

# On Villani's conjecture concerning entropy production for the Kac Master Equation

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February 4, 2011

## Abstract

In this paper we take an idea presented in recent paper by Carlen, Carvalho, Le Roux, Loss, and Villani ([?]) and push it one step forward to find an exact estimation on the entropy production. The new estimation essentially proves that Villani's conjecture is correct, or more precisely that a much worse bound to the entropy production is impossible in the general case. <sup>1</sup>

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### 1 Introduction

In his 1956 paper on the Foundations of Kinetic Theory ([?]), Mark Kac proposed a probabilistic model describing a system of  $N$  one dimensional, randomly colliding particles. The description is given by Kac's Master Equation

$$\frac{\partial \psi}{\partial t}(v_1, \dots, v_N, t) = -N(I - Q)\psi(v_1, \dots, v_N, t) \quad (1)$$

where

$$Q\phi(v_1, \dots, v_N) = \frac{1}{2\pi} \cdot \frac{1}{\binom{N}{2}} \sum_{i < j} \int_0^{2\pi} \phi(R_{i,j}(\vartheta)(v_1, \dots, v_N)) d\vartheta$$

with

$$R_{i,j}(\vartheta)(v_1, \dots, v_N) = (v_1, \dots, v_i(\vartheta), \dots, v_j(\vartheta), \dots, v_N)$$

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\*The work presented in this paper was supported by U.S. National Science Foundation grant DMS-0901304

<sup>1</sup>Keywords and phrases: Entropy Production, Villani's Conjecture

$$v_i(\vartheta) = v_i \cos \vartheta + v_j \sin \vartheta, \quad v_j(\vartheta) = -v_i \sin \vartheta + v_j \cos \vartheta .$$

The function  $\psi(v_1, \dots, v_N, t)$  is a probability distribution on the energy sphere and it is formally given by

$$\psi(\cdot, t) = e^{-N(I-Q)t} \psi_0$$

for some initial condition  $\psi_0$ . In the same paper, Kac introduced the notion of chaotic sequences (although he did not call it that way) and showed that this notion is preserved under the time evolution. This property is now called Propagation of Chaos. Kac went further and showed in fact that single particle marginal of the evolved density is a solution of the model Boltzmann equation

$$\frac{\partial f}{\partial t}(v, t) = \frac{1}{2\pi} \int_{\mathbb{R}} d\omega \int_0^{2\pi} d\vartheta (f(v \cos \vartheta + \omega \sin \vartheta, t) f(-v \sin \vartheta + \omega \cos \vartheta, t) - f(v, t) f(\omega, t))$$

and thus giving a cogent derivation of the spatially homogeneous Boltzmann equation. For a detailed review the reader may consult [?].

The equation (??), or rather the operator, is a bounded self-adjoint operator in the space  $L^2(\mathbb{S}^{N-1}(\sqrt{N}), d\sigma^N)$  where  $d\sigma^N$  is the normalized uniform measure on the sphere. It is fairly easy to see that the time evolution defined by (??) is ergodic, i.e., the solution will approach the function  $\psi = 1$  as  $t \rightarrow \infty$ . By the spectral theorem, the rate of approach to the constant function in the sense of  $L^2$  distance is governed by the gap

$$\Delta_N = \inf \{ \langle \varphi, N(I - Q)\varphi \rangle : \langle \varphi, 1 \rangle = 0, \langle \varphi, \varphi \rangle = 1 \}$$

where the infimum is taken over all  $\varphi \in L^2(\mathbb{S}^{N-1}(\sqrt{N}), d\sigma^N)$ . Kac conjectured that

$$\liminf_{N \rightarrow \infty} \Delta_N > 0 .$$

The conjecture was proved to be true by Janvresse in ([?]) and the exact value of  $\Delta_N$  was computed by Carlen, Carvalho, and Loss in ([?]).

The  $L^2$  distance is rather unsatisfactory. For any reasonable density  $\psi$ , in particular a chaotic one, it is easy to see that

$$\|\psi(v_1, \dots, v_N, 0)\|_{L^2(\mathbb{S}^{N-1}(\sqrt{N}), d\sigma^N)} \geq C^N$$

where  $C > 1$  and hence it would take a time of order  $N$  to see a substantial decay of the  $L^2$ . Clearly, this is not what one considers ‘‘approach to equilibrium’’. A more natural quantity to use is the entropy

$$H_N(\psi) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} \psi \log \psi$$

The crucial difference between the  $L^2$  distance and the entropy lies in the extensivity of the entropy, namely that if  $\psi_N(v_1, \dots, v_N, t)$  satisfies  $\psi_N(v_1, \dots, v_N, t) \approx \prod_{i=1}^N f(v_i, t)$  in a weak sense, i.e., chaotic (referred by Kac as 'The Boltzmann Property') then

$$H_N(\psi_N) \approx N \int_{\mathbb{R}} f(v, t) \log \left( \frac{f(v, t)}{\gamma(v)} \right) dv = NH(f(v, t)|\gamma(v))$$

where  $\gamma(v)$  is the normalized Gaussian.

Differentiating the entropy of a solution to the Kac Model gives the time evolution equation:

$$\frac{\partial H_N(\psi_N)}{\partial t} = \langle \log \psi_N, N(I - Q)\psi_N \rangle$$

This, along with a known inequality by Csiszar, Kullback, Leibler and Pinsker and the extensivity property allows us to conclude that

$$\|\psi_N(v_1, \dots, v_N, t)d\sigma^N - d\sigma^N\|_{\text{Total Variation}}^2 \leq 2Ne^{-\Gamma_N t} H(f(v, 0)|\gamma(v))$$

for

$$\Gamma_N = \inf \frac{\langle \log(\psi_N), N(I - Q)\psi_N \rangle}{H_N(\psi_N)}$$

where the infimum is taken over all probability densities  $\psi_N$  on  $\mathbb{S}^{N-1}(\sqrt{N})$  which are symmetric in all their components.  $\Gamma_N$  is called the *entropy production*.

The hope that there exists  $C > 0$  such that  $\Gamma_N \geq C$  was refuted in 2010 in a paper by Carlen, Carvalho, Le Roux, Loss, and Villani ([?]) where the authors managed to find a sequence of probability densities  $\{\phi_N\}_{N \in \mathbb{N}}$  with

$$\limsup_{N \rightarrow \infty} \frac{\langle \log(\phi_N), N(I - Q)\phi_N \rangle}{H_N(\phi_N)} = 0 \tag{2}$$

While this means that the time of convergence to equilibrium is not of logarithm type, an exact estimation on the entropy production might still give a better convergence rate than that of the original Kac model.

The first step towards this goal was done in 2003 by Villani in ([?]) who proved that

$$\Gamma_N \geq \frac{2}{N-1}$$

Villani conjectured that

$$\Gamma_N = O\left(\frac{1}{N}\right)$$

which wouldn't bode well for the approach to equilibrium in the ergodic sense, but poses an interesting mathematical problem.

The main result of this paper is to show that Villani's conjecture is essentially true. More precisely, we will show that For any  $0 < \beta < \frac{1}{6}$  there exists a constant  $C_\beta$  depending only on  $\beta$  such that

$$\Gamma_N \leq \frac{C_\beta \log N}{N^{1-2\beta}} \quad (3)$$

(See Theorem ?? in Section ??).

Both (??) and (??) are proved with the same idea: creating an  $N$  particle symmetric function  $F_N$  from a one particle function  $f$

$$F_N(v_1, \dots, v_N) = \frac{\prod_{i=1}^N f(v_i)}{Z_N(f, \sqrt{N})}$$

where

$$Z_N(f, r) = \int_{\mathbb{S}^{N-1}(r)} \prod_{i=1}^N f(v_i) d\sigma_r^N$$

and  $d\sigma_r^N$  is the uniform probability measure on  $\mathbb{S}^{N-1}(r)$ . The main difference between the two proofs lies in the fact that while in ([?])  $f$  remains fixed, in our paper  $f$  changes with  $N$  via a parameter  $\delta = \delta_N$ .

The paper is structured as follows: Section ?? reviews known results about the normalization function  $Z_N(f, r)$ . Section ?? is our main theoretical part of the paper, dealing with general properties that will allow us to give an asymptotic expression to the normalization function. Section ?? is where we prove our main result. Picking a function which is natural to the problem at hand and using the result of the previous sections along with some involved computation. Section ?? contains a few last remarks and the Appendix has some simple but very useful computation that we use throughout the entire paper.

