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The Need for an Application of Dual-Process Theory to Mathematics Education

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Abstract

Educational researchers have only recently and even then to a negligible extent begun to borrow analytical frameworks from cognitive psychology to address the systematic cognitive biases that pervade student learning. This review seeks to bridge the gap between educational research and cognitive psychology by analysing two mathematics education studies through the framework of dual-process theory of reasoning. The selected studies depict widespread cognitive biases, which dual-process theory can explain and potentially remedy. The first experimental study, which is borrowed from Dooren et al. (2003), reveals how the probabilistic reasoning of secondary school students (aged 15-18) is marked by inadequate responses that are the necessary result of intuitive Type 1 cognitive processes. The second experimental study taken from Lehman and Nisbett (1990) sheds light on the opposite channel of the relationship between reasoning and learning: the beneficial effect statistics education has on undergraduate students' domain-general statistical reasoning. This far-reaching observation is interpreted in terms of structural changes in students' Type 1 and Type 2 cognitive processing. Given that research on cognitive science and mathematics education continues to be the result of separate scholarly communities, there remains a serious lack of empirical studies that apply frameworks from cognitive psychology to issues that are central to mathematics education. By interpreting the results of a dual-process analysis of two contrasting instances of mathematical learning, this review indicates and motivates promising future avenues for a more systematic collaboration between cognitive scientists and mathematics education researchers. This essay concludes by identifying the design-experimental methodology as a viable way through which dual-process theory could increase the effectiveness of classroom interventions. In turn, theory-driven classroom interventions may resolve some of the tensions between current dual-process theories.

Keywords: cognitive processes, mathematics education, Type 1 and Type 2 processes, dual process theory, rational thinking errors

Introduction

Across the wide landscape of mathematics instruction, educators are perplexed and even disheartened by students who despite having the requisite disciplinary knowledge, repeatedly fail to apply formal reasoning. Instead of drawing on their formal knowledge base, students often rely on intuitive problem-solving strategies, which lead to erroneous or formally inadequate responses. Given the immense extent to which intuitive problem-solving conflicts with effective learning and instruction, there is an increasing appreciation for the educational significance of remedying such systematic biases in learners' reasoning (Attridge & Inglis, 2014; Babai, Shalev, & Stavy, 2015; Dooren, Bock, Depaepe, Janssens, & Verschaffel, 2003; Leron & Hazzan, 2006).

To investigate the cognitive roots underlying this educational challenge, dual-process theorists attempt to explain how human reasoning is shaped by the interaction of two distinct types of cognitive processing (Stanovich, 2011). Whereas Type 1 processing is typically coined as heuristic or intuitive and is said to engender automatic first-impression cognitive processes, Type 2 processing is described as analytic requiring cognitive effort that loads heavily on a person's working memory. Given its limited scope, this review will focus on experimental evidence leaving the evaluation of significant theoretical (e.g., Dubinsky, 2002; Lieberman, 2009; Tzur & Simon, 2004) and neuropsychological (e.g., Goel, 2008; Tsuji & Watanabe, 2009) contributions to other reviewers.

Since current empirical evidence of dual processes is primarily restricted to the field of cognitive psychology, dual-process theory severely lacks evidence of its applicability to educational problems (Gigerenzer, 2010; Keren & Schul, 2009). Through dual-process analyses of experimental data drawn from two complementary mathematics education studies, this review directly addresses this major shortcoming of dual-process theory motivating future investigation along this worthy avenue. The ultimate goal of this research initiative is to map pathways for integrating the currently separate projects conducted by dual-process theorists and mathematics education researchers (Dooren et al., 2003; Kryjevskaia, Stetzer, & Grosz, 2014; Leron & Hazzan, 2009).

In the introductory part of this literature review, I will define the concepts that are integral to evaluating the symbiotic relationship between reasoning and mathematical learning. Key concepts that feature in this definitional section are Type 1 and Type 2 cognitive processes, mathematical problem solving, and knowledge acquisition.

The subsequent section introduces the main body of this review and serves three purposes: First, it provides a bird's-eye view of dual-process theory. Second, it explains the disagreement between two influential mappings of cognitive architecture, namely the parallel-competitive and default-interventionist models. Third, it introduces a prominent extension of dual-process theory, namely Stanovich's tri-process model. The latter model will serve as the primary interpretative lens through which I will proceed to evaluate two empirical studies borrowed from the mathematics education literature.

The third section will illuminate two particularly urgent areas of interaction between dual-process theory and mathematics education, namely their conflicting perspectives on erroneous problem solving and how the two types of cognitive processes relate to conceptual and procedural knowledge.

In the final part of this review, I will proceed to interpreting two experimental studies of mathematical reasoning through an application of dual-process theory. In the first study, Dooren et al. (2003) present experimental evidence for how the so-called illusion of proportionality biases secondary school students' probabilistic reasoning. My corresponding dual-process analysis interprets students' biased reasoning as an overreliance on Type 1 cognitive processes. A second empirical case included in this review is the longitudinal study taken from Lehman and Nisbett (1990). The authors establish a connection between university students' improvements in domain-general statistical reasoning and the statistics education that those students received during their four years of undergraduate education. The evidence presented in Lehman and Nisbett is especially suitable for illuminating the opposite, equally pertinent, channel of the relationship between learners' reasoning and knowledge acquisition: the learning benefits that statistics knowledge may have for students' domain-general cognitive processing.

