is only possible if the group anticommutes in the sense that $[A, B] := ABA^{-1}B^{-1} = -1$ when A, B are any two distinct elements of $\{\mathbf{RR}, \mathbf{CRT}, \mathbf{R}_\pi\}$. One thus finds that the appropriate double cover of $\mathbb{Z}_2 \times \mathbb{Z}_2$ for spinors is the dihedral group D_4.[8] We will focus on fields with integer spin representations and not consider half-integer spin fields in detail this thesis, but a similar formalism should also provide a much easier approach to obtain a Unitarity constraint for correlators involving spinors in flat space.

3.4 Antiunitarity of Unitarity

3.4.1 Reflection Positivity

In this section we review how the Reflection Positivity of correlators follows from the assumption of Unitarity, as it is an important element in the proof of the CPT theorem presented in the previous section, and may be unfamiliar to a reader.

Consider a ket state defined on the $\tau = 0$ slice of a Euclidean field theory. This may be defined by (an arbitrary sum/integral of) a time ordered Euclidean correlator, in which all operators are inserted on the side $\tau < 0$:

$$|\psi\rangle = \sum\!\!\!\!\!\!\int f\, \mathcal{O}_n(\tau_n) \ldots \mathcal{O}_2(\tau_2)\mathcal{O}_1(\tau_1)\, |0\rangle \tag{3.4.1}$$

with $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n < 0$. We have suppressed writing the spatial coordinates for notational conciseness as they are left invariant by this transformation. Note that f has an explicit dependence on τ[9] and is an arbitrary distribution over the spatial coordinates and choices of

[8] With the addition of one additional magical fact that Unitarity requires $\mathbf{CRT}^2 = (-1)^{2s}(-1)^F$, which requires a close inspection of the transformation properties of 2-pt correlators to prove, the considerations in this paragraph provide all the necessary ingredients to prove the spin-statistics theorem. But spin-statistics requires at least covariance under 90° rotations, not just 180° rotations, so it is less robust than the CPT theorem.

[9] To avoid cluttered equations in the discussion to follow we will only highlight the dependence on τ, but the correlators continue to depend on the other spatial coordinates.

fields, such that the resulting n-point correlator is finite. Similarly, the corresponding bra is[10]

$$\langle\psi| = \sum\!\!\!\!\!\!\int f^*_{\tau\to-\tau}\,\langle 0|\,\mathcal{O}_1^\dagger(-\tau_1)\mathcal{O}_2^\dagger(-\tau_2)\ldots\mathcal{O}_n^\dagger(-\tau_n)\,. \tag{3.4.2}$$

The positivity of the inner product $\langle\psi|\psi\rangle \geq 0$ therefore implies the principle of Reflection Positivity:

$$\left(\sum\!\!\!\!\!\!\int f^*_{\tau\to-\tau}\,\langle 0|\,\mathcal{O}_1^\dagger(-\tau_1)\mathcal{O}_2^\dagger(-\tau_2)\ldots\mathcal{O}_n^\dagger(-\tau_n)\right)\left(\sum\!\!\!\!\!\!\int f\,\mathcal{O}_n(\tau_n)\ldots\mathcal{O}_2(\tau_2)\mathcal{O}_1(\tau_1)\,|0\rangle\right) \geq 0\,, \tag{3.4.3}$$

a principle which is much simpler in spirit than the number of symbols it takes to write it out. Note that even in a unitary theory, it is possible for the RHS to vanish if the correlator vanishes, for example if we insert the equation of motion in a free theory, or a pure gauge state in a gauge theory. Such null states may be projected out of the theory, leaving a Hilbert space with a positive-definite inner product.

For a fixed number, N, of insertions on each side, we would obtain the positivity of some $2N$-point function. However, it is also allowed to consider states coming from quantum superpositions of different values of N. Because the sums are taken independently on both sides of the $\tau = 0$ surface, Reflection Positivity also places constraints on the correlations with odd numbers of points. It is not, however, allowed for the operators which define the ket to stray "offside" onto the wrong side of the $\tau = 0$ surface, even if you include the corresponding reflected operator.

In Lorentzian signature, Unitarity consists of two statements: (i) there exists a positive norm on the state space, and (ii) this norm is preserved under time evolution. Strictly speaking, Reflection Positivity is the Wick rotation of (i), not (ii).[11] But (ii) follows if we additionally use the τ-translation invariance of the Euclidean path integral. This implies that if we insert an empty strip of width $\Delta\tau$ between two correlators, it doesn't matter which side we insert it on:

$$|\chi\rangle^\dagger\left(e^{-H\Delta\tau}|\psi\rangle\right) = \left(e^{-H\Delta\tau}|\chi\rangle\right)^\dagger|\psi\rangle\,. \tag{3.4.4}$$

from which it follows that $H = H^\dagger$, guaranteeing that the Lorentzian evolution e^{iHt} preserves the norm. (More generally, if one assumes the Euclidean partition function is a covariant functional of the metric, one could deduce that Lorentzian evolution with respect to arbitrary lapse and shift functions will also be unitary.) Some constraints from only having states with positive norm were explored in [91, 92], and explained in a particularly clear way in [93].

[10] In these and subsequent expressions, we define $\mathcal{O}^\dagger(\tau)$ as the operator obtained by *first* acting on $\mathcal{O}(0)$ with the $\dagger$, and *then* translating it to $\tau \neq 0$. These operations do not commute, because $\tau^\dagger = -\tau$ in Euclidean field theory (in equal-time quantisation). It is therefore reasonable to use notation in which $\mathcal{O}(\tau)^\dagger = \mathcal{O}^\dagger(-\tau)$, in order to distinguish cases in which the $\dagger$ acts on the argument, from those in which it does not. To avoid confusion, in the main body of this thesis we will always place the $\dagger$ on the operator, meaning that any reversal of τ coordinate will be stated explicitly in the τ-argument of the operator.

[11] Unitarity also says that pure states evolve to pure states, but we take this for granted here.

3.4.2 Reflection Reality

Since every positive number is real, Reflection Positivity obviously implies the weaker principle that:

$$\left(\sum\!\!\!\!\!\!\int f^*_{\tau\to-\tau}\,\langle 0|\,\mathcal{O}_1^\dagger(-\tau_1)\mathcal{O}_2^\dagger(-\tau_2)\dots\mathcal{O}_n^\dagger(-\tau_n)\right)\left(\sum\!\!\!\!\!\!\int f\,\mathcal{O}_n(\tau_n)\dots\mathcal{O}_2(\tau_2)\mathcal{O}_1(\tau_1)\,|0\rangle\right)\in\mathbb{R}\,. \tag{3.4.5}$$

This is simply the assertion that the inner product is *real* (but not necessarily positive). This is the analogue of (i) in the previous section. The further statement corresponding to (ii), that this (possibly indefinite) real product is preserved by Lorentzian time evolution, follows from the same argument in (3.4.4).

Eq. (3.4.5) does not yet manifestly take the form of a $\mathbb{Z}_2$ symmetry, because of the annoying restrictions that (a) the amplitude must be a product of two conjugate correlators, (b) each of which is restricted to one side of $\tau = 0$. However, it turns out that when we only care about deriving a reality condition, neither of these restrictions are actually required. To see this, simply consider a state of the form

$$|\beta\rangle = |0\rangle + \beta|\psi\rangle\,, \tag{3.4.6}$$

where β is an arbitrary complex number. Now Eq. (3.4.5) tells us that

$$\langle\beta|\beta\rangle = \langle 0|0\rangle + \beta\langle 0|\psi\rangle + \beta^*\langle\psi|0\rangle + |\beta|^2\langle\psi|\psi\rangle\ \in\mathbb{R}\,. \tag{3.4.7}$$

But the first and last terms are also real by Eq. (3.4.5). And the only way the sum of the middle terms can be real for all values of β, is if

$$\langle\psi|0\rangle = \langle 0|\psi\rangle^*\,, \tag{3.4.8}$$

from which it follows that

$$\left(\sum\!\!\!\!\!\!\int f^*_{\tau\to-\tau}\,\langle 0|\,\mathcal{O}_1^\dagger(-\tau_1)\mathcal{O}_2^\dagger(-\tau_2)\dots\mathcal{O}_n^\dagger(-\tau_n)\right)|0\rangle = \langle 0|\left(\sum\!\!\!\!\!\!\int f\,\mathcal{O}_n(\tau_n)\dots\mathcal{O}_2(\tau_2)\mathcal{O}_1(\tau_1)\,|0\rangle\right)^*\,. \tag{3.4.9}$$

Similarly, at this point the restriction that the correlators remain on one side of the $\tau = 0$ surface can be lifted, either by using translation invariance, or by analyticity of the Euclidean position space correlators. It follows that an arbitrary correlator satisfies the following Reflection Reality (**RR**) condition:

$$\langle\,\mathcal{O}_1(\tau_1;\mathbf{x}_1)\mathcal{O}_2(\tau_2;\mathbf{x}_2)\dots\mathcal{O}_n(\tau_n;\mathbf{x}_n)\,\rangle\ =\ \langle\,\mathcal{O}_1^\dagger(-\tau_1;\mathbf{x}_1)\mathcal{O}_2^\dagger(-\tau_2;\mathbf{x}_2)\dots\mathcal{O}_n^\dagger(-\tau_n;\mathbf{x}_n)\,\rangle^* \tag{3.4.10}$$

which now takes the form of a $\mathbb{Z}_2$ symmetry that reverses the sign of Euclidean time τ and also complex conjugates the amplitude. This symmetry is therefore required to hold in any unitary field theory. But it is somewhat weaker than Unitarity as it is compatible with the existence of states with negative-norm.

3.5 Lorentzian Correlators

In Lorentzian signature, **RR** is simply a hermitian conjugation, and implies that self-adjoint fields have real expectation values. As $\phi = \phi^\dagger$, the action of taking the adjoint simply reverses the order of operators, and (in operator ordering) **RR** implies:

$$\langle \phi_1(t_1;x_1)\phi_2(t_2;x_2)\dots\phi_n(t_n;x_n)\,\rangle = \langle \phi_n(t_n;x_n)\dots\phi_2(t_2;x_2)\phi_1(t_1;x_1)\,\rangle^* \tag{3.5.1}$$

but without any positivity conditions. We also now introduce time ordered correlators as the object of consideration in Lorentzian signature,

$$G_n = \langle\, \mathcal{T}\phi_1(t_1;x_1)\phi_2(t_2;x_2)\dots\phi_n(t_n;x_n)\,\rangle\,. \tag{3.5.2}$$

The same group structure then emerges from the transformations:

- Reflection Reality is the statement that:

$$\mathbf{RR} : G(t;x) \to \overline{G}^*(t;x)\,, \tag{3.5.3}$$

$$\mathbf{RR}(G_n) = \langle\, \overline{\mathcal{T}}\phi_1(t_1;x_1)\phi_2(t_2;x_2)\dots\phi_n(t_n;x_n)\,\rangle^*\,, \tag{3.5.4}$$

  where the bar indicates that time ordering has been exchanged with anti-time ordering (in other words, the order would have been reversed if we had been using operator ordering). For equal time correlation functions this relationship therefore enforces that these correlators are real. In other words, **RR** implies the existence of a real norm on the Hilbert space (which need not be positive). As we shall see later, this norm is also preserved under Lorentzian time evolution.

- The 180° rotation for time-ordered correlators becomes:

$$\mathbf{R}_\pi : G(t;x) \to G(-t;-x)\,, \tag{3.5.5}$$

$$\mathbf{R}_\pi(G_n) = \langle\, \mathcal{T}\phi_1(-t_1;-x_1)\phi_2(-t_2;-x_2)\dots\phi_n(-t_n;-x_n)\,\rangle\,, \tag{3.5.6}$$

  where time ordering has been preserved notwithstanding the reversal of the sign of t.

- Finally, **CRT** is the transformation:

$$\mathbf{CRT} : G(t;x) \to \overline{G}^*(-t;-x)\,, \tag{3.5.7}$$

$$\mathbf{CRT}(G_n) = \langle\, \overline{\mathcal{T}}\phi_1(-t_1;-x_1)\phi_2(-t_2;-x_2)\dots\phi_n(-t_n;-x_n)\,\rangle^*\,. \tag{3.5.8}$$

3.6 CPT and the Spin–Statistics Theorem

One of the most important implications of the CPT theorem is its connection to the spin-statistics theorem — the result that particles of integer spin must obey Bose-Einstein statistics, while those of half-integer spin must obey Fermi-Dirac statistics.

The standard spin-statistics theorem is typically derived using the Wightman axioms or in the operator formalism by assuming locality (microcausality), Lorentz invariance, and a positive-definite Hilbert space. However, the CPT theorem offers an alternative route: it tightly constrains the structure of field (anti)commutators by requiring that any local, Lorentz-invariant, unitary QFT must be invariant under the combined action of charge conjugation (**C**), reflection (**R**), and time reversal (**T**).

The key point is that **CRT** symmetry enforces specific transformation properties of fields depending on their spin, which in turn determines their (anti)commutation relations. Historically, this relationship was clarified by Belinfante [159], Schwinger [160], Lüders [161–164], and Pauli [165], who showed that **CPT**-invariance implies the spin-statistics theorem under the standard assumptions of QFT (see also [156, 166] for historical reviews). Conversely, violation of the spin-statistics relation would generically lead to violation of **CPT**-invariance.

In the cosmological setting considered in this thesis, the assumption of full Poincaré invariance is relaxed due to the time-dependent background geometry. Nevertheless, discrete symmetries — including **CRT**— continue to impose powerful constraints, and we will see that their implications can still be traced back to foundational flat-space structures such as the spin–statistics connection. Indeed, if a cosmological model were to admit **CRT**-violating correlators or wavefunction coefficients, it would necessarily imply violations of spin–statistics at some level, barring exotic alternatives such as non-locality or indefinite-norm states. We leave a more explicit proof of the spin-statistics theorem using **CRT** in flat spacetime and cosmological backgrounds for future work.

3.7 Summary

In this chapter, we revisited the CPT theorem in flat spacetime through a novel group-theoretic lens. By identifying an overlooked discrete antilinear symmetry, Reflection Reality (**RR**), arising from the Unitarity structure of vector spaces with a real norm, which is not necessarily positive, we clarified a foundational question: how the antilinear nature of **CRT** can emerge from the 180° rotation in the $\tau - x$-plane which is purely linear. Together with the 180° Euclidean rotation $\mathbf{R}_\pi$, **RR** and **CRT** form a closed $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. This construction not only recovers the traditional CPT theorem but also allows us to rigorously formulate and prove two non-trivial converse statements, uncovering previously unrecognised constraints and interrelations among discrete symmetries in QFT.

Traditionally, the CPT theorem is stated as:

$$\text{Lorentz Invariance} + \text{Unitarity} \implies \textbf{CRT} \text{ invariance} \,.$$

We showed that for integer spin fields, in which we consider two discrete transformations, Reflection Reality (**RR**) and a 180° rotation ($\mathbf{R}_\pi$), that are implied by Unitarity and Lorentz Invariance respectively, these form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group with **CRT** and the identity.[12] This allows us to weaken the premises of the CPT theorem to:

$$180^\circ \text{ rotation} + \text{Reflection Reality} \implies \textbf{CRT} \text{ invariance} \,.$$

[12] While in this thesis we only treat integer spin carefully, if we allow spinors a 360° rotation introduces a factor of $(-1)^{2s}$, requiring that we pass to a double cover: the dihedral group D_4. See the discussion at the end of Section 3.3.

The existence of a 180° rotation follows from an analytic continuation of the existence of a single $SO^+(1,1)$ Lorentz Invariance symmetry, which is itself a subgroup of the full Lorentz group $SO^+(d,1)$. On the other hand, Reflection Reality (**RR**) is a discrete symmetry which holds for all theories with *real* (but not necessarily positive) norm on states, and hence *all* unitary theories have to satisfy non-perturbatively.[13]

This symmetry group structure enables us to make converse statements to the usual CPT theorem, as any two of the transformations implies the third. So we also have:

$$\mathbf{CRT}\text{ invariance} + 180^\circ\text{ rotation} \implies \text{Reflection Reality}\ ;$$

$$\mathbf{CRT}\text{ invariance} + \text{Reflection Reality} \implies 180^\circ\text{ rotation}\ .$$

It is important to note that a reflection real theory is not necessarily unitary, e.g. a theory with negative norm states or spin-statistics violating fields. However, this situation typically arises only in cases where one makes a bad discrete choice as to the matter content of a theory. In many cases (e.g. where you start with a unitary theory and continuously deform the coupling constants in a way that doesn't change the number of degrees of freedom), **RR** is enough to accidentally imply Unitarity. We will argue that this happens for a generic (codimension-0) set of couplings in Section 5.2.

While this fact, that **RR** is often enough to imply Unitarity, is not commonly discussed (see e.g. [167–171] for related discussions), it plays a structural role in many theoretical physics constructions. For example, it explains why field theorists working in Euclidean signature are very careful to ensure their theories are covariant, but don't spend much time worrying about Unitarity — this is because the discrete symmetries in a covariant Euclidean signature theory are usually sufficient to automatically ensure Unitarity upon continuation to Minkowski signature. Another example of this phenomenon was studied in [172], where the covariant and canonical formulations of the Maxwell field were compared in curved spacetime. In this case, covariance suffices to imply Unitarity, but not vice versa. In our language, this is because Unitarity accidentally follows from **CRT** together with a discrete subgroup of covariance. This perspective also clarifies the precise sense in which Unitarity and Lorentz Invariance are required to obtain **CRT**, and what aspects must be absent for its violation.

Moreover, the CPT theorem is intimately connected to the spin–statistics theorem, which relates the spin of particles to their quantum statistics. The **CRT** symmetry enforces spin-dependent transformation properties of quantum fields, dictating whether fields commute or anticommute at spacelike separations. This ensures that integer spin fields obey Bose-Einstein statistics and half-integer spin fields obey Fermi-Dirac statistics. Violation of this connection generally implies a violation of **CRT** invariance. In cosmological spacetimes, where full Poincaré invariance is relaxed, discrete symmetries including **CRT** continue to impose powerful constraints. Any departure from **CRT** symmetry in cosmological correlators or wavefunctions would signal a

[13] It is easy to find theories which are not invariant under $SO^+(1,1)$, but do satisfy the discrete 180° rotation implied by the boost and hence are **CRT**-invariant provided that they satisfy **RR**. For example, consider the interaction $\tilde{\lambda}(\dot{\phi})^4$, where $\tilde{\lambda}$ is a real coupling, ϕ is a real scalar field and dots denote derivatives with respect to time. This is clearly not invariant under the $SO^+(1,1)$ boost (since only covariant derivatives preserve boosts) but is both Unitary (as it has a real coupling) and **CRT** invariant as it invariant under the discrete 180° rotation $\mathbf{R}_\pi$ due to the even number of time derivatives.

breakdown of the spin-statistics relation or the presence of exotic phenomena such as non-locality or indefinite-norm states.

These insights lay the foundation for generalising CPT symmetry to cosmological spacetimes. In Chapter 4, we will show that an isomorphic $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure persists in de Sitter space, with corresponding transformations acting on both global and Poincaré patches. This allows us to extend the logic of the CPT theorem to cosmological correlators and wavefunctions, ultimately opening the door to a cosmological version of the theorem, and identify how bulk unitarity manifests non-perturbatively in the wavefunction — a long-standing question in holographic cosmology and dS/CFT.

"Let the universe have only two dimensions, and let it be the surface of a rubber ball. An observer will see all the others receding from himself, but it does not follow that he is the centre of the universe. The universe has no centre."
— Willem de Sitter, Kosmos, A Course of Six Lectures on the Development of Our Insight Into the Structure of the Universe.

4

CPT in de Sitter

In Chapter 3, we showed that in flat space the CPT theorem in Euclidean signature can be understood through a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group relating the discrete symmetries of Reflection Reality **RR**, the 180° Euclidean rotation $\mathbf{R}_\pi$, and **CRT**. The first two of these symmetries are guaranteed by the Lorentzian properties of Unitarity and $SO^+(1,1)$ invariance.

Analogous properties also exist in de Sitter (dS) space, leading to an isomorphic $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. In this chapter, we determine the action of the three discrete symmetries **RR**, $\mathbf{R}_\pi$, and **CRT** on the coordinates by considering the embedding of de Sitter in a $D+1$-dimensional Minkowski spacetime. This enables us to show that a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group emerges in both global and Poincaré patches of de Sitter, constraining late-time correlation functions in a universal manner.

In global dS, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group is closely analogous to that in flat space discussed in Chapter 3, leaving correlators in Euclidean dS and Lorentzian invariant. These symmetries descend to the future Poincaré patch, relevant for inflationary observables, where the discrete $\mathbf{R}_\pi$ transformation is interpreted as an analytic continuation of dilation symmetry $\mathbf{D}_{-1}$.

We demonstrate that this $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure continues to hold for cosmological correlators computed in the in-in formalism. However, to constrain the Wavefunction of the Universe (WFU) — whose coefficients ψ_n encode cosmological information more fundamentally — these symmetries must be realised as transformations of the local bulk Lagrangian rather than directly on correlators. This distinction is essential because ψ_n arises from a single-sheet path integral and is not directly subject to the same analytic structure as correlators. We will then extend this analysis in Chapter 5 to the WFU by studying local symmetries of the Lagrangian, laying the groundwork for the derivation of concrete non-perturbative constraints on the wavefunction coefficients presented in Chapter 6. This ultimately enables us to establish the Cosmological CPT Theorem, rigorously linking discrete symmetry, Unitarity, and inflationary observables.

4.1 Global de Sitter

Similar to how a Lorentz invariant theory in a $D+1$-dimensional (where $D = d+1$) Minkowski spacetime is invariant under the restricted Lorentz group $SO^+(D+1,1)$, a de Sitter invariant theory in a $d+1$-dimensional de Sitter spacetime is invariant under $SO^+(d+1,1)$, which consists of those de Sitter transformations that preserve both the orientation of space and the direction of time.

In Euclidean signature, the $SO^+(d+1,1)$ de Sitter invariance in $d+1$-dimensional de Sitter spacetime becomes $SO(d+2)$ rotational invariance of a S^{d+1} sphere of radius ℓ embedded in $\mathbb{R}^{D+1}$:

$$\delta_{AB}x^A x^B = \tau^2 + x_I x^I = \ell^2 \,, \tag{4.1.1}$$

and hence its metric is given by,

$$\mathrm{d}s^2 = \mathrm{d}\Theta^2 + \ell^2 \cos^2\left(\frac{\Theta}{\ell}\right)\left(\mathrm{d}\Phi_d^2 + \sin^2(\Phi_d)\,\mathrm{d}\Omega_{d-1}^2\right) . \tag{4.1.2}$$

The embedding space and the coordinates of the Euclidean dS sphere are related by

$$\tau = \ell \sin\left(\frac{\Theta}{\ell}\right) , \tag{4.1.3}$$

$$x^I = \ell \cos\left(\frac{\Theta}{\ell}\right) \sin(\Phi_d) z^i \,, \tag{4.1.4}$$

$$x^D = \ell \cos\left(\frac{\Theta}{\ell}\right) \cos(\Phi_d) \,, \tag{4.1.5}$$

where the z^i describe a unit S^{d-1} satisfying $z_i z^i = 1$ and are expressible in terms of $d-1$ angular variables, Φ_i.

We now act with the same flat space transformations described in Section 3, but now in the τ-x^D plane of the embedding space. In the embedding space the **CRT** transformation and rotations act on these coordinates as

$$\mathbf{CRT} : (\tau, x^1, \ldots, x^d, x^D) \to (\tau, x^1, \ldots, x^d, -x^D) \,, \tag{4.1.6}$$

$$\mathbf{R}_\theta : (\tau, x^1, \ldots, x^d, x^D) \to (\tau\cos(\theta) - x^D \sin(\theta), x^1, \ldots, x^d, x^D\cos(\theta) + \tau\sin(\theta)) \,. \tag{4.1.7}$$

The reason these are identical for $\theta = \pi$, despite representing different transformations when acting on correlators, is that **CRT** additionally includes complex conjugation, which does not affect the coordinates. We also note that in this formulation this rotation can be understood as an analytic continuation of the Lorentz boost to imaginary values of the rapidity, $\theta = i\xi$. The action of the $\mathbb{Z}_2$ transformations from Section 3 on the Euclidean dS correlators is the following:

- Reflection Reality requires invariance under the transformation:

$$\mathbf{RR} : B(\Theta;\Phi) \to B^*(-\Theta;\Phi) \,, \tag{4.1.8}$$

$$\mathbf{RR}(B_n) = \left\langle\, \phi_1(-\Theta_1;\Phi_1)\phi_2(-\Theta_2;\Phi_2)\ldots\phi_n(-\Theta_n;\Phi_n)\,\right\rangle^* , \tag{4.1.9}$$

  where Reflection Reality maps an insertion on the S^{d+1} across the $\Theta = 0$ plane and a global complex conjugation acting on both complex fields and amplitudes/correlators;

- A 180° rotation corresponds to the transformation:

$$\mathbf{R}_\pi : B(\Theta;\Phi) \to B(-\Theta;\pi-\Phi)\,, \tag{4.1.10}$$

$$\mathbf{R}_\pi(B_n) = \langle\, \phi_1(-\Theta_1;\pi-\Phi_1)\phi_2(-\Theta_2;\pi-\Phi_2)\ldots\phi_n(-\Theta_n;\pi-\Phi_n)\,\rangle\,, \tag{4.1.11}$$

  where the rotation amounts to a reflection from a pole on one axis of the sphere to the opposite pole;

- Finally, **CRT** is the transformation:

$$\mathbf{CRT} : B(\Theta;\Phi) \to B^*(\Theta;\pi-\Phi)\,, \tag{4.1.12}$$

$$\mathbf{CRT}(B_n) = \langle\, \phi_1(\Theta_1;\pi-\Phi_1)\phi_2(\Theta_2;\pi-\Phi_2)\ldots\phi_n(\Theta_n;\pi-\Phi_n)\,\rangle^*\,, \tag{4.1.13}$$

  where the transformation is a reflection from a pole on one axis of the sphere to the opposite pole as well as a global complex conjugation acting on both complex fields and amplitudes/correlators.

When we do a Wick rotation $\tau = ix^0$ to Lorentzian signature, this converts the embedding Euclidean space into a Minkowski space:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -(\mathrm{d}x^0)^2 + \mathrm{d}x^I\,\mathrm{d}x^I\,, \quad I = 1,\ldots,D\,. \tag{4.1.14}$$

De Sitter spacetime then corresponds to a hyperboloid in these coordinates [173] satisfying,

$$\eta_{\mu\nu}x^\mu x^\nu = -(x^0)^2 + x^I x^I = \ell^2 = \frac{1}{H^2} = \frac{d(d-1)}{2\Lambda}\,, \tag{4.1.15}$$

where ℓ is the de Sitter radius which is inversely related to the usual Hubble constant H parameterising the expansion rate set by the positive cosmological constant Λ. This hyperboloid can be sliced up in many ways; the slicings we will consider are the closed slicing of global de Sitter (this section) and the Poincaré flat slicing (Section 4.2).[1] The flat slicing coordinates of interest to inflationary cosmology cover only the region with $x_0 + x_D > 0$, whilst the closed slicing of global de Sitter covers the whole hyperboloid (see Figures 4.1 and 4.2).

In Lorentzian global de Sitter coordinates (also referred to as closed slicing), the metric w.r.t. the proper time coordinate T is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}T^2 + \ell^2\cosh^2\left(\frac{T}{\ell}\right)\left(\mathrm{d}\Phi_d^2 + \sin^2(\Phi_d)\,\mathrm{d}\Omega_{d-1}^2\right)\,, \tag{4.1.16}$$

where $T \in \mathbb{R}$ and $\mathrm{d}\Omega_{d-1}^2$ is the round metric on the unit $d-1$-sphere. We can see from the metric that constant T surfaces are d-spheres which shrink from the past asymptotic boundary $\mathcal{I}^-$ at $T = -\infty$ to $T = 0$ and grow from $T = 0$ to the future asymptotic boundary $\mathcal{I}^+$ at $T = +\infty$.

[1] We leave the implications for the de Sitter static patch (see Figure 1.2) to future work [174].

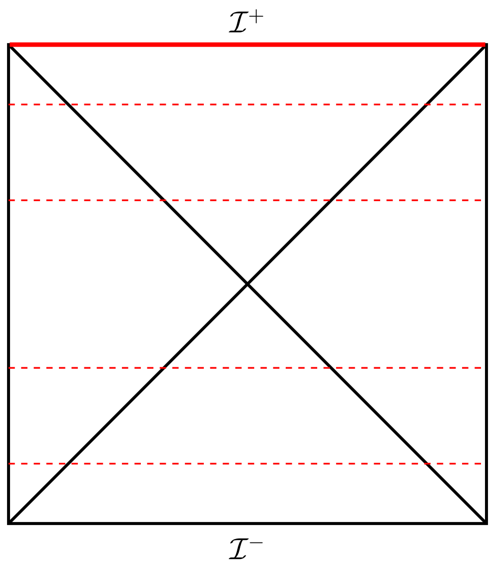

Fig. 4.1 Penrose diagram illustrating global slicing of de Sitter. The red dashed lines represent constant global time slices, which foliate the entire spacetime. Each slice is a spatial $(D-1)$-sphere, reflecting the closed spatial geometry in global coordinates. The thick red line at the top denotes the future boundary $\mathcal{I}^+$, where these spatial slices asymptote. Each point in the diagram's interior corresponds to a $(D-2)$-sphere, whose radius vanishes at the left and right edges—the South Pole and North Pole of the spatial slices, respectively. This representation captures the full causal structure and compact spatial geometry of global de Sitter spacetime.

The embedding space and global de Sitter coordinates are related as

$$x^0 = \ell \sinh\left(\frac{T}{\ell}\right) , \tag{4.1.17}$$

$$x^i = \ell \cosh\left(\frac{T}{\ell}\right) \sin(\Phi_d) z^i , \tag{4.1.18}$$

$$x^D = \ell \cosh\left(\frac{T}{\ell}\right) \cos(\Phi_d) . \tag{4.1.19}$$

The **CRT** transformation and complex rotations act on the embedding space coordinates as

$$\mathbf{CRT} : (x^0, x^1, \ldots, x^d, x^D) \to (-x^0, x^1, \ldots, x^d, -x^D) , \tag{4.1.20}$$

$$\mathbf{R}_\theta : (x^0, x^1, \ldots, x^d, x^D) \to (x^0 \cos(\theta) + i x^D \sin(\theta), x^1, \ldots, x^d, x^D \cos(\theta) + i x^0 \sin(\theta)) . \tag{4.1.21}$$

The **CRT** transformation in Lorentzian global de Sitter is then straightforwardly given by

$$\mathbf{CRT} : (T, \Phi_1, \ldots, \Phi_d) \to (-T, \Phi_1, \ldots, \pi - \Phi_d) , \tag{4.1.22}$$

$$\mathbf{CRT} : G(T; \Phi_d) \to \overline{G}^*(-T; \pi - \Phi_d) , \tag{4.1.23}$$

$$\mathbf{CRT}(G_n) = \langle \overline{\mathcal{T}} \phi_1(-T_1; \pi - \Phi_{d1}) \phi_2(-T_2; \pi - \Phi_{d2}) \ldots \phi_n(-T_n; \pi - \Phi_{dn}) \rangle^* . \tag{4.1.24}$$

Unfortunately, an arbitrary rotation doesn't have such a simple expression; the reason for this is that this rotation in the embedding space formalism misaligns the axis x_0 from the axis of the hyperboloid, so that the slices are now ellipsoids not spheres and attempting to parameterise the resulting slices using spherical coordinates fails. However, for the specific 180° rotation in Section 3.5 we have

$$\mathbf{R}_\pi : (T, \Phi_1, \dots, \Phi_d) \to (-T, \Phi_1, \dots, \pi - \Phi_d)\,, \tag{4.1.25}$$

$$\mathbf{R}_\pi : G(T; \Phi_d) \to G(-T; \pi - \Phi_d)\,, \tag{4.1.26}$$

$$\mathbf{R}_\pi(G_n) = \langle\, \mathcal{T}\phi_1(-T_1; \pi - \Phi_{d1})\phi_2(-T_2; \pi - \Phi_{d2}) \dots \phi_n(-T_n; \pi - \Phi_{dn})\,\rangle\,, \tag{4.1.27}$$

which as before, acts on the coordinates in the same way as the **CRT** transformation, with the difference between the two reappearing when we consider a complex object. Finally we can infer that the **RR** transformation in these coordinates is given by

$$\mathbf{RR} : (T, \Phi_1, \dots, \Phi_d) \to (T, \Phi_1, \dots, \Phi_d)\,, \tag{4.1.28}$$

$$\mathbf{RR} : G(T; \Phi_d) \to \overline{G}^*(T; \Phi_d)\,, \tag{4.1.29}$$

$$\mathbf{RR}(G_n) = \langle\, \overline{\mathcal{T}}\phi_1(T_1; \Phi_{d1})\phi_2(T_2; \Phi_{d2}) \dots \phi_n(T_n; \Phi_{dn})\,\rangle^*\,, \tag{4.1.30}$$

where, once again, we see that for equal time correlation functions this relationship enforces that these correlators are real, i.e. **RR** implies the existence of a real norm on the Hilbert space, which need not be positive.[2]

4.2 Poincaré de Sitter

Having shown in Section 4.1 that global de Sitter inherits the three discrete symmetries of interest in the CPT theorem from the embedding space, we can now consider these transformations in the Poincaré patch, where we perform inflationary calculations. However, things will become much less trivial here, since a *single* Poincaré patch explicitly chooses a specific orientation of time from the perspective of the global time coordinate (see Figure 4.2). Thus, the discrete transformations will each require an analytic continuation outside of the physical domain of the original Poincaré patch. In this section, we will establish the behaviour of this analytic continuation at the level of the coordinates and in Section 4.3 we will discuss the implications for the inflationary observables known as the cosmological correlators and the Wavefunction of the Universe.

In the flat slicing of the Poincaré patch we have the metric,

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + e^{2Ht}\,\mathrm{d}y^i\,\mathrm{d}y^i\,, \quad i = 1, \dots, d\,, \tag{4.2.1}$$

[2]The constraint from **RR** that equal time correlators are real is consistent with what has been found in the literature when exploring the constraints that enforcing a positive norm Hilbert space places on de Sitter correlators [91–93].

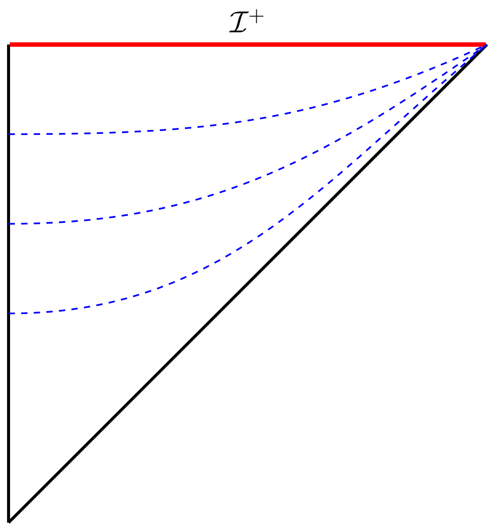

Fig. 4.2 Poincaré patch of de Sitter. The blue dashed lines represent slices of constant Poincaré time, which foliate only half of the spacetime and are conformally flat (4.2.6), making this slicing especially relevant for inflationary cosmology. The Poincaré patch asymptotes to the future boundary $\mathcal{I}^+$, where late-time cosmological correlators are defined. In this framework, we are metaobservers at $\mathcal{I}^+$, with access to primordial information through observations of the cosmic microwave background (CMB) and the large-scale structure (LSS) of the universe. However, things become significantly less trivial in this setting: a *single* Poincaré patch explicitly selects a preferred orientation of time relative to the global slicing of de Sitter. This built-in time asymmetry underpins many conceptual questions in cosmology, including those related to the arrow of time and will have a crucial impact on the role of discrete symmetries in cosmology.

where $t \in \mathbb{R}$ is generally referred to as cosmological time in the literature. These coordinates are related to the embedding space coordinates by

$$x^0 = \frac{1}{H}\sinh(Ht) + \frac{H}{2}y^i y^i e^{Ht}\,, \tag{4.2.2}$$

$$x^D = \frac{1}{H}\cosh(Ht) - \frac{H}{2}y^i y^i e^{Ht}\,, \tag{4.2.3}$$

$$x^i = e^{Ht}y^i, \quad 1 \le i \le d\,, \tag{4.2.4}$$

The transformation $t \to t + \frac{i\pi}{H}$ recovers both parts of the **CRT** transform in these coordinates but it naturally reflects all spatial coordinates rather than just one.[3] It is possible to recover just the transformation in (4.1.20) by additionally transforming all $y^i \to -y^i$. Unfortunately, analytically continuing the time variable is not easily interpreted as a time reversal. However, if

[3]In [175, 176] the authors claim that a transformation which involves flipping both H and t is "CPT" and that this can be used to obtain a unitary formulation of quantum field theory in curved spacetime. However it is clear that this would not map on to the **CRT** transformation defined by the embedding space formalism and thus it is not clear that this transformation has any relation to bulk Unitarity.

we go to conformal time[4], where the metric becomes

$$ds^2 = \left(-\frac{1}{H\eta}\right)^2 (-\,d\eta^2 + dy^i\, dy^i)\,, \quad i = 1, \ldots, d\,, \tag{4.2.6}$$

then our new coordinates are related to those in the embedding space by

$$x^0 = \frac{1}{H}\sinh(Ht) + \frac{H}{2} y_i y_i e^{Ht} = \frac{H^2\eta^2 - 1 - H^2 y^i y^i}{2H^2\eta} = \frac{\eta^2 - 1 - y^i y^i}{2H\eta}\,, \tag{4.2.7}$$

$$x^D = \frac{1}{H}\cosh(Ht) - \frac{H}{2} y^i y^i e^{Ht} = \frac{H^2 y^i y^i - 1 - H^2\eta^2}{2H^2\eta} = \frac{y^i y^i - 1 - \eta^2}{2H\eta}\,, \tag{4.2.8}$$

$$x^i = e^{Ht} y^i = -\frac{y^i}{H\eta}\,, \quad 1 \leq i \leq d\,, \tag{4.2.9}$$

$$\eta = y^0 = \frac{-1}{H(x^0 + x^D)}\,, \quad y^i = \frac{x^i}{x^0 + x^D}\,. \tag{4.2.10}$$

In the final equality for (4.2.7) and (4.2.8) we have used the fact that we can rescale $\eta \to \eta/H$ and $y^i \to y^i/H$ without loss of generality. In order to be consistent with the cosmology literature we will work with the parametrisation in which $\eta \leq 0$ (with $\eta = 0^-$ being the same future asymptotic boundary $\mathcal{I}^+$ in global de Sitter) and $y^i \in \mathbb{R}$, i.e. we will pick the Poincaré patch covering $x^0 + x^D \geq 0$. Such foliations only cover half of de Sitter space; however **CRT** or the 180° rotation will take you from one Poincaré patch to the other.

We see that the **CRT** transformation acts on the coordinates as:

$$\mathbf{CRT} : (\eta, y_1, \ldots, y_d) \to -(\eta, y_1, \ldots, y_d) = (-\eta, -y_1, \ldots, -y_d)\,. \tag{4.2.11}$$

Therefore, in order to recover a **CRT** transformation in a single spatial and time coordinate in the embedding space, or equivalently in global de Sitter slicing, it is necessary to transform all spatial coordinates in the Poincaré patch in addition to analytically continuing our conformal time variable.[5] The analytic continuation in conformal time η is just an artefact of the fact that **T** (or **CRT**) is a transformation within the full global de Sitter which is not captured by a single Poincaré patch. Thus **CRT** involves an analytic continuation since the observables are defined for one Poincaré patch where $\eta \leq 0$. We will discuss in Sections 4.3 how this analytic continuation manifests itself at the level of observables.

