Endogenous Market Turbulence

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Abstract

In this paper I study a nonlinear feedback trading model which can generate stable, unstable, turbulent or chaotic asset returns depending on market conditions. The dynamics are driven by the stochastic price impact of net order flow (inverse market liquidity). If price impact grows beyond exogenous threshold values, liquidity dries up and asset returns become turbulent. In the absence of fundamental factors, the occurence of turbulence and chaos is entirely endogenous. The results highlight the critical role of maintaining stable market-making conditions for averting "liquidity black holes".

Keywords: Feedback Trading, Stochastic Price Impact, Financial Stability, Chaos, Nonlinear Dynamics

JEL classification: G12, G14

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"[There are] *episodes in financial markets where liquidity in a financial instrument vanishes temporarily before re-emerging*". Persaud (2001), xvi.

1   Introduction

Asset liquidity can evaporate in turbulent market conditions, a phenomenon referred to as a "liquidity black hole" by Taleb (1997) and Persaud (2001, 2003). Sharp drops in liquidity associated with discontinuous rises in price impact create market stress and undermine investor confidence.[1] Negative sentiment can, in turn, spread across financial markets and what starts as an asset-specific crisis may end up having very costly systemic repercussions; see Borio (2003), Cifuentes, Ferrucci and Shin (2004), Cohen and Shin (2003b), Longstaff (2001), and Gennotte and Leland (1990).

Thus, research efforts to assess the likelihood of such catastrophic events are important for practitioners and policymakers alike. Morris and Shin (2004) make an important contribution in that regard. These authors show that net selling pressure can arise endogenously from the strategic interaction of long- and short-horizon traders. When the latter have exogenous and privately known trading limits shortening their effective decision horizon, their responses to a liquidity shock can get disproportionately large. Asset sales among the short-horizon traders become mutually self-enforcing and may produce a "liquidity black hole", corresponding to the run outcome in bank run models (Diamond and Dybvig (1984)). Typically, such models have multiple equilibria under complete information. By relaxing common knowledge of the short-horizon liquidity constraints, Morris and Shin obtain the unique trigger point where a liquidity black hole emerges and predict sharp (V-shaped) price reversals, consistent with the empirical findings of

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[1]Such "news-headline" episodes include the 1987 stock market crash, the collapse of the U.S. dollar against the yen on October 7, 1998, and distressed trading in certain fixed income markets during the LTCM crisis in the summer of 1998.

Campbell, Grossman and Wang (1993) and Pastor and Stambaugh (2003).

This paper complements the above global game-theoretic approach by studying the onset of market turbulence in the absence of information asymmetries and liquidity constraints. The theoretical basis is the feedback trading model of Tambakis (2006). In that framework, prices fluctuate deterministically with past returns generating asset demand according to two feedback rules. First, investors buy (sell) the risky asset if returns in the last trading period were positive (negative). Such "momentum" is a prominent feature of portfolio insurance and stop-loss risk management strategies where traders unwind their positions in response to price drops. *Positive feedback* tends to reinforce price movements in either direction.[2]

Second, investors display *risk feedback*: their asset demand falls (rises) following volatile (tranquil) trading periods. Risk feedback amounts to a buy low-sell high trading rule in a mean-variance world where risk and expected return are positively related. The motivation here is quite different. In a high-frequency study of the U.S. Treasury market following Hasbrouck's (1991) VAR methodology, Cohen and Shin (2003a) report that sales pressure grows in last period's realized volatility. The impact of positive and risk feedback trading on order flow is proportional to the level and square of last period's returns.

One-period-ahead expected returns are set to zero at high frequency. This is crucial because it transforms the market maker's pricing rule to a logistic first-order difference equation for actual returns. It also offers a check of the model's predictions by relating them to the efficient market hypothesis (EMH). Specifically, the logistic map has two fixed points; the first is always zero, corresponding to zero expected returns, while the second is generically nonzero, indicating a violation of EMH. Fixed point stability is

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[2]Positive feedback emerges as self-reinforcing equilibrium behavior in strategic models with boundedly rational traders and short horizons; see Abreu and Brunnermeier (2003) and Brunnermeier and Pedersen (2005).

a function of market liquidity, proxied by inverse price impact, and investor diversity, which is inversely related to the intensity of positive feedback. The two feedback rules counteract each other as follows: more positive feedback (risk feedback) tends to destabilize (stabilize) the zero fixed point. At the same time, it stabilizes (destabilizes) the nonzero fixed point. More liquidity and/or investor diversity improve financial stability; less liquidity has the opposite effect.

In this paper I extend Tambakis (2006) with time-varying liquidity and examine the implications for asset return dynamics. The price impact coefficient is assumed to follow a stationary AR(1) process defined by its persistence, proxied by the autoregressive coefficient, and noise, proxied by the variance of price impact shocks. In order to investigate the role of market liquidity in isolation, the intensity of feedback trading and, consequently, investor diversity are constant throughout.

As stochastic price impact and market liquidity fluctuate, the dynamics of the logistic map fall into four non-overlapping states: (i) *Tranquil markets*, when the zero fixed point is stable and the nonzero fixed point is unstable; (ii) *Market stress*, when the reverse is the case; (iii) *Market turbulence*, when both fixed points are unstable. (iv) If price impact grows beyond the turbulence range, the dynamics become *chaotic* and a liquidity black hole emerges. The price impact thresholds defining the four states are exogenous and independent of the shock distribution function.

A clear policy implication of the stochastic framework is that stable market-making conditions are necessary for maintaining financial stability. However, they are not sufficient—market stress or turbulence can always arise from an extreme price impact realization representing a statistical outlier. Positive and risk feedback trading can generate market turbulence despite the price impact coefficient being stationary and bounded. The occurence of market turbulence does not require liquidity to "dry up". Further, the four market states are reversible because price impact can cross a dynamic

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threshold from either direction. Thus, episodes of market instability and turbulence can be said to be *intermittent*, following Mandelbrot's (1974) original use of the term in fluid mechanics. In financial markets, the chaotic state corresponding to a liquidity black hole may lead to a temporary suspension of trading.

