Zero Mach Number Limit of the Compressible Primitive Equations: Ill-prepared Initial Data

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Abstract

In the work, we consider the zero Mach number limit of compressible primitive equations in the domain $\mathbb{R}^2 \times 2\mathbb{T}$ or $\mathbb{T}^2 \times 2\mathbb{T}$. We identify the limit equations to be the primitive equations with the incompressible condition. The convergence behaviors are studied in both $\mathbb{R}^2 \times 2\mathbb{T}$ and $\mathbb{T}^2 \times 2\mathbb{T}$, respectively. This paper takes into account the high oscillating acoustic waves and is an extension of our previous work in [29].

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Contents

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1 Introduction

1.1 Zero Mach number limit

As an example of multi-scale analysis, the zero Mach number limit problem for compressible flows has been an important classical problem in the study of hydrodynamic equations. Pioneered by Klainerman and Majda in [19,20], it is shown that the solutions of inviscid compressible Euler equations converge to that of inviscid incompressible Euler equations for the isentropic flows in $\mathbb{R}^d$ and $\mathbb{T}^d$, $d \in \mathbb{Z}^+$. The initial data are well-prepared and almost incompressible. The result is further studied in bounded domains for non-isentropic flows by Schochet in [38, 39]. In [43], the author establishes the zero Mach number limit with general (ill-prepared) initial data in $\mathbb{R}^d$. As pointed out in [33], in comparison to the case when the initial data are well-prepared, the time derivatives are no longer uniformly bounded with respect to the Mach number when the system is complemented with ill-prepared initial data. This leads to high frequency acoustic waves with large amplitude.

We have studied the zero Mach number limit of compressible primitive equations with well-prepared initial data in [29]. In this work, we are considering such a singular limit problem with ill-prepared initial data.

To present the general idea of singular limit problems with ill-prepared initial data, consider the following equation of the unknown function G,

$$\partial_t G + \frac{1}{\varepsilon} L(\partial_x) G = \mathcal{N}(G), \tag{1.1}$$

where $\varepsilon \in (0, \infty)$ is the singular parameter, $L(\partial_x)$ is an anti-symmetric differential operator with constant coefficients, and $\mathcal{N}(G)$ is the nonlinearity of G. Then one can perform the H^s estimate for some large $s \in \mathbb{Z}^+$, on system (1.1). Since $L(\partial_x)$ is anti-symmetric, $\| G \|_{H^s}$ is uniformly bounded with respect to ε, at least locally in time. If one considers the projection of

(1.1) in the kernel of operator $L(\partial_x)$, which is called the non-singular part of the equation, it follows that

$$\partial_t P_{\ker(L(\partial_x))} G = P_{\ker(L(\partial_x))} \mathcal{N}(G)$$

is uniformly bounded in $H^{s'}$ space for some $s' \in \mathbb{Z}^+ \cup \{0\}$ with respect to ε. On the other hand, $G - P_{\ker(L(\partial_x))} G$ satisfies

$$\begin{aligned} \partial_t(G - P_{\ker(L(\partial_x))} G) &+ \frac{1}{\varepsilon} L(\partial_x)(G - P_{\ker(L(\partial_x))} G) \\ &= \mathcal{N}(G) - P_{\ker(L(\partial_x))} \mathcal{N}(G), \end{aligned} \tag{1.2}$$

which has wave packet solutions with fast oscillations as $\varepsilon \to 0^+$. Consequently, $G - P_{\ker(L(\partial_x))} G$ is oscillating at high frequency. Moreover, as $\varepsilon \to 0^+$, the H^s norm preserving property of the anti-symmetric operator $L(\partial_x)$ implies that the amplitude of the oscillations does not vanish in general. Instead, the limit of the oscillatory part is driven by certain PDEs. To resolve the singular limit problem of (1.1) as $\varepsilon \to 0^+$, one has to study the interactions of the non-singular part and the oscillatory part in the nonlinearity $\mathcal{N}(G)$ in order to identify the limit equations. Such a method of studying the singular limit problem of (1.1) is developed by Schochet in [37] for hyperbolic PDEs with applications to the incompressible limit problem of Euler equations, nonlinear wave equations and the theory of weakly nonlinear geometric optics. Later, this method is further developed for some parabolic equations by Gallagher in [16]. We remark here that, if equation (1.1) is complemented with well-prepared initial data, the amplitude of oscillations is small, and consequently, the interaction between the non-singular part and the oscillatory part is much weaker.

In the study of zero Mach number limit of hydrodynamic equations for isentropic flows, if one writes the equations in the form of (1.1), the corresponding kernel of $L(\partial_x)$ consists the solenoidal velocity field. The equations corresponding to the fast oscillation equations (1.2) are referred to as the acoustic wave equations. Using these terminologies, the theorem developped by Ukai in [43] is basically showing that the acoustic waves decay to zero as $\varepsilon \to 0^+$ in $\mathbb{R}^d$. Such a fact is the consequence of the Strichartz estimates for linear wave equations, as pointed out by Desjardins and Grenier in [10] (see, e.g., [17, 18, 25] about the Strichartz estimate). In $\mathbb{T}^d$, Lions and Masmoudi study the resonance of the high frequency osciallations and show that in the sense of distribution, the solutions to the compressible Navier–Stokes equations converge to that to the incompressible Navier–Stokes equations as the

Mach number goes to zero in [27]. Later in [31], Masmoudi studies the incompressible, inviscid limit, with low Mach number and large Reynolds number, of compressible Navier–Stokes equations and identities the equations in both $\mathbb{R}^d$ and $\mathbb{T}^d$. We refer readers, for further developments, to [6–8, 40].

Additionally, we would like to mention that when considering the non-isentropic flows, the corresponding operator $L(\partial_x)$ in (1.1) has coefficients depending on space and time variables. This causes non-trivial difficulties to study the resonances between the oscillations (see, e.g., [1–3, 33, 34]). Moreover, when taking into account the stratification effect of gravity, the compressible Navier–Stokes equations with gravity may converge to the Oberbeck-Boussinesq equations or the anelastic equations, depending on the strength of the gravity effect. We refer readers, for more discussions of related topics, to [4, 9, 11–15, 32, 35, 36, 44]. Also, for more multi-scale analysis, we refer readers to [21–24, 30].

1.2 The compressible primitive equations

As mentioned before, we aim at studying the low Mach number limit of the compressible primitive equations. As discussed in our previous work [29], this is part of the justification of the PE diagram (see Figure 1 in [29]). We refer readers, for more background of the compressible primitive equations, to [29].

Let $\varepsilon \in (0, \infty)$ denote the Mach number, and let $\rho_\varepsilon \in \mathbb{R}, v_\varepsilon \in \mathbb{R}^2, w_\varepsilon \in \mathbb{R}$ represent the density, the horizontal and the vertical velocities, respectively. Then the compressible primitive equations can be written as, after rescaling the original CPE, similar to that of the compressible Navier–Stokes equations (see, e.g., [14]):

$$\begin{cases} \partial_t \rho_\varepsilon + div_h(\rho_\varepsilon v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon) = 0 & \text{in } \Omega_h \times (0,1), \\ \partial_t(\rho_\varepsilon v_\varepsilon) + div_h(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon v_\varepsilon) & \\ \qquad\qquad + \dfrac{1}{\varepsilon^2}\nabla_h P(\rho_\varepsilon) = div_h \mathbb{S}(v_\varepsilon) + \partial_{zz} v_\varepsilon & \text{in } \Omega_h \times (0,1), \\ \partial_z P(\rho_\varepsilon) = 0 & \text{in } \Omega_h \times (0,1). \end{cases} \tag{CPE}$$

where $P(\rho_\varepsilon) = \rho_\varepsilon^\gamma$, $\mathbb{S}(v_\varepsilon) = \mu(\nabla_h v_\varepsilon + \nabla_h v_\varepsilon^\top) + (\lambda - \mu) div_h v_\varepsilon \mathbb{I}_2$ represent the pressure for isentropic flows and the viscous stress tensor for Newtonian flows, respectively. Here, we assume that $\mu, \lambda > 0$ and $\gamma > 1$. We consider $\Omega_h = \mathbb{R}^2$ or $\mathbb{T}^2$ in this paper, where $\mathbb{T}^2$ represents the periodic domain with period 1 in both directions in $\mathbb{R}^2$. (CPE) is complemented with the stress-free and non-permeable boundary conditions:

$$\partial_z v_\varepsilon\big|_{z=0,1} = 0,\ w_\varepsilon\big|_{z=0,1} = 0. \tag{BC-CPE}$$

Hereafter, we have and will use ∇_h, div_h and Δ_h to represent the horizontal gradient, the horizontal divergence, and the horizontal Laplace operator, respectively; that is,

$$\nabla_h := \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},\ div_h := \nabla_h \cdot,\ \Delta_h := div_h \nabla_h \cdot .$$

Notice, (CPE) with (BC-CPE) is invariant with respect to the following symmetry:

$$v_\varepsilon \text{ and } w_\varepsilon \text{ are even and odd, respectively, in the } z\text{-variable.} \tag{SYM}$$

Owing to such symmetry, in order to study the limit system of (CPE) as $\varepsilon \to 0^+$, it suffices to consider the following system:

$$\begin{cases} \partial_t \rho_\varepsilon + div_h(\rho_\varepsilon v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t(\rho_\varepsilon v_\varepsilon) + div_h(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon v_\varepsilon) & \\ \qquad + \dfrac{1}{\varepsilon^2}\nabla_h P(\rho_\varepsilon) = div_h \mathbb{S}(v_\varepsilon) + \partial_{zz} v_\varepsilon & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z P(\rho_\varepsilon) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \tag{1.3}$$

subject to the periodic boundary condition and symmetry (SYM). Here $2\mathbb{T}$ is the periodic domain with period 2 in $\mathbb{R}$.

We remark that the restrictions of solutions to (1.3) in $\Omega_h \times [0,1]$ solve (CPE) with (BC-CPE), provided that the solutions exist and are regular enough.

We can rewrite $(1.3)_2$ as, provided that $\rho_\varepsilon > 0$,

$$\begin{aligned} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \frac{1}{\varepsilon^2}\nabla_h\Big(\frac{\gamma}{\gamma-1}\rho_\varepsilon^{\gamma-1}\Big) = \frac{\mu}{\rho_\varepsilon}\Delta_h v_\varepsilon \\ + \frac{\lambda}{\rho_\varepsilon}\nabla_h div_h v_\varepsilon + \frac{1}{\rho_\varepsilon}\partial_{zz} v_\varepsilon. \end{aligned}$$

Also from the continuity equation $(1.3)_1$, one can derive

$$\partial_t \rho_\varepsilon^{\gamma-1} + v_\varepsilon \cdot \nabla_h \rho_\varepsilon^{\gamma-1} + (\gamma-1)\rho_\varepsilon^{\gamma-1} div_h v_\varepsilon + (\gamma-1)\rho_\varepsilon^{\gamma-1}\partial_z w_\varepsilon = 0.$$

We consider $\rho_\varepsilon^{\gamma-1}$ with perturbations around the constant state given by $\rho_\varepsilon^{\gamma-1} = \bar\rho^{\gamma-1} \in (0,\infty)$. Then we define the perturbation variable ξ_ε by

$$\xi_\varepsilon := \frac{1}{\varepsilon}\log\Big(\frac{\gamma}{\gamma-1}\frac{\rho_\varepsilon^{\gamma-1}}{c^2}\Big),$$

with $c^2 := \dfrac{\gamma}{\gamma-1}\bar{\rho}^{\gamma-1}$. Let $\alpha := \dfrac{1}{\gamma-1}$, $c_1 := \big(\dfrac{\gamma}{(\gamma-1)c^2}\big)^{\alpha}$. With such notations, $\rho_\varepsilon = c_1^{-1}\exp(\varepsilon\alpha\xi_\varepsilon)$. Without loss of generality, we take $\bar{\rho} \equiv 1$. Hence (1.3) can be written as

$$\begin{cases} \partial_t \xi_\varepsilon + v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \dfrac{\gamma-1}{\varepsilon}(div_h v_\varepsilon + \partial_z w_\varepsilon) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \dfrac{c^2 e^{\varepsilon\xi_\varepsilon}}{\varepsilon}\nabla_h \xi_\varepsilon & \\ \qquad = c_1 e^{-\varepsilon\alpha\xi_\varepsilon}\big(\mu\Delta_h v_\varepsilon + \lambda\nabla_h div_h v_\varepsilon + \partial_{zz} v_\varepsilon\big) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi_\varepsilon = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \tag{1.4}$$

complemented with periodic initial data with symmetry (SYM). In fact, (1.4) is already in the form of (1.1). In this work, we study the asymptotic limit of (1.4) as $\varepsilon \to 0^+$. In fact, we will demonstrate that the existence time of strong solutions to (1.4) has a uniform lower bound, independent of $\varepsilon \in (0, \varepsilon_0)$ for some small $\varepsilon_0 \in (0,1)$. In addition, in the sense of distribution, the limit equations of (1.4), as $\varepsilon \to 0^+$, are the primitive equations (2.9) with the incompressible condition, below. Also, the associated acoustic wave equations for the three dimensional system (1.4) are only two dimensional (see (4.20), below). Consequently, we are able to adopt the strategy of studying the acoustic wave equations for the compressible Navier–Stokes equations in $\mathbb{R}^2$ and $\mathbb{T}^2$ to investigate the oscillatory part of the equations. This is done in section 4, below.

The rest of this work is organized as follows. In the next section, we will introduce some notations as well as some function spaces. Also, the main theorems in this work are stated in both $\Omega_h = \mathbb{R}^2$ and $\Omega_h = \mathbb{T}^2$. Next, in section 3, we establish the uniform local well-posedness of strong solutions to system (1.4). This is done with uniform *a priori* estimates in section 3.1, a local existence theorem in section 3.2 and a continuity argument in section 3.3. In section 4, we first identity the primitive equations, i.e., (2.9), below, as the limit of system (1.4) as $\varepsilon \to 0^+$ in the sense of distribution. Then in section 4.1, we argue that the acoustic waves decay to zero in the case when $\Omega_h = \mathbb{R}^2$; in section 4.2, we study the oscillation equations and identify the limit equations of oscillations in the case when $\Omega_h = \mathbb{T}^2$. Consequently, we conclude the compactness in both cases as stated in the main theorems, below.

2 Preliminaries and main theorems

In this work, we denote by

$$\overline{f}(x,y) := \int_0^1 f(x,y,z')\,dz', \text{ and } \widetilde{f}(x,y,z) := f(x,y,z) - \overline{f}(x,y),$$

as the average and the fluctuation of any function $f = f(x,y,z)$ over the z-variable. We use ∂_h to denote the horizontal derivatives, i.e., $\partial_h \in \{\partial_x, \partial_y\}$. ∂_t and ∂_z denote the time derivative and the vertical derivative, respectively. For any function f, $\partial_g f$ is sometimes denoted as f_g, for $g \in \{t,x,y,z,h\}$. Similar notations are also adopted for higher order derivatives.

As in [26], we introduce the following function spaces. Denote by

$$C_\sigma^\infty := \{u \in C_0^\infty(\Omega_h \times 2\mathbb{T};\mathbb{R}^2) | \int_0^1 div_h u\,dz = 0\},$$
$$C_\tau^\infty := \{u \in C_0^\infty(\Omega_h \times 2\mathbb{T};\mathbb{R}^2) | u = \nabla_h \psi, \text{ for } \psi \in C^\infty(\Omega_h;\mathbb{R}^2)\}.$$

Then with respect to the L^2-inner product, $C_\sigma^\infty \perp C_\tau^\infty$ and $C_0^\infty = C_\sigma^\infty \cup C_\tau^\infty$. We denote the closures in the L^2 norm of $C_\sigma^\infty, C_\tau^\infty$ as $L_\sigma^2 = L_\sigma^2(\Omega_h \times 2\mathbb{T};\mathbb{R}^2), L_\tau^2 := L_\tau^2(\Omega_h \times 2\mathbb{T};\mathbb{R}^2)$, respectively. $\mathcal{D}' = \mathcal{D}'_\sigma \cup \mathcal{D}'_\tau, H^s = H^s_\sigma \cup H^s_\tau$ are the spaces of test functions and Sobolev functions in $\Omega_h \times 2\mathbb{T}$, defined similarly for $s \in \mathbb{Z}^+$. We define the projection operators $\mathcal{P}_\sigma, \mathcal{P}_\tau$ in the following. Let $u \in C_0^\infty(\Omega_h \times 2\mathbb{T};\mathbb{R}^2)$. Consider the elliptic problem

$$\Delta_h \psi_u = \int_0^1 div_h u\,dz,$$
$$\begin{cases} \lim\limits_{|(x,y)|\to\infty} \psi_u = 0 & \text{in the case when } \Omega_h = \mathbb{R}^2, \\ \int_{\Omega_h} \psi_u\,dxdy = 0 & \text{in the case when } \Omega_h = \mathbb{T}^2. \end{cases} \tag{2.1}$$

Let

$$\mathcal{P}_\tau u := \nabla_h \psi_u, \quad \mathcal{P}_\sigma u := u - \nabla_h \psi_u. \tag{2.2}$$

Then $\mathcal{P}_\sigma, \mathcal{P}_\tau$ are the projection operators from C_0^∞ to $C_\sigma^\infty, C_\tau^\infty$, respectively. Also, by a density argument, $\mathcal{P}_\sigma, \mathcal{P}_\tau$ can act on functions in L^2, H^s, $s \in (0,\infty)$. In particular, the standard elliptic estimates yield, provided that $u \in H^s(\Omega_h \times 2\mathbb{T};\mathbb{R}^2)$, $s \in (0,\infty)$,

$$|\psi_u|_{H^{s+1}} \lesssim |\overline{u}|_{H^s} \lesssim \|u\|_{H^s}.$$

Thus $\mathcal{P}_\sigma u \in H^s_\sigma, \mathcal{P}_\tau u \in H^s_\tau$, $u = \mathcal{P}_\sigma u + \mathcal{P}_\tau u$ and

$$\|\mathcal{P}_\sigma u\|_{H^s} \lesssim \|u\|_{H^s}, \quad \|\mathcal{P}_\tau u\|_{H^s} \lesssim \|u\|_{H^s}. \tag{2.3}$$

System (1.4) is complemented with initial data $(\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T};\mathbb{R}) \times H^2(\Omega_h \times 2\mathbb{T};\mathbb{R}^2)$ with the compatibility conditions:

$$\partial_z \xi_0 = 0; \ v_0 \text{ is even in the } z\text{-variable}. \tag{2.4}$$

Also, in the case when $\Omega_h = \mathbb{R}^2$, the far field condition

$$\lim_{|(x,y)|\to\infty} |v_\varepsilon| = 0, \tag{2.5}$$

is also imposed. We denote the constant $M \in (0,\infty)$ satisfying

$$\frac{1}{2} \| v_0 \|_{H^2}^2 + \frac{c^2}{\gamma - 1} \| \xi_0 \|_{H^2}^2 < M, \tag{2.6}$$

to be the bound of initial data (ξ_0, v_0). Now we describe our main theorems in this work. The first theorem is stating that the H^2 norms of the solutions to (1.4) are uniformly bounded with respect to ε, provided that ε is small enough.

Theorem 2.1. *With initial data $(\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T};\mathbb{R}) \times H^2(\Omega_h \times 2\mathbb{T};\mathbb{R}^2)$ satisfying the compatibility conditions in* (2.4)*, let M be the upper bound of the initial data, i.e.,* (2.6)*. Then there are positive constants $\varepsilon_1 \in (0,1), T \in (0,\infty)$ depending only M, such that for any $\varepsilon \in (0, \varepsilon_1)$, there is a unique strong solution $(\xi_\varepsilon, v_\varepsilon)$ to system* (1.4) *in the time interval $[0,T]$. $(\xi_\varepsilon, v_\varepsilon)$ satisfies*

$$\begin{gathered} \xi_\varepsilon \in L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T})), \ \partial_t \xi_\varepsilon \in L^\infty(0,T;H^1(\Omega_h \times 2\mathbb{T})), \\ v_\varepsilon \in L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T})) \cap L^2(0,T;H^3(\Omega_h \times 2\mathbb{T})), \\ \partial_t v_\varepsilon \in L^\infty(0,T;L^2(\Omega_h \times 2\mathbb{T})) \cap L^2(0,T;H^1(\Omega_h \times 2\mathbb{T})). \end{gathered} \tag{2.7}$$

In addition, there exist positive constants $M_0, M_1, M_2 \in (0,\infty)$ independent of ε such that

$$\begin{gathered} \| \xi_\varepsilon \|_{L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T}))}^2 < M_0, \ \| v_\varepsilon \|_{L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T}))}^2 < M_1, \\ \| \nabla v_\varepsilon \|_{L^2(0,T;H^2(\Omega_h \times 2\mathbb{T}))}^2 < M_2. \end{gathered} \tag{2.8}$$

Next, as $\varepsilon \to 0^+$, formally, we expect the solutions v_ε obtained in Theorem 2.1 will converge to a solution to the following primitive equations:

$$\begin{cases} div_h v_p + \partial_z w_p = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t v_p + v_p \cdot \nabla_h v_p + w_p \partial_z v_p + \nabla_h P & \\ \qquad = c_1(\mu \Delta_h v_p + \lambda \nabla_h div_h v_p + \partial_{zz} v_p) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z P = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \tag{2.9}$$

complemented with initial data $v_p|_{t=0} = v_{p,0} = \mathcal{P}_\sigma v_0 \in H^2_\sigma(\Omega_h \times 2\mathbb{T})$. The convergence behaviors are different in the case when $\Omega_h = \mathbb{R}^2$ and the case when $\Omega_h = \mathbb{T}^2$. We summarize the results in the following:

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, as $\varepsilon \to 0^+$, one has*

$$\begin{aligned} v_\varepsilon &\overset{*}{\rightharpoonup} v_p \quad \text{weak-}* \text{ in } L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T})), \\ v_\varepsilon &\rightharpoonup v_p \quad \text{weakly in } L^2(0,T;H^3(\Omega_h \times 2\mathbb{T})), \end{aligned} \tag{2.10}$$

where $v_p \in L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T})) \cap L^2(0,T;H^3(\Omega_h \times 2\mathbb{T}))$ is the unique strong solution to (2.9) *with initial data $v_p|_{t=0} = v_{p,0} = \mathcal{P}_\sigma v_0 \in H^2_\sigma(\Omega_h \times 2\mathbb{T})$. Moreover, the following strong convergence holds,*

$$\mathcal{P}_\sigma v_\varepsilon \to v_p \quad \text{in } C([0,T];H^1_{\sigma,loc}(\Omega_h \times 2\mathbb{T})) \cap L^2(0,T;H^2_{\sigma,loc}(\Omega_h \times 2\mathbb{T})). \tag{2.11}$$

In addition,

- *in the case when $\Omega_h = \mathbb{R}^2$, as $\varepsilon \to 0^+$,*

$$\xi_\varepsilon \to 0,\ \mathcal{P}_\tau v_\varepsilon \to 0 \quad \text{in } L^2(0,T;W^{\frac{1}{2},6}(\mathbb{R}^2)), \tag{2.12}$$

  and therefore,

$$\| v_\varepsilon - v_p \|_{L^2(0,T;L^6_{loc}(\mathbb{R}^2\times 2\mathbb{T}))} + \| \xi_\varepsilon \|_{L^2(0,T;L^6(\mathbb{R}^2\times 2\mathbb{T}))} \to 0; \tag{2.13}$$

- *in the case when $\Omega_h = \mathbb{T}^2$, as $\varepsilon \to 0^+$, only the weak convergences in* (2.10) *hold, and there exists a function $V^o \in L^\infty(0,T;H^2(\mathbb{T}^2)) \cap C(0,T;H^1(\mathbb{T}^2))$ and a constant g^o, satisfying equation* (4.71)*, below in section 4.2, such that, as $\varepsilon \to 0^+$,*

$$\| \begin{pmatrix} \xi_\varepsilon - g^o \\ v_\varepsilon - v_p \end{pmatrix} - \mathcal{L}(\frac{t}{\varepsilon})V^o \|_{L^\infty(0,T;H^1(\mathbb{T}^2\times 2\mathbb{T}))} \to 0 \tag{2.14}$$

  where $\mathcal{L}$ is the solution operator to equation (4.27)*, defined in* (4.26)*, below in section 4.*

Theorem 2.1 is the direct consequence of Proposition 3, below. Theorem 2.2 is the consequence of Propositions 4, 5, 6, below.

3 Uniform stability

In this section, we will establish the uniform local existence of solutios to (1.4) with respect to $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 \in (0, \infty)$ small enough. This is done via a series of *a priori* estimates, a local well-posedness theorem and a continuity argument. To simplify the presentation, in this section, we shorten the notations by dropping the subscript ε. That is, we denote $\xi = \xi_\varepsilon, v = v_\varepsilon, w = w_\varepsilon$.

3.1 *A priori* estimates

We first establish some *a priori* estimates, which are independent of ε. Indeed, for ε small enough, the *a priori* estimates obtained in this subsection allow us to establish a uniform existence time in subsection 3.3. We remind readers system (1.4):

$$\begin{cases} \partial_t \xi + v \cdot \nabla_h \xi + \dfrac{\gamma - 1}{\varepsilon}(div_h v + \partial_z w) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t v + v \cdot \nabla_h v + w \partial_z v + \dfrac{c^2 e^{\varepsilon \xi}}{\varepsilon} \nabla_h \xi & \\ \qquad = c_1 e^{-\varepsilon \alpha \xi} \big(\mu \Delta_h v + \lambda \nabla_h div_h v + \partial_{zz} v \big) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi = 0 & \text{in } \Omega_h \times 2\mathbb{T}. \end{cases} \tag{1.4}$$

We will show the following:

Proposition 1. *Let* (ξ, v) *be a local strong solution to* (1.4) *in the time interval* $[0, T]$, $T \in (0, \infty)$, *and* $(\xi, v)\big|_{t=0} = (\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$ *with the compatibility conditions in* (2.4). *Then the following inequality holds: for any* $\delta \in (0, 1)$ *with corresponding* $C_\delta \simeq \delta^{-1}$,

$$\begin{aligned} & \sup_{0 \le t \le T} \left\{ \frac{1}{2} \| v(t) \|_{H^2}^2 + \frac{c^2}{2(\gamma - 1)} e^{-\varepsilon \| \xi(t) \|_{H^2}} \| \xi(t) \|_{H^2}^2 \right\} \\ & + c_1 \int_0^T \bigg(e^{-\varepsilon \alpha \| \xi(t) \|_{H^2}} \big(\mu \| \nabla_h v(t) \|_{H^2}^2 + \lambda \| div_h v(t) \|_{H^2}^2 \\ & + \| \partial_z v(t) \|_{H^2}^2 \big) \bigg) \, dt \le \frac{1}{2} \| v_0 \|_{H^2}^2 + \frac{c^2 e^{\varepsilon \| \xi(0) \|_{H^2}}}{2(\gamma - 1)} \| \xi_0 \|_{H^2}^2 \\ & + \delta \int_0^T \| \nabla v(t) \|_{H^2}^2 \, dt + \varepsilon \int_0^T \Big(\mathcal{H}_1(\| \xi(t) \|_{H^2}, \| v(t) \|_{H^2}) \\ & \times \| \nabla v(t) \|_{H^2}^2 \Big) \, dt + C_\delta \int_0^T \mathcal{H}_2(\| \xi(t) \|_{H^2}, \| v(t) \|_{H^2}) \, dt, \end{aligned} \tag{3.1}$$

where $\mathcal{H}_1(\cdot), \mathcal{H}_2(\cdot)$ are smooth and bounded functions of the arguments. Also, $\mathcal{H}_1(0) = 0, \mathcal{H}_2(0) = 0$. Moreover, with the same notations, below, we have the inequalities:

$$\|\partial_t e^{\varepsilon\xi}\|_{H^1} \leq \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \tag{3.2}$$

$$\|\partial_t \mathcal{P}_\sigma v\|_{L^2} \leq \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \tag{3.3}$$

$$\begin{aligned}\|\partial_t \mathcal{P}_\sigma v\|_{H^1} &\leq \mathcal{H}_1(\|\xi\|_{H^2}, \|v\|_{H^2})\|\nabla v\|_{H^2} \\ &\quad + \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}),\end{aligned} \tag{3.4}$$

$$\|\partial_t \xi\|_{H^1} \leq \varepsilon^{-1}\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \tag{3.5}$$

$$\|\partial_t v\|_{L^2} \leq (1+\varepsilon^{-1})\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \tag{3.6}$$

$$\begin{aligned}\|\partial_t v\|_{H^1} &\leq \mathcal{H}_1(\|\xi\|_{H^2}, \|v\|_{H^2})\|\nabla v\|_{H^2} \\ &\quad + (1+\varepsilon^{-1})\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}).\end{aligned} \tag{3.7}$$

In order to establish such *a priori* estimates, we first represent the vertical velocity w in terms of (ξ, v). In fact, after averaging $(1.4)_1$ in the z-variable, one has

$$\partial_t \xi + \overline{v} \cdot \nabla_h \xi + \frac{\gamma - 1}{\varepsilon} div_h \overline{v} = 0,$$

and consequently, after comparing the above equation with $(1.4)_1$, it follows that

$$\widetilde{v} \cdot \nabla_h \xi + \frac{\gamma - 1}{\varepsilon}(div_h \widetilde{v} + \partial_z w) = 0.$$

Then the following representations of the vertical velocity and its derivatives hold (recall that $\alpha = 1/(\gamma - 1)$):

$$w = -\int_0^z \left(\varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi + div_h \widetilde{v}\right) dz, \tag{3.8}$$

$$w_z = -\varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi - div_h \widetilde{v}, \tag{3.9}$$

$$w_h = -\int_0^z \left(\varepsilon\alpha(\widetilde{v} \cdot \nabla_h \xi)_h + div_h \widetilde{v}_h\right) dz, \tag{3.10}$$

$$w_{zz} = -\varepsilon\alpha\widetilde{v}_z \cdot \nabla_h \xi - div_h \widetilde{v}_z, \tag{3.11}$$

$$w_{hz} = -\varepsilon\alpha\widetilde{v}_h \cdot \nabla_h \xi - \varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi_h - div_h \widetilde{v}_h. \tag{3.12}$$

In the following, we separate the proof of Proposition 1 in three parts: estimates on the horizontal derivatives; estimates on the vertical derivatives; and estimates on the time derivatives.

