Corrigendum to “Efficiency and stability under substitutable priorities with ties” [J. Econ. Theory 184 (2019) 104950]*

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Abstract

We identify an error in Proposition 3 by Erdil and Kumano (2019). We then recover the result by replacing the substitutability condition on priority structures with a new condition which we call *bridging*. We further show that the priorities in both applications in Erdil and Kumano (2019) satisfy the bridging property, which in turn ensures that their Propositions 5 and 6 are still valid.

1 Introduction

Partly motivated by diversity goals in school choice, Erdil and Kumano (2019) introduce a model where schools compare alternative cohorts of student intake according to some distributional metric (possibly a measure of diversity). Importantly, to maintain generality and flexibility, they allow such comparisons to have ties between many alternatives, and show how *substitutable priorities with ties* can capture these comparisons (i.e., priority rankings over sets of students). They introduce a modified deferred acceptance

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procedure (MDA) to show that a *stable* (i.e., one that *respects priorities*) assignment exists.

In general, there are multiple stable assignments, and some assignments might serve students better than other assignments in the sense of Pareto dominance. A stable assignment is called *constrained efficient* if no other stable assignment Pareto dominates it (from the students' perspective). Noting that the outcome of MDA is not necessarily constrained efficient, they explore the possibility of recovering constrained efficiency by using a construction which they call *potentially stable improvement cycle (PSIC).* They claim, in their Proposition 3, that given acceptant and substitutable school priorities, if a stable assignment does not admit a PSIC, then it is constrained efficient. They then rely on this sufficiency claim to establish an algorithm to compute a constrained efficient assignment in two distinct classes of applications; namely *assignment with distributional constraints* and *admissions by a committee.*

We illustrate, by way of counterexample, that their Proposition 3 is not correct. Since they rely on this proposition to establish the relevant results for their applications (Propositions 5 and 6, and Corollaries 1 and 2), failure of the proposition leads to a gap in the proofs of these latter results.

To complete that gap and recover these results, we introduce a new technical condition on priorities: the *bridging property.* This allows us to revise the statement of Proposition 3 as follows. Given acceptant priorities which satisfy the bridging property, if a stable assignment does not admit a PSIC, then it is constrained efficient. We then show that priorities in both classes of applications satisfy the bridging property. Hence, we verify that Propositions 5 and 6, and Corollaries 1 and 2 in Erdil and Kumano (2019) are still valid. We also show that bridging is a necessary condition for Proposition 3 to hold: if bridging is not satisfied for at least one school's priorities, then it is possible for a stable assignment to be constrained inefficient even though it does not admit a PSIC.

2 Model

Let N and X be a set of students and a set of schools, respectively. There are q_x seats at school x, for $x \in X$. Each student can be assigned to at most one school. Students have strict preferences over the set of schools and being unassigned. Formally, student i's preferences are denoted by a linear order R_i over $X \cup \{i\}$, where i stands for staying unassigned. P_i denotes the strict part of R_i.

Each school x is endowed with its own priorities over subsets of the set of students, which are captured by an admission rule, $\mathcal{A}_x : 2^N \to 2^{2^N}$. Given a school x facing a set S of applicants, each $S' \in \mathcal{A}_x(S)$ is considered to be of highest priority among all possible subsets of S, and is called an **admissible subset** of S. A **priority structure** $\mathcal{A}$ is a vector of admission rules: $(\mathcal{A}_x)_{x \in X}$. Given an admission rule $\mathcal{A}_x$, we define its **rejection rule** $\mathcal{R}_x : 2^N \to 2^{2^N}$ as follows,

$$\mathcal{R}_x(S) = \{S'' \subseteq S \mid S'' = S \setminus S' \text{ for some } S' \in \mathcal{A}_x(S)\}.$$

Each $S'' \in \mathcal{R}_x(S)$ is called a **rejectable subset of** S.

$\mathcal{A}_x$ is **admission monotonic (AM)** if given any $S, T \subseteq N$ with $S \subseteq T$ and $T' \in \mathcal{A}_x(T)$, we have $T' \cap S \subseteq S'$ for some $S' \in \mathcal{A}_x(S)$. $\mathcal{A}_x$ is **rejection monotonic (RM)** if given any $S, T \subseteq N$ with $S \subseteq T$ and $S'' \in \mathcal{R}_x(S)$, we have $S'' \subseteq T''$ for some $T'' \in \mathcal{R}_x(T)$. An admission rule is called **substitutable** if it is both admission monotonic and rejection monotonic. Moreover, we say a priority structure $\mathcal{A}$ is substitutable if $\mathcal{A}_x$ is substitutable for all $x \in X$. $\mathcal{A}$ is called **acceptant** if for each $x \in X$, for each $S \subseteq N$, and for each $S' \in \mathcal{A}_x(S)$, we have $|S'| = \min\{|S|, q_x\}$.