The paper draws on three research fronts. First, empirical evidence from many financial markets suggests liquidity is time-varying, especially during market stress; see Dufour and Engle (2000), Farmer et al. (2005), and Goldreich et al. (2005). Plantin et al. (2004) find that less liquid asset markets are more vulnerable to episodes of turbulence. For U.S. Treasury securities in particular, Furfine and Remolona (2005) report that the market regains composure quite rapidly after experiencing discontinuous price changes. In electronic currency trading, Danielsson and Payne (2001) find that intermittent "liquidity gaps" are a common feature of active markets. Second, the paper relates to the literature on feedback trading rules. Models where asset prices are driven by nonfundamentals can be traced to Shiller (1984); see also Cutler, Poterba and Summers (1990), DeLong et al. (1990), Sentana and Wadhwani (1992), and Shiller (2005). Third, the logistic map is a workhorse of nonlinear dynamical systems. Initially restricted to biological models of population growth—where positive feedback (destabilizing) corresponds to the reproduction tendency and risk feedback (stabilizing) to the inhibiting factor—the logistic's fame grew when its potential for complex dynamics was realized (May (1973)). The seminal reference on chaos in low-dimensional nonlinear systems is Li and Yorke (1975). The relevant applications to financial risk management are reviewed by Mandelbrot (1997).

In the remainder of the paper, Section 2 reviews Tambakis (2006) and extends the feedback trading model with time-varying price impact; Section 3 discusses the Monte Carlo methodology; Section 4 explores the stochastic transition from stability to instability and turbulence with two numerical experiments; and Section 5 concludes.

2 The model

In Tambakis (2006) a single risky asset is traded at high frequency. Denote the actual return at time t by $r_t = r(t, \Delta t) = \log P(t + \Delta t) - \log P(t)$, where $P(t)$ is the log price level. Empirical applications usually fix the time scale to $\Delta t = 5\,\text{min}$. Under a weak martingale property, the conditional expectation at $t-1$ of returns h-intervals-ahead is zero: $E_{t-1} r_{t+h} = 0$ for all $h \geq 0$. For $h = 0$, defining excess returns at t as $x_t \equiv r_t - E_{t-1} r_t$ implies $r_t = x_t$, so actual and excess returns (price increments) coincide.

Asset demand is driven by two feedback trading rules, applied under perfect and complete information. Under positive feedback, investors buy (sell) the asset at time t if they observe positive (negative) returns at $t-1$. With risk feedback, they buy (sell) the asset at t if they observe lower (higher) conditional volatility at $t-1$. Let $\gamma > 0$ be a constant scalar mapping returns at $t-1$ to asset units demanded at t. The order flow generated from feedback trading then is

$$\omega_t = \gamma x_{t-1} - \gamma x_{t-1}^2 \tag{1}$$

The sign of ω_t depends on the realized return and its volatility in the previous trading period, respectively proxied by the level and square of x_{t-1}. Tambakis (2006) models the case where the feedback intensity, γ, mapping $t-1$-returns to period-t asset demand is different for each feedback rule. In particular, the intensity of positive (risk) feedback is assumed to be decreasing (increasing) in the heterogeneity or *diversity* of investor opinion. By contrast, in this paper positive and risk feedback intensity are fixed at $\gamma = 1$, wlog, and the focus is on time-varying liquidity. Upon receiving ω_t, a risk-neutral market maker adjusts prices from $t-1$ to t according to the linear pricing rule

$$x_t = r_t = \lambda_t \, \omega_t \tag{2}$$

The sign of x_t depends on the aggregate volume imbalance: it is positive (negative) if there is net buying (selling) pressure. Price impact coefficient

$\lambda_t > 0$ measures the effect on price of a unit change in order flow. It proxies for market depth and is inversely related to asset liquidity.[3]

Let price impact follow the AR(1) process

$$\lambda_t = \overline{\lambda} + \theta\lambda_{t-1} + \eta_t \ , \quad \eta_t \sim N(0, \sigma) \tag{3}$$

where $\overline{\lambda} > 0$ is a positive constant. Price impact dynamics are driven by the magnitude of $\overline{\lambda}$, the sign and magnitude of θ, and the distribution of the innovation term η. Normality is not essential to the results. The properties of λ_t are straightforward: it is covariance-stationary if $\mid \theta \mid < 1$ and nonstationary otherwise; its unconditional mean and variance are $\overline{\lambda}$ and $\sigma^2/1 - \theta^2$; the conditional mean and variance are $\overline{\lambda} + \theta\lambda_{t-1}$ and σ^2; and persistence at lag k equals the relevant autocorrelation coefficient, $\rho(k) = \theta^k$. Combining equations (1), (2) and (3) yields

$$x_{t+1} \ = \ h_\lambda(x_t) \ = \ \lambda_t x_t (1 - x_t) \tag{4}$$

Stochastic difference equation (4) summarizes the returns' nonlinear dependence. Adjacent returns are deterministically linked by logistic (quadratic) map h_λ. Its fixed points, defined as $x = h_\lambda(x_t) = x_{t+1}$, are just

$$x^1 = 0 \quad , \quad x_t^2 = 1 - \frac{1}{\lambda_t} \neq 0 \tag{5}$$

x^1 is always zero so its time-subscript can be omitted. The zero fixed point corresponds to zero actual and excess returns; it is consistent with the (weak form) efficient market hypothesis that high-frequency expected returns should be (linearly and nonlinearly) unpredictable. By contrast, fixed point x_t^2 decreases with market liquidity. The magnitude of x_t^2 ranges from infinitely negative when price impact tends to zero, to one as $\lambda_t \to \infty$. It is nonzero unless $\lambda_t = 1$.[4]

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[3]Following Kyle (1985), depth is one of three dimensions of liquidity. The others are tightness, which is proportional to the bid-ask spread, and resiliency, that is the speed at which prices adjust back to equilibrium following a large trade.

[4]Note that the two fixed points coincide when $\lambda_t = 1$. However, if price impact is continuously distributed then $\lambda_t = 1$ has zero measure, so x_t^2 is generically nonzero.

Asset return dynamics turn on fixed point stability. A fixed point is *stable*, or *attracting* (unstable, or repelling) if and only if the absolute value of the slope of (4) evaluated at that fixed point is *smaller* (greater) than one. The slope of h_λ with respect to x is $h'_\lambda(x_t) = \lambda_t(1 - 2x_t)$, and its absolute value at x^1 and x^2_t is

$$| \ h'_\lambda(x^1) \ |=| \ \lambda_t \ | \tag{6}$$
$$| \ h'_\lambda(x^2_t) \ |=| \ 2 - \lambda_t \ |$$

Note that at most one fixed point is stable for any parameter value and at any point in time. Specifically, if $x^1 = 0$ is stable then $x^2_t \neq 0$ is unstable, and *vice versa*. In the first case, expected returns are zero so the initial assumption is confirmed. The relevant price impact range is $0 < \lambda_t < 1$, from equations (6). In the second case, occuring when $1 < \lambda_t < 3$, x^1 is unstable and x^2_t is stable. Note that in that price impact range $x^2_t > 0$ so expected returns are nonzero, indicating EMH is violated. Further, $\partial x^2_t / \partial \lambda_t = 1/\lambda^2_t > 0$, thus higher λ_t (less liquidity) implies a bigger violation.

