

Maximum and coupling of the sine-Gordon field

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Abstract

For $0 < \beta < 6\pi$, we prove that the distribution of the centred maximum of the ε -regularised continuum sine-Gordon field on the two-dimensional torus converges to a randomly shifted Gumbel distribution as $\varepsilon \rightarrow 0$. Our proof relies on a strong coupling at all scales of the sine-Gordon field with the Gaussian free field, of independent interest, and extensions of existing methods for the maximum of the lattice Gaussian free field.

1 Introduction and main results

1.1. Main results. We consider the (continuum) sine-Gordon field on the two-dimensional torus $\Omega = \mathbb{T}^2$ with mass $m = 1$ and coupling constants $z \in \mathbb{R}$ and $0 < \beta < 6\pi$. Regularised using a lattice of mesh ε such that $1/\varepsilon$ is an integer, its distribution is the probability measure on $\varphi : \Omega_\varepsilon \rightarrow \mathbb{R}$ given by

$$(1.1) \quad \nu^{\text{SG}_\varepsilon}(d\varphi) = \frac{1}{Z_\varepsilon} \exp \left[-\varepsilon^2 \sum_{\Omega_\varepsilon} \left(\frac{1}{2} \varphi(-\Delta^\varepsilon \varphi) + \frac{1}{2} \varphi^2 + 2z\varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta}\varphi) \right) \right] \prod_{x \in \Omega_\varepsilon} d\varphi(x)$$

where $\Omega_\varepsilon = \mathbb{T}^2 \cap \varepsilon\mathbb{Z}^2$ is the discretised unit torus and Δ^ε is the discretised Laplacian, i.e., $\Delta^\varepsilon f(x) = \varepsilon^{-2} \sum_{y \sim x} (f(y) - f(x))$ with $x \sim y$ denoting that x and y are neighbours in $\varepsilon\mathbb{Z}^2$.

Under suitable assumptions on z and β , it is known that the probability measure $\nu^{\text{SG}_\varepsilon}$ converges weakly as $\varepsilon \downarrow 0$ to a non-Gaussian probability measure ν^{SG} on $H^{-\kappa}(\mathbb{T}^2)$. For $\beta < 6\pi$ and $z \in \mathbb{R}$, this convergence along with a strong coupling to the Gaussian free field (GFF) is also a by-product of our results, see Theorem 1.2. The limiting sine-Gordon field is an example of a superrenormalisable (or subcritical) Euclidean field theory. This means that its short-distance behaviour is that of the GFF (in a suitable sense). In particular, the limiting field Φ^{SG} is not a random function, but only a generalised function of negative regularity, with short-distance correlations diverging logarithmically. Such local regularity properties are by now relatively well understood in a number of examples of superrenormalisable field theories.

In this article, we study more global probabilistic properties of the sine-Gordon field, which go beyond weak convergence of the field. In particular, in the following theorem, we determine the distribution of the centred maximum of the regularised field as $\varepsilon \downarrow 0$. Such extremal behaviour of log-correlated random fields is a topic on which a lot of progress has been made in the last few years. For Gaussian fields, a very complete understanding now exists, but much less is known for non-Gaussian fields and understanding the distributional properties of their extrema (as in our result below) is a mostly open problem. For further discussion and references, see Section 1.2.

Theorem 1.1. *Let $0 < \beta < 6\pi$ and $z \in \mathbb{R}$. Then the centred maximum of the ε -regularised sine-Gordon field $\Phi^{\text{SG}_\varepsilon} \sim \nu^{\text{SG}_\varepsilon}$ converges in law to a randomly shifted Gumbel distribution:*

$$(1.2) \quad \max_{\Omega_\varepsilon} \Phi^{\text{SG}_\varepsilon} - \frac{1}{\sqrt{2\pi}} \left(2 \log \frac{1}{\varepsilon} - \frac{3}{4} \log \log \frac{1}{\varepsilon} + b \right) \rightarrow \frac{1}{\sqrt{8\pi}} X + \frac{1}{\sqrt{8\pi}} \log Z^{\text{SG}},$$

where Z^{SG} is a nontrivial positive random variable (depending on β and z), X is an independent standard Gumbel random variable, and b is a deterministic constant.

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The result for the GFF, namely $z = 0$, was proved in [16] (see also [11, 27, 41]). The constant in (1.2) is the same as in the analogous statement for the GFF, but the distribution of the random variable Z^{SG} is different from the version for the GFF; see Remark 1.3 below. Note that there is also a trivial difference in the factors involving 2π that results from different normalisations of the fields. Our normalisation is such that $\text{Var}(\Phi^{\text{SG}_\varepsilon}(0)) \sim \frac{1}{2\pi} \log \frac{1}{\varepsilon}$.

Our proof of Theorem 1.1 relies on a strong coupling between the sine-Gordon field and the GFF, based on the methods of [7, 17], combined with the methods developed for the study of the maximum of the GFF from [16]. This coupling, which is of independent interest, provides uniform control over the difference between the two fields on any scale. Concretely, we construct the sine-Gordon field as the final solution Φ_0^{SG} to the $H^{-\kappa}(\mathbb{T}^2)$ -valued (backward) SDE

$$(1.3) \quad \Phi_t^{\text{SG}} = - \int_t^\infty \dot{c}_s \nabla v_s(\Phi_s^{\text{SG}}) ds + \Phi_t^{\text{GFF}}, \quad (t \in [0, \infty]),$$

where $\dot{c}_s \nabla v_s$ is the gradient in the field direction of the *renormalised potential* defined in Section 2, and where Φ_t^{GFF} is the decomposed GFF defined by

$$(1.4) \quad \Phi_t^{\text{GFF}} = \int_t^\infty e^{\frac{1}{2}\Delta s - \frac{1}{2}m^2 s} dW_s, \quad (t \in [0, \infty]),$$

where W is a cylindrical Brownian motion in $L^2(\mathbb{T}^2)$, Δ is the Laplace operator on \mathbb{T}^2 and $m = 1$. This scale-by-scale coupling, constructed in Section 3, implies in particular the following global coupling for the limiting continuum sine-Gordon field with the continuum Gaussian free field.

Theorem 1.2. *Let $0 < \beta < 6\pi$ and $z \in \mathbb{R}$. Then the ε -regularised sine-Gordon field $\Phi^{\text{SG}_\varepsilon} \sim \nu^{\text{SG}_\varepsilon}$ converges weakly to a (non-Gaussian) limiting field Φ^{SG} in $H^{-\kappa}(\mathbb{T}^2)$. This continuum sine-Gordon field Φ^{SG} on \mathbb{T}^2 can be coupled to a continuum Gaussian free field Φ^{GFF} on \mathbb{T}^2 such that $\Phi^\Delta := \Phi^{\text{SG}} - \Phi^{\text{GFF}}$ satisfies the following Hölder continuity estimates:*

$$(1.5) \quad \begin{aligned} \max_{x \in \mathbb{T}^2} |\Phi^\Delta(x)| + \max_{x \in \mathbb{T}^2} |\partial \Phi^\Delta(x)| + \max_{x, y \in \mathbb{T}^2} \frac{|\partial \Phi^\Delta(x) - \partial \Phi^\Delta(y)|}{|x - y|^{1-\beta/4\pi}} &\leq O_\beta(|z|), & (0 < \beta < 4\pi), \\ \max_{x \in \mathbb{T}^2} |\Phi^\Delta(x)| + \max_{x, y \in \mathbb{T}^2} \frac{|\Phi^\Delta(x) - \Phi^\Delta(y)|}{|x - y|(1 + |\log |x - y||)} &\leq O_\beta(|z|), & (\beta = 4\pi), \\ \max_{x \in \mathbb{T}^2} |\Phi^\Delta(x)| + \max_{x, y \in \mathbb{T}^2} \frac{|\Phi^\Delta(x) - \Phi^\Delta(y)|}{|x - y|^{2-\beta/4\pi}} &\leq O_\beta(|z|), & (4\pi < \beta < 6\pi), \end{aligned}$$

with deterministic constants $O_\beta(|z|)$.

Theorem 1.2 provides an analogue for the (non-Gaussian) sine-Gordon field of results of [36] for Gaussian log-correlated fields. In particular, Theorem 1.2 implies (with essentially the same proofs) analogues of the corollaries of [36, Theorem A] such as the convergence of the derivative martingale measure (which is not a martingale here) or the critical (non-Gaussian) multiplicative chaos measure associated with the sine-Gordon field.

The tightness of the left-hand side of (1.2) along with Gumbel type tail bounds on its distribution are also immediate from Theorem 1.2 and the corresponding results for the GFF from [16, 26, 27]. However, to obtain the actual convergence of the centred maximum to a randomly shifted Gumbel distribution, we use more than what is given by Theorem 1.2. Our full multiscale coupling, stated in Theorems 3.1–3.4 in Section 3, provides finer control and in particular approximate independence of small scales from the large scales, which we make use of to approximate the field by a GFF on small scales.

The distribution of the sine-Gordon field is not expected to be absolutely continuous with respect to the GFF when $\beta \geq 4\pi$ (while it is absolutely continuous with respect to the GFF for $\beta < 4\pi$, assuming the volume is finite as in our setting). For $\beta < 6\pi$, Theorem 1.2 shows that the sine-Gordon field can nonetheless be coupled to a GFF up to a random Hölder continuous

difference. It remains a very interesting problem to find a best possible coupling in the full regime $\beta < 8\pi$ for which the sine-Gordon model exists [25].

Our coupling bears some resemblance with the recent constructions of pathwise solutions to singular SPDEs (see, e.g., [23, 31, 32]). Note however that our coupling provides uniform global bounds, and is based on the renormalisation group dynamics (1.3) rather than Glauber dynamics. In principle, it is possible to derive couplings based on Glauber dynamics as well. These do not provide an equally useful notion of scale, but in particular for the Φ_2^4 model, it would be interesting to investigate if these (see, for example, [44, 49]) can be used for the study of the distribution of the maximum nonetheless. Other approaches introduced for continuum Φ^4 models that have the potential to be useful in our context (though not immediately) are those of [6] or [19].

Remark 1.3. We expect that arguments along the lines of [11, 14] would allow to characterise the random variable Z^{SG} in Theorem 1.1 as the total mass of the derivative martingale measure $Z^{\text{SG}}(dx)$ associated to Φ^{SG} . Using the above coupling, as briefly discussed below the statement of the theorem, this random measure can be represented as

$$(1.6) \quad Z^{\text{SG}}(dx) = e^{\sqrt{8\pi}\Phi^\Delta(x)} Z^{\text{GFF}}(dx)$$

where $Z^{\text{GFF}}(dx)$ is the suitably normalised derivative martingale measure associated to the Gaussian free field Φ^{GFF} on \mathbb{T}^2 . Alternatively, we expect that one can show that Z^{SG} is the distributional limit of the random variables

$$(1.7) \quad Z^{\text{SG}_\varepsilon} = \varepsilon^2 \sum_{\Omega_\varepsilon} \left(\frac{2}{\sqrt{2\pi}} \log \frac{1}{\varepsilon} - \Phi^{\text{SG}_\varepsilon} \right) e^{-2 \log \frac{1}{\varepsilon} + \sqrt{8\pi}\Phi^{\text{SG}_\varepsilon}}.$$

More generally, we expect that the full extremal process of the sine-Gordon field is characterised in terms of $Z^{\text{SG}}(dx)$, as in [11, 13] for the GFF.

Remark 1.4. In Theorem 1.1, we consider the maximum of the lattice approximation of the sine-Gordon field as $\varepsilon \downarrow 0$. As in [1], it would be possible to prove an analogous statement for $\Phi^{\text{SG}} * \eta_\varepsilon$ where Φ^{SG} is the limiting sine-Gordon field and η is a mollifier.

1.2. Literature. The extremal behaviour of the GFF and more general Gaussian log-correlated fields is now rather well understood. For the maximum of the GFF, we refer in particular to [16] as well as [41] for closely related results, and also to [27] for a general class of Gaussian log-correlated fields that includes the GFF. There are also earlier results on branching Brownian motion and branching random walks, see [3, 15] and references therein and also [18]. The complete extremal process of the GFF was understood in [12–14]; see in particular the excellent review [11]. Further works on related Gaussian fields include [28, 48].

For several non-Gaussian log-correlated fields, progress on the order of the maximum has been made as well. The following ones are most relevant to our work; for none of these, the limiting distribution of the centred maximum has been identified. For gradient interface models with uniformly convex interaction, the leading logarithmic order of the maximum has been identified [8] and subsequential tightness of the centred maximum has been established as well [51], but the understanding of the subleading orders remains open. For the (integer-valued) Discrete Gaussian model, it has been shown that the maximum is of logarithmic order [50]. There are also results for the maximum of the logarithm of the characteristic polynomial of certain random matrices, see for example [4, 21, 22, 46]. Like these random matrix ensembles, the sine-Gordon model is closely related to a Coulomb gas type particle ensemble, but our point of view is completely different.

The continuum sine-Gordon field is a prototypical example of a Euclidean field theory. It is an interesting problem to show that an analogous result for the maximum holds also for other such examples like the continuum Φ_2^4 model.

There are also lattice versions of these models. We emphasise that these are not the same as the lattice regularisations of the continuum models. In physics terminology, the continuum

version is related to the ultraviolet problem, while the lattice model is related to the infrared problem. These problems differ by the scaling of the coupling constants with the lattice spacing, and the expected distribution of the maximum is also different.

We also mention that recent results on the (continuum) sine-Gordon model include [7], [35,37], and older ones [10,45], [17], [9], and [24,25]. The dynamics of sine-Gordon model has been studied in [7,20,33]. The sine-Gordon model is also closely related to the two-component plasma [29,40].

Finally, we comment on some recent works that use methods that bear some relation to the ones we develop. Continuous scale decomposition of the GFF and methods of stochastic analysis also have been used in the context of the Gaussian multiplicative chaos (see [38,47] and references), in recent constructions of the Φ_d^4 model, [5,6,34], and of the boundary sine-Gordon model [37]. Despite the shared use of the continuous decomposition of the GFF, these constructions have a flavour quite different from ours (they use the decomposition to construct the moment generating function of the field rather than a pathwise coupling of the field itself). For Gaussian log-correlated fields, couplings more in the spirit of ours have been constructed in [36]; see also the discussion below Theorem 1.2. The above mentioned results for gradient interface models with uniformly convex potential also make use of a coupling [43] (of a somewhat different flavour though).

1.3. Outline. In Section 2 we give detailed estimates on the renormalised potential v_t of the sine-Gordon field, extending the estimates from [7,17]. Using this renormalised potential, in Section 3, we then construct the sine-Gordon field as the solution to the SDE (1.3). In Section 4, we use this coupling together with the methods developed to determine the distribution of the maximum of the GFF from [16] to prove Theorem 1.1.

1.4. Notation. We use the standard Landau big- O and little- o notation, and emphasise by writing, for example, O_p that the implied constant depends on a parameter p . We also write $A \lesssim B$ to denote that $A \leq O_\beta(B)$, and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

2 Convergence of the renormalised potential

2.1. Renormalised potential. Let $\Omega = \mathbb{T}^2$ be the continuum unit torus, and let $\Omega_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^2$ be its lattice approximation where we assume from now on that $1/\varepsilon$ is an integer. Let $X_\varepsilon = \{\varphi : \Omega_\varepsilon \rightarrow \mathbb{R}\}$. For emphasis, we will sometimes denote the side length of the torus by $L = 1$. Given $z \in \mathbb{R}$, we define the microscopic potential v_0^ε of the sine-Gordon field for $\varphi = (\varphi(x))_{x \in \Omega_\varepsilon} \in X_\varepsilon$ by

$$(2.1) \quad v_0^\varepsilon(\varphi) = \varepsilon^2 \sum_{x \in \Omega_\varepsilon} 2z\varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta}\varphi(x)).$$

For $t > 0$, we define the renormalised potential $v_t^\varepsilon : X_\varepsilon \rightarrow \mathbb{R}$ as follows. Let

$$(2.2) \quad \dot{c}_t^\varepsilon = e^{t\Delta^\varepsilon} e^{-m^2 t}, \quad c_t^\varepsilon = \int_0^t \dot{c}_s^\varepsilon ds,$$

where Δ^ε is the discretised Laplacian acting on X_ε as defined below (1.1). Then for any $t > s > 0$,

$$(2.3) \quad e^{-v_t^\varepsilon(\varphi)} = \mathbf{E}_{c_t^\varepsilon} \left(e^{-v_0^\varepsilon(\varphi+\zeta)} \right) = \mathbf{E}_{c_t^\varepsilon - c_s^\varepsilon} \left(e^{-v_s^\varepsilon(\varphi+\zeta)} \right),$$

where \mathbf{E}_c denotes the expectation of the Gaussian measure with covariance c . The second equality holds since the convolution of two Gaussian measures with covariances c_1 and c_2 is Gaussian with covariance $c_1 + c_2$. Equivalently to (2.3), v_t^ε is the unique solution to the Polchinski equation:

$$(2.4) \quad \partial_t v_t^\varepsilon = \frac{1}{2} \Delta_{\dot{c}_t^\varepsilon} v_t^\varepsilon - \frac{1}{2} (\nabla v_t^\varepsilon)_{\dot{c}_t^\varepsilon}^2 = \frac{1}{2} \varepsilon^4 \sum_{x,y \in \Omega_\varepsilon} \dot{c}_t^\varepsilon(x,y) \left[\frac{\partial^2 v_t^\varepsilon}{\partial \varphi(x) \partial \varphi(y)} - \frac{\partial v_t^\varepsilon}{\partial \varphi(x)} \frac{\partial v_t^\varepsilon}{\partial \varphi(y)} \right]$$

where the matrix $\check{c}_t^\varepsilon(x, y)$ associated to \check{c}_t^ε is normalised such that

$$(2.5) \quad \check{c}_t^\varepsilon f(x) = \varepsilon^2 \sum_{y \in \Omega_\varepsilon} \check{c}_t^\varepsilon(x, y) f(y).$$

Later, we will also use the notation

$$(2.6) \quad \langle f, g \rangle \equiv \langle f, g \rangle_{\Omega_\varepsilon} \equiv \varepsilon^2 \sum_{x \in \Omega_\varepsilon} f(x) \bar{g}(x), \quad \int_{\Omega_\varepsilon} dx \equiv \varepsilon^2 \sum_{\Omega_\varepsilon}.$$

Both representations of v_t^ε , that as a solution to the Polchinski equation (2.4), and that in terms of Gaussian convolution (2.3), are useful. For their equivalence, see [7, 17].

2.2. Statement of estimates. To state the estimates for the renormalised potential, we define

$$(2.7) \quad L_t = \sqrt{t} \wedge 1/m, \quad Z_t = z L_t^{-\beta/4\pi}, \quad \mathbf{Z}_t = L_t^2 Z_t = z L_t^{2-\beta/4\pi}, \quad \theta_t = e^{-\frac{1}{2}m^2 t},$$

and also set $Z_t^\varepsilon = Z_{t\sqrt{\varepsilon^2}}$ and $\mathbf{Z}_t^\varepsilon = \mathbf{Z}_{t\sqrt{\varepsilon^2}}$. We also recall that $L = 1$ is the side length of the torus. Moreover, while the results in Section 1 are stated for the mass parameter $m = 1$, we will allow $m > 0$ throughout this section. By rescaling one could then also recover the general L case.

The following estimates can be extracted from [7, Section 3] (which follows the method of [17]). Compared to [7, Section 3], we have rescaled $t \rightarrow t/\varepsilon^2$ and $x \rightarrow x/\varepsilon$ in preparation for the limit $\varepsilon \downarrow 0$ which we will take later. This rescaling is summarised in Section 2.7 below.

Theorem 2.1. *Let $\beta < 6\pi$, $m^2 > 0$, $z \in \mathbb{R}$, and $\varepsilon > 0$. There is $t_0 = t_0(\beta, z, m) > 0$ independent of $\varepsilon > 0$ such that for all $\varphi \in X_\varepsilon$, $t \geq 0$,*

$$(2.8) \quad L_t^2 \|\check{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi)\|_{L^\infty(\Omega_\varepsilon)} \leq O_\beta(\theta_t |\mathbf{Z}_t^\varepsilon|) + \mathbf{1}_{t > t_0} O_{\beta, m}(\theta_t |\mathbf{Z}_t^\varepsilon|),$$

$$(2.9) \quad L_t^2 \|\check{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi) - \check{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi')\|_{L^\infty(\Omega_\varepsilon)} \leq (O_\beta(\theta_t |\mathbf{Z}_t^\varepsilon|) + \mathbf{1}_{t > t_0} O_{\beta, m, z}(\theta_t)) \|\varphi - \varphi'\|_{L^\infty(\Omega_\varepsilon)}.$$

Since we will not directly apply Theorem 2.1 and extend the estimates in Theorem 2.2 below, we do not give a precise reference at this point.

Note that, from the macroscopic point of view, the microscopic potential v_0^ε blows up as $\varepsilon \downarrow 0$. On the other hand, when normalised with respect to the relevant scale with a factor L_t^2 as in the above theorem, the gradients of the renormalised potential v_t^ε are bounded uniformly in ε and in fact tend to 0 since $\mathbf{Z}_t \rightarrow 0$ as $t \rightarrow 0$.

The main result of this section is Theorem 2.2 below. It extends the above estimates and also shows the convergence of v_t^ε as $\varepsilon \downarrow 0$. To this end, we first note that the lattice covariance decomposition \check{c}_t^ε from (2.2) and (2.5) converges to its continuous counterpart, given by

$$(2.10) \quad \check{c}_t^0(x, y) = e^{t\Delta}(x, y) e^{-m^2 t} = \sum_{n \in \mathbb{Z}^2} \frac{e^{-|x-y+Ln|^2/4t - m^2 t}}{4\pi t}, \quad L = 1,$$

where Δ is the Laplacian on \mathbb{T}^2 . Moreover, we note that v_t^ε is a function from X_ε to \mathbb{R} , so for each $\varphi \in X_\varepsilon$, $\check{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi)$ is an element of $X_\varepsilon = \{\Omega_\varepsilon \rightarrow \mathbb{R}\}$ and we denote this function by

$$(2.11) \quad \check{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi, x) = [\check{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi)](x), \quad x \in \Omega_\varepsilon, \varphi \in X_\varepsilon.$$

Here ∇ denotes the gradient with respect to the field $\varphi \in X_\varepsilon = \mathbb{R}^{\Omega_\varepsilon}$. We will also need (discrete) gradients in the variable x and denote these by $\partial = \partial_\varepsilon$. More precisely, for e one of the $2d$ unit directions in \mathbb{Z}^d , we set $\partial_\varepsilon^e f(x) = \varepsilon^{-1}(f(x + \varepsilon e) - f(x))$, and for $k \in \mathbb{N}$ we then denote by $\partial_\varepsilon^k f$ the matrix-valued function consisting of all iterated discrete gradients $\partial_\varepsilon^{e_1} \cdots \partial_\varepsilon^{e_k} f$ where e_1, \dots, e_k are unit directions in \mathbb{Z}^d . For $\varepsilon = 0$, we similarly denote true spatial derivatives by ∂ .

Finally, for $x \in \Omega$, we will denote by x^ε the point in Ω_ε that is closest to x , i.e., the unique element of Ω_ε such that $x \in x^\varepsilon + (-\varepsilon/2, \varepsilon/2]^2$.

Theorem 2.2. *Let $\beta < 6\pi$, $m^2 > 0$, $z \in \mathbb{R}$. For all $t > 0$, there exist bounded functions $\dot{c}_t^0 \nabla v_t^0 : C(\Omega) \rightarrow C^\infty(\Omega)$ such that $\dot{c}_t^\varepsilon \nabla v_t^\varepsilon$ converges to $\dot{c}_t^0 \nabla v_t^0$ in the sense that, for any $\varphi \in C(\Omega)$ and any $\varphi^\varepsilon \in X_\varepsilon$ such that $\sup_{x \in \Omega} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \rightarrow 0$,*

$$(2.12) \quad L_t^2 \|\dot{c}_t^0 \nabla v_t^0(\varphi) - \dot{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi^\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0 \quad (\varepsilon \rightarrow 0).$$

The following estimates hold (for $\varepsilon > 0$ and in the limit $\varepsilon = 0$), uniformly in φ, φ' , for any $k \in \mathbb{N}$:

$$(2.13) \quad L_t^{2+k} \|\partial^k \dot{c}_t \nabla v_t(\varphi)\|_{L^\infty(\Omega)} \leq O_{\beta,k}(\theta_t |Z_t|) + \mathbf{1}_{t>t_0} O_{\beta,m,k}(\theta_t |Z_t|),$$

$$(2.14) \quad L_t^{2+k} \|\partial^k \dot{c}_t \nabla v_t(\varphi) - \partial^k \dot{c}_t \nabla v_t(\varphi')\|_{L^\infty(\Omega)} \leq (O_{\beta,k}(\theta_t |Z_t|) + \mathbf{1}_{t>t_0} O_{\beta,m,z,k}(\theta_t)) \|\varphi - \varphi'\|_{L^\infty(\Omega)},$$

where $t_0 = t_0(\beta, z, m) > 0$ is a positive constant independent of ε .

Note that we consider $\dot{c}_t \nabla v_t$ as one object rather than as a composition of \dot{c}_t with ∇v_t (which we do not define in the continuum limit). Also, as in this example for $\dot{c}_t \nabla v_t$, we will often omit the index ε , in which case we make the convention that it refers to both cases, $\varepsilon > 0$ and the limiting case $\varepsilon = 0$, simultaneously.

The remainder of this section is devoted to the proof of Theorem 2.2. The theorem provides the main input for the coupling of the sine-Gordon field with the GFF in Section 3.

2.3. Yukawa gas representation of renormalised potential. To prove Theorem 2.2, we first recall the construction of the renormalised potential v_t^ε from [7, Section 3] and [17]. As discussed above, compared to [7, Section 3], we use the continuum rescaling $t \rightarrow t/\varepsilon^2$ that is more convenient in our context; the trivial relation between these scalings is summarised in Section 2.7 below.

For $\varepsilon > 0$ and $\xi_i = (x_i, \sigma_i) \in \Omega_\varepsilon \times \{\pm 1\}$, we define $\tilde{v}_t^\varepsilon(\xi_1, \dots, \xi_n) = \tilde{v}_t^{n,\varepsilon}(\xi_1, \dots, \xi_n)$ by

$$(2.15) \quad \tilde{v}_t^\varepsilon(\xi_1) = e^{-\frac{1}{2}\beta c_t^\varepsilon(0)} \varepsilon^{-\beta/4\pi} z = e^{-\frac{1}{2}\beta(c_t^\varepsilon(0) + (\log \varepsilon^2)/4\pi)} z$$

$$(2.16) \quad \tilde{v}_t^\varepsilon(\xi_1, \dots, \xi_n) = \frac{1}{2} \int_0^t ds \sum_{I_1 \dot{\cup} I_2 = [n]} \sum_{i \in I_1, j \in I_2} \dot{u}_s^\varepsilon(\xi_i, \xi_j) \tilde{v}_s^\varepsilon(\xi_{I_1}) \tilde{v}_s^\varepsilon(\xi_{I_2}) e^{-(w_t^\varepsilon - w_s^\varepsilon)(\xi_1, \dots, \xi_n)},$$

where the second equation is for $n \geq 2$ (see [7, (3.32)]), $\xi_I = (\xi_i)_{i \in I}$ is identified with an element of $(\Omega_\varepsilon \times \{\pm 1\})^{|I|}$ (the order does not matter), $\dot{\cup}$ denotes the disjoint union, $[n] = \{1, \dots, n\}$, and

$$(2.17) \quad \dot{u}_t^\varepsilon(\xi_1, \xi_2) = \beta \sigma_1 \sigma_2 \dot{c}_t^\varepsilon(x_1, x_2), \quad (w_t^\varepsilon - w_s^\varepsilon)(\xi_1, \dots, \xi_n) = \frac{\beta}{2} \sum_{i,j=1}^n \int_s^t \sigma_i \sigma_j \dot{c}_r^\varepsilon(x_i, x_j) dr.$$

In particular, $\tilde{v}_t^{n,\varepsilon}$ is determined inductively by $\tilde{v}_s^{k,\varepsilon}$ with $k < n$ and $s < t$, and the above equations are well defined for all n and t whenever $\varepsilon > 0$.

As we will see further below, for $\varepsilon > 0$, $\beta < 6\pi$ and $t < t_0 = t_0(z, \beta, m)$ independent of $\varepsilon > 0$, the following Fourier series in fact converges absolutely uniformly in $\varphi \in X_\varepsilon$:

$$(2.18) \quad v_t^\varepsilon(\varphi) = \sum_{n=0}^{\infty} v_t^{n,\varepsilon}(\varphi),$$

where

$$(2.19) \quad v_t^{n,\varepsilon}(\varphi) = \frac{1}{n!} \int_{(\Omega_\varepsilon \times \{\pm 1\})^n} d\xi_1 \dots d\xi_n \tilde{v}_t^{n,\varepsilon}(\xi_1, \dots, \xi_n) e^{i\sqrt{\beta} \sum_{i=1}^n \sigma_i \varphi(x_i)}$$

and the discrete integral over $(\Omega_\varepsilon \times \{\pm 1\})^n$ is defined as a sum analogously to (2.6). Moreover, (2.18)–(2.19) then gives the unique solution to the Polchinski equation (2.4); see [7, Section 3.3].

