On the Variational Theory of Yang-Mills-Higgs Energies and the Structure of the Singular Set of $\mathbb{Z}_2$-Harmonic Spinors

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. In particular, the results from Chapter 2 are obtained in collaboration with Alessandro Pigati and Daniel Stern, and appear in [106]. Chapter 3 is based on unpublished work.

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Abstract

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This dissertation investigates Yang-Mills-Higgs energies and $\mathbb{Z}_2$-harmonic spinors, two classes of objects at the interface between analysis and geometry.

In the first part, we present joint work with Alessandro Pigati and Daniel Stern on the variational theory of the self-dual $U(1)$-Yang-Mills-Higgs functionals on a closed Riemannian manifold (M,g). This natural family of energies associated with sections $u\colon M \to L$ and metric connections ∇ of Hermitian line bundles has long been studied in differential geometry and theoretical physics. We show how its variational theory is related to the one of the $(n-2)$-area functional by establishing a Γ-convergence result in the spirit of Modica and Mortola. With this in hand, we study the comparison between the corresponding min-max theories. Therefore, we relate the classical theory for C^1-functionals to the min-max theory introduced by Almgren and Pitts in the setting of geometric measure theory. In particular, we prove that min-max values for the latter always provide a lower bound for the former. En route to proving this comparison, we introduce the gradient flow of the Yang-Mills-Higgs energies and establish a Huisken-type monotonicity result along the flow. We complement this by studying the long-time existence, uniqueness and continuous dependence on the initial data of the flow.

In the second part of the dissertation, we focus on the notion of $\mathbb{Z}_2$-harmonic spinors. These objects were introduced in foundational work of Taubes when studying the compactification of moduli spaces of flat $\mathrm{PSL}(2,\mathbb{C})$-connections over 3-manifolds. Their role is to abstract various limiting phenomena and they also appear in other contexts, for instance when dealing with the moduli space of solutions to the Kapustin-Witten equations, the Vafa-Witten equations, and the Seiberg-Witten equations with multiple spinors. In all of these cases, the role played by the zero loci of $\mathbb{Z}_2$-harmonic spinors is crucial. Based on the pioneering techniques of Simon in the setting of minimal submanifolds, we obtain structural results on the singular set of $\mathbb{Z}_2$-harmonic spinors, subject to the validity of frequency monotonicity (a condition implied by, for instance, an appropriate regularity assumption). More precisely, we prove uniqueness of the blow-ups for every point, excluding an exceptional set of zero 2-dimensional Hausdorff measure, hence answering a question left open in the work of Taubes. From this, we infer 2-rectifiability of the singular set and the branch set. We conclude by analysing the setting of lowest frequency value, in which case we have that, locally, the branch set is a $C^{1,\alpha}$-submanifold.

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Chapter 1

Introduction

This dissertation investigates properties of stationary points of geometric variational problems and of solutions to partial differential equations of geometric type. These are analytical problems exhibiting the common feature of giving rise to objects of special geometric interest. The prototypical, and arguably most studied, class of objects to have in mind is the one of minimal surfaces, which are critical points of the functional that takes a surface as input and outputs its area. The history of this part of Mathematics dates back at least to the 18th century with the works of Lagrange, one of the founders of the modern calculus of variations. The rigourous development of the field occupied mathematicians for a long time and was responsible for the birth of geometric measure theory.

More precisely, in this dissertation we are going to study the variational theory of the Yang-Mills-Higgs energies and their relation with the $(n-2)$-area functional, as well as the regularity and structure of the singular set of $\mathbb{Z}_2$-harmonic spinors. These two problems fall broadly into two synergistic lines of research. The main goal of the first one is to understand qualitatively stationary points of geometric variational problems, and solutions to geometric PDEs. The usual starting question for this class is the one of existence and under what conditions and in which class of objects we have it. Once the existence issue is settled, a wealth of other questions can be raised. Do we have uniqueness? What can be said about the geometry and topology of the ambient manifold? What is the behaviour of the associated parabolic flows? The problems fitting in this category are usually of global nature. On the other hand, the second axis is interested in understanding the local properties of the objects under consideration, especially at those points in which the solution of the PDE or the stationary point is not regular. Questions of interest are: under what conditions can we exclude the presence of singularities, and, if we cannot exclude them, what can be said about them? This parallels the classical treatment of weak solutions of partial differential equations. Indeed, one is first interested in establishing their existence and basic properties, and then develop a regularity theory understanding under what conditions a weak solution is in fact smooth strong.

A theme permeating this thesis is the discrepancy between *scalar* problems and *vectorial* ones. For instance, in the setting of minimal submanifolds, this is embodied by the difference between minimal hypersurfaces, for which we have a satisfactory and developed theory, and minimal surfaces of higher codimensions, of which little is known. This is precisely the setting of our analysis of the variational theory of the Yang-Mills-Higgs energies. Indeed, this analysis bears more technical difficulties and is more challenging compared with the codimension one counterpart. In the context of $\mathbb{Z}_2$-harmonic spinors, and more generally when studying the regularity theories of various ojects, the vectorial nature of the problem complicates considerably the analysis.

Finally, a unifying, albeit unexpected, thread of the whole thesis is the interplay between *geometric measure theory* and *gauge theory*. In this perspective, the two chapters have opposite and complementary approaches. When studying the Yang-Mills-Higgs energies, elements from gauge theory in the form of functionals and equations are drawn to answer deep questions arising in geometric measure theory on the nature of minimal submanifold of higher codimension. On the other hand, while studying $\mathbb{Z}_2$-harmonic spinors, methods and techniques from geometric measure theory are used to answer questions arising in gauge theory.

The rest of this introduction is organised as follows. We start by introducing the notions from geometric measure theory in Section 1.1 relevant for this dissertation. In Section 1.2 we introduce the context surrounding the Yang-Mills-Higgs energies. We state in Subsection 1.2.3 and Subsection 1.2.4 the main resuls, whose proof can be found in Chapter 2. Section 1.3 of this Introduction is then dedicated to introducing $\mathbb{Z}_2$-harmonic spinors, together with our main contributions to this topic, see Subsection 1.3.2. The reader can find the proofs in Chapter 3 of this dissertation, together with further developments on the topic.

1.1 Preliminaries of Geometric Measure Theory

For the convenience of the reader, we start by recalling some terminology and notations from geometric measure theory used throughout the dissertation. We follow [121] and we refer the reader to it for further details, especially for more elementary measure theory.

We shall now introduce the class of maps we are mainly going to work with. For the sake of simplicity we assume to be working in $\mathbb{R}^n$. Let $\mathscr{A}_2(\mathbb{R}^m)$ denote the space of unordered pairs $\{a_1,a_2\}$, with $a_1,a_2 \in \mathbb{R}^m$ not necessarily distinct. This set can be equipped with the following metric

$$\mathscr{G}(a,b) = \min\left\{\sqrt{|a_1-b_1|^2+|a_2-b_2|^2}, \sqrt{|a_1-b_2|^2+|a_2-b_1|^2}\right\}, \tag{1.1}$$

for $a=\{a_1,a_2\}$ and $b=\{b_1,b_2\}$ in $\mathscr{A}_2(\mathbb{R}^m)$. Letting $b=\{0,0\}$, we have

$$|a| := \mathscr{G}(a,\{0,0\}) = \sqrt{|a_1|^2+|a_2|^2}.$$

A *two-valued function* on a set $\Omega\subset\mathbb{R}^n$ is a map $u\colon\Omega\to\mathscr{A}_2(\mathbb{R}^m)$. Henceforth, $\Omega\subset\mathbb{R}^n$ will denote an open set. To each 2-valued function we can associate the single-valued average $u_a\colon\Omega\to\mathbb{R}^m$ given by $u_a(X) := (u_1(X)+u_2(X))/2$, and a symmetric part $u_s\colon\Omega\to\mathscr{A}_2(\mathbb{R}^m)$ given by $u_s(X)=\{\pm(u_1(X)-u_2(X))/2\}$. Thus, we will define a symmetric two-valued function u to be one with vanishing average part, in which case the function coincides with its symmetric part. We will only be interested in symmetric two-valued maps in this work. Giving the set $\mathscr{A}_2(\mathbb{R}^m)$ a metric structure allows to define the space of continuous two-valued functions, denoted $C^0(\mathbb{R}^n,\mathscr{A}_2(\mathbb{R}^m))$. Furthermore, for each $\mu\in(0,1]$ one can introduce the space of μ-Hölder continuous functions $C^{0,\mu}(\mathbb{R}^n,\mathscr{A}_2(\mathbb{R}^m))$ to be the space of two-valued continuous functions u such that

$$[u]_{\mu;\Omega} = \sup_{X,Y\in\Omega, X\neq Y}\frac{\mathscr{G}(u(X),u(Y))}{|X-Y|^\mu} < +\infty.$$

A two-valued function $u\colon\Omega\to\mathscr{A}_2(\mathbb{R}^m)$ is said to be differentiable at $Y\in\Omega$ if there exists a two-valued affine function $\ell_Y\colon\mathbb{R}^n\to\mathscr{A}_2(\mathbb{R}^m)$ such that

$$\lim_{X\to Y}\frac{\mathscr{G}(u(X),\ell_Y(X))}{|X-Y|} = 0.$$

Such an ℓ_Y is unique if it exists and it is referred to as the *affine approximation* of u at Y. Consequently, let $Du(Y) = \{A_1^Y, A_2^Y\}$ denote the corresponding *derivative* of u at Y (here A_1^Y denotes an $m \times n$ matrix), and define the space $C^1(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ as the space of all maps $u\colon \Omega \to \mathscr{A}_2(\mathbb{R}^m)$ such that u is differentiable at every point of Ω and

$$\lim_{Y \to Y_0} \sup_{X \in B_1(0)} \mathscr{G}(\ell_Y(X), \ell_{Y_0}(X)) = 0,$$

for all $Y_0 \in \Omega$ and where ℓ_Y and ℓ_{Y_0} are as above. As in the single-valued case, if $u \in C^1(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ then, both u and Du are continuous. Convergence in $C^1(\Omega, \mathscr{A}_2(\mathbb{R}^m)$ is defined as

$$\lim_{k \to \infty} \sup_{Y \in \Omega} \sup_{X \in B_1(0)} \mathscr{G}(\ell_{k,Y}(X), \ell_Y(X)) = 0,$$

for given $u_k, u \in C^1(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ and where ℓ_k and ℓ are the corresponding affine approximations. The C^1-norm is defined as usual. We introduce next the space $L^2(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ of Lebesgue measurable two-valued functions $u\colon \Omega \to \mathscr{A}_2(\mathbb{R}^m)$ such that $\|u\|_{L^2(\Omega)} := \|\mathscr{G}(u(x), 0)\|_{L^2(\Omega)} < \infty$. We equip this space with the metric

$$d^2(u,v) = \int_\Omega \mathscr{G}(u(x), v(x))^2.$$

One can define more generally two-valued L^p-spaces. From this one can then define the Sobolev space $W^{1,2}(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ via Almgren bilipschitz embedding technique, see [9, Section 2], or using De Lellis and Spadaro's equivalent formulation, see [42, Definition 0.5]. Finally, let $u \in C^1(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ be a symmetric two-valued function, smooth outside of $\mathscr{K}_u = \{X \in \Omega;\ u(X) = 0, Du(X) = 0\}$, in the sense that for every $B \subset \Omega \setminus \mathscr{K}_u$ we have a selection $u = \{\pm u_1\}$, for some smooth function $u_1\colon \Omega \to \mathbb{R}^m$. We shall say that $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ provided $D^2 u \in L^2(\Omega' \setminus \mathscr{K}_u, \mathscr{A}_2(\mathbb{R}^m))$ for every open set $\Omega' \Subset \Omega$ and we extend $D^2 u = \{0,0\}$ on $\mathscr{K}_u$. It follows that $Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathscr{A}_2(\mathbb{R}^m))$ as defined by Almgren.

All of the above extend straightforwardly when replacing $\mathbb{R}^m$ with any other vector space and to the setting of Riemannian manifolds. One just needs to require the various properties to be satisfied in a given chart. Generalizing further, and in accordance with our setting, one can define multi-valued sections of vector bundles as maps sending an element $p \in M$, the base, to an element of $\mathscr{A}_2(\mathscr{V}_p)$, where $\mathscr{V}_p$ denotes the fiber at $p \in M$. Continuity is then defined with respect to local trivializations of $\mathscr{V}$.

Henceforth, $\mathscr{H}^k$ will denote the k-dimensional Hausdorff measure, from which one can derive the associated Hausdorff dimension. We will often work with Radon measure, we refer the reader to [121, Chapter 1, Section 4] for a thorough treatment of it. We will also work with countably k-rectifiable sets, these are sets that are contained, up to an $\mathscr{H}^k$-measure zero set, in a countable union of k-dimensional C^1-regular, embedded submanifolds. Countably k-rectifiable sets are sometimes defined using Lipschitz submanifolds instead of C^1-regular ones, these two definitions are equivalent, see [121, Lemma 1.2]. We can now proceed to more involved notions of geometric measure theory. A *k-dimensional varifold V* on an open subset $\Omega \subset \mathbb{R}^n$ is a Radon measure on the Grassmannian bundle $G_k(\Omega) = \Omega \times G(k,n)$, where $G(k,n)$ denotes the space of k-dimensional unoriented planes in $\mathbb{R}^n$. We endow the space of varifolds with the topology induced by Radon measures. We will denote by $\|V\|$ the weight measure of the varifold V, i.e. $\|V\|(A) = V(\pi^{-1}(A))$, where π denotes the natural projection $G_k(\Omega) \to \Omega$ and where $A \subset \Omega$. Furthermore, denoting the support of V by $\operatorname{spt} V$ we can then introduce the two most relevant subsets of it, namely the regular and singular set. We will say $X \in \operatorname{reg}(V)$ if and only if $X \in \Omega \cap \operatorname{spt} V$ and there exists an open neighborhood of X such that the support of the varifold restricted to it is a smooth, connected, embedded hypersurface whose boundary is contained in the boundary of the neighborhood of p. Define then the singular set $\mathrm{sing}(V)$ as $(\mathrm{spt}\, V \setminus \mathrm{reg}(V)) \cap \Omega$. The regular set is open and consequently, the singular set is closed.

Remark 1. The regular and singular sets just introduced should be more accurately referred to as *interior* regular and singular set. However, we shall not need the distinction for this work.

Given a $\Sigma \subset \Omega$ a countably k-rectifiable set, with locally finite $\mathscr{H}^k$-measure, and $\theta\colon \Sigma \to \mathbb{R}$ a positive locally $\mathscr{H}^k$-integrable function, we denote by (Σ, θ) the varifold obtained by integration

$$V(\phi) := \int_\Sigma \phi(p, T_p\Sigma)\theta(p)\, \mathrm{d}\mathscr{H}^k(p), \qquad \text{for all } \phi \in C^0_c(G_k(\Omega)).$$

The function θ is usually referred to as the *multiplicity* of V and we will denote by $|\Sigma|$ the multiplicity one varifold associated to Σ, viz. when $\theta = 1$ for $\mathscr{H}^k$-a.e. p. An important class of varifolds we are going to be interested in is the one of stationary varifolds, meaning the ones with vanishing first variation with respect to compactly supported variations. This condition can be equivalently characterised by the following requirement

$$\int_{G_k(B^n_1(0))} \mathrm{div}_S(\Upsilon(p))\, \mathrm{d}V(p,S) = 0, \qquad \text{for all } \Upsilon \in C^1_c(B^n_1(0);\mathbb{R}^n),$$

where div_S denotes the divergence with respect to $S \in G(k,n)$. More precisely, for any $S \in G(k,n)$ we have

$$\mathrm{div}_S X = \sum_{i=1}^k \langle \tau_i, D_{\tau_i} X\rangle,$$

where $\tau_1, \tau_2, \ldots, t_k$ is an orthonormal basis for for S. Furthermore, we considered varifolds defined on the unit ball $B^n_1(0)$ for the sake of simplicity. Choosing a suitable radial vector field in the above identity one obtains a monotonicity formula for the mass ratio $\|V\|(B_r(p))/\omega_k r^k$, where ω_k denotes as usual the Lebesgue measure of the unit ball. A consequence of this monotonicity formula is that the mass ratio function $r \mapsto \|V\|(B_r(p))/\omega_k r^k$ is monotone where defined, which in turn allows to define the *density* of V at p as the limit $r \to 0$ of the mass ratios. We shall denote such limit as $\Theta^k(V,p)$. Note that the limit exists by virtue of the monotonicity formula. This function is of the utmost importance in the theory of varifolds and enjoys several properties, among which upper-semicontinuity and being $\|V\|$-measurable. Moreover, a consequence of the Rectifiability Theorem [121, Theorem 42.2] is that for a stationary varifold V with $\Theta^k(V,p) > 0$, $\|V\|$-a.e. we have $\theta = \Theta$ for $\mathscr{H}^k$-a.e. point p. Thus, we can now give the following definition: a stationary integral k-varifold V is said to be *integral* if $\Theta(V,\cdot) \in \mathbb{N}$, $\mathscr{H}^k$ almost everywhere.

One of the main reasons for enlarging the class of smooth minimal submanifolds and considering varifolds is their compactness properties, making them a versatile class of objects to investigate. We refer the reader to [121, Theorem 42.7] and to [5, Section 6] for further details on it. The main drawback of this approach is that it shifts a considerable amount of effort on the regularity side, viz. obtaining a smooth minimal submanifold once a stationary integral varifold is obtained. This parallels the theory of weak solutions of partial differential equations extremely closely. We will return to this issue and the regularity theory of varifolds later in this chapter and we will give a better account of the literature and the various results at play. Further properties and notions on varifolds will be introduced throughout this work when needed.

We shall now move to another notion of generalized submanifold ubiquitous in geometric measure theory. An k-dimensional current is a continuous linear functional on the space of compactly supported differential k-forms. Assuming to be working on a Riemannian manifold

(M,g), we denote the space of such objects by $\mathscr{D}_k(M)$. Motivated by the classical Stokes' theorem one can define the notion of boundary of a k-current T:

$$\langle \partial T, \omega \rangle := \langle T, d\omega \rangle, \quad \text{for } \omega \in \mathscr{D}^{k-1}(M),$$

where the notation $\langle \cdot, \cdot \rangle$ means the pairing between linear form and an element of the underlying space. Note that ∂T is, therefore, a $(k-1)$-current with zero boundary. Another notations also in use for currents are $T[\omega]$ or $T(\omega)$. Having a well defined notion of boundary is a crucial feature of the theory of currents, and one of the main reasons why they were the right tool for the Plateau problem. A k-current will be said to be integer rectifiable if its action on differential forms can be written by integration

$$\langle T, \omega \rangle = \int_{\Sigma} \langle \xi(x), \omega(x) \rangle \theta(x) \, \mathrm{d}\mathscr{H}^k(x),$$

where Σ is an $\mathscr{H}^k$-measurable, countably k-rectifiable subset of M, and ξ is an orientation of Σ, viz. $\xi(x)$ is a unit k-vector representing the approximate tangent space of Σ at x for $\mathscr{H}^k$-a.e. $x \in \Sigma$. Finally, θ is the so-called *multiplicity*, an integer-valued, non-negative, measurable function defined a.e. in Σ. From this definition one can infer that examples of k-currents are given by integration over oriented k-dimensional submanifolds. Note that in the case $k = 0$, currents are just Schwartz distributions. We denote by $\mathscr{I}_k(M;\mathbb{Z})$ the space of *integer rectifiable k-currents with finite mass*. Recall that an *integral k-current* is an integer rectifiable k-current whose *boundary has finite mass* (and, as a consequence, is itself an integer rectifiable $(k-1)$-current). We denote by $\mathbf{I}_k(M;\mathbb{Z})$ the space of k-dimensional integral currents in M and by $\mathscr{Z}_k(M;\mathbb{Z})$ the subset of those currents $T \in \mathbf{I}_k(M;\mathbb{Z})$ satisfying $\partial T = 0$. We shall refer to currents with zero boundary as *k-cycles*.

Given $T \in \mathbf{I}_k(M;\mathbb{Z})$ we denote by $|T|$ the associated integral varifold and by $\|T\|$ the induced Radon measure on M. The definition of mass used in this dissertation is

$$\mathbb{M}(T) := \sup\{T(\phi) \mid \phi \in \Omega^k(M), \|\phi\|_{C^0(M)} \leq 1\},$$

where the last norm is understood with respect to the Euclidean norm on k-covectors. Setting $\mathbb{M}(S,T) := \mathbb{M}(S-T)$ for $S,T \in \mathscr{I}_k(M;\mathbb{Z})$ we obtain a metric on $\mathscr{I}_k(M;\mathbb{Z})$ known as the *mass metric*. We can topologize the space $\mathscr{I}_k(M;\mathbb{Z})$ differently via the so-called *flat distance*

$$\mathscr{F}(S,T) := \inf\{\mathbb{M}(P) + \mathbb{M}(Q) \mid S - T = P + \partial Q, P \in \mathscr{I}_k(M;\mathbb{Z}), Q \in \mathscr{I}_{k+1}(M;\mathbb{Z})\},$$

for $S,T \in \mathscr{I}_k(M;\mathbb{Z})$. Writing $\mathscr{F}(T) = \mathscr{F}(T,0)$, note that we trivially have

$$\mathscr{F}(T) \leq \mathbb{M}(T) \quad \text{for all } T \in \mathscr{I}_k(M;\mathbb{Z}),$$

from which one can deduce that the mass topology is finer that the flat topology. As in the case of varifolds, one can introduce a regular and singular set for currents.

Lastly, we shall briefly overview the notion of sets of finite perimeter, the one that was historically more relevant for the level set approach in codimension one and the study of the Allen-Cahn functional. For $\Omega \subset (M,g)$ an open set, recall that $u \in L^1(\Omega)$ is in $\mathrm{BV}(\Omega)$ if Du is a TM-valued Radon measure. A measurable set $E \subset \Omega$ is a set of finite perimeter, or Caccioppoli set in honor of Italian Mathematician Renato Caccioppoli who first introduced the concept in 1927 if $\chi_E \in \mathrm{BV}(\Omega)$, where χ_E denotes the characteristic function of E, in which case one can define the *perimeter* of E to be

$$\mathrm{Per}(E;\Omega) = \int_{\Omega} |D\chi_E|.$$

Caccioppoli sets have a notion of measure-theoretic boundary called the *reduced boundary* and a celebrated theorem of De Giorgi establishes several structural results on it. We refer the reader to [121, Theorem 14.3] and [95] for a comprehensive treatment of the theory of sets of finite perimeter. Some further concepts from geometric measure theory relevant to the min-max comparison are introduced in Section 2.4 of Chapter 2.

1.2 The Level Set Method

The purpose of this section is to review the literature and provide a historical perspective on the level set method, especially its recent successes in applications to geometry. We shall focus on the Allen-Cahn functional in the codimension one setting and then move to codimension two with the Ginzburg-Landau and Yang-Mills-Higgs energies. This last one will be the main object of study of Chapter 2.

1.2.1 The Codimension One Setting

The mathematical theory that we attempt to describe here originated in the seventies, even though it appeared in the physical community long before that. It goes by the name of *gradient theory of phase transitions* and considers a homogeneous, isothermal fluid confined into a container, i.e. a bounded region Ω, whose total mass m is fixed. This in particular implies that given a density distribution $u\colon \Omega \to \mathbb{R}$ it has to satisfy $\int_\Omega u = m$. The total energy of the system is then given by the functional $u \mapsto \int_\Omega W(u)$, where W denotes the energy per unit volume and it is a non-convex function given by the Van der Waals-Cahn-Hilliard theory. We can assume that the liquid supports two phases $a < b$, meaning that W is a double-well potential with zero set given by a and b. Equilibrium configurations in the sense of Gibbs rendering the body stable are then given by u solving the following minimisation problem

$$\min\left\{\int_\Omega W(u);\ u\colon \Omega \to \mathbb{R},\ \int_\Omega u = m\right\}. \tag{1.2}$$

Without loss of generality, we can assume that the two phases are given by ± 1. Indeed, we can simply apply an affine change of variables to the double-well potential, which does not affect the minimum in (1.2) as it would simply add a constant. Assuming $|\Omega| = 1$ and $-1 < m < 1$, then any measurable set satisfying $|E| = (m-1)/2$ together with corresponding function $u = \chi_E - \chi_{\Omega\setminus E}$ are a minima of (1.2). Here χ_A denotes the characteristic function of $A \subset \Omega$. In other words, minimisers of Gibbs' variational problem are given by all functions vanishing on $\{\pm 1\}$ and satisfying the required mass constraint. This set of solutions is too large and, furthermore, lacks physical interpretation as the interface region between the two phases could, theoretically, be dense in Ω. This is due to the fact that the interface between the two regions is not penalised by the total energy. Thus, we ougth to expect solutions of (1.2) to arise as limits of a theory that penalises the surface tension between the two phases. This is what is observed in physical systems as well. Consequently, we have to add to the functional $u \mapsto \int_\Omega W(u)$ a regularising term depending on higher derivatives of u. This is precisely what is done by the van der Walls–Cahn–Hilliard theory of phase transitions in which the minimum problem under consideration becomes

$$\min\left\{\int_\Omega W(u) + \varepsilon^2|\nabla u|^2;\ u\colon \Omega \to \mathbb{R},\ \int_\Omega u = m\right\}, \tag{1.3}$$

where the power ε^2 is due to dimensional considerations. In the early eighties Gurtin conjectured that mininimers of (1.3) should converge, as $\varepsilon \to 0$ to minimisers of (1.2), with the associated interface minimising area among all other competitors. Additionally, he conjectured that the regularised functional should behave like ε times the area of the limiting interface. Therefore, normalising the functional by ε^{-1} and letting $\varepsilon \to 0$ we expect to recover the area of the limiting minimizing interface.

We will return to Gurtin's conjecture in the next paragraphs. Before that, we introduce some notations. We will mainly consider domains $\Omega \subset \mathbb{R}^n$ or $\Omega \subset M$, where M is a closed, meaning compact and boundaryless, Riemannian manifold of dimension n. The Allen-Cahn functional is given by

$$E_\varepsilon(u;\Omega) := \int_\Omega \left(\varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \mathrm{dvol}_g, \tag{1.4}$$

where $W \in C^3(\mathbb{R})$ is the double-well potential satisfying $W \geq 0$, having precisely three critical points, two of which are minima at ± 1 with $W(\pm 1) = 0$, and $W''(\pm 1) > 0$ and the third local maximum between -1 and $+1$. An example of such a potential is $W(t) = (1-t^2)^2/4$. The Euler-Lagrange equation of (1.4) is

$$\Delta u = \frac{1}{\varepsilon^2} W'(u). \tag{1.5}$$

Note now that the contributions coming from the two terms in (1.4) have the same order. The role of the second term is to penalize values of u away from the two phases ± 1, whilst the effect of the first term is to penalize unnecessary interfaces. Mathematically, both terms can be interpreted as Dirac delta functions over the transition region. It might be instructive to look at the one dimensional case on the real line, where (1.5) reduces to ordinary differential equations $u'' = \frac{1}{\varepsilon^2}(u^3 - u)$. We considered $W(t) = (1-t^2)^2/4$ for the sake of simplicity. In this case, an explicit solution is given by $\mathbb{H}_\varepsilon(t) := \tanh(t/(\varepsilon\sqrt{2}))$. Letting $\varepsilon \to 0$ we have that $\mathbb{H}_\varepsilon$ converges to the function constantly equal to one on $\mathbb{R}_{>0}$ and constantly equal to -1 on $\mathbb{R}_{<0}$, thus presenting a jump discontinuity at the origin, which can be artificially thought of as a minimal submanifold of codimension one in $\mathbb{R}$.

The process that leads to the proof of Gurtin's conjecture [67] in the late 1970s revealed deep connections between minimal hypersurfaces and the Allen-Cahn equations, thus opening up a rich line of investigation, shedding light on the structure of solutions of semilinear elliptic equations and the existence theory of minimal hypersurfaces. A recurring theme in the study of the correspondence between solutions of (1.5) and minimal hypersurfaces is the convergence not only of critical points but of the *variational theory* for the functionals E_ε to that of the $(n-1)$-area on the space of $(n-1)$-boundaries as $\varepsilon \to 0$. The earliest results in this direction were obtained by Modica and Mortola [100] who established the asymptotic convergence of E_ε to a constant multiple of the perimeter functional for Caccioppoli sets, in the framework of Γ-*convergence* introduced a few years earlier by De Giorgi [38]. De Giorgi's Γ-convergence provides a natural weak notion of convergence for variational problems involving a singular perturbation, well-suited to establishing convergence of minimizers to minimizers. See [24] and [34] for a contemporary treatment of Γ-convergence, and [1] for its application to the study of phase transitions. By now Γ-convergence has found applications in a wealth of mathematical fields: homogenization problems, dimension reduction problems, free-discontinuity problems, image reconstructions, and fracture mechanics to name a few, see [24, 34]. The work of Modica and Mortola was later generalized by Modica [99] and Sternberg [129], in their resolution of Gurtin's conjecture [67], thus bridging rigorously the Van der Waals-Cahn-Hilliard theory of phase transitions and Gibbs' approach presented above.

While the Γ-convergence results of [99], [100], and [129] imply that E_ε-minimizing solutions of (1.5) (rather, their level sets and energy measures) converge to area-minimizing hypersurfaces, a series of results obtained over the last five years [51, 64, 66] show that the min-max theory for the Allen-Cahn functionals E_ε likewise converges to the min-max theory for the area functional on $(n-1)$-boundaries in the geometric measure theory framework developed by Almgren and Pitts [109]. Building on the analytic work of [79] and [141], these and related results have established the min-max theory for the Allen-Cahn functionals as a valuable regularization of the Almgren-Pitts min-max construction of minimal hypersurfaces. This last approach was started in the 1960s when Almgren [9] investigated the problem of constructing a critical point of the n-dimensional area functional in an $(n+k)$-dimensional Riemannian manifold. He was able to construct a type of measure-theoretic weak solution V, called a stationary integral n-varifold. This is a generalized surface that can have a singular set, denoted $\mathrm{Sing}(V)$, of points at which V is not smooth and embedded. In codimension one (i.e. $k=1$), the subsequent efforts of Pitts [109], and Schoen and Simon [119] established that this weak solution was actually smooth, except for a small, generally unavoidable, singular set of codimension 7. We refer the reader to Section 1.3 for a more careful treatment of these regularity issues, and the end of this introduction for the various definitions of geometric measure theory. Despite being effective, Almgren's technique was notoriously complex and intricate, which is why the emergence of this new PDE method based on the Allen-Cahn equation to establish the existence of stationary integral varifolds, and consequently optimally regular minimal hypersurfaces, has attracted a lot of interest. This method has brought a novel understanding of minimal surfaces and was the driving force for spectacular advancements in the field. See, for instance, works of Gaspar and Guaraco [64] on the Weyl Law for the volume spectrum, and Chodosh and Mantoulidis' proof in [32] of the multiplicity one conjecture in 3-manifolds of Marques and Neves [96]. See also works of Bellettini [12] and [11] for more on the multiplicity one conjecture, [16, 14, 15, 13] for other applications of the Allen-Cahn approach to the setting of constant and prescribed mean curvature hypersurfaces, and [63, 74] for results on the behavior of the index.

The above discussion was on elliptic partial differential equations and (either classical or distributional notions of) minimal surfaces arising from them. There is however a rich theory on the time-dependent case. Considering the negative L^2-gradient flow of the Allen-Cahn functional (1.4), one obtains a semi-linear parabolic partial differential equation reading

$$\partial_t u = \Delta u - \varepsilon^{-2} W'(u). \tag{1.6}$$

Mimicking the time-independent case one naturally expects the singular limit of the parabolic equation to approximate a weak mean curvature flow of codimension one. This intuition was proven correct when Ilmanen established in [80] that in the singular limit $\varepsilon \searrow 0$ the varifolds associated with (1.6) converge to a Brakke flow, a measure-theoretic weak solution of mean curvature flow. Ilmanen obtained more structural properties on the limiting flow by proving that it is an enhanced flow. One of the main questions that was left out from this work was integrality of the limiting varifolds. Tonegawa bridged this gap in [140] by proving that, modulo multiplication by the surface energy density, $\theta(V_t,\cdot) \in N(t,\cdot)$, for almost every time $t>0$ and for almost every point. This result further strengthened Ilmanen's comment that Allen-Cahn limits yield the same structure as the one arising from limits of elliptic regularisations as introduced in [81]. Another question left unanswered in [80] regarded the cancellation in the BV-convergence of Allen-Cahn limits and it was answered positively (meaning no cancellation) by Nguyen and Wang in [104]. Interestingly, this result was obtained as a consequence of a more general ε-regularity result for the parabolic Allen-Cahn equation, a diffused analog of Brakke's celebrated ε-regularity theorem, proven in the same article by the authors. An ingredient in their proof is a gap theorem for entire eternal parabolic for Allen-Cahn equations. We refer the reader to [104] for further details.

In the next few paragraphs, we make a digression on the fundamental role played by Wickramasekera's regularity theory to obtain existence of (optimally regular) minimal hypersurfaces in the Allen-Cahn setting. While this PDE based method greatly simplifies the existence of weak solutions part of the proof, it necessitates a stronger regularity theory to infer optimal regularity of the corresponding varifold.

Wickramasekera's theory deals with stationary integral varifolds with stable regular part, meaning that the second variation with respect to the area is non-negative for all normal deformations compactly supported on the regular set, see [144, Section 3] for a precise statement. Note that this definition does not require the varifold to be of codimension one. However, an alternative characterization of stability by means of the so-called *stability inequality* is especially useful in the codimension one setting. It reads

$$\int_{\operatorname{reg} V} |A|^2 \phi^2 \leq \int_{\operatorname{reg} V} |\nabla^{\operatorname{reg} V} \phi|^2,$$

for all $\phi \in C^1_c(\operatorname{reg} V)$, where A denotes the classical second fundamental form of $\operatorname{reg} V$, $|A|$ its length and $\nabla^{\operatorname{reg} V}$ the gradient on $\operatorname{reg} V$. This inequality turns out to be extremely useful in practice as it implies L^2-estimates on the second fundamental form. One of the main challenges in understanding the regularity of varifolds in higher codimension is to exploit the corresponding inequality, which has however a less tractable form. To the best of the author's knowledge, there is no clear way yet to take advantage of stability in higher codimension. Instrumental to Wickramasekera's regularity theory is the following definition.

Definition 2 (α-Structural Hypothesis)**.** For each given $p \in \operatorname{sing}(V)$, there exists no $\rho > 0$ such that $\operatorname{spt}\|V\| \cap B_\rho(p)$ is equal to a finite number of embedded $C^{1,\alpha}$-hypersurfaces with boundary of $B_\rho(p)$, all having a common $C^{1,\alpha}$-boundary in $B_\rho(p)$ containing p and no two intersecting except along their common boundary.

Singularities appearing in this definition are referred to as *classical singularities*. In other words, the α-structural hypothesis holds if there are no classical singularities. We are now able to give a rough version of the main theorem of [144].

Theorem 3. *Let V be a stationary integral n-varifold with stable regular part on a smooth $(n+1)$-dimensional Riemannian manifold M. Then, if for some $\alpha \in (0,1/2)$, V satisfies the α-Structural Hypothesis, then $\operatorname{sing}(V) = \emptyset$ if $1 \leq n \leq 6$, $\operatorname{sing}(V)$ discrete if $n = 7$ and $\mathcal{H}^{n-7+\gamma}(\operatorname{sing}(V)) = 0$ for each $\gamma > 0$ if $n \geq 8$. In other words, the support of V is an optimally regular minimal hypersurface.*

The main scope of [141] is then to verify the α-structural hypothesis for those varifolds arising as limits of level sets of stable solutions of the Allen-Cahn equation. The key step from which the main theorem of [141] readily follows is to prove that for a varifold V there exists a, possibly non-empty, Borel set $Z \subset \operatorname{spt}\|V\|$ with $\mathcal{H}^{n-2}(Z) = 0$ such that for each $x \in \operatorname{spt}\|V\| \setminus Z$ and each tangent cone $\mathbf{C}$ to V at x, we have that $\operatorname{spt}\|C\|$ is not equal to a union of three or more half-hyperplanes of $\mathbb{R}^n$ meeting along an $(n-2)$-dimensional affine subspace. Theorem 3 is a remarkable sharpening of the Schoen-Simon regularity theory, which requires knowing a priori that the singular set is sufficiently small. We shall return later in this introduction to the Schoen-Simon regularity theory.

1.2.2 The Codimension Two Setting

In view of these applications, it is natural to seek an analogous correspondence between certain geometric elliptic systems and minimal submanifolds of higher codimension. In [107], Pigati and

Stern proposed a natural analog in codimension two, with the role of the Allen-Cahn equations taken on by a well-studied family of elliptic systems from gauge theory. Specifically, [107] considers the *self-dual* $U(1)$*-Yang-Mills-Higgs* energies: the gauge-invariant functionals $E_\varepsilon(u,\nabla)$ acting on a section $u \in \Gamma(L)$ and metric-compatible connection ∇ on a Hermitian line bundle $L \to M$ by

$$E_\varepsilon(u,\nabla) := \int_M \left(|\nabla u|^2 + \varepsilon^2 |F_\nabla|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\right) \mathrm{dvol}_g,$$

which are simply rescalings of

$$E(u,\nabla) := \int_M \left(|\nabla u|^2 + |F_\nabla|^2 + \frac{1}{4}(1-|u|^2)^2\right) \mathrm{dvol}_g.$$

The functionals E_ε are distinguished from *formally* similar functionals—such as

$$\int_M (|\nabla u|^2 + \lambda |F_\nabla|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2)$$

for $\lambda \neq \varepsilon^2$—by their so-called *self-duality*: namely, E_ε enjoys additional symmetry properties, such that minimizers of E_ε for bundles $L \to \Sigma^2$ over a Riemann surface Σ^2 satisfy a special first-order system known as the *vortex equations*. More precisely, given a complex line bundle $L \to \Sigma^2$, over a Riemann surface equipped with a conformal metric, we can decompose the connection ∇ into its ∂_A and $\overline{\partial}_A$ components, so that the curvature becomes $F_\nabla u = -(\partial_A \overline{\partial}_A u + \overline{\partial}_A \partial_A u)$. Rewrite then the unscaled Yang-Mills-Higgs functional as follows

$$E(u,A) = \int_{\Sigma^2} 2|\overline{\partial}_A u|^2 + \left(*F_\nabla - \frac{1}{2}(1-|u|^2)\right)^2 \mathrm{dvol}_g + 2\pi \deg L, \tag{1.7}$$

where $\deg L := c_1(L)[\Sigma^2]$ is the degree of the line bundle. Consequently, assuming either $\deg L \geq 0$, or $\deg L \leq 0$, we have that minimizers of (1.7) are realised precisely for solutions of

$$\begin{cases} \nabla_{\partial_1} u \pm i \nabla_{\partial_2} u = 0, \\ *F_\nabla = \pm \frac{1}{2}(1-|u|^2), \end{cases} \tag{1.8}$$

the vortex equations. Note interestingly that by integrating the second vortex equation yields the inequality

$$2\pi \deg L = 2\pi \int_{\Sigma^2} *F_\nabla \leq \frac{1}{2}|\Sigma^2|,$$

giving a necessary (topological) condition for solvability. Here $|\Sigma^2|$ denotes the volume of Σ^2.

The study of these functionals has a long history, which we do not attempt to survey here. In his thesis work [133, 134], Taubes classified finite-energy critical points of E_ε for the trivial bundle $L \cong \mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}^2$, showing that all such critical points satisfy the first-order vortex equations, are determined—up to gauge equivalence—by the finite zero set $u^{-1}\{0\} \subset \mathbb{C}$ (counted with multiplicity), and have quantized energy $E_\varepsilon(u,\nabla) = 2\pi N \in 2\pi\mathbb{N}$ corresponding to the mass of the zero set $N = |u^{-1}\{0\}|$ (see [133], [134], and [82] for details). The asymptotic analysis as $\varepsilon \to 0$ of the rescaled functionals E_ε was first taken up by Hong, Jost, and Struwe, who showed in [77] that for minimizers $(u_\varepsilon, \nabla_\varepsilon)$ of E_ε on line bundles $L \to \Sigma^2$ over a Riemann surface Σ, energy and curvature concentrate (subsequentially) as $\varepsilon \to 0$ at a collection of $|\deg(L)|$ points in Σ, outside of which u_ε converges to a unit section u_0 and ∇_ε to a flat connection ∇_0 for which $\nabla_0 u_0 = 0$. Bradlow studies the unscaled vortex equations on Kähler manifolds, see [23].

The results of [107] provide a far-reaching generalization of Hong–Jost–Struwe's analysis, characterizing the limiting behavior of arbitrary critical points on line bundles over a base manifold M^n of general dimension. Namely, it is shown in [107] that for sequences $(u_\varepsilon, \nabla_\varepsilon)$ of critical points satisfying a uniform energy bound $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq C$, the energy densities

$$e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) := |\nabla_\varepsilon u_\varepsilon|^2 + \varepsilon^2 |F_{\nabla_\varepsilon}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}$$

converge subsequentially weakly in $(C^0)^*$ to (the weight measure of) a *stationary integral* $(n-2)$*-varifold* V in M—i.e., a (possibly singular) minimal variety of codimension two. In particular, this gives a codimension-two analog to the results of Hutchinson–Tonegawa [79] for the Allen–Cahn equations, showing that critical points for E_ε converge cleanly to critical points of the $(n-2)$-area functional in the $\varepsilon \to 0$ limit. We note, moreover, that the analysis in [107] depends strongly on the specific choice of coupling constants in the definition of E_ε, suggesting that the *self-dual* $U(1)$-Yang–Mills–Higgs energies provide more or less the *unique* codimension-two analog for the Allen–Cahn energies, at least among similar functionals of Yang–Mills–Higgs type.

Remark 4. In particular, the convergence behavior for critical points $(u_\varepsilon, \nabla_\varepsilon)$ of E_ε in the $O(1)$ energy regime is considerably simpler than its counterpart for the non-gauged Ginzburg–Landau energies

$$G_\varepsilon : W^{1,2}(M, \mathbb{C}) \to \mathbb{R}, \quad G_\varepsilon(u) := \int_M \left(|du|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} \right)$$

in the $O(|\log \varepsilon|)$ energy regime, whose critical points in general exhibit *partial* energy concentration along a stationary, *rectifiable* (not necessarily integral) $(n-2)$-varifold (cf. [19, 20, 31, 69, 76, 92, 93, 115, 128, 130] for details of the asymptotic analysis of the complex Ginzburg–Landau equations, as well as [17, 18, 111–113, 117] for related results for other functionals of Yang–Mills–Higgs type whose behavior resembles that of G_ε). As remarked in [107], the variational theory for the functionals G_ε is best understood as a relaxation of that for the Dirichlet energy on singular S^1-valued maps, and its relation to geometric measure theory and minimal submanifolds is subtle, and qualitatively quite different from that of the Allen–Cahn or self-dual Yang–Mills–Higgs energies. Recent work of Pigati and Stern [108] has confirmed that the limiting stationary varifold needs not be integral.

Building on the ideas of [107], the aim of Chapter 2 is to understand to what extent the *variational theory* for the functionals E_ε converges to that of the $(n-2)$-area, in the spirit of similar results for the Allen–Cahn functionals. Our chief analytic result provides a key step toward answering this question, establishing the Γ*-convergence* of the functionals E_ε for pairs (u, ∇) on a Hermitian line bundle $L \to M$ to the mass functional on the space of integral $(n-2)$-cycles dual to $c_1(L)$. This convergence result—whose precise formulation we give in the following subsection—may be thought of as a *codimension-two analog of the classical results of Modica and Mortola*; and despite the very different setting, its proof bears a surprising resemblance to the original arguments in [100]. In addition to implying the convergence of E_ε-minimizing pairs $(u_\varepsilon, \nabla_\varepsilon)$ to area-minimizing $(n-2)$-cycles, the Γ-convergence framework—together with some topological arguments—allows us to compare the energy of min-max critical points for E_ε to the areas of corresponding min-max minimal varieties, along the lines of the comparison results for the Allen–Cahn and Almgren–Pitts min-max constructions obtained in [64, 66].

1.2.3 Convergence results for the self-dual Yang–Mills–Higgs energies

Let $L \to M^n$ be a Hermitian line bundle over a closed, oriented Riemannian manifold (M^n, g). Given a metric connection ∇ on L, recall that the curvature $F_\nabla \in \Omega^2(M) \otimes \mathfrak{so}(L)$ is given by

$$F_\nabla(X,Y)u := [\nabla_X, \nabla_Y]u - \nabla_{[X,Y]}u = -i\omega_\nabla(X,Y)u \tag{1.9}$$

for some two-form $\omega_\nabla \in \Omega^2(M)$, which we will frequently identify with F_∇ when there is no confusion. Given a pair (u, ∇) consisting of a section $u \in \Gamma(L)$ and metric connection ∇, we define as in [107] the two-form $\psi(u,\nabla) \in \Omega^2(M)$ by

$$\psi(u,\nabla)(X,Y) := 2\langle i\nabla_X u, \nabla_Y u\rangle,$$

which is easily seen to satisfy the pointwise bound $|\psi(u)| \leq |\nabla u|^2$ (cf. [107, Section 2]). For the Γ-convergence results, we will be particularly interested in the two-forms

$$J(u,\nabla) := \psi(u,\nabla) + (1-|u|^2)\omega_\nabla = d\langle \nabla u, iu\rangle + \omega_\nabla, \tag{1.10}$$

whose role should be compared to that of the one-forms $\sqrt{2W(v)} \cdot dv$ for real-valued functions $v : M \to \mathbb{R}$ in the work of Modica–Mortola [100].

As with any Γ-convergence result, our main theorem consists of two parts. First, we show that for any family of pairs $(u_\varepsilon, \nabla_\varepsilon)$ with

$$\sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda < \infty,$$

there exists a subsequence $(u_{\varepsilon_j}, \nabla_{\varepsilon_j})$, with $\varepsilon_j \to 0$, to which we can associate a limiting integral $(n-2)$-cycle Γ with $2\pi\mathbb{M}(\Gamma) \leq \Lambda$. Second, we show that any integral $(n-2)$-cycle dual to $c_1(L)$ can be obtained in this way. More precisely, we have the following result, whose proof can be found in Chapter 2, Section 2.2 and Section 2.3.

Theorem 5 (Γ-convergence)**.** *For a Hermitian line bundle $L \to M$ as above, the following hold:*

(i) Liminf inequality. *Given a family $(u_\varepsilon, \nabla_\varepsilon)$ of smooth sections with $|u_\varepsilon| \leq 1$ and metric connections with uniformly bounded energies $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda$, there exists an integral $(n-2)$-cycle Γ Poincaré dual to the Euler class $c_1(L) \in H^2(M;\mathbb{Z})$ such that, up to a subsequence,*

$$J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup 2\pi\Gamma, \quad \text{as } \varepsilon \to 0,$$

as $(n-2)$-currents. Moreover, the following liminf inequality holds:

$$2\pi\mathbb{M}(\Gamma) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon).$$

(ii) Recovery sequence. *Given an integral $(n-2)$-cycle Γ whose homology class $[\Gamma] \in H_{n-2}(M;\mathbb{Z})$ is Poincaré dual to $c_1(L) \in H^2(M;\mathbb{Z})$, there exists a family $(u_\varepsilon, \nabla_\varepsilon)$ of smooth sections and connections on L such that*

$$J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup 2\pi\Gamma, \quad \text{as } \varepsilon \to 0,$$

as currents, and

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) = 2\pi\mathbb{M}(\Gamma).$$

Remark 6. Since the curvature forms $\omega_\varepsilon := iF_{\nabla_\varepsilon}$ satisfy $J(u_\varepsilon,\nabla_\varepsilon) = \omega_\varepsilon + d\langle\nabla_\varepsilon u_\varepsilon, iu_\varepsilon\rangle$, if $E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) = O(1)$, the boundedness of $\langle\nabla_\varepsilon u_\varepsilon, iu_\varepsilon\rangle$ in $L^2(M)$ together with part (i) above implies that the curvatures ω_ε also have a subsequential limit as currents. Simple examples show that this limit may fail to coincide with $2\pi\Gamma$ under our assumptions, for instance one can consider $u_\varepsilon = 1$ and $\nabla_\varepsilon = d - i\alpha$ for a fixed one-form α with $d\alpha \neq 0$ on the trivial bundle $\mathbb{C}\times M$. However, assuming that ∇_ε is critical for the energy $E_\varepsilon(u_\varepsilon,\cdot)$—hence, a minimizer by convexity of E_ε in ∇_ε—the corresponding Euler–Lagrange equation (2.3) gives $\langle\nabla_\varepsilon u_\varepsilon, iu_\varepsilon\rangle \rightharpoonup 0$, since $\varepsilon^2\omega_\varepsilon \to 0$ in $L^2(M)$. Thus, in this case

$$2\pi\Gamma = \lim_{\varepsilon\to 0}\omega_\varepsilon$$

as currents. Together with Corollary 7 below, this implies that for a sequence of *minimizers* $(u_\varepsilon,\nabla_\varepsilon)$, the curvature forms $\frac{1}{2\pi}\omega_\varepsilon$ converge subsequentially to an integral, area-minimizing cycle Γ whose associated varifold agrees with the energy concentration varifold V from [107, Theorem 1.1] (up to a subsequence). This answers a question raised in [107].

Readers familiar with the Γ-convergence theory developed for the normalized Ginzburg–Landau functionals $\frac{G_\varepsilon}{|\log\varepsilon|}$ in recent decades (see in particular [2–4, 25, 84]) will notice some formal similarities between the above result and analogs for the functionals $\frac{G_\varepsilon}{|\log\varepsilon|}$. Namely, the results of [3] and [84] show that for any complex-valued map $u : M \to \mathbb{C}$ with

$$G_\varepsilon(u) \leq 2\pi\Lambda\log(1/\varepsilon)$$

and $0 < \varepsilon \ll 1$ sufficiently small, the Jacobian 2-form

$$Ju := 2du^1 \wedge du^2$$

(which coincides with both $\psi(u,\nabla)$ and $J(u,\nabla)$ when ∇ is the standard flat connection on the trivial bundle) is weakly close to (2π times) an integral $(n-2)$-boundary Γ of mass $\mathbb{M}(\Gamma) \leq \Lambda + o(1)$. The proof requires some delicate analysis: in particular, the mass $\|Ju\|_{L^1}$ of the Jacobians themselves is *not* bounded in general by the energy $\frac{G_\varepsilon(u)}{|\log\varepsilon|}$ for small ε, and the proof of the associated Γ-convergence result relies instead on a subtle application of the degree estimates of Sandier [114] and Jerrard [83] (see also [116, 120]).

In our setting, by contrast, the two-forms $J(u,\nabla)$ are easily seen to enjoy a pointwise bound

$$|J(u,\nabla)| \leq |\nabla u|^2 + (1-|u|^2)|F_\nabla| \leq |\nabla u|^2 + \varepsilon^2|F_\nabla|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \tag{1.11}$$

by the energy integrand $e_\varepsilon(u,\nabla)$, so that

$$\|J(u,\nabla)\|_{L^1(M)} \leq E_\varepsilon(u,\nabla) \tag{1.12}$$

automatically. As a consequence, to prove part (i) of Theorem 5, the only challenge lies in showing that the limiting $(n-2)$-cycle Γ is integer rectifiable (and lies in the correct homology class).

To achieve this, we establish a compactness result for sections $u_\varepsilon \in \Gamma(L)$ with $E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) = O(1)$, showing that they converge subsequentially (after change of gauge) to a singular unit section, whose topological singular set Γ coincides with the limit of $\frac{1}{2\pi}J(u_\varepsilon,\nabla_\varepsilon)$. These singular sets of unit sections (modulo the action of the gauge group) provide a natural codimension-two analog of Caccioppoli sets, and it is not difficult to see that their topological singular sets are integral $(n-2)$-cycles (indeed, this is a consequence of results in [3] and [85]). Again, we note that the broad outlines of the argument are very much reminiscent of those in [100] for the Allen–Cahn energies, with the bound (1.11) playing the role of the simple estimate

$$|\sqrt{2W(v)}\cdot dv| \leq \frac{\varepsilon}{2}|dv|^2 + \frac{W(v)}{\varepsilon}, \tag{1.13}$$

for real-valued functions $v : M \to \mathbb{R}$.

1.2.4 Applications to the study of minimizers and min-max constructions

As an immediate corollary of Theorem 5, we see that minimizers of E_ε converge to homologically area-minimizing $(n-2)$-cycles, answering a question raised in [107].

Corollary 7. *Let $L \to M$ be a nontrivial Hermitian line bundle over a closed, oriented n-manifold (M^n, g). If $(u_\varepsilon, \nabla_\varepsilon)$ minimize $E_\varepsilon(u, \nabla)$ among all pairs (u, ∇) on L, then*

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) = 2\pi \min\{\mathbb{M}(\Gamma) \mid \Gamma \in \mathscr{Z}_{n-2}(M;\mathbb{Z}) \text{ Poincaré dual to } c_1(L)\}, \tag{1.14}$$

and along a subsequence $\varepsilon = \varepsilon_j \to 0$, we have weak convergence

$$\lim_{\varepsilon\to 0} J(u_\varepsilon, \nabla_\varepsilon) = \lim_{\varepsilon\to 0} \omega_{\nabla_\varepsilon} = 2\pi\Gamma$$

of $J(u_\varepsilon, \nabla_\varepsilon)$ and the curvatures $\omega_{\nabla_\varepsilon}$ to an $(n-2)$-cycle Γ minimizing mass in the homology class dual to $c_1(L)$.

With Theorem 5 in place, the proof of the corollary follows standard lines: by part (i) of the theorem, we know that the forms $J(u_\varepsilon, \nabla_\varepsilon)$ for a minimizing family $(u_\varepsilon, \nabla_\varepsilon)$ converge subsequentially to an integral $(n-2)$-cycle Γ, in the correct homology class, of mass

$$\mathbb{M}(\Gamma) \leq \frac{1}{2\pi} \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon),$$

providing one inequality in (1.14). The opposite inequality follows from part (ii) of the theorem, which guarantees the existence of a recovery sequence $(u_\varepsilon, \nabla_\varepsilon)$ for a mass-minimizing cycle Γ. The convergence of the curvature two-forms $\omega_{\nabla_\varepsilon}$ follows from the discussion in Remark 6.

For the min-max applications, we will focus on the trivial bundle $L = \mathbb{C} \times M \to M$ over a given closed, oriented (M^n, g). We then consider a Banach space X consisting of pairs $(u, \nabla = d - i\alpha)$, equipped with an appropriate norm, with respect to which E_ε is a smooth functional satisfying a variant of the Palais–Smale condition (as in Section 5 of Chapter 2 below or Section 7 of [107]). Removing from X the degenerate set

$$X_0 := \{(u, \nabla) \in X : u \equiv 0\}$$

(on which $E_\varepsilon \sim 1/\varepsilon^2$ blows up as $\varepsilon \to 0$), we see that the action of the gauge group of maps $\mathscr{G} = \text{Maps}(M, S^1)$ given by

$$\phi \cdot (u, \nabla) := (\phi \cdot u, \nabla - i\phi^*(d\theta))$$

restricts to an action on the complement $X \setminus X_0$. Here $\phi \in \mathscr{G}$, meaning that $\phi : M \to S^1$, after identifying S^1 with the unit circle in $\mathbb{C}$. In particular, θ denotes the coodinate on the circle S^1.

For the purposes of intuition, we can view the gauge-invariant functionals E_ε as functions on the moduli space

$$\mathscr{M} := (X \setminus X_0)/\mathscr{G},$$

whose topology may be compared with that of the space

$$Z := \partial \mathbf{I}_{n-1}(M;\mathbb{Z}) \subseteq \mathscr{Z}_{n-2}(M;\mathbb{Z})$$

of integral $(n-2)$-boundaries in M, equipped with the flat metric. Indeed, we claim (see Chapter 2) that there are geometrically natural isomorphisms between the homotopy groups

$$\Phi : \pi_k(\mathscr{M},*) \to \pi_k(Z,0), \tag{1.15}$$

where $* \in \mathscr{M}$ denotes the collection of pairs $(u,\nabla) \in X$ with $|u| \equiv 1$ and $\nabla u = 0$, and $0 \in Z$ is the 0-cycle. Intuitively, one can think of this isomorphism as being induced by the zero locus map $(u,\nabla) \mapsto u^{-1}\{0\}$, but of course this will not define a continuous map into Z in practice. This isomorphism is nontrivial only when $k = 1$ or 2.

For $k = 1,2$, to any class $\alpha \in \pi_k(Z,0)$, one can associate a min-max width

$$\mathbf{W}(\alpha) := \inf_{\psi \in \alpha} \sup_{x \in S^k} \mathbb{M}(\psi(x)) \tag{1.16}$$

for the $(n-2)$-area functional. Note that in the above min-max expression we have $\psi\colon S^k \to Z$. In practice, we work with the (essentially equivalent) discretized variant $\mathbf{W}^*(\alpha)$ of these min-max widths introduced by Almgren and Pitts (see [109], or Section 5 below), which correspond to the masses of stationary $(n-2)$-varifolds. Likewise, for each nontrivial class $\beta \in \pi_k(\mathscr{M},*)$ and $\varepsilon > 0$, one can consider the min-max energies

$$\mathscr{E}_\varepsilon(\beta) := \inf_{F \in \beta} \max_{x \in S^k} E_\varepsilon(F(x)),$$

which are realized as critical values of the functionals E_ε. In practice, rather than working with families in $\pi_k(\mathscr{M},*)$, in Section 5 we work equivalently with the families $[0,1] \to X$ and $\bar{D}^2 \to X$ giving their lifts in the Banach space X. In rough terms, the results of Section 2.4 of Chapter 2 yield the following comparison.

Theorem 8 (Min-max comparison)**.** *Let $\mathscr{M}$ be the moduli space of pairs (u,∇) with $u \not\equiv 0$ and Z the space of integral $(n-2)$-boundaries as above. With respect to the aforementioned isomorphism $\Phi : \pi_k(\mathscr{M},*) \to \pi_k(Z,0)$, the min-max energies satisfy*

$$\liminf_{\varepsilon \to 0} \mathscr{E}_\varepsilon(\beta) \geq \mathbf{W}^*(\Phi(\beta)) \tag{1.17}$$

for any $\beta \in \pi_k(\mathscr{M},*)$. In particular, the mass of the stationary integral $(n-2)$-varifold V_{YMH} associated to the critical points $(u_\varepsilon,\nabla_\varepsilon)$ by the results of [107] is bounded below by the mass of the corresponding min-max $(n-2)$-varifold V_{GMT} produced by Almgren's min-max construction.

While we have restricted our attention here to the comparison of one- and two-parameters min-max constructions associated to the homotopy groups of $\mathscr{M}$ and Z, we believe that the techniques used in the proof of Theorem 8 should apply to all natural min-max constructions for the energies E_ε, with appropriate modifications to the topological part of the argument. In particular, while Theorem 8 can be compared to [66, Proposition 8.19] in the Allen–Cahn setting, we expect that the same ideas can be used to prove an analog of [64, Theorem 6.1] treating higher-parameter families detecting cohomology classes in $H^*(\mathscr{M};\mathbb{Z})$ of higher degree.

Moreover, let us point out that in the Allen–Cahn setting, Akashdeep Dey has recently succeeded in proving a bound in the opposite direction [51], concluding that the min-max energies for the Allen–Cahn functionals in fact *coincide* with the corresponding Almgren–Pitts widths in the $\varepsilon \to 0$ limit. Though establishing a codimension-two analog of Dey's bound for the self-dual Yang–Mills–Higgs functionals lies beyond the scope of the present dissertation, we optimistically conjecture that such an estimate should hold, giving equality in (2.20).

A key element in the proof of the min-max comparison theorem is the L^2 gradient flow associated to the Yang–Mills–Higgs energies: i.e., the following system of coupled nonlinear parabolic equations

$$\begin{cases} \partial_t u_t = -\nabla_t^* \nabla_t u_t + \frac{1}{2\varepsilon^2}(1-|u_t|^2)u_t, \\ \partial_t \alpha_t = -d^* d\alpha_t + \varepsilon^{-2}\langle iu_t, \nabla_t u_t\rangle, \end{cases} \tag{1.18}$$

subject to some initial data $(u_0, \nabla_0 = d - i\alpha_0)$. The necessity of its introduction is due to some technical difficulties emerging in the proof of Theorem 8 when passing from maps continuous in the flat norm, which are given by the Γ-convergence theory, to maps continuous in the mass norm, the relevant ones in the Almgren–Pitts setting. Indeed, the former can exhibit a phenomenon called *concentration of mass* whereby the energy density accumulates at small scales, preventing a direct application of the so-called *interpolation theory* developed by Marques, Neves and collaborators, which would give a corresponding continuous map in the mass norm.

Since we expect the gradient flow of E_ε to approximate a (weak) mean curvature flow of codimension two, a Huisken-type monotonicity formula should be expected to hold, thus providing the desired $(n-2)$-energy density bounds at all scales after running the flow for a fixed amount of time (uniformly in ε). This provides us with a canonical regularization preventing concentration of mass, without increasing the total energy. At the end of Chapter 2, in Section 2.5, we check that the flow satisfies long-time existence, uniqueness, and continuous dependence on the initial data.

1.3 Regularity Theories

Stationary points of geometric functionals and solutions of PDEs of geometric type exhibit in general a singular set, i.e., the set of points at which they are not smooth. For instance, a minimal surface might have a self-intersection or could be badly contorted and pinched at certain points. The main goal of this line of investigation is to understand the structure of the singular set. Given that singularities do occur, what do they look like in general? Moreover, are there natural hypotheses that prevent singularities? The ideal goal would then be to produce a regularity theory robust enough to be applicable to the largest class of geometric objects. Understanding the regularity of solutions is of the utmost importance as the presence of singularities is often the major stumbling block in their application to Geometry, Topology, and Physics. The type of problems fitting in this category is usually of local nature.

Attempting to summarise the regularity theory of minimal hypersurfaces, and weak generalizations thereof would be beyond the scope of this dissertation. However, we shall briefly mention some of the most foundational results in the field, explain more in details Simon's foundational work on cylindrical tangent cones in Subsection 1.3.1, and then move in Subsection 1.3.2 to the main topic of the second part of this dissertation: $\mathbb{Z}_2$-harmonic spinors. Chapter 3 will develop further this last notion and there the reader can find the proofs of the results presented below.

