

INFINITE LOOP SPACES AND POSITIVE SCALAR CURVATURE IN THE PRESENCE OF A FUNDAMENTAL GROUP

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ABSTRACT. This is a continuation of our previous work with Botvinnik on the nontriviality of the secondary index invariant on spaces of metrics of positive scalar curvature, in which we take the fundamental group of the manifolds into account. We show that the secondary index invariant associated to the vanishing of the Rosenberg index can be highly nontrivial, for positive scalar curvature Spin manifolds with torsionfree fundamental groups which satisfy the Baum–Connes conjecture. This gives the first example of the non-triviality of the group C^* -algebra valued secondary index invariant on higher homotopy groups. As an application, we produce a compact Spin 6-manifold such that its space of positive scalar curvature metrics has each rational homotopy group infinite dimensional.

At a more technical level, we introduce the notion of “stable metrics” and prove a basic existence theorem for them, which generalises the Gromov–Lawson surgery technique, and we also give a method for rounding corners of manifold with positive scalar curvature metrics.

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1. INTRODUCTION AND STATEMENT OF RESULTS

It is a simple observation that if M is a closed manifold, then the product $M \times S^2$ admits a metric of positive scalar curvature, so the fundamental group by itself is not an obstruction against the existence of a metric of positive scalar curvature (at least in the high-dimensional regime). Nevertheless, the fundamental group has played an important role in the theory of positive scalar curvature since the work of Gromov and Lawson [24, 25], the point being that there are obstructions to the existence of psc metrics on M which lie in abelian groups depending on $\pi_1(M)$. The most important is the *Rosenberg index* [43] which is defined for a closed d -dimensional spin manifold M . It is defined in terms of the real K -theory of the reduced group C^* -algebra $\mathbf{C}_r^*(\pi_1(M))$, as an element $\alpha_r(M) \in KO_d(\mathbf{C}_r^*(\pi_1(M)))$.

If M has a psc metric then the Rosenberg index vanishes, but in an appropriate sense the positivity of the scalar curvature also gives a *reason* for it to vanish: therefore, following the recipe described in [6, §3.3.2], if $\mathcal{R}^+(M)$ denotes the space of psc metrics on M then there is an associated secondary index invariant

$$\text{inndiff}^{\pi_1(M)} : \mathcal{R}^+(M) \times \mathcal{R}^+(M) \longrightarrow \Omega^{\infty+d+1}\mathbf{KO}(\mathbf{C}_r^*(\pi_1(M)))$$

to an appropriate infinite loop space of the real K -theory spectrum of $\mathbf{C}_r^*(\pi_1(M))$. More generally, for a spin manifold W with collared boundary, one may fix a metric $g \in \mathcal{R}^+(\partial W)$ and consider the space $\mathcal{R}^+(W)_g$ of psc metrics on W which are of the form $g + dt^2$ on the collar. Then, given a discrete group G and a map $f : W \rightarrow BG$, there is a map

$$\text{inndiff}^G : \mathcal{R}^+(W)_g \times \mathcal{R}^+(W)_g \longrightarrow \Omega^{\infty+d+1}\mathbf{KO}(\mathbf{C}_r^*(G))$$

(which depends on the reference map f). Usually, we fix a basepoint $h_0 \in \mathcal{R}^+(W)_g$ and consider the map $\text{inndiff}_{h_0}^G(-) = \text{inndiff}^G(h_0, -)$. On homotopy groups, $\text{inndiff}_{h_0}^G$ induces a map

$$\text{inndiff}_{h_0}^G : \pi_k(\mathcal{R}^+(W), h_0) \longrightarrow \pi_k(\Omega^{\infty+d+1}\mathbf{KO}(\mathbf{C}_r^*(G))) = KO_{d+k+1}(\mathbf{C}_r^*(G))$$

to the KO -theory groups of $\mathbf{C}_r^*(G)$.

In previous work with Botvinnik [6] we studied the nontriviality of this map when $G = 1$, in which case the target is just the real topological K -theory of a point. Our goal in this paper is to extend this to more general groups G , when the target can be much richer.

To state our results, we must introduce some notation from stable homotopy theory. The Madsen–Tillmann–Weiss spectrum associated with the map $B\mathrm{Spin}(d) \rightarrow BO(d)$ is denoted $\mathrm{MTSpin}(d)$. The Atiyah–Bott–Shapiro orientation of spin vector bundles yields a spectrum map $\lambda_{-d} : \mathrm{MTSpin}(d) \rightarrow \Sigma^{-d}\mathbf{KO}$. For a discrete group G , there is the Novikov assembly map $\nu : \mathbf{KO} \wedge BG_+ \rightarrow \mathbf{KO}(\mathbf{C}_r^*(G))$ which we will recall in Definition 5.6.6 below.

Theorem A. *Let W^{2n} be a compact connected spin manifold with boundary ∂W , let G be a discrete group and let $f : W \rightarrow BG$ be a map. Furthermore, let $g \in \mathcal{R}^+(\partial W)$ and $h_0 \in \mathcal{R}^+(W)_g$. Assume that*

- (i) $n \geq 3$, and
- (ii) the homomorphism $\pi_1(f) : \pi_1(W) \rightarrow \pi_1(BG) = G$ is split surjective.

Then there exists a map $\Psi : \Omega^{\infty+1}(\mathrm{MTSpin}(2n) \wedge BG_+) \rightarrow \mathcal{R}^+(W)_g$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \Omega^{\infty+1}(\mathrm{MTSpin}(2n) \wedge BG_+) & \xrightarrow{\Psi} & \mathcal{R}^+(W)_g \\ \Omega^{\infty+1}\lambda_{-2n} \wedge \mathrm{id} \downarrow & & \downarrow \mathrm{inndiff}_{h_0}^G \\ \Omega^{\infty+2n+1}(\mathbf{KO} \wedge BG_+) & \xrightarrow{\Omega^{\infty+2n+1}\nu} & \Omega^{\infty+2n+1}\mathbf{KO}(\mathbf{C}_r^*(G)). \end{array}$$

This theorem will allow us to show that the index difference map $\mathrm{inndiff}_{h_0}^G$ is highly nontrivial on higher homotopy groups. Before this, only its nontriviality on π_0 or when $G = 1$ has been established.

Remarks 1.0.1.

- (i) The case $G = 1$ of Theorem A is precisely our previous result with Botvinnik [6, Theorem B].
- (ii) In [6], we also proved a similar result for odd-dimensional manifolds, derived from [6, Theorem B]. The analogous result for nontrivial G can be derived from Theorem A, and this will be done in the forthcoming Münster PhD thesis of Lukas Buggisch. See Remark 5.3.1 for more details.
- (iii) There is also a maximal group C^* -algebra $\mathbf{C}_m^*(G)$, with a homomorphism $\omega : \mathbf{C}_m^*(G) \rightarrow \mathbf{C}_r^*(G)$. The secondary Rosenberg index can be made to have target $\Omega^{\infty+2n+1}\mathbf{KO}(\mathbf{C}_m^*(G))$, and the Novikov assembly map naturally factors through a maximal version $\nu_m : \mathbf{KO} \wedge BG_+ \rightarrow \mathbf{KO}(\mathbf{C}_m^*(G))$. Theorem A in fact holds when the bottom right-hand corner is replaced with $\Omega^{\infty+2n+1}\mathbf{KO}(\mathbf{C}_m^*(G))$. Although this is a stronger theorem, we prefer to state it in terms of the reduced group C^* -algebra as it is this to which the Baum–Connes conjecture applies, as we will now discuss.

1.1. Applications. To draw computational consequences out of Theorem A as in [6, §5], one needs another assumption, namely that G is torsionfree and satisfies the Baum–Connes conjecture, i.e. that the Baum–Connes assembly map

$$\mu : KO_*^G(\underline{EG}) \longrightarrow KO_*(\mathbf{C}_r^*(G))$$

is an isomorphism. For torsionfree G , we may identify $KO_*^G(\underline{EG})$ with $KO_*(BG)$ and the Baum–Connes assembly map with the map induced by the Novikov assembly map ν on homotopy groups. The Baum–Connes conjecture predicts that μ is an isomorphism, so for torsionfree G it predicts that the Novikov assembly map is a weak homotopy equivalence. The Baum–Connes conjecture has been proven for vast classes of groups; see [38, §2.6] for a slightly outdated survey.

1.1.1. *Bott-stabilised integral surjectivity.* Let B^8 be a Bott manifold, that is a Spin 8-manifold having $\widehat{\mathcal{A}}(B) = \beta \in KO_8(*)$. By the work of Joyce [32, §6], we may choose such a B having a metric g_B of holonomy Spin(7), which is then Ricci-flat and hence scalar-flat. For a closed manifold M there are then induced maps

$$\mathcal{R}^+(M) \xrightarrow{-\times(B, g_B)} \mathcal{R}^+(M \times B) \xrightarrow{-\times(B, g_B)} \mathcal{R}^+(M \times B \times B) \longrightarrow \dots$$

and we write $\mathcal{R}^+(M)[B^{-1}]$ for the homotopy colimit. As $\widehat{\mathcal{A}}(B)$ acts invertibly on $\mathbf{KO}(C^*(G))$, if M is Spin and $f : M \rightarrow BG$ is a reference map then there is an extension

$$\text{inndiff}_{h_0}^G[B^{-1}] : \mathcal{R}^+(M)[B^{-1}] \longrightarrow \Omega^{\infty+d+1}\mathbf{KO}(C_r^*(G)) \quad (1.1.1)$$

of the secondary Rosenberg index map for M .

Theorem B. *If G is a torsionfree discrete group satisfying the Baum–Connes conjecture, $f_* : \pi_1(M) \rightarrow G$ is split surjective, and $d = \dim(M)$ is even¹, then the map (1.1.1) is surjective on all homotopy groups.*

1.1.2. *Rational surjectivity.* The spectrum map λ_{-2n} factors through the desuspension of the connective KO -theory spectrum as

$$\lambda_{-2n} : \text{MTSpin}(2n) \xrightarrow{\lambda'_{-2n}} \Sigma^{-2n}\mathbf{ko} \xrightarrow{\Sigma^{-2n}\text{per}} \Sigma^{-2n}\mathbf{KO},$$

and hence the spectrum map $\nu \circ (\lambda_{-2n} \wedge \text{id})$ may be written as the composition

$$\begin{aligned} \text{MTSpin}(2n) \wedge BG_+ &\xrightarrow{\lambda'_{-2n} \wedge \text{id}} \Sigma^{-2n}\mathbf{ko} \wedge BG_+ \xrightarrow{\Sigma^{-2n}\text{per} \wedge \text{id}} \Sigma^{-2n}\mathbf{KO} \wedge BG_+ \\ &\xrightarrow{\Sigma^{-2n}\nu} \Sigma^{-2n}\mathbf{KO}(C_r^*(G)). \end{aligned}$$

It follows that

$$\text{Im}(\pi_r(\text{inndiff}_{h_0}^G)) \supset \text{Im}(\pi_{r+1}(\Sigma^{-2n}\nu \circ \Sigma^{-2n}(\text{per} \wedge \text{id}) \circ \lambda'_{-2n} \wedge \text{id})).$$

By a standard characteristic class computation (see e.g. [6, Theorem 5.2.1]), the map $\pi_{r+1}(\lambda'_{-2n}) \otimes \mathbb{Q}$ is surjective, and by the (collapsing) Atiyah–Hirzebruch spectral sequence it follows that $\pi_{r+1}(\lambda'_{-2n} \wedge \text{id}) \otimes \mathbb{Q}$ is too. If G is torsionfree and satisfies the Baum–Connes conjecture then as we explained above the map ν is a weak equivalence. The map $\text{per} \wedge \text{id} : \mathbf{ko} \wedge BG_+ \rightarrow \mathbf{KO} \wedge BG_+$ is not rationally surjective, but if the rational homological dimension of the group G is finite, say equal to q , then

$$(\text{per} \wedge \text{id})_* : \pi_s(\mathbf{ko} \wedge BG_+) \otimes \mathbb{Q} \longrightarrow \pi_s(\mathbf{KO} \wedge BG_+) \otimes \mathbb{Q}$$

is surjective for $s \geq q$. From these facts, we derive the following.

Theorem C. *Let W be a spin manifold of dimension $2n \geq 6$, possibly with boundary. Let G be a group and let $f : W \rightarrow BG$ be a map such that $\pi_1(f) : \pi_1(W) \rightarrow G$ is split surjective. Assume that $h_0 \in \mathcal{R}^+(W)_g \neq \emptyset$, and that*

¹This hypothesis may be dropped once the work of Remark 1.0.1 (ii) is complete.

- (i) G satisfies the (rational) Baum–Connes conjecture,
- (ii) G is torsionfree and has finite rational homological dimension q , and
- (iii) $r \geq q - 2n - 1$.

Then the image of

$$(\text{inddiff}_{h_0}^G)_* : \pi_r(\mathcal{R}^+(W)_g) \longrightarrow KO_{2n+1+r}(C^*(G)) \otimes \mathbb{Q}$$

generates the target as a \mathbb{Q} -vector space.

Remark 1.1.2. Some rational consequences can also be obtained without the torsionfree hypothesis. When G is a finite group the map

$$KO_*(BG) \otimes \mathbb{Q} = KO_*^G(EG) \otimes \mathbb{Q} \longrightarrow KO_*^G(\underline{EG}) \otimes \mathbb{Q}$$

is split injective (as $KO_*(BG) \otimes \mathbb{Q} = \mathbb{Q}$, so a splitting may be given by the augmentation of the real representation ring of G), and a spectral sequence argument in equivariant KO -theory show that the analogous map is split injective for any group G , see [39, Corollary A.3]. Thus if G satisfies the (rational injectivity part of the) Baum–Connes conjecture then the Novikov assembly map is also rationally injective. Thus, in the situation of Theorem C but omitting the torsionfree hypothesis, for $r \geq 2$ one finds that $\pi_r(\mathcal{R}^+(W)_g) \otimes \mathbb{Q}$ is at least as large as $KO_{2n+1+r}(BG) \otimes \mathbb{Q}$ (we leave formulating the awkward conclusion of this argument for $r < 2$ to the reader).

On the other hand, consider the homotopy fibre F of the Novikov assembly map. On homotopy groups, the fibre sequence $F \rightarrow KO \wedge BG_+ \xrightarrow{\nu} KO(\mathbf{C}_r^*(G))$ induces a long exact sequence, which may be identified with the analytical surgery sequence of Higson and Roe [29]

$$\cdots \longrightarrow KO_d(BG) \xrightarrow{\nu_*} KO_d(\mathbf{C}_r^*(G)) \xrightarrow{\partial} S_{d-1}(G) \longrightarrow \cdots$$

Recently, Xie–Yu–Zeidler [53, Corollary 1.3] have shown that the image of

$$\pi_0(\mathcal{R}^+(M^d)) \xrightarrow{\text{inddiff}_{h_0}^G} KO_{d+1}(\mathbf{C}_r^*(G)) \xrightarrow{\partial} S_d(G) \longrightarrow S_d(G) \otimes \mathbb{Q}$$

generates the target group, if G satisfies the Baum–Connes conjecture and has rational homological dimension at most $d - 3$ (e.g. if G is finite). This supersedes previously known results by Botvinnik–Gilkey [7], Piazza–Schick [42], Weinberger–Yu [52] (which were all for odd d) and Bárcenas and Zeidler [2, Corollary 1.5]. By virtue of our construction, we can only construct classes in the homotopy of $\mathcal{R}^+(W)$ which become trivial in $S_d(G)$.

1.1.3. *A large example.* Let F_r denote the free group of rank r . The Stallings–Bieri group SB_n is defined as the kernel of the homomorphism

$$\phi : (F_2)^n \longrightarrow \mathbb{Z}$$

sending each generator to 1. This group is of type (F_{n-1}) (i.e., there exists a $K(SB_n, 1)$ -complex with finite $(n - 1)$ -skeleton), but has $H_n(SB_n; \mathbb{Q})$ countably infinite dimensional (see [4, p. 37]). Furthermore, it has homological dimension n , being a subgroup of the n -dimensional group $(F_2)^n$. Thus the group

$$G = SB_3 * SB_4 * SB_5 * SB_6$$

is torsionfree, has homological dimension 6 and has type (F_2) (so is finitely presented). The class of groups satisfying the Baum–Connes conjecture with coefficients is closed under (amalgamated) free products, finite direct products, passing

to subgroups, and HNN extensions [38, Theorem 5.2], so the Baum–Connes conjecture holds for G . Therefore by the Atiyah–Hirzebruch spectral sequence we have that

$$KO_{7+r}(\mathbf{C}_r^*(G)) \otimes \mathbb{Q} \cong KO_{7+r}(BG) \otimes \mathbb{Q} \cong \bigoplus_{i \geq 0} H_{7+r-4i}(BG; \mathbb{Q})$$

is a countably infinite dimensional vector space for each $r \geq 0$.

Take a finite 2-skeleton of BG , embed it into \mathbb{R}^5 , take a regular neighborhood N and consider $M = \partial N$. This is a 4-dimensional spin manifold with $\pi_1(M) = G$. The 6-manifold $W = M \times S^2$ has a psc metric. This puts us in a position to apply Theorem C, showing that $\pi_r(\mathcal{R}^+(W)) \otimes \mathbb{Q}$ is countably infinite dimensional for $r \geq 2$, the same for $r = 1$ after abelianising, and that $\pi_0(\mathcal{R}^+(W))$ maps onto an infinite rank abelian group. As a separable Fréchet manifold (such as $\mathcal{R}^+(W)$) has the homotopy type of a countable CW complex [27, Corollary 4] and hence countable homotopy groups, this is as large as homotopy groups of a space of psc metrics on a compact manifold can possibly be.

1.2. Stable metrics: extension of the Gromov–Lawson surgery method.

The main technical tool we shall develop, which was not available to us in [6], is an extension of the Gromov–Lawson surgery method [23] and in particular of Chernysh’s cobordism invariance theorem [9] (see also [50]) to the case of manifolds with boundaries. One such extension has been given by Walsh [51], but we need to go one step further. Let us first recall some notation and describe this result, which may be of interest in its own right.

If $W : M_0 \rightsquigarrow M_1$ is a cobordism with collared boundaries and $g_i \in \mathcal{R}^+(M_i)$, $i = 0, 1$, we let $\mathcal{R}^+(W)_{g_0, g_1}$ be the space of all metrics of positive scalar curvature on W which are equal to $g_i + dt^2$ near M_i , with respect to the given collar. For any $h \in \mathcal{R}^+(W)_{g_0, g_1}$, there are composition maps

$$\mu(h, -) : \mathcal{R}^+(V)_{g_1, g_2} \longrightarrow \mathcal{R}^+(W \cup V)_{g_0, g_2}$$

and

$$\mu(-, h) : \mathcal{R}^+(V')_{g_{-1}, g_0} \longrightarrow \mathcal{R}^+(V' \cup W)_{g_{-1}, g_1}$$

defined for all cobordisms $V : M_1 \rightsquigarrow M_2$, $V' : M_{-1} \rightsquigarrow M_0$ and boundary conditions $g_{-1} \in \mathcal{R}^+(M_{-1})$ and $g_2 \in \mathcal{R}^+(M_2)$.

Definition 1.2.1. Let $W : M_0 \rightsquigarrow M_1$ be a cobordism and let $h \in \mathcal{R}^+(W)_{g_0, g_1}$. Then h is called *left-stable* if the map $\mu(-, h) : \mathcal{R}^+(V)_{g_{-1}, g_0} \rightarrow \mathcal{R}^+(V \cup W)_{g_{-1}, g_1}$ is a weak equivalence for all cobordisms $V : M_{-1} \rightsquigarrow M_0$ and all boundary conditions g_{-1} . Dually, h is *right-stable* if the map $\mu(h, -) : \mathcal{R}^+(V)_{g_1, g_2} \rightarrow \mathcal{R}^+(W \cup V)_{g_0, g_2}$ is a weak equivalence for all cobordisms $V : M_1 \rightsquigarrow M_2$ and all boundary conditions g_2 . Finally, h is *stable* if it is both left- and right-stable.

We prove two results about the existence of such metrics. The first shows that right-stable metrics exist in abundance.

Theorem D. *Let $d \geq 6$ and let $W : M_0 \rightsquigarrow M_1$ be a d -dimensional cobordism. Assume that the inclusion map $M_1 \rightarrow W$ is 2-connected. Then for each $g_0 \in \mathcal{R}^+(M_0)$, there is a $g_1 \in \mathcal{R}^+(M_1)$ and a right-stable $h \in \mathcal{R}^+(W)_{g_0, g_1}$.*

Note that the metric $g_1 \in \mathcal{R}^+(M_1)$ is part of the conclusion of the theorem, not of the hypothesis. In other words, the theorem does *not* say that given g_0, g_1 such that $\mathcal{R}^+(W)_{g_0, g_1} \neq \emptyset$, then there is a metric $h \in \mathcal{R}^+(W)_{g_0, g_1}$ which is right stable

(let alone that *any* such h has these properties). The theorem applies to the case $M_0 = \emptyset$ (because the unique Riemannian metric on \emptyset has positive scalar curvature). The main result of [9] may be viewed as the special case $W = N \times D^k : \emptyset \rightsquigarrow N \times S^{k-1}$ of Theorem D (for $k \geq 3$), but it took us some time to arrive at the formulation of Theorem D as the correct generalisation of this result.

The second result shows that right-stable metrics are often automatically also left-stable.

Theorem E. *Let $d \geq 6$ and let $W : M_0 \rightsquigarrow M_1$ be a d -dimensional cobordism. Assume that both inclusion maps are 2-connected. Then every right-stable $h \in \mathcal{R}^+(W)_{g_0, g_1}$ is also left-stable.*

This theorem has no analogue in [9], as it does not hold in the situation considered there: a metric $h \in \mathcal{R}^+(W)_g$ on the cobordism $W = N \times D^k : \emptyset \rightsquigarrow N \times S^{k-1}$ being left-stable would mean that

$$\mu(-, h) : \{*\} = \mathcal{R}^+(\emptyset) \longrightarrow \mathcal{R}^+(W)_g$$

is a weak equivalence, which is not generally true. For later use we record the combination of Theorems D and E as follows.