By differentiation in φ (denoted ∇) and then discrete differentiation in x (denoted ∂), we further obtain that, for any $k \in \mathbb{N}$,

$$(2.20) \quad \begin{aligned} & \partial_\varepsilon^k \dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi, x) \\ &= \frac{i\sqrt{\beta}}{n!} \int_{(\Omega_\varepsilon \times \{\pm 1\})^n} d\xi_1 \dots d\xi_n \left(\sum_{i=1}^n \partial_\varepsilon^k \dot{c}_t^\varepsilon(x, x_i) \sigma_i \right) \tilde{v}_t^{n,\varepsilon}(\xi_1, \dots, \xi_n) e^{i\sqrt{\beta} \sum_{i=1}^n \sigma_i \varphi(x_i)}. \end{aligned}$$

Denoting by Hess the Hessian matrix in φ , for any $g : \Omega_\varepsilon \rightarrow \mathbb{R}$, also

$$(2.21) \quad \begin{aligned} & [\partial_\varepsilon^k \dot{c}_t^\varepsilon \text{Hess } v_t^{n,\varepsilon}(\varphi)g](x) \\ &= \frac{-\beta}{n!} \int_{(\Omega_\varepsilon \times \{\pm 1\})^n} d\xi_1 \dots d\xi_n \left(\sum_{i=1}^n \partial_\varepsilon^k \dot{c}_t^\varepsilon(x, x_i) \sigma_i \right) \left(\sum_{i=1}^n g(x_i) \sigma_i \right) \tilde{v}_t^{n,\varepsilon}(\xi_1, \dots, \xi_n) e^{i\sqrt{\beta} \sum_{i=1}^n \sigma_i \varphi(x_i)}, \end{aligned}$$

and analogous expressions for higher derivatives hold as well.

We will define continuum versions $\tilde{v}_t^0(\xi_1, \dots, \xi_n)$ by passing to the limit $\varepsilon \downarrow 0$ in (2.15)–(2.16). To define these limits, for $t \geq s > 0$, we first define u_t^0 and $w_t^0 - w_s^0$ as in (2.17), only replacing c_t^ε by c_t^0 defined in (2.10). We emphasise that (with slight abuse of notation) we only define the difference $w_t^0 - w_s^0$ and not w_t^0 and w_s^0 individually. We will show in Lemma 2.3 below that there exist constants $\gamma_t = \gamma_t(m)$ such that

$$(2.22) \quad e^{-\frac{1}{2}\beta(c_t^0(0) + (\log \varepsilon^2)/4\pi)} \rightarrow \gamma_t^\beta L_t^{-\beta/4\pi}.$$

For all distinct $\xi_1, \xi_2, \dots \in \Omega \times \{\pm 1\}$, we then define inductively

$$(2.23) \quad \tilde{v}_t^0(\xi_1) = L_t^{-\beta/4\pi} \gamma_t^\beta z$$

$$(2.24) \quad \tilde{v}_t^0(\xi_1, \dots, \xi_n) = \frac{1}{2} \int_0^t ds \sum_{I_1 \cup I_2 = [n]} \sum_{i \in I_1, j \in I_2} \dot{u}_s^0(\xi_i, \xi_j) \tilde{v}_s^0(\xi_{I_1}) \tilde{v}_s^0(\xi_{I_2}) e^{-(w_t^0 - w_s^0)(\xi_1, \dots, \xi_n)}.$$

To see that this is well-defined we note that, for any distinct ξ_i , one has $\dot{u}_s^0(\xi_i, \xi_j) = \sigma_i \sigma_j \dot{c}_s^0(x_i, x_j) = O_{x_i, x_j}(s^N)$ as $s \rightarrow 0$ for any N by the heat kernel estimate (2.10), and also $e^{-(w_t - w_s)(\xi_1, \dots, \xi_n)} \leq 1$ since \dot{c}_t is positive definite. From this, it follows easily by induction that the integrals on the right-hand side of (2.24) converge absolutely for all ξ_i distinct.

Finally, when

$$(2.25) \quad \frac{1}{n!} \int_{(\Omega \times \{\pm 1\})^n} d\xi_1 \dots d\xi_n |\tilde{v}_t^{n,0}(\xi_1, \dots, \xi_n)| < \infty$$

respectively

$$(2.26) \quad \frac{1}{n!} \int_{(\Omega \times \{\pm 1\})^n} d\xi_1 \dots d\xi_n \left| \sum_{i=1}^n \partial^k \dot{c}_t^0(x, x_i) \sigma_i \tilde{v}_t^{n,0}(\xi_1, \dots, \xi_n) \right| < \infty$$

we define $v_t^{n,0} : C(\Omega) \rightarrow \mathbb{R}$ respectively $\partial^k c_t^0 \nabla v_t^{n,0} : C(\Omega) \rightarrow C(\Omega)$ as in (2.18)–(2.20) with $\varepsilon = 0$. We will provide conditions for (2.25) and (2.26) below.

2.4. Convergence of covariances and Gaussian fields. The following lemma provides the convergence of the covariance terms and heat kernel terms in (2.15)–(2.16) to those in (2.23)–(2.24). For $x \in \Omega$, recall that we denote by x^ε the unique element of Ω_ε such that $x \in x^\varepsilon + (-\varepsilon/2, \varepsilon/2]^2$, and correspondingly, for $\xi = (x, \sigma) \in \Omega \times \{\pm 1\}$, we write $\xi^\varepsilon = (x^\varepsilon, \sigma)$. For $f : \Omega_\varepsilon^n \rightarrow \mathbb{R}$ we further denote by $E_\varepsilon f$ the piecewise constant extension of f to Ω^n , i.e., $E_\varepsilon f(x_1, \dots, x_n) = f(x_1^\varepsilon, \dots, x_n^\varepsilon)$, and we denote the extension of $f : (\Omega_\varepsilon \times \{\pm 1\})^n \rightarrow \mathbb{R}$ analogously.

Lemma 2.3. *There exists a function $\gamma_t = \gamma_t(m)$ of $(t, m) \in [0, \infty) \times (0, \infty)$ with $\gamma_t = \gamma_0 + O(m^2 t)$ as $t \rightarrow 0$ such that*

$$(2.27) \quad e^{-\frac{1}{2}c_t^\varepsilon(0,0)} \varepsilon^{-1/4\pi} \rightarrow \gamma_t L_t^{-1/4\pi}.$$

For all $x \neq y \in \Omega$ and $t > 0$, the integral $c_t^0(x, y) = \int_0^t \dot{c}_s^0(x, y) ds$ exists and uniformly on compact subsets of $x \neq y$,

$$(2.28) \quad c_t^\varepsilon(x^\varepsilon, y^\varepsilon) \rightarrow c_t^0(x, y).$$

For all $t \geq s \geq \varepsilon^2$, uniformly in $\xi = (\xi_1, \dots, \xi_n) \in (\Omega \times \{\pm 1\})^n$,

$$(2.29) \quad e^{-(w_t^\varepsilon(\xi^\varepsilon) - w_s^\varepsilon(\xi^\varepsilon))} \rightarrow e^{-(w_t^0(\xi) - w_s^0(\xi))}.$$

For all $t \geq \varepsilon^2$, and all $k \in \mathbb{N}$,

$$(2.30) \quad L_t^k \sup_x \|\partial_\varepsilon^k \dot{c}_t^\varepsilon(x, \cdot)\|_{L^1(\Omega_\varepsilon)} \leq O_k(\theta_t), \quad \sup_x \|E_\varepsilon \dot{c}_t^\varepsilon(x, \cdot) - \dot{c}_t^0(x, \cdot)\|_{L^1(\Omega)} \leq O\left(\frac{\varepsilon^2}{t}\right)\theta_t.$$

The proof of the lemma essentially follows from the convergence of the lattice heat kernel to its continuum counterpart. We have collected the required heat kernel statements in the appendix, in Lemmas A.1–A.2, to which we will refer in the proof of the above lemma below.

Proof of (2.27). To emphasise the effect of periodic boundary conditions and the mass term, recall that we write L for the side length of the torus. Then, denoting the torus heat kernel by $p_t^{\varepsilon, L}(x)$, $x \in \Omega_\varepsilon$, as in Appendix A, our goal is to estimate $c_t^{\varepsilon, L}(0) = \int_0^t p_s^{\varepsilon, L}(0) e^{-m^2 s} ds$. We will first estimate its infinite volume version, i.e., with $L = \infty$, given by

$$(2.31) \quad \begin{aligned} c_t^{\varepsilon, \infty}(0) &= \int_0^t p_s^\varepsilon(0) e^{-m^2 s} ds \\ &= \int_0^{t \wedge 1/m^2} p_s^\varepsilon(0) ds + \int_0^{t \wedge 1/m^2} p_s^\varepsilon(0) (e^{-m^2 s} - 1) ds + \int_{t \wedge 1/m^2}^t p_s^\varepsilon(0) e^{-m^2 s} ds, \end{aligned}$$

where $p_t^\varepsilon(x)$, $x \in \varepsilon\mathbb{Z}^2$ is the heat kernel on $\varepsilon\mathbb{Z}^2$, see Appendix A. The first term on the right-hand side is estimated as follows. We denote the unit lattice heat kernel by \tilde{p}_s so that $p_s^\varepsilon(x) = \varepsilon^{-2} \tilde{p}_{s/\varepsilon^2}(x/\varepsilon)$ for $x \in \varepsilon\mathbb{Z}^2$. By Lemma A.1, $\tilde{p}_s(0) = 1/(4\pi s) + O(1/s^2)$, and thus

$$(2.32) \quad \int_0^t \tilde{p}_s(0) ds = \int_0^1 \tilde{p}_s(0) ds + \int_1^t \frac{ds}{4\pi s} + \int_1^t \left(\tilde{p}_s(0) - \frac{1}{4\pi s} \right) ds$$

and

$$(2.33) \quad \int_0^t \tilde{p}_s(0) ds = \frac{1}{4\pi} \log t + \tilde{c} + O\left(\frac{1}{t}\right), \quad \tilde{c} = \int_0^1 \tilde{p}_s(0) ds + \int_1^\infty \left(\tilde{p}_s(0) - \frac{1}{4\pi s} \right) ds.$$

Equivalently, rescaling $(s, x) \rightarrow (s/\varepsilon^2, x/\varepsilon)$,

$$(2.34) \quad \int_0^t p_s^\varepsilon(0) ds + \frac{1}{4\pi} \log \varepsilon^2 = \frac{1}{4\pi} \log t + \tilde{c} + O\left(\frac{\varepsilon^2}{t}\right).$$

For the second term on the right-hand side of (2.31), since $p_t^0(0) = 1/(4\pi t)$ and $p_t^0(0) - p_t^\varepsilon(0) = O(\varepsilon^2/t^2 \wedge 1/t)$, we have

$$(2.35) \quad \int_0^t p_s^\varepsilon(0) (1 - e^{-m^2 s}) ds = \int_0^t \frac{1}{4\pi s} (1 - e^{-m^2 s}) ds + O(\varepsilon^2 m^2) \int_{\varepsilon^2}^{1/m^2} \frac{ds}{s} + O(m^2 \varepsilon^2),$$

where the middle term is $O(\varepsilon^2 m^2 |\log(\varepsilon^2 m^2)|)$. The third term on the right-hand side of (2.31) is

$$(2.36) \quad \int_{1/m^2}^t p_s^\varepsilon(0) e^{-m^2 s} ds = \int_{1/m^2}^t \frac{1}{4\pi s} e^{-m^2 s} ds + O(\varepsilon^2 m^2)$$

where we used $\int_{1/m^2}^\infty \frac{\varepsilon^2}{s^2} e^{-m^2 s} ds = O(\varepsilon^2 m^2)$. Define

$$(2.37) \quad \tilde{c}(m, t) = \tilde{c} + \int_0^{t \wedge 1/m^2} \frac{1}{4\pi s} (e^{-m^2 s} - 1) ds + \int_{t \wedge 1/m^2}^t \frac{1}{4\pi s} e^{-m^2 s} ds$$

and note that

$$(2.38) \quad \tilde{c}(m, t) = \tilde{c} + O(m^2 t) \quad (t \rightarrow 0), \quad \tilde{c}(m, t) = \tilde{c}(m, \infty) + O(e^{-m^2 t}) \quad (t \rightarrow \infty).$$

In summary, with $L_t = \sqrt{t} \wedge 1/m$, we have shown that

$$(2.39) \quad c_t^{\varepsilon, \infty}(0) + \frac{1}{4\pi} \log \varepsilon^2 = \frac{1}{2\pi} \log L_t + \tilde{c}(m, t) + O\left(\frac{\varepsilon^2}{t} + \varepsilon^2 m^2 |\log \varepsilon^2 m^2|\right)$$

and therefore with $\gamma_t(m) = e^{-\frac{1}{2}\tilde{c}(m, t)}$, we have shown that

$$(2.40) \quad \varepsilon^{-1/4\pi} e^{-c_t^{\varepsilon, \infty}(0)/2} = e^{-\frac{1}{2}(c_t^{\varepsilon, \infty}(0) + (\log \varepsilon^2)/4\pi)} = L_t^{-1/4\pi} (\gamma_t(m) + O(\varepsilon^2/t + m^2 \varepsilon^2 |\log m^2 \varepsilon^2|)).$$

Finally, we return to the torus version of c_t . Using (A.5)–(A.6), similar elementary computations show that there is $\kappa > 0$ such that

$$(2.41) \quad |c_t^{\varepsilon, \infty}(0) - c_t^{\varepsilon, L}(0)| \lesssim 1$$

$$(2.42) \quad |(c_t^{0, \infty}(0) - c_t^{0, L}(0)) - (c_t^{\varepsilon, \infty}(0) - c_t^{\varepsilon, L}(0))| \lesssim \varepsilon^\kappa.$$

Thus with $\tilde{c}(m, L, t) = \tilde{c}(m, t) - (c_t^{0, \infty}(0) - c_t^{0, L}(0))$ and $\gamma_t(m, L) = e^{-\frac{1}{2}\tilde{c}(m, L, t)}$, we have shown

$$(2.43) \quad \varepsilon^{-1/4\pi} e^{-c_t^{\varepsilon, L}(0)/2} = L_t^{-1/4\pi} (\gamma_t(m, L) + O(\frac{\varepsilon^2}{t} + O_{m, L}(\varepsilon^\kappa))).$$

The claim is the special case $L = m = 1$. □

Proof of (2.28). It follows directly from the definition of $c_t(x, y)$ that, for any $\kappa \in (0, 1)$,

$$(2.44) \quad |c_t^0(x, y) - c_t^\varepsilon(x^\varepsilon, y^\varepsilon)| \leq \int_0^{\kappa^2|x-y|^2} (p_s^0(x, y) + p_s^\varepsilon(x^\varepsilon, y^\varepsilon)) ds + \int_{\kappa^2|x-y|^2}^\infty |p_s^0(x, y) - p_s^\varepsilon(x^\varepsilon, y^\varepsilon)| e^{-m^2 s} ds.$$

By (A.2) respectively (A.3), the two terms on the right-hand side are bounded by multiples of

$$(2.45) \quad \int_0^{\kappa^2|x-y|^2} e^{-c|x-y|/\sqrt{t}} \frac{dt}{t} \lesssim e^{-c/(2\kappa)}, \quad \varepsilon^2 \int_{\kappa^2|x-y|^2}^\infty t^{-2} dt \lesssim \frac{\varepsilon^2}{\kappa^2|x-y|^2},$$

and choosing say $\kappa = c/(4 \log(|x-y|/\varepsilon))$ gives the claim. □

Proof of (2.29). In the definition of $(w_t - w_s)(\xi_1, \dots, \xi_n)$ there are n^2 pairings between the charges ξ_1, \dots, ξ_n . We ignore the signs of the charges and bound each pairing simply using (A.5)–(A.6): For $t \geq s \geq \varepsilon^2$, this gives

$$(2.46) \quad \int_s^t |\dot{u}_r^\varepsilon(\xi_i^\varepsilon, \xi_j^\varepsilon) - \dot{u}_r^0(\xi_i, \xi_j)| dr \lesssim \int_s^t \left(\frac{\varepsilon^2}{r}\right)^{1-\kappa} \left(\frac{1}{r} + \frac{1}{L^2}\right) \theta_r dr \lesssim \left(\frac{\varepsilon^2}{s}\right)^{1-\kappa},$$

uniformly in ξ_i and ξ_j . Summing over all $O(n^2)$ pairs of charges, therefore

$$(2.47) \quad \sup_{\xi_1, \dots, \xi_n} |w_t^\varepsilon(\xi^\varepsilon) - w_s^\varepsilon(\xi^\varepsilon) - (w_t^0(\xi) - w_s^0(\xi))| \lesssim n^2 \left(\frac{\varepsilon^2}{s}\right)^{1-\kappa}$$

and (2.29) follows. □

Proof of (2.30). The first bound in (2.30) is immediate from (A.5) and (A.2). For the convergence bound in (2.30), we can take the sum of (A.4) and (A.3) over $x \in \Omega_\varepsilon$ to see that, for $t \geq \varepsilon^2$,

$$(2.48) \quad \varepsilon^2 \sum_{x \in \Omega_\varepsilon} |p_t^{\varepsilon,L}(x) - p_t^{0,L}(x)| \leq \varepsilon^2 \sum_{x \in \varepsilon\mathbb{Z}^2} |p_t^\varepsilon(x) - p_t^0(x)| \lesssim \frac{\varepsilon^2}{t} + \frac{\varepsilon^{2k}}{t^k} \lesssim \frac{\varepsilon^2}{t}.$$

For $t \leq \varepsilon^2$ the left-hand side is trivially bounded by $O(1) \leq O(\varepsilon^2/t)$. \square

We also record the following lemma providing some basic estimates for the regularised GFF. Here recall that for $\varepsilon > 0$ we use $\mathbf{E}_{c^\varepsilon}$ to denote the expectation of the Gaussian measure on X_ε with covariance matrix c^ε . Moreover, when c^0 is a smooth covariance kernel on Ω , we denote by \mathbf{E}_{c^0} the Gaussian measure supported on $C^\infty(\Omega)$ with covariance c^0 .

Lemma 2.4. *Let $t_0 > 0$. Then (with all constants depending on t_0 and m but uniform in $\varepsilon \geq 0$)*

$$(2.49) \quad \sup_{1 \geq \varepsilon \geq 0} \sup_{t \geq t_0} \mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon} \left(e^{O(\|\partial_\varepsilon \zeta\|_{L^\infty(\Omega_\varepsilon)})} \right) \lesssim 1.$$

Assume that $F_\varepsilon: X_\varepsilon \rightarrow \mathbb{R}$ and $F_0: C(\Omega) \rightarrow \mathbb{R}$ satisfy the uniform bounds $|F_\varepsilon(\zeta)| \leq e^{O(\|\partial_\varepsilon \zeta\|_{L^\infty(\Omega_\varepsilon)})}$ and moreover $|F_0(\zeta)| \leq e^{O(\|\partial \zeta\|_{L^\infty(\Omega)})}$ and $F_\varepsilon(\varphi^\varepsilon) \rightarrow F_0(\varphi)$ as $\varepsilon \rightarrow 0$ if $\max_{x \in \Omega} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \rightarrow 0$. Then

$$(2.50) \quad \mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon} F_\varepsilon - \mathbf{E}_{c_t^0 - c_{t_0}^0} F_0 \rightarrow 0.$$

Similarly, if $F_\varepsilon: X_\varepsilon \rightarrow X_\varepsilon$ and $F_0: C(\Omega) \rightarrow C(\Omega)$ satisfy $\|F_\varepsilon(\zeta)\|_{L^\infty(\Omega_\varepsilon)} \leq e^{O(\|\partial_\varepsilon \zeta\|_{L^\infty(\Omega_\varepsilon)})}$ and $\|F_0(\zeta)\|_{L^\infty(\Omega)} \leq e^{O(\|\partial \zeta\|_{L^\infty(\Omega)})}$ and moreover $\|F_\varepsilon(\varphi^\varepsilon) - F_0(\varphi)\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ if $\max_{x \in \Omega} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \rightarrow 0$, then

$$(2.51) \quad \|\mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon} F_\varepsilon - \mathbf{E}_{c_t^0 - c_{t_0}^0} F_0\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0.$$

For the proof, the following coupling of the Gaussian measures with expectations \mathbf{E}_{c^0} and $\mathbf{E}_{c^\varepsilon}$ where $c^\varepsilon = c_t^\varepsilon - c_{t_0}^\varepsilon$ and $c^0 = c_t^0 - c_{t_0}^0$ is convenient. Let $\Omega^* = 2\pi\mathbb{Z}^2$ and $\Omega_\varepsilon^* = \{k \in 2\pi\mathbb{Z}^2 : -\pi/\varepsilon < k_i \leq \pi/\varepsilon\}$ be the Fourier duals of Ω and Ω_ε . Let $(X(k))_{k \in 2\pi\mathbb{Z}^2}$ be a collection of independent complex standard Gaussian random variables subject to $X(k) = \overline{X(-k)}$ for $k \neq 0$ and $X(0)$ is a real standard Gaussian random variable (all defined on a common probability space). Let $\hat{q}^0(k)$ denote the Fourier coefficients of $q^0: \Omega \rightarrow \mathbb{R}$ where $[q^0 * q^0](x-y) = c^0(x,y)$ and $*$ denoting the convolution of two functions. Then the random variable Φ defined by

$$(2.52) \quad \Phi(x) = \sum_{k \in \Omega^*} \hat{q}^0(k) e^{ik \cdot x} X(k)$$

takes almost surely values in $C^\infty(\Omega)$ and is Gaussian with mean zero and covariance c^0 , i.e.,

$$(2.53) \quad \mathbb{E}[\langle f, \Phi \rangle \langle g, \Phi \rangle] = \langle c^0 f, g \rangle$$

holds for all $f, g \in L^2(\Omega)$. Moreover, since c^0 is a positive real symmetric function, $\hat{q}^0(k) = \sqrt{\hat{c}^0(k)}$. Similarly, let $q^\varepsilon: \Omega_\varepsilon \rightarrow \mathbb{R}$ be such that $[q^\varepsilon * q^\varepsilon](x-y) = c^\varepsilon(x,y)$, where $*$ now denotes convolution on the lattice, and let $\hat{q}^\varepsilon(k)$ be the Fourier coefficients of q^ε , i.e. for $x \in \Omega_\varepsilon$,

$$(2.54) \quad q^\varepsilon(x) = \sum_{k \in \Omega_\varepsilon^*} \hat{q}^\varepsilon(k) e^{ik \cdot x}.$$

Then the random function Φ^ε defined by

$$(2.55) \quad \Phi^\varepsilon(x) = \sum_{k \in \Omega_\varepsilon^*} \hat{q}^\varepsilon(k) e^{ik \cdot x} X(k)$$

restricted to X_ε is multivariate Gaussian with covariance matrix c^ε .

Proof. Let $-\hat{\Delta}^0(k) = |k|^2$ and $-\hat{\Delta}^\varepsilon(k) = \varepsilon^{-2} \sum_{i=1}^d (2 - 2 \cos(\varepsilon k_i))$ be the Fourier multipliers of the continuum and lattice Laplacians, respectively, and let $\Omega_\varepsilon^* = \{k \in 2\pi\mathbb{Z}^2 : -\pi/\varepsilon < k_i \leq \pi/\varepsilon\}$ be the Fourier dual of Ω_ε . For $k \in \Omega_\varepsilon^*$, then

$$(2.56) \quad 0 \leq -\hat{\Delta}^0(k) + \hat{\Delta}^\varepsilon(k) = \sum_{i=1}^d (k_i^2 - \varepsilon^{-2}(2 - 2 \cos(\varepsilon k_i))) \leq |k|^2 h(\varepsilon k)$$

where $h(x) = \max_{i=1,2}(1 - x_i^{-2}(2 - 2 \cos(x_i)))$ satisfies $h(x) \in [0, 1 - \kappa]$ with $\kappa = 4/\pi^2$ for $|x| \leq \pi$ and $h(x) = O(|x|^2)$.

Denote $c^\varepsilon = c_t^\varepsilon - c_{t_0}^\varepsilon$ and $c^0 = c_t^0 - c_{t_0}^0$. Since, for $\varepsilon \geq 0$,

$$(2.57) \quad 0 \leq \hat{c}^\varepsilon(k) = \hat{c}_t^\varepsilon(k) - \hat{c}_{t_0}^\varepsilon(k) = \int_{t_0}^t e^{s\hat{\Delta}^\varepsilon(k) - sm^2} ds,$$

we see from (2.56) that

$$(2.58) \quad 0 \leq \hat{c}^0(k) \leq \frac{1}{|k|^2 + m^2} e^{-t_0(|k|^2 + m^2)}, \quad 0 \leq \hat{q}^0(k) \leq \frac{1}{\sqrt{|k|^2 + m^2}} e^{-(t_0/2)(|k|^2 + m^2)},$$

$$(2.59) \quad 0 \leq \hat{c}^\varepsilon(k) \leq \frac{1}{\kappa|k|^2 + m^2} e^{-t_0(\kappa|k|^2 + m^2)}, \quad 0 \leq \hat{q}^\varepsilon(k) \leq \frac{1}{\sqrt{\kappa|k|^2 + m^2}} e^{-(t_0/2)(\kappa|k|^2 + m^2)}.$$

Similar, but somewhat more tedious computations give

$$(2.60) \quad 0 \leq \hat{q}^\varepsilon(k) - \hat{q}^0(k) \lesssim e^{-(t_0/2)(\kappa|k|^2 + m^2)} h(\varepsilon k)^{1/2}.$$

To prove (2.49) we note that $\mathbf{E}_{c^\varepsilon} \|\zeta\|_{H^\alpha(\Omega_\varepsilon)}^2 \leq O_\alpha(1)$ for any $\alpha \in \mathbb{R}$, where

$$(2.61) \quad \|\zeta\|_{H^\alpha(\Omega_\varepsilon)}^2 = \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^\alpha |\hat{\zeta}(k)|^2.$$

Indeed, with Φ^ε as in (2.54),

$$(2.62) \quad \mathbf{E}_{c^\varepsilon} \|\zeta\|_{H^\alpha(\Omega_\varepsilon)}^2 = \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^\alpha \mathbb{E} |\hat{q}^\varepsilon(k) X(k)|^2 = \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^\alpha |\hat{q}^\varepsilon(k)|^2 \leq O_\alpha(1)$$

where we used (2.58). By the Sobolev inequality $\|\partial_\varepsilon f\|_{L^\infty(\Omega_\varepsilon)} \leq C_\alpha \|f\|_{H^\alpha(\Omega_\varepsilon)}$ which holds for any $\alpha > d/2 + 1$ (uniformly in $\varepsilon \geq 0$) and Gaussian concentration (for example, [2, Theorem 2.1.1]), hence

$$(2.63) \quad \mathbf{E}_{c^\varepsilon} e^{O(\|\partial_\varepsilon \zeta\|_{L^\infty(\Omega_\varepsilon)})} \lesssim \mathbf{E}_{c^\varepsilon} e^{O(\|\zeta\|_{H^\alpha(\Omega_\varepsilon)})} \lesssim 1.$$

In fact, under the coupling (2.55), one has

$$(2.64) \quad \mathbb{E} \left(\sup_{\varepsilon > 0} e^{O(\|\partial_\varepsilon \Phi^\varepsilon\|_{L^\infty(\Omega_\varepsilon)})} \right) \lesssim 1.$$

This can be seen from $\|\partial_\varepsilon \Phi^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C_\alpha \|\Phi^\varepsilon\|_{H^\alpha(\Omega_\varepsilon)}$ and then bounding $\|\Phi^\varepsilon\|_{H^\alpha(\Omega_\varepsilon)}^2 \lesssim \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^\alpha |\hat{q}^\varepsilon(k)|^2 |X(k)|^2$ which is bounded uniformly in ε by (2.59).

To prove (2.50) and (2.51), we will use that almost surely under the coupling introduced above the statement of the lemma,

$$(2.65) \quad \sup_{x \in \Omega} |\Phi^\varepsilon(x^\varepsilon) - \Phi(x)| \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Indeed,

$$(2.66) \quad |\Phi^\varepsilon(x^\varepsilon) - \Phi(x)| \leq |\Phi^\varepsilon(x^\varepsilon) - \Phi(x^\varepsilon)| + |\Phi(x^\varepsilon) - \Phi(x)| \\ \leq \sum_{k \in \Omega_\varepsilon^*} |\hat{q}^\varepsilon(k) - \hat{q}^0(k)| |X(k)| + \sum_{k \in \Omega^* \setminus \Omega_\varepsilon^*} |\hat{q}^0(k)| |X(k)| + \sum_{k \in \Omega^*} |\hat{q}^0(k) X(k)| |k| |x^\varepsilon - x|$$

and the second and the third sum converge to 0 almost surely since

$$(2.67) \quad \sum_{k \in \Omega^*} |\hat{q}^0(k) X(k)| < \infty \text{ a.s. and } \sum_{k \in \Omega^*} |\hat{q}^0(k)| |k| |X(k)| < \infty \text{ a.s.}$$

These hold by (2.58) since

$$(2.68) \quad \mathbb{E} \left[\sum_{k \in \Omega^*} (1 + |k|) |\hat{q}^0(k) X(k)| \right] \lesssim \sum_{k \in \Omega^*} (1 + |k|) |\hat{q}^0(k)| < \infty.$$

To see that also the first sum converges to 0 almost surely, we use the estimate (2.60) together with the Borel-Cantelli lemma. Let $E_k = \{|X(k)| > |k|\}$. Then $\mathbb{P}(E_k) \leq e^{-|k|^2/2}$ and hence $\sum_{k \in \Omega^*} \mathbb{P}(E_k) < \infty$. By the Borel-Cantelli lemma, $\mathbb{P}(E_k \text{ infinitely often}) = 0$. Let M be the smallest natural number such that E_k^c occurs for all $k > M$. Note that on the event $\{E_k \text{ finitely often}\}$ we have $M < \infty$ and hence

$$(2.69) \quad \sum_{k \in \Omega_\varepsilon^*} |\hat{q}^\varepsilon(k) - \hat{q}^0(k)| |X(k)| \lesssim \sum_{k \in \Omega_\varepsilon^*} e^{-(t_0/2)(\kappa|k|^2 + m^2)} h(\varepsilon k) |X(k)| \\ \leq \sum_{|k| \leq M} e^{-(t_0/2)(\kappa|k|^2 + m^2)} h(\varepsilon k) |X(k)| + \sum_{|k| > M} e^{-(t_0/2)(\kappa|k|^2 + m^2)} h(\varepsilon k) |k|.$$

Taking $\varepsilon \rightarrow 0$, the first sum converges to 0 as it has only finitely many terms. The same holds for the second sum by dominated convergence with respect to the counting measure on Ω^* .