We'd like to conclude the introduction by thanking Michael Loss for his helpful remarks and discussions, making this paper possible.

## 2 The Function $Z_N(f, r)$

The key to the computation of the entropy production lies with the normalization function  $Z_N(f, r)$ . In this short section we'll find a simple probabilistic interpretation to it, along with a formula that will serve us in the following sections and the final computation. This section is a short review of known results from ([?]). Let  $f$  be a density function for the real valued random variable  $V$ . Then the density function of the random variable  $V^2$  is given by

$$h(u) = \frac{f(\sqrt{u}) + f(-\sqrt{u})}{2\sqrt{u}}$$

For any function  $\varphi = \varphi(|x|) = \varphi(r)$  we find that

$$\mathbb{E}\varphi = \int_0^\infty \varphi(r) \cdot (f(r) + f(-r)) dr$$

on the other hand

$$\mathbb{E}\varphi = \int_0^\infty \varphi(\sqrt{t}) h(t) dt = \int_0^\infty \varphi(r) \cdot 2r \cdot h(r^2) dr$$

Since  $\varphi$  was arbitrary we find that

$$2r \cdot h(r^2) = f(r) + f(-r)$$

and the result follows. Let  $V_1, \dots, V_N$  be independent real valued random variables with identical density function  $f(v)$ . Then the density function for  $S_N = \sum_{i=1}^N V_i^2$  is given by  $s_N(u) = \frac{|\mathbb{S}^{N-1}|}{2} u^{\frac{N}{2}-1} Z_N(f, \sqrt{u})$ . Similar to Lemma ?? for any  $\varphi = \varphi(r)$  we find that

$$\mathbb{E}\varphi = \int_0^\infty \varphi(r) \left( \int_{\mathbb{S}^{N-1}(r)} f(v_1) \dots f(v_N) ds_r^N \right) dr = \int_0^\infty \varphi(r) |\mathbb{S}^{N-1}| r^{N-1} Z_N(f, r) dr$$

on the other hand

$$\mathbb{E}\varphi = \int_0^\infty \varphi(\sqrt{x}) s_N(x) dx = \int_0^\infty \varphi(r) \cdot 2r \cdot s_N(r^2) dr$$

Since  $\varphi$  is arbitrary

$$2r \cdot s_N(r^2) = |\mathbb{S}^{N-1}| r^{N-1} Z_N(f, r)$$

which implies the result. (Expression for  $Z_N(f, r)$ ) Under the conditions of Lemma ??

$$Z_N(f, \sqrt{r}) = \frac{2h^{*N}(r)}{|\mathbb{S}^{N-1}| r^{\frac{N}{2}-1}}$$

where  $h^{*N}$  is the  $N$ -fold convolution of  $h$ , defined in Lemma ?. This follows immediately from Lemma ?, Lemma ? and a known probability fact.

### 3 Central Limit Theorem

In order for us to be able to compute the entropy production an asymptotic behavior for  $Z_N(f, r)$  is needed. As seen in Section ?? the function  $Z_N(f, r)$  is closely related to the  $N$ -fold convolution of the density function  $h(u)$  and as such we'll employ standard techniques to estimate it. The specific function we'll construct as a test function for the entropy production has the property that the Fourier transform of its one particle function splits the line into two natural domains: One where we can use analytic expansion, and one where the decay is dominated by exponential functions. The radius of the separating circle would depend on a parameter  $\delta = \delta_N$  that we'll exploit later on to get the final conclusion.

While this is the case arising in our specific construction, we believe that it's a natural way to view the problem. Even though we have yet to attempt any different test functions we think that similar situation would happen in a larger class of functions created from one particle function. As such, a generalization of our computation was made and is presented in this section.

The reader should keep in mind the following intuition while reading this section:  $g(\xi)$  represents the Fourier transform of the function  $h(u)$ , connected to the one particle function via Lemma ???. The first lemma of the section explores the domain outside the radius of analiticity while the second explores the domain where analytic expansion is possible. Lastly, the parameter  $\delta$  is a function of  $N$  that goes to zero as  $N$  goes to infinity. Let  $g_\delta(\xi) = g_{\delta_N}(\xi)$  be such that

- (i) for  $|\xi| > c\delta$   $|g_\delta(\xi)| \leq 1 - \alpha(\delta)$ , where  $\alpha(\delta) > 0$ .
- (ii)  $|g_\delta(\xi)| \leq 1$  for all  $\xi$ .

Then

$$\begin{aligned} & \int_{|\xi| > c\delta} |g_\delta^N(\xi) - \gamma_1^N(\xi)| d\xi \\ & \leq 2 \int_{|\xi| > c\delta} |g_\delta(\xi)|^{N-1} d\xi + \frac{(1 - \alpha(\delta))^{\frac{N}{2}-1}}{\pi c \delta \Sigma_\delta^2} + \frac{1}{\pi c \delta \Sigma_\delta^2} \cdot e^{-(1+N)\pi^2 c^2 \delta^2 \Sigma_\delta^2} \end{aligned}$$

where  $\gamma_1(\xi) = e^{-2\pi i \zeta} \cdot e^{-2\pi^2 \xi^2 \Sigma_\delta^2}$ . We have that

$$\begin{aligned} \int_{|\xi| > c\delta} |g_\delta^N(\xi) - \gamma_1^N(\xi)| d\xi &= \int_{|\xi| > c\delta} |g_\delta(\xi) - \gamma_1(\xi)| \cdot \left| \sum_{k=0}^{N-1} g_\delta^{N-k-1}(\xi) \gamma_1^k(\xi) \right| d\xi \\ &\leq 2 \int_{|\xi| > c\delta} \sum_{k=0}^{N-1} |g_\delta^{N-k-1}(\xi)| |\gamma_1^k(\xi)| d\xi \\ &\leq 2 \int_{|\xi| > c\delta} |g_\delta(\xi)|^{N-1} d\xi + 2 \sum_{k=1}^{N-1} (1 - \alpha(\delta))^{N-k-1} \int_{|\xi| > c\delta} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi \end{aligned}$$

Using Lemma ?? and ?? in the Appendix we find that

$$\begin{aligned} \sum_{k=k_0}^{N-1} \int_{|\xi| > c\delta} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi &\leq \sum_{k=k_0}^{N-1} \frac{\sqrt{2\pi} \cdot e^{-\frac{4k\pi^2 c^2 \delta^2 \Sigma_\delta^2}{2}}}{\sqrt{4k\pi^2 \Sigma_\delta^2}} \\ &\leq \frac{1}{2\pi c \delta \Sigma_\delta^2} \cdot e^{-2k_0 \pi^2 c^2 \delta^2 \Sigma_\delta^2} \end{aligned}$$

Hence

$$\int_{|\xi| > c\delta} |g_\delta^N(\xi) - \gamma_1^N(\xi)| d\xi$$

$$\begin{aligned}
&\leq 2 \int_{|\xi|>c\delta} |g_\delta(\xi)|^{N-1} d\xi + 2(1-\alpha(\delta))^{N-\lceil\frac{N}{2}\rceil-1} \sum_{k=1}^{\lceil\frac{N}{2}\rceil} \int_{|\xi|>c\delta} e^{-2k\pi^2\xi^2\Sigma_\delta^2} d\xi \\
&\quad + 2 \sum_{k=\lceil\frac{N}{2}\rceil+1}^{N-1} \int_{|\xi|>c\delta} e^{-2k\pi^2\xi^2\Sigma_\delta^2} d\xi \\
&\leq 2 \int_{|\xi|>c\delta} |g_\delta(\xi)|^{N-1} d\xi + \frac{(1-\alpha(\delta))^{\frac{N}{2}-1}}{\pi c\delta\Sigma_\delta^2} + \frac{1}{\pi c\delta\Sigma_\delta^2} \cdot e^{-(1+N)\pi^2c^2\delta^2\Sigma_\delta^2}
\end{aligned}$$

Let  $g_\delta(\xi) = g_{\delta_N}(\xi)$  be such that

(i) there exist  $M_0, M_1, M_2 > 0$  such that  $\sup_{|\xi|<c\delta} |g_\delta(\xi) - \gamma_1(\xi)| \leq \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right) |\xi|^3$ .