An **assignment** is a function $\mu : N \to X \cup N$ such that $\mu(i) \in X \cup \{i\}$ for all $i \in N$ and $|\mu^{-1}(x)| \leq q_x$ for all $x \in X$. We say μ is **stable** (i.e., **respects priorities**) if

- $\forall i \in N$, $\mu(i) R_i i$, and
- $\forall x \in X$ and $\forall S \subseteq N$ such that $\mu^{-1}(x) \subseteq S \subseteq \{i \in N \mid x R_i \mu(i)\}$, we have $\mu^{-1}(x) \in \mathcal{A}_x(S)$.

An assignment μ' Pareto dominates another assignment μ if $\mu'(i) R_i \mu(i)$ for all $i \in N$, and $\mu'(j) P_j \mu(j)$ for some $j \in N$. An assignment is **constrained efficient** if it is stable and is not Pareto dominated by any other stable assignment.

Erdil and Kumano (2019) show, via the modified deferred acceptance process (MDA), that if priority structure $\mathcal{A}$ is substitutable, then there exists a stable assignment. MDA is in fact a family of algorithms, because it does not specify how choices are made (i.e., how ties are broken) when $\mathcal{A}_x$ is not singleton valued. Depending on tie-breaking, the outcome of the MDA might be constrained inefficient. Their Proposition 3 claimed to identify a sufficiency condition on a given stable assignment to verify its constrained efficiency. We discuss this claim next.

3 Counterexample

Define the set of students who envy j at assignment μ as:

$$\mathcal{E}_j^\mu \quad = \quad \{i \in N \mid \mu(j) P_i \mu(i)\}.$$

Define the set of students who can replace student j at μ as

$$E_j^\mu \quad = \quad \left\{ i \in \mathcal{E}_j^\mu \;\middle|\; \{i\} \cup [\mu^{-1}(\mu(j)) \setminus \{j\}] \in \mathcal{A}_{\mu(j)}(\mathcal{E}_j^\mu \cup [\mu^{-1}(\mu(j)) \setminus \{j\}]) \right\}.$$

A **potentially-stable improvement cycle (PSIC)** consists of $n \geq 2$ distinct students enumerated as $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$ so that $i_\ell \in E^\mu_{i_{\ell+1}}$ for all $\ell = 0, \ldots, n-1$.

Erdil and Kumano (2019) claimed the following.

Proposition 3 (Original) *Given an acceptant and substitutable priority structure, if a stable assignment does not admit a PSIC, then it is constrained efficient.*

We show by way of example that the above statement is not correct.

Counterexample. Let $N = \{1, 2, \ldots, 5\}$ and $X = \{x, y, z\}$. Let

R_1	R_2	R_3	R_4	R_5
z	z	x	y	z
x	y	z	z	x
y	x	y	x	y

School capacities are given by $q_x = q_y = 1$ and $q_z = 2$.

The admission rules for schools x and y specify every student as admissible. That is, given any set $S \neq \varnothing$ of applicants, $\mathcal{A}_x(S) = \mathcal{A}_y(S) = \{\ \{i\} \mid i \in S\}$. It is automatic to verify that both $\mathcal{A}_x$ and $\mathcal{A}_y$ are acceptant and substitutable.

The admission rule for z admits all applicants if there are no more than two applicants, that is, $\mathcal{A}_z(S) = \{S\}$ if $|S| \leq 2$. When there are more than two applicants, all admissible subsets are of size two. Moreover $\mathcal{A}_z$ satisfies the following: given any S with $|S| \geq 3$, a subset S' with $|S'| = 2$ is not admissible if and only if

(i) $1 \in S'$ and $2 \in S \setminus S'$, or

(ii) $3 \in S'$, $4 \notin S'$ and $5 \in S \setminus S'$.

$\mathcal{A}_z$ is clearly acceptant. We will now verify that it satisfies both AM and RM, and thus conclude that it is substitutable.

[AM] If $|S| \leq 2$, then $\{S\} \in \mathcal{A}_z(S)$ and thus $T' \cap S \subseteq S \in \mathcal{A}_z(S)$ is trivial. Hence, we assume $|S| \geq 3$. Since $T' \in \mathcal{A}_z(T)$, $|T'| \leq 2$ and thus $|T' \cap S| \leq 2$. First, if $|T' \cap S| = 0$, then $T' \cap S = \emptyset \subseteq S' \in \mathcal{A}_z(S)$. Second, suppose $|T' \cap S| = 1$. Let $\{i\} = T' \cap S$. By (i) and (ii), if $i = 2$, 4 or 5, then $\{i, j\} \in \mathcal{A}_z(S)$ for any $j \in S \setminus \{i\}$. Then, first, suppose $i = 1$. By (i), if $2 \in S$, then $\{1, 2\} \in \mathcal{A}_z(S)$, and otherwise, then $\{1, j\} \in \mathcal{A}_z(S)$ for any $j \in S \setminus \{1\}$. Next, suppose $i = 3$. By (ii), if $5 \in S$, then $\{3, 5\} \in \mathcal{A}_z(S)$, and otherwise, then $\{3, j\} \in \mathcal{A}_z(S)$ for any $j \in S \setminus \{3\}$. Thus, in this case, $T' \cap S = \{i\} \subseteq S'$ for some $S' \in \mathcal{A}_z(S)$. Third, suppose $|T' \cap S| = 2$. Then, $|T'| = 2$ and thus $T' \cap S = T'$. Therefore, showing $T' \in \mathcal{A}_z(S)$ is sufficient. Since $T' \in \mathcal{A}_z(T)$ and (i) is satisfied, $1 \notin T'$ or $2 \notin T \setminus T'$ is satisfied. Further, since $T' \in \mathcal{A}_z(T)$ and (ii) is satisfied, $3 \notin T'$, or $4 \in T'$ or $5 \notin T \setminus T'$ is satisfied. Then, since $S \subseteq T$, $2 \notin T \setminus T'$ and $5 \notin T \setminus T'$ imply $2 \notin S \setminus T'$ and $5 \notin S \setminus T'$, respectively. Thus, we have both $1 \notin T'$ or $2 \notin S \setminus T'$ and $3 \notin T'$, or $4 \in T'$ or $5 \notin S \setminus T'$ are satisfied. Therefore, $T' \in \mathcal{A}_z(S)$.