Now the statement (2.50) follows from

$$(2.70) \quad \mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon} F_\varepsilon = \mathbb{E} F_\varepsilon(\Phi^\varepsilon) \rightarrow \mathbb{E} F_0(\Phi) = \mathbf{E}_{c_t^0 - c_{t_0}^0} F_0$$

where we used the dominated convergence theorem. Note that by (2.65) and the assumption on F_ε and F_0 , we have $F_\varepsilon(\Phi^\varepsilon) \rightarrow F_0(\Phi)$ almost surely and moreover

$$(2.71) \quad |F_\varepsilon(\Phi^\varepsilon)| \lesssim e^{O(\|\partial_\varepsilon \Phi^\varepsilon\|_{L^\infty(\Omega_\varepsilon)})} \lesssim e^{O(\sup_\varepsilon \|\partial_\varepsilon \Phi^\varepsilon\|_{L^\infty(\Omega)})}$$

where the right-hand side is integrable by (2.64).

To prove (2.51) we use the embedding E_ε and obtain

$$(2.72) \quad \|E_\varepsilon \mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon} F_\varepsilon - \mathbf{E}_{c_t^0 - c_{t_0}^0} F_0\|_{L^\infty(\Omega)} \leq \mathbb{E} \|E_\varepsilon F_\varepsilon(\Phi^\varepsilon) - F_0(\Phi)\|_{L^\infty(\Omega)} \rightarrow 0,$$

where the convergence follows from an extension of the dominated convergence theorem to Banach space valued functions. Note that in this case, we have

$$(2.73) \quad E_\varepsilon F_\varepsilon(\Phi^\varepsilon) \rightarrow F_0(\Phi) \text{ a.s. in } L^\infty(\Omega)$$

by (2.65) and the properties of F_ε and F_0 , and moreover

$$(2.74) \quad \|E_\varepsilon F_\varepsilon(\Phi^\varepsilon)\|_{L^\infty(\Omega)} = \|F_\varepsilon(\Phi^\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} \lesssim e^{O(\|\partial_\varepsilon \Phi^\varepsilon\|_{L^\infty(\Omega_\varepsilon)})} \lesssim e^{O(\sup_\varepsilon \|\partial_\varepsilon \Phi^\varepsilon\|_{L^\infty(\Omega_\varepsilon)})}$$

where the right-hand side is integrable by (2.64). This shows that the assumption for the dominated convergence theorem are satisfied. \square

2.5. Proof of Theorem 2.2 for $\beta < 4\pi$. We first prove Theorem 2.2 in the simpler case $\beta < 4\pi$. To this end, recall the definitions of $\tilde{v}_t^{n,\varepsilon}$ from (2.15)–(2.16) and those of $\tilde{v}_t^{n,0}$ from (2.23)–(2.24), as well as the notation $E_\varepsilon f$ for the piecewise constant extension of a function f on Ω_ε introduced above Lemma 2.3.

Lemma 2.5. *Let $\beta < 4\pi$. Then for all $\varepsilon \geq 0$, $n \geq 2$, and $t \geq 0$,*

$$(2.75) \quad L_t^2 \sup_{\xi} \|\tilde{v}_t^{n,\varepsilon}(\xi, \cdot)\|_{L^1((\Omega_\varepsilon \times \{\pm 1\})^{n-1})} \leq n^{n-2} (C_\beta |Z_t^\varepsilon|)^n.$$

Moreover, for any $t > 0$, as $\varepsilon \rightarrow 0$,

$$(2.76) \quad L_t^2 \sup_{\xi} \|E_\varepsilon \tilde{v}_t^{n,\varepsilon}(\xi, \cdot) - \tilde{v}_t^{n,0}(\xi, \cdot)\|_{L^1((\Omega \times \{\pm 1\})^{n-1})} \rightarrow 0.$$

Proof. The bound (2.75) is proved in [7, Proposition 3.5 and (3.49)]. Recall that we use a different normalisation here than in [7]; see Section 2.7 below for translation. Moreover, the argument there is stated for $\varepsilon > 0$, but all estimates are uniform in ε and hold without changes also when $\varepsilon = 0$. More precisely, the following is shown in the proof of [7, Proposition 3.5]. Writing (2.16) as

$$(2.77) \quad \tilde{v}_t^{n,\varepsilon}(\xi_1, \dots, \xi_n) = \int_0^t r_{s,t}^{n,\varepsilon}(\xi_1, \dots, \xi_n) ds,$$

where

$$(2.78) \quad r_{s,t}^{n,\varepsilon}(\xi_1, \dots, \xi_n) = \sum_{I_1 \cup I_2 = [n]} \sum_{i \in I_1, j \in I_2} \beta \sigma_i \sigma_j \dot{c}_s^\varepsilon(x_i, x_j) \tilde{v}_s^\varepsilon(\xi_{I_1}) \tilde{v}_s^\varepsilon(\xi_{I_2}) e^{-(w_t^\varepsilon - w_s^\varepsilon)(\xi_1, \dots, \xi_n)},$$

there are $R_{s,t}^n > 0$ such that

$$(2.79) \quad \sup_{\varepsilon \geq 0} \sup_{\xi_1} \|E_\varepsilon r_{s,t}^{n,\varepsilon}(\xi_1, \cdot)\|_{L^1((\Omega \times \{\pm 1\})^{n-1})} \leq R_{s,t}^n$$

and

$$(2.80) \quad L_t^2 \int_0^t R_{s,t}^n ds \leq n^{n-2} (C_\beta |Z_t|)^n.$$

We now deduce the convergence (2.76) as follows. Assume by induction that (2.76) holds for all $t > 0$ and $n \leq k$. For $k = 1$, this follows from (2.27) and the definitions of $v_t^{1,\varepsilon}$ in (2.15) and of $v_t^{1,0}$ in (2.23). To advance the induction, note that the definition of $r_{s,t}^{n,\varepsilon}$ above depends on $\tilde{v}_s^{m,\varepsilon}$ only with $m < n$. Therefore, using the inductive assumption for the \tilde{v}_s terms in (2.78), and Lemma 2.3 for $e^{-(w_t - w_s)}$ and \dot{c}_s in (2.78), it follows that, for any $0 < s < t$, as $\varepsilon \rightarrow 0$,

$$(2.81) \quad \sup_{\xi_1} \|E_\varepsilon r_{s,t}^{k+1,\varepsilon}(\xi_1, \cdot) - r_{s,t}^{k+1,0}(\xi_1, \cdot)\|_{L^1((\Omega \times \{\pm 1\})^k)} \rightarrow 0.$$

By dominated convergence for the s -integral, using (2.80), it then follows that, as $\varepsilon \rightarrow 0$,

$$(2.82) \quad \|E_\varepsilon \tilde{v}_{s,t}^{k+1,\varepsilon}(\xi_1, \cdot) - \tilde{v}_{s,t}^{k+1,0}(\xi_1, \cdot)\|_{L^1((\Omega \times \{\pm 1\})^k)} \rightarrow 0,$$

and the induction is advanced. \square

Lemma 2.6. *Let $\beta < 4\pi$, $z \in \mathbb{R}$. Then for all $n \geq 2$ and $k \in \mathbb{N}$, uniformly in φ and $\varepsilon \geq 0$,*

$$(2.83) \quad |v_t^{n,\varepsilon}(\varphi)| \leq (C_\beta |Z_{t \wedge \varepsilon}|)^n,$$

$$(2.84) \quad L_t^{2+k} \|\partial_\varepsilon^k \dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi)\|_{L^\infty(\Omega_\varepsilon)} \leq C_k (C_\beta |Z_{t \wedge \varepsilon}|)^n,$$

$$(2.85) \quad L_t^{2+k} \|\partial_\varepsilon^k \dot{c}_t^\varepsilon \text{Hess } v_t^{n,\varepsilon}(\varphi) g\|_{L^\infty(\Omega_\varepsilon)} \leq C_k \|g\|_{L^\infty(\Omega_\varepsilon)} (C_\beta |Z_{t \wedge \varepsilon}|)^n.$$

Moreover, for any $t > 0$ and $\max_{x \in \Omega_\varepsilon} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \rightarrow 0$ as $\varepsilon \rightarrow 0$,

$$(2.86) \quad |v_t^{n,\varepsilon}(\varphi^\varepsilon) - v_t^{n,0}(\varphi)| \rightarrow 0,$$

$$(2.87) \quad L_t^2 \|\dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi^\varepsilon) - \dot{c}_t^0 \nabla v_t^{n,0}(\varphi)\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0.$$

Proof. The bounds (2.83)–(2.85) follow easily by substituting (2.75) into (2.18)–(2.21), exactly as in [7, Section 3]. We consider this argument in more detail for (2.84). To this end, let

$$(2.88) \quad r_t^\varepsilon(x, \xi_1, \dots, \xi_n) = \frac{i\sqrt{\beta}}{n!} \left(\sum_{i=1}^n \partial_\varepsilon^k \dot{c}_t^\varepsilon(x, x_i) \sigma_i \right) \tilde{v}_t^{n,\varepsilon}(\xi_1, \dots, \xi_n)$$

so that (2.20) can be written as

$$(2.89) \quad \partial_\varepsilon^k \dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi, x) = \int_{(\Omega_\varepsilon \times \{\pm 1\})^n} d\xi_1 \cdots d\xi_n r_t^\varepsilon(x, \xi_1, \dots, \xi_n) e^{i \sum_{i=1}^n \sigma_i \varphi(x_i)}.$$

Using that $n^n/n! \leq e^n$, it then follows from (2.30) and (2.75) that

$$(2.90) \quad L_t^{2+k} \sup_x \|r_t^\varepsilon(x, \cdot)\|_{L^1(\Omega_\varepsilon \times \{\pm 1\})} \lesssim \frac{n}{n!} L_t^k \sup_x \|\partial_\varepsilon^k \dot{c}_t^\varepsilon(x, \cdot)\|_{L^1((\Omega_\varepsilon \times \{\pm 1\}))} L_t^2 \sup_{\xi_1} \|\tilde{v}_t^{n,\varepsilon}(\xi_1, \cdot)\|_{L^1((\Omega_\varepsilon \times \{\pm 1\})^n)} \lesssim C_k (C_\beta |Z_t|)^n.$$

To show the convergence (2.87), recall the notation ξ^ε from Section 2.4. Then (where now $k = 0$)

$$(2.91) \quad \dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi, x) = \int_{(\Omega \times \{\pm 1\})^n} d\xi_1 \cdots d\xi_n r_t^\varepsilon(x, \xi_1^\varepsilon, \dots, \xi_n^\varepsilon) e^{i \sum_{i=1}^n \sigma_i \varphi(x_i^\varepsilon)},$$

and $\sup_x |\dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi, x) - \dot{c}_t^0 \nabla v_t^{n,0}(\varphi, x)|$ is bounded by the sum of the following two terms:

$$(2.92) \quad \sup_x \int_{(\Omega \times \{\pm 1\})^n} d\xi_1 \cdots d\xi_n \left| r_t^\varepsilon(x, \xi_1^\varepsilon, \dots, \xi_n^\varepsilon) - r_t^0(x, \xi_1, \dots, \xi_n) \right|,$$

$$(2.93) \quad \sup_x \int_{(\Omega \times \{\pm 1\})^n} d\xi_1 \cdots d\xi_n \left| r_t^\varepsilon(x, \xi_1^\varepsilon, \dots, \xi_n^\varepsilon) \left| e^{i \sum_{i=1}^n \sigma_i \varphi(x_i^\varepsilon)} - e^{i \sum_{i=1}^n \sigma_i \varphi(x_i)} \right| \right|.$$

The first term (2.92) converges to 0 as $\varepsilon \rightarrow 0$ by (2.30) and (2.76). For the second term (2.93), we use the assumption $\sup_{x \in \Omega} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \leq a_\varepsilon$ with $a_\varepsilon \rightarrow 0$ which implies

$$(2.94) \quad \left| e^{i \sum_{i=1}^n \sigma_i \varphi(x_i^\varepsilon)} - e^{i \sum_{i=1}^n \sigma_i \varphi(x_i)} \right| \leq O(na_\varepsilon).$$

Hence (2.93) is bounded by $O(na_\varepsilon) \sup_x \|r^\varepsilon(x, \cdot)\|_{L^1((\Omega_\varepsilon \times \{\pm 1\})^n)}$ and the claim follows from (2.90). \square

Proof of Theorem 2.2 for $\beta < 4\pi$. In order to treat the two cases $\varepsilon > 0$ and $\varepsilon = 0$ simultaneously, we will typically omit the superscript ε , and we also make the convention that for $\varepsilon > 0$ the space $C(\Omega_\varepsilon)$ simply refers to the finite dimensional space X_ε . We define $t_0 = t_0(\beta, z, m)$ as the largest $t_0 > 0$ with $C_\beta |Z_{t_0}| \leq \alpha_0$ for a sufficiently small constant α_0 ; this t_0 is independent of ε . We then start with showing the bounds in the case $t \leq t_0$. In this case, by (2.84), the sum on the right-hand side of

$$(2.95) \quad \dot{c}_t \nabla v_t(\varphi) = \sum_{n=0}^{\infty} \dot{c}_t \nabla v_t^n(\varphi)$$

converges in $C(\Omega)$, uniformly in $\varphi \in C(\Omega)$, and for any n_0 , we have

$$(2.96) \quad \dot{c}_t \nabla v_t(\varphi) = \sum_{n=0}^{n_0} \dot{c}_t \nabla v_t^n(\varphi) + O(L_t^{-2} |Z_t|^{n_0})$$

where again the error $O(L_t^{-2} |Z_t|^{n_0})$ is bounded in $C(\Omega)$ uniformly in φ . In particular, with $n_0 \rightarrow \infty$, the bound (2.13) with $k = 0$ follows for $t \leq t_0$. The argument for $k \in \mathbb{N}$ is analogous,

and the bound (2.14) also follows analogously from (2.85). To show the convergence (2.12), for $t \leq t_0$, we apply (2.87) to see from the above that

$$(2.97) \quad \begin{aligned} L_t^2 \|\dot{c}_t^\varepsilon \nabla v_t^\varepsilon(\varphi^\varepsilon) - \dot{c}_t^0 \nabla v_t^0(\varphi)\|_{L^\infty(\Omega_\varepsilon)} &\leq \sum_{n=0}^{n_0} L_t^2 \|\dot{c}_t^\varepsilon \nabla v_t^{n,\varepsilon}(\varphi^\varepsilon) - \dot{c}_t \nabla v_t^{n,0}(\varphi)\|_{L^\infty(\Omega_\varepsilon)} + O(|Z_t|^{n_0}) \\ &\leq o_{n_0}(1) + O(|Z_t|^{n_0}) \rightarrow 0 \end{aligned}$$

by taking $\varepsilon \rightarrow 0$ and then $n_0 \rightarrow \infty$. This establishes the convergence (2.12) for $t \leq t_0$.

We next extend the bounds to $t > t_0$. Since t_0 is of order 1, $\|\dot{c}_{t_0} \nabla v_{t_0}(\varphi)\|_{L^\infty(\Omega)}$ is uniformly bounded, and the same argument shows that $|v_{t_0}(\varphi)|$ is likewise uniformly bounded, and that $v_{t_0}^\varepsilon(\varphi^\varepsilon) \rightarrow v_{t_0}^0(\varphi)$ when $\varphi^\varepsilon \rightarrow \varphi$ as in the statement of the theorem. From this, a simple argument suffices. Indeed, v_t can be obtained from v_{t_0} using the convolution representation (2.3) as

$$(2.98) \quad e^{-v_t(\varphi)} = \mathbf{E}_{c_t - c_{t_0}}(e^{-v_{t_0}(\varphi + \zeta)}),$$

$$(2.99) \quad \partial^k \dot{c}_t \nabla v_t(\varphi) = [\mathbf{P}_{t_0, t}(\partial^k \dot{c}_t \nabla v_{t_0})](\varphi),$$

where

$$(2.100) \quad \mathbf{P}_{t_0, t} F(\varphi) = e^{+v_t(\varphi)} \mathbf{E}_{c_t - c_{t_0}}(e^{-v_{t_0}(\varphi + \zeta)} F(\varphi + \zeta)).$$

This representation holds for any $\varepsilon > 0$ and we use it as our definition in the limiting case $\varepsilon = 0$. To see that this is well-defined note that, by Jensen's inequality,

$$(2.101) \quad v_t(\varphi) = -\log \mathbf{E}_{c_t - c_{t_0}}(e^{-v_{t_0}(\varphi + \zeta)}) \leq \mathbf{E}_{c_t - c_{t_0}}(v_{t_0}(\varphi + \zeta)) \leq O_{t_0, t}(1),$$

and thus e^{+v_t} is bounded. By taking maxima over φ and x , (2.99) also immediately implies that

$$(2.102) \quad \sup_{\varphi} \|\partial^k \dot{c}_t \nabla v_t(\varphi)\|_{L^\infty(\Omega)} \leq \sup_{\varphi} \|\partial^k \dot{c}_t \nabla v_{t_0}(\varphi)\|_{L^\infty(\Omega)} \lesssim \theta_{t-t_0} \sup_{\varphi} \|\partial^k \dot{c}_{t_0} \nabla v_{t_0}(\varphi)\|_{L^\infty(\Omega)}$$

which gives (2.13) for $t \geq t_0$ since t_0 is of order 1 meaning that $L_t \simeq L_{t_0}$ and $Z_t \simeq Z_{t_0}$ (with constants depending on m, L). The bound (2.14) also follows similarly; see [7, Section 3.8].

To see the convergence (2.12) for $t > t_0$, we apply (2.50). Indeed, by the above analysis for $t \leq t_0$, when $\sup_{x \in \Omega} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \rightarrow 0$, we have $v_{t_0}^\varepsilon(\varphi^\varepsilon) \rightarrow v_{t_0}^0(\varphi)$ and $\dot{c}_t^\varepsilon \nabla v_{t_0}^\varepsilon(\varphi^\varepsilon, x^\varepsilon) \rightarrow \dot{c}_t^0 \nabla v_{t_0}^0(\varphi, x)$ uniformly in x . By (2.50), therefore

$$(2.103) \quad e^{-v_t^\varepsilon(\varphi)} = \mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon}(e^{-v_{t_0}^\varepsilon(\varphi + \zeta)}) \rightarrow \mathbf{E}_{c_t^0 - c_{t_0}^0}(e^{-v_{t_0}^0(\varphi + \zeta)}) = e^{-v_t^0(\varphi)},$$

$$(2.104) \quad \mathbf{E}_{c_t^\varepsilon - c_{t_0}^\varepsilon}(\dot{c}_t^\varepsilon \nabla v_{t_0}^\varepsilon(\varphi + \zeta) e^{-v_{t_0}^\varepsilon(\varphi + \zeta)}) \rightarrow \mathbf{E}_{c_t^0 - c_{t_0}^0}(\dot{c}_t^0 \nabla v_{t_0}^0(\varphi + \zeta) e^{-v_{t_0}^0(\varphi + \zeta)}).$$

Since $e^{-v_t^\varepsilon(\varphi)}$ is bounded below uniformly in $\varepsilon \geq 0$ by (2.101), the convergence of

$$(2.105) \quad \dot{c}_t \nabla v_t = e^{+v_t(\varphi)} \mathbf{E}_{c_t - c_{t_0}}(\partial^k \dot{c}_t \nabla v_{t_0}(\varphi + \zeta) e^{-v_{t_0}(\varphi + \zeta)})$$

then follows. \square

2.6. Proof of Theorem 2.2 for $\beta < 6\pi$. Finally, we extend the proof of Theorem 2.2 to the more subtle regime $\beta < 6\pi$. In this case, the $n = 2$ term requires a more careful treatment as it turns out that, for $\beta \geq 4\pi$,

$$(2.106) \quad \|\tilde{v}_t^{2,0}(\xi, \cdot)\|_{L^1(\Omega \times \{\pm 1\})} = \infty.$$

Indeed, explicitly by (2.15)–(2.16),

$$(2.107) \quad \begin{aligned} \tilde{v}_t(\xi_1, \xi_2) &= \int_0^t ds \dot{u}_s(\xi_1, \xi_2) \tilde{v}_s(\xi_1) \tilde{v}_s(\xi_2) e^{-\frac{1}{2} \int_s^t \dot{u}_r(\xi_1, \xi_1) - \frac{1}{2} \int_s^t \dot{u}_r(\xi_2, \xi_2) - \int_s^t dr \dot{u}_r(\xi_1, \xi_2)}, \\ &= \tilde{v}_t(\xi_1) \tilde{v}_t(\xi_2) \int_0^t ds \left(\frac{\partial}{\partial s} e^{-\int_s^t dr \dot{u}_r(\xi_1, \xi_2)} \right) = \tilde{v}_t(\xi_1) \tilde{v}_t(\xi_2) (1 - e^{-\beta \sigma_1 \sigma_2 c_t(x_1, x_2)}). \end{aligned}$$

In the neutral case $\sigma_1 + \sigma_2 = 0$, the term $e^{+\beta c_t(x_1, x_2)}$ diverges as $|x_1 - x_2|^{-\beta/2\pi}$ as $|x_1 - x_2| \rightarrow 0$, when $\varepsilon = 0$, and is therefore not integrable for $\beta \geq 4\pi$. The remedy for this is as follows. First, the *charged* part $\tilde{v}^2(\xi_1, \xi_2) \mathbf{1}_{\sigma_1 + \sigma_2 \neq 0}$ remains integrable, and we will see that it satisfies the same estimates as for $\beta < 4\pi$. Second, while (as observed above) the *neutral* part $\tilde{v}^2(\xi_1, \xi_2) \mathbf{1}_{\sigma_1 + \sigma_2 = 0}$ is not integrable, the following weaker estimates hold and we will see that these are sufficient to prove the theorem. We write

$$(2.108) \quad \delta c_t(x_1, x_2, x_3) = c_t(x_1, x_2) - c_t(x_2, x_3),$$

$$(2.109) \quad \partial^k \delta c_t(x_1, x_2, x_3) = \partial_{x_1}^k c_t(x_1, x_2) - \partial_{x_1}^k c_t(x_1, x_3),$$

where all quantities can depend on $\varepsilon \geq 0$ (as always).

Lemma 2.7. *Let $\beta < 6\pi$. For $n = 2$ and all $t > 0$, the charged part $\tilde{v}_t^2(\xi_1, \xi_2) \mathbf{1}_{\sigma_1 + \sigma_2 \neq 0}$ satisfies the bounds (2.75)–(2.76), while the neutral part $\tilde{v}_t^2(\xi_1, \xi_2) \mathbf{1}_{\sigma_1 + \sigma_2 = 0} =: \tilde{s}_t(x_1, x_2)$ satisfies the following replacement: for any $k \in \mathbb{N}$ and $\varepsilon \geq 0$,*

$$(2.110) \quad L_t^{2+k} \sup_x \|\partial_\varepsilon^k \delta c_t^\varepsilon(x, \cdot) \tilde{s}_t\|_{L^1(\Omega_\varepsilon^2)} \leq C_{\beta, k} Z_t^2 \theta_t^2,$$

$$(2.111) \quad L_t^2 \sup_x \left\| \frac{|x - \cdot|}{L_t} \tilde{s}_t^\varepsilon(x, \cdot) \right\|_{L^1(\Omega_\varepsilon)} \leq C_{\beta, k} Z_t^2.$$

Moreover, for any $t > 0$, as $\varepsilon \rightarrow 0$,

$$(2.112) \quad L_t^2 \sup_x \|E_\varepsilon \delta c_t^\varepsilon(x, \cdot) E_\varepsilon s_t^\varepsilon - \delta c_t^0(x, \cdot) \tilde{s}_t^0\|_{L^1(\Omega^2)} \rightarrow 0,$$

$$(2.113) \quad L_t^2 \sup_x \left\| \frac{|x - \cdot|}{L_t} (E_\varepsilon s_t^\varepsilon(x, \cdot) - \tilde{s}_t^0(x, \cdot)) \right\|_{L^1(\Omega)} \rightarrow 0.$$

Proof. The estimates all essentially follow from the formula (2.107) and heat kernel estimates. In more detail, for the charged part, the estimate (2.75) is derived in [7, (3.65)]. For the neutral part, the bound (2.110) is [7, (3.76)] when $k = 0$ and, as in (2.90), the generalisation to $k \in \mathbb{N}$ is exactly the same since $L_t^k \partial^k c_t$ satisfies the same estimates as c_t by (2.30). The proof of (2.111) can similarly be seen from the representation (2.107) together with [7, Lemma A.4].

For the convergence statements, we can again start from the formula (2.107). By (2.28),

$$(2.114) \quad \sup_{|x_1 - x_2| \geq \kappa} \max_{\sigma_1, \sigma_2 \in \{\pm 1\}} |\tilde{v}_t^{2, \varepsilon}(\xi_1^\varepsilon, \xi_2^\varepsilon) - \tilde{v}_t^{2, 0}(\xi_1, \xi_2)| \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

for any $\kappa > 0$. In the charged case, (2.76) follows because then

$$(2.115) \quad \sup_{\xi_1} \int_{\Omega \times \{\pm 1\}} \left(|\tilde{v}_t^{2, \varepsilon}(\xi_1^\varepsilon, \xi_2^\varepsilon)| + |\tilde{v}_t^{2, 0}(\xi_1, \xi_2)| \right) \mathbf{1}_{|x_1 - x_2| \leq \kappa} d\xi_2 \rightarrow 0 \quad (\kappa \rightarrow 0).$$

The statements for the neutral case, (2.112), and (2.113), follow similarly. \square

For $n \geq 3$, the recursive definition of $\tilde{v}(\xi_1, \dots, \xi_n)$ in (2.16) and (2.24) depends on the neutral part of \tilde{v}^2 only in the combination

$$(2.116) \quad \delta c_t(x_1, x_2, x_3) \tilde{v}_t^2(\xi_2, \xi_3) \mathbf{1}_{\xi_2 + \xi_3 = 0} = \delta c_t(x_1, x_2, x_3) \tilde{s}_t(x_2, x_3).$$

From this, one obtains the following estimates generalising Lemmas 2.5–2.6.

Lemma 2.8. *Let $\beta < 6\pi$. Then (2.75)–(2.76) hold for $n \geq 3$.*

Proof. For $n \geq 3$ and $\beta < 6\pi$, the bounds in (2.75) with $k = 0$ are shown in [7, Proposition 3.7], and again their extension with general $k \in \mathbb{N}$ is exactly the same since $L_t^k \partial^k c_t$ satisfies the same estimates as c_t . Moreover, the $\varepsilon \rightarrow 0$ convergence (2.76) for $n \geq 3$ holds by similarly arguments as in the proof of Lemma 2.5 for $\beta < 4\pi$. \square

The upshot of the above considerations is that, for all $n \geq 3$, we can define and analyse $v_t^{n,0}(\varphi)$ in exactly the same way for $\beta < 6\pi$ as for $\beta < 4\pi$. On the other hand, while the definition of $v_t^{2,0}(\varphi)$ does not make sense when $\beta \geq 4\pi$, it turns out that $v_t^{2,0}(\varphi)$ is actually well-defined up to an divergent additive constant. More precisely, the $\varepsilon \rightarrow 0$ limits of the differences $v_t^{2,\varepsilon}(\varphi + \zeta) - v_t^{2,\varepsilon}(\varphi)$ and of the φ -derivative $\dot{c}_t^\varepsilon \nabla v_t^{2,\varepsilon}(\varphi, x)$ continue to exist. Abusing notation, we will denote this limit by $v_t^{2,0}(\varphi + \zeta) - v_t^{2,0}(\varphi)$, for example, with the implicit understanding that only the difference of $v_t^{2,0}$ is defined. The following lemma then provides the required replacement of Lemma 2.6.