Corollary 1.2.2. *Let $d \geq 6$ and let $W : M_0 \rightsquigarrow M_1$ be a d -dimensional cobordism such that both inclusions $M_i \rightarrow W$ are 2-connected. Then, given $g_0 \in \mathcal{R}^+(M_0)$, there is a $g_1 \in \mathcal{R}^+(M_1)$ and a stable $h \in \mathcal{R}^+(W)_{g_0, g_1}$. \square*

Remark 1.2.3. In the language used here, the main result of Walsh's paper [51, Theorem A] says that if W is as in Corollary 1.2.2 then for each $g_0 \in \mathcal{R}^+(M_0)$, there is a $g_1 \in \mathcal{R}^+(M_1)$ and a left-stable $h \in \mathcal{R}^+(W)_{g_0, g_1}$. That work arose from discussions we had with Walsh during the preparation of [6] (though the final version of [6] did not use it). For the purpose of the present paper, even the results of [51] are not sufficient.

To prove Theorems D and E, we introduce several technical tools, such as spaces of psc metrics on manifolds with corners and a procedure for rounding corners of manifolds with psc metrics. This is done in the fairly elementary Section 2. The proof of Theorems D and E is given in Section 3.

1.3. Outline of the proof of Theorem A. We follow the same general strategy as the argument for [6, Theorem B]. The proof is given in Sections 4 and 5. In those sections, we shall assume familiarity with [6, §2,3,4] and focus on those parts of the argument which exhibit essential differences. The construction in [6] rested on three pillars: Index theory, the Gromov–Lawson surgery method, and the high-dimensional Madsen–Weiss theorem of Galatius and the second named author.

Theorem A has two distinct parts: one concerns the construction of Ψ , and the other the commutativity of the diagram. We have decided to separate these parts more cleanly than in [6]. For the first part, the spin hypothesis does not play a role. Let W^{2n} be a manifold and consider the Gauss map $\tau : W \rightarrow BO(2n)$. Let $W \xrightarrow{\ell} X \xrightarrow{\theta} BO(2n)$ be the second stage of the Moore–Postnikov tower of τ . Recall that θ is a 2-coconnected fibration and ℓ is a 2-connected map. If W is a spin manifold, then $X \simeq B\text{Spin}(2n) \times B\pi_1(M)$ (a complete classification of the second Moore–Postnikov stages of classifying maps of vector bundles can be found in [46]).

To the fibration θ , we have the associated Madsen–Tillmann–Weiss spectrum $\text{MT}\theta$ and, if $\partial W \rightarrow W$ is 2-connected, there is a parameterised Pontrjagin–Thom map

$$\alpha_W : \text{BDiff}_\partial(W) \longrightarrow \Omega_0^\infty \text{MT}\theta.$$

The construction of the map Ψ is a consequence of the following general theorem.

Theorem F. *Let W^{2n} , $n \geq 3$, be a connected compact manifold viewed as a cobordism $W : \emptyset \rightsquigarrow \partial W$. Assume that $(W, \partial W)$ is $(n-1)$ -connected. Let $g \in \mathcal{R}^+(\partial W)$ be such that there exists a right stable $h \in \mathcal{R}^+(W)_g$. Then there exists a fibration $p : T_\infty^+ \rightarrow \Omega_0^\infty \text{MT}\theta$ and a homotopy cartesian diagram*

$$\begin{array}{ccc} \mathcal{R}^+(W)_g // \text{Diff}_\partial(W) & \longrightarrow & T_\infty^+ \\ \downarrow & & \downarrow p \\ \text{BDiff}_\partial(W) & \xrightarrow{\alpha_W} & \Omega_0^\infty \text{MT}\theta. \end{array}$$

Remark 1.3.1. We shall show in [16] that this theorem continues to hold under the weaker assumption that $(W, \partial W)$ is 2-connected. This also follows from work of Perlmutter [40].

If W is spin with $\pi_1(W) = G$ then $\text{MT}\theta = \text{MTSpin}(2n) \wedge BG_+$. Taking the fibre transport of the fibration p at a specific basepoint yields the map Ψ in Theorem A. As in [6] the construction of the fibration $T_\infty^+ \rightarrow \Omega_0^\infty \text{MT}\theta$ is by obstruction-theory, and it is here that Theorems D and E are used.

The second part of Theorem A is index-theoretic. The main difference to [6] is that the secondary index invariant takes place in the K -theory of $\mathbf{C}_r^*(G)$ and not of \mathbb{R} , and uses elliptic operators with coefficients in $\mathbf{C}_r^*(G)$. The necessary elliptic regularity theory is developed in [14]. With these results, the index-theoretic part of the argument is largely the same as in [6], with the exception of the analogue of the Atiyah–Singer index theorem. In Section 5, we review the necessary changes.

At this point, we have established Theorem A for certain W (those which are $(n-1)$ -connected relative to their boundary) and certain boundary conditions (those which extend to a stable metric on W). To extend this to general manifolds and boundary conditions, we need the additivity theorem for the index and an embedding trick; stable metrics are used again here.

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2. SPACES OF PSC METRICS ON MANIFOLDS WITH BOUNDARIES AND CORNERS

2.1. Spaces of psc metrics on manifolds with boundaries. For a closed manifold M , we let $\mathcal{R}(M)$ be the space of all Riemannian metrics, equipped with the usual Fréchet topology and we let $\mathcal{R}^+(M) \subset \mathcal{R}(M)$ be the open subspace of all Riemannian metrics with positive scalar curvature. Let W be a compact manifold with boundary M . We assume that the boundary of W comes equipped with a collar $c : M \times [0, 1] \rightarrow W$. The collar identifies $M \times [0, 1]$ with a subset of W and we usually use this identification without further mention.

For $1 \geq \epsilon > 0$, we denote by $\mathcal{R}^+(W)^\epsilon$ the space of all Riemannian metrics h on W with positive scalar curvature such that $c^*h = g + dt^2$ on $M \times [0, \epsilon]$ for some metric g on M . It is topologised as a subspace of the space of smooth symmetric $(2, 0)$ -tensor fields, with the usual Fréchet topology.

If the scalar curvature of h is positive, then $g \in \mathcal{R}^+(M)$, and assigning to h its boundary value g defines a map

$$\text{res}_\epsilon : \mathcal{R}^+(W)^\epsilon \longrightarrow \mathcal{R}^+(M)$$

which is continuous. We define $\mathcal{R}^+(W)_g^\epsilon := \text{res}_\epsilon^{-1}(g) \subset \mathcal{R}^+(W)^\epsilon$, the space of all psc metrics on W which on $M \times [0, \epsilon]$ are of the form $g + dt^2$.

For $1 \geq \delta > \epsilon$, the inclusion $\mathcal{R}^+(W)_g^\delta \rightarrow \mathcal{R}^+(W)_g^\epsilon$ is a closed embedding, and it is a homotopy equivalence by [6, Lemma 2.1]. As in [6], when the collar length does not play a role we abbreviate

$$\mathcal{R}^+(W) := \mathcal{R}^+(W)^\epsilon \quad \text{and} \quad \mathcal{R}^+(W)_g := \mathcal{R}^+(W)_g^\epsilon$$

for implicitly fixed values of ϵ .

2.2. Smooth manifolds with corners. The proof of Theorem D requires to come to grips with spaces of psc metrics on *manifolds with corners* (whereas Theorem E is a more formal consequence of Theorem D). In the following section, we describe in detail what we mean by this and prove a corner rounding result, which is a new key tool.

Definition 2.2.1. A *smooth d -manifold with acute corners* W is a (second countable, Hausdorff) topological space locally modelled on $\mathbb{R}^{d-2} \times [0, \infty)^2$ and the local diffeomorphisms which preserve the sets $\mathbb{R}^{d-2} \times \{0\}$, $\mathbb{R}^{d-2} \times [0, \infty) \times \{0\}$, and $\mathbb{R}^{d-2} \times \{0\} \times [0, \infty)$.

The (topological) boundary ∂W is decomposed into two codimension zero pieces M_0^{d-1} corresponding to $\mathbb{R}^{d-2} \times [0, \infty) \times \{0\}$ in local coordinates, and M_1^{d-1} corresponding to $\mathbb{R}^{d-2} \times \{0\} \times [0, \infty)$ in local coordinates, which intersect along a closed submanifold M_{01}^{d-2} corresponding to $\mathbb{R}^{d-2} \times \{0\}$ in local coordinates, which is the common boundary of M_0 and M_1 .

A *smooth d -manifold with obtuse corners* is analogous but locally modelled on $\mathbb{R}^{d-2} \times (\mathbb{R}^2 \setminus (0, \infty)^2)$.

We write the following constructions and results for manifolds with acute corners, but the case of obtuse corners can be treated in the same way with only notational differences.

Choose collars $b : M_{01} \times [0, 1] \times \{0\} \rightarrow M_0$ and $c : M_{01} \times \{0\} \times [0, 1] \rightarrow M_1$, so that $b(x, 0, 0) = c(x, 0, 0) = x$ for all $x \in M_{01}$. The manifolds $M_0 \cup_{M_{01}} (M_{01} \times (-\infty, 0])$ and $M_1 \cup_{M_{01}} (M_{01} \times (-\infty, 0])$, where we use the collars b and c to define their smooth structures, are called the elongations of M_0 and M_1 . The collars extend to embeddings

$$\begin{aligned} b' : M_{01} \times (-\infty, 1] \times \{0\} &\longrightarrow M_0 \cup_{M_{01}} (M_{01} \times (-\infty, 0]) \\ c' : M_{01} \times \{0\} \times (-\infty, 1] &\longrightarrow M_1 \cup_{M_{01}} (M_{01} \times (-\infty, 0]) \end{aligned}$$

For each pair $-\infty \leq u \leq 0 \leq v \leq 1$ we may form the pushout

$$\begin{array}{ccc} M_{01} \times [u, v]^2 & \xrightarrow{i_{M_0}} & (M_0 \cup_{M_{01}} M_{01} \times [u, 0]) \times [u, v] \\ \downarrow i_{M_1} & & \downarrow j_{M_0} \\ (M_1 \cup_{M_{01}} M_{01} \times [u, 0]) \times [u, v] & \xrightarrow{j_{M_1}} & K^{[u, v]}, \end{array}$$

where $i_{M_0}(x, s, t) := (b'(x, t), s)$, $i_{M_1}(x, s, t) := (c'(x, s), t)$.

Definition 2.2.2. A *bicollar* of W is an embedding $e : K^{[0,1]} \rightarrow W$ such that $e(j_{M_0}(x, 0)) = x$ for all $x \in M_0$ and $e(j_{M_1}(y, 0)) = y$ for all $y \in M_1$. The smooth manifold $\widetilde{W} := W \cup_{K^{[0,1]}} K^{(-\infty, 1]}$ is called the *elongation* of W .

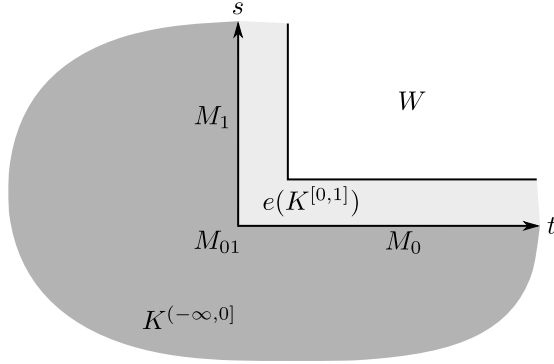


FIGURE 1. The boundary structure of a manifold with (acute) corners

There is a completely analogous model for an obtuse corner, which we shall omit.

2.3. Rounding corners of smooth manifolds. We first need to introduce some notation.

Definition 2.3.1. A *special curve* in \mathbb{R}^2 is a pair (B, b) , with $b \in [0, \infty)$ and $B \subset \mathbb{R}^2$ a topological submanifold homeomorphic to \mathbb{R} , such that

- (i) $B \cap ([0, \infty) \times \mathbb{R}) = [0, \infty) \times \{-b\}$, and
- (ii) $B \cap (\mathbb{R} \times [0, \infty)) = \{-b\} \times [0, \infty)$.

We say that $(B_0, b_0) < (B_1, b_1)$ for two special curves if $B_0 \cap B_1 = \emptyset$ and $b_0 < b_1$. In that case, we denote by $[B_0, B_1]$ the region that lies between these two curves.

If (B, b) is a special curve then the number b is determined by the curve B , so we abuse notation slightly by just writing B for a special curve. For $b \in [0, \infty)$ we let

$$C_b := \{-b\} \times [-b, \infty) \cup [-b, \infty) \times \{-b\} \subset \mathbb{R}^2.$$

Then (C_b, b) is a special curve, and the only special curve with $b = 0$ is C_0 . The part of $[B_0, B_1]$ outside the third quadrant consists of two strips. If $(B_0, b_0) < (B_1, b_1)$, we write

$$V_{B_0, B_1} := (M_{01} \times [B_0, B_1]) \cup (M_0 \times [-b_1, -b_0]) \cup (M_1 \times [-b_1, -b_0]) \subset K^{(-\infty, 1]},$$

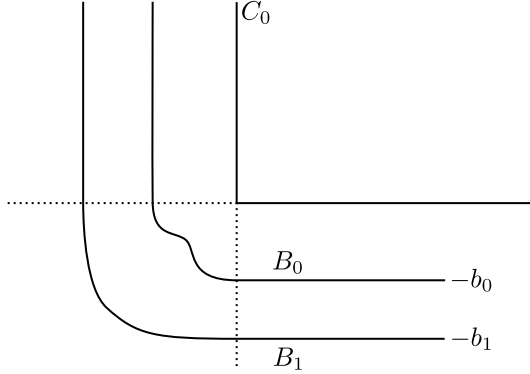


FIGURE 2. Some special curves

and for each special curve B and bicollared manifold with acute corners W , we let

$$W_B := W \cup_{\partial W} V_{C_0, B} \subset \widetilde{W},$$

which is a smooth codimension zero submanifold with boundary of the elongation \widetilde{W} if B is smooth.

Definition 2.3.2. If B is a smooth special curve, then W_B is the *result of rounding the corners of W* .

The smooth manifold with boundary W_B is independent of B up to diffeomorphism, because the region $[C_0, B] \subset \mathbb{R}^2$ is independent of B up to diffeomorphism relative to C_0 . Later we shall have to make more specific choices of B . If $a > 0$ is chosen large enough so that $C_a > B$, then

$$W_B \cup_{\partial W_B} V_{B, C_a} \subset \widetilde{W}$$

is a manifold with corners and it is diffeomorphic as such to W . In this way, we recover the manifold W from W_B .

2.4. Collars. We shall consider psc metrics on the manifolds V_{B_0, B_1} , so must clarify how we shall treat boundary conditions on such manifolds. If $B_0 = C_a$, then the manifold V_{B_0, B_1} has an obtuse corner in the boundary B_0 , and this has an evident bicollar; similarly, if $B_1 = C_a$ then the manifold V_{B_0, B_1} has an acute corner in the boundary B_1 , and this again has an evident bicollar. If instead B_i is a smooth special curve then

$$(M_{01} \times B_i) \cup (M_0 \times \{-b_i\}) \cup (M_1 \times \{-b_i\}) \subset K^{(-\infty, 1]}$$

is a codimension 1 submanifold and a choice of two-sided collar of B_i (agreeing with the evident collar of each $\{-b_i\} \times [0, \infty)$ and $[0, \infty) \times \{-b_i\}$) determines a two-sided collar of this submanifold.

2.5. Spaces of psc metrics on manifolds with corners. We will now define spaces of metrics of positive scalar curvature on a manifold with corners W equipped with a bicollar $e : K^{[0, 1]} \rightarrow W$. To ease notation, we will not mention the embedding e of the bicollar into W and pretend that $K^{[0, 1]} \subset W$.

Definition 2.5.1. Let $g_{01} \in \mathcal{R}^+(M_{01})$, $0 < \epsilon < 1$, $g_0 \in \mathcal{R}^+(M_0)_{g_{01}}^\epsilon$ and $g_1 \in \mathcal{R}^+(M_1)_{g_{01}}^\epsilon$, and write $g_{\partial W} := (g_0, g_1) \in \mathcal{R}^+(M_0)^\epsilon \times_{\mathcal{R}^+(M_{01})} \mathcal{R}^+(M_1)^\epsilon$. Let g_K be the Riemannian metric on $K^{(-\infty, \epsilon]}$ defined by

$$g_K = \begin{cases} g_{01} + dt^2 + ds^2 & \text{on } M_{01} \times (-\infty, \epsilon]^2, \\ g_0 + ds^2 & \text{on } M_0 \times (-\infty, \epsilon], \\ g_1 + dt^2 & \text{on } M_1 \times (-\infty, \epsilon]. \end{cases}$$

The space of all psc metrics h on W such that $h|_{K^{[0, \epsilon]}} = g_K$ is denoted $\mathcal{R}^+(W)_{g_{\partial W}}^\epsilon$.

The following collar stretching lemma is the analogue of [6, Lemma 2.1]. In particular it implies that the homotopy type of $\mathcal{R}^+(W)_{g_{\partial W}}^\epsilon$ is independent of ϵ , which allows us to neglect this from the notation when it is not important.

Lemma 2.5.2. *If $0 < \epsilon' < \epsilon$ then the inclusion map $\mathcal{R}^+(W)_{g_{\partial W}}^\epsilon \rightarrow \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'}$ is a homotopy equivalence.*

Proof. By breaking up this inclusion as a sequence of inclusions

$$\mathcal{R}^+(W)_{g_{\partial W}}^\epsilon \longrightarrow \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon_1} \longrightarrow \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon_2} \longrightarrow \cdots \longrightarrow \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'},$$

without loss of generality we may suppose that $\epsilon - \epsilon' < 1 - \epsilon$ and that $2\epsilon' > \epsilon$. Choose $0 < \delta < 2\epsilon' - \epsilon$ and a diffeomorphism $\phi_1 : [0, 1] \rightarrow [0, 1]$ such that

- (i) $\phi_1(t) \leq t$,
- (ii) $\phi_1(\epsilon) = \epsilon'$,
- (iii) $\phi_1(0) = 0$ and $\phi_1(1) = 1$,
- (iv) $\phi_1' \equiv 1$ near $[0, \delta] \cup \{1\} \cup [\epsilon', \epsilon + (\epsilon - \epsilon')]$.

The only condition which it is not obvious to satisfy is that $\phi_1' \equiv 1$ near $[\epsilon', \epsilon + (\epsilon - \epsilon')]$, which is where the conditions above enter in order to guarantee that

$$\phi_1(\epsilon + (\epsilon - \epsilon')) = \phi_1(\epsilon) + (\epsilon - \epsilon') = \epsilon' + (\epsilon - \epsilon') = \epsilon < 1$$

and

$$\phi_1(\epsilon') = \phi_1(\epsilon - (\epsilon - \epsilon')) = \epsilon' - (\epsilon - \epsilon') = 2\epsilon' - \epsilon > \delta.$$

Now let

$$\phi_u(t) = u \cdot \phi_1(t) + (1 - u) \cdot t$$

be the linear interpolation, an isotopy from the identity map to ϕ_1 . This induces an isotopy of $K^{[0, 1]}$, by taking products, and hence an isotopy Φ_u of W supported inside $e(K^{[0, 1]})$.

Let $f_u : [0, 1] \rightarrow [0, \infty)$ be the function

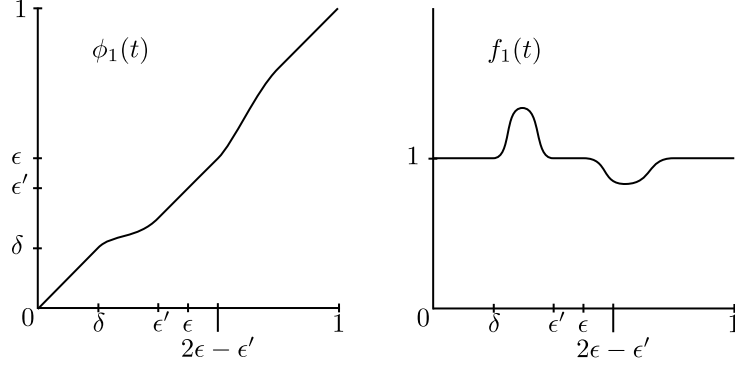
$$f_u(t) := \frac{1}{(\phi_u'(\phi_u^{-1}(t)))^2}$$

(note that $f_u \equiv 1$ near $[0, \delta] \cup [\epsilon', \epsilon] \cup \{1\}$ for all u). Then $\phi_u^*(f_u(t)dt^2) = dt^2$.

For a $h \in \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'}$, define a Riemannian metric h^u on W by

$$h^u := \begin{cases} g_{01} + f_u(s)ds^2 + f_u(t)dt^2 & \text{on } M_{01} \times [0, \epsilon'] \times [0, \epsilon'] \\ g_0 + f_u(s)ds^2 & \text{on } (M_0 \setminus b(M_{01} \times [0, \epsilon'] \times \{0\})) \times [0, \epsilon'] \\ g_1 + f_u(t)dt^2 & \text{on } (M_1 \setminus c(M_{01} \times \{0\} \times [0, \epsilon'])) \times [0, \epsilon'] \\ h & \text{elsewhere} \end{cases}$$

(this is not yet the homotopy we wish to construct). This is a smooth Riemannian metric, as the g_i are ϵ -collared, h is ϵ' -bicollared, and $f_u \equiv 1$ near ϵ' . Furthermore,

FIGURE 3. The function $\phi_1(t)$ and its associated function $f_1(t)$.

it depends continuously on $u \in [0, 1]$, is δ -bicollared, and (as $f_u(t)dt^2$ is flat) it has positive scalar curvature.

Now we define a homotopy $F : \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'} \times [0, 1] \rightarrow \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'}$ by the formula $F(h, u) = \Phi_u^* h^u$, which satisfies

$$\Phi_u^* h^u = \begin{cases} g_{01} + ds^2 + dt^2 & \text{on } M_{01} \times [0, \phi_u^{-1}(\epsilon')] \times [0, \phi_u^{-1}(\epsilon')] \\ g_0 + ds^2 & \text{on } (M_0 \setminus b(M_{01} \times [0, \phi_u^{-1}(\epsilon')] \times \{0\})) \times [0, \phi_u^{-1}(\epsilon')] \\ g_1 + dt^2 & \text{on } (M_1 \setminus c(M_{01} \times \{0\} \times [0, \phi_u^{-1}(\epsilon')])) \times [0, \phi_u^{-1}(\epsilon')] \\ \Phi_u^* h & \text{elsewhere.} \end{cases}$$

This is a smooth Riemannian metric, is $\phi_u^{-1}(\epsilon')$ -bicollared, depends continuously on $u \in [0, 1]$, and has positive scalar curvature. As $F(h, 0) = h$ and $F(h, 1)$ is ϵ -bicollared, this gives a homotopy ending in the subspace $\mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon} \subset \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'}$. Furthermore, as $f_u \equiv 1$ near $[\epsilon', \epsilon]$, if h is ϵ -bicollared then so is $\Phi_u^* h^u$ for all u , so this homotopy preserves the subspace $\mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon}$. Thus the homotopy F is a weak deformation retraction of $\mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon'}$ into $\mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon}$. \square

There are several consequences of this lemma which we shall use more often than the result itself. Firstly, we can interpret adding an external collar as shrinking collar length.

