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# Coherence for bicategorical cartesian closed structure

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**Abstract**
We prove a strictification theorem for cartesian closed bicategories. First, we adapt Power's proof of coherence for bicategories with finite bilimits to show that every bicategory with bicategorical cartesian closed structure is biequivalent to a 2-category with 2-categorical cartesian closed structure. Then we show how to extend this result to a Mac Lane-style 'all pasting diagrams commute' coherence theorem: precisely, we show that in the free cartesian closed bicategory on a graph, there is at most one 2-cell between any parallel pair of 1-cells. The argument we employ is reminiscent of that used by Čubrić, Dybjer, and Scott to show normalisation for the simply-typed lambda calculus (Čubrić et al., 1998). The main results first appeared in a conference paper (Fiore and Saville, 2020) but for reasons of space many details are omitted there; here we provide the full development.

**Keywords:** Bicategories, cartesian closed, coherence, strictification.

## 1. Introduction

Bicategories arise naturally throughout mathematics and theoretical computer science. Examples appear in topology (Leinster, 2004), categorical logic (Fiore et al., 2007), categorical algebra (Bénabou, 1967), semantics of computation (Cattani et al., 1998), and datatype semantics (Abbott, 2003), to name but a few. Much of this work owes a debt to the success of the 'Australian school' of the 1970s and 1980s, which emphasised the fruitfulness of studying categorical constructions in the bicategorical setting (*e.g.* Street (1972, 1980); Blackwell et al. (1989)). A crucial part of this work, in which Power played an important role, are fundamental *coherence* results. These include coherence for bicategories with finite bilimits (Power, 1989*a*) and a general result in the framework of 2-dimensional universal algebra (Power, 1989*b*).

Why are coherence results—which can often have a rather dry, technical feel—so crucial? Without them, calculations in higher categories quickly become intractable. Higher category theory entails layers of complexity that do not exist at the 1-categorical level: morphisms (more generally, $k$-cells) satisfying axioms up to some higher cell may exist in new relationships, and specifying their behaviour leads to intimidating lists of axioms. Proofs then become purgatorial exercises in drawing pasting diagram after pasting diagram, or diagram chases in which an intuitively-clear kernel is dominated by endless structural isomorphisms shifting data back and forth. Even at the level $k = 2$, Lack—certainly a member of the higher-categorical *cognoscenti*—refers to (strict) 2-category theory as a "middle way", avoiding "some of the technical nightmares of bicategories" (Lack, 2010).

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A small example highlights how the step from categories to bicategories blows up the length of a proof. Consider the following lemma, which is an elementary exercise in working with cartesian closed categories.

**Lemma 1.1.**

(1) Every object $X$ in a category with finite products $(\mathbb{C}, \times, 1)$ has a canonical structure as a commutative comonoid, namely $\big(1 \xleftarrow{!} X \xrightarrow{\Delta} X \times X\big)$.

(2) Every endo-exponential $[X \Rightarrow X]$ in a cartesian closed category $(\mathbb{C}, \times, 1, \Rightarrow)$ has a canonical structure as a monoid, namely $\big(1 \xrightarrow{\mathrm{id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\big)$.

Following the principle that higher categories behave in roughly the same manner as 1-categories so long as care is taken to specify the behaviour of the higher cells, one expects a version of this result to hold for cartesian closed bicategories. The bicategorical notion of monoid is called a *pseudomonoid* or *monoidale* (Day and Street, 1997). In a bicategory $\mathscr{B}$ with finite products $(\times, 1)$, this is a structure $(1 \xrightarrow{e} M \xleftarrow{m} M \times M)$ equipped with invertible 2-cells $\alpha, \lambda$ and $\rho$ witnessing the categorical unit and associativity laws:

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times M"] \arrow[dr, bend right, "\simeq"'] \arrow[dr, phantom, shift left=3, "\overset{\lambda}{\cong}"] & M \times M \arrow[d, dashed, "m"] & M \times 1 \arrow[l, "M \times e"'] \arrow[dl, bend left, "\simeq"] \arrow[dl, phantom, shift right=3, "\overset{\rho}{\cong}"] \\
 & M &
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \times M) \times M \arrow[r, "\simeq"] \arrow[d, "m \times M"'] & M \times (M \times M) \arrow[r, "M \times m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

These 2-cells are required to satisfy two coherence laws, corresponding to the triangle and pentagon axioms for a monoidal category. Indeed, the prototypical example—obtained by instantiating the definition in **Cat**—is of monoidal categories. Comparing with our categorical lemma suggests the following.

**Conjecture 1.2.**

(1) Every object $X$ in a bicategory with finite products $(\mathscr{B}, \times, 1)$ has a canonical structure as a commutative pseudocomonoid, with 1-dimensional structure $\big(1 \xleftarrow{!} X \xrightarrow{\Delta} X \times X\big)$.

(2) Every endo-exponential $[X \Rightarrow X]$ in a cartesian closed bicategory $(\mathscr{B}, \times, 1, \Rightarrow)$ has a canonical structure as a pseudomonoid, with 1-dimensional structure

$$1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X].$$

Moreover, in each case the 2-cells witnessing the 1-categorical axioms are canonical choices arising from the cartesian (closed) structure of $\mathscr{B}$.

Constructing the witnessing 2-cells $\alpha, \lambda$ and $\rho$ is relatively straightforward: roughly speaking, one can translate each equality used in the categorical proof into a 2-cell, and then compose these together. The difficulty arises in checking the coherence laws, which entails a series of long diagram chases unfolding the properties of these composites. It is this extra work that makes bicategorical calculations more burdensome than their strict counterparts: it is not enough to merely witness the axioms—which corresponds to checking them in a strict setting—one must also check the witnesses are themselves *coherent*.

Not only do these checks entail extra work, they are often extremely tedious. Generally one does not have to apply clever tricks or techniques, only plough through diagram chases until the result falls out. This is the case, for example, when one sits down to verify the coherence laws for Conjecture 1.2. This leads to a false sense of security: it is tempting to believe that the coherence axioms 'must' work out as expected, and that these extra checks may be omitted. As Power put it as long ago as 1989 (Power, 1989*a*):

> The verification is almost always routine, and one's intuition is almost always vindicated; but to check the detail is often a very tedious job. Of course, one should still do it... [ignoring such details] can be dangerous, as illustrated in (Bénabou, 1985), because on rare occasions, one's intuition fails...

Power considers three strategies for doing away with this tedium: ignore it, check each coherence diagram by hand as it arises, or—the "preferable" approach—prove a wholesale *coherence theorem*. Such theorems can be roughly divided into two classes. A *coherence-by-strictification* result proves that every weak structure (for example, a bicategory) is weakly-equivalent to a strict structure (for example, a 2-category). On the other hand, *Mac Lane-style coherence*—named for Mac Lane's pithy slogan "all diagrams commute" (Mac Lane, 1963)—isolates a class of diagrams and shows that every diagram in this class commutes. Mac Lane-style coherence can be derived from coherence-by-strictification in all examples that we know of but, as we show in Section 5, a non-trivial argument maybe required.

In either form, the importance of coherence theorems is attested to by their proliferation. Since Mac Lane's classic result for monoidal categories (Mac Lane, 1963), a great number of coherence theorems have been proven, in various guises: notable examples include those of Mac Lane and Paré (1985); Power (1989*a*,b); Joyal and Street (1993) and Gordon et al. (1995). Such results often rely on the Yoneda embedding, which allows one to embed a weak structure (such as a bicategory) into a strict structure (such as the 2-category of **Cat**-valued pseudofunctors), or on the sophisticated machinery of 2-dimensional universal algebra. Rewriting theory provides an alternative, syntactic, approach; see for example (Houston, 2007) and (Forest and Mimram, 2018).

In this paper we are concerned with a class of bicategories with particularly good structure, namely cartesian closed bicategories. A *cartesian closed bicategory*, or *cc-bicategory*, is a bicategory equipped with finite products defined as bicategorical limits (*bilimits*) and exponentials defined as a bicategorical right adjoint (*biadjoint*) to every pseudofunctor $(-) \times A$.[a] Examples include the bicategories of generalised species (Fiore et al., 2007) and cartesian distributors (Fiore and Joyal, 2015), as well as bicategories of operads (Gambino and Joyal, 2017) and concurrent games (Paquet, 2020).

Informally, one may think of cc-bicategories as cartesian closed categories 'up to isomorphism'. To construct cc-bicategorical structure one takes the simply-typed lambda calculus (equivalently, cartesian closed structure) and replaces each $\beta\eta$-equality with an invertible 2-cell witnessing the reduction. For products, the $\eta$-law $f = \langle \pi_1 \circ f, \ldots, \pi_n \circ f \rangle$ and $\beta$-law $\pi_i \circ \langle f_1, \ldots, f_n \rangle = f_i$ are respectively replaced by natural isomorphisms

$$f \overset{\cong}{\Rightarrow} \langle \pi_1 \circ f, \ldots, \pi_n \circ f \rangle$$

$$\pi_i \circ \langle f_1, \ldots, f_n \rangle \overset{\cong}{\Rightarrow} f_i \ \ (i = 1, \ldots, n)$$

These isomorphisms are subject to equations—namely, the triangle laws of an adjunction—which express natural equalities: for instance, if one $\eta$-expands then $\beta$-reduces, the composite rewrite is the identity. A similar story holds for exponentials.

Our main result (Theorem 5.11) is that the free cc-bicategory on a graph has at most one 2-cell between any parallel pair of 1-cells; this result was first conjectured by Ouaknine (1997). In terms of Conjecture 1.2 it guarantees that, once one has constructed the required structural isomorphisms

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[a] It is an unfortunate accident of terminology that there is no connection to 'cartesian bicategories' (Carboni and Walters, 1987; Carboni et al., 2008), nor to 'closed cartesian bicategories' (Frey, 2019).

$\alpha$, $\lambda$ and $\rho$, then the coherence axioms must hold and that the definitions of $\alpha$, $\lambda$ and $\rho$ are unique. In fact, by relating the free cc-bicategory to the free cartesian closed category (Section 5.2.2), we shall make precise the informal relationship between categorical and bicategorical cartesian closed structure outlined above, and substantiate the following principle.

**Principle 1.3.** To show that a pseudo structure can be constructed in every cc-bicategory, it suffices to show that its categorical counterpart—that is, the version in which one considers only the 1-cells—may be constructed in any cartesian closed category (equivalently, in the simply-typed lambda calculus).

For example, this principle entails that Conjecture 1.2 follows immediately from Lemma 1.1.

### *1.1 Outline of the paper*

It is convenient to define bicategorical closed structure using the *biuniversal arrows* of T. Fiore (2006, Chapter 9). We therefore spend Section 2 developing a little of their basic theory. Then, in Section 3, we instantiate this to define cc-bicategories. This is enough machinery to prove coherence-by-strictification, which we do in Section 4; in passing, we also observe a similar approach applies to closed monoidal bicategories. Then, in Section 5, we deduce Mac Lane-style coherence and substantiate Principle 1.3.

We assume familiarity with the basic definitions of bicategory theory, including pseudofunctors, biequivalences, and the Yoneda lemma. These are summarised in (Leinster, 1998); for a more extensive introduction see *e.g.* (Bénabou, 1967; Borceux, 1994).

**Notation 1.4.**

- We make free use of the coherence theorem for bicategories (Mac Lane and Paré, 1985), writing simply $\cong$ for composites of structural isomorphisms in diagrams. When we need to be explicit, we denote the structural constraints of a bicategory by $\mathsf{a}_{h,g,f} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$, $\mathsf{l}_f : \mathrm{Id} \circ f \Rightarrow f$ and $\mathsf{r}_g : g \circ \mathrm{Id} \Rightarrow g$.
- We write $\mathrm{Hom}(\mathscr{B}, \mathscr{C})$ for the bicategory of pseudofunctors, pseudonatural transformations, and modifications. The Yoneda pseudofunctor $\mathscr{B} \to \mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$ is denoted $\mathrm{Y}$.
- We denote the structural constraints of a pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ by $\psi_X : \mathrm{Id}_{FX} \overset{\cong}{\Rightarrow} F\mathrm{Id}_X$ and $\phi_{f,g} : F(f) \circ F(g) \overset{\cong}{\Rightarrow} F(f \circ g)$. When there is no risk of ambiguity, we refer to the triple $(F, \psi, \phi)$ simply as $F$.
- We follow the notational convention of Lack (2010) for pseudonatural transformations. Thus, a pseudonatural transformation $(\mathsf{k}, \overline{\mathsf{k}}) : F \Rightarrow G : \mathscr{B} \to \mathscr{C}$ consists of a family of 1-cells $\{\mathsf{k}_X : FX \to GX\}_{X \in \mathscr{B}}$ together with an invertible 2-cell $\overline{\mathsf{k}}_f : \mathsf{k}_Y \circ Ff \Rightarrow Gf \circ \mathsf{k}_X$ for every $f : X \to Y$ in $\mathscr{B}$, subject to the usual axioms.

## 2. Biuniversal arrows

Mac Lane's classic textbook (Mac Lane, 1998) makes precise the notion of universal property by introducing *universal arrows*. The Yoneda Lemma, limits and adjunctions are then all characterised in these terms. We adopt a similar approach using the *biuniversal arrows* of T. Fiore (2006). As well as providing a uniform way to describe bilimits and biadjunctions—and so products and exponentials—this perspective is particularly amenable to syntactic description. This will simplify the construction of the free cc-bicategory in Section 5.2.

We begin by recapitulating the notion of universal arrow and its bicategorical counterpart.

**Definition 2.1.** Let $F : \mathbb{B} \to \mathbb{C}$ be a functor and $C \in \mathbb{C}$. A *universal arrow from $F$ to $C$* is a pair $(R \in \mathbb{B}, u : FR \to C)$ such that, for any $B \in \mathbb{B}$ and $f : FB \to C$, there exists a unique $f^\dagger : B \to R$ such that $u \circ Ff^\dagger = f$.

It is an exercise to show that every universal arrow $(R, u)$ from $F$ to $C$ is equivalently a chosen family of natural isomorphisms $\mathbb{B}(-, R) \cong \mathbb{C}(F(-), C)$, or—equivalently again—a terminal object in the comma category $(F \downarrow C)$. It follows that a right adjoint to $F : \mathbb{B} \to \mathbb{C}$ is determined by a choice of object $UC$ and universal arrow $\varepsilon_C : FUC \to C$ for every $C \in \mathbb{C}$. The mapping $U$ extends to a functor with $Uf := (f \circ \varepsilon_C)^\dagger$ for $f : C \to C'$. The counit is then $\varepsilon$ and the unit $\eta$ arises by applying the universal property to the identity: $\eta_B := (\mathrm{id}_{FB})^\dagger : B \to UFB$. If both $\varepsilon$ and $\eta$ are invertible, the result is an adjoint equivalence.

To define biuniversal arrows, one weakens the isomorphisms defining a universal arrow to equivalences. We take particular care in choosing how we spell these out. It is generally convenient to require adjoint equivalences; by the well-known lifting theorem (*e.g.* Leinster (2004), Prop. 1.5.7) this entails no loss of generality, while providing a more structured object to work with. We also go beyond T. Fiore's definition by requiring that each adjoint equivalence is determined by a choice of universal arrow.

**Definition 2.2** (*c.f.* T. Fiore (2006))**.** Let $F : \mathscr{B} \to \mathscr{C}$ be a pseudofunctor and $C \in \mathscr{C}$. A *biuniversal arrow from $F$ to $C$* consists of a pair $(R \in \mathscr{B}, u : FR \to C)$ and, for every $B \in \mathscr{B}$, a chosen adjoint equivalence of categories

$$\begin{aligned} \mathscr{B}(B, R) &\xrightarrow{\simeq} \mathscr{C}(FB, C) \\ (B \xrightarrow{h} R) &\mapsto (FB \xrightarrow{Fh} FR \xrightarrow{u} C) \end{aligned}$$

specified by choosing a family of invertible universal 2-cells as the counit.

Explicitly, a biuniversal arrow from $F$ to $C$ consists of the following data:

- A pair $(R \in \mathscr{B}, u : FR \to C)$,
- For every $B \in \mathscr{B}$ and $h : FB \to C$, a map $\theta_B(h) : B \to R$ and an invertible 2-cell $\varepsilon_{B,h} : u \circ F\theta_B(h) \Rightarrow h$, universal in the sense that for any map $f : B \to R$ and 2-cell $\tau : u \circ Ff \Rightarrow h$ there exists a 2-cell $\tau^\dagger : f \Rightarrow \theta_B(h)$, unique such that

$$\begin{tikzcd}
FB \arrow[r, bend left, "Ff"] \arrow[r, bend right, "F\theta_B(h)"'] \arrow[r, phantom, "\Downarrow F\tau^\dagger"] \arrow[rrd, bend right=50, "h"'] & FR \arrow[rd, "u"] & & & & FR \arrow[rd, "u"] \arrow[d, phantom, "\Downarrow \tau"] & \\
& \Downarrow \varepsilon_{B,h} & C & = & FB \arrow[ru, "Ff"] \arrow[rr, "h"'] & & C
\end{tikzcd} \tag{1}$$

with the 2-cell $(\mathrm{id}_{u \circ Ff})^\dagger : f \Rightarrow \theta_B(u \circ Ff)$ invertible for every $f : B \to R$.

