Essays on Bargaining and Markets

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To my grandparents, Chan Hou-chu and Law Lam-hei…

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Abstract

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Chapter 1: Inefficiency in a Frictionless Market

Gale and Sabourian (2006) discuss the existence of inefficient Markov-perfect equilibrium (MPE) in a heterogeneous market. This paper shows that the example they provide cannot be supported as a MPE. Indeed, with two buyers and two sellers, the dispersion of bargaining positions is not sufficiently wide to support any inefficient trades in their setup. I then prove that their conjecture is correct by constructing a continuum of inefficient MPE with three buyers and three sellers, where the dynamics of continuation payoffs is rich enough to support wide dispersion of bargaining positions at the first stage, which in turn renders inefficient trades individually rational. This suggests that the number of players in a market could be essential for allocative inefficiency.

Chapter 2: Re-trading and Efficient Allocation

Inefficient Markov-perfect equilibria (MPE) are prevalent in dynamic matching and bargaining games (DMBG). I observe this inefficiency is caused by a substantial friction commonly assumed in the literature: players must exit the market after trading once. In response, this paper studies a simple heterogeneous market with a finite number of players, where re-trading is allowed and each player can choose to exit the market at any period. I show that, for

sufficiently small search cost, all MPE are allocative efficient. The reasoning is conceptually distinct from the existing literature, and indispensable to models where players are allowed to exit the market amidst the game.

Chapter 3: Redistributive Effects of Search Frictions

van den Berg and van Vuuren (2010) recorded the empirical puzzle that search frictions have a positive impact on the wages of managers, as against the prediction of the canonical model of Pissarides (2000). I argue the discrepancy is caused by the assumption of perfectly elastic vacancy supply in the latter, which is inappropriate for managerial labour markets. In response, I construct a search and matching model with perfectly inelastic vacancy supply, and show that the comparative statics with respect to search frictions are totally different. The results explain the divergent welfare impacts of search frictions on workers at different levels of the corporate hierarchy. They also show that lowering search frictions is not necessarily Pareto-improving even in a homogeneous setting.

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Chapter 1

Inefficiency in a Frictionless Market

1.1 Introduction

Since the start of the dynamic matching and bargaining games (DMBG) literature, a central question has been whether or not the Walrasian outcomes can be micro-founded in some extensive form games in which the market process is decentralized and explicitly modelled. For models with a continuum of agents, Lauermann (2013) provides a nice characterization result which explains why certain DMBG fail to approximate the Walrasian outcomes even as trading frictions vanish (see Rubinstein and Wolinsky (1985) and Rubinstein and Wolinsky (1990)), whereas others succeed with alternative trading protocols (see Gale (1987)).

DMBG with a finite number of agents, nevertheless, remain highly intractable and no similar characterization result has been developed to date. For deterministic matching models with zero friction, Gale and Sabourian (2003) and Gale and Sabourian (2005) show that all Markov-perfect equilibria (MPE) must be perfectly competitive, and the Markov assumption can be justified on the grounds of complexity costs. For random matching models with zero or small friction, Gale and Sabourian (2006) and Elliott and Nava (2019) *argue* that there exists a continuum of MPE that are inefficient, and thus non-competitive.

In this chapter, I first show that the inefficient strategy profile described in Gale and Sabourian (2006) which involves two buyers and two sellers cannot be supported as a MPE. I then proceed to prove a stronger result that any MPE involving only two buyers and two sellers must be efficient in their setup. Finally I construct a continuum of inefficient MPE involving three buyers and three sellers in their setup. The contrasting results are significant as they suggest that random matching alone does not guarantee the existence of inefficient MPE, and that conditions on the number of players may play a role in any characterization results about allocative efficiency.

This chapter is structured as follows. Section 1.2 describes the game setup. Section 1.3 proves several results on markets with two buyers and two sellers. Section 1.4 constructs an example of inefficient MPE allocations, the main result of this chapter. Section 1.5 discusses the idea behind the construction and concludes.

1.2 The market game

The setup is identical to Gale and Sabourian (2006). There are finite, equal number of buyers and sellers in the market. A single indivisible good is exchanged for money and each player wants to consume at most one unit of the good. Buyer i's valuation of the good is denoted by $v_i \geq 0$ and seller j's production cost of the good by $w_j \geq 0$. I assume that these values are all distinct. Trades take place in a sequence of periods $t = 1, 2, \ldots$. There is no discounting. A market comprising n buyers and n sellers is called a $n-$market.

Denote by R^t the set of players remaining in the market in period t. As will be evident, the set R^t must contain equal number of buyers and sellers. At the start of each period t, one buyer and one seller is drawn from R^t uniformly randomly; in other words, the matching probabilities depend only on the set R^t but not on t itself. Within the pair, each player is selected to be the proposer and the other the responder with equal chance. Then the proposer makes his offer, and the responder, having observed the offer, accepts or rejects the offer. If the offer is accepted, the good is traded at the agreed price and both players exit the market. If the offer is rejected, there is no trade and all players in R^t proceed to period $t+1$ with the same endowments. I impose the tie-breaking rule that matched players always reach an agreement whenever indifferent.

I study the Markov-perfect equilibria (MPE) of this game. This requires that proposals depend only on the set of remaining players and the matched pair in that period and that the responses depend only on the set of remaining players, the matched pair, and the proposal in that period. I call any profile of such strategies a Markov strategy profile, and refer to any subgame that begins with the set of remaining players R as a $R-$subgame. Let N denote the full set of players before any trades take place.

1.3 Efficiency of 2-markets

Given any Markov strategy profile f and admissible set of remaining players R, denote by $v_i^R(f)$ and $w_j^R(f)$ respectively the continuation payoff of any buyer i and seller j that belong to R in any $R-$subgame, and by $M^R(f)$ the set of buyer-seller pairs that trade first with positive probability in any $R-$subgame. If f is a MPE, then $M^R(f) = \{(i,j) : v_i - w_j \geq v_i^R(f) + w_j^R(f)\}$; the inequality is weak due to the tie-breaking rule. In the following, their dependence on f might be omitted for notational simplicity whenever unambiguous.

As discussed in Section 1.1, one important question is whether there exists any inefficient MPE in this game. Gale and Sabourian (2006) describe a Markov strategy profile where extra-marginal players trade with positive probability, leading to allocative inefficiency. Now I show that their example cannot be supported as a MPE, and therefore cannot serve to prove the existence of inefficient MPE.

Proposition 1.1. Let $(v_{1'}, v_{2'}) = (1, v)$ and $(w_1, w_2) = (0, 1-v)$ define a 2-market for

$v \in (\frac{1}{8}, \frac{1}{2})$. Suppose there exists a Markov strategy profile f such that

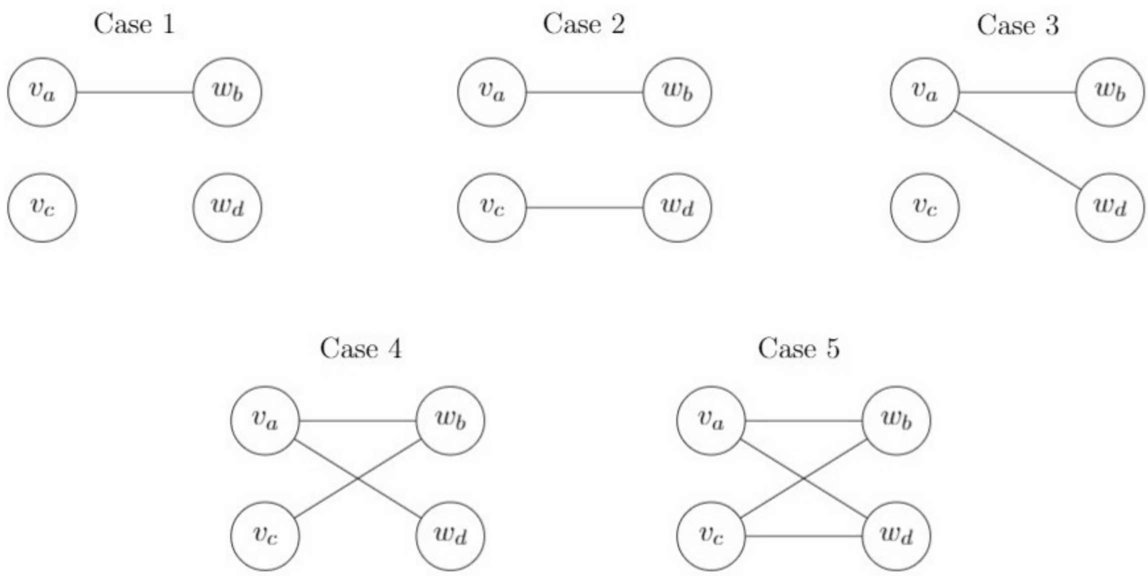
$$v_{1'}^N(f) = w_1^N(f) = \frac{1}{3}v + \frac{5}{24} \quad \text{and} \quad v_{2'}^N(f) = w_2^N(f) = \frac{1}{3}v - \frac{1}{24}$$

Then f cannot be a MPE.

Proof. Suppose one such Markov strategy profile f exists and, for contradiction, is a MPE. Since $v_{2'} < w_2$, obviously it is not individually rational for the pair $(2', 2)$ to trade; but neither is it for the pair $(1', 2)$, because $v_{1'}^N(f) + w_2^N(f) = \frac{2}{3}v + \frac{1}{6} > v = v_{1'} - w_2$. This means seller 2 never trades on the equilibrium path, contradicting the specification that $w_2^N(f) = \frac{1}{3}v - \frac{1}{24} > 0$. □

The question then remains whether or not there exists any inefficient MPE. I next show that all MPE in any 2-markets must be efficient. This justifies my subsequent search for inefficient MPE in 3-markets, the analysis of which is substantially more complicated.

Fig. 1.1 All cases of first-trade possibilities in any 2-market



Proposition 1.2. In any 2-markets, all MPE must be efficient.

Proof. Fix an arbitrary 2-market comprising buyers $\{a,c\}$ and sellers $\{b,d\}$. Fix a MPE. Without ordering these buyers and sellers, there are in total five cases of first-trade possibilities. These are summarized in Figure 1.1, where a link exists between nodes v_i and w_j if and only if $(i,j) \in M^N$.

<u>Case 1: $M^N = \{(a,b)\}$</u>

Suppose for contradiction buyer a is extra-marginal. Individual rationality implies seller b is infra-marginal, so gains from trade of $v_c - w_b > 0$ is available. But $(c,b) \notin M^N$ means that $v_c^N + w_b^N > v_c - w_b$, which violates feasibility. Thus buyer a must be infra-marginal. By symmetric argument, seller b is also infra-marginal. Hence the MPE outcome is efficient.

<u>Case 2: $M^N = \{(a,b),(c,d)\}$</u>

This means $w_b + w_b^N > v_c - v_c^N \geq w_d + w_d^N > v_a - v_a^N$. But then $w_b + w_b^N > v_a - v_a^N$, contradicting $(a,b) \in M^N$. This case is thus ruled out.

<u>Case 3: $M^N = \{(a,b),(a,d)\}$</u>

This means buyer a trades first with probability one, with either seller b or d. The first-stage payoff of buyer a must then satisfy

$$v_a^N = \frac{1}{2}\left[\frac{1}{2}v_a^N + \frac{1}{2}(v_a - w_b - w_b^N)\right] + \frac{1}{2}\left[\frac{1}{2}v_a^N + \frac{1}{2}(v_a - w_d - w_d^N)\right]$$

The first bracketed term, for instance, is a's payoff from trading with b, wherein a is either offered his reservation payoff v_a^N or given the chance to offer b her reservation price $w_b + w_b^N$. Re-arranging the above yields $2(v_a - v_a^N) = (w_b + w_b^N) + (w_d + w_d^N)$. By the hypothesis, it must be that $v_a - v_a^N = w_b + w_b^N = w_d + w_d^N > v_c - v_c^N$. But then

$$w_b^N = \frac{1}{2}\left[\frac{1}{2}w_b^N + \frac{1}{2}(v_a - v_a^N - w_b)\right] + \frac{1}{2}w_b^{\{c,b\}} = \frac{1}{2}w_b^N + \frac{1}{2}w_b^{\{c,b\}} = w_b^{\{c,b\}}$$

$$w_d^N = \frac{1}{2}\left[\frac{1}{2}w_d^N + \frac{1}{2}(v_a - v_a^N - w_d)\right] + \frac{1}{2}w_d^{\{c,d\}} = \frac{1}{2}w_d^N + \frac{1}{2}w_d^{\{c,d\}} = w_d^{\{c,d\}}$$

In other words, the first-stage payoffs of b and d are precisely their respective continuation payoffs from being left alone with c after a has traded and exited the market in the first stage. Let $w_b > w_d$ WLOG. It remains to consider three sub-cases.

Sub-case 3.1: $v_c > w_b$

This means all players are infra-marginal. But then $v_a + v_c - w_b - w_d < v_a^N + v_c^N + w_b^N + w_d^N$, which violates feasibility.

Sub-case 3.2: $w_b > v_c > w_d$

But then $w_d^{\{c,d\}} = w_d^N = w_b + w_b^N - w_d > v_c - w_d$, which violates feasibility.

Sub-case 3.3: $w_d > v_c$

This implies $w_b^{\{c,b\}} = w_d^{\{c,d\}} = 0$ and thus $w_b^N = w_d^N$. But then $w_b = w_d$, contradiction.

Since all three sub-cases lead to contradiction, this case is ruled out.

Case 4: $M^N = \{(a,b),(a,d),(c,b)\}$

This means $v_a - v_a^N \geq w_d + w_d^N > v_c - v_c^N \geq w_b + w_b^N$, which imply $v_a > w_d$, $v_c > w_b$ and $v_a > w_b$. It remains to show $v_c > w_d$, because this will imply all players are infra-marginal, and hence the MPE outcome is efficient. Before proceeding, notice that the MPE outcome in any 1-market must be efficient, and hence $v_i^{\{i,j\}} + w_j^{\{i,j\}} = \max\{0, v_i - w_j\}$ for any buyer i and seller j. This helps me write down the first-stage payoffs.

First, the first-stage payoff of buyer a is given by $\frac{1}{3}[\frac{1}{2}v_a^N + \frac{1}{2}(v_a - w_b - w_b^N)] + \frac{1}{3}[\frac{1}{2}v_a^N + \frac{1}{2}(v_a - w_d - w_d^N)] + \frac{1}{3}[v_a - w_d - w_d^{\{a,d\}}]$. The last bracketed term is a's payoff $v_a^{\{a,d\}}$, in case he fails to trade first, from entering the continuation 1-market with d. Yet if a deviates and always refuses to trade with d in the first stage, for instance by making an unacceptable offer to d and rejecting all proposals from d, his first-stage payoff is instead $\frac{1}{2}[\frac{1}{2}v_a^N + \frac{1}{2}(v_a - w_b - w_b^N)] + \frac{1}{2}[v_a - w_d - w_d^{\{a,d\}}]$. To prevent such deviation, it must be that

$$w_d^{\{a,d\}} - w_d^N \geq \frac{1}{2}[v_a - v_a^N - w_b - w_b^N] \tag{1.1}$$

Second, the first-stage payoff of seller b is given by $\frac{1}{3}[\frac{1}{2}w_b^N + \frac{1}{2}(v_a - v_a^N - w_b)] + \frac{1}{3}[\frac{1}{2}w_b^N + \frac{1}{2}(v_c - v_c^N - w_b)] + \frac{1}{3}[v_c - v_c^{\{c,b\}} - w_b]$. Yet if b deviates and always refuses to trade with c in the first stage, his first-stage payoff is instead $\frac{1}{2}[\frac{1}{2}w_b^N + \frac{1}{2}(v_a - v_a^N - w_b)] + \frac{1}{2}[v_c - v_c^{\{c,b\}} - w_b]$. To prevent such deviation, it must be that

$$v_c^{\{c,b\}} - v_c^N \geq \frac{1}{2}[v_a - v_a^N - w_b - w_b^N] \tag{1.2}$$

Third, the first-stage payoff of buyer c is given by $\frac{1}{3}[\frac{1}{2}v_c^N + \frac{1}{2}(v_c - w_b - w_b^N)] + \frac{1}{3}v_c^{\{c,b\}} + \frac{1}{3}v_c^{\{c,d\}}$. Yet if c deviates and always refuses to trade with b in the first stage, his first-stage payoff is instead $\frac{1}{2}v_c^{\{c,b\}} + \frac{1}{2}v_c^{\{c,d\}}$. To prevent such deviation, it must be that

$$v_c - v_c^{\{c,d\}} - w_b - w_b^N \geq v_c^{\{c,b\}} - v_c^N \tag{1.3}$$

Fourth, the first-stage payoff of seller d is given by $\frac{1}{3}[\frac{1}{2}w_d^N + \frac{1}{2}(v_a - v_a^N - w_d)] + \frac{1}{3}w_d^{\{a,d\}} + \frac{1}{3}w_d^{\{c,d\}}$. Yet if d deviates and always refuses to trade with a in the first stage, his first-stage payoff is instead $\frac{1}{2}w_d^{\{a,d\}} + \frac{1}{2}w_d^{\{c,d\}}$. To prevent such deviation, it must be that

$$v_a - v_a^N - w_d - w_d^{\{c,d\}} \geq w_d^{\{a,d\}} - w_d^N \tag{1.4}$$

Summing inequalities (1.1) through (1.4) yields $v_c - w_d \geq v_c^{\{c,d\}} + w_d^{\{c,d\}} \geq 0$. But then $v_c > w_d$ as desired.