Corollary 2.5.3. *The map*

$$S := -\cup(V_{C_a, C_0}, g_K|_{V_{C_a, C_0}}) : \mathcal{R}^+(W)_{g_{\partial W}}^{\epsilon} \longrightarrow \mathcal{R}^+(W_{C_a})_{g_{\partial W_{C_a}}}^{\epsilon}$$

is a homotopy equivalence.

Proof. The source may be identified with $\mathcal{R}^+(W_{C_a})_{g_{\partial W_{C_a}}}^{a+\epsilon}$, whereupon this map becomes the natural inclusion $\mathcal{R}^+(W_{C_a})_{g_{\partial W_{C_a}}}^{a+\epsilon} \subset \mathcal{R}^+(W_{C_a})_{g_{\partial W_{C_a}}}^{\epsilon}$ which is a homotopy equivalence by Lemma 2.5.2. \square

Secondly, we can add an external collar only to one part of the boundary, say M_0 . Precisely, we can form the manifold $W \cup_{M_0} (M_0 \times [-a, 0])$ whose boundary decomposes as M_0 and $M_1 \cup_{M_{01}} (M_{01} \times [-a, 0])$. We have a boundary condition

$$\tilde{g}_{\partial W} := (g_0, g_1 \cup (g_{01} + ds^2)) \in \mathcal{R}^+(M_0)_{g_{01}}^{\epsilon} \times_{\mathcal{R}^+(M_{01})} \mathcal{R}^+(M_1 \cup_{M_{01}} (M_{01} \times [-a, 0]))_{g_{01}}^{a+\epsilon}$$

for this bicollared manifold.

Corollary 2.5.4. *With the notations introduced above, the gluing map*

$$\mathcal{R}^+(W)_{g_{\partial W}} \longrightarrow \mathcal{R}^+(W \cup_{M_0} M_0 \times [-a, 0])_{\tilde{g}_{\partial W}}$$

which glues in the psc metric $g_0 + ds^2$ is a homotopy equivalence.

Proof. By construction, the map S from Corollary 2.5.3 factors as

$$\mathcal{R}^+(W)_{g_{\partial W}} \longrightarrow \mathcal{R}^+(W \cup_{M_0} M_0 \times [-a, 0])_{\tilde{g}_{\partial W}} \longrightarrow \mathcal{R}^+(W_{C_a})_{g_{\partial W_{C_a}}},$$

and both maps glue in a cylinder along one of the parts of the boundary. We have to prove that the first of those maps is a weak homotopy equivalence, but the second one is of the same type. The conclusion now is as follows: since the composition is a homotopy equivalence, the first map is split monomorphic in the homotopy category, and the second is split epimorphic. Since the second map is of the same type as the first one, it is also split monomorphic, hence a homotopy equivalence. It follows that the first map is a homotopy equivalence as well. \square

2.6. The corner rounding theorem. The main result of this section is as follows. We adopt our convention of omitting the bicollar length.

Theorem 2.6.1. *Let W be as above, $g_{\partial W} = (g_0, g_1) \in \mathcal{R}^+(M_0) \times_{\mathcal{R}^+(M_{01})} \mathcal{R}^+(M_1)$. Then there exists*

- (i) *a collared smooth special curve (B, b) , and a psc metric h' on $V_{C_0, B}$ which is equal to $g_{\partial W}$ over C_0 and is equal to some $g_{\partial W_B}$ over B ,*
- (ii) *a $b' > 0$ such that $C_{b'} > B$, and a psc metric h'' on $V_{B, C_{b'}}$ which is equal to $g_{\partial W_B}$ over B and is equal to $g_{\partial W_{b'}}$ over $C_{b'}$,*

such that the maps

$$\mathcal{R}^+(W)_{g_{\partial W}} \xrightarrow{\cup(V_{B, C_0}, h')} \mathcal{R}^+(W_B)_{g_{\partial W_B}} \xrightarrow{\cup(V_{C_{b'}, B}, h'')} \mathcal{R}^+(W_{C_{b'}})_{g_{\partial W_{C_{b'}}}}$$

are weak homotopy equivalences, and the composition is homotopic to gluing on $(V_{C_{b'}, C_0}, g_K|_{V_{C_{b'}, C_0}})$.

Furthermore, we may suppose that h' and h'' are equal to $g_0 + ds^2$ near $M_0 \times (-\infty, 1]$ and to $g_1 + dt^2$ near $M_1 \times (-\infty, 1]$.

Proof. Let $\varphi : \mathbb{R} \times (0, 4) \rightarrow \mathbb{R}^2$ be a smooth embedding such that

- (i) $\varphi(x, t) = (-t, -1 - x)$ for $x \leq -1$,
- (ii) $\varphi(x, t) = (x - 1, -t)$ for $x \geq 1$,
- (iii) $\varphi(\mathbb{R} \times (0, 4)) = ((-4, \infty) \times (-4, 0)) \cup ((-4, 0) \times (-4, \infty))$,
- (iv) $C_2 \subset \varphi(\mathbb{R} \times (1, 3))$.

Define special curves

$$B_1 := \varphi(\mathbb{R} \times \{1\}) \quad \text{and} \quad B_3 := \varphi(\mathbb{R} \times \{3\}) \subset \mathbb{R}^2.$$

Note that $C_0 < B_1 < C_2 < B_3$, and that φ restricts to a diffeomorphism from $\mathbb{R} \times [1, 3]$ to $[B_1, B_3]$.

Next, choose two isotopies of Riemannian metrics on \mathbb{R}^2 , m_u and n_u , $u \in [0, 1]$, such that

- (i) $m_0 = ds^2 + dt^2$ is the flat metric.
- (ii) For each $u \in [0, 1]$, m_u coincides with $ds^2 + dt^2$ outside $[-2, 0]^2$.
- (iii) $\varphi^* m_1$ is the flat metric on a neighbourhood of $\mathbb{R} \times \{1\}$.

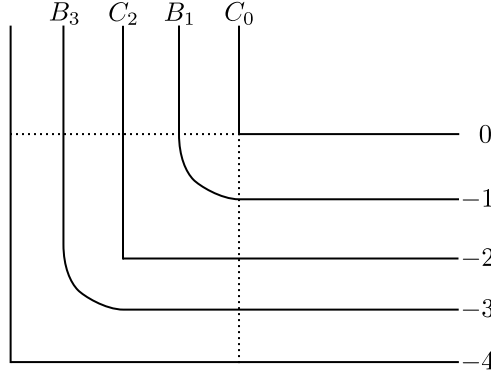


FIGURE 4. Special curves used in the proof of Theorem 2.6.1

- (iv) n_0 agrees with m_1 on a neighbourhood of $[B_1, C_2]$.
- (v) For each $u \in [0, 1]$, n_u coincides with $ds^2 + dt^2$ outside $[-4, 0]^2$, and φ^*n_u is the flat metric on a neighbourhood of $\mathbb{R} \times \{1, 3\}$.
- (vi) φ^*n_1 is the flat metric in a neighbourhood of $\mathbb{R} \times [1, 3]$.

Of course, we cannot require that m_u or n_u have positive or even nonnegative scalar curvature. But because both families of metrics are assumed to be constant (and flat) outside a compact subset of the plane, there exists a $\lambda > 0$ such that

$$\text{scal}(n_u), \text{scal}(m_u) \geq -\lambda$$

for all $u \in [0, 1]$. The following rescaling argument proves that we can make the constant λ as small as we want, at the expense of stretching the region where the metrics n_u and m_u are not standard. More precisely, let $a > 0$. Let $H_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the homothety $H_a(x) := \frac{1}{a}x$. Then $a^2H_a^*(ds^2 + dt^2) = ds^2 + dt^2$. Let

$$m_u^a := a^2H_a^*m_u \quad \text{and} \quad n_u^a := a^2H_a^*n_u.$$

By the well-known scaling identity for the curvature [22, p. 136], we have

$$\text{scal}(n_u^a), \text{scal}(m_u^a) \geq -\frac{1}{a^2}\lambda.$$

These rescaled metrics m_u^a and n_u^a have the same properties as those listed above, but with B_1 replaced by B_a , C_2 by C_{2a} and so on (one also has to rescale the embedding φ). Let

$$\kappa := \inf\{\text{scal}_{g_{01}}, \text{scal}_{g_0}, \text{scal}_{g_1}\} > 0$$

and choose $a \geq \sqrt{\frac{2\lambda}{\kappa}}$. Define $h_u \in \mathcal{R}^+(V_{C_0, C_{2a}})_{g_{\partial W}, g_{\partial W}}$ for $u \in [0, 1]$, by

$$h_u = \begin{cases} g_0 + ds^2 & \text{on } M_0 \times [-2a, 0] \\ g_1 + dt^2 & \text{on } M_1 \times [-2a, 0] \\ g_{01} + m_u^a & \text{on } M_{01} \times [-2a, 0]^2. \end{cases}$$

Over $M_{01} \times [-2a, 0]^2$, this is a product metric, and (using [22, §3.15]) we have

$$\text{scal}(g_{01} + m_u^a) = \text{scal}(g_{01}) + \text{scal}(m_u^a) \geq \kappa - \frac{1}{a^2}\lambda \geq \frac{1}{2}\kappa.$$

Over $M_0 \times [-2a, 0]$ and $M_1 \times [-2a, 0]$, $\text{scal}(h_u) \geq \kappa$, so that h_u is a psc metric.

Gluing on $V_{C_0, C_{2a}}$ with the metric h_u defines a homotopy of maps

$$\mu_{h_u} : \mathcal{R}^+(W)_{g_{\partial W}} \longrightarrow \mathcal{R}^+(W_{C_{2a}})_{g_{\partial W_{C_{2a}}}}.$$

The map μ_{h_0} is a weak homotopy equivalence by Corollary 2.5.3, since it is given by gluing on $V_{C_0, C_{2a}}$ equipped with the metric $m_0 = ds^2 + dt^2$. Hence μ_{h_u} is a weak homotopy equivalence for all $u \in [0, 1]$. By construction, the metric h_1 is collared near B_a so can be split along it, and so is the union of two psc metrics $h' \in \mathcal{R}^+(V_{C_0, B_a})_{g_{\partial W}, g_{\partial W_{B_a}}}$ (this defines $g_{\partial W_{B_a}}$) and $h'' \in \mathcal{R}^+(V_{B_a, C_{2a}})_{g_{\partial W_{B_a}}, g_{\partial W_{C_{2a}}}}$. Therefore, the weak equivalence μ_{h_1} factors as

$$\mathcal{R}^+(W)_{g_{\partial W}} \xrightarrow{\mu_{h'}} \mathcal{R}^+(W_{B_a})_{g_{\partial W_{B_a}}} \xrightarrow{\mu_{h''}} \mathcal{R}^+(W_{C_{2a}})_{g_{\partial W_{C_{2a}}}}.$$

So far, this proves only that $\mu_{h'}$ has a left-inverse and $\mu_{h''}$ has a right-inverse (in the weak homotopy category). To show that $\mu_{h'}$ is a weak equivalence, it suffices to show that $\mu_{h''}$ has a left-inverse as well, and for this, we employ the family n_u constructed above. We define a metric h_u , for $u \in [1, 2]$, on $V_{B_a, B_{3a}}$ by the formula

$$h_u = \begin{cases} g_0 + ds^2 & \text{on } M_0 \times [-3a, -a] \\ g_1 + dt^2 & \text{on } M_1 \times [-3a, -a] \\ g_{01} + n_{u-1}^a & \text{on } M_{01} \times [B_a, B_{3a}]. \end{cases}$$

By construction h_u has positive scalar curvature, and furthermore the metric h_1 is collared near B_{3a} so can be written as the union of the metric h'' and a metric $h''' \in \mathcal{R}^+(V_{C_{2a}, B_{3a}})_{g_{\partial W_{C_{2a}}}, g_{\partial W_{B_{3a}}}}$. So gluing in h_u defines a homotopy

$$\nu_{h_u} : \mathcal{R}^+(W_{B_a})_{g_{\partial W_{B_a}}} \longrightarrow \mathcal{R}^+(W_{B_{3a}})_{g_{\partial W_{B_{3a}}}}, u \in [1, 2].$$

We can write $\nu_{h_1} = \mu_{h'''} \circ \mu_{h''}$. Also by construction, ν_{h_2} glues in a cylinder metric and hence is a weak equivalence by [6, Lemma 2.1]. Thus ν_{h_1} is a weak equivalence, which completes the proof. \square

3. STABLE METRICS

3.1. Summary of the section. If $W : M_0 \rightsquigarrow M_1$ is a cobordism with collared boundaries and $g_i \in \mathcal{R}^+(M_i)$, $i = 0, 1$, we let $\mathcal{R}^+(W)_{g_0, g_1} := \mathcal{R}^+(W)_{g_0} \amalg_{g_1}$ be the space of all metrics of positive scalar curvature on W which are equal to $g_i + dt^2$ near M_i , with respect to the given collars. For any $h \in \mathcal{R}^+(W)_{g_0, g_1}$, there are composition maps

$$\mu(h, -) : \mathcal{R}^+(V)_{g_1, g_2} \longrightarrow \mathcal{R}^+(W \cup V)_{g_0, g_2}$$

and

$$\mu(-, h) : \mathcal{R}^+(V')_{g_{-1}, g_0} \longrightarrow \mathcal{R}^+(V' \cup W)_{g_{-1}, g_1}$$

defined for all cobordisms $V : M_1 \rightsquigarrow M_2$, $V' : M_{-1} \rightsquigarrow M_0$ and boundary conditions $g_{-1} \in \mathcal{R}^+(M_{-1})$ and $g_2 \in \mathcal{R}^+(M_2)$.

Definition 3.1.1. Let $W : M_0 \rightsquigarrow M_1$ be a cobordism and let $h \in \mathcal{R}^+(W)_{g_0, g_1}$. Then h is called *left-stable* if the map $\mu(-, h) : \mathcal{R}^+(V)_{g_{-1}, g_0} \rightarrow \mathcal{R}^+(V \cup W)_{g_{-1}, g_1}$ is a weak equivalence for all cobordisms $V : M_{-1} \rightsquigarrow M_0$ and all boundary conditions g_{-1} . Dually, h is *right-stable* if the map $\mu(h, -) : \mathcal{R}^+(V)_{g_1, g_2} \rightarrow \mathcal{R}^+(W \cup V)_{g_0, g_2}$ is a weak equivalence for all cobordisms $V : M_1 \rightsquigarrow M_2$ and all boundary conditions g_2 . Finally, h is *stable* if it is both, left and right stable.

To state the main result of this section, for a codimension 0 open submanifold $C \subset \text{int}(M)$ which is the interior of a submanifold with boundary $\bar{C} \subset \text{int}(M)$, and a $g_C \in \mathcal{R}^+(C)$, we let

$$\mathcal{R}^+(M, C; g_C) := \{g \in \mathcal{R}^+(M) \mid g|_C = g_C\}.$$

For a cobordism $W : M_0 \rightsquigarrow M_1$ and a codimension 0 open submanifold $[0, 1] \times C \subset W$ such that $M_i \cap ([0, 1] \times C) = \{i\} \times C$, given $g_i \in \mathcal{R}^+(M_i, C; g_C)$ we let

$$\mathcal{R}^+(W, [0, 1] \times C; dt^2 + g_C)_{g_0, g_1} := \{g \in \mathcal{R}^+(W)_{g_0, g_1} \mid g|_{[0, 1] \times C} = dt^2 + g_C\}.$$

Theorem 3.1.2. *Let $W : M_0 \rightsquigarrow M_1$ be a cobordism of dimension ≥ 6 .*

- (i) *Let $[0, 1] \times C \subset W$ be a codimension 0 submanifold as above, and let $g_C \in \mathcal{R}^+(C)$. If (W, M_1) is 2-connected, then for each $g_0 \in \mathcal{R}^+(M_0, C; g_C)$, there is $g_1 \in \mathcal{R}^+(M_1, C; g_C)$ and a right stable $h \in \mathcal{R}^+(W, [0, 1] \times C; dt^2 + g_C)_{g_0, g_1}$.*
- (ii) *If both (W, M_1) and (W, M_0) are 2-connected, then every right stable $h \in \mathcal{R}^+(W)_{g_0, g_1}$ is also left stable.*

We shall only need the absolute ($C = \emptyset$) version in this paper (the full relative version will be used in [16]). The proof is an elaboration of the Gromov–Lawson surgery technique, which we shall first review.

3.2. The theorem of Gromov–Lawson and Chernysh.

Definition 3.2.1. By $g_o^{k-1} \in \mathcal{R}(S^{k-1})$, we denote the round metric on S^{k-1} , i.e. the metric induced from euclidean metric by the standard inclusion $S^{k-1} \subset \mathbb{R}^k$. It has constant scalar curvature $\text{scal}(g_o^{k-1}) = (k-1)(k-2)$ which is positive if $k \geq 3$.

Let $\delta > 0$. A δ -torpedo metric g_{tor}^k on \mathbb{R}^k , $k \geq 3$, is an $O(k)$ -invariant metric such that $\text{scal}(g_{\text{tor}}^k) \geq \frac{1}{\delta^2}$ and such that

$$\varphi^* g_{\text{tor}}^k = dr^2 + \delta g_o^{k-1}$$

near $[1, \infty) \times S^{k-1}$, where $\varphi : (0, \infty) \times S^{k-1} \rightarrow \mathbb{R}^k \setminus \{0\}$ is the diffeomorphism defined by $(r, x) \mapsto rx$.

In this work, we are mostly interested in the case $\delta = 1$. We fix such a torpedo metric once and for all and refer to [49, §2.3] for more details.

Definition 3.2.2. Let $d - k \geq 3$, let W^d be a compact manifold with boundary M and collar $[0, \infty) \times M \subset W$, let V^k be a compact manifold with boundary N and collar $[0, \infty) \times N \subset V$, and let $\phi : V \times \mathbb{R}^{d-k} \rightarrow W$ be an embedding. Assume that $\phi^{-1}(M \times [0, \infty)) = N \times [0, \infty)$, and such that inside the collar ϕ is of the form $(x, t) \mapsto (\varphi(x), t)$ for some embedding $\varphi : N \times \mathbb{R}^{d-k} \rightarrow M$.

Let $h_V \in \mathcal{R}(V)_{g_V}$ be a metric and pick $\delta > 0$ such that $\text{scal}(h_V) + \frac{1}{\delta^2} > 0$ and fix a δ -torpedo metric g_{tor}^{d-k} on \mathbb{R}^{d-k} . By

$$\mathcal{R}^+(W, \phi)^\epsilon \subset \mathcal{R}^+(W)^\epsilon,$$

we denote the space of all ϵ -collared psc metrics h on W such that $\phi^* h = h_V + g_{\text{tor}}^{d-k}$ near $V \times D^{d-k} \subset V \times \mathbb{R}^{d-k}$.

For sufficiently small ϵ , one can view $\mathcal{R}^+(W, \phi)^\epsilon$ as a space of psc metrics on the manifold with corners $W \setminus \phi(V \times \mathring{D}^{d-k})$. Corollary 2.5.4 allows us to neglect the notation ϵ from $\mathcal{R}^+(W, \phi)^\epsilon$. In the cases of interest to us, h_V has nonnegative scalar curvature, in which case we pick $\delta = 1$. The following result due to Chernysh [10, Theorem 1.3] is a sharpening of a classical result by Gromov and Lawson [23], is of crucial importance for this paper:

Theorem 3.2.3 (Chernysh). *Assume that $d-k \geq 3$. Then for each $g \in \mathcal{R}^+(M, \varphi)$, the inclusion*

$$\mathcal{R}^+(W, \phi)_g \longrightarrow \mathcal{R}^+(W)_g$$

is a weak equivalence.

An immediate corollary of Theorem 3.2.3 is the following cobordism invariance result.

Definition 3.2.4. Let W be a compact d -dimensional manifold, possibly with boundary M . A surgery datum (i.e. embedding) $\phi : S^k \times \mathbb{R}^{d-k} \rightarrow \text{int}(W)$ is *admissible* if $2 \leq k \leq d-3$.

We let

$$W_\phi := (W \setminus \phi(S^k \times D^{d-k})) \cup_{S^k \times S^{d-k-1}} D^{k+1} \times S^{d-k-1}$$

be the result of performing a surgery along ϕ .

Corollary 3.2.5. *An admissible surgery datum ϕ determines a preferred homotopy class of weak homotopy equivalences*

$$\text{SE}_\phi : \mathcal{R}^+(W)_g \simeq \mathcal{R}^+(W_\phi)_g,$$

called the surgery equivalence determined by ϕ .

We refer to [6, Theorem 2.5] for the (easy) derivation of this Corollary from Theorem 3.2.3. We remark that there is no actual map between these spaces, but rather a zig-zag of weak equivalences. This is not relevant in the present paper: we only use SE_ϕ to identify the sets of path components of both spaces.

3.3. The stability condition. In this subsection, we collect some fairly straightforward but important facts about stable metrics. The following simple observation is immediate from the definitions and will be used repeatedly.

Lemma 3.3.1. *Let $(W, h) : (M_0, g_0) \rightsquigarrow (M_1, g_1)$ and $(W', h') : (M_1, g_1) \rightsquigarrow (M_2, g_2)$ be psc cobordisms. Then*

- (i) *If (W, h) and (W', h') are left-stable, then so is $(W \cup W', h \cup h')$.*
- (ii) *If (W, h) and (W', h') are right-stable, then so is $(W \cup W', h \cup h')$.*
- (iii) *If (W', h') and $(W \cup W', h \cup h')$ are left-stable, then so is (W, h) .*
- (iv) *If (W, h) and $(W \cup W', h \cup h')$ are right-stable, then so is (W', h') .*

Note that the statements “ (W', h') and $(W \cup W', h \cup h')$ right-stable $\Rightarrow (W, h)$ right-stable” and “ (W', h') and $(W \cup W', h \cup h')$ left-stable $\Rightarrow (W', h')$ left-stable” do *not* follow formally.