Thus, the mapping $\theta_B$ extends to a functor $\mathscr{C}(FB, C) \to \mathscr{B}(B, R)$ defined on 2-cells by $\theta_B(h \overset{\tau}{\Rightarrow} h') := (\tau \circ \varepsilon_{B,h})^\dagger$, and this functor is right adjoint to $u \circ F(-)$. The counit of this adjunction is $\varepsilon_{B,(-)}$ and the unit is $\eta_{B,(-)} := (\mathrm{id}_{u \circ F(-)})^\dagger$. This pattern will be repeated: when asking for an adjoint equivalence in a definition, we shall consistently ask for a *right* adjoint to a given arrow.

**Remark 2.3.** Just as in the categorical case, there is an evident dual notion. A biuniversal arrow from an object $C \in \mathscr{C}$ to a pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ is a pair $(R \in \mathscr{B}, u : C \to FR)$ together with chosen adjoint equivalences $\mathscr{B}(R,B) \xrightarrow{F(-)\circ u} \mathscr{C}(C, FB)$.

On the face of it, a biuniversal arrow is only local structure: the data imposes a requirement on each hom-category, but no global constraints. Global structure arises in the following way.

**Lemma 2.4** (*c.f.* Mac Lane (1998, §III.2))**.** Let $F : \mathscr{B} \to \mathscr{C}$ be a pseudofunctor and $C \in \mathscr{C}$. There exists an equivalence of pseudofunctors $\mathscr{B}(-,R) \simeq \mathscr{C}(F(-),C)$ in $\mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$ if and only if there exists a biuniversal arrow $(R,u)$ from $F$ to $C$.

*Proof.* For every equivalence of pseudofunctors $\mathscr{B}(-,R) \xrightarrow{\gamma} \mathscr{C}(F(-),C)$ one obtains from the Yoneda Lemma an arrow $\gamma_R(\mathrm{Id}_R) : FR \to C$. This arrow is biuniversal: indeed, the image of $\gamma_R(\mathrm{Id}_R)$ under the pseudofunctor $\mathscr{C}(FR,C) \to \mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})\big(\mathscr{B}(-,R), \mathscr{C}(F(-),C)\big)$ given by the Yoneda Lemma is isomorphic to $\gamma$, so $\gamma_R(\mathrm{Id}_R) \circ F(-)$ is an equivalence in $\mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$. This may be promoted to an adjoint equivalence by the usual lifting lemma (*e.g.* Leinster (2004), Prop. 1.5.7); one obtains the required adjoint equivalence $\mathscr{B}(B,R) \simeq \mathscr{C}(FB,C)$ at $B \in \mathscr{B}$ by evaluating at $B$. The converse is Theorem 9.5 of T. Fiore (2006). ☐

Other standard properties of universal arrows extend to biuniversal arrows in the expected way. For example, biuniversal arrows are unique up to equivalence, the $(-)^\dagger$ operation preserves both invertibility and naturality, and a **Cat**-valued pseudofunctor $F$ is birepresentable if and only if there exists a biuniversal arrow from the terminal category to $F$.

### *2.1 Preservation of biuniversal arrows*

The notion of preservation of biuniversal arrows will provide a systematic way to define preservation of bilimits and preservation of biadjoints, and so preservation of cartesian closed structure. We begin by examining preservation of universal arrows. Using the fact that a right adjoint to $F : \mathbb{B} \to \mathbb{C}$ is completely specified by a choice of universal arrow $\big(UC, F(UC) \to C\big)$ for each $C \in \mathbb{C}$—namely, the counit—it is reasonable to define morphisms of universal arrows analogously to morphisms of adjunctions.

**Definition 2.5** (Mac Lane (1998, Chapter IV))**.** Let $F : \mathbb{B} \to \mathbb{C}$ and $F' : \mathbb{B}' \to \mathbb{C}'$ be functors and $(R,u)$ be a universal arrow from $F$ to $C \in \mathbb{C}$. A pair of functors $(K,L)$ *preserves the universal arrow* $(R,u)$ if the following diagram commutes

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "F"] \arrow[d, "L"'] & \mathbb{C} \arrow[d, "K"] \\
\mathbb{B}' \arrow[r, "F'"'] & \mathbb{C}'
\end{tikzcd}$$

and $F'LR = KFR \xrightarrow{Ku} KC$ is a universal arrow from $F'$ to $KC$.

Equivalently, one can ask that the functor $(F \downarrow C) \to (F' \downarrow KC)$ defined by $(B, h : FB \to C) \mapsto (LB, F'LB = KFB \xrightarrow{Kh} KC)$ preserves the terminal object. This is a slight weakening of Mac Lane's definition, which asks that the unit (or counit) be preserved on the nose.

The bicategorical translation is as one would expect.

**Definition 2.6.** Let $F : \mathscr{B} \to \mathscr{C}$ and $F' : \mathscr{B}' \to \mathscr{C}'$ be pseudofunctors and $(R, u)$ be a biuniversal arrow from $F$ to $C \in \mathscr{C}$. Consider pseudofunctors $K$ and $L$ related by a pseudonatural transformation $\rho$ as in the following diagram:

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "F"] \arrow[d, "L"'] & \mathscr{C} \arrow[d, "K"] \\
\mathscr{B}' \arrow[r, "F'"'] \arrow[ur, phantom, "\overset{\rho}{\Rightarrow}"] & \mathscr{C}'
\end{tikzcd} \tag{2}$$

The triple $(K, L, \rho)$ *preserves the biuniversal arrow* $(R, u)$ if $F'LR \xrightarrow{\rho_R} KFR \xrightarrow{Ku} KC$ is a biuniversal arrow from $F'$ to $KC$.

There are two ways of formulating that a functor $F$ preserves limits: one can ask that the image of the terminal cone is also a terminal cone, or that the canonical map $F(\lim H) \to \lim(FH)$ is an isomorphism. Similar considerations apply to preservation of biuniversal arrows.

**Lemma 2.7.** Consider a square of pseudofunctors $K, L, F, F'$ related by a pseudonatural transformation $\rho : F'L \Rightarrow KF$ as in (2). For every pair of biuniversal arrows $(R, u)$ and $(R', u')$ from $F$ to $C \in \mathscr{C}$ and $F'$ to $KC \in \mathscr{C}'$, respectively, the following are equivalent:

(1) $(K, L, \rho)$ preserves the biuniversal arrow $(R, u)$,
(2) The canonical map $\theta'_{LR}(Ku \circ \rho_R) : LR \to R'$ is an equivalence, where we write $\theta'_{B'}$ for the chosen pseudo-inverse to $u' \circ F'(-) : \mathscr{B}'(B', R') \to \mathscr{C}'(F'B', KC)$.

*Proof.* Suppose first that $\theta'_{LR}(Ku \circ \rho_R)$ is an equivalence. Since pseudofunctors preserve equivalences, the composite $\mathscr{B}'(B', LR) \xrightarrow{\theta'_{LR}(Ku\circ\rho_R)\circ(-)} \mathscr{B}'(B', R') \xrightarrow{u'\circ F'(-)} \mathscr{C}'(F'B', KC)$ is an equivalence. Post-composing the counit $\varepsilon'_{LR,Ku\circ\rho_R}$ with the the canonical isomorphism yields a natural isomorphism

$$u' \circ F'\big(\theta'_{LR}(Ku \circ \rho_R) \circ (-)\big) \cong \big(u' \circ F'(\theta'_{LR}(Ku \circ \rho_R))\big) \circ F'(-) \cong \big(Ku \circ \rho_R\big) \circ F'(-)$$

Hence $Ku \circ \rho_R$ is also a biuniversal arrow.

The converse follows from universality: if $(LR, Ku \circ \rho_R)$ and $(R', u')$ are both biuniversal arrows from $F'$ to $KC$, then the canonical arrows $LR \to R'$ and $R' \to LR$ obtained from the universal property must form an equivalence. □

Just as an equivalence of categories preserves all 'categorical' properties, so a biequivalence preserves all 'bicategorical' properties. In particular, biequivalences always preserve biuniversal arrows.

In Section 5.2 we shall construct the free cc-bicategory on a graph. Following Gurski (2013), we shall work with strict universal properties as far as possible: as well as being easier to work with, this avoids the complexities of the tricategorical setting. Thus, it will be useful to define *strict preservation* of biuniversal arrows. The aim of this definition is to ensure that the chosen structure witnessed by a biuniversal arrow (*e.g.* a bilimit) is taken to exactly the chosen structure in the target.

**Definition 2.8.** Let $F : \mathscr{B} \to \mathscr{C}$ and $F' : \mathscr{B}' \to \mathscr{C}'$ be pseudofunctors and suppose $(R, u)$ and $(R', u')$ are biuniversal arrows from $F$ to $C \in \mathscr{C}$ and from $F'$ to $C' \in \mathscr{C}'$, respectively. A pair of pseudofunctors $(K, L)$ is a *strict morphism of biuniversal arrows* from $(R, u)$ to $(R', u')$ if

(1) $K$ and $L$ are strict pseudofunctors such that $KF = F'L$,
(2) The data of the biuniversal arrow is preserved: $LR = R'$, $KC = C'$ and $Ku = u'$,

(3) The mappings $\theta_B : \mathscr{C}(FB, C) \to \mathscr{B}(B, R)$ and $\theta'_{B'} : \mathscr{C}'(F'B', C') \to \mathscr{B}'(B', R')$ are preserved, so that $L\theta_B(f) = \theta'_{LB}K(f)$ for every $f : FB \to C$,

(4) For every $B \in \mathscr{B}$ and chosen equivalence $u \circ F(-) : \mathscr{B}(B, R) \leftrightarrows \mathscr{C}(FB, C) : \theta_B$ the universal arrow $\varepsilon_{B,h} : u \circ F\theta_B(h) \Rightarrow h$ is strictly preserved, in the sense that $K_{FB,C}(\varepsilon_{B,h}) = \varepsilon'_{LB,Kh}$.

In bicategory theory it is usually good practice to specify data up to equivalence, as pseudofunctors preserve equivalences but may not preserve isomorphisms or equalities. The preceding definition abuses this convention, and so is not 'bicategorical' in style. A consequence is that an arbitrary biequivalence may not strictly preserve biuniversal arrows. This level of strictness does, however, provide a way to talk about free bicategories-with-structure using the language of 1-category theory (*c.f.* Gurski (2006), Proposition 2.10).

**Remark 2.9.** We distinguish between *preservation* of biuniversal arrows in the sense of Definition 2.6 and a *morphism* of biuniversal arrows as in the preceding definition on the following basis. In Definition 2.6 we require that the image of the given biuniversal arrow is a biuniversal arrow, but do not specify its exact nature. In the preceding definition, by contrast, we require that the pair $(K, L)$ takes the biuniversal arrow specified in the source, together with all its chosen adjoint equivalences, to exactly the corresponding structure specified in the target. Thus, while preservation of biuniversal arrows is akin to preservation of limits, a morphism of biuniversal arrows is more like a homomorphism between algebraic structures (*e.g.* a group homomorphism).

Strict preservation of a biuniversal arrow implies preservation of the corresponding universal property, in the following sense.

**Lemma 2.10.** Let $F : \mathscr{B} \to \mathscr{C}$ and $F' : \mathscr{B}' \to \mathscr{C}'$ be pseudofunctors and suppose $(R, u)$ and $(R', u')$ are biuniversal arrows from $F$ to $C \in \mathscr{C}$ and from $F'$ to $C' \in \mathscr{C}'$, respectively. If $(K, L)$ is a strict morphism from $(R, u)$ to $(R', u')$, then for every $B \in \mathscr{B}$, $h : B \to R$ and $\tau : u \circ Fh \Rightarrow f$, $L(\tau^\dagger) = (K\tau)^\dagger$.

*Proof.* It suffices to show that $L(\tau^\dagger)$ satisfies the universal property of $(K\tau)^\dagger$. For this one observes that

$$\begin{aligned} \varepsilon'_{LB,Kf} \bullet F'L(\tau^\dagger) &= K(\varepsilon_{B,f}) \bullet KF(\tau^\dagger) && \text{by strict preservation} \\ &= K(\varepsilon_{B,f} \bullet F\tau^\dagger) \\ &= K\tau \end{aligned}$$

as required. □

A strict morphism of biuniversal arrows $(K, L)$ defines a morphism of adjunctions at every hom-category. Indeed, the diagram below commutes for every $B \in \mathscr{B}$ by Definition 2.8(1) while each functor $K_{FB,C}$ preserves the counit by Definition 2.8(4).

$$\begin{tikzcd}
\mathscr{B}(B, R) \arrow[rrr, "u_C \circ F(-)"] \arrow[d, "L_{B,R}"'] & & & \mathscr{C}(FB, C) \arrow[d, "K_{FB,C}"] \\
\mathscr{B}'(LB, LR) \arrow[r, equal] & \mathscr{B}'(LB, R') \arrow[r, "u'_{LB} \circ F'(-)"'] & \mathscr{C}'(F'LB, C') \arrow[r, equal] & \mathscr{C}'(KFB, KC)
\end{tikzcd}$$

### *2.2 Bilimits*

We are now in a position to define bilimits and preservation of bilimits in terms of biuniversal arrows. Note first that for every pair of bicategories $(\mathscr{J}, \mathscr{B})$ one has a *diagonal pseudofunctor* $\Delta : \mathscr{B} \to \mathrm{Hom}(\mathscr{J}, \mathscr{B})$ taking $B \in \mathscr{B}$ to the constant pseudofunctor at $B$. Explicitly, $\Delta B : \mathscr{J} \to \mathscr{B}$ takes a 2-cell $\tau : h \Rightarrow h' : j \to j'$ to the identity 2-cell $\mathrm{id}_B : \mathrm{Id}_B \Rightarrow \mathrm{Id}_B : B \to B$. The 2-cell $\psi_j : \mathrm{Id}_{(\Delta B)(j)} \Rightarrow (\Delta B)(\mathrm{Id}_j)$ is the identity and for a composite $j \xrightarrow{g} j' \xrightarrow{f} j''$ in $\mathscr{J}$ the 2-cell $\phi_{f,g} : (\Delta B)(f) \circ (\Delta B)(g) \Rightarrow (\Delta B)(f \circ g)$ is $\mathfrak{l}_{\mathrm{Id}_B} : \mathrm{Id}_B \circ \mathrm{Id}_B \Rightarrow \mathrm{Id}_B$. A bilimit is then a biuniversal arrow.

**Definition 2.11.** A *bilimit* for $F : \mathscr{J} \to \mathscr{B}$ is a biuniversal arrow from the diagonal pseudofunctor $\Delta : \mathscr{B} \to \mathrm{Hom}(\mathscr{J}, \mathscr{B})$ to $F$.

By Lemma 2.4 this definition can be rephrased as a pseudonatural family of adjoint equivalences $\mathscr{B}(B, \mathrm{bilim}\, F) \simeq \mathrm{Hom}(\mathscr{J}, \mathscr{B})(\Delta B, F)$. It therefore coincides with that of Street (1980) in terms of birepresentations. We say that a bicategory $\mathscr{B}$ is *bicomplete* or *admits all bilimits* if for every small bicategory $\mathscr{J}$ and pseudofunctor $F : \mathscr{J} \to \mathscr{B}$ the bilimit $\mathrm{bilim}\, F$ exists in $\mathscr{B}$.

We now define preservation of bilimits as preservation of the corresponding biuniversal arrows, via the following lemma.