1. Key Terminology

1.1 Two Separable Types of Cognitive Processes

Table T1, which considerably overlaps with "Table 1" in Evans and Stanovich (2013, p. 225) and "Table I.I" in Stanovich (2011, p. 18), juxtaposes the two types of cognitive processes in terms of their features, functional attributes, correlates, and their appearance in human life. I added four examples of how the two processing types may be used both within and outside the classroom to demonstrate their ubiquity. Finally, the evolutionary perspective of the two processing types, which is seen as a prominent contribution to dual-process theory, has also been included.

Table T1. *Features, Correlates, and Attributes of the Two Cognitive Processing Types*

Type 1 processes (intuitive)	Type 2 processes (analytic)

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Defining features	
Autonomous processing	Cognitive decoupling
Independent of working memory	Limited by working memory capacity
Associated with the limbic system	Associated with the prefrontal cortex
Parallel operation possible	Only serial operation possible
Weightier among novice learners	Weightier among expert learners
Functional attributes	
Fast	Slow
Autonomous	Algorithmic
Automatic	Conscious
Contextualised	Abstract/decontextualised
Implicit/tacit	Explicit
Intuitive	Rational
Reflexive	Reflective
Informal	Formal
Associative	Rule-based
Stimulus-bound	Higher order
Cognitively effortless	Cognitively effortful
Correlates	
Biased/modal responses	Normative/formal responses
Independent of cognitive ability	Correlated with cognitive ability
Examples of Use	
Recognising a friend’s face	Memorising a new phone number
Applying a familiar algorithm or rule	Applying a new algorithm or rule
Describing a graph	Interpreting a graph
Speaking in one’s mother tongue	Speaking in a foreign language
Alternative Terminology	

System 1	System 2
Heuristic processes	Analytic processes
Evolutionary Perspective	
Similar to animal cognition	Distinctively human

In light of the immense variety of labels associated with the two types of cognitive processes, I will pay special attention to justifying my choice of terminology. With the previously prevalent distinction between two types of *cognitive systems* (Stanovich, 1999) considered inadequate, there has been a recent trend to substitute *"system 1/system 2"* (Kahneman, 2013) with Type 1 and Type 2 processing (Evans & Stanovich, 2013). I accorded with this definitional trend recognising that the labels system 1 and system 2 may wrongly imply that each of a person's cognitive processes originate from a singular system 1 or system 2. By using terms as restrictive as system 1 and system 2, one excludes the possibility of a given cognitive process resulting from multiple cognitive systems and neural underpinnings (Evans & Stanovich, 2013). This restriction is especially problematic in the context of system 1 as Stanovich (2009) demonstrated in his analysis of the autonomous set of systems (TASS). Through using the labels Type 1 and Type 2 one can effectively distinguish between two types of cognitive processing without making a claim about the number of cognitive or neural systems underlying those types of processing.

1.2 Problem-Solving in the Context of Mathematics

When using the term mathematical problem solving, I refer to a "situation that proposes a mathematical question whose solution is not immediately accessible to the solver" (Callejo, 2009, p. 112). According to this definition, problem solving is tied to Type 2 rather than Type 1 processing. This has to do with the assumption that Type 1 processing generates responses that are automatic and immediate - the opposite of what is thought to constitute problem solving according to the above definition.

1.3 Knowledge Acquisition in Terms of Mathematical Learning

In this review, I will typically use the general term learning instead of narrower alternatives such as response acquisition, knowledge acquisition or knowledge construction (Mayer, 1992). As such, no claim is made about the relative merits or weaknesses of the competing attempts to conceptualise the way knowledge becomes manifest in a learner's mind. Given this paper's limited scope, I will primarily deal with the subset of literature that illuminates the connection between cognition and learning in the mathematical domain. Occasionally, I will present cases in which learning implications for other Science, Technology, Engineering and Mathematics (STEM) domains are discussed.

2. Convergence and Conflict between Current Dual-Process Theories

2.1 A Bird's-Eye View of Dual-Process Theory of Reasoning

Table T2 summarises the scientific synergy accomplished by experimental and neuropsychological approaches to dual-process theory. The depicted variety of scientific subfields for which separable types of cognitive processes were proposed gives weight to my hypothesis of the existence of qualitatively distinct types of cognitive processes.

Table T2. *Methodological Approaches to Dual-Process Theory*

Methodology	Key Publications	Key Findings
Experimental	Attridge & Inglis, 2014; De Neys, 2006 (a & b); De Neys, W., Schaeken, W., & d'Ydewalle, 2005; Evans, 1996; Evans & Curtis-Holmes, 2005; Evans et al., 2010; Kahneman, 2011; Kahneman & Tversky, 2000; Roberts & Newton, 2001; Stanovich, 2011; Stevenson & Over, 1995; Van Hoof et al., 2013	• Administering a challenging task prior to a cognitive reflection test (CRT) makes it more likely that participants will inhibit Type 1 processing • Belief bias increases, while logical accuracy decreases under (a) time constraints and (b) working memory load (inhibiting Type 2 processes) • Conjunction fallacy more common among those who give quick responses (linked with default Type 1 processes) • People's ability to engage the reflective mind determines whether Type 2 processes will override Type 1 processes

		• Type 2 processing positively correlated with cognitive ability and formally correct responses
Neuropsychological	De Neys, Vartanian, & Goel, 2008; Goel, 2008; Greene, Nystrom, Engell, Darley, & Cohen, 2004); Lieberman, 2007; Tsuji & Watanabe, 2009;	• Belief-based responses were systematically linked with different brain areas compared to reason-based responses • When belief-based reasoning was inhibited, right prefrontal cortex areas were activated • Prefrontal and frontal cortical areas were activated when participants made monetary decisions on the basis of deferred rather than immediate reward • By contrast, when immediate reward was chosen, this was associated with the limbic system • Greater activity of prefrontal cortex and parietal lobes when deontological reasoning was overridden by consequentialist reasoning • Despite being aware of a conflict between their intuitive and normative reasoning, participants give intuitive responses on a decision-making task

2.2 Cognitive Architecture: Parallel-Competitive versus Default-Interventionist Structure

Apart from knowing the distinctive roles and features of Type 1 and Type 2 processing, an important project pursued by dual-process theorists is to map the ways in which these two types interact in a given brain. There are currently two major mappings suggested by dual-process researchers: the parallel-competitive and the default-interventionist structure (Evans & Stanovich, 2013).