When $\lambda_t > 3$, $h'_\lambda(\cdot)$ exceeds one at both fixed points. In that case, both fixed points are unstable and the dynamics become *turbulent*. Finally, if $\lambda_t > 4$ the logistic map can be shown to display sensitivity to initial conditions, there are periodic points (return trajectories) of all orders and the dynamics become *chaotic*. An analytic proof of the emergence of chaos lies beyond the scope of this paper; see Devaney (1989) and Holmgren (1997).

To summarize, smaller price impact (more market liquidity) improves the stability of $x^1 = 0$ while bigger price impact (less liquidity) tends to destabilize it, and conversely for $x^2_t \neq 0$. The next two Sections demonstrate the complex dynamics resulting from feedback trading applying Monte Carlo methods.

3 Monte Carlo design

There are two numerical experiments, in the time and price impact domains. First, the parameters of the AR(1) process for price impact $(\overline{\lambda}, \theta, \sigma)$ are set for four possible "market states" in a one-to-one correspondence with the range of λ_t. The states are:

(i) *Tranquil markets*, where 1-period-ahead expected returns are zero and weak-form EMH holds, corresponding to the price impact range $0 < \lambda_t \leq 1$.

(ii) *Market stress*, where 1-period-ahead expected returns are positive and EMH is violated $(1 < \lambda_t \leq 3)$.

(iii) *Market turbulence*, where feedback trading generates unstable return dynamics $(3 < \lambda_t < 4)$.

(iv) A *liquidity black hole*, when asset returns become chaotic $(\lambda_t \geq 4)$.

Given the extreme computational intensity involved in visualizing chaotic dynamics, the simulations focus on the transition between states (i)-(iii); the onset of chaos is discussed in Section 4.3. For the same reason, the simulations use zero-mean, constant-variance Gaussian price impact shocks. Simulations using the uniform and other nonnormal shock distributions did not affect the essence of the results; they are available from the author upon request.

The unconditional mean of price impact is fixed at $\overline{\lambda} = 0.5$ throughout, i.e. at the midpoint of the tranquil market range $(0, 1)$, and $\{\lambda_t\}$ is initialized at $\lambda_0 = \overline{\lambda}$. The time series of returns $\{x_t\}$ is initialized at $x_0 = 0.5$, i.e. at the unique critical pointof the logistic map.[5] The two starting values are fed to logistic equation (4) mapping net order flow to asset returns to yield x_1. In turn, the price impact coefficient for the next trading interval (λ_1) is computed from (3). The values of x_1 and λ_1 are then fed to the logistic map to determine x_2, etc. In this way, feedback trading generates $t = 1, ..., T$ observations for $\{x_t\}$, each using a different price impact realization $\{\lambda_t\}$.

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[5]This choice is justified by Fatou's Theorem, which states that if the trajectories of a quadratic polynomial have an attracting fixed or periodic point, then the critical point is in the stable set of one of the points in each trajectory; see Holmgren (1997).

Figures 1, 3, and 5 present $t = 1, ..., T = 500$ price impact and return realizations for each state. The choice of T conforms to high-frequency trading in the U.S. Treasury market. Daily trading in U.S. Treasuries opens at 7:00am and closes at 5:00pm. Assuming, wlog, that each minute in this 8-hour period contains one trade yields 480 trading intervals.[6] Thus, the simulated asset returns can be viewed as "snapshots" of a particular trading day and state of the market. To assess the characteristics of the simulated return probability density functions, the above 1-day process is run $n = 1, ..., N = 100$ times, yielding over 3 months of artificial high-frequency data. For each state, Tables 1, 2, and 3 report the simulated pdf's first four moments averaged over all N simulated 1-day paths. Following standard practice in nonlinear dynamic simulations, the first 200 feedback returns of each path $\{x_t\}_{t=1}^{200}$ are discarded to ensure convergence has occured. The simulated moments are then computed from $\{x_{t=201}^{T=500}\}_1^{N=100}$ return realizations.

The second numerical study is in the price impact domain. It employs bifurcation diagrams, a powerful nonlinear dynamics tool for assessing the evolution of fixed point stability. Starting with $\lambda_0 = 0.5$ as before, Figures 2, 4, and 6 are based on $i = 1, ..., 400$ price impact realizations $\{\lambda_i\}$ drawn from equation (3). For every λ_i, each asset return trajectory (orbit of iterations) is initialized at $x_{i0} = 0.5$. This value is justified by Fatou's Theorem (see footnote 5 above) stating that an attracting periodic trajectory always includes the critical point. Each single λ_i is fed to logistic map (4) and produces $j = 1, ..., 300$ return trajectories $(\lambda_i, h_\lambda^i(0.5)_{j=1}^{300})_{i=1}^{400}$ under combined positive and risk feedback trading.

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[6]Cohen and Shin (2003a) and Fleming (2001) provide extensive background on the microstructure of the U.S. Treasury market.

4 Feedback return dynamics

4.1 Tranquil markets: $0 < \lambda_t \leq 1$

When $0 < \lambda_t \leq 1$, return dynamics are tranquil. To illustrate this state of
the market, the persistence of price impact is set to $\theta = 0.2$, and shocks η_t
are iid normal with zero mean and 0.05 standard deviation. The simulated
$\{\lambda_t\}$ and $\{x_t\}$ data are shown in Figure 1.

FIGURE 1 HERE

Note that price impact is always less than one so the asset market is firmly
in the tranquil state. As described above, 100 return paths each containing
500 trading periods are simulated—over 3 months of artificial 1 min price
increments—for the same $\overline{\lambda}$, θ and σ values. The risk-adjusted mean (sample
mean/standard deviation), skewness and kurtosis of high-frequency returns
averaged over all $\{x_{t=201}^{T=500}\}_1^{N=100}$ return distributions are reported in Table 1.