Lemma 2.9. *Let $\beta < 6\pi$. For $n \geq 3$ and $t > 0$, the bounds (2.84)–(2.87) continue to hold. For $n = 2$ and $t > 0$, the bounds (2.84)–(2.85) and (2.87) also continue to hold, and the following replacements for (2.83) and (2.86) hold: for any φ and ζ ,*

$$(2.117) \quad |v_t^2(\varphi + \zeta) - v_t^2(\varphi)| \lesssim |Z_t|^2(1 + L_t \|\partial\zeta\|_{L^\infty(\Omega)}),$$

and for any $t > 0$, and $\max_{x \in \Omega} |\varphi^\varepsilon(x^\varepsilon) - \varphi(x)| \rightarrow 0$ and $\max_{x \in \Omega} |\zeta^\varepsilon(x^\varepsilon) - \zeta(x)| \rightarrow 0$, as $\varepsilon \rightarrow 0$,

$$(2.118) \quad |[v_t^{2,0}(\varphi + \zeta) - v_t^{2,0}(\varphi)] - [v_t^{2,\varepsilon}(\varphi + \zeta) - v_t^{2,\varepsilon}(\varphi)]| \rightarrow 0.$$

Proof. For $n \geq 3$ and the charged part of $n = 2$, since (2.75)–(2.76) continue to hold in these cases for $\beta < 6\pi$ by Lemma 2.7–2.8, the argument is the same as that for $\beta < 4\pi$ in Lemma 2.6. It thus suffices to consider the neutral part of $v_t^2(\varphi + \zeta) - v_t^2(\varphi)$ which is given by

$$(2.119) \quad \varepsilon^4 \sum_{x_1, x_2 \in \Omega_\varepsilon} \tilde{v}_t^2((x_1, +1), (x_2, -1)) (\cos(\varphi(x_1) - \varphi(x_2) + \zeta(x_1) - \zeta(x_2)) - \cos(\varphi(x_1) - \varphi(x_2))).$$

Using that $|\cos(a + b) - \cos(a)| \leq |b|$ and (2.111), this is bounded uniformly in φ by

$$(2.120) \quad L_t^2 \sup_x \left\| \frac{x - \cdot}{L_t} \tilde{v}_t^2((x, +1), (\cdot, -1)) \right\|_{L^1(\Omega_\varepsilon)} \left[\sup_{x_1, x_2} \frac{|\zeta(x_1) - \zeta(x_2)|}{|x_1 - x_2|/L_t} \right] \lesssim |Z_t|^2 L_t \|\partial\zeta\|_{L^\infty(\Omega_\varepsilon)}.$$

This proves (2.117). The convergence (2.118) is proved analogously using (2.113). \square

Proof of Theorem 2.2 for $\beta < 6\pi$. For $t \leq t_0$, the proof is again identical to that for $\beta < 4\pi$ since (2.84)–(2.85) and (2.87) continue to hold for $n \geq 2$ by Lemma 2.9.

Thus let $t \geq t_0$. As discussed at the beginning of Section 2.6, $v_{t_0}(\varphi)$ is now divergent as $\varepsilon \rightarrow 0$, but differences $v_{t_0}(\varphi + \zeta) - v_{t_0}(\varphi)$ continue to make sense in the limit. Indeed, by (2.117) for the $n = 2$ term and (2.83) for the $n \geq 3$ terms, for any fixed $t_0 > 0$,

$$(2.121) \quad e^{-v_{t_0}(\varphi + \zeta) + v_{t_0}(\varphi)} \lesssim e^{O(\|\partial\varphi\|_{L^\infty(\Omega)})}.$$

Moreover, for any fixed $t_0 > 0$ and $t > t_0$, the estimated for v_{t_0} used above, Jensen's inequality, and $\mathbf{E}_{c_t - c_{t_0}} \|\partial\zeta\|_{L^\infty(\Omega)} \leq O_{t_0}(1)$ which holds by Lemma 2.4, imply

$$(2.122) \quad \begin{aligned} v_t(\varphi) - v_{t_0}(\varphi) &= -\log \mathbf{E}_{c_t - c_{t_0}} (e^{-v_{t_0}(\varphi + \zeta) + v_{t_0}(\varphi)}) \\ &\leq \mathbf{E}_{c_t - c_{t_0}} (v_{t_0}(\varphi + \zeta) - v_{t_0}(\varphi)) \leq O_{t_0}(1). \end{aligned}$$

Hence $e^{+v_t(\varphi) - v_{t_0}(\varphi)}$ is bounded. We then again start from (2.99), now interpreted for $\varepsilon = 0$ as

$$(2.123) \quad \mathbf{P}_{t_0, t} F(\varphi) = e^{+v_t(\varphi) - v_{t_0}(\varphi)} \mathbf{E}_{c_t - c_{t_0}} (e^{-v_{t_0}(\varphi + \zeta) + v_{t_0}(\varphi)} F(\varphi + \zeta)).$$

From this, the proofs of the bounds (2.13)–(2.14) using the already established case $t = t_0$ are identical to those in case $\beta < 4\pi$, and the convergence (2.12) is also similar from (2.50). \square

2.7. Summary of rescaling. To compare with the estimates from [7, Section 3], we here summarise how the definitions change under the rescaling $t \rightarrow t/\varepsilon^2$ and $x \rightarrow x/\varepsilon$ that we use here.

To distinguish both normalisations, we will here denote objects in unit lattice normalisation by capital letters and their versions in continuum normalisation by lower case letters. For $\xi = (x, \sigma) \in \Lambda \times \{\pm 1\}$, where Λ is a subset of the unit lattice \mathbb{Z}^2 as in [7, Section 3], below we write $\xi/\varepsilon = (x/\varepsilon, \sigma) \in \Omega \times \{\pm 1\}$, and then note the correspondence

$$(2.124) \quad c_t^\varepsilon(x_1, x_2) = C_{t/\varepsilon^2}^\varepsilon(x_1/\varepsilon, x_2/\varepsilon),$$

$$(2.125) \quad \dot{c}_t^\varepsilon(x_1, x_2) = \varepsilon^{-2} \dot{C}_{t/\varepsilon^2}^\varepsilon(x_1/\varepsilon, x_2/\varepsilon),$$

$$(2.126) \quad u_t(\xi_1, \xi_2) = U_{t/\varepsilon^2}(\xi_1/\varepsilon, \xi_2/\varepsilon),$$

$$(2.127) \quad \dot{u}_t(\xi_1, \xi_2) = \varepsilon^{-2} \dot{U}_{t/\varepsilon^2}(\xi_1/\varepsilon, \xi_2/\varepsilon),$$

$$(2.128) \quad \tilde{v}_t(\xi_1, \dots, \xi_n) = \varepsilon^{-2n} \tilde{V}_{t/\varepsilon^2}(\xi_1/\varepsilon, \dots, \xi_n/\varepsilon),$$

$$(2.129) \quad w_t(\xi_1, \dots, \xi_n) = W_{t/\varepsilon^2}(\xi_1/\varepsilon, \dots, \xi_n/\varepsilon),$$

where the functions on the right-hand sides are as in [7, Section 3], with the only difference that we write U_t instead of u_t because we reserve lower case letters for continuum objects. Moreover,

$$(2.130) \quad L_t = \varepsilon \ell_{t/\varepsilon^2} = \sqrt{t} \wedge 1/m, \quad Z_t^\varepsilon = \varepsilon^{-2} z_{t/\varepsilon^2} = z \varepsilon^{-\beta/4\pi} e^{-\beta c_t^\varepsilon(0)/2} \approx Z_t,$$

$$(2.131) \quad Z_t^\varepsilon = L_t^2 Z_t^\varepsilon \asymp \ell_{t/\varepsilon^2}^2 z_{t/\varepsilon^2}^\varepsilon = z_{t/\varepsilon^2}^\varepsilon, \quad \theta_t = \vartheta_{t/\varepsilon^2}^\varepsilon = e^{-\frac{1}{2}m^2 t}.$$

Note that under the identification of functions on Λ^n with functions on Ω^n that are constant on squares of side length ε , the norms $\|\cdot\|$ used in [7, Section 3], i.e.,

$$(2.132) \quad \|F\| = \max_{\xi_1} \sum_{\xi_2, \dots, \xi_n} |F(\xi_1, \dots, \xi_n)|,$$

for functions $F : (\Lambda \times \{\pm 1\})^n \rightarrow \mathbb{R}$ can be written as

$$(2.133) \quad \begin{aligned} \varepsilon^{2(n-1)} \|F\| &= \sup_{\xi_1} \int_{(\Omega_\varepsilon \times \{\pm 1\})^{n-1}} d\xi_2 \dots d\xi_{n-1} |F(\xi_1, \dots, \xi_n)| \\ &= \sup_{\xi_1} \|F(\xi_1, \cdot)\|_{L^1((\Omega_\varepsilon \times \{\pm 1\})^{n-1})}. \end{aligned}$$

3 Coupling of the sine-Gordon field with the GFF

Using the estimates for the renormalised potential given in Section 2, we now couple the sine-Gordon field with the GFF. The main results of this section, Theorems 3.1–3.4 below, immediately imply Theorem 1.2 but are stronger in an important way exploited in Section 4.

3.1. Decomposed Gaussian free field. For the construction of the decomposed lattice GFF on Ω_ε , let $(W^\varepsilon(x))_{x \in \Omega_\varepsilon}$ be independent Brownian motions $W^\varepsilon(x) = (W_t^\varepsilon(x))_{t \geq 0}$ with quadratic variations $t/\varepsilon^d = t/\varepsilon^2$. In other words, W^ε is a standard Brownian motion with values in X_ε equipped with the inner product (2.6). In terms of the (discrete) heat operator $\dot{c}_t^\varepsilon = e^{t\Delta_\varepsilon - m^2 t}$ from (2.2), we then define the decomposed lattice GFF by

$$(3.1) \quad \Phi_t^{\text{GFF}_\varepsilon} = \int_t^\infty q_u^\varepsilon dW_u^\varepsilon, \quad \text{where } q_t^\varepsilon = \dot{c}_{t/2}^\varepsilon.$$

In particular, $\Phi_0^{\text{GFF}_\varepsilon}$ is then a realisation of the massive Gaussian free field on Ω_ε , i.e., a Gaussian field with covariance $\int_0^\infty \dot{c}_t^\varepsilon dt = (-\Delta_\varepsilon + m^2)^{-1}$.

The decomposed continuum GFF is constructed analogously. Let W now be a cylindrical Brownian motion in $L^2(\Omega)$ defined on some probability space. Thus, $W_t = \sum_{k \in 2\pi\mathbb{Z}^2} e^{ik \cdot (\cdot)} \hat{W}_t(k)$

where the $\hat{W}(k)$ are independent complex standard Brownian motions subject to $\hat{W}(k) = \overline{\hat{W}(-k)}$ for $k \neq 0$ and $\hat{W}(0)$ is a real standard Brownian motion, and where the sum over $k \in 2\pi\mathbb{Z}^2$ converges in $C([0, \infty), H^{-d/2-\kappa}(\Omega))$.

The decomposed continuum GFF $(\Phi_t^{\text{GFF}_0})_{t \geq 0}$ is then defined analogously as in (3.1), by replacing the lattice heat kernel \check{c}^ε by its continuum counterpart \check{c}^0 defined in (2.10), i.e.,

$$(3.2) \quad \Phi_t^{\text{GFF}_0} = \int_t^\infty q_u^0 dW_u \equiv \sum_{k \in 2\pi\mathbb{Z}^2} e^{ik \cdot (\cdot)} \int_t^\infty \hat{q}_u^0(k) d\hat{W}_u(k), \quad \text{where } q_t^0 = \check{c}_{t/2}^0.$$

The process Φ^{GFF_0} takes values in $C([0, \infty), H^{-\kappa}(\Omega))$ for any $\kappa > 0$ almost surely since $\int_0^\infty \sum_k (1+|k|^2)^{-\kappa} |\hat{q}_u^0(k)|^2 du < \infty$, and, for any $t_0 > 0$, also $\Phi^{\text{GFF}_0} \in C([t_0, \infty), C^\infty(\Omega))$ almost surely.

To take the limit $\varepsilon \rightarrow 0$, it will be convenient to couple the Brownian motions $W^\varepsilon(x)$ for all $\varepsilon > 0$ to W in the following standard way. Recall that any $f: \Omega_\varepsilon \rightarrow \mathbb{R}$ has the Fourier representation $f(x) = \sum_{k \in \Omega_\varepsilon^*} \hat{f}(k) e^{ik \cdot x}$ where $\Omega_\varepsilon^* = \{k \in 2\pi\mathbb{Z}^2 : -\pi/\varepsilon < k_i \leq \pi/\varepsilon\}$. For $x \in \Omega_\varepsilon$, we set $W_t^\varepsilon(x) = \Pi_\varepsilon W_t(x)$ with Π_ε the restriction to the Fourier coefficients in Ω_ε^* , i.e.,

$$(3.3) \quad W_t^\varepsilon(x) = \Pi_\varepsilon W_t(x) = \sum_{k \in \Omega_\varepsilon^*} e^{ik \cdot x} \hat{W}_t(k), \quad x \in \Omega_\varepsilon.$$

Then $(W^\varepsilon(x))_{x \in \Omega_\varepsilon}$ are independent Brownian motions indexed by Ω_ε with quadratic variation t/ε^2 . Indeed, clearly W^ε is a Gaussian process, and its covariance is

$$(3.4) \quad \begin{aligned} \mathbb{E}[W_t^\varepsilon(x) W_s^\varepsilon(y)] &= \sum_{k \in \Omega_\varepsilon^*} \sum_{k' \in \Omega_\varepsilon^*} \mathbb{E}[\hat{W}_t(k) \hat{W}_s(-k')] e^{i(k \cdot x - k' \cdot y)} \\ &= \sum_{k \in \Omega_\varepsilon^*} (s \wedge t) e^{ik \cdot (x-y)} = \varepsilon^{-2} (s \wedge t) \mathbf{1}_{x=y}. \end{aligned}$$

On the last line, we used that the Fourier series of $\mathbf{1}_{x_0}: \Omega_\varepsilon \rightarrow \mathbb{R}$ is given by

$$(3.5) \quad \mathbf{1}_{x_0}(x) = \sum_{k \in \Omega_\varepsilon^*} \langle \mathbf{1}_{x_0}, e^{ik \cdot (\cdot)} \rangle_{\Omega_\varepsilon} e^{ik \cdot x} = \varepsilon^2 \sum_{k \in \Omega_\varepsilon^*} e^{ik(x-x_0)}.$$

Note also that the lattice field $\Phi_t^{\text{GFF}_\varepsilon}$ admits a Fourier representation analogous to (3.2) using the Brownian motions $W^\varepsilon(x)$:

$$(3.6) \quad \Phi_t^{\text{GFF}_\varepsilon} = \sum_{k \in \Omega_\varepsilon^*} e^{ik \cdot (\cdot)} \int_t^\infty \hat{q}_u^\varepsilon(k) d\hat{W}_u(k), \quad \text{where } q_t^\varepsilon = \check{c}_{t/2}^\varepsilon.$$

We will use the above coupling from now on. The associated forward and backward filtrations (\mathcal{F}_t) and (\mathcal{F}^t) are defined by completing the σ -algebras generated by the past $\{W_s - W_0 : s \leq t\}$ respectively the future $\{W_s - W_t : s \geq t\}$. We emphasise that the processes $\Phi^{\text{GFF}} = (\Phi_t^{\text{GFF}})_{t > 0}$ (with either $\varepsilon > 0$ or $\varepsilon = 0$) are adapted to the *backward* filtration (\mathcal{F}^t) .

3.2. Decomposed sine-Gordon field. The following four theorems are the main results of Section 3. These all refer to the decomposed GFF from (3.1)–(3.2) and the renormalised potential v_t^ε from (2.4) when $\varepsilon > 0$ and Theorem 2.2 when $\varepsilon = 0$.

The first theorem shows the well-posedness of the (backward) SDEs that we subsequently show to construct the sine-Gordon field. In its statement and afterwards, C_0 denotes the space of continuous functions vanishing at infinity.

Theorem 3.1. *For $\varepsilon > 0$, there is a unique \mathcal{F}^t -adapted process $\Phi^{\text{SG}_\varepsilon} \in C([0, \infty), X_\varepsilon)$ such that*

$$(3.7) \quad \Phi_t^{\text{SG}_\varepsilon} = - \int_t^\infty \check{c}_u^\varepsilon \nabla v_u^\varepsilon(\Phi_u^{\text{SG}_\varepsilon}) du + \Phi_t^{\text{GFF}_\varepsilon}.$$

Analogously, there is a unique \mathcal{F}^t -adapted process Φ^{SG_0} with $\Phi^{\text{SG}_0} - \Phi^{\text{GFF}_0} \in C_0([0, \infty), C(\Omega))$ such that

$$(3.8) \quad \Phi_t^{\text{SG}_0} = - \int_t^\infty \dot{c}_u^0 \nabla v_u^0(\Phi_u^{\text{SG}_0}) du + \Phi_t^{\text{GFF}_0}.$$

In particular, for any $t > 0$, $\Phi_0^{\text{GFF}} - \Phi_t^{\text{GFF}}$ is independent of Φ_t^{SG} (in either case).

The next theorem shows that $\Phi_0^{\text{SG}_\varepsilon}$ is really the sine-Gordon field (1.1) when $\varepsilon > 0$.

Theorem 3.2. *Let $\varepsilon > 0$. Then $\Phi_0^{\text{SG}_\varepsilon}$ is distributed as the sine-Gordon field on Ω_ε defined in (1.1).*

For $\varepsilon = 0$, the sine-Gordon field has no direct definition and is instead defined as the $\varepsilon \rightarrow 0$ limit of the regularised fields. Thus, by showing that $\Phi^{\text{SG}_\varepsilon} \rightarrow \Phi^{\text{SG}_0}$, the next theorem justifies that Φ^{SG_0} is the continuum sine-Gordon field.

We denote the difference of the GFF and the sine-Gordon field under the coupling of Theorem 3.1 by

$$(3.9) \quad \Phi_t^\Delta := \Phi_t^{\text{SG}} - \Phi_t^{\text{GFF}} = - \int_t^\infty \dot{c}_u \nabla v_u(\Phi_u^{\text{SG}}) du.$$

Theorem 3.3. *Under the couplings of Theorems 3.1 (with the same Brownian motion for lattice and continuum versions as in Section 3.1), for any $t_0 > 0$,*

$$(3.10) \quad \mathbb{E} \sup_{t \geq t_0} \|\Phi_t^{\Delta_\varepsilon} - \Phi_t^{\Delta_0}\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

In particular, for any $t \geq 0$, the lattice field $\Phi_t^{\text{SG}_\varepsilon}$ converges weakly to $\Phi_t^{\text{SG}_0}$ in $H^{-\kappa}(\Omega)$ as $\varepsilon \downarrow 0$, when $\kappa > 0$, where we have identified $\Phi_t^{\text{SG}_\varepsilon}$ with the element of $C^\infty(\Omega)$ with the same Fourier coefficients for $k \in \Omega_\varepsilon^*$ and vanishing Fourier coefficients for $k \notin \Omega_\varepsilon^*$.

Finally, with the estimates for $\dot{c}_t \nabla v_t$ from Theorem 2.2, we obtain the following strong estimates for the difference of the GFF and the sine-Gordon field.

Theorem 3.4. *The following estimates hold for $\varepsilon > 0$ or $\varepsilon = 0$ with all constants deterministic and independent of ε . For any $t \geq 0$, the difference field Φ^Δ satisfies the bound*

$$(3.11) \quad \max_x |\Phi_0^\Delta(x) - \Phi_t^\Delta(x)| \leq O_\beta(|Z_t|),$$

as well as the following Hölder continuity estimates:

$$(3.12) \quad \begin{aligned} \max_x |\Phi_t^\Delta(x)| + \max_x |\partial \Phi_t^\Delta(x)| + \max_{x,y} \frac{|\partial \Phi_t^\Delta(x) - \partial \Phi_t^\Delta(y)|}{|x-y|^{1-\beta/4\pi}} &\leq O_\beta(|z|), & (0 < \beta < 4\pi), \\ \max_x |\Phi_t^\Delta(x)| + \max_{x,y} \frac{|\Phi_t^\Delta(x) - \Phi_t^\Delta(y)|}{|x-y|(1+|\log|x-y||)} &\leq O_\beta(|z|), & (\beta = 4\pi), \\ \max_x |\Phi_t^\Delta(x)| + \max_{x,y} \frac{|\Phi_t^\Delta(x) - \Phi_t^\Delta(y)|}{|x-y|^{2-\beta/4\pi}} &\leq O_\beta(|z|), & (4\pi \leq \beta < 6\pi). \end{aligned}$$

In addition, for any $t > 0$ and $k \in \mathbb{N}$, the following cruder bounds on all derivatives hold:

$$(3.13) \quad L_t^k \|\partial^k \Phi_t^\Delta\|_{L^\infty(\Omega)} \leq O_{\beta,k}(|z|),$$

and in particular $\Phi_t^\Delta \in C^\infty(\Omega)$ for any $t > 0$.

In the remainder of Section 3, we prove the above four theorems. Theorem 1.2 is then an immediate consequence.

Proof of Theorem 1.2. By Theorem 3.3, the lattice approximation to the sine-Gordon field converges to the solution of (3.8). The stated estimates are immediate from Theorem 3.4. \square

3.3. Well-posedness of SDEs: proof of Theorem 3.1. We first show that the SDEs (3.7) and (3.8) that construct the sine-Gordon field are well-posed. This is essentially the standard argument for SDEs and uses the Lipschitz continuity of $\dot{c}_s \nabla v_s$ given by Theorems 2.1 and 2.2.

Proof of Theorem 3.1. The same argument applies to $\varepsilon > 0$ and to $\varepsilon = 0$, but for concreteness, we will discuss the case $\varepsilon = 0$. For $D, E \in C([t, \infty), C(\Omega))$, let

$$(3.14) \quad F_t^0(D, E) = - \int_t^\infty \dot{c}_s^0 \nabla v_s^0(E_s + D_s) ds.$$

By (2.14), the following estimate holds uniformly in $E \in C([t, \infty), C(\Omega))$:

$$(3.15) \quad \|F_t^0(D, E) - F_t^0(\tilde{D}, E)\|_{L^\infty(\Omega)} \leq \int_t^\infty O(\theta_s |Z_s|) \|D_s - \tilde{D}_s\|_{L^\infty(\Omega)} ds.$$

Using that $0 < \beta < 8\pi$, it follows from (2.7) that

$$(3.16) \quad \int_0^t \theta_s |Z_s| ds \leq |z| \int_0^t L_s^{-\beta/4\pi} ds \leq O_\beta(|z| L_t^{-\beta/4\pi}) = O_\beta(|Z_t|), \quad (t \leq 1/m^2),$$

$$(3.17) \quad \int_0^\infty \theta_s |Z_s| ds \leq O_\beta(|Z_{1/m^2}|) + |Z_{1/m^2}| \int_{1/m^2}^\infty \theta_s ds \leq O_\beta(|z| m^{-2+\beta/4\pi}) = O_{\beta,m}(|z|),$$

where on the second line we used that Z_t is decreasing in t and that $\theta_t = e^{-\frac{1}{2}m^2 t}$.

Suppose first that there are two solutions Φ^{SG} and $\tilde{\Phi}^{\text{SG}}$ to (3.8) satisfying $D := \Phi^{\text{SG}} - \Phi^{\text{GFF}} \in C_0([t_0, \infty), L^\infty(\Omega))$ and $\tilde{D} := \tilde{\Phi}^{\text{SG}} - \Phi^{\text{GFF}} \in C_0([t_0, \infty), L^\infty(\Omega))$. Then by (3.15),

$$(3.18) \quad \begin{aligned} \|D_t - \tilde{D}_t\|_{L^\infty(\Omega)} &= \|F_t^0(D, \Phi^{\text{GFF}}) - F_t^0(\tilde{D}, \Phi^{\text{GFF}})\|_{L^\infty(\Omega)} \\ &\leq \int_t^\infty O_{\beta,m}(\theta_s |Z_s|) \|D_s - \tilde{D}_s\|_{L^\infty(\Omega)} ds. \end{aligned}$$

Thus $f(t) = \|D_t - \tilde{D}_t\|_{L^\infty(\Omega)}$ is bounded with $f(t) \rightarrow 0$ as $t \rightarrow \infty$ and it satisfies

$$(3.19) \quad f(t) \leq a + \int_t^\infty O_{\beta,m}(\theta_s |Z_s|) f(s) ds, \quad \text{with } a = 0.$$

Since $\int_t^\infty O(\theta_s |Z_s|) ds < \infty$ by (3.17) a version of Gronwall's inequality (which is derived in the same way as the standard version) implies that

$$(3.20) \quad f(t) \leq a \exp\left(\int_t^\infty O_\beta(\theta_s |Z_s|) ds\right) = 0.$$

Hence $D = \tilde{D}$.

That a solution as above exists also follows by the standard argument for SDEs, i.e., by Picard iteration. Let $\|D\|_t = \sup_{s \geq t} \|D_s\|_{L^\infty(\Omega)}$. For any $t > 0$ and $E \in C([t, \infty), C(\Omega))$ fixed, set $D^0 = 0$ and $D^{n+1} = F(D^n, E)$. Using (3.17) then

$$(3.21) \quad \|D^1\|_t = \|F^0(0)\|_t \leq \int_t^\infty ds O_{\beta,m}(\theta_s |Z_s|) = O_{\beta,m}(|z|),$$

$$(3.22) \quad \|D^{n+1} - D^n\|_t \leq \int_t^\infty ds O_{\beta,m}(\theta_s |Z_s|) \|D^n - D^{n-1}\|_s,$$

and from the elementary identity

$$(3.23) \quad \int_t^\infty ds g(s) \left(\int_s^\infty ds' g(s')\right)^{k-1} = \frac{1}{k} \left(\int_t^\infty ds g(s)\right)^k$$

applied with $g(s) = O_{\beta,m}(\theta_s|Z_s|)$, we conclude that

$$(3.24) \quad \|D^{n+1} - D^n\|_t \leq \frac{1}{n!} (O_{\beta,m}(|z|))^n.$$

Since the right-hand side is summable, we have that $D^n \rightarrow D$ for some $D = D^*(E) \in C_0([t, \infty), C(\Omega))$ in $\|\cdot\|_t$, and the limit satisfies $F_s^0(D^*(E), E) = D^*(E)_s$ for $s \geq t$. By uniqueness, $D^*(E)_t$ is consistent in t and we can thus define $D^*(E) \in C_0((0, \infty), L^\infty(\Omega))$. In summary, $\Phi_t^{\text{SG}} = D^*(\Phi^{\text{GFF}})_t + \Phi_t^{\text{GFF}}$ is the desired solution for $t > 0$. Further noticing that, for $t > 0$,

$$(3.25) \quad D^*(\Phi^{\text{GFF}})_t = - \int_t^\infty \dot{c}_s \nabla v_s(\Phi_s^{\text{SG}}) ds,$$

we may extend the solution to $t = 0$ by continuity. Indeed, by (3.16), as $t \rightarrow 0$,

$$(3.26) \quad \left\| \int_0^t \dot{c}_s \nabla v_s(\Phi_s^{\text{SG}}) ds \right\|_{L^\infty(\Omega)} \leq \int_0^t O_\beta(\theta_s|Z_s|) ds \leq O_\beta(|Z_t|) \rightarrow 0,$$

and thus the limit $\lim_{t \downarrow 0} D^*(\Phi^{\text{GFF}})_t$ exists in $C(\Omega)$. \square

3.4. Coupling of fields: proof of Theorem 3.2. We next show that (3.7) defines a realisation of the sine-Gordon field on Ω_ε , thus proving Theorem 3.2.