The parallel-competitive model proposed by Sloman (1996) suggests that Type 1 and 2 processing operate independently providing their individual solutions to problems. Despite the word competitive being part of its label, the parallel-competitive structure presupposes a cooperation between Type 1 and 2 processing, which is captured by the analogy of "two experts who are working cooperatively to compute sensible answers" (Sloman, 1996, p. 6). Thus, although Type 1 and 2 processing are assumed to proceed autonomously, an agent's eventual response may result from work performed by both types of processing (Leron & Hazzan, 2006). The simultaneous operation of Type 1 and Type 2 processing characteristic of the parallel-competitive model makes it especially challenging to pinpoint the type of processing that generated a response or solution in a given educational instance.

By contrast, according to the default-interventionist model originally coined by Evans (2007) rapid and automatic Type 1 processing provides a default response to a given task or problem, while Type 2 processing has as a monitory function and may promote an alternative response if the default response generated by Type 1 processing is deemed inadequate. According to Evans (2007), this creates three possible processing outcomes:

1. The default response R generated by Type 1 processing is chosen without the intervention of Type 2 processing,

2. The default response R generated by Type 1 processing is chosen despite the intervention of Type 2 processing,

3. The alternative response A is chosen after an intervention of Type 2 processing.

In contrast to the parallel-competitive model, the default-interventionist structure allows one to pinpoint the processing type that determined a person's response in all three of the above cases. In the first outcome, the agent would lack awareness of the process that guided her response, which suggests that Type 1 processing was at work. In the second outcome, the agent will remember both the intervention of Type 2 processing and her decision to favour a response, which wasn't consciously generated. In the third case, the agent will explain her response in terms of the intervention of Type 2 processing whose response was favoured by the agent.

2.3 An Extension of Conventional Dual-Process Theory: Stanovich's Tri-Process Model

As illustrated in Figure 1, Stanovich's tri-process model of the mind (2009; 2011) consists of the *autonomous mind* (Type 1 processing), the *algorithmic mind* (Type 2 processing), and the *reflective mind*. The latter construct cannot be attributed to either of the two processing types.

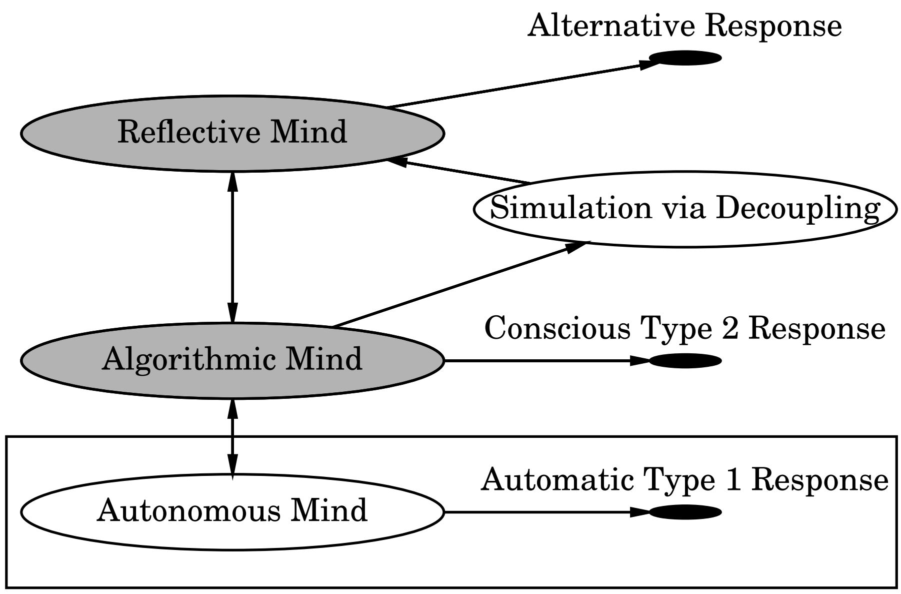

Figure 1. *Stanovich's Tri-Process Model of the Mind*

At the heart of Stanovich's model is the concept of "cognitive decoupling" (2011, p. 48), which describes a person's ability to simulate secondary representations that are abstractions of the person's primary representations of the surrounding world. Cognitive decoupling enables hypothetical thinking and is indispensable for departing from the automatic Type 1 responses generated by the autonomous mind. For a person to perform cognitive decoupling, an intricate interplay between the algorithmic and reflective minds has to take place: First, the reflective mind signals the need to engage in cognitive simulation to the algorithmic mind, which may inhibit and override Type 1 processing. In turn, the algorithmic mind is responsible for sustaining the effortful cognitive decoupling. Provided a successful decoupling operation, the agent will perform the Type 2 response generated by the algorithmic mind or the alternative response generated by the reflective mind. Figure 1’s bidirectional arrows connecting the autonomous, algorithmic, and reflective minds illustrate that the three minds can send as well as receive information.