(ii) for  $c\delta^{1+\beta} < |\xi| < c\delta$   $|g_\delta(\xi)| \leq 1 - \alpha_\beta(\delta)$  where  $\alpha_\beta(\delta) > 0$ .

(iii)  $|g_\delta(\xi)| \leq 1$  for all  $\xi$ .

Then

$$\begin{aligned}
&\int_{|\xi|<c\delta} |g_\delta^N(\xi) - \gamma_1^N(\xi)| d\xi \leq \frac{c^4\delta^2 (M_0 + M_1\delta + M_2\delta^2)}{2} \\
&+ \frac{c^3\delta\sqrt{N} (M_0 + M_1\delta + M_2\delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2}-1}}{\sqrt{\pi\Sigma_\delta^2}} + \frac{c^3\delta^{1-\beta} (M_0 + M_1\delta + M_2\delta^2) e^{-\pi^2(N-1)c^2\delta^{2+2\beta}\Sigma_\delta^2}}{2\pi c\delta\Sigma_\delta^2 \cdot \sqrt{1 - e^{-2\pi^2Nc^2\delta^2\Sigma_\delta^2}}} \\
&\quad + \frac{2c^3 (M_0 + M_1\delta + M_2\delta^2) \sqrt{N}\delta^{1+3\beta}}{\sqrt{2\pi\Sigma_\delta^2}}
\end{aligned}$$

where  $\gamma_1(\xi) = e^{-2\pi i\xi} \cdot e^{-2\pi^2\xi^2\Sigma_\delta^2}$ . The coefficients  $M_0, M_1$  and  $M_2$  play a major role in the estimation. Notice that we can get a better result if have that  $M_0 = 0$  and an even better result if both  $M_0$  and  $M_1$  are zero. Similar to Lemma ?? we find that

$$\begin{aligned}
&\int_{|\xi|<c\delta} |g_\delta^N(\xi) - \gamma_1^N(\xi)| d\xi \leq \sum_{k=0}^{N-1} \int_{|\xi|<c\delta} |g_\delta(\xi) - \gamma_1(\xi)| |g_\delta(\xi)|^{N-k-1} |\gamma_1(\xi)|^k d\xi \\
&\leq \int_{|\xi|<c\delta} \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right) |\xi|^3 d\xi + \sum_{k=1}^{N-1} \int_{|\xi|<c\delta} \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right) |\xi|^3 |g_\delta(\xi)|^{N-k-1} |\gamma_1(\xi)|^k d\xi \\
&= \frac{c^4\delta^2 (M_0 + M_1\delta + M_2\delta^2)}{2} + \sum_{k=1}^{N-1} \int_{c\delta^{1+\beta} < |\xi| < c\delta} \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right) |\xi|^3 |g_\delta(\xi)|^{N-k-1} |\gamma_1(\xi)|^k d\xi \\
&\quad + \sum_{k=1}^{N-1} \int_{|\xi|<c\delta^{1+\beta}} \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right) |\xi|^3 |g_\delta(\xi)|^{N-k-1} |\gamma_1(\xi)|^k d\xi
\end{aligned}$$

We have that

$$\begin{aligned}
& \sum_{k=1}^{N-1} \int_{c\delta^{1+\beta} < |\xi| < c\delta} \left( \frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2 \right) |\xi|^3 |g_\delta(\xi)|^{N-k-1} |\gamma_1(\xi)|^k d\xi \\
& \leq c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) \sum_{k=1}^{N-1} (1 - \alpha_\beta(\delta))^{N-k-1} \int_{c\delta^{1+\beta} < |\xi| < c\delta} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi \\
& \leq c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2}-1} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \int_{|\xi| < c\delta} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi \\
& \quad + c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \int_{c\delta^{1+\beta} < |\xi| < c\delta} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi \\
& \leq c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2}-1} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{\sqrt{1 - e^{-4\pi^2 k c^2 \delta^2 \Sigma_\delta^2}}}{\sqrt{2\pi \Sigma_\delta^2 k}} \\
& \quad + c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \left( \int_{|\xi| < c\delta} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi - \int_{|\xi| < c\delta^{1+\beta}} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi \right) \\
& \leq c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2}-1} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{\sqrt{2\pi \Sigma_\delta^2 k}} \\
& \quad + c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{\left( \sqrt{1 - e^{-4\pi^2 k c^2 \delta^2 \Sigma_\delta^2}} - \sqrt{1 - e^{-2\pi^2 k c^2 \delta^{2+2\beta} \Sigma_\delta^2}} \right)}{\sqrt{2\pi k \Sigma_\delta^2}} \\
& \leq \frac{c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2}-1}}{\sqrt{2\pi \Sigma_\delta^2}} \cdot \sqrt{4 \left\lfloor \frac{N}{2} \right\rfloor} \\
& \quad + \frac{c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2)}{\sqrt{2\pi \Sigma_\delta^2}} \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{1}{\sqrt{k}} \cdot \frac{e^{-2\pi^2 k c^2 \delta^{2+2\beta} \Sigma_\delta^2} - e^{-4\pi^2 k c^2 \delta^2 \Sigma_\delta^2}}{\left( \sqrt{1 - e^{-4\pi^2 k c^2 \delta^2 \Sigma_\delta^2}} + \sqrt{1 - e^{-2\pi^2 k c^2 \delta^{2+2\beta} \Sigma_\delta^2}} \right)} \\
& \leq \frac{c^3 \delta \sqrt{N} (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2}-1}}{\sqrt{\pi \Sigma_\delta^2}}
\end{aligned}$$



$$\begin{aligned}
& + \frac{c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2)}{\sqrt{2\pi \Sigma_\delta^2}} \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{1}{\sqrt{k}} \cdot \frac{e^{-2\pi^2 k c^2 \delta^2 + 2\beta \Sigma_\delta^2}}{\sqrt{1 - e^{-4\pi^2 k c^2 \delta^2 \Sigma_\delta^2}}} \\
& \leq \frac{c^3 \delta \sqrt{N} (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2} - 1}}{\sqrt{\pi \Sigma_\delta^2}} \\
& + \frac{c^3 \delta (M_0 + M_1 \delta + M_2 \delta^2)}{\sqrt{2\pi \Sigma_\delta^2} \cdot \sqrt{1 - e^{-2\pi^2 N c^2 \delta^2 \Sigma_\delta^2}}} \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{e^{-2\pi^2 k c^2 \delta^2 + 2\beta \Sigma_\delta^2}}{\sqrt{k}} \\
& \leq \frac{c^3 \delta \sqrt{N} (M_0 + M_1 \delta + M_2 \delta^2) (1 - \alpha_\beta(\delta))^{\frac{N}{2} - 1}}{\sqrt{\pi \Sigma_\delta^2}} + \frac{c^3 \delta^{1-\beta} (M_0 + M_1 \delta + M_2 \delta^2) e^{-\pi^2 (N-1) c^2 \delta^2 + 2\beta \Sigma_\delta^2}}{2\pi c \delta \Sigma_\delta^2 \cdot \sqrt{1 - e^{-2\pi^2 N c^2 \delta^2 \Sigma_\delta^2}}}
\end{aligned}$$

Next we find that

$$\begin{aligned}
& \sum_{k=1}^{N-1} \int_{|\xi| < c\delta^{1+\beta}} \left( \frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2 \right) |\xi|^3 |g_\delta(\xi)|^{N-k-1} |\gamma_1(\xi)|^k d\xi \\
& \leq c^3 (M_0 + M_1 \delta + M_2 \delta^2) \delta^{1+3\beta} \cdot \sum_{k=1}^{N-1} \int_{|\xi| < c\delta^{1+\beta}} e^{-2k\pi^2 \xi^2 \Sigma_\delta^2} d\xi \\
& \leq c^3 (M_0 + M_1 \delta + M_2 \delta^2) \delta^{1+3\beta} \cdot \sum_{k=1}^{N-1} \frac{\sqrt{1 - e^{-4k\pi^2 c^2 \delta^2 + 2\beta \Sigma_\delta^2}}}{\sqrt{2\pi k \Sigma_\delta^2}} \\
& \leq \frac{c^3 (M_0 + M_1 \delta + M_2 \delta^2) \delta^{1+3\beta}}{\sqrt{2\pi \Sigma_\delta^2}} \cdot \sum_{k=1}^{N-1} \frac{1}{\sqrt{k}} \\
& \leq \frac{2c^3 (M_0 + M_1 \delta + M_2 \delta^2) \sqrt{N} \delta^{1+3\beta}}{\sqrt{2\pi \Sigma_\delta^2}}
\end{aligned}$$