We can similarly consider how the Lorentzian $SO^+(1,1)$ boost in the embedding space manifests in the Poincaré patch, since it is this boost which, when analytically continued, becomes $SO(2)$ in Euclidean signature and gives us the necessary $\mathbf{R}_\pi$ rotation in the $\mathbb{Z}_2 \times \mathbb{Z}_2$

[4] In cosmology conformal time and cosmological time are related by $dt = a\, d\eta$ where for de Sitter we have $a(t) = e^{Ht}$ and thus

$$\int_{-\infty}^{\eta} d\eta' = \int_{-\infty}^{t} e^{-Ht'}\, dt' \implies \eta = -\frac{1}{H} e^{-Ht}\,. \tag{4.2.5}$$

[5] From the perspective of flat space, global de Sitter, and any spacetimes with the full $SO(1,1)$ symmetry, rather than the restricted $SO^+(1,1)$, **CRT** is a trivial crossing symmetry which involves no analytic continuation.

group. From the perspective of the Poincaré patch the generator of the $SO^+(1,1)$ boost becomes:

$$L_{0D} \equiv x_0 \frac{\partial}{\partial x^D} - x_D \frac{\partial}{\partial x^0} \tag{4.2.12}$$

$$= \eta \frac{\partial}{\partial \eta} + y^i \frac{\partial}{\partial y^i} \equiv D\,, \tag{4.2.13}$$

where we have used the fact that

$$\frac{\partial}{\partial x^0} \equiv \frac{\partial y^\mu}{\partial x^0} \frac{\partial}{\partial y^\mu} = H\eta^2 \frac{\partial}{\partial \eta} + H\eta y^i \frac{\partial}{\partial y^i} \tag{4.2.14}$$

$$\frac{\partial}{\partial x^D} \equiv \frac{\partial y^\mu}{\partial x^D} \frac{\partial}{\partial y^\mu} = H\eta^2 \frac{\partial}{\partial \eta} + H\eta y^i \frac{\partial}{\partial y^i}\,, \tag{4.2.15}$$

Hence we find that the $SO^+(1,1)$ boost acts as spacetime dilatations (which we will refer to as just Dilatations for the sake of simplicity) on the Poincaré patch, which in the cosmology literature is generally referred to as bulk Scale Invariance.[6] The action of the rotation operator on the coordinates in embedding space is:

$$(x_0, x_1, \ldots, x_d, x_D) = \left(-\frac{1}{H^2\eta} + \eta - \frac{y_i y_i}{2\eta}, -\frac{y_1}{H\eta}, \ldots, -\frac{y_d}{H\eta}, -\frac{1}{H^2\eta} - \eta + \frac{y_i y_i}{2\eta}\right)\,, \tag{4.2.16}$$

$$\mathbf{R}_\theta : (x_0, x_1, \ldots, x_d, x_D) \to (x_0 \cos(\theta) + i x_D \sin(\theta), x_1, \ldots, x_D \cos(\theta) + i x_0 \sin(\theta)) \tag{4.2.17}$$

$$= \left(-\frac{e^{i\theta}}{H^2\eta} + \frac{\eta}{e^{i\theta}} - \frac{y_i y_i}{2\eta e^{i\theta}}, -\frac{y_1}{H\eta}, \ldots, -\frac{y_d}{H\eta}, -\frac{e^{i\theta}}{H^2\eta} - \frac{\eta}{e^{i\theta}} + \frac{y_i y_i}{2\eta e^{i\theta}}\right)\,. \tag{4.2.18}$$

From this we can read off that this rotation is equivalent to the Poincaré patch coordinate transformation,

$$\mathbf{R}_\theta : (\eta, y_1, \ldots, y_d) \to e^{-i\theta}(\eta, y_1, \ldots, y_d)\,. \tag{4.2.19}$$

Just as the flat space transformation was an analytic continuation of Lorentz boosts to imaginary rapidity, this rotation has become the analytic continuation of the Dilatation operator to complex scales. In particular the 180° rotation acts on the coordinates as a negative Dilatation:

$$\mathbf{D}_{-1} : (\eta, y_1, \ldots, y_d) \to e^{-i\pi}(\eta, y_1, \ldots, y_d) = (e^{-i\pi}\eta, e^{-i\pi}y_1, \ldots, e^{-i\pi}y_d)\,. \tag{4.2.20}$$

Note that the action on coordinates is the same as for **CRT**. This is expected as the same thing happens in flat space; the key differences for the 180° rotation are i) it does not involve any complex conjugation, and ii) the correlator continues to be time-ordered. This is particularly interesting in the context of inflation where we often break de Sitter boosts (these act as special conformal transformations at the boundary) and so we lose the full set of de Sitter isometries. Here we need only to invoke an analytic continuation of Dilatations, which comes naturally from the embedding space formalism.

[6] Note that this is different from the usual notion of Scale Invariance in the CFT literature which is invariance under the generator $D = x^i \partial_i$. However we see that in the limit of $\eta \to 0^-$, $\lim_{\eta \to 0^-} D_{\eta,y} = D$, i.e. they are equivalent at the asymptotic time boundary of de Sitter. In actual fact it is well known in the literature that the group of **ALL** the bulk Killing symmetries (for which the corresponding Ward identities are sometimes referred to as the anomalous Ward identities in the literature) of de Sitter becomes the Euclidean conformal group $SO(d+1,1)$ at the asymptotic time boundary $\eta = 0^-$. This is the fundamental intuition for a “dS/CFT”.

In momentum space we would have almost the same relations:

$$\mathbf{CRT} : (\eta, k_1, \ldots, k_d) \to -(\eta, -k_1, \ldots, -k_d) = (-\eta, k_1, \ldots, k_d)\,, \tag{4.2.21}$$

$$\mathbf{D}_{-1} : (\eta, k_1, \ldots, k_d) \to e^{-i\pi}(\eta, k_1, \ldots, k_d) = (e^{-i\pi}\eta, e^{i\pi}k_1, \ldots, e^{i\pi}k_d)\,. \tag{4.2.22}$$

apart from the fact that the complex conjugation in **CRT** in position space reverses the sign of k, due to the Fourier transform, cancelling out the reflection of each spatial coordinate. The Fourier transform also reverses the direction of rotation of k in $\mathbf{D}_{-1}$.

We will now move on to discussing the constraints these discrete symmetries impose on the inflationary observables known as cosmological correlators.

4.3 Inflationary Observables

In the cosmological context the purpose of studying de Sitter spacetime is to understand the behaviour of quantum fluctuations at the end of inflation. These perturbations then grow over the history of the universe to produce all the structure that we see. Due to the randomness inherent to the quantum nature of these fluctuations and the single universe that we have to observe we study them through their correlation functions. In this section we focus on cosmological correlators calculated using the in-in formalism, since constraints for them only require the *spacetime symmetries* considered in Section 4.2. In this thesis, we will focus only on observables involving fields with integer spin representations and will not consider spinor fields in detail. However, a similar formalism should provide a much easier approach to deriving symmetry and Unitarity constraints for observables involving half-integer spin fields in cosmology (see e.g. [132, 177–189] for discussions of spinors in de Sitter space and cosmology), which has yet to be done in the cosmology literature.

(Later, we will revisit these same observables using the Wavefunction of the Universe (WFU) which introduces a new type of n-pt functions known as the wavefunction coefficients. However, as we will discuss in Section 4.3.2 this requires consideration of a different type of symmetry, namely the *local Lagrangian symmetries* discussed in Section 5.1, as it is those which will impose non-trivial constraints on the WFU and wavefunction coefficients.)

The correlators of interest in cosmology are equal time correlation functions evaluated on some time slice, usually taken to be the asymptotic future of de Sitter or some (quasi-)de Sitter inflationary spacetime,

$$\langle \mathcal{O}(\eta) \rangle = \langle \Psi(\eta) | \mathcal{O}(\eta) | \Psi(\eta) \rangle\,. \tag{4.3.1}$$

We do not (a priori) know the state $|\Psi(\eta)\rangle$. However, we assume that the universe was in a vacuum of the full theory, $|\Psi\rangle = |\Omega\rangle$. Just like in flat space it will prove very helpful to move to the interaction picture where operators evolve according to the free Hamiltonian whilst the evolution of states is dictated by the interacting Hamiltonian. Then these correlators are

$$\langle \mathcal{O}(\eta) \rangle = \langle \Omega(\eta_i) | U_{\rm I}^\dagger(\eta_{\rm i}, \eta) \mathcal{O}_{\rm I}(\eta) U_{\rm I}(\eta_{\rm i}, \eta) | \Omega(\eta_i) \rangle\,, \tag{4.3.2}$$

which are known as "in-in" correlators due to the fact that both vacuum states are defined in the past at η_i. This is in opposition to flat space amplitudes where the ket state corresponds to the future "out" vacuum.

We could perform a Dyson expansion for this expression, using the fact that this time evolution operator is

$$U(\eta_\mathrm{i},\eta) = \mathcal{T}\exp\left[-i\int_{\eta_\mathrm{i}}^{\eta} \mathrm{d}\eta H_\mathrm{int}[\phi,\phi']\right] . \tag{4.3.3}$$

However, $|\Omega(\eta_\mathrm{i})\rangle$ is the in vacuum of the full theory (despite being expressed in the interaction picture). This is relevant as the operators, $\mathcal{O}$, will be constructed from the interaction picture creation and annihilation operators, which act straightforwardly on the free (Bunch-Davies) vacuum, $|0\rangle$. By taking η_i to the infinite past we can turn off the interactions adiabatically so that the free and interacting vacua coincide. We will discuss the procedure by which this is done more extensively in Section 6.4.1.

4.3.1 Symmetries of the Cosmological Correlators

In this section we explore how the symmetry transformations **CRT** and $\mathbf{D}_{-1}$ from Section 4.2 act on in-in cosmological correlators and thus infer the constraint from Reflection Reality (**RR**). The cosmological correlators can be expressed as

$$B_n(\eta_0;\mathbf{y}_1\ldots\mathbf{y}_n) = \langle\phi_1(\eta_0;\mathbf{y}_1)\ldots\phi_n(\eta_0;\mathbf{y}_n)\rangle \tag{4.3.4}$$

$$= \langle\Omega(\eta_0)|\,\phi_1(\eta_0;\mathbf{y}_1)\ldots\phi_n(\eta_0;\mathbf{y}_n)\,|\Omega(\eta_0)\rangle \tag{4.3.5}$$

$$= \langle\Omega(\eta_\mathrm{i})|\,U^\dagger(\eta_\mathrm{i},\eta_0)\phi_1(\eta_0;\mathbf{y}_1)\ldots\phi_n(\eta_0;\mathbf{y}_n)U(\eta_\mathrm{i},\eta_0)\,|\Omega(\eta_\mathrm{i})\rangle \ , \tag{4.3.6}$$

where η_i is some time at which we specify the initial conditions at the start of inflation and η_0 is the reheating surface at the end.

Importantly, for a correlator to be invariant under a transformation, we require several different properties of the theory and background in which it lives. To illustrate this, let us consider the action of each of the symmetries individually. For **CRT** we find

$$\mathbf{CRT}: B_n(\eta;\mathbf{y}_1\ldots\mathbf{y}_n) \to B_n^*(-\eta;-\mathbf{y}_1,\ldots,-\mathbf{y}_n) \tag{4.3.7}$$

$$= \mathbf{CRT}(\langle\Omega(\eta_\mathrm{i})|)U^\dagger(\eta_\mathrm{i},\eta)\phi_n^*(-\eta;-\mathbf{y}_n)\ldots\phi_1^*(-\eta;-\mathbf{y}_1)U(\eta_\mathrm{i},\eta)\mathbf{CRT}(|\Omega(\eta_\mathrm{i})\rangle)\,, \tag{4.3.8}$$

$$= \mathbf{CRT}(\langle\Omega(\eta_\mathrm{i})|)U^\dagger(\eta_\mathrm{i},\eta)\phi_1^*(-\eta;-\mathbf{y}_1)\ldots\phi_n^*(-\eta;-\mathbf{y}_n)U(\eta_\mathrm{i},\eta)\mathbf{CRT}(|\Omega(\eta_\mathrm{i})\rangle)\,, \tag{4.3.9}$$

where we have used the fact that the time evolution operator defined in (4.3.3) is invariant under **CRT** and that all equal-time products of fields in real space are Hermitian

$$(\phi(\mathbf{y}_1)\ldots\phi(\mathbf{y}_n))^\dagger = \phi(\mathbf{y}_1)\ldots\phi(\mathbf{y}_n)\,, \tag{4.3.10}$$

because $\phi(\mathbf{y})$ is Hermitian (given that we are working in a basis of real fields) and the equal-time fields commute[7] (see e.g. [118, 190]). Thus, we recover our original correlator if:

- The state is **CRT**-invariant, $\mathbf{CRT}(|\Omega(\eta_\mathrm{i})\rangle) = |\Omega(\eta_i)\rangle$, which is true for the Bunch-Davies vacuum in which the in-in correlators are usually defined;
- The theory is **CRT**-invariant, $\mathbf{CRT} : H \to H$ which ensures that $U(\eta_i, \eta)$ is also **CRT** 0invariant;
- The fields are **CRT**-invariant, $\mathbf{CRT}(\phi(\eta;\mathbf{y})) \equiv \phi^*(-\eta;-\mathbf{y}) = \phi(\eta;\mathbf{y})$. This is actually ensured by the fact that their equation of motion and the initial Bunch-Davies vacuum state is **CRT**-invariant. However we will see in Section 6.4.4 that at the boundary the **CRT** transformation acts more non-trivially on the fields.

For the Dilatation operator, we will use the intuition from Section 4.2 to assume we can access $\lambda = -1$ despite the fact that it usually defined with $\lambda \in \mathbb{R} \geq 0$ and then address this analytic continuation in more detail in Section 6.1,

$$\mathbf{D}_\lambda : B_n(\eta;\mathbf{y}_1,\ldots,\mathbf{y}_n) \to B_n(\lambda\eta;\lambda\mathbf{y}_1,\ldots,\lambda\mathbf{y}_n) \tag{4.3.11}$$

$$= \langle\Omega(\lambda\eta_\mathrm{i})|\, U^\dagger(\lambda\eta_\mathrm{i},\lambda\eta)\phi_1(\lambda\eta;\lambda\mathbf{y}_1)\ldots\phi_n(\lambda\eta;\lambda\mathbf{y}_n)U(\lambda\eta_\mathrm{i},\lambda\eta)\,|\Omega(\lambda\eta_\mathrm{i})\rangle\ . \tag{4.3.12}$$

Thus we recover our original correlator if:

- The state is scale-invariant, $|\Omega(\lambda\eta_\mathrm{i})\rangle = |\Omega(\eta_i)\rangle$, which is true for the Bunch-Davies vacuum in which the in-in correlators are usually defined, since the Bunch-Davies vacuum (as well as any other α-vacua) is invariant under all the de Sitter isometries;
- The theory is scale-invariant, $\mathbf{D}_\lambda : H \to H$ which ensures that $U(\lambda\eta_i,\lambda\eta) = U(\eta_i,\eta)$;
- The fields are scale-invariant, $\mathbf{D}_\lambda(\phi(\eta;\mathbf{y})) \equiv \phi(\lambda\eta;\lambda\mathbf{y}) = \phi(\eta;\mathbf{y})$. This is actually ensured by their scaling properties, if they were not scale invariant themselves then this transformation would also involve transforming the fields as we will see in Section 6.4.4.

We can then use **CRT** and $\mathbf{D}_{-1}$ to infer **RR**, and thus find in analogy with Sections 3.4.2 and 3.5, that **RR** constrains B_n to be real by ensuring that the time evolution operator is unitary, $U^\dagger U = \mathbb{1}$. Hence we are justified in labelling this property and its associated transformation **RR**.

We are now equipped to construct a symmetry group from the action of these transformations,

$$\mathbf{CRT} :\ B_n(\eta;\mathbf{y}) \ = \ B_n^*(-\eta;-\mathbf{y})\,; \tag{4.3.13}$$

$$\mathbf{D}^{\pm}_{-1} :\ B_n(\eta;\mathbf{y}) \ = \ B_n(-\eta;-\mathbf{y})\,; \tag{4.3.14}$$

$$\mathbf{RR} :\ B_n(\eta;\mathbf{y}) \ = \ B_n^*(\eta;\mathbf{y})\,, \tag{4.3.15}$$

[7] Equal-time bosonic fields commute because they are spacelike separated. In local quantum field theory, microcausality requires that fields commute at spacelike separation: $[\phi(x),\phi(y)] = 0$ when $(x-y)^2 > 0$, where $x \equiv x^\mu = (x^0,\mathbf{x})$ and $y \equiv y^\mu = (y^0,\mathbf{y})$. At equal time $x^0 = y^0$, the separation is purely spatial, so $(x-y)^2 = |\mathbf{x}-\mathbf{y}|^2 > 0$, and $[\phi(\eta,\mathbf{x}),\phi(\eta,\mathbf{y})] = 0$. For fermionic fields, which satisfy anticommutation relations, microcausality instead implies $\{\psi(x),\psi(y)\} = 0$ at spacelike separation, so $\{\psi(\eta,\mathbf{x}),\psi(\eta,\mathbf{y})\} = 0$. In cosmology, this ensures that equal-time correlators of bosonic fields are insensitive to operator ordering, but one must take operator ordering into account for correlators of fermionic fields.

Just as in flat space these three transformations form a closed group and therefore, any two of them imply the last.

Thus we are not asserting that these are guaranteed to be symmetries of an inflationary theory, just that one of them will come for free if the other two are satisfied. Similar to how a theory can be invariant under the discrete 180° rotation $\mathbf{R}_\pi$ but not invariant under $SO^+(1,1)$ flat space, it is relatively easy to come up with theories in de Sitter that accidentally satisfy this discrete ($\lambda = -1$) scaling transformation without actually respecting continuous scale transformations ($\lambda > 0$). For example, consider the interaction $ga^2\phi^4$ in $D = 3+1$-spacetime dimensions, where g is a real coupling. This is clearly not scale invariant (as a scale-invariant non-derivative interaction would contain four powers of a rather than two) but is both Unitary (since it has a real coupling) and **CRT** invariant as it does have Discrete Scale Invariance due to the even number of extra scale factors.

The focus here has been on equal-time correlators due to their importance to cosmology. However, Unitarity also has implications for unequal time correlators. In this case the result is almost identical to the flat space results expressed in Lorentzian signature:

$$\mathbf{CRT}: \ \mathcal{T}B_n(\eta;\mathbf{y}) = \overline{\mathcal{T}}B_n^*(-\eta;-\mathbf{y})\,; \tag{4.3.16}$$
$$\mathbf{D}^{\pm}_{-1}: \ \mathcal{T}B_n(\eta;\mathbf{y}) = \mathcal{T}B_n(-\eta;-\mathbf{y})\,; \tag{4.3.17}$$
$$\mathbf{RR}: \ \mathcal{T}B_n(\eta;\mathbf{y}) = \overline{\mathcal{T}}B_n^*(\eta;\mathbf{y})\,, \tag{4.3.18}$$

where the bar indicates that time ordering has been exchanged with anti-time ordering. The main difference between flat space and here being that it is all the spatial coordinates that are inverted rather than just one. This is the case in any dimension due to the way the global de Sitter transformation is projected onto the Poincaré patch.

4.3.2 Wavefunction of the Universe

The same cosmological correlation functions discussed in Sections 4.3 and 4.3.1 can be calculated by using a basis of ϕ field eigenstates for both the bra and the ket:

$$\langle\mathcal{O}(\eta)\rangle = \int \mathrm{D}\phi \mathrm{D}\tilde{\phi}\langle\Psi(\eta)|\phi;\eta\rangle\langle\phi;\eta|\mathcal{O}(\eta)|\tilde{\phi};\eta\rangle\langle\tilde{\phi};\eta|\Psi(\eta)\rangle \tag{4.3.19}$$
$$= \int \mathrm{D}\phi \mathrm{D}\tilde{\phi}\Psi^*[\phi;\eta]\Psi[\tilde{\phi};\eta]\langle\phi;\eta|\mathcal{O}(\eta)|\tilde{\phi},\eta\rangle \tag{4.3.20}$$
$$= \int \mathrm{D}\phi\Psi^*[\phi;\eta]\Psi[\phi;\eta]\mathcal{O}(\phi;\eta)\,, \tag{4.3.21}$$

where $\mathcal{O}(\phi;\eta)$ is obtained by replacing instances of $\hat{\phi}$ with the corresponding eigenvalue, ϕ, of the state $|\phi;\eta\rangle$, and the Wavefunction of the Universe (WFU) is defined as [8]

$$\Psi[\phi;\eta] \equiv \langle\phi;\eta|\Psi(\eta)\rangle = \langle\phi;\eta|\Omega\rangle\,. \tag{4.3.22}$$

[8] See Appendix A of [50] for a pedagogical review of the WFU approach to the computation of cosmological correlators. This closely follow Weinberg's discussion of Path-Integral methods in [150] and an unpublished manuscript from Garrett Goon.

From (4.3.22) it already becomes apparent that the same analysis used for the cosmological correlators in Section 4.3.1 will not apply to the WFU, since the WFU is not an overlap of two vacuum states $|\Omega\rangle$. We will explain how to deal with this problem in Section 5.1.

The WFU satisfies the Schrödinger equation,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}\eta}\Psi[\phi;\eta] - H(\phi(\eta;\mathbf{x});\eta)\Psi[\phi;\eta] = \mathcal{D}\Psi[\phi;\eta] = 0\,, \tag{4.3.23}$$

subject to the Bunch-Davies initial condition and can be parameterised as

$$\Psi[\phi;\eta] \equiv \langle\phi;\eta|\Omega\rangle = \exp\left\{-\sum_{n=2}^{\infty}\frac{1}{n!}\int\left[\prod_{a=1}^{n}\frac{\mathrm{d}^d k_a}{(2\pi)^d}\phi_{\mathbf{k}_a}\right]\psi_n(\eta;\mathbf{k}_1,\ldots,\mathbf{k}_n)\delta^d\left(\sum_{a=1}^{n}\mathbf{k}_a\right)\right\}\,. \tag{4.3.24}$$

Despite not being directly observable themselves, it has become customary to discuss the role of symmetries and fundamental principles on the terms in this expansion, ψ_n, the so called coefficients of the WFU. The relations between the cosmological correlators B_n and ψ_n can be found in various parts of the literature, see e.g. [1, 115, 116]. In particular we have developed our understanding of Unitarity within the cosmological context in terms of its implications on these wavefunction coefficients. This is due to the discovery of the Cosmological Optical Theorem [50–52, 67] where Unitarity of the time evolution operator was used to derive a constraint on contact wavefunction coefficients (Feynman-Witten amplitudes with only external lines):

$$\psi_n(\eta_0;|\mathbf{k}|,\mathbf{k}) + \psi_n^*(\eta_0;-|\mathbf{k}|,-\mathbf{k}) = 0\,, \tag{4.3.25}$$

where this Unitarity constraint involves an analytic continuation of the magnitude of the *external* momenta (we will sometimes refer to the magnitudes as "energies") from the positive real axis through the lower half of the complex plane. However, for more complicated diagrams with internal lines the right hand side picks up a combination of lower point functions, i.e. in analogy with the Cutkosky cutting rules in flat space Unitarity in cosmology relates different orders in perturbation theory [52, 67]. It was later shown in [130, 131] that if one also analytically continues the *internal* energies (4.3.25) would hold for all tree-level contact and exchange diagrams with some caveats for certain types of *non-local* interactions. Unfortunately, these results are understood only perturbatively [52, 67], whereas the symmetry perspective that we present here is valid non-perturbatively and, as such provides new insights on these objects.

4.4 Summary

In this chapter, we have shown that the discrete symmetry structure underlying the flat space CPT theorem persists in de Sitter space. By embedding de Sitter into a higher-dimensional Minkowski spacetime, we established that the discrete symmetries of Reflection Reality **RR**, the 180° Euclidean rotation $\mathbf{R}_\pi$, and **CRT** generate an isomorphic $\mathbb{Z}_2 \times \mathbb{Z}_2$ group acting on both global and Poincaré patches of de Sitter. This structure provides a new lens through which to understand the symmetry constraints on cosmological correlators in the in-in formalism.

Crucially, we identified that while this group structure continues to constrain late-time observables, its implications are more subtle at the level of the Wavefunction of the Universe.

Since the wavefunction coefficients ψ_n arise from a single-sheet Lorentzian path integral, they are not subject to the same analytic continuation arguments as correlators. As such, the discrete symmetries must be interpreted as acting locally on the Lagrangian.

This shift in perspective lays the foundation for a new symmetry principle in cosmology. In Chapter 5, we will systematically examine how the discrete symmetries **CRT** and $\mathbf{R}_\pi$ can be realised as invariances of the local action, enabling us to infer the form of **RR** and how these transformations propagate through to constrain the structure of the wavefunction coefficients. These results serve as a critical stepping stone towards the formulation and proof of the Cosmological CPT Theorem, which we will present in Chapter 6.

"There ought to be something special about the boundary conditions of the Universe, and what can be more special than that there is no boundary? And there should be no boundary to human endeavour."
— Stephen W. Hawking,
London 2012 Paralympics opening ceremony

5

Symmetries of the Wavefunction

To lay the groundwork for the results presented in Chapter 6, this chapter focuses on an important conceptual distinction: the difference between symmetries of spacetime and symmetries of the local bulk Lagrangian. While the former act on global structures such as correlators or spacetime embeddings, only the latter are capable of establishing non-trivial constraints on the Wavefunction of the Universe (WFU) in a *single* Poincaré patch. This distinction will be central to the formulation of non-perturbative symmetry constraints on cosmological wavefunction coefficients.

We will show that the WFU coefficients ψ_n, unlike cosmological correlators, cannot be meaningfully constrained using analytic continuations or global embeddings alone, due to their origin in a single-sheet path integral. Instead, useful constraints must follow from symmetries of the Lagrangian, implemented locally in Lorentzian spacetime. In particular, we will demonstrate how discrete transformations such as **CRT** and $\mathbf{D}_{-1}$ descend to local field transformations that preserve the form of the action.

Moreover, we highlight a key result: in many generic situations, the discrete symmetry of Reflection Reality (**RR**) is itself sufficient to imply Unitarity of the bulk theory. This elevates the importance of the converse Cosmological CPT Theorem, which shows that **CRT** invariance combined with Discrete Scale Invariance implies **RR**. In this way, we gain a deeper understanding of how symmetry principles constrain physical consistency in quantum cosmology.

5.1 Local Lagrangian Symmetries

To set the scene, let us describe the notion of a local Lagrangian symmetry, and explain why this type of symmetry is needed to obtain a useful constraint on wavefunction coefficients in a single Poincaré patch.

5.1.1 Distinction from Spacetime Symmetries

In order to proceed, we must now make a distinction between two classes of symmetries:

1. A *spacetime symmetry*, which acts on the entire manifold $\mathcal{M}$ by moving points around by means of some map $m : x \to y$. (In addition to some local transformation of the fields $\phi(x)$, and possibly a complex conjugation of the amplitude.)

2. A *local Lagrangian symmetry*, which is an assertion that the Lagrangian L at a point $p \in \mathcal{M}$ respects some symmetry. Here, the symmetry acts on the tangent space $T_p(\mathcal{M})$ (in addition to transforming the fields $\phi(p)$ and possibly complex conjugating).

The relation between these kinds of symmetry is somewhat subtle.

A spacetime symmetry can sometimes give rise to a local Lagrangian symmetry. This happens whenever there is a *fixed point* of the map m, that is a point p for which $m : p \to p$. In this case there will be an induced map on the tangent space T_p, and this will normally[1] give rise to a symmetry of the local Lagrangian.

As translations in Minkowski do not have fixed points, they do not give rise to any local Lagrangian symmetry by themselves. But they do, of course, ensure that if there is a local Lagrangian symmetry at one point p, it can be translated to any other point q.

There can also exist local Lagrangian symmetries which do not arise from any spacetime symmetry of $\mathcal{M}$. A classic example of this, is the standard assumption that a field theory has local Lorentz Invariance at each point p, even on a spacetime manifold which has no Killing symmetry. (This example also shows that the definition of "local" has nothing to do with gauge invariance, as we typically assume local Lorentz Invariance even when doing field theory on a fixed, non-gravitational background.) Another simple example would be a time-varying Hamiltonian $H(t)$, in a case where at each moment of time t the Hamiltonian is time-reversal invariant, but there exists no time t_0 for which $H(t_0 + t) = H(t_0 - t)$.

Let us now consider the implications for the discrete symmetries in our $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. In the case of Reflection Reality (**RR**), the fixed points include the entire *real* part of the Lorentzian manifold $\mathcal{M}$. Hence, if we are working in Lorentzian signature, we are automatically entitled to extend **RR** to a symmetry of the local Lagrangian at each point p.

The same is not true of an individual **CRT** symmetry, since in this case $t \to -t$ and not all points are fixed by a given **CRT**. But in a manifold with time translation symmetry (e.g. Minkowski) we can always translate the story to find *a* **CRT** symmetry at any given point. Alternatively, if we have local Lorentz Invariance, this can be Wick rotated to obtain a local 180° rotation, and then **RR** (which is always local by the argument above) implies local **CRT**.

5.1.2 Flat Space Wavefunction

Below, we will be interested in using discrete symmetries to constrain vacuum wavefunction coefficients. But, as we have alluded to, it will turn out that the local Lagrangian versions of this symmetry are much more useful than the spacetime versions.

Let us consider the spacetime version of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group, starting in flat spacetime. Recall that the vacuum wavefunction Ψ at $\tau = it = 0$, is given by a path integral in the lower half of Euclidean space, that is it is a one-sided path integral over a spacetime with $\tau < 0$ (see Figure

[1] The additional assumptions required are that the theory *has* a local Lagrangian, and somewhat more restrictively we are assuming that the symmetry preserves the Lagrangian and not just the action.

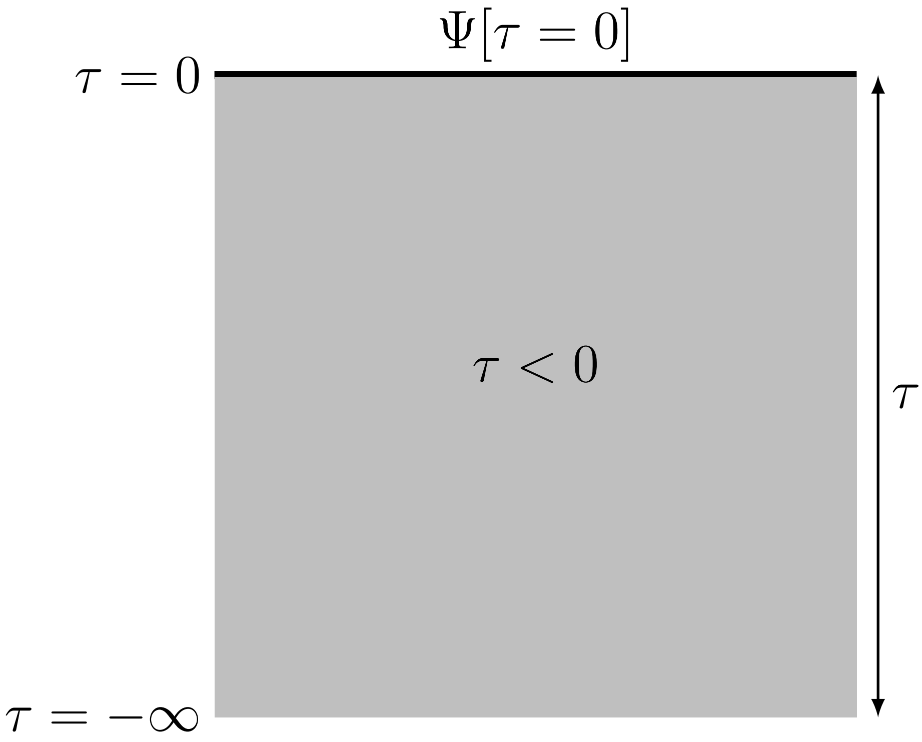

Fig. 5.1 The vacuum wavefunction $\Psi[\tau = 0]$ is given by a path integral over one half of the Euclidean τ plane, with the usual conventions integrating over $-\infty < \tau \leq 0$.

5.1):

$$\Psi[\phi(x)] \equiv \langle\phi(x)|\Psi\rangle = \int_{\tau<0} \mathrm{D}\phi\, e^{-I[\phi(x,\tau)]} \tag{5.1.1}$$

(This might be followed by some amount of Lorentzian time evolution, which will not affect the basic points below.)

There is no problem with identifying the constraint that **CRT** places on the flat space wavefunction. Assuming ϕ is real, and at fixed t, it is simply:

$$\mathbf{CRT} : \Psi[\phi(x)] = \Psi^*[\phi(-x)] \tag{5.1.2}$$

where x is the single spatial coordinate that gets flipped. Nor does it matter much if we think of this as the spacetime or the local form of **CRT**, as long as it holds fixed the t-slice on which the wavefunction is evaluated. (Assuming time translation symmetry, Ψ is independent of t.)

But what is the implication for **RR** on the wavefunction? Naively, there isn't one. **RR** tells you that correlators satisfy hermitian conjugation (3.5.1):

$$\mathbf{RR} : \langle\Psi|\, \phi_1\, \phi_2 \ldots \phi_n\, |\Psi\rangle = \langle\Psi|\, \phi_n \ldots \phi_2\, \phi_1\, |\Psi\rangle^*, \tag{5.1.3}$$

but this holds for *every* possible value of Ψ (that lives in a state space which admits a real inner product) regardless of the functional form of $\Psi[\phi]$. What went wrong? The problem is that the one-sided path integral is done over negative τ values only. Hence, a spacetime symmetry which flips $\tau \to -\tau$ does not place a meaningful constraint on Ψ. Because it reverses the order of operators, it maps a ket state $|\Psi\rangle$ to a bra state $\langle\Psi|$, rather than placing any constraint on the ket itself.

A similar problem occurs for the 180° rotation, assuming we attempt to apply it in isolation and not as part of a bigger group of symmetries.[2]

The story above also applies, without any essential change, to global de Sitter where **CRT** allows us to write a concise non-perturbative constraint for the full de Sitter wavefunction at equal-T:

$$\Psi[T;\phi(\Phi_d)] = \mathbf{CRT}(\Psi[T;\phi(\Phi_d)]) \equiv \Psi^*[-T;\phi(\pi-\Phi_d)]\,, \tag{5.1.4}$$

where we have used the fact that (as discussed in Section 4.1) for equal-time quantities, such as the de Sitter wavefunction, **CRT** simplifies to a complex conjugation, a flip in the global time coordinate and a rotation on the $d-1$-sphere. Note that at $T=0$, **CRT** acts to constrain the wavefunction to be its own conjugate after a spatial reflection.

5.1.3 Poincaré Wavefunction of the Universe

In Poincaré de Sitter, things naively get even worse. As the Poincaré patch explicitly breaks time-reversal invariance, it seems that we can't even constrain the wavefunction using **CRT**. This is because the standard Poincaré **CRT** (described in Section 4.2) maps the future patch P^+ to the past patch P^-. But then it seems to place no constraints on P^+ considered in isolation.

However, this is too quick. In fact, as we shall see, discrete symmetries can place quite strong constraints on the wavefunction Ψ. This is so for two basic reasons:

- We can use the local Lagrangian form of discrete symmetries such as **CRT** and **RR**, and
- We can make use of the fact that Ψ, the Bunch-Davies state, satisfies various analytic continuation properties.

Regarding the first point, let us consider a much more powerful form of local Lagrangian **CRT**:

LL CRT: For every point $p \in P^+$, there exists a **CRT** symmetry of its tangent space.

This is a much more powerful principle. In fact though, we can obtain it from Poincaré **CRT** if we also assume invariance under the full de Sitter group.

$$\text{Poincaré } \mathbf{CRT} + SO^+(d+1,1) \implies \mathbf{LL\ CRT}.$$

Conversely, *if* we stipulate that all local Lagrangian symmetries must arise from spacetime symmetries, and also spatial translations, we find that the full de Sitter group is implied:

$$\mathbf{LL\ CRT} \text{ from spacetime + spatial translations} \implies SO^+(d+1,1)$$

because the conjugation of translations by the local **CRT** symmetries can get you special conformals, and from there one generates the whole group. Of course, the Poincaré version of **CRT** is an easy corollary.

We therefore find, that in the context of homogeneous cosmology, local Lagrangian **CRT** can arise from spacetime symmetries only in the case of de Sitter. These wavefunction constraints will be considered in Chapter 6 (and in more specific detail in Sections 6.1.1–6.2).

[2]If we have both **RR** and the 180°, we can obviously compose them to get **CRT** which then takes kets to kets again. Or, if we impose the full continuous Lorentz Invariance, we can check if Ψ satisfies properties associated with Lorentz Invariance such as the Unruh effect.

On the other hand, local Lagrangian **RR**, which does *not* reflect a Lorentzian time direction, can be meaningfully applied in any real FLRW cosmology, regardless of the existence of any Killing symmetry. In the case where space is flat, we can use this to prove a more general version of the Cosmological Optical Theorem (in Section 6.3).

But before this, in the next Section 5.2, we will examine the close relationship between the local Lagrangian version of **RR**, and Unitarity in Lorentzian signature.

5.2 Accidental Unitarity

While **RR** is weaker than full Unitarity, it turns out it gets you a lot closer to full Unitarity than you might think! In this section, we will argue that, among the set of QFTs satisfying **RR**, a codimension-0 subset are also fully Unitary.

5.2.1 Perturbative Equivalence of Reflection Reality and Unitarity

Suppose we start with a unitary QFT, and consider deforming it by some small perturbation δL to the Lagrangian density L.[3] Let us assume that this small perturbation does not change the number of configuration degrees of freedom in the theory.[4] It should then be possible to view it as a change to the Hamiltonian δH. If we consider all possible terms in the Hamiltonian (satisfying some set of symmetries) up to some degree of irrelevance, this should give us a finite-dimensional space of perturbations to the couplings, so $H \in \mathbb{C}^N$ for some finite dimension N. Now the subspace of perturbations that preserve Unitarity should be those satisfying:

$$\delta H = \delta H^\dagger \,, \tag{5.2.1}$$

where $\dagger$ is defined using the positive norm of the original theory's state space. This can be regarded as imposing a real structure on the space of couplings, i.e. it restricts you to a real subspace $\delta H \in \mathbb{R}^N \subset \mathbb{C}^N$ such that if a vector v is in the space $\mathbb{R}^N$, iv is not in the $\mathbb{R}^N$. In other words all of its basis vectors should be (complex)-linearly independent. Since Unitary implies Reflection Positivity and hence Reflection Reality, for any $v \in \mathbb{R}^N$, $\mathbf{RR}(v) = v$. Since **RR** is antilinear this means that $\mathbf{RR}(iv) = -iv$. We now know how **RR** acts on the entire space, and it is clear that the space of **RR** symmetric perturbations is no larger than the space of unitary ones. Hence, for small perturbations of this sort, **RR** and Unitarity are equivalent conditions.

For a bosonic field theory in Lorentzian signature, such perturbations are in one-to-one correspondence with *real* contributions to δL.[5] If however, we consider perturbations that involve derivative couplings, in Lorentzian signature there can also be (typically divergent) measure factors required to restore Unitarity, that are tantamount to fixing certain imaginary terms in

[3] Even if a QFT does not come from a Lagrangian, perturbations to the theory still do.

[4] If the perturbation involves higher derivative couplings, it may be necessary to assume it comes with powers of the UV cutoff, to prevent additional degrees of freedom from appearing. Throughout this section, we are working in the regime of naive QFT perturbation theory, where we assume the Hilbert space is independent of the choice of coupling constants. This is not really true, but we expect that the conclusions of this section would not be affected by a more sophisticated analysis.

[5] There is, indeed, one class of imaginary contributions to δL that are compatible with Unitarity, namely imaginary total derivatives. These correspond to imaginary canonical transformations of the state space, which can be viewed as rescaling the definition of the inner product. But because the action appears inside of an exponential, such rescalings are multiplicative and thus cannot spoil Unitarity.

the action. However, upon passing to Euclidean signature, such terms are fixed by **RR** alone, without using any additional feature of Unitarity. This is one way in which Euclidean field theory is nicer than Lorentzian field theory.

5.2.2 How to get Theories with Negative Norm States

Does this mean that all **RR** satisfying theories are also unitary? Clearly not, since there exist theories with negative norm states. But to get them you have to break one of the assumptions in the argument above. Since the argument above refers only to *continuous* variations of the Lagrangian, one way to obtain such theories is to make a bad *discrete* choice of field content, which is incompatible with Unitarity. One easy example of this is a negative-norm scalar field. Here we start with a field with negative propagator term, e.g:

$$L = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\,. \tag{5.2.2}$$

Although this action looks unbounded below in Euclidean signature, this problem can be resolved if we define a new field $\chi = i\phi$ and do our functional integrals along the real χ direction. But then $\chi^\dagger = -\chi$, so any terms in $\hat{V}(\chi) = V(i\phi)$ with odd powers of χ will have an unexpected imaginary sign, relative to the requirements of Unitarity. This is because the states with an odd number of χ quanta have negative norm.[6] This is an example of a bad discrete choice which leads to a **RR** but not Unitary theory. Another example of such a bad choice is a spin-statistics violating theory, e.g:

$$L = i\partial_\mu\psi\partial^\mu\overline{\psi}\,, \tag{5.2.3}$$

where ψ and $\overline{\psi}$ are scalar fermions. In this case the 1-particle state space has indefinite signature and there is no way to extend **RR** to full Unitarity.

Finally we can also consider choices which change the number of degrees of freedom. An illuminating example is the Proca Lagrangian:

$$L = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu\,. \tag{5.2.4}$$

In D spacetime dimensions, the number of propagating degrees of freedom is $D-1$ when $m^2 \neq 0$, but only $D-2$ when $m^2 = 0$. The latter is, of course, due to the usual gauge symmetry of a massless photon field. Since the gauge theory has a bunch of null sates, which have 0 inner product with any other vector, it can be a boundary between a unitary theory, and a theory containing negative norm states. And in fact, in the tachyonic case where $m^2 < 0$, there is a propagating temporal mode which has negative norm. If then, you have a unitary theory with a gauge symmetry, and you want to deform it to another unitary theory, the safest thing to do is to restrict attention to only gauge-invariant perturbations to the Lagrangian.

Although the above counterexamples show that **RR** does not imply Unitarity, it is encouraging that we had to work hard to find such counterexamples—generic perturbations to a Lagrangian cannot do the trick. Put another way, once we impose **RR**, we have a decent chance of

[6] If V is an even function, there exists an additional $\mathbb{Z}_2$ symmetry $\chi \to -\chi$, and the inner product can be redefined in a way that restores Unitarity. But this amounts to redefining **RR** using a different $\mathbb{Z}_2$ generator.

accidentally ending up with a unitary theory by pure luck. There is not going to be any need to fine-tune further parameters with infinite precision to end up in the Unitary case.

5.2.3 Unitarity from CRT

It is similarly possible to argue that Unitarity can accidentally follow from a local **CRT** invariance of the action.

If we restrict to locally Lorentz invariant bulk Lagrangians, all such covariant terms have a 180° rotation symmetry,[7] and hence **CRT** implies **RR** implies (for small perturbations as above) Unitarity. Thus **CRT** is also sufficient to accidentally imply Unitarity, in the absence of bad discrete choices for the field content.

It should be noted that the above argument refers to *local* **CRT** invariance of the Lagrangian L at *each* point. In Chapter 4, we discussed a global **CRT** transformation that flips a future Poincaré-dS to the past one. This global **CRT** symmetry will not, by itself, suffice to prove local **CRT** invariance of the Lagrangian, as it only flips **CRT** around one particular static patch bifurcation surface. But, global **CRT** together with the full rotational symmetry $SO^+(d+1,1)$ implies local **CRT** at any point.

This helps to motivate the crucial importance of **CRT** in de Sitter as a check on holographic ideas. Suppose somebody hands you a claimed instance of dS/CFT and you have already checked that it is conformally invariant. But you want to know if it is dual to a bulk unitary theory. According to this argument, checking **CRT** invariance gets you nearly all the way to deciding Unitarity. As long as the low energy bulk fields don't violate spin-statistics or have negative-norm states, it is then reasonably likely that you have a Unitary theory on your hands. On the other hand, if you slap together some partition function without paying any regard to discrete symmetries, it would be infinitely unlikely that you will end up with a Unitary bulk theory. We will now explore how the symmetries described in Sections 4-5.1 impose constraints on the coefficients of the Wavefunction of the Universe.