In 1972, Allard proved in [5] the following result: if in a given ball, a stationary integral varifold is close to a multiplicity one plane, then on a smaller ball, that varifold is given by the graph of a $C^{1,\alpha}$-function over the plane. Furthermore, this theorem comes with estimates on the $C^{1,\alpha}$-norm of the function. A consequence of this result is that the regular set is an open and dense subset of the support of the varifold. Standard elliptic regularity theory allows to upgrade $C^{1,\alpha}$-regularity to smoothness. The proof of Allard's regularity theorem is flexible enough to allow for the case of bounded distributional mean curvature and mean curvature in L^p for $p > n$. Besides, a consequence of Allard regularity is the fact that a point in $\operatorname{spt}\|V\|$ having a multiplicity one tangent plane as a tangent cone is necessarily regular. This regularity result is to date, one of the most general and powerful established in the context of stationary integral varifolds. Requiring further structure of the underlying weak minimal submanifold allows for stronger structural results, which brings us to the regularity theory of area-minimizing currents. Compared to the varifold setting it is possible to bound the size of the singular set and obtain further structural results. More precisely, the result is the following. Given a locally area-minimizing n-dimensional integral current T of codimension one in the unit ball, we have the bound $dim_{\mathscr{H}}(\mathrm{Sing}(T)) \leq n-7$. Furthermore, thanks to the work of Bombieri, De Giorgi and Giusti [21], we know that the cone over $S^3 \times S^3 \subset S^7$, by now referred to as Simons' cone, given by

$$\mathbf{C}^{3,3} = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4;\ |x| = |y|\}$$

is a 7-dimensional area-minimizing hypercones with an isolated singularity at the origin, thus yielding sharpness of the regularity result of area-minimizing currents in codimension one. Much more can be said in this setting, for instance, the singular set is countably $(n-7)$-rectifiable, and in the case of seven-dimensional hypersurfaces the singular points are isolated. This theory was one of the crowning achievements of modern analysis and geometry, and it is due to the combined efforts of Federer [59], Fleming [61], Almgren [7], De Giorgi [35–37], Simons [126], Simon [123], and, as in the example, Bombieri and Giusti [21]. See also work of Reifenberg [110], and Triscari [142]. For further details on this result and a more detailed account of it, we refer the reader to the surveys of De Lellis [39] and Wickramasekera [145].

The natural generalization to the higher codimension setting proved challenging. We have seen that this discrepancy between the codimension one case and the higher codimension counterpart permeates the thesis, first with the analysis of the variational theory of the Yang-Mills-Higgs energies, more involved than Modica and Mortola's analysis and later in the subsequent chapter on the regularity theory of $\mathbb{Z}_2$-harmonic spinors, which effectively are higher codimension multi-valued maps. As far as area-minimizing currents of higher codimension are concerned, Almgren in [9] published posthumously, proved that the singular set is of Hausdorff codimension two, except in the case of area-minimizing curves in which case the singular set is empty. Works of Chang [27] refined Almgren's result in the two-dimensional setting proving that singular points are isolated. We refer the reader to [46, 45, 48, 49, 127] for a further generalization of Almgren and Chang's results in two dimensions for semi-calibrated currents. These works build on [40, 43, 41, 42, 44] in which the authors revisit and streamline Almgren's original proof. The sharpness of this theory is guaranteed by so-called irreducible complex analytic varieties, e.g.

$$\Gamma := \{(z,w) \in \mathbb{C} \times \mathbb{C};\ z^2 = w^3\}.$$

In particular, Γ is a 2-dimensional area-minimizing current with an isolated singularity of branch type, or simply a branch point, at the origin. More precisely, the tangent cone at the origin is given by the plane $z = 0$ with multiplicity two, compare with Allard's regularity theorem on multiplicity one tangent cones. We shall discuss more in-depth Almgren's technique later in this introduction when outlining the proof of our results on $\mathbb{Z}_2$-harmonic spinors. Indeed, the strategy is similar to the one of Simon [122], which in turn borrows elements from [9].

Another major advancement in the field was the regularity theory brought forward by Schoen and Simon and later sharpened by Wickramasekera for stationary, integral, n-dimensional varifolds V with stable regular part. This last condition requires the second variation of the area to be positive definite on the regular set of the varifold. Besides, it readily rules out pathological cases of catenoidal type. As a consequence of this theory, we know that the support of V as above is an optimal regular minimal hypersurface, meaning that the singular set of V is of codimension 7 as in the case of area-minimizing currents. Instrumental to the proofs in both cases are sheeting theorems implying that if V has sufficiently small height excess, i.e. it is sufficiently close to a plane, then the support of V splits into the graph of k different $C^{1,\alpha}$-functions. Overall, the key difference between Schoen and Simon's proof and Wickramasekera's one was that the former required a priori the singular set to be small, whereas the latter applies to varifolds V with $\mathscr{H}^{n-1}(\mathrm{Sing}(V)) = 0$, or equivalently, with no classical singularities. As already pointed out previously, Wickramasekera's regularity theory was instrumental to establish the existence of optimally regular minimal hypersurfaces in Riemannian manifolds.

We conclude this brief survey by pointing out that the above developments were not presented in chronological order. We decided to follow a linear exposition to facilitate the reader's understanding.

1.3.1 The Work of Simon on Cylindrical Tangent Cones

To understand the results on $\mathbb{Z}_2$-harmonic spinors of Chapter 3, it is instructive to look at the foundational work of Leon Simon [122] on the rectifiability of the singular set in the context of multiplicity one class $\mathscr{M}$. Elements belonging to $\mathscr{M}$ are n-dimensional minimal submanifolds that are stationary with respect to the n-dimensional area functional. For the sake of simplicity, each element $M \in \mathscr{M}$ is assumed to lie in an open set $U_M \subset \mathbb{R}^{n+m}$. Further, assume that the class $\mathscr{M}$ is closed under homotheties, ambient rigid motions, and, perhaps even more crucially, under varifold convergence, meaning that measure theoretic limits of elements of $\mathscr{M}$ are still inside the class. From this hypothesis, and as the name suggest, the appearance of branch point singularities, i.e. singular points admitting a higher multiplicity tangent plane, is ruled out a priori. This requirement simplifies the analysis, allowing to deduce fine properties of the singular sets of elements of $\mathscr{M}$.

Note that the existence of non-trivial, singular tangent cones at singular points of minimal submanifolds in this multiplicity one class follows from the classical monotonicity formula, combined with a lower bound on the density. With this in hand, one of the main ingredients of Simon's proof is an excess decay improvement lemma that can be broadly summarized as follow. Consider a cone of the form $\mathbf{C}^{(0)} = \mathbf{C}_0^{(0)} \times \mathbb{R}^k$ in $\mathscr{M}$. Cones of this type are usually referred to as *cylindrical*. Notice that k is given by the maximal dimension of the subspace along which every cone of $\mathscr{M}$ is translation invariant. In particular, $k \in \{0,1,2,\dots,n-1\}$. An additional *integrability hypothesis* has to be satisfied in case $k \neq n-3$. In the cases $k = n-1$ and $k = n-2$ we automatically have integrability because of the special structure of the corresponding cross sections of the cones in the corresponding dimensions. Assume that $M \in \mathscr{M}$ and $\mathbf{C}$ are sufficiently close in L^2 distance depending on $\mathbf{C}^{(0)}$ and provided the masses (areas) are sufficiently close to the one of $\mathbf{C}^{(0)}$ at scale one. Then, Lemma 1 of [122] guarantees that, under the hypothesis that the minimal submanifold M we started with has enough singularities of density at least the density of $\mathbf{C}$ at the origin, there is a new cone $\mathbf{C}^{(1)}$ and a scaled $\theta \in (0,1/2)$ fixed, such that the L^2 distance between M and this newly found $\mathbf{C}^{(1)}$ has improved by at least a half. The dichotomy lies in the fact that either there are not enough singularities as in the previous paragraph, or the excess improvement holds.

Once this result is established, the main results of [122] follow by iteration of this lemma, considering at each step the alternatives that it gives. For clarity, we shall summarize below the main steps of the proof of the lemma as appearing in [122]. Consider two sequences $\{M_j\}_j \subset \mathscr{M}$

and $\{\mathbf{C}_j\}$, both converging to $\mathbf{C}^{(0)}$. Away from a tubular neighborhood of the axis, sometimes referred to as *spine*, of $\mathbf{C}^{(0)}$ we have that the height of M_j of $\mathbf{C}_j$ is given by solutions of the minimal surface system u_j. This follows from Allard regularity theorem, which can be invoked by virtue of the multiplicity one hypothesis and provided the height of M_j with respect to $\mathbf{C}_j$ is small enough. Solving the minimal surface system implies that the u_j satisfy estimates away from the spine, which can be used to blow-up the sequence u_j by

$$E_j = \sqrt{\int_{M_j \cap B_1} \operatorname{dist}^2(X, \mathbf{C}_j)\, d\mathscr{H}^n}.$$

Thus, we have existence of v such that $u_j/E_j \to v$ in $L^2_{\mathrm{loc}}(\mathbf{C}^{(0)} \cap B_1)$, where $v\colon \operatorname{reg}(\mathbf{C}^{(0)}) \cap B_1(0) \to (\operatorname{reg}(\mathbf{C}^{(0)}))^{\perp}$. The fact that convergence is in $L^2_{\mathrm{loc}}(\mathbf{C}^{(0)} \cap B_1)$ follows from several key integral estimates established in the central part of [122]. Among them, the most important ones are

$$\int_{B_\gamma(0)\cap U} R^{2-n} \left(\frac{\partial (u/R)}{\partial R} \right)^2 \leq C \int_{M \cap B_1(0)} \operatorname{dist}(X, \mathbf{C})^2,$$

together with one implying that the excess E_j does not concentrate near the axis of $\mathbf{C}_j$. Using Fourier analysis, Simon then proceeds to prove that homogeneous degree 1 blow-ups are of cylindrical type, and that a blow-up converges to a unique one such homogeneous degree one 1 object upon rescaling. With this result and the excess non-concentration estimate, the result follows.

1.3.2 $\mathbb{Z}_2$-Harmonic Spinors and Uniqueness of Blow-ups

One of the great realizations made in differential geometry roughly forty years ago is that solutions to gauge theoretic equations provided a new set of differential topological invariants of manifolds. This picture was advanced thanks to the contributions of Donaldson, Uhlenbeck, Taubes, and many others. By now, it is a classical theory with ongoing ramifications and it was the driving force for spectacular advancements in the field. As is often the case, many of its features were first introduced in physics and then emerged later as interesting and challenging mathematical problems.

One of the most studied and familiar gauge theories is Yang-Mills theory, whose physical objects of interest are connections A that are critical for the action $A \mapsto \int_M |F_A|^2$, where F_A is the curvature associated to A. The great insight of this theory was that topological features of the moduli space these Yang-Mills connections that were additionally assumed to be anti-self-dual, modulo gauge group, are invariants for the underlying manifold. This is the case when the manifold is sufficiently low dimensional, even though there are recent developments in higher dimensions, see [58, 56]. Thus, extracting numerical information from this geometric object can be useful to understand the manifold under consideration. Note that a priori the moduli space depends on the metric and the smooth structure. We refer the reader to [57] and references therein for further details on this topic. Other examples of well-studied gauge theories are Seiberg-Witten equations on manifolds of dimension 4, and the monopole equations, or Bogomolny equations, on $\mathbb{R}^3$.

A delicate point one has to take into account when dealing with gauge theories is the possible non-compactness of the gauge group, usually more difficult to handle compared to the compact counterpart. The setting of the present work belongs to the non-compact setting. A prototypical case exhibiting many interesting features is provided by the Hitchin equations, see [75], which could be understood as the two-dimensional reductions of the anti self-dual Yang-Mills equations.

Consider Σ a Riemann surface with $E \to \Sigma$ a rank two complex vector bundle with a Hermitian metric. The field of interest in this case is (A, ϕ), where A is a $\mathrm{SU}(2)$-connection and $\phi = \varphi dz$ is a Higgs field, where φ is a trace-free 2×2 complex matrix. The Hitchin equations are then given by

$$F_A + [\phi \wedge \phi^*] = 0 \qquad \text{and} \qquad \overline{\partial}_A \phi = 0, \tag{1.19}$$

where we note that F_A is automatically trace-free as A is a $\mathrm{SU}(2)$-connection. In other words, we are requiring the Higgs field to be holomorphic and the deviation of the connection from being flat, meaning with vanishing curvature, to be measured by the commutator $[\phi \wedge \phi^*]$. One can equivalently understand solutions of the Hitchin equations as (non-unitary) flat connections, which in turn correspond to monodromy representations. This point of view is better adapted to our purposes. Indeed, using this correspondence one can deduce that the Hitchin moduli space is equivalent to the non-compact character variety $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$, and admits a Hitchin fibration. We are not going to attempt to survey this rich theory here and refer the reader to [62] for a thorough treatment. To investigate solutions of (1.19) one might as well work directly with the character variety or, equivalently, the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections modulo complex gauge equivalence. If (A, ϕ) is a solution to (1.19), then the complex connection $\mathscr{A} = A + \phi + \phi^*$ that is additionally assumed flat satisfies

$$F_{\mathscr{A}} = d\mathscr{A} + [\mathscr{A}, \mathscr{A}] = F_A + [\phi \wedge \phi^*] + \overline{\partial}_A \phi + \partial_A \phi^* = 0.$$

In proving a generalisation of Uhlenbeck compactness theorem for unitary connections with L^2-curvature bounds, Taubes established in [135] the following remarkable result about sequences $\mathscr{A}_j$ of such connections: provided we have a uniform bound $\|\phi_j\|_{L^2} \leq C$, we can extract a converging subsequence. As an interesting converse to this statement, assuming the $\|\phi_j\|_{L^2}$ diverge, the following set of equations are satisfied

$$\frac{1}{\|\phi_j\|^2_{L^2}} F_{A_j} - [\psi_j, \psi_j] = 0 \qquad \text{and} \qquad d_{A_j} \psi_j = d_{A_j} * \psi_j = 0.$$

where $\psi_j = \phi_j / \|\phi_j\|_{L^2}$. At this point one can still take a subsequential limit (A, ψ) that satisfies the decoupled equations $F_A = 0$, and $d_A \psi = d_A * \psi = 0$ with $[\psi, \psi] = 0$, on the complement of a negligible subset $\mathscr{Z} \subset M$. Solutions to these limiting equations are $\psi = \omega \otimes \sigma$, for a harmonic 1-form ω and a parallel section σ. It is crucial to notice here that ω and σ are defined only up to a sign, as both $\pm\sigma$ and $\pm\omega$ are valid solutions, thus exhibiting multivaluedness. Besides, $\mathscr{Z}$ is the branching set, where σ and ω change sign. In two dimensions this set corresponds to the zero set of a holomorphic quadratic differential. The interested reader may consult [98] for a two-dimensional, and consequently more refined, version of this decoupling result.

Motivated by this, Taubes introduced the notion of $\mathbb{Z}_2$-harmonic spinor in his pioneering work [137] to abstract the phenomenon presented above and to deal with similar limits arising in the compactification of the moduli space of flat $\mathrm{PSL}_2(\mathbb{C})$-connections over 3-manifolds, see [135]. This notion also appears when dealing with the moduli spaces of solutions of the Kapustin-Witten equations [136], the Vafa-Witten equations [139], and the Seiberg-Witten equations with multiple spinors [72, 138]. In all of these cases crucial is the role played by the zero locus $\mathscr{Z}$ of the corresponding $\mathbb{Z}_2$-harmonic spinor. Before giving the precise definition of these objects, we clarify the setting we are working with and set the notation. We refer the reader to Section 3.1 of Chapter 3. Let (M, g) be a 4-dimensional Riemannian manifold, with no boundary and not necessarily compact, endowed with a Clifford bundle $\mathscr{V}$ over it, i.e. a bundle equipped with a Clifford structure. More precisely, $\mathscr{V}$ is a unitary vector bundle equipped with an extra structure called the *Clifford multiplication*, $\rho \in \mathrm{Hom}(TM, \mathrm{Hom}(\mathscr{V}, \mathscr{V}))$, such that $\rho(e)^2 = -\|e\|^2 \cdot \mathrm{Id}$ and

$\|\rho(e)(u)\| = \|e\| \cdot \|u\|$ for every tangent vector $e \in T_pM$ and $u \in \mathscr{V}_p$. Let ∇ be a connection on $\mathscr{V}$ compatible with ρ, meaning that for every pair of vector fields e, e', and every smooth section u of $\mathscr{V}$, we have $\nabla_e(\rho(e') \cdot u) = \rho(\nabla_e e') \cdot u + \rho(e') \cdot \nabla_e u$. One can then define the Dirac operator on the Clifford bundle $\mathscr{V}$ to be

$$\mathscr{D}(u) = \sum_{i=1}^{4} \rho(e_i) \nabla_{e_i} u,$$

for a local othornormal frame $\{e_i\}$ of TM. We refer the reader to Section 3.1 of Chapter 3 for a more detailed discussion on conditions ensuring the existence of a Dirac operator, and for further properties on this differential operator, e.g. its spectral theory, regularity, unique continuation properties etc. For the present introduction we will not need the full extent of these properties. Any section that annihilates such operator is called a solution of the Dirac equation. We can now give the precise definition of $\mathbb{Z}_2$-harmonic spinor we are going to work with.

Definition 9. Let U be a continuous 2-valued section of $\mathscr{V}$. Then, U is called a $\mathbb{Z}_2$-harmonic spinor if the following conditions hold.

1. The section U is not identically $\{0,0\}$.
2. Let $\mathscr{Z}_U$ be the set of U where $U = \{0,0\}$. For every $p \in M \setminus \mathscr{Z}_U$, there exists a neighborhood of p such that on this neighborhood u can be written as $U = \{u, -u\}$, where u is a smooth section of $\mathscr{V}$ satisfying $\mathscr{D}(u) = 0$.
3. Near a point $p \in M \setminus \mathscr{Z}_U$, write U as $\{u, -u\}$, then the function $|\nabla u|$ is a well defined smooth function on $M \setminus \mathscr{Z}_U$. Moreover, the section U satisfies

$$\int_{M \setminus \mathscr{Z}_U} |\nabla u|^2 < +\infty.$$

We will additionally assume for our results that the $\mathbb{Z}_2$-harmonic spinor U is C^1 as a two-valued section and, as Taubes, we request U to satisfy (3.1).

Remark 10. Note that the definition just given is slightly different from the one given by Taubes in [137]. See [146] for more details on the equivalence between these two definitions and [53] as well for the definition of $\mathbb{Z}_2$-harmonic spinors.

The set $\mathscr{Z}$ can be interpreted as the branching set of a holomorphic multi-valued function and it is the analog of zeroes of holomorphic quadratic differential on Riemann surfaces. Analytically, $\mathscr{Z}$ shares similar features with the singular set of various geometric partial differential equations and with the free boundary in certain variational problems. As such, in monumental work [137], Taubes investigated the size of $\mathscr{Z}$ and managed to prove that it has codimension at least two with respect to the Hausdorff dimension. This aligns with similar results on the singular set in other scenarios (e.g. stable minimal hypersurfaces, Yang-Mills connections, solutions to semilinear elliptic equations, and harmonic maps to name just a few). Taubes exploited ideas and techniques from geometric measure theory, with crucial roles played by the frequency function and its (almost) monotonicity, as well as the study of blow-ups.

Zhang later improved this regularity result by establishing $(n-2)$-rectifiability of $\mathscr{Z}$, viz. $\mathscr{Z}$ can be covered by countably many Lipschitz submanifolds up to a measure zero set, and local Minkowski bounds. To do so he exploited far-reaching techniques recently introduced by Naber and Valtorta in [102] to study the singular sets of various geometric partial differential equations and variational problems. See also [103] for an application to stationary varifolds and minimal currents. These ideas stemmed from earlier works of Cheeger and Naber on the quantitative stratification [30, 29], a finer decomposition of the singular set that allows for stronger results compared to the usual Federer's stratification. To prove his main result, Zhang applied the techniques of Naber and Valtorta, especially the ones that emerged from the work of De Lellis, Marchese, Spadaro, and Valtorta [47] on the $(n-2)$-rectifiability of multi-valued Dirichlet energy minimizing functions. More precisely, a key role is played by the following excess quantity

$$D^2_\mu(x,r) := \inf_L \frac{1}{r^4}\int_{B_r(x)} \operatorname{dist}(y,L)^2\, d\mu(y) \tag{1.20}$$

for a certain Radon measure μ, where the infimum is taken over the set of 2-dimensional affine subspaces. Subsequently, Zhang established a distortion estimate for D^2_μ, meaning a bound of the form

$$D^2_\mu(x,r) \leq \frac{C}{r^4}\int_{B_r(x)} \left(W^{4r}_{r/4}(z) + Cr^2\right) d\mu(z),$$

where W^r_s is the difference between the frequency at scale r and at scale s. The main result then follows from the Rectifiable-Reifenberg theorem established by Naber and Valtorta assuring that under L^2-bounds for the excess (1.20) with the discrete measure $\mu = \sum_i r_i^2 \delta_{x_i}$, for $\{B_{r_i}(x_i)\}_i$ a collection of subballs of a larger $B_{2R}(0)$, one has the measure bound $\mu(B_R(0)) \leq CR^2$. Rectifiability then follows. The surprising aspect of the techniques of Naber and Valtorta is that, under remarkably general hypothesis, they give strong results. Moreover, this method does not seem to require any knowledge of the asymptotic behaviour of the objects near their singularities or uniqueness of their tangents.

In light of this discussion, it seems natural to conjecture that more can be said about $\mathscr{Z}$ and this is precisely where this work arises. We are interested in establishing further regularity and structural properties on the singular set $\mathscr{Z}$. We start with the question of uniqueness of blow-ups left open from the work of Taubes. We answer it positively under the extra assumption that the spinor is C^1-regular in the sense described in Section 1.1. We refer the reader to Section 3.1 of Chapter 3 for further details.

Theorem 11 (Uniqueness of blow ups)**.** *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g). Then, for $\mathscr{H}^2$-a.e. point Z in the branch set $\mathscr{B}_U$ of U, there exist an orthogonal rotation $q_Z \in \mathrm{SO}(4)$, a positive integer $k_Z \in \mathbb{N}$, and a positive real number $r_Z > 0$ such that for every $X \in B_{r_Z}(Z)$, we have*

$$U(Z + q_Z X) = \{f(X) + h_Z(X), -f(X) - h_Z(X)\}, \tag{1.21}$$

where the error term $h_Z(X)$ satisfies the bound

$$\frac{1}{\sigma^4}\int_{B_\sigma(Z)} |h_Z|^2 \leq C_Z \sigma^{k_Z+\gamma_Z},$$

for all $\sigma \in (0, r_Z)$ and constants $C_Z, \gamma_Z > 0$ independent of σ and where f is given by $\{\pm(c_1 z^\alpha + c_2 \bar{z}^\alpha)\}$, for constants c_1 and c_2.

A proof of this theorem will be given in Section 3.8 of Chapter 3. Note that f in the above has to be homogeneous of degree $k_Z/2$. This result can be considered a first-order asymptotic expansion of U near a point $Z \in \mathscr{B}_U$ and, broadly speaking, this work can be seen as aimed at studying the behavior of U on approach to its singularities. From this theorem, rectifiability of the singular set follows, thus recovering Zhang's result [146].

Theorem 12 (Rectifiability of the singular set)**.** *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g). Then Σ_U is countably 2-rectifiable. Furthermore, for every compact set K we have*

$$\Sigma_U \cap K \cap \{X;\ \mathscr{N}(X;V) = \alpha \text{ and } V \text{ has a cylindrical blow up at } X\} \neq \emptyset$$

only if α takes one of finitely many values belonging to the set $\mathbb{N} \cup \frac{1}{2}\mathbb{N}$. For each $\alpha \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$, there is an open set O_α such that

$$\{X;\ \mathscr{N}(X;V) = \alpha \text{ and } V \text{ has a cylindrical blow up at } X\} \subset O_\alpha,$$

and $O_\alpha \cap \{X;\ \mathscr{N}(X;V) \geq \alpha\}$ has locally finite $\mathscr{H}^2$-measure.

A proof of this theorem will be given in Section 3.8 of Chapter 3. An immediate corollary of this theorem is the rectifiability of the branch set, together with the dichotomy of either empty or positive $\mathscr{H}^2$-Hausdorff measure for an even more regular spinor.

Corollary 13 (Rectifiability of the branch set)**.** *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional manifold Riemannian (M,g). Let B_U be the branch set of U. Then, for each closed ball $B \subset M$, either $B \cap \mathscr{B}_U$ is empty or $B \cap \mathscr{B}_U$ has positive 2-dimensional Hausdorff measure and is equal to the union of a finite number of pairwise disjoint, locally compact sets each of which is locally 2-rectifiable. In particular, it has locally finite 2-dimensional Hausdorff measure. Moreover, requiring U to be of class $C^{1,\mu}$, for a certain μ, we conclude that either $\mathscr{B}_U = \emptyset$ or the Hausdorff dimension of $\mathscr{B}_U$ is 2 and the 2-dimensional Hausdorff measure of $\mathscr{B}_U$ is positive.*

A proof of this corollary will be given in Section 3.8 of Chapter 3. The next two results deal with the limiting case of minimal degree of homogeneity. This situation is more delicate and requires a more thorough analysis. However, we can prove stronger structural results and obtain that in the neighborhood of any such singularity, part of $\mathscr{Z}_U$ looks like a $C^{1,\alpha}$-submanifold.

Theorem 14. *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g). If $\mathscr{N}(Z;U) = 3/2$, then $Z \in \mathscr{B}_U$ and there exists $r > 0$ such that $\mathscr{B}_U \cap B_r(Z)$ is an 2-dimensional $C^{1,\alpha}$-submanifold of $B_r(Z)$, for some $\alpha \in (0,1)$.*

A proof of this theorem will be given in Section 3.8 of Chapter 3. As a corollary, we have the following strengthening of Theorem 11.

Corollary 15. *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g) and $k_{Z_0} = 1$ for some point $Z_0 \in \mathscr{B}_U$ at which the asymptotic expansion (1.21) is valid. Then, the same expansion is valid for every $Z \in \mathscr{B}_U \cap B_{r_{Z_0}/2}(Z_0)$, where r_{Z_0} is as in Theorem 11, with the parameters*

$$k_Z = k_{Z_0}, \quad C_Z = C_{Z_0}, \quad \gamma_Z = \gamma_{Z_0}, \quad \text{and} \quad r_z = r_{Z_0}/4.$$

Furthermore, the error term in the expansion (1.21) satisfies

$$\sup_{B_\sigma(Z_0)} |h_Z|^2 \leq C_0 \sigma^{k_{Z_0} + \gamma_{Z_0}/8},$$

for every $Z \in \mathscr{B}_U \cap B_{r_{Z_0}/2}(Z_0)$ and $\sigma \in (0, r_{Z_0}/4)$ and where the constant C_0 is independent of Z and σ. Moreover, we have that $\mathscr{B}_U \cap B_{r_{Z_0}/2}(Z_0)$ is a 2-dimensional $C^{1,\alpha}$-submanifold for some $\alpha = \alpha_{Z_0} \in (0,1)$.

A proof of this theorem will be given in Section 3.8 of Chapter 3. Thanks to the above results we can answer positively a question of Taubes, see [137, Page 6], provided that the $\mathbb{Z}_2$-harmonic spinors under consideration are C^1-regular.

Corollary 16. *Uniqueness of blow-ups of, C^1-regular, $\mathbb{Z}_2$-harmonic spinors holds up to an $\mathscr{H}^2$-measure zero set.*

The above theorems and corollaries hold on 3-manifolds as well. The proof of this last corollary follows from Theorem 11. Indeed, outside of the branch set $\mathscr{B}_U$ we have a local decomposition, whence the uniqueness of the blow-ups there. Thus, the only difficulty lies in dealing with $\mathscr{B}_U$, which is precisely the role of Theorem 11. Uniqueness was previously known by Taubes for the open and dense set of strongly continuous points of the frequency function. Despite this, we had little information about the actual (measure-theoretic) size of the set of points with a unique blow-up, and the complement of such set could have arbitrarily large measure.

A fruitful, and related, line of investigation is concerned with the following question: what is the dependence on the metric g? More precisely, if (M,g) supports a $\mathbb{Z}_2$-harmonic spinor with branching set $\mathscr{Z}$, then do nearby metrics, in some reasonable topology, support $\mathbb{Z}_2$-harmonic spinors as well? Furthermore, can we infer stronger regularity results generically in g? In the three-dimensional case, Takahashi proved that there is a Fredholm deformation theory using the Nash-Moser implicit function theorem. Moreover, he computed the index of the deformation to be -1, meaning that $\mathbb{Z}_2$-harmonic spinors are expected to exist on 3-manifolds only for metrics g belonging to a codimension one submanifold in the space of metrics. Crucial to Takahashi's proof was the assumption that $\mathscr{Z}$ is given by a smooth curve. We refer the reader to [131, 132] for further details. This type of result is related to wall-crossing. Consider the space of Riemannian metrics g over a certain manifold M. Away from those metrics admitting (singular) $\mathbb{Z}_2$-harmonic spinors one can proceed as in the more classical gauge theories and find topological invariants by counting. The numerical information extracted will stay (locally) constant upon corresponding changes in the metric. However, problems might arise as one crosses the codimension one submanifolds of metrics admitting (singular) $\mathbb{Z}_2$-harmonic spinors, from which the name wall-crossing. It seems hopeful to expect these objects to help our understanding of the change of invariants as this phenomenon occurs. We refer the reader to the work of Doan and Walpuski [53] for new results in this direction, and a more careful explanation of the wall-crossing phenomenon. See also work of He [73].

Remark 17. Note in this setting the following analytical analogy. If we have a $\mathbb{Z}_2$-harmonic spinor, then $M \setminus \mathscr{Z}$ supports a solution of an elliptic equation with vanishing Cauchy data. This is reminiscent of Serrin's problem of finding domains $\Omega \subset \mathbb{R}^n$ admitting solution u to the problem $\Delta u = 1$ in Ω, and $u = 0$ as well as $\partial_\nu u = 0$ on the boundary $\partial\Omega$. This is an overdetermined problem and nontrivial solutions are rare. See also [55] for a similar comparison.

1.3.3 Outline of the proof

We outline here the proofs of the main theorems just presented. The actual proofs can be found in Chapter 3, Section 3.8. This work is heavily inspired by Simon's pioneering works [122] on the rectifiability of the singular set of minimal submanifolds belonging to certain compact, multiplicity one classes, in which occurrence of higher multiplicity is ruled out a priori. We also rely on more recent contributions on two-valued, and more generally multi-valued, Dirichlet energy minimizing, $C^{1,\mu}$-harmonic maps and minimal submanifolds, by the same author with Wickramasekera and by this last one with Krummel, cf. [90, 89, 125, 91]. These results were in turn influenced by Almgren's fundamental work [9] on the interior regularity of area minimizing rectifiable currents of codimension higher than one, objects in which the occurrence of branch point singularities, where tangent cones are higher multiplicity planes, is not ruled out a priori. To attack this problem Almgren introduced Dirichlet energy minimizing functions and proved that they have a singular set of Hausdorff codimension two, outside of which they are given by Q single-valued harmonic functions, no two of which have common values unless they are identical. With this regularity result in hand, the second part of [9] was then devoted to the proof, via an approximating procedure, that area-minimizing rectifiable currents of codimension higher than one have a codimension two singular set. Taubes carried out the corresponding result for the setting of $\mathbb{Z}_2$-harmonic spinors that we are interested in. Thus, laying the foundation for a more systematic study of the fine properties of $\mathscr{Z}_U$ and its subsets.

One of the (many) great contributions of Almgren, also exploited by Taubes, was introducing the *frequency function*

$$N(Z,r;u) := \frac{rD(Z,r)}{H(Z,r)} = \frac{r\int_{B_r(Z)} |Du|^2}{\int_{\partial B_r(Z)} |u|^2}$$

and showing that it is monotone non-decreasing in r, thus proving that

$$\mathscr{N}(Z;u) = \lim_{r\to 0} N(Z,r;u)$$

exists. Besides, if u vanishes at Z, then any sequence of rescalings $u(Z+r_jX)/\|u(Z+r_j\cdot)\|_{L^2(B_1(0))}$ with $r_j \to 0$, converges subsequentially to a non-trivial *tangent function* $\varphi^{(Z)}$, also referred as *blow-up*, homogeneous of degree $\alpha = \mathscr{N}(0;\varphi^{(Z)}) = \mathscr{N}(Z;u)$. Note that this follows broadly the same strategy for minimal surfaces submanifolds, for which one can use the standard monotonicity formula, from which existence of non-trivial, singular, tangent cones at every singular point is inferred. Besides, in our setting the frequency plays effectively the role played by the density for minimal submanifolds.

This is essentially where our proof starts. We exploit the (almost) monotonicity of frequency for $\mathbb{Z}_2$-harmonic spinors to infer the the following identity for the radial derivative of the Weiss functional

$$\frac{d}{dr}\left(\frac{1}{r^{2\alpha}}\left(r^{2-n}D(x,r) - \alpha r^{1-n}H(x,r)\right)\right) = 2r^{2-n}\int_{\partial B_r(x)} \left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2 + \mathscr{O}(r^{2-n-2\alpha}H(x,r)),$$

where $D(x,r)$ is the Dirichlet energy and $H(x,r)$ is the height. See Section 3.2 for notation. Note that α in this setting can take integers and half-integer values, and corresponds to the multiplicity, in constrast with Simon's multiplicity one setting.

The next step is to establish a priori L^2-type estimates, whose main consequence is a non-concentration excess result, Theorem 91. En route to establishing this result we prove several corollaries used in the proof of the main theorems. Crucial to all these results are (appropriately modified versions of) the squash

$$\int_{B_r(p)} |Du|^2 = \int_{\partial B_r(p)} u \cdot D_r v, \tag{1.22}$$

and the squeeze

$$(n-2)\int_{B_r(p)} |Du|^2 = r\int_{\partial B_r(p)} |Du|^2 - 2r\int_{\partial B_r(x)} |D_r u|^2 \tag{1.23}$$

identities. Note that in the setting of multi-valued Dirichlet minimizing functions these are consequences of the more general

$$\int_\Omega |Du|^2\zeta = -\int_\Omega u^k D_i u^k D_i \zeta \tag{1.24}$$

and

$$\int_{\Omega}\left(\frac{1}{2}|Du|^2\delta_{ij} - D_i u^k D_j u^k\right) D_i\zeta^j = 0, \tag{1.25}$$

where $\zeta, \zeta^1, \dots, \zeta^n \in C^1_c(\Omega, \mathbb{R})$ and where $u^k = u \cdot e_k$ is the two-valued k-th coordinate of u, i.e. $u^k(X) = \{u_1(X)\cdot e_k, u_2(X)\cdot e_k\}$, where $\{e_1, e_2, \dots, e_m\}$ denotes the standard basis of $\mathbb{R}^m$ and where we assumed to be in Euclidean space to simplify the exposition. Note that (1.22) and (1.23) underpin the monotonicity of the frequency function. Indeed, a simple computation shows

$$\frac{d}{dr}\left(\frac{r\int_{B_r(Z)}|Du|^2}{\int_{\partial B_r(Z)}|u|^2}\right) = \frac{r}{\int_{\partial B_r(Z)}|u|^2}\left(\left(\int_{\partial B_r(Z)}|D_r u|^2\right)\left(\int_{\partial B_r(Z)}|u|^2\right) - \int_{B_r(Z)} u\cdot D_r u\right) \geq 0,$$

where the inequality follows from Cauchy-Schwarz.

With these L^2-type estimates in hand, we can establish another key aspect of the proof: an excess improvement lemma. Broadly speaking, it says that whenever φ is a tangent function sufficiently close to a $\mathbb{Z}_2$-harmonic spinor in L^2-distance at scale 1, depending on a fixed $\varphi^{(0)}$, and U has enough singularities with frequency at least the frequency of φ at the origin, there is a new tangent function $\varphi^{(1)}$ and a fixed smaller scale $\theta \in (0, 1/2)$ such that the L^2-distance between U and $\varphi^{(1)}$ at scale θ has improved, meaning that it is at most half of the L^2 distance between U and φ at scale 1. The dichotomy lies in the fact that either there are not enough singularities as in the previous paragraph, or the excess improvement holds. Once this result is established, the proof of the main theorems and corollaries follow by iteratively applying it, considering at each stage the two alternatives it gives.

The techniques employed in this work have the potential to be applied to other settings, including multi-valued ones. Furthermore, adapting the work of Krummel [88] exploiting partial Legendre-type transformations based on those of Kinderlehrer, Nirenberg, and Spruck [86], should imply further regularity properties, analyticity in fact, on a subset of $\mathscr{B}_U$. We hope to take up this line of investigation in future work.

We conclude by mentioning other fruitful interactions between geometric measure theory and gauge theory, of which this work is an example. Results and techniques from the former have found recent success in tackling problems in the latter. For instance, Doan, Ionel, and Walpuski proved in [54] the finiteness case of the Gopakumar–Vafa conjecture exploiting a landmark result in geometric measure theory: Allard's regularity [5]. The regularity theory of semi-calibrated currents played a role in Doan and Walpuski's proof that k-rigidity implies a Castelnuovo bound [52]. We believe that the techniques of Simon could open up the door for further developments in the field.

1.4 Guide to the thesis

The rest of this dissertation is divided into two chapters. In Chapter 2 we present joint work with Alessandro Pigati and Daniel Stern on the variational theory of the Yang-Mills-Higgs energies. More specifically, we start by presenting the notation and preliminaries needed for the chapter, Section 2.1. We then move to the proof of the liminf inequality in Section 2.2 and conclude in the subsequent section, cf. Section 2.3, the proof of Theorem 5 by constructing a recovery sequence. Section 2.4 is then dedicated to the proof of the min-max comparison, Theorem 8. As we have highlighted in previous paragraphs, the gradient flow of the Yang-Mill-Higgs energies plays a crucial role in the proof of the min-max comparison. We devote Section 2.5 to it and prove there that the flow satisfies long-time existence, uniqueness, and continuous dependence on the initial data.

In Chapter 3 we present unpublished work on the structure of the singular set of $\mathbb{Z}_2$-harmonic spinors. As in the previous chapter, we start in Section 3.1 with some preliminary notions, setting the stage for the next sections. In particular, we recall some well-known results on the Dirac equation and introduce the notion of $\mathbb{Z}_2$-harmonic spinors we are going to work with, together with some properties. In Section 3.2 we define Almgren's celebrated frequency function, and present some immediate consequences of its almost monotonicity. Section 3.3 is dedicated to the proof of existence of blow-ups. We establish several additional properties on them. Section 3.4 borrows some elements of Simon's cylindrical tangent cones and adapts them to our purposes. There we state the main results of the Chapter, Proposition 81 saying that either there are no singular points with a certain frequency, or a certain self-improving inequality holds. The subsequent section establishes a very important result that will be used in the sequel: away from the spine, we can parametrize $\mathbb{Z}_2$-harmonic spinors. The central part of the chapter is dedicated to Section 3.6, where we establish several a priori estimates culminating in non-concentration of the excess type result, Theorem 91. The results of this section will be used later in the proof of the main theorems. In Section 3.7 we analyze and classify homogeneous blow-ups. Finally, in Section 3.8 we combined all the results from the previous sections to prove the main theorems of the chapter. More precisely, we prove Proposition 81, from which Proposition 96 follows by iteration. With these two propositions in hand we can prove Theorem 11, Theorem 12, from which we deduce Corollary 13. We also prove Theorem 14, from which we infer Corollary 15.

Chapter 2

Convergence of the self-dual $U(1)$-Yang-Mills-Higgs energies to the $(n-2)$-area functional

Aim of this Chapter is to study the variational theory of the Yang-Mills-Higgs energies and how it relates to the one of the $(n-2)$-area functional as explained in Section 1.2 of the Introduction. We are going to prove Theorem 5 on the Γ-convergence of E_ε and Theorem 8 comparing the min-max theories of E_ε and the $(n-2)$-area functional. We will recall the statements of the main theorems when needed. The content of this Chapter is based on joint work with Alessandro Pigati and Daniel Stern, see [106].

2.1 Notation and preliminaries

We start by setting the stage of this Chapter. Let (M^n, g) be a closed, oriented Riemannian manifold and let $L \to M$ be a complex line bundle over M, endowed with a Hermitian structure $\langle \cdot, \cdot \rangle$. In this chapter we will consider the real part of the Hermitian structure just introduced to simplify some computations. Henceforth, assume that the structure just defined is real valued. We could have written $\Re(\langle \cdot, \cdot \rangle)$, but to lighten the notation we will simply use $\langle \cdot, \cdot \rangle$. We will denote by $W : L \to \mathbb{R}$ the nonlinear potential

$$W(u) := \frac{1}{4}(1-|u|^2)^2,$$

and for a Hermitian connection ∇ on L, a section $u \in \Gamma(L)$, and a parameter $\varepsilon \in (0,1)$, we denote by $E_\varepsilon(u, \nabla)$ the scaled Yang–Mills–Higgs energy

$$E_\varepsilon(u,\nabla) := \int_M (|\nabla u|^2 + \varepsilon^2 |F_\nabla|^2 + \varepsilon^{-2} W(u)) \,\mathrm{dvol}_g = \int_M e_\varepsilon(u,\nabla)\,\mathrm{dvol}_g, \tag{2.1}$$

where dvol_g denotes the volume form on M, $e_\varepsilon(u,\nabla)$ is the energy density and F_∇ is the curvature of ∇. As discussed in the introduction, working with $U(1)$-connections allows us to identify F_∇ with the real, closed, two-form $\omega = \omega_\nabla$ via

$$F_\nabla(X,Y)u = [\nabla_X, \nabla_Y]u - \nabla_{[X,Y]}u = -i\omega_\nabla(X,Y)u. \tag{2.2}$$

The Euler–Lagrange equations for critical points of (2.1) are given by

$$\begin{cases} \nabla^*\nabla u = \frac{1}{2\varepsilon^2}(1-|u|^2)u, \\ \varepsilon^2 d^*\omega_\nabla = \langle \nabla u, iu \rangle. \end{cases} \tag{2.3}$$

Here ∇^* denotes the formal adjoint of ∇ and d^* the formal adjoint of d. We refer to [107, Section 2] for further details and to the appendix of the same paper for the regularity of solutions to these equations. We follow the convention

$$|\omega|^2 = \sum_{1\leq j<k\leq n} \omega(e_j,e_k)^2 = \frac{1}{2}\sum_{j,k=1}^{m} \omega(e_j,e_k)^2$$

A key feature of the energies E_ε is their *gauge-invariance*: that is, for any $\phi \in \mathscr{G} = \mathrm{Maps}(M,S^1)$, the energy $E_\varepsilon(u,\nabla)$ is invariant under the change of gauge

$$\phi\cdot(u,\nabla) = (\phi u, \nabla - i\phi^*(d\theta)),$$

corresponding to a fiberwise rotation of L and where θ denotes the coodinate on the circle S^1 (after indentifying S^1 with the circle in $\mathbb{C}$). For the above gauge invariance ϕ is an arbitrary element of $\mathrm{Maps}(M,S^1)$. Note that this invariance can be rewritten as $E_\varepsilon(u,\nabla) = E_\varepsilon(e^{i\theta}u, \nabla - id\theta)$, for $\theta\colon M\to\mathbb{R}$. As discussed in the introduction, an important first step in understanding the Γ-convergence theory for E_ε is identifying an appropriate gauge-invariant analog of the Jacobian two-form $2du^1\wedge du^2$ for complex-valued maps. To this end, for a pair (u,∇), we consider the two-forms $\psi(u,\nabla)$ given by

$$\psi(u,\nabla)(X,Y) := 2\langle i\nabla_X u, \nabla_Y u\rangle,$$

for vector fields X and Y, and define the *gauge-invariant Jacobians*

$$J(u,\nabla) := \psi(u,\nabla) + (1-|u|^2)\omega_\nabla.$$

A straightforward computation shows that

$$d\langle\nabla u, iu\rangle = \psi(u,\nabla) - |u|^2\omega_\nabla = J(u,\nabla) - \omega_\nabla, \tag{2.4}$$

from which we deduce that $J(u,\nabla)$ is closed and cohomologous to $-\omega_\nabla$. Indeed, recall that the exterior derivative of a one-form is given by

$$d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]),$$

for smooth vector fields X and Y, whence

$$\begin{aligned} d\langle\nabla u, iu\rangle(e_j,e_k) &= e_j\left(\langle\nabla_{e_k}u, iu\rangle\right) - e_k\left(\langle\nabla_{e_j}u, iu\rangle\right) - \langle\nabla_{[e_j,e_k]}u, iu\rangle \\ &= \langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle - \langle i\nabla_{e_k}u, \nabla_{e_j}u\rangle + \langle iu, F_\nabla(e_j,e_k)u\rangle, \end{aligned}$$

from which the desired identity follows. Moreover, as mentioned in the introduction, it is easy to check that $\psi(u,\nabla)$ satisfies the pointwise estimate $|\psi(u,\nabla)| \leq |\nabla u|^2$, which together with Young's inequality implies

$$|J(u,\nabla)| \leq |\nabla u|^2 + \varepsilon^2|\omega_\nabla|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 = e_\varepsilon(u,\nabla), \tag{2.5}$$

so that $J(u,\nabla)$ has L^1 norm bounded above by $E_\varepsilon(u,\nabla)$. Throughout the chapter, we identify $J(u,\nabla)$ with an $(n-2)$-current, with the assignment

$$\langle J(u,\nabla), \eta\rangle := \int_M J(u,\nabla)\wedge\eta$$

for all $\eta \in \Omega^{n-2}(M)$; under this identification, note that the mass of $(n-2)$-current corresponding to $J(u,\nabla)$ is precisely

$$\mathbb{M}(J(u,\nabla)) = \|J(u,\nabla)\|_{L^1(M)} \le E_\varepsilon(u,\nabla).$$

Finally, given a smooth reference connection ∇_0 on L with associated curvature two-form ω_0, it will be useful to note that, by (2.4), we can write

$$J(u,\nabla) = d(\beta(u,\nabla)) + \omega_0 \tag{2.6}$$

where

$$\beta(u,\nabla = \nabla_0 - i\alpha) := \langle \nabla u, iu\rangle + \alpha = \langle \nabla_0 u, iu\rangle + (1-|u|^2)\alpha, \tag{2.7}$$

implicitly using the fact that ∇ can be written as $\nabla_0 - i\alpha$, for $\alpha \in \Omega^1(M)$, so that $\omega_\nabla = \omega_0 + d\alpha$.

2.2 The liminf inequality

We will prove in this section the liminf part of the Γ-convergence result presented in Section 1.2 of the Introduction. More precisely, the following is the result we are proving.

Theorem 18 (Γ-convergence, liminf part)**.** *For a Hermitian line bundle $L \to M$ as above, the following hold. Given a family $(u_\varepsilon, \nabla_\varepsilon)$ of smooth sections with $|u_\varepsilon| \le 1$ and Hermitian connections with uniformly bounded energies $E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) \le \Lambda$, there exists an integral $(n-2)$-cycle Γ Poincaré dual to the Euler class $c_1(L) \in H^2(M;\mathbb{Z})$ such that, up to a subsequence,*

$$J(u_\varepsilon,\nabla_\varepsilon) \rightharpoonup 2\pi\Gamma, \quad \text{as } \varepsilon \to 0,$$

as currents. Moreover, the following liminf inequality holds:

$$2\pi\mathbb{M}(\Gamma) \le \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon,\nabla_\varepsilon).$$

2.2.1 The distributional gauge-invariant Jacobian and singular unit sections

In the classical Γ-convergence theory for the Allen–Cahn energies, it is important to identify the space of $(n-1)$-boundaries in M with the distributional derivatives of functions in $BV(M,\{-1,1\})$, which arise as limits of the functions $\Phi(v_\varepsilon)$ for real-valued functions $v_\varepsilon : M \to \mathbb{R}$ with $F_\varepsilon(v_\varepsilon) = O(1)$, where $\Phi(s) := \int_0^s \sqrt{2W(t)}\,dt \,/\, \int_0^1 \sqrt{2W(t)}\,dt$. Similarly, the study of Γ-convergence for functionals of Ginzburg–Landau type is closely related to the theory of distributional Jacobians for circle-valued (and, more generally, sphere-valued [2, 3, 85]) maps, but the structure theory of these Jacobians does not play a direct role in the Γ-convergence proofs, since these results are not typically accompanied by compactness results for the given sequence of complex-valued maps.

For our results, it will likewise be useful to identify the space $\mathscr{Z}_{n-2}(M;\mathbb{Z})$ of integral $(n-2)$-cycles in M with the topological singularities (distributional Jacobians) of certain singular unit sections of Hermitian line bundles on M, arising as a limit of the two-forms $J(u,\nabla)$ for smooth pairs (u,∇). To this end, we seek to extend the definition of the $(n-2)$-current $J(u,\nabla)$ to a larger class of pairs (u,∇) of lower regularity, generalizing the distributional Jacobian for complex-valued maps.

First, we need to understand the continuity of $J(u,\nabla)$ as a map into the space of $(n-2)$-currents $\mathscr{D}_{n-2}(M)$ with the $(C^1)^*$ metric. Given $p \in (1,\infty)$ and a fixed reference connection ∇_0 on $L \to M$, we introduce the norm

$$\|(u,\nabla)\|_p := \|u\|_{L^p(M)} + \|\nabla_0 u\|_{L^p(M)} + \|\nabla - \nabla_0\|_{L^p(M)}$$

on the space of smooth pairs $u\in\Gamma(L)$ and $\nabla=\nabla_0-i\alpha$, and denote by $X_p(L)$ the metric space obtained as the completion of the space of smooth pairs

$$(u,\nabla)=(u,\nabla_0-i\alpha),\ \text{where } |u|\le 1$$

with respect to the norm $\|\cdot\|_p$. Note that, in a local trivialization, elements of $X_p(L)$ can be identified with pairs (u,α) where α is a one-form in L^p and u is a $W^{1,p}$ map to the unit disk $\bar{D}\subset\mathbb{C}$. The precise definition of the norm $\|\cdot\|_p$ is somewhat arbitrary, and other equivalent norms would work just as well. With respect to this norm, it is not difficult to check that the assignment $(u,\nabla)\mapsto J(u,\nabla)$ satisfies the desired continuity properties, summarized in the following proposition.

Proposition 19. *For a fixed reference connection ∇_0 on $L\to M$ and $p\in(1,2)$, given pairs (u,∇) and (v,∇') satisfying $|u|\le 1$, $|v|\le 1$, and $\|(u,\nabla)-(v,\nabla')\|_p\le 1$, we see that the one-forms $\beta(u,\nabla)$ and $\beta(v,\nabla')$ given by (2.7) satisfy*

$$\|\beta(u,\nabla)-\beta(v,\nabla')\|_{L^1(M)}\le C(p)(1+\|(u,\nabla)\|_p)\|(u,\nabla)-(v,\nabla')\|_p^{p-1}. \tag{2.8}$$

Consequently, the assignment $(u,\nabla)\mapsto J(u,\nabla)$ extends continuously to a map

$$(X_p(L),\|\cdot\|_p)\to(\mathscr{D}_{n-2}(M),(C^1)^*)$$

where $(\mathscr{D}_{n-2}(M),(C^1)^*)$ denotes the space of $(n-2)$-currents equipped with the $(C^1(M))^*$ norm.

Proof. Writing $\nabla=\nabla_0-i\alpha$ and $\nabla'=\nabla_0-i\eta$ for $\alpha,\eta\in\Omega^1(M)$, it follows from (2.7) that

$$\begin{aligned}\beta(u,\nabla)-\beta(v,\nabla')&=\langle\nabla_0u,iu\rangle-\langle\nabla_0v,iv\rangle+(1-|u|^2)\alpha-(1-|v|^2)\eta\\&=\langle\nabla_0(u-v),iu\rangle+\langle\nabla_0v,i(u-v)\rangle\\&\quad+(1-|u|^2)(\alpha-\eta)+(|v|^2-|u|^2)\eta.\end{aligned}$$

In particular, since $|u|\le 1$ and $|v|\le 1$, letting p' denote the Hölder conjugate of p, we deduce that

$$\begin{aligned}\int_M|\beta(u,\nabla)-\beta(v,\nabla')|&\le\int_M(|\nabla_0(u-v)|+|\nabla_0v||u-v|+|\alpha-\eta|+2|\eta||u-v|)\\&\le\|\nabla_0(u-v)\|_{L^1(M)}+\|\nabla_0v\|_{L^p(M)}\|u-v\|_{L^{p'}(M)}\\&\quad+\|\nabla-\nabla'\|_{L^1(M)}+2\|\nabla'-\nabla_0\|_{L^p(M)}\|u-v\|_{L^{p'}(M)}\\&\le C[\|(u,\nabla)-(v,\nabla')\|_p+(\|(u,\nabla)\|_p+\|(v,\nabla')\|_p)\|u-v\|_{L^{p'}(M)}]\\&\le C[\|(u,\nabla)-(v,\nabla')\|_p+(\|(u,\nabla)\|_p+\|(v,\nabla')\|_p)\|u-v\|_{L^p(M)}^{p-1}]\end{aligned}$$

for a constant $C=C(p,M)$, where we used the fact that $\|u-v\|_{L^\infty(M)}\le 2$ in the last inequality. Assuming that $\|(u,\nabla)-(v,\nabla')\|_p\le 1$, the estimate (2.8) easily follows.

Now, by the characterization (2.6) of $J(u,\nabla)$, for any $\zeta\in\Omega^{n-2}(M)$, we have

$$\begin{aligned}|\langle J(u,\nabla)-J(v,\nabla'),\zeta\rangle|&=\left|\int_M d(\beta(u,\nabla)-\beta(v,\nabla'))\wedge\zeta\right|\\&=\left|\int_M(\beta(u,\nabla)-\beta(v,\nabla'))\wedge d\zeta\right|\\&\le\|\beta(u,\nabla)-\beta(v,\nabla)\|_{L^1(M)}\|d\zeta\|_{C^1(M)}.\end{aligned}$$

The second equality follows from Stokes' theorem. Together with the estimate (2.8), this implies that

$$\|J(u,\nabla)-J(v,\nabla')\|_{(C^1(M))^*}\leq C(p,M)(1+\|(u,\nabla)\|_p)\|(u,\nabla)-(v,\nabla')\|_p$$

when $\|(u,\nabla)-(v,\nabla')\|_p\leq 1$. In particular, the assignment $(u,\nabla)\mapsto J(u,\nabla)$ is continuous with respect to the norms $\|\cdot\|_p$ and $(C^1(M))^*$, and therefore admits the desired extension

$$(X_p(L),\|\cdot\|_p)\to(\mathscr{D}_{n-2}(M),(C^1)^*).\qquad\square$$

Consider now the subset of $X_p(L)$ given by

$$\mathscr{V}_p(L):=\{(u,\nabla)\in X_p(L):|u|\equiv 1\text{ almost everywhere}\},$$

i.e., the set of pairs $(u,\nabla)\in X_p(L)$ where u belongs to the space

$$\mathscr{U}_p(L):=\{u\in W^{1,p}(M,L):|u|\equiv 1\text{ almost everywhere}\}$$

of $W^{1,p}$ unit sections. Note that for any $(u,\nabla)\in\mathscr{V}_p(L)$ we have

$$\beta(u,\nabla)=\beta(u,\nabla_0),$$

so we can view both β and $J=d\beta+\omega_0$ as functions on $\mathscr{U}_p(L)$, *independent of the connection* ∇. Notice that the definition of $\beta(u)$ still depends on the initial choice of reference connection ∇_0, but of course the assignment $\mathscr{U}_p\ni u\mapsto J(u)$ remains gauge-invariant and independent of ∇_0. In particular, in any local trivialization—in which u becomes identified with a $W^{1,p}$ map to S^1 and $\nabla_0=d-i\alpha_0$—we have $\beta(u)=\langle du,iu\rangle-\alpha_0$, and $J(u)=d\langle du,iu\rangle$ coincides with *the standard distributional Jacobian.*

The remainder of the subsection is devoted to recording some key properties of the operator $J:\mathscr{U}_p(L)\to\mathscr{D}_{n-2}(M)$. At the local level, note that this reduces to the study of topological singularities for maps in $W^{1,p}(M,S^1)$, and the arguments that follow are largely drawn from [3] and [85].

Proposition 20. *For any $u,v\in\mathscr{U}_p(L)$, there exists an integer rectifiable $(n-1)$-current $S\in\mathscr{I}_{n-1}(M;\mathbb{Z})$ of mass*

$$\mathbb{M}(S)\leq\frac{1}{2\pi}\int_M|\nabla_0(u+v)||u-v|$$

such that

$$J(u)-J(v)=2\pi\partial S,$$

as currents. Moreover, $J(u)=J(v)$ if and only if $u=\phi e^{i\psi}v$ for some $\phi:M\to S^1$ harmonic and $\psi\in W^{1,p}(M,\mathbb{R})$—i.e., if u and v differ by a change of gauge.

Proof. To prove the first statement, we introduce the map

$$\Phi:\mathscr{U}_p(L)\times\mathscr{U}_p(L)\to W^{1,p}(M,S^1)$$

given by setting

$$\Phi(u,v):=e^{-i\langle u,iv\rangle}u\bar{v}$$

in any local trivialization. Recall here for the computations that follow that $\langle\cdot,\cdot\rangle$ is real-valued (we abuse notation by using the same notation as the Hermitian structure). Indeed, note that the complex-valued map $u\bar{v}$ is invariant under change of gauge. By direct computation, one can check that the map $w:=\Phi(u,v)$ satisfies the identity

$$\langle dw,iw\rangle=\beta(u)-\beta(v)-d\langle u,iv\rangle=\langle\nabla_0(u+v),i(u-v)\rangle.$$

Hence

$$\int_M |dw| \leq \int_M |\nabla_0(u+v)||u-v|,$$

and the distributional Jacobian $Jw = d\langle dw, iw\rangle$ satisfies

$$Jw = d[\beta(u) - \beta(v)] = J(u) - J(v).$$

By [2, Theorem 3.8], we can appeal to the coarea formula for maps in $W^{1,1}(M,S^1)$ to deduce the existence of an integer rectifiable current $S \in \mathscr{I}_{n-1}(M;\mathbb{Z})$ of mass

$$\mathbb{M}(S) \leq \frac{1}{2\pi}\int_M |dw| \leq \int_M |\nabla_0(u+v)||u-v|$$

such that

$$2\pi\partial S = Jw = J(u) - J(v),$$

proving the first part of the proposition.

For the latter statement, note that $J(u) - J(v) = 0$ if and only if the map $w = \Phi(u,v) \in W^{1,p}(M,S^1)$ satisfies $Jw = 0$. But it is easy to check (cf. [50]) that a map $w \in W^{1,p}(M,S^1)$ satisfies $Jw = 0$ if and only if $w = \phi e^{i\psi}$ for some $\phi : M \to S^1$ harmonic and $\psi \in W^{1,p}(M,\mathbb{R})$. Indeed, if $Jw = 0$ then the one-form $\langle dw, iw\rangle$ is closed, and thus decomposes as $h + d\psi$ with h harmonic, so that $\phi = e^{-i\psi}w$ is a harmonic map. The reverse direction is immediate. □

Corollary 21. *If $u \in \mathscr{U}_p(L)$ is such that $J(u)$ has finite mass, then $\frac{1}{2\pi}J(u)$ is an integral $(n-2)$-cycle in the homology class dual to $c_1(L) \in H^2(M;\mathbb{Z})$.*

Proof. By Proposition 26 below, note that there exists at least one $u_0 \in \mathscr{U}_p(L)$ such that $\frac{1}{2\pi}J(u_0)$ is given by a prescribed integral (in fact, polyhedral) cycle $P \in \mathscr{Z}_{n-2}(M;\mathbb{Z})$ dual to $c_1(L)$. As a consequence, for any $u \in \mathscr{U}_p(L)$, it follows from Proposition 20 that

$$\frac{1}{2\pi}(J(u) - J(u_0)) = \partial S$$

for an integer rectifiable $S \in \mathscr{I}_{n-1}(M;\mathbb{Z})$ of finite mass.

In particular, if $\mathbb{M}(J(u)) < \infty$, then it follows that $\mathbb{M}(S) + \mathbb{M}(\partial S) < \infty$, and we can deduce from [121, Theorem 30.3] that ∂S is itself an integral $(n-2)$-cycle. In particular,

$$\frac{1}{2\pi}J(u) = \frac{1}{2\pi}J(u_0) + \partial S = P + \partial S$$

is then an integral $(n-2)$-cycle homologous to P, proving the claim. □

2.2.2 Proof of Theorem 5, part (i)

To complete the proof of the lim inf part of the Γ-convergence theorem, it remains to establish a compactness result for sections u_ε coming from couples $(u_\varepsilon, \nabla_\varepsilon)$ in X_p (modulo gauge transformations) under the assumption of a uniform energy bound $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda$. As in the previous section, we will continue to work with a fixed smooth reference connection ∇_0 on the line bundle $L \to M$.

Lemma 22. *Let (u,∇) satisfy $|u| \leq 1$ and $E_\varepsilon(u,\nabla) \leq \Lambda$. Then there is a gauge-equivalent pair (u',∇') for which*

$$\|\nabla' - \nabla_0\|_{L^p(M)} + \|\nabla_0 u'\|_{L^p(M)} \leq C(p,M,L,\Lambda)$$

for all $p \in (1, \frac{n}{n-1})$.

Proof. Writing the initial connection as

$$\nabla = \nabla_0 - i\eta$$

for a one-form $\eta \in \Omega^1(M)$, consider the Hodge decomposition

$$\eta = d^*\xi + d\psi + h(\eta),$$

where $\xi \in \Omega^2(M)$, $\psi \in C^\infty(M)$, and $h(\eta)$ is harmonic. Since the gradients of S^1-valued harmonic maps form a lattice in the space $\mathscr{H}^1(M)$ of harmonic one-forms, note that we can find a harmonic map $f : M \to S^1$ such that

$$\|f^*(d\theta) - h(\eta)\|_{L^\infty(M)} \leq C(M).$$

Now, letting

$$\phi := e^{i\psi} f : M \to S^1,$$

we see that

$$\alpha := \eta - \phi^*(d\theta) = \eta - f^*(d\theta) - d\psi = d^*\xi + [h(\eta) - f^*(d\theta)].$$

Thus, making the change of gauge

$$(u', \nabla') := (\phi^{-1} \cdot u, \nabla + i\phi^*(d\theta)),$$

we see that the new connection ∇' is given by

$$\nabla' = \nabla_0 - i\alpha,$$

where α is co-closed, and the harmonic component $h(\alpha) = h(\eta) - f^*(d\theta)$ of the Hodge decomposition $\alpha = d^*\xi + h(\alpha)$ satisfies $\|h(\alpha)\|_{L^\infty(M)} \leq C$.

To obtain the desired bound for $\|\nabla' - \nabla_0\|_{L^p(M)} = \|\alpha\|_{L^p(M)}$, it remains to estimate the co-exact component $d^*\xi$. To this end, note that ξ can be assumed exact and is given by

$$\xi = \Delta_H^{-1}(d\eta),$$

by definition of the Hodge decomposition. By the L^p regularity theory for the Hodge Laplacian and a standard duality argument, we have an automatic bound of the form

$$\|d^*\xi\|_{L^p(M)} \leq C(p,M)\|d\eta\|_{W^{-1,p}(M)} = C(p,M)\|d\eta\|_{(W^{1,p'}(M))^*} \tag{2.9}$$

for any $p \in (1, \infty)$.