Table 1: Tranquil markets

Simulated moments

μ/s	0.000
Skewness	8.82
Kurtosis	87

Risk-adjusted expected returns are zero, upto 8 decimal places. High-frequency
returns for an average day are very right-skewed and fat-tailed. The reason
for the asymmetric distribution is that $x_t^2 = 1 - \frac{1}{\lambda_t} \leq 0$ for all $\lambda_t \in (0,1]$.
As the (unstable) nonzero fixed point is always negative, actual returns lie
between that and zero hence skewness is positive. High excess kurtosis is
broadly consistent with the empirical evidence, though it exceeds the values
reported by Cont (2001) for the S&P 500 (15.95), U.S. dollar/Deutschmark
(74) and U.S. dollar/Swiss Frank (60) futures markets.

Figure 2 is a bifurcation diagram showing the behavior of asset return trajectories, $(\lambda_i, h_\lambda^i(0.5)_{j=1}^{300})_{i=1}^{400}$, arising in tranquil market conditions.

<div align="center">**FIGURE 2 HERE**</div>

Stochastic price impact λ_i ($i = 1, ...100$) ranges from 0.45 to around 0.75, thus the logistic map h_λ is firmly in the stable (unstable) range of $x^1 = 0$ ($x_t^2 \neq 0$). The key feature of Figure 2 is that, at each λ_i, the feedback return trajectories x_{ij} ($j = 1, ..., 300$) become more dense towards zero. Clearly, there also trajectories corresponding to positive asset returns. They are unstable, however, in the sense that continued feedback trading—i.e., continued iteration of $\{x_{ij}\}$ given i—tends to attract them back to zero. This property of tranquil markets confirms the model's assumption of zero expected returns. A second interesting feature is that, as price impact grows and market liquidity drops, the unstable trajectories slope upwards. This reflects the nonzero fixed point's growing attraction as the logistic dynamics edge towards the market stress threshold.

4.2   Market stress: $1 < \lambda_t \leq 3$

When $1 < \lambda_t \leq 3$ the relative stability of the two fixed points is reversed: $x^1 = 0$ ($x_t^2 \neq 0$) becomes unstable (stable). The combination of positive and risk feedback trading can then be said to create *market stress*. To generate asset returns under market stress, $\overline{\lambda} = 0.5$, the AR(1) coefficient is raised to $\theta = 0.5$, and the standard deviation of price impact shocks is doubled, $\eta_t \sim N(0, 0.10)$. More persistent and/or more volatile price impact is more likely to exceed the market stress threshold. Figure 3 below shows the artificial $\{\lambda_t\}$ and $\{x_t\}$ series.

<div align="center">**FIGURE 3 HERE**</div>

In the bottom panel, the threshold is indicated with a horizontal line at $\lambda = 1$. Price impact is less than one 45.2 percent of time (500 trading periods over

100 1-day paths), and above one for the remaining 54.8 percent. The nonzero fixed point is now positive and stable, attracting neighbouring points towards it and away from zero, which is unstable. Asset returns are still close to zero (notice the 10^{-8} scale on the vertical axis) but are much more volatile than when the market is tranquil.

The risk-adjusted mean, skewness and kurtosis of returns averaged over all $\{x_{t=201}^{T=500}\}_1^{N=100}$ return distributions are reported in Table 2.

Table 2: Market stress

Simulated moments	
μ/s	0.0011
Skewness	1.52
Kurtosis	5.21

Now risk-adjusted returns are positive, reflecting the fact that when $\lambda_t \in (1, 3]$ the nonzero fixed point attracts nearby points. However, returns are much less right-skewed and leptokurtic than in the tranquil market state.

To the right of $\lambda = 1$, the nonzero fixed point abruptly switches from being unstable (repelling) to being stable (attracting). The bifurcation diagram in Figure 4 displays feedback return trajectories, $(\lambda_i, h_\lambda^i(0.5)_{j=1}^{300})_{i=1}^{400}$, undergoing this discontinuous dynamic transition.

FIGURE 4 HERE

The price impact coefficient now ranges from 0.75 to 1.30. Here, also, at each λ_i $(i = 1, ...100)$ the return trajectories become more dense towards zero. However, unlike the tranquil market state, the logistic's dynamics shift if stochastic price impact crosses one. The dynamics enter the range of market stress and 1-period-ahead expected returns become positive. Recall that $x_t^2 > 0$ increases in λ_t (decreases in market liquidity); this is indicated in Figure 2 by the return trajectories sloping upwards beyond $\lambda = 1$.

4.3 Market turbulence: $3 < \lambda_t < 4$

If both fixed points become unstable, asset returns enter the *turbulence* region. From Section 2 this occurs when $\mid h'_\lambda(\cdot) \mid > 1$ at both fixed points, that is when price impact is in the range $\lambda_t > 3$. To illustrate, consider maintaining $\overline{\lambda} = 0.5$ and $\eta_t \sim N(0, 0.10)$, and increasing the AR(1) coefficient to $\theta = 0.85$. The simulated $\{\lambda_t\}$ and $\{x_t\}$ series are shown in Figure 5.

FIGURE 5 HERE

Now only 0.4 percent of price impact realizations fall in the "tranquil" $(0, 1]$ region; 6.8 percent are in the "stress" range $(1, 3]$; and 92.8 percent are in the turbulent range $3 < \lambda_t < 4$. There are no outcomes in the chaotic region $\lambda_t \geq 4$. Average (not risk-adjusted) 1-period-ahead expected returns is about 0.65%, and there is strong volatility clustering even at this very high frequency.

To prevent feedback returns from diverging, the number of daily paths is now reduced from $N = 100$ to 20, and the first 300 returns of each day are discarded. The average risk-adjusted mean, skewness and kurtosis of returns over all $\{x_{t=301}^{T=500}\}_1^{N=20}$ return distributions are reported in Table 3.

Table 3: Market turbulence

Simulated moments	
μ/s	0.0064
Skewness	-0.258
Kurtosis	1.837

High-frequency risk-adusted returns are now very high, the artificial return distributions are left-skewed and there is negative excess kurtosis. A bifurcation diagram of feedback return trajectories, $(\lambda_i, h_\lambda^i(0.5)_{j=1}^{300})_{i=1}^{400}$, arising in market turbulence is presented in Figure 6.