Case 5: $M^N = \{(a,b),(a,d),(c,b),(c,d)\}$

This implies all players are infra-marginal, hence the MPE outcome is efficient. □

To construct inefficient MPE, I need to ensure the continuation payoff of some infra-marginal player is low enough to render trade with some extra-marginal player profitable. This requires the reservation prices of various infra-marginal players to be significantly disperse and away from the *competitive interval*. However, such dispersion in any 2-markets is severely restricted, because (i) a player's current reservation price is a convex combination of all players' continuation reservation prices in the next stage on the equilibrium path, but (ii) the continuation reservation prices of any buyer i and seller j necessarily coincide in their continuation 1-market ($v_i - v_i^{\{i,j\}} = w_j + w_j^{\{i,j\}}$) if $v_i > w_j$. Indeed, Case 4 in the proof of Proposition 1.2 has precisely exploited this coincidence. This special feature of 2-markets explains the impossibility of inefficient trades.

1.4 Existence of inefficient MPE

In response, I aim to construct inefficient MPE in 3-markets by exploiting the dispersion of reservation prices in continuation 2-markets. In other words, such MPE must support more than one reservation price in certain continuation 2-markets. With this observation in mind, I now construct the desired MPE. For ease of understanding, the valuation and strategies on both sides of the market are constructed to be symmetrical.

Proposition 1.3. Let $(v_{1'}, v_{2'}, v_{3'}) = (2, 2-\psi, 1-\psi)$ and $(w_1, w_2, w_3) = (0, \psi, 1+\psi)$ define a 3-market for $\psi \in (0, 1/30)$. Then there exists an inefficient MPE.

Proof. Since $v_{1'} > v_{2'} > w_3 > v_{3'} > w_2 > w_1$, buyer $3'$ and seller 3 are the only extra-marginal players in the described 3-market. Since inefficiency cannot arise in any continuation 2-markets, my aim is to construct a MPE such that these extra-marginal players trade in the first stage with positive probability.

The proposed Markov strategy profile can be completely described by specifying, for each admissible $R-$subgame, (i) M^R, and (ii) $v_i - v_i^R$ and $w_j + w_j^R$ for all $i, j \in R$. Part (ii) represents the reservation prices of all players in the subgame, and pins down the response rules in the usual way. Part (i) represents the possible first-trades in the subgame, and pin down the proposal rules of all players as follows: if a pair belongs to M^R, then the proposer offers the responder's reservation price; otherwise, the proposer offers his own reservation price. To verify the proposed strategy profile is indeed a MPE, it suffices to check, for each admissible $R-$subgame, (1) v_i^R and w_j^R are non-negative and attained by the continuation strategy profile for all $i, j \in R$, and (2) $M^R = \{(i, j) : v_i - w_j \geq v_i^R + w_j^R\}$.

I now proceed to describe the continuation strategy profile in each of the continuation markets (subgames), starting from those with only one buyer and one seller.

Continuation 1-markets (9 in total)

For each buyer-seller pair $(i, j) \neq (3', 3)$, it suffices to specify $q^{\{i,j\}} \equiv v_i - v_i^{\{i,j\}} \in [w_j, v_i]$, because efficiency pins down $w_j + w_j^{\{i,j\}} = v_i - v_i^{\{i,j\}}$; for $(i, j) = (3', 3)$, trivially $M^{\{3',3\}} = \emptyset$ and $v_{3'}^{\{3',3\}} = w_3^{\{3',3\}} = 0$. I propose the following:

1. $q^{\{1',1\}} = q^{\{1',2\}} = q^{\{2',1\}} = q^{\{2',2\}} = 1$

2. $q^{\{3',2\}} = \psi$

3. $q^{\{2',3\}} = 2 - \psi$

4. $q^{\{3',1\}} = \frac{1}{2}(1 + \psi)$

5. $q^{\{1',3\}} = \frac{1}{2}(3 - \psi)$

Since all of the above satisfy $q^{\{i,j\}} \in [w_j, v_i]$, the induced continuation payoffs are non-negative, and thus constitute a MPE in the respective subgames.

Continuation 2-markets (9 in total)

$N \setminus \{i', j\}-$subgame for $i, j \in \{1,2\}$

For $i, j \in \{1,2\}$, the subgame consists of $v_{(3-i)'} > w_3 > v_{3'} > w_{3-j}$. By Proposition 1.2, the MPE of any 2-markets must be efficient, thus I propose $M^{N\setminus\{i',j\}} = ((3-i)', 3-j)$. To ensure the extra-marginal players do not trade, I propose $v_{(3-i)'} - v_{(3-i)'}^{N\setminus\{i',j\}} = w_{3-j} + w_{3-j}^{N\setminus\{i',j\}} = 1$. But then buyer 3' and seller 3 will never trade, pinning down $v_{3'}^{N\setminus\{i',j\}} = w_3^{N\setminus\{i',j\}} = 0$. It can be easily checked that the above constitute a MPE in the subgame.

$N \setminus \{3', 3\}-$subgame

The subgame consists of $v_{1'} > v_{2'} > w_2 > w_1$, such that all four players are infra-marginal. I propose $M^{N\setminus\{3',3\}} = \{(1',1),(1',2),(2',1),(2',2)\}$ and $v_{1'} - v_{1'}^{N\setminus\{3',3\}} = v_{2'} - v_{2'}^{N\setminus\{3',3\}} = w_1 + w_1^{N\setminus\{3',3\}} = w_2 + w_2^{N\setminus\{3',3\}} = 1$. Evidently the proposed reservation prices support $M^{N\setminus\{3',3\}}$ and induce non-negative payoffs. To see such payoffs are attained by the proposal, notice that $q^{\{1',1\}} = q^{\{1',2\}} = q^{\{2',1\}} = q^{\{2',2\}} = 1$, so it cannot be mutually individually rational for any pair to trade in the first stage at any other prices.

$N \setminus \{3', j\}$−subgame for $j \in \{1,2\}$

For $j \in \{1,2\}$, the subgame consists of $v_{1'} > v_{2'} > w_3 > w_{3-j}$. I propose $M^{N\setminus\{3',j\}} = \{(1', 3-j), (2', 3-j), (2', 3)\}$ and

$$v_{2'} - v_{2'}^{N\setminus\{3',j\}} = w_3 + w_3^{N\setminus\{3',j\}} = \frac{1}{4}(7-3\psi)$$

$$v_{1'} - v_{1'}^{N\setminus\{3',j\}} = w_{3-j} + w_{3-j}^{N\setminus\{3',j\}} = \frac{1}{4}(5-\psi)$$

Evidently the proposed reservation prices support $M^{N\setminus\{3',j\}}$ and imply non-negative payoffs. It remains to verify that the payoffs are attained by the proposal.

Payoff of buyer 1':

$$v_{1'}^{N\setminus\{3',j\}} = \frac{1}{3}\left[\frac{1}{2}v_{1'}^{N\setminus\{3',j\}} + \frac{1}{2}\left(v_{1'} - \frac{1}{4}(5-\psi)\right)\right] + \frac{1}{3}\sum_{\ell\in\{3-j,3\}}(v_{1'} - q^{\{1',\ell\}})$$

The first bracketed term is his payoff from trading first with $3-j$; if instead $2'$ trades first, then $1'$ proceeds to a continuation 1-market with either $3-j$ or 3. Re-arranging the expression and substituting terms yields the desired result.

$$v_{1'} - v_{1'}^{N\setminus\{3',j\}} = \frac{1}{20}(5-\psi) + \frac{2}{5}(q^{\{1',3-j\}} + q^{\{1',3\}}) = \frac{1}{4}(5-\psi)$$

Payoff of buyer 2':

$$\begin{aligned} v_{2'}^{N\setminus\{3',j\}} &= \frac{1}{3}\left[\frac{1}{2}v_{2'}^{N\setminus\{3',j\}} + \frac{1}{2}\left(v_{2'} - \frac{1}{4}(7-3\psi)\right)\right] \\ &\quad + \frac{1}{3}\left[\frac{1}{2}v_{2'}^{N\setminus\{3',j\}} + \frac{1}{2}\left(v_{2'} - \frac{1}{4}(5-\psi)\right)\right] + \frac{1}{3}(v_{2'} - q^{\{2',3\}}) \\ \Leftrightarrow\ v_{2'} - v_{2'}^{N\setminus\{3',j\}} &= \frac{1}{4}\left(\frac{1}{4}(7-3\psi) + \frac{1}{4}(5-\psi)\right) + \frac{1}{2}q^{\{2',3\}} = \frac{1}{4}(7-3\psi) \end{aligned}$$

Payoff of seller $3-j$:

$$w_{3-j}^{N\setminus\{3',j\}} = \frac{1}{3}\left[\frac{1}{2}w_{3-j}^{N\setminus\{3',j\}} + \frac{1}{2}\left(\frac{1}{4}(7-3\psi) - w_{3-j}\right)\right]$$
$$+\frac{1}{3}\left[\frac{1}{2}w_{3-j}^{N\setminus\{3',j\}} + \frac{1}{2}\left(\frac{1}{4}(5-\psi) - w_{3-j}\right)\right] + \frac{1}{3}(q^{\{1',3-j\}} - w_{3-j})$$
$$\Leftrightarrow w_{3-j} + w_{3-j}^{N\setminus\{3',j\}} = \frac{1}{4}\left(\frac{1}{4}(7-3\psi) + \frac{1}{4}(5-\psi)\right) + \frac{1}{2}q^{\{1',3-j\}} = \frac{1}{4}(5-\psi)$$

Payoff of seller 3:

$$w_{3}^{N\setminus\{3',j\}} = \frac{1}{3}\left[\frac{1}{2}w_{3}^{N\setminus\{3',j\}} + \frac{1}{2}\left(\frac{1}{4}(7-3\psi) - w_{3}\right)\right] + \frac{1}{3}\sum_{k\in\{1,2\}}(q^{\{k',3\}} - w_3)$$
$$\Leftrightarrow w_3 + w_{3}^{N\setminus\{3',j\}} = \frac{1}{20}(7-3\psi) + \frac{2}{5}(q^{\{1',3\}} + q^{\{2',3\}}) = \frac{1}{4}(7-3\psi)$$

Therefore the induced payoffs are indeed attained by the proposal, and hence the above constitute a MPE in the subgame.

$N\setminus\{i',3\}-$subgame for $i\in\{1,2\}$

For $i\in\{1,2\}$, the subgame consists of $v_{(3-i)'} > v_{3'} > w_2 > w_1$. I propose $M^{N\setminus\{i',3\}} = \{((3-i)',1),((3-i)',2),(3',2)\}$ and

$$v_{(3-i)'} - v_{(3-i)'}^{N\setminus\{i',3\}} = w_1 + w_1^{N\setminus\{i',3\}} = \frac{1}{4}(3+\psi)$$
$$v_{3'} - v_{3'}^{N\setminus\{i',3\}} = w_2 + w_2^{N\setminus\{i',3\}} = \frac{1}{4}(1+3\psi)$$

Evidently the proposed reservation prices support $M^{N\setminus\{i',3\}}$ and imply non-negative payoffs. It can be similarly verified that the payoffs are indeed attained by the proposal, and hence the above constitute a MPE in the subgame.

The 3-market

I propose $M^N = \{(1',2),(2',1),(2',2),(2',3),(3',2)\}$ and

$$\begin{bmatrix} v_{1'} - v_{1'}^N \\ v_{2'} - v_{2'}^N \\ v_{3'} - v_{3'}^N \\ w_3 + w_3^N \\ w_2 + w_2^N \\ w_1 + w_1^N \end{bmatrix} = \frac{1}{210} \begin{bmatrix} 205 + 3\psi \\ 255 - 27\psi \\ 170 - 102\psi \\ 250 + 102\psi \\ 165 + 27\psi \\ 215 - 3\psi \end{bmatrix} \text{ and thus } \begin{bmatrix} v_{1'}^N \\ v_{2'}^N \\ v_{3'}^N \\ w_3^N \\ w_2^N \\ w_1^N \end{bmatrix} = \frac{1}{210} \begin{bmatrix} 215 - 3\psi \\ 165 - 183\psi \\ 40 - 108\psi \\ 40 - 108\psi \\ 165 - 183\psi \\ 215 - 3\psi \end{bmatrix}$$

Evidently the proposed reservation prices induce non-negative continuation payoffs. First I check they also support M^N:

$$\begin{aligned}
v_{1'} - v_{1'}^N - w_1 - w_1^N &= \frac{-1}{210}(10 - 6\psi) < 0 \\
v_{1'} - v_{1'}^N - w_2 - w_2^N &= \frac{1}{210}(40 - 24\psi) > 0 \\
v_{1'} - v_{1'}^N - w_3 - w_3^N &= \frac{-1}{210}(45 + 99\psi) < 0 \\
v_{2'} - v_{2'}^N - w_1 - w_1^N &= \frac{1}{210}(40 - 24\psi) > 0 \\
v_{2'} - v_{2'}^N - w_2 - w_2^N &= \frac{1}{210}(90 - 54\psi) > 0 \\
v_{2'} - v_{2'}^N - w_3 - w_3^N &= \frac{1}{210}(5 - 129\psi) > 0 \\
v_{3'} - v_{3'}^N - w_1 - w_1^N &= \frac{-1}{210}(45 + 99\psi) < 0 \\
v_{3'} - v_{3'}^N - w_2 - w_2^N &= \frac{1}{210}(5 - 129\psi) > 0 \\
v_{3'} - v_{3'}^N - w_3 - w_3^N &= \frac{-1}{210}(80 + 204\psi) < 0
\end{aligned}$$

Hence the proposed reservation prices are consistent with M^N. It remains to verify the payoffs are attained by the proposal.