Proof of Theorem 3.2. Throughout this proof, $\varepsilon > 0$ is fixed and dropped from the notation. We will show that the sine-Gordon field (1.1) can be constructed from the Polchinski semigroup as follows. Let ν_t be the renormalised measure defined by $\mathbb{E}_{\nu_t} f = e^{v_t(0)} \mathbf{E}_{c_t}(e^{-v_t(\zeta)} f(\zeta))$ where \mathbf{E}_{c_t} is the expectation of the Gaussian measure with covariance c_t and v_t is as in Section 2.1. In [7, Section 2], it is shown that for every bounded and smooth function $f : X_\varepsilon \rightarrow \mathbb{R}$,

$$(3.27) \quad \mathbb{E}_{\nu_0} f = \mathbb{E}_{\nu_t} \mathbf{P}_{0,t} f,$$

where

$$(3.28) \quad \mathbf{P}_{s,t} f(\phi) = e^{v_t(\phi)} \mathbf{E}_{c_t - c_s}(e^{-v_s(\phi + \zeta)} f(\phi + \zeta)).$$

By [7, Proposition 2.1], this semigroup is characterised by its infinitesimal generator

$$(3.29) \quad \mathbf{L}_t = \frac{1}{2} \Delta_{\dot{c}_t} - (\nabla v_t, \nabla)_{\dot{c}_t},$$

in the sense that

$$(3.30) \quad \frac{\partial}{\partial s} \mathbf{P}_{s,t} f = -\mathbf{P}_{s,t} \mathbf{L}_s f, \quad \frac{\partial}{\partial t} \mathbf{P}_{s,t} f = +\mathbf{L}_t \mathbf{P}_{s,t} f.$$

Therefore, as in [7, Remark 2.2], the semigroup $\mathbf{P}_{s,t}$ has the following stochastic representation. (As in Section 2 note that we have here rescaled t by $1/\varepsilon^2$ compared to [7] both in the definition of the renormalised measure and the Polchinski semigroup; see Section 2.7 for a translation of these normalisations.) Let (W_t) be an X_ε -valued Brownian motion normalised so that $(W_t(x))$ is a Brownian motion with quadratic variation t/ε^2 for each $x \in \Omega_\varepsilon$, with completed forward and backward filtrations (\mathcal{F}_t) and (\mathcal{F}^t) . For any $T > 0$, the time-reversed process $\tilde{W}_t^T = W_T - W_{T-t}$ is then again a Brownian motion. Let $\tilde{\Phi}^T$ be the unique strong solution to the following (forward) SDE with Lipschitz coefficients:

$$(3.31) \quad d\tilde{\Phi}_t^T = -\dot{c}_{T-t} \nabla v_{T-t}(\tilde{\Phi}_t^T) dt + q_{T-t} d\tilde{W}_t^T, \quad (0 \leq t \leq T).$$

By Itô's formula, we see that

$$(3.32) \quad \frac{\partial}{\partial t} \mathbb{E}_{\tilde{\Phi}_0^T = \phi} f(\tilde{\Phi}_t^T) = \mathbb{E}_{\tilde{\Phi}_0^T = \phi} (L_{T-t} f(\tilde{\Phi}_t^T)).$$

Therefore $t \mapsto \mathbb{E}_{\tilde{\Phi}_0^T = \phi} f(\tilde{\Phi}_t^T)$ defines an (inhomogeneous) Markov semigroup with generator L_{T-t} and uniqueness of such semigroups implies $\mathbf{P}_{T-t, T} f(\phi) = \mathbb{E}_{\tilde{\Phi}_0^T = \phi} f(\tilde{\Phi}_t^T)$. Indeed, denoting the last right-hand side by $\tilde{\mathbf{P}}_{T-t, T} f(\phi)$, for any bounded smooth function f , one has

$$(3.33) \quad \begin{aligned} \tilde{\mathbf{P}}_{T-t, T} f - \mathbf{P}_{T-t, T} f &= \int_0^t \frac{\partial}{\partial s} (\tilde{\mathbf{P}}_{T-s, T} \mathbf{P}_{T-t, T-s} f) ds \\ &= \int_0^t (\tilde{\mathbf{P}}_{T-s, T} (L_{T-s} - L_{T-s}) \mathbf{P}_{T-t, T-s} f) ds = 0. \end{aligned}$$

In particular,

$$(3.34) \quad \mathbf{P}_{0, T} f(\phi) = \mathbb{E}_{\tilde{\Phi}_0^T = \phi} f(\tilde{\Phi}_T^T).$$

We now reverse the time direction. We thus set $\Phi_t^T = \tilde{\Phi}_{T-t}^T$ so that, with the change of variable $s \mapsto T - s$ on the second line of the following display,

$$(3.35) \quad \begin{aligned} \Phi_t^T &= \Phi_T^T - \int_0^{T-t} \dot{c}_{T-s} \nabla v_{T-s}(\tilde{\Phi}_s^T) ds + \int_0^{T-t} q_{T-s} d\tilde{W}_s^T \\ &= \Phi_T^T - \int_t^T \dot{c}_s \nabla v_s(\Phi_s^T) ds + \int_t^T q_s dW_s. \end{aligned}$$

Then (Φ_t^T) is adapted to the descending filtration \mathcal{F}^t , and

$$(3.36) \quad \mathbf{P}_{0, T} f(\phi) = \mathbb{E}_{\Phi_0^T = \phi} f(\Phi_0^T).$$

In particular, by (3.27), if Φ_T^T is distributed according to ν_T then Φ_0^T is distributed according to the sine-Gordon measure $\nu_0 = \nu^{\text{SG}_\varepsilon}$.

It therefore suffices to show that, as $T \rightarrow \infty$, the solution (3.35) with Φ_T^T distributed according to ν_T converges to the solution $(\Phi_t)_{t \geq 0}$ to (3.7) constructed using the same Brownian motion W . This will essentially follow from the fact that $\nu_T \rightarrow \delta_0$. Let $\Phi_\infty = 0$. We start from

$$(3.37) \quad \Phi_t - \Phi_t^T = (\Phi_\infty - \Phi_T^T) - \int_t^T [\dot{c}_s \nabla v_s(\Phi_s) - \dot{c}_s \nabla v_s(\Phi_s^T)] ds + \int_T^\infty \dot{c}_s \nabla v_s(\Phi_s) ds + \int_T^\infty q_s dW_s.$$

Since $\varepsilon > 0$ is fixed, we may use any norm on X_ε and denote it by $\|\cdot\|$. The first, third, and fourth terms on the right-hand side above are independent of t and we claim that they converge to 0 in probability as $T \rightarrow \infty$. For the first term this follows from the weak convergence of the measure ν_T to δ_0 , e.g., in the sense of [7, (1.6)]. By (2.8), the third term is bounded by

$$(3.38) \quad \int_T^\infty \|\dot{c}_s \nabla v_s(\Phi_s)\| ds \leq \int_T^\infty O_{\beta, m}(\theta_s Z_s) ds \rightarrow 0.$$

The fourth term is a Gaussian field on Ω_ε with covariance matrix $c_\infty - c_T \rightarrow 0$ as $T \rightarrow \infty$. Since Ω_ε is finite, it is a trivial consequence that this Gaussian field converges to 0.

In summary, we have shown that there is R_T such that $\|R_T\| \rightarrow 0$ in probability, and

$$(3.39) \quad \Phi_t - \Phi_t^T = - \int_t^T [\dot{c}_s \nabla v_s(\Phi_s) - \dot{c}_s \nabla v_s(\Phi_s^T)] ds + R_T.$$

By the Lipschitz continuity of $\dot{c}_s \nabla v_s$, see (2.8), with $M_t = O_\beta(\theta_t |Z_t|) + O_{\beta, m, z}(\theta_t \mathbf{1}_{t \geq t_0})$,

$$(3.40) \quad \int_t^T \|\dot{c}_s \nabla v_s(\Phi_s) - \dot{c}_s \nabla v_s(\Phi_s^T)\| ds \leq \int_t^T M_s \|\Phi_s - \Phi_s^T\| ds.$$

Thus we have shown that $D_t = \Phi_t - \Phi_t^T$ satisfies

$$(3.41) \quad \|D_t\| \leq \|R_T\| + \int_t^T M_s \|D_s\| ds.$$

Since $\int_0^\infty M_s ds < \infty$, the same version of Gronwall's inequality as in the proof of Theorem 3.1 implies that

$$(3.42) \quad \|D_t\| \leq \|R_T\| \exp\left(\int_t^T M_s ds\right) \leq \|R_T\| \exp\left(\int_0^\infty M_s ds\right) \lesssim \|R_T\|.$$

This bound is uniform in t and hence $\sup_{t \in [0, T]} \|D_t\| \rightarrow 0$ in probability as $T \rightarrow \infty$. \square

3.5. Lattice convergence: proof of Theorem 3.3. We now prove that $\Phi^{\text{SG}_\varepsilon} \rightarrow \Phi^{\text{SG}_0}$ as $\varepsilon \rightarrow 0$. To this end, we will need the following crude estimates for the convergence of the decomposed GFF. Let $I_\varepsilon : L^2(\Omega_\varepsilon) \hookrightarrow L^2(\Omega)$ be the standard isometric embedding, i.e., given $f : \Omega_\varepsilon \rightarrow \mathbb{R}$, the function $I_\varepsilon f \in C^\infty(\Omega)$ is defined to have the same Fourier coefficients as f for $k \in \Omega_\varepsilon^*$ and to have vanishing Fourier coefficients for $k \in \Omega^* \setminus \Omega_\varepsilon^*$.

Lemma 3.5. *For any $t_0 > 0$, and $1 \leq p < \infty$,*

$$(3.43) \quad \mathbb{E} \left[\sup_{t \geq t_0} \|\partial \Phi_t^{\text{GFF}}\|_{L^\infty(\Omega)}^p \right] \leq O_{p, t_0}(1)$$

$$(3.44) \quad \mathbb{E} \left[\sup_{t \geq t_0} \|\Phi_t^{\text{GFF}_\varepsilon} - \Phi_t^{\text{GFF}_0}\|_{L^\infty(\Omega_\varepsilon)}^p \right] \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

Moreover, with I_ε the isometric embedding $L^2(\Omega_\varepsilon) \hookrightarrow L^2(\Omega)$ as above, for any $t \geq 0$ and $\kappa > 0$,

$$(3.45) \quad \mathbb{E} \left[\|I_\varepsilon \Phi_t^{\text{GFF}_\varepsilon} - \Phi_t^{\text{GFF}_0}\|_{H^{-\kappa}(\Omega)}^2 \right] \rightarrow 0.$$

Proof. Similarly to the proof of Lemma 2.4, we will use the Fourier coefficients of $\Phi_t^{\text{GFF}_\varepsilon}$ and $\Phi_t^{\text{GFF}_0}$ to prove the statements. Let q_t^ε and q_t^0 be as in (3.1) and (3.2). Then as in (2.58)–(2.59),

$$(3.46) \quad 0 \leq \hat{q}_t^0(k) = e^{-(t/2)(|k|^2+m^2)}, \quad 0 \leq \hat{q}_t^\varepsilon(k) \leq e^{-(t/2)(\frac{1}{4}|k|^2+m^2)},$$

and, similarly, with h as below (2.56),

$$(3.47) \quad \begin{aligned} 0 \leq \hat{q}_t^\varepsilon - \hat{q}_t^0 &= e^{(t/2)(\hat{\Delta}^\varepsilon(k)-m^2)} (1 - e^{(t/2)(-\hat{\Delta}^0(k)+\hat{\Delta}^\varepsilon(k))}) \\ &\leq e^{-(t/2)(\frac{1}{4}|k|^2+m^2)} (t/2)|k|^2 h(\varepsilon k). \end{aligned}$$

By the Sobolev inequality $\|f\|_{L^\infty(\Omega_\varepsilon)} \leq C_\alpha \|f\|_{H^\alpha(\Omega_\varepsilon)}$, $\alpha > d/2$, which holds for $\varepsilon = 0$ and $\varepsilon > 0$, it suffices to prove (3.43)–(3.44) with the norms replaced by H^α norms for α large enough.

For any $\alpha \in \mathbb{R}$ and $t_0 > 0$, the field $(\Phi_t^{\text{GFF}})_{t \geq t_0}$ is an $H^\alpha(\Omega)$ -valued backward martingale with respect to the backward filtration $\mathcal{F}^t = \sigma(W_u : u \geq t)$ and quadratic variation

$$(3.48) \quad \int_t^\infty \|q_u\|_{H^\alpha(\Omega)}^2 du = \sum_{k \in \Omega^*} (1 + |k|^2)^\alpha \int_t^\infty |\hat{q}_u(k)|^2 du \leq O_{t_0, \alpha}(1),$$

where we used (2.58)–(2.59) to obtain the uniform bound on the right-hand side.

Thus Burkholder's inequality (for Hilbert space valued martingales, see, e.g., [42]) implies

$$(3.49) \quad \mathbb{E} \left[\sup_{t \geq t_0} \|\Phi_t^{\text{GFF}}\|_{H^\alpha(\Omega)}^p \right] \leq O_{t_0, \alpha, m, p}(1).$$

To prove (3.44) we use a similar argument together with the Fourier representations (3.2) and (3.6). Observe that

$$(3.50) \quad \|\Phi_t^{\text{GFF}\varepsilon} - \Phi_t^{\text{GFF}0}\|_{L^\infty(\Omega_\varepsilon)}^p \lesssim \|\Phi_t^{\text{GFF}\varepsilon} - \Pi_\varepsilon \Phi_t^{\text{GFF}0}\|_{L^\infty(\Omega_\varepsilon)}^p + \|\Pi_\varepsilon \Phi_t^{\text{GFF}0} - \Phi_t^{\text{GFF}0}\|_{L^\infty(\Omega)}^p.$$

Similarly as above the processes $(\Phi_t^{\text{GFF}\varepsilon} - \Pi_\varepsilon \Phi_t^{\text{GFF}0})_{t \geq t_0}$ and $(\Pi_\varepsilon \Phi_t^{\text{GFF}0} - \Phi_t^{\text{GFF}0})_{t \geq t_0}$ are backward martingales taking values in $H^\alpha(\Omega_\varepsilon)$ and $H^\alpha(\Omega)$ with quadratic variations

$$(3.51) \quad [\Phi_t^{\text{GFF}\varepsilon} - \Pi_\varepsilon \Phi_t^{\text{GFF}0}]_t = \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^\alpha \int_t^\infty |\hat{q}_u^\varepsilon(k) - \hat{q}_u^0(k)|^2 du$$

$$(3.52) \quad [\Pi_\varepsilon \Phi_t^{\text{GFF}0} - \Phi_t^{\text{GFF}0}]_t = \sum_{k \in \Omega^* \setminus \Omega_\varepsilon^*} (1 + |k|^2)^\alpha \int_t^\infty |\hat{q}_u^0(k)|^2 du.$$

Since both converge to 0 as $\varepsilon \rightarrow 0$, the claim follows again by Burkholder's inequality for Hilbert space valued martingales.

Finally, we prove (3.45). By the definition of the Sobolev norm, we have

$$(3.53) \quad \|I_\varepsilon \Phi_t^{\text{GFF}\varepsilon} - \Phi_t^{\text{GFF}0}\|_{H^{-\kappa}(\Omega)}^2 = \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^{-\kappa} \left| \int_t^\infty (\hat{q}_u^\varepsilon(k) - \hat{q}_u^0(k)) d\hat{W}_u(k) \right|^2 \\ + \sum_{k \in \Omega^* \setminus \Omega_\varepsilon^*} (1 + |k|^2)^{-\kappa} \left| \int_t^\infty \hat{q}_u^0(k) d\hat{W}_u(k) \right|^2.$$

Taking expectation the corresponding second sum on the right-hand side is finite for $\kappa > 0$ and hence converges to 0 as $\varepsilon \rightarrow 0$.

For the first sum on the right-hand side, we may of course assume that $\kappa \leq 4$. Then $h(\varepsilon k)^2 \leq O(\varepsilon|k|)^\kappa$ and the expectation of the first sum is bounded by

$$(3.54) \quad \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^{-\kappa} \int_0^\infty |\hat{q}_u^\varepsilon(k) - \hat{q}_u^0(k)|^2 du \leq \frac{1}{4} \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^{-\kappa} |k|^4 h^2(\varepsilon k) \int_0^\infty e^{-t(\frac{1}{4}|k|^2 + m^2)} t^2 dt \\ \lesssim \varepsilon^\kappa \sum_{k \in \Omega_\varepsilon^*} (1 + |k|^2)^{-\kappa} |k|^{4+\kappa} \frac{1}{(|k|^2 + m^2)^3} \lesssim \varepsilon^\kappa$$

as needed. \square

Proof of Theorem 3.3. We will first show (3.10), i.e., that for any fixed $t_0 > 0$,

$$(3.55) \quad \sup_{t \geq t_0} \|\Phi_t^{\Delta\varepsilon} - \Phi_t^\Delta\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0 \quad \text{in } L^1(\mathbb{P}) \text{ as } \varepsilon \downarrow 0.$$

To this end, define F^0 as in (3.14) and F^ε analogously with $\hat{c}_t^0 \nabla v_t^0$ replaced by $\hat{c}_t^\varepsilon \nabla v_t^\varepsilon$. Let $D = \Phi^\Delta$ be the corresponding fixed points with $E = \Phi^{\text{GFF}}$, i.e.,

$$(3.56) \quad \Phi^{\Delta\varepsilon} = F^\varepsilon(\Phi^{\Delta\varepsilon}, \Phi^{\text{GFF}\varepsilon}), \quad \Phi^{\Delta 0} = F^0(\Phi^{\Delta 0}, \Phi^{\text{GFF}0}).$$

Then (again identifying $\Phi_t^{\Delta 0}$ with its restriction to Ω_ε)

$$(3.57) \quad \Phi^{\Delta 0} - \Phi^{\Delta\varepsilon} = F^0(\Phi^{\Delta 0}, \Phi^{\text{GFF}0}) - F^\varepsilon(\Phi^{\Delta\varepsilon}, \Phi^{\text{GFF}\varepsilon}) \\ = [F^\varepsilon(\Phi^{\Delta 0}, \Phi^{\text{GFF}0}) - F^\varepsilon(\Phi^{\Delta\varepsilon}, \Phi^{\text{GFF}\varepsilon})] + [F^0(\Phi^{\Delta 0}, \Phi^{\text{GFF}0}) - F^\varepsilon(\Phi^{\Delta 0}, \Phi^{\text{GFF}0})].$$

The first term in (3.57) is bounded as in (3.15) by

$$(3.58) \quad \|F_t^\varepsilon(\Phi^{\Delta 0}, \Phi^{\text{GFF}0}) - F_t^\varepsilon(\Phi^{\Delta\varepsilon}, \Phi^{\text{GFF}\varepsilon})\|_{L^\infty(\Omega_\varepsilon)} \\ \leq \int_t^\infty O(\theta_s |Z_s|) \|\Phi_s^{\Delta 0} - \Phi_s^{\Delta\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} ds + \int_t^\infty O(\theta_s |Z_s|) \|\Phi_s^{\text{GFF}0} - \Phi_s^{\text{GFF}\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} ds.$$

By Lemma 3.5, the second term on the right-hand side converges to 0 in $L^1(\mathbb{P})$. To bound the second term in (3.57), write

$$(3.59) \quad \|F_t^0(\Phi^{\Delta_0}, \Phi^{\text{GFF}_0}) - F_t^\varepsilon(\Phi^{\Delta_0}, \Phi^{\text{GFF}_0})\|_{L^\infty(\Omega_\varepsilon)} \\ \leq \int_t^\infty \|\dot{c}_s^0 \nabla v_s^0(\Phi_s^{\Delta_0} + \Phi_s^{\text{GFF}_0}) - \dot{c}_s^\varepsilon \nabla v_s^\varepsilon(\Phi_s^{\Delta_0} + \Phi_s^{\text{GFF}_0})\|_{L^\infty(\Omega_\varepsilon)} ds.$$

By (3.43), $\Phi_t^{\text{GFF}_0}$ is smooth for all $t \geq t_0 > 0$, almost surely, and by Theorem 3.4 (which is presented after the current theorem but whose proof is independent of it), Φ_t^Δ is also smooth for all $t > 0$. Therefore, by (2.12)–(2.13) and dominated convergence,

$$(3.60) \quad \mathbb{E} \sup_{t \geq t_0} \|F_t^0(\Phi^{\Delta_0}, \Phi^{\text{GFF}_0}) - F_t^\varepsilon(\Phi^{\Delta_0}, \Phi^{\text{GFF}_0})\|_{L^\infty(\Omega_\varepsilon)} \rightarrow 0 \quad (\varepsilon \rightarrow 0).$$

In summary, for every $t > 0$, there is a random variable R_t^ε converging to 0 as $\varepsilon \rightarrow 0$ in $L^1(\mathbb{P})$ such that

$$(3.61) \quad \|\Phi_t^{\Delta_0} - \Phi_t^{\Delta_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} \leq R_t^\varepsilon + \int_t^\infty O(\theta_s Z_s) \|\Phi_s^{\Delta_0} - \Phi_s^{\Delta_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} ds.$$

The same version of Gronwall's inequality as in the proof of Theorem 3.1 thus implies (3.55).

We now conclude the proof of the convergence $I_\varepsilon \Phi^{\text{SG}_\varepsilon} \rightarrow \Phi^{\text{SG}_0}$ in $H^{-\kappa}(\Omega)$. Since $I_\varepsilon \Phi^{\text{GFF}_\varepsilon} \rightarrow \Phi^{\text{GFF}_0}$ in $H^{-\kappa}(\Omega)$ by Lemma 3.5, it is more than sufficient to show that the following right-hand side converges to 0 in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0$:

$$(3.62) \quad (\Phi_0^{\text{SG}_0} - I_\varepsilon \Phi_0^{\text{SG}_\varepsilon}) - (\Phi_0^{\text{GFF}_0} - I_\varepsilon \Phi_0^{\text{GFF}_\varepsilon}) = (\Phi_0^{\Delta_0} - \Phi_t^{\Delta_0}) + I_\varepsilon(\Phi_t^{\Delta_\varepsilon} - \Phi_0^{\Delta_\varepsilon}) + (\Phi_t^{\Delta_0} - I_\varepsilon \Phi_t^{\Delta_\varepsilon}).$$

By (3.11), the first and second terms on the right-hand side are bounded as $\|\Phi_t^\Delta - \Phi_0^\Delta\|_{L^\infty(\Omega)} = O(Z_t)$ which converges to 0 as $t \rightarrow 0$. For the third term, since $\Phi_t^{\Delta_0}$ satisfies the Hölder continuity estimate (3.11), for $\kappa < 2 - \beta/4\pi$ we have

$$(3.63) \quad \|\Phi_t^{\Delta_0} - I_\varepsilon \Phi_t^{\Delta_\varepsilon}\|_{L^\infty(\Omega)} \leq \|\Phi_t^{\Delta_0} - \Phi_t^{\Delta_\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} + O(\varepsilon^\kappa).$$

This converges to 0 in $L^1(\mathbb{P})$ as $\varepsilon \rightarrow 0$ for any $t > 0$ by (3.10). Thus taking first $\varepsilon \rightarrow 0$ and then $t \rightarrow 0$ gives the required convergence. \square

3.6. Estimates: proof of Theorem 3.4. To prove Theorem 3.4, we start from

$$(3.64) \quad \Phi_t^\Delta = \Phi_t^{\text{SG}} - \Phi_t^{\text{GFF}} = - \int_t^\infty \dot{c}_s \nabla v_s(\Phi_s^{\text{SG}}) ds.$$

As previously, we often omit the subscript ε which can be positive or zero. By (2.13), the following estimates for $\dot{c}_t \nabla v_t(\varphi)$ hold with constants uniform in $\varphi \in C(\Omega)$ and $x \in \Omega$:

$$(3.65) \quad L_t^2 |\partial^k \dot{c}_t \nabla v_t(\varphi, x)| = O_{\beta, k}(\theta_t \frac{|Z_t|}{L_t^k}),$$

$$(3.66) \quad L_t^2 |\partial^k \dot{c}_t \nabla v_t(\varphi, x) - \partial^k \dot{c}_t \nabla v_t(\varphi, x')| = O_{\beta, k}(\theta_t \frac{|Z_t|}{L_t^k}) \left(1 \wedge \frac{|x - x'|}{L_t}\right).$$

Theorem 3.4 now follows by integrating these bounds as follows.

Proof of (3.11). By (3.64) and (3.65), for $0 \leq \beta < 8\pi$,

$$(3.67) \quad |\Phi_0^\Delta(x) - \Phi_t^\Delta(x)| \lesssim \int_0^t \theta_s |Z_s| \frac{ds}{L_s^2} \lesssim |z| \int_0^t s^{1-\beta/8\pi} \theta_s \frac{ds}{s} \lesssim |Z_t|. \quad \square$$

Proof of (3.12) for $\beta < 4\pi$. By (3.64) and (3.65), for $\beta < 4\pi$,

$$(3.68) \quad |\partial\Phi_0^\Delta(x)| \lesssim \int_0^\infty \frac{|Z_t|}{L_t} \theta_t \frac{dt}{L_t^2} \lesssim |z| \int_0^\infty t^{1/2-\beta/8\pi} \theta_t \frac{dt}{t} \lesssim |z|.$$

Moreover, by (3.64) and (3.66),

$$(3.69) \quad |\partial\Phi_0^\Delta(x) - \partial\Phi_0^\Delta(x')| \lesssim \int_0^\infty \theta_t \frac{|Z_t|}{L_t} \left(1 \wedge \frac{|x-x'|}{L_t}\right) \frac{dt}{L_t^2}.$$

We split the integral into the two contribution from $t \leq |x-x'|^2$ and $t \geq |x-x'|^2$. For $\beta < 4\pi$, the contribution from $t \leq |x-x'|^2$ is bounded by

$$(3.70) \quad \int_0^{|x-x'|^2} \frac{|Z_t|}{L_t} \frac{dt}{L_t^2} \lesssim |z| \int_0^{|x-x'|^2} t^{1/2-\beta/8\pi} \frac{dt}{t} \lesssim |z| |x-x'|^{1-\beta/4\pi},$$

and the contribution from $t \geq |x-x'|^2$ likewise gives

$$(3.71) \quad \int_{|x-x'|^2}^\infty \frac{|Z_t|}{L_t} \frac{|x|}{L_t} \theta_t \frac{dt}{t} \lesssim |z| |x-x'| \int_{|x|^2}^\infty t^{-\beta/8\pi} \theta_t \frac{dt}{t} \lesssim |z| |x-x'|^{1-\beta/4\pi}. \quad \square$$

Proof of (3.12) for $4\pi \leq \beta < 6\pi$. By (3.64) and (3.66),

$$(3.72) \quad |\Phi_0^\Delta(x) - \Phi_0^\Delta(x')| \lesssim \int_0^\infty \theta_t |Z_t| \left(1 \wedge \frac{|x-x'|}{L_t}\right) \frac{dt}{L_t^2}.$$

The integral is bounded in the same way as in the previous proof by splitting the integral into the two contribution from $t \leq |x-x'|^2$ and $t \geq |x-x'|^2$. The contribution from $t \leq |x-x'|^2$ is bounded by

$$(3.73) \quad \int_0^{|x-x'|^2} |Z_t| \frac{dt}{L_t^2} \lesssim |z| \int_0^{|x-x'|^2} t^{1-\beta/8\pi} \frac{dt}{t} \lesssim |z| |x-x'|^{2-\beta/4\pi}.$$

For the contribution due to $t > |x-x'|^2$, for $\beta > 4\pi$, we also have

$$(3.74) \quad \int_{|x-x'|^2}^\infty |Z_t| \frac{|x-x'|}{L_t} \theta_t \frac{dt}{t} \lesssim |z| |x-x'| \int_{|x|^2}^\infty t^{1/2-\beta/8\pi} \theta_t \frac{dt}{t} \lesssim |z| |x-x'|^{2-\beta/4\pi}.$$

For $\beta = 4\pi$, similarly,

$$(3.75) \quad \int_{|x-x'|^2}^\infty |Z_t| \frac{|x-x'|}{L_t} \theta_t \frac{dt}{t} = |z| |x-x'| \int_{|x-x'|^2}^\infty \theta_t \frac{dt}{t} \lesssim |z| |x-x'| |\log |x-x'||.$$

The claim follows by taking the sum over the $t \leq |x-x'|^2$ and $t > |x-x'|^2$ contributions. \square

Proof of (3.13). By (3.64) and (3.65), for $0 \leq \beta < 8\pi$,

$$(3.76) \quad \|\partial^k \Phi_t^\Delta\|_{L^\infty(\Omega)} \lesssim \int_t^\infty \theta_s \frac{|Z_s|}{L_s^k} \frac{ds}{L_s^2} \leq \frac{1}{L_t^k} \int_t^\infty \theta_s |Z_s| \frac{ds}{L_s^2} \lesssim \frac{|z|}{L_t^k}. \quad \square$$

4 Convergence in law of the maximum

In this section, we show the convergence of the maximum of the regularised sine-Gordon field, Theorem 1.1. To this end, we start from the multiscale coupling of the sine-Gordon field with the GFF given by Theorem 3.1. In the first step, we show that the non-Gaussian part of the sine-Gordon field may be removed at small scales. We then connect our coupling to the decomposition of the GFF used in [16], by decomposing the Gaussian part of our field once more into a ‘fine’

and a ‘coarse’ field analogous to those in [16]. In fact, we will choose exactly the same Gaussian fine field as in [16] so that the key results from that reference immediately transfer to our setting. Together with regularity estimates for our Gaussian coarse field analogous to those in [16] and the regularity estimates from Theorem 3.4 for the non-Gaussian part of the sine-Gordon field, the convergence then follows along similar lines as in [16].

Throughout this section, we always work on a probability space on which the fields $\Phi_t^{\text{SG}\varepsilon}$ and $\Phi_t^{\text{GFF}\varepsilon}$ are constructed simultaneously for all $\varepsilon \geq 0$ and $t \geq 0$ as in Section 3.1. As before, we will often drop the index ε from the notation in $\Phi^{\text{SG}\varepsilon}$ and $\Phi^{\text{GFF}\varepsilon}$. While all estimates are uniform in ε , throughout this section, we will always assume that $\varepsilon > 0$ is strictly positive (unless emphasised otherwise) because we are studying the limiting behaviour of the maximum which is singular as $\varepsilon \rightarrow 0$. The ε -dependent part of the mean of the maximum will be denoted

$$(4.1) \quad m_\varepsilon = \frac{1}{\sqrt{2\pi}} \left(2 \log \frac{1}{\varepsilon} - \frac{3}{4} \log \log \frac{1}{\varepsilon} \right).$$

Some statements below refer to the Lévy distance d on the set of probability measures on \mathbb{R} . This is a metric for the topology of weak convergence, defined for any two probability measures ν_1, ν_2 on \mathbb{R} by

$$(4.2) \quad d(\nu_1, \nu_2) = \min\{\kappa > 0: \nu_1(B) \leq \nu_2(B^\kappa) + \kappa \text{ for all open sets } B\}$$

where $B^\kappa = \{y \in \mathbb{R}: \text{dist}(y, B) < \kappa\}$. We will use the convention that when a random variable appears in the argument of d , we refer to its distribution on \mathbb{R} . Note that if two random variables X and Y can be coupled with $|X - Y| \leq \kappa$ with probability $1 - \kappa$ then $d(X, Y) \leq \kappa$.