Which completes the proof. Let  $h_\delta(x) = h_{\delta_N}(x)$  be a function such that  $g_\delta(\xi) = \widehat{h}_\delta(\xi)$  satisfies

- (i) for  $|\xi| > c\delta_N$   $|g_{\delta_N}(\xi)| \leq 1 - \alpha(\delta_N)$ , where  $\alpha(\delta_N) > 0$
- (ii) there exist  $M_0, M_1, M_2 > 0$  such that  $\sup_{|\xi| < c\delta_N} |g_{\delta_N}(\xi) - \gamma_1(\xi)| \leq \left( \frac{M_0}{\delta_N^2} + \frac{M_1}{\delta_N} + M_2 \right) |\xi|^3$
- (iii) for  $c\delta_N^{1+\beta} < |\xi| < c\delta_N$   $|g_{\delta_N}(\xi)| \leq 1 - \alpha_\beta(\delta_N)$  where  $\alpha_\beta(\delta_N) > 0$
- (vi)  $|g_{\delta_N}(\xi)| \leq 1$  for all  $\xi$

and if

$\delta_N, \alpha(\delta_N)$  and  $\alpha_\beta(\delta_N)$  are of dominated by powers of  $N$

$$\begin{aligned}
\alpha(\delta_N)N &\xrightarrow[N \rightarrow \infty]{} \infty \\
\alpha_\beta(\delta_N)N &\xrightarrow[N \rightarrow \infty]{} \infty \\
\Sigma_{\delta_N}^2 \delta_N^{2+2\beta} N &\xrightarrow[N \rightarrow \infty]{} \infty \\
\delta_N^{1+3\beta} N &\xrightarrow[N \rightarrow \infty]{} 0 \\
\sqrt{N}\Sigma_{\delta_N} \int_{|\xi| > c\delta_N} |g_{\delta_N}(\xi)|^{N-1} d\xi &\xrightarrow[N \rightarrow \infty]{} 0 \\
\delta_N^{\frac{3}{2}(1-\beta)} \Sigma_{\delta_N} &\text{ is bounded}
\end{aligned} \tag{4}$$

then

$$\sup_x \left| h_{\delta_N}^{*N}(x) - \frac{1}{\sqrt{N}\Sigma_{\delta_N}} \cdot \frac{e^{-\frac{(x-N)^2}{2N\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} \right| \leq \frac{\epsilon(N)}{\sqrt{N}\Sigma_{\delta_N}}$$

where  $h_{\delta_N}^{*N}(x)$  is the  $N$ -fold convolution and  $\epsilon(N) \xrightarrow[N \rightarrow \infty]{} 0$ . It is easy to check that

$$\frac{1}{\sqrt{N}\Sigma_{\delta}} \cdot \frac{e^{-\frac{(x-N)^2}{2N\Sigma_{\delta}^2}}}{\sqrt{2\pi}}(\xi) = \gamma_1^N(\xi)$$

Using Lemma ?? and ?? we find that

$$\begin{aligned}
&\sup_x \left| h_{\delta}^{*N}(x) - \frac{1}{\sqrt{N}\Sigma_{\delta}} \cdot \frac{e^{-\frac{(x-N)^2}{2N\Sigma_{\delta}^2}}}{\sqrt{2\pi}} \right| \leq \int_{\mathbb{R}} |g_{\delta}^N(\xi) - \gamma_1^N(\xi)| d\xi \\
&= \int_{|\xi| < c\delta} |g_{\delta}^N(\xi) - \gamma_1^N(\xi)| d\xi + \int_{|\xi| > c\delta} |g_{\delta}^N(\xi) - \gamma_1^N(\xi)| d\xi \\
&\leq \frac{1}{\sqrt{N}\Sigma_{\delta}} \left( \frac{c^4 \sqrt{N} \delta^{1+3\beta} \delta^{\frac{3}{2}(1-\beta)} \Sigma_{\delta} (M_0 + M_1\delta + M_2\delta^2)}{2} \right. \\
&\quad + \frac{c^3 \delta N (M_0 + M_1\delta + M_2\delta^2) (1 - \alpha_{\beta}(\delta))^{\frac{N}{2}-1}}{\sqrt{\pi}} \\
&\quad + \frac{c^3 \sqrt{N} \delta^{1-\beta} (M_0 + M_1\delta + M_2\delta^2) e^{-\pi^2(N-1)c^2\delta^2+2\beta\Sigma_{\delta}^2}}{2\pi c\delta\Sigma_{\delta} \cdot \sqrt{1 - e^{-2\pi^2 N c^2 \delta^2 \Sigma_{\delta}^2}}} \\
&\quad + \frac{2c^3 (M_0 + M_1\delta + M_2\delta^2) N \delta^{1+3\beta}}{\sqrt{2\pi}} + 2\sqrt{N}\Sigma_{\delta} \int_{|\xi| > c\delta} |g_{\delta}(\xi)|^{N-1} d\xi \\
&\quad \left. + 2(1 - \alpha(\delta))^{\frac{N}{2}-1} \cdot \frac{\sqrt{N}}{2\pi c\delta\Sigma_{\delta}} + \frac{\sqrt{N}}{\pi c\delta\Sigma_{\delta}} \cdot e^{-(1+N)\pi^2 c^2 \delta^2 \Sigma_{\delta}^2} \right)
\end{aligned}$$

Conditions (??) insure the desired conclusion. A careful look at the proof of Theorem ?? shows that for a fixed  $j$  if  $\lim_{N \rightarrow \infty} \sqrt{N-j} \Sigma_{\delta_N} \int_{|\xi| > c\delta_N} |g_{\delta_N}(\xi)|^{N-j-1} d\xi = 0$  and conditions (??) are satisfied (with the obvious change) then

$$\sup_x \left| h_{\delta_N}^{*N-j}(x) - \frac{1}{\sqrt{N-j} \Sigma_{\delta_N}} \cdot \frac{e^{-\frac{(x-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} \right| \leq \frac{\epsilon_j(N)}{\sqrt{N-j} \Sigma_{\delta_N}}$$

where  $\epsilon_j(N) \xrightarrow{N \rightarrow \infty} 0$ .

## 4 Entropy Production and Villani's Conjecture

In this section we'll find an exact estimation for the entropy production. The idea behind this estimation is to use superposition of stationary solutions for the Boltzmann equation: the Maxwellian densities  $M_a(v) = \frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$ . This idea was exploited by Carlen, Carvalho, Le Roux, Loss, and Villani ([?]) and Bobylev and Cercignani ([?]) before them.

The basic one particle function would be

$$f_{\delta_N}(v) = f_{\delta}(v) = \delta M_{\frac{1}{2\delta}}(v) + (1-\delta) M_{\frac{1}{2(1-\delta)}}(v)$$

This function has the property that both its parts have the same energy

$$\int_{\mathbb{R}} \delta M_{\frac{1}{2\delta}}(v) dv = \int_{\mathbb{R}} (1-\delta) M_{\frac{1}{2(1-\delta)}}(v) dv = \frac{1}{2}$$

while as  $\delta$  gets smaller the number of particles represented by  $\delta M_{\frac{1}{2\delta}}(v)$  is far smaller than those represented by  $(1-\delta) M_{\frac{1}{2(1-\delta)}}(v)$ . The fact that we have a small number of very energetic particles and a large number of very stable particles trying to equilibrate will cause slow decay into equilibrium. That physical intuition is indeed true as would be seen shortly. Let  $h_{\delta}(u) = \frac{f_{\delta}(\sqrt{u}) + f_{\delta}(-\sqrt{u})}{2\sqrt{u}} = \frac{f_{\delta}(\sqrt{u})}{\sqrt{u}}$  then

$$(i) \int_0^{\infty} h_{\delta}(u) du = 1$$

$$(ii) \int_0^{\infty} u h_{\delta}(u) du = 1$$

$$(iii) \Sigma_{\delta}^2 = \int_0^{\infty} u^2 h_{\delta}(u) du - \left( \int_0^{\infty} u h_{\delta}(u) du \right)^2 = \frac{3}{4\delta(1-\delta)} - 1$$

$$(iv) \widehat{h}_{\delta}(\xi) = \frac{\delta}{\sqrt{1 + \frac{2\pi i \xi}{\delta}}} + \frac{1-\delta}{\sqrt{1 + \frac{2\pi i \xi}{1-\delta}}} \quad (i) - (iii) \text{ follow immediately from the fact that}$$

$$\int_0^{\infty} u^m h_{\delta}(u) du = \int_{\mathbb{R}} x^{2m} f_{\delta}(x) dx \text{ and the fact that}$$

$$\int_{\mathbb{R}} M_a(u) du = 1, \quad \int_{\mathbb{R}} u^2 M_a(u) du = a, \quad \int_{\mathbb{R}} u^4 M_a(u) du = 3a^2$$

We're only left with proving (iv).