Lemma 3.3.2. *Let $g_0, g_1 \in \mathcal{R}^+(M)$ lie in the same path component. Then there is a stable metric $h \in \mathcal{R}^+(M \times [0, 1])_{g_0, g_1}$.*

Proof. Let g_t , $t \in [0, 1]$ be a smooth isotopy from g_0 to g_1 . A result of Gajer [19, p. 184] shows that there is $\Lambda > 0$ such that whenever $f : \mathbb{R} \rightarrow [0, 1]$ is a smooth function with $|f'|, |f''| \leq \Lambda$, the metric $dt^2 + g_{f(t)}$ on $\mathbb{R} \times M$ has positive scalar curvature. There exists a $c > 0$ and a smooth function $f : [0, 2c] \rightarrow [0, 1]$ such that $f \equiv 0$ near 0 and $2c$, $f \equiv 1$ near c , and such that $|f'|, |f''| \leq \Lambda$. The formula

$$h_s = dt^2 + g_{sf(t)}$$

thus defines a curve in the space $\mathcal{R}^+([0, 2c] \times M)_{g_0, g_0}$. But $h_0 = dt^2 + g_0$ is stable by [6, Corollary 2.2], and so h_1 is a stable metric as well. We can write $h_1 = h' \cup h''$, with $h' \in \mathcal{R}^+([0, c] \times M)_{g_0, g_1}$ and $h'' \in \mathcal{R}^+([c, 2c] \times M)_{g_1, g_0}$. Then the map $\mu(-, h')$ has a left inverse in the homotopy category, and $\mu(-, h'')$ has a right inverse. Let $h''' = dt^2 + g_{f(t-2c)}$, a psc metric in $\mathcal{R}^+([2c, 3c] \times M)_{g_1, g_0}$. A similar homotopy proves that $h'' \cup h'''$ is stable. Therefore, the above map $\mu(-, h'')$ also has a left-inverse in the homotopy category. Hence $\mu(-, h'')$ is a weak equivalence, and so is $\mu(-, h')$; in other words, h' is left stable. An analogous argument shows that h' is also right stable. Reparametrising the interval $[0, c]$ gives the desired stable $h \in \mathcal{R}^+([0, 1] \times M)_{g_0, g_1}$. \square

The next result shows the invariance of stable metrics under surgery equivalences.

Lemma 3.3.3. *Let $W : M_0 \rightsquigarrow M_1$ be a cobordism and $\phi : S^k \times D^{d-k} \rightarrow \text{int}(W)$ be an admissible surgery datum (i.e. $2 \leq k \leq d-3$). Let $[h] \in \pi_0 \mathcal{R}^+(W)_{g_0, g_1}$ and $[h'] \in \pi_0 \mathcal{R}^+(W_\phi)_{g_0, g_1}$ correspond under the weak equivalence SE_ϕ . Then h' is left stable (right stable) iff h is left stable (right stable).*

Proof. Exactly as the argument in the third paragraph of the proof of [6, Theorem 2.6]. \square

3.4. Construction of stable metrics. We now prove Theorem 3.1.2. The key step is the case of an elementary cobordism W . Let us introduce some notation. Throughout this section, M_i will be a closed $(d-1)$ -manifold and cobordisms between those are typically denoted $W : M_0 \rightsquigarrow M_1$. When W is such a cobordism, we denote by $W^{op} : M_1 \rightsquigarrow M_0$ the same manifold, but viewed as a cobordism in the other direction.

Let

$$\varphi : S^{k-1} \times \mathbb{R}^{d-k} \longrightarrow M_0$$

be a smooth embedding. We can view $\varphi|_{S^{k-1} \times D^{d-k}}$ as a surgery datum on M_0 , as well as the attaching map for a k -handle. Let M_1 be the result of performing surgery on φ , and let T_φ be the trace of the surgery. Then $T_\varphi : M_0 \rightsquigarrow M_1$ is a cobordism (an *elementary cobordism of index k*). There is an embedding

$$\phi^- : (D^k \times D^{d-k}, S^{k-1} \times D^{d-k}) \longrightarrow (T_\varphi, M_0)$$

such that $\phi^-|_{S^{k-1} \times D^{d-k}} = \varphi$. The image of ϕ^- is denoted H_φ^- and called the *incoming handle*. Moreover, there is a dual embedding

$$\phi^+ : (D^{d-k} \times D^k, S^{d-k-1} \times D^k) \longrightarrow (T_\varphi, M_1)$$

with image H_φ^+ , called the *outgoing handle*.

It is important for us to consider the following decomposition of T_φ . For $r > 1$, we let D_r^n be the closed n -disc of radius r and let S_r^{n-1} be its boundary. Let

$$\varphi_0 : S^{k-1} \times D^{d-k} \longrightarrow S^{k-1} \times D_2^{d-k}$$

be the inclusion and let U be the trace of a surgery on φ_0 . This is a manifold with corners. The boundary of U is decomposed into three pieces: $\partial_0 U := S^{k-1} \times D_2^{d-k}$, $\partial_1 U$ is the result of a surgery along φ_0 and $\partial_2 U = S^{k-1} \times S_2^{d-k-1} \times [0, 1]$. Let $L := M_0 \setminus \varphi(S^{k-1} \times D_2^{d-k}) \subset M_0$. Then $T_\varphi = (L \times [0, 1]) \cup_{S^{k-1} \times S_2^{d-k} \times [0, 1]} U$, see Figure 5. The incoming and the outgoing handle are disjoint from $L \times [0, 1]$.

Denote $\varphi^{op} := \phi^+|_{S^{d-k-1} \times D^k}$. By surgery on φ^{op} , one recovers M_0 from M_1 . Note furthermore that there is an identification $T_{\varphi^{op}} \cong (T_\varphi)^{op}$, relative to the

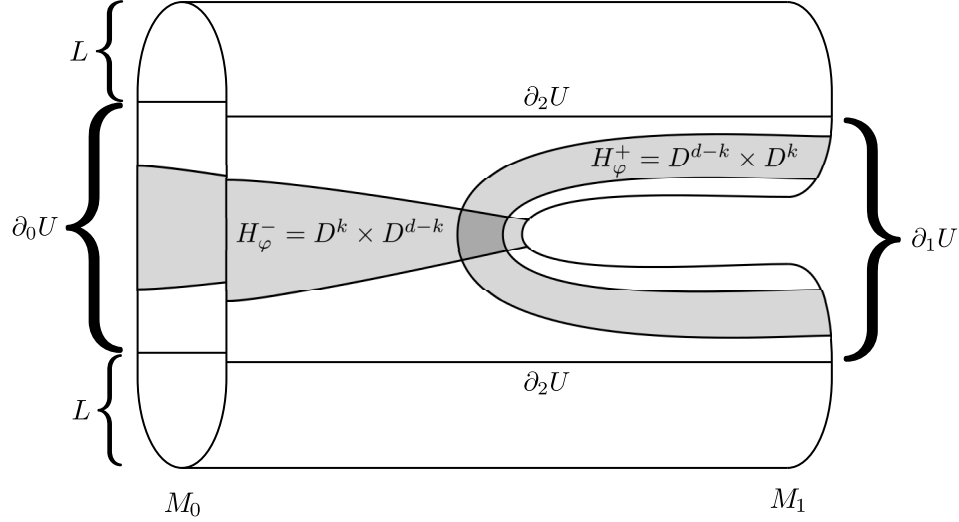


FIGURE 5. Decomposition of an elementary cobordism

boundaries. Inside the composed cobordism $T_\varphi \cup T_{\varphi^{op}}$, we find an embedding $\phi = \phi^+ \cup \phi^-$ of $S^{d-k} \times D^k$. Doing surgery on ϕ results in a manifold diffeomorphic to the cylinder $M_0 \times [0, 2]$ relative to the canonical diffeomorphism $\partial(T_\varphi \cup T_{\varphi^{op}}) \cong \partial(M_0 \times [0, 2])$.

Lemma 3.4.1. *Let $k \leq d - 3$. Then there is a psc metric h on the manifold with corners U , such that*

$$h = \begin{cases} g_\circ^{k-1} + g_\circ^{d-k-1} + dt^2 + ds^2 & \text{near } \partial_2 U \\ g_\circ^{k-1} + g_{\text{tor}}^{d-k} + dt^2 & \text{near } \partial_0 U \end{cases}$$

and such that the following property holds. If M^{d-1} is a closed manifold, $g_0 \in \mathcal{R}^+(M)$ and $\varphi : S^{k-1} \times \mathbb{R}^{d-k} \rightarrow M$ an embedding such that $\varphi^* g_0 = g_\circ^{k-1} + g_{\text{tor}}^{d-k}$ near $\varphi(S^{k-1} \times D_2^{d-k})$, then the psc metric $h \cup (g_0|_L + dt^2)$ on $U \cup (L \times [0, 1]) = T_\varphi$ is right-stable.

Proof. We construct a right stable metric on T_φ and justify during the course of the proof why it has the particular form stated in the lemma. The proof depends on the decomposition of T_φ described above and shown in Figure 5.

Here we let $H_\varphi^- \subset U \subset W = T_\varphi$ be the image of the embedding $\phi^- : (D^k \times D^{d-k}, S^{k-1} \times D^{d-k}) \rightarrow (T_\varphi, M_0)$, and let $S = T_\varphi \setminus \text{int}(H_\varphi^-)$. Note that S is a manifold with (acute) corners in the sense of Definition 2.2.1, and that it is diffeomorphic to a manifold of the form V_{B_1, C_2} considered in the proof of Theorem 2.6.1. More precisely, the left boundary of S is decomposed as

$$(M_0 \setminus \text{int}(S^{k-1} \times D^{d-k})) \cup_{S^{k-1} \times S^{d-k-1}} (S^{k-1} \times D^k)$$

and the right boundary of S is the result of corner smoothing. Let

$$g_\partial = (g_0|_{M_0 \setminus \text{int}(S^{k-1} \times D^{d-k})}, g_\circ^{k-1} + g_{\text{tor}}^k) \\ \in \mathcal{R}^+(M_0 \setminus \text{int}(S^{k-1} \times D^{d-k})) \times_{\mathcal{R}^+(S^{k-1} \times S^{d-k-1})} \mathcal{R}^+(S^{k-1} \times D^k).$$

We now define $g_1 \in \mathcal{R}^+(M_1)$ as $g_{\partial W_B}$, as explained in the proof of Theorem 2.6.1. By that theorem, there is a metric $h' \in \mathcal{R}^+(S)_{g_{\partial}, g_1}$ which is right stable in a slightly generalised sense: if $V : M_1 \rightsquigarrow M_2$ is a cobordism and $g_2 \in \mathcal{R}^+(M_2)$, then $\mathcal{R}^+(V)_{g_1, g_2} \rightarrow \mathcal{R}^+(S \cup V)_{g_{\partial}, g_2}$ is a weak equivalence. Moreover, by the last statement of Theorem 2.6.1, we can assume that h' is of the form $g_0|_L + dt^2$ over $L \times [0, 1]$, and $h'|_{S \setminus (L \times [0, 1])}$ does not depend on the ambient manifold M .

Let now $\mathcal{R}^+(W, H_{\varphi}^-)_{g_0, g_1} \subset \mathcal{R}^+(W)_{g_0, g_1}$ be the subspace of those psc metrics h such that $(\phi^-)^*h = g_{\text{tor}}^k + g_{\text{tor}}^{d-k}$. By Theorem 3.2.3 (iii), the inclusion

$$\mathcal{R}^+(W, H_{\varphi}^-)_{g_0, g_1} \longrightarrow \mathcal{R}^+(W)_{g_0, g_1}$$

is a weak homotopy equivalence because $d - k \geq 3$. The same is true when W is replaced by $W \cup V$, for any other cobordism $V : M_1 \rightsquigarrow M_2$ and any boundary condition $g_2 \in \mathcal{R}^+(M_2)$. Now we consider the psc metric $h = (g_{\text{tor}}^k + g_{\text{tor}}^{d-k}) \cup h'$ on $W = H_{\varphi}^- \cup S$. Let $V : M_1 \rightsquigarrow M_2$ be any other cobordism and $g_2 \in \mathcal{R}^+(M_2)$. Then the map $(W, h) \cup - : \mathcal{R}^+(V)_{g_1, g_2} \rightarrow \mathcal{R}^+(W \cup V)_{g_0, g_2}$ is equal to the composition

$$\mathcal{R}^+(V)_{g_1, g_2} \xrightarrow{(S, h') \cup -} \mathcal{R}^+(S \cup V)_{g_{\partial}, g_2} \cong \mathcal{R}^+(W \cup V, H_{\varphi}^-)_{g_0, g_2} \longrightarrow \mathcal{R}^+(W \cup V)_{g_0, g_2}$$

of two weak homotopy equivalences, so (W, h) is right stable. \square

Proof of Theorem 3.1.2 (i). By handle cancellation theory (see e.g. [35, 48]), the cobordism W admits a decomposition into elementary cobordisms of index $k \leq d-3$ (relative to M_0). Therefore, we may assume that $W = T_{\varphi}$, for some embedding $\varphi : S^{k-1} \times \mathbb{R}^{d-k} \rightarrow M_0 \setminus C$ with $d - k \geq 3$.

The next step is to turn the metric g_0 into some standard form. Because $d - k \geq 3$, there is a smooth family $g_t \in \mathcal{R}^+(M_0)$ such that $\varphi|_{S^{k-1} \times D^{d-k}} g_1 = g_{\circ}^{k-1} + g_{\text{tor}}^{d-k}$, by Chernysh's theorem [9, Theorem 1.1] (the weaker result by Gajer [19] also suffices for this purpose), and this may be supported on a small neighbourhood of $\varphi(S^{k-1} \times D^{d-k})$ and hence disjointly from C , so gives a smooth family $g_t \in \mathcal{R}^+(M_0, C)$. By Lemma 3.3.2, there is a stable metric $h_1 \in \mathcal{R}^+([0, 1] \times M_0, [0, 1] \times C; dt^2 + g_C)_{g_0, g_1}$ (this h_1 and g_1 are *not* yet the metrics we are looking for). Lemma 3.4.1 gives a right-stable metric $h_2 \in \mathcal{R}^+(T_{\varphi})_{g_1, g_2}$ which is cylindrical over $[0, 1] \times L$ and so in particular over $[0, 1] \times C$. The composition

$$h_1 \cup h_2 \in \mathcal{R}^+([0, 1] \times M_0 \cup_{M_0} T_{\varphi})_{g_0, g_2} \cong \mathcal{R}^+(T_{\varphi})_{g_0, g_2}$$

is then right-stable by Lemma 3.3.1 (ii), and is cylindrical over $[0, 1] \times C$ as required. \square

Proof of Theorem 3.1.2 (ii). Let $W : M_0 \rightsquigarrow M_1$ be a cobordism of dimension $d \geq 6$, assume that $M_i \rightarrow W$ are 2-connected and let $h \in \mathcal{R}^+(W)_{g_0, g_1}$ be a right stable psc metric on W . Using handle cancellation theory [35, 48], W has a handle decomposition (relative to M_0) using handles of index $3 \leq k \leq d-2$ (in fact, if $d \geq 7$, one can assume that $3 \leq k \leq d-3$, but if $d = 6$, this can fail, for example if W is an h -cobordism with nontrivial Whitehead torsion).

We have noted above that if $T_{\varphi} : M \rightsquigarrow M'$ is an elementary cobordism of index k , then the composite $T_{\varphi} \cup T_{\varphi}^{op}$ contains a surgery datum $\phi : S^{d-k} \times D^k \rightarrow T_{\varphi} \cup T_{\varphi}^{op}$ in its interior so that performing surgery on ϕ results in the cylinder $M \times [0, 2]$. It follows that $W \cup W^{op}$ can be surgered to $M_0 \times [0, 2]$, and this can be done using admissible surgeries, provided that all handles of W have index $3 \leq k \leq d-2$,

which is what we arranged in the first step of the proof. Hence there is a surgery equivalence

$$\text{SE} : \mathcal{R}^+(M_0 \times [0, 2])_{g_0, g_0} \simeq \mathcal{R}^+(W \cup W^{op})_{g_0, g_0}.$$

Let $h'' \in \mathcal{R}^+(W \cup W^{op})_{g_0, g_0}$ be in the component of the cylinder metric $g_0 + dt^2$ under SE. By Lemma 3.3.3 and [6, Corollary 2.2], h'' is stable.

Now $h \in \mathcal{R}^+(W)_{g_0, g_1}$ is right stable by assumption, so there is a psc metric $h' \in \mathcal{R}^+(W^{op})_{g_1, g_0}$ such that $h \cup h'$ and h'' lie in the same path component of $\mathcal{R}^+(W \cup W^{op})_{g_0, g_0}$ (just because $\mu(h, -) : \mathcal{R}^+(W^{op})_{g_1, g_0} \rightarrow \mathcal{R}^+(W \cup W^{op})_{g_0, g_0}$ is a weak homotopy equivalence). Since h and $h \cup h' \simeq h''$ are right stable, so is h' , by Lemma 3.3.1.

Let us summarise what we proved so far. Given a cobordism W such that both inclusions $M_i \rightarrow W$ are 2-connected, and given a right stable $h \in \mathcal{R}^+(W)_{g_0, g_1}$, we find another right stable $h' \in \mathcal{R}^+(W^{op})_{g_1, g_0}$ such that $h \cup h'$ is stable.

The conclusion of the argument is formal. Namely, $\mu(-, h \cup h') = \mu(-, h') \circ \mu(-, h)$ is a weak equivalence, and so $\mu(-, h)$ is “split monomorphic in the weak homotopy category”. The argument leading to that conclusion only used that h is a right stable psc metric on a cobordism whose inclusion maps $M_i \rightarrow W$ are both 2-connected. Because W^{op} and h' also satisfy these conditions, we can apply the same argument to h' and get that $\mu(-, h')$ is split monomorphic in the weak homotopy category as well. Since $\mu(-, h') \circ \mu(-, h)$ is a weak equivalence, $\mu(-, h')$ is also split epimorphic. So $\mu(-, h')$ is a weak equivalence, and so is $\mu(-, h)$, which is what we had to prove. \square

4. PROOF OF THE MAIN FACTORIZATION RESULT

In this section we will prove Theorem F.

4.1. A commutativity result. Let W^d be a compact manifold with boundary. The group $\text{Diff}_\partial(W)$ of diffeomorphisms which are equal to the identity near ∂W acts by pullback on the space $\mathcal{R}^+(W)_g$ of psc metrics. This action defines a homomorphism $\text{Diff}_\partial(W) \rightarrow \text{hAut}(\mathcal{R}^+(W)_g)$ to the monoid of homotopy automorphisms of $\mathcal{R}^+(W)_g$. A key step of the paper [6] (namely Theorem 4.1) is that under some circumstances the induced homomorphism $\pi_0(\text{Diff}_\partial(W)) \rightarrow \pi_0 \text{hAut}(\mathcal{R}^+(W)_g)$ has an abelian group as its image. This is to say: if $f_0, f_1 \in \text{Diff}_\partial(W)$, then $f_0^* f_1^*$ and $f_1^* f_0^* : \mathcal{R}^+(W)_g \rightarrow \mathcal{R}^+(W)_g$ are homotopic. We introduce the following language.

Definition 4.1.1. Let X be a space and let G be a topological group which acts on X . This gives rise to an H -space map $G \rightarrow \text{hAut}(X)$. The action of G on X is called *homotopy abelian* if the image of the induced map $\pi_0(G) \rightarrow \pi_0(\text{hAut}(X))$ is an abelian group.

Theorem 4.1.2. *Let $W^d : \emptyset \rightsquigarrow \partial W$ be a cobordism. Assume that $d \geq 6$ and that the inclusion map $\partial W \rightarrow W$ is 2-connected. Assume that $g \in \mathcal{R}^+(\partial W)$ is such that there exists a right stable $h \in \mathcal{R}^+(W)_g$. Then the action of $\text{Diff}_\partial(W)$ on $\mathcal{R}^+(W)_g$ is homotopy abelian.*

Remark 4.1.3. Theorem 4.1 of [6] applies under the following hypotheses:

- (i) $d \geq 5$,
- (ii) W is simply connected and spin,

²As in [6, §4.1], one should replace $\mathcal{R}^+(W)_g$ by a space having the homotopy type of a CW complex; we do so implicitly here.

- (iii) $\partial W = S^{d-1}$ and g is the round metric on S^{d-1} ,
- (iv) W is spin cobordant to D^d , relative to its boundary.

Thus except in the case $d = 5$, [6, Theorem 4.1] follows from Theorem 4.1.2.

The proof of Theorem 4.1.2 is quite similar to the proof of [6, Theorem 4.1]. The first step is actually almost the same.

Lemma 4.1.4. *Let $V : M_0 \rightsquigarrow M_1$ be a cobordism such that both boundary inclusions $M_i \rightarrow V$ are 2-connected. Let $g_i \in \mathcal{R}^+(M_i)$ so that there is a right stable $h \in \mathcal{R}^+(V)_{g_0, g_1}$. Then the action of $\text{Diff}_\partial(V)$ on $\mathcal{R}^+(V)_{g_0, g_1}$ is homotopy abelian.*

Proof. By Theorem E, h is also left-stable. One then uses the formal result [6, Lemma 4.2] in the same way as it was used in the proof of [6, Theorem 4.1]. \square

The next step is to reduce Theorem 4.1.2 to Lemma 4.1.4. This is significantly more complicated than the corresponding step in [6, p. 801 f].

Lemma 4.1.5. *Let W be a manifold of dimension $d \geq 6$ and assume that $\partial W \rightarrow W$ is 2-connected. Pick a handle decomposition of W without handles of index $\geq d - 2$ and let K be the union of all handles of index ≤ 2 . Then there is a collar C of ∂W in W which contains K in its interior.*

Proof. We prove the following equivalent statement: if B is an arbitrary collar, then there is an isotopy f_t of W , relative to the boundary, so that $f_0 = \text{id}$ and $f_1(K) \subset B$. If that is proven, then $C := f_1^{-1}(B)$ does the job.

We make use of the Phillips submersion theorem [41]. Let $\text{Sub}(K; W)$ be the space of all submersions $K \rightarrow W$ and let $\text{Epi}(TK; TW)$ be the space of bundle epimorphisms $TK \rightarrow TW$. The submersion theorem states that the differential map $D : \text{Sub}(K; W) \rightarrow \text{Epi}(TK; TW)$ is a weak homotopy equivalence. The space $\text{Epi}(TK; TW)$ is the space of cross-sections in the fibre bundle $\text{Fr}(TK) \times_{\text{GL}_d(\mathbb{R})} \text{Fr}(TW) \rightarrow K$, where $\text{Fr}(V)$ denotes the frame bundle of the vector bundle V .