5.3 Summary

In this chapter, we advanced our program of understanding cosmological symmetry constraints by formulating how discrete transformations act at the level of the local bulk Lagrangian in an inflationary spacetime. While the discrete $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure inherited from de Sitter geometry continues to constrain correlators computed via the in-in formalism, we demonstrated that the Wavefunction of the Universe requires a fundamentally different treatment. Since wavefunction coefficients ψ_n are computed via a single-sheet path integral, they cannot be constrained by analytic continuation alone.

To address this, we investigated how discrete symmetries such as $\mathbf{R}_\pi$ and **CRT** can be promoted to invariances of the local action within a single Poincaré patch. We showed how these symmetries can be used to define consistent transformations of fields and couplings that

[7] An exception might occur in theories where Lorentz Invariance is spontaneously broken, e.g. if the action is defined relative to a unit timelike vector field u^a as in Einstein-Aether theory [191, 192]. But even these theories can sometimes be accidentally unitary, so long as all the terms in the action are even powers of u^a, the 180° rotation symmetry remains.

act directly on the path integral defining the wavefunction, thereby enabling non-perturbative constraints on ψ_n. In doing so, we clarified the distinct but complementary roles of symmetry, Unitarity, and the analytic structure of observables in a cosmological setting.

The results of this chapter pave the way for the central development of this thesis: the derivation and proof of the **Cosmological CPT Theorem**. In Chapter 6, we bring together the insights from flat space, de Sitter symmetry, and local Lagrangian symmetries to establish how Unitarity and Scale Invariance together imply **CRT** symmetry of the wavefunction, as well as the two converse statements implying **RR** and Discrete Scale Invariance. This culminates in a novel characterisation of **CPT** in cosmology, revealing a deeper interplay between discrete symmetries and the quantum gravitational structure of inflationary spacetime.

"There was no 'before' the beginning of our universe,
because once upon a time there was no time."
— John D. Barrow

6

The Cosmological CPT Theorem

In Chapters 3 to 5, we developed the necessary theoretical framework to understand the role of discrete symmetries in quantum field theory (QFT) in curved spacetimes. Beginning with the structure of **CPT** in flat space, we identified an underlying $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group relating discrete rotations, Reflection Reality (**RR**), and **CRT** invariance. We then extended this construction to global and Poincaré patches of de Sitter space, showing that analogous discrete symmetries constrain both cosmological correlators and the structure of inflationary observables. Finally, we clarified that, while these discrete symmetries act geometrically on correlation functions, constraining the Wavefunction of the Universe (WFU) requires a local analysis of the Lagrangian in a single Poincaré patch. This distinction is crucial: since the WFU arises from a single-sheet path integral, it cannot in general be constrained by the same analytic continuation arguments that apply to correlators.

With these ingredients in place, we are now ready to present the main result of this thesis: the **Cosmological CPT Theorem**. This theorem provides a non-perturbative characterisation of CPT invariance in an inflationary spacetime, showing how **CRT** combines with Discrete Scale Invariance to ensure the Unitarity of the wavefunction.

The Cosmological CPT theorem says that:

$$\text{Scale Invariance} + \text{Unitarity} \implies \textbf{CRT}\text{ invariance} \quad ,$$

as well as three closely related statements:

$$\text{Discrete Scale Invariance} + \text{Reflection Reality} \implies \textbf{CRT}\text{ invariance} \quad ;$$

$$\textbf{CRT}\text{ invariance} + \text{Discrete Scale Invariance} \implies \text{Reflection Reality} \quad ;$$

$$\textbf{CRT}\text{ invariance} + \text{Reflection Reality} \implies \text{Discrete Scale Invariance} \quad .$$

Each of these implications can be understood within a single Poincaré patch, where traditional flat-space arguments fail to directly apply. Another central insight presented in this chapter is that Reflection Reality — a discrete symmetry which **ALL** unitary theories have to satisfy non-perturbatively — plays a privileged role in cosmology: it is the only one of the three symmetries that is automatically satisfied by any local, unitary QFT on *any flat FLRW background*. This chapter is devoted to understanding the deep interplay between discrete symmetries, analytic continuation, to proving the Cosmological CPT theorem both non-perturbatively, as well as to all orders in perturbation theory.

6.1 Analytic Continuation in Cosmology

Similar to how the physical domain of Mandelstam variables in amplitudes is defined for $s, t \geq 0$ and crossing symmetry involves analytically continuing to $s, t \in \mathbb{C}$ (see [193] for recent progress regarding crossing in flat space amplitudes), we will now discuss how our discrete transformations in Section 4.2 will involve analytically continuing the conformal time outside of the physical domain of $\eta \leq 0$ and in Fourier space will also involve analytically continuing the magnitude of the momenta $k = \sqrt{\mathbf{k} \cdot \mathbf{k}} \geq 0$.

6.1.1 Non-perturbative Analytic Continuation

To perform the analytic continuation explained in Section 4.2 more carefully, let us do QFT on a fixed spacetime Poincaré patch metric.[1] We take the metric to have the slightly more general form:

$$\mathrm{d}s^2 = (A\,\mathrm{d}\zeta)^2 + (B\,e^{H\zeta}\,\mathrm{d}y_i)^2 = A^2\,\mathrm{d}\zeta^2 + B^2 e^{2H\zeta}\,\mathrm{d}y_i\,\mathrm{d}y_i\,, \tag{6.1.1}$$

where A and B are constants. It should be noted that because a typical matter action has a factor of $\sqrt{g}$ out front, the sign of A and B matters, not just A^2 and B^2. Some special cases of this metric are:

- $A = B = +1$: Normal Euclidean hyperbolic space H^0, with $\zeta = \tau$;
- $A = +i$, $B = +1$: Future dS-Poincaré P^+, with $\zeta = t$;
- $A = -i$, $B = +1$: Past dS-Poincaré P^-, with $\zeta = -t$;
- $A = B = +i$: Funky Euclidean hyperbolic space H^+ with all minus signature, with $\zeta = \tau$.

The funky hyperbolic space H^+ does not in general have the same properties as a normal field theory: massive fields become tachyons, and (depending on the theory) it need not satisfy Reflection Reality.

Correct path via Wick rotation of g_{tt}: From this it can be seen that a *correct* way to analytically continue from P^+ to P^- is to hold $B = +1$ fixed and take A along the path:

$$A = i^{1-2q} = e^{(1-2q)i\pi/2}\,, \qquad q \in [0,1] \tag{6.1.2}$$

where $q = 0$ corresponds to P^+ and $q = 1$ corresponds to P^-. The halfway point, $q = \frac{1}{2}$, corresponds to normal Euclidean hyperbolic space H^0. The advantage of this path is that a

[1] Note as we previously explained that in global de Sitter there is no need for an analytic continuation as the **CRT** transformation is well-defined within those coordinates. The author thanks Carlos Duaso Pueya and Ciaran McCulloch for fruitful discussions regarding analytic continuations in cosmology.

well-behaved QFT (e.g. an arbitrary p-form field) has an action whose real part is bounded below everywhere in the interior of the q-interval. As usual the Lorentzian path integral, being oscillatory, is on the edge of convergence. (If we define the initial condition by taking A to have a small positive part at early times, we obtain the Bunch-Davies state; hence, the initial condition is independent of the choice of $q \in [0, 1]$.)

Geometric path via η continuation: However, the correct path does not have any obvious geometric interpretation in terms of analytically continuing the time coordinate; that is, the path does not lie on the complexification of P^+. If we choose instead to rotate the η coordinate, this corresponds to starting in P^+ and shifting $Ht \to Ht + i\pi$. This time we are holding $A = +i$ fixed and taking B along the path:

$$B = i^{2q} = e^{i\pi q}, \qquad q \in [0, 1] \tag{6.1.3}$$

At $q = \frac{1}{2}$, this contour passes through funky hyperbolic space H^+ where the QFT need not converge. Worse still, at $q = 1$, it ends up at $A = +i$, $B = -1$ which is not the same as P^-!

However, the two paths are very similar. For each value of q, they differ only by an overall rescaling of the metric. That is, the geometrical path can be converted back into the correct path if, for each value of $q \in [0, 1]$, we additionally rotate the metric by

$$g_{ab} \to \Omega^2 g_{ab}, \qquad \Omega = (-i)^{2q} = e^{-i\pi q} \tag{6.1.4}$$

This looks very similar to an imaginary Weyl transformation, although we do not rescale any of the matter fields. At $q = \frac{1}{2}$, this puts us back to normal hyperbolic space H^0, and at $q = 1$, it yields P^-, after applying $\Omega = -1$. Then, $\Omega^2 = +1$ and we have simply rotated the metric g_{ab} a full cycle in the complex plane. However, in general the amplitudes become singular as $g_{ab} \to 0$, and hence we have rotated around a singularity. So we are on a new Riemann sheet, and we do not necessarily get the same physics as if we had not done the Weyl rescaling!

If we write our amplitudes as an explicit function of Ω, we can thus conclude that we can continue from Bunch-Davies in P^+ (evolving forwards in time) to Bunch-Davies in P^- (evolving backwards in time) by doing the transformation:

$$\mathcal{A}^-(\eta; y_1, \ldots, y_d; \Omega) = \mathcal{A}^+(e^{-i\pi}\eta; y_1, \ldots, y_d; e^{-i\pi}\Omega), \tag{6.1.5}$$

where in this expression, the use of $e^{i\pi}$ rather than -1 is a mnemonic indicating in which direction one should rotate in the complex plane to reach -1 (all such rotations are to be done simultaneously, in this case holding the ratio of η and Ω fixed). But P^+ and P^- are also related by **CRT** symmetry, which tells us that P^+ and P^- are also related by sending $y_i \to -y_i$ (which is explicitly the *reflection* **R**) and complex conjugating the amplitude. Hence, invariance under **CRT** also requires P^+ and P^- to also be related by the transformation:

$$\mathcal{A}^-(\eta; y_1, \ldots, y_d; \Omega) = \mathcal{A}^+(\eta; -y_1, \ldots, -y_d; \Omega)^*. \tag{6.1.6}$$

where a minus sign that is written out explicitly, represents a mere reflection of the coordinate (no analytic continuation). Combining these together, it follows that, in a **CRT** invariant theory, the wavefunction coefficients in P^+ must obey:

$$\mathbf{CRT} : \psi_n^*(\eta; y_1, \ldots, y_d; \Omega) = \psi_n(e^{-i\pi}\eta; -y_1, \ldots, -y_d; e^{-i\pi}\Omega)\,. \tag{6.1.7}$$

In momentum space we would have almost the same relation:

$$\mathbf{CRT} : \psi_n^*(\eta; k_1, \ldots, k_d; \Omega) = \psi_n(e^{-i\pi}\eta; k_1, \ldots, k_d; e^{-i\pi}\Omega)\,, \tag{6.1.8}$$

apart from the fact that complex conjugation in position space reverses the sign of k, cancelling out the reflection of each spatial coordinate.

Geometric path via y continuation: There is another way to achieve a similar result, by analytically continuing the y variables rather than the η variable. If we analytically continue $y_i \to e^{i\pi q} y_i$ (for all of the y coordinates at once, and for $q \in [0, 1]$ as before), then to hold the metric fixed, we end up doing the exact same rotation of B as in (6.1.3). As before, to convert this to the correct contour for the A parameter, we will need to rotate Ω to obtain equivalence to the correct path. Also, as η keeps the same sign, we will need to invoke some discrete symmetry which remains within P^+. The obvious choice here is **RR**.

The simplest way to write down a correct constraint on the wavefunction coefficient is to start with **CRT** in the form (6.1.7), and also act with a 180° rotation. As the latter is an analytic continuation of the $SO^+(1, 1)$ scaling symmetry, it takes the form:

$$\mathbf{D}^{\pm}_{-1} : \psi_n(\eta; y_1, \ldots, y_d; \Omega) = \psi_n(e^{\pm i\pi}\eta; e^{\pm i\pi} y_1, \ldots, e^{\pm i\pi} y_d; \Omega)\,, \tag{6.1.9}$$

where the $\pm$ sign depends on the direction of the 180° rotation. When considering wavefunction coefficients, it is no longer an *a priori* truth that the 360° rotation is a trivial transformation. This enhances the symmetry group to $\mathrm{Aut}(\mathbb{Z})$, see the discussion surrounding Table 6.1 in Section 6.4.2. Hence, the combined statement (which should follow from Reflection Reality) implies the following result for the y-continued wavefunction coefficient:

$$\mathbf{RR} : \psi_n^*(\eta; y_1, \ldots, y_d; \Omega) = \psi_n(\eta; -e^{i\pi} y_1, \ldots, -e^{i\pi} y_d; e^{-i\pi}\Omega)\,, \tag{6.1.10}$$

where, similarly to previous expressions, multiplication by $-e^{i\pi}$ is shorthand for an analytic continuation followed by a reflection, and cannot validly be replaced with $+1$!

In momentum space, these relations become:

$$\mathbf{D}^{\pm}_{-1} : \psi_n(\eta; k_1, \ldots, k_d; \Omega) = (-1)^{\pm d}\psi_n(e^{\pm i\pi}\eta; e^{\mp i\pi} k_1, \ldots, e^{\mp i\pi} k_d; \Omega)\,, \tag{6.1.11}$$

and

$$\mathbf{RR} : \psi_n^*(\eta; k_1, \ldots, k_d; \Omega) = (-1)^d \psi_n(\eta; e^{-i\pi} k_1, \ldots, e^{-i\pi} k_d; e^{i\pi}\Omega)\,. \tag{6.1.12}$$

where the prefactor of $(-1)^d$ arises because of the scaling of the overall delta function $\delta^d\left(\sum_a^n \mathbf{k}_a\right)$ due to momentum conservation, which transforms like k^{-d}, and which we (purely conventionally) exclude from our definition of ψ_n. (We would not need to explicitly include this factor if we were manipulating ψ'_n, which is defined to include the delta function.) In cases involving fractional dimensions (e.g. dim. reg.), this prefactor becomes $e^{i\pi d}$ in the case of **RR** and $e^{\pm i\pi d}$ in the case of $\mathbf{D}^{\pm}_{-1}$, transforming in the opposite direction as the k argument.

To summarise, we obtain the following non-perturbative results for the wavefunction coefficients in position space:

$$\mathbf{CRT} : \psi^*_n(\eta; y_1, \dots, y_d; \Omega) = \psi_n(e^{-i\pi}\eta; -y_1, \dots, -y_d; e^{-i\pi}\Omega)\,; \tag{6.1.13}$$

$$\mathbf{D}^{\pm}_{-1} : \psi_n(\eta; y_1, \dots, y_d; \Omega) = \psi_n(e^{\pm i\pi}\eta; e^{\pm i\pi}y_1, \dots, e^{\pm i\pi}y_d; \Omega)\,; \tag{6.1.14}$$

$$\mathbf{RR} : \psi^*_n(\eta; y_1, \dots, y_d; \Omega) = \psi_n(\eta; -e^{i\pi}y_1, \dots, -e^{i\pi}y_d; e^{-i\pi}\Omega)\,. \tag{6.1.15}$$

and in momentum space:

$$\mathbf{CRT} : \psi^*_n(\eta; k_1, \dots, k_d; \Omega) = \psi_n(e^{-i\pi}\eta; k_1, \dots, k_d; e^{-i\pi}\Omega)\,; \tag{6.1.16}$$

$$\mathbf{D}^{\pm}_{-1} : \psi_n(\eta; k_1, \dots, k_d; \Omega) = e^{\pm i\pi d}\,\psi_n(e^{\pm i\pi}\eta; e^{\mp i\pi}k_1, \dots, e^{\mp i\pi}k_d; \Omega)\,; \tag{6.1.17}$$

$$\mathbf{RR} : \psi^*_n(\eta; k_1, \dots, k_d; \Omega) = e^{i\pi d}\,\psi_n(\eta; e^{-i\pi}k_1, \dots, e^{-i\pi}k_d; e^{-i\pi}\Omega)\,. \tag{6.1.18}$$

To avoid confusion, in the expressions above that involve complex conjugation, the arguments η, k, Ω on the LHS should be taken to be real, so that analytic continuation occurs only on the RHS.[2]

Similar to the constraint (5.1.4) on the global de Sitter wavefunction, **CRT** also allows us to write a concise non-perturbative constraint for the full de Sitter wavefunction in the Poincaré patch:

$$\Psi[\eta; \Omega; \phi(\mathbf{y})] = \mathbf{CRT}\,(\Psi[\eta; \Omega; \phi(\mathbf{y})]) \equiv \Psi[e^{-i\pi}\eta; e^{-i\pi}\Omega; \phi(-\mathbf{y})]^*\,, \tag{6.1.19}$$

$$\Psi[\eta; \Omega; \phi(\mathbf{k}_a)] = \mathbf{CRT}\,(\Psi[\eta; \Omega; \phi(\mathbf{k}_a)]) \equiv \Psi[e^{-i\pi}\eta; e^{-i\pi}\Omega; \phi(\mathbf{k}_a)]^*\,, \tag{6.1.20}$$

where η, Ω and ϕ are real, and it is important that the complex conjugation on the RHS is the *final step*, after evaluating the rest of the expression. Note that at $\eta = 0$, **CRT** acts to constrain the wavefunction to be its own conjugate after the spatial reflection and Weyl rotation.

As all of the above analytic continuations are equivalent to the correct Wick rotation, which should work for any fixed background QFT, Eqs. (6.1.7)–(6.1.12) will always be valid *non-perturbatively*, at least in the non-gravitational coupling constants.[3] In the next section,

[2] If we interpret the $*$ on the LHS as global complex conjugation, we would get a contradiction as it is not possible to equate an anti-holomorphic function on the LHS with a holomorphic function on the RHS, unless the functions are constant. Hence we may only take the arguments on the LHS to be complex, if we define the conjugate function $f^*(z)$ as the *holomorphic* extension of the complex conjugate acting on the real domain $f : \mathbb{R} \to \mathbb{C}$, i.e. taking the complex conjugate of the Taylor coefficients of f. This would need to be distinguished from both the anti-holomorphic extension of the original function $f(z^*)$, or the global complex conjugate $f(z)^* = f^*(z^*)$. This is analogous to the notation for the † discussed in footnote 10.

[3] In the case of gravity, at least two additional problems present themselves: (1) the conformal mode problem, which means that the path integral is not convergent for any value of q, and (2) we cannot simply write $g_{ab} = g^0_{ab} + h_{ab}$ and treat h_{ab} like any other field, because the gauge conditions of the graviton no longer make sense unless we allow Ω to act on h_{ab} as well. Furthermore, as H^0 reverses the sign of Λ, there is reason to expect that the result here cannot make sense non-perturbatively, as in many approaches to quantum gravity (e.g. the string landscape), Λ is not a freely adjustable parameter. That said, when working perturbatively in the gravitational coupling, in all the cases we have checked we get the same phases as in the case of a scalar field.

we will see how to rewrite these results perturbatively. In that context, the **RR** conditions (6.1.10) and (6.1.12) will become the statement of Hermitian analyticity, and (as expected from past work [50, 51, 130, 131]) they are actually valid for *all* FLRW spacetimes, to all orders in perturbation theory (and for Bunch-Davies). This makes sense because unlike the other two transformations, **RR** does not require the existence of a map from P^+ to P^-.

6.2 Perturbative Discrete Symmetries

The transformations in Section 6.1.1 are somewhat obscure, due to their explicit dependence on the Weyl factor Ω. If, however, we are doing *unrenormalised* perturbation theory around a free, covariant, parity-even bosonic theory, a remarkable simplification appears. This is because the action takes the form:

$$I = \int \mathrm{d}^D x \sqrt{g}\, \mathcal{L}_0[\phi, g_{ab}]\,, \tag{6.2.1}$$

where, apart from the overall factor of $\sqrt{g}$, only *integer* powers of the metric appear in $\mathcal{L}$.[4] In this section, we are defining I with the conventions of the Euclidean action, so that the amplitude always goes like e^{-I}, even in Lorentzian signature; thus the Lorentzian action is imaginary.

It follows that when we analytically continue $\Omega \to -\Omega$ (regardless of which direction we go around $\Omega = 0$), the sole effect is to multiply the $\sqrt{g}$ factor by a minus sign for each spacetime dimension: $(-1)^D$. This is then equivalent to acting on the Lagrangian with the transformation:

$$\mathcal{L}(e^{-i\pi}\Omega) = (e^{-i\pi})^D \mathcal{L}(\Omega)\,. \tag{6.2.2}$$

So when $D = d+1$ is even, there is actually no difference between the correct path and the paths where we analytically continue η or y_i. We can just (in unrenormalised perturbation theory) ignore the $\Omega \to -\Omega$ instruction.

On the other hand, when $D = d + 1$ is odd, the Lagrangian changes sign: $\mathcal{L} \to -\mathcal{L}$. In any Feynman-Witten diagram, this provides an extra minus sign for every vertex V, and also every internal edge I (because the sign of the propagator term also reverses). In any connected diagram, the Euler formula $V - I = 1 - L$ allows us to write the change of any unrenormalised Feynman-Witten amplitude $\mathcal{A}$ in terms of the number of loops L:

$$\mathcal{A}^{(L)}(e^{-i\pi}\Omega) = e^{i\pi(d+1)(L-1)}\mathcal{A}^{(L)}(\Omega)\,, \tag{6.2.3}$$

where $\mathcal{A}^{(L)}$ denotes a perturbative amplitude computed at some loop order L (e.g. $L = 0$ corresponds to tree-level and $L = 1$ corresponds to 1-loop order), and we allow d to be non-integer. The reason we keep saying "unrenormalised", is that this pattern does not continue to hold when doing renormalisation theory. The reason is that the UV cutoff ϵ has units of length, and so it implicitly depends on the value of g_{ab} and hence Ω. If, for example, one has a log divergence of the form $C \log(\epsilon)$, then when $\Omega \to e^{-i\pi}\Omega$, this will also send $\epsilon \to e^{-i\pi}\epsilon$, thus

[4] For spinor fields, the propagator term contains a factor of γ_μ which scales like the square-root of a metric. So in theories with fermions, we expect an additional power of (-1) in **CRT** for each fermion propagator, in any dimension D.

producing the shift:

$$C \log\left(e^{i\pi}\epsilon\right) = C\left[\,\log(\epsilon) - i\pi\right], \tag{6.2.4}$$

so that any log-divergent term produces an imaginary shift in the corresponding finite term (this shift has no explicit dependence on D). Of course, this effect does not matter for tree-level diagrams, as these are UV-finite and thus do not require renormalisation. Nor does it affect any loop diagrams that happen to be finite.[5] Also, the coefficient of the log divergence itself still satisfies the phase rule (except when considering a diagram with higher powers of the $\log(\epsilon)$, in which case this remark would apply to the leading order log divergence).[6]

A conceptually similar effect comes if we are doing bulk perturbation theory, not around a free theory, but around an interacting CFT where various operators have non-integer anomalous dimensions. In this case, the conformal perturbation theory can have essentially arbitrary non-integer powers of the metric determinant g, and so $\Omega \to -\Omega$ can introduce strange phases even when $D = d+1$ is even. To say it as a slogan, the **CRT** transformation takes the simple form (6.2.3) only for amplitudes that involve no bulk quantum anomalies.

In cases without such quantum anomalies, the only extra factor to worry about is $(-1)^{D(1-L)}$, and so, combining our results with those of the previous section, we therefore obtain the following perturbative results for the wavefunction coefficients in position space:

$$\begin{aligned}
\mathbf{CRT}: \quad \psi_n^{(L)*}(\eta; y_1, \ldots, y_d) &= e^{i\pi(d+1)(L-1)}\,\psi_n^{(L)}(\,e^{-i\pi}\eta; -y_1, \ldots, -y_d)\,; && (6.2.5)\\
\mathbf{D}^{\pm}_{-1}: \quad \psi_n^{(L)}(\eta; y_1, \ldots, y_d) &= \psi_n^{(L)}(e^{\pm i\pi}\eta; e^{\pm i\pi}y_1, \ldots, e^{\pm i\pi}y_d)\,; && (6.2.6)\\
\mathbf{RR}: \quad \psi_n^{(L)*}(\eta; y_1, \ldots, y_d) &= e^{i\pi(d+1)(L-1)}\,\psi_n^{(L)}(\;\eta\;; -e^{i\pi}y_1, \ldots, -e^{i\pi}y_d)\,, && (6.2.7)
\end{aligned}$$

where $\psi_n^{(L)}$ denotes a perturbative wavefunction coefficient computed at some loop order L. Having these constraints in position space may prove to be quite powerful in a position-space cosmological bootstrap program, which is yet to be explored in the literature but could be more well-motivated in terms of connecting to cosmological observations, since imprints of physical principles, such as locality, in the late-time non-Gaussianity is obscured in Fourier space [194].[7] In momentum space we obtain:

$$\begin{aligned}
\mathbf{CRT}: \quad \psi_n^{(L)*}(\eta; k_1, \ldots, k_d) &= e^{i\pi(d+1)(L-1)}\;\psi_n^{(L)}(\,e^{-i\pi}\eta; k_1, \ldots, k_d)\,; && (6.2.8)\\
\mathbf{D}^{\pm}_{-1}: \quad \psi_n^{(L)}(\eta; k_1, \ldots, k_d) &= e^{\pm i\pi d}\;\psi_n^{(L)}(e^{\pm i\pi}\eta; e^{\mp i\pi}k_1, \ldots, e^{\mp i\pi}k_d)\,; && (6.2.9)\\
\mathbf{RR}: \quad \psi_n^{(L)*}(\eta; k_1, \ldots, k_d) &= e^{i\pi((d+1)L-1)}\,\psi_n^{(L)}(\,\eta\,; e^{-i\pi}k_1, \ldots, e^{-i\pi}k_d)\,. && (6.2.10)
\end{aligned}$$

As this section considers only the case of a parity-even Lagrangian, technically the arguments in this section don't care about whether we reflect the y-coordinates in **CRT** or **RR**. However,

[5] In the case of power law divergences, one tends to obtain, for any divergent loop, a power ϵ^{-D+2n} where n is an integer, and hence when $\epsilon \to -\epsilon$ there is an extra factor of $(-1)^D$ for each such divergence. This is actually good since it means that counterterm needed to renormalise a loop diagram has the same sign as the tree level Lagrangian. But, we can also just choose to use a regulator scheme which automatically eliminates power law divergences, and not worry about them.

[6] A similar looking imaginary shift will appear in Section 6.4.5 for IR log divergent quantities that go like $\log(-\eta)$, when we take the $\eta \to 0$ limit of **CRT** at future infinity.

[7] The author thanks Carlos Duaso Pueyo for making him aware of this potential application of the position-space constraints.

we have put the signs in the correct place for parity-odd theories as well, as can be deduced in Section 6.3 where we do not restrict to parity-even theories.

The analytic continuation of the conformal time coordinate η may result in an interesting connection to the recent work in [195], where the in-in correlators were related to an in-out prescription involving evolution through a pair of Poincaré patches stitched together by their $\eta = 0$ surfaces. The authors of [195] refer to the transformation on the Feynman-Witten diagrams used in this construction as a time reversal which is not quite correct as it is really a **CRT** transformation.

6.3 Cosmological Optical Theorem for Flat FLRW

Next, we show how to generalise the **RR** results above to a general spatially-flat FLRW cosmology with arbitrary scale factor $a(\zeta)$. The metric is thus

$$\mathrm{d}s^2 = (A\ \mathrm{d}\zeta)^2 + (B\, a(\zeta)\, \mathrm{d}y_i)^2 \,, \tag{6.3.1}$$

and we continue to define $P^{\pm}$ as before, even though in the general case there is no way to go from P^+ to P^- by an extension of the *real* spacetime manifold. So here, P^- is more of an abstraction representing what would happen if you solve the same QFT but on a contracting spacetime instead of an expanding one. Recall that in Section 6.1.1 there were two ways to go from P^+ to P^-: by analytically continuing η or y. While the former depended on the dS form of the scale factor $a(\zeta)$, the latter requires only that space is Euclidean. Hence, we can derive (6.1.10) and (6.1.12) directly, as a non-perturbative form of the Cosmological Optical Theorem (COT) [50–52, 67].

It is also illuminating to derive the perturbative form of these relations, (6.2.7) and (6.2.10), directly from perturbation theory. In this section, we drop the stipulations that the Lagrangian is Lorentz-covariant, or that it is invariant with respect to time or space reversal. This may be done by allowing the action to depend covariantly on the 1-index *purely temporal* and d-index *purely spatial* permutation symbols:

$$\boldsymbol{\epsilon}^t \,, \qquad \boldsymbol{\epsilon}^{i_1 i_2 \ldots i_d} \,, \tag{6.3.2}$$

where we define each permutation symbol so that it is invariant under all real coordinate transforms in (time / space) except those for which det Jacobian $= -1$, in which case the (time / space) permutation symbol flips sign.[8] However, for any analytic continuation that is continuously connected to the identity, the $\boldsymbol{\epsilon}$'s have to remain the same sign throughout, as the analytic continuation of 1 is 1. Now let a given amplitude be labelled $T = \pm 1$ according to whether it depends on $\boldsymbol{\epsilon}^t$ an (even/odd) number of times, and let it be labelled $X = \pm 1$ according to whether it depends on $\boldsymbol{\epsilon}^{i_1 i_2 \ldots i_d}$ an (even/odd) number of times.

Reflection Reality (**RR**) may now be stated as the principle that on the Euclidean cosmology, each term in the Lagrangian satisfies $\mathcal{L} = T\mathcal{L}^*$, i.e. the τ-reversal even terms are real and the τ-reversal odd terms are imaginary. From this it follows that, if we do the y-rotation described

[8] If a Lorentz covariant parity-odd theory is desired, one simply requires the epsilon symbols to come together in the form of the usual spacetime permutation symbol: $\boldsymbol{\epsilon}^{a_0 a_1 \ldots a_d} = (d+1)\boldsymbol{\epsilon}^{[a_0}\boldsymbol{\epsilon}^{a_1 a_2 \ldots a_d]}$.

in Section 6.1.1, we obtain:

$$\psi_n^*(\eta; y_1, \ldots, y_d; \Omega) = T\psi_n(\eta; e^{i\pi}y_1, \ldots, e^{i\pi}y_d; e^{-i\pi}\Omega)\,, \tag{6.3.3}$$

but note that we are not yet reflecting the y-coordinates here. Doing that reflection in each coordinate gives us an additional factor when d and the parity are both odd:[9]

$$\psi_n(\eta; -y_1, \ldots, -y_d; \Omega) = X^d\psi_n(\eta; y_1, \ldots, y_d; \Omega)\,. \tag{6.3.4}$$

Furthermore, since $\boldsymbol{\epsilon}^t$ substitutes for a factor of $\sqrt{g_{tt}}$ in the Lagrangian $\mathcal{L}$, and $\boldsymbol{\epsilon}^{i_1 i_2 \ldots i_d}$ substitutes for a factor of $\sqrt{g^{(d)}}$ in $\mathcal{L}$ — this is equivalent to saying you don't need the metric to integrate a p-form in p dimensions — the Ω continuation is also modified from (6.2.2) and becomes:[10]

$$\psi_n^{(L)}(\eta; y_1, \ldots, y_d; e^{i\pi}\Omega) = TX^d(-1)^{D(L-1)}\psi_n^{(L)}(\eta; y_1, \ldots, y_d; \Omega) = TX^d e^{i\pi D(L-1)}\psi_n^{(L)}(\eta; y_1, \ldots, y_d; \Omega)\,. \tag{6.3.5}$$

By combining together Eqs. (6.3.3)–(6.3.5), we observe that the dependence on T and X cancels out, and so we verify the relation from the previous section:

$$\textbf{RR}: \qquad \psi_n^{(L)*}(\eta; y_1, \ldots, y_d) = e^{i\pi(d+1)(L-1)}\psi_n^{(L)}(\eta; -e^{i\pi}y_1, \ldots, -e^{i\pi}y_d)\,. \tag{6.3.6}$$

In momentum space we thus recover (6.2.10). This is the same as the *Hermitian Analyticity* component of the Cosmological Optical Theorem (COT), see e.g [130, 131], i.e. the part of the COT which does not involve cutting rules.

6.4 Perturbative Derivation of The Cosmological CPT Theorem

In Section 6.1, we already outlined the necessary analytic continuations needed to define these three symmetries {**CRT**, **RR**, **D**}. In Sections 6.4.1–6.4.3, we will rederive the perturbative form of these symmetries, from a more concrete and detailed perspective. In Section 6.4.4 we will take the limit as $\eta \to 0$ in order to derive the form of the symmetries at the boundary, including the phase formula. Section 6.4.5 will then check this result by comparing it to specific calculations in the literature.

[9] We do not attempt to extend this formula to non-integer d. It is problematic to write down parity-odd terms in non-integer dimensions, as $\boldsymbol{\epsilon}^{i_1 i_2 \ldots i_d}$ would have a non-integral number of indices and therefore cannot be fully contracted with normal tensors. If we instead consider the effect of flipping *all* spatial dimensions, this is equivalent to a rotation in $SO(d)$ when $d =$ even, and so one expects it have no effect on the correlator for rotationally invariant theories that are defined in general d.

[10] Some readers may be wondering why the X and T factors in (6.3.5) do not depend on the number of loops L, even though our parity-even result did depend on the number of loops. But that was because flipping the sign of the Lagrangian changes both the interactions and the propagators, leading to an invocation of the Euler formula. On the other hand, T and X just count the individual number of odd items in the Feynman-Witten diagram, without regard to whether the items are vertices or edges. In an action where *all* terms were odd under X or T, this would provide a factor that also depends on the number of loops, but we make no such assumption here, as in general it is more natural for there to be both odd and even terms in L.

6.4.1 Unitarity and the Bunch-Davies Vacuum

In this section we discuss our approach to enforcing the Bunch-Davies vacuum in a way that is consistent with Unitarity. Establishing such consistency is particularly important here as it enforces the direction in which we must rotate the spacetime coordinates through the complex plane. This extends the discussion in Section 4.3 where we defined the correlation functions in terms of the vacuum of the full theory, $|\Omega\rangle$. In the presence of interactions perturbation theory requires that we rewrite this state in terms of the free (Bunch-Davies) vacuum. It is standard to equate these two states in the early time limit by introducing a small imaginary contribution to the time, $\eta_{\mathrm{i}} = -\infty(1-i\varepsilon)$, rotating the contour off the negative real axis and turning off the interactions. To see this consider the free vacuum expanded in the basis of energy eigenstates of the full theory (see [190] for a more pedagogical derivation),

$$|0\rangle = \sum_N |N\rangle\langle N|0\rangle\,. \tag{6.4.1}$$

We then consider the evolution of this state in the full theory,

$$\begin{aligned} e^{-iH(\eta-\eta_{\mathrm{i}})}|0\rangle &= \sum_n e^{-iE_n(\eta-\eta_{\mathrm{i}})}|n\rangle\langle n|0\rangle & (6.4.2)\\ &= e^{-iE_\Omega(\eta-\eta_{\mathrm{i}})}|\Omega\rangle\langle\Omega|0\rangle + \sum_{N\neq\Omega} e^{-iE_N(\eta-\eta_{\mathrm{i}})}|N\rangle\langle N|0\rangle & (6.4.3)\\ &= e^{-iE_\Omega(\eta-\eta_{\mathrm{i}})}\left(|\Omega\rangle\langle\Omega|0\rangle + \sum_{N\neq\Omega} e^{-i(E_N-E_\Omega)(\eta-\eta_{\mathrm{i}})}|N\rangle\langle N|0\rangle\right)\,. & (6.4.4)\end{aligned}$$

The vacuum is, by definition, the lowest energy state in the theory and so $E_N - E_\Omega > 0$. Therefore, in the limit $\eta_{\mathrm{i}} \to -\infty(1-i\varepsilon)$ only the first term remains and

$$e^{-iH(\eta-\eta_{\mathrm{i}})}|0\rangle = e^{-iH(\eta-\eta_{\mathrm{i}})}|\Omega\rangle\langle\Omega|0\rangle\,. \tag{6.4.5}$$

This is just the Hamiltonian time evolution operator and so

$$\langle\mathcal{O}(\eta)\rangle = \langle 0|U_{\mathrm{I}}^\dagger(\eta)\mathcal{O}_{\mathrm{I}}(\eta)U_{\mathrm{I}}(\eta)|0\rangle \tag{6.4.6}$$

However, as was noted in [196, 197], this approach spoils the Unitarity of the time evolution operator,

$$U_{\mathrm{I}}^{-1}(\eta) = \overline{\mathcal{T}}\exp\left[-i\int_{-\infty(1-i\varepsilon)}^{\eta} \mathrm{d}\eta H_{\mathrm{int}}[\phi,\phi']\right] \neq U_{\mathrm{I}}^\dagger(\eta) = \overline{\mathcal{T}}\exp\left[-i\int_{-\infty(1+i\varepsilon)}^{\eta} \mathrm{d}\eta H_{\mathrm{int}}[\phi,\phi']\right]\,. \tag{6.4.7}$$

Therefore, following [196], we instead introduce the shifted Hamiltonian,

$$H^\varepsilon = He^{\varepsilon\eta}\,, \tag{6.4.8}$$

which was shown to produce the same results perturbatively as the more standard rotation. Furthermore, it has the same effect of turning off the interactions in the infinite past so that interacting and free vacua agree thus we remain in the Bunch-Davies vacuum state. The reason for this is that, in both cases the ε-prescription contributes a negative real part to the exponent of the Hamiltonian in the infinite past thereby ensuring that it vanishes,

$$\lim_{\eta\to-\infty} H^{\varepsilon} \sim \lim_{\eta\to-\infty} e^{ik_T\eta+\varepsilon\eta} = \lim_{\eta\to-\infty} e^{i(k_T-i\varepsilon)\eta} = 0\,. \tag{6.4.9}$$

From (6.4.9) we can see that this ε-prescription is equivalent to introducing a small negative imaginary part to the energies. In terms of this new Hamiltonian the expectation value is

$$\langle\mathcal{O}(\eta)\rangle = \langle 0|U_{\varepsilon}^{\dagger}(\eta)\mathcal{O}_{\mathrm{I}}(\eta)U_{\varepsilon}(\eta)|0\rangle, \quad U_{\varepsilon}(\eta) = \mathcal{T}\exp\left[-i\int_{-\infty}^{\eta}\mathrm{d}\eta H_{\mathrm{int}}^{\varepsilon}[\phi,\phi']\right], \tag{6.4.10}$$

which is now a well defined, unitary operator.

When we act with each of our three transformations it is important to ensure that we remain in the same vacuum state (else they cannot be symmetries of the theory). This will require that we rotate each of our coordinates in a particular direction through the complex plane when moving from the positive to the negative real axis (or vice versa). To see this we first consider **RR**, which involves a rotation of the momenta through the negative half of the complex plane,

$$\mathbf{k} \to e^{-i\theta}\mathbf{k}\,. \tag{6.4.11}$$

From these momenta we generate a set of energies[11], $k = \sqrt{\mathbf{k}\cdot\mathbf{k}}$, which are the variables that are typically used to express both correlation functions and wavefunction coefficients. Under this rotation the energy transforms as

$$k \to \sqrt{\mathbf{k}\cdot\mathbf{k}\,e^{-2i\theta}} = \sqrt{\mathbf{k}\cdot\mathbf{k}}\sqrt{e^{-2i\theta}}\,, \tag{6.4.12}$$

where the final line follows as k is real. To ensure the single-valuedness of k during this rotation we need to continuously follow a Riemann sheet of the square root (i.e. we will not add any factors of 2π to the exponent). Therefore, we have that

$$k \to e^{-i\theta}k\,. \tag{6.4.13}$$

In order for the interactions to vanish in the early time limit and thus to remain in the Bunch-Davies vacuum we must ensure that

$$\mathrm{Re}(ik\eta+\varepsilon\eta) < 0 \Rightarrow \lim_{\eta\to-\infty} e^{ik\eta+\varepsilon\eta} = 0\,. \tag{6.4.14}$$

Under this rotation in the energies this condition becomes,

$$\mathrm{Re}(ike^{-i\theta}\eta+\varepsilon\eta) = k\sin(\theta)\eta+\varepsilon\eta > 0 \tag{6.4.15}$$

which is true for positive θ (thus retroactively justifying our decision to rotate into the lower half of the complex plane). The direction of this rotation (as well as the fact that the factor ε is left unchanged) was noticed in the discussion of the Cosmological Optical Theorem [50] where

[11] Although there is an absence of time translation symmetry in cosmology the literature refers to the magnitude of a spatial momentum vector as "energy", as it plays an analogous role to energy in flat space.

the $i\varepsilon$ prescription was included on the energy, $k \to k - i\varepsilon$. In that case it was necessary to complex conjugate the energy when sending $k \to -k^*$ to keep the sign of ε constant. Although here the ε is treated independently of k the same behaviour is observed, k is reflected and ε remains unchanged.

Next, we consider the **CRT** transformation acting on this Hamiltonian. In this case the momentum is unchanged and we just rotate the time coordinate, $\eta \to \eta e^{-i\theta}$ so that,

$$\mathrm{Re}(ik\eta e^{-i\theta} + \varepsilon\eta e^{-i\theta}) = k\sin(\theta)\eta + \varepsilon\eta\cos(\theta)\,. \tag{6.4.16}$$

This is a linear combination of sines and cosines and so could equivalently be represented as a shifted sine function which is non-zero at $\theta = 0$. Therefore, this function is guaranteed to change sign over the range $0 \leq \theta \leq \pi$ (or $-\pi \leq \theta \leq 0$) and we must also rotate ε alongside η, $\varepsilon \to \varepsilon e^{i\theta}$, so that the exponent is

$$\mathrm{Re}(ik\eta e^{-i\theta} + \varepsilon\eta) = k\sin(\theta)\eta + \varepsilon\eta \tag{6.4.17}$$

which is negative just as for the rotation in k.