4.1. Regularisation of the non-Gaussian part. From Theorem 3.1 and (3.1) it follows that

$$(4.3) \quad \Phi_0^{\text{SG}} = \Phi_s^{\text{SG}} + (\Phi_0^{\text{GFF}} - \Phi_s^{\text{GFF}}) + R_s$$

where

$$(4.4) \quad R_s = - \int_0^s \dot{c}_t \nabla v_t(\Phi_t^{\text{SG}}) dt.$$

Note that the first two terms in (4.3) are independent, by Theorem 3.1, and that $\max_{\Omega_\varepsilon} |R_s|$ is of order $|Z_s|$ by (3.11) and thus tends to 0 as $s \rightarrow 0$ (recall from (2.7) that $|Z_s| \leq |z|s^{1-\beta/8\pi}$). Denoting the main contribution of the field by

$$(4.5) \quad \tilde{\Phi}_s^{\text{SG}} \equiv \Phi_s^{\text{SG}} + (\Phi_0^{\text{GFF}} - \Phi_s^{\text{GFF}}),$$

we can rewrite (4.3) as

$$(4.6) \quad \Phi_0^{\text{SG}} = \tilde{\Phi}_s^{\text{SG}} + R_s.$$

By the following lemma, it suffices to prove Theorem 1.1 with Φ^{SG} replaced by $\tilde{\Phi}_s^{\text{SG}}$ with $s > 0$.

Lemma 4.1. *Assume that the limiting law $\tilde{\mu}_s$ of $\max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon$ as $\varepsilon \rightarrow 0$ exists for every $s > 0$, and that there are positive random variables Z_s (on the above common probability space) such that*

$$(4.7) \quad \tilde{\mu}_s((-\infty, x]) = \mathbb{E}[e^{-\alpha^* Z_s e^{-\sqrt{8\pi}x}}],$$

for some constant $\alpha^* > 0$. Then the law of $\max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon$ converges weakly to some probability measure μ_0 as $\varepsilon \rightarrow 0$ and $\tilde{\mu}_s \rightarrow \mu_0$ weakly as $s \rightarrow 0$. Moreover, there is a positive random variable Z^{SG} such that

$$(4.8) \quad \mu_0((-\infty, x]) = \mathbb{E}[e^{-\alpha^* Z^{\text{SG}} e^{-\sqrt{8\pi}x}}].$$

Proof. By (4.6), on the underlying coupled probability space we have

$$(4.9) \quad \max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon = \max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon + O(\max_{\Omega_\varepsilon} |R_s|)$$

with $\max_{\Omega_\varepsilon} |R_s| = O(|Z_s|) \rightarrow 0$ as $s \rightarrow 0$. Since $(\max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon)_\varepsilon$ is tight by assumption, it follows that also $(\max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon)_\varepsilon$ is tight. By the latter tightness, there is a subsequence $\varepsilon \rightarrow 0$ such that the law of $(\max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon)_\varepsilon$ converges to μ_0 . Thus along this subsequence,

$$(4.10) \quad \begin{aligned} d(\tilde{\mu}_s, \mu_0) &\leq \limsup_{\varepsilon=\varepsilon_k \rightarrow 0} \left[d(\max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon, \tilde{\mu}_s) + d(\max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon, \mu_0) \right. \\ &\quad \left. + d(\max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon, \max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon) \right] \leq O(|Z_s|), \end{aligned}$$

where we recall that d denotes the Lévy distance. In particular, by taking $s \rightarrow 0$, it follows that μ_0 is unique, and therefore now along any sequence $s \rightarrow 0$,

$$(4.11) \quad \tilde{\mu}_s((-\infty, x]) \rightarrow \mu_0((-\infty, x]).$$

Next, we prove that $(Z_s)_s$ is tight. Denote by μ_0^{GFF} the limiting law of the centred maximum of Φ_0^{GFF} which exists by [16, 27]. Since $\tilde{\Phi}_s^{\text{SG}} - \Phi_0^{\text{GFF}} = \Phi_s^\Delta$ is uniformly bounded, it follows from (4.7) that

$$(4.12) \quad \mu_0^{\text{GFF}}((-\infty, x - C]) \leq \mathbb{E}[e^{-\alpha^* Z_s e^{-\sqrt{8\pi}x}}]$$

for some constant $C > 0$ which is independent of s . Assume $(Z_s)_s$ is not tight. Then,

$$(4.13) \quad \exists \kappa > 0: \forall M > 0: \exists s_M: \mathbb{P}(Z_{s_M} > M) > \kappa.$$

It follows that

$$(4.14) \quad \mathbb{E}[e^{-\alpha^* Z_{s_M} e^{-\sqrt{8\pi}x}}] \leq e^{-\alpha^* M e^{-\sqrt{8\pi}x}} + \mathbb{P}(Z_{s_M} \leq M) \leq e^{-\alpha^* M e^{-\sqrt{8\pi}x}} + (1 - \kappa).$$

Sending $M \rightarrow \infty$, (4.12) implies that

$$(4.15) \quad \mu_0^{\text{GFF}}((-\infty, x - C]) \leq 1 - \kappa$$

which is a contradiction when sending $x \rightarrow \infty$.

This means that $(Z_s)_s$ is tight and by (4.11) the Laplace transforms converge as $s \rightarrow 0$. Thus, Lévy's continuity theorem for Laplace transforms implies that there is a random variable Z^{SG} such that $Z_s \rightarrow Z^{\text{SG}}$ and

$$(4.16) \quad \mathbb{E}[e^{-\alpha^* Z_s e^{-\sqrt{8\pi}x}}] \rightarrow \mathbb{E}[e^{-\alpha^* Z^{\text{SG}} e^{-\sqrt{8\pi}x}}]$$

as $s \rightarrow 0$, proving (1.2). That $Z^{\text{SG}} > 0$ a.s. follows from the tightness of $\max_{\Omega_\varepsilon} \Phi_0^{\text{SG}} - m_\varepsilon$. \square

4.2. Approximation of small scale field. As a next step we decompose our Gaussian small scale field $\Phi_s^{\text{GFF}} - \Phi_0^{\text{GFF}}$ (recall the convention that we omit the index $\varepsilon > 0$) further in order to connect to the setup of [16]. First observe that its covariance can be written as

$$(4.17) \quad \int_0^s e^{\Delta t - m^2 t} dt = (-\Delta + m^2 + 1/s)^{-1} + g_s(-\Delta + m^2)$$

where

$$(4.18) \quad g_s(\lambda) = \frac{1}{\lambda}(1 - e^{-\lambda s}) - \frac{1}{\lambda + 1/s} \geq 0.$$

Next, we divide Ω along the grid Γ , which is a union of horizontal and vertical lines intersecting at the vertices $\frac{1}{K}\mathbb{Z}^2 \cap \Omega$, into boxes $V_i \subset \Omega$, $i = 1, \dots, K^2$ of (macroscopic) side length $1/K$ such that $\Omega = \cup_{i=1}^{K^2} V_i \cup \Gamma$. We will always assume that $1/K$ is a multiple of ε such that Γ can be regarded as a subset of Ω_ε and $\Omega_\varepsilon = \cup_{i=1}^{K^2} (V_i \cap \Omega_\varepsilon) \cup \Gamma$. When no confusion can arise, we use the notation V_i for both the subset of Ω and the corresponding lattice version as subset of Ω_ε .

Now, let Δ_Γ be the Laplacian on Ω with Dirichlet boundary conditions on Γ , and let Δ be the Laplacian with periodic boundary conditions on Ω . The domain of Δ is the space of 1-periodic functions, and that of Δ_Γ is the smaller space of 1-periodic functions vanishing on Γ . This implies that $-\Delta_\Gamma \geq -\Delta$ and $(-\Delta + m^2 + 1/s)^{-1} \geq (-\Delta_\Gamma + m^2 + 1/s)^{-1}$ as quadratic form inequalities. (The form inequality for the Green function is equivalent to the use of the Markov property in [16].)

Hence we can decompose $\Phi_0^{\text{GFF}} - \Phi_s^{\text{GFF}}$ in distribution as

$$(4.19) \quad \Phi_0^{\text{GFF}} - \Phi_s^{\text{GFF}} \stackrel{d}{=} \tilde{X}_{s,K}^f + \tilde{X}_{s,K}^c + X_s^h,$$

where the three fields on the right-hand side are independent Gaussian fields with covariances

$$(4.20) \quad \text{Cov}(\tilde{X}_{s,K}^f) = (-\Delta_\Gamma + m^2 + 1/s)^{-1}$$

$$(4.21) \quad \text{Cov}(\tilde{X}_{s,K}^c) = (-\Delta + m^2 + 1/s)^{-1} - (-\Delta_\Gamma + m^2 + 1/s)^{-1}$$

$$(4.22) \quad \text{Cov}(X_s^h) = g_s(-\Delta + m^2)$$

where the function g_s is as in (4.18).

By definition, when restricted to any of the boxes V_i , the Laplacian Δ_Γ coincides with the Dirichlet Laplacian on V_i , and $(-\Delta_\Gamma + m^2 + 1/s)^{-1}(x, y) = 0$ for $x \in V_i$ and $y \in V_j$ with $i \neq j$. Therefore the Gaussian fields $\tilde{X}_{s,K}^f|_{V_i}$ and $\tilde{X}_{s,K}^f|_{V_j}$ are independent for $i \neq j$. Moreover, $\tilde{X}_{s,K}^f$ is essentially a massless GFF with Dirichlet boundary conditions on Γ . Indeed, we will later choose $1/K^2 \ll (m^2 + 1/s)^{-1}$, and the mass term $m^2 + 1/s$ will then be irrelevant. To connect directly to the setup of [16], it is convenient to replace $\tilde{X}_{s,K}^f$ by a field X_K^f which is in fact exactly a massless GFF with Dirichlet boundary conditions on Γ , i.e., a centred Gaussian field with covariance

$$(4.23) \quad \text{Cov}(X_K^f) = (-\Delta_\Gamma)^{-1}.$$

The next two simple lemmas show that the distribution of the maximum of $\Phi_0^{\text{GFF}} - \Phi_s^{\text{GFF}}$ is indeed well approximated by that of $X_K^f + \tilde{X}_{s,K}^c + X_s^h$ and that X_s^h is Hölder continuous.

Lemma 4.2. *Let G be a centred Gaussian field with covariance $(-\Delta_\Gamma)^{-1} - (-\Delta_\Gamma + m^2 + 1/s)^{-1}$. Then for $K^2 \geq m^2 + 1/s$,*

$$(4.24) \quad \mathbb{P}\left(\max_{\Omega_\varepsilon} G > t\right) \lesssim K^2 \exp\left(-\frac{ct^2}{\sigma^2}\right), \quad \sigma^2 = \frac{m^2 + 1/s}{K^2} + e^{-cK^2s}.$$

In particular, $\mathbb{P}(\max G > u) \rightarrow 0$ for any $u > 0$ as $K \rightarrow \infty$ with m and s fixed.

Proof. Let C_G be the covariance of G . We first verify the following estimates:

$$(4.25) \quad C_G(x, x) + C_G(y, y) - 2C_G(x, y) \lesssim \sigma^2(K|x - y|)$$

$$(4.26) \quad C_G(x, x) \lesssim \sigma^2.$$

These follow from standard estimates on the Dirichlet heat kernel given in Lemma A.3. Indeed, to see (4.26), we use $e^{t\Delta_\Gamma}(x, y) \lesssim (1/t)e^{-ctK^2}$ which holds by (A.8) with $\alpha = 0$ to get

$$(4.27) \quad C_G(x, x) = \int_0^\infty (1 - e^{-(m^2+1/s)t}) p_t^\Gamma(x, x) dt \lesssim \int_0^\infty \frac{1 - e^{-(m^2+1/s)t}}{t} e^{-K^2t} dt \lesssim \sigma^2.$$

To see (4.25), we similarly apply (A.8) with $\alpha = 1$ to see that the left-hand side of (4.25) is

$$(4.28) \quad \int_0^\infty (1 - e^{-(m^2+1/s)t})(p_t^\Gamma(x, x) + p_t^\Gamma(y, y) - 2p_t^\Gamma(x, y)) dt \\ \lesssim |x - y| \int_0^\infty \frac{1 - e^{-(m^2+1/s)t}}{t} t^{-1/2} e^{-cK^2 t} dt \lesssim \sigma^2(K|x - y|).$$

As a consequence of (4.25) and Fernique's criterion (see, for example, [16, Lemma 3.5] applied to G/σ), we obtain $\mathbb{E}[\max_{V_i} G] \lesssim \sigma$ uniformly in ε . The tail estimate then follows from a union bound over all K^2 boxes and the Borell-Tsirelson concentration inequality for the maximum (as, for example, stated in [16, Lemma 3.4]). Indeed, using (4.26), this gives

$$(4.29) \quad \mathbb{P}(\max_{x \in \Omega_\varepsilon} G > u) \leq K^2 \mathbb{P}(\max_{V_i} G > u) \leq 2K^2 \mathbb{P}(|\max_{V_i} G - \mathbb{E}[\max_{V_i} G]| > u - O(\sigma)) \\ \leq 2K^2 \exp\left(-\frac{(u - O(\sigma))^2}{O(\sigma^2)}\right) \lesssim K^2 \exp\left(-\frac{cu^2}{\sigma^2}\right)$$

which gives the claim. \square

The next lemma is the Hölder continuity of X_s^h .

Lemma 4.3. *Let X_s^h be a Gaussian field with covariance (4.22). There is $\alpha > 0$ such that*

$$(4.30) \quad \sup_{\varepsilon \geq 0} \mathbb{E} \left(\sup_{x \in \Omega_\varepsilon} |X_s^h(x)| + \sup_{x \neq y \in \Omega_\varepsilon} \frac{|X_s^h(x) - X_s^h(y)|}{|x - y|^\alpha} \right) \leq O_{s,m}(1).$$

Proof. Let C^h be the covariance (4.22) and note that by translation invariance we may identify $C^h(x, y)$ with a function $C^h(x - y)$. Since $g_s(\lambda) = O_s(1/(1 + \lambda^2))$ it follows that

$$(4.31) \quad |\hat{C}^h(k)| \leq O_s(1/(1 + |k|^4)).$$

In particular, $C^h(x, y)$ is Lipschitz continuous uniformly in ε , and Kolmogorov's continuity criterion implies that (4.30) holds. \square

4.3. Coarse field regularity. In what follows, as in [16], it suffices to consider the maximum of the fields restricted to points that are of macroscopic distance to the grid Γ . For $\delta \in (0, 1)$, define

$$(4.32) \quad V_i^\delta = \{x \in V_i : \text{dist}(x, \Gamma) \geq \delta/K\}, \quad \Omega^\delta = \bigcup_{i=1}^{K^2} V_i^\delta, \quad \Omega_\varepsilon^\delta = \Omega^\delta \cap (\varepsilon\mathbb{Z}^2).$$

Note that, differently from [16], dist here denotes the macroscopic distance on $\Omega_\varepsilon \subset \Omega$ rather than the microscopic distance which differs by a factor ε . The following lemma provides estimates for the covariance of $\tilde{X}_{s,K}^c$. It is analogous to [16, Lemma 2.1 and Lemma 3.10], except that we have a mass term and periodic boundary conditions.

Lemma 4.4 (Version of [16, Lemma 2.1 and Lemma 3.10]). *Define the covariance $C^c = C^{c,\varepsilon}$ as in (4.21). Then for $K^2 \geq m^2 + 1/s$ and as $\varepsilon \rightarrow 0$,*

$$(4.33) \quad \max_{x, y \in V_i^\delta} |C^{c,\varepsilon}(x, y) - C^{c,0}(x, y)| \rightarrow 0,$$

and the following estimates hold uniformly in ε and K (with constants depending on δ, s, m):

$$(4.34) \quad C^c(x, y) \lesssim \log K \quad (x, y \in \Omega^\delta),$$

$$(4.35) \quad |C^c(x, y) - C^c(x, y')| \lesssim K|y - y'| \quad (x \in \Omega^\delta, y, y' \in V_i^\delta),$$

$$(4.36) \quad C^c(x, x) + C^c(y, y) - 2C^c(x, y) \lesssim (K|x - y|)^2 \quad (x, y \in V_i^\delta).$$

We remark that in [16, Lemma 3.10], there is also a lower bound for the analogue of the left-hand side of (4.36). This lower bound is however only used in the proof of [16, Proposition 4.1] for which we do not need an analogue given that we can use the already established tail asymptotics for the Dirichlet GFF in a box (Proposition 4.6 below).

Proof. First, observe that C^c can be conveniently expressed as

$$(4.37) \quad C^c(x, y) = \int_0^\infty (e^{\Delta t} - e^{\Delta_{\Gamma} t})(x, y) e^{-(m^2+1/s)t} dt = \int_0^\infty (p_t^L(x, y) - p_t^\Gamma(x, y)) e^{-(m^2+1/s)t} dt.$$

Hence, we will use the standard estimates on heat kernels given in Lemmas A.1–A.3 and first verify (4.34)–(4.36). Indeed, (A.10) and (A.5) give

$$(4.38) \quad \partial^\alpha p_t^\Gamma(x, y) = \partial^\alpha p_t^L(x, y) + O(t^{-d/2-|\alpha|/2} e^{-c\delta/\sqrt{K^2 t}}) + O(t^{-d/2-|\alpha|/2} (\sqrt{t}/L)^d e^{-cL/\sqrt{t}}).$$

For $t \leq 1$ the second error term is smaller than the first one and conversely for $t > 1$.

To see (4.34), we integrate this bound with $d = 2$ and $\alpha = 0$:

$$(4.39) \quad C^c(x, y) \lesssim \int_0^1 t^{-1} e^{-c\delta/\sqrt{K^2 t}} dt + \int_1^\infty e^{-(m^2+1/s)t} dt \lesssim \log(K/\delta) + (m^2 + 1/s)^{-1} \lesssim \log K.$$

To see the upper bound in (4.36), we similarly integrate the above bound with $\alpha = 2$:

$$(4.40) \quad \begin{aligned} C^c(x, x) + C^c(y, y) - 2C^c(x, y) &\lesssim |x - y|^2 \left(\int_0^1 t^{-2} e^{-c\delta/\sqrt{K^2 t}} dt + \int_1^\infty t^{-1} e^{-(m^2+1/s)t} dt \right) \\ &\lesssim (K/\delta)^2 |x - y|^2 \lesssim K^2 |x - y|^2. \end{aligned}$$

The bound (4.35) follows analogously if x is in the same square as y and y' . Otherwise, $e^{\Delta t}(x, y) - e^{\Delta_{\Gamma} t}(x, y) = e^{\Delta t}(x, y)$ and likewise with y' instead of y . Since $|x - y| \wedge |x - y'| \geq \delta$, it therefore again follows from (A.2) that

$$(4.41) \quad \begin{aligned} |C^c(x, y) - C^c(x, y')| &\lesssim |y - y'| \left(\int_0^1 t^{-3/2} e^{-c\delta/\sqrt{t}} dt + \int_1^\infty t^{-3/2} e^{-c\delta/\sqrt{t}} e^{-(m^2+1/s)t} dt \right) \\ &\lesssim |y - y'| \leq K |y - y'|. \end{aligned}$$

The convergence (4.33) follows similarly using (A.3) and (A.9). \square

Finally, we also record the smoothness of Φ_s^{GFF} and Φ_s^{SG} .

Lemma 4.5. *For any $s > 0$, the fields Φ_s^{GFF} and Φ_s^{SG} are smooth uniformly in $\varepsilon \geq 0$. In particular, for $\# \in \{\text{SG}, \text{GFF}\}$,*

$$(4.42) \quad \sup_{\varepsilon \geq 0} \mathbb{E} \left(\max_{x \in \Omega_\varepsilon} |\Phi_s^\#(x)| + \max_{x, y \in \Omega_\varepsilon} \frac{|\Phi_s^\#(x) - \Phi_s^\#(y)|}{|x - y|} \right) < \infty.$$

Proof. The Gaussian field Φ_s^{GFF} has smooth covariance $\int_s^\infty e^{\Delta t - m^2 t} dt$ and the standard estimate follows, e.g., from Lemma 3.5. For $\Phi_s^\Delta = \Phi_s^{\text{SG}} - \Phi_s^{\text{GFF}}$ the estimate follows from (3.13). \square

4.4. Generalisation of required GFF results. In the previous two subsections, we have argued that the maximum of the sine-Gordon field can be replaced with negligible error as $s \rightarrow 0$ by that of the field

$$(4.43) \quad \Phi_s = X_K^f + \tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{SG}},$$

where all four terms on the right-hand side are independent fields. In the remainder of this section we will proceed along similar steps as in [16] to establish convergence in law of the centred

maximum of Φ_s . Since X_K^f is a GFF with Dirichlet boundary conditions, exactly as in [16], most of the steps can be used verbatim from [16]. In this section, we summarise these results for X_K^f , and also generalise the required results from [16] that involve the coarse field as well (which now corresponds to our non-Gaussian field $\tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{SG}}$).

The first result concerns the limiting behaviour of the maximum and maximiser of X_K^f .

Proposition 4.6 (Exactly [16, Propositions 2.2–2.3]). *Let $g(K) \rightarrow \infty$ as $K \rightarrow \infty$. Let X_K^f be a Gaussian field with covariance (4.23) and denote $V^\delta = V_i^\delta$. Then there is $\alpha^* > 0$ and a non-negative continuous function $\psi : (0, 1)^2 \rightarrow \mathbb{R}^2$ with $\int_{(0,1)^2} \psi = 1$ such that, for any $\delta \geq 0$ small enough, with $m_\delta = \int_{(\delta, 1-\delta)^2} \psi$ and $\psi^\delta = \psi/m_\delta$, and $z = z_\delta$ the maximiser of X_K^f in V^δ , the following holds:*

$$(4.44) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{e^{\sqrt{8\pi}g(K)}}{g(K)} \mathbb{P}(\mathcal{A}_{\delta,K,\varepsilon}) = \alpha^* m_\delta, \quad \mathcal{A}_{\delta,K,\varepsilon} = \left\{ \max_{V^\delta} X_K^f \geq m_\varepsilon K + g(K) \right\}.$$

Moreover, for any open set $A \subseteq (\delta, 1 - \delta)^2$ and any sequence $(x_K)_K$ with $x_K \geq 0$ independent of ε ,

$$(4.45) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{e^{\sqrt{8\pi}x_K g(K)}}{g(K) + x_K} \mathbb{P} \left(\max_{V^\delta} X_K^f \geq m_\varepsilon K + g(K) + x_K, Kz_\delta \in A \mid \mathcal{A}_{\delta,K,\varepsilon} \right) \\ = \lim_{K \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \frac{e^{\sqrt{8\pi}x_K g(K)}}{g(K) + x_K} \mathbb{P} \left(\max_{V^\delta} X_K^f \geq m_\varepsilon K + g(K) + x_K, Kz_\delta \in A \mid \mathcal{A}_{\delta,K,\varepsilon} \right) = \int_A \psi^\delta(y) dy.$$

with the convergence being uniform in the sequence $(x_K)_K$.

In the following results, which also involve the coarse field, it will be convenient to separate the Gaussian part of Φ_s . With Φ_s^Δ as in Theorem 3.4, we may write

$$(4.46) \quad \Phi_s = \eta + \Phi_s^\Delta,$$

where

$$(4.47) \quad \eta = X_K^f + (\tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{GFF}}).$$

Here Φ_s^Δ is non-Gaussian, but by Theorem 3.4, $\max_{\Omega_\varepsilon} |\Phi_s^\Delta| = O(1)$ uniformly in ε . Moreover, X_K^f is independent of $\tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{GFF}}$ in (4.47). Hence, using (4.19) we see that

$$(4.48) \quad \eta \stackrel{d}{=} \Phi_0^{\text{GFF}} + X_s^h + G,$$

where G is centred Gaussian with covariance as in Lemma 4.2 and independent of Φ_0^{GFF} .

The first result generalises the robustness result for the maximum of the GFF from [16, Lemma 3.9] and [27, Proposition 1.1] to the maximum of Φ_s .

Lemma 4.7 (Version of [27, Lemma 3.7]). *Let $\{\phi_u, u \in \Omega_\varepsilon\}$ be a collection of random variables independent of Φ_s such that for all $u \in \Omega_\varepsilon$ and all $y \geq 0$,*

$$(4.49) \quad \mathbb{P}(\phi_u \geq 1 + y) \leq e^{-y^2}.$$

Then there exist absolute constants $c, C > 0$ such that for any $\kappa > 0$, $\varepsilon > 0$ and $-c\kappa^{-1/2} \leq x$

$$(4.50) \quad \mathbb{P}(\max_{\Omega_\varepsilon} (\Phi_s + \kappa\phi) \geq m_\varepsilon + x) \leq \mathbb{P}(\max_{\Omega_\varepsilon} \Phi_s \geq m_\varepsilon + x - \sqrt{\kappa})(1 + O(e^{-C^{-1}\kappa^{-1}})).$$

Furthermore, for $\kappa < 1$,

$$(4.51) \quad \mathbb{E}[\max_{\Omega_\varepsilon} (\Phi_s + \kappa\phi)] \leq \mathbb{E}[\max_{\Omega_\varepsilon} \Phi_s] + C\sqrt{\kappa}.$$

Proof. Both inequalities are proved using identical calculations as in the proof of [16, Lemma 3.9] together with the estimate

$$(4.52) \quad \mathbb{P}(\max_A \Phi_s \geq m_\varepsilon + z - y) \lesssim \left(\frac{|A|}{|\Omega_\varepsilon|} \right)^{1/2} z e^{-\sqrt{8\pi}(z-y)} + o_{s,\delta,K}(1)$$

where $z \geq 1$, $y \geq 0$, $A \subseteq \Omega_\varepsilon$ and $o_{s,\delta,K}(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The latter follows from the analogue inequality (3.23) in [16, Lemma 3.8] (or rather [27, Proposition 1.1] applied to the Gaussian field η) and the fact that $\max_{\Omega_\varepsilon} |\Phi_s^\Delta| = O(1)$ uniformly in $\varepsilon > 0$. \square

The next result states that the event that the maximum is not attained close to the grid Γ occurs with high probability.

Proposition 4.8 (Version of [16, Proposition 5.1]). *Let $\Omega_\varepsilon^\delta$ be as in (4.32). Then,*

$$(4.53) \quad \lim_{\delta \rightarrow 0} \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_{\Omega_\varepsilon^\delta} \Phi_s \neq \max_{\Omega_\varepsilon} \Phi_s) = 0.$$

Proof. The proof is essentially the same as that of [16, Proposition 5.1] for the GFF. Indeed, by (4.48) the sequence $(\max_{\Omega_\varepsilon} \eta - m_\varepsilon)_\varepsilon$ is tight and hence, by (4.46), so is the sequence $(\max_{\Omega_\varepsilon} \Phi_s - m_\varepsilon)_\varepsilon$. Thus, using again (4.46) and that Φ_s^Δ is bounded by a deterministic constant independent of ε , the claim follows from the result that, for any fixed $x \in \mathbb{R}$,

$$(4.54) \quad \lim_{\delta \rightarrow 0} \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_{\Omega_\varepsilon \setminus \Omega_\varepsilon^\delta} \eta - m_\varepsilon \geq x) = 0,$$

which holds again by [27, Proposition 1.1] \square

The following result states that, to approximate the maximum of the full field, one may evaluate the field at the local maximizers of the field inside each box.

Proposition 4.9 (Version of [16, Proposition 5.2]). *Let Φ_s be as in (4.46). Let $z_i \in V_i^\delta$ be such that*

$$(4.55) \quad \max_{V_i^\delta} X_K^f = X_K^f(z_i)$$

and let \bar{z} be such that

$$(4.56) \quad \max_i \Phi_s(z_i) = \Phi_s(\bar{z}).$$

Then for any fixed $\kappa > 0$ and small enough $\delta > 0$,

$$(4.57) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_{\Omega_\varepsilon^\delta} \Phi_s \geq \Phi_s(\bar{z}) + \kappa) = 0.$$

Moreover, there is a function $g: \mathbb{N} \rightarrow \mathbb{R}_0^+$ with $g(K) \rightarrow \infty$ as $K \rightarrow \infty$, such that

$$(4.58) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(X_K^f(\bar{z}) \leq m_{\varepsilon K} + g(K)) = 0.$$

Proof. Again, our argument is an adaption of the proof of the analogous statement for the GFF, [16, Proposition 5.2], where it is shown that the event on left-hand side of (4.57) is contained in the union of suitable small events A_0, \dots, A_3 , defined as follows. Let $X_{s,K}^c \equiv \tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{GFF}}$ be the overall coarse part of η . Set $k = \log K$. Similarly as in the proof of [16, Proposition 5.2], we define

$$(4.59) \quad A_0 = \{\max_{\Omega_\varepsilon^\delta} \eta < m_\varepsilon - C\}$$

$$(4.60) \quad A_1 = \{\max_{v \in \Omega_\varepsilon^\delta} \max_{w \in \Omega_\varepsilon^\delta: |v-w| \leq \varepsilon f(k)} |X_{s,K}^c(v) - X_{s,K}^c(w)| \geq \kappa/2\}$$

$$(4.61) \quad A_2 = \{\max_i \max_{u,v \in V_i^\delta} (\eta(u) + X_{s,K}^c(u) - X_{s,K}^c(v)) \geq m_\varepsilon + C'\}$$

$$(4.62) \quad A_3 = \{\exists i, v: 1/K \geq |v - z_i| > \varepsilon f(k), \eta(v) \geq m_\varepsilon - C, \eta(z_i) \geq m_\varepsilon - 2C - C'\}$$

for some constants $C, C' > 0$ and $\kappa > 0$ and where $f(k)$ is a function that grows to ∞ as $k \rightarrow \infty$. We emphasise that, differently from [16], we always use the macroscopic distance $|\cdot|$ on Ω_ε which differs from the microscopic lattice distance used in [16] by a factor ε . As in the proof of [16, Proposition 5.2], we have that

$$(4.63) \quad \lim_{C, C' \rightarrow \infty} \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(A_l) = 0,$$

for $l = 0, \dots, 3$. Indeed, η has precisely the same fine field as in [16], and the events A_l only involve the Gaussian part of our coarse field. The minor difference between the field $\tilde{X}_{s,K}^c$ and the one in [16], i.e., periodic compared to Dirichlet boundary conditions and a mass term, does not matter since the field satisfies the same covariance estimates, by Lemma 4.4, which is all that enters the proof of (4.63). Moreover, using (4.30) and (4.42) it is easy to see that the continuous and K -independent fields Φ_s^{GFF} and X_s^h in the coarse part of Φ_s are irrelevant for the convergence in (4.63). Hence, the conclusion for the limits of $\mathbb{P}(A_l)$ is the same.