We wish to show that the inclusion $h_0 : K \rightarrow W$, a submersion, is regularly homotopic via a regular homotopy h_t to a submersion $h_1 : K \rightarrow B \subset W$. By the submersion theorem this is equivalent to finding a fibrewise homotopy from $D(h_0) : K \rightarrow \text{Fr}(TK) \times_{\text{GL}_d(\mathbb{R})} \text{Fr}(TW)$ to a cross-section with image in $\text{Fr}(TK) \times_{\text{GL}_d(\mathbb{R})} \text{Fr}(TB)$. Using the (fibrewise) homotopy extension property, this may be done by induction over cells of K . Over a single k -cell we may trivialise the fibre bundle, and the problem becomes to show that a certain obstruction in $\pi_k(\text{Fr}(TW), \text{Fr}(TB))$ vanishes. Since the inclusion $B \rightarrow W$ is 2-connected, so is the induced map $\text{Fr}(TB) \rightarrow \text{Fr}(TW)$ on frame bundles and so such relative homotopy groups vanish for $k \leq 2$. On the other hand, because the handle dimension of K is ≤ 2 it is equivalent to a CW-complex only having cells of dimension ≤ 2 : therefore there is no obstruction to finding the required fibrewise homotopy, and so the desired regular homotopy h_t exists.

Since $d = \dim(W) \geq 6$, we can assume, by general position, that each h_t embeds a neighbourhood $U \subset K$ of the cores of the handles of K . There is an embedding $e : K \hookrightarrow U \subset K$ which is isotopic (as a map $K \rightarrow K$) to the identity. By the isotopy extension theorem [30, Theorem 8.1.4], we find an isotopy f_t of W , starting with the identity and such that $f_t \circ e = h_t \circ e$. Finally, the embeddings $K \subset W$ and $K \xrightarrow{e} K \subset W$ are ambiently isotopic, so that altogether, the inclusion $K \rightarrow W$ is ambiently isotopic to an embedding into the collar B . \square

Proof of Theorem 4.1.2. Pick a handlebody decomposition of W using only handles of index $\leq d - 3$. Let K be the union of all handles of index ≤ 2 , and let $V := W \setminus \text{int}(K)$. Let $M_0 := \partial K$ and $M_1 := \partial W$, so that $V : M_0 \rightsquigarrow M_1$. Observe that both inclusions $M_i \rightarrow V$ are 2-connected. According to Lemma 4.1.5, there is a collar C of ∂W in W containing K . Let $\text{Diff}(W, C) \subset \text{Diff}_\partial(W)$ be the subgroup of diffeomorphisms, fixing C . The inclusion $\text{Diff}(W, C) \rightarrow \text{Diff}_\partial(W)$ is a weak equivalence, and it factors through the group $\text{Diff}(W, K) \cong \text{Diff}_\partial(V)$. Therefore, the map $\text{Diff}_\partial(V) \rightarrow \text{Diff}_\partial(W)$ extending diffeomorphisms as the identity over K , induces a surjection on path components. Therefore, it is enough to show that the image of $\pi_0(\text{Diff}_\partial(V)) \rightarrow \pi_0(\text{hAut}(\mathcal{R}^+(W)_g))$ is an abelian group. Write $g_1 := g$. By Theorem D, there is $g_0 \in \mathcal{R}^+(M_0)$, and a right stable $h' \in \mathcal{R}^+(K)_{g_0}$. Pick $h'' \in \mathcal{R}^+(V)_{g_0, g_1}$ so that $h' \cup h''$ and h lie in the same path component of $\mathcal{R}^+(W)_{g_1}$, using that h' is right stable. Since h and h' are right stable, so is $h'' \in \mathcal{R}^+(V)_{g_0, g_1}$, by Lemma 3.3.1.

Because h' is right stable, the map $\mu(h', -) : \mathcal{R}^+(V)_{g_0, g_1} \rightarrow \mathcal{R}^+(W)_{g_1}$ is a weak equivalence, and since $\mu(h', -)$ is $\text{Diff}_\partial(V)$ -equivariant, we have reduced the problem to showing that $\text{Diff}_\partial(V)$ acts on $\mathcal{R}^+(V)_{g_0, g_1}$ through an abelian group. But $h'' \in \mathcal{R}^+(V)_{g_0, g_1}$ is stable, so that by Lemma 4.1.4, the action of $\text{Diff}_\partial(V)$ on $\mathcal{R}^+(V)_{g_0, g_1}$ is homotopy abelian. \square

Corollary 4.1.6. *Let $W : S^{d-1} \rightsquigarrow M$ be a cobordism, $d \geq 6$ and assume that $M \rightarrow W$ is 2-connected. Let $g \in \mathcal{R}^+(M)$ be such that there exists a right stable $h \in \mathcal{R}^+(W)_{g \circ, g}$. Then the action of $\text{Diff}_\partial(W)$ on $\mathcal{R}^+(W)_{g \circ, g}$ is homotopy abelian.*

Proof. The cobordism $V = D^d \cup W : \emptyset \rightsquigarrow M$, with the metric $h' : g_{\text{tor}}^d \cup h$, satisfies the hypotheses of Theorem 4.1.2. Let $f_0, f_1 \in \text{Diff}_\partial(W)$ and let \bar{f}_i be the extension to V by the identity. The diagram

$$\begin{array}{ccc} \mathcal{R}^+(W)_{g \circ, g} & \xrightarrow{f_i^*} & \mathcal{R}^+(W)_{g \circ, g} \\ \downarrow \mu(g_{\text{tor}}^d, -) & & \downarrow \mu(g_{\text{tor}}^d, -) \\ \mathcal{R}^+(V)_g & \xrightarrow{\bar{f}_i^*} & \mathcal{R}^+(V)_g \end{array}$$

commutes and the vertical maps are homotopy equivalences. By Theorem 4.1.2, \bar{f}_0^* and \bar{f}_1^* commute up to homotopy, and hence so do f_0^* and f_1^* . \square

4.2. Starting the proof of Theorem F. Let us first recall the set-up of Theorem F. We are given a $2n$ -manifold W such that $(W, \partial W)$ is $(n - 1)$ -connected, and we form the second stage

$$\tau : W \xrightarrow{\ell_W} X \xrightarrow{\theta} BO(2n)$$

of the Moore–Postnikov tower for the map τ classifying the tangent bundle of W . The map θ classifies a vector bundle $\theta^* \gamma_{2n} \rightarrow X$, and a θ -structure on a $2n$ -dimensional manifold U is a bundle map $\hat{\ell}_U : TU \rightarrow \theta^* \gamma_{2n}$. Similarly, a θ -structure on a $(2n - 1)$ -manifold P is a bundle map $\hat{\ell}_P : \epsilon^1 \oplus TP \rightarrow \theta^* \gamma_{2n}$; if P is the boundary of a collared manifold U , then the collar gives an identification $TU|_P = \epsilon^1 \oplus TP$ and hence a θ -structure on U induces one of P . We call the underlying map $\ell_U : U \rightarrow X$ the *structure map* of the θ -manifold $(U, \hat{\ell}_U)$.

In this section, we construct a certain sequence of manifolds W_k such that $W_0 = W$ and such that $B\text{Diff}_\partial(W_k)$ homologically approximates the infinite loop space $\Omega_0^\infty \text{MT}\theta$. Let us recall the following

Definition 4.2.1 (Wall [47]). A space X is of *type* (F_k) if there exists a finite complex K and a k -connected map $K \rightarrow X$.

Each connected space is of type (F_0) , and being of type (F_1) means that $\pi_1(X)$ is finitely generated. As we have a 2-connected map $\ell_W : W \rightarrow X$ from a compact manifold, the space X is of type (F_2) . If X happens to be of type (F_n) then the proof of Theorem F we shall give can be somewhat simplified: many of the constructions in this section are designed to deal with the more complicated case when X is only of type (F_2) .

As we have said, we need to construct a suitable sequence of manifolds. Before we state the result, we introduce some notation. Let M_p , $p \in \mathbb{Z}$, be closed $(2n-1)$ -dimensional manifolds and $V_{p,p+1} : M_p \rightsquigarrow M_{p+1}$ be cobordisms. We then let, for $p < q$,

$$V_{p,q} := V_{p,p+1} \cup V_{p+1,p+2} \cup \cdots \cup V_{q-1,q}$$

and let $V_{-\infty,q} = \bigcup_{p=-\infty}^q V_{p,q}$ be the infinite composition.

Theorem 4.2.2. *There exist $(2n-1)$ -dimensional θ -manifolds M_p for $p \in \mathbb{Z}$ and θ -cobordisms $V_{p,p+1} : M_p \rightsquigarrow M_{p+1}$, such that*

- (i) $M_0 = \partial W$, and $V_{0,1}$ is the manifold $W \cup_{\partial W} (\partial W \times [0,1] \setminus D^{2n})$, viewed as a θ -cobordism $\partial W \rightsquigarrow S^{2n-1}$.
- (ii) For $p \geq 1$, $M_p = S^{2n-1}$ and $V_{p,p+1} = (S^{2n-1} \times [p,p+1]) \# (S^n \times S^n)$.
- (iii) The inclusion $M_p \rightarrow V_{p,p+1}$ is $(n-1)$ -connected, for all p .
- (iv) The inclusion $M_{p+1} \rightarrow V_{p,p+1}$ is 2-connected, for all $p < 0$.
- (v) The structure map $\ell_{V_{-\infty,p}} : V_{-\infty,p} \rightarrow X$ is n -connected if $p \geq 0$.
- (vi) Each of the manifolds M_p contains an embedded disc D^{2n-1} , and each $V_{p,p+1}$ contains an embedded strip $S_{p,p+1} = D^{2n-1} \times [p,p+1]$ restricting to the embedded discs on both ends. For $p \geq 0$, the disc in $M_p = S^{2n-1}$ is the lower half disc D_-^{2n-1} , and the strip in $V_{p,p+1} = (S^{2n-1} \times [p,p+1]) \# (S^n \times S^n)$ is $D_-^{2n-1} \times [p,p+1]$ (and we assume that the connected sum with $S^n \times S^n$ is taken on the upper half part of $S^{2n-1} \times [p,p+1]$).

We prepare for the proof of Theorem 4.2.2 with some basic homotopy theory.

Lemma 4.2.3. *There is a relative CW-complex (C,W) and an extension $\ell_C : C \rightarrow X$ of $\ell_W : W \rightarrow X$ such that*

- (i) ℓ_C is n -connected, and
- (ii) there exists a filtration $W = C_0 \subset C_1 \subset C_2 \subset \cdots$ of C such that C_{m+1} is obtained from C_m by attaching a single cell of dimension $3 \leq k_m \leq n$.

Proof. First note that the homotopy groups of X are countable, as those of $BO(2n)$ and W are.

For part (i), start by letting $C^{(2)} := W$; the structure map $\ell_{C^{(2)}} := \ell_W : C^{(2)} \rightarrow X$ is 2-connected. Look at the portion

$$\pi_3(X) \longrightarrow \pi_3(X, C^{(2)}) \longrightarrow \pi_2(C^{(2)})$$

of the long exact sequence for the pair $(X, C^{(2)})$. The two outer homotopy groups are countable, so $\pi_3(X, C^{(2)})$ is too, and hence we can attach countably many 3-cells to $C^{(2)}$ to arrive at a countable complex $C^{(3)}$ with a 3-connected map to X . Repeating this process up to dimension n , we obtain a countable complex $C = C^{(n)}$ with an n -connected map $\ell_C : C \rightarrow X$.

For part (ii), note that C has countably many k -cells for $3 \leq k \leq n$. Now we use the well-known property that the attaching map of each k -cell goes into a finite subcomplex of $C^{(k-1)}$. Using this property, we find a bijection from \mathbb{N} to the set of cells of $C \setminus C^{(2)}$, such that the attaching map of the m th cell goes into the subcomplex of C given by the union of the first $m-1$ cells with $C^{(2)}$. Now let C_m be the union of $C^{(2)}$ with the first m cells. \square

Lemma 4.2.4. *Let $\theta : Y \rightarrow BO(2n)$ be a fibration, assume that Y is of type (F_n) and let $k \leq n$. Let W be a $2n$ -manifold with a θ -structure $\ell_W : W \rightarrow Y$ which is $(k-1)$ -connected. Furthermore, assume that the inclusion $M := \partial W \rightarrow W$ is $(n-1)$ -connected. Then there is a θ -cobordism $V : M' \rightsquigarrow M$ such that (V, M) is $(k-1)$ -connected, (V, M') is $(n-1)$ -connected, and the structure map $\ell_V \cup \ell_W : V \cup_M W \rightarrow Y$ is n -connected.*

Proof. By induction it enough to find a V as in the statement of the lemma such that $\ell_V \cup \ell_W : V \cup_M W \rightarrow Y$ is k -connected. This is because composing two cobordisms which are $(l-1)$ -connected relative to their left (say) hand ends gives a cobordism which is again $(l-1)$ -connected relative to its left hand end; similarly relative to their right hand ends.

By [47, Theorem A], there are finitely many elements $x_1, \dots, x_r \in \pi_k(Y, W)$ so that attaching k -cells to W along all of those x_i and extending the map to Y yields an n -connected map $W \cup \bigcup_{i=1}^r D^k \rightarrow Y$. The long exact sequence of the triple $M \rightarrow W \rightarrow Y$ yields the piece

$$\pi_k(Y, M) \longrightarrow \pi_k(Y, W) \longrightarrow \pi_{k-1}(W, M) = 0.$$

Hence each of the elements x_i can be represented by a square of the form

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{g} & M \\ \downarrow & & \downarrow \ell_W|_M \\ D^k & \xrightarrow{f} & Y. \end{array}$$

Since $k \leq n$, we can perturb g so that it becomes an embedding, by general position. The normal bundle of g is stably trivial because $\nu_g \oplus TS^{k-1} \oplus \mathbb{R} \cong g^* \ell_W|_M^* \theta^* \gamma_{2n} \cong f^* \theta^* \gamma_{2n}$. Again because $k \leq n$, the normal bundle of g is actually trivial, so that g extends to an embedding $\hat{x}_i : S^{k-1} \times D^{2n-k} \rightarrow M$, which may be taken to be disjoint for different i . Now attach k -handles along these embeddings. The θ -structure on W can be extended over these handles, see e.g. [20, §4.1], and the cobordism V may be taken to be the composition of the elementary cobordisms of these r handle attachments. The connectivity conditions are clearly satisfied, as the cobordism only has k -handles relative to M . \square

Proof of Theorem 4.2.2. For $p \geq 0$, the manifolds M_p and $V_{p,p+1}$ are already prescribed in the Theorem. Note that $V_{0,1} \cong W \setminus D^{2n}$.

Let $\theta_m : Y_m \rightarrow X$ be the fibrant replacement of the map $\ell_C|_{C_m} : C_m \rightarrow X$ from Lemma 4.2.3 and let $f_m : Y_m \rightarrow Y_{m+1}$ be the map induced by the inclusion $C_m \subset C_{m+1}$. Note that f_m is a map over X and hence over $BO(2n)$. Furthermore, each Y_m is of type (F_n) and each f_m is 2-connected.

The manifold W has a θ_0 -structure with n -connected structure map $W \rightarrow Y_0$, by construction, and $\partial W \rightarrow W$ is $(n-1)$ -connected. Using the map f_0 , we can view W as a θ_1 -manifold, and the structure map $W \rightarrow Y_1$ is 2-connected. Applying Lemma

4.2.4, we find a θ_1 -cobordism $V_{-1,0} : M_{-1} \rightsquigarrow M_0 = \partial W$ such that the structure map $V_{-1,0} \cup_{M_0} W \rightarrow Y_1$ is n -connected, such that $M_0 \rightarrow V_{-1,0}$ is 2-connected and such that $M_{-1} \rightarrow V_{-1,0}$ is $(n-1)$ -connected. Continuing in this way, we construct a sequence of cobordisms

$$\dots \overset{V_{-3,-2}}{\rightsquigarrow} M_{-2} \overset{V_{-2,-1}}{\rightsquigarrow} M_{-1} \overset{V_{-1,0}}{\rightsquigarrow} M_0$$

such that $V_{-p-1,-p}$ has a Y_{p+1} -structure and such that the structure map $V_{p,0} \cup_{M_0} W \rightarrow Y_p$ is n -connected. This property persists if a disc is removed from the interior of W , so we have constructed the required manifolds.

It remains to embed strips $D^{2n-1} \times [p, p+1]$ into $V_{p,p+1}$, for all $p \leq -1$. This can be done inductively, using the connectivity of the cobordisms with respect to both ends. \square

4.3. Application of high-dimensional Madsen–Weiss theory. We continue to let $\theta : X \rightarrow BO(2n)$ be the second Moore–Postnikov stage of the structure map $W \rightarrow BO(2n)$. In Theorem 4.2.2, we constructed certain manifolds M_p and cobordisms $V_{p,p+1} : M_p \rightsquigarrow M_{p+1}$.

Next let

$$\ell_{V_{-p,\infty}} : V_{-p,\infty} \overset{\ell_{V_{-p,\infty}}^p}{\dashrightarrow} X_p \xrightarrow{u_p} X$$

be the n th stage of the Moore–Postnikov factorization of the θ -structure on $V_{-p,\infty}$, and define $\theta_p = \theta \circ u_p : X_p \rightarrow BO(2n)$. The maps

$$V_{-p,q} \subset V_{-p,\infty} \overset{\ell_{V_{-p,\infty}}^p}{\dashrightarrow} X_p \xrightarrow{u_p} X$$

form the n th stage of the Moore–Postnikov factorization of the θ -structure on $V_{-p,q}$ for any $q \geq 1$. By the naturality of the Moore–Postnikov factorization, we obtain a commutative diagram

$$\begin{array}{ccccccc} V_{0,1} & \longrightarrow & V_{-1,1} & \longrightarrow & V_{-2,2} & \longrightarrow & \cdots \longrightarrow V_{-\infty,\infty} \\ \downarrow \ell_0 & & \downarrow \ell_1 & & \downarrow \ell_2 & & \downarrow \ell \\ X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & \cdots \longrightarrow X \\ \downarrow \theta_0 & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta \\ BO(2n) & \equiv & BO(2n) & \equiv & BO(2n) & \equiv & \cdots \equiv BO(2n). \end{array}$$

Note that each space X_p is of type (F_n) , as $\ell_p := \ell_{V_{-p,\infty}}^p|_{V_{-p,p}} : V_{-p,p} \rightarrow X_p$ is an n -connected map from a compact manifold, and that θ_p is n -coconnected. Moreover, the natural map $\text{hocolim}_p X_p \rightarrow X$ is a weak equivalence and each of the maps $f_p : X_p \rightarrow X_{p+1}$ are 2-connected.

Now let $\text{Diff}_\partial(V_{p,q}, S_{p,q})$ be the group of diffeomorphisms of $V_{p,q}$ which are the identity near the boundary and on the embedded strip $S_{p,q} \subset V_{p,q}$. Let $\mathcal{I} = (\mathbb{N}_0, \leq)$ and $\mathcal{J} = (\mathbb{N}, \leq)$ as directed sets. The classifying spaces of these groups induce a directed system of spaces, indexed by $\mathcal{I} \times \mathcal{J}$, namely

$$(p, q) \longmapsto B\text{Diff}_\partial(V_{-p,q}, S_{-p,q})$$

with the maps induced by extending diffeomorphism as the identity. For a space A and points $a_0, a_1 \in A$ we write $\Omega_{a_0, a_1} A$ for the space of paths in A starting at a_0 and ending at a_1 . The parametrised Pontrjagin–Thom construction provides us

with points $M_{-p}, M_q \in \Omega^{\infty-1}\text{MT}\theta_{\mathfrak{p}}$ given by the Pontrjagin–Thom collapse maps of the $(2n-1)$ -manifolds M_{-p} and M_q , and a map

$$\alpha_{p,q} : \text{BDiff}_{\partial}(V_{-p,q}, S_{-p,q}) \longrightarrow \Omega_{M_{-p}, M_q} \Omega^{\infty-1}\text{MT}\theta_{\mathfrak{p}},$$

which under gluing of the cobordism $V_{q,q+1}$ participates in the commutative diagram

$$\begin{array}{ccc} \text{BDiff}_{\partial}(V_{-p,q}, S_{-p,q}) & \xrightarrow{\alpha_{p,q}} & \Omega_{M_{-p}, M_q} \Omega^{\infty-1}\text{MT}\theta_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ \text{BDiff}_{\partial}(V_{-p,q+1}, S_{-p,q+1}) & \xrightarrow{\alpha_{p,q+1}} & \Omega_{M_{-p}, M_{q+1}} \Omega^{\infty-1}\text{MT}\theta_{\mathfrak{p}}, \end{array}$$

where the right hand vertical is the concatenation with the path induced by the Pontrjagin–Thom collapse map of $V_{q,q+1}$.

Remark 4.3.1. Strictly speaking this does not make sense, as the parametrised Pontrjagin–Thom, or “scanning”, construction depends on choices e.g. of tubular neighbourhoods, so in principle just gives a homotopy class of map. There are several standard solutions to this problem. One is that in fact the Pontrjagin–Thom construction depends on a *contractible* space of choices and one can simply build such choices in to a model for the classifying space $\text{BDiff}_{\partial}(V_{-p,q}, S_{-p,q})$ used. Another solution is to replace the spectrum $\text{MT}\theta_{\mathfrak{p}}$ with a homotopy equivalent spectrum $\text{GRW}\theta_{\mathfrak{p}}$ constructed from all manifolds in \mathbb{R}^{∞} , not just affine ones: this receives a canonical scanning map when $\text{BDiff}_{\partial}(V_{-p,q}, S_{-p,q})$ is modelled as a space of θ_p -manifolds in \mathbb{R}^{∞} . Either solution works for our purposes, though we slightly prefer the second option. An extensive discussion of this option is given Section 2.5 of [15].