**Lemma 2.12.** For any bicategory $\mathscr{J}$ and pseudofunctor $H : \mathscr{B} \to \mathscr{C}$ the following diagram commutes up to canonical isomorphism:

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "\Delta^{\mathscr{B}}"] \arrow[d, "H"'] \arrow[dr, phantom, "\overset{\cong}{\Leftarrow}"] & \mathrm{Hom}(\mathscr{J}, \mathscr{B}) \arrow[d, "H \circ (-)"] \\
\mathscr{C} \arrow[r, "\Delta^{\mathscr{C}}"'] & \mathrm{Hom}(\mathscr{J}, \mathscr{C})
\end{tikzcd} \tag{3}$$

*Proof.* Let us write $H_* := H \circ (-)$. Unwinding the respective definitions, $(H_* \circ \Delta^{\mathscr{B}})B : \mathscr{J} \to \mathscr{C}$ is the pseudofunctor sending every $j \in \mathscr{J}$ to $HB$, every $p : j \to j'$ to $H\mathrm{Id}_B$ and every 2-cell $\sigma : p \Rightarrow p'$ to the identity. This coincides with $(\Delta^{\mathscr{C}} \circ H)B$ everywhere except that $(\Delta^{\mathscr{C}} \circ H)(B)(j \xrightarrow{p} j') = \mathrm{Id}_{HB}$. So for every $B \in \mathscr{B}$ there exists a pseudonatural isomorphism $\alpha_B : (H_* \circ \Delta^{\mathscr{B}})B \Rightarrow (\Delta^{\mathscr{C}} \circ H)B$ with components $\alpha_B(j) := \mathrm{Id}_{HB}$ for all $j \in \mathscr{J}$. The witnessing 2-cell is the evident composite of $H$'s unit constraint $\psi^H$ with structural isomorphisms. Thus one obtains an invertible 1-cell $\alpha_B$ in $\mathrm{Hom}(\mathscr{J}, \mathscr{C})$ for every $B \in \mathscr{B}$. To extend this to a pseudonatural isomorphism, one takes $\overline{\alpha}_f : \alpha_{B'} \circ H_*(\Delta^{\mathscr{B}} f) \Rightarrow \Delta^{\mathscr{C}}(Hf) \circ \alpha_B$ (for $f : B \to B'$) to be the invertible modification with components given by the structural isomorphism $\mathrm{Id}_{HB'} \circ Hf \overset{\cong}{\Rightarrow} Hf \circ \mathrm{Id}_{HB}$. Then $(\alpha, \overline{\alpha})$ is the required isomorphism. ☐

Thus, assuming the bilimit exists in $\mathscr{C}$, we say that $H$ *preserves the bilimit* of $F : \mathscr{J} \to \mathscr{B}$ if $(H_*, H, (\alpha, \overline{\alpha}))$ preserves the biuniversal arrow $(\mathrm{bilim}\, F, \lambda)$. By Lemma 2.7, this condition is equivalent to requiring that the canonical map $H(\mathrm{bilim}\, F) \to \mathrm{bilim}(HF)$ is an equivalence.

### *2.3 Biadjunctions*

Recalling that an adjunction is specified by a choice of universal arrows, we define a *biadjunction* (Gray, 1974) by a choice of biuniversal arrows (*c.f.* Power (1998)).

**Definition 2.13.** Let $F : \mathscr{B} \to \mathscr{C}$ be a pseudofunctor. To specify a *right biadjoint* to $F$ is to specify a biuniversal arrow $(UC, u_C : FUC \to C)$ from $F$ to $C$ for every $C \in \mathscr{C}$.

Spelling out the definition, to give a right biadjoint $U : \mathscr{C} \to \mathscr{B}$ to $F$ is to give:

- A mapping $U : ob(\mathscr{C}) \to ob(\mathscr{B})$,
- A family of 1-cells $(u_C : FUC \to C)_{C \in \mathscr{C}}$,
- For every $B \in B$ and $h : FB \to C$ a 1-cell $\theta_B(h) : B \to UC$ and an invertible 2-cell $\varepsilon_{B,h} : u_C \circ F\theta_B(h) \Rightarrow h$ that is universal, such that the unit $\eta_h := (\mathrm{id}_{u_C \circ Fh})^{\dagger} : h \Rightarrow \theta_B(u_C \circ Fh)$ is invertible for every $h$.

Notice that the global structure of a right biadjoint is determined by purely local data. Indeed, Definition 2.2 immediately yields a family of adjoint equivalences $u_C \circ F(-) : \mathscr{B}(B, UC) \leftrightarrows \mathscr{C}(FB, C) : \theta_B$ with $u_C \circ F(-) \dashv \theta_B$, and one obtains the right biadjoint $U : \mathscr{C} \to \mathscr{B}$ by setting $U(C) := UC$ on objects, $U(C \xrightarrow{g} C') := \theta_{UC}(g \circ u_C)$ and $U(g \overset{\sigma}{\Rightarrow} g') := ((\sigma \circ u_C) \bullet \varepsilon_{UC,g})^{\dagger}$. By Lemma 2.4, this definition is in turn equivalent to the more common definition of biadjunction (*e.g.* Street (1980)), namely a pair of pseudofunctors $F : \mathscr{B} \leftrightarrows \mathscr{C} : U$ together with a pseudonatural family of equivalences $\mathscr{B}(B, UC) \simeq \mathscr{C}(FB, C)$.

The biuniversal arrow formulation of biadjoints, relying as it does on universal properties at each level, is perhaps easiest to work with when it comes to calculations. As we shall see in Section 5.2, it is also particularly amenable to being expressed syntactically because we shall be able to construct right biadjoints—and hence cartesian closed structure—without needing global pseudo-naturality conditions.

In order to define preservation of exponentials, we extract the definition of preservation of biadjunctions from the definition of preservation of biuniversal arrows.

**Definition 2.14.** For any biadjoint pair $F : \mathscr{B} \leftrightarrows \mathscr{C} : U$ and pseudofunctor $F' : \mathscr{B}' \to \mathscr{C}'$, we say that the triple $(K, L, \rho)$ as below

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "F"] \arrow[d, "L"'] & \mathscr{C} \arrow[d, "K"] \arrow[dl, phantom, "\overset{\rho}{\Rightarrow}"] \\
\mathscr{B}' \arrow[r, "F'"'] & \mathscr{C}'
\end{tikzcd}$$

*preserves the biadjunction* if $(K, L, \rho)$ preserves each biuniversal arrow $u_C : FUC \to C$ in the sense of Definition 2.6. If $\rho$ is the identity and $(K, L)$ is a strict morphism of each biuniversal arrow in the sense of Definition 2.8, we call $(K, L)$ a *strict morphism of biadjunctions*.

## 3. cc-Bicategories

We now instantiate the definitions of the preceding section to define cc-bicategories. To avoid confusion with the 'cartesian bicategories' of Carboni and Walters (1987) we call a bicategory with finite products an *fp-bicategory*. Our definition asks for a right biadjoint to the diagonal pseudofunctor $\Delta^{(n)} : \mathscr{B} \to \mathscr{B}^{\times n}$ $(n \in \mathbb{N})$ by requiring a choice of biuniversal arrow $(\pi_1, \ldots, \pi_n) : \Delta^{(n)}(\prod_n(A_1, \ldots, A_n)) \to (A_1, \ldots, A_n)$ for every $A_1, \ldots, A_n \in \mathscr{B}$. We choose to work immediately with $n$-ary products for all $n$—a so-called 'unbiased' definition, *c.f.* (Leinster, 2004, §3.1)—as opposed to the 'biased' definition requiring binary products and a terminal object. This avoids having to repeatedly distinguish between the unary and nullary cases. Our strictification theorem (Proposition 4.1) will entail that these two approaches are equivalent: from a biased structure one can define an unbiased structure by induction, while from an unbiased structure one can restrict to a biased structure, and these operations are mutually inverse (up to biequivalence).

**Notation 3.1.** We write $A_\bullet$ for a finite sequence $A_1, \dots, A_n$ $(n \in \mathbb{N})$.

**Definition 3.2.** An *fp-bicategory* $(\mathscr{B}, \Pi_n(-))$ is a bicategory $\mathscr{B}$ equipped with the following data for all $A_1, \dots, A_n \in ob(\mathscr{B})$ $(n \in \mathbb{N})$ and $k = 1, \dots, n$:

(1) A chosen object $\prod_n(A_1, \dots, A_n)$,
(2) Chosen *projections* $\pi_k : \prod_n(A_1, \dots, A_n) \to A_k$,
(3) For every $X \in ob(\mathscr{B})$ an adjoint equivalence

$$\begin{tikzcd} \mathscr{B}(X, \prod_n(A_1, \dots, A_n)) \arrow[r, bend left, "(\pi_1 \circ -, \dots, \pi_n \circ -)"] \arrow[r, phantom, "\perp\simeq"] & \prod_{i=1}^n \mathscr{B}(X, A_i) \arrow[l, bend left, "\langle (-)_1, \dots, (-)_n \rangle"] \end{tikzcd} \tag{4}$$

specified by a choice of universal arrows with components $\varpi^{(i)}_{f_\bullet} : \pi_i \circ \langle f_1, \dots, f_n \rangle \overset{\cong}{\Rightarrow} f_i$ for $i = 1, \dots, n$.

We call the right adjoint $\langle (-)_1, \dots, (-)_n \rangle$ the *n-ary tupling*.

An fp-bicategory has *strict products* if every equivalence (4) is an isomorphism. When the underlying bicategory is a 2-category, one recovers the 2-categorical (**Cat**-enriched) definition of finite products.

Unwrapping the definition as we did for Definition 2.2 yields the following universal property. For any finite family of 1-cells $(f_i : X \to A_i)_{i=1,\dots,n}$ there exists a 1-cell $\langle f_1, \dots, f_n \rangle : X \to \prod_n(A_1, \dots, A_n)$ and a family of invertible 2-cells $\left(\varpi^{(i)}_{f_\bullet} : \pi_i \circ \langle f_1, \dots, f_n \rangle \Rightarrow f_i\right)_{i=1,\dots,n}$. These are universal in the sense that, for any 1-cell $g : X \to \prod_n(A_1, \dots, A_n)$ and family of 2-cells $\left(\alpha_i : \pi_i \circ g \Rightarrow f_i : X \to A_i\right)_{i=1,\dots,n}$, there exists a 2-cell $\mathsf{p}^\dagger(\alpha_1, \dots, \alpha_n) : g \Rightarrow \langle f_1, \dots, f_n \rangle : X \to \prod_n(A_1, \dots, A_n)$, unique such that the following diagram commutes for $i = 1, \dots, n$:

$$\begin{tikzcd} & \pi_i \circ \langle f_1, \dots, f_n \rangle \arrow[dr, "\varpi^{(i)}_{f_\bullet}"] & \\ \pi_i \circ g \arrow[ur, "\pi_i \circ \mathsf{p}^\dagger(\alpha_1, \dots, \alpha_n)"] \arrow[rr, "\alpha_i"'] & & f_i \end{tikzcd}$$

**Notation 3.3.** We adopt standard categorical notation where possible. For instance, we write $A \times B$ for $\prod_2(A, B)$ and $f \times g$ (resp. $\tau \times \sigma$) for the pseudofunctorial action of the product on 1-cells (resp. 2-cells). We denote the *terminal object* $\prod_0()$ by $\mathbf{1}$.

**Remark 3.4.** We shall assume throughout that the unary product $\prod_1(-)$ is the identity, *i.e.* that $\prod_1(A) = A$, $\pi_1^A = \mathrm{Id}_A$, $\langle f \rangle = f$ and $\varpi_f = \mathsf{l}_f : \mathrm{Id} \circ f \Rightarrow f$.

**Example 3.5.** The bicategory of spans over a lextensive category has finite biproducts: finite bicategorical products which coincide with finite bicategorical coproducts (Lack et al., 2010, Theorem 6.2). Biproduct structure is defined using the coproduct structure of the underlying category (*c.f.* the biproduct structure of the category of relations).

To define cartesian closed structure on an fp-bicategory $(\mathscr{B}, \Pi_n(-))$, we specify a biadjunction $(-) \times A \dashv (A \Rightarrow -)$ for every $A \in \mathscr{B}$. Following Definition 2.13, we define this by requiring an object $(A \Rightarrow B)$ and a biuniversal arrow $\mathrm{eval}_{A,B} : (A \Rightarrow B) \times A \to B$ for every $A, B \in \mathscr{B}$.

**Definition 3.6.** A *cartesian closed bicategory* or *cc-bicategory* is an fp-bicategory $(\mathscr{B}, \Pi_n(-))$ equipped with the following data for every $A, B \in ob(\mathscr{B})$:

1. A chosen object $(A \Rightarrow B)$,
2. A chosen 1-cell $\mathrm{eval}_{A,B} : (A \Rightarrow B) \times A \to B$,
3. For every $X \in ob(\mathscr{B})$, an adjoint equivalence

$$\begin{tikzcd} \mathscr{B}(X, A \Rightarrow B) \arrow[r, bend left, "\mathrm{eval}_{A,B}\circ(-\times A)"] \arrow[r, phantom, "\perp\simeq"] & \mathscr{B}(X \times A, B) \arrow[l, bend left, "\lambda"] \end{tikzcd} \tag{5}$$

specified by a choice of universal arrows $\varepsilon_f : \mathrm{eval}_{A,B} \circ (\lambda f \times A) \overset{\cong}{\Rightarrow} f$.

We call the functor $\lambda(-)$ *currying* and refer to $\lambda f$ as the *currying of* $f$.

A cc-bicategory is *strictly cartesian closed* if it has strict products and every equivalence (5) is an isomorphism. When the underlying bicategory is a 2-category, one recovers the definition of **Cat**-enriched cartesian closed categories (*e.g.* Hirschowitz (2013), §6), which we call *2-cc 2-categories*; the prototypical example is **Cat** with its familiar cartesian closed structure.

**Notation 3.7.** As for products, we adopt standard categorical notation such as $f \Rightarrow g$ (resp. $\alpha \Rightarrow \beta$) where possible.

The adjoint equivalences (5) give rise to the following universal property. For every 1-cell $f : X \times A \to B$ there exists a 1-cell $\lambda f : X \to (A \Rightarrow B)$ and an invertible 2-cell $\varepsilon_f : \mathrm{eval}_{A,B} \circ (\lambda f \times A) \Rightarrow f$. This is universal in the sense that for any 1-cell $g : X \to (A \Rightarrow B)$ and 2-cell $\alpha : \mathrm{eval}_{A,B} \circ (g \times A) \Rightarrow f$ there exists a 2-cell $\mathrm{e}^{\dagger}(\alpha) : g \Rightarrow \lambda f$, unique such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{eval}_{A,B} \circ (\lambda f \times A) \arrow[dr, "\varepsilon_f"] & \\ \mathrm{eval}_{A,B} \circ (g \times A) \arrow[ur, "\mathrm{eval}_{A,B}\circ(\mathrm{e}^{\dagger}(\alpha)\times A)"] \arrow[rr, "\alpha"'] & & f \end{tikzcd}$$

**Remark 3.8.** If $\alpha$ is invertible then so is the transpose $\mathrm{e}^{\dagger}(\alpha)$; likewise if $\alpha_1, \ldots, \alpha_n$ are all invertible then so is $\mathrm{p}^{\dagger}(\alpha_1, \ldots, \alpha_n)$. These both follow from the general theory of (bi)universal arrows: if $(R, u)$ is a biuniversal arrow from $F : \mathscr{B} \to \mathscr{C}$ to $C \in \mathscr{C}$ and $\tau : u \circ Ff \Rightarrow h : FB \to C$ is invertible, then so is $\tau^{\dagger} : f \Rightarrow \theta_B(h) : B \to R$. To see this, observe (1) that $\theta_B$ is a functor; (2) that the unit $\eta_B$ of the adjunction $u \circ F(-) \dashv \theta_B$ is invertible; and (3) that $\tau^{\dagger} = \theta_B(\tau) \bullet \eta_B$.

By Lemma 2.4, the definition of cc-bicategories above may be rephrased to parallel the 'hom-set' definition of cartesian closed categories. For every fp-bicategory $(\mathscr{B}, \Pi_n(-))$ one obtains pseudonatural equivalences $\mathscr{B}(X, \prod_{i=1}^{n} A_i) \simeq \prod_{i=1}^{n} \mathscr{B}(X, A_i)$ (for $X, A_1, \ldots, A_n \in ob(\mathscr{B})$ and $n \in \mathbb{N}$) and for every cc-bicategory $(\mathscr{B}, \Pi_n(-), \Rightarrow)$ one obtains pseudonatural equivalences $\mathscr{B}(X, A \Rightarrow B) \simeq \mathscr{B}(X \times A, B)$ (for $X, A, B \in ob(\mathscr{B})$).

Cartesian closed bicategories were first studied by Makkai (1996), who introduced a cartesian closed bicategory of categories, 'anafunctors', and natural transformations. Other examples include the bicategory of operads (Gambino and Joyal, 2017), bicategories of concurrent games (Paquet, 2020), and—in the style of models of linear logic—the *Kleisli bicategory* of a suitably-structured pseudocomonad on a symmetric monoidal closed bicategory with finite products (Paquet, 2020, Theorem 2.58); the bicategory of generalised species (Fiore et al., 2007) arises in this manner. For our purposes, the following further example will be crucial.