Estimates on the horizontal derivatives

Denote by $\partial_h \in \{\partial_x, \partial_y\}$. Then after applying $\partial_h^2 =$ to (1.4), it follows

$$\begin{cases} \partial_t \xi_{hh} + \dfrac{\gamma-1}{\varepsilon}(div_h v_{hh} + \partial_z w_{hh}) = \mathcal{G} & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t v_{hh} + \dfrac{c^2 e^{\varepsilon\xi}}{\varepsilon}\nabla_h \xi_{hh} - c_1 e^{-\varepsilon\alpha\xi}(\mu\Delta_h v_{hh} \\ \quad + \lambda\nabla_h div_h v_{hh} + \partial_{zz} v_{hh}) = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \tag{3.13}$$

where we have denoted by

$$\begin{aligned} &\mathcal{G} := -(v\cdot\nabla_h \xi)_{hh}, \\ &\mathcal{F}_1 := -(v\cdot\nabla_h v)_{hh} - (w\partial_z v)_{hh}, \\ &\mathcal{F}_2 := -c^2 e^{\varepsilon\xi}(\xi_{hh} + \varepsilon\xi_h^2)\nabla_h\xi - 2c^2 e^{\varepsilon\xi}\xi_h\nabla_h\xi_h, \\ &\mathcal{F}_3 := -2\varepsilon c_1\alpha e^{-\varepsilon\alpha\xi}\xi_h(\mu\Delta_h v_h + \lambda\nabla_h div_h v_h + \partial_{zz} v_h), \\ &\mathcal{F}_4 := -\varepsilon c_1\alpha e^{-\varepsilon\alpha\xi}(\xi_{hh} - \varepsilon\alpha\xi_h^2)(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v). \end{aligned}$$

After taking the L^2-inner product of $(3.13)_2$ with v_{hh} in $\Omega_h \times 2\mathbb{T}$, it holds

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{2}\int |v_{hh}|^2\,d\vec{x}\Big\} + \int \frac{c^2 e^{\varepsilon\xi}}{\varepsilon}\nabla_h\xi_{hh}\cdot v_{hh}\,d\vec{x} + c_1\int\Big(e^{-\varepsilon\alpha\xi}\big(\mu|\nabla_h v_{hh}|^2 \\ &\quad + \lambda|div_h v_{hh}|^2 + |v_{hhz}|^2\big)\Big)\,d\vec{x} = \int \mathcal{F}_1\cdot v_{hh}\,d\vec{x} + \int \mathcal{F}_2\cdot v_{hh}\,d\vec{x} \\ &\quad + \int \mathcal{F}_3\cdot v_{hh}\,d\vec{x} + \int \mathcal{F}_4\cdot v_{hh}\,d\vec{x} + \varepsilon c_1\alpha\int\Big(e^{-\varepsilon\alpha\xi}\big(\mu(\nabla_h\xi\cdot\nabla_h v_{hh})\cdot v_{hh} \\ &\quad + \lambda(v_{hh}\cdot\nabla_h\xi)div_h v_{hh}\big)\Big)\,d\vec{x} =: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

On the other hand, after applying integration by parts and substituting $(3.13)_1$, one has

$$\begin{aligned} &\int \frac{c^2 e^{\varepsilon\xi}}{\varepsilon}\nabla_h\xi_{hh}\cdot v_{hh}\,d\vec{x} = -\int \frac{c^2 e^{\varepsilon\xi}}{\varepsilon}\xi_{hh} div_h v_{hh}\,d\vec{x} \\ &\quad - \int c^2 e^{\varepsilon\xi}\xi_{hh}(v_{hh}\cdot\nabla_h\xi)\,d\vec{x} = \int \frac{c^2 e^{\varepsilon\xi}}{\gamma-1}\xi_{hh}\big(\partial_t\xi_{hh} - \mathcal{G} - \frac{\gamma-1}{\varepsilon}\partial_z w_{hh}\big)\,d\vec{x} \\ &\quad - \int c^2 e^{\varepsilon\xi}\xi_{hh}(v_{hh}\cdot\nabla_h\xi)\,d\vec{x} = \frac{d}{dt}\Big\{\frac{c^2}{2(\gamma-1)}\int e^{\varepsilon\xi}|\xi_{hh}|^2\,d\vec{x}\Big\} \\ &\quad - \frac{c^2}{2(\gamma-1)}\int e^{\varepsilon\xi}\varepsilon\xi_t|\xi_{hh}|^2\,d\vec{x} - \frac{c^2}{\gamma-1}\int e^{\varepsilon\xi}\xi_{hh}\mathcal{G}\,d\vec{x} \end{aligned}$$

$$- c^2 \int e^{\varepsilon\xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi) \, d\vec{x}.$$

Then after writing

$$I_6 := \frac{c^2}{2(\gamma-1)} \int e^{\varepsilon\xi} \varepsilon \xi_t |\xi_{hh}|^2 \, d\vec{x}, \qquad I_7 := \frac{c^2}{\gamma - 1} \int e^{\varepsilon\xi} \xi_{hh} \mathcal{G} \, d\vec{x},$$
$$I_8 := c^2 \int e^{\varepsilon\xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi) \, d\vec{x},$$

one has

$$\frac{d}{dt} \left\{ \frac{1}{2} \| v_{hh} \|_{L^2}^2 + \frac{c^2}{2(\gamma-1)} \| e^{\varepsilon\xi/2} \xi_{hh} \|_{L^2}^2 \right\} + c_1 \big(\mu \| e^{-\varepsilon\alpha\xi/2} \nabla_h v_{hh} \|_{L^2}^2$$
$$+ \lambda \| e^{-\varepsilon\alpha\xi/2} div_h v_{hh} \|_{L^2}^2 + \| e^{-\varepsilon\alpha\xi/2} v_{hhz} \|_{L^2}^2 \big) = \sum_{i=1}^{8} I_i. \tag{3.14}$$

Now we will estimate the right-hand side of (3.14). After applying integration by parts, I_1 can be written as

$$I_1 = \int \left(\frac{1}{2} div_h v |v_{hh}|^2 - (2 v_h \cdot \nabla_h v_h + v_{hh} \cdot \nabla_h v) \cdot v_{hh} \right) d\vec{x}$$
$$+ \int \frac{1}{2} w_z |v_{hh}|^2 \, d\vec{x} + \int \Big(w_h \partial_z v_h \cdot v_{hh} + w_h \partial_z v \cdot v_{hhh}$$
$$- 2 w_h \partial_z v_h \cdot v_{hh} \Big) d\vec{x} =: I_1' + I_1'' + I_1'''.$$

Then it follows, after substituting (3.9) and (3.10), that

$$I_1' \lesssim \int |\nabla_h v| |\nabla_h^2 v|^2 \, d\vec{x} \lesssim \| \nabla_h v \|_{L^2} \| \nabla_h^2 v \|_{L^3} \| \nabla_h^2 v \|_{L^6}$$
$$\lesssim \| \nabla_h v \|_{L^2} \big(\| \nabla_h^2 v \|_{L^2}^{1/2} \| \nabla \nabla_h^2 v \|_{L^2}^{1/2} + \| \nabla_h^2 v \|_{L^2} \big) \big(\| \nabla_h^2 v \|_{L^2}$$
$$+ \| \nabla \nabla_h^2 v \|_{L^2} \big) \lesssim \delta \| \nabla \nabla_h^2 v \|_{L^2}^2 + C_\delta \big(1 + \| \nabla_h v \|_{L^2}^4 \big) \| \nabla_h^2 v \|_{L^2}^2,$$
$$I_1'' = -\frac{1}{2} \int \big(\frac{\varepsilon}{\gamma - 1} \widetilde{v} \cdot \nabla_h \xi + div_h \widetilde{v} \big) |v_{hh}|^2 \, d\vec{x} = \frac{1}{2} \int \Big(\big(\frac{\varepsilon}{\gamma-1} \xi div_h \widetilde{v}$$
$$- div_h \widetilde{v} \big) |v_{hh}|^2 \Big) d\vec{x} + \int \frac{\varepsilon}{\gamma - 1} \xi (\widetilde{v} \cdot \nabla_h v_{hh}) \cdot v_{hh} \, d\vec{x}$$
$$\lesssim \varepsilon \int |\xi| (|\nabla_h v| |\nabla_h^2 v|^2 + |v| |\nabla_h^2 v| |\nabla_h^3 v|) \, d\vec{x} + \int |\nabla_h v| |\nabla_h^2 v|^2 \, d\vec{x}$$
$$\lesssim \varepsilon \| \xi \|_{L^\infty} \| \nabla_h^2 v \|_{L^3} \big(\| \nabla_h v \|_{L^2} \| \nabla_h^2 v \|_{L^6} + \| v \|_{L^6} \| \nabla_h^3 v \|_{L^2} \big)$$

$$
\begin{aligned}
&+ \|\nabla_h v\|_{L^2} \|\nabla_h^2 v\|_{L^3} \|\nabla_h^2 v\|_{L^6} \lesssim \|v\|_{H^1} (1 + \varepsilon \|\xi\|_{H^2}) \\
&\times \big(\|\nabla_h^2 v\|_{L^2}^{1/2} \|\nabla\nabla_h^2 v\|_{L^2}^{1/2} + \|\nabla_h^2 v\|_{L^2} \big) \big(\|\nabla\nabla_h^2 v\|_{L^2} + \|\nabla_h^2 v\|_{L^2} \big) \\
&\lesssim \delta \|\nabla\nabla_h^2 v\|_{L^2}^2 + C_\delta (\varepsilon^4 \|\xi\|_{H^2}^4 + 1) \big(\|v\|_{H^1}^4 + 1 \big) \|\nabla_h^2 v\|_{L^2}^2, \\
I_1''' &= -\int \Bigg[\int_0^z \bigg(\frac{\varepsilon}{\gamma - 1} (\widetilde{v} \cdot \nabla_h \xi)_h + div_h \widetilde{v}_h \bigg) \, dz \times \bigg(\partial_z v_h \cdot v_{hh} + \partial_z v \cdot v_{hhh} \\
&- 2\partial_z v_h \cdot v_{hh} \bigg) \Bigg] \, d\vec{x} \lesssim \int_0^1 \varepsilon \big(|v|_{L^\infty} |\nabla_h^2 \xi|_{L^2} + |\nabla_h v|_{L^4} |\nabla_h \xi|_{L^4} \big) \, dz \\
&\times \int_0^1 \big(|\partial_z v_h|_{L^4} |v_{hh}|_{L^4} + |\partial_z v|_{L^\infty} |v_{hhh}|_{L^2} \big) \, dz + \int_0^1 |\nabla_h^2 v|_{L^3} \, dz \\
&\times \int \big(|\partial_z v_h|_{L^3} |v_{hh}|_{L^3} + |\partial_z v|_{L^6} |v_{hhh}|_{L^2} \big) \, dz \lesssim \int_0^1 \varepsilon |v|_{H^2} |\xi|_{H^2} \, dz \\
&\times \int_0^1 \bigg((|\partial_z v_h|_{L^2}^{1/2} |\nabla_h \partial_z v_h|_{L^2}^{1/2} + |\partial_z v_h|_{L^2}) (|v_{hh}|_{L^2}^{1/2} |\nabla_h v_{hh}|_{L^2}^{1/2} + |v_{hh}|_{L^2}) \\
&+ |\partial_z v|_{H^2} |v_{hhh}|_{L^2} \bigg) \, dz + \int_0^1 \big(|\nabla_h^2 v|_{L^2}^{2/3} |\nabla_h^3 v|_{L^2}^{1/3} + |\nabla_h^2 v|_{L^2} \big) \, dz \\
&\times \int_0^1 \bigg((|\partial_z v_h|_{L^2}^{2/3} |\partial_z v_{hh}|_{L^2}^{1/3} + |\partial_z v_h|_{L^2}) (|v_{hh}|_{L^2}^{2/3} |\nabla_h v_{hh}|_{L^2}^{1/3} \\
&+ |v_{hh}|_{L^2}) + |\partial_z v|_{H^1} |v_{hhh}|_{L^2} \bigg) \, dz \lesssim \varepsilon \|v\|_{H^2} \|\xi\|_{H^2} \\
&\times \big(\|\nabla v_{hh}\|_{L^2}^2 + \|v\|_{H^2}^2 \big) + \|v\|_{H^2}^2 \|\nabla v_{hh}\|_{L^2} + \|v\|_{H^2}^3 \\
&+ \|v\|_{H^2}^{5/3} \|\nabla v_{hh}\|_{L^2}^{4/3} \lesssim \big(\varepsilon \|v\|_{H^2} \|\xi\|_{H^2} + \delta \big) \|\nabla v_{hh}\|_{L^2}^2 \\
&+ \big(\varepsilon \|v\|_{H^2} \|\xi\|_{H^2} + \|v\|_{H^2} + \|v\|_{H^2}^3 \big) \|v\|_{H^2}^2,
\end{aligned}
$$

where we have applied the Minkowski, Hölder, Sobolev embedding, and Young inequalities. Hence we have

$$
\begin{aligned}
I_1 \lesssim &\big(\varepsilon \|v\|_{H^2} \|\xi\|_{H^2} + \delta \big) \|\nabla v_{hh}\|_2^2 + \big(\varepsilon \|v\|_{H^2} \|\xi\|_{H^2} \\
&+ \varepsilon^4 \|v\|_{H^1} \|\xi\|_{H^2}^4 + \varepsilon^4 \|v\|_{H^1}^4 \|\xi\|_{H^2}^4 + \|v\|_{H^2}^4 \\
&+ 1 \big) \|v\|_{H^2}^2.
\end{aligned} \tag{3.15}
$$

Next, to estimate I_3, I_4, I_5, applying the Hölder, Sobolev embedding, and Young inequalities yields,

$$
I_3 = -2\varepsilon c_1 \alpha \int e^{-\varepsilon\alpha\xi} \xi_h \bigg((\mu \Delta_h v_h + \lambda \nabla_h div_h v_h) \cdot v_{hh} - \partial_z v_h \cdot v_{hhz} \bigg) \, d\vec{x}
$$

$$
\begin{aligned}
&\lesssim \varepsilon \int_0^1 e^{\varepsilon\alpha \| \xi \|_{L^\infty}} |\xi_h|_{L^4} (|\nabla_h^2 v_h|_{L^2} |v_{hh}|_{L^4} + |\partial_z v_h|_{L^4} |v_{hhz}|_{L^2})\, dz \\
&\lesssim \varepsilon \| \xi \|_{H^2} e^{\varepsilon\alpha \| \xi \|_{L^\infty}} \int_0^1 |\nabla \nabla_h^2 v|_{L^2} (|\nabla v_h|_{L^2}^{1/2} |\nabla v_{hh}|_{L^2}^{1/2} + |\nabla v_h|_{L^2})\, dz \\
&\lesssim \delta \| \nabla \nabla_h^2 v \|_{L^2}^2 + C_\delta (\varepsilon^4 \| \xi \|_{H^2}^4 + \varepsilon^2 \| \xi \|_{H^2}^2) e^{4\varepsilon\alpha \| \xi \|_{H^2}} \| v \|_{H^2}^2, \\
I_4 &= -\varepsilon c_1 \alpha \int \Big(e^{-\varepsilon\alpha\xi} (\xi_{hh} - \varepsilon\alpha\xi_h^2) \cdot ((\mu \Delta_h v + \lambda \nabla_h div_h v) \cdot v_{hh} \\
&\quad - \partial_z v \cdot v_{hhz}) \Big)\, d\vec{x} \lesssim \varepsilon \int_0^1 \Big(e^{\varepsilon\alpha \| \xi \|_{L^\infty}} (|\xi_{hh}|_{L^2} + \varepsilon |\xi_h|_{L^4}^2)(|\nabla_h^2 v|_{L^4}^2 \\
&\quad + |\partial_z v|_{L^\infty} |v_{hhz}|_{L^2}) \Big)\, dz \lesssim \varepsilon e^{\varepsilon\alpha \| \xi \|_{L^\infty}} (\| \xi \|_{H^2} + \varepsilon \| \xi \|_{H^2}^2) \\
&\quad \times \int_0^1 \Big(|\nabla_h^2 v|_{L^2} |\nabla_h^3 v|_{L^2} + |\nabla_h^2 v|_{L^2}^2 + |\nabla_h^2 \partial_z v|_{L^2}^2 + |\partial_z v|_{H^1}^2 \Big)\, dz \\
&\lesssim \varepsilon (\| \xi \|_{H^2} + \varepsilon \| \xi \|_{H^2}^2) e^{\varepsilon\alpha \| \xi \|_{H^2}} (\| \nabla \nabla_h^2 v \|_{L^2}^2 + \| v \|_{H^2}^2), \\
I_5 &\lesssim \varepsilon \int_0^1 e^{\varepsilon\alpha \| \xi \|_{L^\infty}} |\nabla_h \xi|_{L^4} |\nabla_h v_{hh}|_{L^2} |v_{hh}|_{L^4}\, dz \lesssim \varepsilon e^{\varepsilon\alpha \| \xi \|_{L^\infty}} \| \xi \|_{H^2} \\
&\quad \times \int_0^1 |\nabla_h v_{hh}|_{L^2} (|v_{hh}|_{L^2}^{1/2} |\nabla_h v_{hh}|_{L^2}^{1/2} + |v_{hh}|_{L^2})\, dz \lesssim \delta \| \nabla v_{hh} \|_{L^2}^2 \\
&\quad + C_\delta (\varepsilon^4 \| \xi \|_{H^2}^4 + \varepsilon^2 \| \xi \|_{H^2}^2) e^{4\varepsilon\alpha \| \xi \|_{H^2}} \| v \|_{H^2}^2.
\end{aligned}
$$

In order to estimate I_6, I_7, after substituting $(1.4)_1$ and applying integration by parts, it holds

$$
\begin{aligned}
I_6 &= \frac{c^2\alpha}{2} \int e^{\varepsilon\xi} |\xi_{hh}|^2 (-\varepsilon v \cdot \nabla_h \xi - (\gamma - 1) div_h v)\, d\vec{x}, \\
I_7 &= -c^2\alpha \int e^{\varepsilon\xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi + 2 v_h \cdot \nabla_h \xi_h)\, d\vec{x} \\
&\quad + \frac{c^2\alpha}{2} \int e^{\varepsilon\xi} |\xi_{hh}|^2 (div_h v + \varepsilon v \cdot \nabla_h \xi)\, d\vec{x}.
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
I_6 + I_7 + I_8 &= -c^2\alpha \int e^{\varepsilon\xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi + 2 v_h \cdot \nabla_h \xi_h)\, d\vec{x} \\
&\quad + \frac{(2-\gamma)c^2\alpha}{2} \int e^{\varepsilon\xi} |\xi_{hh}|^2 div_h v\, d\vec{x} + c^2 \int e^{\varepsilon\xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi)\, d\vec{x} \\
&\lesssim e^{\varepsilon \| \xi \|_{L^\infty}} \int_0^1 |\xi_{hh}|_{L^2} (|v_{hh}|_{L^4} |\nabla_h \xi|_{L^4} + |\nabla_h v|_{L^\infty} |\nabla_h \xi_h|_{L^2})\, d\vec{x}
\end{aligned}
$$

$$\lesssim e^{\varepsilon \|\xi\|_{L^\infty}} \int_0^1 |\xi|_{H^2}^2 (|\nabla_h^3 v|_{L^2} + |v|_{H^2})\, d\vec{x} \lesssim \delta \|\nabla_h^3 v\|_{L^2}^2$$
$$+ C_\delta e^{2\varepsilon \|\xi\|_{H^2}} \|\xi\|_{H^2}^2 (\|\xi\|_{H^2}^2 + \|v\|_{H^2}),$$

where we have applied the Hölder, Sobolev embedding and Young inequalities. Similarly, I_2 can be estimated as following:

$$\begin{aligned} I_2 &\lesssim e^{\varepsilon \|\xi\|_{L^\infty}} \int_0^1 (|\nabla_h \xi_h|_{L^2} + \varepsilon |\xi_h|_{L^4}^2) |\nabla_h \xi|_{L^4} |v_{hh}|_{L^4}\, dz \\ &\lesssim e^{\varepsilon \|\xi\|_{L^\infty}} \int_0^1 (|\xi|_{H^2}^2 + |\xi|_{H^2}^3)(|\nabla_h v_{hh}|_{L^2}^{1/2} |v_{hh}|_{L^2}^{1/2} + |v_{hh}|_{L^2})\, dz \\ &\lesssim \delta \|\nabla v_{hh}\|_{L^2}^2 + C_\delta (\|\xi\|_{H^2}^{8/3} + \|\xi\|_{H^2}^4) e^{4\varepsilon/3 \|\xi\|_{H^2}} \\ &\qquad \times \|v\|_{H^2} + (\|\xi\|_{H^2}^2 + \|\xi\|_{H^2}^3) e^{\varepsilon \|\xi\|_{H^2}} \|v\|_{H^2}. \end{aligned}$$

Therefore, (3.14) implies, after summing up the estimates above with $\partial_h \in \{\partial_x, \partial_y\}$,

$$\begin{aligned} &\frac{d}{dt} \left\{ \frac{1}{2} \|\nabla_h^2 v\|_{L^2}^2 + \frac{c^2}{2(\gamma - 1)} \|e^{\varepsilon\xi/2} \nabla_h^2 \xi\|_{L^2}^2 \right\} \\ &\quad + c_1 \big(\mu \|e^{-\varepsilon\alpha\xi/2} \nabla_h^3 v\|_{L^2}^2 + \lambda \|e^{-\varepsilon\alpha\xi/2} \nabla_h^2 div_h v\|_{L^2}^2 \\ &\quad + \|e^{-\varepsilon\alpha\xi/2} \nabla_h^2 v_z\|_{L^2}^2\big) \leq \delta \|\nabla\nabla_h^2 v\|_{L^2}^2 \\ &\quad + \varepsilon \mathcal{H}_1(\|\xi\|_{H^2}, \|v\|_{H^2}) \|\nabla\nabla_h^2 v\|_{L^2}^2 + C_\delta \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \end{aligned} \tag{3.16}$$

where

$$\mathcal{H}_1 = \mathcal{H}_1(\|\xi\|_{H^2}, \|v\|_{H^2}),\ \mathcal{H}_2 = \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \tag{3.17}$$

are two regular functions of the arguments and $\mathcal{H}_1(0) = \mathcal{H}_2(0) = 0$. We will adopt the same notations for functions with such properties in this work.

After taking the L^2-inner product of $(1.4)_2$, and the horizontal derivative of $(1.4)_2$ with v, v_h, respectively, repeating similar arguments as above yields the following estimate:

$$\begin{aligned} &\frac{d}{dt} \left\{ \frac{1}{2} \|v\|_{L^2}^2 + \frac{1}{2} \|\nabla_h v\|_{L^2}^2 + \frac{c^2}{2(\gamma - 1)} \|e^{\varepsilon\xi/2} \xi\|_{L^2}^2 \right. \\ &\quad \left. + \frac{c^2}{2(\gamma - 1)} \|e^{\varepsilon\xi/2} \nabla_h \xi\|_{L^2}^2 \right\} + c_1 \big(\mu \|e^{-\varepsilon\alpha\xi/2} \nabla_h v\|_{L^2}^2 \\ &\quad + \mu \|e^{-\varepsilon\alpha\xi/2} \nabla_h^2 v\|_{L^2}^2 + \lambda \|e^{-\varepsilon\alpha\xi/2} div_h v\|_{L^2}^2 \\ &\quad + \lambda \|e^{-\varepsilon\alpha\xi/2} \nabla_h div_h v\|_{L^2}^2 + \|e^{-\varepsilon\alpha\xi/2} v_z\|_{L^2}^2 \\ &\quad + \|e^{-\varepsilon\alpha\xi/2} \nabla_h v_z\|_{L^2}^2\big) \leq \delta \|\nabla v\|_{H^2}^2 + C_\delta \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}). \end{aligned} \tag{3.18}$$

Estimates on the vertical derivatives

After applying ∂_z to $(1.4)_2$, it holds

$$\begin{aligned}\partial_t v_z - c_1 e^{-\varepsilon\alpha\xi}(\mu\Delta_h v_z + \lambda\nabla_h div_h v_z + \partial_{zz}v_z) = -v_z\cdot\nabla_h v \\ - v\cdot\nabla_h v_z - w_z\partial_z v - w\partial_z v_z.\end{aligned} \tag{3.19}$$

Again, applying ∂_z to (3.19) yields,

$$\begin{aligned}\partial_t v_{zz} - c_1 e^{-\varepsilon\alpha\xi}(\mu\Delta_h v_{zz} + \lambda\nabla_h div_h v_{zz} + \partial_{zz}v_{zz}) = -v\cdot\nabla_h v_{zz} \\ - 2v_z\nabla_h v_z - v_{zz}\cdot\nabla_h v - w_{zz}\partial_z v - 2w_z\partial_z v_z - w\partial_z v_{zz}.\end{aligned} \tag{3.20}$$

Next, we take the L^2-inner produce of (3.20) with v_{zz}. After applying integration by parts, one has

$$\begin{aligned}\frac{d}{dt}\Big\{\frac{1}{2}\big\|v_{zz}\big\|_{L^2}^2\Big\} + c_1\big(\mu\big\|e^{-\varepsilon\alpha\xi/2}\nabla_h v_{zz}\big\|_{L^2}^2 + \lambda\big\|e^{-\varepsilon\alpha\xi/2}div_h v_{zz}\big\|_{L^2}^2 \\ + \big\|e^{-\varepsilon\alpha\xi/2}v_{zzz}\big\|_{L^2}^2\big) = I_9 + I_{10} + I_{11},\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}&I_9 := -\int (v\cdot\nabla_h v_{zz} + 2v_z\nabla_h v_z + v_{zz}\cdot\nabla_h v)\cdot v_{zz}\,d\vec{x},\\ &I_{10} := -\int (w_{zz}\partial_z v + \frac{3}{2}w_z\partial_z v_z)\cdot v_{zz}\,d\vec{x},\\ &I_{11} := c_1\varepsilon\alpha\int e^{-\varepsilon\alpha\xi}(\mu(\nabla_h\xi\cdot\nabla_h v_{zz})\cdot v_{zz} + \lambda(v_{zz}\cdot\nabla_h\xi)div_h v_{zz})\,d\vec{x}.\end{aligned}$$