The reflective mind is argued to overlap with the category of Type 2 processing, while being functionally different from what is typically associated with Type 2 processing: whereas the algorithmic mind has been empirically linked to fluid intelligence and executive functioning (Stanovich, 2011; Tsuji & Watanabe, 2009), conventional dual-process theory failed to account for individual differences in critical thinking skills prior to Stanovich’s introduction of the reflective mind. According to Stanovich and West (1997), the reflective mind encompasses a person's rational thinking dispositions, which is a latent construct that they inferred from a total of 11 variables with notable examples such as participants' openness to ideas, their counterfactual thinking or their degree of dogmatism. Critical thinking skills or thinking dispositions were underscored to be both separable of cognitive ability and predicting considerable variation in participants' argument evaluation performance (Stanovich & West, 1997).

One significant example of how the algorithmic mind can harm learning in the absence of reflective thought is one-sided thinking. One-sided thinking describes situations in which people fail to consider alternative approaches or information that might be crucial for revising the response generated by their Type 2 processing or algorithmic mind to reference Stanovich's terminology (2009). Thus, Stanovich promotes a concept of rationality that is more encompassing than the conventional account of Type 2 processing: people's thinking dispositions constitute a second integral component and driver of Type 2 processing in addition to cognitive ability. Stanovich depicts the mutual dependence between the algorithmic and reflective mind by visualising the algorithmic mind as the mechanical foundation of rationality, which is a necessary, but insufficient condition for rationality: rational thought and action remain contingent on the agent's thinking dispositions, which are intrinsic to her reflective mind (Stanovich, 2009).

The substantial effect of thinking dispositions on task performance is supported by Toplak's and Stanovich's (2002) results from regression analyses of data on six problem-solving and decision-making tasks: the two thinking dispositions variables predicted twice of the unique variance (11.8%) of participants' task performance relative to the cognitive ability measure (5.2%). Meanwhile, the value of thinking dispositions as a variable of special interest to educational researchers hinges on learners' thinking dispositions being malleable, which is the case according to Baron's investigation (2005).

3. Interaction between Dual-Process Theory and Mathematics Education Research

Having outlined the cornerstones of recent advances in dual-process theory, I will turn to exposing notable areas of overlap and tension between views held by dual-process theorists and mathematics education researchers. As such, the present section will indicate avenues through which dual-process theory may productively inform mathematics education.

3.1 Two Conflicting Perspectives on Erroneous Problem Solving

According to the first perspective associated with mathematics teachers and education researchers, errors in students' problem solving are bugs that can be explained by one or a combination of the following learning circumstances:

1. Students lack the requisite conceptual or procedural knowledge for formal reasoning and problem solving (Leron & Hazzan, 2009),
2. Students lack the requisite cognitive ability for formal reasoning (Kryjevskaia et al., 2014).

Whereas the first explanation allows for the possibility that students will eventually acquire the disciplinary knowledge that is necessary for formal reasoning, the second explanation entails that some students may be generally incapable of processing formal reasoning. A seminal example of the first explanation is the Van Hiele model of geometric thought (Crowley, 1987), which comprises five hierarchical levels of understanding. This model implies that through progressing to more advanced levels of geometric understanding, learners' initially superficial and informal reasoning will tend to become more formal.

For example, a novice to geometry tends to describe geometric shapes such as triangles or rectangles in terms of their overall shape rather than their formal characteristics like the number of sides or angles. At the basic levels of geometric understanding described by Van Hiele's visualisation or analysis level (Crowley, 1987), learners are yet unable to apply formal reasoning. Once learners achieve more sophisticated levels of understanding such as deduction or rigour, learners are expected to consistently apply formal rather than intuitive reasoning. Assuming that students possess both the disciplinary knowledge and the requisite cognitive hardware, which are collectively considered sufficient conditions for formal reasoning, this perspective cannot account for students' inconsistent use of formal reasoning.

Dual-process theory, which is largely ignored by mathematics educators and education researchers (Kryjevskaia et al., 2014), meaningfully fills the above explanatory gap. This account suggests that in addition to the two learning circumstances already discussed in this section, two additional conditions need to be fulfilled for formal reasoning to manifest in students' problem solving (Stanovich, 2011):

1. Students need to consciously engage the specific Type 2 processes that allow them to process their acquired formal understanding,

2. The activated Type 2 processes need to override students' default Type 1 processes, which offer an informal, often intuitively appealing, approach to problem solving.

The human tendency to act as a "*cognitive miser"* (Stanovich, 2011, p. 21) exerting as little cognitive effort as possible explains instances in which learners engage in cognitively effortful Type 2 processing in an insufficient manner or avoid it altogether. Crucially, a richer domain-specific understanding can only attenuate this educationally troubling tendency to miserly processing, which persists even among expert learners. Under both of the above scenarios, the lack of cognitive effort often means that students default to responses that can be traced back to automatic Type 1 processes. Given that both educator and student can only observe the erroneous problem solving that results from miserly Type 1 processing, students' disciplinary knowledge and ability to formal reasoning may remain hidden and therefore unknown. This phenomenon constitutes the unfortunate source of the first perspective, which is solely based on students' responses and thus neglects the cognitive processes that underlie those responses.