**Lemma 3.9** (Saville (2020, Chapter 6))**.** For any small bicategory $\mathscr{B}$ the 2-category $\mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$ has all bilimits, given pointwise, and admits a cartesian closed structure with exponentials $[P, Q](-) := \mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})(\mathrm{Y}(-) \times P, Q)$.

## 4. Coherence-by-strictification for cc-bicategories

We are already in a position to prove a coherence-by-strictification result for cc-bicategories. The argument is a small refinement of Power's proof of coherence for bicategories with finite bilimits (Power, 1989*a*, Theorem 4.1). The proof does not go through verbatim, because the exponentials in $\mathrm{Hom}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$ are not generally strict. The solution is to first strictify the bicategory $\mathscr{B}$ to a 2-category $\mathscr{C}$, then pass to the 2-category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ of 2-functors, 2-natural transformations, and modifications. This is cartesian closed as a 2-category—and hence as a bicategory—by general enriched category theory (Day, 1970, Example 5.2).

**Proposition 4.1.** For any small cc-bicategory $(\mathscr{B}, \Pi_n(-), \Rightarrow)$ there exists a 2-cc 2-category $(\mathscr{C}, \Pi_n(-), \Rightarrow)$ such that $\mathscr{B} \simeq \mathscr{C}$.

*Proof.* By Power's result we may assume without loss of generality that $(\mathscr{B}, \Pi_n(-), \Rightarrow)$ is a 2-category with 2-categorical products and pseudo-exponentials. In particular, the assumption that $\mathscr{B}$ is a 2-category means it admits a *2-categorical* Yoneda embedding $\mathrm{Y} : \mathscr{B} \hookrightarrow [\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]$. Let $\overline{\mathscr{B}}$ denote the closure of $ob(\mathrm{Y}\mathscr{B})$ under equivalences and factor the Yoneda embedding as $\mathscr{B} \xrightarrow{i} \overline{\mathscr{B}} \overset{j}{\hookrightarrow} [\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]$, for $j$ the inclusion. By the 2-categorical Yoneda lemma, $i$ is a biequivalence.

It remains to show that $\overline{\mathscr{B}}$ inherits a cartesian closed structure from $[\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]$: we mimic Power's approach. For any $P, Q \in ob(\overline{\mathscr{B}})$ the strict exponential $(jP \Rightarrow jQ)$ exists in $[\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]$. By definition of $\overline{\mathscr{B}}$ there exist $B, C \in ob(\mathscr{B})$ such that $P \simeq \mathrm{Y}B$ and $Q \simeq \mathrm{Y}C$. Then, since Y certainly preserves exponentials, $(jP \Rightarrow jQ) \simeq (j\mathrm{Y}B \Rightarrow j\mathrm{Y}C) \simeq \mathrm{Y}(B \Rightarrow C)$ and the exponential $(jP \Rightarrow jQ)$ is in $\overline{\mathscr{B}}$, as required. ☐

Although not the main focus of this paper, we pause briefly to observe that the preceding argument can be adapted to closed monoidal bicategories. A monoidal bicategory can be simply characterised as a one-object tricategory: for a full definition see (Stay, 2016). Closed structure is then defined in the usual manner.

**Definition 4.2** (Day and Street (1997, Definition 5))**.** A *closed monoidal bicategory* is a monoidal bicategory $(\mathscr{B}, \otimes, I)$ equipped with a choice of right biadjoint $(-) \otimes A \dashv (A \Rightarrow -)$ for every $A \in \mathscr{B}$.

Our strategy is to replace enrichment over **Cat** with enrichment over Gray, the category of 2-categories and 2-functors equipped with the Gray tensor product (Gray, 1974, Theorem I.4.9).

**Definition 4.3.**

(1) A Gray-*monoid* is a monoid in Gray.
(2) A *pseudo-closed* Gray-monoid is a Gray-monoid $(\mathbb{C}, \otimes, I)$ equipped with a choice of right biadjoint $(-) \otimes A \dashv (A \Rightarrow -)$ for every $A \in \mathbb{C}$.
(3) A Gray-*closed* Gray-monoid is a Gray-monoid $(\mathbb{C}, \otimes, I)$ equipped with a choice of Gray-enriched right adjoint $(-) \otimes A \dashv (A \Rightarrow -)$ for every $A \in \mathbb{C}$.

Thus, a Gray-monoid is pseudo-closed if it is equipped with closed structure as a monoidal bicategory. For an explicit definition of Gray-monoids, see *e.g.* (Day and Street, 1997, §1).

**Proposition 4.4.** Every closed monoidal bicategory $(\mathscr{B}, \otimes, I, \Rightarrow)$ is biequivalent, as a closed monoidal bicategory, to a Gray-closed Gray-monoid.

*Proof.* Since a monoidal bicategory is a one-object tricategory, the coherence theorem for tricategories (Gordon et al., 1995) entails that $\mathscr{B}$ is monoidally-biequivalent to a Gray-monoid $\mathscr{G}$. Write this $F : \mathscr{G} \leftrightarrows \mathscr{B} : G$. Then $\mathscr{G}$ acquires a pseudo-closed structure with $(X \Rightarrow Y) := G(FX \Rightarrow FY)$, so without loss of generality we may assume $\mathscr{B}$ is a pseudo-closed Gray-monoid.

By a standard fact of enriched category theory, the underlying category of the Gray-enriched functor category $[\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}]$ is the category of Gray-functors and Gray-natural transformations (*e.g.* (Kelly, 1982)). Since Gray is symmetric monoidal closed, complete and cocomplete, Gray-enriched Day convolution makes $[\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}]$ into a Gray-enriched closed monoidal category $([\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}], \circledast, \mathrm{Y}I, \multimap)$ and the Yoneda functor Y becomes a Gray-enriched strong monoidal functor (Day, 1970).

We now emulate the argument for cc-bicategories. Let $\overline{\mathscr{B}}$ be the closure of $ob(\mathrm{Y}\mathscr{B}) \subset [\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}]$ under equivalences. Factor Y as $\mathscr{B} \xrightarrow{i} \overline{\mathscr{B}} \overset{j}{\hookrightarrow} [\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}]$. By the Yoneda lemma, $i$ is a biequivalence. We claim that $\overline{\mathscr{B}}$ inherits a closed monoidal structure from $[\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}]$.

First, the unit $\mathrm{Y}I$ is certainly in $\overline{\mathscr{B}}$. Next, if $P, Q \in \overline{\mathscr{B}}$ then $P \simeq \mathrm{Y}A$ and $Q \simeq \mathrm{Y}B$ for some $A, B \in \mathscr{B}$. Then $jP \circledast jQ \in [\mathscr{B}^{\mathrm{op}}, \mathrm{Gray}]$ and we have $jP \circledast jQ \simeq \mathrm{Y}A \circledast \mathrm{Y}B \simeq \mathrm{Y}(A \otimes B) \in \overline{\mathscr{B}}$ as required. Now we want to show $(jP \multimap jQ) \in \overline{\mathscr{B}}$. We argue as above, using the fact Y preserves the closed structure: $(jP \multimap jQ) \simeq (\mathrm{Y}A \multimap \mathrm{Y}B) \simeq \mathrm{Y}(A \Rightarrow B) \in \overline{\mathscr{B}}$.

Finally, $i$ preserves the closed monoidal structure because Y preserves this structure. □

## 5. From coherence-by-strictification to Mac Lane-style coherence

We now turn to proving a Mac Lane-style coherence result from Proposition 4.1. The strategy is reminiscent of the normalisation-by-evaluation argument of Čubrić et al. (1998). While Proposition 4.1 requires remarkably little technical machinery, our approach to Mac Lane style coherence requires two further components: a suitable notion of cc-pseudofunctor (Section 5.1) and a construction of the free cc-bicategory (Section 5.2).

### *5.1 Cartesian closed pseudofunctors*

We define preservation of products and exponentials as preservation of the corresponding biuniversal arrows. Thus, while a cartesian closed functor preserves products and exponentials up to isomorphism, its bicategorical counterpart preserves products and exponentials up to equivalence.

**Definition 5.1.**

(1) An *fp-pseudofunctor* $(F, \mathrm{q}^{\times}) : (\mathscr{B}, \Pi_n(-)) \to (\mathscr{C}, \Pi_n(-))$ is a pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ equipped with specified adjoint equivalences

$$\langle F\pi_1, \ldots, F\pi_n \rangle : F(\textstyle\prod_{i=1}^{n} A_i) \leftrightarrows \prod_{i=1}^{n}(FA_i) : \mathrm{q}^{\times}_{A_\bullet} \tag{6}$$

for every $A_1, \ldots, A_n \in ob(\mathscr{B})$ $(n \in \mathbb{N})$.

(2) A *cc-pseudofunctor* $(F, \mathrm{q}^{\times}, \mathrm{q}^{\Rightarrow}) : (\mathscr{B}, \Pi_n(-), \Rightarrow) \to (\mathscr{C}, \Pi_n(-), \Rightarrow)$ is an fp-pseudofunctor $(F, \mathrm{q}^{\times})$ equipped with specified adjoint equivalences

$$\mathrm{s}_{A,B} : F(A \Rightarrow B) \leftrightarrows (FA \Rightarrow FB) : \mathrm{q}^{\Rightarrow}_{A,B} \tag{7}$$

for every $A, B \in ob(\mathscr{B})$, where $\mathsf{s}_{A,B} : F(A \Rightarrow B) \to (FA \Rightarrow FB)$ is the exponential transpose of $F(\mathrm{eval}_{A,B}) \circ \mathsf{q}^{\times}_{A\Rightarrow B,A}$.

As it has been throughout, the extra data is the *right* adjoint: thus, we assume that $\langle F\pi_1, \dots, F\pi_n\rangle \dashv \mathsf{q}^{\times}$ and $\mathsf{s}_{A,B} \dashv \mathsf{q}^{\Rightarrow}$. Instantiating Definition 2.8, we obtain the definition of a strict cc-pseudofunctor.

**Definition 5.2.**

(1) A pseudofunctor $F$ is *strict* if it strictly preserves identities and composition: $F(\mathrm{Id}_X) = \mathrm{Id}_{FX}$ and $F(f \circ g) = Ff \circ Fg$.

(2) An fp-pseudofunctor $(F, \mathsf{q}^{\times})$ is *strict* if $F$ is strict and satisfies

$$\begin{aligned} F(\textstyle\prod_n(A_1, \dots, A_n)) &= \textstyle\prod_n(FA_1, \dots, FA_n) \\ F(\pi_i^{A_1,\dots,A_n}) &= \pi_i^{FA_1,\dots,FA_n} \\ F\langle t_1, \dots, t_n\rangle &= \langle Ft_1, \dots, Ft_n\rangle \\ F\varpi^{(i)}_{t_1,\dots,t_n} &= \varpi^{(i)}_{Ft_1,\dots,Ft_n} \\ \mathsf{q}^{\times}_{A_1,\dots,A_n} &= \mathrm{Id}_{\Pi_n(FA_1,\dots,FA_n)} \end{aligned}$$

with adjoint equivalences canonically induced by the 2-cells $\mathsf{p}^{\dagger}(\mathsf{r}_{\pi_1}, \dots, \mathsf{r}_{\pi_n}) : \mathrm{Id} \overset{\cong}{\Rightarrow} \langle \pi_1, \dots, \pi_n\rangle$.

(3) A cc-pseudofunctor $(F, \mathsf{q}^{\times}, \mathsf{q}^{\Rightarrow})$ is *strict* if $(F, \mathsf{q}^{\times})$ is a strict fp-pseudofunctor and $F$ satisfies

$$\begin{aligned} F(A \Rightarrow B) &= (FA \Rightarrow FB) \\ F(\mathrm{eval}_{A,B}) &= \mathrm{eval}_{FA,FB} \\ F(\lambda t) &= \lambda(Ft) \\ F(\varepsilon_t) &= \varepsilon_{Ft} \\ \mathsf{q}^{\Rightarrow}_{A,B} &= \mathrm{Id}_{FA\Rightarrow FB} \end{aligned}$$

with equivalences canonically induced by the 2-cells

$$\mathsf{e}^{\dagger}(\mathrm{eval}_{FA,FB} \circ \kappa) : \mathrm{Id}_{(FA\Rightarrow FB)} \overset{\cong}{\Rightarrow} \lambda(\mathrm{eval}_{FA,FB} \circ \mathrm{Id}_{(FA\Rightarrow FB)\times FA})$$

for $\kappa$ the canonical isomorphism $\mathrm{Id}_{FA\Rightarrow FB} \times FA \cong \mathrm{Id}_{(FA\Rightarrow FB)\times FA}$.

In the 1-categorical setting, a product-preserving functor $(F, \mathsf{q}^{\times} = \langle F\pi_1, \dots, F\pi_n\rangle^{-1})$ satisfies $\mathsf{q}^{\times} \circ \prod_{i=1}^{n} Ff_i = F(\prod_{i=1}^{n} f_i) \circ \mathsf{q}^{\times}$ for every family $(f_i : A_i \to A_i')_{i=1,\dots,n}$. This extends to the bicategorical setting. Indeed, for any fp-pseudofunctor $(F, \mathsf{q}^{\times}) : (\mathscr{B}, \Pi_n(-)) \to (\mathscr{C}, \Pi_n(-))$ one can translate the 1-categorical proof into a composite of canonical 2-cells $\mathsf{nat}_{f_\bullet} : \mathsf{q}^{\times}_{A'_\bullet} \circ \prod_{i=1}^{n} Ffi \Rightarrow F(\prod_{i=1}^{n} f_i) \circ \mathsf{q}^{\times}_{A_\bullet}$ so that $(\mathsf{q}^{\times}, \mathsf{nat})$ becomes a pseudonatural equivalence $\prod_{i=1}^{n}(F(-), \dots, F(=)) \Rightarrow (F \circ \prod_{i=1}^{n})(-, \dots, =)$ in $\mathrm{Hom}(\prod_{i=1}^{n} \mathscr{B}, \mathscr{C})$. In the next section we make this ‘translation’ precise.

### *5.2 The free cc-bicategory on a graph*

Our statement of Mac Lane-style coherence will be that the free cc-bicategory on a graph is locally an equivalence relation (throughout we say simply *graph* to mean a directed graph). In Section 5.2.1 we construct this cc-bicategory and prove the required freeness universal properties. Then, in Section 5.2.2, we make precise the relationship between the free cc-bicategory on a graph and the free cartesian closed category on the same graph. In Section 5 we shall use this to substantiate Principle 1.3.

### 5.2.1 Constructing the free cc-bicategory

Our construction is in the style of the equational presentation of cartesian closed categories of Lambek and Scott (1986, §I.3). For a type-theoretic presentation, constructed as the syntactic model of a type theory in the style of the simply-typed lambda calculus, see (Saville, 2020, Chapter 5).

The axiomatisation is simplified by the use of biuniversal arrows. An alternative approach would be to encode the projection, pairing, currying, and application operations, together with a unit and counit for each of products and exponentials, subject to naturality, invertibility, the triangle laws, and congruence conditions (*c.f.* (Hilken, 1996) and (Ouaknine, 1997)). Instead, for each of products and exponentials we require only two 2-cell introduction rules and four equations to specify the required adjunctions.

**Notation 5.3.** Let G be a graph. We write $\mathrm{G}_0$ for the set of nodes in G and $\widetilde{\mathrm{G}_0}$ for the set generated by the grammar $A_1, \ldots, A_n, C, D ::= B \mid \prod_n(A_1, \ldots, A_n) \mid C \Rightarrow D$ (where $B \in \mathrm{G}_0$ and $n \in \mathbb{N}$). For $A, B \in \mathrm{G}_0$, write $\mathrm{G}(A, B)$ for the set of edges from $A$ to $B$.

**Construction 5.4.** For any graph G define a cc-bicategory $\mathscr{F}[\mathrm{G}]$ as follows. For objects, set $ob(\mathscr{F}[\mathrm{G}]) := \widetilde{\mathrm{G}_0}$. The 1-cells, 2-cells and equational theory are defined by the deductive system of Figures 1, 2a and 2b. The equational theory makes $(\varpi^{(1)}, \ldots, \varpi^{(n)})$ and $\varepsilon$ universal arrows; to ensure the adjunctions (4) and (5) are adjoint equivalences, we further require each unit is invertible.

We abuse notation by denoting whiskering in the usual manner.