After applying the Hölder, Sobolev embedding and Young inequalities, it holds,

$$\begin{aligned}I_9 &\lesssim \big(\big\|v\big\|_{L^6}\big\|\nabla_h v_{zz}\big\|_{L^2} + \big\|v_z\big\|_{L^6}\big\|\nabla_h v_z\big\|_{L^2} + \big\|v_{zz}\big\|_{L^2}\big\|\nabla_h v\big\|_{L^6}\big)\\ &\quad\times\big\|v_{zz}\big\|_{L^3} \lesssim \big(\big\|v\big\|_{H^1}\big\|\nabla_h v_{zz}\big\|_{L^2} + \big\|v\big\|_{H^2}^2\big)\big(\big\|v_{zz}\big\|_{L^2}^{1/2}\big\|\nabla v_{zz}\big\|_{L^2}^{1/2}\\ &\quad + \big\|v_{zz}\big\|_{L^2}\big) \lesssim \delta\big\|\nabla v_{zz}\big\|_{L^2}^2 + C_\delta\big(\big\|v\big\|_{H^2}^4 + \big\|v\big\|_{H^2}\big)\big\|v\big\|_{H^2}^2,\\ I_{10} &= \varepsilon\alpha\int\Big((\widetilde{v}_z\cdot\nabla_h\xi)(\partial_z v\cdot v_{zz}) + \frac{3}{2}(\widetilde{v}\cdot\nabla_h\xi)(\partial_z v_z\cdot v_{zz})\Big)\,d\vec{x}\\ &\quad + \int\Big(div_h\widetilde{v}_z(\partial_z v\cdot v_{zz}) + \frac{3}{2}div_h\widetilde{v}(\partial_z v_z\cdot v_{zz})\Big)\,d\vec{x}\\ &\lesssim \varepsilon\alpha\big(\big\|v_z\big\|_{L^4}\big\|\nabla_h\xi\big\|_{L^4}\big\|v_z\big\|_{L^3} + \big\|v\big\|_{L^6}\big\|\nabla_h\xi\big\|_{L^6}\big\|v_{zz}\big\|_{L^2}\big)\big\|v_{zz}\big\|_{L^6}\\ &\quad + \big(\big\|\nabla_h v_z\big\|_{L^2}\big\|v_z\big\|_{L^3} + \big\|\nabla_h v\big\|_{L^3}\big\|v_{zz}\big\|_{L^2}\big)\big\|v_{zz}\big\|_{L^6}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \big(\varepsilon\alpha \| \xi \|_{H^2} + 1\big) \| v \|_{H^2}^2 \big(\| \nabla v_{zz} \|_{L^2} + \| v \|_{H^2}\big)\\
&\lesssim \delta \| \nabla v_{zz} \|_{L^2}^2 + C_\delta \big(\varepsilon^2\alpha^2 \| \xi \|_{H^2}^2 + 1\big)\big(\| v \|_{H^2}^2 + \| v \|_{H^2}\big) \| v \|_{H^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
I_{11} &\lesssim \varepsilon\alpha e^{\varepsilon\alpha \| \xi \|_{L^\infty}} \| \nabla_h \xi \|_{L^6} \| v_{zz} \|_{L^3} \| \nabla_h v_{zz} \|_{L^2} \lesssim \varepsilon\alpha e^{\varepsilon\alpha \| \xi \|_{H^2}} \| \xi \|_{H^2}\\
&\times \big(\| v_{zz} \|_{L^2}^{1/2} \| \nabla v_{zz} \|_{L^2}^{1/2} + \| v_{zz} \|_{L^2}\big) \| \nabla_h v_{zz} \|_{L^2} \lesssim \delta \| \nabla v_{zz} \|_{L^2}^2\\
&+ C_\delta \big(\varepsilon^4 \| \xi \|_{H^2}^4 + \varepsilon^2 \| \xi \|_{H^2}^2\big) e^{4\varepsilon\alpha \| \xi \|_{H^2}} \| v \|_{H^2}^2,
\end{aligned}
$$

where we have substituted (3.9) and (3.11) into I_{10}. Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{ \frac{1}{2} \| v_{zz} \|_{L^2}^2 \right\} + c_1\big(\mu \| e^{-\varepsilon\alpha\xi/2} \nabla_h v_{zz} \|_{L^2}^2 + \lambda \| e^{-\varepsilon\alpha\xi/2} div_h v_{zz} \|_{L^2}^2\\
&\quad + \| e^{-\varepsilon\alpha\xi/2} v_{zzz} \|_{L^2}^2\big) \leq \delta \| \nabla v_{zz} \|_{L^2}^2 + C_\delta \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}).
\end{aligned} \tag{3.22}
$$

Next, we establish the estimate of v_{hz}. Apply ∂_h to (3.19). It follows

$$
\begin{aligned}
&\partial_z v_{hz} - c_1 e^{-\varepsilon\alpha\xi}(\mu\Delta_h v_{hz} + \lambda\nabla_h div_h v_{hz} + \partial_{zz} v_{hz}) = -v_{hz}\cdot\nabla_h v\\
&\quad - v_z \cdot \nabla_h v_h - v_h \cdot \nabla_h v_z - v\cdot\nabla_h v_{hz} - w_{hz}\partial_z v - w_z \partial_z v_h\\
&\quad - w\partial_z v_{hz} - w_h \partial_z v_z - c_1\varepsilon\alpha e^{-\varepsilon\alpha\xi}\xi_h(\mu\Delta_h v_z + \lambda\nabla_h div_h v_z\\
&\quad + \partial_{zz} v_z).
\end{aligned} \tag{3.23}
$$

Take the L^2-inner product of (3.23) with v_{hz} and apply integration by parts in the resultant equation. It follows,

$$
\begin{aligned}
&\frac{d}{dt}\left\{ \frac{1}{2} \| v_{hz} \|_{L^2}^2 \right\} + c_1(\mu \| e^{-\varepsilon\alpha\xi/2} \nabla_h v_{hz} \|_{L^2}^2 + \lambda \| e^{-\varepsilon\alpha\xi/2} div_h v_{hz} \|_{L^2}^2\\
&\quad + \| e^{-\varepsilon\alpha\xi/2} v_{hzz} \|_{L^2}^2) = I_{12} + I_{13} + I_{14} + I_{15} + I_{16},
\end{aligned} \tag{3.24}
$$

where

$$
\begin{aligned}
I_{12} &:= -\int (v_{hz}\cdot\nabla_h v + v_z\cdot\nabla_h v_h + v_h\cdot\nabla_h v_z - \frac{1}{2}(div_h v) v_{hz})\cdot v_{hz}\, d\vec{x},\\
I_{13} &:= -\int (w_{hz}\partial_z v + w_z \partial_z v_h - \frac{1}{2}(w_z) v_{hz})\cdot v_{hz}\, d\vec{x},\\
I_{14} &:= -\int w_h(\partial_z v_z \cdot v_{hz})\, d\vec{x},\\
I_{15} &:= -c_1\varepsilon\alpha \int e^{-\varepsilon\alpha\xi}\xi_h(\mu\Delta_h v_z + \lambda\nabla_h div_h v_z + \partial_{zz} v_z)\cdot v_{hz}\, d\vec{x},\\
I_{16} &:= c_1\varepsilon\alpha \int e^{-\varepsilon\alpha\xi}(\mu(\nabla_h\xi\cdot\nabla_h v_{hz})\cdot v_{hz} + \lambda(v_{hz}\cdot\nabla_h\xi) div_h v_{hz})\, d\vec{x}.
\end{aligned}
$$

After substituting (3.12), (3.9) in I_{13} and (3.10) in I_{14}, applying the Hölder, Sobolev embedding, Minkowski and Young inequalities yields,

$$
\begin{aligned}
I_{12} &\lesssim (\|\nabla_h v_z\|_{L^2}\|\nabla_h v\|_{L^3} + \|v_z\|_{L^3}\|\nabla_h v_h\|_{L^2})\|v_{hz}\|_{L^6} \\
&\lesssim \|v\|_{H^2}^2(\|\nabla v_{hz}\|_{L^2} + \|v_{hz}\|_{L^2}) \lesssim \delta\|\nabla v_{hz}\|_{L^2}^2 \\
&\quad + C_\delta\|v\|_{H^2}^4 + \|v\|_{H^2}^3, \\
I_{13} &= \varepsilon\alpha\int\left((\widetilde{v}_h\cdot\nabla_h\xi + \widetilde{v}\cdot\nabla_h\xi_h)(\partial_z v\cdot v_{hz}) + \frac{1}{2}(\widetilde{v}\cdot\nabla_h\xi)(v_{hz}\cdot v_{hz})\right)d\vec{x} \\
&\quad + \int\left(div_h\widetilde{v}_h(\partial_z v\cdot v_{hz}) + \frac{1}{2}div_h\widetilde{v}(v_{hz}\cdot v_{hz})\right)d\vec{x} \\
&\lesssim \varepsilon\alpha\big((\|v_h\|_{L^6}\|\nabla_h\xi\|_{L^2} + \|v\|_{L^6}\|\xi_{hh}\|_{L^2})\|\partial_z v\|_{L^6} \\
&\quad + \|v\|_{L^6}\|\nabla_h\xi\|_{L^6}\|v_{hz}\|_{L^2}\big)\|v_{hz}\|_{L^6} + \big(\|\nabla_h v_h\|_{L^2}\|\partial_z v\|_{L^3} \\
&\quad + \|\nabla_h v\|_{L^3}\|v_{hz}\|_{L^2}\big)\|v_{hz}\|_{L^6} \lesssim (\varepsilon\alpha\|v\|_{H^2}^2\|\xi\|_{H^2} + \|v\|_{H^2}^2) \\
&\quad \times(\|\nabla v_{hz}\|_{L^2} + \|v_{hz}\|_{L^2}) \lesssim \delta\|\nabla v_{hz}\|_{L^2}^2 \\
&\quad + C_\delta(\varepsilon^2\|\xi\|_{H^2}^2\|v\|_{H^2}^2 + \|v\|_{H^2}^2 + 1)\|v\|_{H^2}^2, \\
I_{14} &= \int\left[\int_0^z\left(\varepsilon\alpha(\widetilde{v}_h\cdot\nabla_h\xi + \widetilde{v}\cdot\nabla_h\xi_h) + div_h\widetilde{v}_h\right)dz\times(\partial_z v_z\cdot v_{hz})\right]d\vec{x} \\
&\lesssim \int_0^1\left(\varepsilon(|v_h|_{L^4}|\nabla_h\xi|_{L^4} + |v|_{L^\infty}|\nabla_h\xi_h|_{L^2}) + |\nabla_h v_h|_{L^2}\right)dz \\
&\quad \times\int_0^1|\partial_z v_z|_{L^4}|v_{hz}|_{L^4}\,dz \lesssim \int_0^1(\varepsilon|v|_{H^2}|\xi|_{H^2} + |v|_{H^2})\,dz \\
&\quad \times\int_0^1\bigg((|\partial_z v_z|_{L^2}^{1/2}|\nabla_h\partial_z v_z|_{L^2}^{1/2} + |\partial_z v_z|_{L^2})(|v_{hz}|_{L^2}^{1/2}|\nabla_h v_{hz}|_{L^2}^{1/2} \\
&\quad + |v_{hz}|_{L^2})\bigg)dz \lesssim \delta\|\nabla\nabla_h v_z\|_{L^2}^2 + C_\delta(\varepsilon^2\|\xi\|_{H^2}^2\|v\|_{H^2}^2 \\
&\quad + \|v\|_{H^2}^2 + 1)\|v\|_{H^2}^2, \\
I_{15} &\lesssim \varepsilon e^{\varepsilon\alpha\|\xi\|_{L^\infty}}\|\nabla^3 v\|_{L^2}\|v_{hz}\|_{L^3}\|\xi_h\|_{L^6} \lesssim \delta\|\nabla^3 v\|_{L^2}^2 \\
&\quad + C_\delta\varepsilon^4 e^{4\varepsilon\alpha\|\xi\|_{H^2}}(\|\xi\|_{H^2}^4 + \|\xi\|_{H^2}^2)\|v\|_{H^2}^2, \\
I_{16} &\lesssim \varepsilon e^{\varepsilon\alpha\|\xi\|_{L^\infty}}\|\nabla_h\xi\|_{L^6}\|\nabla v_{hz}\|_{L^2}\|v_{hz}\|_{L^3} \lesssim \varepsilon e^{\varepsilon\alpha\|\xi\|_{H^2}}\|\xi\|_{H^2} \\
&\quad \|\nabla v_{hz}\|_{L^2}(\|v_{hz}\|_{L^2}^{1/2}\|\nabla v_{hz}\|_{L^2}^{1/2} + \|v_{hz}\|_{L^2}) \lesssim \delta\|\nabla v_{hz}\|_{L^2}^2 \\
&\quad + C_\delta(\varepsilon^4 e^{4\varepsilon\alpha\|\xi\|_{H^2}}\|\xi\|_{H^2}^4 + 1)\|v\|_{H^2}^2.
\end{aligned}
$$

Therefore, summing up the estimates above with $\partial_h \in \{\partial_x, \partial_y\}$ leads to,

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|\nabla_h v_z\|_{L^2}^2\right\} + c_1\big(\mu\|e^{-\varepsilon\alpha\xi/2}\nabla_h^2 v_z\|_{L^2}^2 \\
&\quad + \lambda\|e^{-\varepsilon\alpha\xi/2}\nabla_h div_h v_z\|_{L^2}^2 + \|e^{-\varepsilon\alpha\xi/2}\nabla_h v_{zz}\|_{L^2}^2\big) \\
&\quad \leq \delta\|\nabla^3 v\|_{L^2}^2 + C_\delta \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}).
\end{aligned} \tag{3.25}$$

Similarly, after taking the L^2-inner product of (3.19) with v_z and performing estimates as above, one has,

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|v_z\|_{L^2}^2\right\} + c_1\big(\mu\|e^{-\varepsilon\alpha\xi/2}\nabla_h v_z\|_{L^2}^2 + \lambda\|e^{-\varepsilon\alpha\xi/2}div_h v_z\|_{L^2}^2 \\
&\quad + \|e^{-\varepsilon\alpha\xi/2}v_{zz}\|_{L^2}^2\big) \leq \delta\|\nabla^3 v\|_{L^2}^2 + C_\delta \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}).
\end{aligned} \tag{3.26}$$

Consequently, after integrating in the time variable, (3.16), (3.18), (3.22), (3.25), and (3.26) yield (3.1).

Estimates on the time derivatives

Now we establish the final pieces of Proposition 1. After multiplying $(1.4)_1$ with $\varepsilon e^{\varepsilon\xi}$ and averaging the resulting equation in the z-variable, one has

$$\partial_t e^{\varepsilon\xi} + \varepsilon e^{\varepsilon\xi}\overline{v}\cdot\nabla_h\xi + (\gamma-1)e^{\varepsilon\xi}div_h\overline{v} = 0. \tag{3.27}$$

Then applying the triangle inequality and the Sobolev embedding inequality in (3.27) leads to

$$\begin{aligned}
\|\partial_t e^{\varepsilon\xi}\|_{L^2} &\lesssim \varepsilon e^{\varepsilon\|\xi\|_{L^\infty}}\|v\|_{L^\infty}\|\nabla_h\xi\|_{L^2} + e^{\varepsilon\|\xi\|_{L^\infty}}\|\nabla_h v\|_{L^2} \\
&\lesssim \varepsilon e^{\varepsilon\|\xi\|_{H^2}}\|v\|_{H^2}\|\xi\|_{H^2} + e^{\varepsilon\|\xi\|_{H^2}}\|v\|_{H^2} \\
&\lesssim \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}).
\end{aligned} \tag{3.28}$$

Meanwhile, we have, after applying ∂_h to (3.27),

$$\begin{aligned}
&\partial_t(e^{\varepsilon\xi})_h + \varepsilon^2 e^{\varepsilon\xi}\xi_h\overline{v}\cdot\nabla_h\xi + \varepsilon e^{\varepsilon\xi}\overline{v}_h\cdot\nabla_h\xi + \varepsilon e^{\varepsilon\xi}\overline{v}\cdot\nabla_h\xi_h \\
&\quad + (\gamma-1)\varepsilon e^{\varepsilon\xi}\xi_h div_h\overline{v} + (\gamma-1)e^{\varepsilon\xi}div_h\overline{v}_h = 0.
\end{aligned} \tag{3.29}$$

Thus it holds,

$$\begin{aligned}
&\|\partial_t(e^{\varepsilon\xi})_h\|_{L^2} \lesssim e^{\varepsilon\|\xi\|_{L^\infty}}\big(\varepsilon^2\|\nabla_h\xi\|_{L^4}^2\|v\|_{L^\infty} + \varepsilon\|\nabla_h\xi\|_{L^4} \\
&\quad \times \|\nabla_h v\|_{L^4} + \varepsilon\|v\|_{L^\infty}\|\nabla_h\xi_h\|_{L^2} + \|\nabla_h v_h\|_{L^2}\big) \\
&\quad \lesssim e^{\varepsilon\|\xi\|_{H^2}}\big(\varepsilon^2\|\xi\|_{H^2}^2\|v\|_{H^2} + \|v\|_{H^2}\big) \lesssim \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}).
\end{aligned} \tag{3.30}$$

Consequently, (3.28) and (3.30) imply (3.2).

On the other hand, after applying the projection operator $\mathcal{P}_\sigma$ (defined in (2.2)) to $(1.4)_2$, we have the following:

$$\begin{aligned} \partial_t \mathcal{P}_\sigma v = & -\mathcal{P}_\sigma(v \cdot \nabla_h v) - \mathcal{P}_\sigma(w\partial_z v) \\ & + c_1 \mathcal{P}_\sigma(e^{-\varepsilon\alpha\xi}(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v)). \end{aligned} \tag{3.31}$$

In order to estimate the L^2 and H^1 norms of $\partial_t \mathcal{P}_\sigma v$, we apply the Hölder and Sobolev embedding inequalities as follows:

$$\begin{aligned} & \| v \cdot \nabla_h v \|_{L^2} \lesssim \| v \|_{L^3} \| \nabla_h v \|_{L^6} \lesssim \| v \|_{H^2}^2, \\ & \| e^{-\varepsilon\alpha\xi}(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v) \|_{L^2} \lesssim e^{\varepsilon\alpha \| \xi \|_{L^\infty}} \\ & \qquad \times \| \mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v \|_{L^2} \lesssim e^{\varepsilon\alpha \| \xi \|_{H^2}} \| v \|_{H^2}, \\ & \| \partial(v \cdot \nabla_h v) \|_{L^2} \lesssim \| \partial v \cdot \nabla_h v \|_{L^2} + \| v \cdot \nabla_h \partial v \|_{L^2} \lesssim \| v \|_{H^2}^2, \\ & \| \partial(e^{-\varepsilon\alpha\xi}(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v)) \|_{L^2} \\ & \quad \lesssim \varepsilon \| e^{-\varepsilon\alpha\xi} \partial\xi(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v) \|_{L^2} \\ & \qquad + \| e^{-\varepsilon\alpha\xi} \partial(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v) \|_{L^2} \\ & \quad \lesssim e^{\varepsilon\alpha \| \xi \|_{H^2}} \big(\| \xi \|_{H^2} + 1 \big) \| v \|_{H^3}. \end{aligned} \tag{3.32}$$

Here $\partial \in \{\partial_x, \partial_y, \partial_z\}$ denotes the spatial derivatives. To estimate the L^2 and H^1 norms of $w\partial_z v$, we first substitute the identities in (3.8), (3.9), (3.10) to $w, \partial_z w, \partial_h w$, respectively, and write down the following:

$$\begin{aligned} w\partial_z v = & -\int_0^z \big(\varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi + div_h \widetilde{v} \big)\, dz \partial_z v, \\ \partial_h(w\partial_z v) = & \; \partial_h w \partial_z v + w \partial_z v_h \\ & = -\int_0^z \big(\varepsilon\alpha\widetilde{v}_h \cdot \nabla_h \xi + \varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi_h + div_h \widetilde{v}_h \big)\, dz \partial_z v \\ & \quad - \int_0^z \big(\varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi + div_h \widetilde{v} \big)\, dz \partial_z v_h, \\ \partial_z(w\partial_z v) = & \; \partial_z w \partial_z v + w \partial_z v_z \\ & = -(\varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi + div_h \widetilde{v}) \partial_z v - \int_0^z \big(\varepsilon\alpha\widetilde{v} \cdot \nabla_h \xi + div_h \widetilde{v} \big)\, dz \partial_z v_z. \end{aligned}$$

Therefore, after applying the Hölder, Minkowski, and Sobolev embedding inequalities, the following estimates hold:

$$
\begin{aligned}
&\|w\partial_z v\|_{L^2}^2 = \int_0^1 |w\partial_z v|_{L^2}^2\,dz \lesssim \int_0^1 \Bigg[\bigg(\int_0^1 (\varepsilon|v|_{L^8}|\nabla_h\xi|_{L^8} \\
&\quad + |\nabla_h v|_{L^4})\,dz\bigg)^2 \times |\partial_z v|_{L^4}^2\Bigg]\,dz \lesssim \int_0^1 \bigg((\varepsilon^2\|v\|_{H^1}^2\|\xi\|_{H^2}^2 \\
&\quad + \|v\|_{H^2}^2)|\partial_z v|_{H^1}^2\bigg)\,dz \lesssim (\varepsilon^2\|v\|_{H^1}^2\|\xi\|_{H^2}^2 + \|v\|_{H^2}^2)\|v\|_{H^2}^2, \\
&\|\partial_h(w\partial_z v)\|_{L^2}^2 = \int_0^1 |\partial_h(w\partial_z v)|_{L^2}^2\,dz \lesssim \int_0^1 \Bigg[\bigg(\int_0^1 (\varepsilon|v_h|_{L^4}|\nabla_h\xi|_{L^4} \\
&\quad + \varepsilon|v|_{L^\infty}|\nabla_h\xi_h|_{L^2} + |\nabla_h^2 v|_{L^2})\,dz\bigg)^2 \times |\partial_z v|_{L^\infty}^2\Bigg]\,dz \\
&\quad + \int_0^1 \Bigg[\bigg(\int_0^1 (\varepsilon|v|_{L^8}|\nabla_h\xi|_{L^8} + |\nabla_h v|_{L^4})\,dz\bigg)^2 \times |v_{hz}|_{L^4}^2\Bigg]\,dz \\
&\quad \lesssim (\varepsilon^2\|v\|_{H^2}^2\|\xi\|_{H^2}^2 + \|v\|_{H^2}^2)\|v\|_{H^3}^2, \\
&\|\partial_z(w\partial_z v)\|_{L^2}^2 \lesssim \|\partial_z w\partial_z v\|_{L^2}^2 + \int_0^1 |w\partial_z v_z|_{L^2}^2\,dz \\
&\quad \lesssim (\varepsilon\|v\|_{L^6}\|\nabla_h\xi\|_{L^6} + \|\nabla_h v\|_{L^3})^2\|\partial_z v\|_{L^6}^2 \\
&\quad + \int_0^1 \Bigg[\bigg(\int_0^1 (\varepsilon|v|_{L^8}|\nabla_h\xi|_{L^8} + |\nabla_h v|_{L^4})\,dz\bigg)^2 \times |v_{zz}|_{L^4}^2\Bigg]\,dz \\
&\quad \lesssim (\varepsilon^2\|v\|_{H^1}^2\|\xi\|_{H^2}^2 + \|v\|_{H^2}^2)\|v\|_{H^3}^2.
\end{aligned}
\tag{3.33}
$$

Combining the above estimates, together with (2.3), we have shown (3.3) and (3.4).

In addition, notice that $\partial_t\xi = \varepsilon^{-1}e^{-\varepsilon\xi}\partial_t e^{\varepsilon\xi}$. From (3.2), one has

$$
\begin{aligned}
&\|\partial_t\xi\|_{L^2} \lesssim \varepsilon^{-1}e^{\varepsilon\|\xi\|_{H^2}}\|\partial_t e^{\varepsilon\xi}\|_{L^2} \lesssim \varepsilon^{-1}\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}), \\
&\|\partial_t\xi\|_{H^1} \lesssim \|\partial_t\xi\|_{L^2} + \varepsilon^{-1}e^{\varepsilon\|\xi\|_{H^2}}(\|\xi_h\|_{L^3}\|\partial_t e^{\varepsilon\xi}\|_{L^6} + \|\partial_t(e^{\varepsilon\xi})_h\|_{L^2}) \\
&\quad \lesssim \|\partial_t\xi\|_{L^2} + \varepsilon^{-1}e^{\varepsilon\|\xi\|_{H^2}}(\|\xi\|_{H^2}\|\partial_t e^{\varepsilon\xi}\|_{H^1} + \|\partial_t(e^{\varepsilon\xi})_h\|_2) \\
&\quad \lesssim \varepsilon^{-1}\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}).
\end{aligned}
\tag{3.5}
$$

On the other hand, $(1.4)_2$, (3.32), and (3.33) yield

$$
\begin{aligned}
&\|\partial_t v\|_{L^2} \lesssim \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}) + \varepsilon^{-1}\|e^{\varepsilon\xi}\nabla_h\xi\|_{L^2} \\
&\quad \lesssim (1+\varepsilon^{-1})\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\|\partial_t v\|_{H^1} \lesssim\, & \mathcal{H}_1(\|\xi\|_{H^2}, \|v\|_{H^2})\|\nabla v\|_{H^2} \\
& + \mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}) + \varepsilon^{-1}\|e^{\varepsilon\xi}\nabla_h \xi\|_{H^1} \\
\lesssim\, & \mathcal{H}_1(\|\xi\|_{H^2}, \|v\|_{H^2})\|\nabla v\|_{H^2} \\
& + (1+\varepsilon^{-1})\mathcal{H}_2(\|\xi\|_{H^2}, \|v\|_{H^2}),
\end{aligned} \tag{3.7}
$$

where in the last inequality we have used the fact that

$$
\begin{aligned}
\|\partial_h(e^{\varepsilon\xi}\nabla_h\xi)\|_{L^2} & \lesssim \varepsilon\|e^{\varepsilon\xi}\xi_h\nabla_h\xi\|_{L^2} + \|e^{\varepsilon\xi}\nabla_h\xi_h\|_{L^2} \\
& \lesssim e^{\varepsilon\|\xi\|_{H^2}}\big(\varepsilon\|\xi_h\|_{L^3}\|\xi_h\|_{L^6} + \|\xi\|_{H^2}\big) \\
& \lesssim e^{\varepsilon\|\xi\|_{H^2}}\big(\varepsilon\|\xi\|_{H^2}^2 + \|\xi\|_{H^2}\big).
\end{aligned}
$$

This finishes the proof of Proposition 1.

3.2 Local-in-time *a priori* estimates and local well-posedness

In this subsection, we aim at establishing the following proposition:

Proposition 2. *In the case when either $\Omega_h = \mathbb{T}^2$ or $\Omega_h = \mathbb{R}^2$, consider initial data $(\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$, satisfying the compatibility conditions in (2.4). Let M_0, M_1 be two positive constants satisfying*

$$
\|\xi_0\|_{H^2}^2 \leq M_0,\ \|v_0\|_{H^2}^2 \leq M_1. \tag{3.34}
$$

Then for some positive constant $\varepsilon_0 \in (0,1)$ small enough, any $\varepsilon \in (0, \varepsilon_0)$, there exists $T_\varepsilon \in (0,\infty)$ such that there exists a unique strong solution (ξ, v) to (1.4) in the time interval $[0, T_\varepsilon]$ with

$$
\begin{gathered}
\xi \in L^\infty(0,T_\varepsilon; H^2(\Omega_h\times 2\mathbb{T})),\ \partial_t\xi \in L^\infty(0,T_\varepsilon; H^1(\Omega_h\times 2\mathbb{T})), \\
v \in L^\infty(0,T_\varepsilon; H^2(\Omega_h\times 2\mathbb{T})) \cap L^2(0,T_\varepsilon; H^3(\Omega_h\times 2\mathbb{T})), \\
\partial_t v \in L^\infty(0,T_\varepsilon; L^2(\Omega_h\times 2\mathbb{T})) \cap L^2(0,T_\varepsilon; H^1(\Omega_h\times 2\mathbb{T})).
\end{gathered} \tag{3.35}
$$

Moreover, there exist positive constants C_0, C_1 independent of ε, and $C_2 = C_2(\varepsilon, C_0M_0, C_1M_1)$ such that

$$
\begin{gathered}
\sup_{0\leq t\leq T_\varepsilon}\|\xi(t)\|_{H^2}^2 \leq C_0M_0,\quad \sup_{0\leq t\leq T_\varepsilon}\|v(t)\|_{H^2}^2 + \int_0^{T_\varepsilon}\|\nabla v(t)\|_{H^2}^2\,dt \leq C_1M_1, \\
\sup_{0\leq t\leq T_\varepsilon}\{\|\xi_t(t)\|_{H^1}^2 + \|\partial_t v(t)\|_{L^2}^2\} + \int_0^{T_\varepsilon}\|\partial_t v(t)\|_{H^1}^2\,dt \leq C_2.
\end{gathered} \tag{3.36}
$$

Here $C_0 \in (1,\infty)$, C_1, C_2 are determined by (3.57), (3.46), *and* (3.58), *below, and T_ε depends on M_0, M_1 and ε.*

Proposition 2 can be shown by applying the Banach fixed point theorem. In the following, without going into too much details, we will only sketch the proper steps to construct this local strong solution.

Sketch of constructing strong solutions. Let ξ', v' be regular enough functions. Consider the following linear system associated with (1.4):

$$\begin{cases} \partial_t \xi + \overline{v}' \cdot \nabla_h \xi + \dfrac{\gamma - 1}{\varepsilon} div_h \overline{v}' = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t v + v' \cdot \nabla_h v + w' \partial_z v + \dfrac{c^2 e^{\varepsilon \xi'}}{\varepsilon} \nabla_h \xi' & \\ \qquad = c_1 e^{-\varepsilon \alpha \xi'} \big(\mu \Delta_h v + \lambda \nabla_h div_h v + \partial_{zz} v \big) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \end{cases} \tag{3.37}$$

where w' is given by

$$w' := - \int_0^z (\varepsilon \alpha \widetilde{v}' \cdot \nabla_h \xi' + div_h \widetilde{v}') \, dz. \tag{3.38}$$

Then for (ξ', v') with $(\xi', v')\big|_{t=0} = (\xi_0, v_0)$ and satisfying the same regularity and bounds as in (3.35) and (3.36), there is a unique solution to system (3.37) with initial data $(\xi, v)\big|_{t=0} = (\xi_0, v_0)$, after applying the standard existence theory for linear hyperbolic and parabolic equations. Moreover, similar *a priori* estimates as in our previous work [28] show that the solution (ξ, v) satisfies the same regularity and bounds of norms in (3.35) and (3.36). We define the following function framework in order to apply the Banach fixed point theorem.