Now, by definition (2.7) of the one-form $\beta(u, \nabla)$, we have

$$\eta = \beta(u, \nabla) - \langle \nabla u, iu \rangle,$$

while (2.6) gives

$$J(u, \nabla) = d(\beta(u, \nabla)) + \omega_0.$$

We therefore see that

$$d\eta = J(u, \nabla) - \omega_0 - d\langle \nabla u, iu \rangle,$$

and for any $\zeta \in \Omega^2(M)$, it follows that

$$\begin{aligned}\int_M \langle d\eta, \zeta\rangle &= \int_M \langle J(u,\nabla)-\omega_0,\zeta\rangle - \int_M \langle d\langle \nabla u, iu\rangle, \zeta\rangle \\ &= \int_M \langle J(u,\nabla)-\omega_0,\zeta\rangle - \int_M \langle\langle \nabla u, iu\rangle, d^*\zeta\rangle \\ &\leq \|J(u,\nabla)\|_{L^1(M)}\|\zeta\|_{C^0(M)} + \|F_{\nabla_0}\|_{L^1(M)}\|\zeta\|_{C^0(M)} \\ &\quad + C\|\langle \nabla u, iu\rangle\|_{L^2(M)}\|\zeta\|_{W^{1,2}(M)}.\end{aligned}$$

We know already that $\|J(u,\nabla)\|_{L^1(M)} \leq E_\varepsilon(u,\nabla) \leq \Lambda$, and since ∇_0 is a fixed reference connection, we automatically have $\|F_{\nabla_0}\|_{L^1(M)} \leq C(M,L)$ independent of (u,∇). Moreover, since $|u| \leq 1$, we also see that

$$\|\langle \nabla u, iu\rangle\|_{L^2(M)} \leq \|\nabla u\|_{L^2(M)} \leq E_\varepsilon(u,\nabla)^{1/2}.$$

Combining the preceding estimates, it follows that

$$\int_M \langle d\eta,\zeta\rangle \leq C(M,L,\Lambda)(\|\zeta\|_{C^0(M)} + \|\zeta\|_{W^{1,2}(M)}),$$

and by the Sobolev embedding $W^{1,q}(M) \hookrightarrow C^0(M)$ for $q > n$ (as well as the obvious embedding $W^{1,q}(M) \hookrightarrow W^{1,2}(M)$ for $q > n \geq 2$), we deduce in particular that

$$\|d\eta\|_{(W^{1,q}(M))^*} \leq C(q,M,L,\Lambda)$$

for any $q > n$. Together with (2.9), this implies that

$$\|d^*\xi\|_{L^p(M)} \leq C(p,M,L,\Lambda)$$

for all $1 < p < \frac{n}{n-1}$, and consequently

$$\|\nabla' - \nabla_0\|_{L^p(M)} = \|\alpha\|_{L^p(M)} \leq \|d^*\xi\|_{L^p(M)} + \|h(\alpha)\|_{L^p(M)} \leq C(p,M,L,\Lambda) \tag{2.10}$$

for $p \in (1,\frac{n}{n-1})$, giving the desired estimate for $\nabla' - \nabla_0$.

In particular, since $\nabla' u' = \nabla_0 u' - i\alpha u'$, for $1 < p < \frac{n}{n-1}$, it also follows that

$$\|\nabla_0 u'\|_{L^p(M)} \leq \|\nabla' u'\|_{L^p(M)} + \|\alpha\|_{L^p(M)} \leq \|\nabla u\|_{L^p(M)} + \|\alpha\|_{L^p(M)} \leq C(p,M,L,\Lambda),$$

as claimed. □

With the preceding lemma in place, we can now finish the proof of the liminf part of the Γ-convergence statement.

Proof of Theorem 5(i). Given a family $(u_\varepsilon, \nabla_\varepsilon = \nabla_0 - i\alpha_\varepsilon)$ with $|u_\varepsilon| \leq 1$ and uniformly bounded energy $E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) \leq \Lambda$, we may assume without loss of generality that the change of gauge given in the preceding lemma has already been applied to $(u_\varepsilon,\nabla_\varepsilon)$, so that

$$\|\alpha_\varepsilon\|_{L^p(M)} + \|\nabla_0 u_\varepsilon\|_{L^p(M)} \leq C(p,M,L,\Lambda)$$

for $1 < p < \frac{n}{n-1}$. In this case, it follows that the sections u_ε are uniformly bounded in $W^{1,p}$ norm

$$\|u_\varepsilon\|_{W^{1,p}(M)} = \|u_\varepsilon\|_{L^p(M)} + \|\nabla_0 u_\varepsilon\|_{L^p(M)},$$

so by the Rellich–Kondrachov theorem, we can pass to a subsequence such that u_ε converges strongly in $L^p(M,L)$ to a limiting section $u \in W^{1,p}(M,L)$. Moreover, since the sections u_ε satisfy the pointwise bound $|u_\varepsilon| \le 1$, we see that the convergence $u_\varepsilon \to u$ must be strong in $L^q(M,L)$ for every $q \in [1,\infty)$, and therefore the limiting section u must satisfy

$$\int_M (1-|u|^2)^2 = \lim_{\varepsilon\to 0}\int_M (1-|u_\varepsilon|^2)^2 \le \lim_{\varepsilon\to 0} \varepsilon^2 E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) = 0;$$

i.e., $|u| \equiv 1$ almost everywhere, so $u \in \mathscr{U}_p(L)$.

By (2.7) and a straightforward calculation, one can check that

$$\begin{aligned}\beta(u_\varepsilon,\nabla_\varepsilon)-\beta(u) &= \langle \nabla_0 u_\varepsilon, iu_\varepsilon\rangle - \langle \nabla_0 u, iu\rangle + (1-|u_\varepsilon|^2)\alpha_\varepsilon \\ &= \langle \nabla_0(u_\varepsilon+u), i(u_\varepsilon-u)\rangle + (1-|u_\varepsilon|^2)\alpha_\varepsilon + d\langle u_\varepsilon, iu\rangle,\end{aligned}$$

so that the difference $J(u_\varepsilon,\nabla_\varepsilon)-J(u) = d[\beta(u_\varepsilon,\nabla_\varepsilon)-\beta(u)]$ satisfies

$$\begin{aligned}&\|J(u_\varepsilon,\nabla_\varepsilon)-J(u)\|_{(C^1(M))^*}\\ &\le C\|\langle \nabla_0(u_\varepsilon+u), i(u_\varepsilon-u)\rangle + (1-|u_\varepsilon|^2)\alpha_\varepsilon\|_{L^1(M)}\\ &\le C(\|\nabla_0 u_\varepsilon\|_{L^p(M)} + \|\nabla_0 u\|_{L^p(M)})\|u_\varepsilon-u\|_{L^{p'}(M)} + C\|\alpha_\varepsilon\|_{L^p(M)}\|1-|u_\varepsilon|^2\|_{L^{p'}(M)}\\ &\le C(p,M,L,\Lambda)(\|u_\varepsilon-u\|_{L^{p'}(M)} + \|1-|u_\varepsilon|^2\|_{L^{p'}(M)}).\end{aligned}$$

Since $u_\varepsilon \to u$ strongly in $L^{p'}$ for $p>1$, taking the limit as $\varepsilon\to 0$, we have that the right-hand side goes to 0, establishing the desired convergence $J(u_\varepsilon,\nabla_\varepsilon)\to J(u)$ in $(C^1)^*$. Finally, lower semicontinuity of the mass gives the obvious bound

$$\mathbb{M}(J(u)) \le \liminf_{\varepsilon\to 0}\mathbb{M}(J(u_\varepsilon,\nabla_\varepsilon)) \le \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) \le \Lambda,$$

and by Corollary 21, it follows that $\frac{1}{2\pi}J(u)$ defines an integral $(n-2)$-cycle in the correct homology class. □

Remark 23. Alternatively, one can also give another proof of the liminf inequality via techniques similar to those used in Alberti, Baldo and Orlandi [2, 3] for functionals of Ginzburg–Landau type. Though this method would be slightly more involved than the proof given here, the automatic mass bound $\|J(u_\varepsilon,\nabla_\varepsilon)\|_{L^1(M)} \le E_\varepsilon(u_\varepsilon,\nabla_\varepsilon)$ again simplifies several steps, reducing the problem to establishing the integrality of the limiting cycle.

2.3 Recovery sequence

In this section we prove existence of a recovery sequence, thus establishing the other half of the Γ-convergence and finishing the proof of Theorem 5. More precisely, we are going to prove the following theorem.

Theorem 24 (Γ-convergence, recovery sequence part)**.** *For a Hermitian line bundle $L\to M$ as in Section 2.1, the following hold. Given an integral $(n-2)$-cycle Γ whose homology class $[\Gamma]\in H_{n-2}(M;\mathbb{Z})$ is Poincaré dual to $c_1(L)\in H^2(M;\mathbb{Z})$, there exists a family $(u_\varepsilon,\nabla_\varepsilon)$ of smooth sections and connections on L such that*

$$J(u_\varepsilon,\nabla_\varepsilon) \rightharpoonup 2\pi\Gamma, \quad \text{as } \varepsilon\to 0,$$

as currents, and

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon,\nabla_\varepsilon) = 2\pi\mathbb{M}(\Gamma).$$

The proof is constructive in nature and exploits in a crucial way the two-dimensional solutions of the vortex equations appearing in Theorem 29. We start by recalling a few basic facts from algebraic topology.

Proposition 25. *Any cohomology class $\alpha \in H^2(M;\mathbb{Z})$ is the Euler class $c_1(L)$ of some complex line bundle $L \to M$. Also, the Euler class classifies the line bundle up to isomorphism.*

Indeed, it is well known that any complex line bundle arises as the pullback of the canonical line bundle on $\mathbb{CP}^\infty$ by means of a continuous map $f : M \to \mathbb{CP}^\infty$, with a correspondence between the homotopy class $[f]$ and the isomorphism class of the line bundle. For a specific choice of the generator λ of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$, we then have $c_1(L) = f^*\lambda$. On the other hand, $\mathbb{CP}^\infty$ is an Eilenberg–MacLane space $K(\mathbb{Z},2)$: hence, any $\alpha \in H^2(M;\mathbb{Z})$ equals $f^*\lambda$ for a unique homotopy class $[f]$; see, e.g., [71, Theorem 4.57]. For a more elementary proof using the exponential sheaf sequence, see for instance [65, pp. 139–140].

We know from Section 2.2 that the homology class of a limit cycle Γ is dual to the Euler class of the bundle. Conversely, given a Hermitian line bundle $L \to M$ and a cycle Γ whose homology class $[\Gamma]$ is dual to $c_1(L)$, we now show how to realize Γ as the limit of $\frac{1}{2\pi}J(u_\varepsilon, \nabla_\varepsilon)$, for appropriate pairs of sections and connections on L, as in part (ii) of Theorem 5.

The next proposition provides a useful variant of Federer's polyhedral approximation theorem (cf. [60, Lemma 4.2.19]) for our setting, providing a polyhedral approximation of a given cycle Γ, which can be realized as the distributional Jacobian $J(v)$ of an appropriate singular unit section. Locally, this is a simpler version of the main result from [2], with appropriate modifications for the manifold setting.

Proposition 26. *Given an integral $(n-2)$-cycle $\Gamma \in \mathscr{Z}_{n-2}(M;\mathbb{Z})$, there exists a triangulation of M and an integer valued function k on the collection $\{\Delta\}$ of $(n-2)$-simplices of the triangulation, each with a fixed orientation, such that the integral current*

$$P := \textstyle\sum_\Delta k(\Delta)\Delta$$

is a cycle arbitrarily close to Γ with respect to the flat distance with $\mathbb{M}(P)$ arbitrarily close to $\mathbb{M}(\Gamma)$. Also, there exists a section $v \in \mathscr{U}_p(L) \cap C^\infty(M \setminus \mathscr{S}_{n-2})$, for $p \in (1,2)$, such that

$$J(v) = 2\pi P$$

and, with respect to a reference connection ∇_0,

$$|\nabla_0 v| \le C \operatorname{dist}(\cdot, \mathscr{S}_{n-2})^{-1}, \tag{2.11}$$

where $\mathscr{S}_{n-2} = \bigcup \Delta$ is the $(n-2)$-skeleton of the triangulation, and C depends on v.

Proof. In order to approximate Γ, we modify Federer's classic approximation result [60, Lemma 4.2.19] as follows. Given $\delta > 0$, using the same proof we can find a finite collection of disjoint C^1 embeddings $F_j : \bar{B}^{n-2} \to M$ and multiplicities $a_j \in \mathbb{Z}$ such that

$$\mathbb{M}(T) < \delta, \quad \text{where } T := \Gamma - \textstyle\sum_j a_j F_j(\bar{B}^{n-2}).$$

Moreover, we can find a triangulation of M such that each piece $F_j(\bar{B}^{n-2})$ is a subcomplex, for instance triangulating first a tubular neighborhood of each and then extending to a triangulation of the complement, using [101, Theorem 10.6]. We can also refine the triangulation in such a way that each simplex has diameter less than a given $\rho > 0$ and admits a diffeomorphism f to (a scaled copy of) the standard simplex with $\operatorname{Lip}(f) + \operatorname{Lip}(f^{-1}) \le C$, for a universal constant C.

We now argue as in the deformation theorem (see [60, Theorem 4.2.9] or [121, Theorem 29.1]), using our triangulation in place of the Euclidean grid. Since we are in a manifold, we cannot easily average over translations; but, recalling that the simplices are identified with the standard one, we can average instead over the center of the retraction.

Namely, given the standard k-dimensional simplex Δ^k, denote $\frac{1}{2}\Delta^k$ the rescaled simplex with the same center. Since $\frac{1}{2}\Delta^k$ has positive distance from the boundary $\partial\Delta^k$, for any point $p \in \frac{1}{2}\Delta^k$ the radial retraction $r_p : \Delta^k \setminus \{p\} \to \partial\Delta^k$ is locally Lipschitz outside $\{p\}$ and satisfies $|dr_p(x)| \le C(k)|x-p|^{-1}$. Then, for $0 \le m < k$, given a normal rectifiable m-current W on Δ^k, with $C = C(k)$ we have

$$\int_{\frac{1}{2}\Delta^k} \int_{\Delta^k} |dr_p(x)|^m \, d|W|(x) \, d\mathscr{L}^k(p) \le C \int_{\Delta^k} \int_{\frac{1}{2}\Delta^k} |x-p|^{-m} \, d\mathscr{L}^k(p) \, d|W|(x) \le C\mathbb{M}(W).$$

Hence, there exists p such that the inner integral on the left-hand side is bounded by $C(k)\mathbb{M}(W)$ (and $\|W\|(\{p\}) = 0$ if $m = 0$). A standard cut-off argument shows that the pushforward $(r_p)_*W$ is a well-defined current whose mass is bounded by the same quantity. If W has no boundary in the interior of Δ^k, as in the proof of the deformation theorem it is easy to check that the difference $W - (r_p)_*W = \partial V$ for some $(m+1)$-current V with $\mathbb{M}(V) \le C(k)\mathbb{M}(W)$. Scaling by a factor ρ gives the same result for a current W supported on the scaled simplex, with the bounds $\mathbb{M}((r_p)_*W) \le C(k)\mathbb{M}(W)$ and $\mathbb{M}(V) \le C(k)\rho\mathbb{M}(W)$.

The same argument applies to an m-current supported on the k-skeleton of our triangulation, assuming that $0 \le m < k$ and that the boundary of the current is supported on the $(k-1)$-skeleton, since the retractions on each k-simplex paste together. In particular, this holds for the $(n-2)$-current T, with $k = n$, since

$$\partial T = -\textstyle\sum_j a_j \partial(F_j(B^{n-2}))$$

is supported on the $(n-3)$-skeleton. We can thus construct a retraction r to the $(n-1)$-skeleton such that $T' := r_*T$ satisfies $T = T' + \partial R'$, with

$$\mathbb{M}(T') \le C\mathbb{M}(T) \quad \text{and} \quad \mathbb{M}(R') \le C\rho\mathbb{M}(T)$$

where T' is an integral current supported on the $(n-1)$-skeleton. We can repeat the same on the $(n-1)$-skeleton and retract T' to a current T'' supported on the $(n-2)$-skeleton, such that $T' = T'' + \partial R''$, with

$$\mathbb{M}(T'') \le C\mathbb{M}(T') \quad \text{and} \quad \mathbb{M}(R'') \le C\rho\mathbb{M}(T').$$

Since $\partial T'' = \partial T$ vanishes on the interior of each $(n-2)$-simplex, by the constancy theorem T'' is an algebraic sum of the $(n-2)$-simplices. Thus, defining

$$P := T'' + \textstyle\sum_j a_j F_j(B^{n-2}),$$

we have $\Gamma - P = \partial(R' + R'')$ and

$$|\mathbb{M}(P) - \mathbb{M}(\Gamma)| \le \mathbb{M}(P - \Gamma) \le \mathbb{M}(T) + \mathbb{M}(T'') \le C\delta,$$

together with

$$\mathbb{M}(R') + \mathbb{M}(R'') \le C\delta,$$

for ρ small enough.

Let us now fix a smooth section $w_0 : M \to L$ which is transverse to the zero section, the existence of which is guaranteed, for instance, by [87, Theorem IV.2.1]. The implicit function theorem implies then that $S_0 := w_0^{-1}\{0\}$ is a smooth $(n-2)$-submanifold. Moreover, it comes equipped with the canonical orientation such that a positive basis $\{v_3,\dots,v_n\}$ of T_pS_0, extended with $\{v_1,v_2\}$ such that $\{dw_0[v_1],dw_0[v_2]\}$ gives a positive basis of L_p, gives a positive basis $\{v_1,\dots,v_n\}$ of T_pM. With this orientation, letting $v_0 := \frac{w_0}{|w_0|}$, we have $J(v_0) = 2\pi S_0$ and $[M] \frown c_1(L) = [S_0]$, where $\frown$ denotes the Poincaré duality isomorphism, so that we have that $c_1(L)$ and $[S_0]$ are Poincaré dual.

We can then find another triangulation of M such that S_0 is a union of $(n-2)$-simplices. Using [101, Theorem 10.4], viewing the two triangulations as embeddings of simplicial complexes, we can find a subdivision of each complex and a perturbation $(F_t)_{t\in[0,1]}$ of the second embedding in such a way that F_1 agrees with the first embedding, for a suitable identification of the two domain complexes. The perturbation can be chosen to be smooth on each domain simplex.

Note that the perturbed $S_1 = F_1(S_0) \subseteq \mathscr{S}_{n-2}$ is still the zero set of a piecewise smooth section, which we denote w_1, obtained for instance from w_0 by parallel transport along the curves $t \mapsto F_t \circ F_0^{-1}(x)$ for $x \in M$ with respect to some fixed connection ∇_0 on L. Clearly, for this transported section w_1, the singular unit section $v_1 := \frac{w_1}{|w_1|}$ satisfies $J(v_1) = 2\pi S_1$ outside $\mathscr{S}_{n-3}$, the $(n-3)$-skeleton of the triangulation.

The perturbations $J(v_1)$ and S_1 differ from $J(v_0)$ and S_0 by boundaries of rectifiable currents, by Proposition 20. Hence, from $J(v_0) = 2\pi S_0$ we deduce that $J(v_1) - 2\pi S_1$ is the boundary of a rectifiable $(n-1)$-current vanishing outside $\mathscr{S}_{n-3}$. The retraction of the latter to the $(n-1)$-skeleton must then be a linear combination of $(n-1)$-simplices by the constancy theorem, with the same boundary. Hence, $J(v_1) - 2\pi S_1 = 0$. Since $[P] = [\Gamma] = [S_1]$, the difference $[P - S_1]$ is trivial in $H_{n-2}(M;\mathbb{Z})$. Hence,

$$P - S_1 = \partial\left(\textstyle\sum_j k_j R_j\right)$$

for a collection $\{R_j\}$ of $(n-1)$-simplices in the triangulation. We have the following elementary fact.

Lemma 27. *There exists a map* $v' \in C^\infty(M \setminus \bigcup_j \operatorname{spt}(\partial R_j), S^1)$ *with* $J(v') = 2\pi\partial(\sum_j k_j R_j)$ *and* $|dv'| \le C \operatorname{dist}(\cdot, \bigcup_j \operatorname{spt}(\partial R_j))^{-1}$.

The proof is a straightforward application of the techniques in [3, Section 4]. Indeed, for a geodesic ball $\bar{U}_j \subset M$ covering R_j, the arguments of [3] can be applied to obtain a map $v'_j : \bar{U}_j \to S^1$, locally Lipschitz outside $\operatorname{spt}(\partial R_j)$, satisfying $J(v'_j) = 2\pi\partial R_j$ and $|dv'_j| \le C\operatorname{dist}(\cdot, \operatorname{spt}(\partial R_j))^{-1}$; up to regularization, we can assume v'_j smooth outside $\operatorname{spt}(\partial R_j)$. The map v'_j restricts to a contractible map from ∂U_j to S^1, even when $n = 2$ (in which case the latter has degree zero). Hence, it can be extended smoothly in the complement of $\bar{U}_j$. The product $v' := \prod_j (v'_j)^{k_j}$ is the desired map.

We can now conclude the proof of the proposition. Since $v_1 = \frac{w_1}{|w_1|}$ has Jacobian $J(v_1) = 2\pi S_1$, the product $v := v' v_1$ then has

$$J(v) = J(v') + J(v_1) = 2\pi(P - S_1) + 2\pi S_1 = 2\pi P.$$

Thus, the cycle P and the map v have all the desired properties and the proof of Proposition 26 is complete. □

We now show how to obtain a recovery sequence $(u_\varepsilon, \nabla_\varepsilon)$ for any polyhedral approximation P of Γ. Once this is done, the result follows for any integral $(n-2)$-cycle Γ by the preceding proposition and a diagonal argument.

Fix a triangulation of M as in the conclusion of Proposition 26. For an $(n-2)$-simplex Δ, fix a diffeomorphism $\bar{\Delta} \to \Delta$ from the standard simplex $\bar{\Delta}$. For $\delta > 0$ small, we denote by Δ_δ the image of the set of points in $\bar{\Delta}$ with distance at least δ from the boundary. Given $p \in \Delta \setminus \partial\Delta$, we denote by $B_r^\perp(p)$ the ball of radius r in the normal bundle to Δ at p; for a set S of such points, we then set $B_r^\perp(S) := \bigcup_{p\in S} B_r^\perp(p)$. Note that there exists $c' > 0$ independent of δ such that the exponential map is a diffeomorphism from $B_{c'\delta}^\perp(\Delta_\delta)$ to its image and such that, setting

$$V_\delta(\Delta) := \exp(B_{c'\delta}^\perp(\Delta_\delta)),$$

we have $V_\delta(\Delta) \cap V_\delta(\Delta') = \emptyset$. We can also require that the closest point to $\exp_p(v)$ in the $(n-2)$-skeleton $\bigcup \Delta$ is p, whenever $v \in B_{c'\delta}^\perp(p)$ and $p \in \Delta_\delta$. With these preparations in place, we come now to the main result of this section.

Proposition 28. *For $\varepsilon > 0$ small enough, there exists a family of smooth couples $(u_\varepsilon, \nabla_\varepsilon)$ such that*

$$J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup 2\pi P, \quad \text{as } \varepsilon \to 0,$$

as currents, and

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) = 2\pi\mathbb{M}(P).$$

Throughout the proof, we will use the following key fact, for a proof of which we refer the reader to [82, Theorem III.2.3].

Theorem 29. *For the trivial line bundle $L \to \mathbb{C}$, given any integer $k_0 \in \mathbb{Z}$ there exists a smooth couple $(u_\varepsilon, \nabla_\varepsilon)$ which is (locally) critical for the energy E_ε, has $u_\varepsilon^{-1}\{0\} = \{0\}$ and*

$$E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) = 2\pi|k_0|.$$

Moreover, $|u_\varepsilon| \le 1$ and, writing $\nabla_\varepsilon = d - i\alpha_\varepsilon$, we have the decay for gauge invariant quantities

$$|\nabla_\varepsilon u_\varepsilon| + \frac{1-|u_\varepsilon|^2}{\varepsilon} + \varepsilon|d\alpha_\varepsilon| \le \frac{C(k_0)}{\varepsilon} e^{-c(k_0)|z|/\varepsilon}. \tag{2.12}$$

Finally, we can require that $u_\varepsilon = |u_\varepsilon| e^{ik_0\theta}$ for $|z| \ge \varepsilon$, which gives

$$|u_\varepsilon^*(d\theta)| \le C(k_0)|z|^{-1}, \quad |du_\varepsilon| + |\alpha_\varepsilon| \le C(k_0)(\varepsilon^{-1} \wedge |z|^{-1}). \tag{2.13}$$

Note that the pairs $(u_\varepsilon, \nabla_\varepsilon)$ can be obtained from (u_1, ∇_1) by scaling. The exponential decay is proved in [82, Theorem III.8.1]; see also the proof of [107, Corollary 5.4]. As for the last part, by a change of gauge we can assume $u_1/|u_1| = e^{ik_0\theta}$ for $|z| \ge 1$. Observing that

$$\langle \nabla_1 u_1, iu_1 \rangle = |u_1|^2 (u_1^*(d\theta) - \alpha_1),$$

we deduce (2.13) from the smoothness of the pair and the decay for $|\nabla_1 u_1|$; the conclusion for arbitrary ε then follows.

We proceed now to the proof of Proposition 28, from which the final part of the Γ-convergence result stated in Theorem 5 will follow.

Proof of Proposition 28. Let P be a polyhedral cycle and $v \in \mathscr{U}_p(L)$ a singular unit section with $J(v) = 2\pi P$ as in the conclusion of Proposition 26. Fix an $(n-2)$-simplex Δ, a small parameter $\delta > 0$, and set $\lambda := \frac{c'}{3}\delta$. Let $k_0 = k(\Delta)$ be the constant multiplicity with which Δ appears in the polyhedral cycle P. In the sequel, we will identify $V_\delta(\Delta)$ with $\Delta_\delta \times B^2_{3\lambda}$, with respect to a fixed trivialization of the normal bundle to Δ. Also, the vector bundle L is trivial near Δ; hence, we can identify the section v with a smooth S^1-valued map on $V_\delta(\Delta) \setminus \Delta$.

We fix a couple $(u'_\varepsilon, d - i\alpha'_\varepsilon)$ as in Theorem 29, with degree k_0. Note that, for any $p \in P$, v has degree k_0 on the loop $\theta \mapsto (p, \lambda e^{i\theta})$, since $J(v) = 2\pi P$. Hence, identifying u'_ε and α'_ε with their pullback under the projection $V_\delta(\Delta) = \Delta_\delta \times B^2_{3\lambda} \to B^2_{3\lambda} \subset \mathbb{C}$, we can write

$$\frac{u'_\varepsilon}{|u'_\varepsilon|} = e^{if} v \tag{2.14}$$

with $f : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ smooth and depending on ε. We then define the new sections

$$\tilde{u}_\varepsilon := [1 - \chi(1 - |u'_\varepsilon|)] e^{i\chi f} v,$$

and one-forms

$$\tilde{\alpha}_\varepsilon := \chi \alpha'_\varepsilon + (1-\chi)(u'_\varepsilon)^*(d\theta) + d((\chi - 1) f),$$

where $\chi : \mathbb{C} \to \mathbb{R}$ is a smooth cut-off function such that $0 \le \chi \le 1$, $|d\chi| \le 2/\lambda$ and

$$\chi(z) = \begin{cases} 1 & \text{for } |z| \le \lambda, \\ 0 & \text{for } |z| \ge 2\lambda. \end{cases}$$

Note that the newly defined couples of sections and connections reduce to

$$(\tilde{u}_\varepsilon, \tilde{\alpha}_\varepsilon) = \begin{cases} (u'_\varepsilon, \alpha'_\varepsilon) & \text{for } |z| < \lambda, \\ (v, v^*(d\theta)) & \text{for } |z| > 2\lambda. \end{cases}$$

In particular, the energy density $e_\varepsilon(\tilde{u}_\varepsilon, d - i\tilde{\alpha}_\varepsilon)$ of this couple vanishes for $|z| > 2\lambda$. Also, $1 - |\tilde{u}_\varepsilon| = \chi(1 - |u'_\varepsilon|)$, so that the inequality

$$(1 - |\tilde{u}_\varepsilon|^2)^2 \le (1 - |u'_\varepsilon|^2)^2 \tag{2.15}$$

holds. Moreover, on the region $\Omega_\lambda := \{\lambda < |z| < 2\lambda\}$, using that $(u'_\varepsilon)^*(d\theta)$ is closed we compute

$$d\tilde{\alpha}_\varepsilon = \chi d\alpha'_\varepsilon + d\chi \wedge (\alpha'_\varepsilon - (u'_\varepsilon)^*(d\theta)).$$

Since $\langle \nabla'_\varepsilon u'_\varepsilon, i u'_\varepsilon \rangle = |u'_\varepsilon|^2((u'_\varepsilon)^*(d\theta) - \alpha'_\varepsilon)$, in view of (2.12) we can conclude that

$$\varepsilon |d\tilde{\alpha}_\varepsilon| \le \varepsilon |d\alpha'_\varepsilon| + \frac{2\varepsilon}{\lambda} |u'_\varepsilon|^{-1} |\nabla'_\varepsilon u'_\varepsilon| \le C \frac{1 + \varepsilon/\lambda}{\varepsilon} e^{-c\lambda/\varepsilon} \tag{2.16}$$

on Ω_λ, provided that λ/ε is big enough. Also,

$$d\tilde{u}_\varepsilon = O(|d\chi|(1 - |u'_\varepsilon|)) + O(|d|u'_\varepsilon||) + i\tilde{u}_\varepsilon(d(\chi f) + v^*(d\theta)),$$

and recalling that $v^*(d\theta) = (u'_\varepsilon)^*(d\theta) - df$, we conclude that

$$(d - i\tilde{\alpha}_\varepsilon)\tilde{u}_\varepsilon = O(|d\chi|(1 - |u'_\varepsilon|)) + O(|d|u'_\varepsilon||) + i\chi\tilde{u}_\varepsilon((u'_\varepsilon)^*(d\theta) - \alpha'_\varepsilon).$$

Denoting $\tilde{\nabla}_\varepsilon := d - i\tilde{\alpha}_\varepsilon$ and using that $|d|u'_\varepsilon|| \le |\nabla'_\varepsilon u'_\varepsilon|$, we obtain the decay

$$|\tilde{\nabla}_\varepsilon \tilde{u}_\varepsilon| \le C\frac{1+\varepsilon/\lambda}{\varepsilon}e^{-c\lambda/\varepsilon} \tag{2.17}$$

on Ω_λ.

Choose now $\delta = \delta(\varepsilon) := \varepsilon^{3/4}$, so that $\lambda(\varepsilon)/\varepsilon \to \infty$ as $\varepsilon \to 0$. Since the slices $\exp(B^\perp_{c'\delta}(p))$ are orthogonal to Δ and have area comparable with λ^2, we deduce from (2.15), (2.16) and (2.17) that the energy of the couple $(\tilde{u}_\varepsilon, \tilde{\nabla}_\varepsilon)$ is bounded as follows

$$\begin{aligned} E_\varepsilon(\tilde{u}_\varepsilon, \tilde{\nabla}_\varepsilon) &= 2\pi|k_0|\mathscr{H}^{n-2}(\Delta)(1+o(1)) + O(\delta^2\varepsilon^{-2}e^{-c\delta/\varepsilon}) \\ &= 2\pi|k_0|\mathscr{H}^{n-2}(\Delta) + o(1), \end{aligned}$$

with $o(1)$ an infinitesimal term as $\varepsilon \to 0$.

Denote by $K := \bigcup \operatorname{spt}\partial\Delta$ the $(n-3)$-skeleton of the triangulation. Let us choose $C' > 1$ such that $q \in B_{C'\delta}(K)$ whenever $\operatorname{dist}(q, \mathscr{S}_{n-2}) \le c'\delta$ and $q \notin \bigcup_\Delta V_\delta(\Delta)$. Note that the pairs glue together to give a pair $(\tilde{u}_\varepsilon, \tilde{\nabla}_\varepsilon)$ on the set $M \setminus \bar{B}_{C'\delta}(K)$ by declaring that $(\tilde{u}_\varepsilon, \tilde{\nabla}_\varepsilon)$ is given by (v, ∇_v) on the complement of $\bigcup_\Delta V_\delta(\Delta)$, with ∇_v the connection making v a parallel section. In order to have a pair defined on all of M, we pick a smooth cut-off function ρ_δ defined by

$$\rho_\delta = \begin{cases} 0 & \text{on } B_{2C'\delta}(K), \\ 1 & \text{on } M \setminus B_{4C'\delta}(K), \end{cases} \tag{2.18}$$

satisfying the additional bound $|d\rho_\delta| \le \delta^{-1}$. With ∇_0 a fixed reference connection, we claim that the couple

$$(u_\varepsilon, \nabla_\varepsilon) := (\rho_\delta \tilde{u}_\varepsilon, (1-\rho_\delta)\nabla_0 + \rho_\delta \tilde{\nabla}_\varepsilon)$$

has the desired properties. As a first trivial observation, note that near K the pair $(u_\varepsilon, \nabla_\varepsilon)$ is given by $(0, \nabla_0)$, i.e., the trivial section with the reference connection.

Next, since $\operatorname{vol}(B_r(K)) = O(r^3)$, we have the estimate

$$\lim_{\varepsilon\to 0}\int_{B_{4C'\delta}(K)} (|d\rho_\delta|^2 + \varepsilon^{-2}) \le \lim_{\varepsilon\to 0}(\delta(\varepsilon)^{-2} + \varepsilon^{-2}) \cdot C\delta(\varepsilon)^3 = 0, \tag{2.19}$$

since $\delta(\varepsilon) = \varepsilon^{3/4}$. Fixing again a simplex Δ, we write $\nabla_0 = d - i\alpha_\Delta$ with respect to the chosen trivialization near Δ. Thus,

$$\nabla_\varepsilon = d - i(1-\rho_\delta)\alpha_\Delta - i\rho_\delta\tilde{\alpha}_\varepsilon.$$

Note that $\nabla_\varepsilon u_\varepsilon = \tilde{u}_\varepsilon d\rho_\delta + \rho_\delta \nabla_\varepsilon \tilde{u}_\varepsilon$ and that the trivialization can be chosen to guarantee $|\alpha_\Delta| + |d\alpha_\Delta| \le C(M, L)$. In view of (2.19), in order to show that the energy of the couple $(u_\varepsilon, \nabla_\varepsilon)$ on $B_{4C'\delta}(K)$ is infinitesimal, we just have to show that the two quantities

$$\int_{B_{4C'\delta}(K)\setminus\bigcup_\Delta V_\delta(\Delta)} (|\rho_\delta \nabla_0 v|^2 + \varepsilon^2|F_{(1-\rho_\delta)\nabla_0 + \rho_\delta\nabla_v}|^2)$$

and

$$\int_{B_{4C'\delta}(K)\cap V_\delta(\Delta)} (|\rho_\delta \nabla_0 \tilde{u}_\varepsilon|^2 + \varepsilon^2|F_{(1-\rho_\delta)\nabla_0 + \rho_\delta\tilde{\nabla}_\varepsilon}|^2)$$

converge to zero (since the contribution of $\tilde{\nabla}_\varepsilon \tilde{u}_\varepsilon$ is infinitesimal on $B_{4C'\delta}(K)$). The first assertion follows from (2.11) and the fact that the integrand equals $\varepsilon^2|F_{\nabla_0}|^2$ when the distance from $\mathscr{S}_{n-2}$ is at most $c'\delta$, while elsewhere we have the bounds

$$|\rho_\delta \nabla_0 v| \leq C\delta^{-1}$$

and

$$|F_{(1-\rho_\delta)\nabla_0+\rho_\delta\nabla_v}| \leq C + |d\rho_\delta|(C+|v^*(d\theta)|) \leq C\delta^{-2};$$

indeed, these bounds imply that the integral is bounded by

$$[O(\delta^{-2}) + O(\varepsilon^2) + O(\varepsilon^2\delta^{-4})] \cdot O(\delta^3),$$

which is infinitesimal. As for the second assertion, by (2.19) it is enough to prove that, for $p \in \Delta_\delta$,

$$\int_{\{p\}\times B^2_{3\lambda}} \left(|d\tilde{u}_\varepsilon|^2 + \frac{\varepsilon^2}{\delta^2}|\tilde{\alpha}_\varepsilon|^2\right) \leq C\log(\varepsilon^{-1}).$$

Indeed, since $|\nabla_0\tilde{u}_\varepsilon| \leq C + |d\tilde{u}_\varepsilon|$ and

$$|F_{(1-\rho_\delta)\nabla_0+\rho_\delta\tilde{\nabla}_\varepsilon}| \leq C + |d\rho_\delta|(C+|\tilde{\alpha}_\varepsilon|) + |F_{\tilde{\nabla}_\varepsilon}|,$$

the last claim implies that on each slice the integral is at most $O(\log(\varepsilon^{-1}))$, and the conclusion follows since the set of points p whose slice intersects $B_{4C'\delta}(K)$ has volume $O(\delta)$.

However, by (2.11) and (2.13), dv, $d\chi$, α'_ε and $(u'_\varepsilon)^*(d\theta)$ at the point (p,z) are all bounded by $C\delta^{-1}$ on the region $\{|z| > \lambda = \frac{c'}{3}\delta\}$, which implies $|df| \leq C\delta^{-1}$ and $|f| \leq C$ by (2.14). Since this region has area $O(\delta^2)$, its contribution is bounded. On the other hand, $(\tilde{u}_\varepsilon, \tilde{\alpha}_\varepsilon) = (u'_\varepsilon, \alpha'_\varepsilon)$ on $\{|z| \leq \lambda\}$; using again (2.13), the claim follows.

Finally, note that $J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup 2\pi P$ as currents. Indeed, with the same computations as above, we obtain that $\nabla_0 u_\varepsilon$ is bounded in L^p independently of ε, for any $p < 2$. But $u_\varepsilon \to v$ almost everywhere, hence weakly in $W^{1,p}(M,L)$, which gives

$$J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup J(v) = 2\pi P,$$

again as currents as ε goes to 0. □

2.4 Comparison of the min-max constructions

With the Γ-convergence result established, we turn now to the proof of the min-max comparison described in Theorem 8. The outline of the proof is broadly similar to that of the analogous result of Guaraco [66, Proposition 8.19] in the Allen–Cahn setting. First, we employ Theorem 5 to extract from continuous families of pairs (u,∇) discretized families of $(n-2)$-boundaries with mass bounded above by $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) + o(1)$. To complete the proof of Theorem 8, we then have to show that the homotopy class of this associated family of cycles is determined by that of the family of pairs (u,∇) in the desired way.

The details of the proof are somewhat more involved than their codimension-one analog, since the assignment from pairs (u,∇) to the space of $(n-2)$-boundaries is less explicit, and the homotopy groups of the space of $(n-2)$-boundaries are slightly more complicated. In the next subsection, we recall the relevant definitions from Almgren's min-max methods, and define carefully the min-max values to which Theorem 8 applies. For convenience of the reader we recall the rough statement of the theorem we wish to prove.

Theorem 30 (Min-max comparison)**.** *Let $\mathscr{M}$ be the moduli space of pairs (u,∇) with $u\not\equiv 0$ and Z the space of integral $(n-2)$-boundaries as in Section 1.2 of the Introduction. With respect to the isomorphism $\Phi:\pi_k(\mathscr{M},*)\to\pi_k(Z,0)$ appearing in (1.15), the min-max energies satisfy*

$$\liminf_{\varepsilon\to 0}\mathscr{E}_\varepsilon(\beta)\geq \mathbf{W}^*(\Phi(\beta)) \tag{2.20}$$

for any $\beta\in\pi_k(\mathscr{M},*)$. In particular, the mass of the stationary integral $(n-2)$-varifold V_{YMH} associated to the critical points $(u_\varepsilon,\nabla_\varepsilon)$ by the results of [107] is bounded below by the mass of the corresponding min-max $(n-2)$-varifold V_{GMT} produced by Almgren's min-max construction.

2.4.1 Natural min-max constructions for E_ε

Throughout this section, let $L=\mathbb{C}\times M\to M$ be the *trivial* line bundle over a closed, oriented n-manifold (M^n,g) of dimension $n\geq 3$. Fixing a trivialization of L, the space of pairs (u,∇) consisting of sections $u\in\Gamma(L)$ and Hermitian connections ∇ can then be identified with pairs (u,α), where $u:M\to\mathbb{C}$ is a complex-valued map and $\alpha\in\Omega^1(M)$ is a one-form such that $\nabla=d-i\alpha$.

For a fixed $p>n$, we will view E_ε as a functional on the Banach space $\widehat{X}$ consisting of pairs (u,∇) where $u\in[W^{1,2}\cap L^p](M)$ and $\nabla=d-i\alpha$ for $\alpha\in W^{1,2}(M)$ (with topology induced by the norm $\|du\|_{L^2(M)}+\|u\|_{L^p(M)}+\|\alpha\|_{W^{1,2}(M)}$), equipped with the Finsler structure

$$\|(v,\beta)\|_{(u,\nabla)}:=\|v\|_{L^p(M)}+\|\nabla v\|_{L^2(M)}+\|\beta\|_{L^2(M)}+\|D\beta\|_{L^2(M)}, \tag{2.21}$$

where D is the (Levi-Civita) covariant derivative of the one-form β. It is straightforward to check (cf. [107, Section 7]) that the energies E_ε define C^1 functionals on $\widehat{X}$, and an adaptation of the proof of [107, Proposition 7.6] shows that they satisfy a variant (modulo gauge transformations) of the Palais–Smale condition with respect to the Finsler structure (2.21), making $\widehat{X}$ an appropriate setting for the min-max construction of critical points (provided the nonlinear potential W is modified as described in [107, Section 7]).

Remark 31. The Palais–Smale result stated in [107, Proposition 7.6] for E_ε in $\widehat{X}$ is not quite correct as written when the base manifold M has $H^1(M;\mathbb{Q})\neq 0$. This is due to the fact that a sequence (u_j,∇_j) for E_ε which is Palais–Smale with respect to the natural Banach norm on $\widehat{X}$ may fail to yield another Palais–Smale sequence under the change of gauge $(\phi_j u_j,\nabla_j-\phi_j^*(d\theta))$ for a sequence of harmonic map $\phi_j:M\to S^1$. However, it is easy to check that the Palais–Smale property with respect to the Finsler structure (2.21) is preserved under harmonic change of gauge, and [107, Proposition 7.6] holds with the Banach norm replaced by this Finsler structure.

Though the space $\widehat{X}$ itself is topologically trivial, the functionals E_ε have a rich min-max theory in the $\varepsilon\to 0$ limit, owing to the topology of the moduli space

$$\mathscr{M}:=(\widehat{X}\setminus X_0)/\mathscr{G},$$

where $X_0:=\{(u,\alpha)\in\widehat{X}:u\equiv 0\}$ and $\mathscr{G}:=W^{2,2}(M,S^1)$ is the gauge group. Indeed, writing

$$Y:=\{(u,\alpha)\in\widehat{X}:d^*\alpha=0\},$$

note that there is a natural retraction $\rho_C:\widehat{X}\to Y$ given by passing to the Coulomb gauge

$$\rho_C(u,\alpha):=(e^{-i\varphi_\alpha}u,\alpha-d\varphi_\alpha),$$

where $\varphi_\alpha\in W^{2,2}(M,\mathbb{R})$ is the unique solution of

$$d^*d\varphi_\alpha=d^*\alpha \quad\text{and}\quad \int_M\varphi_\alpha=0.$$

It is clear that the quotient map $Y \setminus X_0 \to \mathscr{M}$ is surjective. The elements of $\mathscr{G}$ sending a given couple in $Y \setminus X_0$ to a couple in the same space are precisely the harmonic maps $\mathscr{H} = \mathrm{Harm}(M, S^1)$, so we can identify $\mathscr{M}$ (homeomorphically) with the quotient

$$\mathscr{M} = (Y \setminus X_0)/\mathscr{H}.$$

Moreover, note that the harmonic S^1-valued maps $\mathscr{H}$ contain S^1 as a subgroup (by identification with the constant maps), and the quotient $\mathscr{H}/S^1$ has a natural identification

$$\mathscr{H}/S^1 \cong [M : S^1] \cong H^1(M;\mathbb{Z}),$$

since each homotopy class in $[M : S^1]$ is uniquely represented in $\mathscr{H}$ up to rotations. We can then view $\mathscr{M}$ as the quotient

$$\mathscr{M} = [(Y \setminus X_0)/S^1]/H^1(M;\mathbb{Z}),$$

of the quotient space $(Y \setminus X_0)/S^1$ by the free and properly discontinuous action of $H^1(M;\mathbb{Z})$. Moreover, we have the following facts, allowing to extract the algebraic topology invariants of $\mathscr{M}$.

Proposition 32. *The projection $Y \setminus X_0 \to (Y \setminus X_0)/S^1$ is a fiber bundle and, hence, a weak fibration. The former space has trivial homotopy groups, while the latter is weakly homotopy equivalent to $\mathbb{CP}^\infty$, and is the universal cover of $\mathscr{M}$.*

Proof. Let $Q := (Y \setminus X_0)/S^1$ and denote $\pi : Y \setminus X_0 \to Q$ the projection. Given $(u, \alpha) \in Y \setminus X_0$, we can find a measurable set $E \subseteq M$ such that $\int_E u \neq 0$. In particular, there exists $\delta > 0$ such that $\int_E v \neq 0$ for all couples (v, β) with distance less than δ from the S^1-orbit of (u, ∇)—namely, such that $\|(v, \beta) - e^{i\theta} \cdot (u, \alpha)\|_{\hat{X}} < \delta$ for some $e^{i\theta} \in S^1$. These couples form an open set $\pi^{-1}(U)$, for U open in the quotient Q. It is then easy to check that the map

$$\pi^{-1}(U) \to S^1 \times U, \quad (v, \beta) \mapsto \left(\int_E v / |\int_E v|, \pi((v, \beta)) \right)$$

gives a local trivialization over U. Hence, π is a fiber bundle and thus a weak fibration (see [71, Proposition 4.48]).

To check the second statement, note that Q (deformation) retracts onto $\hat{S}/S^1$, where $\hat{S}$ is the unit sphere of the Banach space $[W^{1,2} \cap L^p](M, \mathbb{C})$, viewed as a subset of $\hat{X}$ with trivial connection component. Given a dense, linearly independent set $\{u_k\}_{k=1}^\infty$ in this Banach space, we denote by H^ℓ the linear span of $\{u_1, \dots, u_\ell\}$ and by $\pi_\ell : [W^{1,2} \cap L^p](M, \mathbb{C}) \to H^\ell$ the nearest point projection, which is well-defined and continuous since H^ℓ is finite-dimensional and the Banach space is strictly convex.

Letting $\hat{S}^\ell := \hat{S} \cap H^\ell$, note that the union $P := \bigcup_\ell (\hat{S}^\ell/S^1)$, endowed with the topology induced by the subspaces $\hat{S}^\ell/S^1$, is homeomorphic to $\mathbb{CP}^\infty$, and the identity map $i : P \to \hat{S}/S^1$ is continuous. We claim that, for any compact set $K \subset \hat{S}/S^1$, the inclusion $K \hookrightarrow \hat{S}/S^1$ can be deformed to a map $K \to \hat{S}_\ell/S^1$ for some ℓ (within maps into $\hat{S}/S^1$). This implies that i induces isomorphisms i_* on homotopy groups, because then any map $S^k \to \hat{S}/S^1$ can be deformed to a map with values in $\hat{S}^\ell/S^1$ for some ℓ (hence i_* is surjective), and a homotopy in $\hat{S}/S^1$ between two maps $S^k \to \hat{S}^\ell/S^1$ can be deformed to a homotopy in $\hat{S}^{\ell'}/S^1$ with $\ell' \geq \ell$ (hence i_* is injective). To prove the claim, note that for any $[u] \in \hat{S}/S^1$ there exists ℓ such that the distance from u to H^ℓ is less than 1, and the same holds on a neighborhood of $[u]$. By compactness of K, we can find ℓ such that this is true for all the elements of K. The map

$$([u], t) \mapsto \frac{(1-t)u + t\pi_\ell(u)}{\|(1-t)u + t\pi_\ell(u)\|_{\hat{X}}}$$

gives the desired deformation. The fact that $\hat{S}$, and hence Y/X_0, have trivial homotopy groups is proved in the same way. The last conclusion follows from the fact that $\mathbb{CP}^\infty$ is simply connected. □

We therefore conclude that the path-connected space $\mathscr{M}$ has $\pi_1(\mathscr{M}) \cong H^1(M;\mathbb{Z})$, $\pi_2(\mathscr{M}) \cong \mathbb{Z}$, and $\pi_k(\mathscr{M}) = 0$ for $k \geq 3$; or equivalently, for $k > 0$,

$$\pi_k(\mathscr{M}) \cong H_{n-2+k}(M;\mathbb{Z}).$$

The results of this section concern the min-max energies associated to the generator of $\pi_2(\mathscr{M})$, and to each class $\lambda \in H_{n-1}(M;\mathbb{Z}) \cong \pi_1(\mathscr{M})$ (with basepoint the trivial pair $(u_0 \equiv 1, \nabla_0 \equiv d) \mod \mathscr{G}$). In practice, we work with their lifts to maps $\bar{D}^2 \to \widehat{X}$ and $[0,1] \to \widehat{X}$.

As in [107], consider the collection

$$\mathscr{C}_2 \subset C^0(\bar{D}^2, \widehat{X})$$

of continuous families

$$\bar{D}^2 \ni y \mapsto (u_y, \nabla_y) \in \widehat{X}$$

parametrized by the closed unit disk $\bar{D}^2 \subset \mathbb{C}$, subject to the boundary condition

$$u_y \equiv y \quad \text{and} \quad \nabla_y \equiv d \quad \text{for } y \in \partial D^2 = S^1.$$

By the long exact sequence for homotopy groups in weak fibrations, families in $\mathscr{C}_2$ (avoiding X_0) descend to the generators of $\pi_2(\mathscr{M})$. It was shown in [107, Section 7] that the associated min-max energies

$$\mathscr{E}_\varepsilon(\mathscr{C}_2) := \inf_{F \in \mathscr{C}_2} \max_{y \in \bar{D}^2} E_\varepsilon(F_y) \tag{2.22}$$

are uniformly bounded from above and below as $\varepsilon \to 0$, arise as the energies $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon)$ of nontrivial critical points $(u_\varepsilon, \nabla_\varepsilon)$ for E_ε, and converge subsequentially to the mass of a (nontrivial) stationary integral $(n-2)$-varifold, up to a factor of 2π. Likewise, for each nontrivial $\lambda \in H_{n-1}(M;\mathbb{Z})$, we can consider the collection

$$\mathscr{C}_\lambda \subset C^0([0,1], \widehat{X})$$

of continuous families $[0,1] \ni t \mapsto (u_t, \nabla_t) \in \widehat{X}$ satisfying

$$(u_0, \nabla_0) \equiv (1, d), \quad (u_1, \nabla_1) \equiv (\phi, d - i\phi^*(d\theta)),$$

where $\phi \in C^\infty(M, S^1)$ is a map in the homotopy class dual to λ (i.e., generic fibers of ϕ are homologous to λ). Families in $\mathscr{C}_\lambda$ (avoiding X_0) descend to loops in $\mathscr{M}$, whose class in $\pi_1(\mathscr{M})$ is determined by λ, and we will likewise consider their min-max energies

$$\mathscr{E}_\varepsilon(\lambda) := \inf_{F \in \mathscr{C}_\lambda} \max_{t \in [0,1]} E_\varepsilon(F_t).$$

Remark 33. Note that a family as above, with energy bounded by a given Λ (fixed), must avoid the degenerate set of couples X_0 for ε small enough. Using Proposition 32, one can check that the min-max values defined above coincide with the corresponding ones for the homotopy groups of $\mathscr{M}$.

2.4.2 Natural min-max constructions for the $(n-2)$-mass functional

By Almgren's thesis [6], we know that the space $Z \subseteq \mathscr{Z}_{n-2}(M;\mathbb{Z})$ of integral $(n-2)$-boundaries in M, equipped with the flat topology, has homotopy groups identical to those of $\mathscr{M}$; namely,

$$\pi_k(Z,0) \cong H_{n-2+k}(M;\mathbb{Z})$$

for $k > 0$, while $\pi_0(Z) = 0$. In [10] (see also [109]), Almgren associates to each class in $\pi_k(\mathscr{Z}_m(M;\mathbb{Z}))$ a stationary integral k-varifold by means of a discretized min-max construction, which replaces continuous families of cycles in the flat topology with discrete families satisfying an approximate continuity condition with respect to the stronger mass topology. For our comparison results, it is convenient to work with discrete families which are fine *in flat norm* and exhibit *no concentration of mass*; by the interpolation arguments of [97, Section 13] and [94, Theorem 2.10], the associated min-max masses coincide with the masses of the stationary varifolds produced by Almgren.

Remark 34. While Theorems 2.10 and 2.11 of [94] are stated for cycles with $\mathbb{Z}/2\mathbb{Z}$ coefficients, the coefficient group plays no role in these arguments.

Following the notation of [94, Section 2], for $m = 1$ or 2, denote by I^m the m-cube $I^m = [0,1]^m$, and for $j \in \mathbb{N}$, denote by $I(1,j)$ the cube complex on I^1 with 1-cells (or edges)

$$[0,3^{-j}],[3^{-j},2\cdot 3^{-j}],\ldots,[1-3^{-j},1]$$

and 0-cells (or vertices) $[0],[3^{-j}],\ldots,[1-3^{-j}],[1]$. Likewise, denote by $I(2,j)$ the cell complex

$$I(2,j) = I(1,j)\otimes I(1,j)$$

on I^2 given by subdividing I^2 into 3^{2j} squares of area 3^{-2j}, and denote by $I(m,j)_k$ the collection of k-cells of $I(m,j)$. Given an assignment $\phi : I(m,j)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z})$, we will say that it has (flat) fineness $\mathbf{f}(\phi) < \delta$ if

$$\mathscr{F}(\phi(x),\phi(y)) < \delta \text{ for all adjacent vertices } x,y \in I(m,j)_0.$$

If $\phi : I(m,j)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z})$ satisfies $\phi(x) = 0$ for $x \in \partial I^m$ and $\mathbf{f}(\phi) < \delta$ for $\delta < \delta_M$ sufficiently small, then Almgren's construction [6] assigns to ϕ a homology class $\Psi(\phi) \in H_{n-2+m}(M;\mathbb{Z})$, as follows. For each (oriented) one-cell $e = [x,y] \in I(m,j)_1$, provided $\delta > 0$ is sufficiently small, we can find an integral $(n-1)$-current $S_e \in \mathbf{I}_{n-1}(M;\mathbb{Z})$ such that

$$\partial S_e = \phi(y) - \phi(x) \quad \text{and} \quad \mathbb{M}(S_e) \leq \varepsilon_M$$

for a given small constant $\varepsilon_M > 0$. If $m = 1$, then summing over all one-cells $e \in I(1,j)_1$ gives an $(n-1)$-cycle

$$S = \textstyle\sum_{e\in I(1,j)_1} S_e \in \mathscr{Z}_{n-1}(M;\mathbb{Z})$$

whose homology class $\Psi(\phi) := [S] \in H_{n-1}(M;\mathbb{Z})$ does not depend on the choice of small-mass fill-ins S_e. If $m = 2$, then for each 2-cell $\square \in I(2,j)_2$ we denote by $S_\square \in \mathscr{Z}_{n-1}(M;\mathbb{Z})$ the $(n-1)$-cycle $S_\square = \sum_{e\in\partial\square} S_e$ given by summing the fill-ins S_e over all oriented edges e of $\partial\square$, and consider the (unique) n-current $Q_\square \in \mathbf{I}_n(M;\mathbb{Z})$ such that

$$\partial Q_\square = S_\square \quad \text{and} \quad \mathbb{M}(Q_\square) < \frac{\mathrm{vol}(M)}{2}.$$

Summing over all 2-cells $\square \in I(2,j)_2$ then gives an n-cycle

$$Q = \textstyle\sum_{\square\in I(2,j)_2} Q_\square \in \mathscr{Z}_n(M;\mathbb{Z})$$

whose homology class $\Psi(\phi) := [Q] \in H_n(M;\mathbb{Z})$ is independent of the choice of small-mass fill-ins S_e.

Now, for $\eta > 0$ and a discrete family

$$\phi : I(m, j)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z}),$$

define the quantity

$$\mathbf{m}(\phi,\eta) := \sup\{\|\phi(x)\|(B_\eta(p)) \mid x \in I(m, j)_0,\ p \in M\},$$

giving the maximum amount of mass of a cycle in the family inside a ball of radius η. For $\delta \in (0,\delta_M)$ and $\lambda \in H_{n-2+m}(M;\mathbb{Z})$, and a constant $C_0 = C_0(M,\lambda) < \infty$ to be chosen later, denote by $\mathscr{A}_\delta(\lambda)$ the collection of families

$$\phi : I(m, j)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z})$$

such that

$$\mathbf{f}(\phi) < \delta, \quad \sup_{r>\delta} \frac{\mathbf{m}(\phi,r)}{r^{n-2}} \le C_0, \tag{2.23}$$

and

$$\Psi(\phi) = \lambda \in H_{n-2+m}(M;\mathbb{Z}).$$

Then consider the approximate min-max widths

$$\mathbf{W}_\delta(\lambda) := \inf\Big\{ \max_{y\in I(m,j)_0} \mathbb{M}(\phi(y)) \mid \phi \in \mathscr{A}_\delta(\lambda)\Big\}, \tag{2.24}$$

and define the min-max width

$$\mathbf{W}(\lambda) := \inf\Big\{ \liminf_{k\to\infty} \max_{y\in I(m,j_k)_0} \mathbb{M}(\phi_k(y))\Big\}, \tag{2.25}$$

where the infimum is taken over all sequences $\phi_k : I(m, j_k)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z})$ such that $\delta_M > \mathbf{f}(\phi_k) \to 0$, $\limsup_{k\to\infty} \mathbf{m}(\phi_k,r) \to 0$ as $r \to 0$, and $\Psi(\phi_k) = \lambda$. Clearly,

$$\mathbf{W}(\lambda) \le \lim_{\delta\to 0} \mathbf{W}_\delta(\lambda) = \sup_{\delta>0} \mathbf{W}_\delta(\lambda). \tag{2.26}$$

Since we are ruling out concentration of mass in the limit, we can appeal to the interpolation arguments of [97, Section 13] and [94, Theorem 2.10] to deduce that the widths $\mathbf{W}(\lambda)$ coincide with Almgren's min-max widths, and are therefore realized as the masses of stationary integral $(n-2)$-varifolds in M.

We can now state a more precise version of Theorem 8.

Theorem 35. *The min-max energies $\mathscr{E}_\varepsilon(\mathscr{C}_2)$ and $\mathscr{E}_\varepsilon(\lambda)$ for $\lambda \in H_{n-1}(M;\mathbb{Z})$ satisfy*

$$\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(\mathscr{C}_2) \ge 2\pi \mathbf{W}([M]) \tag{2.27}$$

and

$$\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(\lambda) \ge 2\pi \mathbf{W}(\lambda). \tag{2.28}$$

The remainder of the section is devoted to its proof.

2.4.3 Taming min-max families to avoid energy concentration

To ensure that the min-max energies $\mathscr{E}_\varepsilon$ are bounded below by the masses of cycles satisfying (2.23), we first argue that the energies $\mathscr{E}_\varepsilon$ are almost achieved as the maximum energy in families (u_y, ∇_y) satisfying a uniform energy density bound

$$\int_{B_r(p)} e_\varepsilon(u_y, \nabla_y) \leq Cr^{n-2}$$

for $\varepsilon(M, \delta) > 0$ sufficiently small and $r \geq \delta$.

Lemma 36. *Given $\delta > 0$ and $\Lambda < \infty$, there exists $C(M, \Lambda) < \infty$ such that the following holds. If $\varepsilon < \delta$, for any family $F \in \mathscr{C}_2 \subset C^0(\bar{D}^2, \widehat{X})$ (or $F \in \mathscr{C}_\lambda \subset C^0([0,1], \widehat{X})$ for $\lambda \in H_{n-1}(M;\mathbb{Z})$) satisfying*

$$\max_y E_\varepsilon(F_y) < \Lambda, \tag{2.29}$$

there exists another family $F' = (u', \nabla') \in \mathscr{C}_2$ (resp. $\mathscr{C}_\lambda$) of smooth couples such that

$$\max_y E_\varepsilon(F'_y) < \Lambda$$

and

$$\max_{y, r \geq \delta, p \in M} \frac{\int_{B_r(p)} e_\varepsilon(u'_y, \nabla'_y)}{r^{n-2}} \leq C(M, \Lambda).$$

Proof. First, given a family $F \in \mathscr{C}_2$ or $F \in \mathscr{C}_\lambda$ satisfying (2.29), we can apply a uniform mollification to obtain a new family $\widetilde{F}$ also satisfying (2.29) that defines a continuous map into the space of smooth pairs (u_y, ∇_y), equipped with the C^∞ topology. Thus, we may assume without loss of generality that the original family F defines a continuous map into the space of smooth pairs.

In Section 2.5 below, we investigate a natural L^2 gradient flow system for the energies E_ε, given by a flow of pairs $(u_t, \nabla_t = d - i\alpha_t)$ satisfying

$$\partial_t u_t = -\nabla_t^* \nabla_t + \frac{1}{2\varepsilon^2}(1 - |u_t|^2) u_t \tag{2.30}$$

and

$$\partial_t \alpha_t = -d^* d\alpha_t + \varepsilon^{-2} \langle iu_t, \nabla_t u_t \rangle. \tag{2.31}$$

As discussed in Section 2.5, it is not difficult to establish long-time existence for the flow, and continuous dependence on smooth initial data. Moreover, it is obvious that minimizers of E_ε are stationary under the flow; as a consequence, given a family $y \mapsto F_y = (u_y, \nabla_y)$ in $\mathscr{C}_2$ (resp. $\mathscr{C}_\lambda$) mapping continuously into the space of smooth pairs as above, we may define a new family $F' \in \mathscr{C}_2$ (resp. $\mathscr{C}_\lambda$) by letting $F'_y = (u'_y, \nabla'_y)$ be the solution of (2.30)–(2.31) at time $t = 2$ with initial data $(u_y, \nabla_y) = F_y$. Since the gradient flow decreases energy, it is obvious that

$$\max_y E_\varepsilon(F'_y) \leq \max_y E_\varepsilon(F_y) < \Lambda.$$

Finally, by Proposition 49 below (the main result of Section 2.5), we have the density estimate

$$\int_{B_r(p)} e_\varepsilon(u'_y, \nabla'_y) \leq C(M, \Lambda) r^{n-2}$$

for all $r \geq \varepsilon$, so that the family F' satisfies the desired properties. □

Remark 37. Note, moreover, that we may always deform an initial family (u_y, ∇_y) to one (v_y, ∇_y) with $|v_y| \leq 1$ pointwise, without increasing the energy, by setting $v_y := \frac{u_y}{\max\{1,|u_y|\}}$. In particular, for the purposes of estimating the min-max energies, we may always assume that our families (u_y, ∇_y) satisfy $|u_y| \leq 1$ pointwise, without loss of generality.