Payoff of buyer 1':

$$v_{1'}^N = \frac{1}{5}\left[\frac{1}{2}v_{1'}^N + \frac{1}{2}\left(v_{1'} - \frac{165+27\psi}{210}\right)\right] + \frac{1}{5}\sum_{(i',j)\in M^N, i\neq 1} v_{1'}^{N\backslash\{i',j\}}$$

$$\Leftrightarrow\ v_{1'} - v_{1'}^N = \frac{1}{9}\times\frac{165+27\psi}{210} + \frac{2}{9}\sum_{(i',j)\in M^N, i\neq 1}(v_{1'} - v_{1'}^{N\backslash\{i',j\}})$$

$$= \frac{165+27\psi}{1890} + \frac{2}{9}\left[1+1+\frac{1}{4}(3+\psi)+\frac{1}{4}(5-\psi)\right]$$

$$= \frac{1}{210}(205+3\psi)$$

Payoff of buyer 2':

$$v_{2'}^N = \frac{1}{5}\left[\frac{1}{2}v_{2'}^N + \frac{1}{2}\left(v_{2'} - \frac{215-3\psi}{210}\right)\right] + \frac{1}{5}\left[\frac{1}{2}v_{2'}^N + \frac{1}{2}\left(v_{2'} - \frac{165+27\psi}{210}\right)\right]$$

$$+\frac{1}{5}\left[\frac{1}{2}v_{2'}^N + \frac{1}{2}\left(v_{2'} - \frac{250+102\psi}{210}\right)\right] + \frac{1}{5}\sum_{(i',j)\in M^N, i\neq 2} v_{2'}^{N\backslash\{i',j\}}$$

$$\Leftrightarrow\ v_{2'} - v_{2'}^N = \frac{1}{7}\left(\frac{215-3\psi}{210} + \frac{165+27\psi}{210} + \frac{250+102\psi}{210}\right) + \frac{2}{7}\sum_{(i',j)\in M^N, i\neq 2}(v_{2'} - v_{2'}^{N\backslash\{i',j\}})$$

$$= \frac{630+126\psi}{1470} + \frac{2}{7}\left[1+\frac{1}{4}(7-3\psi)\right]$$

$$= \frac{1}{210}(255-27\psi)$$

Payoff of buyer 3':

$$
\begin{aligned}
v_{3'}^{N} &= \frac{1}{5}\left[\frac{1}{2}v_{3'}^{N} + \frac{1}{2}\left(v_{3'} - \frac{165+27\psi}{210}\right)\right] + \frac{1}{5}\sum_{(i',j)\in M^N, i\neq 3} v_{3'}^{N\setminus\{i',j\}} \\
\Leftrightarrow\ v_{3'} - v_{3'}^{N} &= \frac{1}{9}\times\frac{165+27\psi}{210} + \frac{2}{9}\sum_{(i',j)\in M^N, i\neq 3}\left(v_{3'} - v_{3'}^{N\setminus\{i',j\}}\right) \\
&= \frac{165+27\psi}{1890} + \frac{2}{9}\left[(1-\psi)+(1-\psi)+(1-\psi)+\frac{1}{4}(1+3\psi)\right] \\
&= \frac{1}{210}(170-102\psi)
\end{aligned}
$$

Payoff of seller 1:

$$
\begin{aligned}
w_{1}^{N} &= \frac{1}{5}\left[\frac{1}{2}w_{1}^{N} + \frac{1}{2}\left(\frac{255-27\psi}{210} - w_1\right)\right] + \frac{1}{5}\sum_{(i',j)\in M^N, j\neq 1} w_{1}^{N\setminus\{i',j\}} \\
\Leftrightarrow\ w_1 + w_{1}^{N} &= \frac{1}{9}\times\frac{255-27\psi}{210} + \frac{2}{9}\sum_{(i',j)\in M^N, j\neq 1}\left(w_1 + w_{1}^{N\setminus\{i',j\}}\right) \\
&= \frac{255-27\psi}{1890} + \frac{2}{9}\left[1+1+\frac{1}{4}(3+\psi)+\frac{1}{4}(5-\psi)\right] \\
&= \frac{1}{210}(215-3\psi)
\end{aligned}
$$

Payoff of seller 2:

$$
\begin{aligned}
w_{2'}^{N} = \; & \frac{1}{5}\left[\frac{1}{2}w_{2'}^{N}+\frac{1}{2}\left(\frac{205+3\psi}{210}-w_2\right)\right]+\frac{1}{5}\left[\frac{1}{2}w_{2}^{N}+\frac{1}{2}\left(\frac{255-27\psi}{210}-w_2\right)\right] \\
& +\frac{1}{5}\left[\frac{1}{2}w_{2}^{N}+\frac{1}{2}\left(\frac{170-102\psi}{210}-w_2\right)\right]+\frac{1}{5}\sum_{(i',j)\in M^N, j\neq 2} w_2^{N\backslash\{i',j\}}
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \; w_2+w_2^N = \; & \frac{1}{7}\left(\frac{205+3\psi}{210}+\frac{255-27\psi}{210}+\frac{170-102\psi}{210}\right)+\frac{2}{7}\sum_{(i',j)\in M^N, j\neq 2}\left(w_2+w_2^{N\backslash\{i',j\}}\right) \\
= \; & \frac{630-126\psi}{1470}+\frac{2}{7}\left[1+\frac{1}{4}(1+3\psi)\right] \\
= \; & \frac{1}{210}(165+27\psi)
\end{aligned}
$$

Payoff of seller 3:

$$
w_3^N = \frac{1}{5}\left[\frac{1}{2}w_3^N+\frac{1}{2}\left(\frac{255-27\psi}{210}-w_3\right)\right]+\frac{1}{5}\sum_{(i',j)\in M^N, j\neq 3} w_3^{N\backslash\{i',j\}}
$$

$$
\begin{aligned}
\Leftrightarrow \; w_3+w_3^N = \; & \frac{1}{9}\times\frac{255-27\psi}{210}+\frac{2}{9}\sum_{(i',j)\in M^N, j\neq 3}\left(w_3+w_3^{N\backslash\{i',j\}}\right) \\
= \; & \frac{255-27\psi}{1890}+\frac{2}{9}\left[(1+\psi)+(1+\psi)+(1+\psi)+\frac{1}{4}(7-3\psi)\right] \\
= \; & \frac{1}{210}(250+102\psi)
\end{aligned}
$$

Hence the induced payoffs are indeed attained by the proposal. Therefore the proposed Markov strategy profile constitutes a MPE in the 3-market. Lastly, note that the total gains from trade is $v_{1'}+v_{2'}-w_2-w_1 = 4-2\psi$, yet

$$
v_{1'}^N+v_{2'}^N+v_{3'}^N+w_1^N+w_2^N+w_3^N = 4-2.8\psi < 4-2\psi
$$

This verifies the inefficiency of the MPE outcomes. □

1.5 Conclusion

In this chapter I have constructed a continuum of inefficient MPE for the market game in Gale and Sabourian (2006). The key to inefficient trade agreement is the dispersion of reservation prices, which must be supported by those in the continuation markets. Proposition 1.2 has demonstrated that inefficiency cannot arise in any 2-markets, because any continuation 1-market with positive gains from trade can support only one reservation price; consequently, even with careful choice of transaction prices in the relevant continuation 1-markets, any 2-market can support at most two distinct first-trade prices (see Proposition 1.4 in Appendix A), insufficient to incentivize inefficient trades. To construct inefficient MPE, therefore, I must study markets with more players, and exploit the fact that the albeit limited dispersion of reservation prices in continuation 2-markets can be used to support further dispersion of first-trade prices in the corresponding 3-market. Suggestively, my proposed MPE in Proposition 1.3 manages to support six distinct first-trade prices in a 3-market.

To illustrate this key insight, I have designed the MPE in Proposition 1.3 such that, among all continuation 2-markets with only one infra-marginal pair, trades take place at the same ‘middle’ price of 1. As a result, the dispersion of first-trade prices in the 3-market is solely driven by those in the continuation 2-markets with two infra-marginal pairs. In particular, I have exploited the dispersion to suppress the payoffs of buyer 2’ and seller 2 in the $N \setminus \{3', 2\}$ and $N \setminus \{2', 3\}$ continuation markets respectively, in order to counteract their favourable first-stage opportunity to trade with two infra-marginal players on the opposite side ($(1', 2), (2', 1), (2', 2) \in M^N$). As a result, their first-stage payoffs are low enough they entertain trades with the extra-marginal players, lest they should enter their unfavourable continuation markets after rejection. Meanwhile, without similar suppression, the first-stage payoffs of buyer 1’ and seller 1 are too high to entertain trades with the extra-marginal players. Indeed, their first-stage payoffs are so high that they would rather exploit seller 2 and buyer 2' respectively than trade with each other in the first stage ($(1', 1) \notin M^N$).

Hence I have proved the conjecture in Gale and Sabourian (2006) regarding the existence of inefficient MPE in a frictionless market. More important, I have illustrated that the number of players could play a role in allocative inefficiency because the latter relies on the dispersion of continuation reservation prices, which in turn increases in the number of players.

Chapter 2

Re-trading and Efficient Allocation

2.1 Introduction

The theory of competitive equilibrium postulates that trades in every market are mediated by a single market price. Though simple and elegant, the theory provides little foundation of how such prices emerge as the mediation device. As a result, economic theorists have attempted to provide a strategic foundation of competitive equilibrium through non-cooperative games. One successful approach is the class of dynamic matching and bargaining games (DMBG). Without imposing any centralized price formation mechanism, DMBG start from more primitive concepts and allow different players to trade at different prices. It is therefore an attractive modelling strategy to lay the strategic foundation of competitive equilibrium.

One line of the DMBG literature studies markets with a continuum of agents (Rubinstein and Wolinsky (1985), Gale (1986), Gale (1987), Lauermann (2013)). Although such models are analytically convenient, Gale (2000) points out that "the continuum assumption is justified by the claim that a continuum of agents is a good approximation to a large but finite number of players." If we are to justify (or refute) the theory of competitive equilibrium from first principles, it is crucial to start from a model with $n < \infty$ players, and examine whether or not the equilibrium outcomes approximate the Walrasian ones as n goes to infinity. In other words, we gain more insights from limit theorems than from theorems in the limit.

This motivates me to consider the other line of the DMBG literature, which investigates markets with a finite number of players. Rubinstein and Wolinsky (1990) shows that restriction to Markov strategies can induce the unique competitive outcome in a homogeneous market, while Sabourian (2004) shows that their Markov assumption can be justified on the grounds of complexity costs. Although homogeneous markets are analytically convenient, Gale and Sabourian (2006) point out that the more general case of heterogeneous markets presents the real challenge to the theory of competitive equilibrium, because even inefficient trades can be individually rational.

It might be thought that inefficient trades can only be sustained by delicate threats and counter-threats in complicated strategy profiles, and will disappear once we restrict to Markov strategies as in Rubinstein and Wolinsky (1990). Surprisingly, the literature has repeatedly shown that inefficient trades persist even in Markov-perfect equilibrium (MPE). Gale and Sabourian (2006) *argue* that there exists a continuum of inefficient MPE in a market with random, exogenous matching. Elliott and Nava (2019) show that all MPE are inefficient unless certain parameter restrictions are satisfied in a market with semi-deterministic, semi-endogenous matching. These two papers are important benchmarks because, like this chapter, they study DMBG with (i) a finite number of heterogeneous players, (ii) perfect and complete information, and (iii) zero or small friction.

The fact that inefficient allocations prevail under such idealized conditions is devastating to the theory of competitive equilibrium, because economic theorists can no longer claim that the latter is but an idealization to be approximated. Nevertheless, I observe that there is a hidden friction in most DMBG in the literature, including Gale and Sabourian (2006) and Elliott and Nava (2019), that each player can trade at most once. In a truly frictionless market, however, players should be allowed to trade as many times as they desire. It is this hidden but substantial friction, I propose, that causes a multitude of inefficiency results in the DMBG literature. Supporting this hypothesis, Agranov and Elliott (2020) implement a DMBG closely related to Elliott and Nava (2019) in a laboratory experiment, and find that allocative efficiency is substantially improved by allowing subjects to renege on agreements and re-match at a small cost.

In light of the above reasons, I am motivated to investigate DMBG where players are allowed to trade multiple times. In my model, time is infinite and players with quasi-linear preferences exchange an indivisible, non-perishable good for money. In each period a single player is drawn at random to be active. The active player can choose to either (i) exit the market and consume his endowment, (ii) do nothing and wait for the next period, or (iii) search for a trading partner at a cost $c \geq 0$. If he chooses (iii), he matches with another player and makes an offer. If the offer is accepted, the pair trades accordingly; otherwise there is no trade. Either way, both players stay in the market and proceed to the next period. The game ends when no more gains from trade are available. Before that, players can stay in the market as long as they desire, and trade multiple times with any remaining players. I show that, given the players' valuation and the matching technology, there exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$, all MPE are allocative efficient.

In the existing DMBG literature, only Gale (2000) has investigated setups with re-trading, and thus is the closest work to this chapter. Like this chapter, he studies DMBG with (i) a finite number of heterogeneous players, (ii) perfect and complete information, (iii) zero or small friction, and (iv) the option to re-trade. He first establishes that all MPE are pairwise efficient, then argues this implies global efficiency under some technical assumption; had he restricted the model to a quasi-linear economy like in this chapter, the latter would not have been needed. The problem, however, lies in the fact that pairwise efficiency is not surprising in the first place given that he assumes all players to stay in the market and re-trade with one another ad infinitum.

Although it is analytically convenient to have a stationary set of players, one may wonder whether or not his (asymptotic) efficiency result depends on this rather extreme assumption. Does it extend to a truly frictionless market, where 'barriers to exit' are absent and players stay only as long as they desire? The contribution of this chapter is to relax this strong assumption. I will elaborate in Section 2.5 why the reasoning in Gale (2000) no longer works in models where players can choose to exit the market amidst the game, and hence why the argument developed in this chapter is a novel contribution to the DMBG literature.

This chapter is structured as follows. Section 2.2 describes the game setup. Section 2.3 defines the notion of infra-marginal players. Section 2.4 proves the efficiency of MPE, the main result of this paper. Section 2.5 discusses related literature and concludes.

2.2 The market game

This is a dynamic game of perfect and complete information. There are K players in the market, indexed by $k = 1, 2, ..., K$. A single indivisible, non-perishable good is exchanged for money and each player wants to consume at most one unit of the good. Let $(v_k)_{k=1}^K \gg 0$ be their values of the good, all distinct. Trades take place at a sequence of periods $t = 1, 2, ...$ There is no time cost, but search cost can be positive as explained below.

I assume each player holds at most one unit of the good at a time. While this creates substantial friction in the re-trading process, the fact that allocative efficiency is attained despite such friction only reinforces the key message of this chapter: allocative efficiency is attainable if we allow re-trading. Meanwhile, this assumption greatly simplifies the analysis.

Let N^t be the set of players remaining in market at the start of period t. In any period t, define the state of the game $(g_k^t, m_k^t)_{k=1}^K$ as follows:

- For each $k \in N^t$, let $g_k^t = 1$ if k owns the good at the start of period t, and $g_k^t = 0$ otherwise; let $m_k^t \in \mathbb{R}$ be k's money balance at the start of period t.

- For each $k \notin N^t$, let $g_k^t = m_k^t = \varnothing$.

It will be evident the state is precisely the minimal set of variables that are directly payoff-relevant. Let Ω be the set of all admissible states, and $\hat{\Omega} \subseteq \Omega$ the set of states with positive gains from trade. This means if $(g_k^t, m_k^t)_{k=1}^K \in \hat{\Omega}$, then $v_i > v_j$ for some remaining players $i, j \in \{k : g_k^t \neq \varnothing\}$ such that they are of 'opposite types', i.e. $g_i^t + g_j^t = 1$. It is important to define the state at this point because the matching technology in the game is stationary, i.e. it depends only on the state but not on the time period.

Now I describe the sequence of events taking place in each period. Fix three constants $c \geq 0$ and $\alpha, \gamma \in (0, 1/K)$. At the start of each period t, there are two possibilities: if the state belongs to $\Omega \setminus \hat{\Omega}$, the game ends and all players exogenously exit the market; if the state $\omega^t \equiv (g_k^t, m_k^t)_{k=1}^K$ belongs to $\hat{\Omega}$, the game proceeds as follows. A remaining player $i \in N^t$ is drawn to be active with probability $\alpha_i^{\omega^t} \geq \alpha$. After that, player i has three choices:

1. He can quit the market, after which $i \notin N^T$ for all $T > t$, and the game proceeds to period $t+1$ with the new state;

2. He can remain in the market and do nothing, after which the game proceeds to period $t+1$ with the same state;

3. He can search for a trading partner at cost c, after which he is randomly matched with a remaining 'opposite type' player $j \in \{k \in N^t : g_k^t = 1 - g_i^t\}$ with probability $\gamma_{ij}^{\omega^t} \geq \gamma$. After (i, j) is matched, i proposes an offer. Having observed the offer, j chooses to either accept or reject. If the offer is accepted, the good is traded at the agreed price and the game proceeds to period $t+1$ with the new state. If the offer is rejected, there is no trade and the game proceeds to period $t+1$ with the same state.

Any player i who exits in period t receives a payoff of $g_i^t v_i + m_i^t - \chi_i^t c$, where $\chi_i^t \in \mathbb{N} \cup \{0\}$ is the number of times that i has searched for a trading partner up to period t. Any player i who never exits the market receives a payoff of $\liminf_{t \to \infty} g_i^t v_i + m_i^t - \chi_i^t c$.