For any graph G the bicategory $\mathscr{F}[\mathrm{G}]$ is *locally groupoidal*: every 2-cell is invertible. This is proven by a straightforward induction on the 2-cells in Figures 1 and 2, using Remark 3.8 for the $\mathsf{p}^{\dagger}(-, \ldots, =)$ and $\mathsf{e}^{\dagger}(-)$ cases.

Restricting to strict cc-pseudofunctors yields a strict free property for $\mathscr{F}[\mathrm{G}]$. Recall that every bicategory $\mathscr{B}$ has an *underlying graph* with nodes the objects of $\mathscr{B}$ and a unique edge $A \rightsquigarrow B$ for each 1-cell $f : A \to B$ in $\mathscr{B}$. We do not distinguish notationally between a bicategory and its underlying graph, and write $\iota$ for the canonical inclusion $\mathrm{G} \hookrightarrow \mathscr{F}[\mathrm{G}]$.

**Lemma 5.5.** For any graph G, cc-bicategory $(\mathscr{C}, \Pi_n(-), \Rightarrow)$ and graph homomorphism $h : \mathrm{G} \to \mathscr{C}$ there exists a unique strict cc-pseudofunctor $h^{\sharp} : \mathscr{F}[\mathrm{G}] \to \mathscr{C}$ such that $h^{\sharp} \circ \iota = h$.

*Proof.* We define $h^{\sharp}$ in the obvious manner. On objects, set

$$\begin{aligned} h^{\sharp}(B) &:= h(B) \qquad \text{for } B \in \mathrm{G}_0 \\ h^{\sharp}(\textstyle\prod_n(A_1, \ldots, A_n)) &:= \textstyle\prod_n\big(h^{\sharp}A_1, \ldots, h^{\sharp}A_n\big) \\ h^{\sharp}(B \Rightarrow C) &:= (h^{\sharp}A \Rightarrow h^{\sharp}B) \end{aligned}$$

The action on constants $c \in \mathrm{G}(A, B)$, identities and composition is determined by the requirement that $h^{\sharp}$ is strict and satisfies $h^{\sharp} \circ \iota = h$. Similarly, for products and exponentials the action of $h^{\sharp}$ is determined everywhere except on $\mathsf{p}^{\dagger}(\alpha_1, \ldots, \alpha_n)$ and $\mathsf{e}^{\dagger}(\alpha)$. For these we set

$$\begin{aligned} h^{\sharp}\big(\mathsf{p}^{\dagger}(\alpha_1, \ldots, \alpha_n)\big) &:= \mathsf{p}^{\dagger}(h^{\sharp}\alpha_1, \ldots, h^{\sharp}\alpha_n) \\ h^{\sharp}\big(\mathsf{e}^{\dagger}(\alpha)\big) &:= \mathsf{e}^{\dagger}(h^{\sharp}\alpha) \end{aligned} \tag{8}$$

**Bicategorical structure, 1-cells**

$$\frac{c \in \mathrm{G}(A,B)}{c \in \mathscr{F}[\mathrm{G}](A,B)} \qquad \frac{}{\mathrm{Id}_A \in \mathscr{F}[\mathrm{G}](A,A)} \qquad \frac{f \in \mathscr{F}[\mathrm{G}](A,B) \qquad g \in \mathscr{F}[\mathrm{G}](X,A)}{f \circ g \in \mathscr{F}[\mathrm{G}](X,B)}$$

**Bicategorical structure, 2-cells**

$$\frac{f \in \mathscr{F}[\mathrm{G}](A,B)}{\mathrm{id}_f \in \mathscr{F}[\mathrm{G}](A,B)(f,f)} \qquad \frac{\tau \in \mathscr{F}[\mathrm{G}](A,B)(f',f'') \qquad \sigma \in \mathscr{F}[\mathrm{G}](A,B)(f,f')}{\tau \bullet \sigma \in \mathscr{F}[\mathrm{G}](A,B)(f,f'')}$$

$$\frac{\tau \in \mathscr{F}[\mathrm{G}](A,B)(f,f') \qquad \sigma \in \mathscr{F}[\mathrm{G}](X,A)(g,g')}{\tau \circ \sigma \in \mathscr{F}[\mathrm{G}](X,B)(f \circ g, f' \circ g')}$$

$$\frac{f \in \mathscr{F}[\mathrm{G}](B,C) \qquad g \in \mathscr{F}[\mathrm{G}](A,B) \qquad h \in \mathscr{F}[\mathrm{G}](X,B)}{\mathsf{a}_{f,g,h} \in \mathscr{F}[\mathrm{G}](X,C)\big(f \circ (g \circ h), (f \circ g) \circ h\big)}$$

$$\frac{f \in \mathscr{F}[\mathrm{G}](A,B)}{\mathsf{r}_f \in \mathscr{F}[\mathrm{G}](A,B)(f, f \circ \mathrm{Id}_A)} \qquad \frac{g \in \mathscr{F}[\mathrm{G}](A,B)}{\mathsf{l}_g \in \mathscr{F}[\mathrm{G}](A,B)(\mathrm{Id}_B \circ g, g)}$$

**Equational theory**

The smallest congruence $\equiv$ such that:

- Every $\mathscr{F}[\mathrm{G}](A,B)$ forms a category with composition the $\bullet$ operation and identity on $f \in \mathscr{F}[\mathrm{G}](A,B)$ given by $\mathrm{id}_f$,
- The operation $(f,g) \mapsto f \circ g$ is functorial with respect to this category structure,
- The families of 2-cells $\mathsf{a}$, $\mathsf{l}$ and $\mathsf{r}$ are invertible, natural and satisfy the triangle and pentagon laws of a bicategory.

**Figure 1**: Rules for $\mathscr{F}[\mathrm{G}]$: bicategorical structure

For uniqueness, it suffices to show that any strict cc-pseudofunctor preserves $\mathsf{p}^{\dagger}(-)$ and $\mathsf{e}^{\dagger}(-)$ as in (8). This follows from Lemma 2.10. ☐

We are now justified in calling $\mathscr{F}[\mathrm{G}]$ the *free cc-bicategory on* G. Its strict freeness universal property entails an up-to-equivalence universal property which will play a crucial role in our proof of Mac Lane-style coherence. We write $t : A_1, \dots, A_n \to B$ and $\tau : t \Rightarrow t' : A_1, \dots, A_n \to B$ for 1-cells and 2-cells in $\mathscr{F}[\mathrm{G}]$, respectively.

**Lemma 5.6.** Let G be a graph, $(\mathscr{B}, \Pi_n(-), \Rightarrow)$ be a cc-bicategory and $h : \mathrm{G} \to \mathscr{C}$ a graph homomorphism. Then for any cc-pseudofunctor $(F, \mathsf{q}^{\times}, \mathsf{q}^{\Rightarrow})$ satisfying $F \circ \iota = h$ there exists a pseudonatural adjoint equivalence $F \simeq h^{\sharp}$ between $F$ and the canonical cc-pseudofunctor extending $h$.

*Proof.* We construct a pseudonatural transformation $(\mathsf{k}, \overline{\mathsf{k}}) : F \Rightarrow h^{\sharp}$ such that each component $\mathsf{k}_X$ forms an adjoint equivalence $\mathsf{k}_X \dashv \mathsf{k}^{\star}_X$; write $\mathsf{v}_X : \mathrm{Id}_{FX} \overset{\cong}{\Rightarrow} \mathsf{k}^{\star}_X \circ \mathsf{k}_X$ and $\mathsf{w} : \mathsf{k}_X \circ \mathsf{k}^{\star}_X \overset{\cong}{\Rightarrow} \mathrm{Id}_{h^{\sharp}X}$ for the unit and counit of this adjunction, respectively. One then obtains the required pseudo-inverse $(\mathsf{k}^{\star}, \overline{\mathsf{k}^{\star}}) : h^{\sharp} \Rightarrow F$, by defining $\overline{\mathsf{k}^{\star}_f}$ to be the *mate* of $\overline{\mathsf{k}}_f$ (see *e.g.* (Lack, 2010, §2.1)). Because $\overline{\mathsf{k}}_f$, $\mathsf{v}_X$

**Finite-product structure, 1-cells**

$$\frac{}{\pi_i^{A_\bullet} \in \mathscr{F}[\mathrm{G}](\textstyle\prod_n(A_1,\dots,A_n),A_i)}\;{\scriptstyle (1\le i\le n)} \qquad \frac{(t_i \in \mathscr{F}[\mathrm{G}](X;A_i))_{i=1,\dots,n}}{\langle t_1,\dots,t_n\rangle \in \mathscr{F}[\mathrm{G}](X;\textstyle\prod_n(A_1,\dots,A_n))}$$

**Finite-product structure, 2-cells**

$$\frac{(t_i \in \mathscr{F}[\mathrm{G}](X;A_i))_{i=1,\dots,n}}{\varpi^{(i)}_{t_\bullet} \in \mathscr{F}[\mathrm{G}](X;A_i)\big(\pi_i^{A_\bullet}\circ\langle t_1,\dots,t_n\rangle, t_i\big)}\;{\scriptstyle (1\le i\le n)}$$

$$\frac{\big(\alpha_i \in \mathscr{F}[\mathrm{G}](X;A_i)(\pi_i^{A_\bullet}\circ u, t_i)\big)_{i=1,\dots,n}}{\mathsf{p}^\dagger(\alpha_1,\dots,\alpha_n)\in \mathscr{F}[\mathrm{G}]\big(X;\textstyle\prod_n(A_1,\dots,A_n)\big)(u,\langle t_1,\dots,t_n\rangle)}$$

**Finite product structure, equational theory**

- $\alpha_k \equiv \varpi^{(i)}_{t_\bullet} \bullet \big(\pi_k \circ \mathsf{p}^\dagger(\alpha_1,\dots,\alpha_n)\big)$ if $(\alpha_i : u \Rightarrow t_i : X \to A_i)_{i=1,\dots,n}$ and $k\in\{1,\dots,n\}$,
- $\gamma \equiv \mathsf{p}^\dagger(\varpi^{(1)}_{t_\bullet}\bullet(\pi_1\circ\gamma),\dots,\varpi^{(n)}_{t_\bullet}\bullet(\pi_n\circ\gamma))$ if $\gamma : u \Rightarrow \langle t_1,\dots,t_n\rangle : X \to \prod_n(A_1,\dots,A_n)$,
- $\mathsf{p}^\dagger(\alpha_1,\dots,\alpha_n)\equiv\mathsf{p}^\dagger(\alpha'_1,\dots,\alpha'_n)$ if $\alpha_i\equiv\alpha'_i : \pi_i^{A_\bullet}\circ u\Rightarrow t_i$ for $i=1,\dots,n$,
- Every $\varpi^{(i)}_{t_\bullet}$ and $\varsigma_t := \mathsf{p}^\dagger(\mathrm{id}_{\pi_1\circ t},\dots,\mathrm{id}_{\pi_n\circ t})$ is invertible.

(a) Finite-product structure

**Closed structure, 1-cells**

To simplify notation, we write $t\times B$ for the (derived) arrow $\langle t\circ\pi_1, \mathrm{Id}\circ\pi_2\rangle$, and likewise on 2-cells.

$$\frac{}{\mathsf{eval}_{B,C}\in\mathscr{F}[\mathrm{G}](B\Rightarrow C\times B;C)} \qquad \frac{t\in\mathscr{F}[\mathrm{G}](X\times B;C)}{\lambda t\in\mathscr{F}[\mathrm{G}](X,B\Rightarrow C)}$$

**Closed structure, 2-cells**

$$\frac{t\in\mathscr{F}[\mathrm{G}](X\times B,C)}{\varepsilon_t\in\mathscr{F}[\mathrm{G}](X\times B,C)\big(\mathsf{eval}_{B,C}\circ(\lambda t\times B),t\big)} \qquad \frac{\begin{array}{c}u\in\mathscr{F}[\mathrm{G}](X,B\Rightarrow C)\\ \alpha\in\mathscr{F}[\mathrm{G}](X\times B,C)\big(\mathsf{eval}_{B,C}\circ(u\times B),t\big)\end{array}}{\mathsf{e}^\dagger(\alpha)\in\mathscr{F}[\mathrm{G}](X,A\Rightarrow B)(u,\lambda t)}$$

**Closed structure, equational theory**

- $\alpha \equiv \varepsilon_t \bullet \big(\mathsf{eval}_{B,C}\circ(\mathsf{e}^\dagger(\alpha)\times B)\big)$ for every $\alpha : \mathsf{eval}_{B,C}\circ u\times B \Rightarrow t : X\times B\to C$,
- $\gamma\equiv \mathsf{e}^\dagger(\varepsilon_t\bullet(\mathsf{eval}_{B,C}\circ(\gamma\times B)))$ for every $\gamma: u\Rightarrow\lambda t : X\to(A\Rightarrow B)$,
- $\mathsf{e}^\dagger(\alpha)\equiv\mathsf{e}^\dagger(\alpha')$ if $\alpha\equiv\alpha' : \mathsf{eval}_{B,C}\circ(u\times B)\Rightarrow t : X\times B\to C$,
- Every $\varepsilon_t$ and $\eta_u := \mathsf{e}^\dagger(\mathrm{id}_{\mathsf{eval}\circ(u\times B)})$ is invertible.

(b) Closed structure

**Figure 2**: Rules for $\mathscr{F}[\mathrm{G}]$: cartesian closed structure

and $\mathsf{w}_X$ are all invertible, so is $\overline{\mathsf{k}^\star_f}$. The equivalence $F \simeq h^\sharp$ witnessed by $(\mathsf{k}, \overline{\mathsf{k}})$ and $(\mathsf{k}^\star, \overline{\mathsf{k}^\star})$ extends to an adjoint equivalence with unit $\mathsf{v}$ and counit $\mathsf{w}$.

We define $\mathsf{k}_X$ and $\mathsf{k}^\star_X$ by mutual induction. Recall that we write $\mathrm{s}_{A,B}$ for the canonical map $\lambda\big(F(\mathrm{eval}_{A,B}) \circ \mathrm{q}^\times_{A\Rightarrow B,A}\big) : F(A \Rightarrow B) \to (FA \Rightarrow FB)$.

$$\mathsf{k}_B := FB \xrightarrow{=} hB \xrightarrow{\mathrm{Id}_{hB}} hB \xrightarrow{=} h^\sharp B \qquad \text{for } B \in \mathbb{B}$$
$$\mathsf{k}^\star_B := h^\sharp B \xrightarrow{=} hB \xrightarrow{\mathrm{Id}_{hB}} hB \xrightarrow{=} FB$$

$$\mathsf{k}_{(\prod_n A_\bullet)} := F(\textstyle\prod_n A_\bullet) \xrightarrow{\langle F\pi_1,\ldots,F\pi_n\rangle} \prod_{i=1}^n F(A_i) \xrightarrow{\prod_{i=1}^n \mathsf{k}_{A_i}} \prod_{i=1}^n h^\sharp A_i$$
$$\mathsf{k}^\star_{(\prod_n A_\bullet)} := \textstyle\prod_{i=1}^n h^\sharp A_i \xrightarrow{\prod_{i=1}^n \mathsf{k}^\star_{A_i}} \prod_{i=1}^n F(A_i) \xrightarrow{\mathrm{q}^\times_{A_\bullet}} F(\prod_n A_\bullet)$$

$$\mathsf{k}_{(X\Rightarrow Y)} := F(X \Rightarrow Y) \xrightarrow{\mathrm{s}_{X,Y}} (FX \Rightarrow FY) \xrightarrow{\mathsf{k}^\star_X \Rightarrow \mathsf{k}_Y} (h^\sharp X \Rightarrow h^\sharp Y)$$
$$\mathsf{k}^\star_{(X\Rightarrow Y)} := (h^\sharp X \Rightarrow h^\sharp Y) \xrightarrow{\mathsf{k}_X \Rightarrow \mathsf{k}^\star_Y} (FX \Rightarrow FY) \xrightarrow{\mathrm{q}^{\Rightarrow}_{X,Y}} F(X \Rightarrow Y)$$

It remains to construct the witnessing 2-cells $\overline{\mathsf{k}}_t : \mathsf{k}_B \circ Ft \Rightarrow h^\sharp(t) \circ \mathsf{k}_A$. The construction is long but not especially enlightening, so we leave the full details for Appendix A. □

### 5.2.2 cc-Bicategories as CCCs ‘up to isomorphism’

We can now make precise the statement that the free cc-bicategory on a graph is a cartesian closed category ‘up to isomorphism’. In conjunction with our main theorem, this will enable us to justify Principle 1.3.