It is easy to check that

$$\frac{d}{d\xi} \int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du = \frac{-2\pi i a}{1 + 4\pi i a \xi} \int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du$$

The initial value problem  $\frac{d}{d\xi} \varphi(\xi) = \frac{-2\pi i a}{1 + 4\pi i a \xi} \varphi(\xi)$ ,  $\xi \in \mathbb{R}$ ,  $\varphi(0) = 1$  has the unique solution

$$\varphi(\xi) = \frac{1}{\sqrt{1 + 4\pi i a \xi}}$$

Thus, the result follows from the definition of  $f_\delta$  and the fact that

$$\widehat{h}_\delta(\xi) = \int_0^\infty h_\delta(u) e^{-2\pi i \xi u} du = \int_{\mathbb{R}} f_\delta(u) e^{-2\pi i \xi u^2} du$$

Let  $g_\delta(\xi) = \widehat{h}_\delta(\xi)$  where  $\delta < \frac{1}{2}$  then

(i) for  $|\xi| > \frac{\delta}{4\pi}$   $|g_\delta(\xi)| \leq 1 - \delta \left(1 - \sqrt[4]{\frac{4}{5}}\right) + \rho_1(\delta)$  where  $\frac{\rho_1(\delta)}{\delta} \xrightarrow{\delta \rightarrow 0} 0$

(ii) there exist  $M_0, M_1, M_2 > 0$  such that  $\sup_{|\xi| < \frac{\delta}{4\pi}} |g_\delta(\xi) - \gamma_1(\xi)| \leq \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right) |\xi|^3$ .

(iii) for  $\frac{\delta^{1+\beta}}{4\pi} < |\xi| < \frac{\delta}{4\pi}$   $|g_\delta(\xi)| \leq 1 - \frac{\delta^{1+2\beta}}{16} + \rho_2(\delta)$  where  $\frac{\rho_2(\delta)}{\delta^{1+2\beta}} \xrightarrow{\delta \rightarrow 0} 0$

(vi)  $|g_\delta(\xi)| \leq 1$  for all  $\xi$ .

(v) for a fixed  $j$   $\int_{|\xi| > \frac{\delta}{4\pi}} |g_{\delta_N}(\xi)|^{N-j-1} d\xi \leq \frac{(1 - \delta(1 - \sqrt[4]{\frac{4}{5}}) + \rho_1(\delta))^{N-j-1}}{\pi} + \frac{2}{\pi(N-j)}$  (i) For  $|\xi| > \frac{\delta}{4\pi}$

$$\begin{aligned} |g_\delta(\xi)| &\leq \frac{\delta}{\sqrt[4]{1 + \frac{4\pi^2 \xi^2}{\delta^2}}} + \frac{1 - \delta}{\sqrt[4]{1 + \frac{4\pi^2 \xi^2}{(1-\delta)^2}}} \leq \frac{\delta}{\sqrt[4]{\frac{5}{4}}} + \frac{1 - \delta}{\sqrt[4]{1 + \frac{\delta^2}{4(1-\delta)^2}}} \\ &= \sqrt[4]{\frac{4}{5}} \delta + (1 - \delta) \left(1 - \frac{\delta^2}{16(1-\delta)^2} + \dots\right) = 1 - \delta \left(1 - \sqrt[4]{\frac{4}{5}}\right) + \rho_1(\delta) \end{aligned}$$

where  $\frac{\rho_1(\delta)}{\delta} \xrightarrow{\delta \rightarrow 0} 0$ .

(ii) Using the expansions for  $\frac{1}{\sqrt{1+x}}$  and  $e^x$  we find that for  $|\xi| < \frac{\delta}{4\pi}$

$$\begin{aligned} |h_\delta(\xi) - \gamma_1(\xi)| &\leq |\xi|^3 \left(\frac{8\pi^3}{\delta^2} \cdot \left|\phi\left(\frac{2\pi i \xi}{\delta}\right)\right| + \frac{8\pi^3}{(1-\delta)^2} \cdot \left|\phi\left(\frac{2\pi i \xi}{1-\delta}\right)\right|\right) \\ &+ \frac{3\pi^3}{\delta(1-\delta)} - 4\pi^3 + 2\pi^4 \left(\frac{3}{4\delta(1-\delta)} - 1\right)^2 |\xi| + \frac{3\pi^4}{\delta(1-\delta)} |\xi| - 4\pi^4 |\xi| \\ &+ 4\pi^5 \left(\frac{3}{4\delta(1-\delta)} - 1\right)^2 |\xi|^2 + 4\pi^6 \left(\frac{3}{4\delta(1-\delta)} - 1\right)^2 |\xi|^3 \end{aligned}$$

$$+8\pi^3 |\psi(-2\pi i\xi)| + 8\pi^6 \left( \frac{3}{4\delta(1-\delta)} - 1 \right)^3 |\xi|^3 |\psi(-2\pi^2 \Sigma_\delta^2 \xi^2)|$$

where  $\phi(x)$  is analytic in  $|x| < \frac{1}{2}$  and  $\psi(x)$  is an entire function. Denoting  $M_\phi = \sup_{|x| \leq \frac{1}{2}} |\phi(x)|$  and  $M_\psi = \sup_{|x| \leq \frac{1}{2}} |\psi(x)|$  we find that

$$|h_\delta(\xi) - \gamma_1(\xi)| \leq \left( \frac{8\pi^3}{\delta^2} M_\phi + \frac{57\pi^3}{8\delta} + \pi^3 \left( 32M_\phi + \frac{141}{64} + \frac{539}{64} M_\psi \right) \right) |\xi|^3$$

(iii) For  $|\xi| > \frac{\delta^{1+\beta}}{4\pi}$

$$\begin{aligned} |g_\delta(\xi)| &\leq \frac{\delta}{\sqrt[4]{1 + \frac{4\pi^2 \xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1 + \frac{4\pi^2 \xi^2}{(1-\delta)^2}}} \leq \frac{\delta}{\sqrt[4]{1 + \frac{\delta^{2\beta}}{4}}} + \frac{1-\delta}{\sqrt[4]{1 + \frac{\delta^{2+2\beta}}{4(1-\delta)^2}}} \\ &= \delta \left( 1 - \frac{\delta^{2\beta}}{16} + \dots \right) + (1-\delta) \left( 1 - \frac{\delta^{2+2\beta}}{16(1-\delta)^2} + \dots \right) = 1 - \frac{\delta^{1+2\beta}}{16} + \rho_2(\delta) \end{aligned}$$

where  $\frac{\rho_2(\delta)}{\delta^{1+2\beta}} \xrightarrow{\delta \rightarrow 0} 0$ .

(iv) This is a general property of the Fourier transform of a density function.