To reiterate, at any state $\omega^t \equiv (g_k^t, m_k^t)_{k=1}^K \in \hat{\Omega}$ in any period t, the matching technology satisfies the following:

- $\alpha_i^{\omega^t} \geq \alpha$ if $i \in N^t$; $\alpha_i^{\omega^t} = 0$ otherwise.
- $\sum_{k=1}^K \alpha_k^{\omega^t} = 1$.
- $\gamma_{ij}^{\omega^t} \geq \gamma$ if $i, j \in N^t$ and $g_i^t + g_j^t = 1$; $\gamma_{ij}^{\omega^t} = 0$ otherwise.
- $\sum_{j=1}^K \gamma_{ij}^{\omega^t} = 1$ for any $i \in N^t$.

I study the Markov-perfect equilibria (MPE) of this game. This requires that (i) the decision to quit, do nothing, or search depends only on the state in that period; (ii) the proposal decision depends only on the state and the responder in that period; (iii) the response decision depends only the state, the proposer, and the proposal in that period. Since Markov strategies depend on the state but not the period, with slight abuse of notation, I henceforth replace the time superscript with the state in that period: for any $\omega \in \Omega$, I instead write $(g_k^\omega, m_k^\omega)_{k=1}^K$ and $N^\omega := \{k : g_k^\omega \neq \varnothing\}$.

2.3 State transition and players' roles

First, I elaborate on the roles of all remaining players at each state. For any $\omega \in \Omega$, define $B^\omega := \{i : g_i^\omega = 0\}$ and $S^\omega := \{j : g_j^\omega = 1\}$; the two sets partition N^ω, and represent the set of 'buyers' and 'sellers' at state ω respectively. For any $\omega \in \hat{\Omega}$, define the function $\beta^\omega : B^\omega \to \{1, 2, ..., |B^\omega|\}$ such that $\beta^\omega(i) > \beta^\omega(j) \iff v_i < v_j$; then, player $i \in B^\omega$ is the $\beta^\omega(i)^{th}$ highest value buyer at state ω. Similarly, for any $\omega \in \hat{\Omega}$, define the function $\sigma^\omega : S^\omega \to \{1, 2, ..., |S^\omega|\}$ such that $\sigma^\omega(i) > \sigma^\omega(j) \iff v_i > v_j$; then, player $j \in S^\omega$ is the $\sigma^\omega(j)^{th}$ lowest value seller at state ω. The functions are well-defined because B^ω and S^ω are nonempty for all $\omega \in \hat{\Omega}$.

These help me define the set of infra-marginal players, a key concept in this chapter. For any $\omega \in \hat{\Omega}$, let $B_I^\omega := \{i \in B^\omega : v_i > v_j \text{ and } \beta^\omega(i) = \sigma^\omega(j) \text{ for some } j \in S^\omega\}$ and $S_I^\omega := \{j \in S^\omega : \beta^\omega(i) = \sigma^\omega(j) \text{ for some } i \in B_I^\omega\}$; they represent the set of infra-marginal buyers and infra-marginal sellers at state ω respectively[1]. For any $\omega \in \Omega \setminus \hat{\Omega}$, let $B_I^\omega = S_I^\omega = \varnothing$ for completion. The set of infra-marginal players is then $N_I^\omega := B_I^\omega \cup S_I^\omega$. On the other hand, the set of extra-marginal players is simply the complement: $B_E^\omega := B^\omega \setminus B_I^\omega$ and $S_E^\omega := S^\omega \setminus S_I^\omega$ represent the set of extra-marginal buyers and extra-marginal sellers at state ω respectively. The set of extra-marginal players is then $N_E^\omega := N^\omega \setminus N_I^\omega$. In other words, at any state with zero gains from trade, any remaining players are categorized as extra-marginal.

I have thus exhaustively categorized all remaining players at any state based on (i) whether or not they own the good, and (ii) whether or not they are within the margin, which in turn depends on the distribution of ownership of goods in the market. Next, I carefully denote the immediate successors of each state with positive gains from trade. There are two types of state transition:

Transition via a player exiting the market

For any $\omega \in \hat{\Omega}$ and $i \in N^\omega$, denote by $\omega(i) \in \Omega$ the unique state satisfying

- $g_i^{\omega(i)} = m_i^{\omega(i)} = \varnothing$
- $(g_k^{\omega(i)}, m_k^{\omega(i)}) = (g_k^\omega, m_k^\omega)$ for all $k \neq i$

In other words, $\omega(i)$ is the state that immediately succeeds ω after player $i \in N^\omega$ has exited the market.

Transition via two players trading with each other

For any $\omega \in \hat{\Omega}$ and $(i, j, p) \in B^\omega \times S^\omega \times \mathbb{R}$, denote by $\omega(i, j, p) \in \Omega$ the unique state satisfying

[1] By definition infra-marginal buyers and infra-marginal sellers come in pairs, i.e. $|B_I^\omega| = |S_I^\omega|$.

- $g_i^{\omega(i,j,p)} = 1$ and $m_i^{\omega(i,j,p)} = m_i^{\omega} - p$
- $g_j^{\omega(i,j,p)} = 0$ and $m_j^{\omega(i,j,p)} = m_j^{\omega} + p$
- $(g_k^{\omega(i,j,p)}, m_k^{\omega(i,j,p)}) = (g_k^{\omega}, m_k^{\omega})$ for all $k \neq i, j$

In other words, $\omega(i,j,p)$ is the state that immediately succeeds ω after buyer $i \in B^{\omega}$ and seller $j \in S^{\omega}$ has traded at price p.

Now I study how the set of infra-marginal players evolves with trading. After a pair of buyer and seller have traded, the ranking of each buyer and seller in the market may change. Indeed, even the competitive price interval in the market may shift. It is therefore not obvious how trading alters the infra-marginality of each remaining player. Proposition 2.1 answers this question.

Proposition 2.1. For any $\omega \in \hat{\Omega}$ and $(i,j,p) \in B^{\omega} \times S^{\omega} \times \mathbb{R}$,

1. $k \in N_I^{\omega} \Longleftrightarrow k \in N_E^{\omega(i,j,p)}$ for $k = i, j$
2. $k \in N_I^{\omega} \Longleftrightarrow k \in N_I^{\omega(i,j,p)}$ for $k \in N^{\omega} \setminus \{i, j\}$

Proof. Fix any $\omega \in \hat{\Omega}$ and $(i,j,p) \in B^{\omega} \times S^{\omega} \times \mathbb{R}$. Let $m \equiv |B_I^{\omega}|$.

Case 1: $i \in B_I^{\omega}$ and $j \in S_I^{\omega}$

I claim $B_I^{\omega} \setminus \{i\}$ are the top $m-1$ buyers in $\omega(i,j,p)$. This is trivial if $m = 1$. Consider the case $m > 1$. Note that $B^{\omega(i,j,p)} = (B^{\omega} \cup \{j\}) \setminus \{i\}$. Since B_I^{ω} are the top m buyers at ω but $v_j < \min_{k \in B_I^{\omega}} v_k$, the claim follows. Similarly, $S_I^{\omega} \setminus \{j\}$ are the bottom $m-1$ sellers at $\omega(i,j,p)$. Since $\min_{k \in B_I^{\omega} \setminus \{i\}} v_k > \max_{\ell \in S_I^{\omega} \setminus \{j\}} v_\ell$, by definition, $B_I^{\omega} \setminus \{i\} \subseteq B_I^{\omega(i,j,p)}$ and $S_I^{\omega} \setminus \{j\} \subseteq S_I^{\omega(i,j,p)}$. But $v_\ell > v_k$ for all $\ell \in S_E^{\omega} \cup \{i\}$ and $k \in B_E^{\omega} \cup \{j\}$, so no other players are infra-marginal at $\omega(i,j,p)$. Thus $B_I^{\omega(i,j,p)} = B_I^{\omega} \setminus \{i\}$ and $S_I^{\omega(i,j,p)} = S_I^{\omega} \setminus \{j\}$.

Case 2: $i \in B_E^\omega$ and $j \in S_E^\omega$

I claim $B_I^\omega \cup \{j\}$ are the top $m+1$ buyers at $\omega(i,j,p)$. This is immediate if $v_j > \min_{k \in B_I^\omega} v_k$. Even if $v_j < \min_{k \in B_I^\omega} v_k$, the claim still follows as $v_j > v_k$ for any $k \in B_E^\omega$ by definition. Similarly, $S_I^\omega \cup \{i\}$ are the bottom $m+1$ sellers at $\omega(i,j,p)$. Since $\min_{k \in B_I^\omega \cup \{j\}} v_k > \max_{\ell \in S_I^\omega \cup \{i\}} v_\ell$, by definition, $B_I^\omega \cup \{j\} \subseteq B_I^{\omega(i,j,p)}$ and $S_I^\omega \cup \{i\} \subseteq S_I^{\omega(i,j,p)}$. But $v_\ell > v_k$ for all $\ell \in S_E^\omega \setminus \{i\}$ and $k \in B_E^\omega \setminus \{j\}$, so no other players are infra-marginal at $\omega(i,j,p)$. Thus $B_I^{\omega(i,j,p)} = B_I^\omega \cup \{j\}$ and $S_I^{\omega(i,j,p)} = S_I^\omega \cup \{i\}$.

Case 3: $i \in B_I^\omega$ and $j \in S_E^\omega$

I claim $(B_I^\omega \setminus \{i\}) \cup \{j\}$ are the top m buyers at $\omega(i,j,p)$. This is immediate if $v_j > \min_{k \in B_I^\omega \setminus \{i\}} v_k$. Even if $v_j < \min_{k \in B_I^\omega \setminus \{i\}} v_k$, the claim still follows as $v_j > v_k$ for any $k \in B_E^\omega$ by definition. Yet $\max_{k \in S_I^\omega} v_k < \min_{\ell \in S_E^\omega \cup \{i\}} v_\ell$, so S_I^ω are the bottom m sellers at $\omega(i,j,p)$. Since $\min_{k \in (B_I^\omega \setminus \{i\}) \cup \{j\}} v_k > \max_{\ell \in S_I^\omega} v_\ell$, by definition, $(B_I^\omega \setminus \{i\}) \cup \{j\} \subseteq B_I^{\omega(i,j,p)}$ and $S_I^\omega \subseteq S_I^{\omega(i,j,p)}$. But $v_\ell > v_k$ for all $\ell \in (S_E^\omega \setminus \{j\}) \cup \{i\}$ and $k \in B_E^\omega$, so no other players are infra-marginal at $\omega(i,j,p)$. Thus $B_I^{\omega(i,j,p)} = (B_I^\omega \setminus \{i\}) \cup \{j\}$ and $S_I^{\omega(i,j,p)} = S_I^\omega$.

Case 4: $i \in B_E^\omega$ and $j \in S_I^\omega$

Similarly as Case 3, $B_I^{\omega(i,j,p)} = B_I^\omega$ and $S_I^{\omega(i,j,p)} = (S_I^\omega \setminus \{j\}) \cup \{i\}$. $\square$

Proposition 2.1 provides a convenient way to trace how each player's binary role evolves with trades. Intuitively, after any trade, any players involved in the trade 'switch' roles, while any players not involved in the trade retain the same roles. For instance, if player i and j are infra-marginal and extra-marginal respectively at state ω, then i and j must become extra-marginal and infra-marginal respectively at state $\omega(i,j,p)$; for any other remaining players $k \in N^{\omega} \setminus \{i, j\}$ not involved in trading, if k is infra-marginal (extra-marginal) at state ω he must remain infra-marginal (extra-marginal) at state $\omega(i, j, p)$.

2.4 Efficiency of MPE

As long as no players have exited the market, there are still chances for all goods to end up with the highest value players. It is therefore crucial to study the condition under which a player endogenously chooses to exit the market, i.e. before the gains from trade are exhausted.

In the following, I impose two tie-breaking rules: (i) players quit the market whenever indifferent, and (ii) players trade with one another whenever indifferent. I argue there is no loss of generality, since this is the most challenging combination of tie-breaking rules for attaining allocative efficiency: rule (i) means players stay in the market only if it is strictly profitable, whilst rule (ii) means players are prone to receiving unprofitable proposals. As a result, an infra-marginal player who expects to receive zero surplus from staying in the market will choose to exit prematurely.

For any search cost $c \geq 0$, let F^c be the corresponding set of MPE. First I analyse the proposal and response decisions in any MPE. For any $c \geq 0$, $f \in F^c$, and $\omega \in \hat{\Omega}$, let $\pi_i^{\omega}(f;c)$ denote the continuation payoff of any player $i \in N^{\omega}$ at the start of ω given c, f, and let $P_{ij}^{\omega}(f;c) := \{p \in \mathbb{R} : \pi_i^{\omega(i,j,p)}(f;c) \geq \pi_i^{\omega}(f;c) \text{ and } \pi_j^{\omega(i,j,p)}(f;c) \geq \pi_j^{\omega}(f;c)\}$ for any pair $(i, j) \in B^{\omega} \times S^{\omega}$; by tie-breaking rule (ii), the pair (i, j) trades whenever matched at ω given c, f if and only if $P_{ij}^{\omega}(f;c) \neq \varnothing$. Thus, $S_i^{\omega}(f;c) := \{\ell \in S^{\omega} : P_{i\ell}^{\omega}(f;c) \neq \varnothing\}$ and $B_j^{\omega}(f;c) := \{k \in B^{\omega} : P_{kj}^{\omega}(f;c) \neq \varnothing\}$ are the set of feasible trading partners of buyer i and seller j respectively at ω given c, f.

Next I analyse the search payoffs in any MPE. For any $c \geq 0$, $f \in F^c$, and $\omega \in \hat{\Omega}$, let $\lambda_i^{\omega}(f;c)$ denote the continuation payoff of any $i \in N^{\omega}$ after he has become active and chosen to search for a trading partner at ω given c, f, and let $p_{ij}^{\omega}(f)$ and $p_{ji}^{\omega}(f)$ denote the offer i proposes to j and that j proposes to i respectively for any pair $(i,j) \in B^\omega \times S^\omega$ whenever matched at ω given f; optimality means $p_{ij}^\omega(f) \in \arg\max_{p\in P_{ij}^\omega(f;c)} \pi_i^{\omega(i,j,p)}(f;c)$ if $j \in S_i^\omega(f;c)$, and $p_{ji}^\omega(f) \in \arg\max_{p\in P_{ij}^\omega(f;c)} \pi_j^{\omega(i,j,p)}(f;c)$ if $i \in B_j^\omega(f;c)$. These help me write down the search payoffs. For any $c \geq 0$, $f \in F^c$, $\omega \in \hat{\Omega}$, and $i \in B^\omega$,

$$\lambda_i^\omega(f;c) = \sum_{j\in S_i^\omega(f;c)} \gamma_{ij}^\omega \pi_i^{\omega(i,j,p_{ij}^\omega(f))}(f;c) + \sum_{j\in S^\omega \setminus S_i^\omega(f;c)} \gamma_{ij}^\omega \pi_i^\omega(f;c) - c$$

For any $c \geq 0$, $f \in F^c$, $\omega \in \hat{\Omega}$, and $j \in S^\omega$,

$$\lambda_j^\omega(f;c) = \sum_{i\in B_j^\omega(f;c)} \gamma_{ji}^\omega \pi_j^{\omega(i,j,p_{ji}^\omega(f))}(f;c) + \sum_{i\in B^\omega \setminus B_j^\omega(f;c)} \gamma_{ji}^\omega \pi_j^\omega(f;c) - c$$

Finally I write down the decision to quit, remain, or search precisely. For any $c \geq 0$, $f \in F^c$ and $\omega \in \hat{\Omega}$, let $Q^\omega(f;c), R^\omega(f;c)$ and $M^\omega(f;c)$ be the sets of remaining players that respectively choose to quit the market, do nothing, and search for a trading partner whenever active at ω given c, f; evidently the three sets partition N^ω. By tie-breaking rule (i) and the previous analysis, $Q^\omega(f;c) = \{k \in N^\omega : g_k^\omega v_k + m_k^\omega \geq \max\{\pi_k^\omega(f;c), \lambda_k^\omega(f;c)\}\}$.

In the following, for each $c \geq 0$, fix a MPE $f^c \in F^c$. Since all subsequent equilibrium analysis is conducted given some $f^c \in F^c$, I henceforth omit the direct dependence on c of previously defined equilibrium sets and continuation payoffs without ambiguity. Now I state a lemma regarding equilibrium trading patterns.