**Definition 5.7.** For any graph $\mathrm{G}$, let $\mathbb{F}[\mathrm{G}]$ denote the free cartesian closed category on $\mathrm{G}$ (see *e.g.* Lambek and Scott (1986, §1.3)). Explicitly, the objects are $ob(\mathbb{F}[\mathrm{G}]) := \widetilde{\mathrm{G}_0}$ and the morphisms are defined inductively by taking the rules defining 1-cells in $\mathscr{F}[\mathrm{G}]$. The equational theory is the smallest congruence containing the identity and associativity laws of a category, together with the rules

$$\begin{aligned} s_i &= \pi_i \circ \langle s_1, \ldots, s_n\rangle & &(\varpi^{(i)}) \\ t &= \langle \pi_1 \circ t, \ldots, \pi_n \circ t\rangle & &(\varsigma) \\ u &= \mathrm{eval}_{A,B} \circ (\lambda u \times A) & &(\varepsilon) \\ v &= \lambda(\mathrm{eval} \circ (v \times A)) & &(\eta) \end{aligned}$$

where $(s_i : X \to A_i)_{i=1,\ldots,n}$, $t : X \to \prod_n(A_1, \ldots, A_n)$, $u : X \times A \to B$, and $v : X \to (A \Rightarrow B)$.

There exist evident mappings $(\!|-|\!) : \mathbb{F}[\mathrm{G}] \leftrightarrows \mathscr{F}[\mathrm{G}] : \overline{(-)}$ taking an object or 1-cell to its correlate.

**Definition 5.8.** For any graph $\mathrm{G}$ and $A, B \in \mathscr{F}[\mathrm{G}]$, define an equivalence relation $\cong_{A,B}$ on $ob(\mathscr{F}[\mathrm{G}](A,B))$ by setting $t \cong_{A,B} t'$ if and only if there exists a (necessarily invertible) rewrite $t \overset{\cong}{\Rightarrow} t'$ in $\mathscr{F}[\mathrm{G}]$.

**Lemma 5.9.** Let $\mathrm{G}$ be a graph. For any $A, B \in \widetilde{\mathrm{G}_0}$, the mappings $(\!|-|\!)$ and $\overline{(-)}$ determine a bijection $ob(\mathscr{F}[\mathrm{G}](A,B))/{\cong_{A,B}} \cong \mathbb{F}[\mathrm{G}](A,B)$ between morphisms in the free cartesian closed category and $\cong_{A,B}$-equivalence classes of 1-cells in the free cc-bicategory.

*Proof.* Suppose $\sigma : t \Rightarrow t' : A \to B$ in $\mathscr{F}[\mathrm{G}]$. We need to show that $\overline{t} \equiv \overline{t'}$. If $\sigma$ is a structural isomorphism, this follows from standard properties of syntactic substitution. If $\sigma$ is a counit $\varpi^{(i)}$ or $\varepsilon$, then $t$ and $t'$ are related by the corresponding equation in Definition 5.7. If $\sigma = \mathsf{p}^{\dagger}(\alpha_1, \ldots, \alpha_n) : u \Rightarrow \langle t_1, \ldots, t_n \rangle$ then, arguing inductively, we have $\pi_i \circ \overline{u} \equiv \overline{t_i}$ for $i = 1, \ldots, n$. But then

$$u \overset{(\varsigma)}{\equiv} \langle \pi_1 \circ u, \ldots, \pi_n \circ u \rangle = \langle t_1, \ldots, t_n \rangle$$

as required. The $\mathsf{e}^{\dagger}(\alpha)$ case is similar, while the $\bullet$ and $\circ$ cases follow directly from the induction hypothesis.

Going the other way, induct on the definition of $t \equiv t'$ in $\mathbb{F}[\mathrm{G}]$. If $\equiv$ is a reflexivity or symmetry law, one takes the identity 2-cell or uses the fact every 2-cell in $\mathscr{F}[\mathrm{G}]$ is invertible. If $\equiv$ arises by transitivity or the congruence laws, the required 2-cell is defined by composing those given by the induction hypothesis. Next, if $\equiv$ is an unit or associativity axiom for a category, the corresponding rewrite $(\!| t |\!) \Rightarrow (\!| t' |\!)$ is a structural isomorphism in $\mathscr{F}[\mathrm{G}]$.

Finally, for the cartesian closed structure, if $\equiv$ is either $(\varpi^{(i)})$ or $(\varepsilon)$ in Definition 5.7, the equation is witnessed by the corresponding 2-cell. If it is either $(\varsigma)$ or $(\eta)$, the equation is witnessed by the relevant unit: either $\mathsf{p}^{\dagger}(\mathrm{Id}_{\pi_1 \circ t}, \ldots, \mathrm{Id}_{\pi_n \circ t})$ or $\mathsf{e}^{\dagger}(\mathrm{id}_{\mathsf{eval} \circ (t \times B)})$. ☐

### 5.3 Proving coherence

Finally we come to proving Mac Lane-style coherence. Fix a graph G, a cc-bicategory $(\mathscr{X}, \Pi_n(-), \Rightarrow)$ and a graph homomorphism $h : \mathrm{G} \to \mathscr{X}$ . Now let $(\mathscr{C}, \Pi_n(-), \Rightarrow)$ be a 2-cc 2-category and $(F, \mathrm{q}^{\times}, \mathrm{q}^{\Rightarrow}) : (\mathscr{X}, \Pi_n(-), \Rightarrow) \to (\mathscr{C}, \Pi_n(-), \Rightarrow)$ be any cc-pseudofunctor. The underlying mapping of $F$ determines a graph homomorphism $F_0 : \mathscr{X} \to \mathscr{C}$ and $F \circ h^{\sharp}$ is a cc-pseudofunctor so, applying Lemma 5.5 and Lemma 5.6, one obtains the following diagram:

$$\begin{tikzcd}
 & \mathscr{C} \\
 & \mathscr{X} \arrow[u, "F"'] \\
\mathrm{G} \arrow[ruu, bend left, "F_0 \circ h"] \arrow[ru, "h"] \arrow[r, hook] & \mathscr{F}[\mathrm{G}] \arrow[u, "h^{\sharp}"'] \arrow[uu, bend right=60, "(F_0 \circ h)^{\sharp}"', "\simeq"]
\end{tikzcd}$$

Denote the equivalence $(F_0 \circ h)^{\sharp} \overset{\simeq}{\Rightarrow} F \circ h^{\sharp}$ by $(\mathsf{k}, \overline{\mathsf{k}})$. For any 1-cell $t : X \to A$ in $\mathscr{F}[\mathrm{G}]$, one therefore obtains an iso-commuting square

$$\begin{tikzcd}
(F \circ h^{\sharp})X \arrow[r, "(F \circ h^{\sharp})t"] \arrow[d, "\mathsf{k}_X"'] \arrow[dr, phantom, "\overset{\overline{\mathsf{k}}_t}{\cong}"] & (F \circ h^{\sharp})A \arrow[d, "\mathsf{k}_A"] \\
(F_0 \circ h)^{\sharp}X \arrow[r, "(F_0 \circ h)^{\sharp}t"'] & (F_0 \circ h)^{\sharp}A
\end{tikzcd}$$

Moreover, the naturality condition on a pseudonatural transformation requires that, for any 2-cell $\tau : t \Rightarrow t' : X \to A$ in $\mathscr{F}[\mathrm{G}]$, the following commutes:

$$\begin{tikzcd}
\mathsf{k}_A \circ (F \circ h^\sharp)(t) \arrow[r, "\mathsf{k}_A \circ (F \circ h^\sharp)(\tau)"] \arrow[d, "\overline{\mathsf{k}}_t"'] & \mathsf{k}_A \circ (F \circ h^\sharp)(t') \arrow[d, "\overline{\mathsf{k}}_{t'}"] \\
(F_0 \circ h)^\sharp(t) \circ \mathsf{k}_X \arrow[r, "(F_0 \circ h)^\sharp(\tau) \circ \mathsf{k}_X"'] & (F_0 \circ h)^\sharp(t') \circ \mathsf{k}_X
\end{tikzcd} \tag{9}$$

But the cartesian closed structure of $\mathscr{C}$ is strict and the definition of the pseudofunctor $(F_0 \circ h)^\sharp$ only employs the canonical 2-cells of the cc-bicategory structure. Hence, arguing by induction on the definition of the extension cc-pseudofunctor defined in Lemma 5.5, one sees that $(F_0 \circ h)^\sharp(\tau)$ is the identity for every 2-cell $\tau$. It follows that (9) degenerates to the following:

$$\begin{tikzcd}
\mathsf{k}_A \circ (F \circ h^\sharp)(t) \arrow[r, "\mathsf{k}_A \circ (F \circ h^\sharp)(\tau)"] \arrow[d, "\overline{\mathsf{k}}_t"'] & \mathsf{k}_A \circ (F \circ h^\sharp)(t') \arrow[d, "\overline{\mathsf{k}}_{t'}"] \\
(F_0 \circ h)^\sharp(t) \circ \mathsf{k}_X \arrow[r, equal] & (F_0 \circ h)^\sharp(t') \circ \mathsf{k}_X
\end{tikzcd} \tag{10}$$

Now, since $(\mathsf{k}, \overline{\mathsf{k}})$ is an equivalence, every component $\mathsf{k}_X$ has a pseudoinverse. Let us denote this by $\mathsf{k}_X^\star$. From (10), one sees that the following commutes:

$$\begin{tikzcd}
(F \circ h^\sharp)(t) \arrow[r, "(F \circ h^\sharp)(\tau)"] \arrow[d, "\cong"'] & (F \circ h^\sharp)(t') \arrow[d, "\cong"] \\
(\mathsf{k}_A^\star \circ \mathsf{k}_A) \circ (F \circ h^\sharp)(t) \arrow[r, "(\mathsf{k}_A^\star \circ \mathsf{k}_A) \circ (F \circ h^\sharp)(\tau)"'] \arrow[d, "\cong"'] & (\mathsf{k}_A^\star \circ \mathsf{k}_A) \circ (F \circ h^\sharp)(t) \arrow[d, "\cong"] \\
\mathsf{k}_A^\star \circ \big(\mathsf{k}_A \circ (F \circ h^\sharp)(t)\big) \arrow[r, "\mathsf{k}_A^\star \circ \big(\mathsf{k}_A \circ (F \circ h^\sharp)(\tau)\big)"'] \arrow[d, "\mathsf{k}_A^\star \circ \overline{\mathsf{k}}_t"'] & \mathsf{k}_A^\star \circ \big(\mathsf{k}_A \circ (F \circ h^\sharp)(t')\big) \arrow[d, "\mathsf{k}_A^\star \circ \overline{\mathsf{k}}_{t'}"] \\
\mathsf{k}_A^\star \circ \Big((F_0 \circ h)^\sharp(t) \circ \mathsf{k}_X\Big) \arrow[r, equal] & \mathsf{k}_A^\star \circ \Big((F_0 \circ h)^\sharp(t') \circ \mathsf{k}_X\Big)
\end{tikzcd}$$

Hence, $(F \circ h^\sharp)\tau$ is completely determined by a composite of 2-cells, none of which depend on $\tau$.

**Proposition 5.10.** Let $(\mathscr{X}, \Pi_n(-), \Rightarrow)$ be a cc-bicategory , $(\mathscr{C}, \Pi_n(-), \Rightarrow)$ be a 2-cc 2-category, and $(F, \mathrm{q}^\times, \mathrm{q}^\Rightarrow) : (\mathscr{X}, \Pi_n(-), \Rightarrow) \to (\mathscr{C}, \Pi_n(-), \Rightarrow)$ be any cc-pseudofunctor. Then if $h : \mathrm{G} \to \mathscr{X}$ is a graph homomorphism and $\tau : t \Rightarrow t'$ is any 2-cell in $\mathscr{F}[\mathrm{G}]$, the 2-cell $(F \circ h^\sharp)(\tau)$ in $\mathscr{C}$ is completely determined by $t$ and $t'$. Hence, for any parallel pair of 2-cells $\tau, \sigma : t \Rightarrow t'$ in $\mathscr{F}[\mathrm{G}]$, one has the equality $(F \circ h^\sharp)(\tau) = (F \circ h^\sharp)(\sigma)$.

Together with Proposition 4.1, one obtains Mac Lane-style coherence.

**Theorem 5.11.** For any graph G and any pair of parallel 2-cells $\tau, \sigma : t \Rightarrow t'$ in $\mathscr{F}[\mathrm{G}]$, the equality $\tau \equiv \sigma$ holds.

*Proof.* Let $h := \iota : \mathrm{G} \hookrightarrow \mathscr{F}[\mathrm{G}]$ be the inclusion graph homomorphism and $F$ be the biequivalence between a cc-bicategory and a 2-cc 2-category arising from Proposition 4.1. By Lemma 5.5 we have $\iota^{\sharp} = \mathrm{id}_{\mathscr{F}[\mathrm{G}]}$, so $F \circ \iota^{\sharp}$ is a composite of biequivalences and hence a biequivalence itself. It follows that $F \circ \iota^{\sharp}$ is locally fully-faithful, so $\tau \equiv \sigma$ if and only if $(F \circ \iota^{\sharp})(\tau) = (F \circ \iota^{\sharp})(\sigma)$. The latter holds by Proposition 5.10, so $\tau \equiv \sigma$ as claimed. □

We finish by substantiating Principle 1.3. Suppose we want to show a pseudo structure **P** can be constructed in any cc-bicategory, and we know its strict counterpart **S** is constructible in every cartesian closed category. By Lemma 5.9 we can translate the sequence of equations showing the existence of **S** into a composite of 2-cells; these form the structural data for **P**. It remains to show the composites thus constructed satisfy the coherence laws on **P**. But this follows immediately from Theorem 5.11.

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## References

**Abbott, M. G.** 2003. *Categories of containers*. PhD thesis, University of Leicester.

**Bénabou, J.** 1967. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pp. 1–77, Berlin, Heidelberg. Springer Berlin Heidelberg.

**Bénabou, J.** 1985. Fibered categories and the foundations of naive category theory. *Journal of Symbolic Logic*, 50(1):10–37.

**Blackwell, R.**, **Kelly, G. M.**, **and Power, A. J.** 1989. Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59(1):1–41.

**Borceux, F.** 1994. *Bicategories and distributors*, volume 1 of *Encyclopedia of Mathematics and its Applications*, 281–324. Cambridge University Press.

**Carboni, A.**, **Kelly, G. M.**, **Walters, R. F. C.**, **and Wood, R. J.** 2008. Cartesian bicategories II. *Theory and Applications of Categories*, 19(6):93–124.

**Carboni, A. and Walters, R. F. C.** 1987. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49(1):11–32.

**Cattani, G. L.**, **Fiore, M.**, **and Winskel, G.** 1998. A theory of recursive domains with applications to concurrency. In *Proceedings of the Thirteenth Annual IEEE Symposium on Logic in Computer Science (LICS 1998)*, pp. 214–225. IEEE Computer Society Press.

**Čubrić, D.**, **Dybjer, P.**, **and Scott, P.** 1998. Normalization and the Yoneda embedding. *Mathematical Structures in Computer Science*.

**Day, B.** 1970. On closed categories of functors. In **Mac Lane, S.**, **Applegate, H.**, **Barr, M.**, **Day, B.**, **Dubuc, E.**, **Phreilambud**, **Pultr, A.**, **Street, R.**, **Tierney, M.**, **and Swierczkowski, S.**, editors, *Reports of the Midwest Category Seminar IV*, pp. 1–38, Berlin, Heidelberg. Springer Berlin Heidelberg.

**Day, B. and Street, R.** 1997. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129(1):99–157.

**Fiore, M.**, **Gambino, N.**, **Hyland, M.**, **and Winskel, G.** 2007. The cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society*, 77(1):203–220.

**Fiore, M. and Joyal, A.** 2015. Theory of para-toposes. Talk at the Category Theory 2015 Conference. Departamento de Matematica, Universidade de Aveiro (Portugal).

**Fiore, M. and Saville, P.** 2020. Coherence and normalisation-by-evaluation for bicategorical cartesian closed structure. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. ACM.

**Fiore, T.** 2006. *Pseudo Limits, Biadjoints, and Pseudo Algebras: Categorical Foundations of Conformal Field Theory*. Memoirs of the American Mathematical Society. AMS.

**Forest, S. and Mimram, S.** 2018. Coherence of Gray categories via rewriting. In *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

**Frey, J.** 2019. A language for closed cartesian bicategories. Category Theory 2019, University of Edinburgh, Edinburgh, UK.

**Gambino, N. and Joyal, A.** 2017. On operads, bimodules and analytic functors. *Memoirs of the American Mathematical Society*, 249(1184):153–192.

**Gordon, R.**, **Power, A. J.**, **and Street, R.** 1995. *Coherence for tricategories*. Memoirs of the American Mathematical Society.