FIGURE 6 HERE

The dynamics clearly begin to change very rapidly beyond the turbulence threshold ($\lambda = 3$). In the range $3 < \lambda_t \leq 3.45$ there are period-doubling bifurcations. From $\lambda \simeq 3.3$, the return trajectories display a "cascade" of period-doubling bifurcations. Every period-2 attracting orbit splits into an attracting period-4 trajectory (orbit) and a repelling period-2 trajectory. Note that when price impact is slightly less than $\lambda = 3.5$, another period-doubling bifurcation occurs: period-4 trajectories split into period-8 and period-16 trajectories, and so on. For $\lambda > 3.6$ it can be shown that there are periodic points of all orders if a period-3 periodic point exists (Li and Yorke (1975)). From $\lambda_t = 4$ and beyond, the dynamics of h_λ become chaotic. The orbits of x_t are period-k for all $k > 0$ and display sensitive dependence to initial conditions. A bifurcation diagram is not computationally feasible when $\lambda \geq 4$; Figure 6 offers a geometric case for the onset of chaos.

The Monte Carlo evidence clearly suggests there is a positive relationship between the values of θ and σ and the likelihood of a stochastic transition from tranquil markets to turbulence and chaos. In the chaotic state ($\lambda \geq 4$), actual trading may temporarily be suspended to prevent a market breakdown. Finally, note that the unconditional mean of price impact was fixed at $\overline{\lambda} = 0.5$ throughout. This confirms Persaud's (2001) observation that, as the average level of market liquidity tends to be quite stable—and improving in recent years—it may be that the high uncertainty (volatility) of liquidity is increasingly responsible for episodes of market turbulence.

5 Concluding remarks

This paper investigated a nonlinear feedback trading model generating stable, unstable, turbulent or chaotic return dynamics depending on market conditions. The dynamics were driven by the price impact of net order flow (inverse market liquidity) on actual returns, suggesting a natural correspondence between fixed point stability and the state of the market. It was found

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that the link between feedback trading and asset returns can change qualitatively despite the absence of fundamentals driving the asset price, and the change is affected by stochastic price impact. If the latter grows beyond exogenous thresholds, asset returns may become turbulent and even chaotic.

The results highlight the critical role of market-making conditions for safeguarding financial stability. Market liquidity should be sufficiently high and stable at each point in time, in the sense of weak persistence and small variance of the price impact process. In particular, financial policy makers and regulators should monitor liquidity fluctuations to keep the price impact coefficient away from the chaos threshold. In the chaotic range ($\lambda \geq 4$), financial distress on the part of market participants may cause a "liquidity black hole" and prompt a temporary suspension of trading.

The results also suggest that the transition between any two dynamic states can go in both directions. Therefore, episodes of market turbulence are shown to be *intermittent* in the sense of Mandelbrot (1974). To quote Shin (2004, p.150), '*occasionally*, financial markets experience episodes of turbulence of such an extreme kind that they appear to stop functioning' [italics added]. In that regard, an interpretation of the high-frequency evidence offered by Morris and Shin (2004) is that active markets can experience "mini" liquidity gaps several times in a day.

Possible applications of the theoretical framework include cumulating the high-frequency returns implied by the model and comparing them to realized volatility estimates (Andersen, Bollerslev and Diebold (2005)). Answers to the question of what intra-day volatility is "typical" could then be traced to the underlying state of market liquidity, which in itself constitutes a key fundamental. The artificial return time series could also be formally tested for chaos by estimating the sign of its Lyapunov exponents (Eckmann et al. (1986)) or more recent binary tests (Gottwald and Melbourne (2005)). More generally, market turbulence and chaos are endogenous so their relative likelihood can be assessed in terms of the price impact distribution.

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FIGURE 1
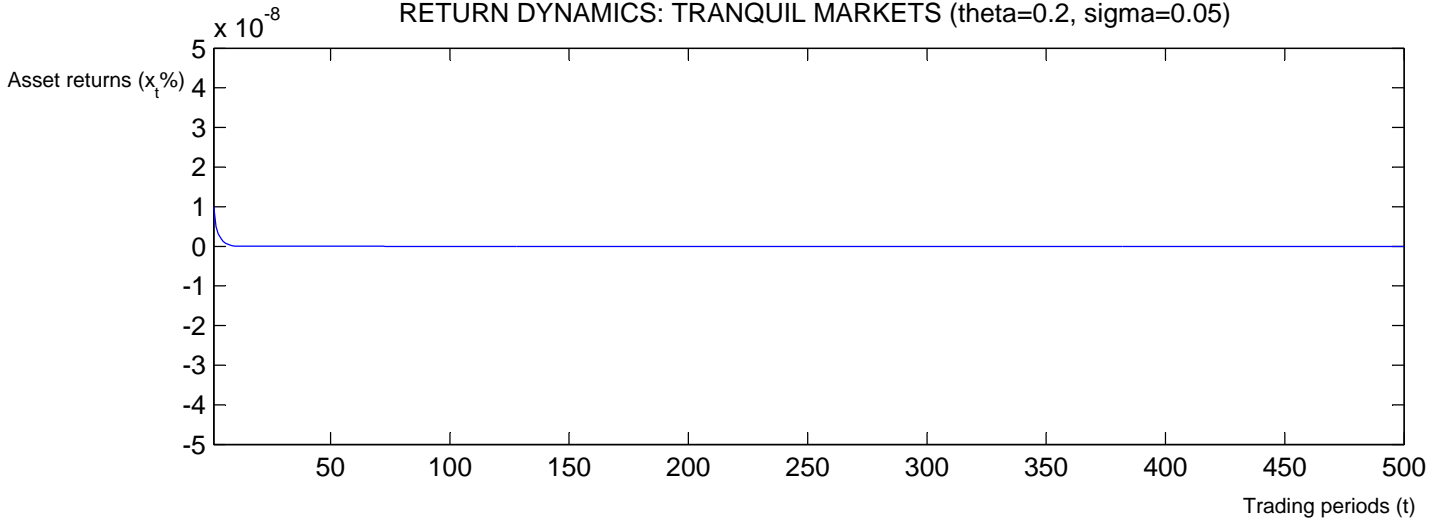RETURN DYNAMICS: TRANQUIL MARKETS (theta=0.2, sigma=0.05)
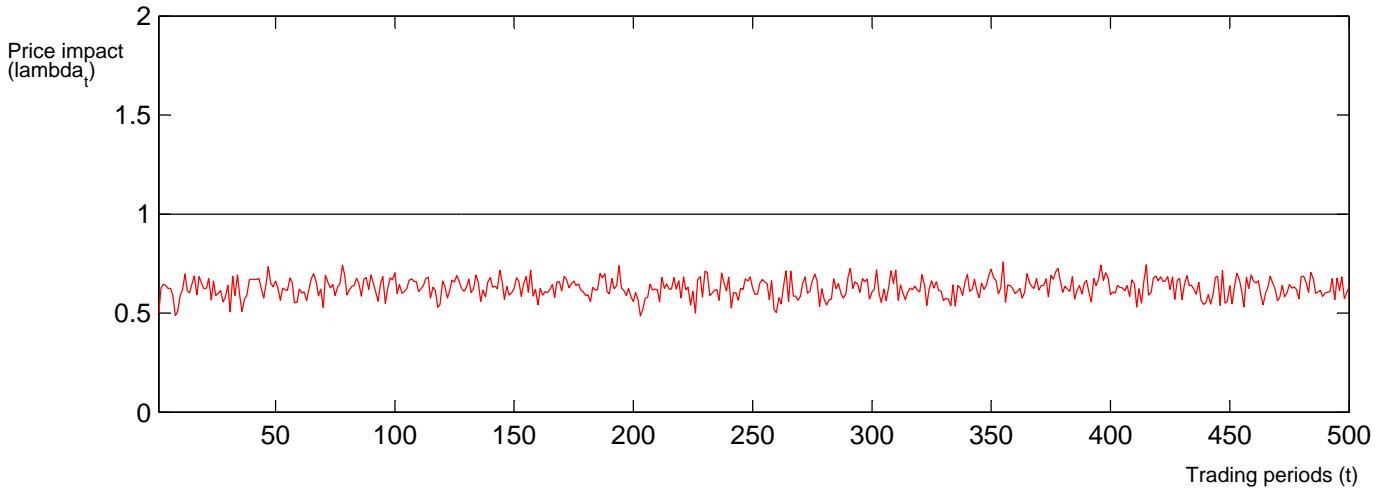
FIGURE 2