Lemma 2.2. There exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$ and $\omega \in \hat{\Omega}$, if $B_I^\omega \cap Q^\omega(f^c) \neq \varnothing$, then there exists $i \in B_I^\omega \cap M^\omega(f^c)$ and $j \in S_I^\omega$ such that $j \in S_i^\omega(f^c)$.

Proof. See Appendix B.

Intuitively, Lemma 2.2 shows that if at any state certain buyer exits the market prematurely – meaning that the exiting player is still infra-marginal, and thus should have stayed to actualize gains from trade – then at least one pair of infra-marginal buyer and seller must trade with positive probability at the same state, provided search cost is sufficiently small. The lemma plays a key role in proving the following important proposition, which asserts that no players will ever exit the market prematurely in equilibrium.

Proposition 2.3. There exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$ and $\omega \in \hat{\Omega}$, $Q^{\omega}(f^c) \subseteq N_E^{\omega}$.

Proof. The proposition is equivalent to the claim that there exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$ and $\omega \in \hat{\Omega}$, (i) $B_I^{\omega} \cap Q^{\omega}(f^c) = \varnothing$ and (ii) $S_I^{\omega} \cap Q^{\omega}(f^c) = \varnothing$. It suffices that I prove (i) because (ii) can be shown by symmetric arguments.

By Lemma 2.2, there exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$ and $\omega \in \hat{\Omega}$, if $B_I^{\omega} \cap Q^{\omega}(f^c) \neq \varnothing$, then there exists $i \in B_I^{\omega} \cap M^{\omega}(f^c)$ and $j \in S_I^{\omega}$ such that $j \in S_i^{\omega}(f^c)$. Fix one such $\bar{c} > 0$.

For each $z = 1, 2, ..., K$, let $\hat{\Omega}_z := \{\omega \in \hat{\Omega} : |B_I^{\omega}| = z\}$; in other words, $\hat{\Omega}_z$ is the set of states with z pairs of infra-marginal players. For each z, define the statement $D(z)$: $B_I^{\omega} \cap Q^{\omega}(f^c) = \varnothing$ for all $c \in [0, \bar{c})$ and $\omega \in \hat{\Omega}_z$. Since evidently $\cup_{z=1}^{K} \hat{\Omega}_z = \hat{\Omega}$, it remains to prove $D(z)$ for all z by mathematical induction.

First, I prove $D(1)$. Suppose not for contradiction. Then there exists $c \in [0, \bar{c})$ and $\omega_c \in \hat{\Omega}_1$ such that $B_I^{\omega_c} \cap Q^{\omega_c}(f^c) \neq \varnothing$. Fix one such c and ω_c. Note that $B_I^{\omega_c}$ is a singleton by definition of $\omega_c \in \hat{\Omega}_1$, and thus $B_I^{\omega_c} \subseteq Q^{\omega_c}(f^c)$. However, $B_I^{\omega_c} \cap M^{\omega_c}(f^c) \neq \varnothing$ by choice of $\bar{c}$. This leads to contradiction as desired.

Next, I prove $D(z) \Longrightarrow D(z+1)$ for all $z < K$. Suppose not for contradiction. Then, $D(Z)$ is true but $D(Z+1)$ is not for some $Z < K$. Fix one such Z. This means there exists $c \in [0, \bar{c})$ and $\omega_c \in \hat{\Omega}_{Z+1}$ such that $\ell_c \in B_I^{\omega_c} \cap Q^{\omega_c}(f^c)$ for some $\ell_c \in N^{\omega_c}$. Fix one such c, ω_c and ℓ_c.

By choice of $\bar{c}$, there exists $i_c \in B_I^{\omega_c} \cap M^{\omega_c}(f^c)$ and $j_c \in S_I^{\omega_c}$ such that $j_c \in S_{i_c}^{\omega_c}(f^c)$. Fix one such i_c and j_c. This means that at ω_c, i_c becomes active, searches for a trading partner, matches with j_c, and reaches an agreement with j_c with probability $\alpha_{i_c}^{\omega_c}\gamma_{i_c j_c}^{\omega_c} \geq \alpha\gamma > 0$. In other words, the state transits from ω_c to $\omega_c^* \equiv \omega_c(i_c, j_c, p_{i_c j_c}^{\omega_c}(f^c))$ with positive probability $\alpha_{i_c}^{\omega_c}\gamma_{i_c j_c}^{\omega_c}$ given f^c.

Meanwhile, because $i_c, j_c, \ell_c \in N_I^{\omega_c}$, Proposition 2.1 implies that $\omega_c^* \in \hat{\Omega}_Z$ and $\ell_c \in B_I^{\omega_c^*}$; but $D(Z)$ in turn implies that $\ell_c \notin Q^{\omega_c^*}(f^c)$ and hence $m_{\ell_c}^{\omega_c^*} < \max\{\pi_{\ell_c}^{\omega_c^*}(f^c), \lambda_{\ell_c}^{\omega_c^*}(f^c)\}$ by definition. Now there are two possibilities: individual rationality of ℓ_c means that either (i) $\pi_{\ell_c}^{\omega_c^*}(f^c) = m_{\ell_c}^{\omega_c^*}$ or (ii) $\pi_{\ell_c}^{\omega_c^*}(f^c) > m_{\ell_c}^{\omega_c^*}$. But (i) leads to contradiction because it implies that $m_{\ell_c}^{\omega_c^*} < \lambda_{\ell_c}^{\omega_c^*}(f^c)$, and thus

$$\pi_{\ell_c}^{\omega_c^*}(f^c) \geq \alpha_{\ell_c}^{\omega_c^*}\lambda_{\ell_c}^{\omega_c^*}(f^c) + (1-\alpha_{\ell_c}^{\omega_c^*})m_{\ell_c}^{\omega_c^*} > m_{\ell_c}^{\omega_c^*}$$

Hence it must be that $\pi_{\ell_c}^{\omega_c^*}(f^c) > m_{\ell_c}^{\omega_c^*}$. This together with the fact that $m_{\ell_c}^{\omega_c} = m_{\ell_c}^{\omega_c^*}$ in turn implies

$$\pi_{\ell_c}^{\omega_c}(f^c) \geq \alpha_{i_c}^{\omega_c}\gamma_{i_c j_c}^{\omega_c}\pi_{\ell_c}^{\omega_c^*}(f^c) + (1-\alpha_{i_c}^{\omega_c}\gamma_{i_c j_c}^{\omega_c})m_{\ell_c}^{\omega_c} > m_{\ell_c}^{\omega_c}$$

But then, $\ell_c \notin Q^{\omega_c}(f^c)$ by definition. This leads to contradiction as desired.

Therefore, by mathematical induction, $D(z)$ is true for all z as desired. □

Corollary 2.4. There exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$, $\omega \in \hat{\Omega}$, and $i \in Q^{\omega}(f^c)$, $N_I^{\omega} = N_I^{\omega(i)}$.

Proposition 2.3 shows that no players exit the market unless they are extra-marginal. Corollary 2.4 provides a convenient way to trace how each player's binary role evolves with exits. Intuitively, after any exit, any remaining players retain the same roles. Together with

Proposition 2.1, I have completely analyzed the evolution of each player's role under any MPE state transition.

I now prove the key result of this chapter. Let $\omega^0 \in \Omega$ be the initial state. An MPE is allocative efficient if and only if the $|S^{\omega^0}|$ units of goods are consumed by the $|S^{\omega^0}|$ highest value players in the market with probability one, i.e. by $B_I^{\omega^0} \cup S_E^{\omega^0}$.

Proposition 2.5. All MPE are allocative efficient.

Proof. This is trivially true if $\omega^0 \in \Omega \setminus \hat{\Omega}$. Consider the case $\omega^0 \in \hat{\Omega}$ and fix any player $i \in B_E^{\omega^0} \cup S_I^{\omega^0}$. Fix any MPE. At any state $\omega \in \Omega$ on the equilibrium path such that $i \in N^{\omega}$, by Proposition 2.1 and Corollary 2.4, $i \in B_E^{\omega} \cup S_I^{\omega}$. But then, by Proposition 2.3, i never exits the market with the goods on the equilibrium path. Since in any MPE the game ends with probability one (see Lemma 2.8 in Appendix B), the goods must be consumed by and only by $B_I^{\omega^0} \cup S_E^{\omega^0}$ with probability one. □

2.5 Conclusion

In this chapter I have developed a dynamic matching and bargaining model with the options of re-trading and exit. The latter remove substantial frictions prevalent in the DMBG literature that (i) players cannot choose to exit the market before trading and (ii) players must exit the market after trading only once. Absent these frictions, I have shown that all Markov-perfect equilibria induce efficient allocation provided search cost is sufficiently small. This key result echoes with Coase Theorem: under quasi-linear preferences (no wealth effects), perfect and complete information (no information asymmetry), and low friction (low transaction costs), decentralized bargaining in simple strategies will lead to efficient outcomes regardless of the initial allocation.

As mentioned in Section 2.1, Gale (2000) is the closest work to this chapter. In his models, players stay in the market and re-trade with one another ad infinitum. Payoff is then defined as the limit inferior of the player's endowment bundle in period t as $t \to \infty$. As admitted in Gale (2000), it sounds almost tautological to show pairwise efficiency in any MPE. Suppose for contradiction that some MPE outcome is not pairwise efficient. Then there exists some players i and j who could mutually benefit from trade. But each pair of players are assumed to meet infinitely often with each other, so i and j would eventually meet and reach a Pareto-improving agreement, and thereby obtain more than their *converged* equilibrium payoffs. This leads to contradiction as desired.

One important question is whether or not this reasoning extends directly to a model with exit option; if it does, this paper would have been a trivial extension of Gale (2000). To see why it does not, consider my model with $c = 0$ for simplicity, and ignore Proposition 2.3 for the moment. Suppose there exists $f^0 \in F^0$ and $\omega \in \hat{\Omega}$ such that $i \in B_I^\omega \cap Q^\omega(f^0)$ for some $i \in N^\omega$ WLOG. In other words, part of the gains from trade is irreversibly destroyed via the premature exit of some player i at some state ω on the equilibrium path of f^0. If I were to adopt a similar proof strategy as in Gale (2000), I must show that, on the equilibrium path of f^0 there exists some state ω^*, equal to or succeeding ω, such that $i \in B^{\omega^*}$, but $\pi_i^{\omega^*(i,j,p_{ij}^{\omega^*}(f^0))}(f^0) > \pi_i^\omega(f^0)$ for some $j \in S_i^{\omega^*}$. In other words, had i stayed in the market longer he could have strictly profit from trading with some seller j, which leads to the desired contradiction.

But how do we know one such state ω^* and seller $j \in S_i^{\omega^*}$ always exist for any such ω and buyer $i \in B_I^\omega \cap Q^\omega(f^0)$? There are plenty of reasons one does not. It could be the case that, for every seller $j \in S^{\omega^*}$ in any state ω^* equal to or succeeding ω, there exists some other buyer $k^{\omega^*}(j) \in B^{\omega^*} \setminus \{i\}$ who is willing to offer a very high price to purchase from j, such that no trade between i and j is mutually individually rational at ω^*. It is extremely difficult to rule out the existence of such 'competing' buyer $k^{\omega^*}(j)$, because $k^{\omega^*}(j)$ need not even be a higher value buyer than i; indeed, seller j may willingly accept a very low price offer from $k^{\omega^*}(j)$, possibly lower than v_j, if such trade agreement can transit the state from ω^* to one in which the continuation play more than compensates for j's momentary loss – for instance, through the expected profit from lucrative prospect of re-trading. At this point, the intractability of such reasoning is evident.

Why this complexity does not arise in Gale (2000)? This is precisely because players there are assumed to stay in the market forever, and payoffs are defined to be their asymptotic endowments. To prove pairwise efficiency, therefore, Gale (2000) can simply start from the *converged* equilibrium payoffs, and argue it is a contradiction if the payoff of any two players can go up further. But this reasoning is inapplicable in models where players can choose to exit the market *midway*, because evidently players' continuation payoffs keep evolving over the course of the game. As demonstrated above, to show certain exiting player i has missed out some trading opportunities, we must first study the incentives of his trading partners and competitors amidst the game, which depends on the continuation play of succeeding states whereof the possibilities are infinitely many and evidently intractable. This shows the technical challenge that I have overcome in this chapter. In particular, the arguments I have developed directly examine the evolution of roles and payoffs of different players amidst the game. The reasoning is both technically novel and conceptually distinct from Gale (2000).

Reasoning aside, another merit of this chapter is that my model is readily comparable with that in Gale and Sabourian (2006), which shows the prevalence of inefficient MPE under perfect, complete information and no discounting. In Gale and Sabourian (2006), a single buyer-seller pair is randomly drawn, and within the pair one player is randomly drawn to be the proposer. To see the similarity of our setups, observe that their matching technology is equivalent to first randomly drawing a player to be the proposer, and then randomly drawing an 'opposite type' player as his responder. Yet our results differ drastically: in their setup, there exists a continuum of inefficient MPE.

The discussion over allocative inefficiency is unsettled in the DMBG literature. In this chapter I have studied the benchmark case where each remaining buyer and seller are matched with positive probability. Yet I conjecture that my reasoning can extend to cover cases where certain pairs of players do not have direct 'links' with each other, as long as the network induced by the matching probabilities is strongly connected. This will lead to the striking result that intermediation could induce efficient allocation, which contrasts sharply with the inefficiency result in the literature (see Manea (2018)).

Chapter 3

Redistributive Effects of Search Frictions

3.1 Introduction

The theory of frictional unemployment attributes unemployment to the frictions of the search and matching process in the labour market – the market persistently fails to clear because hiring and job seeking take time (matching efficiency), whilst idiosyncratic shocks keep dissolving existing job matches (exogenous separation). One natural question to pose within this theory then is how these search frictions affect various labour market outcomes, such as wage, market tightness, unemployment duration, and the welfare of workers. In particular, does lowering search frictions always benefit workers? This question is of policy interest because after the 2008 financial crisis, many firms became insolvent and labour market matching efficiency stayed low for years (Barnichon and Figura (2015), Elsby et al. (2015), Furlanetto and Groshenny (2016)). Presently, the economic recession triggered by the COVID-19 pandemic is expected to heighten search frictions yet again. It is therefore crucial to understand the implications to the welfare of workers.

To study this question it is natural to begin from the canonical frictional unemployment model in Pissarides (2000). Simple derivation yields the following comparative statics. If matching efficiency rises, firms expect to find workers to fill their vacancies faster, which lowers their hiring cost. If (exogenous) separation rate drops, firms expect job matches to last and generate value longer, which raises their hiring benefit. Because vacancy supply is assumed in Pissarides (2000) to be perfectly elastic – known also as the free entry condition – firms in response create more vacancies. This in turn tightens the labour market such that workers can now bargain for a higher wage. But then workers are better off during both employment (from higher wage and lower job loss risk) and unemployment (from shorter unemployment duration). This leads to the unsurprising prediction that lowering search frictions unambiguously improves the welfare of workers.

This reasoning is sound and good, except that it faces some empirical challenge: van den Berg and van Vuuren (2010) exploits a longitudinal dataset on the Danish labour market, and finds that search frictions have heterogeneous impacts on the wages of different workers. In particular, search frictions have a negative and significant impact on the wages of unskilled workers, a negative but insignificant impact on the wages of skilled and office workers, and, most important, a positive and significant on the wages of managers; the last finding evidently contradicts the prediction of Pissarides (2000). Unfortunately, van den Berg and van Vuuren (2010) fail to provide any explanation for their surprising empirical result. Indeed, they seem to downplay it, and proceed to conclude that "the results lend credence to models that predict a negative effect of frictions on wages... [including] many existing so-called equilibrium search and matching models, notably the well-known Burdett-Mortensen and Pissarides models and most of their offsprings."