(v)

$$\begin{aligned} \int_{|\xi| > \frac{\delta}{4\pi}} |g_{\delta_N}(\xi)|^{N-j-1} d\xi &\leq \int_{|\xi| > \frac{\delta}{4\pi}} \left( \frac{\delta}{\sqrt[4]{1 + \frac{4\pi^2 \xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1 + \frac{4\pi^2 \xi^2}{(1-\delta)^2}}} \right)^{N-j-1} d\xi \\ &= \frac{\delta}{2\pi} \int_{|x| > \frac{1}{2}} \left( \frac{\delta}{\sqrt[4]{1 + x^2}} + \frac{1-\delta}{\sqrt[4]{1 + \frac{\delta^2 x^2}{(1-\delta)^2}}} \right)^{N-j-1} dx \\ &\leq \frac{\left( 1 - \delta \left( 1 - \sqrt[4]{\frac{4}{5}} \right) + \rho_1(\delta) \right)^{N-j-1}}{\pi} + \frac{\delta}{\pi} \int_{\frac{1}{8}}^{\infty} \left( \frac{\delta^{\frac{3}{2}}}{\sqrt{\delta x}} + \frac{(1-\delta)^{\frac{3}{2}}}{\sqrt{\delta x}} \right)^{N-j-1} dx \\ &\leq \frac{\left( 1 - \delta \left( 1 - \sqrt[4]{\frac{4}{5}} \right) + \rho_1(\delta) \right)^{N-j-1}}{\pi} + \frac{2}{\pi(N-j-3)} \end{aligned}$$

Note that in our case

$$\begin{aligned} &\sqrt{N-j} \Sigma_{\delta_N} \int_{|\xi| > c\delta_N} |g_{\delta_N}(\xi)|^{N-j-1} d\xi \\ &\leq \frac{3\sqrt{N-j} \left( 1 - \delta \left( 1 - \sqrt[4]{\frac{4}{5}} \right) + \rho_1(\delta) \right)^{N-j-1}}{2\pi\delta} + \frac{3}{\sqrt{N\delta} \cdot \sqrt{1 - \frac{j+3}{N}}} \end{aligned}$$

so as long as the conditions in (??) are satisfied we have that  $\epsilon_j(N)$  defined in Remark ?? would satisfy  $\epsilon_j(N) \xrightarrow{N \rightarrow \infty} 0$ . Let  $f_{\delta_N}(v) = f_\delta(v) = \delta M_{\frac{1}{2\delta}}(v) + (1-\delta)M_{\frac{1}{2(1-\delta)}}(v)$  such that

$$\begin{aligned} \delta_N \text{ is dominated by powers of } N \\ \delta_N^{1+2\beta} \cdot N \xrightarrow{N \rightarrow \infty} \infty \\ \delta_N^{1+3\beta} \cdot N \xrightarrow{N \rightarrow \infty} 0 \end{aligned} \tag{5}$$

then for a fixed  $j$

$$Z_{N-j}(f_{\delta_N}, \sqrt{u}) = \frac{2}{\sqrt{N-j} \cdot \Sigma_{\delta_N} \cdot |\mathbb{S}^{N-j-1}| u^{\frac{N-j}{2}-1}} \left( \frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} + \lambda_j(N-j, u) \right)$$

where  $\sup_{u \in \mathbb{R}} |\lambda_j(N-j, u)| \leq \epsilon_j(N)$  and  $\lim_{N \rightarrow \infty} \epsilon_j(N) = 0$ . This is immediate from Lemma ??, ??, ??, Theorem ?? and Remark ?. We're now ready to compute the entropy production. We'll start by estimating its denominator and numerator. Let  $F_N(v_1, \dots, v_N) = \frac{\prod_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f, \sqrt{N})}$  where  $\delta_N$  satisfies conditions (??). Then

$$\lim_{N \rightarrow \infty} \frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N}{N} = \frac{\log 2}{2}$$

Using the symmetry of the problem, Lemma ?? from the Appendix, Theorem ?? and Stirling's formula we find that

$$\begin{aligned} \int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N &= \frac{1}{Z_N(f_\delta, \sqrt{N})} \cdot \sum_{k=1}^N \int_{\mathbb{S}^{N-1}(\sqrt{N})} (\prod_{i=1}^N f_\delta(v_i)) \log f_\delta(v_k) d\sigma^N - \log Z_N(f_\delta, \sqrt{N}) \\ &= \frac{N |\mathbb{S}^{N-2}|}{N^{\frac{N-2}{2}} |\mathbb{S}^{N-1}|} \int_{-\sqrt{N}}^{\sqrt{N}} f_\delta(v_1) \log f_\delta(v_1) (N - v_1^2)^{\frac{N-3}{2}} \cdot \frac{Z_{N-1}(f_\delta, \sqrt{N - v_1^2})}{Z_N(f_\delta, \sqrt{N})} dv_1 - \log Z_N(f_\delta, \sqrt{N}) \\ &= \frac{N}{\sqrt{1 - \frac{1}{N}} (1 + \sqrt{2\pi} \lambda_0(N, N))} \int_{\mathbb{R}} f_\delta(v_1) \log f_\delta(v_1) \cdot \chi_{[-\sqrt{N}, \sqrt{N}]}(v_1) \\ &\quad \cdot \left( e^{-\frac{(1-v_1^2)^2}{(N-1)\Sigma_\delta^2}} + \sqrt{2\pi} \lambda_1(N-1, N - v_1^2) \right) dv_1 \\ &- \left( \log \left( \sqrt{2} \left( 1 + O\left(\frac{1}{\sqrt{N}}\right) \right) \right) \left( 1 + \sqrt{2\pi} \lambda_0(N, N) \right) \right) - \frac{N}{2} (\log 2\pi + 1) - \frac{1}{2} \cdot \log \left( \frac{3}{4\delta(1-\delta)} - 1 \right) \end{aligned}$$

Since  $0 < f_\delta \leq 1$  we have that

$$\begin{aligned} & \left| f_\delta(v_1) \log f_\delta(v_1) \cdot \chi_{[-\sqrt{N}, \sqrt{N}]}(v_1) \cdot \left( e^{-\frac{(1-v_1^2)^2}{(N-1)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_1(N-1, N-v_1^2) \right) \right| \\ & \leq \left( 1 + \sqrt{2\pi}\epsilon_1(N) \right) (-f_\delta(v_1) \log f_\delta(v_1)) \\ & \leq \left( 1 + \sqrt{2\pi}\epsilon_1(N) \right) \left( -\delta M_{\frac{1}{2\delta}}(v_1) \log \left( \delta M_{\frac{1}{2\delta}}(v_1) \right) - (1-\delta) M_{\frac{1}{2(1-\delta)}}(v_1) \log \left( (1-\delta) M_{\frac{1}{2(1-\delta)}}(v_1) \right) \right) \\ & = g_\delta(v_1) \end{aligned}$$

It is easy to check that

$$g_{\delta_N}(v) \xrightarrow{N \rightarrow 0} -M_{\frac{1}{2}}(v) \log M_{\frac{1}{2}}(v)$$

and

$$\int_{\mathbb{R}} g_{\delta_N}(v) dv \xrightarrow{N \rightarrow 0} - \int_{\mathbb{R}} M_{\frac{1}{2}}(v) \log M_{\frac{1}{2}}(v) dv = \frac{\log \pi}{2} + \frac{1}{2}$$

Since

$$\begin{aligned} & f_{\delta_N}(v_1) \log f_{\delta_N}(v_1) \cdot \chi_{[-\sqrt{N}, \sqrt{N}]}(v_1) \cdot \left( e^{-\frac{4(1-v_1^2)^2 \delta_N(1-\delta_N)}{(N-1)(3-4\delta_N(1-\delta_N))}} + \sqrt{2\pi}\lambda_1(N-1, N-v_1^2) \right) \\ & \xrightarrow{N \rightarrow \infty} M_{\frac{1}{2}}(v_1) \log M_{\frac{1}{2}}(v_1) \end{aligned}$$

we conclude that

$$\frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N}{N} \xrightarrow{N \rightarrow \infty} \int_{\mathbb{R}} M_{\frac{1}{2}}(v_1) \log M_{\frac{1}{2}}(v_1) dv_1 + \frac{1}{2} + \frac{\log 2\pi}{2} = \frac{\log 2}{2}$$

due to the generalized dominated convergence theorem. Let  $F_N(v_1, \dots, v_N) = \frac{\prod_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f, \sqrt{N})}$  where  $\delta_N$  satisfies conditions (??). Then there exists a constant  $C_{type-\delta}$  depending only on the behavior of  $\delta_N$  such that

$$\frac{\langle \log F_N, N(I-Q)F_N \rangle}{N} \leq C_{type-\delta} (-\delta_N \log \delta_N)$$