BIFURCATION DIAGRAM: TRANQUIL MARKETS (theta=0.2, sigma=0.05)

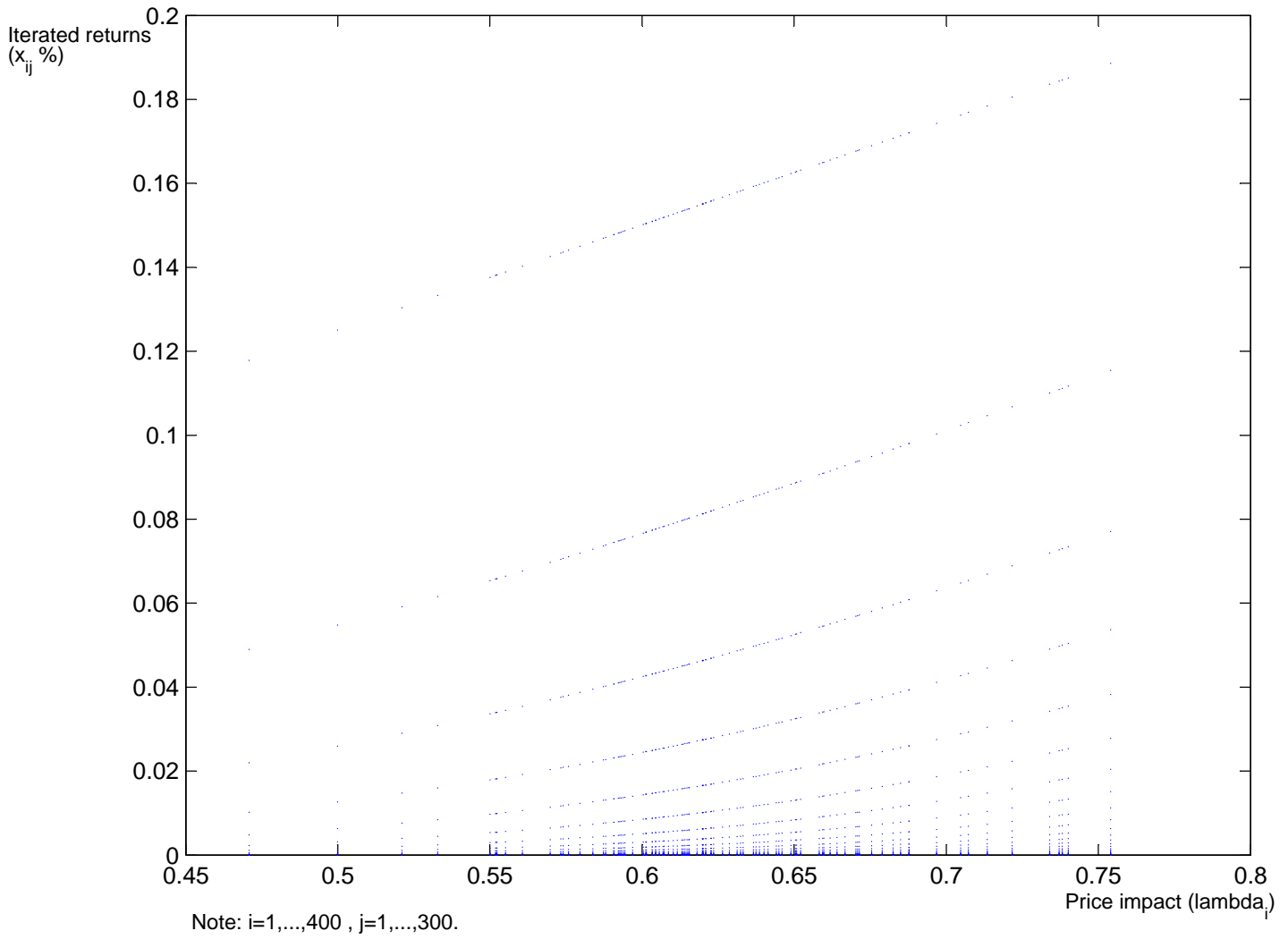Note: i=1,...,400 , j=1,...,300.