This paper takes the above empirical puzzle seriously, and ask if there is something fundamentally distinct about labour markets for senior positions which most existing frictional unemployment models fail to capture. I argue that the discrepancy hinges upon the assumption of *free entry of vacancies*, an equilibrium condition that is adopted in Pissarides (2000) and most of its follow-up studies (see Rogerson et al. (2005)). The free entry condition captures the idea that firms cannot make any positive expected profit (or loss) from having unfilled vacancies; otherwise they will keep creating (or cutting) vacancies until such profit (or loss) is driven to zero. As a result, the supply of vacancies is perfectly elastic. Coles and Kelishomi (2018) however challenges this assumption because it implies that market tightness is orthogonal to unemployment, which contradicts the data; empirical evidence, they argue, suggests that a non-trivial portion of vacancy supply in the labour market should instead be modelled as inelastic.

Indeed, I argue that labour markets for senior positions are precisely characterized by inelastic vacancy supply, for the evident reason that in any organization the number of positions tend to be more rigid the higher they reside up in the hierarchy. For instance, a warehouse hires a flexible number of packers (unskilled) and lorry drivers (skilled) during peak seasons, but there are always only one warehouse manager and one transport director; a farm hires a flexible number of fruit pickers (unskilled) during harvest seasons, but there is always only one farm manager; a university department hires a flexible number of research assistants (skilled) depending on projects, but there are only so many professors. Notice that the discrepancy stems from the seniority of the positions rather than the required skill levels; the latter but not the former has been well-studied in the labour economics literature.

Based on this observation, I develop a search and matching model that incorporates many salient features of Pissarides (2000), except that both the supply of vacancy and labour are assumed to be perfectly inelastic. One contribution of this chapter is to demonstrate that, with this alternative assumption of inelastic vacancy supply, the comparative statics of several labour market outcomes such as market tightness and wage become totally opposite to those obtained in Pissarides (2000), which in turn shows that labour markets for senior positions behave very differently from those for junior positions. In particular, I show that a rise in search frictions surprisingly uplifts the bargaining position and wage of senior workers, which resolves the empirical puzzle in van den Berg and van Vuuren (2010).

Perfect inelasticity is far from a novel assumption in the search and matching literature. For instance, Rubinstein and Wolinsky (1985) and Shimer and Smith (2000) assume fixed population sizes on both sides of the market, whereas Lauermann (2012) and Lauermann (2013) assume exogenous population *inflows* on both sides of the market. What is notable about the above literature is that, except for Rubinstein and Wolinsky (1985), they study settings with heterogeneous agents where higher surplus is generated from matching with higher types. Consequently, when frictions decline, it is unsurprising that agents become more selective of their partners such that lower types become excluded and worse off. On the other hand, in the setting of Rubinstein and Wolinsky (1985) where all agents are homogeneous, such mechanism does not exist, and consequently all agents become better off when frictions decline.

Another contribution of this paper is to demonstrate that, even in a setting with homogeneous agents, surprisingly some agents could become worse off when frictions decline. To make the novel mechanism transparent, the model is constructed in such a way that the two sides of the market are almost symmetric except only for their exogenous population inflows. With this single difference, I show that the short side of the market (firms) always benefits from lower search frictions, whereas the long side (workers) may sometimes suffer from it. In other words, search frictions may have significant redistributive effect on the welfare of different agents in the labour market even in the absence of selective or assortative matching.

This chapter is structured as follows. Section 3.2 describes the model setup. Section 3.3 pins down the wage bargaining solution. Section 3.4 analyzes the steady state. Section 3.5 discusses the redistributive implications of the results and concludes.

3.2 The model setup

This is an infinite-horizon, continuous time model of complete information. Firms are homogeneous and each has an inelastic supply of one vacancy. Workers are homogeneous and each can fill at most one vacancy at any point in time. Each matched firm-worker pair produces one unit of output per unit time. Any unmatched firms and workers are unproductive. As any continuous time model is but the limit of its discrete time counterpart as the period length vanishes, I first write down the discrete time version with period length $\Delta \in (0,1)$.

At the start of each period t, denote by u_t^ℓ the mass of unmatched workers, by u_t^f the mass of unmatched firms, by n_t^ℓ the mass of matched workers, and by n_t^f the mass of matched firms. Each period t consists of the following three stages.

Stage 1: Search and matching

The mass of firm-worker pair *newly* matched in the period t is given by $\Delta\varphi m(u_t^\ell, u_t^f)$, where $\varphi \in (0,1)$ is the matching efficiency parameter, and $m : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the matching function assumed to be increasing in both arguments and homogeneous of degree one. The firms and workers newly matched are randomly drawn from the pool of unmatched firms and unmatched workers. In other words, each unmatched firm fills its vacancy with probability $\Delta\varphi m(u_t^\ell, u_t^f)/u_t^f$, and each unmatched worker finds a job with probability $\Delta\varphi m(u_t^\ell, u_t^f)/u_t^\ell$. Firms and workers that remain unmatched are inactive throughout the period, and receive zero period payoffs.

Stage 2: Bargaining and production

All matched pairs which have either survived from period $t-\Delta$ or been newly formed in Stage 1 of period t negotiate wages before production. Within each matched pair of worker i and firm j, the wage rate w_{ijt} is determined by the Nash bargaining solution with equal bargaining power from both sides. In other words, the two sides of each match split the surplus equally. After production, worker i and firm j receive a period payoff of Δw_{ijt} and $\Delta(1-w_{ijt})$ respectively.

Stage 3: Separation and turnover

After production and consumption, a fraction $\Delta\sigma$ of all matched pairs is randomly drawn to dissolve, and the corresponding firms and workers become unmatched; $\sigma \in (0,1)$ is called the (exogenous) separation rate. After that, a fraction $\Delta\rho$ of all players is randomly drawn to exit the market permanently; $\rho \in (0,1)$ is called the (exogenous) exit rate. Within any surviving match, if one player exits the market but the other does not, then the latter becomes unmatched. Finally, new firms and workers of mass $\Delta\lambda^f$ and $\Delta\lambda^\ell > \Delta\lambda^f$ respectively enter the market unmatched.

This completes the description of the sequence of events that takes place in each period. Following the model description, I now derive the laws of motions of the state variables. First, the mass of matched workers satisfies

$$n^\ell_{t+\Delta} = (n^\ell_t + \Delta\varphi m(u^\ell_t, u^f_t))(1-\Delta\sigma)(1-\Delta\rho)^2$$

Re-arranging the expression and taking limits yields

$$\begin{aligned}
\dot{n}^\ell_t &= \lim_{\Delta\to 0} \frac{n^\ell_{t+\Delta} - n^\ell_t}{\Delta} \\
&= \lim_{\Delta\to 0} \varphi m(u^\ell_t, u^f_t)(1-\Delta\sigma)(1-\Delta\rho)^2 - n^\ell_t(2\rho + \sigma(1-\Delta\rho)^2 - \Delta\rho^2) \\
&= \varphi m(u^\ell_t, u^f_t) - (\sigma + 2\rho)n^\ell_t
\end{aligned}$$

Second, the mass of unmatched workers satisfies

$$\begin{aligned}
u^\ell_{t+\Delta} = [u^\ell_t &- \Delta\varphi m(u^\ell_t, u^f_t) \\
&+ (n^\ell_t + \Delta\varphi m(u^\ell_t, u^f_t))\,(\Delta\sigma + (1-\Delta\sigma)\Delta\rho)\,](1-\Delta\rho) + \Delta\lambda^\ell
\end{aligned}$$

Re-arranging the expression and taking limits yields

$$\begin{aligned}
\dot{u}^\ell_t &= \lim_{\Delta\to 0} \frac{u^\ell_{t+\Delta} - u^\ell_t}{\Delta} \\
&= \lim_{\Delta\to 0} \lambda^\ell - \rho u^\ell_t + n^\ell_t (1-\Delta\rho)(\sigma + (1-\Delta\sigma)\rho) \\
&\qquad\qquad\qquad\qquad - \varphi m(u^\ell_t, u^f_t)(1-\Delta\sigma)(1-\Delta\rho)^2 \\
&= \lambda^\ell - \rho u^\ell_t + (\sigma+\rho) n^\ell_t - \varphi m(u^\ell_t, u^f_t)
\end{aligned}$$

By applying similar reasoning to the mass of matched and unmatched firms, I obtain the following laws of motions of the state variables:

$$\begin{aligned}
\dot{n}^f_t &= \varphi m(u^\ell_t, u^f_t) - (\sigma + 2\rho) n^f_t \\
\dot{n}^\ell_t &= \varphi m(u^\ell_t, u^f_t) - (\sigma + 2\rho) n^\ell_t \\
\dot{u}^f_t &= \lambda^f - \rho u^f_t + (\sigma+\rho) n^f_t - \varphi m(u^\ell_t, u^f_t) \\
\dot{u}^\ell_t &= \lambda^\ell - \rho u^\ell_t + (\sigma+\rho) n^\ell_t - \varphi m(u^\ell_t, u^f_t)
\end{aligned} \tag{3.1}$$

Notice that the mass of firms and workers in the labour market, matched or unmatched, evolves almost symmetrically except that the inflow of new workers persistently exceeds that of new firms ($\lambda^f < \lambda^\ell$).

Now I define market tightness $\theta_t := u^f_t / u^\ell_t$, a crucial variable in the model. Intuitively, when θ_t is large, there is a large mass of unfilled vacancies relative to the mass of unemployed workers, and thus it is relatively hard for firms to fill their vacancies. Indeed, market tightness closely relates to the vacancy filling rate. Because the matching function m is homogeneous of degree one

$$\frac{m(u^\ell_t, u^f_t)}{u^f_t} = m(\frac{1}{\theta_t}, 1) =: q(\theta_t)$$

This means the vacancy filling rate of firms can be re-written as $\varphi q(\theta_t)$. Meanwhile

$$\frac{m(u_t^\ell, u_t^f)}{u_t^\ell} = m(1,\theta_t) = \theta_t q(\theta_t)$$

This means the job finding rate of workers can be re-written as $\varphi \theta_t q(\theta_t)$. I am going to show how market tightness θ_t summarizes the role that all state variables play in the continuation payoffs of workers and firms through the vacancy filling rate and job finding rate.

3.3 Wage determination

From the model, it is evident all workers face symmetric prospect when unemployed, and thus the continuation payoff of being unmatched at any time t is the same for all workers; for similar reason, the continuation payoff of being unmatched at any time t is the same for all firms. Since wage is set by the Nash bargaining solution, this together with the homogeneous productivity assumption in turn imply that wage $w_{ijt} \equiv w_t$ at any time t is the same across all matched pairs (i, j). Since the flow payoff is respectively the same for all matched workers and all matched firms, the continuation payoff of being matched at any time t is also respectively the same for all workers and all firms.

As in Section 3.2, I first apply the above observation to the discrete time analogue of the model with period length $\Delta \in (0,1)$. At the start of each period t, denote by V_t^ℓ and J_t^ℓ the continuation payoff of any workers if unmatched and matched respectively, and by V_t^f and J_t^f the continuation payoff of any firms if unmatched and matched respectively. Following the model description, I now derive the laws of motions of the jump variables. First, the continuation payoffs of matched workers satisfies

$$J_t^\ell = \Delta w_t + (1-\Delta\sigma)(1-\Delta\rho)^2 J_{t+\Delta}^\ell + (1-\Delta\rho)(\Delta\sigma + (1-\Delta\sigma)\Delta\rho) V_{t+\Delta}^\ell$$

Re-arranging the expression and taking limits yields

$$\begin{aligned}
-\dot{J}_t^\ell &= \lim_{\Delta\to 0} \frac{J_t^\ell - J_{t+\Delta}^\ell}{\Delta} \\
&= \lim_{\Delta\to 0} w_t - (2\rho + \sigma(1-\Delta\rho)^2 - \Delta\rho^2)J_{t+\Delta}^\ell \\
&\qquad\qquad + (1-\Delta\rho)(\sigma + (1-\Delta\sigma)\rho)V_{t+\Delta}^\ell \\
&= w_t - (\sigma+2\rho)J_t^\ell + (\sigma+\rho)V_t^\ell
\end{aligned}$$

Second, the continuation payoffs of unmatched workers satisfies

$$V_t^f = \frac{\Delta\varphi m(u_t^\ell, u_t^f)}{u_t^f} J_t^f + \left(1 - \frac{\Delta\varphi m(u_t^\ell, u_t^f)}{u_t^f}\right)(1-\Delta\rho)V_{t+\Delta}^f$$

Re-arranging the expression and taking limits yields

$$\begin{aligned}
-\dot{V}_t^\ell &= \lim_{\Delta\to 0} \frac{V_t^\ell - V_{t+\Delta}^\ell}{\Delta} \\
&= \lim_{\Delta\to 0} \frac{\varphi m(u_t^\ell, u_t^f)}{u_t^f}(J_t^f - (1-\Delta\rho)V_{t+\Delta}^f) - \rho V_{t+\Delta}^f \\
&= \frac{\varphi m(u_t^\ell, u_t^f)}{u_t^f}(J_t^f - V_t^f) - \rho V_t^f
\end{aligned}$$

By applying similar reasoning to the continuation payoffs of matched and unmatched firms, I obtain the following laws of motions of the jump variables:

$$\begin{aligned}
-\dot{J}_t^f &= 1 - w_t - (\sigma+2\rho)J_t^f + (\sigma+\rho)V_t^f \\
-\dot{J}_t^\ell &= w_t - (\sigma+2\rho)J_t^\ell + (\sigma+\rho)V_t^\ell \\
-\dot{V}_t^f &= \frac{\varphi m(u_t^\ell, u_t^f)}{u_t^f}(J_t^f - V_t^f) - \rho V_t^f \\
-\dot{V}_t^\ell &= \frac{\varphi m(u_t^\ell, u_t^f)}{u_t^\ell}(J_t^\ell - V_t^\ell) - \rho V_t^\ell
\end{aligned} \tag{3.2}$$

Notice that the continuation payoffs of firms and workers in the labour market, matched or unmatched, evolves almost symmetrically except that flow payoffs of matched workers w_t may differ from that of matched firms $1-w_t$.

Next I pin down the flow payoffs by solving for the bargained wage. Re-arranging the differential equations in (3.2) yields the employment surplus of workers and firms.

$$J_t^\ell - V_t^\ell = \frac{w_t + \dot{J}_t^\ell - \dot{V}_t^\ell}{\sigma + 2\rho + \varphi\theta_t q(\theta_t)}$$

$$J_t^f - V_t^f = \frac{1 - w_t + \dot{J}_t^f - \dot{V}_t^f}{\sigma + 2\rho + \varphi q(\theta_t)}$$

Since wage w_t^* is set by the Nash bargaining solution with equal bargaining power on the two sides, it must satisfy $w_t^* = \arg\max_{w_t} (J_t^\ell - V_t^\ell)(J_t^f - V_t^f)$. First order condition yields $J_t^\ell - V_t^\ell = J_t^f - V_t^f$; in other words, wage w_t^* is set such that firms and workers obtain equal employment surplus. Re-arranging the first order condition yields the bargained wage.

$$w_t^* = \frac{\sigma + 2\rho + \varphi\theta_t q(\theta_t) - (\dot{J}_t^\ell - \dot{V}_t^\ell)\varphi(1-\theta_t)q(\theta_t)}{2(\sigma + 2\rho) + \varphi(1+\theta_t)q(\theta_t)}$$

This completes the derivation of the wage bargaining outcome. Now I proceed to study how the steady state of the dynamic system varies with the two key parameters of search frictions in the model: matching efficiency φ and separation rate σ.

3.4 Steady state analysis

In the following, I remove the time subscript of all state and jump variables to denote their steady state values. To study the steady state of the model, set $\dot{n}_t^f = \dot{n}_t^\ell = \dot{u}_t^f = \dot{u}_t^\ell = 0$. Applying this to the differential equations in (3.1) and re-arranging, I obtain an implicit function for steady state market tightness:

$$\varphi\theta q(\theta)(\lambda^{\ell}-\lambda^{f})+(\sigma+2\rho)(\lambda^{\ell}\theta-\lambda^{f})=0 \tag{3.3}$$

Proposition 3.1. Market tightness θ decreases in matching efficiency φ.