3.2. Procedural and Conceptual Knowledge in Relation to the Two Processing Types

The mathematics educational distinction between conceptual and procedural knowledge exhibits a striking and potentially informative overlap with this paper's distinction between Type 1 and Type 2 cognitive processes. Conceptual knowledge is described as a "connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert, 1986, pp. 3-4). Such knowledge is independent of specific problem types and can be retrieved and known outside a particular learning context. By contrast, procedural knowledge or skill refers to "knowledge of procedures, including action sequences and algorithms used in problem solving" (Star & Stylianides, 2013, p. 6) and usually can only be retrieved by the learner when solving the specific set of problems for which this knowledge was acquired. The relationship between conceptual and procedural is suggested to be reciprocal and iterative (Rittle-Johnson & Alibali, 1999): It is reciprocal in the sense that gains in one type of knowledge usually engender gains in the other knowledge type. The relationship is iterative as the two types of knowledge are said to develop in a tandem. Strikingly, my dual-process analysis of the empirical case presented in Section 4.2 proposes an analogous reciprocal relationship between the two cognitive processing types: through learning to sustain the Type 2 cognitive processes that are necessary for processing formal statistical reasoning, the quality of students' default Type 1 responses (see Figure 1) is suggested to improve over the course of their undergraduate education. When students had little or no statistics training, effortful Type 2 processes were necessary for processing the algorithm that would lead to the correct solution. If, at that time, students only relied on their Type 1 processing they would be most likely unable to arrive at the correct solution. After receiving prolonged statistical training, which formed a compulsory part of students' undergraduate education, the same students may have been able to process the formally correct algorithm through the use of automatic and effortless Type 1 processing.

One crucial implication of the above example is that the acquisition of statistics knowledge may change the very content of students' cognitive processing. Substantial changes in students' domain-specific understanding are likely to trigger structural changes in students' cognition. The fundamental nature of these cognitive changes suggests that the operation of the two types of cognitive processing will change in a way that leads to new problem-solving responses. It seems impossible and skewed to inextricably tie procedural and conceptual knowledge to a single type of cognitive processing. Instead, the use of both procedural and conceptual knowledge seems to require different types of processing depending on students' level of domain-specific understanding. This important, but complex relationship between knowledge and processing types illustrates the need for a closer investigation of learning progressions through collaboration between cognitive scientists and mathematics education researchers. Through this scientific synergy one may tap into the cognitive mechanisms that can explain and predict learners' gains in procedural and conceptual knowledge in mathematical domains other than statistics. Such insights would increase the effectiveness of classroom interventions that are designed to facilitate students' acquisition of these two types of knowledge. There seem to be, at least, two conditions that such a research project will need to fulfil:

1. It needs to rely on repeated continuous rather than categorical measures of participants' growth in conceptual and procedural knowledge (Rittle-Johnson & Alibali, 1999),

2. It needs to account for any qualitative changes in participants' cognitive processing, which are expected to manifest as participants progress in their learning. These changes may be measured through administering a standard or modified version of the Cognitive Reflection Test (Frederick, 2005; Toplak, West, & Stanovich, 2011; Toplak, West, & Stanovich, 2014).

The above examples are only a small subset of significant areas of overlap between dual-process theory and mathematics education research. To gain greater understanding of these areas of overlap as well as their educational implications, researchers in the two fields need to pursue collaborative rather than independent projects. This collaboration may culminate in the rise of journals, conferences, and research centres that span the two fields. Finally, there is an urgent need for researchers who combine expertise in dual-process theory and mathematics education.

4. Dual-Process Analyses of Mathematical Reasoning

Table T3 sketches only a few examples of previous dual-process analyses of mathematical reasoning, which served as a springboard for this section’s two empirical studies. The depicted diversity of mathematical domains for which dual-process theory was suggested to be a meaningful analytical framework strengthens my hypothesis of its applicability to issues pertaining to mathematics education.

Table T3. *Thumbnail Summary of Existing Applications of Dual-Process Theory to Mathematics Education*

Publications	**Mathematical Area**	**Observed Cognitive Bias**
Obersteiner et al., 2013	Number Theory	Natural Number Bias (NNB)
Clement et al., 1981; Leron & Hazzan, 2006	Algebra	• Word order matching • Static comparison

Babai et al., 2015; Stavy & Tirosh, 1996	Geometry	Larger shape area is associated with a larger perimeter
Dooren et al., 2003; Leron & Hazzan, 2009; Kahneman, 2013	Probability and statistics	• Proportional/linear reasoning • Base rate neglect/fallacy • Conjunction fallacy

4.1 Empirical Study I: The Illusion of Proportionality in Probabilistic Reasoning

4.1.1 Study Background and Reasons for Inclusion

The experimental study in Dooren et al. (2003) has been selected from a sparse body of mathematics education studies that tap into the cognitive roots of students’ fallacious reasoning. The study provides evidence that participants solve probabilistic tasks by routinely applying a proportional reasoning strategy: participants' approach to problem solving aligns with the proportions outlined in the task instructions. Since this proportional strategy instantly becomes participants' default problem-solving strategy, this implies that Type 1 cognitive processes prevailed whenever participants applied the proportional strategy. The experiment's underlying connection between observed patterns in participants' problem solving and the cognitive architecture inherent to dual-process theory justifies its appearance in this review. Another reason is that probability theory is widely regarded as a domain of mathematics for which there is conclusive empirical evidence on the misleading nature of people's intuitions and preconceptions (e.g., Dooren et al., 2003; Kahneman, 2013), which I will link with the predominant role that Type 1 processes play in participants’ cognitive processing. Meanwhile, the illusion of proportionality is a notable example of fallacious reasoning in the study of both probability and geometry (Dooren et al., 2003). Another striking feature of this study is that students' proportional reasoning led to varying test outcomes across distinct types of test questions. Whereas being guided by a proportional strategy would typically lead to finding the correct solution in the case of a qualitative item, this strategy was the prime reason for mistakes on quantitative questions (Dooren et al., 2003).