[RM] If $|S| \leq 2$, then $\mathcal{R}_z(S) = \emptyset$. Moreover, if $|S| = |T|$, then $S = T$. Thus, we assume $3 \leq |S| < |T|$. Suppose $|S| = 3$ and $|T| = 4$. In this case, $|S''| = 1$ for any $S'' \in \mathcal{R}_z(S)$. By (i) and (ii), if there is $T''' \notin \mathcal{R}_z(T)$, then $2 \in T'''$ or $5 \in T'''$. First, if $2 \in T'''$, then $1 \in T$. In this case, since $|T| = 4$, $T \setminus \{1, 2\} \in \mathcal{A}_z(T)$. Thus, $\{1, 2\} \in \mathcal{R}_z(T)$. Second, if $5 \in T'''$, then $3 \in T$. In this case, since $|T| = 4$, $T \setminus \{3, 5\} \in \mathcal{A}_z(T)$ and thus $\{3, 5\} \in \mathcal{R}_z(T)$. Hence, if $|S| = 3$ and $|T| = 4$, then for any $S'' \in \mathcal{R}_z(S)$, there is $T'' \in \mathcal{R}_z(T)$ satisfying $S'' \subseteq T''$. Finally, suppose $|S| = 4$ and $|T| = 5$. Then, for all $S'' \in \mathcal{R}_z(S)$ and all $T'' \in \mathcal{R}_z(T)$, $|S''| = 2$ and $|T''| = 3$. As stated earlier, $T''' \notin \mathcal{R}_z(T)$ implies $2 \in T'''$ or $5 \in T'''$. Moreover, since $\{1, 2\}$, $\{3, 4\}$ and $\{3, 5\}$ are in $\mathcal{A}_z(T)$, for any $S'' \subseteq S$ with $|S''| = 2$, there is some $T'' \in \mathcal{A}_z(T)$ such that $S'' \subseteq T''$.

Now consider the following two assignments:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & y & z & z \end{pmatrix} \qquad \nu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ z & z & x & y \end{pmatrix}$$

They are both stable, and ν Pareto dominates μ. Therefore μ is not a constrained efficient assignment.

However, note that in any PSIC for μ, 1 cannot be involved, because $1 \notin E_3^\mu \cup E_4^\mu$ by $2 \in \{1, 2, 3, 5\} \cap \{1, 2, 4, 5\}$. Therefore, since 3 can only improve his assignment by receiving $x = \mu(1)$, she cannot be involved in a PSIC either. Finally, since $\{2,3\} \notin \mathcal{A}_z(\{1,2,3,5\})$, we have $2 \notin E_4^{\mu}$, and therefore 2 and 4 cannot form a PSIC. Hence, μ does not admit a PSIC, contradicting the conclusion of Proposition 3. $\Diamond$

Let us now briefly illustrate where the proof in Erdil and Kumano (2019) fails. Suppose μ and ν are two stable assignments such that ν Pareto dominates μ. Fix some $i \in N$ such that $\mu(i) \neq \nu(i)$ and let $\mu(i) = x$. Let

$$D'_x = \{j \in N \mid xP_j\mu(j) \text{ and } \mu(j) \neq \nu(j)\}.$$

Erdil and Kumano (2019) state that there is $i' \in D'_x$ such that

$$\{i'\} \cup \left(\mu^{-1}(x) \setminus \{i\}\right) \in \mathcal{A}_x(\{j \in N \mid xR_j\mu(j)\} \setminus \{i\}).$$

This is not correct. In fact, in the example above, $\mu(4) \neq \nu(4) = z$, $D'_z = \{1,2\}$, $\{j \in N | zR_j\mu(j)\} \setminus \{4\} = \{1,2,3,5\}$, and $\mu^{-1}(z) \setminus \{4\} = \{3\}$. However, from condition (ii) of the description of $\mathcal{A}_z$, neither $\{1\} \cup \{3\}$ nor $\{2\} \cup \{3\}$ is included in $\mathcal{A}_z(\{1,2,3,5\})$.