Having (4.63) for the Gaussian part of our field in place, it remains to consider the effect of the non-Gaussian part Φ_s^Δ . For this, it suffices to prove that with

$$(4.64) \quad B := \left\{ \max_{V_j^\delta} \eta \geq \eta(z_j) + \kappa/2, V_j^\delta \text{ the box where } \Phi_s|_{\Omega^\delta} \text{ is maximal} \right\},$$

we have:

- (i) For any given $\kappa > 0$ and K large enough, $\{\max_{\Omega_\varepsilon^\delta} \Phi_s \geq \Phi_s(\bar{z}) + \kappa\} \subseteq B$.
- (ii) $B \subseteq A_0 \cup A_1 \cup A_2 \cup A_3$ where the A_l are as in (4.59)–(4.62).

The event B is defined similarly as in the proof of [16, Proposition 5.2], except that compared to their argument we consider the box where Φ_s is maximal rather than the one where η is maximal. In the proof of both statements we will use that Theorem 3.4 implies that there is $c > 0$, such that for $x, y \in V_i$,

$$(4.65) \quad |\Phi_s^\Delta(x) - \Phi_s^\Delta(y)| \leq c_{s,K} = O_s(K^{-c}).$$

Proof of (i): Using (4.65) and since V_j^δ is the box on which $\Phi_s|_{\Omega^\delta}$ is maximal, we have

$$(4.66) \quad \max_{V_j^\delta} \eta = \max_{V_j^\delta} (\Phi_s - \Phi_s^\Delta) \geq \max_{V_j^\delta} \Phi_s - \Phi_s^\Delta(z_j) - c_{s,K} = \max_{\Omega_\varepsilon^\delta} \Phi_s - \Phi_s^\Delta(z_j) - c_{s,K}.$$

Thus assuming that the event $\{\max_{\Omega_\varepsilon^\delta} \Phi_s \geq \Phi_s(\bar{z}) + \kappa\}$ occurs, we can further estimate

$$(4.67) \quad \max_{V_j^\delta} \eta \geq \Phi_s(\bar{z}) + \kappa - \Phi_s^\Delta(z_j) - c_{s,K} \geq \eta(z_j) + \kappa - c_{s,K} \geq \eta(z_j) + \kappa/2.$$

This completes the proof of (i).

Proof of (ii): The inclusion for B follows similarly as in the proof of [16, Proposition 5.2]. Fix some constants C, C' . We will first show that

$$(4.68) \quad B \subseteq \tilde{A}_0 \cup A_1 \cup A_2 \cup A_3$$

where A_1, A_2, A_3 are as in (4.59)–(4.62) and where

$$(4.69) \quad \tilde{A}_0 = \{\max_{V_j^\delta} \eta < m_\varepsilon - C\}.$$

Let $\tau \in V_j^\delta$ be the maximiser of η in V_j^δ , recall that V_j^δ is the box where Φ_s attains its global maximum on $\Omega_\varepsilon^\delta$. To see that (4.68) holds, we first claim that $(\tilde{A}_0 \cup A_2 \cup A_3)^c \subseteq \{|\tau - z_j| \leq \varepsilon f(k)\}$. Otherwise, on $(\tilde{A}_0 \cup A_2)^c$, we have

$$(4.70) \quad X_{s,K}^c(\tau) - X_{s,K}^c(z_j) = (\eta(\tau) + X_{s,K}^c(\tau) - X_{s,K}^c(z_j)) - \eta(\tau) \leq m_\varepsilon + C' - (m_\varepsilon - C) = C + C'$$

and thus, since z_j maximises $X_K^f = \eta - X_{s,K}^c$ in V_j^δ ,

$$(4.71) \quad \eta(z_j) \geq \eta(\tau) - (X_{s,K}^c(\tau) - X_{s,K}^c(z_j)) \geq m_\varepsilon - C - (C + C') = m_\varepsilon - 2C - C'.$$

This is not consistent with being in A_3^c . Consequently,

$$(4.72) \quad (\tilde{A}_0 \cup A_2 \cup A_3)^c \subseteq \{|\tau - z_j| \leq \varepsilon f(k)\}.$$

But then, we also have

$$(4.73) \quad \begin{aligned} (\tilde{A}_0 \cup A_1 \cup A_2 \cup A_3)^c &\subseteq A_1^c \cap \{|\tau - z_j| \leq \varepsilon f(k)\} \subseteq \{|X_{s,K}^c(\tau) - X_{s,K}^c(z_j)| < \kappa/2\} \\ &\subseteq \{\eta(\tau) - \eta(z_j) < \kappa/2\} = B^c \end{aligned}$$

which is (4.68).

It remains to show that $\tilde{A}_0 \subseteq A_0$, where A_0 is as in (4.59) but for a possibly different choice of the constant C . Denote by $\zeta \in V_j^\delta$ the global maximiser of Φ_s in $\Omega_\varepsilon^\delta$. Since η is maximal at τ in V_j^δ and using the continuity of Φ_s^Δ , see (4.65), we obtain for K large enough that

$$(4.74) \quad \max_{\Omega_\varepsilon^\delta} \Phi_s - \Phi_s(\tau) = \Phi_s(\zeta) - \Phi_s(\tau) = \eta(\zeta) - \Phi_s^\Delta(\zeta) - (\eta(\tau) - \Phi_s^\Delta(\tau)) \leq \Phi_s^\Delta(\tau) - \Phi_s^\Delta(\zeta) < 1.$$

Hence, we have with $\tilde{C} > 0$ such that $\max_{\Omega_\varepsilon} |\Phi_s^\Delta| \leq \tilde{C}$ by Theorem 3.4,

$$(4.75) \quad \begin{aligned} \tilde{A}_0 = \{\eta(\tau) < m_\varepsilon - C\} &\subseteq \{\Phi_s(\tau) \leq m_\varepsilon - C + \tilde{C}\} \subseteq \{\max_{\Omega_\varepsilon^\delta} \Phi_s \leq m_\varepsilon - C + \tilde{C} + 1\} \\ &\subseteq \{\max_{\Omega_\varepsilon^\delta} \eta \leq m_\varepsilon - C + 2\tilde{C} + 1\} \end{aligned}$$

where we used in the last step that the global maxima of Φ_s and η differ by at most \tilde{C} . This completes the proof of (ii).

It remains to prove (4.58). While our fine field X_K^f is the same as in [16], the only difference to the analogous equation in [16, Proposition 5.2] is that our \bar{z} is defined via Φ_s rather than via η . We first fix a constant $\gamma > 0$, which will be adjusted throughout the argument. From (4.63) for A_0 and again $\max_{\Omega_\varepsilon} |\Phi_s^\Delta| \leq \tilde{C}$, we obtain

$$(4.76) \quad \lim_{K \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \mathbb{P}(\max_{\Omega_\varepsilon^\delta} \Phi_s \geq m_\varepsilon - \gamma \log k) = 1$$

where $k = \log K$. By (4.57), and adjusting γ , we also have

$$(4.77) \quad \lim_{K \rightarrow \infty} \liminf_{\varepsilon \rightarrow 0} \mathbb{P}(\Phi_s(\bar{z}) \geq m_\varepsilon - \gamma \log k) = 1.$$

Hence, since $m_\varepsilon - m_{\varepsilon K} = \frac{2}{\sqrt{2\pi}}k + o_K(1)$ with $o_K(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$ for fixed K , it suffices to find an appropriate function $g(K)$, such that

$$(4.78) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_i X_{s,K}^c(z_i) \geq \frac{2}{\sqrt{2\pi}}k - \gamma \log k - g(K)) = 0.$$

Since our Gaussian coarse field $\tilde{X}_{s,K}^c$ satisfies the same covariance estimates as the coarse field in [16] and since the continuous and K -independent fields Φ_s^{GFF} and X_s^h are again irrelevant, exactly the same argument as in the proof of the analogous result in [16, Proposition 5.2] shows that $g(K) = \alpha \log k$ for some $\alpha > 0$ suffices. Finally, writing

$$(4.79) \quad \begin{aligned} \mathbb{P}(X_K^f(\bar{z}) \leq m_{\varepsilon K} + g(K)) &= \mathbb{P}(\Phi_s(\bar{z}) - X_{s,K}^c(\bar{z}) - \Phi_s^\Delta(\bar{z}) \leq m_{\varepsilon K} + g(K)) \\ &\leq \mathbb{P}(\max_i X_{s,K}^c(z_i) \geq \frac{2}{\sqrt{2\pi}}k + C_\varepsilon(K) - \gamma \log K - g(K)) + \mathbb{P}(\Phi_s(\bar{z}) < m_\varepsilon - \gamma \log K), \end{aligned}$$

(4.58) follows by taking first $\varepsilon \rightarrow 0$ and then $K \rightarrow \infty$, as the first probability vanishes by (4.78) and that the second probability vanishes by (4.76). \square

4.5. Approximation of the maximum. In this section, we adapt the approximation of the centred maximum by ε -independent random variables from [16, Section 2.3]. Let

$$(4.80) \quad \Phi_s^* = \max_{\Omega_\varepsilon} \Phi_s.$$

Following [16, Section 2.3] we will approximate $\Phi_s^* - m_\varepsilon$ by $G_{s,K}^* = \max_i G_{s,K}^i$ where

$$(4.81) \quad G_{s,K}^i = \rho_K^i (Y_K^i + g(K)) + Z_{s,K}^{c,0}(\mathbf{u}_\delta^i) - \frac{2}{\sqrt{2\pi}} \log K,$$

and also define

$$(4.82) \quad Z_{s,K} = m_\delta \frac{1}{K^2} \sum_{i=1}^{K^2} \left(\frac{2}{\sqrt{2\pi}} \log K - Z_{s,K}^{c,0}(\mathbf{u}_\delta^i) \right) e^{-2 \log K + \sqrt{8\pi} Z_{s,K}^{c,0}(\mathbf{u}_\delta^i)}$$

Here the sequence $g(K)$ is as in Proposition 4.6 and the random variables in (4.81) and (4.82) are all independent and defined as follows.

- The random variables $\rho_K^i \in \{0, 1\}$, $i = 1, \dots, K^2$, are independent Bernoulli random variables with $\mathbb{P}(\rho_K^i = 1) = \alpha^* m_\delta g(K) e^{-\sqrt{8\pi} g(K)}$ with α^* and m_δ as in Proposition 4.6.
- The random variables $Y_K^i \geq 0$, $i = 1, \dots, K^2$, are independent and characterised by $\mathbb{P}(Y_K^i \geq x) = \frac{g(K)+x}{g(K)} e^{-\sqrt{8\pi} x}$ for $x \geq 0$.
- The random field $Z_{s,K}^{c,0}(x)$, $x \in \Omega$, is the limit of the overall coarse field $Z_{s,K}^c \equiv \tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{SG}}$ as $\varepsilon \rightarrow 0$. The existence of this limit is guaranteed by Theorem 3.3 and Lemma 4.4.
- The random variables $\mathbf{u}_\delta^i \in V_i^\delta$, $i = 1, \dots, K^2$, have the limiting distribution of the maximisers z_i of X_K^f in V_i^δ as $\varepsilon \rightarrow 0$. Thus \mathbf{u}_δ^i takes values in the i -th subbox of $\Omega = \mathbb{T}^2$ and, scaled to the unit square, its density is ψ^δ as in Proposition 4.6.

Note that that the correction in (4.81) can be understood from

$$(4.83) \quad m_{\varepsilon K} - m_\varepsilon = -\frac{2}{\sqrt{2\pi}} \log K + O_K(\varepsilon).$$

Theorem 4.10 (Extension of [16, Theorem 2.4]). *With the notation introduced around (4.2),*

$$(4.84) \quad \lim_{\delta \rightarrow 0} \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} d(\max_{\Omega_\varepsilon} \Phi_s - m_\varepsilon, G_{s,K}^*) = 0.$$

The remainder of this section is devoted to the proof of Theorem 4.10, following [16, Section 6.1] closely. The approach of the proof is to couple the field Φ_s with the random variables $G_{s,K}^i$. In [16, Section 6.1] such a coupling is constructed between the independent random variables $(\rho_K^i, Y_K^i, \mathbf{u}_\delta^i)$ and the values and locations of the maxima of the fine field X_K^f . This is done for each box independently, so we restrict for now to the first box V_1^δ and the random variables $(\rho_K^1, Y_K^1, \mathbf{u}_\delta^1) \equiv (\rho, Y, \mathbf{u}_\delta)$. Since the field Φ_s has the same fine field, we may use the same coupling to prove Theorem 4.10. We first recall this coupling by stating the following three results for X_K^f from [16]. Recall that we set $k = \log K$.

Lemma 4.11 (Exactly [16, Lemma 6.1]). *There exists a constant $C^* > 0$ such that, for all K ,*

$$(4.85) \quad \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_{V_1} X_K^f \geq m_{\varepsilon K} + C^* k) \leq K^{-3}.$$

Set $\theta_K(x) = e^{\sqrt{8\pi}(x+g(K))} / (g(K) + x)$ for $x \geq 0$.

Lemma 4.12 (Exactly [16, Lemma 6.2]). *There exists a sequence $(\varepsilon_K)_K$, $\varepsilon_K \rightarrow 0$ as $K \rightarrow \infty$ and a sequence of numbers $\alpha_{1,K,\varepsilon} < \alpha_{2,K,\varepsilon} < \dots < \alpha_{C^*k,K,\varepsilon}$ satisfying*

$$(4.86) \quad |\alpha_{j,K,\varepsilon} - (g(K) + j - 1)| \leq \varepsilon_K$$

such that for all $j = 1, \dots, C^*k$,

$$(4.87) \quad \theta_K(0)^{-1} \mathbb{P}(Y_K^i \in [j - 1, j]) = \mathbb{P}(\max_{V_1} X_K^f - m_{\varepsilon_K} \in [\alpha_{j,K,\varepsilon}, \alpha_{j+1,K,\varepsilon})).$$

Recall that $X_K^f(z_1) = \max_{V_1^\delta} X_K^f$. The following coupling result relates the fine field maximum to the ε -independent random variables in (4.81).

Proposition 4.13 (Exactly [16, Proposition 6.3]). *Let C^* be as in Lemma 4.11 and let $\alpha_{j,K,\varepsilon}$, $j = 1, \dots, C^*k$, be as in Lemma 4.12. There exists a sequence $(\bar{\varepsilon}_K)_K$ depending on δ , $\bar{\varepsilon}_K \rightarrow 0$ as $K \rightarrow \infty$, such that $(\bar{\rho}_{\delta,K,\varepsilon}, X_K^f(z_1) - m_{\varepsilon_K}, z_1)$ and $(\rho, Y, \mathbf{u}_\delta)$ can be constructed on the same probability space with $\rho = \bar{\rho}_{\delta,K,\varepsilon}$ holding with probability 1, and such that, on the event $\{X_K^f(z_1) - m_{\varepsilon_K} \leq \alpha_{C^*k,K,\varepsilon}\}$,*

$$(4.88) \quad \rho |g(K) + Y - (X_K^f(z_1) - m_{\varepsilon_K})| + K |\mathbf{u}_\delta - z_1| \leq \bar{\varepsilon}_K.$$

As in [16] we need a continuity result which shows that the maximum of Φ_s effectively does not change if the coarse field is evaluated close to the fine field maximum. The proof of the GFF analogue of this statement in [16, Lemma 6.4] and [1, Lemma 10.3.3] is based on Gaussian techniques for the coarse field, in particular the Sudakov–Fernique inequality. Our coarse field is non-Gaussian, but we show that using the decomposition (4.43) we may condition on the non-Gaussian part $\Phi_s^{\text{SG}} + X_s^h$ and apply the Sudakov–Fernique inequality with nonzero mean to the conditional measure. Therefore, we denote $\tilde{\eta} = X_K^f + \tilde{X}_{s,K}^c$ so that (4.43) reads $\Phi_s = \tilde{\eta} + \Phi_s^{\text{SG}} + X_s^h$. Note that in both decompositions the terms are independent. Also recall that $Z_{s,K}^c \equiv \tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{SG}}$ is the overall (non-Gaussian) coarse part of Φ_s .

Lemma 4.14 (Version of [16, Lemma 6.4]). *Let z_i be as in Proposition 4.9, let $(\bar{\varepsilon}_K)_K$ be as in Proposition 4.13, and let z'_i , $i = 1, \dots, K^2$, denote a family of independent random variables chosen so that z'_i is measurable with respect to $\sigma(X_K^f(v) : v \in V_i^\delta)$ and that $K|z_i - z'_i| \leq \bar{\varepsilon}_K$. Then*

$$(4.89) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} d(\max_i (X_K^f(z_i) + Z_{s,K}^c(z_i)), \max_i (X_K^f(z_i) + Z_{s,K}^c(z'_i))) = 0.$$

Proof. Note that by Lemma 4.5 and Lemma 4.3 we may immediately replace $\Phi_s^{\text{SG}}(z'_i)$ by $\Phi_s^{\text{SG}}(z_i)$ and $X_s^h(z'_i)$ by $X_s^h(z_i)$ in (4.89) since their Hölder continuity implies that, for any $\kappa > 0$,

$$(4.90) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_i |\Phi_s^{\text{SG}}(z_i) - \Phi_s^{\text{SG}}(z'_i)| > \kappa) = 0,$$

$$(4.91) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\max_i |X_s^h(z_i) - X_s^h(z'_i)| > \kappa) = 0.$$

Thus, with $M = \max_i (X_K^f(z_i) + \Phi_s^{\text{SG}}(z_i) + X_s^h(z_i) + \tilde{X}_{s,K}^c(z_i))$ and $M' = \max_i (X_K^f(z_i) + \Phi_s^{\text{SG}}(z_i) + X_s^h(z_i) + \tilde{X}_{s,K}^c(z'_i))$, it suffices to prove

$$(4.92) \quad \lim_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(|M - M'| > \kappa) = 0.$$

In the remainder of the proof we proceed along similar steps as in the proofs of [16, Lemma 6.4] and [1, Lemma 10.3.3], i.e. we write

$$(4.93) \quad \mathbb{P}(|M - M'| > \tilde{\varepsilon}) \leq \mathbb{P}(M - M' > \kappa) + \mathbb{P}(M' - M > \kappa) \equiv p_1 + p_2$$

and show that both probabilities on the right-hand side converge to 0 when taking first $\varepsilon \rightarrow 0$ and then $K \rightarrow \infty$. In each case we introduce fields which reflect the coarse field perturbation and condition on the non-Gaussian, but independent part Φ_s^{SG} . This allows to connect to corresponding unconditional results proved in [16] and [1]. Define $V_{\bar{\varepsilon}_K/K}^{\times 2} = \{(u, v) : u, v \in \Omega_\varepsilon^\delta, |u - v| \leq \bar{\varepsilon}_K/K\}$.

For p_1 , we use a similar argument as in [16, Lemma 5.4]. Restricting to the box V_j^δ where M is attained, we have

$$(4.94) \quad p_1 \leq \mathbb{P}(\tilde{X}_{s,K}^c(z_j) - \tilde{X}_{s,K}^c(z'_j) > \kappa).$$

Now, recall that $\tilde{\eta} = X_K^f + \tilde{X}_{s,K}^c$ and set

$$(4.95) \quad \xi_1(x, y) = \Phi_s(x) + \tilde{X}_{s,K}^c(x) - \tilde{X}_{s,K}^c(y)$$

$$(4.96) \quad \tilde{\xi}_1(x, y) = \tilde{\eta}(x) + \tilde{X}_{s,K}^c(x) - \tilde{X}_{s,K}^c(y),$$

for $(x, y) \in V_{\bar{\varepsilon}_K/K}^{\times 2}$. Moreover, define $\xi_1^* = \max_{V_{\bar{\varepsilon}_K/K}^{\times 2}} \xi_1$ and similarly $\tilde{\xi}_1^*$. Then (4.94) and Markov's inequality imply

$$(4.97) \quad p_1 \leq \mathbb{P}(\xi_1(z_j, z'_j) - \Phi_s(z_j) > \kappa) \leq \mathbb{P}(\xi_1^* - \Phi_s(z_j) > \kappa) \leq \mathbb{E}[\xi_1^* - \Phi_s(z_j)]/\kappa.$$

Since the event $\{\Phi_s(z_j) \geq \Phi_s^* - \kappa'\}$ occurs with high probability by Proposition 4.9, it suffices to prove that $\mathbb{E}[\xi_1^* - \Phi_s^*] \rightarrow 0$ as first $\varepsilon \rightarrow 0$ and then $K \rightarrow \infty$.

Note that conditional on $\Phi_s^{\text{SG}} + X_s^h$, we have that ξ_1 is a Gaussian field indexed by $V_{\bar{\varepsilon}_K/K}^{\times 2}$ and that

$$(4.98) \quad \xi_1(x, y) = \tilde{\xi}_1(x, y) + \Phi_s^{\text{SG}}(x) + X_s^h(x).$$

Moreover, $\tilde{\xi}_1$ is the analogue to the field Y defined in [16, page 112], except that Y is defined for the set $V_{1/K, \delta}^{\times 2} = \{(u, v) : u, v \in V_i^\delta \text{ for some } i\}$. Choosing K large enough, we can ensure that $V_{\bar{\varepsilon}_K/K}^{\times 2} \subseteq V_{1/K, \delta}^{\times 2}$. Therefore, the estimate in [16, Lemma 5.3] holds for $\tilde{\xi}_1$, i.e. there is a constant c_1 which is independent of ε and K , such that for all $(x, y), (x', y') \in V_{1/K, \delta}^{\times 2}$,

$$(4.99) \quad \mathbb{E}[(\tilde{\xi}_1(x, y) - \tilde{\xi}_1(x', y'))^2] \leq \mathbb{E}[(\tilde{\eta}(x) - \tilde{\eta}(x'))^2] + c_1(\mathbf{1}_{x=x'}(K|y - y'|)^2 + \mathbf{1}_{x \neq x'}).$$

Indeed, examining the proof of [16, Lemma 5.3], we see that only the independence of the decomposition $\tilde{\eta} = X_K^f + \tilde{X}_{s,K}^c$, the estimates in [16, Lemma 3.1], which we use with Φ_0^{GFF} , and the estimates in [16, Lemma 3.10] for the coarse field enter, which are completely analogous to our estimates for $\tilde{X}_{s,K}^c$; see Lemma 4.4.

We want to replace ξ_1^* in (4.97) by a field of the same structure, but with a coarse field perturbation that is independent of Φ_s , so that Lemma 4.7 applies. Hence, we define

$$(4.100) \quad \psi_1(x, y) = \Phi_s + C_1 \bar{Z}(x, y)$$

$$(4.101) \quad \tilde{\psi}_1(x, y) = \tilde{\eta}(x) + C_1 \bar{Z}(x, y),$$

for $(x, y) \in V_{\bar{\varepsilon}_K/K}^{\times 2}$, where $C_1 > 0$ is some constant to be determined later and \bar{Z} is as in [16, page 114] with

$$(4.102) \quad \mathbb{E}[(\bar{Z}(x, y) - \bar{Z}(x', y'))^2] \leq C_\delta K |y - y'| \text{ for } y, y' \in V_i^\delta,$$

$$(4.103) \quad \mathbb{E}[\bar{Z}(x, y)^2] \leq C_\delta K |x - y| \text{ for } x, y \in V_i^\delta.$$

and $(\bar{Z}(x, y))_y$ and $(\bar{Z}(x', y))_y$ being independent for $x \neq x'$. As for ξ_1 , the field ψ_1 is Gaussian conditional on $\Phi_s^{\text{SG}} + X_s^h$ and

$$(4.104) \quad \psi_1(x, y) = \tilde{\psi}_1(x, y) + \Phi_s^{\text{SG}}(x) + X_s^h(x).$$

Moreover, note that $\tilde{\psi}_1$ is the analogue to \bar{Y} in [16, page 114]. Using (4.102)–(4.103) together with (4.99), it is easy to see that we can choose C_1 large enough such that for $(x, y), (x', y') \in V_{\bar{\varepsilon}_K/K}^{\times 2}$, we have

$$(4.105) \quad \mathbb{E}[(\tilde{\xi}_1(x, y) - \tilde{\xi}_1(x', y'))^2] \leq \mathbb{E}[(\tilde{\psi}_1(x, y) - \tilde{\psi}_1(x', y'))^2].$$

By independence in (4.98) and (4.104) and basic properties of conditional expectation, we have that for all $(x, y), (x', y') \in V_{\bar{\varepsilon}_K/K}^{\times 2}$

$$(4.106) \quad \mathbb{E}[(\xi_1(x, y) - \xi_1(x', y'))^2 | \Phi_s^{\text{SG}} + X_s^h] \leq \mathbb{E}[(\psi_1(x, y) - \psi_1(x', y'))^2 | \Phi_s^{\text{SG}} + X_s^h].$$

Hence, by the Sudakov–Fernique inequality for non-centred Gaussian fields as stated in [2, Theorem 2.2.3],

$$(4.107) \quad \mathbb{E}[\xi_1^* | \Phi_s^{\text{SG}} + X_s^h] \leq \mathbb{E}[\psi_1^* | \Phi_s^{\text{SG}} + X_s^h].$$

Thus, taking expectation on both sides, we obtain

$$(4.108) \quad \mathbb{E}[\xi_1^*] \leq \mathbb{E}[\psi_1^*].$$

Going back to (4.97), we see that it suffices to prove that

$$(4.109) \quad \mathbb{E}[\psi_1^* - \Phi_s^*] \rightarrow 0$$

as first $\varepsilon \rightarrow 0$ and then $K \rightarrow 0$. To this end, denote by $B_\kappa(x)$ the ball of size κ around x intersected with $\Omega_\varepsilon^\delta$ and set

$$(4.110) \quad \theta_1(x) = C_1 \max_{y \in B_{\bar{\varepsilon}_K/K}(x)} \bar{Z}(x, y)$$

for $x \in \Omega_\varepsilon^\delta$. Then, by (4.102), we have for $y, y' \in B_{\bar{\varepsilon}_K/K}(x)$

$$(4.111) \quad \mathbb{E}[(\bar{Z}(x, y) - \bar{Z}(x, y'))^2] \leq C_\delta \frac{|y - y'|}{\bar{\varepsilon}_K/K} \bar{\varepsilon}_K.$$

Thus, the Fernique criterion as stated in [16, Lemma 3.5] applied to the Gaussian field $\bar{Z}(x, \cdot)/\sqrt{\bar{\varepsilon}_K}$ in $B_{\bar{\varepsilon}_K/K}(x)$ yields

$$(4.112) \quad \mathbb{E}[\theta_1(x)] \leq C\sqrt{\bar{\varepsilon}_K}$$

for some constant C . Moreover, by (4.103), we have $\mathbb{E}[\bar{Z}(x, y)^2] \leq C_\delta \bar{\varepsilon}_K$ for all $y \in B_{\bar{\varepsilon}_K/K}(x)$. Hence, it follows from the Borell–Tsirelson concentration inequality as stated in [16, Lemma 3.4] that

$$(4.113) \quad \mathbb{P}(\theta_x - \mathbb{E}[\theta_x] > y\sqrt{\bar{\varepsilon}_K}) \leq 2e^{-\frac{y^2}{2C_\delta}}.$$

Setting $\phi_x = \theta_x/C\sqrt{\bar{\varepsilon}_K}$ this implies

$$(4.114) \quad \mathbb{P}(\phi_x - 1 > y) \leq 2e^{-C'y^2}$$

for some absolute constant $C' > 0$. Now an application of Lemma 4.7 (with the maximum over $\Omega_\varepsilon \setminus \Omega_\varepsilon^\delta$ instead of Ω_ε) allows us to deduce that

$$(4.115) \quad \mathbb{E}[\psi_1^*] \leq \mathbb{E}[\Phi_s^*] + \bar{\varepsilon}_K^q$$

for some power $q > 0$ depending on C' . This completes the proof of $p_1 \rightarrow 0$.

To see that p_2 converges to 0, when taking the said limits, we use a similar argument but with the fields involved defined differently. Now, we set

$$(4.116) \quad \xi_2(x, y) = \Phi_s(x) + \tilde{X}_{s,K}^c(y) - \tilde{X}_{s,K}^c(x)$$

$$(4.117) \quad \tilde{\xi}_2(x, y) = \tilde{\eta}(x) + \tilde{X}_{s,K}^c(y) - \tilde{X}_{s,K}^c(x)$$

for $(x, y) \in V_{\bar{\varepsilon}_K/K}^{\times 2}$ and define $\xi_2^* = \max_{V_{\bar{\varepsilon}_K/K}^{\times 2}} \xi_2$ and similarly $\tilde{\xi}_2^*$. Then, Markov's inequality implies

$$(4.118) \quad p_2 \leq \mathbb{P}(\xi_2(z_i, z'_i) - M > \kappa) \leq \mathbb{E}[\xi_2^* - M]/\kappa$$

As above we use Proposition 4.9 to replace the random variable M by the overall maximum Φ_s^* . Hence, it suffices to prove that $\mathbb{E}[\xi_2^* - \Phi_s^*] \rightarrow 0$ as first $\varepsilon \rightarrow 0$ and then $K \rightarrow \infty$.