Taking the above into account, we obtain an induced map

$$\text{hocolim}_q \text{BDiff}_{\partial}(V_{-p,q}, S_{-p,q}) \xrightarrow{\text{hocolim}_q \alpha_{p,q}} \text{hocolim}_q \Omega_{M_{-p}, M_q} \Omega^{\infty-1}\text{MT}\theta_{\mathfrak{p}} \simeq \Omega^{\infty}\text{MT}\theta_{\mathfrak{p}}.$$

Theorem 4.3.2. *This map is acyclic on to the path-component which it hits.*

Proof. This is simply an application of [21, Theorem 1.5]; let us explain how to connect that with the formulation given here. We assume familiarity with the notation of [21].

The manifold $V_{-p,q}^{\circ} := V_{-p,q} \setminus \text{int}(S_{-p,q})$, with rounded corners, has boundary $P := \partial V_{-p,q}^{\circ}$. The inclusion $M_{-p} \subset V_{-p,q}$ is $(n-1)$ -connected by Theorem 4.2.2 (iii) and induction (if $q \geq 1$). It follows that the inclusions $M_{-p} \setminus \text{int}(D^{2n-1}) \subset V_{-p,q}^{\circ}$ and hence $P \subset V_{-p,q}^{\circ}$ are also $(n-1)$ -connected.

The map

$$V_{-p,q}^{\circ} \subset V_{-p,q} \subset V_{-p,\infty} \xrightarrow{\ell_{V_{-p,\infty}}^p} X_p$$

is n -connected, and restricts to a θ_p -structure $\hat{\ell}_P^{(q)} : \epsilon^1 \oplus TP \rightarrow \theta_p^* \gamma_{2n}$ on $P = \partial V_{-p,q}^{\circ}$. The space $\text{Bun}_{n,\partial}^{\theta_p}(TV_{-p,q}^{\circ}; \hat{\ell}_P^{(q)})$ of θ_p -structures which extend $\hat{\ell}_P^{(q)}$ and whose underlying map is n -connected is therefore nonempty, and by obstruction theory (using that $(V_{-p,q}, P)$ is $(n-1)$ -connected and θ_p is n -coconnected) is hence contractible. Thus

$$\text{BDiff}_{\partial}(V_{-p,q}, S_{-p,q}) = \text{BDiff}_{\partial}(V_{-p,q}^{\circ}) \simeq \text{Bun}_{n,\partial}^{\theta_p}(TV_{-p,q}^{\circ}; \hat{\ell}_P^{(q)}) // \text{Diff}_{\partial}(V_{-p,q})$$

is a path component of the space $\mathcal{N}_n^{\theta_p}(P; \hat{\ell}_P^{(q)})$.

The manifold $V_{-p,q+1}^\circ$ is obtained from $V_{-p,q}^\circ$ by gluing on $V_{q,q+1}^\circ$, or in other words by boundary connect-sum with a manifold diffeomorphic to $S^n \times S^n \setminus \text{int}(D^{2n})$, so its boundary can also be identified with P , with a potentially different θ_p -structure $\hat{\ell}_P^{(q+1)}$. (In fact, as θ_p is once-stable, see [20, §5.1], it can be arranged that $\hat{\ell}_P^{(q+1)} = \hat{\ell}_P^{(q)}$, but that does not matter for this argument.) This identifies $\text{hocolim}_q B\text{Diff}_\partial(V_{-p,q}, S_{-p,q})$ with a path-component of the space

$$\text{hocolim}_{q \rightarrow \infty} \mathcal{N}_n^{\theta_p}(P; \hat{\ell}_P^{(q)})$$

and [21, Theorem 1.5] provides an acyclic map from this space to $\Omega^\infty \text{MT}\theta_p$. The proof of [21, Theorem 1.5] shows that it is indeed the map $\text{hocolim}_q \alpha_{p,q}$ which is acyclic. \square

Next, we use naturality of the above discussion with respect to p . Namely, the maps $f_p : X_p \rightarrow X_{p+1}$ induce maps of spectra $\text{MT}\theta_p \rightarrow \text{MT}\theta_{p+1}$ and the diagram

$$\begin{array}{ccc} B\text{Diff}_\partial(V_{-p,q}, S_{-p,q}) & \longrightarrow & \Omega_{M_{-p}, M_q} \Omega^{\infty-1} \text{MT}\theta_p \\ \downarrow & & \downarrow \\ B\text{Diff}_\partial(V_{-p-1,q}, S_{-p,q}) & \longrightarrow & \Omega_{M_{-p-1}, M_q} \Omega^{\infty-1} \text{MT}\theta_{p+1}. \end{array}$$

commutes³. Here the right hand vertical map is the map induced by f_p , followed by path concatenation with the path corresponding to the cobordism $V_{-p-1, -p}$. Altogether, we have two directed systems of spaces indexed by $\mathcal{I} \times \mathcal{J}$, namely

$$(p, q) \longmapsto B\text{Diff}_\partial(V_{-p,q}, S_{-p,q}) \quad \text{and} \quad (p, q) \longmapsto \Omega_{M_{-p}, M_q} \Omega^{\infty-1} \text{MT}\theta_p,$$

and the maps $\alpha_{p,q}$ together define a map of directed systems.

Corollary 4.3.3.

(i) *The map*

$$\text{hocolim}_{p,q} B\text{Diff}_\partial(V_{-p,q}, S_{-p,q}) \xrightarrow{\text{hocolim}_{p,q} \alpha_{p,q}} \text{hocolim}_{p,q} \Omega_{M_{-p}, M_q} \Omega^{\infty-1} \text{MT}\theta_p$$

is acyclic.

(ii) *There is a weak equivalence*

$$\text{hocolim}_{p,q} \Omega_{M_{-p}, M_q} \Omega^{\infty-1} \text{MT}\theta_p \longrightarrow \Omega_0^\infty \text{MT}\theta.$$

(iii) *The diagonal map*

$$\text{hocolim}_p B\text{Diff}_\partial(V_{-p,p}, S_{-p,p}) \longrightarrow \text{hocolim}_{p,q} B\text{Diff}_\partial(V_{-p,q}, S_{-p,q})$$

is a weak equivalence.

(iv) *There is an acyclic map*

$$\text{hocolim}_p B\text{Diff}_\partial(V_{-p,p}, S_{-p,p}) \longrightarrow \Omega_0^\infty \text{MT}\theta.$$

³The discussion of Remark 4.3.1 applies here too.

Proof. Part (i) follows from the fact the maps $\text{hocolim}_q \alpha_{p,q}$ are acyclic (by Theorem 4.3.2) and that directed (homotopy) colimits of acyclic maps are acyclic (which follows from the homological characterization of acyclic maps [26, §1]).

Part (ii) follows from the fact that $\text{hocolim}_p X_p \rightarrow X$ is a weak equivalence, which carries over to the Madsen–Tillmann–Weiss spectra and their infinite loop spaces, and the fact that path spaces (when non-empty) are homotopy equivalent to loop spaces.

Part (iii) follows from the fact that $p \mapsto (p, p) : \mathbb{N} \rightarrow \mathcal{I} \times \mathcal{J}$ is cofinal, and part (iv) follows by combining all the previous parts. \square

4.4. Finishing the proof of Theorem F. After all these preliminaries, we can now prove Theorem F by the same sort of arguments as those in [6, §4]. Because the argument is so similar, we shall be quite brief. We use the notation introduced in the previous section.

Lemma 4.4.1. *Under the hypotheses of Theorem F there are psc metrics $g_p \in \mathcal{R}^+(M_p)$ and $h_{p,p+1} \in \mathcal{R}^+(V_{p,p+1})_{g_p, g_{p+1}}$ such that*

- (i) $h_{p,p+1}$ is stable for all $p \neq 0$,
- (ii) $h_{0,1}$ is left stable.

Proof. Let $p \geq 1$. Then $M_p = S^{2n-1}$ and $V_{p,p+1} = (S^{2n-1} \times [p, p+1]) \# (S^n \times S^n)$. For those p , put $g_p := g_o \in \mathcal{R}^+(M_p) = \mathcal{R}^+(S^{2n-1})$. For $p \geq 1$, we let $h_{p,p+1} \in \mathcal{R}^+(V_{p,p+1})_{g_p, g_{p+1}}$ be a metric which corresponds to $g_p + dt^2$ under the surgery equivalence $\mathcal{R}^+(V_{p,p+1})_{g_p, g_{p+1}} \simeq \mathcal{R}^+(S^{2n-1} \times [p, p+1])_{g_o, g_o}$ (exactly as in [6, Proposition 4.8]). By Lemma 3.3.3, $h_{p,p+1}$ is stable.

Let $p = 0$. Recall that $W = V_{0,1} \cup D^{2n}$. By the hypotheses of Theorem F there is a psc metric $g_0 \in \mathcal{R}^+(M_0)$, and a left stable $h \in \mathcal{R}^+(W)$ (here we view W as a cobordism $M_0 \rightsquigarrow \emptyset$, and turning the direction of a cobordism turns right stable into left stable psc metrics). The torpedo metric on $D^{2n} : S^{2n-1} \rightsquigarrow \emptyset$ is left stable. Hence there is a metric $h_{0,1} \in \mathcal{R}^+(V_{0,1})_{g_0, g_1}$ such that $h_{0,1} \cup g_{\text{tor}} \in \mathcal{R}^+(W)_{g_0}$ lies in the same component as h . By Lemma 3.3.1, $h_{0,1}$ is left stable.

Let $p < 0$. Since the cobordism $V_{p,p+1}$ is 2-connected with respect to either end, the existence of the metrics g_p and $h_{p,p+1}$ with the desired properties follows by a repeated application of Corollary 1.2.2. \square

Lemma 4.4.2. *The action of $\text{Diff}_\partial(V_{-p,q})$ on $\mathcal{R}^+(V_{-p,q})_{g_{-p}, g_q}$ is homotopy abelian, for $p \geq 0$ and $q > 0$.*

Proof. This is clear from Corollary 4.1.6 and Lemma 4.4.1. \square

The rest of the argument is as in [6, §4] and will only be sketched. Let us write

$$B_0 := B\text{Diff}_\partial(V_{0,1}, S_{0,1}); \quad T_0 := E\text{Diff}_\partial(V_{0,1}, S_{0,1}) \times_{\text{Diff}_\partial(V_{0,1}, S_{0,1})} \mathcal{R}^+(V_{0,1})_{g_0, g_1}$$

and, for $p \geq 1$,

$$\begin{aligned} B_p &:= B\text{Diff}_\partial(V_{-p,p}, S_{-p,p}) \\ T_p &:= E\text{Diff}_\partial(V_{-p,p}, S_{-p,p}) \times_{\text{Diff}_\partial(V_{-p,p}, S_{-p,p})} \mathcal{R}^+(V_{-p,p})_{g_{-p}, g_p}. \end{aligned}$$

Gluing on the psc manifolds $(V_{p,p+1}, h_{p,p+1})$ and $(V_{-p-1,-p}, h_{-p-1,p})$ gives commutative and homotopy cartesian diagrams

$$\begin{array}{ccc} T_p & \longrightarrow & T_{p+1} \\ \downarrow \pi_p & & \downarrow \pi_{p+1} \\ B_p & \longrightarrow & B_{p+1}, \end{array}$$

and passing to the homotopy colimit we obtain a fibration

$$\pi_\infty : T_\infty \longrightarrow B_\infty := \operatorname{hocolim}_p B_p$$

from which the fibration $\pi_0 : T_0 \rightarrow B_0$ is pulled back (up to homotopy). The same obstruction theoretic argument as in [6, Proposition 4.12], using Lemma 4.4.2, applied to the acyclic map $\operatorname{hocolim}_p B_p \rightarrow \Omega_0^\infty \mathbf{MT}\theta$ of Corollary 4.3.3 (iv), produces a fibration $T_\infty^+ \rightarrow \Omega_0^\infty \mathbf{MT}\theta$ which pulls back to $T_\infty \rightarrow B_\infty$ and hence to $T_0 \rightarrow B_0$. In other words, we have constructed a diagram

$$\begin{array}{ccccc} T_0 & \longrightarrow & T_\infty & \xrightarrow{\psi'} & T_\infty^+ \\ \downarrow \pi_0 & & \downarrow \pi_\infty & & \downarrow \pi_\infty^+ \\ B_0 & \longrightarrow & B_\infty & \xrightarrow{\psi} & \Omega_0^\infty \mathbf{MT}\theta \end{array} \quad (4.4.3)$$

in which each square is homotopy cartesian. This finishes the proof of Theorem F. For later use, we record:

Remark 4.4.4. The maps ψ and ψ' in (4.4.3) are acyclic.

5. THE SECONDARY ROSENBERG INDEX

In this section, we explain how to extend the results of [6, §3] to the case of a nontrivial fundamental group. In order to avoid repetitions of large portions of loc. cit., we only explain the differences. We will then explain how, together with Theorem F, these results imply Theorem A. The reader is warned that the following pages are not meant to be understandable without reference to [6].

5.1. The Rosenberg–Dirac operator. Let G be a discrete group and let M be a Riemannian spin manifold equipped with a map $M \rightarrow BG$ (or equivalently with a G -Galois cover). With these data, Rosenberg [43] associated a certain Dirac operator on an infinite dimensional bundle over M .

Before we recapitulate the construction, we recall the notion of a *Real* C^* -algebra [34]. This is a C^* -algebra \mathbf{A} over the complex numbers, together with a complex-antilinear automorphism $a \mapsto \bar{a}$ of order 2. Important examples are: \mathbf{R} , which denotes \mathbb{C} with the complex conjugation; $\mathbf{C}_0(X)$, the algebra of all continuous functions $f : X \rightarrow \mathbb{C}$ from a locally compact Hausdorff space which vanish at ∞ , with conjugation $\bar{f}(x) := \overline{f(x)}$. The complexification $\mathbf{Cl}^{p,q}$ of the real Clifford algebra⁴ $\mathbf{Cl}^{p,q}$ is also important for us.

Let $\mathbb{C}[G]$ be the complex group ring, with the involution $(\sum_g a_g g)^* := \sum_g \bar{a}_g g^{-1}$. The regular representation of G on $L^2(G)$ induces an injective ring homomorphism $\rho : \mathbb{C}[G] \rightarrow \operatorname{Lin}(L^2(G))$ which preserves $*$, and we define $\|x\|_r := \|\rho(x)\|$. The *reduced group C^* -algebra* $\mathbf{C}_r^*(G)$ is the completion of $\mathbb{C}[G]$ with respect to the norm

⁴The first p generators have square -1 , the last q generators have square $+1$.

$\|\cdot\|_r$. The *maximal* (or full) group \mathbf{C}^* -algebra $\mathbf{C}_m^*(G)$ is obtained by completing $\mathbb{C}[G]$ with respect to the norm $\|x\|_m := \sup_\lambda \|\lambda(x)\|$, where λ runs over all unitary representations of G on Hilbert spaces [28, 3.7.4]. If G is countable, then both group \mathbf{C}^* -algebras are separable. The conjugation $\overline{\sum_g a_g g} := \sum_g \overline{a_g} g$ on $\mathbb{C}[G]$ extends to Real structures on $\mathbf{C}_r^*(G)$ and $\mathbf{C}_m^*(G)$. We will write $\mathbf{C}^*(G)$ in all statements which apply to both $\mathbf{C}_r^*(G)$ and $\mathbf{C}_m^*(G)$.

We consider $\mathbf{C}^*(G)$ as a right module over itself; the formula $\langle x, y \rangle := x^*y$ turns $\mathbf{C}^*(G)$ into a Hilbert- $\mathbf{C}^*(G)$ -module. The unitary group $U(\mathbf{C}^*(G))$ contains G as a subgroup; hence G acts by left-multiplication on $\mathbf{C}^*(G)$ by $\mathbf{C}^*(G)$ -linear operators preserving the inner product. The *Mishchenko–Fomenko line bundle* is the bundle

$$\mathcal{L}_G := EG \times_G \mathbf{C}^*(G) \longrightarrow BG$$

of (rank 1 free) Hilbert modules. Let (M, g) be a Riemannian spin manifold of dimension d , with spinor bundle \mathfrak{S}_M . This is a $\mathbf{Cl}(TM \oplus \mathbb{R}^{0,d})$ -module (in the terminology of [6, §3.1.3]). If in addition M is equipped with a map $f : M \rightarrow BG$, then

$$\mathfrak{S}_M \otimes f^* \mathcal{L}_G \longrightarrow M$$

is a bundle of $\mathbf{C}^*(G)$ -Hilbert modules (projective, of finite rank) with $\mathbf{C}^*(G)$ -valued inner product $\langle -, - \rangle$, and has a compatible action of $\mathbf{Cl}(TM) \otimes \mathbf{Cl}^{0,d}$. The spinor bundle \mathfrak{S}_M inherits a connection from the Levi-Civita connection on M , and $f^* \mathcal{L}_G$ has a natural flat connection, so $\mathfrak{S}_M \otimes f^* \mathcal{L}_G$ has the tensor product connection. The *Rosenberg–Dirac operator* or *G-Dirac operator*

$$\mathfrak{D}_f = \mathfrak{D}_{f,g} : \Gamma(M; \mathfrak{S}_M \otimes f^* \mathcal{L}_G) \rightarrow \Gamma(M; \mathfrak{S}_M \otimes f^* \mathcal{L}_G)$$

is defined by the classical formula [37, §II.5]. The Schrödinger–Lichnerowicz formula

$$\mathfrak{D}_{f,g}^2 = \nabla^* \nabla + \frac{1}{4} \text{scal}(g)$$

still holds, and relates $\mathfrak{D}_{f,g}$ to positive scalar curvature on M . The G -Dirac operator is formally self-adjoint, odd with respect to the grading on $\mathfrak{S}_M \otimes f^* \mathcal{L}_G$ and $\mathbf{C}^*(G)$ -linear. It *anticommutes* with the action of $\mathbf{Cl}^{0,d}$.

Remark 5.1.1. In [6], we used a slightly different setup in which the Dirac operator was $\mathbf{Cl}^{d,0}$ -linear. The translation between these is explained in [6, p. 773].

We can extend this construction to the family case, as discussed in [6, §3.2.2]. Assume that $\pi : E \rightarrow X$ is a bundle of d -dimensional compact spin manifolds with fibre W equipped with Riemannian metrics and that $f : E \rightarrow BG$ is a fixed map. Assume that the boundary bundle is trivial, $\partial E = X \times \partial W$. As in [6, §3.2], we form the elongation \hat{E} by adding $\partial E \times [0, \infty)$, and extend the spin structure, Riemannian metric and map to BG in the obvious fashion. We obtain a bundle of noncompact manifolds with cylindrical ends, again denoted by E . We denote the fibres of π by E_x , the Riemannian metrics on E_x by g_x etc.

Let us now turn to the analytical properties of the G -Dirac operator, parallel to [6, §3.2.3]. We deviate from the setting used in [6] and rely on the analytical results proven in [14]. Instead of the Hilbert bundles in [6], we use *continuous fields of Hilbert- $\mathbf{C}^*(G)$ -modules*. This notion is defined in [14, §2.1] and is a straightforward adaptation of the notion of a continuous field of Hilbert spaces from [13]. For each

$x \in X$, the space $\Gamma_c(E_x; \mathfrak{E}_x \otimes \mathcal{L}_{f,x})$ of smooth, compactly supported sections has an $\mathbf{C}^*(G)$ -valued inner product

$$\langle s, t \rangle := \int_{E_x} \langle s(y), t(y) \rangle d\text{vol}_{E_x}(y)$$

with completion $L^2(E_x; \mathfrak{E}_x \otimes \mathcal{L}_{f,x})$, a countably generated Real graded Hilbert $\mathbf{C}^*(G)$ -module with an action of $\mathbf{CI}^{0,d}$. The family $(L^2(E_x; \mathfrak{E}_x \otimes \mathcal{L}_{f,x}))_{x \in X}$ assembles to a continuous field of Hilbert- $\mathbf{C}^*(G)$ -modules $L^2_X(E; \mathfrak{E}_E \otimes f^* \mathcal{L}_G)$; see [14, Example 2.12] for further details. The individual G -Dirac operators $\mathfrak{D}_{f,x}$ on E_x are unbounded symmetric operators, and together they form an unbounded operator family \mathfrak{D}_f , in the sense of [14, §2.2]. By [14, Theorem 1.14 and Example 2.28], the closure of this operator family is self-adjoint in the sense of [14, Definition 2.26]. The point here is that the projection $p : \partial E \times [0, \infty) \rightarrow [0, \infty)$ can be extended to what is called a ‘‘coercive function’’ such that $[\mathfrak{D}_f, p]$ is bounded. This is a generalization of the classical result [28, Proposition 10.2.10]. Therefore, we can use functional calculus [14, §2.3] and form the bounded transform $\frac{\mathfrak{D}_f}{(1 + \mathfrak{D}_f^2)^{1/2}}$. This is a bounded self-adjoint operator family on $L^2_X(E; \mathfrak{E}_E \otimes f^* \mathcal{L}_G)$.

Proposition 5.1.2. [14, Theorem 2.41 and Lemma 2.42] *If the scalar curvature of $E \rightarrow X$ is positive on the ends, then $F = (\frac{\mathfrak{D}_{f,x}}{(1 + \mathfrak{D}_{f,x}^2)^{1/2}})_{x \in X}$ is a Fredholm family on $L^2_X(E; \mathfrak{E}_E \otimes f^* \mathcal{L}_G)$ (in the sense of [14, Definition 2.17]). If the scalar curvature is everywhere positive over a closed subspace $Y \subset X$, then $F|_Y$ is invertible.*

5.2. K -theory with coefficients in a \mathbf{C}^* -algebra. The framework for topological K -theory that we use is different from that in [6, §3.1] and is that developed in [14, §3]. Let us quickly recall the definition.

Definition 5.2.1. Let \mathbf{A} be a Real graded \mathbf{C}^* -algebra (the only relevant example for us is $\mathbf{A} = \mathbf{C}^*(G)$) and let (X, Y) be a space pair. The group $KO^{-d}(X, Y; \mathbf{A})$ is the group of equivalence classes of tuples (H, ι, c, F) , where

- (i) H is a continuous field of Real Hilbert- \mathbf{A} -modules on X , with grading ι ,
- (ii) c is a $\mathbf{CI}^{0,d}$ -structure on H (see [14, Definition 3.1]),
- (iii) F is a self-adjoint graded Fredholm family on H , and $c(v)F + Fc(v) = 0$ for all $v \in \mathbb{R}^d \subset \mathbf{CI}^{0,d}$, such that
- (iv) $F|_Y$ is invertible.

Two such tuples are equivalent if they are concordant. The group structure is given by direct sum.