**Gray, J. W.** 1974. *Formal Category Theory: Adjointness for 2-Categories*, volume 391 of *Lecture Notes in Mathematics*. Springer.

**Gurski, N.** 2006. *An Algebraic Theory of Tricategories*. University of Chicago, Department of Mathematics.

**Gurski, N.** 2013. *Coherence in Three-Dimensional Category Theory*. Cambridge University Press.

**Hilken, B.** 1996. Towards a proof theory of rewriting: the simply typed $2\lambda$-calculus. *Theoretical Computer Science*, 170(1):407–444.

**Hirschowitz, T.** 2013. Cartesian closed 2-categories and permutation equivalence in higher-order rewriting. *Logical Methods in Computer Science*, 9:1–22.

**Houston, R.** 2007. *Linear Logic without Units*. PhD thesis, University of Manchester.

**Joyal, A. and Street, R.** 1993. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78.

**Kelly, G. M.** 1982. *Basic Concepts of Enriched Category Theory*. Number 64 in London Mathematical Society Lecture Note Series. Cambridge University Press.

**Lack, S.** 2010. *A 2-Categories Companion*, pp. 105–191. Springer New York, New York, NY.

**Lack, S.**, **Walters, R. F. C.**, **and Wood, R. J.** 2010. Bicategories of spans as cartesian bicategories. *Theory and Applications of Categories*, 24(1):1–24.

**Lambek, J. and Scott, P. J.** 1986. *Introduction to Higher Order Categorical Logic*. Cambridge University Press, New York, NY, USA.

**Leinster, T.** 1998. Basic bicategories. Available at `https://arxiv.org/abs/math/9810017`.

**Leinster, T.** 2004. *Higher operads, higher categories*. Number 298 in London Mathematical Society Lecture Note Series. Cambridge University Press.

**Mac Lane, S.** 1963. Natural associativity and commutativity. *Rice University Studies*.

**Mac Lane, S.** 1998. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag New York, second edition.

**Mac Lane, S. and Paré, R.** 1985. Coherence for bicategories and indexed categories. *Journal of Pure and Applied Algebra*, 37:59 – 80.

**Makkai, M.** 1996. Avoiding the axiom of choice in general category theory. *Journal of Pure and Applied Algebra*, 108(2):109 – 173.

**Ouaknine, J.** 1997. A two-dimensional extension of Lambek's categorical proof theory. Master's thesis, McGill University.

**Paquet, H.** 2020. *Probabilistic concurrent game semantics*. PhD thesis, University of Cambridge.

**Power, A. J.** 1989a. Coherence for bicategories with finite bilimits I. In **Gray, J. W. and Scedrov, A.**, editors, *Categories in Computer Science and Logic: Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference Held June 14–20, 1987 with Support from the National Science Foundation*, volume 92, pp. 341–349. American Mathematical Society.

**Power, A. J.** 1989b. A general coherence result. *Journal of Pure and Applied Algebra*, 57(2):165–173.

**Power, A. J.** 1998. 2-categories. *BRICS Notes Series*.

**Saville, P.** 2020. *Cartesian closed bicategories: type theory and coherence*. PhD thesis, University of Cambridge.

**Stay, M.** 2016. Compact closed bicategories. *Theories and Applications of Categories*, 31(26):755–798.

**Street, R.** 1972. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149 –168.

**Street, R.** 1980. Fibrations in bicategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 21(2):111–160.

## Appendix A. The proof of Lemma 5.6

In the main body we construct adjoint equivalences $\mathsf{k}_X : FX \leftrightarrows h^\sharp X : \mathsf{k}^\star_X$. Denote the unit and counit of these equivalences by $\mathsf{v}_X : \mathrm{Id}_{FX} \Rightarrow \mathsf{k}^\star_X \circ \mathsf{k}_X$ and $\mathsf{w}_X : \mathsf{k}_X \circ \mathsf{k}^\star_X \Rightarrow \mathrm{Id}_{h^\sharp X}$, respectively, and assume without loss of generality that they satisfy the two triangle laws. Moreover, for any cc-pseudofunctor $(F, \mathrm{q}^\times, \mathrm{q}^\Rightarrow)$, write

$$\begin{aligned}
\mathsf{u}^\times_{A_\bullet} &: \mathrm{Id}_{(F\Pi_i A_i)} \Rightarrow \mathrm{q}^\times_{A_\bullet} \circ \langle F\pi_1, \dots, F\pi_n \rangle \\
\mathsf{c}^\times_{A_\bullet} &: \langle F\pi_1, \dots, F\pi_n \rangle \circ \mathrm{q}^\times_{A_\bullet} \Rightarrow \mathrm{Id}_{(\prod_i FA_i)} \\
\mathsf{u}^\Rightarrow_{A,B} &: \mathrm{Id}_{F(A \Rightarrow B)} \Rightarrow \mathrm{q}^\Rightarrow_{A,B} \circ \mathrm{s}_{A,B} \\
\mathsf{c}^\Rightarrow_{A,B} &: \mathrm{s}_{A,B} \circ \mathrm{q}^\Rightarrow_{A,B} \Rightarrow \mathrm{Id}_{(FA \Rightarrow FB)}
\end{aligned}$$

for the 2-cells witnessing the adjoint equivalences (6) and (7).

It remains to construct the witnessing 2-cells $\overline{\mathsf{k}}_t : \mathsf{k}_B \circ Ft \Rightarrow h^\sharp(t) \circ \mathsf{k}_A$. To do this we require names for some of the canonical 2-cells in a cc-bicategory. These 2-cells provide witnesses for standard equations in a cartesian closed category.

**Construction A.1.** Let $(\mathscr{B}, \Pi_n(-))$ be an fp-bicategory. We define the following natural families of invertible 2-cells:

(1) For $(h_i : Y \to A_i)_{i=1,\dots,n}$ and $g : X \to Y$, define

$$\mathsf{post}(h_\bullet; g) : \langle h_1, \dots, h_n \rangle \circ g \Rightarrow \langle h_1 \circ g, \dots, h_n \circ g \rangle$$

to be $\mathsf{p}^\dagger(\alpha_1, \dots, \alpha_n)$, where each $\alpha_k$ is defined to be

$$\pi_k \circ (\langle h_1, \dots, h_n \rangle \circ g) \overset{\cong}{\Rightarrow} (\pi_k \circ \langle h_1, \dots, h_n \rangle) \circ g \xRightarrow{\varpi^{(k)} \circ g} h_k \circ g$$

(2) For $(h_i : A_i \to B_i)_{i=1,\dots,n}$ and $(g_i : X \to A_i)_{i=1,\dots,n}$, define

$$\mathsf{fuse}(h_\bullet; g_\bullet) : (\textstyle\prod_{i=1}^n h_i) \circ \langle g_1, \dots, g_n \rangle \Rightarrow \langle h_1 \circ g_1, \dots, h_n \circ g_n \rangle$$

to be $\mathsf{p}^\dagger(\beta_1, \dots, \beta_n)$, where each $\beta_k$ is defined by the diagram

$$\begin{tikzcd}
\pi_k \circ ((\prod_{i=1}^n h_i) \circ \langle g_1, \dots, g_n \rangle) \arrow[rr, "\beta_k"] \arrow[d, "\cong"'] & & h_k \circ g_k \\
(\pi_k \circ \prod_{i=1}^n h_i) \circ \langle g_1, \dots, g_n \rangle \arrow[r, "\varpi^{(k)} \circ \langle g_1, \dots, g_n \rangle"'] & (h_k \circ \pi_k) \circ \langle g_1, \dots, g_n \rangle \arrow[r, "\cong"'] & h_k \circ (\pi_k \circ \langle g_1, \dots, g_n \rangle) \arrow[u, "h_k \circ \varpi^{(k)}"']
\end{tikzcd}$$

(3) For $(h_i : A_i \to B_i)_{i=1,\dots,n}$ and $(g_j : X_j \to A_j)_{j=1,\dots,n}$ define $\Phi_{h_\bullet, g_\bullet} : (\prod_{i=1}^n h_i) \circ (\prod_{i=1}^n g_i) \Rightarrow \prod_{i=1}^n (h_i g_i)$ to be the composite $\langle \mathsf{a}^{-1}_{h_1, g_1, \pi_1}, \dots, \mathsf{a}^{-1}_{h_n, g_n, \pi_n} \rangle \bullet \mathsf{fuse}(h_\bullet; g_1 \circ \pi_1, \dots, g_n \circ \pi_n)$. This 2-cell witnesses the pseudofunctoriality of $\prod_n (-, \dots, =)$.

For cartesian closed structure we have one further canonical transformation.

**Construction A.2.** Let $(\mathscr{B}, \Pi_n(-), \Rightarrow)$ be a cc-bicategory. For $g : X \to Y$ and $f : Y \times A \to B$ we define $\mathsf{push}(f, g) : \lambda(f) \circ g \Rightarrow \lambda\big(f \circ (g \times A)\big)$ as $\mathsf{e}^\dagger(\tau)$, for $\tau$ the composite

$$\begin{tikzcd}
\mathrm{eval}_{A,B} \circ ((\lambda f \circ g) \times A) \arrow[r, "\tau"] \arrow[d, "\mathrm{eval} \circ (\Phi_{f,g})^{-1}"'] & f \circ (g \times A) \\
\mathrm{eval}_{A,B} \circ ((\lambda f \times A) \circ (g \times A)) \arrow[r, "\cong"'] & (\mathrm{eval}_{A,B} \circ (\lambda f \times A)) \circ (g \times A) \arrow[u, "\varepsilon_f \circ (g \times A)"']
\end{tikzcd}$$

where $\Phi_{f,g} : (f \times A) \circ (g \times A) \Rightarrow (fg \times A)$ is as in Construction A.1(3).

Finally, in a cartesian category it is often useful to 'unpack' an $n$-ary tupling from inside a cartesian functor in the following manner:

$$\begin{aligned}
\langle F\pi_1, , \dots, F\pi_n \rangle \circ F \langle f_1, , \dots, f_n \rangle &= \langle F(\pi_\bullet) \circ F \langle f_1, , \dots, f_n \rangle \rangle \\
&= \langle F(\pi_\bullet \circ \langle f_1, , \dots, f_n \rangle) \rangle \\
&= \langle Ff_1, , \dots, Ff_n \rangle
\end{aligned}$$

In an fp-bicategory, one obtains a natural family of 2-cells we call $\mathsf{unpack}$.

**Construction A.3.** For any fp-pseudofunctor $(F, \mathrm{q}^{\times}) : (\mathscr{B}, \Pi_n(-)) \to (\mathscr{C}, \Pi_n(-))$ the invertible 2-cell $\mathsf{unpack}_{f_\bullet} : \langle F\pi_1, , \dots , F\pi_n\rangle \circ F\langle f_1, , \dots , f_n\rangle \Rightarrow \langle Ff_1, , \dots , Ff_n\rangle : FX \to \prod_{i=1}^{n} FB_i$ is defined to be $\mathsf{p}^{\dagger}(\tau_1, , \dots , \tau_n)$, where $\tau_k$ $(k = 1, , \dots , n)$ is given by the following diagram:

$$\begin{tikzcd}
\pi_k \circ (\langle F\pi_1, , \dots , F\pi_n\rangle \circ F\langle f_1, , \dots , f_n\rangle) \arrow[r, "\tau_k"] \arrow[d, "\cong"'] & Ff_k \\
(\pi_k \circ \langle F\pi_1, , \dots , F\pi_n\rangle) \circ F\langle f_1, , \dots , f_n\rangle \arrow[d, "\varpi^{(k)} \circ F\langle f_1,, \dots, f_n\rangle"'] & \\
F(\pi_k) \circ F\langle f_1, , \dots , f_n\rangle \arrow[r, "\phi^F_{\pi_k, \langle f_\bullet\rangle}"'] & F\,(\pi_i \circ \langle f_1, , \dots , f_n\rangle) \arrow[uu, "F\varpi^{(k)}"']
\end{tikzcd}$$

We now construct $\overline{\mathsf{k}}_t$ by induction on $t$.

**Construction A.4** (Defining $\mathsf{nat}_t$)**.**

$\boxed{\text{Composition and identities.}}$ For identities, the definition is forced upon us by the unit law of a pseudonatural transformation. We define

$$\overline{\mathsf{k}}_{\mathrm{Id}_A} := \mathsf{k}_A \circ F(\mathrm{Id}_A) \xRightarrow{\mathsf{k}_A \circ (\psi^F_A)^{-1}} \mathsf{k}_A \circ \mathrm{Id}_{F(A)} \overset{\cong}{\Rightarrow} \mathrm{Id}_{h^\sharp(A)} \circ \mathsf{k}_A$$

The definition for maps of the form $t \circ u : Z \to B$ is forced by the composition law of a pseudonatural transformation. Using that $h^\sharp$ is a strict pseudofunctor, we define

$$\begin{tikzcd}
\mathsf{k}_B \circ F(t \circ u) \arrow[rr, "\overline{\mathsf{k}}_{t \circ u}"] \arrow[d, "\mathsf{k}_B \circ (\phi^F_{t,u})^{-1}"'] & & \big(h^\sharp(t) \circ h^\sharp(u)\big) \circ \mathsf{k}_Z \\
\mathsf{k}_B \circ (F(t) \circ F(u)) \arrow[d, "\cong"'] & & h^\sharp(t) \circ \big(h^\sharp(u) \circ \mathsf{k}_Z\big) \arrow[u, "\cong"'] \\
(\mathsf{k}_B \circ Ft) \circ Fu \arrow[r, "\overline{\mathsf{k}}_t \circ F(u)"'] & \big(h^\sharp(t) \circ \mathsf{k}_A\big) \circ Fu \arrow[r, "\cong"'] & h^\sharp(t) \circ (\mathsf{k}_A \circ Fu) \arrow[u, "h^\sharp(t) \circ \overline{\mathsf{k}}_u"']
\end{tikzcd}$$

$\boxed{\text{Constants.}}$ Let $c \in \mathrm{G}(A, B)$. Then we must have $F(c) = h(c) = h^\sharp(c)$, so $\mathsf{k}_c$ is the composite of structural isomorphisms

$$\mathsf{k}_B \circ Fc = \mathrm{Id}_{hB} \circ Fc \xrightarrow{\cong} Fc = h^\sharp(c) \xrightarrow{\cong} h^\sharp(c) \circ \mathrm{Id}_{hA} = h^\sharp(c) \circ \mathsf{k}_A$$

Product structure. We define $\overline{\mathsf{k}}_{\pi_k}$ and $\overline{\mathsf{k}}_{\langle t_1, \ldots, t_n\rangle}$ by the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathsf{k}_{A_k} \circ F\pi_k \arrow[rr, "\overline{\mathsf{k}}_{\pi_k}"] \arrow[d, "\mathsf{k}_{A_k} \circ \varpi^{(-k)}"'] & & h^\sharp(\pi_k) \circ \mathsf{k}_{(\prod_n A_\bullet)} \\
\mathsf{k}_{A_k} \circ (\pi_k \circ \langle F\pi_\bullet \rangle) \arrow[dr, bend right, "\cong"'] & & (\pi_k \circ \prod_{i=1}^n \mathsf{k}_{A_i}) \circ \langle F\pi_\bullet \rangle \arrow[u, "\cong"'] \\
& \big(\mathsf{k}_{A_k} \circ \pi_k\big) \circ \langle F\pi_\bullet \rangle \arrow[ur, bend right, "\varpi^{(-k)} \circ \langle F\pi_\bullet \rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\prod_{i=1}^m \mathsf{k}_{A_i} \circ \langle F\pi_\bullet \rangle) \circ F(\langle t_1, \ \ldots \ , t_m \rangle) \arrow[rr, "\overline{\mathsf{k}}_{\langle t_1, \ldots, t_m \rangle}"] \arrow[d, "\cong"'] & & h^\sharp(\langle t_1, \ \ldots \ , t_m \rangle) \circ \mathsf{k}_X \arrow[d, equal] \\
(\prod_{i=1}^m \mathsf{k}_{A_i}) \circ (\langle F\pi_\bullet \rangle \circ F(\langle t_1, \ \ldots \ , t_m \rangle)) \arrow[d, "(\prod_i \mathsf{k}_{A_i}) \circ \mathsf{unpack}"'] & & \langle h^\sharp(t_\bullet) \rangle \circ \mathsf{k}_X \\
(\prod_{i=1}^m \mathsf{k}_{A_i}) \circ \langle F(t_\bullet) \rangle \arrow[r, "\mathsf{fuse}"'] & \langle \mathsf{k}_{A_\bullet} \circ F(t_\bullet) \rangle \arrow[r, "\langle \overline{\mathsf{k}}_{t_1}, \ldots, \overline{\mathsf{k}}_{t_m} \rangle"'] & \langle h^\sharp(t_\bullet) \circ \mathsf{k}_X \rangle \arrow[u, "\mathsf{post}^{-1}"']
\end{tikzcd}$$

The cases for the evaluation map and currying require more work, but are similar in spirit.