Consider the function space

$$\begin{aligned} \mathfrak{Y}_{T_\varepsilon} := \big\{ & (\xi, v) | \xi \in L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T})), \\ & v \in L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T})) \cap L^2(0, T_\varepsilon; H^1(\Omega_h \times 2\mathbb{T})), \\ & \partial_z \xi = 0 \big\} \end{aligned} \tag{3.39}$$

and the following subset of $\mathfrak{Y}_{T_\varepsilon}$

$$\mathfrak{X}_{T_\varepsilon} := \big\{ (\xi, v) | (\xi, v) \in \mathfrak{Y}_{T_\varepsilon} \text{ and the bounds in (3.36) hold} \big\}. \tag{3.40}$$

For any $(\xi, v) \in \mathfrak{Y}_{T_\varepsilon}$, define the norm

$$\begin{aligned} \| (\xi, v) \|_{\mathfrak{Y}_{T_\varepsilon}}^2 := & \| \xi \|_{L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2 + \| v \|_{L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2 \\ & + \| \nabla v \|_{L^2(0, T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2. \end{aligned} \tag{3.41}$$

Then $(\mathfrak{Y}_{T_\varepsilon}, \|\cdot\|_{\mathfrak{Y}_{T_\varepsilon}})$ is a complete metric space. In addition, we define the following map: with same initial data $(\xi', v')\big|_{t=0} = (\xi, v)\big|_{t=0} = (\xi_0, v_0)$,

$$\begin{aligned} \mathcal{T} : (\xi', v') \rightsquigarrow (\xi, v) \quad & \text{mapping } (\xi', v') \in \mathfrak{X}_{T_\varepsilon} \subset \mathfrak{Y}_{T_\varepsilon} \\ & \text{to the solution to system (3.37)} \end{aligned} \tag{3.42}$$

with trivial extension outside the set $\mathfrak{X}_{T_\varepsilon}$.

Then we show that, For $\varepsilon_0 \in (0,1)$ small enough and any $\varepsilon \in (0, \varepsilon_0)$, there exists $T_\varepsilon \in (0, \infty)$, such that $\mathcal{T} : \mathfrak{Y}_{T_\varepsilon} \mapsto \mathfrak{Y}_{T_\varepsilon}$ is a contraction mapping in $\mathfrak{X}_{T_\varepsilon}$. Notice, once this is proved, then the Banach fixed point theorem implies that we have a unique strong solution to (1.4) with $\varepsilon, T_\varepsilon$ as described above.

In the rest of this subsection, we focus on showing that for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough, and some $T_\varepsilon \in (0, \infty)$ depending on ε, we have:

(1) $\mathcal{T}$ maps $\mathfrak{X}_{T_\varepsilon}$ into $\mathfrak{X}_{T_\varepsilon}$; i.e., $(\xi, v) = \mathcal{T}(\xi', v') \in \mathfrak{X}_{T_\varepsilon}$ for any $(\xi', v') \in \mathfrak{X}_{T_\varepsilon}$;

(2) $\mathcal{T}$ is a contraction mapping in $\mathfrak{X}_{T_\varepsilon}$ with respect to the topology of $\mathfrak{Y}_{T_\varepsilon}$; i.e., for any $(\xi'_i, v'_i) \in \mathfrak{X}_{T_\varepsilon} \subset \mathfrak{Y}_{T_\varepsilon}$ and $(\xi_i, v_i) = \mathcal{T}(\xi'_i, v'_i)$, $i = 1, 2$, one has

$$\|(\xi_1 - \xi_2, v_1 - v_2)\|_{\mathfrak{Y}_{T_\varepsilon}} \leq q \|(\xi'_1 - \xi'_2, v'_1 - v'_2)\|_{\mathfrak{Y}_{T_\varepsilon}}, \tag{3.43}$$

for some $q \in (0, 1)$.

We will only show the corresponding *a priori* estimates to show (1) and (2).
Proof of (1): Estimates of ξ. First, we shall derive the estimates of ξ from $(3.37)_1$. We shall only show the highest order estimates. After applying ∂_{hh} to $(3.37)_1$, we have

$$\partial_t \xi_{hh} + \overline{v}' \cdot \nabla_h \xi_{hh} = -2\overline{v}'_h \cdot \nabla_h \xi_h - \overline{v}'_{hh} \cdot \nabla_h \xi - \frac{\gamma - 1}{\varepsilon} div_h \overline{v}'_{hh}. \tag{3.44}$$

Then after taking the L^2-inner product of (3.44) with $2\xi_{hh}$ in Ω_h, it follows

$$\begin{aligned} \frac{d}{dt} |\xi_{hh}|_{L^2}^2 = & \int_{\Omega_h} div_h \overline{v}' |\xi_{hh}|^2 \, dxdy - 4 \int_{\Omega_h} (\overline{v}'_h \cdot \nabla_h \xi_h) \xi_{hh} \, dxdy \\ & - 2 \int_{\Omega_h} (\overline{v}'_{hh} \cdot \nabla_h \xi) \xi_{hh} \, dxdy - \frac{2(\gamma - 1)}{\varepsilon} \int_{\Omega_h} div_h \overline{v}'_{hh} \xi_{hh} \, dxdy \\ \leq & \, C \|\nabla v'\|_{H^2} |\xi|_{H^2}^2 + C\varepsilon^{-1} \|\nabla v'\|_{H^2} |\xi_{hh}|_{L^2}. \end{aligned}$$

Similar arguments also hold for $\left|\xi_h\right|_{L^2}$ and $\left|\xi\right|_{L^2}$, and therefore we have

$$\frac{d}{dt}\left|\xi\right|^2_{H^2} \leq C\left\|\nabla v'\right\|_{H^2}\left|\xi\right|^2_{H^2} + C\varepsilon^{-1}\left\|\nabla v'\right\|_{H^2}\left|\xi\right|_{H^2}$$
$$\leq 2C\left\|\nabla v'\right\|_{H^2}\left|\xi\right|^2_{H^2} + C\varepsilon^{-2}\left\|\nabla v'\right\|_{H^2}.$$

Then applying the Grönwall inequality and the Hölder inequality yields

$$\sup_{0\leq t\leq T_\varepsilon}\left\|\xi(t)\right\|^2_{H^2} \leq e^{\int_0^{T_\varepsilon} 2C\left\|\nabla v'\right\|_{H^2}dt}\Bigg(\left\|\xi_0\right\|^2_{H^2}$$
$$+ C\varepsilon^{-2}\int_0^{T_\varepsilon}\left\|\nabla v'\right\|_{H^2}dt\Bigg) \leq e^{2CT_\varepsilon^{1/2}(\int_0^{T_\varepsilon}\left\|\nabla v'\right\|^2_{H^2}dt)^{1/2}}\Bigg(\left\|\xi_0\right\|^2_{H^2}$$
$$+ C\varepsilon^{-2}T_\varepsilon^{1/2}(\int_0^{T_\varepsilon}\left\|\nabla v'\right\|^2_{H^2}dt)^{1/2}\Bigg)$$
$$\leq e^{2CT_\varepsilon^{1/2}(C_1M_1)^{1/2})}(M_0 + C\varepsilon^{-2}T_\varepsilon^{1/2}(C_1M_1)^{1/2}) \leq C_0M_0, \tag{3.45}$$

where we have chosen T_ε sufficiently small in the last inequality and $C_0 \in (1,\infty)$.

On the other hand, from $(3.37)_1$, we have

$$\left|\partial_t\xi\right|^2_{H^1} \leq C\left|\overline{v}'\cdot\nabla_h\xi\right|^2_{H^1} + \varepsilon^{-2}C\left|\nabla_h\overline{v}'\right|^2_{H^1} \leq C(\left\|\xi\right\|^2_{H^2} + \varepsilon^{-2})\left\|v'\right\|^2_{H^2}$$
$$\leq C(C_0M_0 + \varepsilon^{-2})C_1M_1 < \infty. \tag{3.46}$$

Estimates of v. Next we shall present the estimates of v. Similarly, we will only sketch the highest order estimates. After applying ∂_{hh} to $(3.37)_2$, one obtains

$$\partial_t v_{hh} + v'\cdot\nabla_h v_{hh} + w'\partial_z v_{hh} - c_1 e^{-\varepsilon\alpha\xi'}(\mu\Delta_h v_{hh} + \lambda\nabla_h div_h v_{hh} + \partial_{zz}v_{hh})$$
$$= -\big(\frac{c^2e^{\varepsilon\xi'}}{\varepsilon}\nabla_h\xi'\big)_{hh} - v'_{hh}\cdot\nabla_h v - 2v'_h\cdot\nabla_h v_h - w'_{hh}\partial_z v - 2w'_h\partial_z v_h$$
$$- \varepsilon c_1\alpha e^{-\varepsilon\alpha\xi'}(\xi'_{hh} - \varepsilon\alpha(\xi'_h)^2)(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz}v)$$
$$- 2\varepsilon c_1\alpha e^{-\varepsilon\alpha\xi'}\xi'_h(\mu\Delta_h v_h + \lambda\nabla_h div_h v_h + \partial_{zz}v_h). \tag{3.47}$$

Then after taking the L^2-inner product of (3.47) with $2v_{hh}$ in $\Omega_h\times 2\mathbb{T}$, it follows, after applying integration by parts,

$$\frac{d}{dt}\left\|v_{hh}\right\|^2_{L^2} + 2c_1\int e^{-\varepsilon\alpha\xi'}(\mu|\nabla_h v_{hh}|^2 + \lambda|div_h v_{hh}|^2 + |\partial_z v_{hh}|^2)\,d\vec{x}$$
$$= \int\big(\frac{2c^2e^{\varepsilon\xi'}}{\varepsilon}\nabla_h\xi'\big)_h\cdot v_{hhh}\,d\vec{x} + \int\Bigg(div_h v'|v_{hh}|^2 - 2(v'_{hh}\cdot\nabla_h v)\cdot v_{hh}$$

$$
\begin{aligned}
&- 4(v_h' \cdot \nabla_h v_h) \cdot v_{hh} - div_h \widetilde{v}' |v_{hh}|^2 \Big) \, d\vec{x} - 2 \int \Big(w_h' \partial_z v_h \cdot v_{hh} \\
&- w_h' \partial_z v \cdot v_{hhh} \Big) \, d\vec{x} - \varepsilon\alpha \int (\widetilde{v}' \cdot \nabla_h \xi') |v_{hh}|^2 \, d\vec{x} \\
&- 2\varepsilon c_1 \alpha \int e^{-\varepsilon\alpha\xi'} (\xi_{hh}' - \varepsilon\alpha(\xi_h')^2)(\mu\Delta_h v + \lambda\nabla_h div_h v + \partial_{zz} v) \cdot v_{hh} \, d\vec{x} \\
&- 4\varepsilon c_1 \alpha \int e^{-\varepsilon\alpha\xi'} \xi_h' (\mu\Delta_h v_h + \lambda\nabla_h div_h v_h + \partial_{zz} v_h) \cdot v_{hh} \, d\vec{x} \\
&+ 2\varepsilon c_1 \alpha \int e^{-\varepsilon\alpha\xi'} (\mu(\nabla_h \xi' \cdot \nabla_h v_{hh}) \cdot v_{hh} + \lambda(v_{hh} \cdot \nabla_h \xi')(div_h v_{hh})) \, d\vec{x} \\
&=: L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7,
\end{aligned}
$$

where we have substituted the identity

$$
w_z' = -\varepsilon\alpha\widetilde{v}' \cdot \nabla_h \xi' - div_h \widetilde{v}',
$$

which is obtained by taking ∂_z to (3.38),

Similarly as in the last section, from (3.38), we have

$$
w_h' = -\int_0^z \big(\varepsilon\alpha\widetilde{v}_h' \cdot \nabla_h \xi' + \varepsilon\alpha\widetilde{v}' \cdot \nabla_h \xi_h' + div_h \widetilde{v}_h' \big) \, dz. \tag{3.48}
$$

Then following similar arguments as before, one can derive, for any $\delta \in (0,1)$ and fixed $\varepsilon \in (0,1)$,

$$
\begin{aligned}
&L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 \leq \big(\delta + \varepsilon\mathcal{H}_1(\| \xi' \|_{H^2}, \| v' \|_{H^2}) \big) \\
&\quad \times \| \nabla v \|_{H^2}^2 + \delta \| \nabla v' \|_{H^2}^2 \| v \|_{H^2}^2 + \mathcal{H}_2(\varepsilon, C_\delta, \| \xi' \|_{H^2}, \| v' \|_{H^2}) \| v \|_{H^2}^2 \\
&\quad + \mathcal{H}_3(\varepsilon, C_\delta, \| \xi' \|_{H^2}, \| v' \|_{H^2}),
\end{aligned} \tag{3.49}
$$

where $C_\delta \simeq \delta^{-1}$. We remind readers that we have been using $\{\mathcal{H}_i\}_{i=1,2,3}$ to denote regular functions of the arguments with property $\mathcal{H}_i(0) = 0$. Details are listed below, for readers' reference:

$$
\begin{aligned}
L_1 &\lesssim \| v_{hhh} \|_{L^2} \big(\varepsilon^{-1} e^{\varepsilon \| \xi' \|_{H^2}} \| \xi' \|_{H^2} + e^{\varepsilon \| \xi' \|_{H^2}} \| \xi' \|_{H^2}^2 \big) \\
&\lesssim \delta \| \nabla v \|_{H^2}^2 + C_\delta e^{2\varepsilon \| \xi' \|_{H^2}} \big(\varepsilon^{-2} \| \xi' \|_{H^2}^2 + \| \xi' \|_{H^2}^4 \big), \\
L_2 &\lesssim \| \nabla v' \|_{L^6} \| v_{hh} \|_{L^3} \| v_{hh} \|_{L^2} + \| v_{hh}' \|_{L^2} \| \nabla_h v \|_{L^3} \| v_{hh} \|_{L^6} \\
&\lesssim \| v' \|_{H^2} \big(\| v \|_{H^2}^{3/2} \| \nabla v \|_{H^2}^{1/2} + \| v \|_{H^2} \| \nabla v \|_{H^2} \big) \\
&\lesssim \delta \| \nabla v \|_{H^2}^2 + C_\delta \big(\| v' \|_{H^2}^2 + 1 \big) \| v \|_{H^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
L_3 &\lesssim \int_0^1 \Big(\varepsilon \big|v'_h\big|_{L^4} \big|\xi'_h\big|_{L^4} + \varepsilon \big|v'\big|_{L^\infty} \big|\xi'\big|_{H^2} \Big)\, dz \cdot \int_0^1 \Big(\big|\partial_z v_h\big|_{L^4} \big|v_{hh}\big|_{L^4} \\
&\quad + \big|\partial_z v\big|_{L^\infty} \big|v_{hhh}\big|_{L^2} \Big)\, dz + \int_0^1 \big|v'_{hh}\big|_{L^4}\, dz \cdot \int_0^1 \Big(\big|\partial_z v_h\big|_{L^2} \big|v_{hh}\big|_{L^4} \\
&\quad + \big|\partial_z v\big|_{L^4} \big|v_{hhh}\big|_{L^2} \Big)\, dz \lesssim \int_0^1 \varepsilon \big|v'\big|_{H^2} \big|\xi'\big|_{H^2}\, dz \cdot \int_0^1 \Big(\big|\nabla v\big|_{H^2} (\big|\nabla v\big|_{H^1} \\
&\qquad + \big|\nabla v\big|_{H^2}) \Big)\, dz + \int_0^1 \big|v'\big|_{H^2}^{1/2} \big|\nabla v'\big|_{H^2}^{1/2}\, dz \cdot \int_0^1 \Big(\big|\nabla v\big|_{H^1}^{3/2} \big|\nabla v\big|_{H^2}^{1/2} \\
&\qquad + \big|\nabla v\big|_{H^1} \big|\nabla v\big|_{H^2} \Big)\, dz \lesssim \big(\varepsilon \big\|\xi'\big\|_{H^2} \big\|v'\big\|_{H^2} + \delta\big) \big\|\nabla v\big\|_{H^2}^2 \\
&\quad + \delta \big\|\nabla v'\big\|_{H^2}^2 \big\|v\big\|_{H^2}^2 + \varepsilon \big\|\xi'\big\|_{H^2} \big\|v'\big\|_{H^2} \big\|v\big\|_{H^2}^2 \\
&\quad + C_\delta (\big\|v'\big\|_{H^2}^2 + 1) \big\|v\big\|_{H^2}^2, \\
L_4 &\lesssim \varepsilon \big\|v'\big\|_{H^2} \big\|\nabla_h \xi'\big\|_{L^6} \big\|v_{hh}\big\|_{L^2} \big\|v_{hh}\big\|_{L^3} \lesssim \varepsilon \big\|v'\big\|_{H^2} \\
&\qquad \times \big\|\xi'\big\|_{H^2} \big\|v\big\|_{H^2}^{3/2} \big\|\nabla v\big\|_{H^2}^{1/2} \lesssim \delta \big\|\nabla v\big\|_{H^2}^2 \\
&\quad + C_\delta \varepsilon^{4/3} \big\|v'\big\|_{H^2}^{4/3} \big\|\xi'\big\|_{H^2}^{4/3} \big\|v\big\|_{H^2}^2, \\
L_5 &\lesssim \varepsilon e^{\varepsilon \|\xi'\|_{H^2}} \big(\big\|\xi'\big\|_{H^2} + \varepsilon \big\|\xi'\big\|_{H^2}^2 \big) \big\|\nabla^2 v\big\|_{L^3} \big\|v_{hh}\big\|_{L^6} \lesssim \varepsilon e^{\varepsilon \|\xi'\|_{H^2}} \\
&\qquad \times \big(\big\|\xi'\big\|_{H^2} + \varepsilon \big\|\xi'\big\|_{H^2}^2 \big) \times \big\|v\big\|_{H^2}^{1/2} \big\|\nabla v\big\|_{H^2}^{3/2} \lesssim \delta \big\|\nabla v\big\|_{H^2}^2 \\
&\quad + C_\delta \varepsilon^4 \big(\big\|\xi'\big\|_{H^2}^4 + \varepsilon^4 \big\|\xi'\big\|_{H^2}^8 \big) e^{4\varepsilon \|\xi'\|_{H^2}} \big\|v\big\|_{H^2}^2, \\
L_6 + L_7 &\lesssim \varepsilon e^{\varepsilon \|\xi'\|_{H^2}} \big\|\xi'\big\|_{H^2} \big\|\nabla v\big\|_{H^2}^{3/2} \big\|v\big\|_{H^2}^{1/2} \lesssim \delta \big\|\nabla v\big\|_{H^2}^2 \\
&\quad + \varepsilon^4 \big\|\xi'\big\|_{H^2}^4 e^{4\varepsilon \|\xi'\|_{H^2}} \big\|v\big\|_{H^2}^2.
\end{aligned}
$$

Similar estimates also hold for $\|v_{hz}\|_{L^2}$, $\|v_{zz}\|_{L^2}$, $\|v_h\|_{L^2}$, $\|v_z\|_{L^2}$, $\|v\|_{L^2}$. Then we have arrived at the estimate

$$
\begin{aligned}
\frac{d}{dt} \big\|v\big\|_{H^2}^2 &+ c_1 \min\{\mu, \lambda, 1\} \big\|\nabla v\big\|_{H^2}^2 \le \big(\delta + \varepsilon \mathcal{H}_1(\big\|\xi'\big\|_{H^2}, \big\|v'\big\|_{H^2})\big) \\
&\times \big\|\nabla v\big\|_{H^2}^2 + \delta \big\|\nabla v'\big\|_{H^2}^2 \big\|v\big\|_{H^2}^2 + \mathcal{H}_2(\varepsilon, C_\delta, \big\|\xi'\big\|_{H^2}, \big\|v'\big\|_{H^2}) \big\|v\big\|_{H^2}^2 \\
&+ \mathcal{H}_3(\varepsilon, C_\delta, \big\|\xi'\big\|_{H^2}, \big\|v'\big\|_{H^2}),
\end{aligned} \tag{3.50}
$$

where we have chosen $\varepsilon \in (0, \varepsilon_0)$, with ε_0 small enough such that

$$
\varepsilon\alpha \big\|\xi'\big\|_{H^2} \le \varepsilon\alpha (C_0 M_0)^{1/2} \le \log 2, \quad \text{and thus } 1/2 \le e^{-\varepsilon\alpha\xi'} < 2. \tag{3.51}
$$

Next, we choose ε_0, δ small enough such that

$$\begin{aligned}\delta + \varepsilon_0 \mathcal{H}_1(\|\xi'\|_{H^2}, \|v'\|_{H^2}) &\le \delta + \varepsilon_0 \mathcal{H}_1((C_0M_0)^{1/2}, (C_1M_1)^{1/2}) \\ &\le \frac{c_1 \min\{\mu, \lambda, 1\}}{2}.\end{aligned} \tag{3.52}$$

Then for $\varepsilon \in (0, \varepsilon_0)$, after applying the Grönwall inequality to (3.50), we have

$$\begin{aligned}&\sup_{0\le t\le T_\varepsilon} \|v(t)\|_{H^2}^2 + \frac{c_1 \min\{\mu, \lambda, 1\}}{2} \int_0^{T_\varepsilon} \|\nabla v\|_{H^2}^2 \, dt \\ &\le e^{\delta \int_0^{T_\varepsilon} \|\nabla v'\|_{H^2}^2 \, dt + T_\varepsilon \mathcal{H}_2(\varepsilon, C_\delta, \|\xi'\|_{H^2}, \|v'\|_{H^2})} \\ &\times \big(\|v_0\|_{H^2}^2 + T_\varepsilon \mathcal{H}_3(\varepsilon, C_\delta, \|\xi'\|_{H^2}, \|v'\|_{H^2})\big).\end{aligned} \tag{3.53}$$

Now we let δ small enough such that

$$\delta \int_0^{T_\varepsilon} \|\nabla v'\|_{H^2}^2 \, dt \le \delta C_1 M_1 \le \log 2. \tag{3.54}$$

Then for fixed ε_0, δ satisfying (3.52) and (3.54), and $\varepsilon \in (0, \varepsilon_0)$, let T_ε small enough, depending on ε, δ, such that

$$\begin{aligned}T_\varepsilon \mathcal{H}_2(\varepsilon, C_\delta, \|\xi'\|_{H^2}, \|v'\|_{H^2}) &\le T_\varepsilon \mathcal{H}_2(\varepsilon, C_\delta, (C_0M_0)^{1/2}, (C_1M_1)^{1/2}) < \log 2, \\ T_\varepsilon \mathcal{H}_3(\varepsilon, C_\delta, \|\xi'\|_{H^2}, \|v'\|_{H^2}) &\le T_\varepsilon \mathcal{H}_3(\varepsilon, C_\delta, (C_0M_0)^{1/2}, (C_1M_1)^{1/2}) < \frac{M_1}{2}.\end{aligned} \tag{3.55}$$

Then (3.53) yields

$$\begin{aligned}&\sup_{0\le t\le T_\varepsilon} \|v(t)\|_{H^2}^2 + \frac{c_1 \min\{\mu, \lambda, 1\}}{2} \int_0^{T_\varepsilon} \|\nabla v\|_{H^2}^2 \, dt \le 4(M_1 + M_1/2) \\ &= 6M_1 \le \min\left\{\frac{1}{3}, \frac{c_1 \min\{\mu, \lambda, 1\}}{6}\right\} C_1 M_1.\end{aligned} \tag{3.56}$$

Here we have required C_1 to be sufficiently large such that

$$6 \le \min\left\{\frac{1}{3}, \frac{c_1 \min\{\mu, \lambda, 1\}}{6}\right\} C_1. \tag{3.57}$$

On the other hand, from $(3.37)_2$, we have

$$\begin{aligned}\|\partial_t v\|_{L^2}^2 &\lesssim \big(\|v'\|_{H^2}^2 + e^{2\varepsilon\alpha \|\xi'\|_{H^2}} + \varepsilon^2 \|v'\|_{H^1}^2 \|\xi'\|_{H^2}^2\big) \|v\|_{H^2}^2 \\ &+ \varepsilon^{-2} e^{2\varepsilon \|\xi'\|_{H^2}} \|\xi'\|_{H^1}^2,\end{aligned}$$

$$
\begin{aligned}
\|\partial_t v\|_{H^1}^2 &\lesssim \big(\|v'\|_{H^2}^2 + e^{2\varepsilon\alpha\|\xi'\|_{H^2}} + \varepsilon^2\|v'\|_{H^1}^2\|\xi'\|_{H^2}^2\big)\|v\|_{H^2}^2 \\
&\quad + \big((\|\xi'\|_{H^2}^2+1)e^{2\varepsilon\alpha\|\xi'\|_{H^2}} + \varepsilon^2\|\xi'\|_{H^2}^2\|v'\|_{H^2}^2 + \|v'\|_{H^2}^2\big)\|\nabla v\|_{H^2}^2 \\
&\quad + \varepsilon^{-2}e^{2\varepsilon\|\xi'\|_{H^2}}\|\xi'\|_{H^2}^2 + e^{2\varepsilon\|\xi'\|_{H^2}}\|\xi'\|_{H^2}^4,
\end{aligned}
$$

after applying similar arguments as in (3.32) and (3.33). Then we have the following

$$
\begin{aligned}
&\|v_t\|_{L^2}^2 \leq \mathcal{H}_1(\varepsilon, \|\xi'\|_{H^2}, \|v'\|_{H^2})\|v\|_{H^2}^2 + \mathcal{H}_2(\varepsilon, \|\xi'\|_{H^2}) \\
&\quad \leq \mathcal{H}_1(\varepsilon, (C_0M_0)^{1/2}, (C_1M_1)^{1/2})6M_1 + \mathcal{H}_2(\varepsilon, (C_0M_0)^{1/2}) < \infty, \\
&\int_0^T \|v_t\|_{H^1}^2 \leq \int_0^T \Big(\mathcal{H}_1(\varepsilon, (C_0M_0)^{1/2}, (C_1M_1)^{1/2})6M_1 \\
&\quad + \mathcal{H}_2(\varepsilon, (C_0M_0)^{1/2}, (C_1M_1)^{1/2})\|\nabla v\|_{H^2}^2 + \mathcal{H}_3(\varepsilon, (C_0M_0)^{1/2})\Big)\,dt < \infty.
\end{aligned} \tag{3.58}
$$

This finishes the proof of (1).