Similar to Lemma ?? by using the symmetry of the problem, Lemma ?? from the Appendix, Theorem ?? and Stirling's formula we find that

$$\begin{aligned} & \langle \log F_N, N(I-Q)F_N \rangle \\ & = \frac{1}{Z_N(f_\delta, \sqrt{N})(N-1)\pi} \sum_{k=1}^N \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_\delta(v_k) \end{aligned}$$

$$\cdot \left( \sum_{i < j} \int_0^{2\pi} (f^{\otimes N}(v_1, \dots, v_N) - f^{\otimes N}(R_{i,j}(\vartheta)(v_1, \dots, v_N))) d\vartheta \right) d\sigma^N$$

if  $i$  and  $j$  are different than  $k$  the integral is zero and so

$$\begin{aligned} \langle \log F_N, N(I - Q)F_N \rangle &= \frac{1}{Z_N(f_\delta, \sqrt{N})(N-1)\pi} \sum_{k=1}^N \sum_{j \neq k} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_\delta(v_k) \\ &\quad \cdot \left( \int_0^{2\pi} (f^{\otimes N}(v_1, \dots, v_N) - f^{\otimes N}(R_{k,j}(\vartheta)(v_1, \dots, v_N))) d\vartheta \right) d\sigma^N \\ &= \frac{N}{Z_N(f_\delta, \sqrt{N})\pi} \int_0^{2\pi} d\vartheta \int_{\mathbb{S}^{N-1}(\sqrt{N})} (-\log f_\delta(v_1)) (f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2)) (\prod_{i=3}^N f_\delta(v_i)) d\sigma^N \\ &= \frac{N|\mathbb{S}^{N-3}|}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}\pi} \int_0^{2\pi} d\vartheta \int_{v_1^2+v_2^2 \leq N} (-\log f_\delta(v_1)) (f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2)) \\ &\quad \cdot (N - v_1^2 - v_2^2)^{\frac{N-4}{2}} \frac{Z_{N-2}(f_\delta, \sqrt{N - v_1^2 - v_2^2})}{Z_N(f_\delta, \sqrt{N})} dv_1 dv_2 \\ &= \frac{N}{\pi\sqrt{1 - \frac{2}{N}}} \int_0^{2\pi} d\vartheta \int_{v_1^2+v_2^2 \leq N} (-\log f_\delta(v_1)) (f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2)) \\ &\quad \cdot \frac{e^{-\frac{(2-v_1^2-v_2^2)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2(N - 2.N - v_1^2 - v_2^2)}{1 + \sqrt{2\pi}\lambda_0(N, N)} dv_1 dv_2 \end{aligned}$$

Using rotational symmetry and symmetry in the variables we find that

$$\begin{aligned} &\langle \log F_N, N(I - Q)F_N \rangle \\ &= \frac{N}{4\pi\sqrt{1 - \frac{2}{N}}} \int_0^{2\pi} d\vartheta \int_{v_1^2+v_2^2 \leq N} (\log f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - \log f_\delta(v_1)f_\delta(v_2)) \\ &\quad (f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2)) \cdot \frac{e^{-\frac{(2-v_1^2-v_2^2)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2(N - 2.N - v_1^2 - v_2^2)}{1 + \sqrt{2\pi}\lambda_0(N, N)} dv_1 dv_2 \\ &\leq \frac{N}{4\pi\sqrt{1 - \frac{2}{N}}} \int_0^{2\pi} d\vartheta \int_{\mathbb{R}^2} (\log f_\delta(v_1(\vartheta))f_\delta(v_1(\vartheta)) - \log f_\delta(v_1)f_\delta(v_1)) \\ &\quad \cdot (f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2)) \cdot \frac{1 + \sqrt{2\pi}\epsilon_2(N)}{1 + \sqrt{2\pi}\lambda_0(N, N)} dv_1 dv_2 \end{aligned}$$



$$= \frac{N(1 + \sqrt{2\pi}\epsilon_2(N))}{\pi\sqrt{1 - \frac{2}{N}}(1 + \sqrt{2\pi}\lambda_0(N, N))} \int_0^{2\pi} d\vartheta \int_{\mathbb{R}^2} (-\log f_\delta(v_1)) (f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2)) dv_1 dv_2$$

Since  $M_a(v_1(\vartheta))M_a(v_2(\vartheta)) = M_a(v_1)M_a(v_2)$  we see that

$$\begin{aligned} f_\delta(v_1(\vartheta))f_\delta(v_2(\vartheta)) - f_\delta(v_1)f_\delta(v_2) &= \delta(1-\delta) \left( M_{\frac{1}{2\delta}}(v_1(\vartheta))M_{\frac{1}{2(1-\delta)}}(v_2(\vartheta)) - M_{\frac{1}{2\delta}}(v_1)M_{\frac{1}{2(1-\delta)}}(v_2) \right) \\ &\quad + \delta(1-\delta) \left( M_{\frac{1}{2\delta}}(v_2(\vartheta))M_{\frac{1}{2(1-\delta)}}(v_1(\vartheta)) - M_{\frac{1}{2\delta}}(v_2)M_{\frac{1}{2(1-\delta)}}(v_1) \right) \\ &\leq \delta(1-\delta) \left( M_{\frac{1}{2\delta}}(v_1(\vartheta))M_{\frac{1}{2(1-\delta)}}(v_2(\vartheta)) + M_{\frac{1}{2\delta}}(v_2(\vartheta))M_{\frac{1}{2(1-\delta)}}(v_1(\vartheta)) \right) \end{aligned}$$

and along with

$$-\log f_\delta(v_1) \leq -\log \left( \delta M_{\frac{1}{2\delta}}(v_1) \right) \leq -\frac{3 \log \delta}{2} + \frac{\log \pi}{2} + \delta(v_1^2(\vartheta) + v_2^2(\vartheta))$$

we conclude that

$$\begin{aligned} &\frac{\langle \log F_N, N(I-Q)F_N \rangle}{N} \\ &\leq \frac{4(1 + \sqrt{2\pi}\epsilon_2(N))\delta(1-\delta)}{\sqrt{1 - \frac{2}{N}}(1 + \sqrt{2\pi}\lambda_0(N, N))} \int_{\mathbb{R}^2} \left( -\frac{3 \log \delta}{2} + \frac{\log \pi}{2} + \delta(v_1^2 + v_2^2) \right) M_{\frac{1}{2\delta}}(v_1)M_{\frac{1}{2(1-\delta)}}(v_2) dv_1 dv_2 \\ &\leq \frac{4(1 + \sqrt{2\pi}\epsilon_2(N))}{\sqrt{1 - \frac{2}{N}}(1 + \sqrt{2\pi}\lambda_0(N, N))} \left( \frac{3}{2} - \frac{\log \pi}{2 \log \delta} - \frac{1}{2 \log \delta} - \frac{\delta}{2 \log \delta} \right) (-\delta \log \delta) \end{aligned}$$

The result follows. Let  $F_N(v_1, \dots, v_N) = \frac{\prod_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f, \sqrt{N})}$  where  $\delta_N$  satisfies conditions (??). Then there exists a constant  $C_{type-\delta}$  and an integer  $N_{type-\delta}$  depending only on the behavior of  $\delta_N$  such that for every  $N > N_{type-\delta}$

$$\frac{\langle \log F_N, N(I-Q)F_N \rangle}{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N} \leq C_{type-\delta} (-\delta_N \log \delta_N)$$

This follows immediately from Lemma ?? and ?. Let  $F_N(v_1, \dots, v_N) = \frac{\prod_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f, \sqrt{N})}$  where  $\delta_N = \frac{1}{N^{1-2\beta}}$  and  $0 < \beta < \frac{1}{6}$ . Then there exists a constant  $C_\beta$  and an integer  $N_\beta$  depending only on  $\beta$  such that for every  $N > N_\beta$

$$\frac{\langle \log F_N, N(I-Q)F_N \rangle}{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N} \leq \frac{C_\beta \log N}{N^{1-2\beta}}$$

This follows immediately from Theorem ?? and the fact that  $\delta_N = \frac{1}{N^{1-2\beta}}$  satisfies conditions (??).

From this we conclude our main result: For any  $0 < \beta < \frac{1}{6}$  there exists a constant  $C_\beta$  depending only on  $\beta$  such that

$$\Gamma_N \leq \frac{C_\beta \log N}{N^{1-2\beta}}$$

## 5 Final Remarks

One question we might ask ourselves is: Can we modify the given proof to get the exact value in Villani's conjecture? Looking at the proof we notice that the result we obtained has very tight conditions in terms of  $\beta$ . We needed  $\delta_N^{1+2\beta} N$  to diverge to infinity and  $\delta_N^{1+3\beta} N$  to go to zero. This doesn't leave much room for variations. This leads us to believe that the family of functions constructed here would not be helpful to prove the exact version of Villani's conjecture. Something more clever must be done.