When (X, Y) is a compact Hausdorff pair, the group $KO^{-d}(X, Y; \mathbf{A})$ is naturally isomorphic to Kasparov’s Real KK -groups

$$\begin{aligned} KK(\mathbf{CI}^{0,d}, \mathbf{C}_0(X - Y) \otimes \mathbf{A}) &\cong KK(\mathbf{C}, \mathbf{C}_0(X - Y) \otimes \mathbf{A} \otimes \mathbf{CI}^{d,0}) \\ &\cong KK(\mathbf{C}, \mathbf{C}_0((X - Y) \times \mathbb{R}^d) \otimes \mathbf{A}) \end{aligned}$$

by [14, Proposition 3.12]. The functor $(X, Y) \mapsto KO^{-d}(X, Y; \mathbf{A})$ is the degree $-d$ part of the cohomology theory represented by the real K -theory spectrum of the graded \mathbf{C}^* -algebra \mathbf{A} (the values for positive degrees can be defined using the Clifford algebra $\mathbf{CI}^{d,0}$ instead of $\mathbf{CI}^{0,d}$). As in [6], we shall represent elements in $KO^{-d}(X, Y; \mathbf{A})$ by maps of pairs

$$(X, Y) \longrightarrow (\Omega^{\infty+d} \mathbf{KO}(\mathbf{A}), *).$$

Hence if $\pi : E \rightarrow X$ is a bundle of spin manifolds with cylindrical ends, equipped with a map $f : E \rightarrow BG$ and a fibrewise Riemannian metric which is cylindrical over the ends and has positive scalar curvature, then the G -Dirac operators $\mathfrak{D}_{f,g}$ define an element

$$\text{ind}(\mathfrak{D}_{f,g}) = [L_X^2(E; \mathfrak{S}_E \otimes f^* \mathcal{L}_G), \iota, c, \frac{\mathfrak{D}_{f,g}}{(1 + \mathfrak{D}_{f,g}^2)^{1/2}}] \in KO^{-d}(X; \mathbf{C}^*(G))$$

and if the scalar curvature is positive over a subspace $Y \subset X$, then this refines to an element in the relative group $KO^{-d}(X, Y; \mathbf{C}^*(G))$.

5.3. The index difference. If W is a d -dimensional compact spin manifold with boundary, equipped with a map $f : W \rightarrow BG$, and $g \in \mathcal{R}^+(\partial W)$, then we can define the Hitchin version of the index difference

$$\text{inddiff}^G : \mathcal{R}^+(W)_g \times \mathcal{R}^+(W)_g \longrightarrow \Omega^{\infty+d+1} \mathbf{KO}(\mathbf{C}^*(G)),$$

analogously to [6, §3.3.1]. It should be emphasised that it of course depends on the homotopy class of the map f and not merely on the abstract group G .

Remark 5.3.1. For closed W , we also can define the Gromov–Lawson version of the index difference, as in [6, §3.3.2]. The main result of [17], restated as Theorem 3.10 in [6] shows that both definitions agree if $G = 1$. This was used in [6] to derive a detection theorem for odd-dimensional manifolds (Theorem B loc. cit.) from the even-dimensional case (Theorem A loc. cit.).

The proof given in [17] does not generalise to the case $G \neq 1$. In the forthcoming thesis of Lukas Buggisch, a proof that both versions agree for general G will be given, using Kasparov theory. With the aid of that result, one can prove a version of [6, Theorem C] from Theorem A, following exactly the argument in [6, §3.6].

5.4. The additivity theorem. An important ingredient in [6] was the additivity theorem for the index [6, Theorem 3.12] (a strengthening of a result by Bunke [8]). For the G -index, the additivity theorem continues to hold. When the base space is a point, this was already proven in [8]. For the general case, one follows the proof given in [6], replacing the quotations to [17] and [28] by quoting [14], in particular Propositions 2.34 and 2.36. The proof of [6, Theorem 3.16] (which states an additivity property for the index difference) carries over without change to the case $G \neq 1$.

5.5. The relative index construction. The results of [6, §3.5] are of formal nature and hold verbatim in the present more general framework, replacing KO by KO -theory with coefficients in $\mathbf{C}^*(G)$.

5.6. Some words about KK -theory. One of the key ingredients in [6] was the Atiyah–Singer family index theorem for the Dirac operators. Here, we need the family index theorem for the $\mathbf{C}^*(G)$ -valued index, in the real case. While this can certainly be extracted from e.g. [11], we are not aware of a detailed account. Presenting the details would lead us too far away from the main focus of this paper, and we content ourselves with precise statements and an overview of the main steps in the proof, using KK -theory. We begin by introducing notation for the Kasparov product.

- (i) A homomorphism $\phi : \mathbf{A} \rightarrow \mathbf{B}$ of \mathbf{C}^* -algebras defines a class $[\phi] \in KK(\mathbf{A}, \mathbf{B})$ [5, Example 17.1.2(a)].

- (ii) The product $KK(\mathbf{A}, \mathbf{B}) \times KK(\mathbf{B}, \mathbf{D}) \rightarrow KK(\mathbf{A}, \mathbf{D})$ is denoted $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \# \mathbf{y}$.
- (iii) The exterior product $KK(\mathbf{A}, \mathbf{B}) \times KK(\mathbf{A}', \mathbf{B}') \rightarrow KK(\mathbf{A} \otimes \mathbf{A}', \mathbf{B} \otimes \mathbf{B}')$ is denoted $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \boxtimes \mathbf{y}$.

We will give a homotopy-theoretic formulation of the index theorem, and the formulation uses results by Joachim and Stolz [31]. They constructed spectra $KK(\mathbf{A}, \mathbf{B})$ for each pair of Real graded C^* -algebras and spectrum maps

$$\begin{aligned} KK(\mathbf{A}, \mathbf{B}) \wedge KK(\mathbf{B}, \mathbf{D}) &\longrightarrow KK(\mathbf{A}, \mathbf{D}) \\ KK(\mathbf{A}, \mathbf{B}) \wedge KK(\mathbf{A}', \mathbf{B}') &\longrightarrow KK(\mathbf{A} \otimes \mathbf{A}', \mathbf{B} \otimes \mathbf{B}'), \end{aligned} \quad (5.6.1)$$

as well as an equivalence

$$F(\Sigma^\infty X_+, KK(\mathbf{A}, \mathbf{B})) \simeq KK(\mathbf{A}, C(X) \otimes \mathbf{B}) \quad (5.6.2)$$

(the left hand side denotes the function spectrum) for each compact space X . The equivalence (5.6.2) implies that

$$[\Sigma^\infty X_+, KK(\mathbf{A}, \mathbf{B})] \cong KK(\mathbf{A}, C(X) \otimes \mathbf{B}). \quad (5.6.3)$$

In particular $\pi_0(KK(\mathbf{A}, \mathbf{B})) \cong KK(\mathbf{A}, \mathbf{B})$, and the maps (5.6.1) realise the Kasparov product. Bott periodicity in KK -theory implies that there are natural equivalences

$$KK(CI^{p,q} \otimes \mathbf{A}, \mathbf{B}) \simeq KK(\mathbf{A}, CI^{q,p} \otimes \mathbf{B}) \simeq \Sigma^{p-q} KK(\mathbf{A}, \mathbf{B})$$

and hence isomorphisms

$$KK(CI^{p,q} \otimes \mathbf{A}, \mathbf{B}) \cong \pi_{q-p}(KK(\mathbf{A}, \mathbf{B})).$$

Moreover, there is a natural equivalence

$$KK(\mathbf{R}, \mathbf{A}) \simeq KO(\mathbf{A})$$

for each graded Real C^* -algebra \mathbf{A} . We further write $KO(\mathbf{R}) = KO$. The *analytic K -homology groups* of a compact Hausdorff space X are

$$KO_{p-q}^{\text{an}}(X) := KK(CI^{p,q} \otimes C(X), \mathbf{R}) \cong \pi_{p-q}(KK(C(X), \mathbf{R})).$$

For an arbitrary space Y , one defines

$$RKO_{p-q}(Y) := \operatorname{colim}_{X \subset Y} KO_{p-q}^{\text{an}}(X),$$

where the colimit runs over all compact subspaces of Y . The following result is folklore:

Proposition 5.6.4. *For CW complexes Y , there is a natural isomorphism*

$$KO_k(Y) := \pi_k(\Sigma^\infty Y_+ \wedge KO) \xrightarrow{\cong} RKO_k(Y)$$

from the topologically defined KO -homology groups $KO_k(Y)$.

Sketch of proof. This can be deduced quickly from [31], so we include the proof here. Let X be a finite CW-complex. Under the isomorphism

$$KK(C(X), C(X)) \cong \pi_0(KK(C(X), C(X))) \stackrel{(5.6.2)}{\cong} [\Sigma^\infty X_+, KK(C(X), \mathbf{R})],$$

the identity element $[\text{id}_{C(X)}] \in KK(C(X), C(X))$ corresponds to a map of spectra

$$u_X : \Sigma^\infty X_+ \longrightarrow KK(C(X), \mathbf{R}),$$

which depends naturally on X . Using the \mathbf{KO} -module structure on the \mathbf{KK} -spectra, we obtain a map $v_X : \Sigma^\infty X_+ \wedge \mathbf{KO} \xrightarrow{u_X \wedge \text{Id}} \mathbf{KK}(\mathbf{C}(X), \mathbf{R}) \wedge \mathbf{KO} \rightarrow \mathbf{KK}(\mathbf{C}(X), \mathbf{R})$. On homotopy groups, v_X yields a natural transformation

$$t_X : KO_k(X) \longrightarrow KO_k^{\text{an}}(X).$$

The source is excisive by general homotopy theory, and the target by excision in analytic K -homology. If $X = *$, then $u_X = 1 \in \pi_0(\mathbf{KO}) = \mathbb{Z}$, and this shows that t_* is an isomorphism. Hence t_X is an isomorphism for all finite CW-complexes, and this establishes the claimed result upon taking colimits. \square

The universal Mishchenko–Fomenko line bundle $\mathcal{L}_G \rightarrow BG$ yields a map

$$\mathcal{L}_G : \Sigma^\infty BG_+ \longrightarrow \mathbf{KK}(\mathbf{R}, \mathbf{C}^*(G))$$

of spectra. If $f : E \rightarrow BG$ is a map from a compact space, then $\mathcal{L}_G \circ (\Sigma^\infty f_+) \in [\Sigma^\infty E_+, \mathbf{KK}(\mathbf{R}, \mathbf{C}^*(G))]$ corresponds to an element $[\mathcal{L}_f] \in KK(\mathbf{R}, \mathbf{C}(E) \otimes \mathbf{C}^*(G))$ under the isomorphism (5.6.3). There is the following concrete description for $[\mathcal{L}_f]$:

$$[\mathcal{L}_f] = [\Gamma(E; f^* \mathcal{L}_G), -, 0] \in KK(\mathbf{R}, \mathbf{C}(E) \otimes \mathbf{C}^*(G)) \quad (5.6.5)$$

(the symbol $-$ denotes the usual representation of \mathbf{R} on the sections of $f^* \mathcal{L}_G$).

Definition 5.6.6. The *Novikov assembly map* is the composition

$$\nu : \mathbf{KK}(\mathbf{R}, \mathbf{R}) \wedge BG_+ \xrightarrow{\text{id} \wedge \mathcal{L}_G} \mathbf{KK}(\mathbf{R}, \mathbf{R}) \wedge \mathbf{KK}(\mathbf{R}, \mathbf{C}^*(G)) \longrightarrow \mathbf{KK}(\mathbf{R}, \mathbf{C}^*(G)).$$

As explained in the introduction to [31], the map ν induces the classical assembly map on homotopy groups. More precisely, for each compact subset $Y \subset BG$, we have the class $[\mathcal{L}|_Y] \in KK(\mathbf{R}, \mathbf{C}(Y) \otimes \mathbf{C}^*(G))$, and we get maps

$$\begin{aligned} KK(\mathbf{C}^{p,q} \otimes \mathbf{C}(Y), \mathbf{R}) &\longrightarrow KK(\mathbf{C}^{p,q}, \mathbf{C}^*(G)) \\ \mathbf{x} &\longmapsto ([\text{id}_{\mathbf{C}^{p,q}}] \boxtimes [\mathcal{L}|_Y]) \# (\mathbf{x} \boxtimes [\text{id}_{\mathbf{C}^*(G)}]). \end{aligned}$$

These maps are natural in Y and hence induce a map on the colimit

$$RKO_{p-q}(BG) := \text{colim}_{Y \subset BG} KK(\mathbf{C}^{p,q} \otimes \mathbf{C}(Y), \mathbf{R}) \longrightarrow KO_{p-q}(\mathbf{C}^*(G)), \quad (5.6.7)$$

which is the classical Novikov assembly map.

5.7. The Atiyah–Singer index theorem. Let $\pi : E \rightarrow X$ be a bundle of *closed* spin d -dimensional manifolds on a compact manifold X (X is allowed to have boundary) and let $f : E \rightarrow BG$ be a map. To these data there are associated elements in certain KK -groups, besides the class (5.6.5) given by the Mishchenko–Fomenko line bundle. The ordinary spin Dirac operator defines a class

$$[\mathcal{D}] = [L_X^2(E, \mathcal{S}_E), \mu, \frac{\mathcal{D}}{(1 + \mathcal{D}^2)^{1/2}}] \in KK(\mathbf{C}(E) \otimes \mathbf{C}^{0,d}, \mathbf{C}(X))$$

where μ is the representation of $\mathbf{C}(E)$ by multiplication operators, and the G -Dirac operator defines a class

$$\begin{aligned} [\mathcal{D}_f] &= [L_X^2(E, \mathcal{S}_E \otimes f^* \mathcal{L}_G), \mu, \frac{\mathcal{D}_f}{(1 + \mathcal{D}_f^2)^{1/2}}] \\ &\in KK(\mathbf{C}(E) \otimes \mathbf{C}^{0,d}, \mathbf{C}(X) \otimes \mathbf{C}^*(G)). \end{aligned}$$

The index of \mathfrak{D}_f is recovered from this by the unit homomorphism $u : \mathbf{R} \rightarrow \mathbf{C}(E)$: we have

$$\text{ind}(\mathfrak{D}_f) = (u \boxtimes [\text{id}_{\mathbf{C}\mathbf{I}^{0,d}}]) \# [\mathfrak{D}_f] \in KK(\mathbf{C}\mathbf{I}^{0,d}, \mathbf{C}(X) \otimes \mathbf{C}^*(G)) = KO^{-d}(X; \mathbf{C}^*(G)),$$

essentially by the definition of all these terms (note that the composition product with the class of a homomorphism is easy to compute, and so is the exterior product with the class of the identity). The index theorem describes $\text{ind}(\mathfrak{D}_f)$ in topological terms.

The K -theoretic Thom class of a rank n spin vector bundle $V \rightarrow X$ gives a map $\lambda_V : \text{Th}(V) \rightarrow \Omega^{\infty-n}\mathbf{KO}$ of spaces, which is adjoint to a map $\lambda_V : \text{Th}(V) \rightarrow \Sigma^n\mathbf{KO}$ of spectra. More generally if $V \rightarrow X$ is a stable spin vector bundle of rank r , we get a spectrum map

$$\lambda_V : \text{Th}(V) \longrightarrow \Sigma^r\mathbf{KO}.$$

For example, if V is the additive inverse of the universal vector bundle on $B\text{Spin}(d)$, the corresponding spectrum map is

$$\lambda_{-d} : \text{MTSpin}(d) \longrightarrow \Sigma^{-d}\mathbf{KO}.$$

Let $\pi : E \rightarrow X$ be a bundle of d -dimensional closed spin manifolds, equipped with a map $f : E \rightarrow BG$. The normal bundle $\nu(\pi)$ of $\pi : E \rightarrow X$ is a stable vector bundle of rank $-d$, and the Pontrjagin–Thom collapse defines a map

$$c : \Sigma^\infty X_+ \longrightarrow \text{Th}(\nu(\pi))$$

from the suspension spectrum of X to the Thom spectrum of the normal bundle. Write

$$\Delta : \text{Th}(\nu(\pi)) \longrightarrow \text{Th}(\nu(\pi)) \wedge E_+$$

for the diagonal map. Using the classifying map $E \rightarrow B\text{Spin}(d)$ of the vertical tangent bundle of π we obtain a map $v : \text{Th}(\nu(\pi)) \rightarrow \text{MTSpin}(d)$, and so can form the composition

$$\alpha_\pi^{ad} : \Sigma^\infty X_+ \xrightarrow{c} \text{Th}(\nu(\pi)) \xrightarrow{\Delta} \text{Th}(\nu(\pi)) \wedge E_+ \xrightarrow{v \wedge f_+} \text{MTSpin}(d) \wedge BG_+.$$

and its adjoint

$$\alpha_\pi : X \longrightarrow \Omega^\infty(\text{MTSpin}(d) \wedge BG_+).$$

Theorem 5.7.1 (The index theorem). *If the base space X is compact, then the composition*

$$\Omega^\infty(\Sigma^{-d}\nu \circ (\lambda_{-d} \wedge \text{id}_{BG})) \circ \alpha_\pi : X \longrightarrow \Omega^{\infty+d}\mathbf{KO}(\mathbf{C}^*(G))$$

is equal to $\text{ind}(\mathfrak{D}_f)$ under the identification of the set $[X, \Omega^{\infty+d}\mathbf{KO}(\mathbf{C}^(G))]$ of homotopy classes with $KO^{-d}(X; \mathbf{C}^*(G))$.*

We now give a sketch of the proof. The main part is analytical in nature and is carried out on the level of KK -groups. The first step is to relate the index of the G -Dirac operator to the K -homology class of the ordinary Dirac operator.

Proposition 5.7.2. *The relation*

$$\text{ind}(\mathfrak{D}_f) = ([\mathcal{L}_f] \boxtimes [\text{id}_{\mathbf{C}\mathbf{I}^{0,d}}]) \# ([\mathfrak{D}] \boxtimes [\text{id}_{\mathbf{C}^*(G)}]) \in KK(\mathbf{C}\mathbf{I}^{0,d}, \mathbf{C}(X) \otimes \mathbf{C}^*(G))$$

holds.

The case of $X = *$ (and with complex K -theory) is done in [45, Theorem 5.22]. The linear-algebraic modifications to carry out the Real case are clear (everything in loc.cit. is Real). To deal with the family case, one replaces the references in [45] to the analytical parts of [8] with references to [14, §2.4].

To describe the proof of the index theorem, we have to give explicit descriptions of the Bott maps and the Thom isomorphism. The *Bott class* $\mathbf{b}_n \in KK(\mathbf{R}, \mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n})$ is

$$\mathbf{b}_n = [\mathbf{C}_0(\mathbb{R}^n; \mathbf{Cl}^{0,n}), 1, \frac{\lambda(x)}{(1 + \|x\|^2)^{\frac{1}{2}}}] \in KK(\mathbf{R}, \mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n})$$

where λ is the action by left-multiplication, and x is the identity function on \mathbb{R}^n . The *inverse Bott class* $\mathbf{a}_n \in KK(\mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n}, \mathbf{R})$ is

$$\mathbf{a}_n = [L^2(\mathbb{R}^n; \mathbf{S}_{n,n}), \mu, \frac{D}{(1 + D^2)^{1/2}}] \in KK(\mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n}, \mathbf{R}). \quad (5.7.3)$$

Here $\mathbf{S}_{n,n}$ is the complexified exterior algebra $\Lambda^* \mathbb{R}^n \otimes \mathbb{C}$ with its canonical $\mathbf{Cl}^{n,n}$ -structure (see e.g. [14, Definition 3.2]), μ is the action of $\mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n}$ on $L^2(\mathbb{R}^n; \mathbf{S}_{n,n})$ which combines pointwise multiplication by real-valued functions on \mathbb{R}^n and the restriction of the Clifford structure to $\mathbf{Cl}^{0,n}$, and D is the Dirac operator (which is equal to $d + d^*$ in this case). Kasparov proved that

$$\begin{aligned} \mathbf{b}_n \# \mathbf{a}_n &= 1 = [\text{id}_{\mathbf{R}}] \in KK(\mathbf{R}, \mathbf{R}) \cong \mathbb{Z} \\ \mathbf{a}_n \# \mathbf{b}_n &= [\text{id}_{\mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n}}] \in KK(\mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n}, \mathbf{C}_0(\mathbb{R}^n) \otimes \mathbf{Cl}^{0,n}), \end{aligned} \quad (5.7.4)$$

see [34, Theorem 7] or [12, p. 101 ff.] for more details.

There are parametrised version of these classes: let $p : V \rightarrow Y$ be a real vector bundle of rank n on a locally compact space Y , and let $\mathbf{Cl}(V^-) \rightarrow Y$ be the bundle of Clifford algebras of V^- (the Clifford generators have positive square). By $\mathbf{\Gamma}_0(V; \mathbf{Cl}(V^-))$, we denote the C^* -algebra of sections of $p^* \mathbf{Cl}(V^-) \rightarrow V$ which vanish at infinity. The above elements generalise to

$$\begin{aligned} \mathbf{b}_V &\in KK(\mathbf{C}_0(Y), \mathbf{\Gamma}_0(V; \mathbf{Cl}(V^-))) \\ \mathbf{a}_V &\in KK(\mathbf{\Gamma}_0(V; \mathbf{Cl}(V^-)), \mathbf{C}_0(Y)). \end{aligned}$$

A spin structure on V yields a KK -equivalence $\mathbf{s}_V \in KK(\mathbf{\Gamma}_0(Y; \mathbf{Cl}(V^-)), \mathbf{C}_0(Y) \otimes \mathbf{Cl}^{0,n})$. Namely, let $P \rightarrow Y$ be the underlying $\text{Spin}(n)$ -principal bundle and consider the bundle $P \times_{\text{Spin}(n)} \mathbf{Cl}^{n,n} \rightarrow Y$. It has an action of $\mathbf{Cl}(V)$ and one of $\mathbf{Cl}^{0,n}$ which anticommute. Using the grading, one can turn the $\mathbf{Cl}(V)$ -action into a $\mathbf{Cl}(V^-)$ -action, as in [6, p. 773]. We apply this construction to the pullback $p^* V \rightarrow V$ and obtain the Thom class of the spin bundle V

$$\mathbf{t}_V := \mathbf{b}_V \# \mathbf{s}_{p^* V} \in KK(\mathbf{C}(Y), \mathbf{C}_0(V) \otimes \mathbf{Cl}^{0,n})$$

and the inverse Thom class

$$\mathbf{u}_V := \mathbf{s}_{p^* V}^{-1} \# \mathbf{a}_V \in KK(\mathbf{C}_0(V) \otimes \mathbf{Cl}^{0,n}, \mathbf{C}(Y)).$$

These are mutually inverse KK -equivalences.