4 Corrected results

Definition 1. *We say $\mathcal{A}_x$ satisfies* ***bridging*** *if the following holds. If $T \subseteq S$ with $|T| \geq q_x$, $A \in \mathcal{A}_x(S)$, $B \in \mathcal{A}_x(T)$ and $(T \cap A) \subseteq B$, then for each $i \in A \setminus B$, there exists $i' \in (B \setminus A) \cup ((S \setminus T) \setminus A)$ such that*

$$(A \setminus \{i\}) \cup \{i'\} \quad \in \quad \mathcal{A}_x(S \setminus \{i\}).$$

Now we state the corrected result.

Proposition 3 (Corrected) *Given an acceptant priority structure $\mathcal{A}$, assume $\mathcal{A}_x$ satisfies bridging for each x. If a stable assignment does not admit a PSIC, then it is constrained efficient.*

Proof. We will show that if both μ and ν are stable, and if ν Pareto dominates μ, then μ must admit a PSIC. From this, it will follow that if μ does not admit a PSIC, then it must be constrained efficient.

Setting $N' = \{j \in N \mid \mu(j) \neq \nu(j)\}$, let $i \in N'$ and $\mu(i) = x$.

Let

$$D_x^{\mu} = \{j \in N \mid xP_j\mu(j)\} \qquad \text{and} \qquad D_x^{\nu} = \{j \in N \mid xP_j\nu(j)\},$$

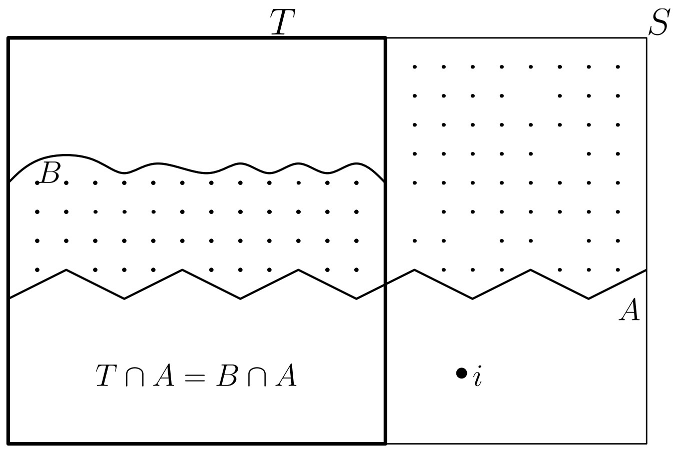

Figure 1: Given $i \in A \setminus B$, there must exist $i' \in (B \setminus A) \cup ((S \setminus T) \setminus A)$ such that $(A \setminus \{i\}) \cup \{i'\} \in \mathcal{A}_x(S \setminus \{i\})$. That is, such i' must be in the dotted area above.

and

$$D'_x = \{j \in N' \mid xP_j\mu(j)\} \qquad \text{and} \qquad D''_x = \{j \in N \setminus N' \mid xP_j\mu(j)\}.$$

Lastly, denoting by $\sqcup$ the disjoint union of sets, set

$$\bar{D}_x \;=\; D'_x \sqcup D''_x \sqcup \mu^{-1}(x) \;=\; D^\mu_x \sqcup \mu^{-1}(x).$$

In order to make it clear how we use the bridging property, let us use the notation of the definition and set

$$S = \bar{D}_x \qquad T = D^\nu_x \sqcup \nu^{-1}(x) \qquad A = \mu^{-1}(x) \qquad B = \nu^{-1}(x).$$

For the bridging property to apply, we need to verify the relevant conditions on the sets S, T, A and B.

First, we will show that $|T| \geq q_x$. Since $\mathcal{A}$ is acceptant and μ is stable, μ must be non-wasteful, that is, if there exist a student j and school y such that $yP_j\mu(j)$, then $|\mu^{-1}(y)| = q_y$. Hence the *reshuffling lemma* (Erdil, 2014) implies that any Pareto improvement over μ can be expressed as a set of students reshuffling their seats assigned under μ. Hence, $|\mu^{-1}(x)| = |\nu^{-1}(x)|$. Moreover, since $i \in N'$, she is part of an improvement, and therefore she must receive someone else's seat, whereas her seat must be reassigned to another student in N'. In other words we have $\mu(i) \in \nu(N')$. In particular, there exists a student in N' who prefers $\mu(i)$ to their match under μ. Since μ is non-wasteful, we must have $|\mu^{-1}(x)| = q_x$, and therefore $|\nu^{-1}(x)| = q_x$. Thus, we conclude that $|T| = |D^\nu_x \sqcup \nu^{-1}(x)| \geq q_x$.

Second, ν Pareto dominates μ, so those students who prefer x to their match under ν, also do so under μ. Therefore $D_x^\nu = \{j \in N \mid xP_j\nu(j)\} \subseteq D_x^\mu$. Moreover, if $j \in \nu^{-1}(x)$, then either $j \in \mu^{-1}(x)$ or $j \in D'_x$. And finally, since $\mu(i) = x$ and $i \in N'$, we know that $i \notin \nu^{-1}(x)$, and $\nu(i)P_ix$. Therefore $i \notin D_x^\nu$. Thus $D_x^\nu \sqcup \nu^{-1}(x) \subseteq D_x^\mu \cup \nu^{-1}(x) \subseteq \bar{D}_x$, which verifies $T \subseteq S$.