To prove Theorem 35, we will use this lemma in concert with the following technical lemma, which follows in a straightforward way from the results of Section 3.

Lemma 38. *Given $\Lambda, C_0 \in (0,\infty)$, for any $\delta > 0$ there exists $\varepsilon_0(M,\Lambda,\delta,C_0)$ such that, if $\varepsilon \in (0,\varepsilon_0)$ and (u,∇) is a smooth pair satisfying $|u| \leq 1$,*

$$E_\varepsilon(u,\nabla) \leq \Lambda,$$

and

$$\max_{r\geq\delta,\, p\in M} r^{2-n} \int_{B_r(p)} e_\varepsilon(u,\nabla) \leq C_0,$$

then there exist a smooth $\phi : M \to S^1$ and a unit section $v \in \mathscr{U}_p(L)$ (i.e., $v \in W^{1,p}(M,S^1)$) for all $p \in (1,\frac{n}{n-1})$, satisfying

$$\|u - v\|_{L^1(M)} \leq \delta, \tag{2.32}$$

$$\|d(\phi^{-1}v)\|_{L^p(M)} \leq C(p,M,\Lambda), \tag{2.33}$$

$$\mathbb{M}(J(v)) \leq \Lambda, \tag{2.34}$$

and

$$\|J(v)\|(B_r(p)) \leq 2C_0 r^{n-2} \tag{2.35}$$

for all $p \in M$ and $r \geq \delta$. Moreover, the map ϕ is chosen such that

$$\|\phi^*(d\theta) - \Pi(\alpha)\|_{L^2(M)} \leq C(M),$$

where $\nabla = d - i\alpha$ and $\Pi(\alpha)$ is the closed component of the Hodge decomposition of α.

Proof. The proof follows a straightforward argument by contradiction, using the analysis of Section 2.2. If the statement were false, then we could find some fixed $\delta > 0$, a sequence $\varepsilon_j \to 0$, and pairs $(u_j, \nabla_j = d - i\alpha_j)$ such that

$$E_{\varepsilon_j}(u_j,\nabla_j) \leq \Lambda, \tag{2.36}$$

and

$$\max_{r\geq\delta,\, p\in M} r^{2-n} \int_{B_r(p)} e_{\varepsilon_j}(u_j,\nabla_j) \leq C_0, \tag{2.37}$$

for which there are no $\phi_j : M \to S^1$ and $v_j \in \mathscr{U}_p(L)$ satisfying (2.32)–(2.35). By Lemma 22 (and its proof), we can find maps $\phi_j : M \to S^1$ such that

$$\|d(\phi_j^{-1}u_j)\|_{L^p(M)} \leq C(p,M,\Lambda) \quad \text{and} \quad \|\alpha_j - \phi_j^*(d\theta)\|_{L^p(M)} \leq C(p,M,\Lambda)$$

for every $p \in (1,\frac{n}{n-1})$, while

$$\|\phi_j^*(d\theta) - \Pi(\alpha_j)\|_{L^2(M)} \leq C(M).$$

In particular, the maps $\phi_j^{-1}u_j$ are uniformly bounded in $W^{1,p}$ for $p \in (1,\frac{n}{n-1})$, and—as discussed in the proof of Theorem 1.2(i)—a subsequence therefore converges strongly in L^1 and weakly in $W^{1,p}$ to a singular unit section $v \in \mathscr{U}_p(L)$ (i.e., $v \in W^{1,p}(M,S^1)$, since L is now trivial), while the gauge-invariant $(n-2)$-currents $J(u_j,\nabla_j)$ converge weakly to $J(v)$. Moreover, by (2.36), (2.37), and the lower semicontinuity of mass under weak convergence, we see that

$$\mathbb{M}(J(v)) \leq \liminf_{j\to\infty} \mathbb{M}(J(u_j,\nabla_j)) \leq E_{\varepsilon_j}(u_j,\nabla_j) \leq \Lambda$$

and

$$\|J(v)\|(B_r(p)) \leq \liminf_{j\to\infty} \|J(u_j,\nabla_j)\|(B_r(p)) \leq \liminf_{j\to\infty} \int_{B_r(p)} e_{\varepsilon_j}(u_j,\nabla_j) \leq C_0 r^{n-2}$$

for all $r \geq \delta$ and $p \in M$. In particular, for j sufficiently large, we see that ϕ_j and $\phi_j v$ satisfy (2.32)–(2.35) (in place of ϕ and v) with respect to u_j, giving the desired contradiction. □

Remark 39. In particular, recall from Corollary 21 that for any $v \in \mathscr{U}_p(L)$ with $\mathbb{M}(J(v)) < \infty$, we have $J(v) = 2\pi\Gamma$ for an integral $(n-2)$-cycle $\Gamma \in \mathscr{Z}_{n-2}(M;\mathbb{Z})$.

2.4.4 Filling in cycles by filling maps

The results of the preceding subsection will allow us to relate min-max families $F \in \mathscr{C}_2$ or $F \in \mathscr{C}_\lambda$ for the energies E_ε to certain discrete families of $(n-2)$-cycles with the desired mass bounds. In what follows, we collect some technical lemmas which will allow us to identify the images of those families of $(n-2)$-cycles under the Almgren isomorphism.

Lemma 40. *Given $u,v \in W^{1,p}(M,S^1)$, for $p \in (1,2)$, there exists $w \in W^{1,p}(M\times[0,1],S^1)$ satisfying the boundary condition*

$$w(x,0) = u(x,0), \quad \text{and} \quad w(x,1) = v(x,1),$$

in the trace sense, for which the estimate

$$\|\partial_t w\|_{L^p(M\times[0,1])} \leq C(p)\|u-v\|_{L^p(M)}$$

holds, and such that the pushforward $\pi_*[J(w)]$ of the distributional Jacobian $J(w)$ under the projection $\pi : M\times[0,1] \to M$ satisfies

$$\mathbb{M}(\pi_*[J(w)]) \leq C\int_M |u-v|(|du|+|dv|).$$

Proof. The proof combines ideas from [26, Section 3] and [70]. First, we mollify u and v to obtain maps $u_\delta, v_\delta \in C^\infty(M,D^2)$ with

$$\|u_\delta - u\|_{W^{1,p}(M)} + \|v_\delta - v\|_{W^{1,p}(M)} < \delta.$$

Let $w_\delta : M\times[0,1] \to D^2$ be the linear interpolation

$$w_\delta(x,t) := (1-t)u_\delta(x) + tv_\delta(x).$$

Consider then the $(n-1)$-currents

$$\Gamma_y^\delta := \pi_*[w_\delta^{-1}\{y\}]$$

given by pushing forward the $(n-1)$-dimensional submanifold $w_\delta^{-1}\{y\}$ for every regular value $y \in D$. Then for any $\zeta \in \Omega^{n-1}(M)$, the coarea formula gives

$$\begin{aligned}\langle \Gamma_y^\delta, \zeta\rangle &= \int_{w_\delta^{-1}\{y\}} \pi^*(\zeta)\\ &= \int_{w_\delta^{-1}\{y\}} *\Big(\zeta \wedge \frac{J(w_\delta)}{|J(w_\delta)|}\Big)\, d\mathscr{H}^{n-1}\\ &= \int_{w_\delta^{-1}\{y\}} *(\zeta \wedge dt \wedge \iota_{\partial_t} J(w_\delta))|J(w_\delta)|^{-1}\, d\mathscr{H}^{n-1}.\end{aligned}$$

In particular, since

$$|\iota_{\partial_t} J(w_\delta)| \le 2|\partial_t w_\delta||dt \wedge dw_\delta| \le 2|u_\delta - v_\delta|(|du_\delta| + |dv_\delta|),$$

it follows that

$$\mathbb{M}(\Gamma_y^\delta) \le \int_{w_\delta^{-1}\{y\}} \frac{|u_\delta - v_\delta|(|du_\delta| + |dv_\delta|)}{\frac{1}{2}|J(w_\delta)|}\, d\mathscr{H}^{n-1},$$

and applying the coarea formula for w_δ, we arrive at

$$\int_D \mathbb{M}(\Gamma_y^\delta) \le \int_M |u_\delta - v_\delta|(|du_\delta| + |dv_\delta|). \tag{2.38}$$

Now, for each $y \in D_{1/4}$, fix a map $\Phi_y \in C^\infty(D_1 \setminus \{y\}, S^1)$ satisfying

$$\Phi_y(z) = \begin{cases} \frac{z-y}{|z-y|} & \text{for } z \in D_{1/4}(y) \subset D_{1/2},\\ \frac{z}{|z|} & \text{for } |z| \ge 3/4,\end{cases} \tag{2.39}$$

and

$$|d\Phi_y(z)| \le \frac{C}{|z-y|} \text{ on } D_1$$

for some fixed constant C. Then, writing

$$w_{\delta,y} := \Phi_y \circ w_\delta,$$

if $y \in D_{1/4}$ is a regular value of w_δ, we see that $w_{\delta,y}$ belongs to $W^{1,p}(M \times [0,1], S^1)$ and satisfies $J(w_{\delta,y}) = 2\pi w_\delta^{-1}\{y\}$, as well as

$$\|dw_{\delta,y}\|_{L^p(M\times[0,1])}^p \le C \int_{M\times[0,1]} |dw_\delta|(x,t)^p |w_\delta(x,t) - y|^{-p}\, dx\, dt$$

and

$$\|\partial_t w_{\delta,y}\|_{L^p(M\times[0,1])}^p \le C \int_{M\times[0,1]} |u_\delta - v_\delta|^p(x) |w_\delta(x,t) - y|^{-p}\, dx\, dt.$$

Integrating the latter two estimates over $y \in D_{1/4}$ and applying Fubini's theorem, we see that

$$\begin{aligned}\int_{D_{1/4}} \|dw_{\delta,y}\|_{L^p(M\times[0,1])}^p\, dy &\le \int_{M\times[0,1]} |dw_\delta(x,t)|^p \Big(\int_{D_{1/4}} |w_\delta(x,t) - y|^{-p}\, dy\Big)\, dx\, dt\\ &\le C(p)\|dw_\delta\|_{L^p(M\times[0,1])}^p,\end{aligned}$$

and similarly

$$\int_{D_{1/4}} \|\partial_t w_{\delta,y}\|_{L^p(M\times[0,1])}^p \le C(p)\|u_\delta - v_\delta\|_{L^p(M)}^p.$$

Combining these estimates together with (2.38), we can find $y = y_\delta \in D_{1/4}$ such that

$$\|dw_{\delta,y}\|_{L^p(M\times[0,1])} \le C(p)\|dw_\delta\|_{L^p(M\times[0,1])}$$

and

$$\|\partial_t w_{\delta,y}\|_{L^p(M\times[0,1])} \le C(p)\|u_\delta - v_\delta\|^p_{L^p(M)},$$

together with

$$\mathbb{M}(\pi_*[J(w_{\delta,y})]) = 2\pi\mathbb{M}(\Gamma^\delta_y) \le C\int_M |u_\delta - v_\delta|(|du_\delta| + |dv_\delta|).$$

Since w_{δ,y_δ} is bounded in $W^{1,p}(M\times[0,1],S^1)$, we may take a subsequential limit

$$w = \lim_{\delta\to 0} w_{\delta,y_\delta}$$

as $\delta \to 0$, to obtain a map $w \in W^{1,p}(M\times[0,1],S^1)$ with the desired properties. □

Remark 41. On a manifold with Lipschitz boundary $(N,\partial N)$ of dimension m (e.g. $N = M\times[0,1]$ or $N = M\times[0,1]^2$ where M is our underlying manifold), given a map $w \in W^{1,p}(N,S^1)\cap W^{1,p}(\partial N,S^1)$, recall that the *(interior) distributional Jacobian* $J(w)$ is the $(m-2)$-current given by

$$\langle J(w),\zeta\rangle := \int_N w^*(d\theta)\wedge d\zeta + \int_{\partial N} w^*(d\theta)\wedge\zeta. \tag{2.40}$$

In the sequel, we endow $M\times[0,1]$ with the orientation such that $M\times\{1\}$ is oriented as M. Using the product orientation on $M\times[0,1]^2$ and the induced one on the boundary $M\times\partial[0,1]^2$, note that $\tau\wedge v$ is positively oriented on the latter manifold when v is a positively oriented n-vector of M and τ is tangent to $\partial[0,1]^2$, pointing counter-clockwise.

Remark 42. The distributional Jacobian interacts well with concatenation of maps. Indeed, for any two $w_1,w_2 \in W^{1,p}(M\times[0,1],S^1)\cap W^{1,p}(M\times\{0,1\},S^1)$, if $w_1 * w_2 : M\times[0,1]\to S^1$ is the usual concatenation, we have that

$$\pi_*[J(w_1 * w_2)] = \pi_*[J(w_1)] + \pi_*[J(w_2)].$$

Reasoning by induction one can then prove that the above identity holds for an arbitrary finite concatenation.

Lemma 43. *Let $I^2 = [0,1]^2$, and let $F \in W^{1,p}(M\times I^2,S^1)\cap W^{1,p}(M\times\partial I^2,S^1)$. Letting $\pi : M\times I^2 \to M$ be the canonical projection, the n-current*

$$\Xi := \pi_*[J(F)] \in \mathscr{D}_n(M)$$

depends only on $F|_{M\times\partial I^2}$, is given by

$$\langle \Xi, \varphi\,\mathrm{dvol}_g\rangle = 2\pi\int_M \varphi(x)\deg\left(F|_{\{x\}\times\partial I^2}\right)dx,$$

and satisfies

$$\mathbb{M}(\Xi) \le \|\partial_t F\|_{L^1(M\times\partial I^2)},$$

where $\partial_t F$ denotes the partial derivative of F along the ∂I^2 direction.

Proof. Since any n-form $\zeta \in \Omega^n(M^n)$ is closed, (2.40) implies

$$\langle J(F), \pi^*\zeta \rangle = \int_{M\times\partial I^2} F^*(d\theta)\wedge\pi^*\zeta = \int_M \zeta(x)\Big(\int_{\{x\}\times\partial I^2} F^*(d\theta)\Big)\,dx,$$

from which the desired results follow. □

Hence, if $F_1, F_2 \in W^{1,p}(M\times I^2, S^1)\cap W^{1,p}(M\times\partial I^2, S^1)$ are two such maps, satisfying

$$F_1(x,1,t) = F_2(x,0,t),$$

and $\Phi = F_1 * F_2$ is the map given by concatenating along one face of the square, i.e.,

$$\Phi(x,s,t) := \begin{cases} F_1(x,2s,t) & \text{on } M\times[0,1/2]\times I, \\ F_2(x,2s-1,t) & \text{on } M\times[1/2,1]\times I, \end{cases} \tag{2.41}$$

we have

$$\pi_*[J(F_1)] + \pi_*[J(F_2)] = \pi_*[J(F_1 * F_2)]. \tag{2.42}$$

Of course, the same statement holds if we define $F_1 * F_2$ by concatenation along any other face of I^2.

2.4.5 One-parameter families corresponding to $\pi_1(\mathscr{Z}_{n-2}(M;\mathbb{Z}),0)$.

We come now to the proof of the second inequality in Theorem 35, comparing the one-parameter min-max constructions for the $U(1)$-Higgs energies and the $(n-2)$-mass. That is, for any $\lambda \in H_{n-1}(M;\mathbb{Z})$, our goal in this section is to prove that

$$\liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(\lambda) \geq 2\pi\mathbf{W}(\lambda). \tag{2.43}$$

To this end, fix $0 \neq \lambda \in H_{n-1}(M;\mathbb{Z})$ and a small constant $\delta > 0$. Let $\psi \in C^\infty(M,S^1)$ be a map whose (regular) fibers lie in $\lambda \in H_{n-1}(M;\mathbb{Z})$. Recall that, by definition of $\mathscr{C}_\lambda$, the endpoints (u_0,∇_0) and (u_1,∇_1) of a family $(u_t,\nabla_t)_{t\in[0,1]}$ in $\mathscr{C}_\lambda$ are given by

$$(u_0,\nabla_0) = (1,d) \quad \text{and} \quad (u_1,\nabla_1) = (\psi, d - i\psi^*(d\theta)).$$

We claim that

$$\Lambda := \liminf_{\varepsilon\to 0} \mathscr{E}_\varepsilon(\lambda) < \infty. \tag{2.44}$$

Proof of (2.44). Since the proof is very similar to the one for two-parameter families, given in [107, Section 7], we just sketch it. Identifying M with a simplicial complex $\tilde{M}$ in some Euclidean space $\mathbb{R}^L$, by means of a triangulation of M, we can find a piecewise affine map $\tilde{\psi} : \mathbb{R}^L \to \mathbb{C}$ such that $\tilde{\psi} = 1$ far from $\tilde{M}$ and $|\tilde{\psi} - \psi| < \frac{1}{2}$ on $\tilde{M}$ (provided the triangulation was chosen fine enough). Let y be a small regular value of $\tilde{\psi}$.

By composing ψ with a piecewise affine homeomorphism of $\mathbb{C}$, we can assume that $y = 0$ and that $\tilde{\psi}^{-1}(\bar{D}_{1/2})$ is an $O(\varepsilon)$-neighborhood of $\tilde{\psi}^{-1}(0)$, with the bound $|d\tilde{\psi}| = O(\varepsilon^{-1})$. In particular, the fiber $\tilde{\psi}^{-1}(0)$ is contained in finitely many affine $(L-2)$-planes P_j. With a slight perturbation of $\tilde{M}$, which does not intersect $\tilde{\psi}^{-1}(\bar{D}_{1/2})$, we can assume that all the simplices in $\tilde{M}$ are transverse to each P_j (when both are translated to the origin).

Now we can apply [107, Proposition 7.13] to (a regularization of) the maps $\tilde{\psi}(\cdot - C(1-t))$, with $t \in [0,1]$ and C big enough. The preimage of $\bar{D}_{1/2}$ under these maps intersects $\tilde{M} \cong M$ in an

$O(\varepsilon)$-neighborhood of $[\tilde{\psi}^{-1}(0)+C(1-t)]\cap\tilde{M}$, which has volume $O(\varepsilon^2)$. Also, for $t=0$ the initial map is constant and equal to 1 (for C big enough).

The aforementioned proposition gives then a family $(u_t,\nabla_t)_{t\in[0,1]}$ with uniformly bounded energy from $(1,d)$ to $(\bar{\psi},d-i\bar{\psi}^*(d\theta))$, for some $\bar{\psi}:M\to S^1$ homotopic to ψ. Concatenating this family with $(\psi_t,d-i\psi_t^*(d\theta))$, for a homotopy ψ_t from $\bar{\psi}$ to ψ, we get a family in C_λ with the same energy, as desired. □

Now, consider a small $\varepsilon\in(0,\delta)$ such that

$$\mathscr{E}_\varepsilon(\lambda)\leq\Lambda+\delta<\Lambda+1. \tag{2.45}$$

By Lemma 36 and Remark 37, we can find a family $[0,1]\ni t\mapsto(u_t,\nabla_t=d-i\alpha_t)$ in $\mathscr{C}_\lambda\subset C^0([0,1],\widehat{X})$ such that $|u_t|\leq 1$,

$$\max_{t\in[0,1]}E_\varepsilon(u_t,\nabla_t)\leq\mathscr{E}_\varepsilon(\lambda)+\varepsilon\leq\Lambda+2, \tag{2.46}$$

and

$$\max_{t\in[0,1],\,r\geq\varepsilon,\,p\in M}r^{2-n}\int_{B_r(p)}e_\varepsilon(u_t,\nabla_t)\leq C_0(M,\Lambda). \tag{2.47}$$

Now, by the continuity of the path $t\mapsto(u_t,\nabla_t=d-i\alpha_t)$ in $\widehat{X}$, we may select a finite sequence of times

$$0=t_0<t_1<\cdots<t_{N=3^k}=1$$

such that

$$\|u_{t_{i+1}}-u_{t_i}\|_{W^{1,2}(M)}+\|\alpha_{t_{i+1}}-\alpha_{t_i}\|_{W^{1,2}(M)}<\delta.$$

In what follows, we write $u_i=u_{t_i}$ and $\alpha_i=\alpha_{t_i}$. Suppose now that $\varepsilon<\varepsilon_0(M,\Lambda+2,\delta,C_0)$ as in Lemma 38, and for each $i=1,\ldots,N=3^k$, let

$$v_i\in W^{1,p}(M,S^1)\quad\text{and}\quad\phi_i:M\to S^1$$

be as in the conclusion of Lemma 38, so that

$$\|u_i-v_i\|_{L^1(M)}\leq\delta,$$

and

$$\|d(\phi_i^{-1}v_i)\|_{L^p(M)}\leq C(p,M,\Lambda) \tag{2.48}$$

for $p\in(1,\frac{n}{n-1})$, while

$$\mathbb{M}(J(v_i))\leq\Lambda+2\delta,$$

together with

$$\max_{r\geq\delta,\,p\in M}\frac{\|J(v_i)\|(B_r(p))}{r^{n-2}}\leq 2C_0,$$

and

$$\|\phi_i^*(d\theta)-\Pi(\alpha_i)\|_{L^2(M)}\leq C(M). \tag{2.49}$$

In this way, we get a sequence

$$1=v_0,v_1,\ldots,v_N=\psi\text{ in }W^{1,p}(M,S^1)$$

such that

$$\|v_{i+1}-v_i\|_{L^1(M)}\leq C\delta$$

and the integral $(n-2)$-cycles $T_i := \frac{1}{2\pi}J(v_i)$ satisfy

$$2\pi\mathbb{M}(T_i) \leq \Lambda + 2\delta$$

and

$$\max_{r\geq\delta,\, p\in M} \frac{\|T_i\|(B_r(p))}{r^{n-2}} \leq C_0.$$

Moreover, for each $i = 0,\ldots,N-1$, the following holds.

Lemma 44. *For $p \in [1, \frac{n}{n-1})$, there exists $w_i \in W^{1,p}(M\times[0,1], S^1)$ with boundary values*

$$w_i(x,0) = v_i(x), \quad w_i(x,1) = v_{i+1}(x),$$

satisfying

$$\|\partial_t w_i\|_{L^p(M\times[0,1])} \leq C(p)\|v_{i+1} - v_i\|_{L^p(M)} \leq C(p)\delta^{1/p}$$

and

$$\mathbb{M}(\pi_*[J(w_i)]) \leq C(p,M,\Lambda)\delta^{1-1/p}.$$

Proof. To begin, apply Lemma 40 with $u = \phi_i^{-1}v_i$ and $v = \phi_i^{-1}v_{i+1}$, to obtain a map $\tilde{w} \in W^{1,p}(M\times[0,1])$ which restricts to $\phi_i^{-1}v_i$ and $\phi_i^{-1}v_{i+1}$ on $M\times\{0,1\}$, and satisfies

$$\|\partial_t\tilde{w}\|_{L^p(M\times[0,1])} \leq C(p)\|\phi_i^{-1}(v_i - v_{i+1})\|_{L^p(M)} = C(p)\|v_{i+1} - v_i\|_{L^p(M)}$$

and

$$\begin{aligned}\mathbb{M}(\pi_*[J(\tilde{w})]) &\leq C\int_M |\phi_i^{-1}(v_i - v_{i+1})|(|d(\phi_i^{-1}v_i)| + |d(\phi_i^{-1}v_{i+1})|)\\ &\leq \|v_i - v_{i+1}\|_{L^{p'}(M)}(\|d(\phi_i^{-1}v_i)\|_{L^p(M)} + \|d(\phi_i^{-1}v_{i+1})\|_{L^p(M)}).\end{aligned}$$

Now, we know that

$$\|v_i - v_{i+1}\|_{L^{p'}(M)} \leq C(p)\|v_{i+1} - v_i\|_{L^1(M)}^{1-1/p} \leq C(p)\delta^{1-1/p}$$

and

$$\|d(\phi_i^{-1}v_i)\|_{L^p(M)} \leq C(p,M,\Lambda),$$

while

$$\begin{aligned}\|d(\phi_i^{-1}v_{i+1})\|_{L^p(M)} &= \|v_{i+1}^*(d\theta) - \phi_i^*(d\theta)\|_{L^p(M)}\\ &\leq \|d(\phi_{i+1}^{-1}v_{i+1})\|_{L^p(M)} + \|\phi_{i+1}^*(d\theta) - \phi_i^*(d\theta)\|_{L^p(M)}\\ &\leq C(p,M,\Lambda) + \|\phi_{i+1}^*(d\theta) - \Pi(\alpha_{i+1})\|_{L^p(M)} + \|\phi_i^*(d\theta) - \Pi(\alpha_i)\|_{L^p(M)}\\ &\quad + \|\Pi(\alpha_i - \alpha_{i+1})\|_{L^p(M)}\\ &\leq C(p,M,\Lambda),\end{aligned}$$

which together with the preceding estimates gives

$$\mathbb{M}(\pi_*[J(\tilde{w})]) \leq C(p,M,\Lambda)\delta^{1-1/p}.$$

Taking $w_i := \phi_i\tilde{w}$, one sees that w_i satisfies the conclusions of the claim, since $J(w_i) = J(\tilde{w})$ and $\partial_t w_i = \phi_i\partial_t\tilde{w}$. $\square$

In particular, by (2.40), we see that the $(n-1)$-currents $\Gamma_i := \frac{1}{2\pi}\pi_*[J(w_i)] \in \mathscr{I}_{n-1}(M;\mathbb{Z})$ give fill-ins

$$\partial\Gamma_i = T_{i+1} - T_i$$

of small mass (taking $p = \frac{n+1}{n}$)

$$\mathbb{M}(\Gamma_i) \leq C(M,\Lambda)\delta^{1/(n+1)}.$$

Thus, the sequence $T_0, T_1, \ldots, T_{N=3^k}$ defines a discrete family

$$\beta : I(1,k)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z})$$

with

$$\mathbf{m}(\beta, r) \leq C_0 r^{n-2} \quad \text{for } r \geq \delta,$$

together with

$$\max_i \mathbb{M}(T_i) \leq \frac{1}{2\pi}(\Lambda + 2\delta),$$

and

$$\mathbf{f}(\beta) \leq C(M,\Lambda)\delta^{1/(n+1)}.$$

Moreover, for $\delta < \delta_0(M,\Lambda)$ sufficiently small, the homology class $\Psi(\beta) \in H_{n-1}(M;\mathbb{Z})$ associated to β by Almgren's isomorphism is given by

$$\Psi(\beta) := [\Gamma],$$

where

$$\Gamma := \Sigma_{i=0}^{N-1}\Gamma_i.$$

Now, by Remark 42, we can identify Γ with the projected Jacobian

$$2\pi\Gamma = \pi_*[J(w_0 * w_1 * \cdots * w_{N-1})] = \pi_*[J(w)]$$

of the concatenated map $w := w_0 * \cdots * w_{N-1} : M \times [0,1] \to S^1$, which satisfies

$$w(x,0) = 1 \quad \text{and} \quad w(x,1) = \psi(x).$$

In particular, for any $\zeta \in \Omega^{n-1}(M^n)$, it follows that

$$2\pi\langle\Gamma,\zeta\rangle = \int_{M\times[0,1]} w^*(d\theta)\wedge d\zeta + \int_M \psi^*(d\theta)\wedge\zeta.$$

Hence, the action of Γ on *closed* $(n-1)$-forms agrees with that of $\frac{1}{2\pi}\int_M \psi^*(d\theta)\wedge\cdot$. In particular, since there is no torsion in $H_{n-1}(M;\mathbb{Z})$, it follows that

$$[\Gamma] = [\psi^{-1}\{\theta\}] = \lambda \in H_{n-1}(M;\mathbb{Z}),$$

as desired.

That is, letting $\eta(\delta) := \max\{\delta, C\delta^{1/(n+1)}\}$, we see that $\beta \in \mathscr{A}_{\eta(\delta)}(\lambda)$, so that

$$\mathbf{W}_{\eta(\delta)}(\lambda) \leq \max_i \mathbb{M}(T_i) \leq \frac{1}{2\pi}(\Lambda + 2\delta) = \frac{1}{2\pi}\liminf_{\varepsilon\to 0}\mathscr{E}_\varepsilon(\lambda) + \frac{1}{\pi}\delta.$$

Finally, taking the limit as $\delta \to 0$ and using (2.26), we get the desired estimate (2.43).

2.4.6 Two-parameter families and the generator of $\pi_2(\mathscr{Z}_{n-2}(M;\mathbb{Z}),0)$.

In this subsection, we complete the proof of Theorem 35, establishing the inequality for the two-parameter families

$$\liminf_{\varepsilon\to 0}\mathscr{E}_\varepsilon(\mathscr{C}_2)\geq 2\pi\mathbf{W}([M]). \tag{2.50}$$

To begin, set

$$\Lambda := \liminf_{\varepsilon\to 0}\mathscr{E}_\varepsilon(\mathscr{C}_2),$$

which is finite (see [107, Section 7]), and fix some small $\delta>0$. Again let $L\to M$ be the trivial line bundle, and consider a two-parameter family

$$\bar{D}^2\ni y\mapsto (u_y,\nabla_y=d-i\alpha_y)$$

belonging to $\mathscr{C}_2\subset C^0(\bar{D}^2,\widehat{X})$, so that

$$(u_\theta,\nabla_\theta)\equiv(\theta,d)\quad \text{for all } \theta\in\partial D=S^1.$$

Choose a small $\varepsilon\in(0,\delta)$ such that

$$\mathscr{E}_\varepsilon(\mathscr{C}_2)\leq\Lambda+\delta;$$

by Lemma 36 and the subsequent remark, we can select our family $\bar{D}\ni y\mapsto(u_y,\nabla_y)$ in $\mathscr{C}_2$ such that $|u_y|\leq 1$,

$$\max_{y\in\bar{D}}E_\varepsilon(u_y,\nabla_y)\leq\Lambda+2\delta,$$

and

$$\max_{y\in\bar{D},\,r\geq\delta,\,p\in M} r^{2-n}\int_{B_r(p)}e_\varepsilon(u_y,\nabla_y)\leq C_0(M,\Lambda).$$

Now, identifying $\bar{D}$ with the square $I^2=[0,1]^2$ in the usual bi-Lipschitz way, by the continuity of the family $I^2\cong\bar{D}\ni y\mapsto(u_y,\nabla_y)\in\widehat{X}$, we can choose k sufficiently large that the discrete assignment

$$I(2,k)_0\ni a\mapsto(u_a,\nabla_a)=(u_a,d-i\alpha_a)\in\widehat{X}$$

satisfies

$$\|u_a-u_b\|_{W^{1,2}(M)}+\|\alpha_a-\alpha_b\|_{W^{1,2}(M)}<\delta$$

for any adjacent vertices $a,b\in I(2,k)_0$. By Lemma 38, for each vertex $a\in I(2,k)_0$, there exist

$$v_a\in W^{1,p}(M,S^1)\quad\text{and}\quad \phi_a:M\to S^1$$

such that

$$\|u_a-v_a\|_{L^1(M)}\leq\delta$$

and

$$\|d(\phi_a^{-1}v_a)\|_{L^p(M)}\leq C(p,M,\Lambda)$$

for $p\in[1,\frac{n}{n-1})$, while

$$\mathbb{M}(J(v_a))\leq\Lambda+2\delta,$$

together with

$$\max_{r\geq\delta,\,p\in M}\frac{\|J(v_a)\|(B_r(p))}{r^{n-2}}\leq 2C_0,$$

and

$$\|\phi_a^*(d\theta)-\Pi(\alpha_a)\|_{L^2(M)}\leq C(M).$$

The following lemma, and its proof, is identical to Lemma 44.

Lemma 45. *For each pair of adjacent vertices $a,b \in I(2,k)_0$, there exists $w_{a,b} \in W^{1,p}(M \times [0,1], S^1)$ satisfying the boundary conditions*

$$w_{a,b}(x,0) = v_a(x) \quad \text{and} \quad w_{a,b}(x,1) = v_b(x),$$

while for every $p \in [1, \frac{n}{n-1})$,

$$\|\partial_t w_{a,b}\|_{L^p(M\times[0,1])} \le C(p)\|v_b - v_a\|_{L^p(M)} \le C(p)\delta^{1/p},$$

and

$$\mathbb{M}(\pi_*[J(w_{a,b})]) \le C(p,M,\Lambda)\delta^{1-1/p}.$$

Remark 46. If the vertices a,b lie on the boundary ∂I^2, so that u_a and u_b are constant maps to S^1, then we take $v_a = u_a$, $v_b = u_b$, and simply let $w_{a,b}$ be the geodesic interpolation in S^1 between the two constants.

In particular, for each pair of adjacent vertices $a,b \in I(2,k)_0$, the $(n-1)$-current

$$\Gamma_{a,b} := \frac{1}{2\pi}\pi_*[J(w_{a,b})] \in \mathscr{I}_{n-1}(M;\mathbb{Z})$$

provides a small-mass fill-in

$$\partial\Gamma_{a,b} = T_b - T_a$$

for the difference of the integral $(n-2)$-cycles $T_a := \frac{1}{2\pi}J(v_a)$; namely, taking $p = \frac{n+1}{n}$ in the preceding lemma, we have

$$\mathbb{M}(\Gamma_{a,b}) \le C(M,\Lambda)\delta^{\frac{1}{n+1}}.$$

Thus, setting $\beta(a) := T_a$ gives a discrete family

$$\beta : I(2,k)_0 \to \mathscr{Z}_{n-2}(M;\mathbb{Z})$$

satisfying

$$\mathbf{m}(\beta,r) \le C_0 r^{n-2} \quad \text{for } r \ge \delta,$$

together with

$$\max_{a\in I(2,k)_0} \mathbb{M}(T_a) \le \frac{1}{2\pi}(\Lambda + 2\delta),$$

and

$$\mathbf{f}(\beta) \le C(M,\Lambda)\delta^{\frac{1}{n+1}}.$$

It remains to show that the homology class $\Psi(\beta) \in H_n(M;\mathbb{Z})$ associated to β by Almgren's isomorphism is the fundamental class $[M]$.

For each 2-cell $\square \in I(2,k)_2$ with vertices a,b,c,d (ordered counter-clockwise), let $F : M \times \partial I^2 \to S^1$ be the concatenation given by $w_{a,b}$ along the edge $[a,b]$ of ∂I^2, $w_{b,c}$ on $[b,c]$, and so on. We apply Lemma 40 to interpolate between F and 1, obtaining an extension $F_\square \in W^{1,p}(M \times I^2, S^1) \cap W^{1,p}(M \times \partial I^2, S^1)$ of the map F, so that

$$\Xi_\square := \frac{1}{2\pi}\pi_*[J(F_\square)] \in \mathscr{I}_n(M;\mathbb{Z})$$

has boundary

$$\partial\Xi_\square = \frac{1}{2\pi}\pi_*[J(F)] = \Gamma_{a,b} + \Gamma_{b,c} + \Gamma_{c,d} + \Gamma_{d,a}.$$

In particular, since $\|\partial_t w_{a,b}\|_{L^p(M\times[a,b])} \leq C(p)\delta^{1/p} = C(n)\delta^{n/(n+1)}$, it follows from Lemma 43 that $\Xi_\Box$ is the (unique) small-mass fill-in of $\Gamma_{a,b}+\cdots+\Gamma_{d,a}$, provided $\delta < \delta_0(M,\Lambda)$ is sufficiently small. In particular, we see that

$$\Psi(\beta) = [\Sigma_{\Box\in I(2,k)_2}\Xi_\Box] \in H_n(M;\mathbb{Z}).$$

By concatenating the maps F_1 and F_2 associated to adjacent boxes $\Box_1,\Box_2$ along the shared edge, we obtain a map $\Phi = F_1 * F_2$ which satisfies

$$\pi_*[J(\Phi)] = \Xi_{\Box_1} + \Xi_{\Box_2}.$$

In particular, concatenating all maps along each row of the grid, we obtain a column of maps, which we may again concatenate to obtain finally a map

$$F \in W^{1,p}(M\times I^2,S^1)\cap W^{1,p}(M\times\partial I^2,S^1)$$

for which

$$\pi_*[J(F)] = 2\pi\Sigma_\Box\Xi_\Box.$$

On the other hand, it is clear from the construction that the restriction of F to $M\times\partial I^2$ has the form

$$F(x,t) = h(t)$$

for a fixed homeomorphism $h:\partial I^2\to S^1$. In particular, it follows that

$$\deg\left(F|_{\{x\}\times\partial I^2}\right) = 1$$

for all $x\in M$, so that

$$2\pi\Sigma_\Box\Xi_\Box = \pi_*[J(F)] = 2\pi[M],$$

by Lemma 43.

Thus, $\Psi(\beta) = [M]$, as desired, and again setting $\eta(\delta) := \max\{\delta, C\delta^{1/(n+1)}\}$, we see that $\beta\in\mathscr{A}_{\eta(\delta)}([M])$, and consequently

$$\mathbf{W}_{\eta(\delta)}([M]) \leq \max_{a\in I(2,k)_0}\mathbb{M}(T_a) \leq \frac{1}{2\pi}(\Lambda+2\delta) = \frac{1}{2\pi}\liminf_{\varepsilon\to 0}\mathscr{E}_\varepsilon(\mathscr{C}_2) + \frac{1}{\pi}\delta.$$

Taking the limit as $\delta\to 0$ and using (2.26), we then get the desired estimate (2.50), completing the proof of Theorem 35.

2.5 Huisken-type monotonicity along the gradient flow

In Lemma 36 of the previous section, we made use of the fact that a continuous family of pairs $y\mapsto(u_y,\nabla_y)$ may be deformed to a family (u'_y,∇'_y) with $E_\varepsilon(u'_y,\nabla'_y)\leq E_\varepsilon(u_y,\nabla_y)$ satisfying uniform bounds on the $(n-2)$-energy densities $r^{2-n}\int_{B_r(p)} e_\varepsilon(u'_y,\nabla'_y)$ in terms of the initial energies $E_\varepsilon(u_y,\nabla_y)$. We achieve this by showing that the natural L^2 gradient flow for these energies satisfies a variant of Huisken's monotonicity formula [78] for the codimension-two mean curvature flow. In addition to its applications above, the result may be of independent interest, in that it provides strong evidence that these gradient flows provide a regularization of the codimension-two Brakke flow—a relationship which we plan to explore further in future work. We also show that this E_ε-gradient flow satisfies long-time existence and continuous dependence on initial data (the fact that we are working with the abelian gauge group $U(1)$ is of course crucial here).

2.5.1 Definition, Bochner identities, and estimates along the gradient flow

Let $L \to M$ be the trivial complex line bundle over a closed, oriented Riemannian manifold (M^n, g). We will assume $n \geq 3$ throughout this section.

We will say that the smooth couples $(u_t, \nabla_t = d - i\alpha_t)_{t\in[0,\infty)}$ solve the gradient flow equations for E_ε if they satisfy the coupled nonlinear heat equations

$$\begin{cases} \partial_t u_t = -\nabla_t^* \nabla_t u_t + \frac{1}{2\varepsilon^2}(1-|u_t|^2)u_t, \\ \partial_t \alpha_t = -d^* d\alpha_t + \varepsilon^{-2}\langle iu_t, \nabla_t u_t\rangle. \end{cases} \tag{2.51}$$

Note that they are formally the gradient flow of $\frac{1}{2}E_\varepsilon$ with respect to the L^2-scalar product

$$\langle (u,\alpha),(v,\beta)\rangle = \int_M (\langle u,v\rangle + \varepsilon^2\langle \alpha,\beta\rangle),$$

where u and v are sections, and α and β are one-forms. We defer the proof of long-time existence, uniqueness and continuous dependence on initial data to the end of the section. In what follows, we will also assume that the initial section $u_0 \in \Gamma(L)$ satisfies $|u_0| \leq 1$ pointwise.

Assuming the initial data (u_0, ∇_0) satisfies the energy bound

$$E_\varepsilon(u_0,\nabla_0) \leq \Lambda, \tag{2.52}$$

it is easy to see that we have

$$E_\varepsilon(u_t,\nabla_t) \leq \Lambda$$

for all $t > 0$, as the energy is decreasing along the flow. Similar to results for the stationary case in [107] (and analogous work of Ilmanen for the parabolic Allen–Cahn equation in codimension one [80]), a key ingredient in establishing the desired monotonicity result will be bounding the discrepancy function

$$\xi_t := \varepsilon|d\alpha_t| - \frac{1-|u_t|^2}{2\varepsilon} \tag{2.53}$$

along the flow.

As in the stationary case [107, Section 3], it is straightforward to check that solutions of (2.51) satisfy the following identities: letting

$$\omega_t := d\alpha_t$$

and

$$\psi(u_t,\nabla_t)(e_j,e_k) := 2\langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle,$$

we have

$$\varepsilon^2(\partial_t + \Delta_H)\omega_t = \psi(u_t,\nabla_t) - |u_t|^2\omega_t, \tag{2.54}$$

from which one obtains the parabolic Bochner identity

$$-\varepsilon^2(\partial_t + d^*d)\frac{1}{2}|\omega_t|^2 = |u_t|^2|\omega_t|^2 + \varepsilon^2|D\omega_t|^2 - \langle \psi(u_t,\nabla_t),\omega_t\rangle + \varepsilon^2\mathscr{R}_2(\omega_t,\omega_t), \tag{2.55}$$

where $\mathscr{R}_2$ denotes the Weitzenböck curvature operator for two-forms. Also,

$$-(\partial_t + d^*d)\frac{1}{2}|u_t|^2 = |\nabla_t u_t|^2 - \frac{1}{2\varepsilon^2}(1-|u_t|^2)|u_t|^2. \tag{2.56}$$

Remark 47. It is an easy consequence of (2.56) and the parabolic maximum principle that $|u_t| \leq 1$ for all $t > 0$, for initial sections u_0 satisfying $|u_0| \leq 1$.

By a combination of (2.55) and (2.56), similarly to [107], we find that the discrepancy function in (2.53) satisfies the weak differential inequality

$$-(\partial_t + d^*d + \varepsilon^{-2}|u_t|^2)\xi_t \geq -C_0(M)\varepsilon|\omega_t|. \tag{2.57}$$

Equivalently, writing

$$\bar{\xi}_t := e^{-C_0 t}\xi_t,$$

we have

$$-(\partial_t + d^*d + \varepsilon^{-2}|u_t|^2)\bar{\xi}_t \geq -C_0 e^{-C_0 t}\frac{1-|u_t|^2}{2\varepsilon} \geq -\frac{C_0}{2\varepsilon}(1-|u_t|^2). \tag{2.58}$$

Now, let $K(t,x,y)$ be the heat kernel of M, so that

$$(\partial_t + d^*d)K(t,\cdot,y) = 0 \quad \text{and} \quad \lim_{t\to 0} K(t,\cdot,y) = \delta_y.$$

Define then

$$\varphi(t,x) := \int_M K(t,x,y)|\xi_0|(y)\,dy$$

and

$$\psi(t,x) := \int_0^t \int_M K(t-s,x,y)\frac{C_0}{2\varepsilon}(1-|u_s|^2)(y)\,dy\,ds.$$

Thus, φ is the nonnegative solution of the heat equation $-(\partial_t + d^*d)\varphi = 0$, with initial condition $\varphi(0,x) = |\xi_0(x)|$. By Duhamel's principle, ψ is the nonnegative solution of the inhomogeneous heat equation

$$-(\partial_t + d^*d)\psi = -\frac{C_0}{2\varepsilon}(1-|u_t|^2),$$

with boundary data $\psi(0,x) = 0$. In particular, it follows from (2.58) that

$$-(\partial_t + d^*d + \varepsilon^{-2}|u_t|^2)(\bar{\xi}_t - \psi_t - \varphi_t) \geq \frac{|u_t|^2}{\varepsilon^2}(\varphi_t + \psi_t) \geq 0, \tag{2.59}$$

while $\xi_0 - \psi_0 - \varphi_0 = \xi_0 - |\xi_0| \leq 0$. Hence, the parabolic maximum principle (for continuous weak solutions) implies the pointwise bound

$$\bar{\xi}_t \leq \varphi_t + \psi_t. \tag{2.60}$$

We now use the following well-known asymptotics for the heat kernel on a compact manifold (see, e.g., [28, Chapter VI]).

Lemma 48. *We have* $(4\pi t)^{n/2}e^{d(x,y)^2/4t}K(t,x,y) \to 1$ *as* $t \to 0^+$, *uniformly in* $x,y \in M$.

In particular, since $|K(t,x,y)| \leq C(\tau,M)$ for any $t \geq \tau > 0$, one has

$$\int_M K(t,x,y)^p\,dy \leq C(p)\max\{t^{(1-p)n/2},1\}.$$

Since $|u_t| \leq 1$ by Remark 47, we have automatically

$$\left\|\frac{1}{\varepsilon}(1-|u_t|^2)\right\|_{L^\infty(M)} \leq \frac{1}{\varepsilon} \quad \text{and} \quad \left\|\frac{1}{\varepsilon}(1-|u_t|^2)\right\|_{L^2(M)} \leq 2\sqrt{\Lambda}$$

for every t, and interpolating we see that

$$\left\|\frac{1}{\varepsilon}(1-|u_t|^2)\right\|_{L^q(M)} \leq C(M,\Lambda)\varepsilon^{(2-q)/q}$$

for $2 \leq q \leq \infty$. It follows that, for $p \in (1, \frac{n}{n-2})$ with Hölder conjugate q,

$$\begin{aligned}\psi(t,x) &\leq \int_0^t \|K(t-s,x,y)\|_{L^p(M)} C\varepsilon^{(2-q)/q}\, ds \\ &\leq C\varepsilon^{(2-q)/q} \int_0^t (t-s)^{\frac{n(1-p)}{2p}}\, ds \\ &\leq C(p,M,\Lambda)\varepsilon^{(2-q)/q}\left(\frac{n}{2p} - \frac{n-2}{2}\right)^{-1} t^{\frac{n}{2p}-\frac{n-2}{2}},\end{aligned}$$

provided that $q \geq 2$. In particular, taking $p := \frac{n-1}{n-2}$ and $q := n-1$, we arrive at an estimate of the form

$$\psi(t,x) \leq C_1(M,\Lambda)\varepsilon^{\frac{3-n}{n-1}} t^{\frac{n-2}{2(n-1)}}.$$

Now, let

$$\eta_t := \bar{\xi}_t - \varphi_t \leq \psi_t \leq C_1 \varepsilon^{\frac{3-n}{n-1}} t^{\frac{n-2}{2(n-1)}},$$

and setting

$$f_t := \eta_t - C_1 \varepsilon^{\frac{3-n}{n-1}} t^{\frac{n-2}{2(n-1)}} (1-|u_t|^2),$$

note that

$$f_t \leq C_1 \varepsilon^{\frac{3-n}{n-1}} t^{\frac{n-2}{2(n-1)}} |u_t|^2$$

pointwise. On the other hand, recalling (2.58), note that f_t satisfies

$$\begin{aligned}-(\partial_t + d^*d) f_t &\geq \frac{|u_t|^2}{\varepsilon^2}\bar{\xi}_t - \frac{C_0}{2\varepsilon}(1-|u_t|^2) + C_1 \varepsilon^{\frac{3-n}{n-1}} t^{\frac{n-2}{2(n-1)}} (2|\nabla_t u_t|^2 - \varepsilon^{-2}(1-|u_t|^2)|u_t|^2) \\ &\geq \frac{|u_t|^2}{\varepsilon^2} f_t - \frac{C_0}{2\varepsilon}(1-|u_t|^2),\end{aligned}$$

and since $|u_t|^2 \geq c\varepsilon^{\frac{n-3}{n-1}} t^{\frac{2-n}{2(n-1)}} f_t$, it follows that on $\{f > 0\}$ we have

$$-(\partial_t + d^*d) f_t \geq \varepsilon^{-2}(c\varepsilon^{\frac{n-3}{n-1}} t^{\frac{2-n}{2(n-1)}} f_t^2 - C_0\varepsilon).$$

Note that $f_0 = \xi_0 - |\xi_0| \leq 0$. For any $\tau > 0$, if f has a positive maximum on $[0,\tau] \times M$ at some point (t,x) with $t > 0$, then the last weak subequation implies that here

$$c\varepsilon^{\frac{n-3}{n-1}} t^{\frac{2-n}{2(n-1)}} f_t^2 - C_0\varepsilon \leq 0,$$

or equivalently

$$f_t \leq C\varepsilon^{\frac{1}{n-1}} t^{\frac{n-2}{4(n-1)}} \leq C\varepsilon^{\frac{1}{n-1}} \tau^{\frac{n-2}{4(n-1)}}.$$

The same inequality holds then on all of $[0,\tau] \times M$. Since τ was arbitrary, we obtain

$$f_t \leq C\varepsilon^{\frac{1}{n-1}} t^{\frac{n-2}{4(n-1)}}$$

for all $t \geq 0$. Recalling the definitions of f, η, $\bar{\xi}$ and φ, the preceding estimate tells us that

$$\xi_t \leq Ce^{Ct}\left(\varphi_t + \varepsilon^{\frac{1}{n-1}} + \varepsilon^{\frac{2}{n-1}} \frac{1-|u|^2}{\varepsilon}\right), \tag{2.61}$$

where φ is the solution of the heat equation with initial data $\varphi_0 = |\xi_0|$, for a constant $C = C(M,\Lambda)$. Finally, noting that

$$\varphi_t \leq C\|\xi_0\|_{L^1(M)} \leq C(M,\Lambda) \quad \text{for } t \geq 1,$$

it follows from the above that

$$\xi_t \leq Ce^{Ct}(1 + \varepsilon^{\frac{2}{n-1}} \sqrt{e_\varepsilon(u_t,\nabla_t)}) \quad \text{for } t \geq 1. \tag{2.62}$$

2.5.2 Huisken-type monotonicity and $(n-2)$-energy-density bounds along the flow

As above, let (u_t,∇_t) be a solution of the gradient flow with $E_\varepsilon(u_0,\nabla_0)\le\Lambda$ and $|u_0|\le 1$. Mimicking the computations leading to Huisken's monotonicity for the mean curvature flow [78], let us introduce $h(t,x)$, a positive solution of the *backward* heat equation

$$\partial_t h = d^*dh$$

on $[0,T)\times M$, with $\int_M h=1$. Write $e_t := e_\varepsilon(u_t,\nabla_t)$ to lighten the notation and set

$$\Phi_h(t) := \int_M he_t.$$

Integration by parts combined with the gradient flow equations allows us to deduce that

$$\begin{aligned}\Phi_h'(t) &= \int_M(\partial_t he_t + h\partial_t e_t)\\ &= \int_M[(d^*dh)e_t + h(2\langle\nabla\dot{u}-i\dot{\alpha}u,\nabla u\rangle + 2\varepsilon^2\langle d\dot{\alpha},d\alpha\rangle - \varepsilon^{-2}(1-|u|^2)\langle u,\dot{u}\rangle)]\\ &= \int_M[\langle dh,de_t\rangle - 2h(|\dot{u}|^2+\varepsilon^2|\dot{\alpha}|^2) - 2(\langle\nabla_{dh}u,\dot{u}\rangle + \varepsilon^2 d\alpha(dh,\dot{\alpha}))]\end{aligned}$$

(where we dropped the subscript t from u_t, α_t, $\dot{u}_t$, and $\dot{\alpha}_t$). Next, recall from [107, Section 4] the stress-energy tensor

$$T_\varepsilon(u,\nabla) := e_\varepsilon(u,\nabla)g - 2\nabla u^*\nabla u - 2\varepsilon^2 d\alpha^* d\alpha,$$

and note (cf. [107, Section 4]) that we have the identities

$$\begin{aligned}\operatorname{div}(T_\varepsilon) &= 2\langle\nabla u,\nabla^*\nabla u\rangle + d\frac{W(u)}{\varepsilon^2} + 2\omega(\langle iu,\nabla u\rangle,\cdot) - 2\varepsilon^2\omega(d^*\omega,\cdot)\\ &= -2\langle\nabla u,\dot{u}\rangle - 2\varepsilon^2 d\alpha(\cdot,\dot{\alpha}),\end{aligned}$$

where the second equality follows from (2.51). We can now rewrite the term $\langle dh,de_t\rangle$ in our computation of $\Phi_h'(t)$ as

$$\langle dh,de_t\rangle = \langle dh,\operatorname{div}(T_\varepsilon) + 2\operatorname{div}(\nabla u^*\nabla u + \varepsilon^2 d\alpha^* d\alpha)\rangle,$$

and apply the formula for $\operatorname{div}(T_\varepsilon)$ to see that

$$\begin{aligned}\Phi_h'(t) &= 2\int_M\langle dh,\operatorname{div}(\nabla u^*\nabla u+\varepsilon^2 d\alpha^* d\alpha)\rangle\\ &\quad -2\int_M h(|\dot{u}|^2+\varepsilon^2|\dot{\alpha}|^2) - 4\int_M(\langle\nabla_{dh}u,\dot{u}\rangle + \varepsilon^2 d\alpha(dh,\dot{\alpha}))\\ &= -2\int\langle D^2h,\nabla u^*\nabla u + \varepsilon^2 d\alpha^* d\alpha\rangle\\ &\quad -2\int_M(h|\dot{u}+h^{-1}\nabla_{dh}u|^2 + \varepsilon^2 h|\dot{\alpha}+h^{-1}\iota_{dh}d\alpha|^2)\\ &\quad +2\int_M h^{-1}(|\nabla_{dh}u|^2+\varepsilon^2|\iota_{dh}d\alpha|^2)\\ &\le -2\int\langle D^2h,\nabla u^*\nabla u+\varepsilon^2 d\alpha^* d\alpha\rangle + 2\int_M h^{-1}(|\nabla_{dh}u|^2+\varepsilon^2|\iota_{dh}d\alpha|^2).\end{aligned}$$

Now, setting
$$P_t := \nabla u^* \nabla u + \varepsilon^2 d\alpha^* d\alpha,$$
so that the stress-energy tensor $T_\varepsilon(u,\nabla)$ becomes simply $e_\varepsilon(u,\nabla)g - 2P_t$, we can rewrite the preceding inequality as

$$\Phi_h'(t) \le -2\int_M \langle P_t, D^2 h - h^{-1} dh \otimes dh\rangle. \tag{2.63}$$

On the other hand, by Hamilton's *matrix Harnack estimate* for the heat equation, see [68, p. 132], there exist constants $C(M)$ and $B(M)$ such that, for $t \in [T-1,T)$,

$$D^2 h - \frac{dh \otimes dh}{h} + \frac{1}{2(T-t)} hg \ge -C[(1 + h\log(B/(T-t)^{n/2})]g.$$

Applying this in (2.63), we see that for $t \in [T-1,T)$ the following inequality holds:

$$\Phi_h'(t) \le \int_M \Big(\frac{h}{T-t} + C + Ch\log(B/(T-t)^{n/2})\Big)\langle P_t, g\rangle.$$

Now, recalling (2.62), observe that

$$\begin{aligned}
\langle P_t, g\rangle &= |\nabla u|^2 + 2\varepsilon^2 |d\alpha|^2 \\
&= e_t + \varepsilon^2 |d\alpha|^2 - \frac{(1-|u|^2)^2}{4\varepsilon^2} \\
&= e_t + \xi_t\Big(\varepsilon|d\alpha| + \frac{1}{2\varepsilon}(1-|u|^2)\Big) \\
&\le (1 + Ce^{Ct}\varepsilon^{\frac{2}{n-1}})e_t + Ce^{Ct}\sqrt{e_t}
\end{aligned}$$

for $t \ge 1$. In particular, setting $\alpha_n := \frac{2}{n-1}$, for $T \in [2,3]$ and $t \in [T-1,T)$ it then follows that

$$\begin{aligned}
\Phi_h'(t) &\le \frac{1 + C\varepsilon^{\alpha_n}}{T-t}\Phi_h(t) + \frac{C}{T-t}\int_M h\sqrt{e_t} + C\int_M e_t + C\log(B/(T-t)^{n/2})\Phi_h(t) \\
&\le \frac{1 + C_2\varepsilon^{\alpha_n}}{T-t}\Phi_h(t) + \frac{C_2}{T-t}\Phi_h(t)^{1/2} + C_2 + C_2\log(B/(T-t)^{n/2})\Phi_h(t)
\end{aligned}$$

for some $C_2(M,\Lambda)$, where we also used the trivial inequality $\langle P_t, g\rangle \le 2e_t$. Thus, setting

$$\Psi_h(t) := (T-t)^{1+C_2\varepsilon^{\alpha_n}} e^{\zeta(t)}\Phi_h(t),$$

where $|\zeta(t)| \le C(M,\Lambda)$ is the bounded function on $[T-1,T)$ given by

$$\zeta(t) := -\int_1^t C_2\log(B/(T-s)^{n/2})\,ds,$$

we see that
$$\Psi_h'(t) \le C(T-t)^{-1/2}\Psi_h(t)^{1/2} + C$$
for $t \in [T-1,T) \subseteq [1,3)$. From this differential inequality, we can conclude that

$$\Psi_h(t) \le C(M,\Lambda)(\Psi_h(T-1) + 1), \tag{2.64}$$

for any $t \in [T-1,T) \subseteq [1,3)$.

Specializing, fix $T \in [2,3]$ and $x_0 \in M$, and let

$$h(t,x) = h_{T,x_0}(t,x) := K(T-t,x,x_0),$$

where K is the heat kernel on M. Then, for $t \in [T-1,T)$, the inequality in (2.64) leads to an estimate of the form

$$\begin{aligned}(T-t)^{1+C\varepsilon^{\alpha_n}} & \int_M K(T-t,x,x_0)e_\varepsilon(u_t,\nabla_t)\,dx \\ & \le C\int_M K(1,x,x_0)e_\varepsilon(u_{T-1},\nabla_{T-1}) + C \\ & \le CE_\varepsilon(u_{T-1},\nabla_{T-1}) + C \\ & \le C(M,\Lambda).\end{aligned}$$

In particular, taking $t := 2$ and $T := 2+\delta^2$ for $\delta \in (0,1]$, we see that

$$\delta^{2+2C\varepsilon^{\alpha_n}} \int_M K(\delta^2,x,x_0)e_\varepsilon(u_2,\nabla_2)\,dx \le C(M,\Lambda). \tag{2.65}$$

Since

$$\inf_{\varepsilon\in(0,1]} \varepsilon^{2C\varepsilon^{\alpha_n}} = c(M,\Lambda) > 0,$$

it follows that

$$\max_{\varepsilon\le\delta\le 1}\left(\delta^2\int_M K(\delta^2,x,x_0)e_\varepsilon(u_2,\nabla_2)\,dx\right) \le C(M,\Lambda). \tag{2.66}$$

Finally, using again Lemma 48, it follows that

$$\delta^{2-n}\int_{B_\delta(x_0)} e_\varepsilon(u_2,\nabla_2) \le C(M,\Lambda) \quad \text{for } \varepsilon \le \delta \le 1.$$

Thus, we have arrived at the following bound.

Proposition 49. *If (u_t,∇_t) is a solution of the gradient flow (2.51) for E_ε with initial energy bound $E_\varepsilon(u_0,\nabla_0) \le \Lambda$, then at time 2 the pair (u_2,∇_2) satisfies*

$$\int_{B_r(x_0)} e_\varepsilon(u_2,\nabla_2) \le C(M,\Lambda)r^{n-2}, \tag{2.67}$$

for all $r \in [\varepsilon,1]$ and $x_0 \in M$.

Since (u_2,∇_2) depends continuously on the initial couple (u_0,∇_0), this provides in particular the regularization that we needed in the previous section.

Remark 50. Note that, in analogy with the monotonicity formula for critical couples, if we just used the trivial bound $\langle P_t, g\rangle \le 2e_t$ we would have obtained

$$\Phi_h'(t) \le \frac{2}{T-t}\Phi_h(t) + C + C\log(B/(T-t)^{n/2})\Phi_h(t),$$

leading to

$$(T-t)^2\int_M h_{T,x_0}(t,\cdot)e_t \le C(M,\Lambda)$$

and hence a non-sharp bound $C\delta^{n-4}$ for the energy of (u_2,∇_2) on a ball $B_\delta(x_0)$. This would have sufficed for our present purposes (of ruling out concentration of mass in the min-max families) only when $n > 4$.

2.5.3 Long-time existence of the gradient flow

In this last part we show long-term existence, uniqueness and continuous dependence on initial conditions for the gradient flow of E_ε, on the trivial line bundle. To do so, it is convenient to pass to the Coulomb gauge. Namely, given a smooth couple (u,α), we can always find a change of gauge

$$(v,\beta)=(e^{i\theta}u,\alpha+d\theta)\quad \text{with } d^*\beta=0. \tag{2.68}$$

Indeed, it is enough to take a solution $\theta : M\to\mathbb{R}$ of $d^*\alpha+d^*d\theta=0$, i.e., $\Delta_H\theta=-d^*\alpha$. The solution is unique once we impose $\int_M\theta=0$.

In the sequel, we denote $Q:=-\Delta_H^{-1}d^*:\Omega^1(M)\to\Omega^0(M)$ the corresponding operator, with values into mean-zero functions. By standard elliptic regularity, this operator maps $H^k(M)$ continuously into $H^{k+1}(M)$, for any $k\in\mathbb{N}$.

Given a smooth solution (u_t,α_t) to the gradient flow equations, let $\theta_t=Q\alpha_t$. Omitting the time dependence and passing to the Coulomb gauge as in (2.68) we get $\dot\theta=\varepsilon^{-2}Q\langle iu,\nabla u\rangle$. Thus, setting $\tilde\nabla:=d-i\beta=\nabla-id\theta$, we obtain

$$\begin{aligned}\dot\beta&=\dot\alpha+d\dot\theta\\&=-d^*d\alpha+\varepsilon^{-2}(\langle iu,\nabla u\rangle+dQ\langle iu,\nabla u\rangle)\\&=-\Delta_H\beta+\varepsilon^{-2}(\langle iv,\tilde\nabla v\rangle+dQ\langle iv,\tilde\nabla v\rangle),\end{aligned}$$

since by gauge invariance $d^*d\alpha=d^*d\beta=\Delta_H\beta$ and $\langle iu,\nabla u\rangle=\langle iv,\tilde\nabla v\rangle$. Similarly,

$$\begin{aligned}\dot v&=e^{i\theta}\dot u+ie^{i\theta}\dot\theta u\\&=-\tilde\nabla^*\tilde\nabla v+\frac{1}{2\varepsilon^2}(1-|v|^2)v+\varepsilon^{-2}(Q\langle iv,\tilde\nabla v\rangle)iv.\end{aligned}$$

Let $P:\Omega^1(M)\to\Omega^1(M)$ denote the Hodge projection on the co-closed part of a one-form. Since $-dQ\lambda$ equals the exact part of λ, we have $\lambda+dQ\lambda=P\lambda$ for any $\lambda\in\Omega^1(M)$. Thus, expanding $\tilde\nabla^*\tilde\nabla$ in terms of β, the equations (2.51) give the new system

$$\begin{cases}\dot v+d^*dv=-2i\langle\beta,dv\rangle-|\beta|^2v+\frac{1}{2\varepsilon^2}(1-|v|^2)v+\varepsilon^{-2}(Q\langle iv,dv-i\beta v\rangle)iv,\\ \dot\beta+\Delta_H\beta=\varepsilon^{-2}P\langle iv,dv-i\beta v\rangle.\end{cases} \tag{2.69}$$

Conversely, given a couple (u_0,α_0) and setting $\theta_0:=Q\alpha_0$, from a smooth solution (v_t,β_t) of (2.69) with initial condition $(e^{i\theta_0}u_0,\alpha_0+d\theta_0)$ one recovers a smooth solution (u_t,α_t) to the original system (2.51), by letting $\theta=\theta_t$ solve $\dot\theta_t=\varepsilon^{-2}Q\langle iv_t,(d-i\beta_t)v_t\rangle$, and setting $(u,\alpha):=(e^{-i\theta}v,\beta-d\theta)$.