An educationally troubling observation is that participants' proportional reasoning was the direct result of the prior mathematics education they received at school. Since their mathematics instruction emphasised and valued proportional reasoning, students began to routinely rely on this type of reasoning without exerting cognitive effort to assess whether it is most or even minimally applicable in a specific learning circumstance. The following dual-process analysis will reveal how prior knowledge of proportionality may entice learners to employ Type 1 instead of Type 2 processing. The experimental data that the following analysis relies on is taken from Dooren et al. (2003).

4.1.2 Dual-Process Analysis

In case of the qualitative test items, which received 88.9% correct answers, my theory proposes that participants' Type 1 processing enabled students to arrive at the correct solution. There are, at least, two reasons for supposing that participants were employing Type 1 rather than Type 2 processing in those instances:

Firstly, as emphasised by Dooren et al. (2003), the concept of proportionality is a centrepiece of participants' mathematics education prior to the experiment. Analogous to the driving case presented in Leron and Hazzan (2006), the use of proportional reasoning probably constituted a cognitively effortful Type 2 process before students began to extensively practise this approach in the classroom. Through repeated application and due to the intuitiveness of proportional reasoning (Dooren et al., 2003), its application became a default Type 1 process to students.

Secondly, by knowing that the experimental participants employed proportional thinking when solving the quantitative test problems, there is little reason to suppose that their cognitive processing changed within a single experiment: both the quantitative and qualitative test items were testing a single type of probabilistic reasoning.

The same proportional reasoning strategy that led to 88.9% correct answers on qualitative problems, misguided participants on quantitative problems with the latter receiving only 22.4% correct answers. What is revealing is that most participants perceived the incorrect application of linear thinking as self-evident. For instance, a 12th grader named Anneken remarked: “In the second case, you get double as much occasions to obtain double as much fives. Therefore, logically speaking, the statement should be correct” (p.131).

Similarly, a 10th grader called Karen explained his fallacious reasoning in the following manner: "The occasions are halved, but also the goals. It is still directly proportional" (p. 129). Crucially, the varying test and learning outcomes triggered by the participants' use of Type 1 processing across qualitative and quantitative test items, is in line with dual-process theory. There is nothing about Type 1 processes that guarantees them to lead to correct problem solving for a specific category of mathematical problems. Thus, it would be erroneous to infer from the above experiment that Type 1 processing generally yields correct answers on qualitative test items. Similarly, there is no ground for establishing that Type 1 processing always or even typically misguides students in the case of quantitative problems.

Rather, Type 1 processing's learning implications depend on the quality of those cognitive processes in a given learner. The results of the probabilistic experiment imply that most of the participants' Type 1 processing is at a level that allows correct problem-solving on the experiment's qualitative items, but rules out correct problem-solving on the experiment's quantitative items. In illustration, it is worth consulting a specific set of test items. For that purpose, I will consult experimental variable p, which exhibits the most drastic variation of correct answers between qualitative and quantitative items among all the study's variables. Whereas the qualitative items received, on average, 95.6% correct answers, only an average of 16% of participants gave the correct answer on the quantitative item.

The qualitative test item on variable p:

"I roll a fair die several times. The chance to have at least once an even number if I can roll two times is

(a) larger than,

(b) smaller than or,

(c) equal to

the chance to have at least once a six if I can roll two times" (p. 131).

The corresponding quantitative item:

"I roll a fair die several times. The chance to have at least once an even number if I can roll three times is three times as large as the chance to have at least once a five if I can roll three times.

(a) This is true,

(b) This is not true" (p. 131).

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By consulting participants' written explanations, Dooren et al. were able to verify whether students' problem solving was characterised by proportional thinking. From that data, Dooren et al. inferred that 95.6% of 10th graders and 86.9% of 12th graders applied proportional thinking when giving an incorrect answer on the above quantitative item. An explanation offered by Mathilde who is one of the 10th graders participating in the experiment will illustrate how using proportionality as a default strategy can lead to a mistake:

"The odd numbers on a die are 1,3,5 = 3/6

The chance to get one of the three numbers is three times as large as the chance to get a five = 1/6 (p. 131)".

Strikingly, although Mathilde was able to correctly determine the probability of getting an odd number on a single try, she ignored an essential piece of information mentioned in the instruction: the die is rolled three times rather than once. Dooren et al. promote this type of error being the result of pupils' linear thinking producing an automatic mental response. Thus, the immediate availability of this response predetermined pupils' problem solving. Crucially, the way Dooren et al. interpret participants' behaviour extensively overlaps with the definition of Type 1 processing featured at an earlier stage of this review. Supporting our hypothesis that Type 1 processes informed participants' reasoning is that Mathilde's act of calculating the probability of a simplified event is compatible with what Kahneman (2013) denotes as substitution.

Being presented with a cognitively challenging probabilistic problem, people tend to subconsciously substitute a difficult problem with an easier heuristic one. Thus, Mathilde gives a correct answer to a wrong problem, which was set up by her Type 1 processing. Through the creation of an easy-to-solve problem with an immediately accessible solution, Mathilde managed to maintain cognitive ease (Kahneman, 2013): a situation perceived as familiar and comfortable without the need for effortful engagement of Type 2 processing. Mathilde's erroneous problem solving is argued to be common among novices who tend to focus on surface-level features of a task (Berliner & Calfee, 1996). By contrast, more experienced learners tend to integrate task elements into abstract schemas that are elusive to novices (Berliner & Calfee, 1996).