Third, stability of μ implies $\mu^{-1}(x) \in \mathcal{A}_x(\bar{D}_x)$, which verifies $A \in \mathcal{A}_x(S)$.

Fourth, stability of ν implies $\nu^{-1}(x) \in \mathcal{A}_x(D_x^\nu \sqcup \nu^{-1}(x))$, which verifies $B \in \mathcal{A}_x(T)$.

Finally, $(T \cap A) \subseteq B$, for otherwise, there is a j in $T \cap A = (D_x^\nu \sqcup \nu^{-1}(x)) \cap \mu^{-1}(x)$, but not in $B = \nu^{-1}(x)$. That means $\mu(j) = xP_j\nu(j)$, which contradicts with ν Pareto dominating μ.

Now, since $i \in \mu^{-1}(x) \setminus \nu^{-1}(x)$, i.e., $i \in A \setminus B$, the bridging property implies there must exist

$$i' \quad \in \quad (B \backslash A) \cup ((S \backslash T) \backslash A)$$

such that

$$(\mu^{-1}(x) \setminus \{i\}) \cup \{i'\} \quad \in \quad \mathcal{A}_x(S \setminus \{i\}).$$

Since $S \setminus \{i\} = \mathcal{E}_i^\mu \cup (\mu^{-1}(x) \setminus \{i\})$ and $i' \in \mathcal{E}_i^\mu$, we conclude that $i' \in E_i^\mu$.

Note that $B \setminus A = \{j \mid \nu(j) = xP_j\mu(j)\} \subseteq D'_x$. Also observe that

$$D''_x = \{j \mid \mu(j) = \nu(j) \text{ and } xP_j\mu(j)\} \quad \subseteq \quad D_x^\nu \cup \nu^{-1}(x) \cup \mu^{-1}(x) = T \cup A$$

and therefore $(S \setminus T) \setminus A = S \setminus (T \cup A) \subseteq S \setminus D''_x = D'_x \sqcup \mu^{-1}(x)$.

Thus,

$$(B \backslash A) \cup ((S \backslash T) \backslash A) \quad \subseteq \quad D'_x$$

which, in turn, implies $i' \in D'_x$, and in particular $i' \in N'$.

Now construct a directed graph with N' being its set of vertices. For every $i \in N'$, the above argument shows that there exists $i' \in N'$ such that $i' \in E_i^\mu$, so draw a directed edge $i' \to i$. Since this is a finite graph with every vertex having an incoming edge, there must be a cycle. By construction, this is a PSIC. $\square$

The following remark shows that the bridging property is also necessary for the conclusion of Proposition 3 to remain valid.

Remark. *Suppose $\mathcal{A}_x$ is acceptant, but violates the bridging property. Then there exists a problem where there exists a stable matching which does not admit a PSIC even though it is not constrained efficient, and all schools but x have strict responsive priorities.*

Proof. $\mathcal{A}_x$ violating the bridging property means there exists $T \subseteq S$ with $|T| \geq q_x$, $A \in \mathcal{A}_x(S)$, $B \in \mathcal{A}_x(T)$, $(T \cap A) \subseteq B$, and $i \in A \setminus B$ such that for all $i' \in (B \setminus A) \cup ((S \setminus T) \setminus A)$, we have

$$(A \setminus \{i\}) \cup \{i'\} \notin \mathcal{A}_x(S \setminus \{i\}).$$

Let $(S \setminus T) \setminus A = \{1, 2, \ldots, n\}$. Now consider $n+1$ schools $y, z_1, \ldots, z_n$ such that $q_y = |A \setminus B|$, and $q_{z_k} = 1$ for all $k \in \{1, \ldots, n\}$. Let these schools have strict responsive priorities, where their rankings over individual students satisfy:

$$\begin{array}{lllll} j & \succ_y & i & \succ_y & j' \quad\quad \text{for all } j \in B \setminus A, \text{and for all } j' \in A \setminus \{i\} \\ 1 & \succ_{z_1} & i & \succ_{z_1} & n \\ k & \succ_{z_k} & k-1 & \succ_{z_k} & j' \quad \text{for all } k \in \{2, \ldots, n\} \end{array}$$

Setting $z_{n+1} = z_1$, let students' preferences be:

	$j \in A \setminus (B \cup \{i\})$	$j \in A \cap B$	$j \in B \setminus A$	$j \in T \setminus B$	$k \in (S \setminus T) \setminus A$
P_i	P_j	P_j	P_j	P_j	P_k
y	y	x	x	x	z_{k+1}
z_1	x	j	y	j	x
x	j		j		z_k
i					k

Now consider the following two matchings μ and ν:

	i	$A \setminus (B \cup \{i\})$	$A \cap B$	$B \setminus A$	$T \setminus B$	$k \in (S \setminus T) \setminus A$
μ	x	x	x	y	$\varnothing$	z_k
ν	y	y	x	x	$\varnothing$	z_{k+1}

Note that both μ and ν are stable, and ν Pareto dominates μ. Therefore μ is not constrained efficient.