Now choose an embedding $E \rightarrow X \times \mathbb{R}^n$ over X . Let $V \rightarrow E$ be the normal bundle of this embedding, and pick an open embedding $j : V \rightarrow X \times \mathbb{R}^n$ over X as a tubular neighbourhood. Extension by zero gives a homomorphism $j_! : \mathbf{C}_0(V) \rightarrow \mathbf{C}_0(X \times \mathbb{R}^n)$.

Proposition 5.7.5. *The relation*

$(\mathbf{u}_V \boxtimes [\text{id}_{\mathbf{C}\mathbf{I}^{0,d}}]) \# [\mathcal{D}] = ([j_! \boxtimes \text{id}_{\mathbf{C}\mathbf{I}^{0,n}}]) \# (\mathbf{a}_n \boxtimes [\text{id}_{\mathbf{C}(X)}]) \in KK(\mathbf{C}_0(V) \otimes \mathbf{C}\mathbf{I}^{0,n}, \mathbf{C}(X))$
holds.

Sketch of proof. By homotopy invariance of KK -theory, it is enough to prove this equation in $KK(\mathbf{C}_0(V_0) \otimes \mathbf{C}\mathbf{I}^{0,n}, \mathbf{C}(X))$, where $V_0 \subset V$ is the open unit disc bundle. The given Riemannian metric on E , a bundle metric on V and a connection on V together define a complete Riemannian metric on the total space V . Using the formula for \mathbf{u}_V , one may check that $(\mathbf{u}_V \boxtimes [\text{id}_{\mathbf{C}\mathbf{I}^{0,d}}]) \# [\mathcal{D}]$ is represented by the continuous field of Hilbert spaces $L_X^2(V; \mathfrak{S}_V)$, with the Clifford action by the (trivial!) vertical tangent bundle of V , and the Dirac operator on V . The spin Dirac operator with respect to the euclidean metric on the manifold bundle $X \times \mathbb{R}^n \rightarrow X$ represents the element $\mathbf{a}_n \boxtimes [\text{id}_{\mathbf{C}(X)}]$, by (5.7.3). The same is true if we deform the metric, as long as the deformation is constant near infinity. Pick a Riemannian metric on the bundle $X \times \mathbb{R}^n \rightarrow X$ which coincides with the euclidean metric near infinity and with the metric of V on V_0 . So the two classes we claim are equal are represented by operators which are equal on V_0 (but are defined on widely different domains). Using the techniques of [28, Proposition 10.8.2 and Lemma 10.8.4] (and the analytical results of [14, §2.4] to deal with the family case) finishes the proof. \square

Let us introduce more notation. For a rank n spin vector bundle $V \rightarrow Y$ on a compact Hausdorff space and \mathbf{C}^* -algebras \mathbf{A}, \mathbf{B} , the *Thom isomorphism* is

$$\begin{aligned} \tau_V : KK(\mathbf{A}, \mathbf{C}(Y) \otimes \mathbf{B}) &\longrightarrow KK(\mathbf{A}, \mathbf{C}_0(V) \otimes \mathbf{C}\mathbf{I}^{0,n} \otimes \mathbf{B}) \\ \mathbf{x} &\longmapsto \mathbf{x} \# (\mathbf{t}_V \boxtimes [\text{id}_{\mathbf{B}}]). \end{aligned}$$

The Thom isomorphism of the trivial bundle $Y \times \mathbb{R}^n \rightarrow Y$ is the *Bott map* $\beta_n := \tau_{Y \times \mathbb{R}^n}$. The inclusion $j : U \rightarrow Y$ of an open subspace induces a homomorphism $j_! : \mathbf{C}_0(U) \rightarrow \mathbf{C}_0(Y)$ and we write

$$\begin{aligned} j_! : KK(\mathbf{A}, \mathbf{C}_0(U) \otimes \mathbf{B}) &\longrightarrow KK(\mathbf{A}, \mathbf{C}_0(Y) \otimes \mathbf{B}) \\ \mathbf{x} &\longmapsto \mathbf{x} \# ([j_! \boxtimes \text{id}_{\mathbf{B}}]). \end{aligned}$$

With these short notations, Propositions 5.7.2 and 5.7.5, together with the relations (5.7.4) and the formal properties of the Kasparov product [5, §18.6 and 18.7] imply the formula

$$\beta_n(\text{ind}(\mathcal{D}_f)) = j_!(\tau_V([\mathcal{L}_f])) \in KK(\mathbf{C}\mathbf{I}^{0,d}, \mathbf{C}(X \times \mathbb{R}^n) \otimes \mathbf{C}\mathbf{I}^{0,n} \otimes \mathbf{C}^*(G)). \quad (5.7.6)$$

Let $c : S^n \wedge X_+ \rightarrow \text{Th}(V)$ be the Pontrjagin–Thom collapse, and $\Delta : \text{Th}(V) \rightarrow \text{Th}(V) \wedge E_+$ be the diagonal map. The K -theory classes $\lambda_V \in KO^{n-d}(\text{Th}(V), \infty)$ and $\mathcal{L}_f \in KO(E_+, +; \mathbf{C}^*(G))$ together give a K -theory class

$$\lambda_V \wedge \mathcal{L}_f \in KO^{n-d}(\text{Th}(V) \wedge E_+, +; \mathbf{C}^*(G))$$

which is represented by a based map

$$\lambda_V \wedge \mathcal{L}_f : \text{Th}(V) \wedge E_+ \longrightarrow \Omega^{\infty+d-n} \mathbf{K}\mathbf{O}(\mathbf{C}^*(G)).$$

In homotopy theoretic terms, the identification (5.7.6) can be reformulated by saying that the adjoint

$$((\lambda_V \wedge \mathcal{L}_f) \circ \Delta \circ c)^{ad} : X \longrightarrow \Omega^{\infty+d} \mathbf{K}\mathbf{O}(\mathbf{C}^*(G))$$

of the composition $(\lambda_V \wedge \mathcal{L}_f) \circ \Delta \circ c$ represents the index class $\text{ind}(\mathcal{D}_f)$. Finally, we let n tend to ∞ and use the map $f : E \rightarrow BG$ and the classifying map $E \rightarrow B\text{Spin}(d)$

of the vertical tangent bundle of E and arrive at the formulation given in Theorem 5.7.1. This finishes our sketch of the proof of the index theorem.

5.8. Proof of Theorem A. We now explain how to deduce Theorem A from what we have shown so far. Let $f : W \rightarrow BG$ and $h \in \mathcal{R}^+(W)_g$ be as in the statement of Theorem A.

By [47, Lemma 1.3], as $\pi_1(f) : \pi_1(W) \rightarrow G$ is a split surjection from a finitely presented group, G is finitely presented. By embedding a finite presentation 2-complex for G into \mathbb{R}^{2n} and taking a regular neighbourhood, we obtain a Spin manifold P with fundamental group G , whose structure map $P \rightarrow B\text{Spin}(2n) \times BG$ is 2-connected, and such that $(P, \partial P)$ is $(2n - 3)$ -connected. (Alternatively, P could be constructed by appealing to Lemma 4.2.4). We consider P as a cobordism $P : \emptyset \rightsquigarrow \partial P$.

By Theorem D, there exists a $g_0 \in \mathcal{R}^+(\partial P)$ and a right stable $h_0 \in \mathcal{R}^+(P)_{g_0}$. The pair (P, g_0) satisfies the hypotheses of Theorem F, and so we obtain the left half (which is homotopy cartesian) of a homotopy commutative diagram

$$\begin{array}{ccccc} \mathcal{R}^+(P)_g // \text{Diff}_\partial(P) & \longrightarrow & T_\infty^+ & \longrightarrow & * \\ \downarrow & & \downarrow \pi_\infty^+ & & \downarrow \\ B\text{Diff}_\partial(P) & \xrightarrow{\alpha_P} & \Omega_0^\infty \text{MTSpin}(2n) \wedge BG_+ & \xrightarrow{\eta} & \Omega^{\infty+2n} \text{KO}(\mathbf{C}_r^*(G)). \end{array} \quad (5.8.1)$$

The right half of this diagram is constructed using index theory, parallel to [6]. In a little more detail, one first uses the Lichnerowicz–Schrödinger formula to construct a class $\beta_p \in KO^{-2n}(\pi_p; \mathbf{C}_r^*(G))$ for each $p \geq 0$, precisely as in [6, §3.8.4]. The classes β_p for various p are compatible by the additivity theorem (as in [6, Proposition 3.33]). Hence the same argument with Milnor’s \lim^1 -sequence as in [6, Proposition 4.9] applies and yields a class $\beta_\infty \in KO^{-2n}(\pi_\infty; \mathbf{C}_r^*(G))$ restricting to β_p for each p . The maps ψ and ψ' in (4.4.3) are acyclic by Remark 4.4.4, hence induce isomorphisms in $KO^*(; \mathbf{C}_r^*(G))$. Therefore, there exists $\beta_\infty^+ \in KO^{-2n}(\pi_\infty^+; \mathbf{C}_r^*(G))$ which restricts to β_∞ . The formalities of [6, §3.5], in particular [6, Remark 3.21] produce the right half of the diagram (5.8.1) out of the class β_∞^+ . By [6, Remark 3.21, Proposition 3.33], the induced map on vertical homotopy fibres of (5.8.1) is homotopic to inddiff_{h_0} . The bottom map η in (5.8.1) is homotopic to $\Omega^\infty(\Sigma^{-2n}\nu \circ (\lambda_{-2n} \wedge \text{id}_{BG}))$. This follows from family index theorem that was discussed in §5.7, in the same way as in [6, Proposition 4.13].

Once the commutative diagram (5.8.1) is established, fibre transport for the middle column acting on $h_0 \in \mathcal{R}^+(P)_{g_0}$ gives a map

$$\Psi = \Psi_{(P, h_0)} : \Omega^{\infty+1} \text{MTSpin}(2n) \wedge BG_+ \longrightarrow \mathcal{R}^+(P)_{g_0}$$

such that

$$\text{inddiff}^G \circ \Psi \simeq \Omega^{\infty+1}(\Sigma^{-2n}\nu \circ (\lambda_{-2n} \wedge \text{id}_{BG})).$$

This proves Theorem A for the manifold P . To prove it for arbitrary W and boundary conditions g we shall embed P into W , for which we use the following result.

Theorem 5.8.2. *Let W^d , $d \geq 5$, be a connected compact spin manifold equipped with a map $f : W \rightarrow BG$ which is split surjective on π_1 . Let P^d be a spin manifold with a map $h : P \rightarrow BG$ which is 2-connected, and assume that P is built from \emptyset*

by attaching handles of dimension ≤ 2 . Then there exists an embedding $g : P \rightarrow W$ preserving the spin structure and such that $f \circ g \simeq h$.

Proof. First note that it is really necessary that $\pi_1(f)$ is split surjective. The first part of the proof is pure homotopy theory. Write $B = B\text{Spin}(d) \times BG$, with universal vector bundle $V \rightarrow B$. The product of the map f with the map classifying the (spin structure on the) tangent bundle of W gives a map $f' : W \rightarrow B$; similarly, there is a $h' : P \rightarrow B$. Observe that P is homotopy equivalent to a 2-dimensional complex $K \subset P$. We claim that the lifting problem

$$\begin{array}{ccc} & & W \\ & \nearrow^{g|_K} & \downarrow f' \\ K & \xrightarrow{h'|_K} & B \end{array}$$

can be solved (up to homotopy). To do this, let $\sigma : G \rightarrow \pi_1(W)$ be a splitting of the map $\pi_1(f)$. We identify G and $\pi_1(K)$ using the isomorphism $\pi_1(h)$. We can assume that $K^{(0)} = *$ and that the attaching maps $S^1 \rightarrow K^{(1)}$ for all 2-cells are pointed. The complex K is the presentation complex for some finite presentation $\langle S|R \rangle$ of the group G , with S indexing the 1-cells and R indexing the 2-cells. First, we construct the lift $g|_K$ on the 1-skeleton $K^{(1)}$. Each 1-cell α_s , $s \in S$, is mapped to a loop in W representing $\sigma(s) \in \pi_1(W)$. This defines $g|_{K^{(1)}}$. Let β_r , $r \in R$ be a 2-cell with attaching map $\phi_r : S^1 \rightarrow K^{(1)}$. Now $g|_{K^{(1)}} \circ \phi_r : S^1 \rightarrow W$ represents the element $\sigma(r) \in \pi_1(W)$. Since σ is a homomorphism, $\sigma(r) = 1$, and so $g|_{K^{(1)}} \circ \phi_r$ is nullhomotopic. Thus we can extend $g|_{K^{(1)}}$ over each 2-cell of K , and this shows the existence of the lift $g|_K$.

Since P is homotopy equivalent to K , the map $g|_K$ extends to a map $g : P \rightarrow W$ such that $f' \circ g \simeq h'$. But then

$$TP \cong (h')^*V \cong g^*(f')^*V \cong g^*TW.$$

Therefore, g is covered by a bundle map $TP \rightarrow TW$. Since P has no closed component, we can apply Phillips' submersion theorem [41] and find a submersion $P \rightarrow W$ homotopic to g . Thus, we may assume that g is a submersion.

By general position, we can assume that g , restricted to the core of each handle of P , is self-transverse. As the cores of the handles are at most 2-dimensional, g will embed all handles, since $\dim(W) \geq 5$. Then g embeds a small neighbourhood $U \subset P$ into W . But there is a diffeomorphic copy of P contained in U , and hence we have found the desired embedding. \square

Now take an embedding $g : P \rightarrow W$, and view the cobordism $W : \emptyset \rightsquigarrow \partial W$ as the composition $\emptyset \xrightarrow{P} \partial P \xrightarrow{K} \partial W$ of two cobordisms. By hypothesis, $h \in \mathcal{R}^+(W)_g \neq \emptyset$. Since $h_0 \in \mathcal{R}^+(P)_{g_0}$ is right-stable, we find a psc metric $h' \in \mathcal{R}^+(K)_{g_0, g}$ such that $h_0 \cup h$ lies in the same component of $\mathcal{R}^+(W)_g$ as h . The composition

$$\Omega^{\infty+1} \text{MTSpin}(2n) \wedge BG_+ \xrightarrow{\Psi_{(P, h_0)}} \mathcal{R}^+(P)_{g_0} \xrightarrow{\mu_{(-, h')}} \mathcal{R}^+(W)_g$$

gives the map Ψ whose existence is asserted by Theorem A. To compute the composition with the index difference, one uses the additivity theorem for the index, along the same lines as in [6, Theorem 3.16].

5.9. The Baum–Connes conjecture. Here we recall the statement of the Baum–Connes conjecture and the relation to the Novikov assembly map, mostly referring to the literature. Here we only use the reduced group C^* -algebra $\mathbf{C}_r^*(G)$. For a proper and locally compact G -space X and $p, q \in \mathbb{N}_0$, one defines an analytical assembly map in complex K -theory

$$K_{p-q}^{\text{an},G}(X) := KK^G(\mathbf{C}(X) \otimes \mathbf{C}\mathbf{I}^{p,q}, \mathbf{C}) \xrightarrow{\mu_X} K_{p-q}(\mathbf{C}_r^*(G)) := KK(\mathbf{C} \otimes \mathbf{C}\mathbf{I}^{p,q}, \mathbf{C}_r^*(G))$$

from the G -equivariant analytic K -homology of X to the K -theory of $\mathbf{C}_r^*(G)$. If $\underline{E}G$ is the universal G -space with finite isotropy, we obtain a map

$$\mu : RK_i^G(\underline{E}G) := \text{colim}_{X \subset \underline{E}G} K_i^G(X) \longrightarrow K_i(\mathbf{C}_r^*(G))$$

for all $i \in \mathbb{Z}$. The *Baum–Connes conjecture for G* predicts that μ is an isomorphism for all $i \in \mathbb{Z}$. The classical Novikov assembly map

$$\nu^r = \nu : KO_i(BG) = RKO_i(BG) \longrightarrow KO_i(\mathbf{C}_r^*(G))$$

from (5.6.7) has a complex analogue

$$\nu^c : K_i(BG) = RK_i(BG) \longrightarrow K_i(\mathbf{C}_r^*(G)).$$

Proposition 5.9.1. *If G is torsionfree and the Baum–Connes conjecture holds for G , then the real Novikov assembly map $\nu^r : KO_i(BG) \rightarrow KO_i(\mathbf{C}_r^*(G))$ is an isomorphism.*

Proof. This is by juxtaposing several results from the literature. If G is torsionfree, then $\underline{E}G = EG$ and so the source of μ is just $RK_i^G(EG)$. There is a commutative diagram

$$\begin{array}{ccc} RK_i^G(EG) & \xrightarrow{\mu} & K_i(\mathbf{C}_r^*(G)) \\ \downarrow \cong & \nearrow \nu^c & \\ RK_i(BG) & & \end{array}$$

which was established rigorously in recent work by Land [36]. Therefore, if μ is an isomorphism, then so is ν^c . But there is a Galois descent property: if ν^c is an isomorphism (in all degrees), then so is ν^r , see [3], [33] and [44, Theorem 2.14] for a particularly simple proof. \square

Similarly, rational injectivity results for the complex Baum–Connes assembly map imply rational injectivity results for the real Novikov assembly map [44, Corollary 2.13 and Theorem 2.14].

5.10. Proof of Theorem B. Recall that we assume that G is a torsionfree group satisfying the Baum–Connes conjecture, and $f : M \rightarrow BG$ is a reference map from an even-dimensional manifold such that $f_* : \pi_1(M) \rightarrow G$ is split surjective, and we wish to show that

$$(\text{inndiff}_{h_0}^G)_* : \pi_i(\mathcal{R}^+(M)[B^{-1}]) \longrightarrow KO_{i+2n+1}(\mathbf{C}_r^*(G)) \quad (5.10.1)$$

is surjective for each $i \geq 0$. By the Baum–Connes conjecture, the map

$$\nu_* : \pi_{i+2n+1}(\mathbf{KO} \wedge BG_+) \longrightarrow KO_{i+2n+1}(\mathbf{C}_r^*(G))$$

is surjective. As $\pi_*(\mathbf{KO} \wedge -)$ is a homology theory, any class is carried on a finite complex, so given a class $x \in KO_{i+2n+1}(\mathbf{C}^*(G))$ there is a map $f : X \rightarrow BG$ from a finite CW-complex such that x is in the image of

$$\pi_{i+2n+1}(\mathbf{KO} \wedge X_+) \xrightarrow{f_*} \pi_{i+2n+1}(\mathbf{KO} \wedge BG_+) \xrightarrow{\nu_*} KO_{i+2n+1}(\mathbf{C}^*(G)).$$

In [6, Proof of Theorem 5.5] there is constructed a class $\mathbf{b} \in \pi_4(\mathbf{MTSpin}(4))$ such that $\widehat{\mathcal{A}}(\mathbf{b}) = \beta \in ko_8(*)$ is the Bott class. Multiplication by this class gives a map of spectra

$$S^4 \wedge \mathbf{MTSpin}(d-4) \xrightarrow{\mathbf{b} \wedge \text{id}} \mathbf{MTSpin}(4) \wedge \mathbf{MTSpin}(d-4) \xrightarrow{-\mu} \mathbf{MTSpin}(d),$$

which can be iterated and smashed with X_+ , whence it gives a map

$$\mathbf{b}^r \cdot - : \Sigma^{4r} \mathbf{MTSpin}(d-4r) \wedge X_+ \longrightarrow \mathbf{MTSpin}(d) \wedge X_+.$$

For parameters $k \geq \ell$, to be tuned later, consider the diagram

$$\begin{array}{ccc} \pi_{i+2n+8(k-2\ell)+1}(\mathbf{MSpin} \wedge X_+) & \longrightarrow & \pi_{i+2n+8(k-2\ell)+1}(\mathbf{ko} \wedge X_+) \\ \uparrow st & & \downarrow per \\ \pi_{i+1-8\ell}(\mathbf{MTSpin}(2n+8(k-\ell)) \wedge X_+) & \longrightarrow & \pi_{i+2n+8(k-2\ell)+1}(\mathbf{KO} \wedge X_+) \\ \downarrow \mathbf{b}^{2\ell} \cdot - & & \sim \downarrow \beta^{2\ell} \cdot - \\ \pi_{i+1}(\mathbf{MTSpin}(2n+8k) \wedge X_+) & \longrightarrow & \pi_{i+2n+8k+1}(\mathbf{KO} \wedge X_+) \\ \downarrow f_* & & \downarrow f_* \\ \pi_{i+1}(\mathbf{MTSpin}(2n+8k) \wedge BG_+) & \longrightarrow & \pi_{i+2n+8k+1}(\mathbf{KO} \wedge BG_+) \\ \downarrow \Psi_* & & \downarrow \nu_* \\ \pi_i(\mathcal{R}^+(M \times B^k)) & \longrightarrow & KO_{i+2n+8k+1}(\mathbf{C}_r^*(G)) \\ \downarrow & & \sim \uparrow \beta^k \cdot - \\ \pi_i(\mathcal{R}^+(M)[B^{-1}]) & \xrightarrow{(5.10.1)} & KO_{i+2n+1}(\mathbf{C}_r^*(G)) \end{array}$$

in which all horizontal maps are the evident ones, and all squares commute for trivial reasons apart from the fourth, which commutes by Theorem A.

The top map is surjective, by the Anderson–Brown–Peterson [1] splitting of $\mathbf{MSpin}_{(2)}$, and the Baas–Sullivan description of $ko_*(-)[\frac{1}{2}]$ in terms of $\Omega_*^{Spin}(-)[\frac{1}{2}] = \Omega_*^{SO}(-)[\frac{1}{2}]$, see e.g. [18]. The map *per* is an isomorphism as long as

$$2n + 8(k - 2\ell) + 1 > \dim(X),$$

by considering the Atiyah–Hirzebruch spectral sequence. The map *st* is an isomorphism as long as $i + 1 - 8\ell < 0$.

Thus, for the given class $x \in KO_{i+2n+1}(\mathbf{C}_r^*(G))$ carried on the finite complex X , we may first choose ℓ so that $8\ell > i + 1$, and then choose k so that $8k > \dim(X) - 2n - 1 + 16\ell$ and $k \geq \ell$. Then the commutativity of the above diagram shows that x is in the image of the map (5.10.1) as required.

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