It is timely to recall the disagreement among dual-process theorists whether a person's cognitive architecture is best described by a parallel-competitive (Sloman, 1996) or a default-interventionist model (Evans, 2007). Verifying the applicability of the parallel-competitive model, the written explanations collected from the 10th and 12th graders participating in Dooren's experiment (2003) show no indication for a conflict or a parallel interplay between distinct types of processing. Rather, pupils seem to be guided by a default response that was unconsciously generated in their minds (Dooren et al., 2003). Instead of reflecting on the validity and suitability of the automatic Type 1 responses generated by the autonomous mind (see Figure 1), an overwhelming majority of pupils routinely implemented the default responses when answering the quantitative test items. There is no sign that participants engage their algorithmic or reflective minds whose combined operation is necessary for generating alternative responses. The only compatible among the three possible processing outcomes proposed by Evans (2007) is the case in which Type 1 processing generates a response without any intervention by Type 2 processing. This interpretation gives weight to the default-interventionist model of cognitive architecture.

4.2 Empirical Study II: The Cognitive Effects of Undergraduate Education on Statistical Reasoning

4.2.1 Study Background and Reasons for Inclusion

Complementing Study I from the previous section, Study II (Lehman and Nisbett, 1990) has been drawn from the miniscule cognitive science literature that analyses longitudinal changes in students' statistical reasoning. The data found in Lehman and Nisbett (1990) provide an empirical foundation for illuminating the reverse channel of the relationship between reasoning and learning performance: the effect of knowledge acquisition on learners' use of Type 1 and Type 2 processing and its effectiveness. There are several pertinent reasons for including this specific study: First, the study's results suggest a particularly significant improvement in students' statistical-methodological reasoning, which is of central concern to both dual-process theorists and mathematics education researchers. Second, participants' statistical-methodological reasoning was measured both in a domain-specific and domain-general setup: the study's participants completed a test that combined scientific and everyday-life problems. This variety of mathematical problems intentionally designed by Lehman and Nisbett indicates that the observed changes in students' reasoning hold for domains that can be regarded as mutually independent. For that reason, I find this study's results more credible than alternative ones, which verify changes in people's reasoning in a single or otherwise more narrow context (e.g., Lem, 2015; Morsanyi, Busdraghi, & Primi, 2014).

An additional strength of this study is its sample data, which features a total of 121 students across a wide range of subjects including the natural sciences, humanities, social sciences, and psychology. The richness of the sample makes it possible to verify the presence of heterogeneous treatment effects. In other words, it is possible to compare changes in statistical reasoning not only within individuals, but also between students with varying degrees of statistics training. The experimental data that the following analysis relies on is taken from Lehman and Nisbett (1990).

4.2.2 Dual-Process Analysis

My conjecture is that there are two possible explanations for the substantial improvement in statistical reasoning that experimental participants achieved throughout their undergraduate studies: First, after having received statistics training in the course of their undergraduate education, the now final (fourth) year students were more likely to approach statistical problems through the use of Type 2 processing. Thus, compared to when the same students were in their very first semester, their reasoning was more likely to be dominated by Type 2 processes. In other words, as a consequence of their statistics training, students were increasingly able to critically reflect on the default responses generated by Type 1 processing with Type 2 processing interfering when necessary (Stanovich & West, 1997).

A second explanation is that through their statistical training students' Type 1 processing improved in a way that would be compatible with the observed improvements in statistical reasoning (Lehman & Nisbett, 1990). What previously would require effortful Type 2 processes was now performed by means of the immediate responses generated by automatic Type 1 processes. Thus, even without interventions of Type 2 processing, students' refined Type 1 processing may explain the highlighted improvements in statistical reasoning.

Meanwhile, these two dual-process explanations are by no means mutually exclusive. Through repeated practice learners' performance is said to become more automatic, whereby an increasing number of previously controlled cognitive processes becomes available for abstract processing (Gagné, 1985). Gagné maintains that this process of unitisation is marked by people's reorganisation of knowledge: the skills that are necessary to complete a given task or problem are organised into larger and more efficient units enabling more effective problem solving. Extrapolating from Gagné's theory of unitisation, one may argue that the students with enhanced statistical reasoning featuring in the sample collected by Lehman and Nisbett, were more likely to exhibit two cognitive trends:

1. Following their statistical training, students' Type 1 processing became more efficient compared to their Type 1 processing prior to taking statistics courses and thus engendered the observed improvements in reasoning,

2. Having already practiced with numerous statistical problems, the statistical reasoning test administered by Lehman and Nisbett created less cognitive strain among students compared to when the students were new to statistical problems (Kahneman, 2013). As a consequence of the reduced cognitive strain, students were more likely to employ abstract Type 2 processing, which may explain their improved test performance.

In order to further evaluate the relationship between knowledge acquisition and reasoning one should consider the significant correlations between the number of statistics courses taken by students and the associated improvement in their "statistical-methodological reasoning" (Lehman & Nisbett, 1990, p. 957). Such a correlation was observed across three out of the four subject areas: psychology, social sciences, and natural sciences with the respective correlation coefficients amounting to .23, .23, and .28. Due to the small number of statistics courses taken by humanities students, Lehman and Nisbett regarded the sample of humanities students as insufficient for calculating a correlation. The observed correlation coefficients give weight to the general hypothesis that, by working on statistics courses, students not only acquire statistical knowledge, but also enhance their overall statistical reasoning. An even more notable observation is that students' improvements in reasoning are contingent on the number of courses they have taken throughout their studies. The latter relationship can be explained by expanding on the two dual-process explanations that I elucidated earlier in this section.