To explore whether μ admits a PSIC or not, we need to identify which agent can unilaterally replace which other agent without violating stability.

First note that $\mathcal{E}_i^\mu = (B \setminus A) \cup (T \setminus B) \cup ((S \setminus T) \setminus A)$ and $\mathcal{E}_i^\mu \cup \mu^{-1}(x) = S$. By definition, $E_i^\mu \subseteq \mathcal{E}_i^\mu$.

We know $\mathcal{A}_x$ is acceptant, and we have assumed for all $i' \in (B \setminus A) \cup ((S \setminus T) \setminus A)$ that we have

$$(A \setminus \{i\}) \cup \{i'\} \notin \mathcal{A}_x(S \setminus \{i\}).$$

Therefore $E_i^\mu \subseteq T \setminus B$.

We also have, for all $j \in B \setminus A$, $E^{\mu}_j = \{i\}$, $E^{\mu}_1 = \{i\}$, and for all $k \in \{2, \dots, n\}$, $E^{\mu}_k = \{k-1\}$. This exhausts all possibilities as to who can replace whom, and no cycle can therefore be formed. Hence, μ does not admit a PSIC. $\square$

4.1 The two applications in Erdil and Kumano (2019)

Erdil and Kumano (2019) discuss two classes of applications after Proposition 3, and in these settings, characterize constrained efficiency of an assignment by its not admitting any PSIC (Propositions 5 and 6). However, the "if" part of the proofs of these two statements relies on the conclusion of the original Proposition 3. Below, we recover their Propositions 5 and 6 by showing that priority structures in both applications satisfy the bridging property.

4.1.1 Assignment with distributional constraints

Suppose students are sorted into different types, and for each type, each school reserves some of its seats for students of that type. When some of those reserves cannot be filled with students of the intended types, admission choice evenly distributes the remaining seats as evenly as possible across types of remaining students. Formally, let $\mathcal{T} = \{\tau_1, \dots, \tau_m\}$ be the set of types, and $\tau : N \to \mathcal{T}$ be a function such that $\tau(i)$ indicates student i's type. School x has q_x seats and its own vector of reserves $\boldsymbol{q}_x = (q_x^{\tau_1}, \dots, q_x^{\tau_m})$, where $\sum_{k=1}^{m} q_x^{\tau_k} \leq q_x$.

Given a school x with a reserve vector $\boldsymbol{q}_x$, we define the distance $\mathrm{ed}(\boldsymbol{q}_x, S)$ between $\boldsymbol{q}_x$ and set S of students as

$$\mathrm{ed}(\boldsymbol{q}_x, S) \;=\; \sum_{\tau \in \mathcal{T}} (|S_\tau| - q_x^\tau)^2.$$

We say an admission rule $\mathcal{A}_x$ **evenly distributes (ED)** surplus seats if there exists a reflexive and transitive order $\succsim_x$ on subsets of N which is consistent with $\mathcal{A}_x$ and satisfies the following property: for every S, S' such that $|S| = |S'| = q_x$,

$$\mathrm{ed}(\boldsymbol{q}_x, S) \;\leq\; \mathrm{ed}(\boldsymbol{q}_x, S') \quad \Leftrightarrow \quad S \succsim_x S'.$$

Each school x has its own exogenous priority ranking $\succsim_x^{exo}$. We say that $\mathcal{A}_x$ is **ED-constrained responsive (EDCR)** to $\succsim^{exo}$ if there exists a reflexive and transitive order $\succsim_x$ on subsets of N which is consistent with $\mathcal{A}_x$ and satisfies:

(i) whenever $|S| = |S'| = q_x$, we have $\text{ed}(\boldsymbol{q}_x, S) < \text{ed}(\boldsymbol{q}_x, S') \;\Rightarrow\; S \succ_x S'$, and

(ii) whenever $|S \cup \{s'\}| = |S \cup \{s''\}| = q_x$ and $\text{ed}(\boldsymbol{q}_x, S \cup \{s'\}) = \text{ed}(\boldsymbol{q}_x, S \cup \{s''\})$,

$$S \cup \{s'\} \succsim_x S \cup \{s''\} \quad \Leftrightarrow \quad s' \succsim_x^{exo} s''.$$

Claim 1. *If an acceptant admission rule $\mathcal{A}_x$ satisfies EDCR, then it also satisfies bridging.*

Proof. Suppose $T \subseteq S$ with $|T| \geq q_x$, $A \in \mathcal{A}_x(S)$ and $B \in \mathcal{A}_x(T)$. Let $i \in A \setminus B$. We need to show that there exists $i' \in (B \setminus A) \cup ((S \setminus T) \setminus A)$ such that $(A \setminus \{i\}) \cup \{i'\} \in \mathcal{A}_x(S \setminus \{i\})$.

Proposition 4 of Erdil and Kumano (2019) implies that $\mathcal{A}_x$ is substitutable. Since $\mathcal{A}_x$ is also acceptant, we know that there exists $j \in S \setminus A$ such that

$$(A \setminus \{i\}) \cup \{j\} \in \mathcal{A}_x(S \setminus \{i\})$$

If such a j is not in $T \setminus (A \cup B)$, then it is necessarily in $(B \setminus A) \cup ((S \setminus T) \setminus A)$, and hence we are done.