Thus, we reduce ourselves to establishing the desired long-term existence, uniqueness and continuous dependence for (2.69). We will use the following classical fact from the theory of linear parabolic equations.

Lemma 51. *Given $f_t\in\Omega^\ell(M)$ smooth on $[0,T]\times M$, with $0<T\le 1$, the (unique) solution w_t to $\partial_t w_t+\Delta_H w_t=f_t$ with initial condition $w_0=0$ satisfies*

$$\|w\|_{C^0([0,T],H^{k+1}(M))}\le C(k,\ell,M)\|f\|_{L^2([0,T],H^k(M))},$$

where the norms are shorthand for $\max_{t\in[0,T]}\|w_t\|_{H^{k+1}(M)}$ and $(\int_0^T\|f_t\|^2_{H^k(M)}dt)^{1/2}$.

As a consequence, we get a well-defined operator

$$T_{\ell,k} : L^2([0,T],H^k(M)) \to C^0([0,T],H^{k+1}(M))$$

mapping f to w.

Using this lemma, short-time existence and uniqueness easily follow using the Banach fixed-point theorem. Namely, fix an integer $k > \frac{n}{2}$ and, given a smooth initial condition (v_0, β_0), let w^0 denote the constant couple $w_t^0 = (v_0, \beta_0)$. For $R > 0$, the subset S of

$$Y_T := C^0([0,T],H^{k+1}(M) \times H^k(M)),$$

given by the couples w_t with initial value $w_0 = (v_0, \beta_0)$ and $\|w - w^0\|_{Y_T} \le R$, forms a complete metric space with the distance induced by Y_T. To any $w = (v, \beta) \in S$ we can associate the solution $F(w) = (v', \beta')$ of

$$\begin{cases} \dot{v}' + d^*dv' = -2i\langle\beta, dv\rangle - |\beta|^2 v + \frac{1}{2\varepsilon^2}(1-|v|^2)v + \varepsilon^{-2}(Q\langle iv, dv - i\beta v\rangle)iv, \\ \dot{\beta}' + \Delta_H \beta' = \varepsilon^{-2} P\langle iv, dv - i\beta v\rangle. \end{cases}$$

Denoting $G(w_t)$ and $H(w_t)$ the right-hand sides of the two equations, note that they belong to $C^0([0,T],H^k(M))$, since $H^k(M)$ is an algebra and P and Q map $H^k(M)$ into itself. Hence, $F(w) \in Y_T$ is well-defined. For the same reason, letting $R' := R + \|v_0\|_{H^{k+1}(M)} + \|\beta_0\|_{H^k(M)}$, note that for a fixed $t \in [0,T]$ we have

$$\|G(w_t^1) - G(w_t^2)\|_{H^k(M)} \le C(M,R')\|w_t^1 - w_t^2\|_{H^{k+1}(M)\times H^k(M)}$$

and similarly

$$\|H(w_t^1) - H(w_t^2)\|_{H^k(M)} \le C(M,R')\|w_t^1 - w_t^2\|_{H^{k+1}(M)\times H^k(M)},$$

whenever $w^1, w^2 \in S$. As a consequence, Lemma 51 gives

$$\|F(w^1) - F(w^2)\|_{Y_T} \le C(M,R')\sqrt{T}\|w^1 - w^2\|_{Y_T}.$$

Hence, for T small enough, we have $\|F(w^1) - F(w^2)\|_{Y_T} \le \frac{1}{2}\|w^1 - w^2\|_{Y_T}$ and, by continuity, $\|F(w^0) - w^0\|_{Y_T} \le R/2$; in particular,

$$\|F(w) - w^0\|_{Y_T} \le \|F(w) - F(w^0)\|_{Y_T} + \|F(w^0) - w^0\|_{Y_T} \le R$$

for $w \in S$, and thus $F(w) \in S$ as well. The Banach fixed-point theorem applies and gives a unique $w \in S$ with $F(w) = w$, as desired. Since R was arbitrary, this also establishes uniqueness in this regularity class.

Let $[0, \bar{T})$ be the maximal time of existence in the same class. From standard L^2 regularity theory for linear parabolic equations, it then follows that the solution (v, β) is smooth on $[0, \bar{T}) \times M$.

We shall now prove long-time existence of the flow. Assume by contradiction that $\bar{T} < \infty$. As we already saw above, the corresponding solution (u, α) to the original system (2.51) satisfies

$$\sup_{[0,\bar{T})\times M} |d\alpha| < \infty.$$

In a similar fashion, we can derive a bound for $|\nabla u|$. Indeed, as in [107, Section 3], we have the Bochner identity

$$-(\partial_t + dd^*)\frac{1}{2}|\nabla u|^2 = |\nabla^2 u|^2 + \frac{3|u|^2 - 1}{2\varepsilon^2}|\nabla u|^2 - 2\langle\omega, \psi(u,\nabla)\rangle + \mathscr{R}_1(\nabla u, \nabla u)$$

and, in particular, using the bound $|\psi(u,\nabla)| \le |\nabla u|^2$, we easily deduce the weak subequation

$$-(\partial_t + d^*d)|\nabla u| \ge \frac{3|u|^2-1}{2\varepsilon^2}|\nabla u| - 2|\omega||\nabla u| - C(M)|\nabla u|.$$

Recalling that

$$-(\partial_t + d^*d)\frac{1-|u|^2}{\varepsilon} = \frac{|u|^2}{\varepsilon^2}\frac{1-|u|^2}{\varepsilon} - \frac{2}{\varepsilon}|\nabla u|^2,$$

we obtain for the difference $w := |\nabla u| - \frac{1-|u|^2}{\varepsilon}$ that

$$-(\partial_t + d^*d)w \ge \frac{|u|^2}{\varepsilon^2}w + |\nabla u|\Big(\frac{2}{\varepsilon}|\nabla u| - \frac{1-|u|^2}{2\varepsilon^2} - 2|\omega| - C(M)\Big).$$

For any $0 < \tau < \bar{T}$, if w attains a positive maximum on $[0,\tau]\times M$ at some point (t,x) with $t>0$, it then follows that here

$$\frac{2}{\varepsilon}|\nabla u| \le \frac{1-|u|^2}{2\varepsilon^2} + 2|\omega| + C(M).$$

Hence,

$$|\nabla u| \le \frac{1}{\varepsilon} + \sup_{[0,\bar{T})\times M} w \le \frac{2}{\varepsilon} + \varepsilon \sup_{[0,\bar{T})\times M}|\omega| + \frac{\varepsilon}{2}C(M) + \|\nabla_0 u_0\|_{L^\infty(M)}$$

on all of $[0,\bar{T})\times M$. By gauge invariance, we then get

$$\sup_{[0,\bar{T})\times M}|d\beta| < \infty \quad \text{and} \quad \sup_{[0,\bar{T})\times M}|dv - i\beta v| < \infty.$$

In particular, the co-exact part of β_t is also bounded. From (2.51) it follows that

$$\int_M (|\dot{u}_t|^2 + \varepsilon^2|\dot{\alpha}_t|^2) = -\frac{1}{2}\frac{d}{dt}E_\varepsilon(u_t,\alpha_t),$$

from which we deduce the bound $\int_0^{\bar{T}}\int_M |\dot{\alpha}|^2 < \infty$ just by integrating the above expression. In particular, $\dot{\alpha} \in L^1([0,\bar{T}],L^2(M))$, giving $\alpha \in C^0([0,\bar{T}],L^2(M))$. Thus, the harmonic part α_t^h in the Hodge decomposition of α_t stays bounded. Since $\beta_t^h = \alpha_t^h$ and β has no exact part, this implies that

$$\sup_{[0,\bar{T})\times M}|\beta| < \infty.$$

Also, note that $|v| = |u| \le 1$ as a simple application of the maximum principle to the equation satisfied by $|u|^2$, provided $|u_0| \le 1$, implying

$$\sup_{[0,\bar{T})\times M}|dv| < \infty.$$

From L^p regularity theory (see, e.g., [118]), it follows that $v,\beta \in L^p([0,\bar{T}],W^{k,p}(M))$ for all $k\in\mathbb{N}$, $1<p<\infty$ and, hence, v and β extend smoothly to $[0,\bar{T}]\times M$. Since we can extend the solution past $\bar{T}$, we arrive at a contradiction. This shows that $\bar{T}=\infty$. Finally, continuous dependence (in the smooth topology) on the initial condition for the system (2.51) follows from the same property for (2.69).

Chapter 3

Uniqueness of blow-ups of $\mathbb{Z}_2$-harmonic spinors

This chapter investigates the regularity theory and structure of the singular set of $\mathbb{Z}_2$-harmonic spinors outlined in Section 1.3 of the Introduction. We are going to prove Theorem 11 on the uniqueness of blowups, Theorem 12 on the rectifiability of the singular set, from which it readily follows Corollary 13 on the rectifiability of the branch set. We then analyse the case when the frequency drops below one: Theorem 14, from which we infer Corollary 15. The content of this Chapter is based on unpublished work.

3.1 Preliminary Notions

The main purpose of this section is to collect the various definitions and results relevant to the present chapter.

3.1.1 Clifford Bundles and Dirac Equation

We are going to work on a 4-dimensional Riemannian manifold (M,g), with no boundary and not necessarily compact, endowed with a Clifford bundle $\mathscr{V}$ over it, meaning a bundle equipped with a Clifford structure. More precisely, $\mathscr{V}$ is a unitary vector bundle equipped with an extra structure $\rho \in \mathrm{Hom}(TM, \mathrm{Hom}(\mathscr{V},\mathscr{V}))$, such that $\rho(e)^2 = -\|e\|^2 \cdot \mathrm{Id}$ and $\|\rho(e)(u)\| = \|e\| \cdot \|u\|$ for every tangent vector $e \in T_pM$ and $u \in \mathscr{V}_p$; ρ is usually referred to as *Clifford multiplication*. Consider now a connection ∇ on $\mathscr{V}$ compatible with ρ, which means that for every pair of vector fields e, e', and every smooth section u of $\mathscr{V}$, we have

$$\nabla_e \left(\rho(e') \cdot u\right) = \rho(\nabla_e e') \cdot u + \rho(e') \cdot \nabla_e u.$$

One can then define the Dirac operator on the Clifford bundle $\mathscr{V}$ to be

$$\mathscr{D}(u) = \sum_{i=1}^{4} \rho(e_i) \nabla_{e_i} u,$$

for a local othornormal frame $\{e_i\}$ of TM. Note that the existence of a Dirac operator is equivalent to the bundle admitting a Clifford structure. Any section that annihilates such operator is called a solution of the Dirac equation. It can be shown that the Dirac operator is elliptic and formally self-adjoint, this last one meaning that given compactly supported sections s_1 and s_2 of $\mathscr{V}$, we have

$$\int_M \langle \mathscr{D}s_1, s_2 \rangle = \int_M \langle s_1, \mathscr{D}s_2 \rangle.$$

We shall now recall some well-known results on the Dirac equation that are used throughout this work. The definition of $W^{k,2}$-section of a Hermitian, or Euclidean, vector bundle can be found in [105, Chapter 4]. Alternatively, we refer the reader to [143, Chapter 15], from which the next three theorems are taken. Moreover, as in the classical settings, these Sobolev sections satisfy a Rellich-Kondrachov compactness result, as well as the Morrey-Sobolev embedding. Concerning the Dirac operator, we have the following regularity result. For the statement of the next theorem, we assume (M,g) to be compact. Otherwise, when in a non-compact setting one needs to assume to be working in locally in a coordinate ball with appropriate boundary conditions.

Theorem 52. *Let $k \in \mathbb{N} \cup \{0\}$. There exists a constant $C > 0$ such that the following elliptic estimate holds*

$$\|s\|_{W^{k+1,2}} \leq C\left(\|\mathscr{D}s\|_{W^{k,2}} + \|s\|_{L^2}\right).$$

Moreover, the image of $\mathscr{D}$ is closed and its kernel is finite-dimensional.

Before stating the next elliptic regularity result we have to recall some standard notation. For $k \geq 1$ and $\psi \in W^{-k+1,2}\Gamma(\mathscr{V})$, we define $\mathscr{D}\psi \in W^{-k,2}\Gamma(\mathscr{V})$ by duality as

$$\langle \mathscr{D}\psi, \phi \rangle_{W^{-k,2}, W^{k,2}} := \langle \psi, \mathscr{D}\phi \rangle_{W^{-k+1,2}, W^{k-1,2}},$$

where $\langle \cdot, \cdot \rangle_{\mathscr{H}^*, \mathscr{H}}$ is the usual pairing for a Hilbert space $\mathscr{H}$. For all $l \geq k \geq 0$, we have the inclusion $W^{l,2}\Gamma(\mathscr{V}) \subset W^{k,2}\Gamma(\mathscr{V})$, so that by passing to the duals, we deduce

$$W^{-k,2}\Gamma(\mathscr{V}) = \left(W^{k,2}\Gamma(\mathscr{V})\right)^* \subset \left(W^{l,2}\Gamma(\mathscr{V})\right)^* = W^{-l,2}\Gamma(\mathscr{V})$$

and consequently, $W^{l,2}\Gamma(\mathscr{V}) \subset W^{k,2}\Gamma(\mathscr{V})$, for all $l \geq k$. We then have the following, where, as in the previous theorem, we assume (M,g) to be compact. Otherwise, when in a non-compact setting one needs to assume to be working in locally in a coordinate ball with appropriate boundary conditions.

Theorem 53. *If ψ is an S-valued distribution, meaning elements of $\bigcup_{k\in\mathbb{Z}} W^{k,2}\Gamma(\mathscr{V})$, and $\mathscr{D}(\psi) \in W^{k,2}\Gamma(\mathscr{V}) \subset W^{-\infty,2}\Gamma(\mathscr{V})$, then we infer more regularity on ψ. More precisely, $\psi \in W^{k+1,2}\Gamma(\mathscr{V})$.*

As a corollary, we have that $\ker(\mathscr{D}\colon W^{k+1,2}\Gamma(\mathscr{V}) \to W^{k,2}\Gamma(\mathscr{V}))$ consists of smooth sections. We will now recall briefly the spectral theory of Dirac operators needed for this work. As in the two previous theorems, we assume (M,g) to be compact. Otherwise, when in a non-compact setting one needs to assume to be working in locally in a coordinate ball with appropriate boundary conditions.

Theorem 54. *There exist a complete orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ of $L^2\Gamma(\mathscr{V})$, where this last space is the one of L^2-sections consisting of smooth sections of $\mathscr{V}$, and a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ such that*

$$\mathscr{D}(\phi_n) = \lambda_n \phi_n \qquad \textit{and} \qquad \lim_{n\to\infty} |\lambda_n| = \infty.$$

Thus, for any $\phi \in L^2\Gamma(S)$, we can write

$$\phi = \sum_{n\in\mathbb{N}} \phi_{\lambda_n},$$

where ϕ_λ denotes the L^2-orthogonal projection on the λ_n-eigenspace of $\mathscr{D}$. The series on the right-hand side has to be understood as converging in $L^2\Gamma(\mathscr{V})$.

We conclude with the unique continuation property (UCP) that is used throughout this work. An operator $\mathscr{A}$ over a smooth manifold M, with or without boundary, is said to satisfy the (UCP), if any solution s of $\mathscr{A}s = 0$, which vanishes on an open subset of M vanishes on the whole connected component of the manifold. From the Cauchy-Kovalevskaya it follows that all classical, euclidean, Dirac operators satisfy the UCP. Indeed, they are analytic operators. More generally, we have the following result, whose proof is based on establishing a specific Carleman-type estimate.

Theorem 55. *Given a compact Riemannian manifold (M,g) endowed with a Clifford bundle $\mathscr{V}$. Then, the (UCP) is valid for the corresponding Dirac operator between smooth sections.*

For further details on the (UCP) for Dirac operators, we refer the reader to [22, Chapter 8].

3.1.2 $\mathbb{Z}_2$-Harmonic Spinor

We can now give the definition of $\mathbb{Z}_2$-harmonic spinor we are going to work with, notice that it is different from Taubes' original one appearing in [137]. See [146] for further details.

Definition 56. Let U be a continuous 2-valued section of $\mathscr{V}$. Then, U is called a $\mathbb{Z}_2$-harmonic spinor if the following conditions hold.

1. The section U is not identically $\{0,0\}$.
2. Let $\mathscr{Z}_U$ be the set of U where $U = \{0,0\}$. For every $p \in M \setminus \mathscr{Z}_U$, there exists a neighborhood of p such that on this neighborhood u can be written as $U = \{u, -u\}$, where u is a smooth section of $\mathscr{V}$ satisfying $\mathscr{D}(u) = 0$.
3. Near a point $p \in M \setminus \mathscr{Z}_U$, write U as $\{u, -u\}$, then the function $|\nabla u|$ is a well defined smooth function on $M \setminus \mathscr{Z}_U$. Moreover, the section U satisfies
$$\int_{M\setminus\mathscr{Z}_U} |\nabla u|^2 < +\infty.$$

We also require the $\mathbb{Z}_2$-harmonic spinors in this work to satisfy the following additional Campanato-like assumption. There exists $\varepsilon > 0$ such that the following holds. For every $x \in M$ with $U(x) = \{0,0\}$, there exists constant $C, r_0 > 0$, depending on x such that

$$\int_{B_r(x)} |U(y)|^2 \, dy \le C r^{4+\varepsilon}, \tag{3.1}$$

for every $r \in (0, r_0)$. We will refer to a harmonic $\mathbb{Z}_2$-spinor as *singular* if $\mathscr{Z}_U \neq \emptyset$. Note that the norm of a harmonic $\mathbb{Z}_2$-spinor extends to a Hölder continuous sections on the whole of M as well, see [137, Theorem 1.5]. We shall now give a few examples, taken from [137], to illustrate the definition of harmonic $\mathbb{Z}_2$-spinor.

1. Consider $\mathbb{R}^4 \cong \mathbb{C} \times \mathbb{C}$ and use complex coordinates (z, w). Define
$$\mathscr{Z}_U = \{z = 0\} \cup \{w = 0\}$$
and $\mathscr{I}$ to be the restriction to $\mathbb{C}^2 \setminus \mathscr{Z}_U$ of a non-trivial real line bundle over the complement of the $\{z = 0\}$ plane. Finally let
$$U = \Re\left(\frac{3}{2}\sqrt{z}w^2 dz + 2wz^{3/2} dw\right).$$

This is a harmonic 1-form that is $\mathscr{I}$-valued on $\mathbb{C}^2 \setminus \mathscr{Z}_U$ and vanishes on $\mathscr{Z}_U$. Note that $\mathscr{N}((0,0);U) = 5/2$ and

$$\mathscr{N}(\cdot;U) = \begin{cases} 1/2, & \text{on } (\mathbb{C}^2 \setminus \{0\}) \cap \{z = 0\}, \\ 1, & \text{on } (\mathbb{C}^2 \setminus \{0\}) \cap \{w = 0\}. \end{cases}$$

2. Consider $\mathbb{R}^4 \cong \mathbb{C} \times \mathbb{R}^2$ again, and use coodinates $(z,(s,t))$. Let then

$$\mathscr{Z}_U = \{z = 0\} \cup \{s = 0\} \cup \{\Re(z^{3/2}) = 0\}$$

and define $\mathscr{I}$ to be the restriction to $\mathbb{R}^4 \setminus \mathscr{Z}_U$ of a non-trivial real line bundle over the complement of $\{z = 0\}$. Then, the one form

$$U = \Re\left(\frac{3}{2}\sqrt{z}s dz + z^{3/2} ds\right)$$

is harmonic, $\mathscr{I}$-valued on $\mathbb{R}^4 \setminus \mathscr{Z}_U$ and vanishes on $\mathscr{Z}_U$. The values of the frequency function are given by

$$\mathscr{N}(\cdot;U) = \begin{cases} 1/2, & \text{on } (\mathbb{R}^4 \setminus \{0\}) \cap \{z = 0\}, \\ 1, & \text{on } (\mathbb{R}^4 \setminus \{0\}) \cap (\{s = 0\} \cup \{\Re(z^{3/2}) = 0\}), \\ 3/2, & \text{on } \{z = s = 0\}. \end{cases}$$

Mimicking the works of Almgren on Dirichlet minimizing multi-valued functions we introduce the following sets.

Definition 57. Let U be a $\mathbb{Z}_2$-harmonic spinor. We can then define the following sets.

(i) The *singular set* Σ_U is the set of points Y such that there is no $r > 0$ with the following property: there exist singled-valued smooth sections u_1 and u_2 defined on the ball $B_r(Y)$ with either $u_1 \equiv u_2$, or $u_1(X) \neq u_2(X)$ for each $X \in B_r(Y)$, such that $\mathscr{D}(u_1) = \mathscr{D}(u_2) = 0$ and $U(X) = \{u_1(X), u_2(X)\}$ for $\mathscr{H}^n$-a.e. $X \in B_r(Y)$. Note that such u_1, u_2 are unique if they exist. Moreover, by virtue of the symmetry of U we have that $u_1 = -u_2$.

(ii) The *branch set* $\mathscr{B}_U$ is set to be the set of points $y \in M$ such that there exists no $r > 0$ with the property that $U(x) = \{u_1(X), u_2(X)\} = \{u(X), -u(X)\}$, for each $X \in B_r(Y)$ and two single valued smooth sections satisfying $\mathscr{D}(u) = 0$.

We followed Almgren's definition of singular set, see [9, Theorem 2.14]. Clearly, both Σ_U and $\mathscr{B}_U$ are closed subsets of M and, additionally, we have $\mathscr{B}_U \subset \Sigma_U \subset \mathscr{Z}_U$. Furthermore, note that on $M \setminus \mathscr{B}_U$ we can write the spinor as two smooth single-valued sections satisfying the Dirac equation. Furthermore, for every open ball $B \subset M \setminus \mathscr{B}_U$ we have $U(X) = \{u(X), -u(X)\}$ where $\mathscr{D}(u) = 0$, from which we deduce $\mathscr{D}^2(u) = \nabla^*\nabla u + \mathscr{R} u = 0$, i.e. $\Delta u = \mathscr{R} u$ on B, where $\Delta = \mathrm{tr}_g(\nabla^2)$.

Remark 58. In the case of Dirichlet energy minimizing and $C^{1,\mu}$-harmonic maps, even if one were interested in studying fine properties of the branch set $\mathscr{B}$ it is more convenient to study the full singular set Σ_U. The reason for this is a phenomenon called *persistence of singularities* that can be explained as follows. Consider a sequence of Dirichlet energy minimizing maps $\{u_j\}_j$ converging locally in L^2, or uniformly on compact subsets, to another Dirichlet energy minimizing u. Then, for every $\Omega \Subset M$, either $\Omega \cap \Sigma_{u_j} = \emptyset$ for all sufficiently large j, or $\Sigma_u \cap \Omega \neq \emptyset$. Unfortunately, this property does not hold with Σ_u replaced by $\mathscr{B}_u$.

To further clarify the notion of $\mathbb{Z}_2$-harmonic spinor and multi-valued sections we have the following result.

Proposition 59. *Let U be a $\mathbb{Z}_2$-harmonic spinor, then $U \in W^{1,2}(M, \mathscr{A}_2(\mathscr{V}))$. i.e. harmonic $\mathbb{Z}_2$-spinors are Sobolev sections, and $\mathscr{D}(U) = 0$ in the distributional sense. Moreover, there exists a sequence of smooth sections U_i, such that $U_i = -U_i$ and $U_i \to U$ in $W^{1,2}$.*

Proof. The idea is similar to the one in [146, Lemma 2.1]. Consider the smooth non-decreasing function χ on $\mathbb{R}$ defined as follows: $\chi(t) = 0$, for $t \geq 2$ and $\chi(t) = 1$, when $t \leq 1$. For $s \in (0,1)$, introduce the map $\tau_s = \chi(\ln(|U|)/\ln(s))$ which satisfies $\tau_s = 1$, provided $|U| \geq s$ and $\tau_s = 0$ if and only if $|U| \leq s^2$ (note that for this range of s the term $\ln(s)$ is negative so some inequalities are flipped, also note that for the same range of s we have $s^2 < s$). In other words, the function τ_s cuts off the singular set $\mathscr{Z}_U$. Moreover, $\tau_s U$ is a smooth 2-valued section of $\mathscr{V}$ (recall that a multi-valued section is said to be smooth if for every $p \in M$, there exists a neighborhood of p on which $\tau_s U$ can be written as $\{(\tau_s U)^1, (\tau_s U)^2\}$ for smooth sections τ_s^1, τ_s^2 of $\mathscr{V}$). We can now compute the $W^{1,2}$-norm of $\tau_s U$:

$$\|\tau_s U\|^2_{W^{1,2}(M)} = \int_M |\tau_s U|^2 + |\nabla(\tau_s U)|^2 = \sqrt{2}\int_M |\tau_s|^2|u|^2 + |u\nabla\tau_s + \tau_s\nabla u|^2,$$

where we used the fact that outside of $\mathscr{Z}_U$ the $\mathbb{Z}_2$-harmonic spinor decomposes into two smooth branches: $U = \{\pm u\}$. Now, note the bound

$$|\nabla\tau_s||u| \leq \frac{1}{|\ln(s)|}\sup|\chi'||\nabla u|,$$

holds, so that it converges to zero in $L^2(M)$ as $s \to 0$. Thus,

$$\lim_{s\to 0}\|\tau_s U\|^2_{W^{1,2}(M)} = \sqrt{2}\int_{M\setminus Z} |u|^2 + |\nabla u|^2,$$

from which we infer $W^{1,2}$-boundedness of the $\tau_s U$'s and, consequently, we know that there exists a, not-relabelled, subsequence and a element U' such that $\tau_s U$ converges to U' weakly in $W^{1,2}$. Furthermore, because of the uniform convergence $\tau_s U \to U$, we conclude $U' = U$, and then $U \in W^{1,2}(M, \mathscr{A}_2(\mathscr{V}))$. This last condition, combined with the fact that the Dirac operator $\mathscr{D}$ is a smooth first order differential operator allows us to conclude that $\mathscr{D}(U) \in L^2_{\text{loc}}(M)$. To finish the proof of the proposition we need to establish that U is a distributional solution over the whole manifold M, i.e. that it extends across Z. Indeed, $\mathscr{D}(U) = 0$ on $M \setminus \mathscr{Z}_U$ by definition of $\mathbb{Z}_2$-harmonic spinor. The fact that U being a Sobolev map entails that $\mathscr{D}(U)|_{\mathscr{Z}_U} = 0$ a.e. A proof of this follows from [44, Section 2.2.1]. Thus, we have that the U is a distributional solution of the Dirac equation on the whole M. The approximation result stated in the proposition follows as a byproduct of the above proof, see [146, Lemma 2.3]. □

Remark 60. Note that the above result is a good indication that we might expect further properties on the singular set of U. Indeed, the result is trivially false for second-order operators as the case of functions with large zero set and harmonic away from it show. In this case, we need to introduce further additional properties, of variational type like being Dirichlet minimizing or of regularity type, e.g. being $C^{1,1/2}$, to conclude that a large zero set cannot be present and that we have further structural results on it.

We also recall the following estimates due to Taubes, see Lemma 2.1 and Lemma 2.3 of [137].

Lemma 61. *For any two open sets $A \Subset B \subset M$ there exists a constant C depending on them, as well as on the norms of the curvatures of M and $\mathscr{V}$, such that the following C^0-estimate holds*

$$\sup_A |u|^2 \le C \int_B |u(x)|^2$$

It follows from a rescaling argument that $r^4|u(x)| \le C \int_{B_r(x)} |u|^2$. Similarly, we have

$$\int_A |\nabla u|^2 \le C \int_B |u|^2.$$

In the following, we prove that $\mathbb{Z}_2$-harmonic spinors satisfying extra regularity assumptions are in fact locally $W^{2,2}$, a fact that will allow us to justify various computations later in this work.

Lemma 62. *Suppose that U is a $\mathbb{Z}_2$-harmonic spinor of C^1-regularity on M. Then, $D^2U \in L^2(B_r(p))$, meaning $DU \in L^2(B_r(p))$, for each ball $B_r(p) \Subset M$. Furthermore, we have the estimate,*

$$r^{2-n} \int_{B_{r/2}(p)} |\nabla^2 u|^2 \le Cr^{-n} \int_{B_r(p)} |\nabla u|^2 + Cr^{-2-n} \int_{B_r(p)} |u|^2,$$

again for all balls $B_r(p) \Subset M$.

Proof. The argument is similar to [125, Lemma 2.1], we sketch below the main steps of the argument. Recall that in any ball away from the singular set Σ_U we can write $U = \{\pm u\}$, so that the Bochner-Weitzenböck formula implies

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 s|^2 - \sum_{k=1}^{4} \langle \nabla^* \nabla(\nabla_{e_k} s), \nabla_{e_k} s\rangle = |\nabla^2 s|^2 + \sum_{k=1}^{4} \langle \mathrm{tr}_g \nabla^2(\nabla_{e_k} s), \nabla_{e_k} s\rangle,$$

where $\{e_k\}_{k=1}^4$ is a local orthonormal frame. Now, we can commute the connections in the last term of the above expression, modulo introducing curvature and derivative of the curvature terms. We refer the reader to [**?** , Section 2.4] for similar computations. Now, if γ_δ is a smooth non-negative convex function on $\mathbb{R}$ with $\gamma_\delta \equiv 0$ in a neighborhood of the origin and $\gamma'_\delta(t) \equiv 1$ on $[\delta, \infty)$, then we can estimate the term $\Delta(\gamma_\delta(|Du|^2))$ from below by $2\gamma'_\delta(|Du|^2)$ times the expression for $\Delta\big((|\nabla u|^2)\big)$ found above. Multiplying now the resulting expression by a cut-off function supported in $B_r(p)$ which is additionally identically 1 on $B_{r/2}(p)$, and then integrate over $B_r(p)$. Letting $\delta \to 0$ we obtain the required $W^{2,2}$-estimate. A scaling argument allows to infer the right power of r in front of each term. □

The following theorem summarises the main results of [146].

Theorem 63 (Theorem 1.3 and Theorem 1.4 of [146])**.** *If U is a $\mathbb{Z}_2$-harmonic spinor satisfying (3.1), then the Hausdorff dimension of $\mathscr{Z}_U$ is at most 2. Moreover, $\mathscr{Z}_U$ is 2-rectifiable and has locally finite 2-dimensional Minkowski content. The latter means that for every compact set $A \subset M$, there exist a constant C and r_0 depending on A and $\mathscr{Z}_U$, such that for every $r < r_0$,*

$$\mathrm{vol}(\{x; \mathrm{dist}(x, A \cap \mathscr{Z}_U) < r\}) \le Cr^2.$$

Note that for the above theorem Taubes and Zhang do not need to impose C^1-regularity of the $\mathbb{Z}_2$-harmonic spinor.

3.2 Almgren's Frequency Function and its consequences

A classical tool to understand rectifiability and the singular set is the so-called frequency function. First introduced by Almgren in [8] it is by now a classical tool to study the regularity theory of partial differential equations.

Before introducing it we need to set some notation. For $q \in M$ let $\exp\colon B_3(0) \subset T_qM \to M$ be the exponential map and denote by $J\exp$ its Jacobian. We will use in the next computations the following fact: $\partial_r J\exp(rz) = \mathscr{O}(1)$. Furthermore, notice that $d_g(\exp(y), q) = |y|$, where we denoted by d_g the distance induced by the Riemannian metric. Fixing now $q \in M$, take $r_0 > 0$ such that the injectivity radius of M is greater than $2r_0$ for every point in the ball $B_{r_0}(q)$. Consider then $p \in B_{r_0}(q)$ and use normal coordinates centered at p to identify the balls $B_r(p)$ with the euclidean ones centered at the origin of $\mathbb{R}^4$, for $r \leq r_0$. For the rest of this section we let $x \in B_r(q)$, with $r \in (0, r_0]$. Denoting $g_q(\cdot)$ the function of metric matrices on the ball centered at the origin in the Euclidean space corresponding to $B_r(q)$, we have, provided r_0 is sufficiently small, for every $p \in B_r(q)$,

$$\frac{1}{\Lambda^2} \leq \lambda_q(\cdot) \leq \Lambda_q(\cdot) \leq \Lambda^2, \tag{3.2}$$

where Λ is a constant sufficiently close to one and λ_q, respectively Λ_q, denote the smallest, respectively larger, eigenvalue of g_q. Note that the variable $\cdot$ in the above inequality refers to points in the ball centered at the origin of Euclidean space. Other relevant geometric notions will be introduced when need be. We can now define the height function

$$H(x,r) = \int_{\partial B_r(x)} |u|^2.$$

and the Dirichlet energy

$$D(x,r) = \int_{B_r(x)} |\nabla u|^2.$$

The frequency function then reads

$$N(x,r) = \frac{rD(x,r)}{H(x,r)}. \tag{3.3}$$

When we want to emphasize the dependence on U we shall write $N(x,r;U)$. If no source of confusion should arise we shall keep the notation as simple as possible. Work of Taubes, [137, Lemma 3.1], guarantees that the function H is always strictly positive, so that the above expression is well-defined. We shall recall this result here for the convenience of the reader.

Lemma 64. *There exists a constant $\kappa > 1$ with the following significance. Fix $p \in M$ to define the height function $H(p,r)$ on $[0,r_0]$. Then, this function is strictly positive on $(0,r_0]$ and, in addition, if $s \in (0,r_0]$ and if $r \in [s,\kappa^{-1}]$, we have*

$$H(p,r) \geq \left(\frac{r}{s}\right)^{d-1} e^{-\kappa(r^2-s^2)} H(p,s).$$

Taubes again established the following monotonicity properties.

Proposition 65. *The Dirichlet energy is absolutely continuous with respect to r and there exists a constant $\kappa > 0$, depending only on the norms of the curvatures of X and $\mathscr{V}$ on the ball $B_{2r_0}(x_0)$, such that*

$$\frac{\partial}{\partial r} H(x,r) \geq \frac{3}{r} H(x,r) - \kappa r H(x,r), \tag{3.4}$$

and

$$\left(\frac{N(x,r)}{r} + \kappa r\right) \frac{H(x,r)}{r^3} \geq \frac{\partial}{\partial r}\left(\frac{H(x,r)}{r^3}\right) \geq \left(\frac{N(x,r)}{r} - \kappa r\right) \frac{H(x,r)}{r^3}.$$

For the frequency function we have the following.

Proposition 66. *The frequency function is an absolutely continuous function of r. Moreover, we have the lower bound*

$$\frac{\partial}{\partial r}N(x,r) \geq \frac{2}{rH(x,r)}\int_{\partial B_r(x)}\left|\nabla_r u - \frac{1}{r}N(x,r)u\right|^2 - \kappa r(1+N(x,r)), \tag{3.5}$$

where, as in the previous proposition, the constant κ depends only on the curvatures of X and $\mathcal{V}$ on the ball $B_{2r_0}(x_0)$, and the above inequality holds for $r \leq r_0$.

Remark 67. Note that the first term on the right-hand side of (3.5) measures how far u is from being homogenous of degree $N(x,r)$.

Proof. A proof of this can be found in [137, Lemma 3.2]. Alternatively, we refer the reader to Section 3 of [**?**] or Sections 3 and 4 of [146] for similar computations. We shall repeat here the main steps of the proof for the convenience of the reader. Doing this also serves the purpose of collecting identities and inequalities useful for later purposes.

Instead of working with the above defined frequency function one can work with a smoothed version of it, which was fisrt introduced in [42]. We refer the reader to it and to [47] for further details. Let $\chi\colon \mathbb{R} \to [0,1]$ be a non-increasing smooth function satisfying $\chi(t) = 1$ for $t \leq 1/2$ and $\chi(t) = 0$ on $[1,\infty)$. We can then define

$$D_\chi(x,r) = \int_M |\nabla u(y)|^2 \chi\left(\frac{d_g(x,y)}{r}\right)$$

to be the smoothed Dirichlet energy, and

$$H_\chi(x,r) = -\int_M |u(y)|^2 d_g(x,y)^{-1}\chi'\left(\frac{d_g(x,y)}{r}\right)$$

the smoothed height function. Define accordingly the smoothed frequency function as

$$N_\chi(x,r) = \frac{rD_\chi(x,r)}{H_\chi(x,r)},$$

and the smoothed energy

$$E_\chi(x,r) = -\int_M |\nabla_{\nu_x} u(y)|^2 d_g(x,y)\chi'\left(\frac{d_g(x,y)}{r}\right),$$

where ν_x is the gradient vector field of the distance function $d_g(x,\cdot)$. Note that letting χ converge to the indicator function of the interval $[0,1)$, we recover the corresponding classical Dirichlet, height, frequency and energy functions. As the names suggest, D_χ, H_χ, N_χ and E_χ are smooth in both variables. One can then compute the following

$$\begin{aligned}
D_\chi(x,r) &= -\frac{1}{r}\int_M \chi'\left(\frac{d_g(x,y)}{r}\right)\nabla_{\nu_x}u(y)\cdot u(y) + \mathcal{O}(rH_\chi(x,r)),\\
\frac{d}{dr}D_\chi(x,r) &= \frac{2}{r}D_\chi(x,r) + \frac{2}{r^2}E_\chi(x,r) + \mathcal{O}(H_\chi(x,r)),\\
\partial_\nu D_\chi(x,r) &= -\frac{2}{r}\int_M \chi'\left(\frac{d_g(x,y)}{r}\right)\nabla_{\nu_x}u(y)\cdot\nabla_\nu u(y) + \mathcal{O}(H_\chi(x,r))\\
\frac{d}{dr}H_\chi(x,r) &= \frac{3}{r}H_\chi(x,r) + 2D_\chi(x,r) + \mathcal{O}(rH_\chi(x,r))\\
\partial_\nu H_\chi(x,r) &= -2\int_M u(y)\cdot\nabla_\nu u(y)\frac{1}{d_g(x,y)}\chi'\left(\frac{d(x,y)}{r}\right) + \mathcal{O}(rH_\chi(x,r)),
\end{aligned} \tag{3.6}$$

where in the third and fifth identities $v \in T_xM$. where we used the following bounds

$$\int_{B_r(x)} |u|^2 = \mathcal{O}(rH_\chi(x,r)),$$
$$\int_{B_r(x)} |u||\nabla u| = \mathcal{O}(H_\chi(x,r)),$$
$$\int_{B_r(x)} |\nabla u|^2 = \mathcal{O}(H_\chi(x,r)/r),$$

for all r small enough depending on the injectivity radius. In the particular case in which both the base manifold M and the bundle $\mathscr{V}$ are flat the above reduce to

$$\begin{aligned}
D_\chi(x,r) &= -\frac{1}{r}\int_M \chi'\left(\frac{d_g(x,y)}{r}\right)\nabla_{v_x}u(y)\cdot u(y),\\
\frac{d}{dr}D_\chi(x,r) &= \frac{2}{r}D_\chi(x,r) + \frac{2}{r^2}E_\chi(x,r)\\
\partial_v D_\chi(x,r) &= -\frac{2}{r}\int_M \chi'\left(\frac{d_g(x,y)}{r}\right)\nabla_{v_x}u(y)\cdot\nabla_v u(y)\\
\frac{d}{dr}H_\chi(x,r) &= \frac{3}{r}H_\chi(x,r) + 2D_\chi(x,r)\\
\partial_v H_\chi(x,r) &= -2\int_M u(y)\cdot\nabla_v u(y)\frac{1}{d_g(x,y)}\chi'\left(\frac{d(x,y)}{r}\right),
\end{aligned}$$

where, as above, in the third and fifth identities $v \in T_xM$. An identity used in the proof of the proof of the above formulae is the following: let f be a smooth function whose support is contained in the ball $B_r(x)$, $\{e_i\}$ be an orthonormal basis of T_xM and F the curvature of $\mathscr{V}$. Then, there holds

$$\begin{aligned}
&\int_M |\nabla u|^2\partial_v f\\
=&2\int_M \langle df\otimes\nabla_v u,\nabla u\rangle - 2\int_M\sum_i f\langle F(v,e_i)u,\nabla_{e_i}u\rangle\\
&-2\int_M\sum_i f\langle\nabla_{[v,e_i]}u,\nabla_{e_i}u\rangle - \int_M|\nabla u|^2 f\,\mathrm{div}(v)\\
&+2\int_M\sum_i f\langle\nabla_v u,\nabla_{\nabla_{e_i}e_i}u\rangle + 2\int_M\sum_i f\langle\nabla_v u,\nabla_{e_i}u\rangle\,\mathrm{div}(e_i) + 2\int_M f\langle\nabla_v u,\mathscr{R}_0\rangle,
\end{aligned}$$

where $\mathscr{R}_0$ is the curvature term from the Weitzenböck formula and

$$\langle df\otimes\nabla_v u,\nabla u\rangle = \sum_i(\nabla_{e_i}f)\langle\nabla_v u,\nabla u\rangle.$$

The key idea to prove the above identity is to work with a sequence of smooth harmonic $\mathbb{Z}_2$-spinors approximating the given one, instead of just working with this last one.

We are now able to prove the inequality appearing in the statement of the proposition. Let $\eta_x(y) = d_g(x,y)\cdot v_x(y)$, we then have

$$\frac{d}{dr}N_\chi(x,r) = -\frac{2}{rH_\chi(x,r)}\int_M\frac{1}{d_g(x,y)}\chi'\left(\frac{d_g(x,y)}{r}\right)\left|\nabla_{\eta_x}u(y) - N_\chi(x,r)u(y)\right|^2 + \mathcal{O}(r)$$

as well as,

$$\partial_v N_\chi(x,r) = -\frac{2}{H_\chi(x,r)}\int_M\frac{1}{d_g(x,y)}\chi'\left(\frac{d_g(x,y}{r}\right)\left(\nabla_{\eta_x}u(y) - N_\chi(x,r)u(y)\right)\cdot\nabla_v u(y) + \mathcal{O}(r),$$

with respect to any $v \in T_xM$. To prove the former, compute

$$\frac{d}{dr}N_\chi(x,r) = \frac{2}{rH_\chi(x,r)}\left(E_\chi(x,r) - \frac{r^2 D_\chi^2(x,r)}{H_\chi(x,r)}\right) + \mathscr{O}(r),$$

which combined with

$$\begin{aligned} E_\chi(x,r) - \frac{r^2 D_\chi^2(x,r)}{H_\chi(x,r)} \\ &= E_\chi(x,r) - 2rD_\chi(x,r)N_\chi(x,r) + N_\chi^2(x,r)H_\chi(x,r) \\ &= -\int_M \chi'\left(\frac{d_g(x,y)}{r}\right)\frac{1}{d_g(x,y)}\left|\nabla_{\eta_x}u(y) - N_\chi(x,r)u(y)\right|^2 + \mathscr{O}(r^2 H_\chi(x,r)) \end{aligned}$$

gives the desired identity. The latter equality follows in a similar way. Repeating the same computations with the integral identities appearing in Subsection 3.2.1 we deduce

$$\frac{d}{dr}N(x,r) = \frac{2}{rH(x,r)}\int_{\partial B_r(x)}\left|\nabla_{v_x}u(y) - \frac{1}{r}N(x,r)u(y)\right|^2 + \mathscr{O}(r),$$

from which the desired estimate follows simply by keeping track of the $\mathscr{O}(r)$ terms. □

Remark 68. Note that an alternative proof of the almost monotonicity of the frequency function follows from the following equation for the norm of a harmonic $\mathbb{Z}_2$-spinor, which in turn is a consequence of the definition of Dirac operator:

$$\frac{1}{2}\Delta|u|^2 + |\nabla u|^2 + \langle \mathscr{R}u, u\rangle = 0,$$

It then follows by integration by parts,

$$\int_U \frac{1}{2}\Delta\eta\cdot|u|^2 + \eta|\nabla u|^2 = -\int_U \eta\langle\mathscr{R}u,u\rangle + \frac{1}{2}\int_{\partial U}\eta\partial_\nu|u|^2 - \partial_\nu\eta\cdot|u|^2, \tag{3.7}$$

for U an open subset of M with smooth boundary and $\chi \in C^\infty(\overline{U})$ and for u a solutions of the Dirac equation. Taking $\eta = 1$ and $U = B_r(x)$ one could then start estimating the various terms appearing in the definition of frequency function. See [**?**] for further details.

From (3.5) we deduce

$$\frac{\partial}{\partial r}N(x,r) \geq -\kappa r(1 + N(x,r)) \geq -Cr,$$

from which, by integration, we can then immediately infer the following corollary.

Corollary 69. *There exists a constant $\kappa > 0$, such that when $s < r < r_0$,*

$$N(x,r) \geq e^{-\kappa(r^2 - s^2)}N(x,s) - \kappa(r^2 - s^2).$$

Moreover, it follows that N is bounded for small enough r. Finally, we also have for $s \leq t$ the following inequality

$$N(x,s) \leq (1 + Cr^2)N(x,r) + Cr^2.$$

We can push the analysis further to obtain

$$N(x,r) \geq N(x,s) - C(r^2 - s^2).$$

Indeed, as $N(x,r_0)$ is a countinuous function of x we can assume that $N(x,r)$ is bounded for all $x \in B_{r_0}(x_0)$ and $r \leq r_0$, from which the desired inequality follows. In other words the frequency function is almost monotonic. Thus, we can take limits as $r \to 0^+$ to obtain

$$\mathscr{N}(x;U) := N(x,0;U) = \lim_{r\to 0^+} N(x,r;U),$$

where we know that the limit exists by almost monotonicity, of course provided that there exists some $t > 0$ for which the denominator in the frequency function is non zero for all $r \in (0,t)$, fact that is guaranteed by Lemma 64. We call $\mathscr{N}(x;U)$ the *frequency of U at x.*

We collect a few other results on the frequency function, the first lemma is a consequence of the almost monotonicity.

Lemma 70. *The frequency of U is upper semicontinuous in the sense that if U_k are harmonic $\mathbb{Z}_2$ such that $U_k \to U$ in $L^2(\Omega, \mathscr{A}_2(\mathscr{V}))$, for all $\Omega \Subset M$, and if $p_k, p \in M$ are such that $p_k \to p$, then*

$$\limsup_{k\to\infty} \mathscr{N}(p_k;U_k) \leq \mathscr{N}(p;U).$$

[Lemma 6.7 of [137]]

Lemma 71. *Suppose that $p \in \mathscr{Z}_U$ and that $\mathscr{N}(p;U)$ is not half of a positive integer. Then there exists a sequence $\{p_i\}_i \subset \mathscr{Z}_U$ that converges to p with the limit $\lim_i \mathscr{N}(p_i;U)$ being half of a positive integer.*

Combining the above two lemmas we obtain the following result. Note that we have the same bound as in the Dirichlet minimizing case, thus strengthening the parallelism between this case and the one we are considering.

Corollary 72 (Lemma 6.4 of [137])**.** *We have the bound $\mathscr{N}(X;U) \geq 1/2$ for every $X \in \mathscr{Z}_U$.*

3.2.1 Integral Identities

We finish this section by recalling some useful integral identities proven by Taubes in [137]. We have

$$\int_{\partial B_r(x)} \langle \nabla_{\nu_x} u, u\rangle = \int_{B_r(x)} |\nabla u|^2 + \int_{B_r(x)} \langle u, \mathscr{R}u\rangle,$$

where $\mathscr{R}$ is a bounded curvature term coming from the Weitzenböck formula. Note that this is the analogue of the squash identity in (1.22). The curvature term can be bounded by

$$\int_{B_r(x)} \langle u, \mathscr{R}u\rangle \leq CrH(x,r), \tag{3.8}$$

where we used the the monotonicity appearing in Lemma 64 after writing the integral in polar coordinates. In particular, we can write the above identity as

$$D(x,r) = \int_{\partial B_r(x)} \langle \nabla_{\nu_x} u, u\rangle + \mathscr{O}(rH(x,r)).$$

Furthermore, there holds

$$\int_{\partial B_r(x)} |\nabla u|^2 = 2\int_{\partial B_r(x)} |\nabla_{\nu_x} u|^2 + \frac{2}{r}\int_{B_r(x)} |\nabla u|^2 + \frac{2}{r}\int_{B_r(x)} \langle u, \mathscr{R}u\rangle - \int_{\partial B_r(x)} \langle u, \mathscr{R}^{\perp} u\rangle,$$

where $\mathscr{R}, \mathscr{R}^{\perp}$ are smooth bounded tensors. Note that the above identity can be rewritten as

$$2\int_{B_r(x)} |\nabla u|^2 = r\int_{\partial B_r(x)} |\nabla u|^2 - 2r\int_{\partial B_r(x)} |\nabla_{v_x} u|^2 - 2\int_{B_r(x)} \langle u, \mathscr{R}u\rangle + r\int_{\partial B_r(x)} \langle u, \mathscr{R}^{\perp}u\rangle,$$

thus showing that, modulo some curvature error terms, we have the analogue of the squeeze identity as well. Note that the curvature terms can be bound as follows:

$$r\int_{\partial B_r(x)} \langle u, \mathscr{R}^{\perp}u\rangle \leq CrH(x,r),$$

and the first curvature error term can be bounded as in (3.8). In other words, we have the same bounds for the error terms of our modified squeeze identity:

$$2\int_{B_r(x)} |\nabla u|^2 = r\int_{\partial B_r(x)} |\nabla u|^2 - 2r\int_{\partial B_r(x)} |\nabla_{v_x} u|^2 + \mathscr{O}(rH(x,r)).$$

Note that this identity translates to

$$\frac{d}{dr}D(x,r) = \frac{2}{r}D(x,r) + \frac{2}{r^2}E(x,r) + \mathscr{O}(H(x,r)).$$

Finally, we also have

$$\begin{aligned}\partial_s H(x,s) &= \frac{3}{s}H(x,s) + 2D(x,s) + \int_{B_s(x)} \langle u, \mathscr{R}u\rangle + \int_{\partial B_s(x)} \mathfrak{t}|u|^2 \\ &= \frac{3}{s}H(x,s) + 2D(x,s) + \mathscr{O}(sH(x,s)),\end{aligned} \tag{3.9}$$

where $\mathscr{R}$ is the curvature term from the Weitzenböck formula. The function $\mathfrak{t}(y)$ comes from the mean curvature of the geodesic boundaries $\partial B_s(x)$ and satisfies $|\mathfrak{t}(y)| = O(d(x,y))$. Note that the above simplifies to

$$\partial_s\left(s^{-3}H(x,s)\right) = \frac{2}{s^3}D(x,s) + \mathscr{O}(s^{-2}H(x,s)). \tag{3.10}$$

We will sometimes need to work with the normalised height and Dirichlet energy, whence the notation

$$\overline{D}(x,r) = r^{2-n}D(x,r) \quad \text{and} \quad \overline{H}(x,r) = r^{1-n}H(x,r),$$

and similarly for the smoothed height and Dirichlet energy. Thus, we can write the frequency, and similarly for the smoothed version, simply as $N(x,r) = \overline{D}(x,r)/\overline{H}(x,r)$.

3.3 Existence of Blow-ups and their Properties

This section aims to introduce the compactness result for harmonic $\mathbb{Z}_2$-spinors relevant to this work, as well as give a definition of blow-ups together with their main properties.

3.3.1 Compactness

We start by modifying the compactness result from Section 5 of [146]. Consider the closed ball $\Omega = B_2(0) \subset \mathbb{R}^4$ and let $\mathscr{V}$ be a fixed trivial vector bundle over Ω. Assume furthermore that $(g_n)_n$ is a sequence of Riemannian metrics on Ω, that $(A_n)_n$ is a sequence of connection forms on $\mathscr{V}$ and $(\rho_n)_n$ is a sequence of Clifford bundle structures of $\mathscr{V}$. Suppose that $(g_n, A_n, \rho_n)_n$ are compatible and assume that they converge to a triple (g, A, ρ) in the C^{∞} topology, where g is the Euclidean metric on Ω. Consequently, we can assume without loss of generality that the injectivity radius at any point $B_1(0)$ is at least 1 for every n. Fix then $\varepsilon > 0$ and consider a sequence of $(U_n)_n$ of 2-valued sections of $\mathscr{V}$ on Ω such that the following properties hold

1. The section U_n is a $\mathbb{Z}_2$-harmonic spinor on Ω with respect to (g_n, A_n, ρ_n);
2. The U_n's satisfy the Campanato-like assumption in (3.1) with respect to ε;
3. The U_n's have a uniform L^2-bound on $\nabla_{A_n} U_n$ in the interior of $B_2(0)$;
4. There holds $\|U_n\|_{L^2(B_1(0))} = 1$, for all $n \in \mathbb{N}$;

In place of the third assumption, one could alternatively require a uniform upper bound on the frequency function of the U_n's. We then have the following compactness result.

Proposition 73. *Let $\{U_n\}_n$ be a sequence of harmonic $\mathbb{Z}_2$-spinors as above. Then, there exists a subsequence converging strongly in $W^{1,2}(B_1(0))$ to a section U, which is a harmonic $\mathbb{Z}_2$-spinor with respect to (g, A, ρ). Moreover, U satisfies the assumption (3.1) with a possibly smaller ε and the subsequence converges uniformly in $B_1(0)$.*

Sketch of the proof. The upper bounds on the L^2-norm of the U_n, together with the uniform bound on the L^2-norm of $\nabla_{A_n} U_n$ in the interior of $B_2(0)$ implies that the sequence U_n is uniformly bounded in $W^{1,2}(B_1(0))$. Thus, by Rellich-Kondrakov we can pass to a subsequence converging strongly in L^2 and weakly in $W^{1,2}$. In particular, $\|U\|_{L^2(B_1(0))} = 1$, from which we infer that U is not identically $\{0,0\}$. Now, [137, Section 3(e)] we infer a uniform bound on the $C^{0,\alpha}$-norm of the U_n, so that an application of Arzela-Ascoli implies that the limit is $C^{0,\alpha}$ and that convergence to the limit is uniform as well. In particular, from Hölder continuity we infer that (3.1) is satisfied and, since solutions to the Dirac equation are closed under C^0 limits we have that U_n is a harmonic $\mathbb{Z}_2$-spinor as well. We are left with checking that we have convergence of the full $W^{1,2}$-energy, namely

$$\lim_{n\to\infty} \int_{B_1(0)} |\nabla_{A_n} U_n|^2 = \int_{B_1(0)} |\nabla_A U|^2.$$

This can be proven using a contradiction argument and appealing to (3.9). □

Remark 74. When working with Dirichlet minimizing map, a good example to keep in mind is the following: $U_k = z^k$ on the unit ball of the complex plane $\mathbb{C}$. Rescale the U_k by their L^2-energy. Each of the U_k is energy minimizing. However, as $k \to \infty$, the order of vanishing becomes infinite at the origin so that by unique continuation for harmonic functions we would have the sequence converging to zero. On the other hand, the normalization ensures that the limit has L^2-norm one. In other words, more and more energy is accumulating over the boundary of the ball as k goes to infinity.

3.3.2 Blow ups

Denote by $R_p \in (0, \infty]$ the injectivity radius of the point $p \in M$ and consider a normal coordinate ball $B_r(p)$ of radius $r \in (0, R_p)$. Consider then $(\exp_p^{-1})^* U$ restricted to the ball $B_r(q)$. We have that $\widetilde{U}$ is a harmonic $\mathbb{Z}_2$-spinor on the euclidean ball $B_r(0) \subset T_pM \cong \mathbb{R}^4$. Note that working on the ball $B_r(q)$ we have a local trivialisation at our disposal. We can now introduce blow-ups, one of the main tools to investigate the singular set. To do so we introduce the dilation map

$$\eta_{Y,r} \colon \mathbb{R}^4 \to \mathbb{R}^4, \quad \text{given by} \quad \eta_{Y,r}(X) = \frac{1}{r}(X - Y)$$

for $Y \in \mathbb{R}^4$ and $r > 0$. Keeping the same notation as before, define then $\widetilde{U} = \eta_{0,r} \circ (\exp_p^{-1})^* U$, where U is restricted to the normal coordinate ball $B_r(p)$ as before, and where $r \in (0, R_q)$. Given a harmonic $\mathbb{Z}_2$-spinor U, let

$$U_{Y,r} = \frac{\widetilde{U}}{r^{-n/2}\|\tilde{U}\|_{L^2(B_r(0))}} = \frac{\eta_{0,r}\circ(\exp_Y^{-1})^*U}{r^{-n/2}\|\eta_{0,r}\circ(\exp_Y^{-1})^*U\|_{L^2(B_r(0))}} \tag{3.11}$$

where $Y \in \mathscr{Z}_U$ and $r \in (0,R_Y)$. Considering now a sequence of r_j's converging to zero we have the following result. We refer the reader to Proposition 4.1 of [137] for a similar statement. However, the one that follows resembles more closely the results in [89, Section 3].

Proposition 75. *If r_j is a sequence of numbers with $r_j \to 0^+$, there exists a symmetric two-valued function $\varphi \in W^{1,2}_{\text{loc}}(\mathbb{R}^4,\mathscr{A}_2(\mathscr{V}))\cap C^{0,\alpha}(\mathbb{R}^4,\mathscr{A}_2(\mathscr{V}))$ for certain $\alpha\in(0,1)$, such that, after passing to a not relabelled subsequence,*

$$U_{Y,r_j} \to \varphi,$$

locally uniformly in $\mathbb{R}^4$ and locally in the $W^{1,2}$-topology.

We call such a function φ a *blow up of U at Y* (or tangent cone following Taubes terminology in [137] or tangent function as appearing in [47]). Note that at this point *uniqueness* of the blow-up, upon changing the sequence r_j, is not clear and, indeed, it ends up being a delicate issue.

The definition of blow-up given by Taubes requires additional pieces of information. In the paragraphs that follow and leading to Proposition 76 we shall digress on Taubes' definition of blow-up. We will refer the reader to it when needed and for a detailed exposition. Given a point $p\in M$, an oriented orthonormal frame L at T_pM and a number $r\in(0,R_p]$ one considers then the data $T=(p,L,r)$. The frame L determines then a normal coordinate chart centered at p whose differential at the origin of $\mathbb{R}^4$ sends a fixed orthonormal frame to the given L. Repeating then the same scaling argument as above one obtain a map Φ_T mapping the ball $B_{r_p/r}(0)\subset\mathbb{R}^4$ to M. (This map is simply given by the scaling map composed with the Gaussian coordinate chart map). Denoting by $\mathbb{V}^+$ and $\mathbb{V}^-$ the fibers at the point p of the bundles $\mathbb{S}^+$ and $\mathbb{S}^-$, where $\mathscr{V}=\mathbb{S}^+\oplus\mathbb{S}^-$ (in other words $\mathbb{S}^+$ and $\mathbb{S}^-$ are a grading of $\mathscr{V}$) one can identity $\Phi_T^*\mathbb{S}^+$ over the ball $B_{r_p/r}(0)$ with the product $\mathbb{V}^+$ bundle over the same ball, and similarly for $\Phi_T^*\mathbb{S}^-$. Furthermore, cf. [137, page 27], the principal symbol of $\mathscr{D}$ at p defines an elliptic operator from $C^\infty(\mathbb{R}^4,\mathbb{V}^+)$ to $C^\infty(\mathbb{R}^4,\mathbb{V}^-)$. Recall further that a real line bundle that is defined on the complement of a scale-invariant subset is canonically isomorphic to its pull-back by the map $\psi_\lambda:\mathbb{R}^n\to\mathbb{R}^n$ defined by $\psi_\lambda(x)=\lambda x$, for $\lambda>0$. One can then denote by $\mathscr{Z}_T$ the pullback of $\mathscr{Z}$ by Φ_T and by $\mathscr{I}_T$ the pullback of $\mathscr{I}$ by Φ_T.

Letting then $r_j\to 0$ we obtain the same blow-up as in Proposition 75, together with a scale-invariant subset $\mathscr{Z}_*$ of $\mathbb{R}^4$, a Euclidean line bundle $\mathscr{I}_*$ defined on the complement of it and such that the blow-up is a harmonic section of $\mathbb{V}^+$ over $\mathbb{R}^4\setminus\mathscr{Z}_*$. It follows that the $\mathscr{Z}_*$ is the zero set of the resulting blow-up. Note that $\mathbb{R}^4$ is endowed with the Euclidean metric and the pulled back ones on the balls $B_{r_p/r_j}(p)$ converges to it on compact subsets. We refer the reader to [137, Proposition 4.1] for further details on the blow-up construction. The case we will be interested in is when Taubes' data $T=(p,L,r)$ have fixed the point p and the frame L, and where only the radii converge to zero, precisely as in Proposition 75.

Using the frequency function we can go beyond existence and establish several other properties of blow-ups, we collect them in the following proposition and lemmas.