Proof of (2)**:** Denote by $(\xi_{12}, v_{12}) := (\xi_1 - \xi_2, v_1 - v_2)$ and $(\xi'_{12}, v'_{12}) := (\xi'_1 - \xi'_2, v'_1 - v'_2)$. Then (ξ_{12}, v_{12}) satisfies the following system:

$$
\begin{cases}
\partial_t \xi_{12} = -\overline{v}'_1 \cdot \nabla_h \xi_{12} - \overline{v}'_{12} \cdot \nabla_h \xi_2 - \dfrac{\gamma-1}{\varepsilon} div_h \overline{v}'_{12} & \text{in } \Omega_h \times 2\mathbb{T}, \\
\partial_t v_{12} - c_1\big(\mu\Delta_h v_{12} + \lambda\nabla_h div_h v_{12} + \partial_{zz} v_{12}\big) & \\
\quad = -v'_1 \cdot \nabla_h v_{12} - v'_{12} \cdot \nabla_h v_2 - w'_1 \partial_z v_{12} - (w'_1 - w'_2)\partial_z v_2 & \\
\qquad - \dfrac{c^2 e^{\varepsilon\xi'_1}}{\varepsilon}\nabla_h \xi'_{12} - \dfrac{c^2 e^{\varepsilon\xi'_1} - c^2 e^{\varepsilon\xi'_2}}{\varepsilon}\nabla_h \xi'_2 + c_1\big(e^{-\varepsilon\alpha\xi'_1} - 1\big) & \\
\qquad\quad \times \big(\mu\Delta_h v_{12} + \lambda\nabla_h div_h v_{12} + \partial_{zz} v_{12}\big) & \\
\qquad + c_1\big(e^{-\varepsilon\alpha\xi'_1} - e^{-\varepsilon\alpha\xi'_2}\big)\big(\mu\Delta_h v_2 + \lambda\nabla_h div_h v_2 + \partial_{zz} v_2\big) & \text{in } \Omega_h \times 2\mathbb{T},
\end{cases} \tag{3.59}
$$

with $(\xi_{12}, v_{12})\big|_{t=0} = 0$. Then after taking the L^2-inner product of $(3.59)_1$ with $2\xi_{12}$ in Ω_h, one has, for any $\delta \in (0,1)$ with corresponding $C_\delta \simeq \delta^{-1}$,

$$
\begin{aligned}
\frac{d}{dt}|\xi_{12}|_{L^2}^2 &= \int_{\Omega_h} div_h \overline{v}'_1 |\xi_{12}|^2\,dxdy - 2\int_{\Omega_h} (\overline{v}'_{12} \cdot \nabla_h \xi_2)\xi_{12}\,dxdy \\
&\quad - \frac{2(\gamma-1)}{\varepsilon}\int_{\Omega_h} div_h \overline{v}'_{12}\xi_{12}\,dxdy \lesssim |\nabla_h \overline{v}'_1|_{L^\infty}|\xi_{12}|_{L^2}^2 \\
&\quad + |\overline{v}'_{12}|_{L^4}|\nabla_h \xi_2|_{L^4}|\xi_{12}|_{L^2} + |\nabla_h \overline{v}'_{12}|_{L^2}|\xi_{12}|_{L^2} \lesssim \delta\|\nabla v'_{12}\|_{L^2}^2 \\
&\quad + C_\delta\big(1 + \|\nabla v'_1\|_{H^2} + |\xi_2|_{H^2}^2\big)|\xi_{12}|_{L^2}^2 + C_\delta\|v'_{12}\|_{L^2}^2,
\end{aligned}
$$

where we have applied the Hölder, Sobolev embedding and Young inequalities. Therefore, applying the Grönwall inequality in the above inequality yields,

$$\begin{aligned}
&\sup_{0\leq t\leq T_\varepsilon} \left\| \xi_{12} \right\|_{L^2}^2 \leq e^{C_\delta \int_0^{T_\varepsilon} (1+\left\| \nabla v_1' \right\|_{H^2} + \left\| \xi_2 \right\|_{H^2}^2)\, dt} \\
&\quad \times \int_0^{T_\varepsilon} \big(\delta \left\| \nabla v_{12}' \right\|_{L^2}^2 + C_\delta \left\| v_{12}' \right\|_{L^2}^2\big)\, dt \\
&\quad \leq e^{C_\delta (1+C_0 M_0) T_\varepsilon + C_\delta (C_1 M_1)^{\frac{1}{2}} T_\varepsilon^{\frac{1}{2}}} \big(\delta \left\| \nabla v_{12}' \right\|_{L^2(0,T_\varepsilon;L^2(\Omega_h\times 2\mathbb{T}))}^2 \\
&\quad + C_\delta T_\varepsilon \left\| v_{12}' \right\|_{L^\infty(0,T_\varepsilon;L^2(\Omega_h\times 2\mathbb{T}))}^2\big).
\end{aligned} \tag{3.60}$$

On the other hand, after taking the L^2-inner product of $(3.59)_2$ with $2v_{12}$ in $\Omega_h \times 2\mathbb{T}$ and applying integration by parts, one has

$$\begin{aligned}
&\frac{d}{dt} \left\| v_{12} \right\|_{L^2}^2 + c_1 \big(\mu \left\| \Delta_h v_{12} \right\|_{L^2}^2 + \lambda \left\| div_h v_{12} \right\|_{L^2}^2 + \left\| \partial_z v_{12} \right\|_{L^2}^2\big) \\
&\quad = \int (div_h v_1' - \varepsilon\alpha \widetilde{v}_1' \cdot \nabla_h \xi_1' - div_h \widetilde{v}_1') |v_{12}|^2 \, d\vec{x} - \int (v_{12}' \cdot \nabla_h v_2) \cdot v_{12} \, d\vec{x} \\
&\quad + \int \int_0^z \big(\varepsilon\alpha \widetilde{v}_{12}' \cdot \nabla_h \xi_1' + div_h \widetilde{v}_{12}'\big)\, dz \times (\partial_z v_2 \cdot v_{12}) \, d\vec{x} \\
&\quad - \int \bigg(\int_0^z (\varepsilon\alpha (div_h \widetilde{v}_2') \xi_{12}') \, dz \times (\partial_z v_2 \cdot v_{12}) \\
&\qquad + \int_0^z (\varepsilon\alpha \xi_{12}' \widetilde{v}_2') \, dz \cdot \nabla_h (\partial_z v_2 \cdot v_{12}) \bigg) \, d\vec{x} \\
&\quad + \int \xi_{12}' \big(\frac{c^2 e^{\varepsilon \xi_1'}}{\varepsilon} div_h v_{12} + c^2 e^{\varepsilon \xi_1'} v_{12} \cdot \nabla_h \xi_1' \big) \, d\vec{x} \\
&\quad - \int \frac{c^2 (e^{\varepsilon\xi_1'} - e^{\varepsilon \xi_2'})}{\varepsilon} \nabla_h \xi_2' \cdot v_{12} \, d\vec{x} - c_1 \int \bigg(\big(e^{-\varepsilon\alpha\xi_1'} - 1\big) \big(\mu |\nabla_h v_{12}|^2 \\
&\quad + \lambda |div_h v_{12}|^2 + |\partial_z v_{12}|^2\big) \bigg) \, d\vec{x} + c_1 \varepsilon\alpha \int \bigg(e^{-\varepsilon\alpha\xi_1'} \big(\mu (\nabla_h \xi_1' \cdot \nabla_h v_{12}) \cdot v_{12} \\
&\quad + \lambda (v_{12} \cdot \nabla_h \xi_1')(div_h v_{12}) \bigg) \, d\vec{x} + c_1 \int \bigg(\big(e^{-\varepsilon\alpha\xi_1'} - e^{-\varepsilon\alpha\xi_2'}\big) \\
&\quad \times \big(\mu \Delta_h v_2 + \lambda \nabla_h div_h v_2 + \partial_{zz} v_2\big) \cdot v_{12} \bigg) \, d\vec{x} =: L_8 + L_9 + L_{10} + L_{11} \\
&\quad + L_{12} + L_{13} + L_{14} + L_{15} + L_{16},
\end{aligned}$$

where we have substituted (3.38) for w_i', $i = 1, 2$. Consequently, after applying similar arguments as before, we have the following estimates of the terms on the right-hand side: for any $\delta \in (0,1)$ and some constant $C_\delta \simeq \delta^{-1}$,

$$
\begin{aligned}
L_8 &\lesssim \big(\| \nabla_h v_1' \|_{L^3} + \| v_1' \|_{L^6} \| \nabla_h \xi_1' \|_{L^6} \big) \| v_{12} \|_{L^2} \| v_{12} \|_{L^6} \\
&\lesssim \big(\| v_1' \|_{H^2} + \| v_1' \|_{H^1} \| \xi_1' \|_{H^2} \big) \| v_{12} \|_{L^2} \big(\| \nabla v_{12} \|_{L^2} + \| v_{12} \|_{L^2} \big) \\
&\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, C_0 M_0, C_1 M_1) \| v_{12} \|_{L^2}^2, \\
L_9 &\lesssim \| v_{12}' \|_{L^2} \| \nabla_h v_2 \|_{L^6} \| v_{12} \|_{L^3} \lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + C_\delta \| v_{12} \|_{L^2}^2 \\
&\quad + C_\delta \| v_2 \|_{H^2}^2 \| v_{12}' \|_{L^2}^2, \\
L_{10} &\lesssim \int_0^1 \big(\varepsilon |v_{12}'|_{L^4} |\nabla_h \xi_1'|_{L^4} + |\nabla_h v_{12}'|_{L^2} \big) \, dz \times \int_0^1 |\partial_z v_2|_{L^4} |v_{12}|_{L^4} \, dz \\
&\lesssim \big(\varepsilon \| v_{12}' \|_{L^2}^{1/2} \| v_{12}' \|_{H^1}^{1/2} \| \xi_1' \|_{H^2} + \| \nabla_h v_{12}' \|_{L^2} \big) \\
&\quad \times \| \partial_z v_2 \|_{H^1} \| v_{12} \|_{L^2}^{1/2} \| v_{12} \|_{H^1}^{1/2} \lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \delta \| \nabla v_{12}' \|_{L^2}^2 \\
&\quad + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \big(\| v_{12} \|_{L^2}^2 + \| v_{12}' \|_{L^2}^2 \big), \\
L_{11} &\lesssim \varepsilon \int_0^1 |\nabla_h v_2'|_{L^8} |\xi_{12}'|_{L^2} \, dz \times \int_0^1 |\partial_z v_2|_{L^8} |v_{12}|_{L^4} \, dz \\
&\quad + \varepsilon \int_0^1 |\xi_{12}'|_{L^2} |v_2'|_{L^\infty} \, dz \times \int_0^1 \big(|\partial_z v_2|_{L^\infty} |\nabla_h v_{12}|_{L^2} \\
&\quad + |\nabla_h \partial_z v_2|_{L^4} |v_{12}|_{L^4} \big) \, dz \lesssim \varepsilon \| v_2' \|_{H^2} \| \xi_{12}' \|_{L^2} \\
&\quad \times \big(\| v_2 \|_{H^2} \| v_{12} \|_{L^2}^{1/2} \| v_{12} \|_{H^1}^{1/2} + \| \partial_z v_2 \|_{L^2}^{1/2} \| \partial_z v_2 \|_{H^2}^{1/2} \| \nabla_h v_{12} \|_{L^2} \\
&\quad + \| \nabla^2 v_2 \|_{L^2}^{1/2} \| \nabla^2 v_2 \|_{H^1}^{1/2} \| v_{12} \|_{L^2}^{1/2} \| v_{12} \|_{H^1}^{1/2} \big) \lesssim \delta \| \nabla_h v_{12} \|_{L^2}^2 \\
&\quad + \| v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \| \nabla v_2 \|_{H^2} \| \xi_{12}' \|_{L^2}^2, \\
L_{12} &\lesssim \| \xi_{12}' \|_{L^2} \big(\varepsilon^{-1} e^{\varepsilon \| \xi_1' \|_{H^2}} \| \nabla_h v_{12} \|_{L^2} + e^{\varepsilon \| \xi_1' \|_{H^2}} \| v_{12} \|_{L^3} \| \nabla_h \xi_1' \|_{L^6} \\
&\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \big(\| \xi_{12}' \|_{L^2}^2 + \| v_{12} \|_{L^2}^2 \big), \\
L_{13} &\lesssim e^{\varepsilon (\| \xi_1' \|_{H^2} + \| \xi_2' \|_{H^2})} \| \xi_{12}' \|_{L^2} \| \nabla_h \xi_2' \|_{L^6} \| v_{12} \|_{L^3} \\
&\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \big(\| \xi_{12}' \|_{L^2}^2 + \| v_{12} \|_{L^2}^2 \big), \\
L_{14} &\lesssim \varepsilon e^{\varepsilon \alpha \| \xi_1' \|_{H^2}} \| \xi_1' \|_{H^2} \| \nabla v_{12} \|_{L^2}^2 \lesssim \varepsilon e^{\varepsilon \alpha (C_0 M_0)^{1/2}} (C_0 M_0)^{1/2} \\
&\qquad \times \| \nabla v_{12} \|_{L^2}^2, \\
L_{15} &\lesssim \varepsilon e^{\varepsilon \alpha \| \xi_1' \|_{H^2}} \| \nabla_h \xi_1' \|_{L^6} \| \nabla v_{12} \|_{L^2} \| v_{12} \|_{L^3} \lesssim \delta \| \nabla v_{12} \|_{L^2}^2 \\
&\quad + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0) \| v_{12} \|_{L^2}^2, \\
L_{16} &\lesssim \varepsilon e^{\varepsilon \alpha (\| \xi_1' \|_{H^2} + \| \xi_2' \|_{H^2})} \| \xi_{12}' \|_{L^2} \| \nabla^2 v_2 \|_{L^3} \| v_{12} \|_{L^6}
\end{aligned}
$$

$$\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \delta \| v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0)(\| \nabla v_2 \|_{H^2} + 1) \| \xi'_{12} \|_{L^2}^2,$$

where $\mathcal{H}(\cdot)$, as before, is a regular function of the arguments. Consequently, one has

$$\begin{aligned}
&\frac{d}{dt} \| v_{12} \|_{L^2}^2 + c_1 \big(\mu \| \nabla_h v_{12} \|_{L^2}^2 + \lambda \| div_h v_{12} \|_{L^2}^2 + \| \partial_z v_{12} \|_{L^2}^2 \big) \\
&\quad \le (\delta + \varepsilon e^{\varepsilon \alpha (C_0 M_0)^{1/2}} (C_0 M_0)^{1/2}) \| \nabla v_{12} \|_{L^2}^2 + \delta \| \nabla v'_{12} \|_{L^2}^2 \\
&\quad\quad + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \big(\| v_{12} \|_{L^2}^2 + \| v'_{12} \|_{L^2}^2 + \| \xi'_{12} \|_{L^2}^{\prime} \big) \\
&\quad\quad + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \| \nabla v_2 \|_{H^2} \| \xi'_{12} \|_{L^2}^2.
\end{aligned}$$

Notice that C_0, C_1 are independent of ε. For $\varepsilon_0, \delta \in (0,1)$ small enough and $\varepsilon \in (0, \varepsilon_0]$, applying the Grönwall inequality in the above inequality yields

$$\begin{aligned}
&\sup_{0 \le t \le T_\varepsilon} \| v_{12} \|_{L^2}^2 + \frac{c_1 \min\{\mu, \lambda, 1\}}{2} \int_0^{T_\varepsilon} \| \nabla v_{12} \|_{L^2}^2 \, dt \\
&\quad \le e^{T_\varepsilon \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1)} \big(\delta \| \nabla v'_{12} \|_{L^2(0,T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2 \\
&\quad\quad + T_\varepsilon \| v'_{12} \|_{L^\infty(0,T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2 + T_\varepsilon \| \xi'_{12} \|_{L^\infty(0,T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2 \\
&\quad\quad + T_\varepsilon^{1/2} \| \nabla v_2 \|_{L^2(0,T_\varepsilon; H^2(\Omega_h \times 2\mathbb{T}))} \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \\
&\quad\quad \times \| \xi'_{12} \|_{L^\infty(0,T_\varepsilon; L^2(\Omega_h \times 2\mathbb{T}))}^2 \big).
\end{aligned} \tag{3.61}$$

Then for fixed $\varepsilon \in (0, \varepsilon_0)$, by first choosing δ small enough, and then T_ε small enough, (3.60) and (3.61) yield inequality (3.43) with $q = \frac{1}{2}$. This finishes the proof of (2) and completes the proof of Proposition 2. □

3.3 Uniform stability

In this section, we will show that the existence time T_ε of the strong solutions constructed in Propositive 2 is uniform in ε provided $\varepsilon \in (0, \varepsilon_1)$ for some $\varepsilon_1 \in (0, \varepsilon_0)$. In order to show this, it suffices to show that there is a positive constant ε_1 such that the H^2-norms of (ξ, v) remain bounded in a time interval $(0, T)$ with $T \in (0, \infty)$ independent of ε, provided $\varepsilon \in (0, \varepsilon_1)$ with $\varepsilon_1 \in (0,1)$ small enough. We perform a continuity argument in the following.

Recall that we are given initial data $(\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$, and a positive constant $M \in (0, \infty)$ satisfying

$$\frac{1}{2} \| v_0 \|_{H^2}^2 + \frac{c^2}{\gamma - 1} \| \xi_0 \|_{H^2}^2 < M. \tag{2.6}$$

Denote by

$$M_0 := \frac{8(\gamma-1)}{c^2}M, \ M_1 := 4M. \tag{3.62}$$

Then it is obvious that

$$\|\xi_0\|_{H^2}^2 < M_0, \ \|v_0\|_{H^2}^2 < M_1.$$

From Proposition 2, there is a strong solution (ξ, v) satisfying (3.35) and (3.36) in the time interval $[0, T_\varepsilon]$, for some $T_\varepsilon \in (0, \infty)$. Then for any $t \in [0, T_\varepsilon]$, we have

$$\|\xi(t)\|_{H^2}^2 \le C_0 M_0 = \frac{8(\gamma-1)}{c^2}C_0 M, \ \|v(t)\|_{H^2}^2 \le C_1 M_1 = 4C_1 M, \tag{3.63}$$

where C_0, C_1 are given in Proposition 2. We remind readers that T_ε depends on M_0, M_1 and ε. On the other hand, consider ε_1 satisfying

$$\sup_{0\le t\le T_\varepsilon} \max\{1,\alpha\}\varepsilon_1 \|\xi(t)\|_{H^2} \le \max\{1,\alpha\}\varepsilon_1 \times \big(C_0 M_0\big)^{1/2} \le \log 2,$$

$$\sup_{0\le t\le T_\varepsilon} \varepsilon_1 \mathcal{H}_1(\|\xi(t)\|_{H^2}, \|v(t)\|_{H^2}) \le \varepsilon_1 \mathcal{H}_1(C_0M_0, C_1M_1) \le \frac{\min\{\mu,\lambda,1\}c_1}{8},$$

where $\mathcal{H}_1$ is as in the right-hand side of (3.1). Then the *a priori* estimate in (3.1) implies that, for $\varepsilon \in (0, \varepsilon_1)$ and $t \in [0, T_\varepsilon]$,

$$\begin{aligned}
&\frac{1}{2}\|v(t)\|_{H^2}^2 + \frac{c^2}{4(\gamma-1)}\|\xi(t)\|_{H^2}^2 + \frac{\min\{\mu,\lambda,1\}c_1}{2}\int_0^t \|\nabla v(s)\|_{H^2}^2\,ds \\
&\quad \le \frac{1}{2}\|v_0\|_{H^2}^2 + \frac{c^2}{\gamma-1}\|\xi_0\|_{H^2}^2 + \delta\int_0^t \|\nabla v(s)\|_{H^2}^2\,ds \\
&\qquad + \frac{\min\{\mu,\lambda,1\}c_1}{8}\int_0^t \|\nabla v(s)\|_{H^2}^2\,ds + C_\delta \int_0^t \mathcal{H}_2(C_0M_0, C_1M_1)\,ds.
\end{aligned}$$

Hence after substituting (2.6) and choosing

$$\delta = \delta' := \frac{\min\{\mu,\lambda,1\}c_1}{8}, \text{ and } T := \frac{M}{C_{\delta'}\mathcal{H}_2(C_0M_0, C_1M_1)}, \tag{3.64}$$

it follows

$$\begin{aligned}
&\frac{1}{2}\|v(t)\|_{H^2}^2 + \frac{c^2}{4(\gamma-1)}\|\xi(t)\|_{H^2}^2 + \frac{\min\{\mu,\lambda,1\}c_1}{4}\int_0^t \|\nabla v(s)\|_{H^2}^2\,ds \\
&\quad \le M + C_{\delta'}\mathcal{H}_2(C_0M_0, C_1M_1)t < 2M,
\end{aligned} \tag{3.65}$$

for

$$t \in [0, \min\{T_\varepsilon, T\}].$$

Notice that T is independent of ε once $\varepsilon \in (0, \varepsilon_1)$. Also from (3.65), we have

$$\begin{aligned} \sup_{0 \leq t \leq \min\{T_\varepsilon, T\}} \| \xi(t) \|_{H^2}^2 < \frac{8(\gamma - 1)}{c^2} M = M_0, \\ \sup_{0 \leq t \leq \min\{T_\varepsilon, T\}} \| v(t) \|_{H^2}^2 < 4M = M_1, \\ \int_0^{\min\{T_\varepsilon, T\}} \| \nabla v(t) \|_{H^2}^2 \, dt \leq \frac{8M}{c_1 \min\{\mu, \lambda, 1\}} =: M_2. \end{aligned} \tag{3.66}$$

Therefore, supposed that $T_\varepsilon < T$, we will have

$$\| \xi(T_\varepsilon) \|_{H^2}^2 < M_0, \; \| v(T_\varepsilon) \|_{H^2}^2 < M_1.$$

Then by setting $(\xi(T_\varepsilon), v(T_\varepsilon))$ as the new initial data, applying Proposition 2 again, estimate (3.63) holds in $[0, 2T_\varepsilon]$. Thus the arguments between (3.63) and (3.65) hold with T_ε replaced by $2T_\varepsilon$, without needing to choose the smallness of ε_1 and T. Then estimate (3.65) holds in the time interval $[0, \min\{2T_\varepsilon, T\}]$. Repeat such arguments n times, $n \in \mathbb{Z}^+$, until $nT_\varepsilon \geq T$. This extends the existence time of the local strong solution (ξ, v) to (1.4) to $T > 0$, which is independent of ε provided $\varepsilon \in (0, \varepsilon_1)$ with ε_1 given as above. Consequently, we conclude that:

Proposition 3. *Consider* $(\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$ *with the compatibility conditions in* (2.4)*, bounded as in* (2.6)*. Then there is a positive constant* $\varepsilon_1 \in (0, 1)$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$*, there is a unique strong solution to* (1.4) *in the time interval* $[0, T]$ *satisfying the regularity* (3.35)*. Here* $T \in (0, \infty)$ *is a positive time which is independent of* ε*. Moreover, there are positive constants* $M_0, M_1, M_2 \in (0, \infty)$*, given in* (3.62) *and* (3.66) *and independent on* ε*, such that*

$$\begin{gathered} \| \xi \|_{L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T}))}^2 < M_0, \;\; \| v \|_{L^\infty(0,T;H^2(\Omega_h \times 2\mathbb{T}))}^2 < M_1, \\ \| \nabla v \|_{L^2(0,T;H^2(\Omega_h \times 2\mathbb{T}))}^2 < M_2. \end{gathered} \tag{3.67}$$

4 The limit equations

In this section, we will identify the limit equations of (1.4) as $\varepsilon \to 0^+$ in the distribution sense. Recall that,

$$\begin{cases} \partial_t \xi_\varepsilon + v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \dfrac{\gamma - 1}{\varepsilon}(div_h v_\varepsilon + \partial_z w_\varepsilon) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \dfrac{c^2 e^{\varepsilon \xi_\varepsilon}}{\varepsilon} \nabla_h \xi_\varepsilon & \\ \qquad = c_1 e^{-\varepsilon\alpha\xi_\varepsilon}\big(\mu \Delta_h v_\varepsilon + \lambda \nabla_h div_h v_\varepsilon + \partial_{zz} v_\varepsilon\big) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi_\varepsilon = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\ w_\varepsilon = 0 & \text{on } \Omega_h \times \mathbb{Z}, \end{cases} \tag{1.4}$$

where the initial data are taken as $(\xi_\varepsilon, v_\varepsilon)\big|_{t=0} = (\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$ satisfying the compatibility conditions in (2.4), bounded as in (2.6). We first list the uniform estimates obtained in the previous section. In fact, from (3.67) and (3.2)–(3.4), the following estimates hold uniformly in $\varepsilon \in (0, \varepsilon_1)$ for some constant $C, T \in (0, \infty)$ independent of ε and depending only on the initial data,

$$\begin{aligned} \sup_{0 \leq t \leq T} \big\{ \big\| v_\varepsilon(t) \big\|_{H^2} + \big\| \xi_\varepsilon(t) \big\|_{H^2} + \big\| \partial_t (e^{\varepsilon \xi_\varepsilon})(t) \big\|_{H^1} + \big\| \partial_t \mathcal{P}_\sigma v_\varepsilon(t) \big\|_{L^2} \big\} & \\ + \big\| \nabla v_\varepsilon \big\|_{L^2(0,T;H^2)} + \big\| \partial_t \mathcal{P}_\sigma v_\varepsilon \big\|_{L^2(0,T;H^1)} < C. & \end{aligned} \tag{4.1}$$

Then as $\varepsilon \to 0^+$, there exist $\xi_p, v_p, \zeta_p, v_{p,\sigma}$ with

$$\begin{gathered} \xi_p \in L^\infty(0,T;H^2), \quad v_p \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \\ \zeta_p \in L^\infty(0,T;H^1), \quad v_{p,\sigma} \in L^\infty(0,T;H^2_\sigma) \cap L^2(0,T;H^3_\sigma), \\ \partial_t v_{p,\sigma} \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma), \end{gathered} \tag{4.2}$$

such that,

$$\begin{aligned} & \xi_\varepsilon \overset{*}{\rightharpoonup} \xi_p \quad \text{weak-}* \text{ in } L^\infty(0,T;H^2), \\ & v_\varepsilon \text{ (and } \mathcal{P}_\sigma v_\varepsilon) \overset{*}{\rightharpoonup} v_p \text{ (respectively } v_{p,\sigma}) \\ & \qquad \text{weak-}* \text{ in } L^\infty(0,T;H^2) \text{ (respectively } L^\infty(0,T;H^2_\sigma)), \\ & v_\varepsilon \text{ (and } \mathcal{P}_\sigma v_\varepsilon) \rightharpoonup v_p \text{ (respectively } v_{p,\sigma}) \\ & \qquad \text{weakly in } L^2(0,T;H^3) \text{ (respectively } L^2(0,T;H^3_\sigma)), \\ & \partial_t (e^{\varepsilon \xi_\varepsilon}) \overset{*}{\rightharpoonup} \zeta_p \quad \text{weak-}* \text{ in } L^\infty(0,T;H^1), \\ & \partial_t \mathcal{P}_\sigma v_\varepsilon \overset{*}{\rightharpoonup} \partial_t v_{p,\sigma} \quad \text{weak-}* \text{ in } L^\infty(0,T;L^2_\sigma), \\ & \partial_t \mathcal{P}_\sigma v_\varepsilon \rightharpoonup \partial_t v_{p,\sigma} \quad \text{weakly in } L^2(0,T;H^1_\sigma). \end{aligned} \tag{4.3}$$

Also applying the Aubin compactness lemma (see, e.g., [42, Theorem 2.1] and [5, 41]) yields, as $\varepsilon \to 0^+$,

$$\mathcal{P}_\sigma v_\varepsilon \to v_{p,\sigma} \quad \text{in } L^2(0,T;H^2_{\sigma,loc}) \cap C([0,T];H^1_{\sigma,loc}). \tag{4.4}$$

Moreover, from (4.1), we have

$$\varepsilon \left\| \xi_\varepsilon \right\|_{H^2} < \varepsilon C \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{4.5}$$

Thus we have, as $\varepsilon \to 0^+$,

$$\begin{gathered} e^{\varepsilon \xi_\varepsilon} \to 1 \quad \text{in } L^\infty(0,T;H^2), \\ \text{and} \quad \partial_t (e^{\varepsilon \xi_\varepsilon}) \to 0 = \zeta_p \quad \text{in the sense of distribution.} \end{gathered} \tag{4.6}$$

On the other hand, recall that the vertical velocity is represented by

$$w_\varepsilon = - \int_0^z \left(\frac{\varepsilon}{\gamma - 1} \widetilde{v}_\varepsilon \cdot \nabla_h \xi_\varepsilon + div_h \widetilde{v}_\varepsilon \right) dz. \tag{3.8}$$

Therefore, a direct calculation yields that $w_\varepsilon, \partial_z w_\varepsilon \in L^\infty(0,T;H^1)$ and we have the following uniform bounds:

$$\begin{aligned} & \left\| w_\varepsilon \right\|_{L^\infty(0,T;H^1)} + \left\| \partial_z w_\varepsilon \right\|_{L^\infty(0,T;H^1)} \\ & \quad \lesssim \varepsilon \left\| v_\varepsilon \right\|_{L^\infty(0,T;H^2)} \left\| \xi_\varepsilon \right\|_{L^\infty(0,T;H^2)} + \left\| v_\varepsilon \right\|_{L^\infty(0,T;H^2)} < C. \end{aligned} \tag{4.7}$$

Therefore, there exists

$$w_p \in L^\infty(0,T;H^1) \quad \text{with } \partial_z w_p \in L^\infty(0,T;H^1) \tag{4.8}$$

such that,

$$w_\varepsilon \text{ (and } \partial_z w_\varepsilon) \overset{*}{\rightharpoonup} w_p \text{ (respectively } \partial_z w_p) \quad \text{weak-}* \text{ in } L^\infty(0,T;H^1), \tag{4.9}$$

as $\varepsilon \to 0^+$. Notice, after applying the trace theorem, $\{w_\varepsilon|_{z\in\mathbb{Z}}\}_{\varepsilon\in(0,\varepsilon_1)} \subset L^\infty(0,T;H^{1/2}(\Omega_h))$ and is uniformly bounded. Then it follows from (3.8) that $w_\varepsilon|_{z\in\mathbb{Z}} = 0$ and hence $w_p|_{z\in\mathbb{Z}} = 0$. Moreover, after multiplying $(1.4)_1$ with $\varepsilon e^{\varepsilon\xi_\varepsilon}$, one has

$$\partial_t e^{\varepsilon \xi_\varepsilon} + \varepsilon e^{\varepsilon \xi_\varepsilon} v_\varepsilon \cdot \nabla_h \xi_\varepsilon + (\gamma - 1) e^{\varepsilon \xi_\varepsilon} (div_h v_\varepsilon + \partial_z w_\varepsilon) = 0. \tag{4.10}$$

Then, (4.1) and (4.6) imply that as $\varepsilon \to 0^+$, (4.10) converges to

$$div_h v_p + \partial_z w_p = 0 \quad \text{in the sense of distribution.}$$

Consequently, we have shown that

$$div_h v_p + \partial_z w_p = 0, \quad \text{and} \quad w_p\big|_{z\in\mathbb{Z}} = 0. \tag{4.11}$$