Finally, we must consider the simultaneous rotation of k and η this comes from the analytic continuation of the scaling transformation and so when we send $\eta \to \lambda\eta$ we must simultaneously send $k \to \lambda^{-1}k$. Therefore, the two rotate in opposite directions through the complex plane such that the combination $k\eta$ is unchanged. As a result of this the exponent in the infinite time limit of the Hamiltonian has a real part given by,

$$\mathrm{Re}(ik\eta + \varepsilon\eta e^{i\theta}) = \varepsilon\eta\cos(\theta)\,. \tag{6.4.18}$$

Just as before this will change sign and so we must rotate ε in the same direction as k. This doesn't enforce a particular direction to the rotation of k or η however.

6.4.2 The Loop Momentum Integral

In Section 6.2 we established the transformation properties of the wavefunction coefficients from the analytic continuation of the coordinates and metric. Here we present an alternative perspective to reach the same conclusions by looking at the explicit calculation of the wavefunction coefficients.

In order to achieve this goal we first break up the wavefunction into distinct parts that are easier to tackle on their own. As explained in [118] the wavefunction coefficients can be calculated by performing a series of nested time integrals plus, potentially, some loop integrals,

$$\psi_n'^{(L)}(\eta_0;\mathbf{k};\varepsilon) = i^V \int^{\eta_0} \prod_v^V a_v^{d+1}\,\mathrm{d}\eta_v e^{\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\mathbf{P}_l \prod_a^n K_{k_a}(\eta) \prod_i^I G_{p_i}(\eta,\eta')\delta^d\left(\sum_a^n \mathbf{k}_a\right)\,, \tag{6.4.19}$$

where the prime in $\psi_n'^{(L)}$ is used to explicitly highlight that these wavefunction coefficients contain the δ-function whilst those without the δ-function will be denoted by $\psi_n^{(L)}$, and we have used the shorthand notation $\psi_n^{(L)}(\eta_0;\mathbf{k};\varepsilon) \equiv \psi_n^{(L)}(\eta;k_1,\ldots,k_d;\epsilon)$. The external energies are indicated by $k_a = |\mathbf{k}_a|$ whilst the p_i are the internal energies which are built from the $\mathbf{k}_a$ plus potentially some of the loop momenta, $\mathbf{P}_l$. Note that here we have made the ε prescription explicit and that there are no derivatives acting on any of our propagators. This is purely for the sake of notational simplicity and adding any derivatives will not add any technical complications.

We do not explicitly consider UV divergences in this section (which can lead to some additional corrections with a relative i sign, as discussed in Section 6.2). However, as our results are valid in general dimensions, it is straightforward to renormalise these divergences by dimensional regularisation and this will be discussed explicitly in Section 6.4.5.

The first component we will look at is the time integral which runs from $-\infty$ to η_0 when we have a transformation involving the time coordinate it is these limits that pick up a minus sign. To remove this we redefine the variables inside the integral with $\eta_v \to -\eta_v$[12]. The effect of this on the measure factor is then straightforward as everything is just some power of η.

The loop integral is slightly more involved. To treat this we first change variables so that rather than doing the loop integral over some d dimensional vectors we instead perform it over some set of loop energies. This requires a change of basis which was elucidated for arbitrary loops in [198]. For our purposes, the most important fact about this determinant is that it is related to the volume of a simplex with edges that are the lengths of squares of energies. Therefore, its zeros and functional form are unchanged under a transformation which sends $k \to -k$. Explicitly, for a single loop, the integration measure becomes

$$\int \mathrm{d}^d\mathbf{P}_l = \frac{N}{\sqrt{\mathrm{Vol}^2(s)}} \int_\Gamma \prod_i^{I_l} 2p_i\, \mathrm{d}p_i \left[\frac{\mathrm{Vol}^2(p,s)}{\mathrm{Vol}^2(s)}\right]^{\frac{d-I_l-1}{2}} , \tag{6.4.20}$$

where I_l is the number of internal lines in the loop, l; $\mathbf{P}_l$ is the momentum running through this loop; and N accounts for the integral over the $d - I_L$ sphere plus an extra, unimportant, proportionality constant. Under the rescalling $p \to \lambda p$ and $s \to \lambda s$ this transforms as

$$\lambda^d \int \mathrm{d}^d\mathbf{P}_l = \frac{N\lambda^{2I_l}}{\sqrt{\lambda^{2I_l-2}}\sqrt{\mathrm{Vol}^2(s)}} \int_\Gamma \prod_i^{I_l} 2p_i\, \mathrm{d}p_i \left[\frac{\lambda^{2I_l}\mathrm{Vol}^2(p,s)}{\lambda^{2I_l-2}\mathrm{Vol}^2(s)}\right]^{\frac{d-I_l-1}{2}} . \tag{6.4.21}$$

For positive λ we can see that the scaling on each side agrees. Naively, on the left hand side continuing $\lambda \to -\lambda$ would appear to have no impact. However, from the right hand side we can see how our analytic continuation in the energies alters this conclusion, instead recovering a factor of $(-1)^d$.[13] To see this, we first note that the region, Γ, depends only on the squares of the energies and so is unchanged. However, each of our square roots of energies does pick up a factor of -1. As we now have a potentially non-integer exponent in the dimension of the integral d we will need to be careful with $e^{i\pi d}$ vs $e^{-i\pi d}$. For **RR**, in accordance with the discussion in Section 6.4.1, all rotations in the energy must be taken in the lower half plane and so $\lambda = \lambda e^{-i\pi}$.

[12] Note that whether this minus sign is $e^{i\pi}$ or $e^{-i\pi}$ will be important but, as it can rotate in a different way depending on which transformation we are considering, we will keep things vague at this point. When we write explicit expressions for the transformations we will be careful to make clear in which direction the rotation goes.

[13] Now that we have identified how to analytically continue loop energies, this removes the barrier which prevented the authors of [67] from deriving the cosmological equivalent of cutting rules from generalised Unitarity (see [199] for a pedagogical review of flat space generalised Unitarity).

Therefore we have,

$$(-\lambda)^d \int \mathrm{d}^d\mathbf{P}_l = \frac{Ne^{-i\pi d}\lambda^d}{\sqrt{\mathrm{Vol}^2(s)}} \int_\Gamma \prod_i^{I_l} p_i \,\mathrm{d}p_i \left[\frac{\mathrm{Vol}^2(p,s)}{\mathrm{Vol}^2(s)}\right]^{\frac{d-I_l-1}{2}} . \tag{6.4.22}$$

We next explore the transformations of the propagators. To do this we consider the differential equation,

$$\mathcal{O}_\mathbf{k}(\eta) = a^{d-1}\frac{\partial^2}{\partial\eta^2} + (d-1)a'a^{d-2}\frac{\partial}{\partial\eta} + m^2a^{d+1} + a^{d-1}k^2 \Rightarrow \begin{cases} \mathcal{O}_\mathbf{k}(\eta)G_k(\eta,\eta') & = -i\delta(\eta-\eta')\,, \\ \mathcal{O}_\mathbf{k}(\eta)K_k(\eta) & = 0\,. \end{cases} \tag{6.4.23}$$

The propagators additionally satisfy the boundary conditions

$$\lim_{\eta\to\eta_0} K_k(\eta) = 1\,, \qquad \lim_{\eta\to-\infty} K_k(\eta,\eta') \propto e^{ik\eta}\,, \tag{6.4.24}$$

$$\lim_{\eta,\eta'\to\eta_0} G_k(\eta,\eta') = 0\,, \qquad \lim_{\eta,\eta'\to-\infty} G_k(\eta,\eta') \propto e^{ik\eta}\,. \tag{6.4.25}$$

These boundary conditions at early time will ensure that our $i\varepsilon$ prescription will cause the early time integrand to vanish. Let us first consider three relationships that our differential operator satisfies. These will later be linked to symmetries of the wavefunction and are labeled accordingly,

$$\mathbf{RR}: \mathcal{O}^*_\mathbf{k}(\eta) = \mathcal{O}_{e^{-i\pi}\mathbf{k}}(\eta)\,, \tag{6.4.26}$$

$$\mathbf{CRT}: \mathcal{O}^*_\mathbf{k}(\eta) = e^{-i\pi(d+1)}\mathcal{O}_\mathbf{k}(e^{-i\pi}\eta)\,, \tag{6.4.27}$$

$$\mathbf{D}^\pm_{-1}: \mathcal{O}_\mathbf{k}(\eta) = e^{\pm i\pi(d+1)}\mathcal{O}_{e^{\mp i\pi}\mathbf{k}}(e^{\pm i\pi}\eta)\,. \tag{6.4.28}$$

Here, the direction of the rotations in $\mathbf{k}$ and η are dictated by the asymptotic time behaviour, preserving the Bunch-Davies vacuum discussed in Section 6.4.1 and the $\pm$ sign in $\mathbf{D}^\pm_{-1}$ represents the ambiguity in the direction of rotation of $\eta \to e^{\pm i\pi}\eta$[14].

We start by considering Reflection Reality,

$$\mathcal{O}^*_\mathbf{k}(\eta)K^*_k(\eta) = 0, \qquad \lim_{\eta\to\eta_0} K^*_k(\eta) = 1\,, \qquad \lim_{\eta\to-\infty} K^*_k(\eta) \propto e^{-ik\eta}\,, \tag{6.4.29}$$

$$\mathcal{O}_{e^{-i\pi}\mathbf{k}}(\eta)K_{e^{-i\pi}k}(\eta) = 0, \qquad \lim_{\eta\to\eta_0} K_{e^{-i\pi}k}(\eta) = 1\,, \qquad \lim_{\eta\to-\infty} K_{e^{-i\pi}k}(\eta) \propto e^{-ik\eta}\,. \tag{6.4.30}$$

In light of (6.4.26) these two differential equations are identical to each other as are the boundary conditions therefore,

$$\mathbf{RR}: \; K^*_k(\eta) = K_{e^{-i\pi}k}(\eta). \tag{6.4.31}$$

[14] Note that the ambiguity in the sign of the Scale Invariance transformation is meaningless in integer dimension but in non-integer dimension it introduces a seemingly meaningful choice which we will discuss more later.

Similarly, the bulk-bulk propagator satisfies

$$-\mathcal{O}^*_{\mathbf{k}}(\eta)G^*_k(\eta,\eta') = -i\delta(\eta-\eta'), \qquad \lim_{\eta\to\eta_0} -G^*_k(\eta,\eta') = 0\,, \qquad \lim_{\eta\to-\infty} -G^*_k(\eta,\eta') \propto e^{-ik\eta}\,, \tag{6.4.32}$$

$$\mathcal{O}_{e^{-i\pi}\mathbf{k}}(\eta)G_{e^{-i\pi}k}(\eta,\eta') = -i\delta(\eta-\eta'), \quad \lim_{\eta\to\eta_0} G_{e^{-i\pi}k}(\eta,\eta') = 0\,, \quad \lim_{\eta\to-\infty} G_{e^{-i\pi}k}(\eta,\eta') \propto e^{-ik\eta}\,, \tag{6.4.33}$$

which gives a very similar expression for this propagator,

$$\mathbf{RR}:\ G^*_k(\eta,\eta') = -G_{e^{-i\pi}k}(\eta,\eta'). \tag{6.4.34}$$

Notice, that we are free to introduce a minus sign in the early time limit in (6.4.32), this is because the boundary condition when we introduce ε is that the propagator vanishes, so it is just the sign of the imaginary exponent that needs to be fixed.

The **CRT** transformation acts similarly,

$$\mathcal{O}^*_{\mathbf{k}}(\eta)K^*_k(\eta) = 0, \qquad \lim_{\eta\to\eta_0} K^*_k(\eta) = 1\,, \qquad \lim_{\eta\to-\infty} K^*_k(\eta) \propto e^{-ik\eta}\,, \tag{6.4.35}$$

$$\mathcal{O}_{\mathbf{k}}(e^{-i\pi}\eta)K_k(e^{-i\pi}\eta) = 0, \qquad \lim_{\eta\to\eta_0} K_k(e^{-i\pi}\eta) = 1\,, \qquad \lim_{\eta\to-\infty} K_k(e^{-i\pi}\eta) \propto e^{-ik\eta}\,. \tag{6.4.36}$$

The additional phase introduced to the differential operator can simply be dropped when considering the bulk-boundary propagator as the right hand side of the differential equation is zero and so,

$$\mathbf{CRT}:\ K^*_k(\eta) = K_k(e^{-i\pi}\eta). \tag{6.4.37}$$

For the Green's function this is not the case, but the boundary conditions can have an arbitrary phase introduced,

$$\mathcal{O}^*_{\mathbf{k}}(\eta)G^*_k(\eta,\eta') = i\delta(\eta-\eta'), \qquad \mathcal{O}_{\mathbf{k}}(e^{-i\pi}\eta)G_k(e^{-i\pi}\eta, e^{-i\pi}\eta') = i\delta(\eta-\eta'), \tag{6.4.38}$$

$$\lim_{\eta\to\eta_0} G^*_k(\eta,\eta') = 0\,, \qquad \lim_{\eta\to\eta_0} e^{i\pi(d+1)}G_k(e^{-i\pi}\eta, e^{-i\pi}\eta') = 0\,, \tag{6.4.39}$$

$$\lim_{\eta\to-\infty} G^*_k(\eta,\eta') \propto e^{-ik\eta}\,, \qquad \lim_{\eta\to-\infty} e^{i\pi(d+1)}G_k(e^{-i\pi}\eta, e^{-i\pi}\eta') \propto e^{-ik\eta}\,, \tag{6.4.40}$$

where the time reversal has introduced an extra minus sign into the delta function due to its action on the limits of the integral. To see this consider,

$$\int_{-\infty}^{\eta'} \mathrm{d}\eta\,\delta(\eta-\eta') \to \int_{\infty}^{-\eta'} \mathrm{d}\eta\,\delta(\eta+\eta') = -\int_{-\infty}^{\eta'} \mathrm{d}\eta\,\delta(-\eta+\eta'). \tag{6.4.41}$$

Once again, we can see that (6.4.27) allows us to extract a relationship between the complex conjugate of the bulk-bulk propagator and its analytic continuation,

$$\mathbf{CRT}:\ G^*_k(\eta,\eta') = e^{i\pi(d+1)}G_k(e^{-i\pi}\eta, e^{-i\pi}\eta'). \tag{6.4.42}$$

Next we consider dilatations. This symmetry simply introduces a phase to both equations whilst leaving the boundary conditions unchanged. Therefore,

$$\mathbf{D}_{-1}^{\pm}: K_k(\eta) = K_{e^{\mp i\pi}k}(e^{\pm i\pi}\eta), \text{ and } G_k(\eta,\eta') = e^{\mp i\pi d} G_{e^{\mp i\pi}k}(e^{\pm i\pi}\eta, e^{\pm i\pi}\eta'). \tag{6.4.43}$$

Thus, we see that the bulk-boundary propagators pick up very simple relationships from each of our symmetries. The bulk-bulk propagators have similar relationships but additionally introduce a dimension dependent phase.

Finally, we must consider the transformation of the delta function. This will behave similarly to the right hand side of the differential equation for the bulk-bulk propagator (6.4.41),

$$\delta^d\left(e^{i\theta}\sum_a^k \mathbf{k}_a\right) = e^{-i\theta d}\delta^d\left(\sum_a^k \mathbf{k}_a\right). \tag{6.4.44}$$

This is unlike what we expect for the delta function which, for real numbers, transforms as

$$\delta(ax) = \frac{1}{|a|}\delta(x). \tag{6.4.45}$$

However, just as for the loop and time integrals, the analytic continuation that we perform will also rotate the integrals over the external momenta by $e^{i\theta d}$. This is then matched by the analytic continuation of the delta function.

We now have all the ingredients to understand how our three transformations work on a wavefunction coefficient. The first symmetry we will consider is Reflection Reality which relates the complex conjugate to the rotation of all momenta through the lower half of the complex plane,

$$\mathbf{RR}: \left[\psi_n'^{(L)}(\eta_0;\mathbf{k};\varepsilon)\right]^* \tag{6.4.46}$$

$$= (-i)^V \int^{\eta_0} \prod_v^V a_v^{d+1}\, \mathrm{d}\eta_v e^{\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\mathbf{P}_l \prod_a^n K_{k_a}^*(\eta) \prod_i^I G_{p_i}^*(\eta,\eta')\delta^d\left(\sum_a^n \mathbf{k}_a\right) \tag{6.4.47}$$

$$= (-i)^V \int^{\eta_0} \prod_v^V a_v^{d+1}\, \mathrm{d}\eta_v e^{\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\mathbf{P}_l \prod_a^n K_{e^{-i\pi}k_a}(\eta) \prod_i^I -G_{e^{-i\pi}p_i}(\eta,\eta')\delta^d\left(\sum_a^n \mathbf{k}_a\right) \tag{6.4.48}$$

$$= e^{i\pi(I-V+d(L-1))} i^V \int^{\eta_0} \prod_v^V a_v^{d+1}\, \mathrm{d}\eta_v e^{\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\left(e^{-i\pi}\mathbf{P}_l\right) \prod_a^n K_{e^{-i\pi}k_a}(\eta) \prod_i^I G_{e^{-i\pi}p_i}(\eta,\eta')\delta^d\left(e^{-i\pi}\sum_a^n \mathbf{k}_a\right) \tag{6.4.49}$$

$$= e^{i\pi(1+d)(L-1)}\psi_n'^{(L)}(\eta_0;e^{-i\pi}\mathbf{k};\varepsilon). \tag{6.4.50}$$

where in the penultimate line we have employed the Feynman-Euler relationship, $V - I + L = 1$ and the rotation on $\mathbf{k}$ applies to all internal and external energies. Next we investigate Scale

Invariance which rotates all momenta magnitudes as well as the conformal time,

$$\mathbf{D}^{\pm}_{-1} : \psi'^{(L)}_n(\eta_0; \mathbf{k}; \varepsilon) = i^V e^{\pm i\pi d(V+L-I-1)} \int^{\eta_0} \prod_v^V e^{\mp i\pi d} a_v^{d+1} \,\mathrm{d}\eta_v e^{\varepsilon \eta_v} \prod_l^L \mathrm{d}^d \left(e^{\mp i\pi}\mathbf{P}_l\right) \tag{6.4.51}$$

$$\times \prod_a^n K_{e^{\mp i\pi}k_a}(e^{\pm i\pi}\eta) \prod_i^I G_{e^{\mp i\pi}p_i}(e^{\pm i\pi}\eta, e^{\pm i\pi}\eta')\delta^d\left(e^{\mp i\pi}\sum_a^n \mathbf{k}_a\right) \tag{6.4.52}$$

$$= i^V \int^{e^{\pm i\pi}\eta_0} \prod_v^V a_v^{d+1}\,\mathrm{d}\eta_v e^{e^{\mp i\pi}\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\left(e^{\mp i\pi}\mathbf{P}_l\right) \prod_a^n K_{e^{\mp i\pi}k_a}(\eta) \prod_i^I G_{e^{\mp i\pi}p_i}(\eta,\eta')\delta^d\left(e^{\mp i\pi}\sum_a^n \mathbf{k}_a\right) \tag{6.4.53}$$

$$= \psi'^{(L)}_n(e^{\pm i\pi}\eta_0; e^{\mp i\pi}\mathbf{k}; e^{\mp i\pi}\varepsilon)\,. \tag{6.4.54}$$

The final transformation we will consider is **CRT** which flips the conformal time with an overall complex conjugation of the wavefunction coefficient,

$$\mathbf{CRT} : \left[\psi'^{(L)}_n(\eta_0; \mathbf{k}; \varepsilon)\right]^* \tag{6.4.55}$$

$$= (-i)^V \int^{\eta_0} \prod_v^V a_v^{d+1}\,\mathrm{d}\eta_v e^{\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\mathbf{P}_l \prod_a^n K^*_{k_a}(\eta) \prod_i^I G^*_{p_i}(\eta,\eta')\delta^d\left(\sum_a^n \mathbf{k}_a\right) \tag{6.4.56}$$

$$= (-i)^V e^{-i\pi dV} \int^{\eta_0} \prod_v^V e^{i\pi d} a_v^{d+1}\,\mathrm{d}\eta_v e^{\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\mathbf{P}_l \prod_a^n K_{k_a}(e^{-i\pi}\eta) \prod_i^I e^{i\pi(d+1)} G_{p_i}(e^{-i\pi}\eta, e^{-i\pi}\eta')\delta^d\left(\sum_a^n \mathbf{k}_a\right) \tag{6.4.57}$$

$$= e^{i\pi(I-V)(d+1)} i^V \int^{-\eta_0} \prod_v^V a_v^{d+1}\,\mathrm{d}\eta_v e^{e^{-i\pi}\varepsilon\eta_v} \prod_l^L \mathrm{d}^d\mathbf{P}_l \prod_a^n K_{k_a}(\eta) \prod_i^I G_{p_i}(\eta,\eta')\delta^d\left(\sum_a^n \mathbf{k}_a\right) \tag{6.4.58}$$

$$= e^{i\pi(1+d)(L-1)} \psi'^{(L)}_n(e^{-i\pi}\eta_0; \mathbf{k}; e^{i\pi}\varepsilon)\,. \tag{6.4.59}$$

	1	**CRT**	**RR**
1	1	**CRT**	**RR**
CRT	**CRT**	1	$\mathbf{D}^{+}_{-1}$
RR	**RR**	$\mathbf{D}^{-}_{-1}$	1

Table 6.1 Table showing how the transformations combine for $\psi^{(L)}_n$. In this table the first transformation is in the column heading and then the row heading second. We see that **CRT** and **RR** each square to 1. Furthermore, they combine to give $\mathbf{D}^{\pm}_{-1}$. Since *a priori* (before imposing dilation symmetry) the operations $\mathbf{D}^{+}_{-1}$ and $\mathbf{D}^{-}_{-1}$ act differently, we no longer have the $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure as the dilatation operator doesn't square to itself. Instead this group has the same structure as the group $\mathrm{Aut}(\mathbb{Z})$ of automorphisms of the integers.

An interesting observation one can make is that for $\psi'^{(L)}_n$, **CRT** and **RR** introduce equal phases, and in even spacetime dimensions all three transformations leave any real $\psi'^{(L)}_n$ invariant at any loop order, provided $\psi'^{(L)}_n$ is UV-finite. This also highlights the fact that in the original contact Cosmological Optical Theorem [50, 51] the minus sign picked up between $\psi^{(L)}_n$ and its reflected counterpart can be understood as an artefact of the fact that the δ-function has been stripped off. Indeed, when we consider $\psi_n^{(L)}$ with this delta function stripped off we find

$$\mathbf{CRT}: \left[\psi_n^{(L)}(\eta_0;\mathbf{k};\varepsilon)\right]^* = e^{i\pi(d+1)(L-1)}\psi_n^{(L)}(e^{-i\pi}\eta_0;\mathbf{k};e^{i\pi}\varepsilon)\,, \tag{6.4.60}$$

$$\mathbf{D}^{\pm}_{-1}: \psi_n^{(L)}(\eta_0;\mathbf{k};\varepsilon) = e^{\pm i\pi d}\psi_n^{(L)}(e^{\pm i\pi}\eta_0;e^{\mp i\pi}\mathbf{k};e^{\mp i\pi}\varepsilon)\,, \tag{6.4.61}$$

$$\mathbf{RR}: \left[\psi_n^{(L)}(\eta_0;\mathbf{k};\varepsilon)\right]^* = e^{i\pi((d+1)L-1)}\psi_n^{(L)}(\eta_0;e^{-i\pi}\mathbf{k};\varepsilon)\,. \tag{6.4.62}$$

These relationships each imply an operator that leaves the wavefunction coefficient unchanged up to an overall phase in the presence of the corresponding symmetry,

$$\mathbf{CRT}\left(\psi_n^{(L)}\right) = \left[\psi_n^{(L)}(e^{-i\pi}\eta_0,\mathbf{k},e^{i\pi}\varepsilon)\right]^* = e^{i\pi(d+1)(L-1)}\psi_n(\eta_0;\mathbf{k};\varepsilon)\,, \tag{6.4.63}$$

$$\mathbf{D}^{\pm}_{-1}\left(\psi_n^{(L)}\right) = \psi_n^{(L)}(e^{\pm i\pi}\eta_0;e^{\mp i\pi}\mathbf{k};e^{\mp i\pi}\varepsilon) = e^{\mp i\pi d}\psi_n(\eta_0;\mathbf{k};\varepsilon)\,, \tag{6.4.64}$$

$$\mathbf{RR}\left(\psi_n^{(L)}\right) = \left[\psi_n^{(L)}(\eta_0;e^{-i\pi}\mathbf{k};\varepsilon)\right]^* = e^{i\pi((d+1)L-1)}\psi_n(\eta_0;\mathbf{k};\varepsilon)\,, \tag{6.4.65}$$

where, for clarity, we have kept the ε-dependence to explicitly show how it rotates. Although, in the final wavefunction coefficient $\psi_n^{(L)}$ we take $\varepsilon\to 0$ so that this rotation, ultimately, will not influence these transformations[15]. These symmetry transformations can then be combined, as presented in Table 6.1. Just as in the flat space case the combination of **CRT** and **RR** generates the other transformations and they square to themselves. However, here we do not recover the $\mathbb{Z}_2\times\mathbb{Z}_2$ group but instead the group Aut($\mathbb{Z}$) of automorphisms of the integers. It is important to note that we need both $\mathbf{D}^{\pm}_{-1}$ for this group to close as (in arbitrary dimensions) they are not generically self inverse. This is not an obstacle as both transformations are implied by the Scale Invariance of the theory due to the fact that the Bunch-Davies condition is preserved by both rotations (this condition is what restricts us from considering rotations in the other directions for either **CRT** or **RR**). Interestingly, the ambiguity in this rotation direction appears absent from $\psi_n^{\prime(L)}$ where $\mathbf{D}^{\pm}_{-1}$ doesn't introduce a phase. However, this is merely an artifact of the dilatation symmetry, without imposing it the two transformations act differently. Furthermore, to connect **RR** to **CRT** we still need to be able to rotate in both directions using our dilatation operator. To see this consider how $\mathbf{D}^{+}_{-1}$ acts on an already transformed wavefunction coefficient,

$$\mathbf{D}^{+}_{-1}\left(\mathbf{RR}\left(\psi_n^{(L)}\right)\right) = \mathbf{D}^{+}_{-1}\left(e^{i\pi(-1+(d+1)L)}\psi_n^{(L)}\right) = e^{i\pi(d+1)(L-1)}\psi_n^{(L)} = \mathbf{CRT}\left(\psi_n^{(L)}\right)\,, \tag{6.4.66}$$

$$\mathbf{D}^{+}_{-1}\left(\mathbf{CRT}\left(\psi_n^{(L)}\right)\right) = \mathbf{D}^{+}_{-1}\left(e^{i\pi(d+1)(L-1)}\psi_n^{(L)}\right) = e^{i\pi((d+1)L-2d-1)}\psi_n^{(L)} \neq \mathbf{RR}\left(\psi_n^{(L)}\right)\,. \tag{6.4.67}$$

We can see that this agrees with the group structure in Table 6.1,

$$\mathbf{CRT}\cdot\mathbf{RR} = \mathbf{D}^{+}_{-1} \Rightarrow \begin{cases} \mathbf{D}^{+}_{-1}\cdot\mathbf{RR} = \mathbf{CRT}\,, \\ \mathbf{D}^{+}_{-1}\cdot\mathbf{CRT} = \mathbf{CRT}\cdot\mathbf{RR}\cdot\mathbf{CRT} \neq \mathbf{RR}\,, \end{cases} \tag{6.4.68}$$

where the second implication follows from the fact that the **CRT** transformation squares to one.

We will now move on to discuss how these transformations can be used to identify constraints of the $\mathcal{I}^+$ boundary correlators where $\eta = 0^-$ which will be of particular importance for dS/CFT!

[15] Of course, ε is important in these transformations as it dictates how $\mathbf{k}$ and η_0 transform but when considering the final wavefunction it can be safely dropped if we keep this rotation direction in mind.

6.4.3 Spinning Fields

The analysis above also applies to spinning fields. To see this it is convenient to use the following notations and conventions. In $d+1$-dimensional spacetime, for traceless[16], integer spin fields we use the free action developed in [200] and discussed in the context of the cosmological bootstrap in [52, 201]:

$$S = \int \mathrm{d}^{d+1}x \, [a(\eta)]^{d-1} \frac{1}{2s!} \left[(\sigma'_{i_1 \dots i_s})^2 - c_s^2 (\partial_j \sigma_{i_1 \dots i_s})^2 - \delta c_s^2 \left(\partial^j \sigma_{j i_2 \dots i_s} \right)^2 - m^2 a^2 (\sigma_{i_1 \dots i_s})^2 \right] . \tag{6.4.69}$$

The totally-symmetric, traceless tensor $\sigma_{i_1 \dots i_s}$ has spatial indices $i_1 = 1, \dots, d$ which span the d-dimensional spacelike hypersurface orthogonal to the η coordinate.[17] $\sigma_{i_1 \dots i_s}$ has $(2s+1)$ components, which each create states ("particles") with helicities $0, \pm 1, \dots, \pm s$, and we have enforced invariance under dilatations by including inverse factors of the scale factor for each coordinate derivative. Following [52, 201] we Fourier transform and diagonalise this using the helicity modes, σ_h, defined by:

$$\sigma_{i_1 \dots i_s}(\eta; x) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{h=-S}^{S} \mathbf{e}^h_{i_1 \dots i_s}(\mathbf{k}) \sigma_h(\eta; \mathbf{k}) \,, \tag{6.4.70}$$

These helicity tensors are defined as an outer product of helicity vectors,

$$\mathbf{e}^h_{i_1 \dots i_s} = \mathbf{e}^{h_1}_{i_1} \dots \mathbf{e}^{h_s}_{i_s} \,, \tag{6.4.71}$$

which satisfy the following relations:

$$\mathbf{e}^h_i(\mathbf{k}) \left[\mathbf{e}^{h'}_i(\mathbf{k}) \right]^* - 4\delta_{hh'} = 0 \qquad \text{(orthogonality and normalisation)} \,, \tag{6.4.72}$$

$$\left[\mathbf{e}^h_i(\mathbf{k}) \right]^* - \mathbf{e}^h_i(-\mathbf{k}) = 0 \qquad (\sigma_{i_1 \dots i_s}(x) \text{ is real}) \,. \tag{6.4.73}$$

Note that these fields are not assumed to be transverse, h is allowed to take d different values including 0 where $\mathbf{e}^0$ is proportional to the momentum. The contributions from the other helicity modes are therefore transverse by the orthogonality condition.

We parameterise the WFU, Ψ, at conformal time η_0 in terms of the helicities of the integer spin field as

$$\Psi[\eta_0; \sigma(\mathbf{k})] = \exp \left[- \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{h_i = \pm} \int_{\mathbf{k}_1, \dots, \mathbf{k}_n} \psi_n^{h_1 \dots h_n}(\eta_0; \mathbf{k}) (2\pi)^d \delta^d \left(\sum \mathbf{k}_a \right) \sigma_{h_1}(\eta_0; \mathbf{k}_1) \dots \sigma_{h_n}(\eta_0; \mathbf{k}_n) \right] . \tag{6.4.74}$$

[16] To consider fields with a non-zero trace, one can subtract the trace and treat it as an additional scalar field.

[17] The Effective Field Theory of Inflation (EFToI) [68] is derived by considering a theory which is only invariant under spatial diffeomorphisms, for which there is a preference for the co-ordinate choice used in unitary gauge, where the time coordinate is chosen to coincide with the surfaces of constant value of the field $\sigma_{i_1 \dots i_s}$. The EFToI thus encapsulates generic class of models of inflation where spatial diffeomorphisms are preserved. (6.4.69) can be written in a covariant way by using the Goldstone boson π of time translations to upgrade the spatial tensor $\sigma_{i_1 \dots i_s}$ to a covariant spacetime tensor. The coupling of $\sigma_{i_1 \dots i_s}$ to π is also dictated by this constructions but we will not need this here.

Spatial translations and spatial rotations ensure that wavefunction coefficients can be written as a product of a *helicity factor*, which is an $SO(d)$ invariant function of helicity vectors and spatial momenta, multiplied by a *trimmed wavefunction coefficient* which is only a function of the magnitudes of the momenta in the literature:

$$\psi_n^{h_1 \ldots h_n} = (\text{tensor structure}) \times (\text{trimmed wavefunction coefficient})\,. \tag{6.4.75}$$

We take all coefficients appearing in the tensor structure to be real and therefore include any factors of i that might appear when converting to momentum space, or simply as part of the Feynman rules, in the trimmed part which we will denote as ψ_n for brevity. For a general $\psi_n^{h_1 \ldots h_n}$, we have

$$\psi_n^{h_1 \ldots h_n} = \left[\mathbf{e}^{h_1}(\mathbf{k}_1) \ldots \mathbf{e}^{h_n}(\mathbf{k}_n)\, \mathbf{k}_1^{\alpha_1} \ldots \mathbf{k}_n^{\alpha_n}\right] \psi_n, \tag{6.4.76}$$

for some integer α_i. Note that we can choose for *both parity-even and parity-odd* tensor structures to be invariant with respect to the discrete symmetries of **CRT**, **RR** and **D** which we discuss in this thesis.[18] Hence, all our results extend directly from the scalar ψ_n discussed in Section 6.4.2 to the tensor case.

6.4.4 Constraints on Boundary Wavefunction Coefficients

All of the results that we have presented in this section so far rely on an analytic continuation, either in momenta or time. However, by considering the Wavefunction of the Universe at the future boundary of dS, where the time dependence often trivialises, we can derive direct constraints on the wavefunction coefficients. This future boundary is where the reheating surface is understood to live in the inflationary paradigm. Therefore, such constraints are of particular cosmological relevance.

In the late time limit the bulk fields take the form

$$\lim_{\eta \to 0^-} \phi(\eta, \mathbf{y}) = \overline{\phi}_+(\mathbf{y})\eta^{\Delta^+} + \overline{\phi}_-(\mathbf{y})\eta^{\Delta^-}\,, \tag{6.4.77}$$

$$= \overline{\phi}_+(\mathbf{y})\eta^{\Delta} + \overline{\phi}_-(\mathbf{y})\eta^{d-\Delta}\,, \tag{6.4.78}$$

where for the case of scalars, the mass $m^2/H^2 = m^2\ell^2 = \Delta(d-\Delta)$ is related to the conformal dimension in the usual way $\Delta = d/2+\nu$, $\nu = \sqrt{d^2/4 - m^2/H^2}$.[19] For massive spin-$s$ fields the mass and spin are related to the conformal dimension by $\Delta = d/2+\mu$, $\mu = \sqrt{(d+2s-4)^2/4 - m^2/H^2}$. We define $\overline{\phi}_+$ with $\Delta^+ \equiv \Delta$ and $\overline{\phi}_-$ (with $\Delta^- = d - \Delta$). For heavy scalar fields ($m > dH/2$), Δ is complex and so these boundary operators do not represent a self adjoint basis[20]. We will therefore restrict our discussion to light fields and leave a more comprehensive examination, including heavy fields, to future work. In this case, Δ is real and positive so the $\overline{\phi}_-$ term

[18] Had we used another convention where the tensor structures scaled in non-trivial way with Dilatations **D**, then the scalar component of the field would have to scale in the inverse way, in order to leave the field $\sigma_{i_1 \ldots i_s}$ invariant.

[19] Note for AdS this relation is $m^2\ell^2_{AdS} = \Delta(\Delta - d)$. In AdS $\Delta^\pm \in \mathbb{R}$ for scalar fields of any mass. In dS for scalar fields with mass $m^2/H^2 > d^2/4$, $\Delta^\pm \in \mathbb{C}$ which is known as the principal series; fields with mass $m^2/H^2 \leq d^2/4$, $\Delta^\pm \in \mathbb{R}$ which is known as the complementary series.

[20] Of course, we could have made the choice to expand the fields in a self-adjoint way but this would sacrifice boundary conformal invariance as such an expansion would necessarily mix terms of different weights. See e.g. [120–129] for discussions regarding the principal series.

dominates. Starting from the expansion of the wavefunction in terms of its coefficients, (4.3.24), we can similarly define a "boundary" wavefunction[21] which is a functional of these boundary fields $\overline{\phi}_-$ and has no explicit η-dependence

$$\overline{\Psi}[\overline{\phi}_-(\mathbf{k}_a)] = \exp\left\{-\sum_{n=2}^{\infty}\frac{1}{n!}\int\left[\prod_{a=1}^{n}\frac{\mathrm{d}^d\mathbf{k}_a}{(2\pi)^d}\overline{\phi}_-(\mathbf{k}_a)\right]\overline{\psi}_n(\mathbf{k})\delta^d\left(\sum_{a=1}^{n}\mathbf{k}_a\right)\right\}, \tag{6.4.79}$$

where we have introduced boundary wavefunction coefficients[22] denoted by $\overline{\psi}_n$ to distinguish them from the bulk wavefunction coefficients ψ_n.[23] The choice to Taylor expand the wavefunction in the field operator with dimension $d-\Delta$ rather than using the seemingly more sensible choice of labeling this operator's dimension Δ is purely conventional. It is standard to refer to Δ^+ as *the dimension* of a field in cosmology (for example a massless field has dimension $\Delta = d$) but it turns out that this is not the part of the field that survives on the boundary.

Let us now count the scalings of each term in the exponent one by one to see how $\overline{\psi}_n(\mathbf{k}_a)$ scales under $\mathbf{k}\to\lambda\mathbf{k}$ which in turn will scale the internal and external energies $k\to\lambda k$. The d-dimensional $\mathrm{d}^d k$ measures each scale with volume and thus they will scale with an overall factor of λ^{nd}. This cancels with an equivalent scaling of the Fourier transformed $\overline{\phi}_-(\mathbf{k}_a)$ which give an overall scaling of λ^{-nd}.[24] Each boundary field must contribute an extra scaling like $\eta^{\Delta-d}$ to ensure the Scale Invariance of the bulk field (6.4.78). The result of this is that in addition to the factor coming from the Fourier Transform the fields contribute a further $\lambda^{dn-\sum_\alpha\Delta_\alpha}$. Finally, the $\delta^d(\mathbf{k}_1+\ldots+\mathbf{k}_n)$ scales with inverse volume λ^{-d}.[25] We know that the exponent of (6.4.79) must be dimensionless, thus by dimensional analysis, any IR-finite boundary wavefunction coefficient will scale as

$$\overline{\psi}_n(\lambda\mathbf{k}) = \lambda^d\lambda^{-nd}\lambda^{\sum_\alpha\Delta_\alpha}\overline{\psi}_n(\mathbf{k}) = \lambda^{d(1-n)+\sum_\alpha\Delta_\alpha}\overline{\psi}_n(\mathbf{k})\,. \tag{6.4.81}$$

We will now look at the constraints from **CRT**, **RR** and $\mathbf{D}_{-1}$ on the boundary wavefunction coefficients $\overline{\psi}_n$, where this dimensional analysis exercise will prove to be useful. Let us look at each transformation individually.

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[21]Although this is referred to as a "boundary" wavefunction in the literature, this is not really a wavefunction living at the $\mathcal{I}^+$ where $\eta = 0^-$, since from (6.4.77) it is clear that only massless fields would survive in the limit of $\eta = 0^-$. The more accurate description is that the coefficients of the wavefunction have had their η-dependence stripped off.

[22]These can be interpreted as the correlators of the operators of some conjectured CFT living at the boundary, i.e. dS/CFT.

[23]To obtain the boundary wavefunction coefficient we differentiate the boundary wavefunction $\overline{\Psi}$ with respect to the sources $\overline{\phi}_-$ which have dimension $d-\Delta$

$$\overline{\psi}_n(\mathbf{k}) \equiv \lim_{\overline{\phi}_-\to 0}\frac{\delta^n}{\delta^n\overline{\phi}_-}\overline{\Psi}[\overline{\phi}_-(\mathbf{k}_a)]\,. \tag{6.4.80}$$

[24]This is in the exact same way that in (4.3.24) the original bulk field $\phi_{\mathbf{k}_a}$ scaled as λ^d resulting in the bulk wavefunction coefficient scaling inversely to the $\delta^d\left(\sum\mathbf{k}_a\right)$.

[25]Usually in the context of computations in de Sitter, the $\delta(\mathbf{k}_1+\ldots+\mathbf{k}_n)$-function is omitted but we remind the reader of its importance in terms of arguments involving Scale Invariance, as it scales with inverse volume, as well as its necessity for ensuring momentum conservation which comes from translation invariance in position space.