FIGURE 3
RETURN DYNAMICS: MARKET STRESS (theta=0.5, sigma=0.10)
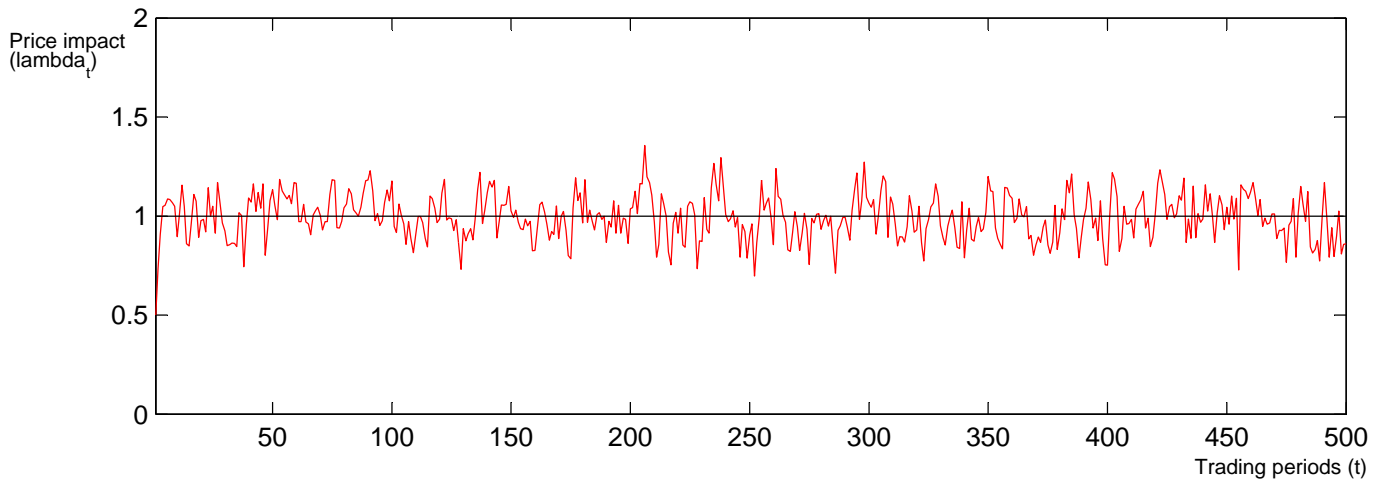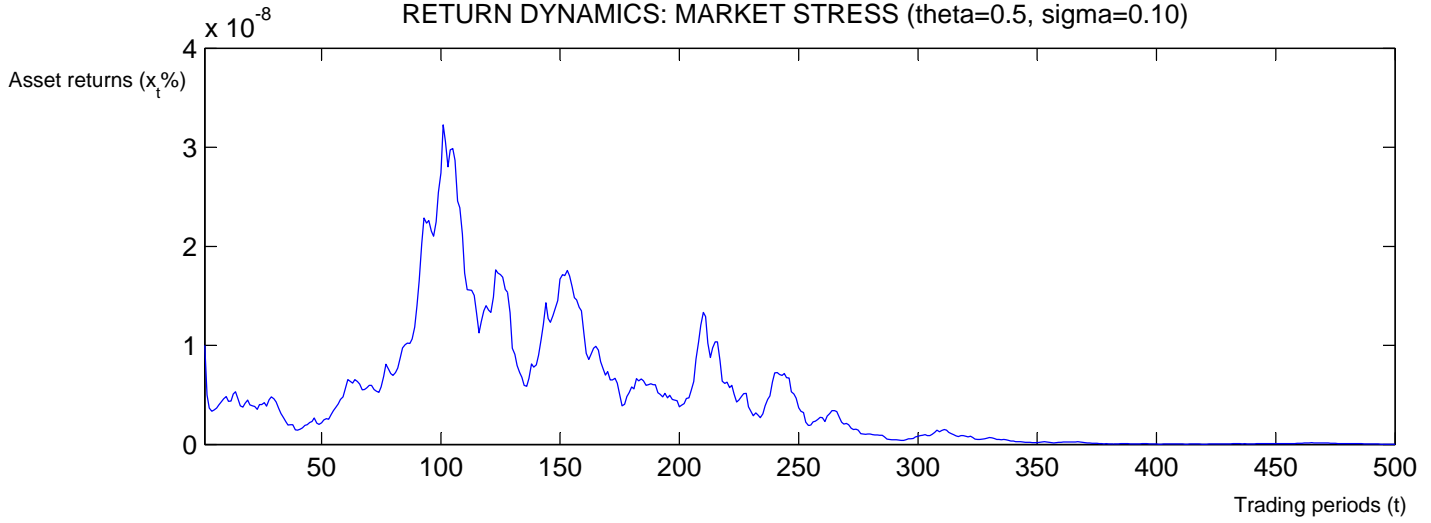
FIGURE 4

BIFURCATION DIAGRAM: MARKET STRESS (theta=0.5, sigma=0.10)

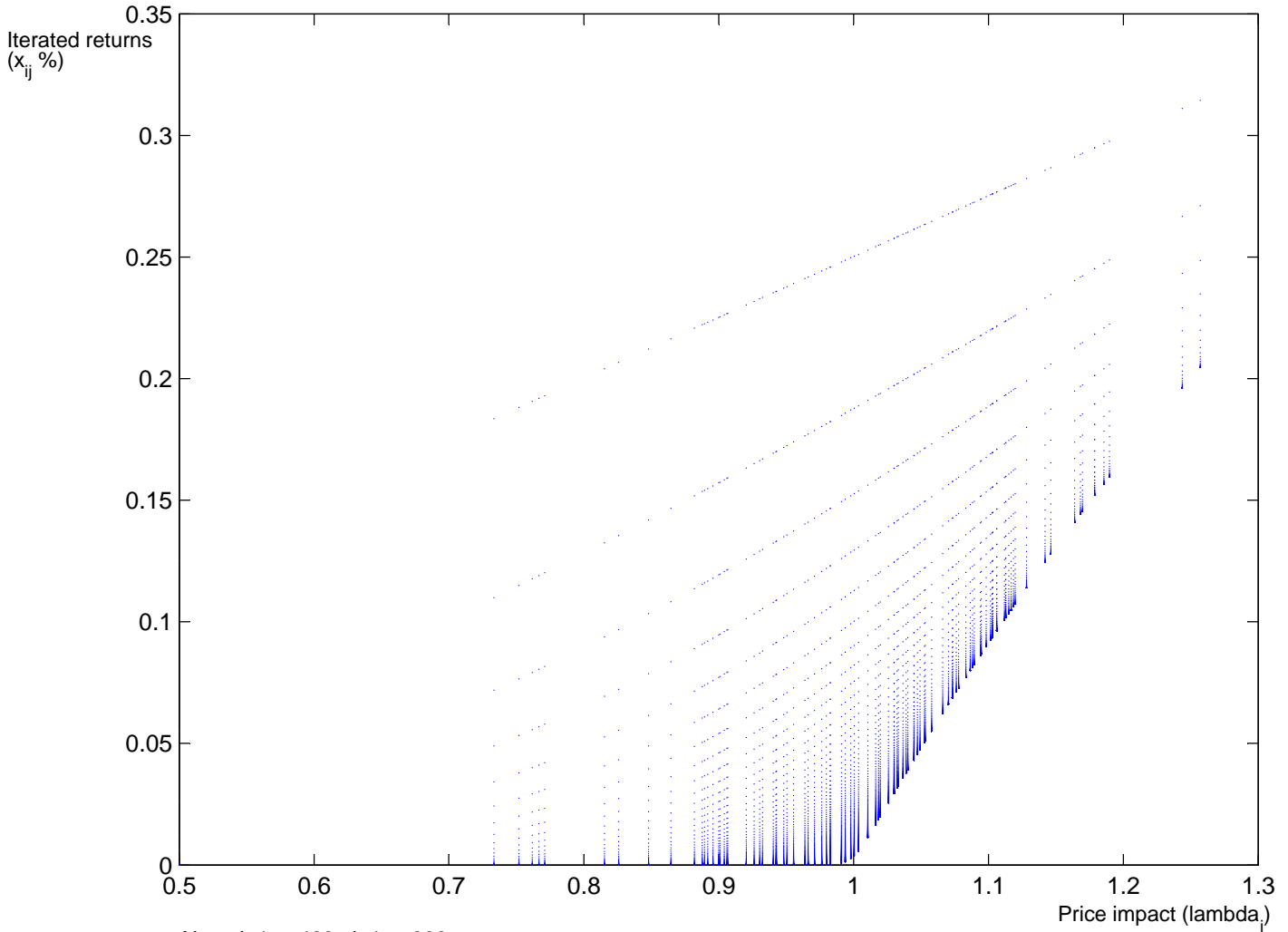Note: i=1,...,400 , j=1,...,300.