To this end, we follow the ideas in the proof of [1, Lemma 10.3.3], i.e. we subdivide $\Omega_\varepsilon^\delta$ into subboxes of sidelength $\bar{\varepsilon}_K/K$ and construct Gaussian fields Z^l , $l = 1, 2$ on Ω^δ independent of Φ_s and independent of each other which satisfy

$$(4.119) \quad \text{Var}(Z^l(x)) = C\bar{\varepsilon}_K \text{ for } x \in V_i^\delta$$

$$(4.120) \quad c_2K|x - y| \leq \mathbb{E}[(Z^l(x) - Z^l(y))^2] \leq C_2K|x - y|$$

where x, y in the same subbox of sidelength $\bar{\varepsilon}_K/K$ and $c_2, C_2, C > 0$ are absolute constants to be chosen later. Then we set for $(x, y) \in V_{\bar{\varepsilon}_K/K}^{\times 2} \cap V_{1/K, \delta}^{\times 2}$

$$(4.121) \quad \psi_2(x, y) = \Phi_s(x) + Z^1(y) + Z^2(x)$$

$$(4.122) \quad \tilde{\psi}_2(x, y) = \tilde{\eta}(x) + Z^1(y) + Z^2(x)$$

and define $\psi_2^* = \max_{V_{\bar{\varepsilon}_K/K}^{\times 2}} \psi_2$, and similarly $\tilde{\psi}_2^*$. As before we first replace the expectation of ξ_2^* in (4.118) by that of ψ_2^* conditioning on $\Phi_s^{\text{SG}} + X_s^h$ and using the Sudakov-Fernique inequality. Here, the analogue of (4.105) is proved in [1, Lemma 10.3.3] by distinguishing different cases depending on the locations of x and y . The same calculations continue to hold for the fields $\tilde{\xi}_2$ and $\tilde{\psi}_2$, as the only assumptions that enter are the estimates of Lemma 4.4.

From there on, the rest of the argument is completely analogous: using the Borell-Tsirelson inequality together with the Fernique criterion and (4.119)–(4.120), we see that Lemma 4.7 applies, and hence that

$$(4.123) \quad \mathbb{E}[\psi_2^*] \leq \mathbb{E}[\Phi_s^*] + \bar{\varepsilon}_K^q$$

for some $q > 0$. This concludes the proof of Lemma 4.14. \square

Proof of Theorem 4.10. Let $\Psi^* = \max\{\Phi_s(z_i) : i = 1, \dots, K^2, X_K^f(z_i) \geq m_\varepsilon K + g(K)\}$, with $g(K)$ as in (4.58). Then $\Psi^* \leq \Phi_s(\bar{z})$. On the other hand,

$$(4.124) \quad \forall \kappa > 0: \exists \delta > 0, K_0 \in \mathbb{N}, \varepsilon_0 > 0: \forall K \geq K_0, \forall \varepsilon \leq \varepsilon_0: \mathbb{P}(\Phi_s^* > \Psi^* + \kappa) \leq \kappa,$$

which follows using (4.57), Proposition 4.8, and (4.58) since

$$(4.125) \quad \mathbb{P}(\Phi_s^* > \Psi^* + \kappa) \leq \mathbb{P}(\max_{\Omega_\varepsilon^\delta} \Phi_s > \Phi_s(\bar{z}) + \kappa) + \mathbb{P}(\Phi_s^* \neq \max_{\Omega_\varepsilon^\delta} \Phi_s) + \mathbb{P}(\Phi_s(\bar{z}) > \Psi^*).$$

Let ν_K be the law of $\Psi^* - m_\varepsilon$. Then by (4.124) and the definition of the Lévy distance (4.2),

$$(4.126) \quad \forall \kappa > 0: \exists \delta > 0, K_0 \in \mathbb{N}, \varepsilon_0 > 0: \forall K \geq K_0, \varepsilon \leq \varepsilon_0: d(\Phi_s^* - m_\varepsilon, \nu_K) < \kappa.$$

Now, we use the coupling from Proposition 4.13 and Lemma 4.14 for each box V_i to relate the various maxima of Φ_s to the random variables $G_{s,K}^*$ from (4.81). Let $(\rho_K^i, Y_K^i, \mathbf{u}_\delta^i)_i$ be independent

and distributed as $(\rho, Y, \mathbf{u}_\delta)$. Recall that $Z_{s,K}^c \equiv \tilde{X}_{s,K}^c + X_s^h + \Phi_s^{\text{SG}}$ is the overall lattice coarse field of Φ_s and let $\nu_{K,\varepsilon}$ be the law of

$$(4.127) \quad \max_{\{i: \rho_K^i=1\}} Y_K^i + Z_{s,K}^c(\mathbf{u}_\delta^i) - (m_\varepsilon - m_{\varepsilon K}) + g(K).$$

Here, by writing $Z_{s,K}^c(x)$ for $x \in \Omega$ we refer to the coarse field evaluated at the vertex x^ε defined as in section 2.4. Then

$$(4.128) \quad \forall \kappa > 0: \exists \delta > 0, K_1 \in \mathbb{N}, \varepsilon_1 > 0: \forall K \geq K_1, \forall \varepsilon \leq \varepsilon_1: d(\nu_{K,\varepsilon}, \nu_K) \leq \kappa.$$

Indeed, recalling that $\rho_K^i = 1 \iff X_K^f(z_i) \geq m_{\varepsilon K} + \alpha_{1,K,\varepsilon}$ almost surely, it follows from the monotonicity of the $\alpha_{j,K,\varepsilon}$ that, up to an event of probability 0,

$$(4.129) \quad \{\rho_K^i = 0\} \subseteq \{X_K^f(z_i) < m_{\varepsilon K} + \alpha_{C^*k,K,\varepsilon}\} \equiv A_i.$$

Note that on $\{\rho_K^i = 1\} \cap A_i$ we have by (4.88)

$$(4.130) \quad |g(K) + Y_K^i - (X_K^f(z_i) - m_{\varepsilon K})| \leq \bar{\varepsilon}_K.$$

Hence, for any open $B \subseteq \mathbb{R}$,

$$(4.131) \quad \begin{aligned} \nu_{K,\varepsilon}(B) &\leq \mathbb{P}\left(\max_{\{i: \rho_K^i=1\}} X_K^f(z_i) - m_{\varepsilon K} + Z_{s,K}^c(\mathbf{u}_\delta^i) - (m_\varepsilon - m_{\varepsilon K}) \in B^{\bar{\varepsilon}_K}, \cap_i A_i\right) + \mathbb{P}((\cap_i A_i)^c) \\ &\leq \mathbb{P}\left(\max_{\{i: \rho_K^i=1\}} X_K^f(z_i) + Z_{s,K}^c(\mathbf{u}_\delta^i) - m_\varepsilon \in B^{\bar{\varepsilon}_K}\right) + \mathbb{P}((\cap_i A_i)^c). \end{aligned}$$

By Lemma 4.14, we may now replace $Z_{s,K}^c(\mathbf{u}_\delta^i)$ by $Z_{s,K}^c(z_i)$, and we may also replace $\{i: \rho_K^i = 1\}$ by $\{i: X_K^f(z_i) \geq m_{\varepsilon K} + g(K)\}$ on the event $A \equiv \{X_K^f(\bar{z}) \geq m_{\varepsilon K} + g(K) + \varepsilon_K\}$ with ε_K as in Lemma 4.12, as both random variables agree on this event. Thus

$$(4.132) \quad \nu_{K,\varepsilon}(B) \leq \nu_K(B^{\bar{\varepsilon}_K + \kappa/2}) + \kappa/2 + \mathbb{P}((\cap_i A_i)^c) + \mathbb{P}(A^c).$$

Using Lemma 4.11 and a union bound we see that

$$(4.133) \quad \begin{aligned} \mathbb{P}(\cup_i A_i^c) &= \mathbb{P}(\exists i: X_K^f(z_i) \geq m_{\varepsilon K} + \alpha_{C^*k,K,\varepsilon}) \\ &\leq K^2 \mathbb{P}(X_K^f(z_i) \geq m_{\varepsilon K} + C^*k) \leq 1/K. \end{aligned}$$

Thus, by (4.133), (4.58) and the convergence of the sequences $(\varepsilon_K)_K$ and $(\bar{\varepsilon}_K)_K$, (4.128) follows.

Next we observe that $d(\nu_{K,\varepsilon}, \mu_{s,K}) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed, by Lemmas 4.4–4.5, we have the weak convergence of $Z_{s,K}^c(x)$ to $Z_{s,K}^{c,0}(x)$ as $\varepsilon \rightarrow 0$, for every fixed $x \in \Omega_\varepsilon$. Using independence of ρ_K^i, Y_K^i and \mathbf{u}_δ^i from $Z_{s,K}^c$, it follows that, for any open $B \subseteq \mathbb{R}$,

$$(4.134) \quad \begin{aligned} \nu_{K,\varepsilon}(B) &= \mathbb{P}\left(\max_{\{i: \rho_K^i=1\}} g(K) + Y_K^i + Z_{s,K}^c(\mathbf{u}_\delta^i) - (m_\varepsilon - m_{\varepsilon K}) \in B\right) \\ &\leq \mathbb{P}\left(\max_{\{i: \rho_K^i=1\}} g(K) + Y_K^i + Z_{s,K}^{c,0}(\mathbf{u}_\delta^i) - (m_\varepsilon - m_{\varepsilon K}) \in B^\kappa\right) + \kappa, \end{aligned}$$

which, using (4.83), is the claimed convergence.

In summary, by (4.126) and (4.128), we conclude that for any $\kappa > 0$ there is $\delta = \delta(\kappa)$ such that

$$(4.135) \quad \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} d(\Phi_s^* - m_\varepsilon, \mu_{s,K}) < 2\kappa,$$

which proves (4.84), when sending $\kappa \rightarrow 0, \delta \rightarrow 0$. \square

4.6. Limiting distribution of regularised field. In this section, we apply the previous preparation to generalise the proof of [16, Theorem 2.5] and prove the following theorem. The proof is entirely analogous, but we include it for completeness.

Theorem 4.15 (Analogue of [16, Theorem 2.5]). *Let $\mu_{s,K}$ be the law of $G_{s,K}^*$, let $Z_{s,K}$ be as in (4.82), and let α^* be as in Proposition 4.6. Then*

$$(4.136) \quad \lim_{\delta \rightarrow 0} \lim_{K \rightarrow \infty} \frac{\mu_{s,K}((-\infty, x])}{\mathbb{E}[e^{-\alpha^* Z_{s,K} e^{-\sqrt{8\pi}x}}]} = 1.$$

In particular, there is a probability measure $\tilde{\mu}_s$ on \mathbb{R} and a positive random variable Z_s such that $\mu_{s,K} \rightarrow \tilde{\mu}_s$ weakly and

$$(4.137) \quad \tilde{\mu}_s((-\infty, x]) = \mathbb{E}[e^{-\alpha^* Z_s e^{-\sqrt{8\pi}x}}].$$

Remark 4.16. The constant α^* is independent of s . Indeed, it is characterised by Proposition 4.6 which only involves the fine field. Hence, this value is the same as in the corresponding GFF result.

The following result will be needed in the course of the proof.

Lemma 4.17 (Analogue of [16, Lemma 6.5]). *There exists $\gamma > 0$ such that*

$$(4.138) \quad \lim_{K \rightarrow \infty} \mathbb{P}(\max_i Z_{s,K}^{c,0}(\mathbf{u}_\delta^i) \geq \frac{2}{\sqrt{2\pi}} \log K - \gamma \log \log K) = 0.$$

Proof. This is the analogue of [16, Lemma 6.5] and the proof is identical since our fields are coupled up to order 1 with those of the GFF. \square

Proof of Theorem 4.15. In order to demonstrate (4.136) we first establish the analogue of (6.9) in [16]. Let $\mathcal{F}^c = \sigma(Z_{s,K}^{c,0}(\mathbf{u}_\delta^i) : i = 1, \dots, K^2)$ and denote $S_{s,K}^i = Z_{s,K}^{c,0}(\mathbf{u}_\delta^i) - \frac{2}{\sqrt{2\pi}} \log K$. Then, for $x \in \mathbb{R}$,

$$(4.139) \quad \mathbb{P}(G_{s,K}^* \leq x) = \mathbb{E} \left[\prod_{i=1}^{K^2} \left(1 - \mathbb{P}(\rho_K^i(Y_K^i + g(K))) > x - S_{s,K}^i \mid \mathcal{F}^c \right) \right].$$

We will prove this equation by first conditioning on \mathcal{F}^c and then using independence of $(\rho_K^i, Y_K^i)_i$ and \mathcal{F}^c . By the definition of $G_{s,K}^*$ we have

$$(4.140) \quad \mathbb{P}(G_{s,K}^* \leq x) = \mathbb{E} \left[\mathbb{P}(\forall i = 1, \dots, K^2 : \rho_K^i(Y_K^i + g(K)) \leq x - S_{s,K}^i \mid \mathcal{F}^c) \right].$$

Since $(\rho_K^i, Y_K^i)_i$ are independent of $S_{s,K}^i$, a standard result on conditional expectation allows us to write (4.140) as

$$(4.141) \quad \mathbb{P}(G_{s,K}^* \leq x) = \mathbb{E} \left[\mathbb{P}(\forall i = 1, \dots, K^2 : (Y_K^i + g(K))\rho_K^i \leq x - w_i \mid w_i = S_{s,K}^i) \right].$$

Since the random variables Y_K^i, ρ_K^i are independent for different i , we can write the probability as a product, i.e.

$$(4.142) \quad \begin{aligned} \mathbb{P}(G_{s,K}^* \leq x) &= \mathbb{E} \left[\prod_{i=1}^{K^2} \mathbb{P}((Y_K^i + g(K))\rho_K^i \leq x - w_i \mid w_i = S_{s,K}^i) \right] \\ &= \mathbb{E} \left[\prod_{i=1}^{K^2} (1 - \mathbb{P}((Y_K^i + g(K))\rho_K^i > x - S_{s,K}^i \mid \mathcal{F}^c)) \right] \end{aligned}$$

which proves (4.139).

Next we exploit the independence of ρ_K^i and Y_K^i to further simplify the probabilities within the expectation. To this end, we consider the events $D_K^i = \{x - S_{s,K}^i \geq g(K)\}$ and $D_K = \bigcap_{i=1}^{K^2} D_K^i$. By Lemma 4.17 we may assume that D_K occurs. Using the definition of the ρ_K^i and Y_K^i we have

$$(4.143) \quad \begin{aligned} \mathbf{1}_{D_K^i} \mathbb{P}(\rho_K^i(Y_K^i + g(K)) > x - S_{s,K}^i \mid \mathcal{F}^c) &= \mathbf{1}_{D_K^i} \mathbb{P}(\rho_K^i = 1) \mathbb{P}(Y_K^i + g(K) > x - S_{s,K}^i \mid \mathcal{F}^c) \\ &= \mathbf{1}_{D_K^i} \alpha^* m_\delta(x - S_{s,K}^i) e^{-\sqrt{8\pi}(x - S_{s,K}^i)}. \end{aligned}$$

By the definition of D_K^i the right-hand side in (4.143) converges to 0 as $K \rightarrow \infty$. Using $1 - x < e^{-x}$ for $x > 0$, we thus obtain that, on D_K ,

$$(4.144) \quad \begin{aligned} 1 - \mathbb{P}(\rho_K^i(Y_K^i + g(K)) > x - S_{s,K}^i \mid \mathcal{F}^c) &\leq \exp\left(-\alpha^* m_\delta(x - S_{s,K}^i) e^{-\sqrt{8\pi}(x - S_{s,K}^i)}\right) \\ &\leq \exp\left(-(1 - \varepsilon_K) \alpha^* m_\delta(-S_{s,K}^i) e^{-\sqrt{8\pi}(x - S_{s,K}^i)}\right) \end{aligned}$$

for an x -dependent sequence $(\varepsilon_K)_K$ with $\varepsilon_K \rightarrow 0$ as $K \rightarrow \infty$. On the other hand, using $1 - x > e^{-(1+\kappa)x}$ for $\kappa > 0$ and all $x > 0$ small enough,

$$(4.145) \quad 1 - \mathbb{P}(\rho_K^i(Y_K^i + g(K)) > x - S_{s,K}^i \mid \mathcal{F}^c) \geq \exp\left(-(1 + \varepsilon_K) \alpha^* m_\delta(-S_{s,K}^i) e^{-\sqrt{8\pi}(x - S_{s,K}^i)}\right)$$

for a possibly different sequence $(\varepsilon_K)_K$ with $\varepsilon_K \rightarrow 0$ as $K \rightarrow \infty$. Hence, using these bounds together with (4.139), the limit in (4.136) follows. \square

4.7. Proof of Theorem 1.1. By collecting the previous results, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.1, it suffices to show that $\max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon \rightarrow \tilde{\mu}_s$, where

$$(4.146) \quad \tilde{\mu}_s((-\infty, x]) = \mathbb{E}[e^{-\alpha^* Z_s e^{-\sqrt{8\pi}x}}]$$

and Z_s are as in Theorem 4.15. Note that (4.8) can equivalently be stated as

$$(4.147) \quad \mu_0 \stackrel{D}{=} \frac{1}{\sqrt{8\pi}} (X + \log Z^{\text{SG}} + \log \alpha^*).$$

where X is an independent standard Gumbel variable. Moreover, by (4.43) and Lemma 4.2,

$$(4.148) \quad \lim_{\delta \rightarrow 0} \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} d(\max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon, \Phi_s^* - m_\varepsilon) = 0,$$

by Theorem 4.10,

$$(4.149) \quad \lim_{\delta \rightarrow 0} \limsup_{K \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} d(\Phi_s^* - m_\varepsilon, G_{s,K}^*) = 0,$$

and by Theorem 4.15,

$$(4.150) \quad \lim_{\delta \rightarrow 0} \lim_{K \rightarrow \infty} d(G_{s,K}^*, \tilde{\mu}_s) = 0,$$

so that $d(\max_{\Omega_\varepsilon} \tilde{\Phi}_s^{\text{SG}} - m_\varepsilon, \tilde{\mu}_s) \rightarrow 0$, as needed. \square

A Heat kernel estimates

We denote by $p_t^\varepsilon(x)$ the heat kernel on $\varepsilon\mathbb{Z}^d$ and by $p_t^0(x)$ the continuous heat kernel on \mathbb{R}^d :

$$(A.1) \quad p_t^\varepsilon(x) = \varepsilon^{-d} \tilde{p}_{t/\varepsilon^2}(x/\varepsilon), \quad p_t^0(x) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{d/2}},$$

where \tilde{p}_t is the heat kernel on the unit lattice \mathbb{Z}^d . We will also write $\partial f = \partial_\varepsilon f$ for the vector of lattice gradients of a function $f : \varepsilon\mathbb{Z}^d \rightarrow \mathbb{R}$. Thus if $\alpha = (\alpha_1, \dots, \alpha_{|\alpha|})$ is a sequence of $|\alpha|$ unit directions in \mathbb{Z}^d , i.e., $\alpha_i \in \{(\dots, 0, \pm 1, 0, \dots)\}$ then $\partial^\alpha = \prod_{i=1}^{|\alpha|} \partial^{\alpha_i}$ where $\partial^{\alpha_i} f(x) = \varepsilon^{-1}(f(x + \alpha_i) - f(x))$.

Lemma A.1. *The heat kernel p_t^ε on $\varepsilon\mathbb{Z}^d$ satisfies the following upper bounds for $t \geq \varepsilon^2$, $x \in \mathbb{Z}^d$, and all sequences of unit vectors α :*

$$(A.2) \quad |\partial^\alpha p_t^\varepsilon(x)| \leq O_\alpha(t^{-d/2-|\alpha|/2} e^{-c|x|/\sqrt{t}}).$$

Moreover, for all $x \in \varepsilon\mathbb{Z}^d$ and $k \geq 4$,

$$(A.3) \quad |p_t^\varepsilon(x) - p_t^0(x)| \leq O\left(\frac{1}{t^{d/2}}\right) \times \frac{\varepsilon^2}{t} \left[\left(\left(\frac{|x|}{\sqrt{t}}\right)^k + 1 \right) e^{-|x|^2/t} + \left(\frac{\varepsilon^2}{t}\right)^{(k-3)/2} \right].$$

Proof. For the upper bound (A.2), see for example [7, Lemma A.1] and rescale the statements there by $(t, x) \rightarrow (t/\varepsilon^2, x/\varepsilon)$. The convergence estimate (A.3) is [39, Theorem 2.1.3] again after the previous rescaling. \square

The heat kernel on a torus of side length L is given by

$$(A.4) \quad p_t^{\varepsilon,L}(x) = \sum_{y \in \mathbb{Z}^d} p_t^\varepsilon(x + yL).$$

Lemma A.2. *The torus heat kernel satisfies, for $t \geq \varepsilon^2$, $|x|_\infty \leq L/2$,*

$$(A.5) \quad \partial^\alpha p_t^{\varepsilon,L}(x) = \partial^\alpha p_t^\varepsilon(x) + O_\alpha(t^{-|\alpha|/2} L^{-d} e^{-cL/\sqrt{t}}).$$

Moreover, for any $t > 0$, as $\varepsilon \rightarrow 0$,

$$(A.6) \quad \sup_{x \in \Omega_\varepsilon} |p_t^{\varepsilon,L}(x) - p_t^{0,L}(x)| \lesssim (t^{-d/2} + L^{-d}) \frac{\varepsilon^2}{t} \log\left(\frac{\varepsilon^2}{t}\right)^d.$$

Proof. The estimate (A.5) is straightforward from (A.4) and Lemma A.1. Moreover, with (A.2) for $|y| \leq N$ and (A.3) for $|y| > N$, we obtain from (A.4) that

$$(A.7) \quad \begin{aligned} |p_t^{\varepsilon,L}(x) - p_t^{0,L}(x)| &\lesssim \inf_{N \geq 1} \left[\sum_{0 < |y| \leq N} t^{-d/2} \frac{\varepsilon^2}{t} + \sum_{|y| > N} t^{-d/2} e^{-c|y|L/\sqrt{t}} \right] \\ &\lesssim \inf_{N \geq 1} \left[\left(\frac{N}{\sqrt{t}}\right)^d \frac{\varepsilon^2}{t} + \left(\frac{1}{L}\right)^d e^{-cNL/\sqrt{t}} \right] \lesssim (t^{-d/2} + L^{-d}) \frac{\varepsilon^2}{t} \log\left(\frac{\varepsilon^2}{t}\right)^d \end{aligned}$$

where in the last inequality we have chosen $N = (1/c) \log(t/\varepsilon^2) \sqrt{t}/L + 1$. \square

We also need the heat kernel on a square with Dirichlet boundary conditions. Thus let $\Gamma \subset \mathbb{R}^2$ be the union of horizontal and vertical lines that form a regular grid of spacing $1/K$, centred so that 0 lies at a vertex of this grid. Let Δ_Γ be the Laplacian with Dirichlet boundary conditions on Γ . We denote the square containing 0 bounded by Γ by V and set $V^\delta = \{x \in V : \text{dist}(x, \Gamma) \geq \delta/K\}$. Moreover, if $\Gamma \subset \varepsilon\mathbb{Z}^2$ these definitions have obvious ε -dependent versions. In the following, we will often omit the index ε and write, for example, $p_t^\Gamma(x, y)$ instead of $p_t^{\varepsilon,\Gamma}(x, y)$ when either $\varepsilon = 0$ or $\varepsilon > 0$.

Lemma A.3. *The Dirichlet heat kernel $p_t^\Gamma(x, y) = e^{t\Delta_\Gamma}(x, y)$ (where $\varepsilon = 0$ or $\varepsilon > 0$) satisfies*

$$(A.8) \quad \sup_{x, y \in V} |\partial^\alpha p_t^\Gamma(x, y)| \leq O_\alpha(t^{-d/2-|\alpha|/2} e^{-ctK^2})$$

and, for any $t > 0$, as $\varepsilon \rightarrow 0$,

$$(A.9) \quad \sup_{x, y \in V} |p_t^{\Gamma, \varepsilon}(x, y) - p_t^{\Gamma, 0}(x, y)| \rightarrow 0.$$

Moreover,

$$(A.10) \quad \max_{x, y \in V^\delta} |\partial^\alpha p_t^\Gamma(x, y) - \partial^\alpha p_t(x, y)| \leq O_\alpha(t^{-d/2-|\alpha|/2} e^{-c\delta/\sqrt{K^2 t}}).$$

Proof. Restricted to V , the Dirichlet Laplacian $-\Delta_\Gamma^\varepsilon$ has eigenfunctions and eigenvalues

$$(A.11) \quad e_k(x) = 2^{d/2} K^d \prod_{i=1}^d \sin(k_i x_i), \quad \lambda^\varepsilon(k) = \varepsilon^{-2} \sum_{i=1}^d (2 - 2 \cos(\varepsilon k_i)) \sim |k|^2,$$

where $k \in V^* = K\pi\{l \in \mathbb{Z}^2 : 0 < l_i < 1/\varepsilon\}$; see, for example, [30, Section 9.5]. Therefore

$$(A.12) \quad p_t^\Gamma(x, y) = \sum_{k \in V^*} e^{-\lambda^\varepsilon(k)t} e_k(x) e_k(y),$$

and it follows that

$$(A.13) \quad |\partial^\alpha p_t^\Gamma(x, y)| \lesssim K^d \sum_{k \in V^*} e^{-c|k|^2 t} |k|^{|\alpha|} \lesssim K^d \sum_{q \in \mathbb{Z}^2 \setminus 0} e^{-cK^2 |q|^2 t} (K|q|)^{|\alpha|} \lesssim t^{-d/2-|\alpha|/2} e^{-cK^2 t}$$

which is (A.8).

To see (A.9) and (A.10), we start from the following alternative representation for the Dirichlet heat kernel. For $y \in \mathbb{R}^2 \setminus \Gamma$, let $\{y_j\}$ be the set of points obtained by reflecting $y = y_0$ about the lines in Γ , and denote by σ_j the number of reflections needed to obtain y_j from y_0 . Then

$$(A.14) \quad e^{t\Delta_\Gamma}(x, y) = \begin{cases} \sum_{j=0}^{\infty} (-1)^{\sigma_j} e^{t\Delta}(x, y_j) & (x, y \text{ in the same connected component of } \mathbb{R}^2 \setminus \Gamma) \\ 0 & (\text{otherwise}). \end{cases}$$

Indeed, the proof of (A.14) is analogous to that in [30, Section 7.5] for the Green function: Denoting the right-hand side of (A.14) by $p_t^\Gamma(x, y)$, one has $\partial_t p_t^\Gamma(x, y) = \Delta p_t^\Gamma(x, y)$, and the reflections ensure that $p_t^\Gamma(x, y) = 0$ if $y \in \Gamma$. Therefore, for x and y in the same connected component of $\mathbb{R}^2 \setminus \Gamma$,

$$(A.15) \quad p_t(x, y) - p_t^\Gamma(x, y) = - \sum_{j \neq 0} (-1)^{\sigma_j} p_t(x, y_j).$$

The convergence (A.9) then follows analogously to (A.7). To see (A.10), notice that the condition $\text{dist}(x, \Gamma), \text{dist}(y, \Gamma) \geq \delta/K$ implies that $\text{dist}(x, y_j) \geq 2\delta/K$ and hence that $|\partial^\alpha p_t(x, y_j)| \lesssim t^{-d/2-|\alpha|/2} e^{-c\delta/\sqrt{K^2 t}}$. More generally, the sum over all reflected points is bounded by

$$(A.16) \quad \sum_{j \neq 0} |\partial^\alpha p_t(x, y_j)| \lesssim t^{-d/2-|\alpha|/2} \sum_{q \in \mathbb{Z}^2 \setminus 0} e^{-c\delta|q|/\sqrt{K^2 t}} \lesssim t^{-d/2-|\alpha|/2} (\sqrt{K^2 t}/\delta)^d e^{-c\delta/\sqrt{K^2 t}}.$$

This gives

$$(A.17) \quad |\partial^\alpha p_t^\Gamma(x, y) - \partial^\alpha p(x, y)| \leq O_\alpha(t^{-d/2-|\alpha|/2} (\sqrt{K^2 t}/\delta)^d e^{-c\delta/\sqrt{K^2 t}}).$$

On the other hand, by (A.2) and (A.8) also

$$(A.18) \quad |\partial^\alpha p_t^\Gamma(x, y) - \partial^\alpha p_t(x, y)| \leq |\partial^\alpha p_t^\Gamma(x, y)| + |\partial^\alpha p_t(x, y)| \leq O_\alpha(t^{-d/2-|\alpha|/2}).$$

Using (A.17) if $K^2 t/\delta \leq 1$ and (A.18) if $K^2 t/\delta > 1$ gives (A.10). \square

Acknowledgements

We thank Marek Biskup and Pierre-François Rodriguez for essential discussions that inspired the maximum problem studied in this article and the approach through a coupling, and for important feedback on a preliminary version of this manuscript. We thank Thierry Bodineau and Christian Webb for very helpful general discussions related to the sine-Gordon model, and we thank Ofer Zeitouni for a helpful discussion which brought [1] to our attention. RB gratefully acknowledges funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 851682 SPINRG). MH was partially supported by the UK EPSRC grant EP/L016516/1 for the Cambridge Centre for Analysis.

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