Now, suppose $j \in T \setminus (A \cup B)$ and $(A \setminus \{i\}) \cup \{j\} \in \mathcal{A}_x(S \setminus \{i\})$.

<u>*Case 1.*</u> There exists $i' \in B \setminus A$ such that $\tau(j) = \tau(i')$ or there is no over-subscribed type in B.

If there is no over-subscribed type in B, B's distance from the target is zero, because $\mathcal{A}_x$ is acceptant, $|T| \geq q_x$, and $B \in \mathcal{A}_x(T)$. So must be the distance of $(A \setminus \{i\}) \cup \{j\}$ and A since they are both chosen from larger sets. Then, $|B_{\tau(j)}| = |((A \setminus \{i\}) \cup \{j\})_{\tau(j)}| = |A_{\tau(j)}| = q_x^{\tau(j)}$. So $\tau(i) = \tau(j)$, and since $i \in A \setminus B$, there must be an $i' \in B \setminus A$ with $\tau(i') = \tau(j)$.

In either case, we must have $i' \succsim_x^{exo} j$, since otherwise we would have $(B \setminus \{i'\}) \cup \{j\} \succ_x B$, a contradiction with $B \in \mathcal{A}_x(T)$ (because $(B \setminus \{i'\}) \cup \{j\} \subseteq T$.)

Now, $i' \succsim_x^{exo} j$ and $\tau(j) = \tau(i')$ imply $(A \setminus \{i\}) \cup \{i'\} \succsim_x (A \setminus \{i\}) \cup \{j\}$, and we are done.

<u>*Case 2.*</u> There is no $i' \in B \setminus A$ such that $\tau(i') = \tau(j)$ and there must be over-subscribed types in B.

We will first establish four facts.

1. $\tau(j)$ surplus in B is $b \geq 0$, and $\tau(j)$ surplus in $(A \setminus \{i\}) \cup \{j\}$ is at least $b + 1$.

Since there is no $i' \in B \setminus A$ with $\tau(i') = \tau(j)$, all type-$\tau(j)$ elements of B are also in A. All elements of $A \cap B$ are automatically in A. And since $i \notin B$, all type-$\tau(j)$ elements of B are also in $A \setminus \{i\}$.

In B, type-$\tau(j)$ must be weakly over-subscribed, since otherwise replacing an over-subscribed type student $j' \in B \subseteq T$ with j would yield $(B \setminus \{j'\}) \cup \{j\} \succ_x B$, contradicting with $B \in \mathcal{A}_x(T)$. So let type-$\tau(j)$ surplus in B be $b \geq 0$.

Type-$\tau(j)$ surplus in $A \setminus \{i\}$ is at least b because all type-$\tau(j)$ elements of B are in $A \setminus \{i\}$. Thus type-$\tau(j)$ surplus in $(A \setminus \{i\}) \cup \{j\}$ is at least $b+1$.

2. If there is no $i' \in B \setminus A$ such that $(A \setminus \{i\}) \cup \{i'\} \in \mathcal{A}_x(S \setminus \{i\})$, then over-subscribed types in $B \setminus A$ with surplus $b+1$ in B have surplus $b+1$ in $A \setminus \{i\}$, and remaining over-subscribed types in $B \setminus A$ have surplus b both in B and in $A \setminus \{i\}$.

   For every type in B, the surplus is less than or equal to $b+1$, since otherwise we could replace a student of type with surplus of $b+2$ or more with j (whose type is of surplus b in B) to get a q_x-element set in T higher ranked than B.

   Therefore, for every type in A, the surplus is less than or equal to $b+1$. (Since a surplus vector with entries less than or equal to $b+1$ can be achieved by B when choosing from a smaller set T.)

   Each type in $B \setminus A$ must be of surplus b or $b+1$ in $A \setminus \{i\}$, since otherwise we can take a student j' in $B \setminus A$ whose type is of surplus less than or equal to $b-1$ in $A \setminus \{i\}$, and replace j in $(A \setminus \{i\}) \cup \{j\}$ with j' to get $(A \setminus \{i\}) \cup \{j'\} \succ_x (A \setminus \{i\}) \cup \{j\}$ due to type $\tau(j)$ being of surplus at least $b+1$ in $(A \setminus \{i\}) \cup \{j\}$.

   If there is a type in $B \setminus A$ of surplus $b+1$ in B, then all such students have weakly higher $\succsim^{exo}$-priority than j, since otherwise we could replace one of those students with lower $\succsim^{exo}$-priority with j to keep the surplus vector equidistant from the target, but achieve a higher $\succsim^{exo}$-priority cohort. So, now, for every j' in $B \setminus A$ so that type $\tau(j')$-surplus in B is $b+1$, we must have $j' \succsim_x^{exo} j$.

   If type $\tau(j')$-surplus in $A \setminus \{i\}$ is less than or equal to b, we can replace j in $(A \setminus \{i\}) \cup \{j\}$ with j', and we'd get $(A \setminus \{i\}) \cup \{j'\} \succsim_x (A \setminus \{i\}) \cup \{j\} \in \mathcal{A}_x(S \setminus \{i\})$. And we'd be done.