RR does not care about the time at which the wavefunction is evaluated and so the resulting constraint is unchanged:[26]

$$\mathbf{RR}:\left[\overline{\psi}_n^{(L)}(\mathbf{k})\right]^* = e^{i\pi[(d+1)L-1]}\overline{\psi}_n^{(L)}(e^{-i\pi}\mathbf{k})\,. \tag{6.4.82}$$

Scale Invariance in the bulk requires that

$$\psi_n^{(L)}(\eta/\lambda;\lambda\mathbf{k}) = \lambda^d\psi_n^{(L)}(\eta;\mathbf{k})\,. \tag{6.4.83}$$

As was derived in (6.4.81), Scale Invariance of the boundary theory at $\mathcal{I}^+$ where $\eta = 0^-$ tells us that the wavefunction coefficients must scale as

$$\overline{\psi}_n^{(L)}(\lambda\mathbf{k}) = \lambda^{d(1-n)+\sum_\alpha\Delta_\alpha}\overline{\psi}_n^{(L)}(\mathbf{k})\,, \tag{6.4.84}$$

where Δ_α are the dimensions of the external fields. The Cosmological CPT theorem tells us that the $SO^+(1,1)$ boost which is the important continuous symmetry in the flat space CPT theorem acts in the Poincaré patch as dilatations $\mathbf{D}$, i.e. Scale Invariance $\lambda\in\mathbb{R}^+$, which we can analytically continue to $SO(2)$, provided we have a Hamiltonian bounded from below. Under analytic continuation of $\lambda\in\mathbb{R}^+$ through the complex plane $\lambda\in\mathbb{C}$,[27] we can then access $\lambda = e^{-i\pi}$, to conclude that:

$$\overline{\psi}_n^{(L)}(e^{-i\pi}\mathbf{k}) = (-1)^{d(1-n)+\sum_\alpha\Delta_\alpha}\overline{\psi}_n^{(L)}(\mathbf{k})\,, \tag{6.4.85}$$

where for cases with fractional d and Δ we have

$$\mathbf{D}^\pm_{-1}:\overline{\psi}_n^{(L)}(e^{\mp i\pi}\mathbf{k}) = e^{\mp i\pi[d(1-n)+\sum_\alpha\Delta_\alpha]}\overline{\psi}_n^{(L)}(\mathbf{k})\,. \tag{6.4.86}$$

We need to take into account that $\overline{\psi}_n^{(L)}$ is analytic through the lower half-plane $\mathbb{C}^{-i}$. Given that the **RR** (6.4.82) transformation rotates $\overline{\psi}_n^{(L)}$ counter-clockwise, we must use the $\mathbf{D}^+_{-1}$ in (6.4.86) to land on

$$\mathbf{D}^+_{-1}\bullet\mathbf{RR}:\left[\overline{\psi}_n^{(L)}(\mathbf{k})\right]^* = e^{i\pi[(d+1)L-1-d(1-n)-\sum_\alpha\Delta_\alpha]}\overline{\psi}_n^{(L)}(\mathbf{k})\,. \tag{6.4.87}$$

We can also derive the same expression independently using **CRT**, which tells us directly that

$$\left[\psi_n^{(L)}(\eta;\mathbf{k})\right]^* = e^{i\pi(d+1)(L-1)}\psi_n^{(L)}(e^{-i\pi}\eta;\mathbf{k})\,, \tag{6.4.88}$$

$$\left[\overline{\psi}_n^{(L)}(\mathbf{k})\right]^* = e^{i\pi[(d+1)(L-1)+dn-\sum_\alpha\Delta_\alpha]}\overline{\psi}_n^{(L)}(\mathbf{k})\,, \tag{6.4.89}$$

where for (6.4.88) we have to continue $1/\eta \propto k \equiv |\mathbf{k}|$ in the lower-half plane $\mathbb{C}^{-i}$, and as previously explained the factor of $\eta^{\sum_\alpha d-\Delta_\alpha}$ comes from the fields scaling as $\eta^{\Delta-d}\propto\lambda^{d-\Delta}$ at the boundary. We have now identified the discrete symmetries which the boundary wavefunction

[26] We remind the reader that the conventions we use in cosmology where the conformal time on conformally flat Poincaré slicing $\eta\in[-\infty,0^-]$ is negative in the expanding branch and increases from $-\infty$ in the infinite past to $\mathcal{I}^+ = 0^-$.

[27] Given that $\overline{\psi}_n^{(L)}$ is only analytic in the lower half-place $\overline{\psi}_n^{(L)}\in\mathbb{C}^{-i}$, we can see from the group structure in Table 6.1 we must act with $\mathbf{D}^+_{-1}$ after **RR** in order to obtain **CRT**; or alternatively act with **RR** after $\mathbf{D}^-_{-1}$.

coefficients $\overline{\psi}_n^{(L)}$ satisfy:

$$\begin{aligned}
\mathbf{CRT}: \quad \overline{\psi}_n^{(L)}(\mathbf{k}) &= e^{-i\pi[(d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha]} \left[\overline{\psi}_n^{(L)}(\mathbf{k})\right]^*, && (6.4.90)\\
\mathbf{D}_{-1}^{\pm}: \quad \overline{\psi}_n^{(L)}(\mathbf{k}) &= e^{\pm i\pi[d(1-n)+\sum_\alpha \Delta_\alpha]}\, \overline{\psi}_n^{(L)}(e^{\mp i\pi}\mathbf{k}), && (6.4.91)\\
\mathbf{RR}: \quad \overline{\psi}_n^{(L)}(\mathbf{k}) &= e^{-i\pi[(d+1)L-1]} \left[\overline{\psi}_n^{(L)}(e^{-i\pi}\mathbf{k})\right]^*, && (6.4.92)
\end{aligned}$$

where the last condition from **RR** holds for boundary correlators in any flat FLRW spacetime with a future conformal boundary. We can use (6.4.90) to solve for the phase of $\overline{\psi}_n^{(L)}$ directly, obtaining the following phase formula for the boundary wavefunction coefficients:

$$e^{i\arg(\overline{\psi}_n^{(L)})} \equiv \frac{\overline{\psi}_n^{(L)}(\mathbf{k})}{|\overline{\psi}_n^{(L)}(\mathbf{k})|} = \pm\sqrt{\frac{\overline{\psi}_n^{(L)}(\mathbf{k})}{\overline{\psi}_n^{(L)*}(\mathbf{k})}} = \pm(-i)^{(d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha}, \tag{6.4.93}$$

where there is a $\pm$ out front because **CRT** cannot determine the overall real sign, because obtaining the phase involves taking a square root.[28] Hence, we obtain the result (1.42) quoted in Chapter 1:

$$\arg(\overline{\psi}_n^{(L)}) = -\frac{\pi}{2}\left((d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha\right) + \pi\mathbb{N}. \tag{6.4.94}$$

Remarkably (6.4.93) and (6.4.94) will hold for any $\overline{\psi}_n^{(L)}$ computed in cosmology provided that:

- spacetime is de Sitter (possibly with boost-breaking terms);
- the Lagrangian is locally **CRT**-invariant;
- the amplitude is UV- and IR-finite;
- and involves fields in representations with integer spin and real Δ (no spinors or principal series);
- all fields satisfy the Bunch-Davies vacuum.

It is important to note, however, that the conformal dimensions Δ_α appearing in (6.4.93) and (6.4.94) are not, in general, protected quantities: they can receive radiative corrections and hence run with couplings. The phase relation is thus only exact when evaluated with respect to the effective (loop-corrected) conformal dimensions at a given order in perturbation theory[29].

In particular, if any of the Δ_α acquire non-trivial scale dependence, i.e. anomalous dimensions, this manifests through logarithmic violations of scaling symmetry, introducing terms such as $\log(k)$, $\log(k\eta)$, etc., in the wavefunction coefficients. For example, a de Sitter-invariant 2-point wavefunction coefficient ψ_2, i.e. one invariant under all isometries including special conformal transformations, takes the form (see e.g. [118])

$$\langle \mathcal{O}_1(\mathbf{k})\mathcal{O}_2(-\mathbf{k})\rangle \equiv \psi_2 = k^{2\Delta-d}. \tag{6.4.95}$$

[28] Although, for the 2-point function of a field which is weakly coupled in the bulk, this sign is fixed by normalisability.

[29] The author thanks Tarek Anous and Alejandra Castro for helpful discussions regarding anomalous dimensions and renormalisation in perturbation theory.

However, a 1-loop renormalised ψ_2, depending on the choice of renormalisation scheme (see e.g. [67, 76]), can acquire logarithmic corrections of the form

$$\psi_2 = k^{2\Delta_0 - d}\left[1 + 2g\ln(k) + \mathcal{O}(g^2)\right] = k^{2\Delta_0 - d}e^{2g\ln(k)} = k^{2(\Delta_0+g)-d}\,, \tag{6.4.96}$$

where Δ_0 is the tree-level conformal dimension and g is a small coupling constant. This can be interpreted as a radiative correction to the conformal dimension:

$$\Delta(g) = \Delta_0 + g + \mathcal{O}(g^2)\,, \tag{6.4.97}$$

so that the renormalised 2-point coefficient becomes

$$\psi_2 = k^{2\Delta(g)-d}\,. \tag{6.4.98}$$

Such logs thus give rise to additional imaginary contributions upon analytic continuation, modifying the simple phase rule (6.4.94). They reflect the running of coupling-dependent dimensions and will be discussed further in Chapter 7 and [137].

Nevertheless, for UV- and IR-finite diagrams where such logarithms are absent, and the theory admits approximate scale invariance with fixed Δ_α, the phase formula (6.4.93) and (6.4.94) remains valid and provides a powerful constraint on the imaginary parts of wavefunction coefficients — independent of the details of the bulk interactions. For instance, different choices of derivative couplings or operator structure yield the same phase as long as the external fields and their conformal dimensions are unchanged.

It is instructive to contrast this result with the flat space setting. In flat space, the CPT theorem guarantees that the S-matrix is invariant under a **CRT** transformation, but it does not constrain the individual phases of the S-matrix elements. This is because Lorentz boosts, including those in the $SO^+(1,1)$ subgroup, act non-trivially on the flat spacetime boundary at null infinity — they shift the asymptotic states and mix positive and negative energy modes. Hence, **CRT** in flat space relates entire in- and out-states but does not fix relative phases of amplitudes.

By contrast, in de Sitter space, the analogue of a $SO^+(1,1)$ Lorentz boost becomes a conformal scaling at future infinity, which preserves the future boundary slice. In this cosmological setting, $SO^+(1,1)$ acts as a dilation symmetry on the late-time boundary and thus leaves the support of the wavefunction unchanged. As a result, the Cosmological CPT theorem (6.4.90) can be used to derive precise constraints on the phases of individual wavefunction coefficients, such as in (6.4.93) and (6.4.94).

This difference is subtle but significant: whereas the flat space CPT theorem applies to scattering amplitudes and relates asymptotic states defined at different times (and thus different spatial slices), the Cosmological CPT theorem applies directly to the late-time wavefunction defined on a single boundary slice. Consequently, the Cosmological CPT theorem leads to stronger constraints than its flat space counterpart.

Finally, we emphasise that the phase formula (6.4.93) and (6.4.94) above applies to the IR-finite part of the amplitude. In cases with IR divergences, such as terms proportional to $C\log(-\eta)$, rotating $\eta \to e^{-i\pi}\eta$ yields additional shifts of the form $-i\pi C$, which can modify the finite part of the amplitude. Even so, the leading-order IR-divergent piece still obeys (6.4.94). A concrete example of this behaviour will be discussed in Section 6.4.5.

6.4.5 Explicit Checks of the Phase Formula

We will now provide a list of non-trivial checks of the phase formula (6.4.93) and (6.4.94) for various specific Feynman-Witten diagrams in cosmology.

The first five calculations **i**–**v** below all involve conformally coupled scalars (denoted by φ), while the remaining two **vi** and **vii** involve massless fields (denoted by ϕ). For the calculations in **iii**–**vii**, we work in non-integer spatial dimensions for the purposes of dimensional regularisation (dim-reg) using the prescription described in Appendix C of [202]. In this approach, the mass of the field is also renormalised to keep the order of the Hankel function ν fixed, which ensures that the integrals can be computed analytically in the dim-reg parameter δ. An alternative prescription proposed in [76] does not renormalise the mass of the field, resulting in the order of the Hankel function for the case of massless fields in non-integer $d = 3 + \delta$ spatial dimensions becoming $\nu = d/2 = (3 + \delta)/2$. However, this prevents the integrals from being computed analytically. Consequently, the authors carry out an expansion in δ prior to computing the integrals which introduces non-trivial corrections at various points in the calculation.

It will be shown in [137] that by using the analysis given in (6.4.95)-(6.4.98), we can see that the two choices of regularisation scheme are in fact equivalent!

i. Conformal tree-level IR-finite coefficient $\overline{\psi}^{\text{tree}}_{3,\varphi\varphi\tilde{\sigma}}$

The phase of the contact three-point wavefunction coefficient $\overline{\psi}^{\text{tree}}_{3,\varphi\varphi\tilde{\sigma}}$, generated via a simple cubic interaction $\varphi\varphi\tilde{\sigma}$ in $3 + 1$-dimensions, involving two conformally-coupled scalars (with $\Delta = 2$) and a massive scalar $\tilde{\sigma}$ with mass $m < \sqrt{2}H$ (corresponding to $2 < \Delta \leq 3$), in order for the time integral to converge in the IR (i.e. when $\eta = 0^-$), is found in Appendix B of [50] to be

$$\arg(\overline{\psi}^{\text{tree}}_{3,\varphi\varphi\tilde{\sigma}}) = \frac{\pi}{2}\left(\nu + \frac{1}{2}\right) . \tag{6.4.99}$$

This agrees with (6.4.93) and (6.4.94) as it correctly predicts the generically complex phase of a wavefunction coefficient involving a light field with a mass $m < \sqrt{2}H$.

ii. Conformal tree-level IR-divergent coefficient $\overline{\psi}^{\text{tree}}_{3,\varphi\varphi\varphi}$

The contact three-point wavefunction coefficient $\overline{\psi}^{\text{tree}}_{3,\varphi\varphi\varphi}$ for the interaction involving three conformally-coupled scalars, generated via a simple cubic interaction $\varphi\varphi\varphi$ in $3 + 1$-dimensions, where the time integral diverges in the IR (i.e. when $\eta = 0^-$), is found to be

$$\overline{\psi}^{\text{tree}}_{3,\varphi\varphi\varphi} = \frac{i(\log(-i(k_1 + k_2 + k_3)\eta_0) + \gamma)}{(\eta_0)^3 H^4} + O\left(\frac{1}{\eta_0^2}\right) , \tag{6.4.100}$$

i.e. it has a $\log(-\eta)$ IR-divergence. However, as we previously explained (6.4.93) and (6.4.94) continue to hold for the phase of the coefficient of the leading $\log(-\eta)$ IR-divergence and thus we find correctly that there is an i in front of the $\log(-\eta)$ term. This is another non-trivial check of the phase formula as it correctly predicts the imaginary phase of the $\log(-\eta)$ in the IR-divergent wavefunction coefficient.

iii. Conformal tree-level IR-finite coefficient in non-integer dimension $d = 3+\delta$, $\overline{\psi}^{\text{tree}}_{6,\varphi^6}$

The phase of the contact six-point wavefunction coefficient $\overline{\psi}^{\text{tree}}_{6,\varphi^6}$ for the interaction involving six conformally-coupled scalars, generated via a simple 6th-order interaction φ^6 in non-integer $d = 3+\delta$ spatial dimensions (which rescales the conformal dimension of the conformally coupled scalar $\Delta = d/2 + 1/2 = (4+\delta)/2$), was computed in [75]. However, an incorrect phase was found, which will be corrected in [137]. For this particular Feynman-Witten diagram one finds

$$\overline{\psi}^{\text{tree}}_{6,\varphi^6} = -\frac{e^{-i\pi\delta}\Gamma(3+2\delta)}{H^{4+\delta}\left(k_T^{(6)}\right)^{3+2\delta}} \,. \tag{6.4.101}$$

Once again (6.4.93) and (6.4.94) correctly predict the generically complex phase of the wavefunction coefficient in non-integer spatial dimensions $d = 3+\delta$.

iv. Conformal 1-loop UV-finite coefficient in non-integer dimension $d = 3+\delta$, $\overline{\psi}^{(L=1)}_{4,\varphi^6}$

The phase of the 1-site 1-loop four-point wavefunction coefficient $\overline{\psi}^{(L=1)}_{4,\varphi^6}$ for the interaction involving four conformally-coupled scalars, generated via a 6th-order interaction φ^6 in non-integer $d = 3+\delta$ spatial dimensions (which, as previously explained, rescales the conformal dimension of the conformally coupled scalar $\Delta = d/2 + 1/2 = (4+\delta)/2$), was computed in [75]. However, an incorrect phase was found, which will be corrected in [137]. For this particular Feynman-Witten diagram one finds

$$\overline{\psi}^{(L=1)}_{4,\varphi^6} = \frac{\lambda e^{-i\pi\delta}}{2^{5+\delta}\pi^{\frac{4+\delta}{2}}H^2\left(k_T^{(4)}\right)^{1+\delta}}\Gamma(1+\delta)\Gamma\left(\frac{2+\delta}{2}\right) \,. \tag{6.4.102}$$

Since $\overline{\psi}^{(L=1)}_{4,\varphi^6}$ comes from a UV-finite diagram we see that is is analytic in the k_T-energy pole as $\delta \to 0$. As predicted we find $\lim_{\delta\to 0}\overline{\psi}^{(L=1)}_{4,\varphi^6} \in \mathbb{R}$. Additionally (6.4.93) and (6.4.94) correctly predicts the generically complex phase of $\overline{\psi}^{(L=1)}_{4,\varphi^6}$ when δ is kept finite. This will be important when considering the next examples, which are UV-divergent.

v. Conformal 1-loop UV-divergent coefficient in non-integer dimension $d = 3+\delta$, $\overline{\psi}^{(L=1)}_{n,\varphi^{(n+4)/2}}$

The 2-site, 1-loop de Sitter wavefunction coefficient for arbitrary number n of conformally coupled fields in external legs in non-integer $d = 3+\delta$ spatial dimensions will be computed in upcoming work [203, 204][30] where the author generalises the recently developed differential equation approach [207, 208] to computing flat FLRW cosmological wavefunction coefficients to loop diagrams. The result for the phase of the UV-divergent term is

$$\arg\left(\overline{\psi}^{(L=1)}_{n,\varphi^{(n+4)/2}}\right) = -\frac{n\pi(2+\delta)}{4} + \pi\mathbb{N} \,, \tag{6.4.103}$$

which is again compatible with our phase formula. Note that the phase for $n = 4$ matches the phase of (6.4.102) because the phase formula depends only on the external legs and not the internal structure of the Feynman-Witten diagram.

[30] The author thanks Tom Westerdijk for kindly sharing these results and related with us [203–205]. See also similar results in [206].

vi. Massless 1-loop UV-divergent coefficient in non-integer dimension $d = 3 + \delta$, $\overline{\psi}^{(L=1)}_{2,\phi'^3}$

As previously discussed, in cases with a logarithmic UV-divergence, one can renormalise these using dim-reg. In dim-reg, the logarithmic UV-divergence turns into a $1/\delta$ divergence (for IR-finite 1-loop diagrams), which can be canceled by expanding the phase using the simple fact that any generic complex number A can be expressed as $A = |A|e^{i\arg(A)}$. For IR-finite 1-loop diagrams we thus find

$$\lim_{\delta\to 0}\left[|\overline{\psi}^{(L=1)}|e^{i\arg(\overline{\psi}^{(L=1)})}\right] \sim \frac{1}{\delta}(1 + i\pi\delta + O(\delta^2)) = \frac{1}{\delta} + i\pi + O(\delta)\,. \tag{6.4.104}$$

This phenomenon was found in [75], where the authors computed a parity-odd contribution to the scalar trispectrum using the in-in formalism. We have demonstrated that this is in fact a generic feature of UV-divergent loop diagrams and will be explored further in [3].

In [67], the 1-loop four-point wavefunction coefficient $\overline{\psi}^{(L=1)}_{2,\phi'^3}$ for the interaction involving three massless scalars, generated via a boost-breaking derivative interaction ϕ'^3 in non-integer $d = 3 + \delta$ spatial dimensions (which rescales the conformal dimension of the massless scalar $\Delta = d/2 + 3/2 = (6+\delta)/2$), was computed with an incorrect phase that will be corrected in [137]. For this particular Feynman-Witten diagram one finds

$$\overline{\psi}^{(L=1)}_{2,\phi'^3} = H^2\frac{S_{1+\delta}}{(2\pi)^{3+\delta}}k^3(-iH)^{\delta}I(\delta)\,, \tag{6.4.105}$$

where S_{d-2} is the surface area of the $(d-2)$-dimensional unit sphere (i.e. $S_1 = 2\pi$) and $I(\delta)$ is a dimensionless quantity with no contribution to the phase defined in [67]. As we stated in Sections 6.4.2 and 6.4.4 the phase formula still holds for the case of derivative interactions; the phase is also important for the consistency of the cutting rules derived in [67].

vii. Graviton 2-pt function in integer dimension $\psi^{(L)}_2 = \langle T_{ij}(\mathbf{k})T_{lm}(-\mathbf{k})\rangle$

This graviton 2-point function in cosmology is equivalent to calculating the 2-point function of the stress tensor in dS/CFT. Our phase rule also gives the correct prediction in this case. The answer depends on the dimension d and loop order L as follows:

- In even $d+1$-spacetime dimensions, $\langle T_{ij}(\mathbf{k})T_{lm}(-\mathbf{k})\rangle \in \mathbb{R}$ to ALL loop order;
- In odd $d+1$-spacetime dimensions, $\langle T_{ij}(\mathbf{k})T_{lm}(-\mathbf{k})\rangle \in i\mathbb{R}$ at tree level and even loop order (i.e. when $L \in 2\mathbb{Z}$);
- In odd $d+1$-spacetime dimensions, $\langle T_{ij}(\mathbf{k})T_{lm}(-\mathbf{k})\rangle \in \mathbb{R}$ at odd loop order (i.e. when $L \in 2\mathbb{Z}+1$).

Hence for the case of $d = 2$ we expect the central charge, which is equivalent to the coefficient of the 2-point stress tensor correlator, to be imaginary at tree level and real at 1-loop order and so on. This matches what has been found in the literature — refer to Chapter 1 for a list of previous results regarding the central charge in dS/CFT.

In fact, the phase formula offers a natural explanation for how a complex de Sitter wavefunction Ψ — with Lorentzian bulk evolution — can be reconstructed from its modulus-squared

$|\Psi|^2 \equiv \Psi\Psi^*$, which is typically associated with a doubled Euclidean CFT lacking phase information. For example, in $d = 2$ pure gravity, this leads to a theory with $c = 26$. Supposing one is given only the probability distribution $|\Psi|^2$, the phase formula provides precisely the information needed to define a consistent "square root" $\Psi \equiv Z_{\text{CFT}}$, restoring the correct complex structure of the wavefunction. This suggests that, in principle, nearly any $c = 26$ two-dimensional CFT can be used to reconstruct a candidate dS/CFT wavefunction. This perspective also lends support to the recently proposed duality between dS_3 and the complex Liouville string theory with $c = i\frac{3\ell}{2G_N} + 13 + \mathcal{O}(G_N)$ [209–214].

6.5 Summary and Outlook

In this chapter, we formulated and proved the Cosmological CPT Theorem: a non-perturbative statement that establishes **CRT** invariance of the Wavefunction of the Universe as a consequence of Unitarity and Discrete Scale Invariance. Working within a single Poincaré patch, we demonstrated that three fundamental symmetries — **CRT** invariance, Reflection Reality (**RR**), and Discrete Scale Invariance — are intimately related, with any two implying the third. This triangular structure captures the essential symmetry content of any unitary inflationary model with a Bunch-Davies vacuum.

Another central insight was that Reflection Reality, though rarely highlighted in standard treatments, plays a distinguished role in cosmology: it is automatically satisfied by all unitary QFTs on flat FLRW spacetimes, and in many cases, **RR** alone is sufficient to imply full unitarity over a codimension-zero domain of generic couplings. This explains why, in practice, requiring covariance and discrete symmetry in Euclidean theories is often enough to ensure Lorentzian unitarity upon continuation (see e.g. [172]). It also clarifies why the detailed imposition of unitarity in Lorentzian path integrals — which can be technically delicate — is often unnecessary when **CRT** and covariance are preserved. Our analysis further clarified the limitations of correlator-based arguments and the necessity of working directly with local symmetries of the Lagrangian to derive robust constraints on the wavefunction.

We also presented a perturbative derivation of the theorem, showing how **CRT** symmetry is preserved order-by-order in the coupling expansion of ψ_n in inflationary theories, including a general formula for their phase valid to all loop orders. Our non-perturbative **RR** result requires analytically continuing both momenta $\mathbf{k}$ and the bulk Weyl factor Ω, while our perturbative derivation (Section 6.4.2) requires only $\mathbf{k}$. This opens the door to deriving cosmological cutting rules analogous to those in flat space [50–52, 67]. In odd spacetime dimensions, we found that the phase of ψ_n depends on the loop order, and in general spacetime dimensions, logarithmic UV or IR divergences shift this phase by a universal $i\pi$, matching expectations from dimensional regularisation [75]. This provides a powerful consistency condition for constructing viable models of the early universe and ensures that seemingly exotic features of perturbation theory, such as the running of dimensions (often called anomalous dimensions), are appropriately constrained by fundamental principles.

These results also have direct implications for dS/CFT. Since boundary CFTs dual to de Sitter are typically non-unitary, they do not satisfy Reflection Positivity. Our phase rule provides a new tool to determine which boundary theories are compatible with bulk unitarity and suggests the existence of a yet-to-be-formulated positivity condition analogous to Reflection Positivity. Clarifying this condition remains an important open direction.

Several avenues for future work emerge. First, in Chapter 7, we apply the Cosmological CPT Theorem to investigate the viability of parity violation in inflationary correlators. We will prove a new no-go theorem demonstrating that, under broad and physically well-motivated assumptions, parity-odd signatures cannot arise unless additional ingredients beyond the standard inflationary paradigm are introduced. Similar methods could be used to constrain fermionic correlators via projective representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$, requiring the double cover D_4. It would also be valuable to extend our analysis to principal series fields [120–129], whose oscillatory late-time behaviour raises subtle issues for boundary conditions.

Moreover, our phase formula could constrain wavefunctions computed using differential equation methods [203, 204, 207, 208, 215, 216], and its extension to curved FLRW spacetimes (see e.g. [217–229]) is an important direction. One may also explore whether a version of our result applies to the de Sitter S-matrix [230, 231] or in the in-out formalism [195].

Lastly, our derivation assumes vacuum stability. In the string theory landscape, de Sitter vacua are typically metastable, and vacuum decay could invalidate our phase rule. Although these effects may be absorbed into a renormalised $\log Z_{\rm CFT}$ via a real counterterm [13, 14, 19, 87, 136], this would require non-unitary postselection, and whether the phase formula still holds remains unclear. Understanding the interplay between vacuum decay and our results is a promising direction for further investigation.

"It is perhaps difficult for a modern student of Physics to realize the basic taboo of the past period (before 1956) . . . it was unthinkable that anyone would question the validity of symmetries under "space inversion," "charge conjugation" and "time reversal." It would have been almost sacrilegious to do experiments to test such unholy thoughts."
— Chien-Shiung Wu

7

No-go Theorem for Cosmological Parity Violation

The structure of primordial correlation functions carries valuable imprints of the universe's earliest moments. Among these, parity violation — asymmetry between left- and right-handed configurations — offers a unique window into fundamental physics beyond the standard model of cosmology. Since general relativity (GR) and late-time cosmological evolution are parity-conserving, any detection of parity-odd signals would point to new physics in the early universe. Recent observational hints from cosmic microwave background (CMB) polarisation [70, 232, 233] and galaxy clustering [71, 72, 234, 235] further motivate a theoretical investigation into the viability and consistency of such signals. Though these findings are tentative [236], they highlight the importance of understanding how parity violation could manifest in the scalar and tensor sectors of cosmology [237–247].

In this chapter, we examine the conditions under which parity-odd cosmological correlators can arise from the Wavefunction of the Universe (WFU), building on the Cosmological CPT Theorem established in Chapter 6. Focusing on scalar and integer-spin fields, we prove a no-go theorem showing that, in even spacetime dimensions, parity-odd correlators vanish at the $\eta = 0^-$ boundary of inflation for any model with:

- only massless scalars and even-spin fields (e.g. gravitons) in the external legs,
- a Bunch-Davies initial state and locally **CRT**-invariant Lagrangian,
- UV- and IR-finite wavefunction coefficients ψ_n.

This result generalises earlier no-go theorems [74, 130–132] and shows that such parity violation cannot arise at either tree- or loop-level unless one introduces additional ingredients, such as internal massive spinning fields, time-dependent couplings, or non-standard vacua. An interesting connection emerges with Furry's theorem in quantum electrodynamics (QED), which forbids correlators involving an odd number of photons due to charge conjugation symmetry [248]. In the cosmological setting, a similar result holds: if the external legs contain an odd number of massless odd-spin fields (e.g. photons) and the theory respects **CRT**, then the wavefunction coefficients become purely imaginary, leading to a vanishing parity-even cosmological correlator. This implies that a cosmological analogue of Furry's theorem holds in de Sitter space, further constraining the structure of possible observables.

At loop level, we identify a subtle exception: UV-divergent diagrams can introduce an imaginary part to ψ_n through scheme-dependent logarithmic divergences. If the wavefunction phase is complex — as occurs when the field mass is renormalised in dimensional regularisation — then parity-odd contributions can persist even after cancellation of divergences. Whether such contributions are physical or merely artefacts of regularisation [67, 75, 76] remains an open question. Nonetheless, these effects do not arise for protected fields such as massless gauge bosons or gravitons, whose conformal dimensions remain fixed under renormalisation due to underlying gauge or diffeomorphism invariance. In such cases, the phase remains stable, and the cosmological analogue of Furry's theorem continues to apply non-perturbatively.

In odd spacetime dimensions, we find an intriguing reversal: tree-level and even-loop contributions to parity-even correlators vanish under the same assumptions, implying that quantum corrections are essential for constructing consistent inflationary models involving massless scalars and gravitons.

This chapter thus provides a general framework for identifying which models of inflation can produce observable parity-odd signatures. It clarifies when such signatures are prohibited by fundamental symmetries and when they remain viable, guiding both theoretical model-building and the interpretation of forthcoming observations.

7.1 The Wavefunction of the Universe

In this section, we will provide a review of how the cosmological wavefunction (also referred to as the Wavefunction of the Universe in this thesis), and its corresponding wavefunction coefficients, are computed for various types of theories. In this chapter, we will be discussing theories with interactions involving scalars and integer-spin fields and will be working with real fields without loss of generality since any complex field can be written in a basis of real fields. As explained in [2] it should be possible to extend the analysis to half-integer-spin fields (see e.g. [132, 177–189] for discussions of spinors in de Sitter space and cosmology) but we leave this as a direction for future work.

7.1.1 Scalars

The action of a massive scalar in an arbitrary $D = d + 1$-dimensional spacetime background is

$$S_\sigma = \int \mathrm{d}^D x\sqrt{-g}\,\mathcal{L} = \int \mathrm{d}^D x\sqrt{-g}\,(\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}) \tag{7.1.1}$$

$$= \int \mathrm{d}^D x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}m^2\sigma^2 + \mathcal{L}_{\text{int}}\right), \tag{7.1.2}$$

where the interaction terms in the Lagrangian density have been grouped into $\mathcal{L}_{\text{int}}$ and separated from the free part $\mathcal{L}_{\text{free}}$. The background metric for a spatially flat FLRW spacetime in the expanding patch is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathbf{x}^2 \,, \tag{7.1.3}$$

A perfect fluid has an equation of state

$$p = w\rho\,, \tag{7.1.4}$$

where p is the pressure, ρ is the mass density of the fluid in the comoving frame, w is the equation of state parameter, with $w = -1$ for a cosmological constant dominated universe, i.e. de Sitter. When we solve the Friedmann equations for the case of a cosmological constant-dominated universe, we find the scale factor to be

$$a(t) \propto e^{Ht}\,, \tag{7.1.5}$$

where H is the Hubble parameter related to the de Sitter length ℓ and the cosmological constant Λ in the usual way

$$H^2 = \frac{1}{\ell^2} = \frac{2\Lambda}{d(d-1)}\,, \tag{7.1.6}$$

and the cosmological time $t \in (-\infty, \infty)$. Let us now take the background metric to be exact de Sitter and work in the Poincaré (inflationary) patch, for which the metric is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathbf{x}^2 \,, \qquad a(t) = e^{Ht} \,, \tag{7.1.7}$$

and this metric describes one half of the global geometry of de Sitter space using planar spacelike slices growing from the infinite past at $t = -\infty$ to the boundary of de Sitter space $\mathcal{I}^+$ (or equivalently the end of inflation) at $t = \infty$. One can redefine these co-ordinates $\mathrm{d}t = a(\eta)\,\mathrm{d}\eta$ to rewrite (7.1.7) as a conformally flat metric,

$$\mathrm{d}s^2 = a^2(\eta)(-\,\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2) \,, \qquad a(\eta) = -\frac{1}{H\eta} \,, \tag{7.1.8}$$

since computing the late-time wavefunction is more convenient to do so in conformal time $\eta = -e^{-Ht} \in (-\infty, 0^-)$, where the perturbations evolve from the far past at $\eta = -\infty$ to $\mathcal{I}^+$ at $\eta = 0$. The free part of the action for a massive scalar for this particular background metric (7.1.8) is thus given by

$$S_{\sigma,\,\mathrm{free}} = \int \mathrm{d}\eta\,\mathrm{d}^d\mathbf{x}\,\left[a^{d-1}(\eta)\right]\left(\frac{1}{2}(\sigma')^2 - \frac{1}{2}c_s^2\partial_i\sigma\partial_i\sigma - \frac{1}{2}m^2\sigma^2\right) \,, \tag{7.1.9}$$

where primes denote derivatives with respect to conformal time η and we have allowed for an arbitrary, constant speed of sound c_s which signals the fact we are allowing for dS boosts to be spontaneously or explicitly broken.[1]

[1]When the speed of sound differs from the speed of light appearing in the metric, $c_s \neq 1$, the sound cone is not invariant under de Sitter boosts, a fact which can be simply seen in the flat-space limit, where de Sitter boosts reduce to Lorentz boosts.

The equation of motion for a free massive scalar field in de Sitter is thus given by

$$\sigma'' - \frac{d-1}{\eta}\sigma' + c_s^2\nabla^2\sigma + \frac{m^2}{H^2\eta^2}\sigma = 0 \tag{7.1.10}$$

The equation of motion in the late-time limit $\eta \to 0^-$ simplifies to

$$\sigma'' - \frac{(d-1)}{\eta}\sigma' \cancelto{0}{+ c_s^2\nabla^2\sigma} + \frac{m^2}{H^2\eta^2}\sigma = 0\,, \tag{7.1.11}$$

and has simple power-law solutions

$$\lim_{\eta\to 0^-} \sigma(\eta, \mathbf{x}) = \overline{\sigma}_+(\mathbf{x})\eta^{\Delta^+} + \overline{\sigma}_-(\mathbf{x})\eta^{\Delta^-}\,, \tag{7.1.12}$$

$$= \overline{\sigma}_+(\mathbf{x})\eta^{\Delta} + \overline{\sigma}_-(\mathbf{x})\eta^{d-\Delta}\,, \tag{7.1.13}$$

where Δ^+ and Δ^- obey the standard quadratic Casimir relation for massive scalar representations of the Euclidean conformal group $SO(d+1,1)$

$$m^2/H^2 = m^2\ell^2 = \Delta(d-\Delta)\,, \tag{7.1.14}$$

with the conformal dimension defined in the usual way $\Delta = d/2 + \nu$, $\nu = \sqrt{d^2/4 - m^2/H^2}$ by defining $\overline{\sigma}_+$ with $\Delta^+ \equiv \Delta$ and $\overline{\sigma}_-$ (with $\Delta^- \equiv d - \Delta$).

Working in momentum space, one can write the quantum free field operator as

$$\hat{\sigma}(\mathbf{k}, \eta) = \sigma^-(k,\eta)a_{\mathbf{k}} + \sigma^+(k,\eta)a^\dagger_{-\mathbf{k}}\,, \tag{7.1.15}$$

where the mode functions $\sigma^\pm(k,\eta)$ correspond to solutions of the free classical equation of motion and are given by

$$\sigma^+(k,\eta) = i\,\frac{\sqrt{\pi}H}{2}e^{-i\frac{\pi}{2}(\nu+\frac{1}{2})}\left(\frac{-\eta}{c_s}\right)^{\frac{d}{2}} H_\nu^{(2)}(-c_s k\eta)\,, \qquad \sigma^-(k,\eta) = (\sigma^+(k,\eta))^*\,, \tag{7.1.16}$$

where $\nu = \sqrt{d^2/4 - m^2/H^2}$ denotes the order of the Hankel function, and by simply setting $m = 0$, one can find the mode functions for massless fields to be

$$\phi^+(k,\eta) = i\,\frac{\sqrt{\pi}H}{2}e^{-i\frac{\pi}{2}(\frac{d+1}{2})}\left(\frac{-\eta}{c_s}\right)^{\frac{d}{2}} H_{d/2}^{(2)}(-c_s k\eta)\,, \qquad \phi^-(k,\eta) = (\phi^+(k,\eta))^*\,. \tag{7.1.17}$$

7.1.2 Spinning Fields

The mode functions for graviton fluctuations take the same form as (7.1.17) (with $c_s = 1$) with the addition of polarisation tensors $e^h_{ij}(\mathbf{k})$, with $h = \pm 2$, as required by little group scaling. This is because, for each polarisation mode, the equation of motion is that of a massless scalar.

The polarisation tensors satisfy the following conditions:

$$e^h_{ii}(\mathbf{k}) - k^i e^h_{ij}(\mathbf{k}) = 0 \qquad \text{(transverse and traceless)}\,, \tag{7.1.18}$$

$$e^h_{ij}(\mathbf{k}) - e^h_{ji}(\mathbf{k}) = 0 \qquad \text{(symmetric)}\,, \tag{7.1.19}$$

$$e^h_{ij}(\mathbf{k}) e^h_{jk}(\mathbf{k}) = 0 \qquad \text{(lightlike)}\,, \tag{7.1.20}$$

$$e^h_{ij}(\mathbf{k}) e^{h'}_{ij}(\mathbf{k})^* - 4\delta_{hh'} = 0 \qquad \text{(normalisation)}\,, \tag{7.1.21}$$

$$e^h_{ij}(\mathbf{k})^* - e^h_{ij}(-\mathbf{k}) = 0 \qquad (\gamma_{ij}(x) \text{ is real})\,. \tag{7.1.22}$$

For generic spin-s fields, it is convenient to use the following notations and conventions. In $D = d + 1$-dimensional spacetime, for traceless[2], integer spin-s fields we use the free action developed in [200] and discussed in the context of the cosmological bootstrap in [2, 52, 201]:

$$S = \int \mathrm{d}^D x \; [a(\eta)]^{d-1} \frac{1}{2s!} \left[\left(\sigma'_{i_1 \dots i_s}\right)^2 - c_s^2 \left(\partial_j \sigma_{i_1 \dots i_s}\right)^2 - \delta c_s^2 \left(\partial^j \sigma_{j i_2 \dots i_s}\right)^2 - m^2 a^2 \left(\sigma_{i_1 \dots i_s}\right)^2 \right] . \tag{7.1.23}$$

The totally-symmetric, traceless tensor $\sigma_{i_1 \dots i_s}$ has spatial indices $i_1 = 1, \dots, d$ which span the d-dimensional spacelike hypersurface orthogonal to the η coordinate[3]. $\sigma_{i_1 \dots i_s}$ has $(2s+1)$ components, which each create states ("particles") with helicities $0, \pm 1, \dots, \pm s$, and we have enforced invariance under dilatations by including inverse factors of the scale factor for each coordinate derivative. Following [52, 201] we Fourier transform and diagonalise this using the helicity modes, σ_h, defined by:

$$\sigma_{i_1 \dots i_s}(\eta; x) = \int_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}} \sum_{h=-S}^{S} \mathbf{e}^h_{i_1 \dots i_s}(\mathbf{k}) \sigma_h(\eta; \mathbf{k})\,, \tag{7.1.24}$$

These helicity tensors are defined as an outer product of helicity vectors,

$$\mathbf{e}^h_{i_1 \dots i_s} = \mathbf{e}^{h_1}_{i_1} \dots \mathbf{e}^{h_s}_{i_s}\,, \tag{7.1.25}$$

which satisfy the following relations:

$$\mathbf{e}^h_i(\mathbf{k}) \left[\mathbf{e}^{h'}_i(\mathbf{k})\right]^* - 4\delta_{hh'} = 0 \qquad \text{(orthogonality and normalisation)}\,, \tag{7.1.26}$$

$$\left[\mathbf{e}^h_i(\mathbf{k})\right]^* - \mathbf{e}^h_i(-\mathbf{k}) = 0 \qquad (\sigma_{i_1 \dots i_s}(x) \text{ is real})\,. \tag{7.1.27}$$

Note that these fields are not assumed to be transverse, h is allowed to take d different values including 0 where $\mathbf{e}^0$ is proportional to the momentum. The contributions from the other helicity modes are therefore transverse by the orthogonality condition.

[2]To consider fields with a non-zero trace, one can subtract the trace and treat it as an additional scalar field.

[3]The Effective Field Theory of Inflation (EFToI) [68] is derived by considering a theory which is only invariant under spatial diffeomorphisms, for which there is a preference for the co-ordinate choice used in unitary gauge, where the time coordinate is chosen to coincide with the surfaces of constant value of the field $\sigma_{i_1 \dots i_s}$. The EFToI thus encapsulates a generic class of models of inflation where spatial diffeomorphisms are preserved. (7.1.23) can be written in a covariant way by using the Goldstone boson π of time translations to upgrade the spatial tensor $\sigma_{i_1 \dots i_s}$ to a covariant spacetime tensor. The coupling of $\sigma_{i_1 \dots i_s}$ to π is also dictated by this constructions but we will not need this here.

The equation of motion for a massive spin-s $\sigma_{i_1,\ldots,i_s}$ field in exact de Sitter is found to be

$$\sigma''_{i_1\ldots i_s} - \frac{d-1}{\eta}\sigma'_{i_1\ldots i_s} + \left(\frac{m^2/H^2-(s-2)(s+d-2)}{\eta^2} + c_s^2\nabla^2\right)\sigma_{i_1\ldots i_s} = 0\,. \tag{7.1.28}$$

The equation of motion for a free massless spin-s in the late-time limit $\eta \to 0$ simplifies to

$$\sigma''_{i_1\ldots i_s} - \frac{d-1}{\eta}\sigma'_{i_1\ldots i_s} + \left(\frac{m^2/H^2-(s-2)(s+d-2)}{\eta^2}\right)\sigma_{i_1\ldots i_s} \cancelto{0}{+c_s^2\nabla^2\sigma_{i_1\ldots i_s}} = 0\,, \tag{7.1.29}$$

and has simple power-law solutions

$$\lim_{\eta\to 0^-}\sigma_{i_1\ldots i_s}(\eta,\mathbf{x}) = \overline{\sigma}_{i_1\ldots i_s,+}(\mathbf{x})\eta^{\Delta^+} + \overline{\sigma}_{i_1\ldots i_s,-}(\mathbf{x})\eta^{\Delta^-}\,, \tag{7.1.30}$$

$$= \overline{\sigma}_{i_1\ldots i_s,+}(\mathbf{x})\eta^{\Delta} + \overline{\sigma}_{i_1\ldots i_s,-}(\mathbf{x})\eta^{d-\Delta}\,, \tag{7.1.31}$$

where Δ^+ and Δ^- obey the standard quadratic Casimir relation for massive integer spin-s representations of the Euclidean conformal group $SO(d+1,1)$

$$m^2/H^2 = m^2\ell^2 = (\Delta+s-2)(d+s-2-\Delta)\,, \tag{7.1.32}$$

with the conformal dimension defined in the usual way $\Delta = d/2+\mu$, $\mu = \sqrt{(d+2s-4)^2/4 - m^2/H^2}$ by defining $\overline{\sigma}_{i_1\ldots i_s,+}$ with $\Delta^+ \equiv \Delta$ and $\overline{\sigma}_{i_1\ldots i_s,-}$ (with $\Delta^- \equiv d-\Delta$).