Proposition 76. *The blow up φ appearing in the previous proposition is non-zero, satisfies the Dirac equation distributionally on $\mathbb{R}^4$. Furthermore, it satisfies $N(0,\rho;\varphi)=\mathscr{N}(0;\varphi)=\mathscr{N}(y;U)$ for each $\rho>0$. Finally, φ is homogeneus of degree $\mathscr{N}(y;U)$.*

Proof. We only need to establish non-triviality. Recall from Corollary 69 the inequality $N(x,s) \leq (1+Cr^2)N(x,r)+Cr^2$, for $s \leq r$, which essentially follows from

$$\frac{d}{dr}\left(e^{\frac{1}{2}Cr^2}(N(x,r)+1)\right) \geq 0.$$

Combine it with (3.10) to infer

$$\frac{d}{d\sigma}\log \overline{H}(x,\sigma) \leq \frac{2}{\sigma}N(x,\sigma)+C\sigma \leq \frac{2(1+Cr^2)}{\sigma}N(x,r)+\frac{Cr^2}{\sigma},$$

for $\sigma \in [s,r]$. Thus, integrating and exponentiating we deduce

$$\overline{H}(x,r) \leq \left(\frac{r}{s}\right)^{2(1+Cr^2)N(x,r)+Cr^2}\overline{H}(x,s). \tag{3.12}$$

Reasoning similarly, one can obtain the lower bound

$$\frac{d}{d\sigma}\log \overline{H}(x,\sigma) \geq \frac{2(1-Cr^2)}{\sigma}N(x,r)-\frac{Cr^2}{\sigma},$$

from which one can deduce

$$\left(\frac{r}{s}\right)^{2(1-Cr^2)N(x,r)-Cr^2}\overline{H}(x,s) \leq \overline{H}(x,r). \tag{3.13}$$

Integrating (3.12) and (3.13) we obtain

$$\left(\frac{r}{s}\right)^{2(1-Cr^2)N(x,r)-Cr^2} s^{-4}\int_{B_s(x)}|u|^2 \leq r^{-4}\int_{B_r(x)}|u|^2 \tag{3.14}$$

$$\leq \left(\frac{r}{s}\right)^{2(1+Cr^2)N(x,r)+Cr^2} s^{-4}\int_{B_s(x)}|u|^2, \tag{3.15}$$

from which we deduce non-triviality of blow-ups. Note that one can reverse the inequalities to have upper and lower bounds on $s^{-4}\int_{B_s(x)}|u|^2$. Furthermore, by almost monotonicity of the frequency some of the exponents could be rewritten in terms of $\mathscr{N}(x;U)$. □

Remark 77. Note that using the bound $N(x,s) \leq N(x,r)+C(r^2-s^2)$ the inquality in (3.12) can be simplified to

$$\overline{H}(x,r) \leq \left(\frac{r}{s}\right)^{2N(x,r)+Cr^2}\overline{H}(x,s), \tag{3.16}$$

while (3.13) reduces to

$$\left(\frac{r}{s}\right)^{2\mathscr{N}(x;U)}\overline{H}(x,s) \leq \overline{H}(x,r). \tag{3.17}$$

By upper semicontinuity of the frequency we know that for φ satisfies $\mathscr{N}(Y;\varphi) \leq \mathscr{N}(0;\varphi)$ for every $Y \in \mathbb{R}^4$. Thus, we are naturally led to define the set

$$S(\varphi) = \left\{Y \in \mathbb{R}^4;\ \mathscr{N}(Y;\varphi) = \mathscr{N}(0;\varphi)\right\}.$$

The reader can check that the following set is in fact a linear subspace of $\mathbb{R}^4$ on which the blow up φ is constant. It is customary to refer to $S(\varphi)$ as the *spine*. For $j \in \{0,1,2,3\}$ define the following sets

$$\mathscr{S}_U^j = \{Y \in \Sigma_U;\ \dim(S(\varphi)) \leq j \text{ for every blow up } \varphi \text{ of } U \text{ at } Y\}$$

and observe that the following chain of inclusion holds

$$\Sigma_U = \mathscr{S}_U^3 \supseteq \mathscr{S}_U^2 \supseteq \mathscr{S}_U^1 \supseteq \mathscr{S}_U^0.$$

A well-known dimension reducing argument implies that the Hausdorff dimension of the $\mathscr{S}_U^j$ is bounded by j. Furthermore, we have that the set $\{y;\ \mathscr{N}(y;U) = \alpha\} \cap \mathscr{S}_U^0$ is discrete, for $\alpha > 0$. We refer the reader to [9, Theorem 2.16] for a proof of this in the context of stationary integral varifolds. Alternatively, the reader can consult the following work of Simon [124, Chapter 3].

A better understanding of the set $S(\varphi)$ is given by the following lemma.

Lemma 78. *We have $\mathscr{S}_U^{(3)} = \mathscr{S}_U^{(2)}$. Consequently,* $\dim(S(\varphi)) \leq 2$.

Proof. Assume not, there would then exists a point $Y \in \Sigma_U$ and a blow-up ψ at it such that $\dim S(\psi) = 3$, from which we woul infer $\mathscr{H}^3(S(\psi)) = \infty$ and, because of $S(\psi) \subset \Sigma_\psi$, $\mathscr{H}^3(\Sigma_\psi) = \infty$. However, from [137], we know that $\mathscr{H}^{2+\gamma}(\Sigma_\psi) = 0$, where $\gamma > 0$, giving the desired contradiction. ☐

Lemma 79. *For every homogeneous of degree α $\mathbb{Z}_2$-harmonic spinor U such that* $\dim S(\varphi) = n-2$ *(meaning that φ is constant in two directions so that it can be interpreted as a function on $\mathbb{R}^2$ instead of $\mathbb{R}^4$), we have $\alpha = k/2$ for some integer $k \in \mathbb{N}$. Thus, the set of all such α's is discrete. Moreover, using polar coordinates in $\mathbb{R}^2$, they take the form*

$$\varphi(r,\theta) = \{\pm u_* r^\alpha\} = \{\pm r^\alpha(u_1 e^{i\alpha\theta} + u_2 e^{-i\alpha\theta})\} = \{\pm(u_1 z^\alpha + u_2 \bar{z}^\alpha)\}, \tag{3.18}$$

where $u_* \in \Gamma(\mathbb{V}^+ \times \mathscr{I}_*)$ and $|u_*| = 1/2\pi$, together with u_1 and u_2 constants.

We will refer to blow-ups satisfying $\dim S(\varphi) = 2$ as *cylindrical blow ups*. More generally, in general dimension n, one usually refers to blow-ups satisying $\dim S(\varphi) = n-2$, as cylindrical.

Proof. The proof follows [137, Lemma 5.1] and for further details, we refer the reader to it. Due to cylindricality we need only consider blow-ups defined on $\mathbb{R}^2$. By homogeneity, we can write $\varphi = u_*|x|^\alpha$, where $u_* \in \Gamma(\mathbb{V}^+ \times \mathscr{I}_*)$ over S^1. The equation $\mathscr{D}\varphi = 0$ reduces to an ordinary differential equation for u_*, whose solution is given by $u_* = u_1 e^{i\alpha\theta} + u_2 e^{-i\alpha\theta}$ with u_1 and u_2 constants. Moreover, there holds $|u_*|^2 = 1/2\pi$, from which it follows that $\mathscr{Z}_*$ is given by the origin. The other results in the statement of the lemma follow from this description of u_*. ☐

An immediate consequence of the above discussion is that to conclude 2-rectifiability properties of Σ_U it suffices to consider $\Sigma_U \setminus \mathscr{S}_U^{(1)}$. Elements $Y \in \Sigma_U \setminus \mathscr{S}_U^{(1)}$ have the properties that there is a blow-up φ of U at Y such that, after possibly composing with an orthogonal transformation, φ has the form appearing in (3.18).

3.4 Elements from Simon's Cylindrical Tangent Cones

We collect here some results, notions and definitions used later in this work. The next result is inspired by Simon's Lemma 2.4 in [122], see also [89, Lemma 3.3].

Proposition 80. *Let $\alpha \in (0,K]$, there exist functions $\delta\colon (0,1) \to (0,1)$ and $R\colon (0,1) \to (2,\infty)$, depending on K, such that if $\varepsilon \in (0,1)$ and $U \in W^{1,2}(M, \mathscr{A}_2(\mathscr{V}))$ is a $\mathbb{Z}_2$-harmonic spinor with $\overline{B_{R(\varepsilon)}}(0) \subset \Omega$ such that $0 \in \mathscr{Z}_U$, and*

$$N(0,R(\varepsilon);U) - \alpha < \delta(\varepsilon),$$

and if $X_1 \in \mathscr{Z}_U \cap B_1(0)$ with $\mathscr{N}(X_1;U) \geq \alpha$, then the following hold:

(i) *We have* $0 < N(X_1, r; U) - \alpha < \varepsilon^2$ *for* $r < R(\varepsilon) - 1$.

(ii) *For every* $r \in (0,1]$, *there is a* $\mathbb{Z}_2$ *harmonic spinor* $\varphi \in W^{1,2}(\mathbb{R}^4; \mathscr{A}_2(\mathscr{V}))$, *homogeneous of degree* $\mathscr{N}(0;\varphi)$ *such that* $|\mathscr{N}(0;\varphi) - \alpha| < \varepsilon^2$ *and*

$$\int_{B_1(0)} \mathscr{G}(U_{X_1,r}, \varphi)^2 < \varepsilon^2.$$

Recall that $\mathscr{G}$ *in the above denotes the distance between* $U_{X_1,\varphi}$ *and* φ, *see* (1.1).

(iii) *For every* $r \in (0,1]$, *either there is a* $\mathbb{Z}_2$ *harmonic spinor* $\varphi \in W^{1,2}(\mathbb{R}^4; \mathscr{A}_2(\mathscr{V}))$, *homogeneous of degree* $\mathscr{N}(0;\varphi)$ *such that* $|\mathscr{N}(0;\varphi) - \alpha| < \varepsilon^2$ *and* $\dim(S(\varphi)) \leq n-2$, *together with*

$$\int_{B_1(0)} \mathscr{G}(U_{X_1,r}, \varphi)^2 < \varepsilon^2,$$

or

$$\left\{X \in \Sigma_{U_{X_1,r}} \cap \overline{B_1(0)};\ \mathscr{N}(X; U_{X_1,r}) \geq \alpha\right\} \subset \{X;\ \mathrm{dist}(X,L) < \varepsilon\}$$

for some $(n-3)$*-dimensional subspace L.*

Proof. We start by proving the first point. Observe that by virtue of the monotonicity of frequency we have $N(X_1, r; U) + Cr^2 \geq \alpha$, given the hypothesis $\mathscr{N}(X_1; U) \geq \alpha$. Note also that

$$\overline{D}(X_1, r) \leq \left(1 + \frac{|X_1|}{r}\right)^2 \overline{D}(0, r + |X_1|),$$

for all $r \in (0, R(\varepsilon) - 1)$. Compute now

$$\begin{aligned}
\overline{H}(Y,r) &= r^{-3} \int_{\partial B_r(Y)} |u|^2 \\
&= r^{-4} \int_{\partial B_r(Y)} |u|^2 (X-Y) \cdot \frac{(X-Y)}{|X-Y|} \\
&= r^{-4} \int_{B_r(Y)} n|u|^2 + 2RuD_R u \\
&= 4r^{-4} \int_{B_r(Y)} |u|^2 + 2r^{-4} \int_0^r \sigma \int_{B_\sigma(Y)} |Du|^2 + 2r^{-4} \int_0^r \sigma \mathscr{O}(\sigma H(Y,\sigma)) \\
&= 4r^{-4} \int_{B_r(Y)} |u|^2 + 2r^{-4} \int_0^r \sigma \int_{B_\sigma(Y)} |Du|^2 + 2\left(\frac{\sigma}{r}\right)^4 \int_0^r \sigma \mathscr{O}(\overline{H}(Y,\sigma))
\end{aligned}$$

for all $Y \in B_1(0)$ and $r \in (0, R(\varepsilon) - |Y|)$ and where we used the divergence theorem for the third inequality. Furthermore, for $|X_1| < r$ we have

$$\begin{aligned}
\int_0^r \sigma \int_{B_\sigma(X_1)} |Du|^2 &\geq \int_{|X_1|}^r \sigma \int_{B_{\sigma-|X_1|}(0)} |Du|^2 \\
&= \int_0^{r-|X_1|} (\sigma + |X_1|) \int_{B_\sigma(0)} |Du|^2 \geq \int_0^{r-|X_1|} \sigma \int_{B_\sigma(0)} |Du|^2,
\end{aligned}$$

and

$$\begin{aligned}
\overline{H}(X_1,r) &\geq 4r^{-4}\int_{B_{r-|X_1|}(0)}|u|^2+2r^{-4}\int_0^{r-|X_1|}\sigma\int_{B_\sigma(0)}|Du|^2+2r^{-4}\int_0^r\sigma\mathscr{O}(\sigma H(X_1,\sigma))\\
&=\left(1-\frac{|X_1|}{r}\right)^4\overline{H}(0,r-|X_1|)+2r^{-4}\int_{r-|X_1|}^r\sigma\mathscr{O}(\sigma H(X_1,\sigma))\\
&\geq\left(1-\frac{|X_1|}{r}\right)^4\left(\frac{r-|X_1|}{r+|X_1|}\right)^{2N(0,r+|X_1|)+C(r+|X_1|)^2}\overline{H}(0,r+|X_1|)\\
&\qquad+2r^{-4}\int_{r-|X_1|}^r\sigma\mathscr{O}(\sigma H(X_1,\sigma))\\
&\geq\left(1-\frac{|X_1|}{r}\right)^4\left(\frac{r-|X_1|}{r+|X_1|}\right)^{2\alpha+2\delta(\varepsilon)}\overline{H}(0,r+|X_1|)\\
&\qquad+2r^{-n}\int_{r-|X_1|}^r\sigma\mathscr{O}(\sigma H(X_1,\sigma))
\end{aligned}$$

for all $r\in(|X_1|,R(\varepsilon)-1)$, and where we applied (3.16) for the second inequality. From the above and the definition of frequency function we obtain the desired result. Alternatively, one can prove that a bound of the form $\overline{H}(X_1,r)\geq\lambda(U)$, holds arguing by contradiction and using compactness and non-triviality of the blow-up.

The second point follows by combining the first one and the compactness of $\mathbb{Z}_2$-harmonic spinors. Were the last assertion false, for some fixed $\varepsilon\in(0,1)$ there would be sequences $r_j\in(0,1]$, $\alpha_j\in(0,K]$, $\delta_j\downarrow 0$, $R_j\uparrow\infty$, as well as U^j $\mathbb{Z}_2$-harmonic spinors and points $X_j\in B_1(0)$ with $\mathscr{N}(X_j;U)\geq\alpha_j$ such that, with $\widetilde{U}_j:=U^j_{X_j,r_j}$ we would have

$$0\leq N(0,R_j;\widetilde{U}_j)-\alpha_j<\delta_j$$

and

$$\int_{B_1(0)}\mathscr{G}(\widetilde{U}_j,\varphi)^2\geq\varepsilon^2\quad\text{or}\quad|\mathscr{N}(0;\varphi)-\alpha_j|\geq\varepsilon^2,$$

for every φ as in the statement, and such that

$$\left\{X\in\Sigma_{\widetilde{U}_j}\cap\overline{B_1(0)};\ \mathscr{N}(X;\widetilde{U}_j)\geq\alpha\right\}\not\subset\{X;\ \mathrm{dist}(X,L)<\varepsilon\},$$

for all $(m-3)$-dimensional subspaces $L\subset\mathbb{R}^n$. By (ii) we can find a sequence of φ_j's as in the statement such that

$$\int_{B_1(0)}\mathscr{G}(\widetilde{U}_j,\varphi_j)^2\to 0,$$

and $|\mathscr{N}(0;\varphi_j)-\alpha_j|\to 0$, as $j\to\infty$. By compactness, we can extract limits φ_* and U_*, agreeing in the ball $B_1(0)$. Moreover, we can deduce $\alpha_j\to\mathscr{N}(0;\varphi_*)$, as well as $\mathscr{N}(0;\varphi)\to\mathscr{N}(0;\varphi_*)$, together with

$$\int_{B_1(0)}\mathscr{G}(\widetilde{U}_j,\varphi_*)\to 0,\quad\text{as } j\to\infty,$$

from which we infer that φ_* does not have, up to rotation, $\dim(S(\varphi_*))=2$. The reader can check that the same proof that allows us to bound the Hausdorff dimension of $\mathscr{S}^j_U$ by j in the previous section implies the existence of L, an 1-dimensional subspace such that

$$\{X\in\Sigma_{\varphi_*};\ \mathscr{N}(X;\varphi_*)\geq\mathscr{N}(0;\varphi_*)\}\subset L.$$

However, by upper-semicontinuity of the frequency we obtain that $\{X\in B_1(0);\ \mathscr{N}(X,\widetilde{U}_j)\geq\alpha_j\}$ is contained in $\{X;\ \mathrm{dist}(X;L)<\varepsilon\}$, giving the required contradiction and allows to finish the proof of the lemma. □

Before stating the next result we need to introduce some notation. Let $\varphi^{(0)} : \mathbb{R}^4 \to \mathscr{A}_2(\mathscr{V})$ be a fixed, homogeneous of degree $\alpha \geq 1/2$, non-zero harmonic $\mathbb{Z}_2$ spinor of the form (3.18). The number α will be fixed. Requiring $\varphi^{(0)}$ to be a $\mathbb{Z}_2$-harmonic spinor imposes conditions on the degree α and on the constant defining $\varphi^{(0)}$, see Lemma 79. Let now $\varepsilon \in (0,1)$ and denote by $\mathscr{F}_\varepsilon(\varphi^{(0)})$ the set of harmonic $\mathbb{Z}_2$ spinors $U \in W^{1,2}(B_1(0), \mathscr{A}_2(\mathscr{V}))$ satisfying $\|U\|_{L^2(\Omega)} \leq \varepsilon$ and

$$\int_{B_1(0)} \mathscr{G}(U, \varphi^{(0)})^2 \leq \varepsilon^2$$

On the other hand, denote by $\Phi_\varepsilon(\varphi^{(0)})$ the space of symmetric maps $\varphi\colon B_1(0) \to \mathscr{A}_2(\mathscr{V})$, having the same degree of homogeneity as $\varphi^{(0)}$, having the form (3.18) and satisfying

$$\int_{B_1(0)} \mathscr{G}(\varphi, \varphi^{(0)})^2 \leq \varepsilon^2.$$

Note that we are not requiring φ being a $\mathbb{Z}_2$-harmonic spinor, meaning that the constants u_1 and u_1 appearing in (3.18) are not specified. Finally, let us introduce the last class of multi-valued maps we are going to work with. Let $\widetilde{\Phi}_\varepsilon(\varphi^{(0)})$ denote the space of $\mathbb{Z}_2$-harmonic spinor of the form $\varphi(e^A X)$ for some $\varphi \in \Phi_\varepsilon(\varphi^{(0)})$ and a skew symmetric (4×4)-matrix A with $|A| \leq \varepsilon$ and satisfying $a_{ij} = 0$ if $i,j \leq 2$ and $a_{ij} = 0$ if $i,j \geq 3$.

Note that given $\varphi \in \Phi_\varepsilon(\varphi^{(0)})$ we can view the graph of φ restricted to $\mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^2)$ as an immersed, smooth submanifold. Furthermore, if $\varphi^{(0)}$ is a $\mathbb{Z}_2$-harmonic spinor and ε is small enough we have that the submanifold is embedded as well. In case α is a positive integer we have that the graph of φ is given by the union of two embedded submanifolds, the graphs of the values appearing in (3.18). In particular, any function defined on the graph of φ, restricted to $\Omega \subset \mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^2)$ is determined by two functions, one on each branch appearing in (3.18). In the special case $\alpha = q + 1/2$ the graph of φ consists of a single branched submanifold parametrised by $(re^{i\theta}, y, u_1 r^\alpha e^{i\theta} + u_2 r^\alpha e^{-i\alpha})$, where $r \geq 0$, $\theta \in [0, 4\pi]$, and $y \in \mathbb{R}^2$, and any function defined on the graph of φ restricted to Ω can be parametrised over Ω accordingly.

We are now able to state one of the main results of this work, whose proof can be found at the beginning of Section 3.8. This is based on [122, Lemma 1] and we refer the reader to it for comparisons.

Proposition 81. *Let $\theta \in (0,1/4)$ and let $\varphi^{(0)}$ be a homogeneous $\mathbb{Z}_2$-harmonic spinor defined on $\mathbb{R}^4$. Then, there are $\delta_0, \varepsilon_0 \in (0,1/4)$ depending only on the dimension of the manifold, the rank of the bundle, the degree of homogeneity of $\varphi^{(0)}$, the spinor $\varphi^{(0)}$ itself and θ, such that if $\varphi \in \widetilde{\Phi}_\varepsilon(\varphi^{(0)})$ and $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, then*

(i) Either,

$$B_{\delta_0}(0, y_0) = \{x \in B_1(0) \cap Z;\ \mathscr{N}(x;U) \geq \alpha\} = \emptyset,$$

for some $y_0 \in B^2_{1/2}(0)$;

(ii) Or, there is a new $\tilde{\varphi} \in \widetilde{\Phi}_{\gamma\varepsilon}(\varphi^{(0)})$ such that

$$\theta^{-4-2\alpha} \int_{B_\theta(0)} \mathscr{G}(U, \tilde{\varphi})^2 \leq C\theta^{2\mu} \left(\int_{B_1(0)} \mathscr{G}(U, \varphi)^2 + \|U\|_{L^2(B_1(0))} \right),$$

where $\gamma \in [1, \infty)$, $\mu \in (0,1)$ and $C \in (0, \infty)$ are constants depending only on the dimension of the manifold, the rank of the bundle, $\varphi^{(0)}$ and α.

3.5 A coarse representation lemma

The aim of this section is to prove that, for $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$ and $\varphi \in \Phi_{\varepsilon_0}(\varphi^{(0)})$ for ε_0 small, we have that outside of a tubular neighborhood of $\{0\} \times \mathbb{R}^2$ the graph of U is given by the graph of a single-valued function V over the graph of φ, see Proposition 83 below. As a byproduct, we obtain some preliminary coarse estimates satisfied by V.

Lemma 82. *Let $\gamma \in (0,1)$. Suppose ψ is a $\mathbb{Z}_2$-harmonic spinor with empty singular set $\Sigma_\psi = 0$. There exists $\varepsilon = \varepsilon(\gamma, \psi) > 0$ such that the following holds true. Let U, V be $\mathbb{Z}_2$-harmonic spinor on the unit ball such that*

$$\int_{B_1(0)} \mathscr{G}(U,\psi)^2 < \varepsilon^2 \qquad \text{and} \qquad \int_{B_1(0)} \mathscr{G}(V,\psi)^2 < \varepsilon^2.$$

Then, $U = \{u, -u\}$ and $V = \{v, -v\}$ on $B_\gamma(0)$ for some single-valued, smooth, solutions of the Dirac equation u and v defined in $B_\gamma(0)$. Moreover, they satisfy

$$\|u - v\|_{C^3(B_\gamma(0))} \leq C \left(\int_{B_1(0)} \mathscr{G}(U,V)^2 \right)^{1/2},$$

for some constant $C = C(\gamma) \in (0,\infty)$.

Proof. From Lemma 61 we deduce that if U and V are $L^2(B_1(0))$ close to ψ, then they are uniformly close to ψ on $B_{(1+\gamma)/2}(0)$. More precisely, for every $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\psi, \varepsilon_0) > 0$ such that if U is a $\mathbb{Z}_2$-harmonic spinor satisfying

$$\int_{B_1(0)} \mathscr{G}(U,\psi)^2 \leq \varepsilon_0^2,$$

then $U = \{u, -u\}$ on $B_{(1+\gamma)/2}(0)$ for some single-valued smooth section u of $\mathscr{V}$ satisfying $\mathscr{D}(u) = 0$ and

$$\sup_{B_{(1+\gamma)/2}(0)} |u - \psi_1| < \delta,$$

where $\psi = \{\psi_1, -\psi_1\}$ for a single-valued smooth section u of $\mathscr{V}$ satisfying $\mathscr{D}(u) = 0$. Set $\delta = (1/3)\inf_{B_{(1+\gamma)/2}(0)} |\psi_1|$ and consider the corresponding ε_0 as above. Assume then that U and V are given as in the statement of the lemma and note that the above discussion carries over to give single-valued smooth solutions of the Dirac equation u and v. Our choice of δ implies $\mathscr{G}(U,V) = \sqrt{2}|u - v|$ in the ball $B_{(1+\gamma)/2}(0)$, so that standard elliptic regularity entails the desired estimate

$$\|u - v\|_{C^3(B_\gamma(0))} \leq C\|u - v\|_{L^2(B_{(1+\gamma)/2}(0))} \leq C \left(\int_{B_{(1+\gamma)/2}(0))} \mathscr{G}(U,V)^2 \right)^{1/2},$$

for $C = C(\gamma) \in (0,\infty)$. □

The following is the main result of the section.

Proposition 83. *Let $\varphi^{(0)}$ be the fixed $\mathbb{Z}_2$-harmonic spinor with degree of homogeneity α. Let $\beta, \gamma, \tau \in (0,1)$ be arbitrary with $\tau \leq (1-\gamma)/10$. There is $\varepsilon_0 = \varepsilon_0(\varphi^{(0)}, \gamma, \beta, \tau) \in (0,1]$ such that if $\varphi \in \Phi_{\varepsilon_0}(\varphi^{(0)})$ and $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, then there is an open set $O \subset B_1(0) \setminus (\{0\} \times \mathbb{R}^{n-2})$ satisfying*

(i) $(x,y) \in O$ implies $(\tilde{x}, y) \in O$, whenever $|x| = |\tilde{x}|$;

(ii) $\{(x,y) \in B_\gamma(0); |x| > \tau\} \subset O$;

and there is a single-valued V deined on the graph of φ such that

$$U(x) = \{\varphi_1(x) + V(x, \varphi_1(x)), -\varphi_1(x) + V(x, -\varphi_1(x))\},$$

for $x \in U$ and where we write $\varphi(x) = \{\pm\varphi_1(x)\}$. Note that by symmetry of U we have

$$V(x, \varphi_1(x)) = -V(x, -\varphi_1(x)).$$

Moreover, there holds that

$$\hat{V}(x) = \{V(x, \varphi_1(x)), V(x, -\varphi_1(x))\}$$

is a two-valued harmonic function on O, and the estimate

$$\sup_{B_\gamma(0)} r^{-\alpha}|\hat{V}| + \sup_{B_\gamma(0)} r^{1-\alpha}|D\hat{V}| \le \beta,$$

together with

$$\int_O |\hat{V}|^2 + r^2|D\hat{V}| + \int_{B_\gamma(0)\setminus O} \left(|U|^2 + r^2|DU|^2\right) \le C \int_{B_1(0)} \mathscr{G}(u, \varphi)^2,$$

where $r(x,y) = |x|$ for $(x,y) \in B_1(0)$ and $C = C(\varphi^{(0)}, \alpha, \beta, \gamma)$ is a constant independent of τ.

Proof. For each $\zeta \in \mathbb{R}^2$ and $r \in (0,1]$, introduce the torus

$$A_{r,\kappa}(\zeta) = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2;\ (|x| - r)^2 + |y - \zeta|^2 < \kappa^2(1-\gamma)^2 r^2/4\}.$$

Invoking Lemma 82 we have the existence of $\delta = \delta(\varphi^{(0)}, \beta) > 0$ such that if

$$r^2 + |\zeta|^2 < \gamma^2 \quad \text{and} \quad \int_{A_{r,3/4}(\zeta)} \mathscr{G}(U, \varphi)^2 < \delta r^{4+2\alpha}, \tag{3.19}$$

then there exists a single-valued $\mathbb{Z}_2$-harmonic spinor $V_{r,\zeta}$, smooth and defined on graph $\varphi|_{A_{r,1/2}}$. Moreover, we know that such $V_{r,\zeta}$ satisfies

$$U(X) = \{\varphi_1(X) + V_{r,\zeta}(X, \varphi_1(X)), -\varphi_1(X) + V_{r,\zeta}(X, -\varphi_1(X))\},$$

for all $x \in A_{r,1/2}(\zeta)$ and where, as usual, $\varphi(x) = \{\pm\varphi_1(x)\}$ and, by symmetry of U as in the statement of the proposition, $V_{r,\zeta}(\cdot, \varphi_1(\cdot)) = -V_{r,\zeta}(\cdot, -\varphi_1(\cdot))$, as well as

$$\hat{V}_{r,\zeta}(\cdot) = \{V_{r,\zeta}(\cdot, \varphi_1(\cdot)), V_{r,\zeta}(\cdot, -\varphi_1(\cdot))\}$$

which furthermore satisfies the bound

$$r^{-\alpha} \sup_{A_{r,1/2}(\zeta)} |\hat{V}_{r,\zeta}| + r^{1-\alpha} \sup_{A_{r,1/2}(\zeta)} |D\hat{V}_{r,\zeta}| \le 2^{-\alpha}\beta.$$

We define the desired open set O to be the union of the tori $A_{r,1/2}(\zeta)$ such that (3.19) holds. One can then get a well-defined singled valued smooth map V by requiring $V = V_{r,\zeta}$ on graph $\varphi|_{A_{r,1/2}(\zeta)}$ whenever (3.19) holds. As $\varphi \in \Phi_{\varepsilon_0}(\varphi^{(0)})$ and $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, it follows from Lemma 82 that

$\{(x,y) \in B_\gamma(0);\ |x| > \tau\} \subset O$, provided ε_0 is sufficiently small. By the above bound on $\hat{V}_{r,\zeta}$ we obtain

$$\sup_{B_\gamma(0)} |x|^{-\alpha}|\hat{V}(x,y)| + \sup_{B_\gamma(0)\cap U} |x|^{1-\alpha}|D\hat{V}(x,y)| \le \beta,$$

where $\hat{V}(\cdot) = \{V(\cdot,\varphi_1(\cdot)), -V(\cdot,-\varphi_1(\cdot))\}$. Now, for $(\xi,\zeta) \in B_\gamma(0)\cap\partial O$ we have

$$\int_{A_{|\xi|,3/4}(\zeta)} \mathscr{G}(U,\varphi) \ge \delta|\xi|^{4+2\alpha},$$

otherwise $A_{|\xi|,3/4}(\zeta)$ is an open neighborhood of (ξ,ζ) with $A_{|\xi|,1/2}(\zeta) \subset O$ giving a contradiction. We then infer

$$\int_{B_{10|\xi|}(0,\zeta)B_\gamma\cap O} \left(|\hat{V}|^2 + r^2|D\hat{V}|^2\right) \le \frac{C\beta^2}{\delta}\int_{A_{|\xi|,3/4}(\zeta)} \mathscr{G}(U,\varphi)^2,$$

where the constant C depends on α. Note now that

$$\{(x,y) \in B_\gamma(0)\cap U;\ \mathrm{dist}((x,y), B_\gamma\cap\partial O) \le |x|/2\} \subset \bigcup_{(\xi,\zeta)\in B_\gamma\cap\partial O} B_{2|\xi|}(0,\zeta)$$

and that $B_{2|\xi|}(0,\zeta)\cap B_{2|\xi'|}(0,\zeta') = \emptyset$ implies $A_{|\xi|,3/4}(\zeta)\cap A_{|\xi'|,3/4}(\zeta') = \emptyset$. Thus, by the 5-times covering lemma, we deduce

$$\int_{\{(x,y)\in B_\gamma(0)\cap U;\ \mathrm{dist}((x,y),B_\gamma\cap\partial O)\le|x|/2\}} \left(|\hat{V}|^2 + r^2|D\hat{V}|^2\right) \le C\int_{B_1(0)} \mathscr{G}(U,\varphi)^2$$

for a certain constant $C(\varphi^{(0)},\alpha,\beta)$. Define now the Lipschitz function $d\colon U \to [0,\infty)$ by $d(x,y) = \mathrm{dist}((x,y), \partial O\cap B_\gamma(0))$. Introduce moreover $\psi\colon [0,\infty) \to [0,1]$, a smooth function satisfying $\psi(t) = 0$ for $t \le 1/4$, $\psi(t) = 1$ for $t \ge 1/2$ and $0 \le \psi'(t) \le 6$ for all t. Finally, consider $\eta\colon B_1(0) \to [0,1]$ a smooth function such that $\eta = 1$ on $B_\gamma(0)$, $\eta = 0$ on $B_{(1+\gamma)/2}(0)$ and $|D\eta| \le 3/(1-\gamma)$. As $\hat{V}$ solves the Dirac equation on O, we know that locally around points of O, $\hat{V}$ is given by two harmonic spinors with zero average. Multiplying then the equation $\mathscr{D}^2\hat{V} = 0$ by $\hat{V}r^2\psi(d/r)^2\eta^2$ and applying the Weitzenböck formula, where we denote $\mathscr{R}$ the associated curvature operator, combined with a partition of unity argument and integrating by parts we obtain

$$\begin{aligned}\int_O |D\hat{V}|^2 r^2\psi(d/r)^2\eta^2 = -\int_O \hat{V}^k D_i\hat{V}^k \big(&D_i(r^2)\psi(d/r)^2\eta^2 + 2r^2\psi(d/r)^2\eta D_i\eta \\ &+2r^2\psi(d/2)\psi'(d/r)D_i\psi(d/r)D_i(d/r)\big) \\ &+\int_O \langle\mathscr{R}\hat{V},\hat{V}\rangle r^2\psi(d/r)^2\eta^2.\end{aligned}$$

The definitions of ψ and η, together with an application of Cauchy's inequality, imply the bound

$$\int_{\{(x,y)\in O\cap B_\gamma(0);\, d(x,y)\ge|x|/2\}} r^2|D\hat{V}|^2 \le C\int_{O\cap B_{(1+\gamma)/2}(0)} |\hat{V}|^2 + C\int_{O\cap B_{(1+\gamma)/2}(0)} \mathscr{R}r^2|\hat{V}|^2,$$

where the constant C depends on γ. Note that by letting the constant depend on $\mathscr{R}$ as well, we can combine the two terms on the right-hand side. Consequently, we obtain the global bound

$$\int_{O\cap B_\gamma(0)} \left(|\hat{V}|^2 + r^2|D\hat{V}|^2\right) \le C\int_{B_1(0)} \mathscr{G}(U,\varphi)^2.$$

Now observe that by elliptic estimates we have

$$\int_{A_{r,1/2}(\zeta)} \left(|U|^2 + r^2|DU|^2\right) \leq C \int_{A_{r,3/4}(\zeta)} |U|^2,$$

so that if $r^2 + |\zeta|^2 < \gamma^2$ and $\int_{A_{r,3/4}(\zeta)} \mathscr{G}(U,\varphi)^2 \geq \delta r^{4+2\alpha}$, then an application of the triangle inequality together with the fact that φ is α-homogeneous implies

$$\begin{aligned} \int_{A_{r,1/2}(\zeta)} \left(|U|^2 + r^2|DU|^2\right) &\leq C \int_{A_{r,3/4}(\zeta)} \mathscr{G}(U,\varphi)^2 + C \int_{A_{r,3/4}(\zeta)} |\varphi|^2 \\ &\leq C \int_{A_{r,3/4}(\zeta)} \mathscr{G}(U,\varphi)^2 + C r^{n+2\alpha} \leq C\left(1 + \frac{1}{\delta}\right) \int_{A_{r,3/4}(\zeta)} \mathscr{G}(U,\varphi)^2. \end{aligned}$$

Thus, we deduce for all $\zeta \in B^2_\gamma(0)$ and $r \in (0,1/2)$ the following

$$\int_{A_{r,1/2}(\zeta)\setminus O} \left(|U|^2 + r^2|DU|^2\right) \leq C(1+\delta^{-1}) \int_{A_{r,3/4}(\zeta)} \mathscr{G}(U,\varphi)^2,$$

so that a covering argument allows to prove the desired inequality

$$\int_{B_\gamma(0)\setminus O} \left(|U|^2 + r^2|DU|^2\right) \leq C \int_{B_1(0)} \mathscr{G}(U,\varphi)^2,$$

for a certain constant $C = C(\varphi^{(0)}, \alpha, \beta, \gamma)$ and concludes the proof. □

3.6 A priori estimates: non-concentration of excess

We establish here several a priori estimates for the for $\mathbb{Z}_2$-harmonic spinors close to $\varphi^{(0)}$, a fixed non-zero, symmetric two-valued map, homogeneous of degree $\alpha = k/2$ with $k \in \mathbb{N}$. We start by proving a lower bound for the Weiss functional, which is a straightforward consequence of the almost monotonicity of the frequency function, see Proposition 84. We then proceed to prove the main result of the section: Theorem 85. We conclude with various corollaries of this main theorem. These estimates are paramount to the proof of Proposition 81 in Section 3.8. Crucial are the integral estimates appearing at the end of Section 3.2.

Overall, we follow the strategy of Krummel and Wickramasekera, [89, Section 6]. The reader should compare the estimates of this section to the ones in [122, Section 3] and [33, Section 5].

Proposition 84. *Let $\alpha \in \mathbb{R}$ and U a harmonic $\mathbb{Z}_2$ spinor in M. Then, for each $y \in M$ we have the following identity for the height and Dirichlet energy,*

$$\frac{d}{dr}\left(\frac{1}{r^{2\alpha}}\left(r^{-2}D(x,r) - \alpha r^{-3}H(x,r)\right)\right) = 2r^{-2}\int_{\partial B_r(x)} \left|\frac{\partial}{\partial R}\left(\frac{u}{R^\alpha}\right)\right|^2 + \mathscr{O}(r^{-2-2\alpha}H(x,r)).$$

for a.e. $r \in (0,R_X)$ and where $R = |X-Y|$. On the other hand, for the smoothed terms we obtain

$$\begin{aligned} &\frac{d}{dr}\left(\frac{1}{r^{2\alpha}}\left(r^{-2}D_\chi(x,r) - \alpha r^{-3}H_\chi(x,r)\right)\right) \\ &\quad = 2r^{4-2\alpha}\int_M \left(-\frac{\chi'(d_g(x,y)/r)}{d_g(x,y)}\right) |d_g(x,y)\nabla_{\nu_x}u - \alpha u|^2 + \mathscr{O}(r^{-2-2\alpha}H_\chi(x,r)), \end{aligned}$$

for a.e. $r \in (0,R_X)$ and where $R = |X-Y|$.

Proof. Start by computing

$$
\begin{aligned}
r^2 \frac{d}{dr} & \left(r^{-2\alpha-2} D_\chi(x,r) - \alpha r^{-2\alpha-3} H_\chi(x,r)\right) \\
&= (-4\alpha - 4 + 4)\, r^{-2\alpha-1} D_\chi(x,r) + \alpha\,(4 + 2\alpha - 4)\, r^{-2\alpha-2} H_\chi(x,r) \\
&\quad + 2r^{-2\alpha-2} E_\chi(x,r) + \mathscr{O}(r^{-2\alpha} H_\chi(x,r)) \\
&= 2r^{-2\alpha-2}\left(-2\alpha r D_\chi(x,r) + \alpha^2 H_\chi(x,r) + E_\chi(x,r)\right) \\
&\quad + (-4+4)\, r^{-2\alpha-1} D_\chi(x,r) + \alpha\,(4-4)\, r^{-2\alpha-2} H_\chi(x,r) \\
&\quad + \mathscr{O}(r^{-2\alpha} H_\chi(x,r)) \\
&=: \mathrm{I} + \mathrm{II} + \mathscr{O}(r^{-2\alpha} H_\chi(x,r))
\end{aligned}
$$

where we applied the identities appearing in (3.6) for $\partial_r D_\chi(x,r)$ and $\partial_r H_\chi(x,r)$. We now claim that I gives the desired term measuring how far U is from being α-homogeneous, whilst II can be absorbed in the error terms. To prove this claim compute

$$
\begin{aligned}
\mathrm{I} &= 2r^{-2\alpha-2}\left(-2\alpha r D_\chi(x,r) + \alpha^2 H_\chi(x,r) + E_\chi(x,r)\right) \\
&= 2r^{-2\alpha-2}\Bigg(-\int_M d_g^2(x,y)|\nabla_{\nu_x} u|^2 \frac{\chi'(d_g(x,y)/r)}{d_g(x,y)} - \alpha^2 \int_M \frac{\chi'(d_g(x,y)/r)}{d_g(x,y)} |u|^2 \\
&\qquad\qquad + 4\alpha \int_M d_g(x,y) \nabla_{\nu_x} u \cdot u \frac{\chi'(d_g(x,y)/r)}{d_g(x,y)} + \mathscr{O}(r^2 H_\chi(x,r))\Bigg) \\
&= 2r^{-2\alpha-2} \int_M \left(-\frac{\chi'(d_g(x,y)/r)}{d_g(x,y)}\right) |d_g(x,y)\nabla_{\nu_x} u - \alpha u|^2 + \mathscr{O}(r^{-2\alpha} H_\chi(x,r)).
\end{aligned}
\tag{3.20}
$$

We now move to II. We have therefore proven the second identity in the proposition. Repeating the same computation with the identities for the non-smoothed height and Dirichlet energy gives the corresponding statement. Intuitively, one would let χ converge to the characteristic function of $[0,1)$ the right-hand side of the identity we have just proven would provide the first one. However, there is the constant appearing in $\mathscr{O}(r^{-2-2\alpha} H(x,r))$ could depend on χ and blow-up. Thus, one has for the non-smoothed version of I:

$$
2r^{-2\alpha} \int_{\partial B_r(x)} \left|\nabla_{\nu_x} u - \alpha \frac{u}{r}\right|^2 + \mathscr{O}(r^{-2\alpha} H(x,r)),
$$

which in turn can be written as

$$
2\int_{\partial B_r(x)} \left|\frac{\partial}{\partial r}\left(\frac{u}{r^\alpha}\right)\right|^2 + \mathscr{O}(r^{-2\alpha} H(x,r)).
$$

Multiplying back by r^{2-n} gives the desired identity, concluding the proof. □

The above proof can be straightforwardly modified to give the bound

$$
\begin{aligned}
\frac{d}{dr} & \left(\frac{1}{r^{2\alpha}}\left(r^\beta D(y,r) - \alpha r^{\beta-1} H(y,r)\right)\right) \\
&= 2r^\beta \int_{\partial B_r(x)} \left|\frac{\partial}{\partial r}\left(\frac{u}{r^\alpha}\right)\right|^2 + \mathscr{O}\left(r^{\beta-2\alpha}\left(1 + r^{-2}\right) H(x,r)\right),
\end{aligned}
$$

for $\beta \in \mathbb{R}$. The error terms are more involved due to the non-vanishing of II appearing in the proof.

Henceforth, we assume to be working on the ball $B_1(0)$ in Euclidean space instead of M. As explained in Section 3.2 we identify the ball $B_r(q) \subset M$ with one centered at the origin and contained in $B_1(0)$. The pull-back metric satisfies (3.2). However, to simplify some of the computations below and avoid diverting the reader's attention to unnecessary technicalities, we decided to work with the Euclidean metric. In light of (3.2), working with the pull-back metric would amount to introducing further error terms on the right-hand side of each inequality. The following is the main result of this section.

Theorem 85. *Let $\varphi^{(0)}$ be the fixed $\mathbb{Z}_2$-harmonic spinor homogeneous degree $\alpha \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$. Let $\gamma \in (0,1)$. There exists $\varepsilon_0 = \varepsilon_0(\varphi^{(0)}, \gamma, \sigma) > 0$ such that, if $\varphi \in \Phi_{\varepsilon_0}(\varphi^{(0)})$, $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, $0 \in \Sigma_U$ and $\mathscr{N}_U(0) \geq \alpha$, then*

(i) The inequality

$$\int_{B_\gamma(0)} R^{-2} \left| \frac{\partial}{\partial R}(U/R^\alpha) \right|^2 + \int_0^\gamma \mathscr{O}(s^{-2-2\alpha} H(0,s))\, ds \leq C \int_{B_1(0)} \mathscr{G}(U,\varphi)^2,$$

holds.

(ii) Moreover, we have

$$\int_{B_\gamma(0)} |D_y U|^2 + \int_0^\gamma \mathscr{O}(s^{-2-2\alpha} H(0,s))\, ds \leq C \int_{B_1(0)} \mathscr{G}(U,\varphi)^2,$$

where $R = |X|$. Both constants depend on $\varphi^{(0)}, \alpha, \gamma$ and the curvatures of M and $\mathscr{V}$.

Proof. Assume that ε_0 is less than the one appearing in Proposition 83 with $(1+\gamma)/2$ in place of γ, $\tau = (1-\gamma)/20$ and $\beta = 1/2$. From Proposition 84 we obtain

$$\begin{aligned}
&\frac{d}{dr}\left(r^2\left(\overline{D}(0,r) - \alpha\overline{H}(0,r)\right)\right) \\
&= \frac{d}{dr}\left(r^{2+2\alpha} r^{-2\alpha}\left(\overline{D}(0,r) - \alpha\overline{H}(0,r)\right)\right) \\
&= (2+2\alpha) r\left(\overline{D}(0,r) - \alpha\overline{H}(0,r)\right) + 2r^{2\alpha} \int_{\partial B_r(0)} \left| \frac{\partial}{\partial r}\left(\frac{u}{r^\alpha}\right)\right|^2 + \mathscr{O}(H(0,r)).
\end{aligned}$$

Integrating now the identity in Proposition 84 and using the fact that $\mathscr{N}(0,r) \geq \alpha$ we infer

$$r^{-2\alpha}\left(\overline{D}(0,r) - \alpha\overline{H}(0,r)\right) \geq 2\int_{B_r(0)} R^{-2} \left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2 + \int_0^r \mathscr{O}(s^{-2-2\alpha} H(0,s))\, ds,$$

from which we deduce

$$\begin{aligned}
&\frac{d}{dr}\left(r^2\left(\overline{D}(0,r) - \alpha\overline{H}(0,r)\right)\right) \\
&\geq 2\frac{d}{dr}\left(r^{2+2\alpha}\left(\int_{B_r(0)} R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2 + \int_0^r \mathscr{O}(s^{-2-2\alpha} H(0,s))\, ds\right)\right).
\end{aligned} \tag{3.21}$$

Let now $\psi \colon [0,\infty) \to \mathbb{R}$ be a smooth function with $\psi(t) = 1$, for $t \in [0,\gamma]$, $\psi(t) = 0$ for $t \geq (1+\gamma)/2$, and with the following control on the derivative $0 \leq \psi'(t) \leq 3/(1-\gamma)$ for $t \in [0,\infty)$.

Multiply both sides of (3.21) by $\psi(r)^2$ and integrate with respect to r to obtain the following chain of inequalities

$$\begin{aligned}&\int_{\mathbb{R}} \frac{d}{dr}\left(r^2\left(\overline{D}(0,r)-\alpha\overline{H}(0,r)\right)\right)\psi(r)^2\,dr\\&\quad\geq 2\int_{\mathbb{R}}\frac{d}{dr}\left(r^{2+2\alpha}\left(\int_{B_r(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2+\int_0^r\mathscr{O}(s^{-2-2\alpha}H(0,s))\,ds\right)\right)\psi^2(r)\,dr\\&\quad=-4\int_{\mathbb{R}}\left(r^{2+2\alpha}\left(\int_{B_r(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2+\int_0^r\mathscr{O}(s^{-2-2\alpha}H(0,s))\,ds\right)\right)\psi(r)\psi'(r)\,dr\end{aligned}$$

which, in vue of the definition of ψ, becomes

$$-4\int_\gamma^{(1+\gamma)/2}\left(r^{2+2\alpha}\left(\int_{B_r(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2+\int_0^r\mathscr{O}(s^{-2-2\alpha}H(0,s))\,ds\right)\right)\psi(r)\psi'(r)\,dr.$$

This last quantity is bounded from below by

$$-4\gamma^{2+2\alpha}\left(\int_{B_\gamma(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2+\int_0^\gamma\mathscr{O}(s^{2-2\alpha}H(0,s))\,ds\right)\int_\gamma^{(1+\gamma)/2}\psi(r)\psi'(r)\,dr,$$

which in turn is equal to

$$2\gamma^{2+2\alpha}\left(\int_{B_\gamma(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2+\int_0^\gamma\mathscr{O}(s^{-2-2\alpha}H(0,s))\,ds\right),$$

simply by definition of ψ. Now, coarea implies

$$\int_{\mathbb{R}}\frac{d}{dr}(D(0,r))\,\psi^2(r)\,dr=\int_{\mathbb{R}}\int_{\partial B_r}|Du|^2\psi^2(r)\,dr=\int|Du|^2\psi^2(R).$$

Furthermore, reasoning along the same lines and integrating by parts we obtain

$$\begin{aligned}\int_{\mathbb{R}}\frac{d}{dr}\left(r^{-2}H(0,r)\right)\psi^2(r)\,dr&=-2\int_{\mathbb{R}}r^{-2}H(0,r)\psi(r)\psi'(r)\,dr\\&=-2\int_{\mathbb{R}}\int_{\partial B_r(0)}\frac{1}{r}|u|^2\psi(r)\psi'(r)\,dr=-2\int\frac{1}{R}|u|^2\psi(R)\psi'(R).\end{aligned}$$

Combining all of the above we obtain

$$\begin{aligned}\int|Du|^2\psi^2(R)+2\alpha\frac{1}{R}|u|^2\psi(R)\psi'(R)&\\ \geq C\left(\int_{B_\gamma(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U}{R^\alpha}\right)\right|^2+\int_0^\gamma\mathscr{O}(s^{-2-2\alpha}H(0,s))\,ds\right),&\end{aligned}\tag{3.22}$$

where the constant C depends on γ and α. Note that due to homogeneity of φ, integrating by parts we get

$$\int_{B_1^2(0)}\left(|D\varphi|^2\psi_0^2(r)+2\alpha r^{-1}|\varphi|^2\psi_0(r)\psi_0'(r)\right)=0,$$

for any $\psi_0 \in C^1_c([0,1))$, where $r = r(x) = |x|$. Whence, letting $\psi_0(r) = \psi\left(\sqrt{r^2+|y|^2}\right)$ for each $y \in B^2_1(0)$, we infer

$$\int |D\varphi|^2 \psi^2(R) + 2\alpha \frac{1}{R} |\varphi|^2 \psi(R)\psi'(R) = 0,$$

where we used $\psi_0'(r) = (r/R)\psi'(R)$. Letting now the vector field $\zeta = \psi^2(R)\Upsilon$, where Υ is the vector field given by $(x_1, x_2, 0, 0)$, in the generalised squash and squeeze identities in subsection 3.2.1 we obtain

$$\begin{aligned}\int &\left(|Du|^2 - |D_x u|^2 - \mathscr{R}u(rD_r u)\right)\psi^2(R) \\ &= -2\int \left(\frac{r^2}{2}|Du|^2 - r^2|D_r u|^2 - rD_r u(y\cdot D_y u)\right)\frac{1}{R}\psi(R)\psi'(R)\end{aligned}$$

and

$$\int |Du|^2\psi^2(R) + \langle \mathscr{R}u, u\rangle \psi^2(R) = -2\int (ruD_r u + u(y\cdot D_y u))\frac{1}{R}\psi(R)\psi'(R)$$

respectively. We multiply then the latter by α, add it to the former, and then introduce on both sides the term $\int 2\alpha^2 R^{-1}|u|^2\psi(R)\psi'(R)$ to obtain

$$\begin{aligned}\int &\left(\alpha|Du|^2 + \alpha\langle\mathscr{R}u,u\rangle + \frac{1}{2}|D_y u|^2 - \mathscr{R}u(rD_r u)\right)\psi^2(R) + \frac{2\alpha^2}{R}|u|^2\psi(R)\psi'(R) \\ &= -2\int\left(\frac{1}{2}r^2|Du|^2 - rD_r u\,(rD_r u - \alpha u) - \alpha^2|u|^2 - (y\cdot D_y u)(rD_r u - \alpha u)\right)\frac{1}{R}\psi(R)\psi'(R) \\ &\qquad - \int \frac{1}{2}|D_y u|^2\psi^2(R) \\ &\le -2\int\left(\frac{1}{2}r^2|Du|^2 - rD_r u\,(rD_r u - \alpha u) - \alpha^2|u|^2\right)\frac{1}{R}\psi(R)\psi'(R) \\ &\qquad + 2\int |rD_r u - \alpha u|^2\psi'(R)^2 \\ &= -2\int\left(\frac{1}{2}r^2(|Du|^2 - |D\varphi|^2) - rD_r u\,(rD_r u - \alpha u) - \alpha^2(|u|^2 - |\varphi|^2)\right)\frac{1}{R}\psi(R)\psi'(R) \\ &\qquad + 2\int |rD_r u - \alpha u|^2\psi'(R)^2\end{aligned}$$

where we applied Cauchy's inequality for the first inequality, and used in the last step the following identity which can be verified by direct computation

$$\int\left(\frac{r^2}{2}|D\varphi|^2 - \alpha^2|\varphi|^2\right)\frac{1}{R}\psi(R)\psi'(R) = 0.$$

We now claim that

$$\begin{aligned}\left|\int_O\left(\frac{1}{2}r^2(|Du|^2 - |D\varphi|^2) - rD_r u\,(rD_r u - \alpha u) - \alpha^2(|u|^2 - |\varphi|^2)\right)\frac{1}{R}\psi(R)\psi'(R)\right| \\ \le C\int_{B_1(0)} \mathscr{G}(U,\varphi)^2,\end{aligned} \tag{3.23}$$

where C depends on α, γ, β and $\varphi^{(0)}$. Once this estimate is established the main theorem follows. Indeed, notice that

$$\int_{B_{\frac{1+\gamma}{2}}\setminus O} (r^2|D\varphi|^2 - 2\alpha^2|\varphi|^2)\frac{1}{R}\psi(R)\psi'(R) = 0,$$

so that, writing $\hat{V}$ in place of U on O and applying the main estimates of Proposition 83, we infer

$$\int \left(\alpha|Du|^2+\frac{1}{2}|D_yu|^2\right)\psi^2(R)+\frac{2\alpha^2}{R}|u|^2\psi(R)\psi'(R)\leq C\int_{B_1(0)}\mathscr{G}(U,\varphi)^2,$$

where C is a constant with the same dependencies as the one in (3.23). Combining this bound with the one in (3.22) we conclude the proof of the theorem. Note that the terms $\alpha\langle\mathscr{R}u,u\rangle$ and $-\mathscr{R}u(rD_ru)$ can be absorbed on the right-hand side in the term $\int_{B_1(0)}\mathscr{G}(U,\varphi)^2$ using again the estimates in Proposition 83, albeit upon changing the constant C and allowing it to depend on the curvature. Therefore, we are left with establishing (3.23). We divide the proof in two cases, $\alpha=q+1/2$ and $\alpha\in\mathbb{N}$. However, we first recall that $\varphi(re^{i\theta},y)=\{\pm(c_1e^{i\alpha\theta}+c_2e^{-i\alpha\theta})\}$ for constants c_1 and c_2. Introduce then the notation $\varphi_*(r,\theta,y)=c_1e^{i\alpha\theta}+c_2e^{-i\alpha\theta}$, and $V_*(r,\theta,y)=V(re^{i\theta},y,\varphi_*(r,\theta,y))$, for coordinates satisfying $(re^{i\theta},y)\in O$, the open set in Proposition 83. We recall for further applications that O is rotationally symmetric with respect to the x variable.

(a) Case $\alpha=q+1/2$. Recall that φ_* is 4π-periodic as a function of θ and write $U(re^{i\theta},y)=\{\pm(\varphi_*(r,\theta,y)+V_*(r,\theta,y))\}$. Compute then

$$\begin{aligned}&\int_O\left(\frac{1}{2}r^2(|Du|^2-|D\varphi|^2)-rD_ru\,(rD_ru-\alpha u)-\alpha^2(|u|^2-|\varphi|^2)\right)\frac{1}{R}\psi(R)\psi'(R)\\&=\int_0^{4\pi}\int_{\{(r,y);(x,y)\in O\}}\Big(D_\theta\varphi_*D_\theta V_*-\alpha^2\varphi_*V_*+\frac{1}{2}r^2|DV_*|^2\\&\qquad\qquad-rD_rV_*(rD_rV_*-\alpha V_*)-\alpha^2|V_*|^2\Big)\frac{1}{R}\psi(R)\psi'(R)\,drdyd\theta\\&=\int_0^{4\pi}\int_{\{(r,y);(x,y)\in O\}}\left(\frac{1}{2}r^2|DV_*|^2-rD_rV_*(rD_rV_*-\alpha V_*)-\alpha^2|V_*|^2\right)\frac{1}{R}\psi(R)\psi'(R)\,drdyd\theta,\end{aligned}$$

where we used Euler's theorem for homogeneous functions, viz. $rD_r\varphi_*=\alpha\varphi_*$, and integration by parts in the θ-variable. The estimates we have on $\hat{V}=\{\pm V_*\}$ from Proposition 83 allow to conclude.

(b) Case $\alpha\in\mathbb{N}$. In this case φ_* is 2π-periodic in θ. Arguing precisely as in (a) we can conclude the proof of (3.22) and with it the proof of the theorem.

□

We will mostly use the estimates in Theorem 85 with the error term on the right-hand side and with $\gamma=1$. Note that the error term, modulo constant, can be written as

$$\int_0^\gamma s^{-2-2\alpha}H(0,s)\,ds=\int_{B_\gamma(0)}\frac{|U|^2}{R^{n+2\alpha-2}},\quad\text{where } R=|X|.$$

We can further simplify this error term by noticing

$$\begin{aligned}
\int_{B_\gamma(0)} \frac{|U|^2}{R^{4+2\alpha-2}} &\le \sum_{k=0}^{\infty} \int_{B_{2^{-k}\gamma}(0)\setminus B_{2^{-(k+1)}\gamma}(0)} \frac{|U|^2}{R^{4+2\alpha-2}} \\
&\le \sum_{k=0}^{\infty} 2^{-(-2-2\alpha)(k+1)} \int_{B_{2^{-k}\gamma}(0)\setminus B_{2^{-(k+1)}\gamma}(0)} |U|^2 \\
&\le \sum_{k=0}^{\infty} 2^{-(-2-2\alpha)(k+1)} \int_{B_{2^{-k}\gamma}(0)} |U|^2 \qquad (3.24)\\
&\le \sum_{k=0}^{\infty} 2^{-(-2-2\alpha)(k+1)} (2^{-k}\gamma)^{(4+2\alpha)} \int_{B_1(0)} |U|^2 \\
&\le C \sum_{k=0}^{\infty} 2^{-(k+1)} \int_{B_1(0)} |U|^2 \le C \int_{B_1(0)} |U|^2
\end{aligned}$$

where the constant C depends on α and γ and where we used (3.14) combined with (3.17) for the fourth inequality. Note that by the triangle inequality we have $\|U\|^2_{L^2(B_1(0))} \le \|\varphi^{(0)}\|^2_{L^2(B_1(0))} + 1$, so that the last term of (3.24) can be simply bounded by a constant depending on $\varphi^{(0)}$ as well. We shall not need this simplification in the sequel of this work.

The next proposition deals with the delicate case $\alpha = 1/2$, i.e. when the frequency drops below 1. Note that in the case of Q-valued Dirichlet energy minimizing maps with $Q > 2$ this issue becomes even more problematic as the frequency could drop as low as $1/Q$. Imposing C^1-regularity for $\mathbb{Z}_2$-harmonic spinors we rule out this phenomenon so that the following proposition becomes unnecessary. However it is important to understand what happens in this case as well with the ultimate goal of removing the C^1-regularity hypothesis and proving uniqueness of blow-ups for general $\mathbb{Z}_2$-harmonic spinors.

Proposition 86. *Let $\varphi^{(0)}$ be the fixed $\mathbb{Z}_2$-harmonic spinor homogeneous of degree $\alpha = 1/2$. Let $\tau \in (0, 1/20)$. There exists $\varepsilon_0 = \varepsilon_0(\varphi^{(0)}, \tau) > 0$ such that if $\varphi \in \Phi_{\varepsilon_0}$, and $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$ (we do not require U to be C^1-regular) with $0 \in \Sigma_U$ and $\mathscr{N}(0;U) \ge 1/2$, V as in Proposition 83 with $\gamma = 1/2$ and $\beta = 1/2$, $\hat{V}(X) = \{V(X;\varphi(X)), V(X,-\varphi(X))\}$, and*

$$B_r(0,y) \cap \{X \in B_1(0);\ \mathscr{N}(X;U) \ge \mathscr{N}(0;\varphi) = \alpha\} \ne \emptyset \quad \textit{for all } y \in B^2_{1/2}(0).$$

Further assume that the modified squash and squeeze identities introduced in 3.2.1 hold. Then, for each function $\zeta(x,y) = \tilde{\zeta}(|x|,y)$, where $\tilde{\zeta}(r,y) \in C^1_c(B^3_{1/2}(0))$ with $D_r\tilde{\zeta}(r,y) = 0$ whenever

$r \leq \tau$, we have that

$$\begin{aligned}
&\left| \int_{B^{n-2}_{1/2}(0)} \int_\tau^\infty \int_0^{4\pi} D_j(r\widehat{V}^k D_i\varphi^k) D_j D_{y_p}\zeta \, d\theta dr dy \right| \\
&\qquad \leq C\|DD_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))} \frac{1}{\tau^2} \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0)} |U|^2 \right) \\
&\qquad + C\|D_{y_l}D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))} \tau^{2\alpha} \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0)} |U|^2 \right)^{1/2} \\
&\qquad + C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))} \tau^{\alpha} \left(\left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 \right)^{1/2} + \frac{1}{\tau} \int_{B_1(0)} \mathscr{G}(U,\varphi)^2 \right) \\
&\qquad + C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))} \left(1 + \frac{1}{\tau} \right) \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 \right)^{1/2} \\
&\qquad + C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))} \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0)} |U|^2 \right)^{1/2},
\end{aligned}$$

for $i = 1,2$ and for each $p = 1,2$, and where $C = C(\alpha)$.