Proof. Applying the implicit function theorem to equation (3.3) yields

$$\frac{\partial\theta}{\partial\varphi}=-\frac{\theta q(\theta)(\lambda^{\ell}-\lambda^{f})}{\varphi(q(\theta)+\theta q'(\theta))(\lambda^{\ell}-\lambda^{f})+(\sigma+2\rho)\lambda^{\ell}}<0 \tag{3.4}$$

The inequality holds because $q(\theta)+\theta q'(\theta)=\frac{\partial}{\partial\theta}\theta q(\theta)=\frac{\partial}{\partial\theta}m(1,\theta)>0$. □

Proposition 3.1 is the key driving force of all comparative statics with respect to φ. For intuition, consider a labour market with initially 200 vacancies and 300 unemployed workers; by definition, market tightness is $\theta_0=\frac{200}{300}=\frac{2}{3}$. Yet if matching efficiency rises such that 150 more job matches are formed, market tightness now becomes $\theta_1=\frac{200-150}{300-150}=\frac{1}{3}$. But then on average three unemployed workers are now competing for one vacancy instead of two as a result. This intuitively shows how raising matching efficiency depresses market tightness and intensifies the competition among the long side of the market. Indeed, equation (3.3) can be re-written as

$$\theta=\frac{\lambda^{f}-\dfrac{\varphi m(u^{\ell},u^{f})\rho}{\sigma+2\rho}}{\lambda^{\ell}-\dfrac{\varphi m(u^{\ell},u^{f})\rho}{\sigma+2\rho}}$$

Such form corresponds to the intuitive explanation above: raising φ leads to a greater term subtracted from both the numerator and the denominator, but the percentage decrease therefrom is greater in the former because $\lambda^{f}<\lambda^{\ell}$.

Next, I show what Proposition 3.1 implies for the jump variables of the model. Set $\dot{J}_t^f = \dot{J}_t^\ell = \dot{V}_t^f = \dot{V}_t^\ell = 0$. This yields the steady state wage:

$$w^* = \frac{\sigma + 2\rho + \varphi\theta q(\theta)}{2(\sigma + 2\rho) + \varphi(1+\theta)q(\theta)} \tag{3.5}$$

Proposition 3.2. Wage w^* decreases in matching efficiency φ.

Proof. Differentiating equation (3.5) with respect to φ yields

$$\begin{aligned} \frac{\partial w^*}{\partial \varphi} &= \frac{1}{[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]^2} \\ &\quad \times \left[(\sigma+2\rho)[\varphi(q(\theta)-(1-\theta)q'(\theta))\frac{\partial\theta}{\partial\varphi} - (1-\theta)q(\theta)] + (\varphi q(\theta))^2\frac{\partial\theta}{\partial\varphi}\right] < 0 \end{aligned}$$

Note that $q'(\theta) = \frac{\partial}{\partial\theta}m(\frac{1}{\theta},1) < 0$, and thus $q(\theta) - (1-\theta)q'(\theta) > 0$. Together with Proposition 3.1, the big square bracket is evidently negative. □

Proposition 3.2 is a direct consequence of Proposition 3.1. If matching efficiency rises, the competition among unemployed workers is intensified relative to that among firms, which upsets the bargaining position of workers and leads to lower wage. On the other hand, if matching efficiency drops, the labour market becomes tighter and it is harder for firms to fill their vacancies; consequently, firms have to pay a higher wage to retain matched workers.

Having demonstrated the effect of matching efficiency on wages, I proceed to study the ex-ante payoffs of firms and workers. I argue these can be captured by the jump variables V^f and V^ℓ respectively, because firms and workers arrive into the labour market unmatched. Substituting equation (3.5) into the system of differential equations (3.2) and solving for the steady state values of the jump variables yields

$$V^f = \frac{\varphi q(\theta)}{\rho[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]} \tag{3.6}$$

$$V^\ell = \frac{\varphi \theta q(\theta)}{\rho[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]} \tag{3.7}$$

Note that V^ℓ and V^f respectively represents the outside options of matched workers and firms in case wage bargaining breaks down. As market tightness $\theta = V^\ell/V^f$ is precisely the ratio of workers' payoffs to firms' payoffs, the former can also be interpreted as measuring the *bargaining position* of workers relative to that of firms.

Proposition 3.3. Firms' ex-ante payoff V^f increases in matching efficiency φ.

Proof. Differentiating equation (3.6) with respect to φ yields

$$\begin{aligned}\frac{\partial V^f}{\partial \varphi} = &\frac{1}{\rho[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]^2} \\ &\times\left[2(\sigma+2\rho)q(\theta)+[2(\sigma+2\rho)\varphi q'(\theta)-(\varphi q(\theta))^2]\frac{\partial \theta}{\partial \varphi}\right] > 0\end{aligned}$$

Note that $q'(\theta) = \frac{\partial}{\partial \theta} m(\frac{1}{\theta}, 1) < 0$. Together with Proposition 3.1, the big square bracket is evidently positive. □

Proposition 3.3 shows that firms, or more generally the short side of the market, always benefit from higher matching efficiency. One obvious reason is that rising matching efficiency shortens the time it takes for firms to fill their vacancies. To see this formally, note that unmatched firms fill their vacancies following a Poisson process with rate $\varphi q(\theta)$. Thus the mean vacancy duration is given by $1/\varphi q(\theta)$, and

$$\frac{\partial}{\partial\varphi}\varphi q(\theta) = q(\theta) + \varphi q'(\theta)\frac{\partial\theta}{\partial\varphi} > 0$$

Hence, when matching efficiency rises, unmatched firms transit out of unemployment faster; at the same time, once firms successfully match with workers, they only have to pay a lower wage now because of the rise of their bargaining position relative to that of workers, as shown in Proposition 3.2. Therefore, raising matching efficiency benefits firms whether during employment or unemployment.

Proposition 3.4. Workers' ex-ante payoff V^ℓ increases in matching efficiency φ if and only if

$$\Omega(\varphi,\sigma,\rho,\lambda^f/\lambda^\ell) := 2(\sigma+2\rho)^2 - (1-\frac{\lambda^f}{\lambda^\ell})(\varphi q(\theta))^2 > 0$$

Also, $\Omega(\varphi,\sigma,\rho,\lambda^f/\lambda^\ell)$ decreases in φ and increases in $\sigma,\rho,\lambda^f/\lambda^\ell$.

Proof. Differentiating equation (3.7) with respect to φ yields

$$\frac{\partial V^\ell}{\partial\varphi} = \frac{1}{\rho[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]^2}$$
$$\times\left[2(\sigma+2\rho)\theta q(\theta) + [2(\sigma+2\rho)\varphi(q(\theta)+\theta q'(\theta)) + (\varphi q(\theta))^2]\frac{\partial\theta}{\partial\varphi}\right]$$

Substituting equation (3.4) into the above, the big square bracket is positive if and only if

$$\frac{-2(\sigma+2\rho)\theta q(\theta)}{2(\sigma+2\rho)\varphi(q(\theta)+\theta q'(\theta)) + (\varphi q(\theta))^2}$$
$$< \frac{\partial\theta}{\partial\varphi} = \frac{-\theta q(\theta)(\lambda^\ell-\lambda^f)}{\varphi(q(\theta)+\theta q'(\theta))(\lambda^\ell-\lambda^f)+(\sigma+2\rho)\lambda^\ell}$$

After simplification, this is equivalent to $\Omega(\varphi,\sigma,\rho,\lambda^f/\lambda^\ell) > 0$. Finally, applying the implicit function theorem to equation (3.3) shows that θ decreases in φ but increases in $\sigma,\rho,\lambda^f/\lambda^\ell$. Together with the fact that $q'(\theta) < 0$, these in turn imply $\Omega(\varphi,\sigma,\rho,\lambda^f/\lambda^\ell)$ decreases in φ and increases in $\sigma,\rho,\lambda^f/\lambda^\ell$. $\square$

Proposition 3.4 shows that workers, or more generally the long side of the market, do no necessarily benefit from higher matching efficiency. The upside of raising matching efficiency is obviously that it shortens the time it takes for unemployed workers to find jobs. To see this formally, note that unmatched workers find jobs following a Poisson process with rate $\varphi\theta q(\theta)$. Thus the mean vacancy duration is given by $1/\varphi\theta q(\theta)$, and substituting equation (3.3) yields

$$\frac{\partial}{\partial\varphi}\varphi\theta q(\theta) = \frac{\partial}{\partial\varphi}\frac{(\sigma+2\rho)(\lambda^f/\lambda^\ell-\theta)}{1-\lambda^f/\lambda^\ell} = \frac{\sigma+2\rho}{1-\lambda^f/\lambda^\ell}\left(-\frac{\partial\theta}{\partial\varphi}\right) > 0$$

Hence, when matching efficiency rises, unmatched workers transit out of unemployment faster; however, once workers successfully match with firms, they only receive a lower wage now because of the decline of their bargaining position relative to that of firms, as shown in Proposition 3.2. Therefore, raising matching efficiency generates a trade-off between lower unemployment risk and lower employment reward, so its the overall effect on workers' welfare is ambiguous. Nevertheless, if frictions in the labour market are high (high σ,ρ and low φ) and the competition among workers for jobs is mild (high λ^f/λ^ℓ), then raising matching efficiency tends to be Pareto-improving.

Up to this point I have focused on the comparative statics of various labour market outcomes with respect to matching efficiency φ. Now I proceed to derive analogous comparative statics with respect to separation rate σ.

Proposition 3.5. Market tightness θ increases in separation rate σ.

Proof. Applying the implicit function theorem to equation (3.3) yields

$$\begin{aligned}\frac{\partial\theta}{\partial\varphi} &= \frac{\lambda^f - \lambda^\ell\theta}{\varphi(q(\theta)+\theta q'(\theta))(\lambda^\ell - \lambda^f) + (\sigma+2\rho)\lambda^\ell} \\ &= \frac{\varphi\theta q(\theta)(\lambda^\ell - \lambda^f)/(\sigma+2\rho)}{\varphi(q(\theta)+\theta q'(\theta))(\lambda^\ell - \lambda^f) + (\sigma+2\rho)\lambda^\ell} > 0\end{aligned} \tag{3.8}$$

The second equality holds by substituting equation (3.3) into the numerator. The inequality holds because $q(\theta)+\theta q'(\theta) = \frac{\partial}{\partial\theta}\theta q(\theta) = \frac{\partial}{\partial\theta}m(1,\theta) > 0$. □

Proposition 3.5 is the key driving force of all comparative statics with respect to σ. For intuition, consider a labour market with initially 50 vacancies and 150 unemployed workers; by definition, market tightness is $\theta_0 = \frac{50}{150} = \frac{1}{3}$. Yet if separation rate rises such that 150 more job matches are dissolved, market tightness now becomes $\theta_1 = \frac{50+150}{150+150} = \frac{2}{3}$. This intuitively shows how raising separation rate tightens the market and intensifies the competition among the short side. As the remaining results follow for analogous reasons, I omit repetitive elaboration of their intuition.

Proposition 3.6. Wage w^* increases in separation rate σ.

Proof. Differentiating equation (3.5) with respect to σ yields

$$\begin{aligned}\frac{\partial w^*}{\partial\sigma} = &\frac{1}{[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]^2} \\ &\times\left[[(\sigma+2\rho)\varphi(q(\theta)-(1-\theta)q'(\theta))+(\varphi q(\theta))^2]\frac{\partial\theta}{\partial\varphi}+(1-\theta)\varphi q(\theta)\right] > 0\end{aligned}$$

Note that $q'(\theta) = \frac{\partial}{\partial\theta} m(\frac{1}{\theta}, 1) < 0$, and thus $q(\theta) - (1-\theta)q'(\theta) > 0$. Together with Proposition 3.5, the big square bracket is evidently positive. □

Proposition 3.7. Firms' ex-ante payoff V^f decreases in separation rate σ.

Proof. Differentiating equation (3.6) with respect to σ yields

$$\frac{\partial V^f}{\partial \sigma} = \frac{1}{\rho[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]^2} \times \left[[2(\sigma+2\rho)\varphi q'(\theta) - (\varphi q(\theta))^2] \frac{\partial \theta}{\partial \sigma} - 2\varphi q(\theta) \right] < 0$$

Note that $q'(\theta) = \frac{\partial}{\partial\theta} m(\frac{1}{\theta}, 1) < 0$. Together with Proposition 3.5, the big square bracket is evidently negative. □

Proposition 3.8. Workers' ex-ante payoff V^ℓ decreases in separation rate σ if and only if $\Omega(\varphi, \sigma, \rho, \lambda^f/\lambda^\ell) > 0$.

Proof. Differentiating equation (3.7) with respect to σ yields

$$\frac{\partial V^\ell}{\partial \sigma} = \frac{1}{\rho[2(\sigma+2\rho)+\varphi(1+\theta)q(\theta)]^2} \times \left[[2(\sigma+2\rho)\varphi(q(\theta)+\theta q'(\theta)) + (\varphi q(\theta))^2] \frac{\partial \theta}{\partial \sigma} - 2\varphi\theta q(\theta) \right]$$

Substituting (8), the big square bracket is negative if and only if

$$\frac{2\varphi\theta q(\theta)}{2(\sigma+2\rho)\varphi(q(\theta)+\theta q'(\theta))+(\varphi q(\theta))^2}$$

$$> \frac{\partial\theta}{\partial\sigma} = \frac{\varphi\theta q(\theta)(\lambda^\ell-\lambda^f)/(\sigma+2\rho)}{\varphi(q(\theta)+\theta q'(\theta))(\lambda^\ell-\lambda^f)+(\sigma+2\rho)\lambda^\ell}$$

After simplification, this is equivalent to $\Omega(\varphi,\sigma,\rho,\lambda^f/\lambda^\ell)>0$. $\square$

3.5 Conclusion

In this chapter I have developed a search and matching model that incorporates many salient features of Pissarides (2000), except that the supply of vacancy is assumed to be perfectly inelastic. The focus has been the comparative statics of several labour market outcomes with respect to matching efficiency φ and separation rate σ. By simple derivation, I have also obtained analogous results from Pissarides (2000) as summarized below.

For matching efficiency φ:

Steady state variables	This paper	Pissarides (2000)
Market tightness θ	decreases in φ	increases in φ
Wage w^*	decreases in φ	increases in φ
Firms' ex-ante payoff V^f	increases in φ	is set to be zero
Workers' ex-ante payoff V^ℓ	may decrease in φ	increases in φ

For separation rate σ:

Steady state variables	This paper	Pissarides (2000)
Market tightness θ	increases in σ	decreases in σ
Wage w^*	increases in σ	decreases in σ
Firms' ex-ante payoff V^f	decreases in σ	is set to be zero
Workers' ex-ante payoff V^ℓ	may increase in σ	decreases in σ

As argued in Section 3.1, positions occupying the upper part of the corporate hierarchy are characterized by inelastic supply rather than elastic supply. Therefore, I argue that my model is relatively suitable for labour markets for senior positions, whereas *Pissarides* (2000) and its follow-up studies which impose the free entry condition on vacancies are relatively suitable for labour markets for junior positions. The contrasting results then suggest that search frictions have redistributive implications. In particular, raising search frictions in the labour market depresses the bargaining position of junior workers but uplifts that of senior workers, which in turn contributes to worsening income inequality. This observation is of policy interest given the heightening of labour market search frictions presently triggered by the COVID-19 pandemic.

Theoretically, the results are also interesting as they show that lowering search frictions is not necessarily Pareto-improving even in a setting with homogeneous agents. In an almost symmetric model where the two sides of the market differs only in exogenous population inflows, lowering search frictions uplifts the bargaining position of the short side of the market (firms) but depresses that of the long side (workers). Consequently, the short side always gain from lowering search frictions, whereas the long side may sometimes suffer from it. This is another dimension of the redistributive implications of search frictions.

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Appendix A

Omitted proofs of Chapter 1

Proposition 1.4. In any 2-markets, any MPE supports at most two distinct first-trade prices.

Proof. Fix an arbitrary 2-market with buyers $\{a,c\}$ and sellers $\{b,d\}$. Suppose for contradiction there is a MPE with more than two distinct first-trade prices. Fix one such MPE.

First, it must be the case that all players are infra-marginal; otherwise, efficiency implies that only the two infra-marginal players would trade first with positive probability, generating at most two distinct first-trade prices. Second, it is impossible that $v_i - v_i^N \leq w_j + w_j^N$ for all $i \in \{a,c\}$ and $j \in \{b,d\}$; otherwise, feasibility implies that they must all hold with equality, generating a unique first-trade price. Let $v_a - v_a^N > w_b + w_b^N$ WLOG. But then efficiency implies $(v_c - v_c^N) - (w_d + w_d^N) = (w_b + w_b^N) - (v_a - v_a^N) < 0$ and thus $v_c - v_c^N < w_d + w_d^N$.