Another question we don't know the answer to is the fourth moment question. Both in this paper and in ([?]) the family of functions constructed has an unbounded fourth moment. Would restricting the fourth moment lead to a lower bound on the entropy production?

Lastly, can our computation be generalized to a more difficult interaction than Kac's model? Can we try and use the same idea in a different models of the Boltzmann equation?

While we don't know the answers to the proposed questions we hope that this paper shed some light on the entropy production problem and that at least some of the above questions would seem more solvable after reading it.

## A Helpful Computations

The appendix consists of Lemmas that are vital for the computations needed in our paper, and are used extensively in Sections ?? and ??. (Gaussian Integral Estimation)

$$\frac{\sqrt{2\pi}}{a} \cdot \sqrt{1 - e^{-\frac{a\eta^2}{2}}} \leq \int_{|x| < \eta} e^{-\frac{a^2 x^2}{2}} dx \leq \frac{\sqrt{2\pi}}{a} \cdot \sqrt{1 - e^{-a^2 \eta^2}}$$

$$\int_{|x| > \eta} e^{-\frac{a^2 x^2}{2}} dx \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2 \eta^2}{2}}}{a}$$

We have

$$\begin{aligned} \int_{|x| < \eta} e^{-\frac{a^2 x^2}{2}} dx &= \sqrt{\int \int \int_{|x|, |y| < \eta} e^{-\frac{a^2(x^2+y^2)}{2}} dx dy} \leq \sqrt{\int \int_{x^2+y^2 < 2\eta^2} e^{-\frac{a^2(x^2+y^2)}{2}} dx dy} \\ &= \sqrt{\int_0^{2\pi} \int_0^{\sqrt{2}\eta} r e^{-\frac{a^2 r^2}{2}} dr d\vartheta} = \sqrt{2\pi} \cdot \sqrt{\frac{1 - e^{-a^2 \eta^2}}{a^2}} \end{aligned}$$

And

$$\int_{|x|<\eta} e^{-\frac{a^2 x^2}{2}} dx \geq \sqrt{\int \int_{x^2+y^2<\eta^2} e^{-\frac{a^2(x^2+y^2)}{2}} dx dy} = \sqrt{2\pi} \cdot \sqrt{\frac{1 - e^{-\frac{a\eta^2}{2}}}{a^2}}$$

Similarly

$$\begin{aligned} \int_{|x|>\eta} e^{-\frac{a^2 x^2}{2}} dx &= \int_{\mathbb{R}} e^{-\frac{a^2 x^2}{2}} dx - \int_{|x|<\eta} e^{-\frac{a^2 x^2}{2}} dx = \frac{\sqrt{2\pi}}{a} - \int_{|x|<\eta} e^{-\frac{a^2 x^2}{2}} dx \\ &\leq \frac{\sqrt{2\pi}}{a} \left( 1 - \sqrt{1 - e^{-\frac{a^2 \eta^2}{2}}} \right) = \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2 \eta^2}{2}}}{a \left( 1 + \sqrt{1 - e^{-\frac{a^2 \eta^2}{2}}} \right)} \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2 \eta^2}{2}}}{a} \end{aligned}$$

(Special Sums Evaluation)

$$\sum_{k=k_0+1}^m \frac{e^{-\frac{a^2 k}{2}}}{\sqrt{k}} \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2 k_0}{2}}}{a}$$

$$\sum_{k=k_0+1}^m \frac{1}{\sqrt{k}} \leq 2\sqrt{m}$$

We have that

$$\begin{aligned} \sum_{k=k_0+1}^m \frac{e^{-\frac{a^2 k}{2}}}{\sqrt{k}} &\leq \int_{k_0}^m \frac{e^{-\frac{a^2 x}{2}}}{\sqrt{x}} dx = \frac{2}{a} \int_{a\sqrt{k_0}}^{a\sqrt{m}} e^{-\frac{y^2}{2}} dy \leq \frac{2}{a} \int_{a\sqrt{k_0}}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{a} \int_{|y|>a\sqrt{k_0}} e^{-\frac{y^2}{2}} dy \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2 k_0}{2}}}{a} \end{aligned}$$

Similarly

$$\sum_{k=k_0+1}^m \frac{1}{\sqrt{k}} \leq \int_{k_0}^m \frac{dx}{\sqrt{x}} = 2 \left( \sqrt{m} - \sqrt{k_0} \right) \leq 2\sqrt{m}$$

The next set of Lemmas refer to integration over the sphere  $\mathbb{S}^{N-1}(r)$ . (Integration on the Sphere I) Let  $f(v_1, \dots, v_N)$  be a continuous function on  $\mathbb{R}^N$  then

$$\int_{\mathbb{S}^{N-1}(r)} f ds_r^N = \sum_{\epsilon=\{+,-\}} \int_{\sum_{i=1}^{N-1} v_i^2 \leq r^2} \frac{r \cdot f \left( v_1, \dots, v_{N-1}, \epsilon \sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2} \right)}{\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}} dv_1 \dots dv_{N-1}$$

Standard in any Differential Geometry course. (Integration on the Sphere with the Uniform Probability Measure)

$$\int_{\mathbb{S}^{N-1}(r)} f d\sigma_r^N = \frac{1}{|\mathbb{S}^{N-1}|r^{N-2}} \cdot \sum_{\epsilon=\{+,-\}} \int_{\sum_{i=1}^{N-1} v_i^2 \leq r^2} \frac{f\left(v_1, \dots, v_{N-1}, \epsilon \sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}\right)}{\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}} dv_1 \dots dv_{N-1}$$

(Integration on the Sphere II) Let  $f(v_1, \dots, v_j)$  and  $g(v_{j+1}, \dots, v_N)$  be continuous functions on  $\mathbb{R}^j$  and  $\mathbb{R}^{N-j}$  respectively. Then

$$\begin{aligned} & \int_{\mathbb{S}^{N-1}(r)} f(v_1, \dots, v_j) \cdot g(v_{j+1}, \dots, v_N) d\sigma_r^N \\ &= \frac{|\mathbb{S}^{N-j-1}|}{|\mathbb{S}^{N-1}|r^{N-2}} \int_{\sum_{i=1}^j v_i^2 \leq r^2} f(v_1, \dots, v_j) \left( r^2 - \sum_{i=1}^j v_i^2 \right)^{\frac{N-j-2}{2}} \\ & \quad \left( \int_{\mathbb{S}^{N-j-1}(\sqrt{r^2 - \sum_{i=1}^j v_i^2})} g d\sigma_{\sqrt{r^2 - \sum_{i=1}^j v_i^2}}^{N-j} \right) dv_1 \dots dv_j \end{aligned}$$

Using Corollary ?? we find that

$$\begin{aligned} & \int_{\mathbb{S}^{N-1}(r)} f(v_1, \dots, v_j) \cdot g(v_{j+1}, \dots, v_N) d\sigma_r^N \\ &= \frac{\sum_{\epsilon=\{+,-\}}}{|\mathbb{S}^{N-1}|r^{N-2}} \int_{\sum_{i=1}^{N-1} v_i^2 \leq r^2} \frac{f(v_1, \dots, v_j) \cdot g\left(v_{j+1}, \dots, v_{N-1}, \epsilon \sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}\right)}{\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}} dv_1 \dots dv_{N-1} \\ &= \frac{1}{|\mathbb{S}^{N-1}|r^{N-2}} \int_{\sum_{i=1}^j v_i^2 \leq r^2} \frac{f(v_1, \dots, v_j)}{\sqrt{r^2 - \sum_{i=1}^j v_i^2}} \left( \int_{\mathbb{S}^{N-j-1}(\sqrt{r^2 - \sum_{i=1}^j v_i^2})} g ds^{N-j} \right) dv_1 \dots dv_j \\ &= \frac{|\mathbb{S}^{N-j-1}|}{|\mathbb{S}^{N-1}|r^{N-2}} \int_{\sum_{i=1}^j v_i^2 \leq r^2} f(v_1, \dots, v_j) \left( r^2 - \sum_{i=1}^j v_i^2 \right)^{\frac{N-j-2}{2}} \\ & \quad \left( \int_{\mathbb{S}^{N-j-1}(\sqrt{r^2 - \sum_{i=1}^j v_i^2})} g d\sigma_{\sqrt{r^2 - \sum_{i=1}^j v_i^2}}^{N-j} \right) dv_1 \dots dv_j \end{aligned}$$

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