We parameterise the wavefunction, Ψ, at conformal time η_0 in terms of the helicities of the integer spin field as

$$\Psi[\eta_0;\sigma(\mathbf{k})] = \exp\left[-\sum_{n=2}^{\infty}\frac{1}{n!}\sum_{h_i=\pm}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\psi_n^{h_1\ldots h_n}(\eta_0;\mathbf{k})(2\pi)^d\delta^d\left(\sum\mathbf{k}_a\right)\sigma_{h_1}(\mathbf{k}_1)\ldots\sigma_{h_n}(\mathbf{k}_n)\right]. \tag{7.1.33}$$

Spatial translations and spatial rotations ensure that wavefunction coefficients can be written as a product of a *helicity factor*, which is an $SO(d)$ invariant function of helicity vectors and spatial momenta, multiplied by a *trimmed wavefunction coefficient* [1, 38] (see also [30] where the authors introduced a spinor helicity formalism to describe gravitational waves with circular polarisation in cosmology) which is only a function of the magnitudes of the momenta in the literature:

$$\psi_n^{h_1\ldots h_n} = (\text{tensor structure}) \times (\text{trimmed wavefunction coefficient})\,. \tag{7.1.34}$$

We take all coefficients appearing in the tensor structure to be real and therefore include any factors of i that might appear when converting to momentum space, or simply as part of the Feynman rules, in the trimmed part which we will denote as ψ_n for brevity. For a general $\psi_n^{h_1\ldots h_n}$, we have

$$\psi_n^{h_1\ldots h_n} = \left[\mathbf{e}^{h_1}(\mathbf{k}_1)\ldots\mathbf{e}^{h_n}(\mathbf{k}_n)\,\mathbf{k}_1^{\alpha_1}\ldots\mathbf{k}_n^{\alpha_n}\right]\psi_n, \tag{7.1.35}$$

for some integer α_i. Note that we can choose for *both parity-even and parity-odd* tensor structures to be invariant with respect to the discrete symmetries of **CRT**, **RR** and **D**, which we will be discussing in this chapter.[4] Hence, all our results extend directly from the scalar ψ_n to the tensor case.

7.1.3 The Boundary Wavefunction

We are interested in scenarios where dS boosts are broken since it is known that these symmetries could not have been exact in the early universe, and large non-Gaussianities are associated with a large breaking of boosts [66]. We will take the remaining symmetries of the dS group to be exact: spatial translations, spatial rotations and dilations. A general interaction vertex with n fields, scalars and spinning fields, therefore takes the schematic form

$$S_{\text{int}} = \int \mathrm{d}\eta\, \mathrm{d}^d\mathbf{x}\, a(\eta)^{D-N_{\text{deriv}}} \partial^{N_{\text{deriv}}} \varphi^n \,, \tag{7.1.36}$$

where φ can be any spinning field, ∂ stands for either time derivatives ∂_η or spatial derivatives ∂_i, and N_{deriv} is the total number of derivatives. Spatial derivatives and the spinning fields' indices are contracted with the $SO(d)$ invariant objects δ_{ij} and ϵ_{ijk} and the overall number of scale factors is dictated by scale invariance. Here and throughout this and following sections we use $\varphi(\mathbf{k})$ to schematically denote scalars and integer spin-s fields, with $SO(d)$ indices suppressed, and each of these fields satisfies $\varphi(\mathbf{k}) = \varphi(-\mathbf{k})^*$ which follows directly from (7.1.15), (7.1.16) and (7.1.22).

Let us start with defining the late-time wavefunction which is one of the objects of interest in cosmology, wherein the weak coupling approximation can be parameterised in terms of a series expansion

$$\Psi[\eta_0; \varphi] = \int_{\phi_{\text{BD}}(-\infty)=0}^{\phi(\eta_0)=\varphi} \mathrm{D}\phi\, e^{iS[\phi]} \tag{7.1.37}$$

$$= \exp\left\{ -\sum_{n=2}^{\infty} \left[\prod_{a=1}^{n} \int \frac{\mathrm{d}^d k_a}{(2\pi)^d} \varphi(\mathbf{k}_a) \right] \psi_n(\eta_0; \mathbf{k}) \right\}, \tag{7.1.38}$$

where $\{k\}$ collectively denotes the external energies [5] $k_a = |\mathbf{k}_a|$, $\{\mathbf{k}\}$ collectively denotes their spatial momenta, and $\varphi(\mathbf{k})$ collectively represents all fields in the theory with indices suppressed. The wavefunction coefficients are also dependent on their internal energies, which are a function of the external lines spatial momenta as required by momentum conservation; a wavefunction coefficient ψ_n with n external can have an arbitrary number of $m < n$ internal fields. For example, a tree-level ψ_5 coming from the exchange diagram depicted in Figure 7.1 with 2 internal lines and a cubic $(\varphi')^3$ interaction and coupling g (where φ can be a scalar or spinning field with any mass) in each vertex, has the form,

$$\psi_5 \sim g^3 e^{i \arg(\psi_5)} \frac{\text{Poly}_{d+(a+b+c+d)-\alpha}(k_1, k_2, k_3, k_4, k_5)}{(k_T)^a (k_1 + k_2 + |\mathbf{k}_1 + \mathbf{k}_2|)^b (k_3 + |\mathbf{k}_1 + \mathbf{k}_2| + |\mathbf{k}_4 + \mathbf{k}_5|)^c (k_3 + k_4 + |\mathbf{k}_4 + \mathbf{k}_5|)^d}, \tag{7.1.39}$$

[4] Had we used another convention where the tensor structures scaled in a non-trivial way with Dilatations $\mathbf{D}$, then the scalar component of the field would have to scale inversely, in order to leave the field $\sigma_{i_1 \ldots i_s}$ invariant.

[5] The literature refers to the magnitude of a spatial momentum vector as "energy" despite the absence of time translation symmetry in cosmology, since in cosmological amplitudes/observables they play the analogous role to energy in flat-space amplitudes.

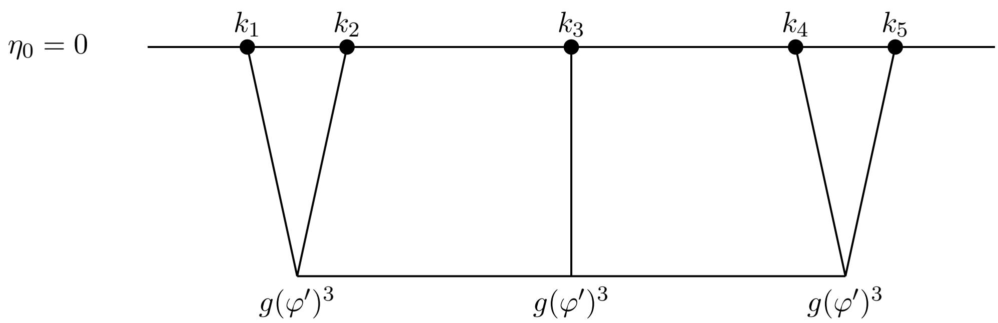

Fig. 7.1 Tree-level exchange Feynman diagram for ψ_5 generated by a $(\varphi')^3$ interaction.

with a being the order of the leading total-energy k_T-pole[6], b, c, d being the order of the partial-energy poles, and $d + (a + b + c + d) - \alpha$ the degree of the polynomial which is fixed by scale invariance. We emphasise this point as we will be utilising the analyticity of both the external $k_a = |\mathbf{k}_a|$ and internal energies $|\sum \mathbf{k}_i| \equiv \sqrt{\sum \mathbf{k}_i}$ [7], where the internal line connects two vertices and $\mathbf{k}_i$ denotes all the external momenta entering one vertex.

Notice that this parameterisation does not require any saddle-point approximation of the bulk path integral that defines Ψ. In fact, the wavefunction coefficient can be found non-perturbatively from

$$\psi_n(\mathbf{k};\eta_0)(2\pi)^d \delta^d\left(\sum_{a=1}^{n} \mathbf{k}_a\right) = -\frac{\delta^n \log \Psi[\varphi, \eta_0]}{\delta\varphi_{\mathbf{k}_1}\cdots\delta\varphi_{\mathbf{k}_n}}\bigg|_{\varphi=0} . \tag{7.1.40}$$

Upon renormalisation, ψ_n can be computed to any desired order in perturbation theory including any number of loops. In this work, we focus on the natural observables of the Poincaré patch of de Sitter and of inflationary cosmology, namely correlation functions of the equal-time product of fields at the future conformal boundary $\eta_0 \to 0$.

In the context of dS/CFT, the wavefunction is treated as a partition function $\Psi[g_{ij}, \varphi, \ldots] = Z[g_{ij}, \varphi, \ldots]$ and satisfies the standard rules of a generating functional, where the wavefunction coefficients are the dS/CFT correlators, e.g. for the 2-point stress tensor correlator we have

$$\langle T_{ij}(\mathbf{k}_1) T_{lm}(\mathbf{k}_2)\rangle_{\eta_0} = -\frac{\delta^2 \log \Psi[g_{ij}, \eta_0]}{\delta g_{ij}(\mathbf{k}_1)\delta g_{lm}(\mathbf{k}_2)}\bigg|_{g_{ij}=0} = \psi_2(\mathbf{k};\eta_0)(2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2) . \tag{7.1.41}$$

We insert here a brief aside about scaling dimensions as it is important for the arguments which follow. By considering the Wavefunction of the Universe at the future boundary of dS, where the time dependence often trivialises, we can derive direct constraints on the wavefunction coefficients. This future boundary is where the reheating surface is understood to live in the inflationary paradigm. Therefore, such constraints are of particular cosmological relevance.

[6]The total-energy is defined as the sum of all the external energies in any given Feynman-Witten diagram. For our example in (7.1.39) the total energy is given by $k_T = k_1 + k_2 + k_3 + k_4 + k_5$.
[7]These are sometimes denoted as $s \equiv |\sum \mathbf{k}_i| \equiv \sqrt{\sum \mathbf{k}_i}$ in the literature.

In the late-time limit, the bulk fields take the form

$$\lim_{\eta\to 0^-} \varphi(\eta,\mathbf{x}) = \overline{\varphi}_+(\mathbf{x})\eta^{\Delta^+} + \overline{\varphi}_-(\mathbf{x})\eta^{\Delta^-}\,, \tag{7.1.42}$$

$$= \overline{\varphi}_+(\mathbf{x})\eta^{\Delta} + \overline{\varphi}_-(\mathbf{x})\eta^{d-\Delta}\,, \tag{7.1.43}$$

where for the case of scalars, the mass $m^2/H^2 = m^2\ell^2 = \Delta(d-\Delta)$ is related to the conformal dimension in the usual way $\Delta = d/2+\nu$, $\nu = \sqrt{d^2/4 - m^2/H^2}$ [8]. For massive spin-s fields, the mass and spin are related to the conformal dimension by $\Delta = d/2+\mu$, $\mu = \sqrt{(d+2s-4)^2/4 - m^2/H^2}$. We define $\overline{\varphi}_+$ with $\Delta^+ \equiv \Delta$ and $\overline{\varphi}_-$ (with $\Delta^- = d-\Delta$). For heavy scalar fields ($m > dH/2$) and heavy spin-s fields ($m > (d+2s-4)H/2$), Δ is complex and so these boundary operators do not represent a self-adjoint basis[9]. We will therefore restrict our discussion to light fields and leave a more comprehensive examination, including heavy fields, to future work. In this case, Δ is real and positive so the $\overline{\varphi}_-$ term dominates. Starting from the expansion of the wavefunction in terms of its coefficients, (7.1.38), we can similarly define a "boundary" wavefunction[10] which is a functional of these boundary fields $\overline{\varphi}_-$ and has no explicit η-dependence

$$\overline{\Psi}[\overline{\varphi}_-(\mathbf{k}_a)] = \exp\left\{-\sum_{n=2}^{\infty}\int\left[\prod_{a=1}^{n}\frac{\mathrm{d}^d k_a}{(2\pi)^d}\overline{\varphi}_-(\mathbf{k}_a)\right]\overline{\psi}_n(\mathbf{k})\delta^d\left(\sum \mathbf{k}_a\right)\right\}\,, \tag{7.1.44}$$

where we have introduced boundary wavefunction coefficients[11] denoted by $\overline{\psi}_n$ to distinguish them from the bulk wavefunction coefficients ψ_n.[12] The choice to Taylor expand the wavefunction in the field operator with dimension $d-\Delta$ rather than using the seemingly more sensible choice of labelling this operator's dimension Δ is purely conventional. It is standard to refer to Δ^+ as *the dimension* of a field in cosmology (for example a massless field has dimension $\Delta = d$) but it turns out that this is not the part of the field that survives on the boundary.

Let us now count the scalings of each term in the exponent one by one to see how $\overline{\psi}_n(\mathbf{k}_a)$ scales under $\mathbf{k} \to \lambda\mathbf{k}$ which in turn will scale the internal and external energies $k \to \lambda k$. The d-dimensional $\mathrm{d}^d k$ measures each scale with volume and thus they will scale with an overall factor of λ^{nd}. This cancels with an equivalent scaling of the Fourier transformed $\overline{\varphi}_-(\mathbf{k}_a)$ which gives

[8] Note for AdS this relation is $m^2\ell_{AdS}^2 = \Delta(\Delta-d)$. In AdS $\Delta^\pm \in \mathbb{R}$ for scalar fields of any mass. In dS for scalar fields with mass $m^2/H^2 > d^2/4$, $\Delta^\pm \in \mathbb{C}$ which is known as the principal series; fields with mass $m^2/H^2 \leq d^2/4$, $\Delta^\pm \in \mathbb{R}$ which is known as the complementary series.

[9] Of course, we could have made the choice to expand the fields in a self-adjoint way but this would sacrifice boundary conformal invariance as such an expansion would necessarily mix terms of different weights. See e.g. [120–129] for discussions regarding the principal series.

[10] Although this is referred to as a "boundary" wavefunction in the literature, this is not really a wavefunction living at the $\mathcal{I}^+$ where $\eta = 0^-$, since from (7.1.42) it is clear that only massless fields would survive in the limit of $\eta = 0^-$. The more accurate description is that the coefficients of the wavefunction have had their η-dependence stripped off.

[11] These can be interpreted as the correlators of the operators of some conjectured CFT living at the boundary, i.e. dS/CFT.

[12] To obtain the boundary wavefunction coefficient we differentiate the boundary wavefunction $\overline{\Psi}$ with respect to the sources $\overline{\varphi}_-$ which have dimension $d-\Delta$

$$\overline{\psi}_n(\mathbf{k}) \equiv \lim_{\overline{\varphi}_-\to 0}\frac{\delta^n}{\delta^n\overline{\varphi}_-}\overline{\Psi}[\overline{\varphi}_-(\mathbf{k}_a)]\,. \tag{7.1.45}$$

an overall scaling of λ^{-nd}.[13] Each boundary field must contribute an extra scaling like $\eta^{\Delta-d}$ to ensure the Scale Invariance of the bulk field (7.1.43). The result of this is that in addition to the factor coming from the Fourier Transform the fields contribute a further $\lambda^{dn-\sum_\alpha \Delta_\alpha}$. Finally, the $\delta^d(\mathbf{k}_1+\ldots+\mathbf{k}_n)$ scales with inverse volume λ^{-d}.[14] We know that the exponent of (7.1.44) must be dimensionless, thus by dimensional analysis, any IR-finite boundary wavefunction coefficient will scale as

$$\overline{\psi}_n(\lambda\mathbf{k}) = \lambda^d\lambda^{-nd}\lambda^{\sum_\alpha \Delta_\alpha}\overline{\psi}_n(\mathbf{k}) = \lambda^{d(1-n)+\sum_\alpha \Delta_\alpha}\overline{\psi}_n(\mathbf{k})\,. \tag{7.1.46}$$

To simplify notation we will drop the bar/overline notation and write $\overline{\psi}$ and $\overline{\phi}$ as ψ and ϕ from this point onwards.

7.2 Cosmological Correlators

A powerful property of the wavefunction is that it contains all the information regarding the quantum processes that occur during inflation and thus fundamentally related to the observations we make on the night sky in cosmology with a specific dictionary translating information from the wavefunction to observations and vice versa. In particular, one can use the wavefunction to compute equal-time expectation values, i.e. cosmological correlators (also known as in-in correlators) using the usual quantum mechanics formula, i.e.

$$B_n(\mathbf{k}) \equiv \langle\varphi(\mathbf{k}_1)\ldots\varphi(\mathbf{k}_n)\rangle = \frac{\int \mathrm{D}\varphi\ \Psi\Psi^*\ \varphi(\mathbf{k}_1)\ldots\varphi(\mathbf{k}_n)}{\int \mathrm{D}\varphi\ \Psi\Psi^*}\,, \tag{7.2.1}$$

Here we have not made a distinction between the dependence of the wavefunction coefficients on the set of spatial momenta $\{\mathbf{k}\}$ and their norms $\{k\}$, but in general we will work away from the physical configuration and treat $\{\mathbf{k}\}$ and $\{k\}$ as independent objects, for reasons that will become clear. Since we will be making a purely non-perturbative statement which is true for all exchange and loop diagrams, it is also important to note that there is a possible dependence on internal energies (sometimes denoted as $\{s\}$ in the literature) which are related to external energies by momentum conservation, as demonstrated in (7.1.39). We are going to use **CRT** to constrain the form of the probability distribution $\Psi\Psi^*$. Now from this perturbative expression

[13]This is in the same way that in (7.1.38) the original bulk field $\varphi_{\mathbf{k}_a}$ scaled as λ^d resulting in the bulk wavefunction coefficient scaling inversely to the $\delta^d\left(\sum \mathbf{k}_a\right)$.

[14]Usually in the context of computations in de Sitter, the $\delta(\mathbf{k}_1+\ldots+\mathbf{k}_n)$-function is omitted but we remind the reader of its importance in terms of arguments involving Scale Invariance, as it scales with inverse volume, as well as its necessity for ensuring momentum conservation which comes from translation invariance in position space.

for the wavefunction, we have

$$
\begin{aligned}
-\log(\Psi\Psi^*) &= \left(\sum_{n=2}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\psi_n(\mathbf{k})\varphi(\mathbf{k}_1)\ldots\varphi(\mathbf{k}_n)\right) \\
&\quad + \left(\sum_{n=2}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\psi_n(\mathbf{k})\varphi(\mathbf{k}_1)\ldots\varphi(\mathbf{k}_n)\right)^* \qquad (7.2.2)\\
&= \left(\sum_{n=2}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\psi_n(\mathbf{k})\varphi(\mathbf{k}_1)\ldots\varphi(\mathbf{k}_n)\right) \\
&\quad + \left(\sum_{n=2}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\psi_n^*(\mathbf{k})\varphi(-\mathbf{k}_1)\ldots\varphi(-\mathbf{k}_n)\right). \qquad (7.2.3)
\end{aligned}
$$

If we change the integration variables on the final line by sending $\{\mathbf{k}\} \to \{-\mathbf{k}\}$ we have

$$
-\log(\Psi\Psi^*) = \sum_{n=2}^{\infty}\frac{1}{n!}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_n}[\psi_n(\mathbf{k}) + \psi_n^*(-\mathbf{k})]\varphi(\mathbf{k}_1)\ldots\varphi(\mathbf{k}_n)\,. \qquad (7.2.4)
$$

It follows from Gaussian integral formulae that the resulting correlators arising from Feynman diagrams in perturbation theory, are given by

$$
B_n(\mathbf{k}) = -\frac{\psi_n(\mathbf{k}) + \psi_n^*(-\mathbf{k})}{\prod_{a=1}^n \psi_2(\mathbf{k}) + \psi_2^*(-\mathbf{k})} + \textit{factorised}\,, \qquad (7.2.5)
$$

where the lower order factorised terms are 0 for the contact diagram contribution to B_n and come in the same type of linear combination of ψ_n's, e.g. for B_4 at tree-level we find

$$
B_4 = -\frac{1}{\prod_{a=1}^4 \psi_2(\mathbf{k}) + \psi_2^*(-\mathbf{k})}\left[\psi_4(\mathbf{k}) + \psi_4^*(-\mathbf{k}) - \frac{(\psi_3(\mathbf{k}) + \psi_3^*(-\mathbf{k}))\,(\psi_3(\mathbf{k}) + \psi_3^*(-\mathbf{k}))}{\psi_2(\mathbf{k}) + \psi_2^*(-\mathbf{k})}\right]. \qquad (7.2.6)
$$

For parity-even interactions, we thus have

$$
B_n(\mathbf{k}) = -\frac{\psi_n(\mathbf{k}) + \psi_n^*(\mathbf{k})}{\prod_{a=1}^n \psi_2(\mathbf{k}) + \psi_2^*(\mathbf{k})} + \textit{factorised}\,, \qquad (7.2.7)
$$

the numerator and denominator are simply 2Re ψ_n' and 2Re ψ_2' respectively in which case our expression matches the one that usually appears in the literature. Let us now define a new variable[15] related to the wavefunction coefficients

$$\rho_n(\mathbf{k}) = \psi_n(\mathbf{k}) + \psi_n^*(-\mathbf{k}) \ . \tag{7.2.10}$$

In perturbation theory, correlators can be computed in terms of the ρ_n's. For example, at tree level we have

$$B_2 \equiv P = \frac{1}{\rho_2} \ , \quad B_4 = -\frac{1}{\prod_{a=1}^4 P(\mathbf{k}_a)}\left[\rho_4 - \frac{\rho_3\rho_3}{P}\right] \ , \quad B_n^{\text{contact}} = -\frac{\rho_n}{\prod_{a=1}^n P(\mathbf{k}_a)} \ , \tag{7.2.11}$$

where P is the power spectrum. However, we are interested in parity-violating correlators, i.e. the parity-odd contribution to the correlator, which is related to wavefunction coefficients via the density matrix coefficients ρ_n. For the parity-even contribution to the correlator we have

$$\rho_n^{\text{PE}}(\mathbf{k}) = \frac{1}{2}\left[\rho_n(\mathbf{k}) + \rho_n(-\mathbf{k})\right] \tag{7.2.12}$$

$$= \frac{1}{2}\left[\psi_n(\mathbf{k}) + \psi_n^*(\mathbf{k}) + \psi_n(-\mathbf{k}) + \psi_n^*(-\mathbf{k})\right] \ , \tag{7.2.13}$$

where $\rho_n^{\text{PE}}(-\mathbf{k}) = \rho_n^{\text{PE}}(\mathbf{k})$. Thus we see that ρ_n^{PE} must be purely real to find a non-vanishing parity-even correlator. For the parity-odd contribution to the correlator we have

$$\rho_n^{\text{PO}}(\mathbf{k}) = \frac{1}{2}\left[\rho_n(\mathbf{k}) - \rho_n(-\mathbf{k})\right] \tag{7.2.14}$$

$$= \frac{1}{2}\left[\psi_n(\mathbf{k}) - \psi_n^*(\mathbf{k}) + \psi_n^*(-\mathbf{k}) - \psi_n(-\mathbf{k})\right] \ , \tag{7.2.15}$$

where $\rho_n^{\text{PO}}(-\mathbf{k}) = -\rho_n^{\text{PO}}(\mathbf{k})$. Thus we see that ρ_n^{PO} must be purely imaginary if we are to find a non-vanishing parity-odd correlator:

$$\rho_n^{\text{PO}} \in i\mathbb{R} \qquad\qquad \rho_n^{\text{PE}} \in \mathbb{R} \ . \tag{7.2.16}$$

From this we can infer that $\psi_n \in \mathbb{R} \implies \rho_n^{\text{PO}} = 0$. Our no-go theorems will thus be based on asking when ψ_n, can be imaginary, specifically that if we find $\psi_n \in \mathbb{R}$ we can infer that the parity-odd cosmological correlator vanishes. We can see from (7.2.15) that although an imaginary part to ψ_n is necessary in order to have a non-zero ρ_n^{PO}, if ψ_n is purely parity-even,

[15] In recent literature [74], the connection between the wavefunction and the density matrix has been explored in terms of their coefficients, as they essentially correspond to the components of a density matrix. This connection becomes explicit in the basis of field eigenstates, which satisfy $\hat{\varphi}(\mathbf{x})\,|\varphi\rangle = \varphi(\mathbf{x})\,|\varphi\rangle$. Here, we explicitly indicate the operator with a hat for this equation, although we will omit it in subsequent expressions for simplicity. Using such eigenstates, we can insert two resolutions of the identity to express the density matrix:

$$\rho = \int \mathrm{D}\varphi\mathrm{D}\bar{\varphi}, |\varphi\rangle\langle\varphi|\rho|\bar{\varphi}\rangle\langle\bar{\varphi}| = \int \mathrm{D}\varphi\mathrm{D}\bar{\varphi}, \rho_{\varphi\bar{\varphi}}, |\varphi\rangle\langle\bar{\varphi}|, . \tag{7.2.8}$$

From this, it follows that the components of the operator in the field basis are given by:

$$\rho_{\varphi\bar{\varphi}} = \langle\varphi|\rho|\bar{\varphi}\rangle = \langle\varphi|\Omega\rangle\langle\Omega|\bar{\varphi}\rangle = \Psi[\varphi]\Psi[\bar{\varphi}]^* \ , \tag{7.2.9}$$

where is the field-theoretic wavefunction corresponding to the state $|\Omega\rangle$. Here, we remind the reader that we are working in a basis of real fields, where the reality condition implies that, in momentum space, $\varphi(\mathbf{k}) = \varphi^*(-\mathbf{k})$.

i.e. $\psi_n(-\mathbf{k}) = \psi_n(\mathbf{k})$, we find

$$\rho_n^{\text{PO}}(\mathbf{k}) = \frac{1}{2}\left[\psi_n(\mathbf{k}) - \psi_n^*(\mathbf{k}) + \psi_n^*(-\mathbf{k}) - \psi_n(-\mathbf{k})\right] \tag{7.2.17}$$

$$= \frac{1}{2}\left[\psi_n(\mathbf{k}) - \psi_n^*(\mathbf{k}) + \psi_n^*(\mathbf{k}) - \psi_n(\mathbf{k})\right] \tag{7.2.18}$$

$$= 0\,. \tag{7.2.19}$$

Hence an imaginary part to ψ_n is *necessary but not sufficient* to ensure a parity-odd correlator, since the imaginary part to ψ_n must itself be parity-odd, i.e. $\psi_n(-\mathbf{k}) = -\psi_n(\mathbf{k})$, by being sourced by a parity-odd interaction.

7.3 Constraints on the Wavefunction

One particular application of the Cosmological CPT Theorem [2] is the constraint of discrete transformations at a *single* asymptotic future boundary $\eta = 0^-$, where the wavefunction coefficients have no dependence on time, i.e. the dS/CFT correlators. Let us look at each transformation individually.

A constraint corresponding to bulk unitary time evolution known as Reflection Reality **RR** was derived in [2]. **RR** does not care about the time at which the wavefunction is evaluated, i.e. it is the same for both boundary and bulk wavefunction coefficients, and when expressed perturbatively:

$$\left[\psi_n^{(L)}(\mathbf{k})\right]^* = e^{i\pi[(d+1)L-1]}\psi_n^{(L)}(e^{-i\pi}\mathbf{k})\,, \tag{7.3.1}$$

where the rotation $e^{-i\pi}\mathbf{k}$ not only rotates the spatial momenta $\{\mathbf{k}\}$, but also rotates their norms $\{k\}$, since under this rotation the norm or "energy" transforms as

$$k \to \sqrt{\mathbf{k}\cdot\mathbf{k}\,e^{-2i\theta}} = \sqrt{\mathbf{k}\cdot\mathbf{k}}\sqrt{e^{-2i\theta}}\,, \tag{7.3.2}$$

where the final line follows as k is real. To ensure the single-valuedness of k during this rotation we need to continuously follow a Riemann sheet of the square root (i.e. we will not add any factors of 2π to the exponent). Similarly the internal energies $\{s\}$ which are related to external energies by momentum conservation which exchange and loop diagrams will generically depend will transform as

$$s \to \sqrt{(\mathbf{k}_1 + \ldots + \mathbf{k}_n)\cdot(\mathbf{k}_1 + \ldots + \mathbf{k}_n)\,e^{-2i\theta}} = \sqrt{(\mathbf{k}_1 + \ldots + \mathbf{k}_n)\cdot(\mathbf{k}_1 + \ldots + \mathbf{k}_n)}\sqrt{e^{-2i\theta}}\,. \tag{7.3.3}$$

As was derived in (7.1.46), Scale Invariance of the boundary theory at $\mathcal{I}^+$ where $\eta = 0^-$ tells us that the wavefunction coefficients must scale as

$$\psi_n^{(L)}(\lambda\mathbf{k}) = \lambda^{d(1-n)+\sum_\alpha \Delta_\alpha}\psi_n^{(L)}(\mathbf{k})\,, \tag{7.3.4}$$

where $\psi_n^{(L)}$ denotes a perturbative wavefunction coefficient computed at some loop order L (e.g. $L = 0$ corresponds to tree-level and $L = 1$ corresponds to 1-loop order), i.e. it is the loop order of the bulk theory, and Δ_α are the dimensions of the external fields. The Cosmological CPT theorem tells us that the $SO^+(1,1)$ boost which is the important continuous symmetry in the flat-space CPT theorem acts in the Poincaré patch as dilatations **D**, i.e. Scale Invariance $\lambda \in \mathbb{R}^+$, which we can analytically continue to $SO(2)$, provided we have a Hamiltonian bounded from below. In [2], it was demonstrated that the three discrete symmetries **RR**, $\mathbf{D}^{\pm}_{-1}$ and **CRT** form a group structure, and thus two of the three discrete symmetries can be combined to obtain the third. Under analytic continuation of $\lambda \in \mathbb{R}^+$ through the complex plane $\lambda \in \mathbb{C}$,[16] we can then access $\lambda = e^{-i\pi}$, to conclude that:

$$\psi_n^{(L)}(e^{-i\pi}\mathbf{k}) = (-1)^{d(1-n)+\sum_\alpha \Delta_\alpha}\psi_n^{(L)}(\mathbf{k})\,, \tag{7.3.5}$$

where for cases with fractional d and Δ we have

$$\mathbf{D}^{\pm}_{-1} : \psi_n^{(L)}(e^{\mp i\pi}\mathbf{k}) = e^{\mp i\pi[d(1-n)+\sum_\alpha \Delta_\alpha]}\psi_n^{(L)}(\mathbf{k})\,. \tag{7.3.6}$$

We need to take into account that $\psi_n^{(L)}$ is analytic through the lower half-plane $\mathbb{C}^{-i}$. Given that the **RR** (7.3.1) transformation rotates $\psi_n^{(L)}$ counter-clockwise, we must use the $\mathbf{D}^{+}_{-1}$ in (7.3.6) to land on

$$\mathbf{D}^{+}_{-1}\boldsymbol{\cdot}\mathbf{RR} : \left[\psi_n^{(L)}(\mathbf{k})\right]^* = e^{i\pi[(d+1)L-1-d(1-n)-\sum_\alpha \Delta_\alpha]}\psi_n^{(L)}(\mathbf{k})\,. \tag{7.3.7}$$

We can also derive the same expression independently using **CRT**, which tells us directly that

$$\left[\psi_n^{(L)}(\mathbf{k})\right]^* = e^{i\pi[(d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha]}\psi_n^{(L)}(\mathbf{k})\,, \tag{7.3.8}$$

where in order to obtain (7.3.8) we have had to continue $1/\eta \propto k \equiv |\mathbf{k}|$ in the lower-half plane $\mathbb{C}^{-i}$, and as previously explained a factor of $\eta^{\sum_\alpha d-\Delta_\alpha}$ arises from the fields scaling as $\eta^{\Delta-d} \propto \lambda^{d-\Delta}$ at the boundary. We have now identified the discrete symmetries which the boundary wavefunction coefficients $\psi_n^{(L)}$ satisfy:

$$\begin{aligned}
\mathbf{CRT}: \quad \psi_n^{(L)}(\mathbf{k}) \quad &= \quad e^{-i\pi[(d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha]}\left[\psi_n^{(L)}(\mathbf{k})\right]^*\,, &(7.3.9)\\
\mathbf{D}^{\pm}_{-1}: \quad \psi_n^{(L)}(\mathbf{k}) \quad &= \quad e^{\pm i\pi[d(1-n)+\sum_\alpha \Delta_\alpha]}\,\psi_n^{(L)}(e^{\mp i\pi}\mathbf{k})\,, &(7.3.10)\\
\mathbf{RR}: \quad \psi_n^{(L)}(\mathbf{k}) \quad &= \quad e^{-i\pi[(d+1)L-1]}\left[\psi_n^{(L)}(e^{-i\pi}\mathbf{k})\right]^*\,, &(7.3.11)
\end{aligned}$$

where the last condition from **RR** holds for boundary correlators in any flat FLRW spacetime with a future conformal boundary. We can use (7.3.9) to solve for the phase of $\psi_n^{(L)}$ directly, obtaining the following phase formula for the boundary wavefunction coefficients:

$$e^{i\arg(\psi_n^{(L)})} \equiv \frac{\psi_n^{(L)}(\mathbf{k})}{|\psi_n^{(L)}(\mathbf{k})|} = \pm\sqrt{\frac{\psi_n^{(L)}(\mathbf{k})}{\psi_n^{(L)*}(\mathbf{k})}} = \pm(-i)^{(d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha}\,, \tag{7.3.12}$$

where there is a $\pm$ out front because **CRT** cannot determine the overall real sign, because obtaining the phase involves taking a square root.[17] Hence, we obtain the result quoted in

[16] Given that $\psi_n^{(L)}$ is only analytic in the lower half-place $\psi_n^{(L)} \in \mathbb{C}^{-i}$, we can see from the group structure discussed in [2] we must act with $\mathbf{D}^{+}_{-1}$ after **RR** in order to obtain **CRT**; or alternatively act with **RR** after $\mathbf{D}^{-}_{-1}$.

[17] Although, for the 2-point function of a field which is weakly coupled in the bulk, this sign is fixed by normalisability.

Chapter 1:

$$\arg(\psi_n^{(L)}) = -\frac{\pi}{2}\left((d+1)(L-1) + dn - \sum_\alpha \Delta_\alpha\right) + \pi\mathbb{N}\,. \tag{7.3.13}$$

Remarkably (7.3.12) and (7.3.13) will hold for any $\psi_n^{(L)}$ computed in cosmology provided that:

- spacetime is de Sitter (possibly with boost-breaking terms);
- the Lagrangian is locally **CRT**-invariant;
- the amplitude is UV- and IR-finite;
- and involves fields in representations with integer spin and real Δ (no spinors or principal series);
- all fields satisfy the Bunch-Davies vacuum.

Crucially, the phase of $\psi_n^{(L)}$ has no dependence on the details of the bulk interactions, e.g. derivative couplings will have the same phase, provided the Feynman-Witten diagrams have the same external legs with the same Δ's. Thus both contact diagrams with the same external legs and those with internal lines (exchange and loop diagrams) will have the same phase, provided the exchanged field is not heavy, i.e. it does not fall within the principal series (see [130] for a more detailed discussion).

The steps leading up to this phase formula implicitly assumed that the amplitudes were IR and UV-finite. In Chapter 6 the $-i\pi$ shift due to UV log divergences was discussed; we now make the corresponding remark for IR divergences. For a typical IR-divergent amplitude, there is a term like $C\log(-\eta)$ which, upon rotating η from 0^- to 0^+, becomes $C\log(-e^{-i\pi}\eta)$ This provides an extra $-i\pi C$ shift that adjusts the IR-finite piece of the amplitude, which does not conform to (7.3.13). But, the leading order IR-divergence will continue to satisfy (7.3.13). An example of such a calculation can be found in [2].

As discussed in Section 6.4.4 of Chapter 6, it is also important to recognise that the conformal dimensions Δ_α appearing in (7.3.13) are not protected quantities. In general, they can receive radiative corrections and thus run with couplings, leading to anomalous dimensions. The phase relation (7.3.13) should therefore be interpreted as applying to the *effective*, i.e. loop-corrected, conformal dimensions at the order of perturbation theory considered.

For example, in loop-corrected wavefunction coefficients, logarithmic scale dependence such as $\log(k)$ or $\log(k\eta)$ appears, signalling a running dimension. As discussed in [67, 76], a renormalised two-point wavefunction coefficient may take the form

$$\psi_2 = k^{2\Delta_0-d}\left[1 + 2g\ln(k) + \mathcal{O}(g^2)\right] = k^{2\Delta_0-d}e^{2g\ln(k)} = k^{2(\Delta_0+g)-d}\,, \tag{7.3.14}$$

where g is a small coupling and Δ_0 the tree-level dimension. The loop-corrected or "effective" conformal dimension is thus

$$\Delta(g) = \Delta_0 + g + \mathcal{O}(g^2)\,. \tag{7.3.15}$$

This running modifies the phase and must be taken into account when applying the phase rule. Similar arguments apply for other wavefunction coefficients with external fields acquiring anomalous dimensions. Nevertheless, in theories where the correlators are free from such logarithmic corrections — e.g. when Δ_α are fixed and no divergences are present — the formula (7.3.13) remains valid and yields powerful constraints on the imaginary parts of cosmological wavefunction coefficients.

7.4 No-go Theorem for Cosmological Parity Violation

By examining (7.3.12) we find that for any $d = 2\mathbb{Z}+1$ (where $\mathbb{Z}$ means integer), $\psi_n^{(L)} \in \mathbb{R}$ when $dn - \sum_\alpha \Delta_\alpha \in 2\mathbb{Z}$ (this result was derived at tree-level independently for scalars, photons and gravitons in [130, 131]). Given that, we know

- $\Delta = d$ for massless scalars;
- $\Delta = (d+1)/2$ for conformally coupled scalars;
- $\Delta = d+s-2$ for massless spinning fields,

we arrive at the following:

No-go Theorem for Cosmological Parity Violation

In $D = d+1$-spacetime dimensions, where $d = 2\mathbb{Z}+1$, parity-odd correlators at the $\eta = 0^-$ boundary of inflation **cannot** be generated at **tree or loop-level** by models with:

★ **only** massless scalar fields and even-spin fields (e.g. gravitons) satisfying the Bunch-Davies vacuum (can also have even number of massless odd-spin fields, e.g. photons, or conformally coupled fields in external legs);

★ locally **CRT**-invariant Lagrangians;

★ IR-finite and UV-finite ψ_n.

If these criteria are met, then a parity-odd correlator B_n^{PO} can only be sourced by *factorised* contributions from internal massive spinning fields (see [130, 131] for a more detailed discussion).

7.4.1 UV Divergences

As discussed previously, UV-finite loop diagrams—such as the one-site loop analysed in [75] do not generate a finite $i\pi$ phase correction to the **CRT** and **RR** constraints, and hence do not contribute to parity-odd correlators.[18] At *loop-level*, we generally encounter UV divergences, requiring us to work in non-integer $d+\delta$ spatial dimensions (where $d \in \mathbb{Z}$ and δ is non-integer), which can allow for a non-vanishing B_n^{PO}.

Massless dim-reg In [76], the authors do not renormalise the mass of the field at the outset. In this case, for massless fields in non-integer $d+\delta$ spatial dimensions, the order of the Hankel

[18] Natural candidates for other UV-finite diagrams in cosmology would be those that give UV-finite amplitudes in flat space (see e.g. [249]). The author thanks Lorenzo di Pietro for helpful discussions regarding this point.

function becomes $\nu = (d+\delta)/2$, so that

$$\Delta = \frac{d+\delta}{2} + \nu = \frac{d+\delta}{2} + \frac{d+\delta}{2} = d + \delta\,. \tag{7.4.1}$$

In the massless dim-reg scheme, since the conformal dimension equals the number of spatial dimensions ($\Delta = d+\delta$), the phase of the one-loop wavefunction coefficient $\psi_n^{(L=1)}$ becomes

$$\arg(\psi_n^{(L=1)}) = \pi\mathbb{N}\,. \tag{7.4.2}$$

Hence, no $i\pi$ phase arises because the wavefunction coefficients stay real. It should also be noted that the loop integrals cannot be computed exactly; instead, δ is expanded before integration, leading to different (non-analytic) behaviour.