Next, we will identify the limit equation of the momentum equation $(1.4)_2$. First, (4.2) and (4.11) imply that $v_p \in L^\infty(0,T;H^2_\sigma) \cap L^2(0,T;H^3_\sigma)$. We first show that $v_{p,\sigma} \equiv v_p$. Let $u \in C_0^\infty(0,T;C_\sigma^\infty(\Omega_h \times 2\mathbb{T}, \mathbb{R}^2))$. Then we have, from (4.3),

$$\begin{aligned} \int_0^T \int u \cdot v_p \, d\vec{x}\, dt &= \lim_{\varepsilon\to 0^+} \int_0^T \int u \cdot v_\varepsilon \, d\vec{x}\, dt \\ &= \lim_{\varepsilon\to 0^+} \int_0^T \int u \cdot \mathcal{P}_\sigma v_\varepsilon \, d\vec{x}\, dt = \int_0^T \int u \cdot v_{p,\sigma} \, d\vec{x}\, dt, \end{aligned}$$

which shows that $v_{p,\sigma} \equiv v_p$. Then (4.4) can be written as

$$\mathcal{P}_\sigma v_\varepsilon \to v_p \ \text{ in } L^2(0,T;H^2_{\sigma,loc}) \cap C([0,T];H^1_{\sigma,loc}), \tag{4.12}$$

as $\varepsilon \to 0^+$. In particular, $v_p(t=0) = \mathcal{P}_\sigma v_0$. Moreover, from (4.3), since $v_{p,\sigma} \equiv v_p$, we have the following: as $\varepsilon \to 0^+$,

$$\begin{aligned} &\mathcal{P}_\tau v_\varepsilon = v_\varepsilon - \mathcal{P}_\sigma v_\varepsilon \overset{*}{\rightharpoonup} 0 \ \text{ weak-}* \text{ in } L^\infty(0,T;H^2_\tau), \\ &\mathcal{P}_\tau v_\varepsilon = v_\varepsilon - \mathcal{P}_\sigma v_\varepsilon \rightharpoonup 0 \ \text{ weakly in } L^2(0,T;H^3_\tau). \end{aligned} \tag{4.13}$$

Also, denote $\mathcal{P}_\tau v_\varepsilon = \nabla_h \psi_\varepsilon$ with ψ_ε defined by the following elliptic problem (see, e.g., (2.1)),

$$\begin{gathered} \Delta_h \psi_\varepsilon = \int_0^1 div_h v_\varepsilon \, dz \ \text{ in } \Omega_h, \\ \begin{cases} \lim\limits_{|(x,y)|\to\infty} \psi_\varepsilon = 0 & \text{in the case when } \Omega_h = \mathbb{R}^2, \\ \int_{\Omega_h} \psi_\varepsilon \, dxdy = 0 & \text{in the case when } \Omega_h = \mathbb{T}^2. \end{cases} \end{gathered} \tag{4.14}$$

Then $\mathcal{P}_\tau \partial_t v_\varepsilon = \nabla_h \partial_t \psi_\varepsilon$ and we have the identity:

$$\begin{aligned} &\mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \mathcal{P}_\tau v_\varepsilon + w_\varepsilon \partial_z \mathcal{P}_\tau v_\varepsilon = \nabla_h \psi_\varepsilon \cdot \nabla_h \nabla_h \psi_\varepsilon + w_\varepsilon \partial_z \nabla_h \psi_\varepsilon \\ &\quad = \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2. \end{aligned}$$

Therefore, one has

$$v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon = \mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h v_\varepsilon + \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \mathcal{P}_\sigma v_\varepsilon + w_\varepsilon \partial_z \mathcal{P}_\sigma v_\varepsilon$$

$$+ \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \mathcal{P}_\tau v_\varepsilon + w_\varepsilon \partial_z \mathcal{P}_\tau v_\varepsilon = \mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h v_\varepsilon + \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \mathcal{P}_\sigma v_\varepsilon$$
$$+ w_\varepsilon \partial_z \mathcal{P}_\sigma v_\varepsilon + \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2.$$

Then $(1.4)_2$ can be written as

$$\begin{aligned}
&\partial_t \mathcal{P}_\sigma v_\varepsilon + \mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h v_\varepsilon + \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \mathcal{P}_\sigma v_\varepsilon + w_\varepsilon \partial_z \mathcal{P}_\sigma v_\varepsilon \\
&\qquad - c_1 e^{-\varepsilon\alpha\xi_\varepsilon} (\mu \Delta_h v_\varepsilon + \lambda \nabla_h div_h v_\varepsilon + \partial_{zz} v_\varepsilon) + c^2 \big(\frac{e^{\varepsilon\xi_\varepsilon} - 1}{\varepsilon} - \xi_\varepsilon \big) \nabla_h \xi_h \\
&= - \big(\nabla_h \partial_t \psi_\varepsilon + \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2 + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon + \frac{c^2}{2} \nabla_h |\xi_\varepsilon|^2 \big).
\end{aligned} \tag{4.15}$$

Then (4.1), (4.7) and (4.15) imply that

$$\begin{aligned}
&\big| \nabla_h \partial_t \psi_\varepsilon + \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2 + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon + \frac{c^2}{2} \nabla_h |\xi_\varepsilon|^2 \big|_{L^\infty(0,T;L^2) \cap L^2(0,T;H^1)} \\
&\quad = \big| \text{left-hand side of } (4.15) \big|_{L^\infty(0,T;L^2) \cap L^2(0,T;H^1)} < C,
\end{aligned}$$

where C is independent of ε. Hence $\exists P \in L^\infty(0,T;H^1(\Omega_h)) \cap L^2(0,T;H^2(\Omega_h))$ such that

$$\begin{aligned}
&\nabla_h \partial_t \psi_\varepsilon + \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2 + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon + \frac{c^2}{2} \nabla_h |\xi_\varepsilon|^2 \\
&\qquad \overset{*}{\rightharpoonup} \nabla_h P \quad \text{weak-}* \text{ in } L^\infty(0,T;L^2), \\
&\nabla_h \partial_t \psi_\varepsilon + \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2 + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon + \frac{c^2}{2} \nabla_h |\xi_\varepsilon|^2 \\
&\qquad \rightharpoonup \nabla_h P \quad \text{weakly in } L^2(0,T;H^1).
\end{aligned} \tag{4.16}$$

Also, using (4.1), we have

$$\big| \frac{e^{\varepsilon\xi_\varepsilon} - 1}{\varepsilon} - \xi_\varepsilon \big|_{L^\infty} \lesssim \varepsilon \big| \xi_\varepsilon \big|_{H^2} \to 0 \quad \text{ as } \varepsilon \to 0^+. \tag{4.17}$$

Then the weak and strong convergences in (4.3), (4.6), (4.9), (4.12), (4.13), (4.16), and (4.17) yield that, as $\varepsilon \to 0^+$, (4.15) converges to

$$\partial_t v_p + v \cdot \nabla_h v_p + w_p \partial_z v_p + \nabla_h P = c_1 \big(\mu \Delta_h v_p + \lambda \nabla_h div_h v_p + \partial_{zz} v_p \big), \tag{4.18}$$

in the sense of distribution, where w_p is determined by (4.11), and $\partial_z P = 0$.

To conclude, we have shown the following:

Proposition 4. *Let* $(\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$ *satisfying the compatibility conditions in* (2.4) *and* $\varepsilon \in (0, \varepsilon_1)$ *with* ε_1 *given in Proposition 3. Then* *the unique strong solutions* $\{(\xi_\varepsilon, v_\varepsilon)\}_{\varepsilon\in(0,\varepsilon_1)}$ *obtained in Proposition 3 in the time interval* $[0,T]$, $T \in (0,\infty)$, *and* w_ε *given as in* (3.8), *converge to* (ξ_p, v_p) *and* w_p, *with the convergences in* (4.3), (4.6), (4.9), (4.12) *and* (4.13), *as* $\varepsilon \to 0^+$. ξ_p, v_p, w_p *satisfy the regularity* (4.2) *and* (4.8). *Also,* (v_p, w_p) *satisfies the primitive equations* (2.9) *with the initial data* $v_p|_{t=0} = v_{p,0} = \mathcal{P}_\sigma v_0 \in H^2_\sigma(\Omega_h \times 2\mathbb{T})$.

In the rest of this section, we will discuss the converging behaviors of ξ_ε and $\mathcal{P}_\tau v_\varepsilon$ as $\varepsilon \to 0^+$ in some strong sense. We will consider such a problem in two cases: $\Omega_h = \mathbb{R}^2$ and $\Omega_h = \mathbb{T}^2$. In fact, we will investigate the following acoustic wave equations associated with (1.4):

$$\begin{cases} \partial_t \xi_\varepsilon + \dfrac{\gamma-1}{\varepsilon} div_h \mathcal{P}_\tau v_\varepsilon = -v_\varepsilon \cdot \nabla_h \xi_\varepsilon \\ \qquad - \dfrac{\gamma-1}{\varepsilon}(div_h \mathcal{P}_\sigma v_\varepsilon + \partial_z w_\varepsilon) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_t \mathcal{P}_\tau v_\varepsilon + \dfrac{c^2}{\varepsilon}\nabla_h \xi_\varepsilon = \dfrac{c^2(1-e^{\varepsilon\xi_\varepsilon})}{\varepsilon}\nabla_h \xi_\varepsilon - \partial_t \mathcal{P}_\sigma v_\varepsilon \\ \qquad - v_\varepsilon \cdot \nabla_h v_\varepsilon - w_\varepsilon \partial_z v_\varepsilon + c_1 e^{-\varepsilon\alpha\xi_\varepsilon}\big(\mu \Delta_h v_\varepsilon \\ \qquad + \lambda \nabla_h div_h v_\varepsilon + \partial_{zz} v_\varepsilon\big) & \text{in } \Omega_h \times 2\mathbb{T}, \\ \partial_z \xi_\varepsilon = 0 & \text{in } \Omega_h \times 2\mathbb{T}. \end{cases} \tag{4.19}$$

To study the acoustic wave, notice that, $\mathcal{P}_\tau v_\varepsilon = \nabla_h \psi_\varepsilon$ is a function independent of the z-variable, where ψ_ε is defined in (4.14). After averaging $(4.19)_1$ in the z-variable and applying projection operator $\mathcal{P}_\tau$ to $(4.19)_2$, system (4.19) is reduced to,

$$\begin{cases} \partial_t \xi_\varepsilon + \dfrac{\gamma-1}{\varepsilon} div_h \nabla_h \psi_\varepsilon = G_1 & \text{in } \Omega_h, \\ \partial_t \nabla_h \psi_\varepsilon + \dfrac{c^2}{\varepsilon}\nabla_h \xi_\varepsilon = G_2 + G_3 & \text{in } \Omega_h, \end{cases} \tag{4.20}$$

where we have used $\nabla_h \psi_\varepsilon$ (defined in (4.14)) to represent $\mathcal{P}_\tau v_\varepsilon$, and

$$\begin{aligned} G_1 &:= -\overline{v}_\varepsilon \cdot \nabla_h \xi_\varepsilon, \\ G_2 &:= \frac{c^2(1-e^{\varepsilon\xi_\varepsilon})}{\varepsilon}\nabla_h \xi_\varepsilon + \mathcal{P}_\tau\big(c_1 e^{-\varepsilon\alpha\xi_\varepsilon}\big(\mu\Delta_h v_\varepsilon + \lambda \nabla_h div_h v_\varepsilon + \partial_{zz} v_\varepsilon\big)\big), \\ G_3 &:= -\mathcal{P}_\tau\big(v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon\big). \end{aligned}$$

Therefore, the acoustic wave system is only two dimensional. From (3.32), (3.33), (4.1), we have

$$\begin{aligned} &\big|G_1\big|_{L^\infty(0,T;H^1(\Omega_h))} + \big|G_2\big|_{L^\infty(0,T;L^2(\Omega_h))} + \big|G_2\big|_{L^2(0,T;H^1(\Omega_h))} \\ &\qquad + \big|G_3\big|_{L^\infty(0,T;L^2(\Omega_h))} + \big|G_3\big|_{L^2(0,T;H^1(\Omega_h))} < C. \end{aligned} \tag{4.21}$$

Now denote by the linear operator L defined on $(\mathcal{D}'(\Omega_h))^3$ as

$$L(\mathcal{U}) := \begin{pmatrix} (\gamma-1)div_h u \\ c^2\nabla_h q \end{pmatrix}, \quad \text{where } \mathcal{U} = \begin{pmatrix} q \\ u \end{pmatrix}, \tag{4.22}$$

where $q \in \mathcal{D}'(\Omega_h),\ u \in (\mathcal{D}'(\Omega_h))^2$. Then (4.20) can be written as

$$\partial_t \mathcal{U}_\varepsilon + \frac{1}{\varepsilon} L\mathcal{U}_\varepsilon = \mathcal{G} \quad \text{in } \Omega_h, \tag{4.23}$$

where

$$\mathcal{G} := \begin{pmatrix} G_1 \\ G_2 + G_3 \end{pmatrix}, \ \mathcal{U}_\varepsilon := \begin{pmatrix} \xi_\varepsilon \\ \nabla_h \psi_h \end{pmatrix} = \begin{pmatrix} \xi_\varepsilon \\ \mathcal{P}_\tau v_\varepsilon \end{pmatrix}.$$

Also, the initial data are given as

$$\mathcal{U}_\varepsilon(t=0) = \mathcal{U}_0 = \begin{pmatrix} \xi_0 \\ \mathcal{P}_\tau v_0 \end{pmatrix}. \tag{4.24}$$

Also, we have

$$\xi_\varepsilon \in L^\infty(0,T;H^2(\Omega_h)),\ \mathcal{P}_\tau v_\varepsilon \in L^\infty(0,T;H^2(\Omega_h)) \cap L^2(0,T;H^3(\Omega_h)),$$
$$\mathcal{G} \in L^\infty(0,T;L^2(\Omega_h)) \cap L^2(0,T;H^1(\Omega_h)),\ \mathcal{U}_0 \in H^2(\Omega_h),$$

and from (2.6), (4.1) and (4.21),

$$\begin{aligned} &\big|\xi_\varepsilon\big|_{L^\infty(0,T;H^2(\Omega_h))} + \big|\mathcal{P}_\tau v_\varepsilon\big|_{L^\infty(0,T;H^2(\Omega_h))\cap L^2(0,T;H^3(\Omega_h))} \\ &+\big|\mathcal{G}\big|_{L^\infty(0,T;L^2(\Omega_h))\cap L^2(0,T;H^1(\Omega_h))} + \big|\mathcal{U}_0\big|_{H^2(\Omega_h)} < C. \end{aligned} \tag{4.25}$$

Define the associated solution operator of L by

$$\mathcal{L} : t \mapsto \mathcal{L}(t) = \exp(-Lt). \tag{4.26}$$

That is, given $\mathcal{V}_0 \in (\mathcal{D}')^3$, $\mathcal{L}(t)\mathcal{V}_0$ satisfies the linear equation:

$$\partial_t(\mathcal{L}(t)\mathcal{V}_0) + L(\mathcal{L}(t)\mathcal{V}_0) = 0. \tag{4.27}$$

Notice that $\mathcal{L}$ is linear and $\mathcal{L}(t_1+t_2) = \mathcal{L}(t_1)\mathcal{L}(t_2)$ for $t_1, t_2 \in (0,\infty)$. Then the solution $\mathcal{U}_\varepsilon$ of (4.23) can be represented as, using the Duhamel principle,

$$\mathcal{U}_\varepsilon(t) = \mathcal{L}(\frac{t}{\varepsilon})\mathcal{U}_0 + \int_0^t \mathcal{L}(\frac{t-s}{\varepsilon})\mathcal{G}(s)\,ds. \tag{4.28}$$

4.1 The case when $\Omega_h = \mathbb{R}^2$: dispersion of the acoustic waves

In this section, we shall establish the strong convergence of $\mathcal{U}_\varepsilon$ in the case when $\Omega_h = \mathbb{R}^2$. To be precise, we will show the following:

Proposition 5. *Under the same assumptions as in Proposition 4, in the case when $\Omega_h = \mathbb{R}^2$, we have*

$$\xi_\varepsilon \to 0,\ \mathcal{P}_\tau v_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0^+ \quad \text{in } L^2(0,T; W^{\frac{1}{2},6}(\mathbb{R}^2)). \tag{2.12}$$

In particular, we have, as $\varepsilon \to 0^+$,

$$\left\| v_\varepsilon - v_p \right\|_{L^2(0,T;L^6_{loc}(\mathbb{R}^2))} + \left\| \xi_\varepsilon \right\|_{L^2(0,T;L^6(\mathbb{R}^2))} \to 0.$$

In order to show Proposition 5, we will use the dispersion estimates of operator $\mathcal{L}$ in the whole space $\mathbb{R}^2$. In particular, we will employ the Strichartz inequalities as in [10], to derive the appropriate decay as $\varepsilon \to 0^+$. Indeed, let $\phi_0 \in (\mathcal{D}'(\mathbb{R}^2))^3$. The following inequality is from [10, equation (3.4)]:

$$\left|\mathcal{L}(t)\phi_0\right|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^2))} \leq C_{p,q}\left|\phi_0\right|_{H^\eta(\mathbb{R}^2)}, \tag{4.29}$$

provided that the right-hand side is finite, where $p, q > 2$ and $\eta \in (0,\infty)$ satisfying

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p} \quad \text{and} \quad \eta q = 3. \tag{4.30}$$

We will make use of (4.29) to derive the strong dispersion. To begin with, the following dispersion inequalities are analogies to inequalities (3.6) and (3.7) in [10].

Lemma 1. *For ϕ_0 and $\phi(s) \in (\mathcal{D}'(\mathbb{R}^2))^3$, p, q, η satisfying (4.30) and $s \geq 0$, we have*

$$\left|\mathcal{L}(\frac{t}{\varepsilon})\phi_0\right|_{L^q(0,T;W^{s,p}(\mathbb{R}^2))} \leq C\varepsilon^{1/q}\left|\phi_0\right|_{H^{\eta+s}(\mathbb{R}^2)}, \tag{4.31}$$

$$\begin{aligned}&\left|\int_0^t \mathcal{L}(\frac{t-s}{\varepsilon})\phi(s)\,ds\right|_{L^2(0,T;W^{s,p}(\mathbb{R}^2))}\\ &\qquad \leq C(1+T)\varepsilon^{1/q}\left|\phi\right|_{L^2(0,T;H^{\eta+s}(\mathbb{R}^2))},\end{aligned} \tag{4.32}$$

provided the right-hand sides are finite.

Proof. Indeed (4.31) is a direct consequence of rescaling the temporal variable and replacing ϕ_0 with $(I-\Delta_h)^{s/2}\phi_0$ in (4.29). In order to show (4.32), notice that after applying the Minkowski and Hölder inequalities, one has

$$\left|\int_0^t \mathcal{L}(\frac{t-s}{\varepsilon})\phi(s)\,ds\right|^2_{W^{s,p}(\mathbb{R}^2)} \leq Ct\int_0^t \left|\mathcal{L}(\frac{t-s}{\varepsilon})\phi(s)\right|^2_{W^{s,p}(\mathbb{R}^2)}\,ds.$$

Then by employing the Fubini theorem, it holds

$$
\begin{aligned}
&\int_0^T \Big|\int_0^t \mathcal{L}\big(\frac{t-s}{\varepsilon}\big)\phi(s)\,ds\Big|^2_{W^{s,p}(\mathbb{R}^2)}\,dt \\
&\quad\leq CT\int_0^T\int_s^T \big|\mathcal{L}\big(\frac{t-s}{\varepsilon}\big)\phi(s)\big|^2_{W^{s,p}(\mathbb{R}^2)}\,dt\,ds \\
&\quad\leq CT\int_0^T (T-s)^{1-2/q}\Big(\int_s^T \big|\mathcal{L}\big(\frac{t-s}{\varepsilon}\big)\phi(s)\big|^q_{W^{s,p}(\mathbb{R}^2)}\,dt\Big)^{2/q}\,ds \\
&\quad\leq \varepsilon^{2/q}CT^{2-2/q}\int_0^T \big|\phi(s)\big|^2_{H^{\eta+s}(\mathbb{R}^2)}\,ds = \varepsilon^{2/q}CT^{2-2/q}\big|\phi\big|^q_{L^2(0,T;H^{\eta+s}(\mathbb{R}^2))}.
\end{aligned}
$$

This finishes the proof of (4.32). □

Proof of Proposition 5. Consider $\eta=\frac{1}{2}, s=\frac{1}{2}, q=6, p=6$ in (4.31) and (4.32), which satisfy (4.30). Then from (4.28), after applying the triangle and Hölder inequalities,

$$
\begin{aligned}
&\big|\mathcal{U}_\varepsilon\big|_{L^2(0,T;W^{\frac{1}{2},6}(\mathbb{R}^2))} \lesssim T^{\frac{1}{3}}\big|\mathcal{L}\big(\frac{t}{\varepsilon}\big)\mathcal{U}_0\big|_{L^6(0,T;W^{\frac{1}{2},6}(\mathbb{R}^2))} \\
&\quad + \Big|\int_0^t \mathcal{L}\big(\frac{t-s}{\varepsilon}\big)\mathcal{G}(s)\,ds\Big|_{L^2(0,T;W^{\frac{1}{2},6}(\mathbb{R}^2))} \lesssim T^{\frac{1}{3}}\varepsilon^{\frac{1}{6}}\big|\mathcal{U}_0\big|_{H^1(\mathbb{R}^2)} \\
&\quad + (1+T)\varepsilon^{\frac{1}{6}}\big|\mathcal{G}\big|_{L^2(0,T;H^1(\mathbb{R}^2))} \lesssim (1+T)\varepsilon^{\frac{1}{6}},
\end{aligned}
$$

where the second and the third inequalities follow from (4.31), (4.32) and (4.25). This yields (2.12) and finishes the proof of Proposition 5. □

4.2 The case when $\Omega_h=\mathbb{T}^2$: the fast oscillation

We will establish the convergence behavior of $\mathcal{U}_\varepsilon$ in the case when $\Omega_h=\mathbb{T}^2$ in this subsection. Indeed, we shall investigate the fast oscillations of the acoustic waves as $\varepsilon\rightarrow 0^+$. This is motivated by [37] (see also [16]).

The oscillation equations and the convergence of oscillations

To begin with, $(4.20)_2$ can be written as

$$
\begin{aligned}
&\partial_t\mathcal{P}_\tau v_\varepsilon + \frac{c^2}{\varepsilon}\nabla_h\xi_\varepsilon - c_1(\mu\Delta_h\mathcal{P}_\tau v_\varepsilon + \lambda\nabla_h div_h\mathcal{P}_\tau v_\varepsilon) \\
&\quad = \frac{c^2(1-e^{\varepsilon\xi_\varepsilon})}{\varepsilon}\nabla_h\xi_\varepsilon - \mathcal{P}_\tau(v_\varepsilon\cdot\nabla_h v_\varepsilon) - \mathcal{P}_\tau(w_\varepsilon\partial_z v_\varepsilon) \\
&\qquad + \mathcal{P}_\tau(c_1(e^{-\varepsilon\alpha\xi_\varepsilon}-1)(\mu\Delta_h v_\varepsilon + \lambda\nabla_h div_h v_\varepsilon + \partial_{zz}v_\varepsilon)).
\end{aligned}
\tag{4.33}
$$

Moreover, for any $u \in (\mathcal{D}'(\mathbb{T}^2 \times 2\mathbb{T}))^2$, consider $\overline{u}(x,y) = \int_0^1 u\,dz \in (\mathcal{D}'(\mathbb{T}^2))^2$ as a function on $\mathbb{T}^2 \times 2\mathbb{T}$. One has $\mathcal{P}_\tau u = \mathcal{P}_\tau \overline{u}$. Then one has, after applying integration by parts and substituting (3.9),

$$\begin{aligned}
&\mathcal{P}_\tau(w_\varepsilon \partial_z v_\varepsilon) = \mathcal{P}_\tau(-\int_0^1 \partial_z w_\varepsilon v_\varepsilon\,dz) \\
&\quad = \mathcal{P}_\tau(\int_0^1 \big((\varepsilon\alpha \widetilde{v}_\varepsilon \cdot \nabla_h \xi_\varepsilon + div_h \widetilde{v}_\varepsilon) v_\varepsilon\big)\,dz) \\
&\quad = \mathcal{P}_\tau(\int_0^1 \big((\varepsilon\alpha \widetilde{\mathcal{P}_\sigma v_\varepsilon} \cdot \nabla_h \xi_\varepsilon)\mathcal{P}_\sigma v_\varepsilon + div_h \widetilde{\mathcal{P}_\sigma v_\varepsilon} \mathcal{P}_\sigma v_\varepsilon\big)\,dz) \\
&\quad = \mathcal{P}_\tau((\varepsilon\alpha \widetilde{\mathcal{P}_\sigma v_\varepsilon} \cdot \nabla_h \xi_\varepsilon)\mathcal{P}_\sigma v_\varepsilon + div_h \widetilde{\mathcal{P}_\sigma v_\varepsilon} \mathcal{P}_\sigma v_\varepsilon), \\
&\mathcal{P}_\tau((e^{-\varepsilon\alpha\xi_\varepsilon} - 1)\partial_{zz} v_\varepsilon) = 0,
\end{aligned} \tag{4.34}$$

where we have used the facts that $\int_0^1 \widetilde{P_\sigma v_\varepsilon}\,dz = 0, \widetilde{v}_\varepsilon = \widetilde{P_\sigma v_\varepsilon}$ and that $\mathcal{P}_\tau v_\varepsilon, \xi_\varepsilon$ are independent of the z-variable.