Proof. Fix $i \in \{1,2\}$. Testing the modified squeeze identity with $\delta_{ij} D_{y_p}\zeta(r,y)$ for a ζ as in the statement we obtain

$$\begin{aligned}
&\int_{B_{1/2}(0)\cap\{r\leq\tau\}} D_i U^k D_{y_l} U^k \cdot D_{y_l} D_{y_p}\zeta - \int_{B_{1/2}(0)\cap\{r\leq\tau\}} \mathscr{R} U^k D_j U^k \delta_{ij} D_{y_p}\zeta \\
=&\frac{1}{2} \int_{B_{1/2}(0)\cap\{r\geq\tau\}} \left(|D\varphi|^2 + 2D_j\varphi^k D_j\hat{V}^k + |D\hat{V}|^2 \right) D_i D_{y_p}\zeta \\
&- \int_{B_{1/2}(0)\cap\{r\geq\tau\}} \left(D_i\varphi^k D_j\varphi^k + D_i\varphi^k D_j\hat{V}^k + D_i\hat{V}^k D_j\varphi^k + D_i\hat{V}^k D_j\hat{V}^k \right) D_j D_{y_p}\zeta \\
&+ \int_{B_{1/2}(0)\cap\{r\geq\tau\}} \mathscr{R}(\varphi^k + V^k)(D_j\varphi^k + D_j V^k) \delta_{ij} D_{y_p}\zeta
\end{aligned}$$

where we used the fact that for $X \in B_{1/2}(0) \cap \{r \geq \tau\}$ we have $U(X) = \{\pm(\varphi_1(X) + V(X,\varphi_1(X)))\}$, where $\varphi(X) = \{\varphi_1(X)\}$ as usual. As φ is independent of y integration by parts implies

$$\int_{B_{1/2}(0)} \frac{1}{2} \left(|D\varphi|^2 + 2D_j\varphi^k D_j\hat{V}^k + |D\hat{V}|^2 \right) D_i D_{y_p}\zeta = 0,$$

from which we deduce that the following holds

$$\begin{aligned}
&\int_{B_{1/2}(0)\cap\{r\geq\tau\}} \left(-D_j\hat{\varphi}^k D_j\hat{V}^k D_i D_{y_p}\zeta + D_i\varphi^k D_j\hat{V}^k D_j D_{y_p}\zeta + D_i\hat{V}^k D_j\varphi^k D_j D_{y_p}\zeta \right) \\
&\quad = \int_{B_{1/2}(0)\cap\{r\geq\tau\}} \left(\frac{1}{2}|D\hat{V}|^2 D_i D_{y_p}\zeta - D_i\hat{V}^k D_j\hat{V}^k D_j D_{y_p}\zeta \right) \\
&\quad - \int_{B_{1/2}(0)\cap\{r\leq\tau\}} D_i U^k D_{y_l} U^k \cdot D_{y_l} D_{y_p}\zeta \\
&\quad + \int_{B_{1/2}(0)\cap\{r\leq\tau\}} \mathscr{R} U^k D_j U^k \delta_{ij} D_{y_p}\zeta \\
&\quad - \int_{B_{1/2}(0)\cap\{r\geq\tau\}} \mathscr{R}(\varphi^k + V^k)(D_j\varphi^k + D_j V^k) \delta_{ij} D_{y_p}\zeta.
\end{aligned} \tag{3.25}$$

The estimates in Proposition 83 imply for a constant $C = C(\alpha, \varphi^{(0)})$,

$$\int_{B_{1/2}(0)\cap\{r\geq\tau\}} r^2|D\hat{V}|^2 \leq C\int_{B_1(0)} \mathscr{G}(U,\varphi)^2,$$

so that we can deduce

$$\int_{B_{1/2}(0)\cap\{r\geq\tau\}} |D\hat{V}|^2|D_iD_{y_p}\zeta| \leq \frac{C}{\tau^2}\|D_iD_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\int_{B_1(0)} \mathscr{G}(U,\varphi)^2.$$

We can now estimate

$$\begin{aligned}\int_{B_{2\tau}(0,y)} |Du|^2 &\leq \int_{B_{3\tau}(Z)} |Du|^2 \leq \frac{C}{\tau^2}\int_{B_{6\tau}(Z)} |u|^2 \\ &\leq \frac{C}{\tau^2}\left(\frac{6\tau}{1/2}\right)^{2\mathscr{N}(Z;U)+n}\int_{B_{1/2}(Z)} |u|^2 \leq \frac{C}{\tau^2}\tau^{4+2\alpha}\int_{B_{1/2}(Z)} |u|^2 \leq C\tau^{2+2\alpha},\end{aligned}$$

where $Z \in B_\tau(0,y)$ satisfies $\mathscr{N}(Z;U) \geq \alpha$. We know such a point Z exists for every $y \in B^2_{1/2}(0)$ by hypothesis. Furthermore, we used (3.17) in the third inequality and for the second one we applied the second inequality in Lemma 61, after a scaling argument has been performed so that we obtain the factor τ^{-2}. A covering argument implies

$$\int_{B_{1/2}(0)\cap\{r\leq\tau\}} |Du|^2 \leq C\tau^{2\alpha},$$

from which we infer, combined with Theorem 85, the bound

$$\begin{aligned}\int_{B_1(0)\cap\{r\leq\tau\}} &|D_iU^kD_{y_l}U^k\cdot D_{y_l}D_{y_p}\zeta| \\ &\leq \left(\int_{B_{1/2}(0)\cap\{r\leq\tau\}} |Du|^2\right)^{1/2}\left(\int_{B_{1/2}(0)} |D_yu|^2\right)^{1/2}\|D_{y_l}D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))} \\ &\leq C\tau^{2\alpha}\left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0)} |U|^2\right)^{1/2}\|D_{y_l}D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}.\end{aligned}$$

As far as the curvature terms are concerned, we obtain

$$\begin{aligned}\int_{B_{1/2}(0)\cap\{r\leq\tau\}} &\mathscr{R}U^kD_jU^k\delta_{ij}D_{y_p}\zeta \\ &\leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\|U\|_{L^2(B_{1/2}(0)\cap\{r\leq\tau\})}\|DU\|_{L^2(B_{1/2}(0)\cap\{r\leq\tau\})} \\ &\leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\tau^\alpha\left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2\right)^{1/2}\end{aligned}$$

and

$$\int_{B_{1/2}(0)\cap\{r\geq\tau\}} \mathscr{R}(V^k+\varphi^k)(D_jV^k+D_j\varphi^k)\delta_{ij}D_{y_p}\zeta =: (\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV}),$$

where we shall estimate each term separately. Start with the first one

$$\begin{aligned}\int_{B_{1/2}(0)\cap\{r\geq\tau\}} &\mathscr{R}V^kD_jV^k\delta_{ij}D_{y_p}\zeta \\ &\leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\|V\|_{L^2(B_{1/2}(0)\cap\{r\geq\tau\})}\|DV\|_{L^2(B_{1/2}(0)\cap\{r\geq\tau\})} \\ &\leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\frac{1}{\tau}\int_{B_1(0)} \mathscr{G}(U,\varphi)^2.\end{aligned}$$

Moving to the second one we infer

$$\int_{B_{1/2}(0)\cap\{r\geq\tau\}} \mathscr{R}\varphi^k D_j V^k \delta_{ij} D_{y_p}\zeta \leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\frac{1}{\tau}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2\right)^{1/2}.$$

The bound for the third is

$$\int_{B_{1/2}(0)\cap\{r\geq\tau\}} \mathscr{R}V^k D_j \varphi^k \delta_{ij} D_{y_p}\zeta \leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2\right)^{1/2}.$$

Finally, the last term can be controlled by

$$\int_{B_{1/2}(0)\cap\{r\geq\tau\}} \mathscr{R}\varphi^k D_j \varphi^k \delta_{ij} D_{y_p}\zeta \leq C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+\int_{B_1(0)}|U|^2\right)^{1/2}.$$

These inequalities further imply that the term on the left-hand side of (3.25) is bounded by

$$\begin{aligned}
&C\|D_{y_l}D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\frac{1}{\tau^2}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+\int_{B_1(0)}|U|^2\right)\\
&\quad+C\|D_{y_l}D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\tau^{2\alpha}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+\int_{B_1(0)}|U|^2\right)^{1/2}\\
&\quad+C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\tau^{\alpha}\left(\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2\right)^{1/2}+\frac{1}{\tau}\int_{B_1(0)}\mathscr{G}(U,\varphi)^2\right)\\
&\quad+C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\left(1+\frac{1}{\tau}\right)\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2\right)^{1/2}\\
&\quad+C\|D_{y_p}\zeta\|_{L^\infty(B_{1/2}(0))}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+\int_{B_1(0)}|U|^2\right)^{1/2}
\end{aligned} \tag{3.26}$$

To conclude the proof we need to bound the quantity in the statement of the Proposition. Start by noting that, by virtue of φ being homogeneous of degree $1/2$ and independent of y, and ζ depending only on r and y, we have

$$D_j(r\hat{V}^k D_i\varphi^k)D_jD_{y_p}\zeta = \frac{1}{2}\hat{V}^k D_i\varphi^k D_r D_{y_p}\zeta + rD_j\hat{V}^k D_i\varphi^k D_j D_{y_p}\zeta$$

so that

$$\begin{aligned}
&\int_{B^2_{1/2}(0)}\int_\tau^\infty\int_0^{4\pi} D_j(r\hat{V}^k D_i\varphi^k)D_jD_{y_p}\zeta\, d\theta dr dy\\
&=\int_{B^2_{1/2}(0)}\int_\tau^\infty\int_0^{4\pi}\left(\frac{1}{2}\hat{V}^k D_i\varphi^k D_r D_{y_p}\zeta + rD_i\hat{V}^k D_j\varphi^k D_i D_{y_p}\zeta - rD_i\hat{V}^k D_j\varphi^k D_j D_{y_p}\zeta\right)+\mathscr{E}
\end{aligned}$$

where the remainder $\mathscr{E}$ is bounded by (3.26) and where we introduced extra terms, the same ones as in the left-hand side of (3.25). Compute now

$$\begin{aligned}
&\int_{B^2_{1/2}(0)}\int_\tau^\infty\int_0^{4\pi} D_j(r\hat{V}^k D_i\varphi^k)D_jD_{y_p}\zeta\, d\theta dr dy\\
&=\int_{B^2_{1/2}(0)}\int_\tau^\infty\int_0^{4\pi}\left(\frac{1}{2}\hat{V}^k D_i\varphi^k + x_i D_j\hat{V}^k D_j\varphi^k - \frac{1}{2}D_i\hat{V}^k\varphi^k\right)D_r D_{y_p}\zeta\, d\theta dr dy+\mathscr{E}\\
&=\int_{B^2_{1/2}(0)}\int_\tau^\infty\int_0^{4\pi}\left(-\frac{1}{2}D_i\left(\hat{V}^k\varphi^k\right)+D_j(x_i\hat{V}^k D_j\varphi^k)-x_i\hat{V}^k\varphi^k\mathscr{R}\right)D_r D_{y_p}\zeta\, d\theta dr dy+\mathscr{E}\\
&=\int_{B_{1/2}(0)\cap\{r\geq\tau\}}\left(-\frac{1}{2}D_i\left(\hat{V}^k\varphi^k\right)+D_j(x_i\hat{V}^k D_j\varphi^k)\right)D_r D_{y_p}\zeta\,\frac{1}{r}d\mathscr{H}^4(X)+\widetilde{\mathscr{E}}
\end{aligned}$$

where the first equality follows since φ is homogeneous of degree $1/2$ and independent of y, and ζ depends only on r and y. The second is equality is due to the fact that away from $\{r=0\}$ the map φ is given by two harmonic spinors, while in the last one we simply absorbed the curvature term in the error $\mathscr{E}$. The new bound for $\tilde{E}$ is simply given by the same one appearing in (3.26) with $\tau^{2\alpha}$ replaced by $(1+\tau^{2\alpha})$, or by $(\tau^{-1}+\tau^{2\alpha})$ if need be. Therefore, integrating by parts in the radial variable, we infer

$$\begin{aligned}
&\int_{B^{n-2}_{1/2}(0)}\int_\tau^\infty\int_0^{4\pi} D_j(r\hat{V}^k D_i\varphi^k)D_jD_{y_p}\zeta\,d\theta drdy\\
&=\int_{B_{1/2}(0)\cap\{r=\tau\}}\left(-\frac{1}{2}D_i\left(\hat{V}^k\varphi^k\right)+D_j(x_i\hat{V}^kD_j\varphi^k)\right)D_rD_{y_p}\zeta\,\frac{1}{r}d\mathscr{H}^4(X)\\
&\quad-\int_{B_{1/2}(0)\cap\{r\geq\tau\}}\left(-\frac{1}{2}\frac{x_i}{r}\hat{V}^k\varphi^k+x_i\hat{V}^kD_j\varphi^k\right)D_rD_{y_p}\zeta\,\frac{1}{r}d\mathscr{H}^4(X)+\widetilde{\mathscr{E}}
\end{aligned}$$

so that by homogeneity of φ, as $rD_r\varphi=\varphi/2$, the two integrals in the right-hand side of the above vanish and the left-hand side is simply equal to the error term $\widetilde{\mathscr{E}}$. From the □

Corollary 87. *Let the hypothesis be as in Theorem 85 and let $\sigma\in(0,1)$. Then,*

$$\int_{B_\gamma(0)}\frac{\mathscr{G}(U,\varphi)^2}{R^{4+2\alpha-\sigma}}\leq C\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+C\int_0^\gamma s^{-2-2\alpha}H(0,s)\,ds,$$

where the constant C depends on $\varphi^{(0)},\alpha,\gamma,\sigma$ and the curvature of M and $\mathscr{V}$.

Proof. Define the vector field $\Upsilon^i=\psi(R)^2\eta_\delta(R)R^{-4+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2X^i$, where for each δ the function $\eta_\delta\in C^\infty([0,\infty))$ is a cutoff satisfying $\eta_\delta(t)=0$ for $t\in[0,\delta/2]$, $\eta_\delta(t)=1$ for $t\in[\delta,\infty)$, and $0\leq\eta'_\delta(t)\leq 3/\delta$ for all $t\in[0,\infty)$. Furthermore, $\psi\colon[0,\infty)\to\mathbb{R}$ satisfies $\psi(t)=1$ for $t\in[0,\gamma]$, and $\psi(t)=0$ for $t\geq(1+\gamma)/2$, together with the bound $0\leq\psi'(t)\leq 3/(1-\gamma)$ for $t\in[0,\infty)$.

Recall now that for every vector field with compact support in $\mathbb{R}^4$ regular enough, just $W^{1,1}_0(\mathbb{R}^n)$ suffices, the divergence theorem entails $\int\operatorname{div}\Upsilon=0$. Whence,

$$\begin{aligned}
\sigma\int\psi(R)^2\eta_\delta(R)R^{-4+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2&=-\int\psi(R)^2\eta_\delta(R)R^{-3+\sigma}D_R\left(R^{-2\alpha}\mathscr{G}(U,\varphi)^2\right)\\
&\quad+\int\psi(R)\psi'(R)\eta_\delta(R)R^{-3+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2+\int\psi(R)^2\eta'_\delta R^{-3+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2.
\end{aligned}$$

Note that we can apply the divergence theorem for the vector field Υ as $\mathscr{G}(U,\varphi)^2\in W^{1,1}_{\mathrm{loc}}(B_1(0))$ as U and φ are $W^{1,2}$-sections and have singular sets of Hausdorff dimension at most $n-2$, making Υ a legitimate candidate. Observe that a.e. in the ball $B_1(0)$, we have

$$|D_R(R^{-2\alpha}\mathscr{G}(U,\varphi))|^2=|D_R\left(\mathscr{G}(U/R^\alpha),\varphi/R^\alpha\right)^2|\leq 2\mathscr{G}(U/R^\alpha,\varphi/R^\alpha)|D_R(U/R^\alpha)|,$$

so that, applying Cauchy's inequalty inequality we infer

$$\begin{aligned}
\int\psi(R)^2\eta_\delta(R)R^{-4+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2&\leq\frac{C}{\sigma^2}\int\psi(R)^2R^{-2+\sigma}|D_R(U/R^\alpha)|^2\\
&\quad+\frac{C}{\sigma^2}\int\psi'(R)R^{-2+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2\\
&\quad+\frac{C}{\sigma}\int\psi(R)^2\eta'_\delta R^{-3+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2.
\end{aligned}$$

Note now that, by definition of the cutoff function η_δ we have

$$\begin{aligned}\int \psi(R)^2\eta_\delta'(R)R^{-4+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2 &\le C\delta^{-4+\sigma-2\alpha}\int_{B_\delta(0)}|U|^2+|\varphi|^2\\ &\le C\delta^\sigma\int_{B_1(0)}|U|^2+|\varphi|^2\le C\delta^\sigma.\end{aligned}$$

Letting now $\delta\to 0^+$ and applying the monotone convergence we infer

$$\int \psi(R)^2R^{-4+\sigma-2\alpha}\mathscr{G}(U,\varphi)^2 \le \frac{C}{\sigma^2}\int \psi(R)^2R^{-2+\sigma}|D_R(U/R^\alpha)|^2+C\int_{B_1(0)}\mathscr{G}(U,\varphi)^2,$$

from which we deduce, thanks to Theorem 85, the desired estimate. □

Before proceeding further with the next two corollaries we need to fix some notation. Consider a point $Z=(\xi,\zeta)\in\Sigma_U\cap B_{1/2}(0)$ at which $\mathscr{N}(Z;U)\ge\alpha$. Fix then $\varphi\in\Phi_{\varepsilon_0}(\varphi^{(0)})$ and $U\in\mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$. For $\theta\in(0,1)$ and $X=(x,y)\in B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$, we know that $\varphi=\{\pm\varphi_1\}$ on $B^2_{\theta|x|}(x)\times\mathbb{R}^2$ for some single valued solution of the Dirac equation $\mathscr{D}(\varphi_1)=0$. Apply now Taylor's theorem to φ_1 in order to deduce the existence of $\theta=\theta(\varphi^{(0)})\in(0,1)$ such that, if $|\xi|\le\theta|x|$, we have

$$\mathscr{G}(U(X),\varphi(X-Z))\le\mathscr{G}(U(X),\varphi(X)-D_x\varphi(X)\cdot\xi)+\mathscr{R}, \tag{3.27}$$

provided that ε_0 small enough. We implicitly used the fact that $\varphi^{(0)}$ is homogeneous of degree α and that ε_0 can be chosen to be sufficiently small. In the above the remainder satisfies $|\mathscr{R}|\le C(\varphi^{(0)})|x|^{\alpha-2}|\xi|^2$. In addition, we have

$$\mathscr{G}(U(X),\varphi(X-Z))\ge|D_x\varphi(X)\cdot\xi|-\mathscr{G}(U(X),\varphi(X))-C|x|^{\alpha-2}|\xi|^2. \tag{3.28}$$

Consequently, we infer the inequalities

$$\begin{aligned}|\mathscr{G}(U(X),\varphi(X-Z))-\mathscr{G}(U(X),\varphi(X))|^2 &\le\mathscr{G}(\varphi(X-Z),\varphi(X))^2\\ &\le|\xi|^2\int_0^1|D\varphi(x-t\xi,y)|^2\,dt,\end{aligned} \tag{3.29}$$

for a.e. $X=(x,y)\in B_1(0)$. Consider now this inequality with $\varphi^{(0)}$ in place of φ to deduce for any harmonic $\mathbb{Z}_2$ spinor $U\in\mathscr{F}_\varepsilon(\varphi^{(0)})$ that

$$\begin{aligned}\frac{1}{4^{4+2\alpha}}\int_{B_{1/4}(Z)}\mathscr{G}(U(X),\varphi^{(0)}(X-Z))^2 &\le C\int_{B_1(0)}\mathscr{G}(U(X),\varphi^{(0)}(X))^2+C|\xi|^2\\ &\le C\varepsilon+C\delta(\varepsilon)<\varepsilon_0.\end{aligned}$$

Here ε_0 is as in Theorem 85 and the second inequality follows from Proposition 83. Note that $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$, from which the last inequality follows.

Corollary 88. *Let $\varphi^{(0)}$ be the fixed $\mathbb{Z}_2$-harmonic spinor homogeneous degree $\alpha\in\mathbb{N}\cup\frac{1}{2}\mathbb{N}$. There exists $\varepsilon_0=\varepsilon_0(\varphi^{(0)},\alpha)>0$ such that if $\varphi\in\Phi_{\varepsilon_0}(\varphi^{(0)})$ and $Z\in\Sigma_U\cap B_{1/2}(0)$, as well as $\mathscr{N}(Z;U)\ge\alpha$, then*

$$\operatorname{dist}^2(Z,\{0\}\times\mathbb{R}^2)+\int_{B_1(0)}\mathscr{G}(U,\varphi(\cdot-Z))^2\le C\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+\int_{B_1(0)}|U|^2\right),$$

for a constant C depending on $\varphi^{(0)}$ and α.

Proof. We start by claiming that there exists $\delta_1 = \delta_1(\varphi^{(0)}) > 0$, and given $\rho \in (0,1/4)$, a constant $\varepsilon_0 = \varepsilon_0(\rho, \varphi^{(0)})$, such that for $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, $\varphi \in \Phi_{\varepsilon_0}(\varphi^{(0)})$, $a \in \mathbb{R}^2$ and $Z = (\xi, \eta) \in \mathscr{B}_U \cap B_{1/2}(0)$ the following holds

$$\mathscr{L}^n \left(\{X \in B_\rho(Z);\ \delta_1 |a||x|^{\alpha-1} \leq |D_x \varphi(X) \cdot a|\} \right) \geq \delta_1 \rho^n. \tag{3.30}$$

Assume for the sake of contradiction that this is not the case. Thus, for every $\delta_1 > 0$, there is $\rho > 0$ such that with $\varepsilon_j = 1/j$, there exists $U_j \in \mathscr{F}_{\varepsilon_j}(\varphi^{(0)})$, $\varphi \in \Phi_{\varepsilon_j}(\varphi^{(0)})$, $Z_j \in \mathscr{B}_{U_j}$, and $a_j \in S^1$ such that the above inequality does not hold. After passing to a subsequence, we have $\varphi_j \to \varphi^{(0)}$ in C^0, as well as $Z_j \to Z \in \{0\} \times \mathbb{R}^2$, and $a_j \to a \in S^1$, so that we can make sure that the above inequality does not hold for the limits as well. By translating and rescaling we can assume $\rho = 1$ and $Z = 0$. Taking now $\delta_1 = 1/k$, we have the existence of $a_k \in S^1$ such that

$$\mathscr{L}^n \big(\{X \in B_1(0);\ |x|^{\alpha-1}/k \leq |D_x \varphi^{(0)}(X) \cdot a_k|\} \big) < 1/k.$$

Passing to a further subsequence, we infer $D_x \varphi^{(0)}(X) \cdot a = 0$ a.e. in $B_1(0)$ for $a \in S^1$ such that $a_k \to a$, which gives the desired contradiction as this is not possible by virtue of (3.18).

We can now prove the corollary. Considering now (3.30) and applying it with $a = \xi$ we infer the existence of a set $S \subset B_\rho(Z)$ such that

$$\int_S |x|^{2\alpha-2} |\xi|^2 \leq \frac{1}{\delta_1^2} \int_{B_\rho(Z)} |D_x \varphi(X) \cdot \xi|^2. \tag{3.31}$$

Applying now (3.28) we have that the right-hand side of the above can be bounded by

$$\begin{aligned} &C \frac{1}{\delta_1^2} \int_{B_\rho(Z)} \mathscr{G}(U(X), \varphi(X - Z))^2 + C \frac{1}{\delta_1^2} \int_{B_1(0)} \mathscr{G}(U(X), \varphi(X))^2 \\ &\quad + C \frac{1}{\delta_1^2} \int_{B_\rho(Z) \cap \{|x| \geq |\xi|/\theta\}} |x|^{2\alpha-4} |\xi|^4 + C \frac{1}{\delta_1^2} \int_{B_\rho(Z) \cap \{|x| \leq |\xi|/\theta\}} |D_x \varphi(X)|^2 |\xi|^2 \\ &\quad =: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}), \end{aligned} \tag{3.32}$$

for all $\rho \in (0, 1/4)$. Now, for some $\kappa = \kappa(n)$ we have

$$\mathscr{L}^4 \left(B^2_{\kappa \delta_1^{1/2} \rho} \times B^2_\rho(0) \right) < \delta_1 \rho^n / 2,$$

so that $\mathscr{L}^4(\{(x,y) \in S;\ |x| \geq \kappa \delta_1^{1/2} \rho\}) \geq \delta_1 \rho^4 / 2$ and, consequently, the left-hand side of (3.31) is bounded from below by $\rho^{4+2\alpha-2} |\xi|^2$. To conclude the proof we need to bound the terms $(\mathrm{I}), (\mathrm{III})$ and (IV) in (3.32). For (I) we obtain

$$\begin{aligned} &\frac{1}{\rho^{4+2\alpha-1/2}} \int_{B_\rho(Z)} \mathscr{G}(U(X), \varphi(X - Z))^2 \\ &\leq C \int_{B_1(0)} \mathscr{G}(U(X), \varphi(X - Z))^2 + C \int_0^1 s^{-2-2\alpha} H(0,s)\, ds \\ &\leq C \int_{B_1(0)} \mathscr{G}(U(X), \varphi(X))^2 + C |\xi|^2 \int_{B_1(0)} \int_0^1 |D\varphi(x - t\xi, y)|^2 + C \int_0^1 s^{-2-2\alpha} H(0,s)\, ds \\ &\leq C \int_{B_1(0)} \mathscr{G}(U(X), \varphi(X))^2 + C |\xi|^2 + C \int_0^1 s^{-2-2\alpha} H(0,s)\, ds, \end{aligned} \tag{3.33}$$

where we applied Corollary 87 with $\sigma = 1/2$ for the first inequality, (3.29) to obtain the second one and we estimated the integral $\int_{B_1(0)}\int_0^1 |D\varphi(x-t\xi,y)|^2$ using the change of variables $x' = x - t\xi$. Here $C = C(\alpha,\varphi^{(0)})$. We shall estimate (III). The reader can check that the following holds

$$\text{(III)} \leq \begin{cases} C\left(\rho^{2\alpha-2} + |\xi|^{2\alpha-2}\right)\rho^{-2}|\xi|^4, & \text{if } \alpha = 1/2 \text{ and } \alpha > 1, \\ C\rho^2|\xi|^4\log(|\xi|+\rho), & \text{if } \alpha = 1. \end{cases}$$

The case $\alpha = 1$ can be further bounded by $C\rho^2|\xi|^{7/2}$. Therefore, we have

$$\text{(III)} \leq C\left(\rho^{2\alpha-5/2} + |\xi|^{2\alpha-5/2}\right)\rho^2|\xi|^4.$$

On the other hand, for (IV) we have

$$\frac{1}{\delta_1^2}\int_{B_\rho(Z)\cap\{|x|\leq|\xi|/\theta\}} |D_x\varphi(X)|^2|\xi|^2 \leq C\rho^2\int_{B^2_{|\xi|/\theta}(0)} |x|^{2\alpha-2}|\xi|^2 \leq C\rho^2|\xi|^{2\alpha+2}.$$

Combining the estimates for (I), (III) and (IV) gives

$$\begin{aligned} \rho^{4+2\alpha-2}|\xi|^2 \leq C\int_{B_1(0)} \mathscr{G}(U(X),\varphi(X))^2 + C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds \\ + C\rho^{2+2\alpha}|\xi|^2\left(\rho^{3/2} + \rho^{-2\alpha}|\xi|^{2\alpha} + \rho^{-5/2}|\xi|^2 + |\xi|^{2\alpha-1/2}\rho^{-2\alpha}\right). \end{aligned}$$

Applying now Proposition 83, for $\tau > 0$ we infer the existence of $\varepsilon_0(\varphi^{(0)},\tau)$ such that $|\xi| \leq \tau$, so that the expression multiplying $\rho^{2+2\alpha}|\xi|^2$ in the above can be made smaller than 1 so that it can be absorbed by the left-hand side, and, consequently,

$$\begin{aligned} |\xi|^2 &\leq C\rho^{-4-2\alpha+2}\left(\int_{B_1(0)} \mathscr{G}(U(X),\varphi(X))^2 + \int_0^1 s^{2-4-2\alpha}H(0,s)\,ds\right) \\ &\leq C\int_{B_1(0)} \mathscr{G}(U(X),\varphi(X))^2 + C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds, \end{aligned}$$

where $C = C(\alpha,\varphi^{(0)})$. Combining this with (3.33) we duduce that the term $\int_{B_1(0)} \mathscr{G}(U,\varphi(\cdot - Z))^2$ has the same bound, thus concluding the proof. ☐

Corollary 89. *Let $\varphi^{(0)}$ be the fixed $\mathbb{Z}_2$-harmonic spinor homogeneous degree $\alpha \in \mathbb{N}\cup\frac{1}{2}\mathbb{N}$. Let $\sigma,\tau,\gamma \in (0,1)$. There are then $\varepsilon_0(\varphi^{(0)},\gamma,\tau) \in (0,1)$ and $\beta_0 = \beta(\varphi^{(0)}) \in (0,1)$ such that if $\varphi \in \Phi_{\varepsilon_0}(\varphi^{(0)})$, $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$ and V is as in Proposition 83 with $\beta = \beta_0$ and $Y = (\xi,\eta) \in \Sigma_U \cap B_{1/2}(0)$ with $\mathscr{N}((\xi,\eta);U) \geq \alpha$, then the following estimate holds*

$$\int_{B_\gamma(0)} R_Y^{-2}\left|\frac{\partial}{\partial R_Y}\left(\frac{U}{R_Y^\alpha}\right)\right|^2 \leq C\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds,$$

where $R_Y = |X-Y|$ and $C = C(\varphi^{(0)},\alpha,\gamma) \in (0,\infty)$. Moreover, we have

$$\begin{aligned} \int_{B_\gamma(0)} \frac{\mathscr{G}(U,\varphi)^2}{|X-Y|^{3-\sigma}} + \int_{B_\gamma(0)\cap\{|x|>\tau\}} \frac{|V(X,\varphi(X)) - \langle D_x\varphi(X),\xi\rangle|^2}{|X-Y|^{4+2\alpha-\sigma}} \\ \leq C_1\left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_0^1 s^{-2-2\alpha}H(0,s)\,ds\right), \end{aligned} \tag{3.34}$$

where the constant C depends on $\varphi^{(0)},\alpha,\gamma$ and σ. In the above inequalities, X denotes the variable of integration.

Remark 90. Note that in the previous corollary the constants C and C_1 are independent of τ.

Proof. We only need to prove the latter inequality as the former follows from previous results. We break it down into two steps, we prove that both terms on the left-hand side of (3.34) can be bounded by $C\int_{B_1(0)}\mathscr{G}(U,\varphi)^2$. The desired result will follow, albeit with a different constant. From (3.29) we obtain

$$\begin{aligned}\int_{B_\gamma(0)}\frac{1}{|X-Z|^{3-\sigma}}\mathscr{G}(U(X),\varphi(X))^2 \\ &\leq C\int_{B_{1/4}(Z)}\frac{1}{|X-Z|^{3-\sigma}}\mathscr{G}(U(X),\varphi(X-Z))^2 \\ &\quad+C\int_{B_{1/4}(Z)}\int_0^1\frac{1}{|X-Z|^{3-\sigma}}|D\varphi(x-t\xi,y)|^2|\xi|^2 \\ &\quad+4^{3-\sigma}\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2\end{aligned} \tag{3.35}$$

where the last term on the right-hand side comes from changing the domain of integration from $B_\gamma(0)$ to $B_{1/4}(Z)$ and bounding the term $|X-Z|^{-4+\sigma+1}$ from above. To conclude we need to estimate the first two terms.

Before proceeding with the proof we make the following observation: $\mathrm{dist}(Z,\{0\}\times\mathbb{R}^2)^2 = |\xi|^2$, so that Corollary 88 implies $|\xi|^2\leq C\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds$. A second observation is the following computation

$$\begin{aligned}&\int_{B_{1/4}(Z)}\frac{\mathscr{G}(U(X),\varphi(X-Z))^2}{|X-Z|^{4+2\alpha-\sigma}}\\ &\leq C\int_{B_{1/2}(Z)}\mathscr{G}(U(X),\varphi(X-Z))^2+C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds\\ &\leq C\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2+C\int_{B_{1/2}(Z)}\int_0^1|D\varphi(x-t\xi,y)|^2|\xi|^2+C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds\\ &\leq C\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2+C\int_{B_{1/2}(Z)}\int_0^1|x-t\xi|^{2\alpha-2}|\xi|^2+C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds\\ &\leq C\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2+C|\xi|^2+C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds\\ &\leq C\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2+C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds,\end{aligned}$$

where the constant C depends on α. The desired bound on the first term on the right-hand side of (3.35) by virtue of the bound $|X-Z|^{-4+1+\sigma}\leq|X-Z|^{-4-2\alpha+\sigma}(1/4)^{2\alpha+1}$ on the ball $B_{1/4}(Z)$. We can now move to the second term on the right-hand side of (3.35), we divide it in two cases. We are also using the bound

$$\int_{B_1(0)}\int_0^1|D\varphi(x-t\xi,y)|^2\,dt\,dx\,dy\leq C\sup_{\partial B_1^2(0)\times\mathbb{R}^2}|D\varphi|^2\int_0^1\int_{B^2_{1+|\xi|}(0)}|x'|^{2\alpha-2}\,dx'\,dt$$

that appeared in the estimates in (3.33) already and where $x'=x-t\xi$.

(i) Assume $\alpha\geq 1$. Then, Corollary 88 implies

$$\int_{B_{1/4}(Z)}\int_0^1\frac{|x-t\xi|^{2\alpha-2}|\xi|^2}{|X-Z|^{3-\sigma}}\leq C\int_{B_{1/4}(Z)}\int_0^1\frac{|\xi|^2}{|X-Z|^{3-\sigma}}\leq C|\xi|^2,$$

from which we conclude.

(ii) Assume now $\alpha = 1/2$ and apply again Corollary 88 to obtain

$$\begin{aligned}\int_{B_{1/4}(Z)}\int_0^1 \frac{|x-t\xi|^{2\alpha-2}|\xi|^2}{|X-Z|^{3-\sigma}} &\leq \int_0^1\int_{B_{1/4}(Z)} \frac{|\xi|^2}{|x-t\xi||x-\xi|^{1-\sigma/2}|y-\zeta|^{2-\sigma/2}}\\ &\leq \int_0^1\int_{B^2_{1/4}(\zeta)}\int_{\{|x-t\xi|\geq|x-\xi|\}} \frac{|\xi|^2}{|x-\xi|^{2-\sigma/2}|y-\zeta|^{2-\sigma/2}}\\ &\quad+\int_0^1\int_{B^2_{1/4}(\zeta)}\int_{\{|x-t\xi|\leq|x-\xi|\}} \frac{|\xi|^2}{|x-t\xi|^{2-\sigma/2}|y-\zeta|^{2-\sigma/2}}\\ &\leq C|\xi|^2,\end{aligned}$$

and we can conclude as before. Thus, the first part of the proof is concluded.

We can now prove the second part of the statement. Proceeding as before and considering ε_0 small enough so that $|\xi| \leq \theta\tau$, we have

$$\begin{aligned}&\int_{B_\gamma(0)\cap\{|x|>\tau\}} \frac{|V(X,\varphi(X)) - \langle D_x\varphi(X),\xi\rangle|^2}{|X-Y|^{4+2\alpha-\sigma}}\\ &\quad\leq C\int_{B_{1/4}(Z)} \frac{\mathscr{G}(U(X),\varphi(X-Z))^2}{|X-Z|^{4+2\alpha-\sigma}}\\ &\qquad+C\tau^{2\alpha-4}|\xi|^4\int_{B_{1/4}(Z)\cap\{|x|>\tau\}}|X-Z|^{-4-2\alpha+\sigma}\\ &\qquad+C(\alpha,\sigma)\int_{(B_\gamma(0)\setminus B_{1/4}(Z))\cap\{|x|>\tau\}}\left(|V(X,\varphi(X))|^2+|x|^{2\alpha-2}|\xi|^2\right),\end{aligned}$$

where all the constants appearing depend on $\varphi^{(0)}$ as well. As seen previously in the proof, the first term on the right-hand side of the above can be bounded by

$$C\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2 + C\int_0^1 s^{-2-2\alpha}H(0,s)\,ds,$$

while the third term can be bounded by $C\int_{B_1(0)}\mathscr{G}(U(X),\varphi(X))^2$ by definition of V, where the constant C depends on α,γ,σ and $\varphi^{(0)}$, the curvature of M and $\mathscr{V}$. To bound the second term, and conclude, note that we can choose ε_0 small enough so that $|\xi|^2 \leq \min\{\tau^2,\tau/4\}$ by Corollary 88. Thus, taking $\varepsilon_0 < \min\{\tau^2,\tau/4\}/(2C)$ so that $|X-Z|^2 \geq \tau/2$ if $X=(x,y)$ with $|x|\geq\tau$ (recall that $Z=(\xi,\eta)$), and $|\xi|^2/\tau^4 < 1$, we have

$$\tau^{2\alpha-4}|\xi|^4\int_{B_{1/4}(Z)\cap\{|x|>\tau\}}|X-Z|^{-4-2\alpha+\sigma} \leq C\frac{|\xi|^4}{\tau^4}\int_{B_{1/4}(Z)\cap\{|x|>\tau\}}|X-Z|^{-4+\sigma} \leq C|\xi|^2,$$

where the constant depends on $\varphi^{(0)}, \alpha$ and σ. This concludes the proof. □

Theorem 91 (Non-concentration of the Excess)**.** *Let $\varphi^{(0)}$ be the fixed $\mathbb{Z}_2$-harmonic spinor homogeneous degree $\alpha \in \mathbb{N}\cup\frac{1}{2}\mathbb{N}$. Let $\sigma,\tau,\delta\in(0,1)$. There are then $\varepsilon_0 = \varepsilon_0(\varphi^{(0)},\tau) > 0$ and $\beta_0 = \beta_0(\varphi^{(0)}) > 0$ such that, if $\varphi \in \Phi_\varepsilon(\varphi^{(0)})$ and $U \in \mathscr{F}_\varepsilon(\varphi^{(0)})$, with $\varepsilon \leq \min\{\varepsilon_0,\delta\}$, and if*

$$B_\delta(0,y)\cap\{z\in B_1(0);\ \mathscr{N}(z,U)\geq\mathscr{N}(0;\varphi)=\alpha\}\neq\emptyset,$$

for each $y\in B^2_{1/2}(0)$, then

$$\int_{B_{1/2}(0)\cap(B^2_{\rho/2}(0)\times\mathbb{R}^2)}\mathscr{G}(U,\varphi)^2 \leq C\rho^{1-\sigma}\left(\int_{B_1(0)}\mathscr{G}(U,\varphi)^2+\int_{B_1(0)}|U|^2\right). \tag{3.36}$$

where $C = C(\varphi^{(0)}, \alpha, \sigma) \in (0, \infty)$*. Moreover,* (3.36) *can be strengthened to*

$$\int_{B_{1/2}(0)} \frac{\mathscr{G}(U,\varphi)^2}{r_\delta^{1-\sigma}} \leq C \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0)} |U|^2 \right),$$

where, as before, $C = C(\varphi^{(0)}, \alpha, \sigma) \in (0, \infty)$ *and* $r_\delta(X) = \max\{|x|, \delta\}$*, where* $X = (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Proof. Note that by Corollary 89 we have that for all $\rho \in (\delta, 1/4]$ the estimate

$$\rho^{-4+1+\sigma} \int_{B_\rho(0,z)} \mathscr{G}(U,\varphi)^2 \leq C \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0} |U|^2 \right)$$

holds. Indeed, recall that by hypothesis we have that for any $y \in B^2_{1/2}(0)$, there exists $(\xi, \eta) \in B_\delta(0,y)$ such that $\mathscr{N}((\xi,\eta);U) \geq \alpha$, which justifies the application of Corollary 89. Note now that we can cover the set $B_{1/2}(0) \cap (B^2_{\rho/2}(0) \times \mathbb{R}^2)$ by a finite collection of ball of the form $B_\rho(0,z_j)$, with $z_j \in B^2_{1/2}(0)$. Furthermore, by a simple covering argument, the bound on the number of such balls is given by $N \leq C\rho^{-2}$. Summing the preceding inequality over these balls we infer

$$\sum_{j=1}^N \rho^{-3+\sigma} \int_{B_\rho(0,z_j)} \mathscr{G}(U,\varphi)^2 \leq C\rho^{-2} \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0} |U|^2 \right),$$

and consequently

$$\frac{1}{\rho^{1-\sigma}} \int_{B_{1/2}(0) \cap (B^2_{\rho/2}(0) \times \mathbb{R}^2)} \mathscr{G}(U,\varphi)^2 \leq C \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0} |U|^2 \right), \tag{3.37}$$

as required. To conclude the proof of the theorem, we need to integrate this last inequality with respect to $\rho \in (\delta, 1/4]$, after replacing σ by $\sigma/2$ and having multiplied both sides by $\rho^{-1+\sigma/2}$:

$$\begin{aligned} &\int_\delta^{1/4} \frac{1}{\rho^{2-\sigma}} \int_{B_{1/2}(0) \cap (B^2_{\rho/2}(0) \times \mathbb{R}^2)} \mathscr{G}(U,\varphi)^2 \\ &\qquad \leq C \int_\delta^{1/4} \rho^{-1+\sigma/2} \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \int_{B_1(0} |U|^2 \right). \end{aligned} \tag{3.38}$$

Indeed, the right-hand side of this last inequality is trivially bounded by $C \int_{B_1(0)} \mathscr{G}(U,\varphi)$, with C depending on σ. On the other hand, the left-hand side can be compared with the term $\int_{B_{1/2}(0)} \mathscr{G}(U,\varphi)^2 / r_\delta^{1-\sigma}$. Rewrite the integral on the left-hand side of (3.38) by introducing the characteristic function of the sets over which we are integrating, apply then Fubini's theorem to change the order of integration and get the desired term $r_\delta(X) = \max\{|x|, \delta\}$. ☐

3.7 Asymptotic decay for blow-ups

This section aims to prove a classification result for homogeneous of degree α two-valued maps, harmonic away from the spine. This is a crucial ingredient for the proof of the main dichotomy result, Proposition 81, proven in the next section. Indeed, such maps arise as blow-ups of sequences of $\mathbb{Z}_2$-harmonic spinors. Moreover, we prove L^2-decay estimates under certain integral hypotheses. We follow the classification result of Krummel and Wickramasekera, see [89, Section 7], which in turn is based on Simon's one [122].

Proposition 92. *Let $\sigma \in (0,1)$ and $\varphi^{(0)}$ the usual $\mathbb{Z}_2$-harmonic spinor, symmetric, homogeneous of degree $\alpha \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$. Let now w belong to $L^2(\operatorname{graph}\varphi^{(0)}|_{B_1(0)\setminus(\{0\}\times\mathbb{R}^{n-2})}, \mathscr{V})$ and $L^2(\operatorname{graph}\varphi^{(0)}|_{B_1(0)}, \mathscr{V}, p^*\mathscr{L}^n)$ be such that*

$$w(X,\varphi^{(0)}(X)) = \{w(X,\varphi_1^{(0)}(X)), w(X,-\varphi_1^{(0)}(X))\},$$

is a homogeneous of degree α, harmonic, symmetric, two-valued function on $B_1(0)\setminus(\{0\}\times\mathbb{R}^{n-2})$. Recall that here $\varphi^{(0)} = \{\pm\varphi_1^{(0)}(X)\}$. Moreover, the map w satisfies

$$\int_{B_1(0)} \frac{1}{r^{2+2\alpha-\sigma}} |w(re^{i\theta},y,\varphi^{(0)(re^{i\theta},y)}) - \kappa(re^{i\theta},y,\varphi^{(0)}(re^{i\theta},y))|^2 < \infty, \tag{3.39}$$

for some map κ taking the form

$$\kappa(re^{i\theta},y,\varphi^{(0)}(re^{i\theta},y)) = \kappa_1(r,y)D_1\varphi^{(0)}(re^{i\theta},y) + \kappa_2(r,y)D_2\varphi^{(0)}(re^{i\theta},y),$$

where $\kappa_1,\kappa_2 \in L^\infty(B_1(0),\mathscr{V})$. If $\alpha = 1/2$, we further assume that

$$\lim_{r\to 0^+} \frac{\partial^2}{\partial r \partial y_p} \int_{S^1} rw(re^{i\theta},y,\varphi^{(0)}(re^{i\theta},y))D_i\varphi^{(0)}(re^{i\theta},y)\,d\theta = 0, \tag{3.40}$$

for $i = 1,2$. Then, the following expansion holds

$$\begin{aligned} & w(re^{i\theta},y,\varphi^{(0)}(re^{i\theta},y)) \\ &\quad = a_0 r^\alpha \cos(\alpha\theta) + b_0 r^\alpha \sin(\alpha\theta) + \sum_{j=1}^{2}\left(a_j D_1\varphi^{(0)}(re^{i\theta},y)y_j + b_j D_2\varphi^{(0)}(re^{i\theta},y)y_j\right), \end{aligned}$$

for certain coefficients a_0, b_b, a_j, b_j, with $j = 1,2,\dots,n-2$.

Proof. Let $\phi_1,\phi_2,\dots$ be 4π-periodic functions on $\mathbb{R}$ whose restriction to $[0,4\pi]$ is an L^2-orthonormal basis satisfying $\phi_k''(\theta) + \lambda_k\phi_k = 0$, where $\lambda_1 \le \lambda_2 \le \dots$. Denote by l_0 the smallest positive integer such that $\lambda_{l_0} = (\alpha-1)^2$ and assume, without loss of generality, that the span of ϕ_{l_0}, respectively ϕ_{l_0+1}, equals $D_1\varphi^{(0)}(e^{i\theta})$, respectively $D_2\varphi^{(0)}(e^{i\theta})$. Decompose now w in Fourier coeffients with respect to θ:

$$w_l(r,y) = \frac{1}{4\pi}\int_0^{4\pi} w(re^{i\theta},y,\varphi^{(0)}(re^{i\theta}))\phi_l(\theta)\,d\theta.$$

The hypothesis on w guarantee that each w_l is smooth, homogeneous of degree α single-valued, satisfying

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_l}{\partial r}\right) - \frac{\lambda_l}{r^2}w_l + \Delta_y w_l = 0. \tag{3.41}$$

Fix $l \in \mathbb{N}$ for the sequel of this proof. From (3.39) we deduce that if ϕ_l is in the span of $D_1\varphi^{(0)}(e^{i\theta})$ and $D_2\varphi^{(0)}$, then $\lambda_l = (\alpha-1)^2$ and we also have

$$\int_{B_1(0)} \frac{|w_l - \kappa_l(r,y)|^2}{r^{2+2\alpha-\sigma}} < \infty, \tag{3.42}$$

where κ_l is the L^∞-function of (r,y) given by the l-th Fourier coefficient of κ. Note that if ϕ_l is not in the span $D_1\varphi^{(0)}(e^{i\theta})$ and $D_2\varphi^{(0)}(e^{i\theta})$, then

$$\int_{B_1(0)} \frac{|w_l|^2}{r^{2+2\alpha-\sigma}} < \infty. \tag{3.43}$$

Write now $w_l(r,y) = r^\alpha \psi(y/r)$, where $\psi(z) = w_l(1,z)$ is smooth, so that (3.41) becomes

$$\Delta_z \psi + z_i z_j D_{z_i z_j} \psi - (2\alpha - 1) z_i D_{z_i} \psi + (\alpha^2 - \lambda_l)\psi = 0, \tag{3.44}$$

which gives the following equation

$$\Delta_z f + z_i z_j D_{z_i z_j} f - (2\alpha - 3) z_i D_{z_i} f + ((\alpha - 1) - \lambda_l) f = 0, \tag{3.45}$$

for $f := D_p \psi$. Fourier expanding $\psi(r\omega) = \sum_{k=0}^{\infty} \gamma_k(r)\chi_k(\omega)$, where $\omega \in S^1$, and χ_k are eigenfunctions of the Laplacian on the sphere S^1, i.e. $\Delta_{S^1}\chi_k + \mu_k(\mu_k + n - 4)\chi_k = 0$ with $\mu_k \geq 0$. From (3.44) we deduce an ordinary differential equation reading

$$(1+r^2)\frac{\partial^2 \gamma_k}{\partial r^2} + \left(\frac{1}{r} - (2\alpha - 1)r\right)\frac{\partial \gamma_k}{\partial r} - \mu_k^2 \frac{\gamma_k}{r^2} + (\alpha^2 - \lambda_l)\gamma_k = 0. \tag{3.46}$$

As ψ is bounded in a neighborhood of 0, we have the bound

$$\sup_{\rho/2 < |z| < \rho} |D\psi| \leq \frac{C}{\rho} \sup_{\rho/4 < |z| < 2\rho} |\psi|.$$

Depending on the value of λ_l and α, we have that the following versions of (3.42) and (3.43) hold

$$\int_{S^1}\int_1^\infty r^{1-\sigma}\left|\frac{\psi(r\omega)}{r} - \kappa(r\omega)\right|^2 dr d\omega \quad \text{and} \quad \int_{S^1}\int_1^\infty \frac{1}{r^{1+\sigma}}|\psi(r\omega)|^2 dr d\omega < \infty. \tag{3.47}$$

We claim that if ϕ_l is in the span of $D_1\varphi^{(0)}(e^{i\theta})$ and $D_2\varphi^{(0)}(e^{i\theta})$, then, if $\alpha = 1/2$, then $\psi(z) = a + \langle b, z\rangle$, for $z \in \mathbb{R}^2$ and $a \in \mathbb{R}$, whereas if $\alpha \geq 1$ then $\psi(z) = \langle b, z\rangle$, again for $b \in \mathbb{R}^2$. If $\alpha \geq 1$ and $\lambda_l = \alpha^2$, then $\psi(z) = ar$, for $a \in \mathbb{R}$. Otherwise, $\psi \equiv 0$ in all the other cases. Once we establish this claim the proof will follow. We divide its proof in the following three cases, which we analyse separately.

(i) Assume $\alpha = 1/2$ and ϕ_l is in the span of $\{D_1\varphi^{(0)}(e^{i\theta}), D_2\varphi^{(0)}(e^{i\theta})\}$. By (3.41) we know that $r^{1/2}w_l(r,y)$ is harmonic on $(0,\infty)\times\mathbb{R}^2$. Define then the even extension in the r variable of $\sqrt{r}D_{y_p}w_l(r,y)$ for every $p = 1,2$. Denote such an extension $\widehat{w}_p(r,y)$ and note that it is simply given by $|r|^{1/2}D_{y_p}w_l(|r|,y)$ for $r \in \mathbb{R}$ and $y \in \mathbb{R}^2$. The hypothesis in (3.40) implies that each $\widehat{w}_p$ is harmonic on $\mathbb{R}^3$. Besides the $\widehat{w}_p$ are homogeneous of degree zero and as such they are all constant. It follows that $r^{1/2}w_l(r,y) = ar + \langle b, y\rangle$, for $a \in \mathbb{R}$ and $b \in \mathbb{R}^3$, from which we deduce $psi(z) = a + \langle b, z\rangle$ as required.

(ii) Assume either $\alpha = 1/2$ and ϕ_l is not in the span of $\{D_1\varphi^{(0)}(e^{i\theta}), D_2\varphi^{(0)}(e^{i\theta})\}$, or $\alpha \geq 1$ and $\lambda_l \geq (l-1)^2$. In this case, write (3.7) as

$$\frac{\partial}{\partial r}\left(g\frac{\partial v}{\partial r}\right) + \frac{g}{r^2(1+r^2)}\Delta_{S^1} v + \frac{((\alpha-1)^2 - \lambda_l)}{1+r^2} v = 0,$$

where $g := r^{n-3}(1+r^2)^{-(2\alpha+n-6)}$. As the term $r^{-1}\nabla_{S^1} v$ is bound as $r \to 0$, we can write the above equation in its weak form

$$\int_0^\infty \int_{S^1}\left(g\frac{\partial v}{\partial r}\frac{\partial \zeta}{\partial r} + \frac{g}{r^2(1+r^2)}\nabla^{S^1} v \cdot \nabla^{S^1}\zeta - \frac{((\alpha-1)^2 - \lambda_l)g}{1+r^2} v\zeta\right) d\omega dr = 0,$$

for all test functions $\zeta \in C^1_c(\mathbb{R}^2)$. Replacing ζ in the weak form by $v\zeta^2$, where ζ is a smooth readial function, constant near the origin and vanishing for r large we infer

$$\int_0^\infty \int_{S^1} \left(g \left| \frac{\partial v}{\partial r} \right|^2 + \frac{g}{r^2(1+r^2)} |\nabla^{S^1} v|^2 + \frac{(\lambda_l - (\alpha-1)^2)g}{1+r^2} v^2 \right) \zeta^2 \, d\omega dr \tag{3.48}$$

$$\leq C \int_0^\infty g|\zeta'(r)|^2 \int_{S^1} v^2 \, d\omega dr, \tag{3.49}$$

after applying Cauchy's inequality. Now, given $\rho \in (0,\infty)$ consider the logarithmic cutoff function ζ given by 1 if $r \leq \rho$, by $-\log(r/\rho^2)/\log(\rho)$ for $r \in (\rho, \rho^2)$ and by 0 if $r \geq \rho^2$. We have two subcases. First, if $\alpha \geq 1$, $\lambda_l = (\alpha-1)^2$, and ϕ_l is in the span of $D_1\varphi^{(0)}(e^{i\theta})$ and $D_2\varphi^{(0)}(e^{i\theta})$, then

$$\begin{aligned} &\int_0^\rho \int_{S^1} \left(g \left| \frac{\partial v}{\partial r} \right|^2 + \frac{g}{r^2(1+r^2)} |\nabla^{S^1} v|^2 \right) d\omega dr \\ &\leq \frac{C}{\log^2(\rho)} \int_\rho^{\rho^2} r^{1-2\alpha} \int_{S^1} v^2 \, d\omega dr \\ &\leq \frac{C}{\log^2(\rho)} \int_{\rho/2}^{2\rho^2} r^{-1-2\alpha} \int_{S^1} \psi^2 \, d\omega dr \\ &\leq \frac{C}{\log^2(\rho)} \int_{\rho/2}^{2\rho^2} r^{1-2\alpha} \int_{S^1} \left| \frac{\psi(r\omega)}{r} - \kappa(r\omega) \right|^2 d\omega dr + \frac{C}{\log(\rho)} \rho^{2-2\alpha} \sup |\kappa|^2 \\ &\leq \frac{C}{\log(\rho)} \left(\frac{1}{\log(\rho)} \rho^{-2\alpha+\sigma} + \rho^{2-2\alpha} \sup |\kappa|^2 \right), \end{aligned} \tag{3.50}$$

that goes to zero as $\rho \to 0$, and where we applied the first growth condition in (3.47). Therefore, v is given by a constant and consequently, $\psi(z) = a + \langle b, z \rangle$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}^2$. Alternatively, if either ϕ_l is not in the span of $D_1\varphi^{(0)}(e^{i\theta})$ and $D_2\varphi^{(0)}(e^{i\theta})$, or $\lambda_l > (\alpha-1)^2$ then estimate the left-hand side of (3.48) as we did (3.50) with the sole exception of applying the second growth restriction in (3.47) instead of the first one, which amount to simplyfying the computation. Thus, $v \equiv 0$, so that $\psi(z) = a$ for some $a \in \mathbb{R}$. By (3.44) we deduce that either $a = 0$ or $\lambda_l = \alpha^2$.

(iii) Assume $\alpha \geq 1$ and $\lambda_l < (\alpha-1)^2$. The reader can check that series expansions near $+\infty$ and $-\infty$ give that a non-zero solution γ_k of (3.46) satisfies

$$\liminf_{r \to \infty} r^{-1-2\alpha+2\sqrt{\lambda_l}} \int_{r/2<s<r} |\gamma_k(s)|^2 \, ds > 0,$$

contradicting the second growth condition in (3.47), implying $\gamma_k \equiv 0$ and, consequently, $\psi(z) = 0$.

□

In the next proposition we will denote by $\mathscr{L}$, and by $\mathscr{L}^\perp$ its L^2-orthogonal complement, the subspace of $L^2(\operatorname{graph} \varphi^{(0)}|_{B_1(0)}, \mathscr{V}, p^*\mathscr{L}^4)$ spanned by the following set of functions

$$\left\{ a_0 r^\alpha \cos(\alpha\theta), b_0 r^\alpha \sin(\alpha\theta), D_i\varphi^{(0)}(re^{i\theta}, y) y_j;\ i \in \{1,2\}, j \in \{1,2\} \right\}.$$

Proposition 93. *Let $\varphi^{(0)}$ the usual $\mathbb{Z}_2$-harmonic spinor, symmetric, homogeneous of degree $\alpha \in \mathbb{N}\cup\frac{1}{2}\mathbb{N}$. Given $\sigma\in(0,1)$ and $\beta_1,\beta_2\in(0,1)$, there exists a constant $\beta=\beta(\alpha,\varphi^{(0)},\sigma,\beta_1,\beta_2)\in(0,\infty)$ such that if $w\in\mathscr{L}^\perp$ is a harmonic two-valued symmetric function on $B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$ satisfying*

$$\int_{B_{1/4}((0,z))}\frac{|w(X,\varphi^{(0)}(X))-\lambda_1(z)D_1\varphi^{(0)}(X)-\lambda_2(X)D_2\varphi^{(0)}(X)|^2}{|X-(0,z)|^{4+2\alpha-\sigma}}\le\beta_1\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2, \tag{3.51}$$

for each $z\in B^2_{1/2}(0)$ and some bounded functions $\lambda_1,\lambda_2\colon B^2_{1/2}(0)\to\mathbb{R}$ with

$$\sup_{B^2_{1/2}(0)}\left(|\lambda_1(z)|^2+|\lambda_2(z)|^2\right)\le\beta_2\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2, \tag{3.52}$$

and if in case $\alpha=1/2$, the limit in (3.40) holds, then

$$\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2\le\beta\int_{B_1(0)\setminus B_{1/4}(0)}\left|\frac{\partial}{\partial R}\left(w(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2.$$

Proof. The proposition will follow once we establish that w satisfies

$$\int_{B_{1/2}(0)}\frac{|w(X,\varphi^{(0)}(X))-\kappa(X,\varphi^{(0)})|^2}{r^{2+2\alpha-\sigma}}\le C(\beta_1)\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2, \tag{3.53}$$

where

$$\kappa(re^{i\theta},y,\varphi^{(0)}(re^{i\theta},y))=\kappa_1(r,y)D_1\varphi^{(0)}(re^{i\theta},y)+\kappa_2(r,y)D_2\varphi^{(0)}(re^{i\theta},y), \tag{3.54}$$

as well as

$$|\kappa_1(r,y)|^2+|\kappa_2(r,y)|^2\le\beta_2\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2, \tag{3.55}$$

and the λ_j's satisfy the estimate

$$\sup_{z_1,z_2\in B^2_{1/2}(0),z_1\neq z_2}\frac{|\lambda_j(z_1)-\lambda_j(z_2)|^2}{|z_1-z_2|^{2-\sigma}}\le C(\beta_1,\beta_2,\sigma)\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2. \tag{3.56}$$

Indeed, assume the proposition false for the sake of contradiction. Thus, for every $k\in\mathbb{N}$ we would have $w_k\in\mathscr{L}^\perp$ and $\lambda_{1,k},\lambda_{2,k}\colon B^3_{1/2}(0)\to\mathbb{R}$ such that

$$\int_{B_1(0)\setminus B_{1/4}(0)}\left|\frac{\partial}{\partial R}\left(w_k(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2<\frac{1}{k}, \tag{3.57}$$

after normalising by $\|w_k(X,\varphi^{(0)}(X))\|^2_{L^2(B_1(0))}$, so that the right-hand side simply becomes $1/k$. Note that w_k satisfies the hypothesis of the lemma, as well as $\lambda_{1,k}$ and $\lambda_{2,k}$. Thanks to (3.53) and (3.54), estimate

$$\int_{B_{1/2}(0)\cap\{|x|\le\delta\}}|w_k(X,\varphi^{(0)}(X))|^2\le\delta^{2+2\alpha-\sigma}\left(C+\int_{B_{1/2}(0)\cap\{|x|\le\delta\}}\frac{1}{r^{2+2\alpha-\sigma}}|\kappa_1(X,\varphi^{(0)}(X))|^2\right) \tag{3.58}$$

$$\le C(\beta_1)\delta^2, \tag{3.59}$$

for every $\delta \in (0,1/4)$. Thus, we have that $\sup_k \|w_k\|_{C^3(K)}$ is bounded independently of k for every compact set $K \subset B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$. Arzelà-Ascoli implies the existence of a two-valued w in $C^2(B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$ such that $w_k \to w$, up to a not relabelled subsequence, in $C^2(K)$ on compact subsets $K \subset B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$ and satisfying $\mathscr{D}w(X,\varphi^{(0)}(X)) = 0$ in $B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$. The local L^2-bound in (3.58) implies L^2-convergence in $B_{1/2}(0)$ as well. Note now that

$$\sup_{B^2_{1/2}(0)} |\lambda_{k,j}(z)|^2 + \sup_{z_1,z_2\in B^2_{1/2}(0), z_1\neq z_2} \frac{|\lambda_{k,j}(z_1)-\lambda_{k,j}(z_2)|^2}{|z_1-z_2|^{2-\sigma}} \leq \beta_2 + C(\beta_1,\beta_2,\sigma) \leq C(\beta_1,\beta_2,\sigma),$$

holds for $j=1,2$ and with C independent of k by hypothesis combined with (3.56). Therefore, it follows that w satisfies (3.51) for each $z \in B^2_{1/2}(0)$ for some bounded functions λ_1 and λ_2 as in (3.52). Note now that (3.57) implies that w is α-homogeneous on $(B_1(0)\setminus B_{1/4}(0))\setminus(\{0\}\times\mathbb{R}^2)$, so that, since $\mathscr{D}w(X,\varphi^{(0)}(X)) = 0$ by the Unique Continuation Principle, cf. Theorem 55, we conclude that w is α-homogeneous everywhere away from the spine. Proposition (92) immediately implies that $w \in \mathscr{L}$. Thus, by L^2-orthogonality of w_k and w we have

$$\int_{B_\sigma(0)\setminus(B^2_\delta(0)\times\mathbb{R}^2)} |w_k(X,\varphi^{(0)}(X))| \leq \|w\|_{L^2(B_\sigma(0)\cap(B^2_\delta\times\mathbb{R}^2))} + \|w\|_{L^2(B_1(0)\setminus B_\sigma(0))},$$

for each $\sigma \in (1/2,1)$ and each $\delta \in (0,1/2)$. We conclude that $w \equiv 0$ by letting $k \to \infty$, $\sigma \to 1$ and $\delta \to 0$ in this order. By the fundamental theorem of calculus, for $r,s \in (1/4,1)$ and $\omega \in S^3$ we have

$$|r^{-\alpha}w_k(r\omega,\varphi^{(0)}(r\omega)) - s^{-\alpha}w_k(s\omega,\varphi^{(0)}(s\omega))| \leq \int_{1/4}^1 \left|\partial_R(w_k(R\omega,\varphi^{(0)}(R\omega))/R^\alpha)\right|$$

and, by triangle inequality and Cauchy-Schwarz

$$|w_k(r\omega,\varphi^{(0)}(r\omega))|^2 \leq C|w_k(s\omega,\varphi^{(0)}(s\omega))| + C\int_{1/4}^1 \left|\partial_R(w_k(R\omega,\varphi^{(0)}(R\omega))/R^\alpha)\right|^2,$$

where the constant depends on α. Multiply both sides of this inequality by r^3 and integrate over $\omega \in S^3$ and then over $r \in (1/4,1)$, then multiply the resulting expression by s^3 and repeat the same process albeit with integration over $s \in (1/4,1/2)$. We obtain

$$\begin{aligned}&\int_{B_1(0)\setminus B_{1/4}(0)} |w_k(X,\varphi^{(0)}(X))|^2\\ &\leq C\int_{B_{1/2}(0)\setminus B_{1/4}(0)} |w_k(X,\varphi^{(0)}(X))|^2 + C\int_{B_1(0)\setminus B_{1/2}(0)} \left|\partial_R(w_k(R\omega,\varphi^{(0)}(R\omega))/R^\alpha)\right|^2\end{aligned}$$

Complete then the integral on the left-hand side to infer

$$1 \leq C\int_{B_{1/2}(0)} |w_k(X,\varphi^{(0)}(X))|^2 + C\int_{B_1(0)\setminus B_{1/2}(0)} \left|\partial_R(w_k(R\omega,\varphi^{(0)}(R\omega))/R^\alpha)\right|^2,$$

yielding the desired contradiction as the right-hand side goes to zero as $k \to \infty$ by (3.57) and the fact that $w_k \to 0$ in $L^2(\operatorname{graph}\varphi^{(0)} \cap B_{1/2}(0))$.