The first explanation, which centres on the increased role of Type 2 processing, entails that students who took a relatively large number of statistics courses were more likely to employ strenuous Type 2 processing compared to students who received relatively little statistical training. Thus, the former group's problem solving was more immune to misleading Type 1 processing compared to the latter group. In terms of the three possible outcomes originating from Evans' default-interventionist model (2007), the former group was more likely than the latter to achieve either the second outcome - the default response R generated by Type 1 processing is chosen despite the intervention of Type 2 processing - or the third outcome - the alternative response A is chosen after an intervention of Type 2 processing (Evans, 2007). Thus, the students with extensive statistical training benefited from

Type 2 processing as an analytical monitor of their Type 1 processing, whereby they managed to deviate from an erroneous default response R.

The second explanation of students' refined statistical reasoning, which proposes a more efficient working of Type 1 processing, implies that the more statistical instruction students receive, the more adept their Type 1 processing becomes at solving any type of statistical problems or tasks. Different from Evans' model (2007), which includes only one possible response R generated by Type 1 processing, this interpretation allows for multiple responses being generated by Type 1 processes accounting for changes in learners' knowledge base. Accordingly, it may be that a participant's original Type 1 response R, which determined her problem-solving during the experiment's first reasoning test (Lehman & Nisbett, 1990) is qualitatively different from the Type 1 response R' appearing in the same participant's mind during the experiment's second reasoning test during her fourth year of undergraduate studies. Whereas response R may have elicited a fallacious solution, response R' may have helped the student to arrive at a correct solution.

A highlight of the experimental analysis carried out by Lehman and Nisbett is that they were able to test both domain-specific and domain-general reasoning ability. Given that domain-general reasoning is an especially pressing concern among educators and educational researchers, tapping into a potential cause of this type of reasoning is an educationally paramount project (e.g., Fuchs et al., 2012; Niaz, 1994). By choosing this form of research design, Lehman and Nisbett accounted for the possibility that the observed changes in domain-specific reasoning may result from students' refined statistical knowledge base rather than fundamental changes in their cognitive processing. Given that participants' statistical reasoning improved on both domain-specific and domain-general test items, it seems that as a result of their statistics education students began to employ more refined statistical reasoning even when dealing with novel and unrelated problems that weren't covered in their statistics classes. What exactly justifies this argumentative leap?

To assert the validity of this argument it is worth looking at an exemplary every-day life problem, which forms part of the statistical reasoning test featuring in Lehman and Nisbett. Participants were asked to explain why rookies of the year in major-league baseball tend not to do as well in their second league year. The experimenters did not mention regression to the mean, which is a statistical principle that facilitates correct problem solving on this test item. In order to solve this problem, participants had to apply the statistical principle they learned in class. Since it is exceedingly unlikely that students learned about regression to the mean by the means of this particular baseball example, the participants' improved test performance indicates that their understanding of regression of the mean transformed the way their cognitive processing would operate when presented with such a or a similar statistical problem. Thus, one would expect the students to successfully identify regression to the mean problems taken from other randomly selected domains with which students had no prior learning experience.

One important question that remained untouched by Lehman and Nisbett is how temporary or permanent the observed improvements in students' reasoning are. As suggested by the theoretical and empirical evidence on Type 1 processing evaluated in this review, Type 1 processes are not fixed algorithms, but evolve in response to individual learning progressions (Lehrer & Schauble, 2012). In light of this malleability associated with Type 1 processing an important educational and theoretical consideration is whether and how changes in cognitive processing that are favourable to people's learning, as for instance the statistical reasoning improvements documented by Lehman and Nisbett, can be sustained over time. Hence, when designing an experimental study similar to the one found in Lehman and Nisbett, it appears advantageous to verify participants' cognitive processing at a third time point at which participants have already completed their statistics education.

Conclusion and Avenues for Future Research

Researchers of mathematics education are increasingly aware of dual-process theory as a valuable analytical framework for evaluating and improving students' mathematical reasoning. While the application of dual-process theory to the study of mathematics education is only at its advent, this research avenue begins to gain ground with increasing speed and impact (see Kryjevskaia et al., 2014; Lem, 2015; Morsanyi et al., 2014).

In this review, I have synthesised recent conceptual trends in dual-process theory of reasoning and subsequently introduced two experimental studies of mathematical learning that merit an application of dual-process theory. My dual-process analyses indicate the immense extent to which the effectiveness of mathematics education depends on learners' use of Type 1 and Type 2 cognitive processes. Thus, this paper has made a case for bridging the two largely separate bodies of literature on dual processing and mathematics education. Such scientific synergy promises to benefit the two scientific communities and in turn to empower mathematics educators and students.

On the one hand, dual-process theorists could markedly refine their accounts by testing the exact interplay between Type 1 and 2 processing across different areas of mathematics education. This form of theory testing promises to advance dual-process theory in two significant regards: testing the accuracy of the parallel-competitive and default-interventionist cognitive mappings (Evans & Stanovich, 2013) and examining the validity of Stanovich's tri-process model, which proposes a complex interaction between the algorithmic and the reflective minds.

On the other hand, dual-process theory offers educational researchers a valuable interpretative lens through which they can tap into and effectively remedy cases in which students are misled by either type of cognitive processing or their combined failure. One promising way to nurture a collaboration between dual-process theorists and mathematics education researchers is to analyse the varying roles Type 1 and 2 processing play across different time points of a mathematical learning unit both within the same and across distinct learners. A potential methodological direction for accomplishing a fruitful collaboration between cognitive psychology and mathematics education research is to conduct longitudinal design-experiments across distinct classroom settings. This methodology combines the practical purpose of improving instruction and the theoretical purpose of refining dual-process theory.

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