On the other hand, notice that

$$\begin{aligned}
v_\varepsilon \cdot \nabla_h \xi_\varepsilon = \mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon = div_h(\xi_\varepsilon \mathcal{P}_\sigma v_\varepsilon) - \xi_\varepsilon div_h \mathcal{P}_\sigma v_\varepsilon \\
+ \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon.
\end{aligned}$$

$(4.20)_1$ can be written as

$$\partial_t \xi_\varepsilon + \frac{\gamma - 1}{\varepsilon} div_h \mathcal{P}_\tau v_\varepsilon = -div_h(\xi_\varepsilon \int_0^1 \mathcal{P}_\sigma v_\varepsilon\,dz) - \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon. \tag{4.35}$$

Additionally, integrating (4.35) in $\mathbb{T}^2$ yields

$$\frac{d}{dt}\int_{\mathbb{T}^2} \xi_\varepsilon\,dxdy = -\int_{\mathbb{T}^2} (\mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon)\,dxdy. \tag{4.36}$$

Consequently, after combining equations (4.33), (4.34), (4.35), (4.36), we have the following system of oscillations:

$$\begin{cases}
\partial_t \xi_\varepsilon^o + \dfrac{\gamma - 1}{\varepsilon} div_h \mathcal{P}_\tau v_\varepsilon = -div_h(\xi_\varepsilon^o \int_0^1 \mathcal{P}_\sigma v_\varepsilon\,dz) \\
\qquad - \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon^o + \int_{\mathbb{T}^2} (\mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon^o)\,dxdy & \text{in } \mathbb{T}^2, \\
\partial_t \mathcal{P}_\tau v_\varepsilon + \dfrac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon^o - c_1(\mu \Delta_h \mathcal{P}_\tau v_\varepsilon + \lambda \nabla_h div_h \mathcal{P}_\tau v_\varepsilon) \\
\qquad = -\mathcal{P}_\tau(v_\varepsilon \cdot \nabla_h v_\varepsilon + (\varepsilon\alpha \widetilde{\mathcal{P}_\sigma v_\varepsilon} \cdot \nabla_h \xi_\varepsilon)\mathcal{P}_\sigma v_\varepsilon) \\
\qquad + \mathcal{P}_\tau(div_h \widetilde{\mathcal{P}_\sigma v_\varepsilon} \mathcal{P}_\sigma v_\varepsilon) + \dfrac{c^2(1 - e^{\varepsilon\xi_\varepsilon})}{\varepsilon} \nabla_h \xi_\varepsilon^o \\
\qquad + \mathcal{P}_\tau(c_1(e^{-\varepsilon\alpha\xi_\varepsilon} - 1)(\mu \Delta_h v_\varepsilon + \lambda \nabla_h div_h v_\varepsilon)) & \text{in } \mathbb{T}^2,
\end{cases} \tag{4.37}$$

where

$$\xi_\varepsilon^o := \xi_\varepsilon - \int_{\mathbb{T}^2} \xi_\varepsilon \, dxdy. \tag{4.38}$$

In the following, denote by

$$\mathcal{U}_\varepsilon^o := \begin{pmatrix} \xi_\varepsilon^o \\ \mathcal{P}_\tau v_\varepsilon \end{pmatrix} = \begin{pmatrix} \xi_\varepsilon - \int_{\mathbb{T}^2} \xi_\varepsilon \, dxdy \\ \nabla_h \psi_\varepsilon \end{pmatrix}, \quad g_\varepsilon := \int_{\mathbb{T}^2} \xi_\varepsilon \, dxdy, \quad \text{and}$$

$$V_\varepsilon^o := \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\mathcal{U}_\varepsilon^o = \begin{pmatrix} \mathcal{L}_1\big(-\frac{t}{\varepsilon}\big)\mathcal{U}_\varepsilon^o \\ \mathcal{L}_2\big(-\frac{t}{\varepsilon}\big)\mathcal{U}_\varepsilon^o \end{pmatrix} \quad \big(\text{so } \mathcal{U}_\varepsilon^o = \mathcal{L}\big(\frac{t}{\varepsilon}\big)V_\varepsilon^o\big), \tag{4.39}$$

where $\mathcal{L}_1 : (\mathcal{D}'(\mathbb{T}^2))^3 \mapsto \mathcal{D}'(\mathbb{T}^2),\ \mathcal{L}_2 : (\mathcal{D}'(\mathbb{T}^2))^3 \mapsto (\mathcal{D}'(\mathbb{T}^2))^2$ will be referred to as the first and the second components of $\mathcal{L}$, respectively. Then V_ε^o satisfies

$$\begin{aligned} &\partial_t V_\varepsilon^o + \mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma v_\varepsilon, V_\varepsilon^o) + \mathcal{Q}_{\varepsilon,2}(V_\varepsilon^o, V_\varepsilon^o) + \mathcal{Q}_{\varepsilon,3}(g_\varepsilon, V_\varepsilon^o) - \mathcal{A}_\varepsilon(D)V_\varepsilon^o \\ &\qquad = \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\begin{pmatrix} 0 \\ K + L_1 + L_2 \end{pmatrix}, \end{aligned} \tag{4.40}$$

where the linear and bi-linear operators are defined as follows: for $V_1, V_2 \in (\mathcal{D}'(\mathbb{T}^2))^3, u \in (\mathcal{D}'(\mathbb{T}^2 \times 2\mathbb{T}))^2, g \in \mathbb{R}$,

$$\begin{aligned} &\mathcal{A}_\varepsilon(D)V_1 := \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\begin{pmatrix} 0 \\ c_1\big(\mu\Delta_h\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1 + \lambda\nabla_h div_h\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1\big) \end{pmatrix}, \\ &\mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma u, V_1) := \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\begin{pmatrix} div_h\big(\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_1 \int_0^1 \mathcal{P}_\sigma u \, dz\big) \\ \mathcal{P}_\tau\big(\mathcal{P}_\sigma u \cdot \nabla_h\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1 + \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1 \cdot \nabla_h\mathcal{P}_\sigma u\big) \end{pmatrix}, \\ &\mathcal{Q}_{\varepsilon,2}(V_1, V_2) \\ &:= \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\begin{pmatrix} \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1 \cdot \nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2 - \int_{\mathbb{T}^2} \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1 \cdot \nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2 \, dxdy \\ \mathcal{P}_\tau\big(\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1 \cdot \nabla_h\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_2 + c^2\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_1\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2\big) \end{pmatrix}, \\ &\mathcal{Q}_{\varepsilon,3}(g, V_1) := \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\begin{pmatrix} 0 \\ \mathcal{P}_\tau\big(c^2 g\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_1\big) \end{pmatrix}. \end{aligned} \tag{4.41}$$

Here,

$$
\begin{aligned}
&K := \mathcal{P}_\tau(-\mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h \mathcal{P}_\sigma v_\varepsilon - div_h \widetilde{\mathcal{P}_\sigma v_\varepsilon} P_\sigma v_\varepsilon),\\
&L_1 := -\varepsilon\alpha \mathcal{P}_\tau((\widetilde{\mathcal{P}_\sigma v_\varepsilon} \cdot \nabla_h \xi_\varepsilon)\mathcal{P}_\sigma v_\varepsilon) + \mathcal{P}_\tau(\frac{c^2(1+\varepsilon\xi_\varepsilon - e^{\varepsilon\xi_\varepsilon})}{\varepsilon}\nabla_h \xi^o_\varepsilon),\\
&L_2 := \mathcal{P}_\tau(c_1(e^{-\varepsilon\alpha\xi_\varepsilon} - 1)(\mu\Delta_h v_\varepsilon + \lambda\nabla_h div_h v_\varepsilon)).
\end{aligned}
\tag{4.42}
$$

Also $g_\varepsilon(t) = \int_{\mathbb{T}^2} \xi_\varepsilon \, dxdy$ satisfies, from (4.36),

$$
\begin{aligned}
\frac{d}{dt} g_\varepsilon &= -\int_{\mathbb{T}^2} ((\mathcal{U}^o_\varepsilon)_2 \cdot \nabla_h (\mathcal{U}^o_\varepsilon)_1) \, dxdy\\
&= -\int_{\mathbb{T}^2} \mathcal{L}_2\big(\frac{t}{\varepsilon}\big) V^o_\varepsilon \cdot \nabla_h \mathcal{L}_1\big(\frac{t}{\varepsilon}\big) V^o_\varepsilon \, dxdy,\\
&\text{with} \quad g_\varepsilon(t=0) = \int_{\mathbb{T}^2} \xi_0 \, dxdy.
\end{aligned}
\tag{4.43}
$$

Now we will address the strong convergence of $V^o_\varepsilon, g_\varepsilon$ as $\varepsilon \to 0^+$. Notice that L defined in (4.22) is anti-symmetric with respect to $A := \mathrm{diag}(c^2, \gamma - 1, \gamma - 1)$ (i.e., $\int V^\top ALU \, d\vec{x} = -\int U^\top ALV \, d\vec{x}$), linear and commutative with $\partial_h \in \{\partial_x, \partial_y\}$. Consequently the standard H^s estimate of the solutions to equation (4.27) implies that the operator $\mathcal{L}$ preserves the H^s norm, i.e.,

$$
\big|\mathcal{L}(t)V\big|_{H^s} \simeq \big|V\big|_{H^s} \text{ for } t \in (0,\infty), s \in \mathbb{N}, V \in (\mathcal{D}'(\mathbb{T}^2))^3. \tag{4.44}
$$

Therefore one has, as the consequence of (4.25) and (4.39),

$$
\big|V^o_\varepsilon\big|_{L^\infty(0,T;H^2(\mathbb{T}^2))} + \big|\mathcal{U}^o_\varepsilon\big|_{L^\infty(0,T;H^2(\mathbb{T}^2))} + \big|(\mathcal{U}^o_\varepsilon)_2\big|_{L^2(0,T;H^3(\mathbb{T}^2))} < C. \tag{4.45}
$$

Then it is easy to see from (4.43) that

$$
\sup_{0\leq t\leq T} \{|g_\varepsilon| + |\frac{d}{dt} g_\varepsilon|\} \leq (1+T)\big|\mathcal{U}^o_\varepsilon\big|^2_{L^\infty(0,T;H^2(\mathbb{T}^2))} < C. \tag{4.46}
$$

Therefore the Arzelà-Ascoli theorem implies that as $\varepsilon \to 0^+$,

$$
g_\varepsilon \to g^o \text{ uniformly for some } g^o \in C([0,T]), \tag{4.47}
$$

and $|g^o| < C$. On the other hand, after applying the Hölder and Sobolev embedding inequalities, (4.1), (4.41), (4.45), and (4.46) imply the following estimates:

$$\begin{aligned}
&\left|\mathcal{A}_\varepsilon(D)V_\varepsilon^o\right|_{L^\infty(0,T;L^2(\mathbb{T}^2))\cap L^2(0,T;H^1(\mathbb{T}^2)} \lesssim \left|V_\varepsilon^o\right|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \\
&\quad + \left|(\mathcal{U}_\varepsilon^o)_2\right|_{L^2(0,T;H^3(\mathbb{T}^2))} < C, \\
&\left|Q_{\varepsilon,1}(\mathcal{P}_\sigma v_\varepsilon, V_\varepsilon^o)\right|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \lesssim \left|V_\varepsilon^o\right|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \\
&\quad \times \left\| \mathcal{P}_\sigma v_\varepsilon \right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))} < C, \\
&\left|Q_{\varepsilon,2}(V_\varepsilon^o, V_\varepsilon^o)\right|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \lesssim \left|V_\varepsilon^o\right|^2_{L^\infty(0,T;H^2(\mathbb{T}^2))} < C, \\
&\left|Q_{\varepsilon,3}(g_\varepsilon, V_\varepsilon^o)\right|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \lesssim \sup_{0\leq t\leq T} |g_\varepsilon| \\
&\quad \times \left|V_\varepsilon^o\right|_{L^\infty(0,T;H^2(\mathbb{T}^2))} < C,
\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}
&\left|K\right|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \lesssim \left\| \mathcal{P}_\sigma v_\varepsilon \right\|^2_{L^\infty(0,T;H^2(\mathbb{T}^2))} < C, \\
&\left|L_1\right|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \lesssim \varepsilon \left\| \mathcal{P}_\sigma v_\varepsilon \right\|^2_{L^\infty(0,T;H^2(\mathbb{T}^2))} \left\| \xi_\varepsilon \right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \\
&\quad + \varepsilon e^{\varepsilon \left\| \xi_\varepsilon \right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))}} \left(\left\| \xi_\varepsilon \right\|^2_{L^\infty(0,T;H^2(\mathbb{T}^2))} + 1 \right) < \varepsilon C, \\
&\left|L_2\right|_{L^\infty(0,T;L^2(\mathbb{T}^2))\cap L^2(0,T;H^1(\mathbb{T}^2))} \lesssim \varepsilon e^{\varepsilon \left\| \xi_\varepsilon \right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))}} \\
&\quad \times \left(\left\| \xi_\varepsilon \right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))} + 1 \right) \left\| v_\varepsilon \right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))\cap L^2(0,T;H^3(\mathbb{T}^2))} \\
&\quad < \varepsilon C.
\end{aligned} \tag{4.49}$$

Therefore, equation (4.40) implies

$$\left|\partial_t V_\varepsilon^o\right|_{L^\infty(0,T;L^2(\mathbb{T}^2))\cap L^2(0,T;H^1(\mathbb{T}^2))} < C.$$

Then the Aubin compactness lemma (see, i.e., [42, Theorem 2.1] and [5,41]) implies that, there is

$$\begin{aligned}
&V^o \in L^\infty(0,T;H^2(\mathbb{T}^2)) \cap C(0,T;H^1(\mathbb{T}^2)) \quad \text{with} \\
&\partial_t V^o \in L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2)),
\end{aligned} \tag{4.50}$$

such that as $\varepsilon \to 0^+$,

$$\begin{aligned}
&V_\varepsilon^o \overset{*}{\rightharpoonup} V^o \quad \text{weak-}* \text{ in } L^\infty(0,T;H^2(\mathbb{T}^2)), \\
&V_\varepsilon^o \to V^o \quad \text{in } L^\infty(0,T;H^1(\mathbb{T}^2)) \cap C(0,T;H^1(\mathbb{T}^2)), \\
&\partial_t V_\varepsilon^o \overset{*}{\rightharpoonup} \partial_t V^o \quad \text{weak-}* \text{ in } L^\infty(0,T;L^2(\mathbb{T}^2)), \\
&\partial_t V_\varepsilon^o \rightharpoonup \partial_t V^o \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^2)).
\end{aligned} \tag{4.51}$$

Also

$$V_\varepsilon^o(t=0) = V^o(t=0) = \begin{pmatrix} \xi_0 - \int_{\mathbb{T}^2} \xi_0 \, dxdy \\ \mathcal{P}_\tau v_0 \end{pmatrix}. \tag{4.52}$$

Consequently, as $\varepsilon \to 0^+$, since $\mathcal{L}$ preserves the H^1 norm,

$$\mathcal{U}_\varepsilon^o - \mathcal{L}\big(\frac{t}{\varepsilon}\big)V^o = \mathcal{L}\big(\frac{t}{\varepsilon}\big)\big(V_\varepsilon^o - V^o\big) \to 0 \ \text{ in } L^\infty(0,T;H^1(\mathbb{T}^2)). \tag{4.53}$$

In conclusion, as $\varepsilon \to 0^+$, (4.12), (4.47) and (4.53) imply that

$$\begin{aligned} &\big\| v_\varepsilon - v_p - \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V^o \big\|_{L^\infty(0,T;H^1(\mathbb{T}^2\times 2\mathbb{T}))} \\ &\quad + \big\| \xi_\varepsilon - g^o - \mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V^o \big\|_{L^\infty(0,T;H^1(\mathbb{T}^2\times 2\mathbb{T}))} + \big\| g_\varepsilon - g^o \big\|_{L^\infty(0,T)} \\ &\lesssim \big\| g_\varepsilon - g^o \big\|_{L^\infty(0,T)} + \big|\mathcal{U}_\varepsilon^o - \mathcal{L}\big(\frac{t}{\varepsilon}\big)V^o\big|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \\ &\quad + \big\| \mathcal{P}_\sigma v_\varepsilon - v_p \big\|_{L^\infty(0,T;H^1(\mathbb{T}^2\times 2\mathbb{T}))} \to 0. \end{aligned} \tag{4.54}$$

Next, we will identify the limit equations of (4.40) and (4.43); that is, the equation satisfied by V^o and g^o.

The limit equations of oscillations

In order to identify the limit equations of (4.40) and (4.43) as $\varepsilon \to 0^+$, we first introduce the Fourier representation of the operators defined in (4.41). Notice that (4.27) implies $\int_{\mathbb{T}^2} V_\varepsilon^o \, dxdy = \int_{\mathbb{T}^2} \mathcal{U}_\varepsilon^o \, dxdy = 0$. It suffices to study in the space consisting of functions in $\mathcal{D}'(\mathbb{T}^2)^3$ with zero average.

Notice that, on the other hand, $\mathcal{U}_\varepsilon^o$ is inside the orthogonal complement of the kernel of operator L with respect to the L^2-inner product, and thanks to (4.27), so is V_ε^o. In fact, $(\ker L)^\perp = \{(q,u) \subset \mathcal{D}'(\mathbb{T}^2) \times (\mathcal{D}'(\mathbb{T}^2))^2 | \int_{\mathbb{T}^2} q \, dxdy = 0, u = \nabla_h \psi, \psi \in \mathcal{D}'(\mathbb{T}^2)\}$. Next, we introduce, as in [6] and [31], the following basis of $(\ker L)^\perp$: for $\mathbf{k} \in 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}$,

$$\begin{aligned} V_{\mathbf{k}}^+(\vec{x}_h) &:= \frac{1}{\sqrt{\gamma - 1 + c^2}|\mathbf{k}|} \begin{pmatrix} \sqrt{\gamma-1}|\mathbf{k}| \\ -c \,\mathrm{sg}(\mathbf{k})\mathbf{k} \end{pmatrix} e^{i\mathbf{k}\cdot\vec{x}_h}, \\ V_{\mathbf{k}}^-(\vec{x}_h) &:= \frac{1}{\sqrt{\gamma - 1 + c^2}|\mathbf{k}|} \begin{pmatrix} \sqrt{\gamma-1}|\mathbf{k}| \\ +c \,\mathrm{sg}(\mathbf{k})\mathbf{k} \end{pmatrix} e^{i\mathbf{k}\cdot\vec{x}_h}, \end{aligned} \tag{4.55}$$

and the corresponding conjugates

$$\begin{aligned} V_{\mathbf{k}}^{*,+}(\vec{x}_h) &:= \frac{\sqrt{\gamma-1+c^2}}{2c\sqrt{\gamma-1}|\mathbf{k}|} \begin{pmatrix} c|\mathbf{k}| \\ -\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{k})\mathbf{k} \end{pmatrix} e^{i\mathbf{k}\cdot\vec{x}_h}, \\ V_{\mathbf{k}}^{*,-}(\vec{x}_h) &:= \frac{\sqrt{\gamma-1+c^2}}{2c\sqrt{\gamma-1}|\mathbf{k}|} \begin{pmatrix} c|\mathbf{k}| \\ +\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{k})\mathbf{k} \end{pmatrix} e^{i\mathbf{k}\cdot\vec{x}_h}. \end{aligned}$$

Here $\operatorname{sg}(\mathbf{k})$ is the generalized sign function for vector $\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}$, defined as,

$$\operatorname{sg}(\mathbf{k}) := \begin{cases} 1 & \text{if and only if } k_1 > 0 \text{ or } k_1 = 0, k_2 > 0, \\ -1 & \text{otherwise.} \end{cases}$$

Notice that $V_{\mathbf{k}}^{\pm} = \overline{V_{-\mathbf{k}}^{\pm}}^c$, where, hereafter, $\overline{\cdot}^c$ represents the complex conjugate.

Then by defining $\varsigma = c\sqrt{\gamma - 1}$, $\{V_{\mathbf{k}}^{\pm}\}_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}}$ are the eigenfunctions of L with the eigenvalues $\{\mp i\varsigma \operatorname{sg}(\mathbf{k})|\mathbf{k}|\}_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}}$, i.e.,

$$LV_{\mathbf{k}}^{+} = -i\varsigma \operatorname{sg}(\mathbf{k})|\mathbf{k}|V_{\mathbf{k}}^{+}, \quad \text{and} \quad LV_{\mathbf{k}}^{-} = i\varsigma \operatorname{sg}(\mathbf{k})|\mathbf{k}|V_{\mathbf{k}}^{-}, \ \mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}.$$

Also any $V \in (\ker L)^{\perp}$ can be represented in terms of Fourier series as

$$V = \sum_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}} \left(a_{\mathbf{k}}^{+} V_{\mathbf{k}}^{+} + a_{\mathbf{k}}^{-} V_{\mathbf{k}}^{-}\right) \tag{4.56}$$

with the Fourier coefficients $\{a_{\mathbf{k}}^{\pm}\}_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}}$ determined by

$$a_{\mathbf{k}}^{\pm} = \langle V, V_{\mathbf{k}}^{*,\pm} \rangle_{\mathbb{C}} := \int_{\mathbb{T}^2} V \cdot \overline{V_{\mathbf{k}}^{*,\pm}}^c \, dxdy. \tag{4.57}$$

If V is real-valued, $a_{-\mathbf{k}}^{\pm} = \overline{a_{\mathbf{k}}^{\pm}}^c$. In addition, one has

$$\mathcal{L}(s)V = \sum_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}} \left(a_{\mathbf{k}}^{+} V_{\mathbf{k}}^{+} e^{i\varsigma \operatorname{sg}(\mathbf{k})|\mathbf{k}|s} + a_{\mathbf{k}}^{-} V_{\mathbf{k}}^{-} e^{-i\varsigma \operatorname{sg}(\mathbf{k})|\mathbf{k}|s}\right). \tag{4.58}$$

Denote by $\mathcal{P}_{(\ker L)^{\perp}} : (\mathcal{D}'(\mathbb{T}^2))^3 \mapsto (\ker L)^{\perp} \subset (\mathcal{D}'(\mathbb{T}^2))^3$, being the projection operator on $(\ker L)^{\perp}$ with respect to the L^2-inner product. Then for any $q \in \mathcal{D}'(\mathbb{T}^2)$ and $u \in (\mathcal{D}'(\mathbb{T}^2 \times 2\mathbb{T}))^2$, it satisfies,

$$\mathcal{P}_{(\ker L)^{\perp}} \begin{pmatrix} q \\ \int_0^1 u \, dz \end{pmatrix} = \begin{pmatrix} q - \int_{\mathbb{T}^2} q \, dxdy \\ \mathcal{P}_{\tau} \int_0^1 u \, dz \end{pmatrix} = \begin{pmatrix} q - \int_{\mathbb{T}^2} q \, dxdy \\ \mathcal{P}_{\tau} u \end{pmatrix}. \tag{4.59}$$

Moreover, as in (4.56) and (4.57), one has

$$\mathcal{P}_{(\ker L)^{\perp}} \begin{pmatrix} q \\ \int_0^1 u \, dz \end{pmatrix} = \sum_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}} \left(b_{\mathbf{k}}^{+} V_{\mathbf{k}}^{+} + b_{\mathbf{k}}^{-} V_{\mathbf{k}}^{-}\right), \tag{4.60}$$

with $\{b_{\mathbf{k}}^{\pm}\}_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}}$ determined by

$$b_{\mathbf{k}}^{\pm} = \langle \begin{pmatrix} q \\ \int_0^1 u \, dz \end{pmatrix}, V_{\mathbf{k}}^{*,\pm} \rangle_{\mathbb{C}} = \int_{\mathbb{T}^2} \begin{pmatrix} q \\ \int_0^1 u \, dz \end{pmatrix} \cdot \overline{V_{\mathbf{k}}^{*,\pm}}^c \, dxdy.$$

We will now investigate the action of the operators in (4.41) on $V_{\mathbf{k}}^{\pm}$. Without further mentioning, we will use the representations (4.59) and (4.60), below. To shorten the notations, the multi-indices $\mathbf{k}, \mathbf{l}, \mathbf{m}, \cdots$ in the following calculations are always subject to the set $2\pi\mathbb{T}^2 \setminus \{(0,0)\}$. For instance,

$$\sum_{\mathbf{k}} = \sum_{\mathbf{k}\in 2\pi\mathbb{T}^2\setminus\{(0,0)\}}, \qquad \text{etc.}$$

Calculation of $\mathcal{A}_\varepsilon(D)$. To calculate $\mathcal{A}_\varepsilon(D)V_{\mathbf{k}}^{\pm}$, notice that,

$$\begin{aligned} & c_1\big(\mu\Delta_h\mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} + \lambda\nabla_h div_h\mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm}\big) \\ & \quad = \pm c_1(\mu+\lambda)\frac{c\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\mathbf{k}}{\sqrt{\gamma-1+c^2}}e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}e^{i\mathbf{k}\cdot\vec{x}_h}. \end{aligned}$$

Denote by

$$\begin{pmatrix} 0 \\ c_1(\mu\Delta_h\mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} + \lambda\nabla_h div_h\mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm}) \end{pmatrix} =: A_{\mathbf{k}} = \alpha_{\mathbf{k}}V_{\mathbf{k}}^{\pm} + \beta_{\mathbf{k}}V_{\mathbf{k}}^{\mp}.$$

Then

$$\begin{aligned} \alpha_{\mathbf{k}} &= \int_{\mathbb{T}^2} A_{\mathbf{k}}\cdot\overline{V_{\mathbf{k}}^{*,\pm}}^c\,dxdy = \frac{\sqrt{\gamma-1+c^2}}{2c\sqrt{\gamma-1}|\mathbf{k}|}e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}} \\ & \quad\times\begin{pmatrix} 0 \\ \pm c_1(\mu+\lambda)\frac{c\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\mathbf{k}}{\sqrt{\gamma-1+c^2}}e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}} \end{pmatrix}\cdot\begin{pmatrix} c|\mathbf{k}| \\ \mp\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{k})\mathbf{k} \end{pmatrix} \\ &= -\frac{c_1(\mu+\lambda)}{2}|\mathbf{k}|^2e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}, \qquad \text{and similarly} \\ \beta_{\mathbf{k}} &= \int_{\mathbb{T}^2} A_{\mathbf{k}}\cdot\overline{V_{\mathbf{k}}^{*,\mp}}^c\,dxdy = \frac{c_1(\mu+\lambda)}{2}|\mathbf{k}|^2e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{A}_\varepsilon(D)V_{\mathbf{k}}^{\pm} = \mathcal{L}\big(-\frac{t}{\varepsilon}\big)A_{\mathbf{k}} = -\frac{c_1(\mu+\lambda)}{2}|\mathbf{k}|^2V_{\mathbf{k}}^{\pm} \\ + \frac{c_1(\mu+\lambda)}{2}|\mathbf{k}|^2V_{\mathbf{k}}^{\mp}e^{\pm 2i\varsigma\,\mathrm{sg}(\mathbf{k})\mathbf{k}\frac{t}{\varepsilon}}. \end{aligned}$$

Calculation of $\mathcal{Q}_{\varepsilon,1}$. To calculate $\mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma u, V_{\mathbf{k}}^{\pm})$, we first represent $\mathcal{P}_\sigma u$ as, after expanding it as Fourier series in the horizontal direction,

$$\mathcal{P}_\sigma u = \sum_{\mathbf{k}} u_{\mathbf{k}}(z,t)e^{i\mathbf{k}\cdot\vec{x}_h}, \quad \int_0^1 \mathcal{P}_\sigma u\,dz = \sum_{\mathbf{k}}\overline{u}_{\mathbf{k}}e^{i\mathbf{k}\cdot\vec{x}_h}, \tag{4.61}$$

satisfying

$$\overline{u}_{\mathbf{k}} \cdot \mathbf{k} = 0. \tag{4.62}$$

Here, recall that $\overline{u}_{\mathbf{k}} = \int_0^1 u_{\mathbf{k}}\, dz$. Then the relations (4.59) and (4.60) allow us the write down the following:

$$\begin{aligned}
&\begin{pmatrix} div_h(\mathcal{L}_1(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm}\int_0^1 \mathcal{P}_\sigma u\, dz) \\ \mathcal{P}_\tau(\mathcal{P}_\sigma u \cdot \nabla_h \mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} + \mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} \cdot \nabla_h \mathcal{P}_\sigma u) \end{pmatrix} \\
&= \mathcal{P}_{(\ker L)^\perp}\begin{pmatrix} div_h(\mathcal{L}_1(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm}\int_0^1 \mathcal{P}_\sigma u\, dz) \\ \int_0^1 \mathcal{P}_\sigma u\, dz \cdot \nabla_h \mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} + \mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} \cdot \nabla_h \int_0^1 \mathcal{P}_\sigma u\, dz) \end{pmatrix} \\
&=: \sum_{\mathbf{m}} (\xi_{\mathbf{m}} V_{\mathbf{m}}^{\pm} + \zeta_{\mathbf{m}} V_{\mathbf{m}}^{\mp}).
\end{aligned}$$

Denote by

$$\begin{aligned}
B_{\mathbf{k}} &:= \begin{pmatrix} div_h(\mathcal{L}_1(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm}\int_0^1 \mathcal{P}_\sigma u\, dz) \\ \int_0^1 \mathcal{P}_\sigma u\, dz \cdot \nabla_h \mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} + \mathcal{L}_2(\frac{t}{\varepsilon})V_{\mathbf{k}}^{\pm} \cdot \nabla_h \int_0^1 \mathcal{P}_\sigma u\, dz) \end{pmatrix} \\
&= \sum_{\mathbf{l}} \begin{pmatrix} i\dfrac{\sqrt{\gamma-1}}{\sqrt{\gamma-1+c^2}}(\mathbf{k}+\mathbf{l})\cdot \hat{u}_{\mathbf{l}} \\ i\dfrac{\mp c\,\mathrm{sg}(\mathbf{k})}{\sqrt{\gamma-1+c^2}|\mathbf{k}|}((\hat{u}_{\mathbf{l}}\cdot\mathbf{k})\mathbf{k} + (\mathbf{k}\cdot\mathbf{l})\hat{u}_{\mathbf{l}}) \end{pmatrix} e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}} e^{i(\mathbf{k}+\mathbf{l})\cdot\vec{x}_h}.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{\mathbf{m}} &= \int_{\mathbb{T}^2} B_{\mathbf{k}} \cdot \overline{V_{\mathbf{m}}^{*,\pm}}^{c}\, dxdy = \frac{\sqrt{\gamma-1+c^2}}{2c\sqrt{\gamma-1}|\mathbf{m}|} e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}} \\
&\times \begin{pmatrix} i\dfrac{\sqrt{\gamma-1}}{\sqrt{\gamma-1+c^2}}(\mathbf{k}+\mathbf{l})\cdot \hat{u}_{\mathbf{l}} \\ i\dfrac{\mp c\,\mathrm{sg}(\mathbf{k})}{\sqrt{\gamma-1+c^2}|\mathbf{k}|}((\hat{u}_{\mathbf{l}}\cdot\mathbf{k})\mathbf{k} + (\mathbf{k}\cdot\mathbf{l})\hat{u}_{\mathbf{l}}) \end{pmatrix} \cdot \begin{pmatrix} c|\mathbf{m}| \\ \mp\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{m})\mathbf{m} \end{pmatrix} \\
&= \frac{i}{2}(\mathbf{k}\cdot\hat{u}_{\mathbf{l}})\big(1 + (\mathbf{m}\cdot\mathbf{k} + \mathbf{k}\cdot\mathbf{l})\frac{\mathrm{sg}(\mathbf{m})\,\mathrm{sg}(\mathbf{k})}{|\mathbf{m}||\mathbf{k}|}\big) e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}, \quad \text{and similarly} \\
\zeta_{\mathbf{m}} &= \int_{\mathbb{T}^2} B_{\mathbf{k}} \cdot \overline{V_{\mathbf{m}}^{*,\mp}}^{c}\, dxdy \\
&= \frac{i}{2}(\mathbf{k}\cdot\hat{u}_{\mathbf{l}})\big(1 - (\mathbf{m}\cdot\mathbf{k} + \mathbf{k}\cdot\mathbf{l})\frac{\mathrm{sg}(\mathbf{m})\,\mathrm{sg}(\mathbf{k})}{|\mathbf{m}||\mathbf{k}|}\big) e^{\pm i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}},
\end{aligned}$$

where $\mathbf{l} = \mathbf{m} - \mathbf{k}$ and we have used (4.62). Therefore

$$\begin{aligned}
\mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma u, V_{\mathbf{k}}^{\pm}) &= \mathcal{L}\big(-\frac{t}{\varepsilon}\big)\sum_{\mathbf{m}}\big(\xi_{\mathbf{m}}V_{\mathbf{m}}^{\pm} + \zeta_{\mathbf{m}}V_{\mathbf{m}}^{\mp}\big) \\
&= \frac{i}{2}\sum_{\mathbf{m}-\mathbf{l}=\mathbf{k}}\Bigg(\big((\mathbf{k}\cdot\hat{u}_{\mathbf{l}})(1 + \frac{\operatorname{sg}(\mathbf{m})\operatorname{sg}(\mathbf{k})}{|\mathbf{m}||\mathbf{k}|}\mathbf{k}\cdot(\mathbf{m}+\mathbf{l}))\big)V_{\mathbf{m}}^{\pm} \\
&\qquad \times e^{\pm i\varsigma(\operatorname{sg}(\mathbf{k})|\mathbf{k}| - \operatorname{sg}(\mathbf{m})|\mathbf{m}|)\frac{t}{\varepsilon}} \\
&\quad + \big((\mathbf{k}\cdot\hat{u}_{\mathbf{l}})(1 - \frac{\operatorname{sg}(\mathbf{m})\operatorname{sg}(\mathbf{k})}{|\mathbf{m}||\mathbf{k}|}\mathbf{k}\cdot(\mathbf{m}+\mathbf{l}))\big)V_{\mathbf{m}}^{\mp} \\
&\qquad \times e^{\pm i\varsigma(\operatorname{sg}(\mathbf{k})|\mathbf{k}| + \operatorname{sg}(\mathbf{m})|\mathbf{m}|)\frac{t}{\varepsilon}}\Bigg).
\end{aligned}$$