In d spatial dimensions, the wavefunction coefficients of scalar fields scale as[19]

$$\langle \mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_n)\rangle' \equiv \psi_n' \equiv \psi_n\, \delta^d\left(\sum_{a=1}^n \mathbf{k}_a\right) = k^{d(1-n)+\sum_\alpha \Delta_\alpha}\, \delta^d\left(\sum_{a=1}^n \mathbf{k}_a\right), \tag{7.4.3}$$

From (7.4.3), we see that in the massless dim-reg scheme in $d+\delta$ spatial dimensions, ψ_n scales as

$$\psi_n = k^{(d+\delta)(1-n)+n(d+\delta)} = k^{(d+\delta)}\,, \tag{7.4.4}$$

In this context, one can understand the appearance of $\log(k)$ terms in loop-corrected correlators as a signal of anomalous dimension running since

$$\psi_n = k^{(d+\delta)} = k^d k^\delta = k^d e^{\delta\ln(k)} = k^d\left[1+\delta\ln(k)+\mathcal{O}(\delta^2)\right], \tag{7.4.5}$$

For instance, the one-loop correction to the two-point function found in [76] has the form

$$\psi_2 = k^{2\Delta_0-d}\left[1+2g\log(k)+\mathcal{O}(g^2)\right] = k^{2(\Delta_0+g)-d} + \ldots\,, \tag{7.4.6}$$

which can be interpreted as a radiative shift in the conformal dimension,

$$\Delta(g) = \Delta_0 + g + \mathcal{O}(g^2)\,, \tag{7.4.7}$$

as in standard QFT. A familiar example is the renormalisation of the electron mass in QED, where the physical (pole) mass receives quantum corrections from photon loops. In schemes such as dimensional regularisation with minimal subtraction, the running of the mass parameter with respect to the renormalisation scale μ is governed by the anomalous dimension of the mass. At one-loop, this results in a scale-dependent bare mass:

$$m(\mu) = m_0\left[1+\frac{\alpha}{\pi}\ln\left(\frac{\mu}{m_0}\right)+\mathcal{O}(\alpha^2)\right], \tag{7.4.8}$$

[19]This follows from dimensional analysis of the Fourier transform of the position-space correlator, which scales like $\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \sim \lambda^{-\sum_\alpha \Delta_\alpha}$ under dilatations $x_i \to \lambda x_i$. Fixing one point (say $x_n = 0$), the Fourier transform gives a factor of $\lambda^{d(n-1)}$ from the $d^{d(n-1)}x_i$ integration, leading to the overall momentum scaling. For example, the 2-point function $\langle\mathcal{O}_1(x)\mathcal{O}_2(0)\rangle \sim 1/|x|^{2\Delta}$ Fourier transforms to $\langle\mathcal{O}_1(\mathbf{k})\mathcal{O}_2(-\mathbf{k}) \sim k^{2\Delta-d}\delta^d\left(\sum_{a=1}^n \mathbf{k}_a\right)$.

where m_0 is the renormalised mass at a reference scale, and $\alpha = e^2/4\pi \approx 1/137$ is the fine-structure constant, which is the loop expansion parameter in QED. Although the physical pole mass remains invariant under changes in μ, this $\ln(\mu/m_0)$ dependence captures how the mass parameter must vary with scale to keep physical observables fixed. This scale dependence is conceptually analogous to the appearance of $\ln(k)$ in cosmological correlators, which reflects the running of operator dimensions due to quantum corrections.

Massive dim-reg An alternative prescription, used in the calculations of [75, 202], the authors used non-integer dimensions for dimensional regularisation (dim-reg) following the prescription detailed in Appendix C of [202]. In this scheme, which we will refer to as **massive dim-reg**, the field mass is renormalised to keep the order of the Hankel function ν fixed, where for scalar (and spin $s = 2$) fields we find

$$\nu = \frac{d}{2} = \sqrt{\frac{(d+\delta)^2}{4} - \frac{m^2}{H^2}} \implies m = H\sqrt{\frac{\delta(2d+\delta)}{4}}\,, \tag{7.4.9}$$

allowing the integrals to be computed analytically in the dim-reg parameter δ. Note that in this choice of regualrisation scheme, working in $d - \delta$ spatial dimensions will result in $m = iH\sqrt{\delta(2d+\delta)/4} \implies m^2 = -H^2\delta(2d+\delta)/4 < 0$, i.e. a tachyonic mass. Hence in the massive dim-reg regularisation scheme one must be careful to analytically continue the spatial dimensions within the region $d + \delta \geq d$, in order to preserve bulk unitarity. Consequently, the conformal dimension for massless scalar fields becomes

$$\Delta = \frac{d+\delta}{2} + \nu = \frac{d+\delta}{2} + \frac{d}{2} = \frac{2d+\delta}{2} = d + \frac{\delta}{2}\,. \tag{7.4.10}$$

As a result, the phase of the 1-loop wavefunction coefficient $\psi_n^{(L=1)}$ becomes

$$\arg(\psi_n^{(L=1)}) = \frac{n\pi\delta}{4} + \pi\mathbb{N}\,. \tag{7.4.11}$$

In dim-reg, the logarithmic UV divergence manifests as a $1/\delta$ pole (assuming the loop diagram is IR-finite), which contributes to the imaginary part of the phase upon expansion of the exponential. Specifically, for any complex number $A = |A|e^{i\arg(A)}$, we find

$$\lim_{\delta\to 0}\left[|\overline{\psi}^{(L=1)}|e^{i\arg(\overline{\psi}^{(L=1)})}\right] \sim \frac{1}{\delta}(1 + i\pi\delta + \mathcal{O}(\delta^2)) = \frac{1}{\delta} + i\pi + \mathcal{O}(\delta)\,. \tag{7.4.12}$$

A manifestation of this generic mechanism for a particular UV-divergent loop diagram was identified in [75], where the authors computed a parity-odd contribution to the scalar trispectrum using the in-in formalism.

From (7.4.3), we see that in the massive dim-reg scheme in $d + \delta$ spatial dimensions, ψ_n scales as

$$\psi_n = k^{(d+\delta)(1-n)+nd+\frac{n\delta}{2}} = k^{d+\delta-\frac{n\delta}{2}}\,, \tag{7.4.13}$$

Hence, the absence of a $\ln(k)$ term for ψ_2 in massive dim-reg is simply explained by the fact that

$$\psi_2 = k^{d+\delta-\frac{2\delta}{2}} = k^d\,, \tag{7.4.14}$$

where $d \in \mathbb{Z}$ and thus has no expansion in the exponent, i.e. a running of dimensions (anomalous dimensions), which can give rise to a $\ln(k)$ term. Further implications of this result will be explored in [137].

Comparison of massless and massive dim-reg The running of dimension (7.4.5) found in the massless dim-reg scheme is completely analogous to the effective shift in Δ introduced by the massive dim-reg prescription (with fixed $\nu = 3/2$), where

$$\Delta = \frac{d}{2} + \nu = \frac{3+\delta}{2} + \frac{3}{2} = \frac{6+\delta}{2} = 3 + \frac{\delta}{2}\,. \tag{7.4.15}$$

On the other hand, the massless dim-reg prescription, where the mass is not adjusted, yields

$$\Delta = \frac{d}{2} + \nu = \frac{3+\delta}{2} + \frac{3+\delta}{2} = 3 + \delta\,, \tag{7.4.16}$$

which corresponds to the presence of a running dimension without absorbing it into the Hankel index ν.

Both schemes ultimately capture the same physical effect: the running of Δ with respect to the coupling or the regulator. What differs is how this running manifests — in the massless scheme, through explicit logarithmic scale dependence; in the massive scheme, through implicit shifts in the conformal dimension via ν. Reconciling these regularisation schemes and clarifying their implications for cosmological parity violation will be explored further in [137].

Overall, resummation of loops can further modify the effective mass and thus the conformal dimension, leading to additional corrections that lie outside the assumptions of our parity no-go theorem:

$$\Delta(g) = \Delta_0 + \mathcal{O}(g)\,. \tag{7.4.17}$$

This reinforces the importance of understanding anomalous dimensions and effective IR behaviour when evaluating the robustness of symmetry-based constraints in cosmology. However, it is important to note that not all fields experience such running. In particular, massless gauge bosons and gravitons are protected by their underlying gauge and diffeomorphism symmetries, respectively [250–258]. In standard QFT, gauge invariance forbids the generation of a mass term for the photon, ensuring that the corresponding conserved current remains exactly conserved, and that the conformal dimension of its dual boundary operator remains fixed at $\Delta = d - 1$ under renormalisation. Similarly, diffeomorphism invariance protects the graviton from acquiring a mass, preserving the conservation and tracelessness of the stress-energy tensor, and thereby fixing its conformal dimension at $\Delta = d$ under renormalisation. These symmetries prevent radiative corrections from altering the tensorial structure or scaling behaviour of the kinetic terms.

By contrast, scalar fields generically acquire corrections to their mass under renormalisation unless protected by an additional symmetry. For example, some inflationary models impose an (approximate) shift symmetry $\phi \to \phi + c$ to suppress quantum corrections to the inflaton potential and ensure radiative stability of the scalar mass[20]. This symmetry restricts the allowed operators in the effective action and protects the flatness of the potential, thereby maintaining

[20] The author thanks Harry Goodhew for helpful discussions on the shift symmetries of the inflaton.

the slow-roll conditions required for inflation. In natural inflation [259] and axion monodromy models [260], the inflaton arises as a pseudo-Nambu–Goldstone boson with a softly broken shift symmetry, while in the Effective Field Theory of Inflation (EFToI) [68], the Goldstone mode associated with broken time translations inherits an approximate shift symmetry that constrains the structure of allowed interactions.

As a result, contributions from photons and gravitons to ψ_n do not experience anomalous dimension running, and the phase rule derived from **CRT** symmetry continues to apply non-perturbatively. This lends further support to the robustness of the no-go theorem in models composed solely of massless spin-1 and spin-2 fields. The status of scalar fields, including the inflaton, depends sensitively on their symmetry structure and must be assessed on a case-by-case basis.

7.4.2 Cosmological Analogue of Furry's Theorem

An interesting observation is that an implication of the no-go theorem relates to a well-known result in QED known as Furry's theorem, which states that any correlation function of $n \in 2\mathbb{Z}+1$ photon operators must vanish [248].

Review of Furry's Theorem in Flat Space

Furry's theorem in flat space asserts that any Feynman diagram containing a closed fermion loop with an odd number of vertices contributes zero to the total amplitude. This result arises if there is a charge conjugation symmetry, under which the charge conjugation operator **C** anticommutes with the photon field $A^\mu(x)$[21]:

$$\mathbf{C}A^\mu(x)\mathbf{C}^\dagger = -A^\mu(x)\,. \tag{7.4.20}$$

Since the vacuum state $|\Omega\rangle$ is invariant under charge conjugation $\mathbf{C}|\Omega\rangle = |\Omega\rangle$, the correlation function of a single photon operator satisfies

$$\langle\Omega|A^\mu(x)|\Omega\rangle = \langle\Omega|\mathbf{C}^\dagger\mathbf{C}A^\mu(x)\mathbf{C}^\dagger\mathbf{C}|\Omega\rangle = -\langle\Omega|A^\mu(x)|\Omega\rangle \implies \langle\Omega|A^\mu(x)|\Omega\rangle = 0\,. \tag{7.4.21}$$

[21] Even though the photon field $A^\mu(x)$ can be expressed in terms of real components, it still transforms with a minus sign under charge conjugation **C**. This transformation property is not determined by whether the field is real or complex in a given basis, but by how it couples to charged matter. **C** is a discrete symmetry operation that inverts the sign of all charges. In particular, it sends particles to antiparticles and reverses the sign of their electromagnetic interactions. To preserve the structure of the theory under this transformation, the gauge field must also transform appropriately. For example, consider the QED interaction term between the photon and a Dirac fermion:

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu A_\mu \psi,\,. \tag{7.4.18}$$

Under **C**, the fermion field transforms to its charge-conjugate $\psi \to \psi^c$, which carries the opposite charge. To ensure that the interaction term remains invariant under **C**, the photon field must flip sign:

$$\mathbf{C}A^\mu(x)\mathbf{C}^\dagger = -A^\mu(x),\,. \tag{7.4.19}$$

This transformation rule holds regardless of the field basis because it reflects the physical requirement that a particle and its antiparticle couple with opposite sign to the electromagnetic field. Thus, the minus sign picked up by A^μ under charge conjugation is a universal feature of gauge theories with conserved charge, and plays a central role in results such as Furry's theorem.

Extending this reasoning, the correlation function for any odd number of photon operators also vanishes:

$$\langle\Omega|A^{\mu_1}(x_1)A^{\mu_2}(x_2)\ldots A^{\mu_{2n+1}}(x_{2n+1})|\Omega\rangle = 0\,. \tag{7.4.22}$$

This reflects the fact that such amplitudes change sign under charge conjugation, and therefore must vanish when evaluated in a **C**-invariant vacuum. In QED, this implies that any process involving an odd number of photon fields and/or currents must vanish — whether the photons are on-shell or off-shell. This significantly simplifies amplitude computations by eliminating entire classes of diagrams. Since Furry's theorem holds non-perturbatively, it also applies order by order in perturbation theory. At leading order, for instance, any fermion loop with an odd number of photon vertices gives a vanishing contribution to the amplitude.

Cosmological Extension and Interpretation

From (7.3.12) and (7.3.13), we observe that in even $D = d+1$-spacetime dimensions, $\psi_n \in i\mathbb{R}$ when the number of external photon legs is odd. Consequently, any parity-even correlator must vanish, suggesting that for any theory which respects parity **P** (such as standard QED), cosmological correlators with an odd number of photons (and no charged particles) also vanish. We thus find a cosmological analogue of Furry's theorem![22]

Moreover, this result is robust under renormalisation. Gauge invariance forbids the generation of a photon mass term, ensuring that the conformal dimension Δ of the boundary operator dual to the bulk gauge field remains protected. In particular, the dimension $\Delta = d-1$ for a conserved spin-1 current is exact and receives no anomalous corrections. Just as in flat space, the transversality of the photon self-energy enforces that its anomalous dimension vanishes to all orders, so the wavefunction renormalisation does not introduce any logarithmic scale dependence. As a result, the phase of any ψ_n involving only photons is also protected under renormalisation, and the vanishing of odd-n correlators in parity-invariant theories is a non-perturbatively stable feature. This underlines the universality of the cosmological analogue of Furry's theorem across energy scales and interaction strengths.

Additionally, although a vanishing flat-space amplitude[23] does not necessarily imply that the corresponding cosmological correlator must vanish, (7.4.22) suggests that if the theory respects charge conjugation **C**, then cosmological correlators with an odd number of photons (and no charged particles) also vanish. This serves as a complementary non-perturbative derivation of Furry's theorem in a cosmological setting.

7.4.3 Odd Spacetime Dimensions

In odd $D = d+1$-spacetime dimensions, parity-even correlators at the $\eta = 0^-$ boundary of inflation *cannot* be generated at *tree level or even-loop level* by models with

[22]The author thanks Aron Wall for extensive discussions regarding Furry's theorem.

[23]This is referred to as the "flat-space limit" in the cosmology literature, where the residue of the total-energy pole for a Feynman-Witten diagram coming from particular interaction in cosmology, is the corresponding massless flat-space amplitude since the amplitude corresponds to the ultraviolet regime of the associated flat-space process, where the masses of the internal propagators are effectively zero. See [261], where the authors introduce a novel massive flat-space limit, in which the internal masses in the corresponding flat-space Feynman graph remain finite.

- *only* massless scalar fields and even-spin fields (e.g. gravitons) satisfying the Bunch-Davies vacuum (can also have even number of massless odd-spin fields, e.g. photons, or conformally coupled fields in external legs);
- locally **CRT**-invariant Lagrangians;
- IR-finite and UV-finite boundary wavefunction coefficients ψ_n.

This result is particularly intriguing, as computing cosmological correlators in odd spacetime dimensions poses significant challenges due to the mode functions being expressed in terms of Hankel functions, $H_\nu(-k\eta)$ where $\nu \in \mathbb{Z}$. Although certain computations, such as the tree-level contribution to the two-point function, can be carried out in general $D = d + 1$-spacetime dimensions for fields of arbitrary mass (see, e.g.,[136]), it would be compelling to explore whether this result could also be established using alternative methods, such as the Wick rotation method employed in [130].

Another noteworthy observation we can make is that in $D = d + 1$-spacetime dimensions, $\psi_n \in i\mathbb{R}$ at tree-level (and any even-loop level) for the case of pure gravity or massless scalars (and any massless even spin-s fields). However, normalisability imposes the condition that $\psi_n \in \mathbb{R} < 0$. This leads to the intriguing conclusion that bulk loop effects are necessary for a consistent theory of massless scalars and gravitons.

More broadly, this work extends and generalises the previous no-go theorems of [74, 130, 132] by providing statements that hold in any spacetime dimension for any massive integer spin-s field, as long as the conformal dimension of the field, Δ, satisfies $\Delta \in \mathbb{R}$.

7.5 Summary and Outlook

In this chapter, we have developed a comprehensive framework for understanding the symmetry-based constraints on parity-violating signals in primordial cosmology, culminating in a novel No-go Theorem for Cosmological Parity Violation. Specifically, we demonstrated that in even $D = d + 1$ spacetime dimensions, parity-odd correlators cannot be generated under the following conditions:

1. the theory contains only massless scalar fields and even-spin fields (e.g. gravitons), and/or an even number of massless odd-spin fields (e.g. photons) or conformally coupled scalars, all satisfying the Bunch-Davies vacuum;
2. the Lagrangian is locally **CRT**-invariant;
3. the wavefunction coefficients ψ_n are both IR-finite and UV-finite.

Under these assumptions, any parity-odd correlator B_n^{PO} must arise solely from factorised contributions of internal massive spinning fields, as shown explicitly in Sections 7.2 and 7.3, and discussed in greater detail in [130, 131].

A crucial aspect of this result is the renormalisation protection of the conformal dimensions of the photon and graviton. Due to gauge invariance and diffeomorphism invariance, respectively, these fields remain massless to all loop orders, and their dual operators retain fixed conformal weights: $\Delta = d - 1$ for photons, and $\Delta = d$ for gravitons. As a result, their contributions to the cosmological wavefunction cannot introduce parity-violating structures via anomalous dimensions or loop-induced phase shifts. This protection follows from the Ward-Takahashi and Slavnov–Taylor identities in gauge theory [250, 251, 253, 254, 257, 258], and from the structure of generally covariant theories for gravitons [252, 255, 256, 258].

Importantly, the inflaton is not generically protected by any symmetry from acquiring loop corrections to its mass. However, in many inflationary models, such as the EFToI [68], the inflaton possesses an approximate or exact shift symmetry, $\phi \to \phi + c$, which suppresses its mass and renders it effectively massless over cosmological timescales. This symmetry plays a crucial role in controlling radiative corrections to the inflaton's potential and maintaining its nearly scale-invariant spectrum. In such cases, although the inflaton is technically light, its dual operator's conformal dimension may not remain exactly fixed, and small departures from scale invariance may introduce anomalous running or logarithmic violations in ψ_n. These subtleties are model-dependent and must be treated carefully when assessing parity-violating effects.

Nevertheless, we identified UV-divergent loop corrections as a distinct mechanism that can give rise to parity-odd signals, primarily through $i\pi$ phase shifts generated by logarithmic divergences in dimensional regularisation. These corrections introduce subtle scheme-dependent effects that alter the analytic structure of ψ_n. For instance, massless and massive dimensional regularisation schemes yield equivalent physical results, but differ in their manifestation of running: the former exhibits explicit $\log(k)$ dependence, while the latter encodes the same physics through a shift in the conformal dimension Δ. These effects, and their implications for parity violation, will be explored further in [137].

While our results underscore the power of symmetry-based constraints in eliminating parity-violating observables in scale-invariant theories with standard vacua, they also point to exciting opportunities to probe symmetry-breaking effects by relaxing certain assumptions. Several concrete scenarios in which these constraints can be bypassed — such as theories with heavy spinning fields, broken scale invariance, or non-standard initial states — were discussed in Section 7.3. We elaborate on these directions further below.

7.5.1 Yes-go Examples

Dynamical Chern-Simons and Axion Inflation

Interactions that induce IR-divergent wavefunction coefficients ψ_n, such as those appearing in Chern-Simons and Axion inflation models, naturally lead to parity-odd correlators. This is due to the presence of $\log(-k\eta)$ terms in ψ_n, as discussed in Section 7.3. Furthermore, most calculations in these models rely on a series expansion of the mode functions in terms of the chemical potential μ (see e.g. [238]), which when truncated modifies the form of the discrete symmetry constraints **CRT**, **RR** and **D**, since only wavefunction coefficients computed in the full theory satisfies the form presented in this chapter. A similar phenomenon was found in the case of resonant non-Gaussianities [56] where the expansion in the dimensionless frequency of oscillations $\alpha = \omega/H$ requires the Cosmological Optical Theorem (COT) [50–52] to be modified

$$\text{Modified COT:} \qquad \psi_n(k, \alpha) + [\psi_n(-k^*, -\alpha)]^* = 0\,, \tag{7.5.1}$$

where $\alpha = \omega/H \in \mathbb{R}^+$, and in the resonance approximation, for which terms that are exponentially suppressed in α are neglected. For such an approximation **CRT** and **RR** would also be modified. In the case of the approximations used for theories with chemical potentials, where terms suppressed in μ are neglected, one would find **RR** and **CRT** to be modified in the following way[24]:

$$\text{Modified } \mathbf{RR} \qquad \left[\psi_n^{(L)}(\mathbf{k};\mu)\right]^* = e^{i\pi[(d+1)L-1]}\psi_n^{(L)}(e^{-i\pi}\mathbf{k}; e^{-i\pi}\mu)\,, \tag{7.5.2}$$

$$\text{Modified } \mathbf{CRT} \qquad \left[\psi_n^{(L)}(\mathbf{k};\mu)\right]^* = e^{i\pi[(d+1)(L-1)+dn-\sum_\alpha \Delta_\alpha]}\psi_n^{(L)}(\mathbf{k}; e^{-i\pi}\mu)\,, \tag{7.5.3}$$

i.e. **CRT** will no longer relate ψ_n to its complex conjugate, and thus can no longer be used to determine the phase of ψ_n. This implies that the discrete symmetry of **CRT** involves an analytic continuation of the chemical potential, despite this not being the case for the original full theory. Hence, this suggests that the complex phase of ψ_n and consequently the non-vanishing parity-odd correlators found in these theories are an artefact of the expansion since these models would fall within the class of models included in the regime of this chapter's no-go theorem. Interestingly, **CRT** symmetry of the original Lagrangian implies that for inflationary models involving a chemical potential, there must be a fundamental relation between the chemical potential μ parametrising **CP**-violation and the slow-roll parameter ξ which is a measure of **T**-violation. Given that we already have bounds on the ξ from the spectral tilt of the 2-point function, this could potentially lead to constraints on baryogenesis for these inflationary models. These all suggest further avenues to refine these analyses to study and probe parity-violating signatures.

Non-BD Vacuum States

The discrete symmetries **CRT**, **RR** and $\mathbf{D}^{\pm}_{\mathbf{-1}}$ defined in this thesis of the cosmological wavefunction hold strictly for the Bunch-Davies vacuum and α-vacua with real Bogoliubov coefficients (see [262] where this was derived perturbatively). Relaxing this condition by allowing for non-BD initial states breaks **CRT** symmetry, resulting in parity-odd signals in the final state correlators. This offers a compelling direction for future investigations into the role of initial conditions in generating parity violation.

Massive Spinning Fields

Parity-odd correlators can emerge through factorised contributions when massive internal spinning fields are present. This phenomenon arises from the universal reality of wavefunction coefficients' total-energy poles under tree-level and UV-finite conditions. Such factorised contributions are a robust signal of parity violation and provide a clean observational target and are discussed in further detail in [130, 131].

[24] The author thanks Harry Goodhew and Tommaso Moretti for helpful discussions regarding chemical potentials during the Parity Violation from Home 2024 conference.

Dimensional Regularisation for Loop Diagrams

We have shown that UV-divergent one-loop diagrams can generate parity-odd contributions to the cosmological wavefunction through an $i\pi$ phase shift in dimensional regularisation, as previously observed in [75]. Importantly, this effect does not indicate a contradiction between different regularisation schemes but rather reflects distinct conventions in how the running of conformal dimensions is treated.

In particular, the scheme used in [76] — which we refer to as massless dim-reg — keeps the field mass fixed, resulting in conformal dimensions that run explicitly with the regulator δ and generate $\log(k)$ dependence in ψ_n. In contrast, the scheme of [67, 75] — referred to as massive dim-reg — renormalises the mass to hold the Hankel index ν fixed, so the running is encoded in a shift of the conformal dimension itself. Both approaches capture the same underlying physics — namely, the scale dependence introduced by quantum corrections — but implement it in structurally distinct ways.

This clarification not only unifies earlier results but also illustrates the importance of understanding how regularisation choices impact the analytic structure of the wavefunction. A detailed analysis of these schemes and their implications for wavefunction phases will be presented in [137].

Extensions to General FLRW Spacetimes

The results derived in flat FLRW spacetimes extend naturally to more general cosmological backgrounds. The solutions to the $\eta \to 0$ limit of the equations of motion for general power-law cosmologies suggest that equivalent constraints on the wavefunction phase hold in broader settings. It would also be interesting to investigate the implications of **CRT** for curved FLRW cosmologies given that the overall curvature of the universe is still an open question in cosmology (see e.g. [217–229] for recent studies on curved inflating universes). Investigating these generalisations will likely uncover new mechanisms for generating parity-violating signals.

7.5.2 Future Directions

This work serves as a foundation for both theoretical and observational advancements in probing parity violation in the early universe. Theoretically, future studies could explore non-standard inflationary scenarios, such as those with time-dependent couplings, ghost condensates (see e.g. [73, 74, 263, 264]), or deviations from exact scale invariance, i.e. slow-roll corrections due to time translation invariance being slightly broken by standard inflation. Observationally, these insights could guide the analysis of forthcoming data from CMB polarisation and large-scale structure surveys, where parity-violating signals may manifest.

By clarifying the conditions under which parity-odd correlators arise and identifying concrete examples of their generation, this chapter contributes to our understanding of the fundamental symmetries of the universe and opens pathways to uncover new physics beyond the standard models of cosmology and particle physics. The interplay between theoretical constraints and observational possibilities underscores the importance of continuing this line of inquiry, with the ultimate goal of unravelling the mysteries of the primordial universe.

"There is an alternative understanding of what hope means. It is one that says hope is less about the future and more about the now; it is an attitude, stubborn and persistent, consistently doing what feels to be right and not giving up, even when things are dark. It is about identifying things that are good, and putting effort into them; it's about finding meaning in all forms of existence. It's about believing that through such actions we may bring something to fulfilment."
— Jocelyn Bell Burnell, A Quaker Astronomer Reflects: Can a Scientist Also Be Religious?

8 Conclusions

The search for a quantum theory of gravity in cosmological settings remains one of the most profound and enduring challenges in theoretical physics. Unlike in flat space and Anti-de Sitter space, where the framework of quantum field theory is well established and string theory offers a rich tapestry of consistent completions, the cosmological context poses unique conceptual and technical difficulties. The presence of a dynamical background, horizons, and a positive cosmological constant renders many traditional tools inapplicable or incomplete. In this thesis, we have adopted a symmetry-first approach to quantum cosmology, guided by the idea that the deepest and most universal features of quantum gravity may be encoded in symmetry principles rather than in specific Lagrangians or UV completions. Central to our analysis is the Wavefunction of the Universe, which acts as a unifying object linking theoretical consistency, observational data, and emergent holographic structures.

By elevating **CRT** symmetry — a composite of charge conjugation **C**, spatial reflection **R**, and time reversal **T**— to a central organising principle, we have shown that powerful constraints can be derived on the analytic properties, phase structure, and observable content of the cosmological wavefunction. These constraints extend to parity-violating processes, loop corrections, and the interpretation of the dS/CFT correspondence, providing a non-perturbative window into the internal consistency of quantum cosmological models. In this way, the results of this thesis contribute to a growing body of work aimed at reformulating quantum gravity in cosmological spacetimes from first principles.

Summary

Throughout this thesis, we have used the term "non-perturbative" to mean that our results hold non-perturbatively for quantum field theory on a fixed cosmological background. That is, the constraints we derive are not limited to any finite loop order, but instead follow from symmetry principles or the analytic structure of the wavefunction and remain valid without truncation. We do not claim to include genuinely non-perturbative quantum gravity effects, such as gravitational instantons, tunnelling, or large-N resummation, unless explicitly stated. In this more restricted but still powerful sense, our use of "non-perturbative" corresponds to the full perturbative expansion of QFT in curved spacetime, rather than non-perturbative dynamics in the gravitational sector itself.

The central result of this thesis is the formulation and proof of the Cosmological CPT Theorem. This theorem provides a general and non-perturbative constraint on the Wavefunction of the Universe in de Sitter space, showing that under assumptions of unitarity and discrete scale invariance (or equivalently **CRT** symmetry), the phase of each wavefunction coefficient is uniquely determined at any loop order and in any spacetime dimension. Remarkably, this phase depends only on the number and conformal dimensions of the operator insertions, and not on the details of the bulk interactions. This result provides a universal organising principle for the analytic structure of the wavefunction and its observable consequences.

These results generalise the familiar CPT theorem of flat-space quantum field theory to cosmological settings and establish a foundational symmetry principle for wavefunction-based approaches to quantum gravity. In de Sitter space, we demonstrated that the requirement of **CRT** invariance leads to a precise and universal phase formula for the wavefunction coefficients. These coefficients encode all cosmological correlation functions and are directly tied to physical observables such as the CMB anisotropies and galaxy correlations. The constraint structure persists at all loop orders for theories that are UV- and IR-finite, and is robust under changes to the detailed bulk interactions, so long as the external field content and their conformal dimensions remain fixed.

The **CRT** phase formula yields a powerful no-go result: under broad and physically motivated assumptions — including a Bunch-Davies vacuum, local **CRT** invariance, and the absence of infrared or ultraviolet divergences — parity-odd cosmological correlators must vanish. This generalises prior work on no-go theorems for parity-odd contributions to the scalar trispectrum and draws a parallel with familiar results such as Furry's theorem in flat-space QFT. In doing so, we have mapped out a symmetry-based obstruction to parity violation in early-universe cosmology, pointing to necessary ingredients for evading the constraint, such as non-standard vacua, principal series fields, or explicit breaking of scale invariance.

Crucially, the Cosmological CPT Theorem also resolves a long-standing conceptual puzzle in holographic cosmology: how can unitary time evolution in the bulk manifest in a dual Euclidean boundary theory without time, such as in dS/CFT? The key insight is that **Reflection Reality** (**RR**) — an antilinear symmetry implied by unitarity — continues to hold even in the absence of Poincaré invariance, as long as the cosmological background is a spatially flat FLRW spacetime. Since the Wavefunction of the Universe is defined via a gravitational path integral on such spacetimes, this result ensures that bulk unitarity imposes concrete analytic and reality constraints on the boundary wavefunction coefficients.

Thus, despite the non-unitarity of the boundary CFT, bulk unitarity encodes itself in the boundary theory as a hidden symmetry **RR**, and the Cosmological CPT Theorem furnishes its exact manifestation. In this way, we clarify and resolve a foundational ambiguity in the interpretation of the dS/CFT correspondence.

We also explored the consequences of these symmetry constraints at loop level. Here, the interplay between regularisation schemes, divergence structures, and analytic continuation becomes subtle but tractable. By analysing dimensional regularisation techniques and identifying how different choices affect the mass renormalisation and conformal dimensions of bulk fields, we clarified how logarithmic corrections manifest in the wavefunction and how they modify or preserve **CRT** constraints. In particular, we have shown that UV-finite contributions do not introduce $i\pi$ phase shifts, whereas logarithmic divergences do, and that these corrections can be cleanly interpreted in terms of running conformal dimensions and anomalous dimensions, in close analogy with renormalisation in flat-space QFT.

Outlook

The full implications of **CRT** for the boundary wavefunction will be explored further in [205]. In this work, we will show how **CRT** can help determine the boundary wavefunction of de Sitter in both tree-level and loop-level examples, and distinguish it from the analytic continuation of AdS wavefunctions. Additionally, we will describe how the real parts of divergent coefficients obey recursion relations that encode the structure of logarithmic divergences across loop orders, potentially opening new avenues for understanding resummation and renormalisation in cosmology.

In a separate paper [136], we show that Reflection Reality (**RR**) and the phase formula are consistent with Cauchy Slice Holography, in which the real-time de Sitter wavefunction is computed on a finite time Cauchy slice. There, we provide a bulk derivation of the same phase structure via saddle-point analysis of the on-shell action, demonstrating that it is not only consistent with **CRT** symmetry, but also required by a well-defined variational problem in real-time cosmology.

Another upcoming paper [265] will use Reflection Reality (**RR**) as the foundational principle to classify viable dS/CFT candidates. The assumptions underlying this classification are minimal yet physically compelling:

- Reflection Reality (**RR**) holds — a necessary condition for bulk unitarity;
- The cosmological background is spatially flat FLRW and admits a large-N genus expansion in the bulk;
- The boundary theory is a $U(N)$ 't Hooftian theory with adjoint matter;
- Both sides are defined perturbatively.

Under these assumptions alone, it is possible to determine sharp constraints on which non-unitary Euclidean CFTs could serve as consistent duals to unitary bulk theories in flat FLRW backgrounds. This provides a practical and symmetry-guided framework for evaluating and constructing dS/CFT correspondences, or more generally speaking FLRW/CFT correspondences.

Closing Remarks

Together, these results suggest that the Wavefunction of the Universe encodes more than just statistical information about fluctuations — it reflects deep, symmetry-protected features of quantum gravity in cosmological spacetimes. By leveraging **CRT** invariance and holographic structure, we have gained new tools for analysing the fundamental constraints on observable physics in the early universe. The Cosmological CPT Theorem not only generalises the flat-space CPT theorem to curved backgrounds, but also provides a concrete mechanism by which bulk unitarity constrains non-unitary boundary theories.

While many questions remain open, the path forward appears increasingly navigable. A synthesis of symmetry, holography, and real-time formulation offers not just technical control, but conceptual clarity. It is my hope that this work contributes a small but meaningful step in that broader program.

A

Simplifying the bulk action for the EFToI

In this appendix, we determine all the relevant curvature and connection data within the EFToI, which will then use in Section 2.2 to show that all EFToI operators contributing to graviton bispectra can be constructed from a minimal set of building blocks: $\tilde{R}ij$, δKij, and their covariant derivatives. In Section 2.2.2, we will take this analysis one step further and demonstrate — for the first time — that all graviton bispectra can, in fact, be derived purely from operators built from the extrinsic curvature.

In our particular framework the metric is

$$ds^2 = -dt^2 + a^2(e^\gamma)_{ij}dx^i dx^j, \tag{A.0.1}$$

where the graviton is symmetric, transverse and traceless: $\gamma_{ii} = 0$, $\partial_i\gamma_{ij} = 0$ with Christoffel symbols

$$\Gamma^0_{00} = 0, \tag{A.0.2}$$
$$\Gamma^0_{0i} = 0, \tag{A.0.3}$$
$$\Gamma^0_{ij} = \frac{1}{2}\partial_t g_{ij}, \tag{A.0.4}$$
$$\Gamma^i_{00} = 0, \tag{A.0.5}$$
$$\Gamma^i_{j0} = \frac{1}{2}g^{ik}\partial_t g_{kj}, \tag{A.0.6}$$
$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk}). \tag{A.0.7}$$

The extrinsic curvature of constant-t hyper-surfaces in unitary gauge is given by $K_{\mu\nu} = \Gamma^0_{\mu\nu}$ [1], so we have $K_{00} = 0 = K_{0i}$ with non-zero components

$$K_{ij} = \Gamma^0_{ij} = \frac{1}{2}\dot{g}_{ij} = a\dot{a}(e^\gamma)_{ij} + \frac{1}{2}a^2\partial_t(e^\gamma)_{ij}. \tag{A.0.8}$$

[1]This follows directly from the expression $K_{\mu\nu} = \nabla_\mu n_\nu = -\nabla_\mu(\partial_\nu\phi/\sqrt{X})$, where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -(\partial\phi)^2$ [68].

Throughout we will actually work with the perturbed extrinsic curvature defined by $\delta K_{\mu\nu} = K_{\mu\nu} - Hh_{\mu\nu}$, where $h_{\mu\nu} = g_{\mu\nu} + \delta^0_\mu \delta^0_\nu$. Hence we can use only the spatial components of $\delta K_{\mu\nu}$ without loss of generality

$$\begin{aligned}\delta K_{ij} &= a\dot{a}(e^\gamma)_{ij} + \frac{1}{2}a^2\partial_t(e^\gamma)_{ij} - Ha^2(e^\gamma)_{ij} \\ &= \frac{1}{2}a^2\partial_t(e^\gamma)_{ij} = \frac{1}{2}a^2\left(\dot{\gamma}_{ij} + \frac{1}{2}\gamma_{ik}\dot{\gamma}_{kj} + \frac{1}{2}\dot{\gamma}_{ik}\gamma_{kj}\right) + \mathcal{O}(\gamma^3). \end{aligned} \tag{A.0.9}$$

We also have $\delta K_{00} = K_{00} - Hh_{00} = 0$ and $\delta K_{0i} = K_{0i} - Hh_{0i} = 0$. This agrees with [36]. Now let's check the trace of the perturbed extrinsic curvature

$$\delta K = g^{\mu\nu}\delta K_{\mu\nu} = g^{ij}\delta K_{ij} = \frac{1}{2}(e^{-\gamma})^{ij}\partial_t(e^\gamma)_{ij}. \tag{A.0.10}$$

Up to cubic order in perturbations we have

$$\delta K = \frac{1}{2}\left(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{ik}\gamma^{kj} + \dots\right)\partial_t\left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma_{kj} + \frac{1}{6}\gamma_{ik}\gamma_{km}\gamma_{mj} + \dots\right). \tag{A.0.11}$$

The spatial metric $g_{ij} = a^2(e^\gamma)_{ij}$ has determinant

$$\sqrt{-g} = \det(a^2 e^\gamma) = a^6 e^{\mathrm{tr}(\gamma)} = a^6. \tag{A.0.12}$$

where we have used Jacobi's formula[2] and the fact that the graviton is traceless $\mathrm{tr}(\gamma) = 0$. Hence γ drops out of $\sqrt{-g}$, so we have

$$\begin{aligned} K = g^{ij}K_{ij} = \frac{1}{2}g^{ij}\dot{g}_{ij} = \frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}) = 6H \\ \implies \delta K = g^{ij}K_{ij} - Hg^{ij}g_{ij} = 6H - 3H = 3H\,. \end{aligned} \tag{A.0.13}$$

Therefore γ drops out of the trace of the perturbed extrinsic curvature and δK is a constant to all orders, i.e. we can ignore the trace when building graviton cubic vertices.

Covariant derivatives of the extrinsic curvature are also useful. We have

$$\nabla_0 K_{ij} = \dot{K}_{ij} - 2K^m{}_i K_{mj}, \tag{A.0.14}$$
$$\nabla_m K_{ij} = \partial_m K_{ij} - \Gamma^n_{mi}K_{nj} - \Gamma^n_{mj}K_{ni}, \tag{A.0.15}$$
$$\nabla_\sigma K_{00} = 0, \tag{A.0.16}$$
$$\nabla_0 K_{0i} = 0, \tag{A.0.17}$$
$$\nabla_m K_{0i} = -K^j{}_m K_{ij}. \tag{A.0.18}$$

Note that these expressions are the derivatives of $K_{\mu\nu}$ rather than of $\delta K_{\mu\nu}$, however since $h_{\mu\nu}$ is covariantly conserved $\nabla_\mu h_{\mu\nu} = 0$ we have $\nabla_\sigma \delta K_{\mu\nu} = \nabla_\sigma K_{\mu\nu}$.

[2] A useful corollary which follows from Jacobi's identity is the relation connecting the trace to the determinant of a matrix exponential $\det(A)$, where $A = e^B$, is $\det(A) = \det(e^B) = e^{\mathrm{tr}(A)}$.

The Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$ is another building block for which we have

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{A.0.19}$$

$$R^{0}{}_{000} = 0, \tag{A.0.20}$$

$$R^{0}{}_{00i} = 0, \tag{A.0.21}$$

$$R^{0}{}_{i0j} = \nabla_0 K_{ij} + K^{m}{}_{i} K_{mj}, \tag{A.0.22}$$

$$R^{0}{}_{0ij} = 0, \tag{A.0.23}$$

$$R^{0}{}_{ijk} = \nabla_j K_{ik} - \nabla_k K_{ij}, \tag{A.0.24}$$

$$R^{i}{}_{jkl} = \tilde{R}^{i}{}_{jkl} + K^{i}{}_{k} K_{lj} - K^{i}{}_{l} K_{kj}. \tag{A.0.25}$$

where we use a tilde to represent three-dimensional objects. These expressions tell us that we don't need to use the full four-dimensional Riemann tensor $R^{i}{}_{jkl}$. A further simplification we can make is by expressing the three-dimensional Riemann tensor $\tilde{R}_{ijkl}$ in terms of the trace-free three-dimensional Weyl tensor $\tilde{C}_{ijkl}$ which vanishes in three dimensions. Hence the three-dimensional Riemann tensor is completely fixed by the three-dimensional Ricci tensor $\tilde{R}_{ij}$. For the interested reader, a proof that the Weyl tensor vanishes in three dimensions using identities by double antisymmetrisation is given in [266]

For the Ricci and Einstein tensor we have

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\sigma\mu\rho\nu} \tag{A.0.26}$$

$$R_{00} = -\nabla_0 (g^{ij} K_{ij}) - K^{ij} K_{ij}, \tag{A.0.27}$$

$$R_{0i} = \nabla^j K_{ji} - \nabla_i (g^{mn} K_{mn}), \tag{A.0.28}$$

$$R_{ij} = \tilde{R}_{ij} + \nabla_0 K_{ij} + (g^{mn} K_{mn}) K_{ij}, \tag{A.0.29}$$

$$R = \tilde{R} + 2\nabla_0 (g^{ij} K_{ij}) + K^{ij} K_{ij} + (g^{ij} K_{ij})^2. \tag{A.0.30}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{A.0.31}$$

$$G_{00} = \frac{1}{2} (\tilde{R} - K^{ij} K_{ij} + (g^{ij} K_{ij})^2), \tag{A.0.32}$$

$$G_{ij} = \tilde{G}_{ij} + \nabla_0 K_{ij} + (g^{mn} K_{mn}) K_{ij} \tag{A.0.33}$$

$$- \frac{1}{2} g_{ij} \left[2\nabla_0 (g^{mn} K_{mn}) + K^{mn} K_{mn} + (g^{mn} K_{mn})^2 \right]. \tag{A.0.34}$$

References

[1] G. Cabass, D. Stefanyszyn, J. Supeł and A. Thavanesan, *On graviton non-Gaussianities in the Effective Field Theory of Inflation*, *JHEP* **10** (2022) 154 [2209.00677].

[2] H. Goodhew, A. Thavanesan and A.C. Wall, *The Cosmological CPT Theorem*, 2408.17406.