3. For those types in B but not in $B \setminus A$, all such students of B are in $B \cap A$, and hence they are necessarily in $A \setminus \{i\}$. In particular, for all such types τ, we have $|B_\tau| \leq |(A \setminus \{i\})_\tau|$.

4. Deficit of each type in B (if any) is weakly less than the deficit of that type in $(A\setminus\{i\})\cup\{j\}$ (if any) since $(A\setminus\{i\})\cup\{j\}$ is a chosen cohort from a larger set, i.e., for those types with $|B_\tau| \leq q_x^\tau$, we must have $|B_\tau| \leq \min\{|((A\setminus\{i\})\cup\{j\})_\tau|, q_x^\tau\}$.

Since $|B| = |(A\setminus\{i\})\cup\{j\}|$, the above facts 1–4 imply that there must be a type in $B\setminus A$ for which the type-surplus in $(A\setminus\{i\})\cup\{j\}$ is less than b.

Taking a student j' of one such type in $B\setminus A$ and replacing j in $(A\setminus\{i\})\cup\{j\}$ with j', we change the surplus distribution to reduce the distance from the target, and hence $(A\setminus\{i\})\cup\{j'\} \succ_x (A\setminus\{i\})\cup\{j\}$, contradicting $(A\setminus\{i\})\cup\{j\} \in \mathcal{A}_x(A\setminus\{i\})$. ☐

Thus we recover Proposition 5 and Corollary 1 of Erdil and Kumano (2019) via the corrected Proposition 3.[1]

4.1.2 Admissions by a committee

Suppose a school has q seats to fill. Let H be its set of referees, each of whom has a linear order over the set of students. A function $\pi : \{1,\dots,q\} \to H$ determines the order in which the referees will take turns to make admission decisions one student at a time. The admission correspondence $\mathcal{A}^H$ is constructed in the following fashion:

$$\mathcal{A}^H(S) \quad = \quad \{\mathcal{A}^\pi(S) \;\mid\; \pi \in \Pi\;\}$$

where Π is the set of all functions $\pi : \{1,\dots,q\} \to H$, and each $\mathcal{A}^\pi$ is such that for any given $S \subseteq N$ with $|S| > q$,

- let i_1 be the highest ranked student in S according to $\pi(1)$, and
- for each $k = 2,\dots,q$, let i_k be the highest ranked student in $S\setminus\{i_1,\dots,i_{k-1}\}$ according to $\pi(k)$,

and $\mathcal{A}^\pi(S) = \{i_1,\dots,i_q\}$.

Claim 2. *$\mathcal{A}^H$ satisfies bridging.*

Proof. Given $T \subseteq S$ with $|T| \geq q$, let $A \in \mathcal{A}(S)$ and $B \in \mathcal{A}(T)$. Suppose $i \in A\setminus B$ and $k \in B\setminus A$.

[1] In a model of assignment with distributional constraints, Kitahara and Okumura (2020) generalize Proposition 5 to a more general setting.

Let π and π' be the sequences of referees such that

$$A = \mathcal{A}^{\pi}(S) \quad \text{and} \quad B = \mathcal{A}^{\pi'}(T).$$

Suppose i is selected by referee $\pi(\ell)$. Define another sequence of refeeres π'' so that

$$\pi''(1) = \pi(1),\ \ldots,\ \pi''(\ell-1) = \pi(\ell-1), \quad \pi''(\ell) = \pi(\ell+1),\ \ldots,\ \pi''(q-1) = \pi(q),$$

and $\pi''(q) = h$, where h is the first referee in π' who chooses student k from T.

The construction of π'' ensures that $A \setminus \{i\} \subseteq \mathcal{A}^{\pi''}(S \setminus \{i\})$.

By acceptance, there exists $j \in S \setminus A$ such that

$$(A \setminus \{i\}) \cup \{j\} = \mathcal{A}^{\pi''}(S \setminus \{i\}).$$

j is picked by referee h and since k is in $S \setminus \{i\}$, not in $A \setminus \{i\}$, it must be that h ranks j at least as high as k.

In order to verify the bridging property, it suffices to show that $j \notin (S \setminus A) \cap (T \setminus B)$. Suppose, for a contradiction, that $j \in (S \setminus A) \cap (T \setminus B)$. That is suppose $j \in T \setminus (A \cup B)$. We know that referee h ranks k above j, because in the sequence of referees π'' picking from T, k is picked by referee h, whereas j is not picked at all despite being in T. That contradicts with h ranking j at least as high as k. $\square$

Now we recover Proposition 6 and Corollary 2 of Erdil and Kumano (2019) via the corrected Proposition 3.

References

[1] Erdil, Aytek. 2014. "Strategy-proof stochastic assignment." *Journal of Economic Theory* 151: 146–162.

[2] Erdil, Aytek, and Taro Kumano. 2019. "Efficiency and stability under substitutable priorities with ties." *Journal of Economic Theory* 184: 104950.

[3] Kitahara, Minoru and Yasunori Okumura. 2020. "Stable improvement cycles in a controlled school choice." Working paper