Thus, we are left with proving (3.53), (3.54), (3.55) and (3.56) to conclude the proof. For the first one note that for each $z \in B^2_{1/2}(0)$ and $r \in (0,1/4)$ we have (3.51) with the domain of integration on the left-hand side replaced by $B_r((0,z))$, so that the factor $|X-(0,z)|^{4+2\alpha-\sigma}$ is replaced by $r^{4+2\alpha-\sigma}$. Now, for each (r,y) satisfying $r^2+|y|^2<1/2$, take $\kappa_1(r,y)$ and $\kappa_2(r,y)$ to realise the following infimum

$$\inf\int_0^{4\pi}|w(re^{i\theta},y,\varphi^{(0)}(e^{i\theta},y))-\mu_1D_1\varphi^{(0)}(re^{i\theta})-\mu_2D_2\varphi^{(0)}(re^{i\theta})|^2\,d\theta,$$

where the infimum is taken over all μ_1,μ_2 satisfying $|\mu_1|^2+|\mu_2|^2\le\beta_2\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2$, so that (3.55) is satisfied by definition. Considering then κ as in (3.54) it follows by definition that

$$\frac{1}{r^{4+2\alpha-\sigma}}\int_{B_r((0,z))}|w(X,\varphi^{(0)}(X))-\kappa(X,\varphi^{(0)}(X))|^2\le\beta_1\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2,$$

holds for each $z\in B^2_{1/2}(0)$ and $r\in(0,1/4)$. Consequently, a covering argument implies

$$\frac{1}{r^{2+2\alpha-\sigma}}\int_{B_{1/2}(0)\cap\{s<r/2\}}|w(X,\varphi^{(0)}(X))-\kappa(X,\varphi^{(0)}(X))|^2\le\beta_1\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2.$$

Replace σ by $\sigma/2$ and multiply both sides of this inequality by $r^{-1+\sigma/2}$. Integrating over $r\in(0,1/4)$ yields

$$\int_0^{1/4}\frac{1}{r^{3+2\alpha-\sigma}}\int_{B_{1/2}(0)\cap\{s<r/2\}}|w(X,\varphi^{(0)}(X))-\kappa(X,\varphi^{(0)}(X))|^2$$
$$\le\int_0^{1/4}\frac{1}{r^{1-\sigma/2}}\beta_1\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2\le C(\beta_1)\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2,$$

from which (3.53) follows. As already mentioned (3.54) and (3.55) hold by definition. In order to see (3.56) consider two points $z_1,z_2\in B^3_{1/2}(0)$ with $0<|z_1-z_2|<1/8$. As we have earlier to prove (3.53) take (3.51) with the domain of integration on the left-hand side replaced by $B_r((0,z))$, so that the factor $|X-(0,z)|^{4+2\alpha-\sigma}$ is replaced by $r^{4+2\alpha-\sigma}$ and infer for $r=2|z_1-z_2|$ the following

$$\frac{1}{|z_1-z_2|^{4+2\alpha-\sigma}}\int_{B_{|z_1-z_2|}(0,z_1)}|(\lambda_1(z_1)-\lambda_2(z_2))D_1\varphi^{(0)}(X)+(\lambda_2(z_1)-\lambda_2(z_2))D_2\varphi^{(0)}(X)|^2$$
$$\le C\int_{B_1(0)}|w(X,\varphi^{(0)}(X))|^2,$$

which, combined with the $L^2(S^1)$-orthogonality of $D_1\varphi^{(0)}|_{r=1}$ and $D_1\varphi^{(0)}|_{r=1}$, together with (3.52), imply the desired result, thus finishing the proof of (3.56) and with it the proof of the proposition. □

Before stating the last result of this section we need to introduce some notation. Given $w\in L^2(\operatorname{graph}\varphi^{(0)}|_{B_1(0)},\mathscr{V},p^*\mathscr{L}^4)$ and $s\in(0,1)$, we let $\psi_s\in\mathscr{L}$ be the function achieving the following infimum

$$\inf_{\psi\in\mathscr{L}}\int_{B_s(0)}|w(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))|^2.$$

Furthermore, for the same given W we let $w_s=w-\psi_s$, so that w_s is L^2-orthogonal to $\mathscr{L}$. As we are simply taking the orthogonal projection on a Hilbert space, existence of ψ_s is guaranteed by standard theory. The following is the last proposition of the section, whose proof follows a straightforward hole-filling argument.

Proposition 94. *Let $\varphi^{(0)}$ the usual $\mathbb{Z}_2$-harmonic spinor, symmetric, homogeneous of degree $\alpha \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$. Let $\theta \in (0,1/8), \sigma \in (0,1)$ and $\beta_1, \beta_2 \in (0,\infty)$.*
If w belongs to both $C^2(\operatorname{graph}\varphi^{(0)}|_{B_1(0)\setminus(\{0\}\times\mathbb{R}^2)}, \mathscr{V})$ and $L^2(\operatorname{graph}\varphi^{(0)}|_{B_1(0)}, \mathscr{V})$ and it is a symmetric, two-valued harmonic spinor on $B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$ and for each $\rho \in [\theta, 1/4]$ and $z \in B^2_{\rho/2}(0)$ additionally satisfying

$$\int_{B_{\rho/4}((0,z))} \frac{|w_\rho(X,\varphi^{(0)}(X)) - \rho\lambda_{1,\rho}(z)D_1\varphi^{(0)}(X) - \rho\lambda_{2,\rho}(X)D_2\varphi^{(0)}(X)|^2}{|X-(0,z)|^{4+2\alpha-\sigma}} \leq \beta_1 \int_{B_\rho(0)} |w_\rho(X,\varphi^{(0)}(X))|^2,$$

for some $\lambda_{1,\rho}, \lambda_{2,\rho} \in L^\infty(B^2_{\rho/2}(0);\mathscr{V})$, with

$$|\lambda_{1,\rho}(z)|^2 + |\lambda_{2,\rho}(z)|^2 \leq \beta_2 \rho^{-4-2\alpha} \int_{B_\rho(0)} |w_\rho(X,\varphi^{(0)}(X))|^2,$$

and

$$\int_{B_{\rho/4}(0)} R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{w(X,\varphi^{(0)}(X))}{R^\alpha}\right)\right|^2 = \int_{B_{\rho/4}(0)} R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{w_\rho(X,\varphi^{(0)}(X))}{R^\alpha}\right)\right|^2 \leq \frac{\beta_2}{\rho^{4+2\alpha}}\int_{B_\rho(0)} |w_\rho(X,\varphi^{(0)}(X))|^2.$$

Moreover, we require (3.40) to hold in case $\alpha = 1/2$. Then,

$$\theta^{-4-2\alpha}\int_{B_\theta(0)} |w_\theta(X,\varphi^{(0)}(X))|^2 \leq C\theta^{2\mu}\int_{B_1(0)} |w_1(X,\varphi^{(0)}(X))|^2,$$

for constants $\mu \in (0,1)$ and $C > 0$, depending on $\alpha, \varphi^{(0)}, \sigma, \beta_1, \beta_2$.

Remark 95. Note that both C and μ do not depend on θ.

Proof. Apply Proposition 93 to infer

$$\frac{1}{r^{4+2\alpha}}\int_{B_r(0)} |w(X,\varphi^{(0)}(X))|^2 \leq \beta \int_{B_r(0)\setminus B_{r/4}(0)} R^2\left|\frac{\partial}{\partial R}\left(w(xX\varphi^{(0)}(X))/R^\alpha\right)\right|^2,$$

for β depending on $\sigma, \alpha, \beta_1, \beta_2$, from which one can deduce

$$\int_{B_{r/4}(0)}\left|\frac{\partial}{\partial R}\left(w(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2 \leq \beta\beta_2\int_{B_r(0)\setminus B_{r/4}(0)}\left|\frac{\partial}{\partial R}\left(w(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2.$$

Thus, by adding on both sides $\beta\beta_2$ multiplied by the integral on the left-hand side we can complete the integral on the right-hand side from an annulus to a full ball

$$\int_{B_{r/4}(0)}\left|\frac{\partial}{\partial R}\left(w(X\varphi^{(0)}(X))/R^\alpha\right)\right|^2 \leq \frac{\beta\beta_2}{1+\beta\beta_2}\int_{B_r(0)\setminus B_{r/4}(0)}\left|\frac{\partial}{\partial R}\left(w(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2,$$

denote for the sequel $\gamma := \beta\beta_2/(1+\beta\beta_2)$. Fix now $\theta \in (0,1/8)$ as in the statement and iterate this last inequality with $r = 4^{-j}$ for $j = 0,1,\ldots,k$, where k is chosen so that $4^{-k-1} < \theta \leq 4^{-k}$ we infer

$$\int_{B_\theta(0)}\left|\frac{\partial}{\partial R}\left(w(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2 \leq \theta^{2\mu}\int_{B_{1/4}(0)}\left|\frac{\partial}{\partial R}\left(w(X,\varphi^{(0)}(X))/R^\alpha\right)\right|^2,$$

where $\mu := -\log(\gamma)/\log(4) \in (0,1)$, whence finishing the proof of the proposition. □

3.8 Proof of the Main Results

We shall prove in this section the main results appearing in the Introduction. We start with the proof of the dichotomy in Proposition 81, from which all the other results will follow by iteration. For the reader's convenience we restate the Proposition.

Proposition 81. *Let $\theta \in (0,1/4)$ and let $\varphi^{(0)}$ be a homogeneous $\mathbb{Z}_2$-harmonic spinor defined on $\mathbb{R}^4$. Then, there are $\delta_0, \varepsilon_0 \in (0,1/4)$ depending only on the dimension of the manifold, the rank of the bundle, the degree of homogeneity of $\varphi^{(0)}$, the spinor $\varphi^{(0)}$ itself and θ, such that if $\varphi \in \widetilde{\Phi}_\varepsilon(\varphi^{(0)})$ and $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, then*

(i) Either,

$$B_{\delta_0}(0,y_0) = \{x \in B_1(0) \cap Z;\ \mathscr{N}(x;U) \geq \alpha\} = \emptyset,$$

for some $y_0 \in B^2_{1/2}(0)$;

(ii) Or, there is a new $\tilde{\varphi} \in \widetilde{\Phi}_{\gamma\varepsilon}(\varphi^{(0)})$ such that

$$\theta^{-4-2\alpha} \int_{B_\theta(0)} \mathscr{G}(U,\tilde{\varphi})^2 \leq C\theta^{2\mu} \left(\int_{B_1(0)} \mathscr{G}(U,\varphi)^2 + \|U\|_{L^2(B_1(0))} \right),$$

where $\gamma \in [1,\infty)$, $\mu \in (0,1)$ and $C \in (0,\infty)$ are constants depending only on the dimension of the manifold, the rank of the bundle, $\varphi^{(0)}$ and α.

Proof of Proposition 81. Fix $\theta \in (0,1/4)$ as in the statement and let $\varphi^{(0)}$ be the usual harmonic $\mathbb{Z}_2$-spinor, homogeneous of degree α. The proof is divided in five steps.

Step 1. In this first step we argue by contradiction and we establish some preliminary results thanks to the estimates from the previous section. Thus, arguing by contradiction, assume there exist $0 < \varepsilon_j \leq \delta_j$, with $\delta_j \to 0$ and harmonic $\mathbb{Z}_2$-spinor $U_j \in \mathscr{F}_{\varepsilon_j}(\varphi^{(0)})$ such that the first point of the lemma does not hold, with δ_j, respectively U_j, replacing δ_0, respectively U, meaning that for every $y_0 \in B^2_{1/2}(0)$ we have

$$B_{\delta_k}(0,y_0) \cap \{X \in B_1(0) \cap \Sigma_{U_j};\ \mathscr{N}(X;U_j) \geq \alpha\} \neq \emptyset.$$

We now wish to prove the second alternative of the statement, namely for $\gamma \in [1,\infty)$, $\mu \in (0,1)$ and a constant C (with the same dependencies as the ones in the statement), and for infinitely many j, there exists $\widetilde{\varphi}_j \in \widetilde{\Phi}_{\gamma\varepsilon_j}(\varphi^{(0)})$ such that

$$\theta^{-4-2\alpha} \int_{B_\theta(0)} \mathscr{G}(U_j,\widetilde{\varphi}_j)^2 \leq C\theta^{2\mu} \left(\int_{B_1(0)} \mathscr{G}(U_j,\varphi_j)^2 + \|U_j\|_{L^2(B_1(0))} \right).$$

Arbitrariness of the sequences ε_j, δ_j and U_j will allow to conclude. Let $\beta = \beta_0(\varphi^{(0)})$ be as in Theorem 91 and Proposition 83. Consider then a sequence $\tau_j \to 0^+$, whose rate of convergence will be given later, such that Proposition 83 and Theorem 85 hold with the following choice of parameters

$$\gamma = 3/4, \quad \tau = \tau_j, \quad \beta = \beta_0, \quad \sigma = 1/2, \quad U = U_j, \quad \text{and} \quad \varphi = \varphi'_j,$$

where $\varphi'_j := \varphi_j(e^{-A_j}\cdot) \in \Phi_{\varepsilon_j}(\varphi^{(0)})$ for some matrix $A_j \in \mathscr{S}$, with $|A_j| \leq \varepsilon$. Proposition 83 then gives a sequence of open subsets O_j of $B_1(0) \setminus (\{0\} \times \mathbb{R}^4)$ and C^2-maps V_j defined on graph $\varphi'_j\big|_{O_j}$ such that $\{(x,y) \in B_{3/4}(0);\ |x| \geq \tau_j\} \subset O_j$, and

$$U_j(e^{-A_j}X) = \left\{\varphi'_{j,1}(X) + V_j(X,\varphi'_{j,1}(X)), -\varphi'_{j,1}(X) + V_j(X,-\varphi'_{j,1}(X))\right\},$$

for all points $X \in O_j$ and $V_j(X, \varphi_j'(X))$ is a two-valued solution of the Dirac equation that, locally away from the spine $\{0\} \times \mathbb{R}^2$, can be written as $\varphi_j' = \{\pm\varphi_{j,1}'\}$ for some single-valued solution of Dirac's equation $\varphi_{j,1}'$. We can now apply Theorems 85 and 91 for φ_j' and $\rho^{-\alpha}U_j(\rho e^{-A_j}X)$ to infer

$$\int_{B_{\rho/2}(0)} R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{U_j}{R^\alpha}\right)\right| \leq C\rho^{-4-2\alpha}\left(\int_{B_\rho(0)} \mathscr{G}(U_j,\varphi_j)^2 + \int_{B_\rho(0)} |U_j|^2\right), \tag{3.60}$$

and

$$\rho^{1/2}\int_{B_{\rho/2}(0)} \frac{\mathscr{G}(U_j,\varphi_j)^2}{r_\delta^{1/2}} \leq C\left(\int_{B_\rho(0)} \mathscr{G}(U_j,\varphi_j)^2 + \int_{B_\rho(0)} |U_j|^2\right), \tag{3.61}$$

where we require $\delta \geq 2\delta_j/\theta$. Both constants appearing in these two inequalities depend on α and $\varphi^{(0)}$. Moreover, for every $z \in B^{n-2}_{\rho/2}(0)$, there exists $Z_j = (\xi_j, \zeta_j) \in B^n_{\delta_j}((0,z))$ with $\mathscr{N}_{U_j}(Z_j) \geq \alpha$, so that by applying Corollary 88 and Corollary 89 we obtain

$$\frac{1}{\rho^2}|\xi_j|^2 \leq C\rho^{-4-2\alpha}\left(\int_{B_\rho(0)} \mathscr{G}(U_j,\varphi_j)^2 + \int_{B_\rho(0)} |U_j|^2\right) \tag{3.62}$$

and

$$\rho^{4+2\alpha-\sigma}\int_{B_{\rho/2}((0,z))\cap\{|x|>\theta^{-1}\tau_j\rho\}} \frac{|V_j(X,\varphi_j(X)) - D_x\varphi_j(X)\cdot\xi_j|^2}{|X-\rho Z_j|^{n+2\alpha-\sigma}} \leq C\left(\int_{B_\rho(0)} \mathscr{G}(U_j,\varphi_j)^2 + \int_{B_\rho(0)} |U_j|^2\right). \tag{3.63}$$

Step 2. We now blow up the maps V_j and conclude some estimates and key identities. Let $W_j = V_j/E_j$, where

$$E_j := \left(\int_{B_1(0)} \mathscr{G}(U_j,\varphi_j)^2 + \|U_j\|_{L^2(B_1(0))}\right)^{1/2}$$

Note that we are not squaring the L^2-error term on purpose, as this will imply $\|U_j\|_{L^2(B_1(0))}/E_k \to 0$ as $k \to 0$. Furthermore, for each X of the form $(re^{i\theta}, y) \in B \subset O_k$, where B denotes a ball, we have that an equation of the form $\Delta W_j = \mathscr{R}V_j/E_j =: f_j$ is satisfied on B. Because $U_j \in \mathscr{F}_{\varepsilon_k}(\varphi^{(0)})$ we have $\|U_j\|_{L^2(B_1(0))} \to 0$ as $k \to 0$. Consequently, we have $|f_j| \leq CE_j$ in O_k by the estimates in Lemma 61 and for some constant C depending on $\varphi^{(0)}$. In particular, $\|f_j\|_{C^0(B)} \to 0$ as $k \to \infty$ for any closed ball B contained in $B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$, i.e. for any closed ball away from the spine. By the elliptic estimates appearing in the Appendix, for every compact set $K \subset B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$, we have

$$\|W_j(X,\varphi_j'(X))\|_{C^3(K)} \leq C(K),$$

for j large enough depending on K. Note now that we can write the two-valued map $\varphi_j'(X)$ in the following way

$$\varphi_j'(X) = \left\{\varphi_1^{(0)}(X) + \psi_j(X,\varphi_1^{(0)}(X)), -\varphi_1^{(0)}(X) + \psi_j(X,-\varphi_1^{(0)}(X)\right\},$$

where ψ_j is a C^2 map over graph $\varphi^{(0)}|_{B_1(0)\setminus(B^2_{\tau_j/2}\times\mathbb{R}^2)}$, satisfying the condition $|\psi_j| \leq C\varepsilon_j$. Thus, we have the existence of W a C^2-map defined on graph $\varphi^{(0)}|_{B_1(0)\setminus(\{0\}\times\mathbb{R}^2)}$, such that the associated two-valued map $W(X,\varphi^{(0)}(X))$ harmonic on $B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$. In addition, for each compact set $K \subset B_1(0)\setminus(\{0\}\times\mathbb{R}^2)$, the map $\psi_j + W_j$ converges to W in C^2 over the domain graph $\varphi^{(0)}|_K$. From (3.61) we further deduce the non-concentration of the excess inequality

$$\int_{B_{1/4}(0)\cap(B^2_\delta(0)\times\mathbb{R}^2)} \mathscr{G}(U_j,\varphi_j)^2 \le C\delta^{1/2}\left(\int_{B_{1/2}(0)} \mathscr{G}(U_j,\varphi_j)^2 + \int_{B_{1/2}(0)} |U_j|^2\right) \le C\delta^{1/2}E_j^2,$$

for $C\delta_j \le \delta < 1/2$, where C depends on α and $\varphi^{(0)}$. Thus, $W_j+\psi_j$ converges in L^2 as well, and we have full convergence of the energy

$$\lim_{j\to\infty}\int_{B_\rho(0)} \frac{\mathscr{G}(U_j,\varphi_j)^2}{E_j^2} = \int_{B_\rho(0)} |W(X,\varphi^{(0)}(X))|^2, \tag{3.64}$$

for all radii $\rho \in (0,1/2]$. Note that this implies in particular

$$\int_{B_1(0)} |W(X,\varphi^{(0)}(X))|^2 = 1.$$

Multiplying (3.60) by E_j^{-2} and applying the above identity we obtain

$$\int_{B_{\rho/2}(0)} R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{W(X,\varphi^{(0)}(X))}{R^\alpha}\right)\right|^2 \le C\rho^{-4-2\alpha}\int_{B_1(0)} |W(X,\varphi^{(0)}(X))|^2, \tag{3.65}$$

for each $\rho \in [\theta,1/4]$. We now claim that for each such ρ and $z \in B^2_{\rho/2}(0)$, there exists $\lambda_\rho(z) = (\lambda^1_\rho(z),\lambda^2_\rho(z)) \in \mathbb{R}^2$ such that the following two inequalities hold

$$\rho^{4+2\alpha}|\lambda_\rho(z)|^2 \le C\int_{B_\rho(0)} |W(X,\varphi^{(0)}(X))|^2 \tag{3.66}$$

and

$$\rho^{4+2\alpha-\sigma}\int_{B_{\rho/2}((0,z))} \frac{|W(X,\varphi^{(0)}(X)) - \rho D_x\varphi^{(0)}(X)\cdot\lambda_\rho(z)|^2}{|X-(0,z)|^{4+2\alpha-\sigma}} \le \widetilde{C}\int_{B_\rho(0)} |W(X,\varphi^{(0)}(X))|^2, \tag{3.67}$$

where both constants depend on α and $\varphi^{(0)}$, while $\widetilde{C}$ additionally depends on σ. To see why this is the case, consider $z \in B^2_{\rho/2}(0)$ and choose $Z_j = (\xi_j,\zeta_j) \in B_{\delta_j}((0,z))$ with $\mathscr{N}(Z_j;U_j) \ge \alpha$, so that, the inequality in (3.62), combined with the fact that up to subsequence we have $\xi_j/E_j \to \lambda_\rho(z) \in \mathbb{R}^2$, implies

$$\frac{1}{\rho^2}|\lambda(z)|^2 \le C\rho^{-4-2\alpha}\int_{B_\rho(0)} |W(X,\varphi^{(0)}(X))|^2,$$

form which we further infer, also thanks to (3.63),

$$\rho^{4+2\alpha-\sigma}\int_{B_{\rho/2}((0,z))} \frac{|W(X,\varphi^{(0)}(X)) - D_x\varphi^{(0)}(X)\cdot\lambda(z)|^2}{|X-(0,z)|^{4+2\alpha-\sigma}} \le \widetilde{C}\int_{B_\rho(0)} |W(X,\varphi^{(0)}(X))|^2.$$

Note that both inequalities are valid for $\rho \in (\theta,1/4]$. Setting now $\lambda_\rho(z) = \rho^{-1}\lambda(z)$ allows to conclude. From the triangle inequality we can also infer uniqueness of $\lambda(z)$ as above. Indeed, assuming that $\lambda(z)$ is not unique we would obtain

$$\int_{B_{1/8}((0,z))} \frac{D_x\varphi^{(0)}(X)\cdot(\lambda(z)-\overline{\lambda}(z))}{|X-(0,z)|^{4+2\alpha-\sigma}} < +\infty,$$

which, after the change of coordinates $X-(0,z)=(x,y-z)=(re^{i\theta},\omega)$, becomes

$$\int_{B^2_{1/4}(0)}\int_0^{1/4}\int_0^{4\pi}\frac{r^{2\alpha-1}|(D_1\varphi^{(0)}\big|_{r=1},D_2\varphi^{(0)}\big|_{r=1})\cdot(\lambda(z)-\overline{\lambda}(z))|^2}{(r^2+|y'|^2)^{(4+2\alpha-\sigma)/2}}d\theta drd\omega<+\infty,$$

which is possible only if $\lambda=\overline{\lambda}$ in view of L^2-orthogonality of $D_1\varphi^{(0)}\big|_{r=1}$ and $D_2\varphi^{(0)}\big|_{r=1}$. Note that the function λ_ρ is well-defined and, by (3.66) with $\rho=1$, it is bounded $|\lambda(z)|\le C$, where C depends on α and $\varphi^{(0)}$.

Step 3. In this step we exploit the classification of degree α-homogeneous blow-ups to refine the estimates in the previous step. Consider an element $\psi\in\mathscr{L}$ defined in the previous section, and assume that $\|\psi\|_{L^\infty(B_1(0))}$ is bounded. We shall determine later such bound. Write ψ as

$$\begin{aligned}\psi(re^{i\theta},y,\varphi^{(0)}(re^{i\theta},y))=&a_0r^\alpha\cos(\alpha\theta)+b_0r^\alpha\sin(\alpha\theta)\\&+\sum_{j=1}^4\left(a_jD_1\varphi^{(0)}(r,e^{i\theta,y})y_j+b_jD_2\varphi^{(0)}(r,e^{i\theta,y})y_j\right),\end{aligned}$$

for constant values a_0,b_0,a_j,b_j. The reader can check that orthogonality implies

$$|a_0|^2+|b_0|^2+\sum_{j=1}^2\left(|a_j|^2+|b_j|^2\right)\le C\|\psi(\cdot,\varphi^{(0)}(\cdot))\|^2_{L^2(B_1(0))} \tag{3.68}$$

Consider then a skew-symmetric matrix $G=(G_{ij})\in\mathscr{S}$ satisfying $G_{ij}=0$, for $1\le i,j\le 2$, and $G_{1j}=-G_{j1}=a_j$ and $G_{2j}=-G_{j2}=b_j$ and $G_{ij}=0$ for $3\le i,j\le 4$. For every j there exists $\widetilde{\varphi}_j\in\widetilde{\Phi}_{\gamma\varepsilon_j}(\varphi^{(0)})$ for $\gamma:=\gamma(\|\psi\|_{L^\infty},\varphi^{(0)})\ge 1$, one can take $\gamma=1+C\|\psi\|_{L^2(B_1(0))}$ in light of (3.68), such that

$$\widetilde{\varphi}_j(X)=\left\{\pm\left(\varphi^{(0)}(X)+\psi(X,\varphi^{(0)}(X))\right)+\mathscr{R}_j\right\},$$

on $B_1(0)\setminus(B_{\tau_j}(0)\times\mathbb{R}^2)$ and where $\mathscr{R}_j/E_j$ converges to zero uniformly as $j\to 0$ on $B_1(0)\setminus(B_\tau(0)\times\mathbb{R}^2)$, for each $\tau\in(0,1/2)$. (Note that more abstractly one could have considered $\gamma\ge 1$ and repeat the same proofs as in Step 1 and Step 2 replacing $\varphi_j\in\widetilde{\Phi}_{\varepsilon_j}(\varphi^{(0)})$ with $\widetilde{\varphi}_j\in\widetilde{\Phi}_{\gamma\varepsilon_j}(\varphi^{(0)})$. One would have obtained the exact same estimate, albeit with different V_j etc). One can in fact refine the above identy and prove that $\widetilde{\varphi}_j(e^{-E_jG}X)$ is actually equal to $\{(c_j+E_ja)z^\alpha\}$, again for all points away from the spine. It follows that we have

$$\widetilde{V}_j(X,\widetilde{\varphi}^{(0)}(X))=V_j(X,\widetilde{\varphi}^{(0)}(X))+\psi(X,\widetilde{\varphi}^{(0)}(X))-E_j\psi(X,\widetilde{\varphi}^{(0)}(X))+\mathscr{R}'_j,$$

again away from the spine, meaning on $B_1(0)\setminus(B_{\tau_j}(0)\times\mathbb{R}^2)$ and where we have that the remainder $\mathscr{R}'_j/E_j\to 0$ as before. As a consequence, repeating the blow-up procedure with $\widetilde{W}_j=\widetilde{V}_j/E_j$ converges to $\widetilde{W}=W-\psi$. In other words, upon reperforming the blow-up procedure we can assume without loss of generality that W is translated by an element ψ of $\mathscr{L}$. Thus, the right-hand side of all the previous identities and inequalities is translated by ψ. More precisely, (3.64) becomes

$$\lim_{j\to 0}\int_{B_\rho(0)}\frac{\mathscr{G}(U_j,\widetilde{\varphi}_j)^2}{E_j^2}=\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))|^2, \tag{3.69}$$

for all $\rho\in[\theta,1/4]$, whereas (3.65) is now

$$\int_{B_{\rho/2}(0)}R^{-2}\left|\frac{\partial}{\partial R}\left(\frac{W(X,\varphi^{(0)}(X))}{R^\alpha}\right)\right|^2\le C\rho^{-4-2\alpha}\int_{B_1(0)}|W(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))|^2, \tag{3.70}$$

and the bound on λ in (3.66) becomes

$$|\lambda_\psi(z)|^2 \le C\rho^{-2-2\alpha}\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))|^2, \tag{3.71}$$

while (3.67) is

$$\begin{aligned}\rho^{4+2\alpha-\sigma}\int_{B_{\rho/2}((0,z))}&\frac{|W(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))-\rho D_x\varphi^{(0)}(X)\cdot\lambda_{\psi,\rho}(z)|^2}{|X-(0,z)|^{4+2\alpha-\sigma}}\\ &\le \widetilde{C}\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))|^2,\end{aligned} \tag{3.72}$$

where the constants $C,\widetilde{C}$ depends on α and $\varphi^{(0)}$, with $\widetilde{C}$ depending on σ as well.

Step 4. There are now two cases: $\alpha\ge 1$ and $\alpha = 1/2$. We shall analyse them separately and start with the former. Before proceeding to it, recall now the notation $\psi_\rho\in\mathscr{L}$ for the homogeneous degree α map achieving the infimum

$$\inf_{\psi\in\mathscr{L}}\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))-\psi(X,\varphi^{(0)}(X))|^2,$$

so that $W-\psi_\rho$ is L^2-orthogonal to $\mathscr{L}$ on $B_\rho(0)$. Now compute

$$\begin{aligned}&\int_{B_1(0)}|\psi_\rho(X,\varphi^{(0)}(X))|^2\\ &\quad=\frac{1}{\rho^{4+2\alpha}}\int_{B_\rho(0)}|\psi_\rho(X,\varphi^{(0)}(X))|^2\\ &\quad\le\frac{2}{\rho^{4+2\alpha}}\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))|^2+\frac{2}{\rho^{4+2\alpha}}\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))-\psi_\rho(X,\varphi^{(0)}(X))|^2\\ &\quad\le\frac{4}{\rho^{4+2\alpha}}\int_{B_\rho(0)}|W(X,\varphi^{(0)}(X))|^2\le 4\theta^{-4-2\alpha},\end{aligned}$$

for all $\rho\in[\theta,1/4]$ and where we used $\int_{B_1(0)}|W(X,\varphi^{(0)}(X))|^2\le 1$. Elliptic estimates combined with homogeneity imply the uniform bound

$$\sup_{B_1(0)}|\psi_\rho(X,\varphi^{(0)}(X))|\le C\theta^{-2-\alpha},$$

again for all $\rho\in(\theta,1/4]$. Thus, by applying Proposition 94 with $\sigma=1/2$, allowed by the fact that (3.69), (3.70), (3.71) and (3.72) hold for ψ_ρ replacing ψ and with $\beta=C\theta^{-2-\alpha}$ as the bound on $\|\psi\|_{L^\infty(B_1(0))}$, we may conclude that the following bounds

$$\theta^{-4-2\alpha}\int_{B_\theta(0)}|W(X,\varphi^{(0)}(X))-\psi_\theta(X,\varphi^{(0)}(X))|^2\le C\theta^{2\mu}\int_{B_1(0)}|W(X,\varphi^{(0)}(X))|^2\le C\theta^{2\mu} \tag{3.73}$$

hold, where the last inequality follows by normalisation $\|W\|_{L^2(B_1(0))}=1$ once again. Note that both C and μ depend on α and $\varphi^{(0)}$. Therefore, for j sufficiently large we obtain as desired

$$\theta^{4-2\alpha}\int_{B_\theta(0)}\mathscr{G}(U_j,\widetilde{\varphi}_j)^2\le 2C\theta^{2\mu}E_j^2,$$

by combining (3.69), with $\rho=\theta$, and (3.73). The proof in the case $\alpha=1$ is therefore concluded.

Step 5. Let us now move to the case $\alpha = 1/2$, the delicate one requiring further results established in the previous section. From Proposition 94 we only need to check the additional hypothesis (3.40). To do so, consider the estimate in Proposition 86 with U_j, respectively φ_j, instead of U, respectively φ_j, and divide by the excess E_j. Let then $j \to \infty$ and then $\tau \to 0$ to infer

$$\int_{B_{1/2}(0)} D_j(rWD_i\varphi^{(0)})D_jD_{y_p}\zeta = 0$$

for each $q = 1,2$ and where ζ is as in the statement of the proposition. The whole proof is therefore concluded. □

We can now move to one most important results of the present work. By iterating the previous lemma, we can prove that the subset of the singular set of points of high enough frequency can be covered by a $C^{1,\mu}$-submanifold and a collection of balls with controlled radii. The former will be defined inductively requiring the first alternative of Proposition 81 to not be satisfied until a certain scale. On the other hand, the latter will consist of all the residual points.

Proposition 96. *Let $\varphi^{(0)}$ be the usual $\mathbb{Z}_2$-harmonic spinor homogeneous of degree $\alpha \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$. There are $\varepsilon, \delta_0, \mu \in (0,1)$ depending only on $\varphi^{(0)}$ such that if $U \in \mathscr{F}_{\varepsilon_0}(\varphi^{(0)})$, then*

$$\{x \in \Sigma_U \cap B_{1/2}(0);\ \mathscr{N}(x;U) \geq \alpha\} \subset S \cup T,$$

where S is contained in a properly embedded 2-dimensional $C^{1,\mu}$-submanifold Γ of $B_{1/2}(0)$ with $\mathscr{H}^2(\Gamma \cap B_{1/2}(0)) \leq \omega_2$ and $T \subset \bigcup_{j=1}^{\infty} B_{\rho_j}(x_j)$ for some family of balls $B_{\rho_j}(x_j)$ with $\sum_j \rho_j^2 \leq 1 - \delta_0$.

Proof. We shall follow the proof of Theorem 1 of [122] and refer the reader to it for further clarifications. We have decided to repeat the argument as it is instrumental for the proof of most of the theorems in the Introduction.

Start by choosing $\theta \in (0, 1/8)$ such that $C\theta^{2\mu} < 1/4$, where C and μ are as in Proposition 81. Always from Proposition 81 we have ε_0 and δ_0, where the former can be assumed, without loss of generality, to be smaller than the ε_0 appearing in Theorem 85, Corollary 87 and Theorem 91, applied with the choice of parameters

$$\gamma = \sigma = 1/2 \quad \text{and} \quad \tau = \frac{1}{100}.$$

Consider further a $\mathbb{Z}_2$-harmonic spinor $U \in \mathscr{F}_{\varepsilon}(\varphi^{(0)})$, where we require $C\varepsilon \leq \varepsilon_0$ for some constant C to be chosen later. Define then the set

$$\Sigma_U^* = \{X \in B_{1/2}(X);\ \mathscr{N}(X;U) \geq \alpha\},$$

as well as, inductively, the sets $\Upsilon_0, \Upsilon_1, \Upsilon_2, \dots, \Upsilon_\infty$ as follows. Let $Y \in \Upsilon_0$ provided $U(Y + \rho\cdot)$ satisfies the first condition of Proposition 81 for some $\rho \in [\theta, 1]$. For $j \geq 1$, define Υ_j to be the set of points $Y \in \Sigma_U^*$ such that $U(Y + \theta^i\cdot)$ does not satisfy the first condition of Proposition 81 for all $i = 0,1,2,\dots,j$, but $U(Y + \theta^{j+1}\cdot)$ does. Eventually, let Υ_∞ be the set of $Y \in \Sigma_U^*$ such that $U(Y + \theta^i\cdot)$ does not satisfy the first alternative of Proposition 81 for all $i \in \mathbb{N} \cup \{0\}$.

We now have two options, either $\Upsilon_0 \neq \emptyset$, or $\Upsilon_0 = \emptyset$. Thanks to Proposition 83 the former is trivial. Indeed, in this case we would have that $\Sigma_U^* \cap B_1(0)$ is contained in a $\delta(\varepsilon)$-neighborhood of the spine, with $\delta(\varepsilon) \to 0^+$. We would then conclude taking $S = \emptyset$ and $T = \Sigma_U^*$. The second case is more involved, we claim that the desired result follows choosing

$$S = \Upsilon_\infty \quad \text{and} \quad T = \Sigma_U^* \setminus \Upsilon_\infty.$$

Assume $Y \in \Upsilon_\infty$. Apply the second alternative of Proposition 81 to obtain

$$\theta^{-(4+2\alpha)} \int_{B_\theta(0)} \mathscr{G}(U(X+Y),\varphi_1(X))^2 \leq \frac{1}{4}\int_{B_1(0)} \mathscr{G}(U(X+Y),\varphi_0(X))^2 + \frac{1}{4}\int_{B_1(0)} |U(X+Y)|^2.$$

Introducing then the notation $\overline{U}(\cdot) = U(\cdot\theta)/\theta^\alpha$, we can rewrite the left-hand side of the above as $\int_{B_1(0)} \mathscr{G}(\overline{U}(X+Y),\varphi_1(X))^2$, so that we have a quantity defined over the ball $B_1(0)$ and we can reapply the same proposition. Thus, we can find φ_2 satisfying

$$\theta^{-(4+2\alpha)} \int_{B_\theta(0)} \mathscr{G}(\overline{U}(X+Y),\varphi_2(X))^2 \leq \frac{1}{4}\int_{B_1(0)} \mathscr{G}(\overline{U}(X+Y),\varphi_1(X))^2 + \frac{1}{4}\int_{B_1(0)} |\overline{U}(X+Y)|^2,$$

meaning that we have an inequality of the form

$$\theta^{-(4+2\alpha)2} \int_{B_{\theta^2}(0)} \mathscr{G}(U(X+Y),\varphi_2(X))^2 \leq \frac{1}{4}\theta^{-(4+2\alpha)}\left(\int_{B_\theta(0)} \mathscr{G}(U(X+Y),\varphi_1(X))^2 + \int_{B_\theta(0)} |U(X+Y)|^2\right),$$

Applying then inductively this inequality for each i, we can find $\varphi_i \in \widetilde{\Phi}_{\gamma\varepsilon_0}(\varphi^{(0)})$ satisfying

$$\theta^{-(4+2\alpha)i} \int_{B_{\theta^i}(0)} \mathscr{G}(U(X+Y),\varphi_i(X))^2 \leq \frac{1}{4}\theta^{-(4+2\alpha)(i-1)}\left(\int_{B_{\theta^{i-1}}(0)} \mathscr{G}(U(X+Y),\varphi_{i-1}(X))^2 + \int_{B_{\theta^{i-1}}(0)} |U(X+Y)|^2\right),$$

where X is the variable of integration. Let $\varphi_0 = \varphi^{(0)}$. Interatively applying this inequality we obtain

$$\begin{aligned}\theta^{-(4+2\alpha)i} &\int_{B_{\theta^i}(0)} \mathscr{G}(U(X+Y),\varphi_i(X))^2 \\ &\leq \frac{1}{4^i}\int_{B_1(0)} \mathscr{G}(U(X+Y),\varphi^{(0)}(X))^2 + \sum_{j=1}^{i}\frac{1}{4^j}\theta^{-(4+2\alpha)(i-j)}\int_{B_{\theta^{i-j}}(0)} |U(X+Y)|^2 \\ &\leq \frac{1}{4^i}\varepsilon^2 + \sum_{j=1}^{i}\frac{1}{4^j}\int_{B_1(0)} |U(X+Y)|^2 \leq C\varepsilon^2,\end{aligned} \tag{3.74}$$

where the second inequality follows from (3.17) and the last one by definition. Consequently, we infer

$$\int_{B_1(0)} \mathscr{G}(\varphi_i,\varphi_{i-1})^2 \leq \frac{C}{4^i}\varepsilon^2, \tag{3.75}$$

thanks to the triangle inequality, as well as

$$\int_{B_1(0)} \mathscr{G}(\varphi_i,\varphi^{(0)})^2 \leq C\varepsilon^2 < \varepsilon_0^2. \tag{3.76}$$

Note that both constants depend on α and $\varphi^{(0)}$ and both inequalities follow from the triangle one. Consequently, $\varphi_i \in \widetilde{\Phi}_{\varepsilon_0}(\varphi^{(0)})$ for all i and by virtue of the above estimates we can find a limit $\varphi_Y \in \widetilde{\Phi}_{\varepsilon_0}(\varphi^{(0)})$ such that φ_i converges to it in L^2 on the ball $B_1(0)$. Moreover, because of (3.74) there hold

$$\int_{B_1(0)} \mathscr{G}(\varphi_i,\varphi_Y)^2 \leq C\frac{\varepsilon^2}{4^i},$$

as well as

$$\theta^{-(4+2\alpha)i}\int_{B_{\theta^i}(0)} \mathscr{G}(U(X+Y),\varphi_Y(X))^2 \leq C\frac{1}{4^i}\varepsilon^2,$$

due to (3.75). Given an arbitrary $\rho\in(0,1]$ let i be such that $\theta^{i-1}<r\leq\theta^i$, so that we can then deduce

$$r^{-4-2\alpha}\int_{B_\rho(0)} \mathscr{G}(U(X+Y),\varphi_Y(X))^2 \leq C\rho^{2\mu}\varepsilon^2, \tag{3.77}$$

where $\mu=-\log(\theta)/\log(2)$. This estimate implies in particular uniqueness of φ_Y for every $Y\in\Upsilon_\infty$. Furthermore, because of (3.76) we have the existence of a rotation q_Y of $\mathbb{R}^4$ such that $\varphi_Y(q_Y\cdot)\in\Phi_{\varepsilon_0}(\varphi^{(0)})$ and $|q_Y-\mathrm{Id}|\leq C\varepsilon$. Combining this with Corollary (88) and (3.77) we have

$$\frac{1}{r^{1+\mu}}\operatorname{dist}\left(q_Y(\Sigma_U^*-Y)\cap B_r(0),\{0\}\times\mathbb{R}^2\right)\leq C\varepsilon, \tag{3.78}$$

yielding, based on a Morrey decay type of argument, the desired regularity for the projections: $|q_Y-q_Z|\leq C\varepsilon|Y-Z|^\mu$, for every $Y,Z\in\Upsilon_\infty$. Consequently, we conclude that

$$S=\Upsilon_\infty=\operatorname{graph}(f)\cap B_{1/2}(0),$$

where $\|f\|_{C^{1,\mu}(B^2_{1/2}(0))}\leq C\varepsilon$. The $\mathscr{H}^2$-bound follows directly from the graph structure.

We shall now move to the second part of the statement, namely covering T. By definition of T we can assume that $Y\in T$ belongs to some $Y\in\Upsilon_j$ for $j\in\mathbb{N}\cup\{0\}$. Applying the second alternative of Proposition 81 gives $\varphi_i\in\widetilde{\Phi}_{\gamma\varepsilon_0}(\varphi^{(0)})$ for $i=1,2,\dots,j$ and $\varphi_0=\varphi^{(0)}$, as well as (3.74) up to j. Taking $\varphi_Y=\varphi_j$ and arguing as before we obtain (3.78) for $\rho\in[\theta^j,1]$. By virtue of the bound

$$|q_Y-\mathrm{Id}|\leq C\varepsilon$$

we obtain

$$\frac{1}{\rho}\operatorname{dist}\left(\Sigma_U^*\cap B_\rho(Y),Y+\{0\}\times\mathbb{R}^2\right)\leq C\varepsilon,$$

for $\rho\in[\theta^j,1]$. Now note that for every $Y\in\Upsilon_j$ there exists $Z\in Y+(\{0\}\times B^2_{\theta^{j+1}}(0))$ such that $B_{\delta_0\theta^{j+1}}(Z)\cap\Sigma_U^*=\emptyset$, where δ_0 is as in Proposition 81. The end of the proof follows precisely like in [122, Theorem 1]. Crucial is the application of [122, Lemma 2.7], in which the author establishes a sufficient condition for a subset F of the unit hypercube Q_0 in $\mathbb{R}^4$, contained in a collection of subcubes of Q_0, to be contained in a related collection of subcubes satisfying some additional properties, the first of which is related to the requirement on the radii $\sum_j\rho_j^2\leq1-\delta_0$ appearing in the statement. The other important steps of the end of the proof are covering by cubes the sets Υ_j, applying the aforementioned lemma, and then relating the collection of cubes to the desired collection of balls. This gives a covering of $\bigcup_{k\in\mathbb{N}}\Upsilon_k$ by a collection of balls $\{B_{\rho_j}(X_j)\}$ satisfying the required bound on the radii. □

We can now give the proof of Theorem 11 from Section 1.3 of the Introduction. We repeat the statement for the reader's convenience.

Theorem 11 (Uniqueness of blow ups)**.** *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g). Then, for $\mathscr{H}^2$-a.e. point Z in the branch set $\mathscr{B}_U$ of U, there exist an orthogonal rotation $q_Z \in \mathrm{SO}(4)$, a positive integer $k_Z \in \mathbb{N}$, and a positive real number $r_Z > 0$ such that for every $X \in B_{r_Z}(Z)$, we have*

$$U(Z + q_Z X) = \{f(X) + h_Z(X), -f(X) - h_Z(X)\}, \tag{1.21}$$

where the error term $h_Z(X)$ satisfies the bound

$$\frac{1}{\sigma^4}\int_{B_\sigma(Z)} |h_Z|^2 \leq C_Z \sigma^{k_Z+\gamma_Z},$$

for all $\sigma \in (0, r_Z)$ and constants $C_Z, \gamma_Z > 0$ independent of σ and where f is given by $\{\pm(c_1 z^\alpha + c_2 \bar{z}^\alpha)\}$, for constants c_1 and c_2.

Proof of Theorem 11. Consider a harmonic $\mathbb{Z}_2$-spinor U and note that for $\mathscr{H}^2$-a.e. point in $\mathscr{B}_U$ there exists a harmonic $\mathbb{Z}_2$-spinor $\varphi^{(Z)}$, homogeneous of some degree, and cylindrical. Furthermore, there exists a number $r_Z \in (0, 1/2)$ such that for the corresponding choice of $\theta \in (0, 1/8)$ we can apply Proposition 81 with $\varphi^{(Z)}$ in place of $\varphi^{(0)}$ and

$$V_{Z,r_Z} = \frac{V(Z + r_Z\cdot)}{\|V(Z + r_Z\cdot)\|_{L^2(B_1(0))}},$$

instead of U. We can rule out the first alternative of Proposition 81 with $\varphi^{(Z)}$ in place of $\varphi^{(0)}$ and $V_{Z,r_Z}(\theta^j\cdot)$ instead of U for each $j = 0, 1, 2, \ldots$. Consequently, arguing as in the proof of Proposition 81 we conclued the desired uniqueness of the blow-ups $\widetilde{\varphi}^{(Z)}$, where we note that, up to a rotation factor, the unique blow up is given by (3.18). The L^2-estimates are a consequence of Proposition 81. □

The proof of Theorem 12 from Section 1.3 of the Introduction now follows. We repeat the statement for the reader's convenience.

Theorem 12 (Rectifiability of the singular set)**.** *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g). Then Σ_U is countably 2-rectifiable. Furthermore, for every compact set K we have*

$$\Sigma_U \cap K \cap \{X;\, \mathscr{N}(X;V) = \alpha \text{ and } V \text{ has a cylindrical blow up at } X\} \neq \emptyset$$

only if α takes one of finitely many values belonging to the set $\mathbb{N} \cup \frac{1}{2}\mathbb{N}$. For each $\alpha \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$, there is an open set O_α such that

$$\{X;\, \mathscr{N}(X;V) = \alpha \text{ and } V \text{ has a cylindrical blow up at } X\} \subset O_\alpha,$$

and $O_\alpha \cap \{X;\, \mathscr{N}(X;V) \geq \alpha\}$ has locally finite $\mathscr{H}^2$-measure.

Proof of Theorem 12. We start by noticing that Federer's dimension reducing implies

$$\dim_{\mathscr{H}} \Sigma_U^{(1)} \leq 1,$$

so that we only have to consider elements Y of the singular set Σ_U satisfying the requirement that at least one blow-up φ_0 at them is cylindrical, i.e. $\dim S(\varphi) = 2$. Denote such set Σ_U^*. Thus, consider one such point Y_0 and a cylindrical blow-up at it, φ. For every $\varepsilon > 0$, we have the existence of $\sigma = \sigma(\varepsilon) > 0$ such that $B_{R(\varepsilon)(0)} \subset B_{(1-|Y_0|)/\sigma}(Y_0)$, together with

$$\int_{B_1(0)} \mathscr{G}(U_{Y_0,\sigma}, \varphi_0)^2 \le \varepsilon^2 \quad \text{and} \quad N(0,R(\varepsilon);U_{Y_0,\sigma}) + CR(\varepsilon)^2 - \alpha < \delta(\varepsilon), \tag{3.79}$$

where R and δ are as Lemma 80. Define then the outer measure

$$\mu_{\rho_0}(A) = \inf \sum_{j=1}^{N} \omega_2 \sigma_i^2,$$

where the infimum is taken over finite covers by balls of radius bounded above by ρ_0. Introduce now the set $\Sigma_\alpha^+ = \Sigma_{U_{Y_0,\sigma}} \cap \{Y;\ \mathscr{N}(Y;U_{Y_0,\varphi_0}) \ge \alpha\}$ and notice that it is closed by upper-semicontinuity of the frequency function, cf. Lemma 70. Choose then a finite collection of balls covering the set $\Sigma_\alpha^+ \cap \overline{B}_1(0)$ and satisfying additionally

$$\sum_{j=1}^{N} \omega_2 \sigma_i^2 \le \mu_{\rho_0}(\Sigma_\alpha^+) + 1.$$

Assume without loss of generality that $B_{\sigma_i}(Y) \cap \Sigma_\alpha^+ \neq \emptyset$ for all $i = 1,2,\ldots,N$, indeed, if it were not the case we could discard them from the collection. Applying again Proposition 80 we infer the following dichotomy: either there exists a homogeneous of degree α, harmonic $\mathbb{Z}_2$-spinor φ_i with $\dim S(\varphi_i) = 2$ and

$$\int_{B_1(0)} \mathscr{G}\left((U_{Y_0,\sigma})_{Y_i,2\rho_i}, \varphi_i\right)^2 < \varepsilon^2$$

or

$$\{X \in \Sigma_{U_{X_0,\sigma}} \cap \overline{B}_{Y_i,2\rho_i};\ \mathscr{N}(X;U_{Y_0,\sigma}) \ge \alpha\} \subset \{X;\ \operatorname{dist}(X, Y_i + L) < \varepsilon\},$$

for a certain 1-dimensional subspace L. All of the above holds provided $\varepsilon > 0$ and for all $i = 1,2,\ldots,N$. We can now conclude applying Lemma 79 and Proposition 96 precisely like in [122]. □

From this theorem it readily follows Corollary 13 from Section 1.3 of the Introduction. We repeat the statement for the reader's convenience.

Corollary 13 (Rectifiability of the branch set)**.** *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional manifold Riemannian (M,g). Let B_U be the branch set of U. Then, for each closed ball $B \subset M$, either $B \cap \mathscr{B}_U$ is empty or $B \cap \mathscr{B}_U$ has positive 2-dimensional Hausdorff measure and is equal to the union of a finite number of pairwise disjoint, locally compact sets each of which is locally 2-rectifiable. In particular, it has locally finite 2-dimensional Hausdorff measure. Moreover, requiring U to be of class $C^{1,\mu}$, for a certain μ, we conclude that either $\mathscr{B}_U = \emptyset$ or the Hausdorff dimension of $\mathscr{B}_U$ is 2 and the 2-dimensional Hausdorff measure of $\mathscr{B}_U$ is positive.*

Proof of Corollary 13. Let B be a closed ball in (M,g). By Theorem 12 there exists a finite set of homogeneities $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset \mathbb{N} \cup \frac{1}{2}\mathbb{N}$ such that if a point $Z \in \Sigma_U \cap B$ satisfies the requirement that V has a cylindrical blow-up at it, then $\mathscr{N}(Z;V) = \alpha_j$ for some $j \in \{1,2,\ldots,k\}$. Let now O_{α_j} be the open set given by Theorem 12. Set then $\alpha_0 = 1/2$ and $\alpha_{k+1} = \infty$, as well as

$$\Gamma_j = \{Z \in B \cap \Sigma_U;\ \alpha_j \le \mathscr{N}(Z;V) < \alpha_{j+1}\} \cap O_{\alpha_j}$$

and

$$\widetilde{\Gamma}_j = \{Z \in B \cap \Sigma_U;\ \alpha_j \le \mathscr{N}(Z;V) < \alpha_{j+1}\} \setminus O_{\alpha_j}.$$

Theorem 12 implies that that Γ_j is locally 2-rectifiable and has locally finite $\mathscr{H}^2$-measure and, by Federer's dimension reducing argument we infer $\dim_{\mathscr{H}} \widetilde{\Gamma}_j \leq 1$. Furthermore, we trivially have

$$B \cap \Sigma_U = \bigcup_{j=0}^{k} \Gamma_j \cup \widetilde{\Gamma}_j,$$

which, combined with the fact that the frequency function $\mathscr{N}(\cdot, V)$ is upper semicontinuous implies that all of the Γ_j and $\widetilde{\Gamma}_j$ are the intersection of a closed and open set, hence locally compact; gives the desired result.

To conclude we only need to prove the last dichotomy result in case U is more regular. Note that if $\mathscr{H}^2(\mathscr{B}_U) = 0$, then for any geodesic ball $B_r(q)$ we have that $B_r(q) \setminus \mathscr{B}_U$ is simply connected. This follows from a more general result for closed sets, whose proof can be found in [125, Appendix]. Consequently, $U|_{B_r(q)\setminus \mathscr{B}_U} = \pm u$ for a pair of smooth solutions to the Dirac equation u and $-u$. As $\mathscr{H}^3(\mathscr{B}_U) = 0$, combined with the fact that U is $C^{1,\mu}$ for a certain μ, we have that both u and $-u$ are in fact weak solutions of the Dirac equation on the whole $B_r(q)$. In particular, they are smooth strong solutions of $\mathscr{D}v = 0$, thus implying $B_r(q) \cap \mathscr{B}_U = \emptyset$. Repeating the same argument for every point of M implies as required that $\mathscr{B}_U = \emptyset$. □

We can now move to the proof of Theorem 14 from Section 1.3 from the Introduction. We repeat the statement for the reader's convenience.

Theorem 14. *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M, g). If $\mathscr{N}(Z;U) = 3/2$, then $Z \in \mathscr{B}_U$ and there exists $r > 0$ such that $\mathscr{B}_U \cap B_r(Z)$ is an 2-dimensional $C^{1,\alpha}$-submanifold of $B_r(Z)$, for some $\alpha \in (0,1)$.*

Proof of Theorem 14. Let $\varphi^{(0)}$ be the usual harmonic $\mathbb{Z}_2$-spinor, homogeneous of degree α. Note that by Corollary 72, the minimum degree possible is $1/2$, precisely like in the Dirichlet minimizing case. We claim that for each $\delta \in (0, 1/2)$, there exists $\varepsilon = \varepsilon(n, m, \delta, \varphi^{(0)}) \in (0, 1/2)$ such that if $U \in \mathscr{F}_\varepsilon(\varphi^{(0)})$, then

$$\{X; \mathscr{N}(X;U) \geq 3/2\} \cap B_\delta(0,y) \neq \emptyset, \quad \text{for each } y \in B_1^2(0). \tag{3.80}$$

Note that we may choose $\varepsilon > 0$ so that $\mathscr{B}_U \cap B_\delta(0,y) \neq \emptyset$ for every $y \in B_1^2(0)$. We then infer

$$\mathscr{H}^2(\mathscr{B}_U \cap B_\delta(0,y)) > 0, \quad \text{for each } y \in B_1^2(0), \tag{3.81}$$

from which we deduce as desired the condition in (3.80). To see why (3.81) follows we argue by contradiction. Assume it is not the case, so that we can find $y \in B_1^2(0)$ and $Z \in \mathscr{B}_U \cap B_\delta(0,y)$ such that for $r \in (0, \operatorname{dist}(Z, \partial B_\delta(0,y)))$ the set $B_r(Z) \setminus \mathscr{B}_U$ is simply connected, so that the restiction of U to $B_r(Z)$ is given by two single-valued smooth solutions of the Dirac equation, giving a contradition. The fact that $B_r(Z) \setminus \mathscr{B}_U$ would be simply connected follows from the following standard fact: if closed subset Γ of $\mathbb{R}^n$ is such that $\mathscr{H}^{n-2}(\Gamma) = 0$, then $B_1 \setminus \Gamma$ is simply connected. For a proof of this we refer the reader to [125, Appendix]. In particular, this is true in our case where $n = 4$.

Consider now $Z \in B_2(0)$, a point with minimal homogeneity $\mathscr{N}(Z;U) = 3/2$ and let $\varphi^{(Z)}$ be any blow-up at it. Because $\mathscr{N}(Z;U)$ is not an integer we have that $Z \in \mathscr{B}_{\varphi^{(Z)}}$, so that $\mathscr{H}^2(\mathscr{B}_{\varphi^{(Z)}} \cap B_1(0)) > 0$ by the same simple connectivity argument as above. Thus, we also have

$$\mathscr{H}^2\left(\{X;\ \mathscr{N}(X;\varphi^{(Z)}) \geq 3/2\} \cap B_1(0)\right) > 0,$$

as well as

$$\mathscr{H}^2\left(\{X;\ \mathscr{N}(X;\varphi^{(Z)})=3/2\}\cap B_1(0)\right)>0,$$

due to the identity $\mathscr{N}(X;\varphi^{(Z)})\leq\mathscr{N}(0;\varphi^{(Z)})=3/2$. In other words, we can find 2 linearly independent vectors $X_1,X_2\in B_1(0)$ such that $3/2=\mathscr{N}(0;\varphi^{(Z)})=\mathscr{N}(X_j;\varphi^{(Z)})$, from we deduce that the blow-up $\varphi^{(Z)}$ is cylindrical. Choosing then θ small enough and taking δ as in the previous paragraph we can now proceed as in Proposition 96 to obtain as desired that $\mathscr{B}_U$ is a 2-dimensional $C^{1,\alpha}$-submanifold in small enough neighborhood of Z, for a certain $\alpha\in(0,1)$. Note that the claims in the previous paragraph are used to rule out the first alternative of Proposition 81 so that the second alternative can be inductively applied. We thus conclude that $\mathscr{B}_U$ is a 2-dimensional $C^{1,\alpha}$-submanifold near Z. □

From this theorem it follows Corollary 15 from Section 1.3 from the Introduction. We repeat the statement of the Corollary for the reader's convenience.

Corollary 15. *Let U be a C^1-regular, $\mathbb{Z}_2$-harmonic spinor on a 4-dimensional Riemannian manifold (M,g) and $k_{Z_0}=1$ for some point $Z_0\in\mathscr{B}_U$ at which the asymptotic expansion (1.21) is valid. Then, the same expansion is valid for every $Z\in\mathscr{B}_U\cap B_{r_{Z_0}/2}(Z_0)$, where r_{Z_0} is as in Theorem 11, with the parameters*

$$k_Z=k_{Z_0},\quad C_Z=C_{Z_0},\quad \gamma_Z=\gamma_{Z_0},\quad \textit{and}\quad r_z=r_{Z_0}/4.$$

Furthermore, the error term in the expansion (1.21) satisfies

$$\sup_{B_\sigma(Z_0)}|h_Z|^2\leq C_0\sigma^{k_{Z_0}+\gamma_{Z_0}/8},$$

for every $Z\in\mathscr{B}_U\cap B_{r_{Z_0}/2}(Z_0)$ and $\sigma\in(0,r_{Z_0}/4)$ and where the constant C_0 is independent of Z and σ. Moreover, we have that $\mathscr{B}_U\cap B_{r_{Z_0}/2}(Z_0)$ is a 2-dimensional $C^{1,\alpha}$-submanifold for some $\alpha=\alpha_{Z_0}\in(0,1)$.

Proof of Corollary 15. Let Z_0 as in the statement of the Corollary. From the proof of Theorem 15 we infer the existence of $r_{Z_0}>0$ such that $\mathscr{B}_U\cap B_{r_{Z_0}}(Z_0)$ is a 2-dimensional $C^{1,\alpha}$-submanifold, and for each $Z\in\mathscr{B}_U\cap B_{r_{Z_0}}(Z_0)$ the conclusions of Theorem 11 hold with the following parameters

$$k_Z=k_{Z_0}=1,\quad C_Z=C_{Z_0},\quad \gamma=\gamma_{Z_0},\quad \text{and}\quad r_Z=r_{Z_0}/4.$$

In particular, for each $Z\in\mathscr{B}_U\cap B_{r_{Z_0}/2}(Z_0)$ we have

$$\frac{1}{\sigma^4}\int_{B_\sigma(0)}|h_Z|^2\leq C\sigma^{k_{Z_0}+\gamma_{Z_0}},\tag{3.82}$$

where the constant C depends on Z_0 and $\sigma\in(0,r_{Z_0})$. To conclude the proof of the corollary we have to establish the desired L^∞-estimate for h_Z. Fix a point $Z\in\mathscr{B}_U\cap B_{r_{Z_0}/2}(Z_0)$ and let $U_Z(X)=U(Z+q_ZX)$, where q_Z is a rotation of $\mathbb{R}^n$. From Corollary 88 and (3.82) we deduce

$$\mathscr{B}_{U_Z}\cap B_{\sigma/2}(0)\subset\left\{X;\ \operatorname{dist}(X;\{0\}\times\mathbb{R}^2)\leq C_{Z_0}\sigma^{1+\gamma_{Z_0}/2}\right\}.$$

Thus, if $X\in B_{\sigma/2}(0)$ and $\operatorname{dist}(X,\{0\}\times\mathbb{R}^2)>\sigma^{1+\gamma_{Z_0}/8}=:\delta$, then $\mathscr{B}_{U_Z}\cap B_{\delta/2}(X)=\emptyset$, provided r_{Z_0} is sufficiently small. Consequently, we have the estimates

$$|h_Z(X)|^2\leq C\frac{1}{\delta^4}\int_{B_{\delta/2}(X)}|h_Z|^2\leq C\left(\frac{\sigma}{\delta}\right)^4\sigma^{-4}\int_{B_\sigma(0)}|h_Z|^2\leq C(n)C_{Z_0}\sigma^{k_{Z_0}+\gamma_{Z_0}/2}.$$

If on the other hand, $X \in B_{\sigma/2}(0)$ and $\operatorname{dist}(X,\{0\}\times\mathbb{R}^2) \leq \delta$, then, with $r = \frac{1}{2}\operatorname{dist}(X,\mathscr{B}_{U_Z})$, we have

$$|h_Z(X)|^2 \leq 2|U_Z(X)|^2 + C_1 \operatorname{dist}(X,\{0\}\times\mathbb{R}^2)^{k_Z} \leq \frac{C}{r^2}\int_{B_r(X)} |U_Z|^2 + C_1\delta^{k_{Z_0}}$$

where we applied Theorem 11 and the elliptic estimates for $\mathbb{Z}_2$-harmonic spinors appearing in Lemma 61. Note that the constant C depends on the dimension of the manifold, whereas C_1 depends on Z_0. Choosing $Y \in \mathscr{B}_{U_Z}$ realising the distance $\operatorname{dist}(Y,\mathscr{B}_{U_Z}) = |X-Y|$ and invoking (3.17) we deduce

$$\frac{1}{r^n}\int_{B_r(X)} |U_Z|^2 \leq \frac{1}{r^4}\int_{B_{2r}(Y)} |U_Z|^2 \leq 2^4\left(\frac{8r}{r_{Z_0}}\right)^{k_{Z_0}}\left(\frac{r_{Z_0}}{4}\right)^{-4}\int_{B_{r_{Z_0}/2}(0)} |U_Z|^2.$$

Note now that we have

$$\int_{B_{r_{Z_0}}(0)} |v_Z|^2 \leq \int_{B_{r_{Z_0}}(Z_0)} |U-h_Z|^2, \tag{3.83}$$

as well as $r \leq \frac{1}{2}\operatorname{dist}(X,\{0\}\times\mathbb{R}^2)$. Combining all of the above we obtain as required

$$|h_Z(X)| \leq C\delta^{k_{Z_0}} = C\sigma^{k_{Z_0}(1+\gamma_{Z_0}/8)} \leq C\sigma^{k_{Z_0}+\gamma_{Z_0}/8},$$

where the constant C does not depend on σ and the point Z. □

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