[3] A. Thavanesan, *No-go Theorem for Cosmological Parity Violation*, 2501.06383.

[4] Planck collaboration, *Planck 2015 results. I. Overview of products and scientific results*, *Astron. Astrophys.* **594** (2016) A1 [1502.01582].

[5] V. Springel et al., *Simulating the joint evolution of quasars, galaxies and their large-scale distribution*, *Nature* **435** (2005) 629 [astro-ph/0504097].

[6] B.S. DeWitt, *Quantum Theory of Gravity. 1. The Canonical Theory*, *Phys. Rev.* **160** (1967) 1113.

[7] C. Kiefer, *The Semiclassical approximation to quantum gravity*, *Lect. Notes Phys.* **434** (1994) 170 [gr-qc/9312015].

[8] J.B. Hartle and S.W. Hawking, *Wave Function of the Universe*, *Phys. Rev. D* **28** (1983) 2960.

[9] J.J. Halliwell and S.W. Hawking, *The Origin of Structure in the Universe*, *Phys. Rev. D* **31** (1985) 1777.

[10] L. Freidel, *Reconstructing AdS/CFT*, 0804.0632.

[11] C. Rovelli, *The strange equation of quantum gravity*, *Class. Quant. Grav.* **32** (2015) 124005 [1506.00927].

[12] V. Shyam, E. Silverstein, R.M. Soni, A. Thavanesan and G. Torroba, *dS/CFT from* $T\overline{T}+\Lambda_d$, 25xx.xxxxx.

[13] G. Araujo-Regado, R. Khan and A.C. Wall, *Cauchy slice holography: a new AdS/CFT dictionary*, *JHEP* **03** (2023) 026 [2204.00591].

[14] G. Araujo-Regado, *Holographic Cosmology on Closed Slices in 2+1 Dimensions*, 2212.03219.

[15] E. Witten, *A note on the canonical formalism for gravity*, *Adv. Theor. Math. Phys.* **27** (2023) 311 [2212.08270].

[16] I.Y. Park, *Foliation, jet bundle and quantization of Einstein gravity*, *Front. in Phys.* **4** (2016) 25 [1503.02015].

[17] R. Khan, *The Semiclassical Approximation: Its Application to Holography and the Information Paradox*, 2309.08116.

[18] V. Godet, *Quantum cosmology as automorphic dynamics*, 2405.09833.

[19] T. Chakraborty, J. Chakravarty, V. Godet, P. Paul and S. Raju, *The Hilbert space of de Sitter quantum gravity*, *JHEP* **01** (2024) 132 [2303.16315].

[20] M. Usatyuk and Y. Zhao, *Closed universes, factorization, and ensemble averaging*, 2403.13047.

[21] A. Castro and A. Maloney, *The Wave Function of Quantum de Sitter*, *JHEP* **11** (2012) 096 [1209.5757].

[22] M.J. Blacker and S.A. Hartnoll, *Cosmological quantum states of de Sitter-Schwarzschild are static patch partition functions*, *JHEP* **12** (2023) 025 [2304.06865].

[23] D. Anninos, *De Sitter Musings*, *Int. J. Mod. Phys. A* **27** (2012) 1230013 [1205.3855].

[24] D. Anninos, T. Hartman and A. Strominger, *Higher Spin Realization of the dS/CFT Correspondence*, *Class. Quant. Grav.* **34** (2017) 015009 [1108.5735].

[25] D. Anninos, F. Denef and D. Harlow, *Wave function of Vasiliev's universe: A few slices thereof*, *Phys. Rev. D* **88** (2013) 084049 [1207.5517].

[26] R.L. Arnowitt, S. Deser and C.W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997 [gr-qc/0405109].

[27] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).

[28] A. Ghosh, N. Kundu, S. Raju and S.P. Trivedi, *Conformal Invariance and the Four Point Scalar Correlator in Slow-Roll Inflation*, *JHEP* **07** (2014) 011 [1401.1426].

[29] E. Pajer, *Building a Boostless Bootstrap for the Bispectrum*, *JCAP* **01** (2021) 023 [2010.12818].

[30] J.M. Maldacena and G.L. Pimentel, *On graviton non-Gaussianities during inflation*, *JHEP* **09** (2011) 045 [1104.2846].

[31] S. Raju, *New Recursion Relations and a Flat Space Limit for AdS/CFT Correlators*, *Phys. Rev. D* **85** (2012) 126009 [1201.6449].

[32] S. Raju, *Four Point Functions of the Stress Tensor and Conserved Currents in* AdS_4/CFT_3, *Phys. Rev. D* **85** (2012) 126008 [1201.6452].

[33] J. Bonifacio, H. Goodhew, A. Joyce, E. Pajer and D. Stefanyszyn, *The graviton four-point function in de Sitter space*, *JHEP* **06** (2023) 212 [2212.07370].

[34] S. Albayrak and S. Kharel, *All plus four point (A)dS graviton function using generalized on-shell recursion relation*, *JHEP* **05** (2023) 151 [2302.09089].

[35] C. Armstrong, H. Goodhew, A. Lipstein and J. Mei, *Graviton trispectrum from gluons*, *JHEP* **08** (2023) 206 [2304.07206].

[36] L. Bordin and G. Cabass, *Graviton non-Gaussianities and Parity Violation in the EFT of Inflation*, *JCAP* **07** (2020) 014 [2004.00619].

[37] G. Cabass, *Zoology of Graviton non-Gaussianities*, 2103.09816.

[38] G. Cabass, E. Pajer, D. Stefanyszyn and J. Supeł, *Bootstrapping large graviton non-Gaussianities*, *JHEP* **05** (2022) 077 [2109.10189].

[39] A.H. Guth and S.H.H. Tye, *Phase Transitions and Magnetic Monopole Production in the Very Early Universe*, *Phys. Rev. Lett.* **44** (1980) 631.

[40] A.A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, *JETP Lett.* **30** (1979) 682.

[41] A.H. Guth, *Inflationary universe: A possible solution to the horizon and flatness problems*, *Physical Review D* **23** (1981) 347.

[42] A.D. Linde, *A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems*, *Physics Letters B* **108** (1982) 389.

[43] D. Baumann, *Inflation*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, pp. 523–686, 2011, DOI [0907.5424].

[44] D. Baumann, *Cosmology*, Cambridge University Press (7, 2022), 10.1017/9781108937092.

[45] N. Arkani-Hamed and J. Maldacena, *Cosmological Collider Physics*, 1503.08043.

[46] N. Arkani-Hamed, D. Baumann, H. Lee and G.L. Pimentel, *The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities*, *JHEP* **04** (2020) 105 [1811.00024].

[47] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee and G.L. Pimentel, *The cosmological bootstrap: weight-shifting operators and scalar seeds*, *JHEP* **12** (2020) 204 [1910.14051].

[48] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee and G.L. Pimentel, *The Cosmological Bootstrap: Spinning Correlators from Symmetries and Factorization*, 2005.04234.

[49] S. Jazayeri, E. Pajer and D. Stefanyszyn, *From Locality and Unitarity to Cosmological Correlators*, 2103.08649.

[50] H. Goodhew, S. Jazayeri and E. Pajer, *The Cosmological Optical Theorem*, 2009.02898.

[51] S. Céspedes, A.-C. Davis and S. Melville, *On the time evolution of cosmological correlators*, *JHEP* **02** (2021) 012 [2009.07874].

[52] H. Goodhew, S. Jazayeri, M.H. Gordon Lee and E. Pajer, *Cutting Cosmological Correlators*, 2104.06587.

[53] S. Jazayeri and S. Renaux-Petel, *Cosmological Bootstrap in Slow Motion*, 2205.10340.

[54] G.L. Pimentel and D.-G. Wang, *Boostless Cosmological Collider Bootstrap*, 2205.00013.

[55] D.-G. Wang, G.L. Pimentel and A. Achúcarro, *Bootstrapping multi-field inflation: non-Gaussianities from light scalars revisited*, *JCAP* **05** (2023) 043 [2212.14035].

[56] C. Duaso Pueyo and E. Pajer, *A cosmological bootstrap for resonant non-Gaussianity*, *JHEP* **03** (2024) 098 [2311.01395].

[57] T. Heckelbacher, I. Sachs, E. Skvortsov and P. Vanhove, *Analytical evaluation of cosmological correlation functions*, 2204.07217.

[58] C. Chowdhury, A. Lipstein, J. Mei, I. Sachs and P. Vanhove, *The Subtle Simplicity of Cosmological Correlators*, 2312.13803.

[59] D. Baumann, D. Green, A. Joyce, E. Pajer, G.L. Pimentel, C. Sleight et al., *Snowmass White Paper: The Cosmological Bootstrap*, in *2022 Snowmass Summer Study*, 3, 2022 [2203.08121].

[60] C. Cheung, *TASI Lectures on Scattering Amplitudes*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics : Anticipating the Next Discoveries in Particle Physics (TASI 2016): Boulder, CO, USA, June 6-July 1, 2016*, R. Essig and I. Low, eds., pp. 571–623 (2018), DOI [1708.03872].

[61] P. Benincasa and F. Cachazo, *Consistency Conditions on the S-Matrix of Massless Particles*, 0705.4305.

[62] H. Elvang and Y.-t. Huang, *Scattering amplitudes in gauge theory and gravity*, Cambridge University Press (2015).

[63] M.P. Hertzberg and J.A. Litterer, *Symmetries from Locality Part 1: Electromagnetism and Charge Conservation*, 2005.01731.

[64] M.P. Hertzberg, J.A. Litterer and M. Sandora, *Symmetries from Locality Part 2: Gravitation and Lorentz Boosts*, 2005.01744.

[65] N. Arkani-Hamed, L. Rodina and J. Trnka, *Locality and unitarity of scattering amplitudes from singularities and gauge invariance*, *Phys. Rev. Lett.* **120** (2018) 231602.

[66] D. Green and E. Pajer, *On the Symmetries of Cosmological Perturbations*, 2004.09587.

[67] S. Melville and E. Pajer, *Cosmological Cutting Rules*, *JHEP* **05** (2021) 249 [2103.09832].

[68] C. Cheung, P. Creminelli, A.L. Fitzpatrick, J. Kaplan and L. Senatore, *The Effective Field Theory of Inflation*, *JHEP* **03** (2008) 014 [0709.0293].

[69] S. Endlich, A. Nicolis and J. Wang, *Solid Inflation*, *JCAP* **10** (2013) 011 [1210.0569].

[70] Y. Minami and E. Komatsu, *New Extraction of the Cosmic Birefringence from the Planck 2018 Polarization Data*, *Phys. Rev. Lett.* **125** (2020) 221301 [2011.11254].

[71] J. Hou, Z. Slepian and R.N. Cahn, *Measurement of parity-odd modes in the large-scale 4-point correlation function of Sloan Digital Sky Survey Baryon Oscillation Spectroscopic Survey twelfth data release CMASS and LOWZ galaxies*, *Mon. Not. Roy. Astron. Soc.* **522** (2023) 5701 [2206.03625].

[72] O.H.E. Philcox, *Probing parity violation with the four-point correlation function of BOSS galaxies*, *Phys. Rev. D* **106** (2022) 063501 [2206.04227].

[73] N. Arkani-Hamed, P. Creminelli, S. Mukohyama and M. Zaldarriaga, *Ghost inflation*, *JCAP* **04** (2004) 001 [hep-th/0312100].

[74] G. Cabass, S. Jazayeri, E. Pajer and D. Stefanyszyn, *Parity violation in the scalar trispectrum: no-go theorems and yes-go examples*, *JHEP* **02** (2023) 021 [2210.02907].

[75] M.H.G. Lee, C. McCulloch and E. Pajer, *Leading loops in cosmological correlators*, *JHEP* **11** (2023) 038 [2305.11228].

[76] L. Senatore and M. Zaldarriaga, *On Loops in Inflation*, *JHEP* **12** (2010) 008 [0912.2734].

[77] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[78] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [hep-th/0603001].

[79] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, *JHEP* **08** (2006) 045 [hep-th/0605073].

[80] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [0806.1218].

[81] E. Witten, *Five-brane effective action in M-theory.*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234].

[82] E. Witten, *AdS/CFT correspondence and topological field theory.*, *JHEP* **12** (1998) 012 [hep-th/9812012].

[83] J.M. Maldacena and A. Strominger, *AdS(3) black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [hep-th/9804085].

[84] J. de Boer, *Six-dimensional supergravity on S**3 x AdS(3) and 2-D conformal field theory*, *Nucl. Phys. B* **548** (1999) 139 [hep-th/9806104].

[85] O. Lunin and S.D. Mathur, *Correlation functions for M**N / S(N) orbifolds*, *Commun. Math. Phys.* **219** (2001) 399 [hep-th/0006196].

[86] A. Strominger, *The dS / CFT correspondence*, *JHEP* **10** (2001) 034 [hep-th/0106113].

[87] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [astro-ph/0210603].

[88] T.S. Bunch and P.C.W. Davies, *Quantum field theory in de sitter space: Renormalization by point-splitting*, *Proc. Roy. Soc. Lond. A* **360** (1978) 117.

[89] G.L. Pimentel, *Inflationary Consistency Conditions from a Wavefunctional Perspective*, *JHEP* **02** (2014) 124 [1309.1793].

[90] S.P. de Alwis, *Wave function of the Universe and CMB fluctuations*, *Phys. Rev. D* **100** (2019) 043544 [1811.12892].

[91] M. Hogervorst, J.a. Penedones and K.S. Vaziri, *Towards the non-perturbative cosmological bootstrap*, 2107.13871.

[92] L. Di Pietro, V. Gorbenko and S. Komatsu, *Analyticity and Unitarity for Cosmological Correlators*, 2108.01695.

[93] K. Salehi Vaziri, *Nonperturbative Quantum Field Theory in Curved Spacetime*, Ph.D. thesis, Ecole Polytechnique, Lausanne, 2022. 10.5075/epfl-thesis-9263.

[94] J. Penedones, K. Salehi Vaziri and Z. Sun, *Hilbert space of quantum field theory in de Sitter spacetime*, *Phys. Rev. D* **111** (2025) 045001 [2301.04146].

[95] M. Loparco, J. Penedones, K. Salehi Vaziri and Z. Sun, *The Källén-Lehmann representation in de Sitter spacetime*, *JHEP* **12** (2023) 159 [2306.00090].

[96] M. Loparco, J. Qiao and Z. Sun, *A radial variable for de Sitter two-point functions*, 2310.15944.

[97] M. Loparco, *RG flows in de Sitter: C-functions and sum rules*, *SciPost Phys.* **17** (2024) 079 [2404.03739].

[98] M. Loparco, J. Penedones and Y. Ulrich, *What is a photon in de Sitter spacetime?*, 2505.00761.

[99] G.S. Ng and A. Strominger, *State/Operator Correspondence in Higher-Spin dS/CFT*, *Class. Quant. Grav.* **30** (2013) 104002 [1204.1057].

[100] C.-M. Chang, A. Pathak and A. Strominger, *Non-Minimal Higher-Spin DS4/CFT3*, 1309.7413.

[101] D. Anninos, R. Mahajan, D. Radičević and E. Shaghoulian, *Chern-Simons-Ghost Theories and de Sitter Space*, *JHEP* **01** (2015) 074 [1405.1424].

[102] D. Anninos, F. Denef, R. Monten and Z. Sun, *Higher Spin de Sitter Hilbert Space*, *JHEP* **10** (2019) 071 [1711.10037].

[103] L. Boyle, K. Finn and N. Turok, *The Big Bang, CPT, and neutrino dark matter*, *Annals Phys.* **438** (2022) 168767 [1803.08930].

[104] L. Boyle, K. Finn and N. Turok, *CPT-Symmetric Universe*, *Phys. Rev. Lett.* **121** (2018) 251301 [1803.08928].

[105] L. Boyle and N. Turok, *Two-Sheeted Universe, Analyticity and the Arrow of Time*, 2109.06204.

[106] N. Turok and L. Boyle, *Gravitational entropy and the flatness, homogeneity and isotropy puzzles*, *Phys. Lett. B* **849** (2024) 138443 [2201.07279].

[107] L. Boyle, M. Teuscher and N. Turok, *The Big Bang as a Mirror: a Solution of the Strong CP Problem*, 2208.10396.

[108] N. Turok and L. Boyle, *A Minimal Explanation of the Primordial Cosmological Perturbations*, 2302.00344.

[109] W.-N. Deng and W. Handley, *Predicting spatial curvature Ω_K in globally CPT-symmetric universes*, 2407.18225.

[110] A. Bzowski, P. McFadden and K. Skenderis, *Holographic predictions for cosmological 3-point functions*, *JHEP* **03** (2012) 091 [1112.1967].

[111] A. Bzowski, P. McFadden and K. Skenderis, *Holography for inflation using conformal perturbation theory*, *JHEP* **04** (2013) 047 [1211.4550].

[112] A. Bzowski, P. McFadden and K. Skenderis, *Implications of conformal invariance in momentum space*, *JHEP* **03** (2014) 111 [1304.7760].

[113] A. Bzowski, P. McFadden and K. Skenderis, *Conformal n-point functions in momentum space*, *Phys. Rev. Lett.* **124** (2020) 131602 [1910.10162].

[114] A. Bzowski, P. McFadden and K. Skenderis, *Renormalisation of IR divergences and holography in de Sitter*, *JHEP* **05** (2024) 053 [2312.17316].

[115] D. Anninos, T. Anous, D.Z. Freedman and G. Konstantinidis, *Late-time Structure of the Bunch-Davies De Sitter Wavefunction*, *JCAP* **11** (2015) 048 [1406.5490].

[116] G. Goon, K. Hinterbichler, A. Joyce and M. Trodden, *Shapes of gravity: Tensor non-Gaussianity and massive spin-2 fields*, *JHEP* **10** (2019) 182 [1812.07571].

[117] A. Abolhasani and H. Goodhew, *Derivative interactions during inflation: a systematic approach*, *JCAP* **06** (2022) 032 [2201.05117].

[118] D. Baumann and A. Joyce, "Lectures on cosmological correlations." https://github.com/ddbaumann/cosmo-correlators, 2023.

[119] S. Céspedes, A.-C. Davis and D.-G. Wang, *On the IR divergences in de Sitter space: loops, resummation and the semi-classical wavefunction*, *JHEP* **04** (2024) 004 [2311.17990].

[120] T. Anous and J. Skulte, *An invitation to the principal series*, *SciPost Phys.* **9** (2020) 028 [2007.04975].

[121] D. Anninos, T. Anous, B. Pethybridge and G. Şengör, *The discreet charm of the discrete series in dS_2*, *J. Phys. A* **57** (2024) 025401 [2307.15832].

[122] E. Joung, J. Mourad and R. Parentani, *Group theoretical approach to quantum fields in de Sitter space. I. The Principle series*, *JHEP* **08** (2006) 082 [hep-th/0606119].

[123] G. Sengör and C. Skordis, *Unitarity at the Late time Boundary of de Sitter*, *JHEP* **06** (2020) 041 [1912.09885].

[124] G. Sengor and C. Skordis, *Scalar two-point functions at the late-time boundary of de Sitter*, *JHEP* **02** (2024) 076 [2110.01635].

[125] G. Sengor and C. Skordis, *Principal and Complementary Series Representations at the Late-Time Boundary of de Sitter*, *Springer Proc. Math. Stat.* **396** (2022) 269 [2205.11550].

[126] G. Sengör, *The de Sitter group and its presence at the late-time boundary*, *PoS* **CORFU2021** (2022) 356 [2206.04719].

[127] G. Şengör, *Particles of a de Sitter Universe*, *Universe* **9** (2023) 59 [2212.10626].

[128] G. Şengör, *Searching for discrete series representations at the late-time boundary of de Sitter*, in *15th International Workshop on Lie Theory and Its Applications in Physics*, 12, 2023 [2312.00363].

[129] I. Dey, K.K. Nanda, A. Roy and S.P. Trivedi, *Aspects of dS/CFT Holography*, 2407.02417.

[130] D. Stefanyszyn, X. Tong and Y. Zhu, *Cosmological correlators through the looking glass: reality, parity, and factorisation*, *JHEP* **05** (2024) 196 [2309.07769].

[131] D. Stefanyszyn, X. Tong and Y. Zhu, *There and Back Again: Mapping and Factorising Cosmological Observables*, 2406.00099.

[132] T. Liu, X. Tong, Y. Wang and Z.-Z. Xianyu, *Probing P and CP Violations on the Cosmological Collider*, *JHEP* **04** (2020) 189 [1909.01819].

[133] H.E. Logan, *Lectures on perturbative unitarity and decoupling in Higgs physics*, 2207.01064.

[134] D.A. Galante, *Modave lectures on de Sitter space & holography*, *PoS* **Modave2022** (2023) 003 [2306.10141].

[135] J. Cotler, K. Jensen and A. Maloney, *Low-dimensional de Sitter quantum gravity*, *JHEP* **06** (2020) 048 [1905.03780].

[136] G. Araujo-Regado, A. Thavanesan and A.C. Wall, *Holographic Cosmology at Finite Time*, 250x.xxxxx.

[137] A. Thavanesan, *Going through phases of UV and IR divergences in Cosmology*, 25xx.xxxxx.

[138] F. Piazza and F. Vernizzi, *Effective Field Theory of Cosmological Perturbations*, *Class. Quant. Grav.* **30** (2013) 214007 [1307.4350].

[139] E. Gourgoulhon, *3+1 formalism and bases of numerical relativity*, gr-qc/0703035.

[140] P. Creminelli, J. Gleyzes, J. Noreña and F. Vernizzi, *Resilience of the standard predictions for primordial tensor modes*, *Phys. Rev. Lett.* **113** (2014) 231301 [1407.8439].

[141] L. Bordin, G. Cabass, P. Creminelli and F. Vernizzi, *Simplifying the EFT of Inflation: generalized disformal transformations and redundant couplings*, *JCAP* **09** (2017) 043 [1706.03758].

[142] J. Guven, B. Lieberman and C.T. Hill, *Schrodinger Picture Field Theory in Robertson-walker Flat Space-times*, *Phys. Rev. D* **39** (1989) 438.

[143] S. Jain, R.R. John, A. Mehta, A.A. Nizami and A. Suresh, *Higher spin 3-point functions in 3d CFT using spinor-helicity variables*, 2106.00016.

[144] D. Meltzer, *The Inflationary Wavefunction from Analyticity and Factorization*, 2107.10266.

[145] D. Green and R.A. Porto, *Signals of a Quantum Universe*, *Phys. Rev. Lett.* **124** (2020) 251302 [2001.09149].

[146] T. Grall, S. Jazayeri and D. Stefanyszyn, *The cosmological phonon: symmetries and amplitudes on sub-horizon scales*, *JHEP* **11** (2020) 097 [2005.12937].

[147] K. Hinterbichler, L. Hui and J. Khoury, *An Infinite Set of Ward Identities for Adiabatic Modes in Cosmology*, *JCAP* **01** (2014) 039 [1304.5527].

[148] D. Ghosh, K. Panchal and F. Ullah, *Mixed graviton and scalar bispectra in the EFT of inflation: Soft limits and Boostless Bootstrap*, *JHEP* **07** (2023) 233 [2303.16929].

[149] E. Pajer, D. Stefanyszyn and J. Supeł, *The Boostless Bootstrap: Amplitudes without Lorentz boosts*, *JHEP* **12** (2020) 198 [2007.00027].

[150] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*, Cambridge University Press (2005).

[151] D. Tong, "Standard model (part iii)." https://www.damtp.cam.ac.uk/user/tong/standardmodel.html, 2024.

[152] D. Harlow and T. Numasawa, *Gauging spacetime inversions in quantum gravity*, 2311.09978.

[153] E. Witten, *Bras and Kets in Euclidean Path Integrals*, 2503.12771.

[154] E. Wigner, *Ueber die operation der zeitumkehr in der quantenmechanik*, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1932** (1932) 546.

[155] R.F. Streater and A.S. Wightman, *PCT, spin and statistics, and all that* (1989).

[156] A.S. Blum and A. Martínez de Velasco, *The genesis of the cpt theorem*, *The European Physical Journal H* **47** (2022) 1.

[157] L. Susskind, *A Paradox and its Resolution Illustrate Principles of de Sitter Holography*, 2304.00589.

[158] K. Doi, N. Ogawa, K. Shinmyo, Y.-k. Suzuki and T. Takayanagi, *Probing de Sitter space using CFT states*, *JHEP* **02** (2025) 093 [2405.14237].

[159] F.J. Belinfante, *The heavy quanta theory of nuclear and cosmic ray phenomena*, in *Theory of Heavy Quanta: Proefschrift*, (Dordrecht), pp. 39–121, Springer Netherlands (1939), DOI.

[160] J.S. Schwinger, *The Theory of quantized fields. 1.*, *Phys. Rev.* **82** (1951) 914.

[161] G. Luders, *On the Equivalence of Invariance under Time Reversal and under Particle-Antiparticle Conjugation for Relativistic Field Theories*, *Kong. Dan. Vid. Sel. Mat. Fys. Med.* **28N5** (1954) 1.

[162] G. Luders, *Proof of the TCP theorem*, *Annals Phys.* **2** (1957) 1.

[163] G. Luders and B. Zumino, *Some Consequences of TCP-Invariance*, *Phys. Rev.* **106** (1957) 385.

[164] G. Luders and B. Zumino, *Connection between Spin and Statistics*, *Phys. Rev.* **110** (1958) 1450.

[165] W. Pauli, *Exclusion principle, lorentz group and reflection of space-time and charge*, in *Wolfgang Pauli: Das Gewissen der Physik*, C.P. Enz and K. v. Meyenn, eds., (Wiesbaden), pp. 459–479, Vieweg+Teubner Verlag (1988), DOI.

[166] A. Blum, *From the necessary to the possible: the genesis of the spin-statistics theorem*, *European Physical Journal H* **39** (2014) 543.

[167] C.M. Bender and S. Boettcher, *Real spectra in nonHermitian Hamiltonians having PT symmetry*, *Phys. Rev. Lett.* **80** (1998) 5243 [physics/9712001].

[168] C.M. Bender, S. Boettcher and P. Meisinger, *PT symmetric quantum mechanics*, *J. Math. Phys.* **40** (1999) 2201 [quant-ph/9809072].

[169] C.M. Bender, *$\mathcal{PT}$-symmetric quantum field theory*, *J. Phys. Conf. Ser.* **1586** (2020) 012004.

[170] C.M. Bender and D.W. Hook, *PT-symmetric quantum mechanics*, 2312.17386.

[171] L. Chen and S. Sarkar, *PT symmetric fermionic particle oscillations in even dimensional representations*, 2407.02036.

[172] W. Donnelly and A.C. Wall, *Unitarity of maxwell theory on curved spacetimes in the covariant formalism*, *Physical Review D* **87** (2013) .

[173] T. Hartman, *Lecture notes on classical de sitter space*, *arXiv preprint arXiv:1205.3855* (2017) .

[174] A. Thavanesan, *CPT in the Static Patch*, 25xx.xxxxx.

[175] K.S. Kumar and J.a. Marto, *Towards a unitary formulation of quantum field theory in curved spacetime I: the case of de Sitter spacetime*, 2305.06046.

[176] K.S. Kumar and J.a. Marto, *Towards a unitary formulation of quantum field theory in curved space-time II: the case of Schwarzschild black hole*, 2307.10345.

[177] Z. Sun, *A note on the representations of $SO(1, d+1)$*, 2111.04591.

[178] B. Pethybridge and V. Schaub, *Tensors and spinors in de Sitter space*, *JHEP* **06** (2022) 123 [2111.14899].

[179] V. Schaub, *Spinors in (Anti-)de Sitter Space*, *JHEP* **09** (2023) 142 [2302.08535].

[180] V. Schaub, *A Walk Through $Spin(1, d+1)$*, 2405.01659.

[181] V.A. Letsios, *The eigenmodes for spinor quantum field theory in global de Sitter space–time*, *J. Math. Phys.* **62** (2021) 032303 [2011.07875].

[182] V.A. Letsios, *(Non-)unitarity of strictly and partially massless fermions on de Sitter space II: an explanation based on the group-theoretic properties of the spin-3/2 and spin-5/2 eigenmodes*, *J. Phys. A* **57** (2024) 135401 [2206.09851].

[183] V.A. Letsios, *New conformal-like symmetry of strictly massless fermions in four-dimensional de Sitter space*, *JHEP* **05** (2024) 078 [2310.01702].

[184] V.A. Letsios, *(Non-)unitarity of strictly and partially massless fermions on de Sitter space*, *JHEP* **05** (2023) 015 [2303.00420].

[185] V.A. Letsios, *Unconventional conformal invariance of maximal depth partially massless fields on dS_4 and its relation to complex partially massless SUSY*, *JHEP* **08** (2024) 147 [2311.10060].

[186] C.M. Sou, X. Tong and Y. Wang, *Chemical-potential-assisted particle production in FRW spacetimes*, *JHEP* **06** (2021) 129 [2104.08772].

[187] X. Tong, Y. Wang, C. Zhang and Y. Zhu, *BCS in the sky: signatures of inflationary fermion condensation*, *JCAP* **04** (2024) 022 [2304.09428].

[188] C. Chowdhury, P. Chowdhury, R.N. Moga and K. Singh, *Loops, Recursions, and Soft Limits for Fermionic Correlators in (A)dS*, 2408.00074.

[189] D. Baumann, G. Mathys, G.L. Pimentel and F. Rost, *A New Twist on Spinning (A)dS Correlators*, 2408.02727.

[190] E. Pajer, "Field theory in cosmology (part iii)." https://www.damtp.cam.ac.uk/user/ep551/field_theory_in_Cosmology.pdf, 2024.

[191] T. Jacobson and D. Mattingly, *Gravity with a dynamical preferred frame*, *Phys. Rev. D* **64** (2001) 024028 [gr-qc/0007031].

[192] T. Jacobson, *Einstein-aether gravity: A Status report*, *PoS* **QG-PH** (2007) 020 [0801.1547].

[193] S. Caron-Huot, M. Giroux, H.S. Hannesdottir and S. Mizera, *Crossing beyond scattering amplitudes*, *JHEP* **04** (2024) 060 [2310.12199].

[194] D. Baumann and D. Green, *The power of locality: primordial non-Gaussianity at the map level*, *JCAP* **08** (2022) 061 [2112.14645].

[195] Y. Donath and E. Pajer, *The in-out formalism for in-in correlators*, *arXiv preprint arXiv:2402.05999* (2024) .

[196] M. Baumgart and R. Sundrum, *Manifestly Causal In-In Perturbation Theory about the Interacting Vacuum*, 2010.10785.

[197] S. Albayrak, P. Benincasa and C. Duaso Pueyo, *Perturbative unitarity and the wavefunction of the Universe*, *SciPost Phys.* **16** (2024) 157 [2305.19686].

[198] P. Benincasa and F. Vazão, *The Asymptotic Structure of Cosmological Integrals*, 2402.06558.

[199] Z. Bern and Y.-t. Huang, *Basics of Generalized Unitarity*, *J. Phys. A* **44** (2011) 454003 [1103.1869].

[200] L. Bordin, P. Creminelli, A. Khmelnitsky and L. Senatore, *Light Particles with Spin in Inflation*, *JCAP* **10** (2018) 013 [1806.10587].

[201] H. Goodhew, *Rational wavefunctions in de Sitter spacetime*, *JCAP* **03** (2023) 036 [2210.09977].

[202] S. Melville, D. Roest and D. Stefanyszyn, *UV Constraints on Massive Spinning Particles: Lessons from the Gravitino*, *JHEP* **02** (2020) 185 [1911.03126].

[203] C. Fevola, G.L. Pimentel, A.-L. Sattelberger and T. Westerdijk, *Algebraic Approaches to Cosmological Integrals*, 2410.14757.

[204] T. Westerdijk and C. Yang, *Bananas are Unparticles: Differential Equations and Cosmological Bootstrap*, 2503.08775.

[205] A. Thavanesan and T. Westerdijk, *Massless Cosmological Correlators from Shift Relations*, 25xx.xxxxx.

[206] P. Benincasa, G. Brunello, M.K. Mandal, P. Mastrolia and F. Vazão, *On one-loop corrections to the Bunch-Davies wavefunction of the universe*, 2408.16386.

[207] N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, H. Lee and G.L. Pimentel, *Differential Equations for Cosmological Correlators*, 2312.05303.

[208] N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, H. Lee and G.L. Pimentel, *Kinematic Flow and the Emergence of Time*, 2312.05300.

[209] S. Collier, L. Eberhardt, B. Mühlmann and V.A. Rodriguez, *The complex Liouville string*, 2409.17246.

[210] S. Collier, L. Eberhardt, B. Mühlmann and V.A. Rodriguez, *The complex Liouville string: the worldsheet*, 2409.18759.

[211] S. Collier, L. Eberhardt, B. Mühlmann and V.A. Rodriguez, *The complex Liouville string: the matrix integral*, 2410.07345.

[212] S. Collier, L. Eberhardt, B. Mühlmann and V.A. Rodriguez, *The complex Liouville string: worldsheet boundaries and non-perturbative effects*, 2410.09179.

[213] S. Collier, L. Eberhardt and B. Mühlmann, *A microscopic realization of dS_3*, *SciPost Phys.* **18** (2025) 131 [2501.01486].

[214] S. Collier, L. Eberhardt and B. Mühlmann, *The complex Liouville string: the gravitational path integral*, 2501.10265.

[215] D. Baumann, H. Goodhew and H. Lee, *Kinematic Flow for Cosmological Loop Integrands*, 2410.17994.

[216] D. Baumann, H. Goodhew, A. Joyce, H. Lee, G.L. Pimentel and T. Westerdijk, *Geometry of Kinematic Flow*, 2504.14890.

[217] E. Di Valentino, A. Melchiorri and J. Silk, *Planck evidence for a closed Universe and a possible crisis for cosmology*, *Nature Astron.* **4** (2019) 196 [1911.02087].

[218] W. Handley, *Curvature tension: evidence for a closed universe*, *Phys. Rev. D* **103** (2021) L041301 [1908.09139].

[219] W. Handley, *Primordial power spectra for curved inflating universes*, *Phys. Rev. D* **100** (2019) 123517 [1907.08524].

[220] G. Avis, S. Jazayeri, E. Pajer and J. Supeł, *Spatial Curvature at the Sound Horizon*, *JCAP* **02** (2020) 034 [1911.04454].

[221] A. Thavanesan, D. Werth and W. Handley, *Analytical approximations for curved primordial power spectra*, *Phys. Rev. D* **103** (2021) 023519 [2009.05573].

[222] Z. Shumaylov and W. Handley, *Primordial power spectra from k-inflation with curvature*, *Phys. Rev. D* **105** (2022) 123532 [2112.07547].

[223] M.I. Letey, Z. Shumaylov, F.J. Agocs, W.J. Handley, M.P. Hobson and A.N. Lasenby, *Quantum initial conditions for curved inflating universes*, *Phys. Rev. D* **109** (2024) 123502 [2211.17248].

[224] Q. Huang, K. Zhang, Z. Fang and F. Tu, *Analytical approximations for primordial power spectra in a spatially closed emergent universe*, *Phys. Dark Univ.* **38** (2022) 101124 [2203.06447].

[225] J. Bel, J. Larena, R. Maartens, C. Marinoni and L. Perenon, *Constraining spatial curvature with large-scale structure*, *JCAP* **09** (2022) 076 [2206.03059].

[226] B. Ratra, *Tilted spatially nonflat inflation*, *Phys. Rev. D* **106** (2022) 123524 [2211.02999].

[227] Q. Vigneron and V. Poulin, *Is expansion blind to the spatial curvature?*, *Phys. Rev. D* **108** (2023) 103518 [2212.00675].

[228] D.D. Dineen and W.J. Handley, *Analytic approximations for the primordial power spectrum with Israel junction conditions*, *Phys. Rev. D* **109** (2024) 083513 [2309.15984].

[229] Q. Vigneron and J. Larena, *A natural model for curved inflation*, 2405.04450.

[230] S. Melville and G.L. Pimentel, *A de Sitter S-matrix for the masses*, 2309.07092.

[231] S. Melville and G.L. Pimentel, *A de Sitter S-matrix from amputated cosmological correlators*, 2404.05712.

[232] G. Orlando, *Probing parity-odd bispectra with anisotropies of GW V modes*, *JCAP* **12** (2022) 019 [2206.14173].

[233] O.H.E. Philcox and M. Shiraishi, *Testing parity symmetry with the polarized cosmic microwave background*, *Phys. Rev. D* **109** (2024) 083514 [2308.03831].

[234] R.N. Cahn, Z. Slepian and J. Hou, *Test for Cosmological Parity Violation Using the 3D Distribution of Galaxies*, *Phys. Rev. Lett.* **130** (2023) 201002 [2110.12004].

[235] O.H.E. Philcox, M.J. König, S. Alexander and D.N. Spergel, *What can galaxy shapes tell us about physics beyond the standard model?*, *Phys. Rev. D* **109** (2024) 063541 [2309.08653].

[236] A. Krolewski, S. May, K. Smith and H. Hopkins, *No evidence for parity violation in BOSS*, *JCAP* **08** (2024) 044 [2407.03397].

[237] X. Niu, M.H. Rahat, K. Srinivasan and W. Xue, *Parity-odd and even trispectrum from axion inflation*, *JCAP* **05** (2023) 018 [2211.14324].

[238] C. Creque-Sarbinowski, S. Alexander, M. Kamionkowski and O. Philcox, *Parity-violating trispectrum from Chern-Simons gravity*, *JCAP* **11** (2023) 029 [2303.04815].

[239] S. Garcia-Saenz, Y. Lu and Z. Shuai, *Scalar-induced gravitational waves from ghost inflation and parity violation*, *Phys. Rev. D* **108** (2023) 123507 [2306.09052].

[240] W.R. Coulton, O.H.E. Philcox and F. Villaescusa-Navarro, *Signatures of a parity-violating universe*, *Phys. Rev. D* **109** (2024) 023531 [2306.11782].

[241] F. Zhang, J.-X. Feng and X. Gao, *Scalar induced gravitational waves in symmetric teleparallel gravity with a parity-violating term*, *Phys. Rev. D* **108** (2023) 063513 [2307.00330].

[242] S. Jazayeri, S. Renaux-Petel, X. Tong, D. Werth and Y. Zhu, *Parity violation from emergent nonlocality during inflation*, *Phys. Rev. D* **108** (2023) 123523 [2308.11315].

[243] T. Fujita, T. Murata, I. Obata and M. Shiraishi, *Parity-violating scalar trispectrum from a rolling axion during inflation*, *JCAP* **05** (2024) 127 [2310.03551].

[244] S. Akama and M. Zhu, *Parity violation in primordial tensor non-Gaussianities from matter bounce cosmology*, *JCAP* **07** (2024) 039 [2404.05464].

[245] K. Inomata, L. Jenks and M. Kamionkowski, *Parity-breaking galaxy 4-point function from lensing by chiral gravitational waves*, 2408.03994.

[246] T. Moretti, N. Bartolo and A. Greco, *Breaking Parity: the case of the Trispectrum from Chiral Scalar-Tensor Theories of Gravity*, 2410.11801.

[247] M. Reinhard, Z. Slepian, J. Hou and A. Greco, *Full Parity-Violating Trispectrum in Axion Inflation: Reduction to Low-D Integrals*, 2412.16037.

[248] W.H. Furry, *A Symmetry Theorem in the Positron Theory*, *Phys. Rev.* **51** (1937) 125.

[249] M.H.G. Lee, *From amplitudes to analytic wavefunctions*, *JHEP* **03** (2024) 058 [2310.01525].

[250] J.C. Ward, *An Identity in Quantum Electrodynamics*, *Phys. Rev.* **78** (1950) 182.

[251] Y. Takahashi, *On the generalized Ward identity*, *Nuovo Cim.* **6** (1957) 371.

[252] S. Deser, *Self-interaction and gauge invariance*, *Gen. Rel. Grav.* **1** (1970) 9 [gr-qc/0411023].

[253] A.A. Slavnov, *Ward Identities in Gauge Theories*, *Theor. Math. Phys.* **10** (1972) 99.

[254] J.C. Taylor, *Ward Identities and Charge Renormalization of the Yang-Mills Field*, *Nucl. Phys. B* **33** (1971) 436.

[255] G. 't Hooft and M.J.G. Veltman, *One loop divergencies in the theory of gravitation*, *Ann. Inst. H. Poincare A Phys. Theor.* **20** (1974) 69.

[256] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, UK (1982), 10.1017/CBO9780511622632.

[257] M.E. Peskin and D.V. Schroeder, *An Introduction to quantum field theory*, Addison-Wesley, Reading, USA (1995).

[258] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge University Press (8, 2013), 10.1017/CBO9781139644174.

[259] K. Freese, J.A. Frieman and A.V. Olinto, *Natural inflation with pseudo - Nambu-Goldstone bosons*, *Phys. Rev. Lett.* **65** (1990) 3233.

[260] E. Silverstein and A. Westphal, *Monodromy in the CMB: Gravity Waves and String Inflation*, *Phys. Rev. D* **78** (2008) 106003 [0803.3085].

[261] S. Cespedes and S. Jazayeri, *The Massive Flat Space Limit of Cosmological Correlators*, 2501.02119.

[262] D. Ghosh, E. Pajer and F. Ullah, *Cosmological cutting rules for Bogoliubov initial states*, *SciPost Phys.* **18** (2025) 005 [2407.06258].

[263] N. Arkani-Hamed, H.-C. Cheng, M.A. Luty and S. Mukohyama, *Ghost condensation and a consistent infrared modification of gravity*, *JHEP* **05** (2004) 074 [hep-th/0312099].

[264] G. Cabass, M.M. Ivanov and O.H.E. Philcox, *Colliders and ghosts: Constraining inflation with the parity-odd galaxy four-point function*, *Phys. Rev. D* **107** (2023) 023523 [2210.16320].

[265] A. Thavanesan and A. Wall, *Holographic Duals to Strings on Time-dependent Backgrounds*, 25xx.xxxxx.

[266] S.B. Edgar and A. Hoglund, *Dimensionally dependent tensor identities by double antisymmetrization*, *J. Math. Phys.* **43** (2002) 659 [gr-qc/0105066].