If $v_a - v_a^N > w_d + w_d^N$, then either (i) $v_a - v_a^N > w_d + w_d^N \geq w_b + w_b^N > v_c - v_c^N$ or (ii) $v_a - v_a^N > w_b + w_b^N > w_d + w_d^N > v_c - v_c^N$. Either way, $M^N = \{(a,b),(a,d)\}$. But this corresponds to Case 3 in Figure 1.1, and I have already shown in the proof of Proposition 1.2 that this first-trade pattern cannot be supported in any MPE. By symmetric reason, it is not possible that $v_a - v_a^N < w_d + w_d^N$.

But then it must be that $v_a - v_a^N = w_d + w_d^N$ and hence $v_a - v_a^N = w_d + w_d^N > v_c - v_c^N = w_b + w_b^N$. In other words, there are only two distinct first-trade prices, contradiction. □

Appendix B

Omitted proofs of Chapter 2

In order to prove Lemma 2.2, I must first prove Lemma 2.6 and 2.7, which relate players' continuation payoffs to the equilibrium trading patterns.

Lemma 2.6. For all $\varepsilon > 0$, there exists $\bar{c}(\varepsilon) > 0$ such that for all $c \in [0, \bar{c}(\varepsilon))$, $\omega \in \hat{\Omega}$, $(i, j) \in B^{\omega} \times S^{\omega}$, if $i \in Q^{\omega}(f^c) \cup R^{\omega}(f^c)$ and $j \in S_i^{\omega}(f^c)$, then

$$\pi_i^{\omega(i,j,p_{ij}^{\omega}(f^c))}(f^c) - \pi_i^{\omega}(f^c) < \varepsilon$$

Proof. Suppose not for contradiction. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $c_n \in [0, 1/n)$, $\omega_n \in \hat{\Omega}$, $(i_n, j_n) \in B^{\omega_n} \times S^{\omega_n}$ such that $i_n \in Q^{\omega_n}(f^{c_n}) \cup R^{\omega_n}(f^{c_n})$, $j_n \in S_{i_n}^{\omega_n}(f^{c_n})$ but $\pi_{i_n}^{\omega_n(i_n,j_n,p_{i_n j_n}^{\omega_n}(f^{c_n}))}(f^{c_n}) - \pi_{i_n}^{\omega_n}(f^{c_n}) \geq \varepsilon$. Fix one such $\varepsilon > 0$ and one such sequence $(c_n, \omega_n, i_n, j_n)_{n\in\mathbb{N}}$. However, for any $n \in \mathbb{N}$,

$$
\begin{aligned}
\lambda_{i_n}^{\omega_n}(f^{c_n}) = & \sum_{k\in S_{i_n}^{\omega_n}(f^{c_n})} \gamma_{i_n k}^{\omega_n} \pi_{i_n}^{\omega_n(i_n,k,p_{i_n k}^{\omega_n}(f^{c_n}))}(f^{c_n}) + \sum_{k\in S^{\omega_n}\setminus S_{i_n}^{\omega_n}(f^{c_n})} \gamma_{i_n k}^{\omega_n} \pi_{i_n}^{\omega_n}(f^{c_n}) - c_n \\
\geq & \ (1-\gamma_{i_n j_n}^{\omega_n})\pi_{i_n}^{\omega_n}(f^{c_n}) + \gamma_{i_n j_n}^{\omega_n}(\pi_{i_n}^{\omega_n}(f^{c_n})+\varepsilon) - c_n \\
\geq & \ (1-\gamma)\pi_{i_n}^{\omega_n}(f^{c_n}) + \gamma(\pi_{i_n}^{\omega_n}(f^{c_n})+\varepsilon) - c_n \\
= & \ \pi_{i_n}^{\omega_n}(f^{c_n}) + \gamma\varepsilon - c_n
\end{aligned}
\tag{B.1}
$$

The first inequality holds by applying the hypothesis to $\pi_{i_n}^{\omega_n(i_n,j_n,p_{i_n j_n}^{\omega_n}(f^{c_n}))}(f^{c_n})$ and the fact that $\pi_{i_n}^{\omega_n(i_n,k,p_{i_n k}^{\omega_n}(f^{c_n}))}(f^{c_n}) \geq \pi_{i_n}^{\omega_n}(f^{c_n})$ for all $k \in S_{i_n}^{\omega_n}(f^{c_n}) \setminus \{j_n\}$; the second by the fact that $\gamma_{i_n j_n}^{\omega_n} \geq \gamma$. However, for large enough n, $c_n < \gamma\varepsilon$ and thus $\lambda_{i_n}^{\omega_n}(f^{c_n}) > \pi_{i_n}^{\omega_n}(f^{c_n})$, which implies $i_n \notin R^{\omega_n}(f^{c_n})$ and thus $i_n \in Q^{\omega_n}(f^{c_n})$. But then, for large enough n,

$$
g_{i_n}^{\omega_n} v_{i_n} + m_{i_n}^{\omega_n} \geq \lambda_{i_n}^{\omega_n}(f^{c_n}) > \pi_{i_n}^{\omega_n}(f^{c_n}) \geq g_{i_n}^{\omega_n} v_{i_n} + m_{i_n}^{\omega_n}
$$

The first inequality holds by definition of $i_n \in Q^{\omega_n}(f^{c_n})$; the second as shown in (B.1); the third by individual rationality. This leads to contradiction as desired. □

Lemma 2.7. For all $\varepsilon > 0$, there exists $\bar{c}(\varepsilon) > 0$ such that for all $c \in [0, \bar{c}(\varepsilon))$, $\omega \in \hat{\Omega}$, $(i, j) \in B^\omega \times S^\omega$, if $i \in Q^\omega(f^c) \cup R^\omega(f^c)$ or $j \notin S_i^\omega(f^c)$, then

$$
\pi_i^\omega(f^c) + \pi_j^\omega(f^c) > v_i + m_i^\omega + m_j^\omega - \varepsilon
$$

Proof. Suppose not for contradiction. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $c_n \in [0, 1/n)$, $\omega_n \in \hat{\Omega}$, $(i_n, j_n) \in B^{\omega_n} \times S^{\omega_n}$ such that $\pi_{i_n}^{\omega_n}(f^{c_n}) + \pi_{j_n}^{\omega_n}(f^{c_n}) \leq v_{i_n} + m_{i_n}^{\omega_n} + m_{j_n}^{\omega_n} - \varepsilon$, but either $i_n \in Q^{\omega_n}(f^{c_n}) \cup R^{\omega_n}(f^{c_n})$ or $j_n \notin S_{i_n}^{\omega_n}(f^{c_n})$. Fix one such

$\varepsilon > 0$ and one such sequence $(c_n, \omega_n, i_n, j_n)_{n\in\mathbb{N}}$. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\pi_{i_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})}(f^{c_n}) &\geq g_{i_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})} v_{i_n} + m_{i_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})} \\
&= v_{i_n} + (m_{i_n}^{\omega_n} - \pi_{j_n}^{\omega_n}(f^{c_n}) + m_{j_n}^{\omega_n}) \\
&\geq \pi_{i_n}^{\omega_n}(f^{c_n}) + \varepsilon
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\pi_{j_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})}(f^{c_n}) &\geq g_{j_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})} v_{j_n} + m_{j_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})} \\
&= 0 + (m_{j_n}^{\omega_n} + \pi_{j_n}^{\omega_n}(f^{c_n}) - m_{j_n}^{\omega_n}) \\
&= \pi_{j_n}^{\omega_n}(f^{c_n})
\end{aligned} \tag{B.3}$$

The first inequalities in (B.2) and (B.3) hold by individual rationality; the second of (B.2) by hypothesis. Thus, for large enough n, $\pi_{j_n}^{\omega_n}(f^{c_n}) - m_{j_n}^{\omega_n} \in P_{i_n j_n}^{\omega_n}(f^{c_n})$, and the fact that $P_{i_n j_n}^{\omega_n}(f^{c_n}) \neq \varnothing$ in turn implies that $j_n \in S_{i_n}^{\omega_n}(f^{c_n})$ and thus $i_n \in Q^{\omega_n}(f^{c_n}) \cup R^{\omega_n}(f^{c_n})$. Hence, for large enough n,

$$\begin{aligned}
\pi_{i_n}^{\omega_n}(f^{c_n}) + \varepsilon &> \pi_{i_n}^{\omega_n(i_n,j_n,p_{i_n j_n}^{\omega_n}(f^{c_n}))}(f^{c_n}) \\
&\geq \pi_{i_n}^{\omega_n(i_n,j_n,\pi_{j_n}^{\omega_n}(f^{c_n})-m_{j_n}^{\omega_n})}(f^{c_n}) \\
&\geq \pi_{i_n}^{\omega_n}(f^{c_n}) + \varepsilon
\end{aligned}$$

The first inequality holds by Lemma 2.6; the second since $\pi_{j_n}^{\omega_n}(f^{c_n}) - m_{j_n}^{\omega_n} \in P_{i_n j_n}^{\omega_n}(f^{c_n})$ and $p_{i_n j_n}^{\omega_n}(f^{c_n}) \in \arg\max_{p \in P_{i_n j_n}^{\omega_n}(f^{c_n})} \pi_{i_n}^{\omega_n(i_n,j_n,p)}(f^{c_n})$; the third as shown in (B.2). This leads to contradiction as desired. □

Finally, I write down the feasibility condition which plays a key role in proving Lemma 2.2. For all $\omega \in \hat{\Omega}$, define

$$T^{\omega} := \sum_{k \in B_I^{\omega} \cup S_E^{\omega}} v_k + \sum_{k \in N^{\omega}} m_k^{\omega}$$

In other words, absent search cost, T^{ω} is the maximum attainable sum of continuation payoffs over all remaining players at ω, and is attained at ω if and only if all $|S^{\omega}|$ units of goods are consumed with probability one by the $|S^{\omega}|$ highest value remaining players at ω, i.e. by $B_I^{\omega} \cup S_E^{\omega}$. This condition must hold across all states in any MPE.

Lemma 2.2. There exists $\bar{c} > 0$ such that for all $c \in [0, \bar{c})$ and $\omega \in \hat{\Omega}$, if $B_I^{\omega} \cap Q^{\omega}(f^c) \neq \varnothing$, then there exists $i \in B_I^{\omega} \cap M^{\omega}(f^c)$ and $j \in S_I^{\omega}$ such that $j \in S_i^{\omega}(f^c)$.

Proof. Suppose not for contradiction. Then for all $n \in \mathbb{N}$, there exists $c_n \in [0, 1/n)$ and $\omega_n \in \hat{\Omega}$ such that $j \notin S_i^{\omega_n}(f^{c_n})$ for all $i \in B_I^{\omega_n} \cap M^{\omega_n}(f^{c_n})$ and $j \in S_I^{\omega_n}$, but $\ell_n \in B_I^{\omega_n} \cap Q^{\omega_n}(f^{c_n})$ for some $\ell_n \in N^{\omega_n}$. Fix one such sequence $(c_n, \omega_n, \ell_n)_{n \in \mathbb{N}}$.

For each $n \in \mathbb{N}$ and $i \in B_I^{\omega_n}$, let $j^{\omega_n}(i) \equiv k \in S_I^{\omega_n}$ such that $\beta^{\omega_n}(i) = \sigma^{\omega_n}(k)$; in other words, $j^{\omega_n}(i)$ is the $\beta^{\omega_n}(i)^{th}$ lowest value seller at state ω_n. Also, let $v \equiv \min_{i, j \neq i} |v_i - v_j| > 0$; in other words, v is the minimum difference in valuations between two distinct players among the K players in the game. Note that for each $n \in \mathbb{N}$ and $i \in B_I^{\omega_n}$, either (i) $i \in Q^{\omega_n}(f^{c_n}) \cup R^{\omega_n}(f^{c_n})$, or (ii) $i \in M^{\omega_n}(f^{c_n})$ and thus $j^{\omega_n}(i) \notin S_i^{\omega_n}(f^{c_n})$ by hypothesis. Either way, for large enough n, Lemma 2.7 implies $\pi_i^{\omega_n}(f^{c_n}) + \pi_{j^{\omega_n}(i)}^{\omega_n}(f^{c_n}) > v_i + m_i^{\omega_n} + m_{j^{\omega_n}(i)}^{\omega_n} - \alpha v/K$ for all $i \in B_I^{\omega_n}$, and therefore

$$\begin{aligned}
\sum_{k \in N_I^{\omega_n}} \pi_k^{\omega_n}(f^{c_n}) &= \sum_{i \in B_I^{\omega_n}} \left(\pi_i^{\omega_n}(f^{c_n}) + \pi_{j^{\omega_n}(i)}^{\omega_n}(f^{c_n})\right) \\
&> \sum_{i \in B_I^{\omega_n}} \left(v_i + m_i^{\omega_n} + m_{j^{\omega_n}(i)}^{\omega_n}\right) - |B_I^{\omega_n}| \frac{\alpha v}{K} \geq \sum_{k \in B_I^{\omega_n}} v_k + \sum_{k \in N_I^{\omega_n}} m_k^{\omega_n} - \alpha v
\end{aligned}$$

The second inequality holds because there are no more than K pairs of infra-marginal players at any state. Meanwhile, by individual rationality, for any $n \in \mathbb{N}$,

$$\sum_{k \in N_E^{\omega_n}} \pi_k^{\omega_n}(f^{c_n}) \geq \sum_{k \in N_E^{\omega_n}} (g_k^{\omega_n} v_k + m_k^{\omega_n}) = \sum_{k \in S_E^{\omega_n}} v_k + \sum_{k \in N_E^{\omega_n}} m_k^{\omega_n}$$

But then, summing the above two chains of inequalities, for large enough n,

$$\sum_{k \in N^{\omega_n}} \pi_k^{\omega_n}(f^{c_n}) > \sum_{k \in B_I^{\omega_n} \cup S_E^{\omega_n}} v_k + \sum_{k \in N^{\omega_n}} m_k^{\omega_n} - \alpha v = T^{\omega_n} - \alpha v$$

Yet ℓ_n exits the market at ω_n with at least probability α, destroying at least v of the gains from trade thereby. Hence the maximum attainable sum of continuation payoffs over all remaining players at ω_n cannot exceed $T^{\omega_n} - \alpha v$. This leads to contradiction as desired. □

Lemma 2.8. In any MPE, the game ends with probability one.

Proof. Supppse not for contradiction. Then there exists $c \geq 0$, $f \in F^c$, and $\omega \in \hat{\Omega}$ on the equilibrium path of f such that $\omega^* \in \hat{\Omega}$ and $N^\omega = N^{\omega^*}$ for all states ω^* that succeeds ω on the equilibrium path of f. Fix one such $c \geq 0$, $f \in F^c$, and $\omega \in \hat{\Omega}$. Let $\hat{\Omega}^\omega(f) := \{\omega^* \in \hat{\Omega} : \omega^*$ succeeds ω on the equilibrium path of $f\}$, let $\omega^0 \in \arg\max_{\omega^* \in \hat{\Omega}^\omega(f)} \sum_{k \in N^\omega} g_k^{\omega^*} v_k$, and let $(\omega^t)_{t=0}^\infty$ denote an arbitrary equilibrium path of states after reaching ω^0 given f. But then

$$\sum_{k \in N^\omega} \pi_k^{\omega^0}(f;c) \leq \sum_{k \in N^\omega} \mathbb{E}_{(\omega^t)_{t=0}^\infty} \liminf_{t \to \infty} g_k^{\omega^t} v_k + m_k^{\omega^t} \leq \sum_{k \in N^\omega} g_k^{\omega^0} v_k + m_k^{\omega^0}$$

The first inequality holds by the definition of payoffs of never exiting and by dropping any incurred search costs; the second by choice of ω^0. However, $N^\omega \cap Q^{\omega^0}(f;c) = \varnothing$ by choice of ω. But then $\max\{\pi_k^{\omega^0}(f;c), \lambda_k^{\omega^0}(f;c)\} > g_k^{\omega^0} v_k + m_k^{\omega^0}$ and thus $\pi_k^{\omega^0}(f;c) > g_k^{\omega^0} v_k + m_k^{\omega^0}$ for all $k \in N^\omega$. This leads to contradiction as desired. □