Calculation of $\mathcal{Q}_{\varepsilon,2}$. To calculate $\mathcal{Q}_{\varepsilon,2}(V_{\mathbf{k}}^{\pm}, V_{\mathbf{l}}^{\pm})$ and $\mathcal{Q}_{\varepsilon,2}(V_{\mathbf{k}}^{\pm}, V_{\mathbf{l}}^{\mp})$, as before, notice that, for $V_1, V_2 \in (\mathcal{D}'(\mathbb{T}^2))^3$, the relation (4.59) implies the following identity:

$$\begin{aligned}
&\begin{pmatrix} \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1\cdot\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2 - \fint_{\mathbb{T}^2}\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1\cdot\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2\,dxdy \\ \mathcal{P}_\tau(\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1\cdot\nabla_h\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_2 + c^2\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_1\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2) \end{pmatrix} \\
&= \mathcal{P}_{(\ker L)^\perp}\begin{pmatrix} \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1\cdot\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2 \\ \mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_1\cdot\nabla_h\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V_2 + c^2\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_1\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V_2 \end{pmatrix}.
\end{aligned}$$

Thus following the same arguments as in the calculation of $\mathcal{Q}_{\varepsilon,1}$ yields:

$$\begin{aligned}
\mathcal{Q}_{\varepsilon,2}(V_{\mathbf{k}}^{\pm}, V_{\mathbf{l}}^{\pm}) &= \mp\frac{ic}{2\sqrt{\gamma-1+c^2}}\big(\frac{\operatorname{sg}(\mathbf{k})\mathbf{k}\cdot\mathbf{l}}{|\mathbf{k}|} + \frac{\operatorname{sg}(\mathbf{m})\operatorname{sg}(\mathbf{k})\operatorname{sg}(\mathbf{l})(\mathbf{l}\cdot\mathbf{k})(\mathbf{l}\cdot\mathbf{m})}{|\mathbf{m}||\mathbf{k}||\mathbf{l}|} \\
&\quad + \frac{(\gamma-1)\operatorname{sg}(\mathbf{m})\mathbf{l}\cdot\mathbf{m}}{|\mathbf{m}|}\big)V_{\mathbf{m}}^{\pm}e^{\pm i\varsigma(\operatorname{sg}(\mathbf{k})|\mathbf{k}| + \operatorname{sg}(\mathbf{l})|\mathbf{l}| - \operatorname{sg}(\mathbf{m})|\mathbf{m}|)\frac{t}{\varepsilon}} \\
&\quad \mp\frac{ic}{2\sqrt{\gamma-1+c^2}}\big(\frac{\operatorname{sg}(\mathbf{k})\mathbf{k}\cdot\mathbf{l}}{|\mathbf{k}|} - \frac{\operatorname{sg}(\mathbf{m})\operatorname{sg}(\mathbf{k})\operatorname{sg}(\mathbf{l})(\mathbf{l}\cdot\mathbf{k})(\mathbf{l}\cdot\mathbf{m})}{|\mathbf{m}||\mathbf{k}||\mathbf{l}|} \\
&\quad - \frac{(\gamma-1)\operatorname{sg}(\mathbf{m})\mathbf{l}\cdot\mathbf{m}}{|\mathbf{m}|}\big)V_{\mathbf{m}}^{\mp}e^{\pm i\varsigma(\operatorname{sg}(\mathbf{k})|\mathbf{k}| + \operatorname{sg}(\mathbf{l})|\mathbf{l}| + \operatorname{sg}(\mathbf{m})|\mathbf{m}|)\frac{t}{\varepsilon}}, \\
\mathcal{Q}_{\varepsilon,2}(V_{\mathbf{k}}^{\pm}, V_{\mathbf{l}}^{\mp}) &= \mp\frac{ic}{2\sqrt{\gamma-1+c^2}}\big(\frac{\operatorname{sg}(\mathbf{k})\mathbf{k}\cdot\mathbf{l}}{|\mathbf{k}|} - \frac{\operatorname{sg}(\mathbf{m})\operatorname{sg}(\mathbf{k})\operatorname{sg}(\mathbf{l})(\mathbf{l}\cdot\mathbf{k})(\mathbf{l}\cdot\mathbf{m})}{|\mathbf{m}||\mathbf{k}||\mathbf{l}|} \\
&\quad + \frac{(\gamma-1)\operatorname{sg}(\mathbf{m})\mathbf{l}\cdot\mathbf{m}}{|\mathbf{m}|}\big)V_{\mathbf{m}}^{\pm}e^{\pm i\varsigma(\operatorname{sg}(\mathbf{k})|\mathbf{k}| - \operatorname{sg}(\mathbf{l})|\mathbf{l}| - \operatorname{sg}(\mathbf{m})|\mathbf{m}|)\frac{t}{\varepsilon}}
\end{aligned}$$

$$
\mp \frac{ic}{2\sqrt{\gamma-1+c^2}}\Big(\frac{\mathrm{sg}(\mathbf{k})\mathbf{k}\cdot\mathbf{l}}{|\mathbf{k}|}+\frac{\mathrm{sg}(\mathbf{m})\,\mathrm{sg}(\mathbf{k})\,\mathrm{sg}(\mathbf{l})(\mathbf{l}\cdot\mathbf{k})(\mathbf{l}\cdot\mathbf{m})}{|\mathbf{m}||\mathbf{k}||\mathbf{l}|}
$$

$$
-\frac{(\gamma-1)\,\mathrm{sg}(\mathbf{m})\mathbf{l}\cdot\mathbf{m}}{|\mathbf{m}|}\Big)V^{\mp}_{\mathbf{m}}e^{\pm i\varsigma(\mathrm{sg}(\mathbf{k})|\mathbf{k}|-\mathrm{sg}(\mathbf{l})|\mathbf{l}|+\mathrm{sg}(\mathbf{m})|\mathbf{m}|)\frac{t}{\varepsilon}},
$$

where $\mathbf{m}=\mathbf{k}+\mathbf{l}$.

Calculation of $\mathcal{Q}_{\varepsilon,3}$. This will be similar to the calculation of $\mathcal{A}_\varepsilon(D)$. The result is

$$
\mathcal{Q}_{\varepsilon,3}(g,V^{\pm}_{\mathbf{k}})=\mp\frac{icg\sqrt{\gamma-1}}{2}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|V^{\pm}_{\mathbf{k}}
\pm\frac{icg\sqrt{\gamma-1}}{2}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|V^{\mp}_{\mathbf{k}}e^{\pm 2i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}.
$$

The limits of the operators $\mathcal{A}_\varepsilon(D),\mathcal{Q}_{\varepsilon,1},\mathcal{Q}_{\varepsilon,2},\mathcal{Q}_{\varepsilon,3}$ as $\varepsilon\to 0^+$. For fixed $\mathbf{k},\mathbf{l}$, in the sense of distribution, we have the following: as $\varepsilon\to 0^+$,

$$
\mathcal{A}_\varepsilon(D)V^{\pm}_{\mathbf{k}}\rightharpoonup-\frac{(\mu+\lambda)c_1}{2}|\mathbf{k}|^2V^{\pm}_{\mathbf{k}}=\frac{(\mu+\lambda)c_1}{2}\Delta_hV^{\pm}_{\mathbf{k}}=:\mathcal{A}(D)V^{\pm}_{\mathbf{k}},\tag{4.63}
$$

$$
\mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma u,V^{\pm}_{\mathbf{k}})\rightharpoonup i\sum_{\substack{\mathbf{m}-\mathbf{l}=\mathbf{k}\\ \mathrm{sg}(\mathbf{k})|\mathbf{k}|=\mathrm{sg}(\mathbf{m})|\mathbf{m}|}}\frac{(\hat u_{\mathbf{l}}\cdot\mathbf{k})(\mathbf{m}\cdot\mathbf{k})}{|\mathbf{k}|^2}V^{\pm}_{\mathbf{m}}
+i\sum_{\substack{\mathbf{m}-\mathbf{l}=\mathbf{k}\\ \mathrm{sg}(\mathbf{k})|\mathbf{k}|=-\mathrm{sg}(\mathbf{m})|\mathbf{m}|}}\frac{(\hat u_{\mathbf{l}}\cdot\mathbf{k})(\mathbf{m}\cdot\mathbf{k})}{|\mathbf{k}|^2}V^{\mp}_{\mathbf{m}}=:\mathcal{Q}_1(\mathcal{P}_\sigma u,V^{\pm}_{\mathbf{k}}),\tag{4.64}
$$

$$
\mathcal{Q}_{\varepsilon,2}(V^{\pm}_{\mathbf{k}},V^{\pm}_{\mathbf{l}})\rightharpoonup\mp\frac{ic(\gamma+1)}{2\sqrt{\gamma-1+c^2}}\,\mathrm{sg}(\mathbf{l})|\mathbf{l}|V^{\pm}_{\mathbf{m}}=:\mathcal{Q}_2(V^{\pm}_{\mathbf{k}},V^{\pm}_{\mathbf{l}}),\tag{4.65}
$$

where $\mathbf{m}=\mathbf{k}+\mathbf{l}$, $\mathrm{sg}(\mathbf{k})|\mathbf{k}|+\mathrm{sg}(\mathbf{l})|\mathbf{l}|=\mathrm{sg}(\mathbf{m})|\mathbf{m}|$, $\mathbf{k}$ and $\mathbf{l}$ are co-linear,

$$
\mathcal{Q}_{\varepsilon,2}(V^{\pm}_{\mathbf{k}},V^{\mp}_{\mathbf{l}})\rightharpoonup 0=:\mathcal{Q}_2(V^{\pm}_{\mathbf{k}},V^{\mp}_{\mathbf{l}}),\tag{4.66}
$$

$$
\mathcal{Q}_{\varepsilon,3}(g_\varepsilon,V^{\pm}_{\mathbf{k}})\rightharpoonup\mp\frac{icg\sqrt{\gamma-1}}{2}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|V^{\pm}_{\mathbf{k}}=:\mathcal{Q}_3(g,V^{\pm}_{\mathbf{k}}),\tag{4.67}
$$

where we have used the facts that

$$
\begin{aligned}
&\big\{\mathbf{m}=\mathbf{k}+\mathbf{l},\ \mathrm{sg}(\mathbf{k})|\mathbf{k}|+\mathrm{sg}(\mathbf{l})|\mathbf{l}|=\mathrm{sg}(\mathbf{m})|\mathbf{m}|\big\}\\
&=\big\{\mathbf{k},\mathbf{l},\mathbf{m}\text{ are co-linear, and }\mathbf{l}\cdot\mathbf{k}=\mathrm{sg}(\mathbf{l})\,\mathrm{sg}(\mathbf{k})|\mathbf{l}||\mathbf{k}|,\\
&\qquad\mathbf{l}\cdot\mathbf{m}=\mathrm{sg}(\mathbf{l})\,\mathrm{sg}(\mathbf{m})|\mathbf{l}||\mathbf{m}|\big\},\\
&\text{and }\big\{\mathbf{m}=\mathbf{k}+\mathbf{l},\ \mathrm{sg}(\mathbf{k})|\mathbf{k}|+\mathrm{sg}(\mathbf{l})|\mathbf{l}|=-\mathrm{sg}(\mathbf{m})|\mathbf{m}|\big\}=\emptyset,\\
&\big\{\mathbf{m}=\mathbf{k}+\mathbf{l},\ \mathrm{sg}(\mathbf{k})|\mathbf{k}|-\mathrm{sg}(\mathbf{l})|\mathbf{l}|=\pm\,\mathrm{sg}(\mathbf{m})|\mathbf{m}|\big\}=\emptyset.
\end{aligned}
$$

The limit equations. Now we have prepared enough to identify the limit equations of equations (4.40) and (4.43) as $\varepsilon \to 0^+$.

We rewrite equation (4.40) in the following fashion:

$$
\begin{aligned}
&\partial_t V_\varepsilon^o + \mathcal{Q}_{\varepsilon,1}(v_p, V^o) + \mathcal{Q}_{\varepsilon,2}(V^o, V^o) + \mathcal{Q}_{\varepsilon,3}(g^o, V^o) - \mathcal{A}_\varepsilon(D)V^o \\
&\quad = -\mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma v_\varepsilon - v_p, V_\varepsilon^o) - \mathcal{Q}_{\varepsilon,1}(v_p, V_\varepsilon^o - V^o) \\
&\quad - \mathcal{Q}_{\varepsilon,2}(V_\varepsilon^o - V^o, V_\varepsilon^o) - \mathcal{Q}_{\varepsilon,2}(V^o, V_\varepsilon^o - V^o) \\
&\quad - \mathcal{Q}_{\varepsilon,3}(g_\varepsilon - g^o, V_\varepsilon^o) - \mathcal{Q}_{\varepsilon,3}(g^o, V_\varepsilon^o - V^o) \\
&\quad + \mathcal{A}_\varepsilon(D)(V_\varepsilon^o - V^o) + \mathcal{L}(-\frac{t}{\varepsilon})\begin{pmatrix} 0 \\ K + L_1 + L_2 \end{pmatrix}.
\end{aligned}
\tag{4.68}
$$

Then, from the regularity in (4.2) and (4.50), the weak convergence of $\partial_t V_\varepsilon^o$ in (4.51), and the convergence of operators in (4.63), (4.64), (4.65), (4.66) and (4.67), as $\varepsilon \to 0^+$, the left-hand side of (4.68) converges in the sense of distribution to

$$
\partial_t V^o + \mathcal{Q}_1(v_p, V^o) + \mathcal{Q}_2(V^o, V^o) + \mathcal{Q}_3(g^o, V^o) - \mathcal{A}(D)V^o.
$$

Indeed, one can replace v_p, V^o with their finite dimensional Fourier truncations on the left-hand side of (4.68), and similar estimates as in (4.48) of the operators for such truncations imply that the actions of the operators on the remainders are bounded by certain norms of the remainders uniformly in ε. Then by letting ε go to zero, since the truncations approximate the identity operator, one can show the convergence of the actions of the operators.

We claim that the right-hand side of (4.68) converges in the sense of distribution to 0. Indeed, from the definition of the operators in (4.41) and the norm preserving property (4.44), applying the Hölder and Sobolev embedding inequalities implies that, as $\varepsilon \to 0^+$,

$$
\begin{aligned}
&\left|\mathcal{Q}_{\varepsilon,1}(\mathcal{P}_\sigma v_\varepsilon - v_p, V_\varepsilon^o)\right|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \lesssim \left|\mathcal{L}(\frac{t}{\varepsilon})V_\varepsilon^o\right|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))} \\
&\quad \times \left\|\nabla_h(\mathcal{P}_\sigma v_\varepsilon - v_p)\right\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} + \left|\nabla_h \mathcal{L}(\frac{t}{\varepsilon})V_\varepsilon^o\right|_{L^\infty(0,T;L^4(\mathbb{T}^2))} \\
&\quad \times \left\|\mathcal{P}_\sigma v_\varepsilon - v_p\right\|_{L^\infty(0,T;L^4(\mathbb{T}^2))} \lesssim \left|V_\varepsilon^o\right|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \\
&\quad \times \left\|\mathcal{P}_\sigma v_\varepsilon - v_p\right\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \to 0, \\
&\left|\mathcal{Q}_{\varepsilon,1}(v_p, V_\varepsilon^o - V^o)\right|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \lesssim \left|V_\varepsilon^o - V^o\right|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \\
&\quad \times \left\|v_p\right\|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \to 0, \\
&\left|\mathcal{Q}_{\varepsilon,2}(V_\varepsilon^o - V^o, V_\varepsilon^o)\right|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \lesssim \left|\mathcal{L}(\frac{t}{\varepsilon})(V_\varepsilon^o - V^o)\right|_{L^\infty(0,T;L^4(\mathbb{T}^2))}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \big|\nabla_h \mathcal{L}(\frac{t}{\varepsilon})V_\varepsilon^o\big|_{L^\infty(0,T;L^4(\mathbb{T}^2))} \lesssim \big|V_\varepsilon^o - V^o\big|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \\
&\quad \times \big|V_\varepsilon^o\big|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \to 0, \\
&\big|\mathcal{Q}_{\varepsilon,2}(V^o, V_\varepsilon^o - V^o)\big|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \lesssim \big|V^o\big|_{L^\infty(0,T;H^2(\mathbb{T}^2))} \\
&\quad \times \big|V_\varepsilon^o - V^o\big|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \to 0, \\
&\big|\mathcal{Q}_{\varepsilon,3}(g_\varepsilon - g^o, V_\varepsilon^o)\big|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \lesssim \big|g_\varepsilon - g^o\big|_{L^\infty(0,T)} \\
&\quad \times \big|V_\varepsilon^o\big|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \to 0, \\
&\big|\mathcal{Q}_{\varepsilon,3}(g^o, V_\varepsilon^o - V^o)\big|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \lesssim \big|g^o\big|_{L^\infty(0,T)} \\
&\quad \times \big|V_\varepsilon^o - V^o\big|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \to 0,
\end{aligned}
$$

where we have applied (4.12), (4.47), (4.45), (4.50), (4.51). On the other hand, the norm preserving property (4.44) implies that, for $t \in (0,\infty)$ and any $V_1, V_2 \in (\mathcal{D}'(\mathbb{T}^2))^3$,

$$
\begin{aligned}
0 &= \big|V_1 + \mathcal{L}(t)V_2\big|^2_{L^2(\mathbb{T}^2)} - \big|\mathcal{L}(-t)V_1 + V_2\big|^2_{L^2(\mathbb{T}^2)} \\
&= 2\int_{\mathbb{T}^2} V_1 \cdot \mathcal{L}(t)V_2 \, dxdy + 2\int_{\mathbb{T}^2} \mathcal{L}(-t)V_1 \cdot V_2 \, dxdy.
\end{aligned}
$$

Consider any $\psi \in (\mathcal{D}'(\mathbb{T}^2))^3$. By making use of the above relation, we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} \mathcal{A}_\varepsilon(D)(V_\varepsilon^o - V^o)\cdot\psi \, dxdy \\
&= -\int_{\mathbb{T}^2} \begin{pmatrix} 0 \\ c_1(\mu\Delta_h \mathcal{L}_2(\frac{t}{\varepsilon})(V_\varepsilon^o - V^o) + \lambda\nabla_h div_h \mathcal{L}_2(\frac{t}{\varepsilon})(V_\varepsilon^o - V^o)) \end{pmatrix} \cdot \mathcal{L}(\frac{t}{\varepsilon})\psi \, dxdy \\
&= c_1 \int_{\mathbb{T}^2} \Big(\mu \nabla_h \mathcal{L}_2(\frac{t}{\varepsilon})(V_\varepsilon^o - V^o)\cdot\nabla_h\mathcal{L}_2(\frac{t}{\varepsilon})\psi \\
&\quad + \lambda(div_h\mathcal{L}_2(\frac{t}{\varepsilon})(V_\varepsilon^o - V^o)) \times (div_h \mathcal{L}_2(\frac{t}{\varepsilon})\psi)\Big)\, dxdy \lesssim \big|\nabla_h\psi\big|_{H^1(\mathbb{T}^2)} \\
&\quad \times \big|V_\varepsilon^o - V^o\big|_{H^1(\mathbb{T}^2)} \to 0 \quad \text{in } L^\infty(0,T), \text{ as } \varepsilon \to 0^+,
\end{aligned}
$$

where we have applied the Hölder and Sobolev embedding inequalities, and (4.51). Consequently, $\mathcal{A}_\varepsilon(V_\varepsilon^o - V^o) \to 0$ as $\varepsilon \to 0^+$ in the sense of distribution.

What is left is to show the convergence of the last term on the right-hand side of (4.68). That is, we will show in the following, as $\varepsilon \to 0^+$, that

$$
\mathcal{F} := \mathcal{L}(-\frac{t}{\varepsilon})\begin{pmatrix} 0 \\ K + L_1 + L_2 \end{pmatrix} \to 0, \quad \text{in the sense of distribution.} \tag{4.69}
$$

On the one hand, the estimates in (4.49) and the norm-preserving property (4.44) yield that,

$$\mathcal{L}(-\frac{t}{\varepsilon})\begin{pmatrix} 0 \\ L_1 + L_2 \end{pmatrix} \to 0, \quad \text{in } L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2)),$$

as $\varepsilon \to 0^+$. On the other hand,

$$\begin{aligned} K &= \mathcal{P}_\tau\big(-v_p \cdot \nabla_h v_p - (div_h \widetilde{v}_p) v_p\big) \\ &\quad + \mathcal{P}_\tau\big(-\mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h(\mathcal{P}_\sigma v_\varepsilon - v_p) - (div_h \widetilde{\mathcal{P}_\sigma v_\varepsilon})(\mathcal{P}_\sigma v_\varepsilon - v_p) \\ &\qquad - (\mathcal{P}_\sigma v_\varepsilon - v_p) \cdot \nabla_h v_p - (div_h(\widetilde{\mathcal{P}_\sigma v_\varepsilon} - \widetilde{v}_p)) v_p\big) =: K' + K'', \end{aligned}$$

where, as the consequence of (4.12) and the Hölder and Sobolev embedding inequalities, we have, as $\varepsilon \to 0^+$,

$$K'' \to 0 \ \text{ in } L^\infty(0,T;L^2(\Omega_h)) \cap L^2(0,T;H^1(\Omega_h)).$$

Therefore, one only has to investigate,

$$\mathcal{L}(-\frac{t}{\varepsilon})\begin{pmatrix} 0 \\ K' \end{pmatrix} = \sum_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}} \big(\hat{K'}^+_{\mathbf{k}} V^+_{\mathbf{k}} e^{-i\varsigma\, \mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}} + \hat{K'}^-_{\mathbf{k}} V^-_{\mathbf{k}} e^{i\varsigma\, \mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}\big), \tag{4.70}$$

where we have substituted the representation (4.58), and it is represented, using the relation (4.59) and (4.60),

$$\begin{pmatrix} 0 \\ K' \end{pmatrix} = \sum_{\mathbf{k} \in 2\pi\mathbb{T}^2 \setminus \{(0,0)\}} \big(\hat{K'}^+_{\mathbf{k}} V^+_{\mathbf{k}} + \hat{K'}^-_{\mathbf{k}} V^-_{\mathbf{k}}\big).$$

Together with the norm preserving property (4.44) and the fact that

$$\big|K'\big|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \leq \big\| v_p \big\|^2_{L^\infty(0,T;H^2(\mathbb{T}^2 \times 2\mathbb{T}))} < \infty,$$

(4.70) yields, as $\varepsilon \to 0^+$,

$$\mathcal{L}(-\frac{t}{\varepsilon})\begin{pmatrix} 0 \\ K' \end{pmatrix} \overset{*}{\rightharpoonup} 0 \ \text{ weak-}* \text{ in } L^\infty(0,T;H^1(\mathbb{T}^2)).$$

This finishes the proof of (4.69). Therefore, we have identified the limit equation of (4.40):

$$\partial_t V^o + \mathcal{Q}_1(v_p, V^o) + \mathcal{Q}_2(V^o, V^o) + \mathcal{Q}_3(g^o, V^o) - \mathcal{A}(D)V^o = 0. \tag{4.71}$$

To identity the limit equation of (4.43), notice that equation (4.43) can be written as,

$$\begin{aligned}\frac{d}{dt}g_\varepsilon &= -\int_{\mathbb{T}^2}\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V^o\cdot\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V^o\,dxdy\\ &\quad+\int_{\mathbb{T}^2}\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)(V^o-V^o_\varepsilon)\cdot\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)V^o\,dxdy\\ &\quad+\int_{\mathbb{T}^2}\mathcal{L}_2\big(\frac{t}{\varepsilon}\big)V^o_\varepsilon\cdot\nabla_h\mathcal{L}_1\big(\frac{t}{\varepsilon}\big)(V^o-V^o_\varepsilon)\,dxdy := M_1+M_2+M_3.\end{aligned}$$

Due to the norm preserving property of $\mathcal{L}$, we have $M_2+M_3\to 0$ as $\varepsilon\to 0^+$ from (4.45), (4.50) and (4.51). To investigate M_1, since V^o is real-valued, we denote, with $\hat{V}^\pm_{-\mathbf{k}}=\overline{\hat{V}^\pm_{\mathbf{k}}}^c$

$$V^o=\sum_{\mathbf{k}\in 2\pi\mathbb{T}^2\backslash\{(0,0)\}}\big(\hat{V}^+_{\mathbf{k}}V^+_{\mathbf{k}}+\hat{V}^-_{\mathbf{k}}V^-_{\mathbf{k}}\big). \tag{4.72}$$

Then after applying (4.58), direct calculations show that

$$\begin{aligned}M_1 &= \sum_{\mathbf{k}\in 2\pi\mathbb{T}^2\backslash\{(0,0)\}}\Bigg(-i\hat{V}^+_{\mathbf{k}}\hat{V}^+_{-\mathbf{k}}\frac{c\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|}{\gamma-1+c^2}\\ &\quad+i\hat{V}^-_{\mathbf{k}}\hat{V}^-_{-\mathbf{k}}\frac{c\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|}{\gamma-1+c^2}\\ &\quad-2i\hat{V}^+_{\mathbf{k}}\hat{V}^-_{-\mathbf{k}}\frac{c\sqrt{\gamma-1}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|}{\gamma-1+c^2}e^{2i\varsigma\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\frac{t}{\varepsilon}}\Bigg)\\ &\overset{*}{\rightharpoonup}\frac{c\sqrt{\gamma-1}}{\gamma-1+c^2}\sum_{\mathbf{k}\in 2\pi\mathbb{T}^2\backslash\{(0,0)\}}\Big(-i\hat{V}^+_{\mathbf{k}}\hat{V}^+_{-\mathbf{k}}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|+i\hat{V}^-_{\mathbf{k}}\hat{V}^-_{-\mathbf{k}}\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\Big)\\ &= i\frac{c\sqrt{\gamma-1}}{\gamma-1+c^2}\sum_{\mathbf{k}\in 2\pi\mathbb{T}^2\backslash\{(0,0)\}}\Big(-|\hat{V}^+_{\mathbf{k}}|^2+|\hat{V}^-_{\mathbf{k}}|^2\Big)\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|\end{aligned}$$

in $L^\infty(0,T)$.

Moreover, direct integrating (4.43) in the temporal variable yields

$$g_\varepsilon(t)-\int_{\mathbb{T}^2}\xi_0\,dxdy=\int_0^t M_1\,dt+\int_0^t(M_2+M_3)\,dt.$$

After taking $\varepsilon\to 0$, we have for any $t\in[0,T]$,

$$\begin{aligned}&g^o(t)-\int_{\mathbb{T}^2}\xi_0\,dxdy\\ &= i\frac{c\sqrt{\gamma-1}}{\gamma-1+c^2}\sum_{\mathbf{k}\in 2\pi\mathbb{T}^2\backslash\{(0,0)\}}\Big(-|\hat{V}^+_{\mathbf{k}}|^2+|\hat{V}^-_{\mathbf{k}}|^2\Big)\,\mathrm{sg}(\mathbf{k})|\mathbf{k}|,\end{aligned}$$

and since g^o is real-valued, this implies

$$g^o \equiv \int_{\mathbb{T}^2} \xi_0 \, dxdy. \tag{4.73}$$

We summarize the discussion in this subsection in the following:

Proposition 6. *Under the same assumptions as in Proposition 4, in the case when* $\Omega_h = \mathbb{T}^2$*, there are a function* V^o *satisfying the regularity in* (4.50) *and a constant* g^o *given by* (4.73)*, such that* (V^o, g^o) *satisfies equation* (4.71)*. The convergence in* (4.54) *holds as* $\varepsilon \to 0^+$*.*

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