FIGURE 5
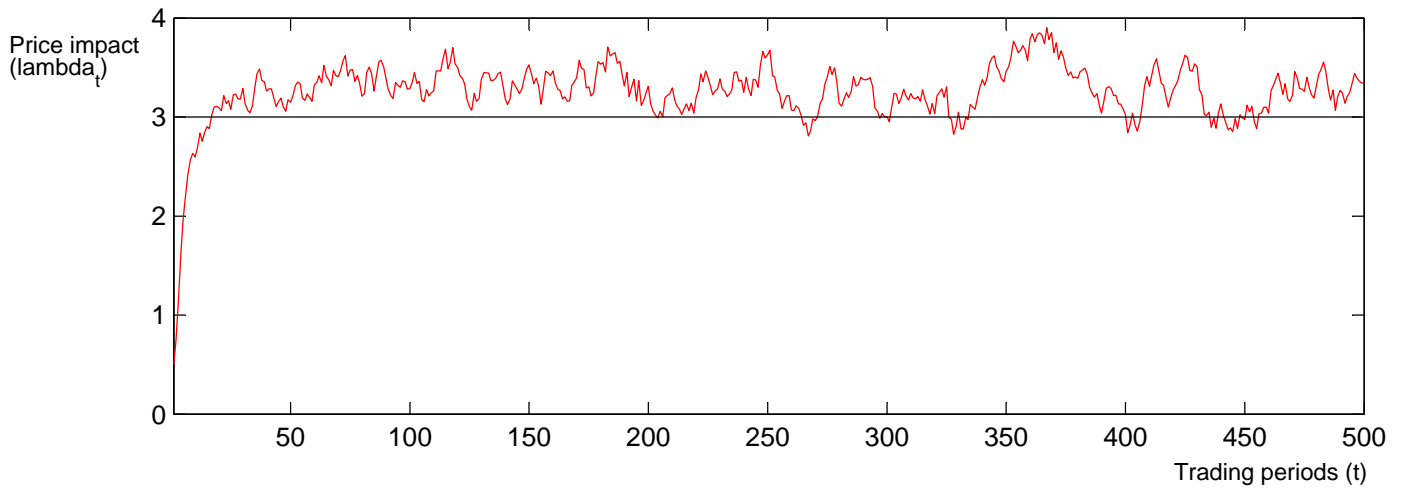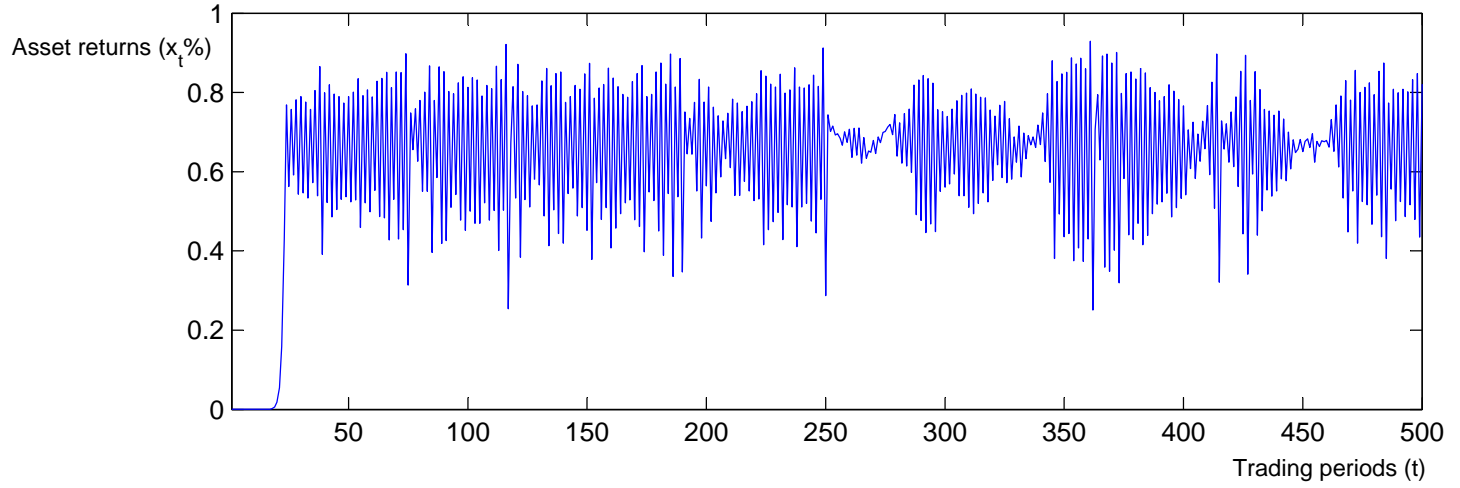RETURN DYNAMICS: MARKET TURBULENCE (theta=0.85, sigma=0.10)

FIGURE 6

BIFURCATION DIAGRAM: MARKET TURBULENCE (theta=0.85, sigma=0.10)

Note: i=1,...,400 , j=1,...,300.