Evaluation map. We are required to give an invertible 2-cell filling the diagram

$$\begin{tikzcd}
F\big((A \Rightarrow B) \times A\big) \arrow[rr, "F\mathsf{eval}_{A,B}"] \arrow[d, "\langle F\pi_1, F\pi_2 \rangle"'] \arrow[dd, bend right=90, "(\mathsf{k}_{(A\Rightarrow B)} \times \mathsf{k}_A) \circ \langle F\pi_1, F\pi_2 \rangle"'] & & FB \arrow[dd, "\mathsf{k}_B"] \\
F(A \Rightarrow B) \times F(A) \arrow[d, "\mathsf{k}_{(A\Rightarrow B)} \times \mathsf{k}_A"'] & \overset{\overline{\mathsf{k}}_{\mathrm{eval}}}{\Leftarrow} & \\
h^\sharp(A \Rightarrow B) \times h^\sharp A \arrow[r, equal] & (h^\sharp A \Rightarrow h^\sharp B) \times h^\sharp A \arrow[r, "\mathrm{eval}"'] & h^\sharp B
\end{tikzcd}$$

To this end, first define an invertible 2-cell $\delta_{A,B}$ applying the counit $\varepsilon$ as far as possible:

$$\begin{tikzcd}
\mathrm{eval}_{h^\sharp A, h^\sharp B} \circ \big(\mathsf{k}_{(A\Rightarrow B)} \times \mathsf{k}_A\big) \arrow[d, equal] \arrow[dddddddr, bend left, "\delta_{A,B}"] & \\
\mathrm{eval}_{h^\sharp A, h^\sharp B} \circ \Big((\mathsf{k}_A^\star \Rightarrow \mathsf{k}_B) \circ \mathrm{s}^F_{A,B} \times \mathsf{k}_A\Big) \arrow[d, "\cong"'] & \\
\Big(\mathrm{eval}_{h^\sharp A, h^\sharp B} \circ \big((\mathsf{k}_A^\star \Rightarrow \mathsf{k}_B) \times h^\sharp A\big)\Big) \circ (\mathrm{s}^F_{A,B} \times \mathsf{k}_A) \arrow[d, "\varepsilon_{(\mathsf{k}\circ \mathrm{eval} \circ (\mathrm{Id}\times \mathsf{k}^\star))} \circ (\mathrm{s}^F_{A,B} \times \mathsf{k}_A)"'] & \\
\big((\mathsf{k}_B \circ \mathrm{eval}_{FA,FB}) \circ \big(\mathrm{Id}_{(FA\Rightarrow FB)} \times \mathsf{k}_A^\star\big)\big) \circ (\mathrm{s}^F_{A,B} \times \mathsf{k}_A) \arrow[d, "\cong"'] & \\
\Big(\mathsf{k}_B \circ \Big(\mathrm{eval}_{FA,FB} \circ \big(\mathrm{s}^F_{A,B} \times FA\big)\Big)\Big) \circ \big(\mathrm{Id}_{(FA\Rightarrow FB)} \times \mathsf{k}_A^\star \mathsf{k}_A\big) \arrow[d, "\mathsf{k} \circ \varepsilon_{(F(\mathrm{eval}) \circ \mathrm{q}^\times)} \circ (\mathrm{Id} \times \mathsf{k}^\star \mathsf{k})"'] & \\
\Big(\mathsf{k}_B \circ \Big(F(\mathrm{eval}_{A,B}) \circ \mathrm{q}^\times_{A\Rightarrow B,A}\Big)\Big) \circ \big(\mathrm{Id}_{(FA\Rightarrow FB)} \times \mathsf{k}_A^\star \mathsf{k}_A\big) \arrow[d, "\mathsf{k} \circ F\mathrm{eval} \circ \mathrm{q}^\times \circ (\mathrm{Id} \times \mathsf{v}_A^{-1})"'] & \\
\Big(\mathsf{k}_B \circ \Big(F(\mathrm{eval}_{A,B}) \circ \mathrm{q}^\times_{A\Rightarrow B,A}\Big)\Big) \circ \big(\mathrm{Id}_{(FA\Rightarrow FB)} \times \mathrm{Id}_{FA}\big) \arrow[r, "\cong"'] & (\mathsf{k}_B \circ F(\mathsf{eval}_{A,B})) \circ \mathrm{q}^\times_{A\Rightarrow B,A}
\end{tikzcd}$$

Then define $\overline{\mathsf{k}}_{\mathsf{eval}}$ to be the composite

$$\begin{tikzcd}
\mathsf{k}_B \circ F(\mathrm{eval}_{A,B}) \arrow[r, "\overline{\mathsf{k}}_{\mathsf{eval}}"] \arrow[d, "\cong"'] & \mathrm{eval}_{h^\sharp A, h^\sharp B} \circ \big(\big(\mathsf{k}_{(A\Rightarrow B)} \times \mathsf{k}_A\big) \circ \langle F\pi_1, F\pi_2\rangle\big) \\
(\mathsf{k}_B \circ F(\mathrm{eval}_{A,B})) \circ \mathrm{Id}_{F((A\Rightarrow B)\times A)} \arrow[d, "(\mathsf{k}_B \circ F(\mathrm{eval}_{A,B})) \circ \mathrm{u}^\times_{A\Rightarrow B,A}"'] & \\
(\mathsf{k}_B \circ F(\mathrm{eval}_{A,B})) \circ \Big(\mathrm{q}^\times_{A\Rightarrow B,A} \circ \langle F\pi_1, F\pi_2\rangle\Big) \arrow[d, "\cong"'] & \\
\Big(\mathsf{k}_B \circ \big(F(\mathrm{eval}_{A,B}) \circ \mathrm{q}^\times_{A\Rightarrow B,A}\big)\Big) \circ \langle F\pi_1, F\pi_2\rangle \arrow[dr, bend right, "\delta^{-1}_{A,B} \circ \langle F\pi_1, F\pi_2\rangle"'] & \\
& \Big(\mathrm{eval}_{h^\sharp A, h^\sharp B} \circ \big(\mathsf{k}_{(A\Rightarrow B)} \times \mathsf{k}_A\big)\Big) \circ \langle F\pi_1, F\pi_2\rangle \arrow[uuuu, "\cong"']
\end{tikzcd}$$

$\boxed{\text{Currying.}}$ Suppose $t : Z \times A \to B$. By induction we are given $\overline{\mathsf{k}}_t$ filling

$$\begin{tikzcd}
F(Z \times A) \arrow[rr, "Ft"] \arrow[d, "\langle F\pi_1, F\pi_2 \rangle"'] \arrow[dd, bend right=90, "(\mathsf{k}_Z \times \mathsf{k}_A) \circ \langle F\pi_1, F\pi_2 \rangle"'] & & FB \arrow[dd, "\mathsf{k}_B"] \\
FZ \times FA \arrow[d, "\mathsf{k}_Z \times \mathsf{k}_A"'] & \overset{\overline{\mathsf{k}}_t}{\Leftarrow} & \\
h^\sharp(Z) \times h^\sharp(A) \arrow[r, equal] & h^\sharp(Z \times A) \arrow[r, "h^\sharp t"'] & h^\sharp B
\end{tikzcd}$$

and we are required to fill the diagram

$$\begin{tikzcd}
FZ \arrow[rr, "F(\lambda t)"] \arrow[dd, "\mathsf{k}_Z"'] & & F(A \Rightarrow B) \arrow[d, "\mathrm{s}^F_{A,B}"] \arrow[dd, bend left=90, "(\mathsf{k}^\star_A \Rightarrow \mathsf{k}_B) \circ \mathrm{s}^F_{A,B}"] \\
 & \overset{\overline{\mathsf{k}}_{\lambda t}}{\Leftarrow} & \big(FA \Rightarrow FB\big) \arrow[d, "(\mathsf{k}^\star_A \Rightarrow \mathsf{k}_B)"] \\
h^\sharp Z \arrow[rr, "h^\sharp(\lambda t)"'] & & (h^\sharp A \Rightarrow h^\sharp B)
\end{tikzcd}$$

Our strategy is the following. Writing $cl$ for the clockwise composite around the preceding diagram, we define a 2-cell

$$\zeta_{A,B} : \mathrm{eval}_{h^\sharp A, h^\sharp B} \circ (cl \times h^\sharp A) \Rightarrow h^\sharp(t) \circ (\mathsf{k}_Z \times h^\sharp A)$$

so that $\mathsf{e}^\dagger(\zeta_{A,B}) : cl \Rightarrow \lambda\left(h^\sharp(t) \circ (\mathsf{k}_Z \times h^\sharp A)\right)$; by Remark 3.8 and the induction hypothesis this is invertible. We then define $\overline{\mathsf{k}}_{\lambda t}$ as the composite

$$cl \xRightarrow{\mathsf{e}^\dagger(\zeta_{A,B})} \lambda\left(h^\sharp(t) \circ (\mathsf{k}_Z \times h^\sharp A)\right) \xRightarrow{\mathsf{push}^{-1}} \lambda\left(h^\sharp t\right) \circ \mathsf{k}_Z = h^\sharp(\lambda t) \circ \mathsf{k}_Z$$

The 2-cell $\zeta_{A,B}$ is defined in stages. First we set $\upsilon_{A,B}$ to be the following composite, where we write

$\cong$ for composites of $\Phi$ and structural isomorphisms:

$$\begin{tikzcd}
\mathrm{eval}_{h^\sharp A,h^\sharp B} \circ (cl \times h^\sharp A) \arrow[d, "\cong"'] \\
\Big(\mathrm{eval}_{h^\sharp A,h^\sharp B} \circ ((\mathsf{k}^\star_A \Rightarrow \mathsf{k}_B) \times h^\sharp A)\Big) \circ \Big(\Big(\mathrm{s}^F_{A,B} \circ F(\lambda t)\Big) \times h^\sharp A\Big) \arrow[d, "\varepsilon_{\mathsf{k}\circ\mathrm{eval}\circ(\mathrm{Id}\times\mathsf{k}^\star)}\circ(\mathrm{s}^F_{A,B}F(\lambda t)\times h^\sharp A)"'] \\
((\mathsf{k}_B \circ \mathrm{eval}_{FA,FB}) \circ (\mathrm{Id}_{(FA\Rightarrow FB)} \times \mathsf{k}^\star_A)) \circ \Big(\Big(\mathrm{s}^F_{A,B} \circ F(\lambda t)\Big) \times h^\sharp A\Big) \arrow[d, "\cong"'] \\
\Big(\mathsf{k}_B \circ \Big(\mathrm{eval}_{FA,FB} \circ (\mathrm{s}^F_{A,B} \times F(A))\Big)\Big) \circ (F(\lambda t) \times \mathsf{k}^\star_A) \arrow[d, "\mathsf{k}_B\circ\varepsilon_{(F(\mathrm{eval})\circ\mathrm{q}^\times)}\circ(F(\lambda t)\times\mathsf{k}^\star)"'] \\
\Big(\mathsf{k}_B \circ \Big(F(\mathrm{eval}_{A,B}) \circ \mathrm{q}^\times_{A\Rightarrow B,A}\Big)\Big) \circ (F(\lambda t) \times \mathsf{k}^\star_A)
\end{tikzcd}$$

Next we define $\theta_{A,B}$ to be the composite

$$\begin{tikzcd}
F(\mathrm{eval}_{A,B}) \circ \Big(\mathrm{q}^\times_{A\Rightarrow B,A} \circ (F\lambda t \times FA)\Big) \arrow[r, "\theta_{A,B}"] \arrow[d, "F(\mathrm{eval})\circ\mathrm{q}^\times\circ(F(\lambda t)\times\psi^F_A)"'] & Ft \circ \mathrm{q}^\times_{Z,A} \\
F(\mathrm{eval}_{A,B}) \circ \Big(\mathrm{q}^\times_{A\Rightarrow B,A} \circ \Big(\lambda t \times F\mathrm{Id}_A\Big)\Big) \arrow[d, "F(\mathrm{eval})\circ\mathsf{nat}"'] & F(\mathrm{eval}_{A,B} \circ (\lambda t \times A)) \circ \mathrm{q}^\times_{Z,A} \arrow[u, "F(\varepsilon_t)\circ\mathrm{q}^\times"'] \\
F(\mathrm{eval}_{A,B}) \circ \Big(F(\lambda t \times A) \circ \mathrm{q}^\times_{Z,A}\Big) \arrow[r, "\cong"'] & (F(\mathrm{eval}_{A,B}) \circ F(\lambda t \times A)) \circ \mathrm{q}^\times_{Z,A} \arrow[u, "\phi^F_{(\mathrm{eval},\lambda t\times A)}\circ\mathrm{q}^\times"']
\end{tikzcd}$$

We can now define $\zeta_{A,B}$ as follows:

$$\begin{tikzcd}
\mathrm{eval}_{h^\sharp A, h^\sharp B} \circ (cl \times h^\sharp A) \arrow[r, "\zeta_{A,B}"] \arrow[d, "\upsilon_{A,B}"'] & h^\sharp(t) \circ \big(\mathsf{k}_Z \times A\big) \\
\Big(\mathsf{k}_B \circ \Big(F\mathrm{eval}_{A,B} \circ \mathrm{q}^{\times}_{A \Rightarrow B, A}\Big)\Big) \circ (F(\lambda t) \times \mathsf{k}^{\star}_{A}) \arrow[d, "\cong"'] & \\
\Big(\mathsf{k}_B \circ \Big(F\mathrm{eval}_{A,B} \circ \Big(\mathrm{q}^{\times}_{A \Rightarrow B, A} \circ (F(\lambda t) \times FA)\Big)\Big)\Big) \circ (FZ \times \mathsf{k}^{\star}_{A}) \arrow[d, "\mathsf{k}_B \circ \theta_{A,B} \circ (FZ \times \mathsf{k}^{\star}_{A})"'] & \\
\Big(\mathsf{k}_B \circ \Big(Ft \circ \mathrm{q}^{\times}_{Z,A}\Big)\Big) \circ (FZ \times \mathsf{k}^{\star}_{A}) \arrow[d, "\cong"'] & \\
(\mathsf{k}_B \circ Ft) \circ \Big(\mathrm{q}^{\times}_{Z,A} \circ (FZ \times \mathsf{k}^{\star}_{A})\Big) \arrow[d, "\overline{\mathsf{k}}_t \circ \mathrm{q}^{\times} \circ (FZ \times \mathsf{k}^{\star}_{A})"'] & \\
\Big(h^\sharp(t) \circ ((\mathsf{k}_Z \times \mathsf{k}_A) \circ \langle F\pi_1, F\pi_2 \rangle)\Big) \circ \Big(\mathrm{q}^{\times}_{Z,A} \circ (FZ \times \mathsf{k}^{\star}_{A})\Big) \arrow[d, "\cong"'] & \\
\Big(\Big(h^\sharp(t) \circ (\mathsf{k}_Z \times \mathsf{k}_A)\Big) \circ \Big(\langle F\pi_1, F\pi_2 \rangle \circ \mathrm{q}^{\times}_{Z,A}\Big)\Big) \circ (FZ \times \mathsf{k}^{\star}_{A}) \arrow[d, "h^\sharp(t) \circ (\mathsf{k}_Z \times \mathsf{k}_A) \circ \mathsf{c}^{\times}_{Z,A} \circ (FZ \times \mathsf{k}^{\star}_{A})"'] & \\
h^\sharp(t) \circ (\mathsf{k}_Z \times \mathsf{k}_A) \circ \mathrm{Id}_{FZ \times FA} \circ \big(FZ \times \mathsf{k}^{\star}_{A}\big) \arrow[dr, "\cong"'] & \\
& h^\sharp(t) \circ (\mathsf{k}_Z \times \mathsf{k}_A \mathsf{k}^{\star}_{A}) \arrow[uuuuuuuu, "h^\sharp(t) \circ (\mathsf{k}_Z \times \mathsf{w}_A)"']
\end{tikzcd}$$

This completes the definition of $\overline{\mathsf{k}}_{\lambda t}$.

To show that $(\mathsf{k}, \overline{\mathsf{k}})$ is indeed a pseudonatural transformation, we need to check the naturality condition and the two axioms. Naturality is a straightforward check for each case outlined above. The two axioms—corresponding to the identity and composition cases—hold by construction. This completes the proof of Lemma 5.6.